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New Formulae to approximate an Infinitesimal Rotation Followed or Preceded by a Large one

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Abstract— **Rotations of the Cartesian reference frames are used extensively in many fields of research such as navigation and guidance systems, and motion tracking systems. Of particular interest is the case when an infinitesimal rotation is followed or preceded by a large rotation. If the order of rotations is required to be changed, there is no formula to describe a good approximation. Instead, researchers usually use dot product of the unit vectors of the coordinate systems before and after the reversal. In this work, we provide a direct formula for such approximation. Further, we show that the proposed formulae can be used to construct the inverse of any transformation comprising two successive rotations about different axes by taking many successive infinitesimal rotations approximated properly. Moreover, we present a case study for which the proposed formulae can be used to find the solution in a direct way compared to other techniques.**

I. INTRODUCTION

Rotation of reference frames is an essential tool in many fields of science and technology such as motion tracking systems and navigation and guidance systems. Of particular interest is the problem of the *infinitesimal* rotation followed or preceded by a large rotation of the reference frame. In many cases, the order of such transformation is required to be reversed. The authors usually use dot product between the unit vectors of the coordinate systems before and after the reversal to determine the error resulting from it. This problem is encountered in the literature on, among other topics, magnetic tracking systems [1], [2], and error analysis of guidance systems [3]. In their work on magnetic tracking system, the authors in [1] proposed the *previous measurement* technique to determine the position of a magnetic sensor moving around a transmitter. By this technique, the position and orientation of the moving sensor are assumed at a small increment from the last measured ones. The incremental changes in elevation and azimuth of the position, and those in Euler angles which describe the orientation are expressed in matrix form. Then, depending on matching the corresponding elements of the sequence of the infinitesimal rotation matrices in different coordinate systems, the increments can be found and the position and orientation can be updated [1]. Further, this problem can be of great interest whenever perturbation of the coordinate systems rotation is to be investigated. For example, the author in [3], studied the error in guidance systems whenever a physical realization of coordinate transformations is required, as with gimbals. In order to analyze those errors, the author in [3] proposed using the *Piogram* technique that is a symbolic representation of rotations which was introduced in [4] and [5]. However, none of the aforementioned references provided a mathematical formulae to describe a good approximation of the infinitesimal rotation followed or preceded by large rotation when the order is to be reversed.

Actually, during our investigation on the sensitivity of the motion tracking systems, we wanted to have a formula that can be used to reverse the order of the infinitesimal rotation followed or preceded by a large rotation without the need for the dot product to simplify the mathematical analysis. In addition, we found that the error resulting from several successive infinitesimal rotations is significant, especially with increasing number of them. Thus, we propose in this work formulae to approximate such transformations. We firstly consider a specific example, and describe the method to obtain this formula. Then, we provide a list of formulae for all possible combinations of rotations, and deduce a general rule. Besides, we show that the proposed formulae can be used to construct, what we call in the sequel, the *inverting transformation*, which is the transformation that reverses the effect of successive large rotations. Furthermore, we use the proposed formulae to solve the example formulated and solved in [3], to show that with those formulae one does not need to use rotation matrices or Piograms.

Although the results presented here can be generalized to vector rotations, that represent the cornerstone of rigid body mechanics [6], we confine our discussion to the rotation of reference frames.

A. Statement of the Problem

It is well known that the transformation matrix that describes the rotation of the reference frame around z-axis by angle α, in the positive direction according to the right-hand rule, is given by:

$$\mathbf{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1}$$

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Similarly, the matrices $\mathbf{R}_y(\beta)$ and $\mathbf{R}_x(\gamma)$ that describe the rotations around y-axis and x-axis by angles β and γ, in the positive direction according to the right-hand rule, are given by:

$$\begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \text{and} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{bmatrix}, \quad (2)$$

respectively. When dealing with small or infinitesimal increments $(\Delta\theta)$, it is usually common to approximate [3]:

$$\sin(\Delta\theta) \approx \Delta\theta \qquad \text{and} \qquad \cos(\Delta\theta) \approx 1. \quad (3)$$

Thus, the corresponding infinitesimal rotation matrices can be described by [1]:

$$\mathbf{R}_z(\Delta\alpha) \approx \begin{bmatrix} 1 & \Delta\alpha & 0 \\ -\Delta\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (4)$$

and similar for infinitesimal rotations about x and y axes.
For the case of successive infinitesimal transformations, it is unanimously accepted to reverse the order of rotations. For example [6]:

$$\mathbf{R}_z(\Delta\alpha)\mathbf{R}_x(\Delta\gamma) \approx \mathbf{R}_x(\Delta\gamma)\mathbf{R}_z(\Delta\alpha). \quad (5)$$

This approximation can be easily verified after neglecting the terms that contain cross products of the small increments like $\Delta\alpha\Delta\gamma$, see e.g. [6]. However, when an infinitesimal rotation is followed or preceded by another considerable rotation, this approximation is dubious and this is the subject of this work. Consider, for example, the transformation that results from an infinitesimal rotation around z-axis by a small angle $\Delta\alpha$ followed by a rotation around y-axis by angle β, described by:

$$\mathbf{E} = \mathbf{R}_y(\beta)\mathbf{R}_z(\Delta\alpha) \quad (6)$$

Reversing the order of rotations results in:

$$\widetilde{\mathbf{E}} = \mathbf{R}_z(\Delta\tilde{\alpha})\mathbf{R}_y(\beta). \quad (7)$$

