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Mapping of eigenvalue performance specifications by real root classification

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***Abstract*— This paper provides a method based on real root classification (RRC) for computation of parameter space regions for LTI systems, guaranteeing certain system performance specifications described by the eigenvalue location. To this end, parameter-dependent state-space system descriptions are considered and the Lyapunov equation is utilized as the basic vehicle to derive the mapping conditions.**

I. INTRODUCTION

This paper provides a slight extension of the method presented in [5] for computation of parameter regions, which – in addition to stability – ensure fixed performance levels as prescribed by the location of the eigenvalues for continuous- or/and discrete-time LTI systems. To this end, the approach presented in [4] based on solving the Lyapunov equation for state-space models serves as the basic vehicle. The preceding paper [5] suggests formulation of the mapping problem as a semi-algebraic system and solving it by the "real root classification" (RRC) computer algebra technique. The present paper extends this approach to mapping the performance specifications. As a further related work, we refer to [3], where the same problem has been studied - though by applying a different mapping procedure, yet, serving as an inspiration for the present one. An advantage of the RRC computational technique is the automatic generation of the mapping boundaries in the symbolic form.

The paper is organized as follows. Section II recaps the basic ideas of investigating feasibility of real Lyapunov equations by means of the RRC, e.g. as discussed in [5]. Section III introduces special modifications of the system matrix which guarantee specific shifting of eigenvalues to a certain region in the complex plane. Section IV discusses the solution methods for the SAS, and, finally, in Section V a numerical example is given.

II. LYAPUNOV APPROACH

The problem under discussion regards the computation of the regions in parameter space of a parameter-dependent LTI system, guaranteeing its stability. This problem is e.g. considered in [5] by using Lyapunov equations in the following manner. Let

$$\dot{\mathbf{x}} = A(\mathbf{k})\mathbf{x} \tag{1}$$

be a parametric LTI system, where

$$\mathbf{x} = \{x_1, x_2, \ldots, x_s\} \in \mathbb{R}^s, \mathbf{k} = \{k_1, k_2, \ldots, k_d\} \in \mathbb{R}^d$$

stand for the state and parameters, respectively. It is an elementary fact, that the Lyapunov-stability for these kind of systems can be determined by checking the feasibility of the corresponding Lyapunov equation. The system (1) is asymptotically stable, if and only if for any symmetric positive definite matrix Q, a positive definite matrix solution P exists to the matrix equation:

$$A(\mathbf{k})^T P + PA(\mathbf{k}) = -Q. \tag{2}$$

The idea consists now in transforming this equation into a semi-algebraic system (SAS). Semi-algebraic systems are defined by a system of s equations, t inequalities and r strict inequalities in the forms:

$$\begin{cases} f_1(\mathbf{u},\mathbf{x}) = 0, f_2(\mathbf{u},\mathbf{x}) = 0, \ldots, f_s(\mathbf{u},\mathbf{x}) = 0, \\ g_1(\mathbf{u},\mathbf{x}) \geq 0, g_2(\mathbf{u},\mathbf{x}) \geq 0, \ldots, g_t(\mathbf{u},\mathbf{x}) \geq 0, \\ h_1(\mathbf{u},\mathbf{x}) > 0, h_2(\mathbf{u},\mathbf{x}) > 0, \ldots, h_r(\mathbf{u},\mathbf{x}) > 0, \end{cases} \tag{3}$$

where each of the terms f, g and h is a real multivariate polynomial in term of the arguments $\mathbf{u}$ (standing for the parameters) and $\mathbf{x}$ (standing for the variables). Observe that the number s of the equality constraints matches the dimension of the unknown variables $\mathbf{x}$. In order to transform the matrix equation (2) into a SAS, it is rewritten in the form $V(\mathbf{k}, P) \overset{!}{=} 0$, with P and $\mathbf{k}$ as variables and parameters, respectively. For simplicity, Q can be replaced by a unit matrix $\mathbb{I}$, yielding

$$V(\mathbf{k}, P) := A(\mathbf{k})^T P + PA(\mathbf{k}) + \mathbb{I}. \tag{4}$$

It is now easy to see that each entry v_{ij} of the matrix $V(\mathbf{k}, P)$ is a polynomial with real coefficients. Thus, for a stable LTI system, a P must exist, such that the polynomials $v_{ij} = 0$ are feasibly, under the additional constraint that P must be positive definite. Positive definiteness of P requires that all leading principle minors $\mu_i(P)$ of P to be strictly greater than zero. Thus, (2) can be converted into the following SAS:

$$\begin{cases} v_{11}(\mathbf{k}, P) = 0, \ldots, v_{1n}(\mathbf{k}, P) = 0 \\ \vdots \qquad\qquad\qquad \vdots \\ v_{n1}(\mathbf{k}, P) = 0, \ldots, v_{nn}(\mathbf{k}, P) = 0 \\ \mu_1(P) > 0, \ldots, \mu_n(P) > 0. \end{cases} \tag{5}$$

Note that the the constructed SAS has the same number of variables as equations, which will be important for solving the SAS, as covered in Section IV.

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If we are to generate a SAS, whose solution guarantees the eigenvalues of the system matrix to lie within a unit circle, then we need to consider the discrete Lyapunov equation:

$$A(\mathbf{k})^T P A(\mathbf{k}) - P = -Q. \tag{6}$$

Again, the corresponding matrix equation $V(\mathbf{k}, P) = A(\mathbf{k})^T P A(\mathbf{k}) - P + \mathbb{I} = 0$ in conjunction with the conditions for the positive definiteness of the matrix P need to be expressed in the form of a SAS (5).

III. Modifying the system matrix

A. Real part specification

To change the real part of all eigenvalues of the system matrix A, the latter is modified to:

$$\widehat{A} = A + \alpha\mathbb{I}. \tag{7}$$

Indeed, with $\mathbf{v}$ being the eigenvector corresponding to the eigenvalue λ, we have:

$$\widehat{A}\mathbf{v} = (A + \alpha\mathbb{I})\mathbf{v} = (\lambda + \alpha)\mathbf{v} = \widehat{\lambda}\mathbf{v}. \tag{8}$$

Thus, it is evident that under this modification, the eigenvalues of $\widehat{A}$ are given by shifted eigenvalues of A by a real number α.

B. The magnitude specification

To scale the eigenvalues of the system matrix by $\rho > 0$, the latter is modified to:

$$\widehat{A} = \frac{1}{\rho}A. \tag{9}$$

Again, let $\mathbf{v}$ be the eigenvector to some eigenvalue λ. Then

$$\widehat{A}\mathbf{v} = \left(\frac{1}{\rho}A\right)\mathbf{v} = \frac{\lambda}{\rho}\mathbf{v} = \widehat{\lambda}\mathbf{v}, \tag{10}$$

this revealing a scaling of eigenvalues of A by a centric aspect ratio $\frac{1}{\rho}$ in the complex frequency domain, i.e. $\widehat{\lambda} = \lambda/\rho$. Note that this modification of the system matrix is relevant to both, continuous- (regarding the bandwidth constraints) and discrete-time systems (regarding the response settling-time). As we are rather focusing on the continuous-time case in the present paper, the modification (9) invokes utilization of the SAS stemming from the discrete Lyapunov equation (6). Then, the solution of the underlying "discrete" SAS w.r.t. $\widehat{A}(\mathbf{k})$ will guarantee that the eigenvalues λ of the original matrix $A(\mathbf{k})$ will lie within a circle with radius ρ (see Fig. 4).

C. Damping ratio specification

Complex eigenvalues sharing the same damping ratio lie on the same straight line passing through the origin. To characterize all eigenvalues that share at least a damping ratio, introduce the θ-sector to be the set of all eigenvalues $\hat{\lambda}$ with $\arg(\hat{\lambda}) \in [\pi/2+\theta, 3\pi/2-\theta]$. Now pick any point on the upper straight halfline of the θ-sector, originatig at the origin at an angle $\pi/2+\theta$. Then the matrix modification, [3]

$$\widehat{A} = e^{-j\theta}A = \cos\theta A - j\sin\theta A \tag{11}$$

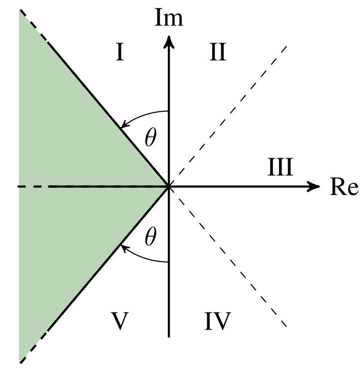

Fig. 1. Different characteristic areas of the s-plane after clockwise rotation of the eigenvalues around the origin by the angle θ.

maps it into a rotated eigenvalue $\widehat{\lambda} = e^{-j\theta}\lambda$ of $\widehat{A}$ lying onto the imaginary axis. Notice that the shaded region in Fig. 1 will rotate by an angle θ in the clockwise direction. In [3] a slightly different modification $\widehat{A} = A - j\tan\theta\, A$ has been suggested. Notice that the latter includes a combined rotation and scaling.

It is important to observe that the Hurwitz stability of the matrix $\widehat{A}(\mathbf{k}) := e^{-j\theta}A(\mathbf{k})$ implies that all eigenvalues of $A(\mathbf{k})$ lie within the θ-sector. Indeed, any original eigenvalues of $A(\mathbf{k})$ lying in the region I and V (see Fig. 1) are denied by the Hurwitz stability of $\widehat{A}(\mathbf{k})$, as after the rotation they enter II. Also, after the rotation no eigenvalues in V will exist, as they are again denied by the Hurwitz stability of $\widehat{A}(\mathbf{k})$. This is inferred from the fact that the corresponding complex-conjugates lying originally in II would enter III.

In contrast to the other cases discussed above, the modified system matrix possesses also complex entries. Therefore, a complex Lyapunov equation has to be taken into the account:

$$A^*(\mathbf{k})P + PA(\mathbf{k}) = -Q, \tag{12}$$

where P and Q are now positive definite hermitian matrices with corresponding dimensions and A^* is the transposed complex conjugate of A. In this case, the real and imaginary parts of this Lyapunov equation must be considered separately. Thus, with $P = P_R + jP_I$, where $P_R = P_R^T$ and $P_I = -P_I^T$ are real matrices we have:

$$\begin{aligned} -Q &= \widehat{A}^*(\mathbf{k})P + P^*\widehat{A}(\mathbf{k}) \\ &= (\cos\theta\, A^T(\mathbf{k}) + j\sin\theta\, A^T(\mathbf{k}))\,(P_R + jP_I) \\ &\quad + (P_R + jP_I)^*\,(\cos\theta\, A(\mathbf{k}) - j\sin\theta\, A(\mathbf{k})). \end{aligned}$$

Next, proceed in the same way as in Section II. First, Q is replaced by a unit matrix $\mathbb{I}$. The resulting expression can be then written as $V(\mathbf{k}, P) = V_R(\mathbf{k}, P) + jV_I(\mathbf{k}, P)$, with

$$\begin{aligned} V_R(\mathbf{k}, P) &= \mathbb{I} + \cos\theta A^T(\mathbf{k})P_R - \sin\theta\, A^T(\mathbf{k})P_I \\ &\qquad + (\cos\theta A^T(\mathbf{k})P_R - \sin\theta\, A^T(\mathbf{k})P_I)^T \\ V_I(\mathbf{k}, P) &= \cos\theta A^T(\mathbf{k})P_I + \sin\theta\, A^T(\mathbf{k})P_R \\ &\qquad - (\cos\theta A^T(\mathbf{k})P_I + \sin\theta\, A^T(\mathbf{k})P_R)^T. \end{aligned}$$

As $X + X^T$ is symmetric and the difference $X - X^T$ skew-symmetric, we have that $V_R(\mathbf{k}, P)$ is a symmetrical matrix and $V_I(\mathbf{k}, P)$ is a skew-symmetric matrix.
Due to the structural properties of the matrices $V_R(\mathbf{k}, P)$, $V_I(\mathbf{k}, P)$ and P_R, P_I there are n^2 independent equations v_{Rij}, v_{Iij} and n^2 independent variables p_{ij}. Additionally, the hermitian form of the matrix P ensures that all leading principle minors are real, an analogous procedure to Section II can be followed. Thus, the Lyapunov equation of the modified system can now also be converted into a solvable SAS. The resulting SAS has the form:

$$\begin{cases} v_{Rij}(\mathbf{k}, P) = 0, \forall i \geq j \\ v_{Iab}(\mathbf{k}, P) = 0, \forall a < b \\ \mu_1(P) > 0, \ldots, \mu_n(P) > 0, \end{cases} \tag{13}$$

where $\mu_i(P)$ is the i-th leading principle minor of the matrix P, $v_{Rij}(\mathbf{k}, P)$ and $v_{Iab}(\mathbf{k}, P)$ are the entries of the matrices $V_R(\mathbf{k}, P)$ and $V_I(\mathbf{k}, P)$.

D. Combinations of various specifications

The performance specifications discussed above can be combined as desired. Clearly the intersection of the corresponding parameter regions will guarantee joint specifications. Other set operations such as complementing, joining and cutting are possible, too. In fact, a semi-algebraic set (i.e. set of feasible solutions of an underlying semi-algebraic system) is built by a concatenation of logical operations applied upon inequality constraints of the form

$$\left\{\mathbf{k} \in \mathbb{R}^d : p(\mathbf{k}) \leq 0\right\} \tag{14}$$

where p stands for a polynomial in $\mathbf{k}$. For example, if the cases from Section III-B and III-C are combined with each other, then the eigenvalues will be forced to lie within a circled sector.

IV. Real root classification technique

The solution of the semi-algebraic set (3) is a subset of the parameter space, where the system of equations consisting of polynomials f_i with real coefficients under the constraints described by g_i, h_i possess at least one real solution. Several solutions strategies and implementations have been proposed in literature. Here we discuss the approach proposed in [7]. Solving a SAS involves an initial decomposition into a number of triangular SAS (TSAS) of the form:

$$\begin{cases} f_1(\mathbf{u}, x_1) = 0, \\ f_2(\mathbf{u}, x_1, x_2) = 0, \\ \cdots \\ f_s(\mathbf{u}, x_1, x_2, \ldots, x_s) = 0, \\ g_1(\mathbf{u}, \mathbf{x}) \geq 0, g_2(\mathbf{u}, \mathbf{x}) \geq 0, \ldots, g_t(\mathbf{u}, \mathbf{x}) \geq 0. \end{cases} \tag{15}$$

The union of the solutions of all TSAS represents the solution to the initial SAS. Hereby the number of variables of a SAS should equal the number of equations, otherwise it can not be decomposed into TSAS and therefore no longer be solvable.

An important concept for solving SAS is that of *border polynomial* of the TSAS. Consider a TSAS Sys as defined in (15). Then, we define:

$$\mathrm{BP}(Sys) = \prod_{1 \leq i \leq s} \mathrm{BP}_{f_i} \prod_{1 \leq j \leq t} \mathrm{BP}_{g_j} \tag{16}$$

with:

$$\begin{aligned} \mathrm{BP}_{f_1} &= \mathrm{res}(f_1(\mathbf{u}, x_1), f_i'(\mathbf{u}, x_1), x_1) \\ \mathrm{BP}_{f_i} &= \mathrm{res}(\mathrm{res}(f_i, f_i', x_i); f_{i-1}, \ldots, f_1) \\ \mathrm{BP}_{g_j} &= \mathrm{res}(g_j(\mathbf{u}, \mathbf{x}); f_s, \ldots, f_1) \end{aligned}$$

where f_i' is the derivative of $f_i = f_i(\mathbf{u}, x_1, x_2, \ldots, x_i)$ with respect to x_i, $\mathrm{res}(f_i, f_i', x_i)$ is the Sylvester-resultant of f_i and f_i' with respect to x_i and $\mathrm{res}(h(\mathbf{x}); f_{i-1}, \ldots, f_1)$ represents the successive Sylvester-resultant of a polynomial $h(\mathbf{x})$ with respect to the system of equations consisting of f_i, as defined in [6]. Then, $\mathrm{BP}(Sys)$ is called border polynomial of the TSAS Sys and if, additionally, $\mathrm{BP}(Sys) \neq 0$ holds true, the system Sys will be called a regular TSAS.
It is sufficient to consider the special case of the regular TSAS. As presented in [8], transforming TSAS into regular TSAS does not affect its solution. It turns out that if Sys is a regular TSAS, then the number of solutions of the equation system $f_i(\mathbf{u}, \mathbf{x})$ under the conditions $g_j(\mathbf{u}, \mathbf{x})$ is constant over a closed region $\mathbf{u}$ in the parameter space, for which it applies that $\mathrm{BP}(Sys) \neq 0$, see [6].
On this basis, closed regions in the parameter space can be determined, in which the number of solutions of the equation system f_i is constant under the constrains g_j. To determine the regions where $\mathrm{BP}(Sys) \neq 0$, a cylindrical algebraic decomposition (CAD) of the $\mathrm{BP}(Sys)$ in the parameter space is performed. The CAD is then output as semi-algebraic sets in the parameter space where $\mathrm{BP}(Sys) \neq 0$. In addition, for each semi-algebraic set, an explicit point is output which is an element of the set. (The CAD algorithm is presented in [2].) It is now possible to determine closed regions in the parameter space, where the number of solutions is constant. Then there follows a check on each semi-algebraic set to see whether at least one solution exists for it. Therefore a sample point of each region with $\mathrm{BP}(Sys) \neq 0$ is computed through the CAD and inserted into the initial SAS, to determine whether the SAS has a solution for a specific point in the parameter space is trivial.

V. Numerical example

We borrow an aircraft model from [1], with the autonomous, continuous LTI system matrix $A(\mathbf{k})$ depending on two parameters k_1 and k_2:

$$\begin{pmatrix} -\frac{851}{500} + \frac{1361}{5}k_1 & \frac{1268}{25} + \frac{1361}{5}k_2 & \frac{527}{2} \\ \frac{2201}{10000} & -\frac{709}{500} & -\frac{3199}{100} \\ -14k_1 & -14k_2 & -14 \end{pmatrix}. \tag{17}$$

First, according to the method described in [5] the region of all Hurwitz-stabilizing parameters is computed to be as depicted in Fig. 2. The resulting stable parameter region

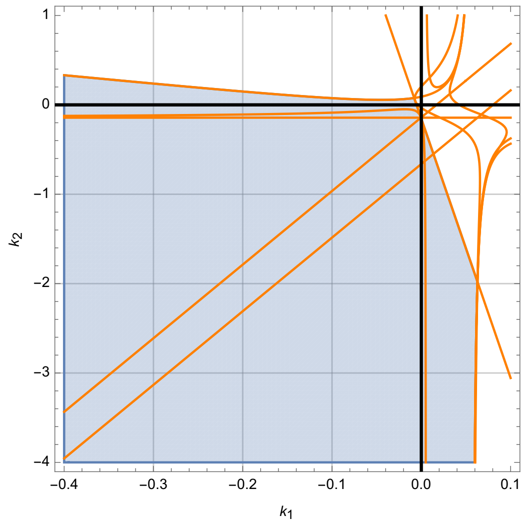

Fig. 2. Stable parameters for the system with the system matrix (17). The lines appearing at a large number represent the border polynomials. They are omitted in Figs. 3 and 4.

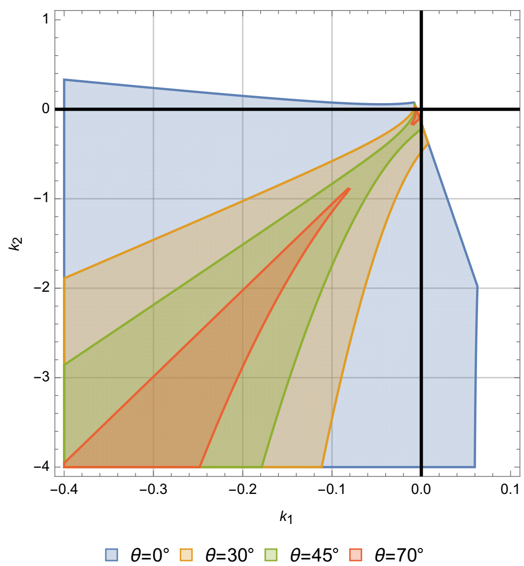

Fig. 3. Various performance regions for the system with the system matrix (17). Observe that in the case of $\theta = 70°$, the region is no longer connected as a small feasible region near the origin exists. In other words, for a value of $45° < \theta < 70°$, a bifurcation in the parameter space (k_1, k_2) emerges.

matches exactly to the one presented in [1]. The outcome of the RRC algorithm for the case study from Fig. 2 reads:

$$10k_1 + 0.34k_2 + 0.05 < 0$$
$$-10k_1^2 - 10k_1k_2 + 0.34k_1 + 0.57k_2 + 0.05 < 0.$$

Figs. 3 and 4 provide the parameter space outcomes with additional specifications on the system behavior as described by the minimal damping specification and the modification introduced in Section III-C. From Fig. 3 it is obvious that the solution set becomes smaller with growing θ and that the solution sets are subsets of each other. Fig. 4 depicts the region of parameters where joint specifications resulting from the combined modifications of the system matrix as given in Sections III-B and III-C. The eigenvalues are then forced to lie within an encircled θ-sector.

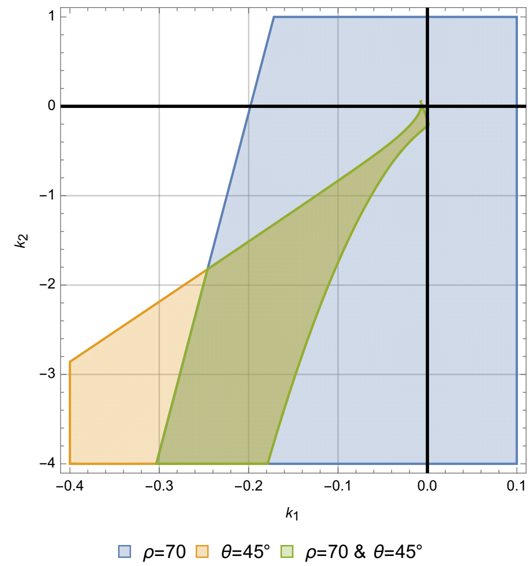

Fig. 4. Stability region for the system with the system matrix (17) under combined damping and bandwidth performance specifications (see Section III-B a. III-C).

VI. Conclusions

This paper adresses the computation of parameter regions under performance-specific constraints for continuous- and (in principle, also) discrete-time linear-time invariant systems. To this end, real root classification techniques have been applied. It has been shown how such parameter regions are determined depending of the systems specifications described by the eigenvalue location in the complex frequency domain. Basic contribution of our study consists in providing guidelines for formulation of suitable semi-algebraic systems, implementing a variety of characteristic eigenvalue domains, as well as a combination thereof. Thereby, we make use of the feasibility condition of the parameterized complex Lyapunov equations. Future work will include handling of nonlinear system specifications by means of the real root classification technique.

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