Note that we use $\Delta\tilde{\alpha}$ for the infinitesimal rotation when the order is reversed to allow for possible compensation to make the difference between the two transformations in (6) and (7) as small as possible in terms of a meaningful metric. Define the error matrix $\Delta\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}} - \mathbf{E}$, which can be obtained from (6) and (7) and the definitions in (1) and (2) to be:

$$\Delta\widetilde{\mathbf{E}} = \begin{bmatrix} 0 & \Delta\tilde{\alpha} - \Delta\alpha\cos\beta & 0 \\ \Delta\alpha - \Delta\tilde{\alpha}\cos\beta & 0 & \Delta\tilde{\alpha}\sin\beta \\ 0 & -\Delta\alpha\sin\beta & 0 \end{bmatrix}. \quad (8)$$

Obviously, there is no choice of $\Delta\tilde{\alpha}$ that can make $\Delta\widetilde{\mathbf{E}}$ zero. Hence, if one has to use the rough approximation in (7), the common consensus among the researchers is to approximate $\Delta\tilde{\alpha}$ by $\Delta\alpha$. However, since the norm of the matrix $\Delta\widetilde{\mathbf{E}}$ is apparently a function of $\Delta\alpha$ and β, a better approximation can be obtained.
In the subsequent section, we present a formula for that better approximation for the case given in (6), and then we generalize it for any similar case. In the third section, we present another argument to support the proposed formulae by showing that they can be used to construct the inverting transformation that will be defined there. Further, we show how the proposed formulae can be used to solve a practical example taken from [3]. Finally, in the last section, we draw some conclusions.

II. THE PROPOSED MODEL

When the order of rotations, in e.g. (6), is required to be reversed, we propose to replace the infinitesimal rotation $\mathbf{R}_z(\Delta\alpha)$ before the rotation $\mathbf{R}_y(\beta)$ with two consecutive infinitesimal rotations $\mathbf{R}_z(\Delta\alpha^*)$ and $\mathbf{R}_x(\Delta\gamma^*)$. Let the matrix $\mathbf{E}^*$ describe these successive rotations as:

$$\mathbf{E}^* = \mathbf{R}_z(\Delta\alpha^*)\mathbf{R}_x(\Delta\gamma^*)\mathbf{R}_y(\beta) \approx \begin{bmatrix} \cos\beta & \Delta\alpha^* & -\sin\beta \\ \Delta\gamma^*\sin\beta - \Delta\alpha^*\cos\beta & 1 & \Delta\alpha^*\sin\beta + \Delta\gamma^*\cos\beta \\ \sin\beta & -\Delta\gamma^* & \cos\beta \end{bmatrix} \quad (9)$$

Note that the terms with cross products $\Delta\alpha^*\Delta\gamma^*$ are neglected in the above matrix, as usual. Now, define the error matrix $\Delta\mathbf{E}^* = \mathbf{E}^* - \mathbf{E}$, as before. By using (6) and (9) and the definitions in (1) and (2), it is easy to verify that $\Delta\mathbf{E}^*$ is given by:

$$\begin{bmatrix} 0 & \Delta\alpha^* - \Delta\alpha \mathrm{C}\beta & 0 \\ \Delta\gamma^*\mathrm{S}\beta - \Delta\alpha^*\mathrm{C}\beta + \Delta\alpha & 0 & \Delta\alpha^*\mathrm{S}\beta + \Delta\gamma^*\mathrm{C}\beta \\ 0 & -\Delta\gamma^* - \Delta\alpha\mathrm{S}\beta & 0 \end{bmatrix} \quad (10)$$

where S(.) and C(.) denote sin(.) and cos(.), respectively. Then, choosing

$$\Delta\alpha^* = \Delta\alpha\cos\beta \qquad \text{and} \qquad \Delta\gamma^* = -\Delta\alpha\sin\beta, \quad (11)$$

obviously makes the matrix $\Delta\mathbf{E}^*$ zero. It is also important here to note that reversing the order of the infinitesimal rotations in (9) to $\mathbf{R}_x(\Delta\gamma^*)\mathbf{R}_z(\Delta\alpha^*)$ will not change the result in (11), as anticipated from (5).
As another example, let us consider the transformation:

$$\mathbf{F} = \mathbf{R}_z(\Delta\alpha)\mathbf{R}_y(\beta). \quad (12)$$

Applying the same technique above, one can verify that, upon reversing the order of rotations, $\mathbf{F}$ can be approximated by:

$$\mathbf{F}^* = \mathbf{R}_y(\beta)\mathbf{R}_x(\Delta\gamma^*)\mathbf{R}_z(\Delta\alpha^*), \quad (13)$$

with

$$\Delta\alpha^* = \Delta\alpha\cos\beta \qquad \text{and} \qquad \Delta\gamma^* = \Delta\alpha\sin\beta. \quad (14)$$

A. General Formula

Now, in order to give the general formula, we need to list all possible combinations of the infinitesimal rotations followed or preceded by large ones, as shown in Table I. Let us denote the set of the coordinate axes by $\{\omega_1, \omega_2, \omega_3\}$ in random order, and the angles of rotations by $\{\theta_1, \theta_2, \theta_3\}$, respectively. Let also the unit vectors along the

TABLE I

LIST OF ALL POSSIBLE COMBINATIONS OF INFINITESIMAL ROTATIONS FOLLOWED OR PRECEDED BY LARGE ONES

Followed by		
Rotation	Approximation	Sign of middle angle
$\mathbf{R}_y(\beta)\mathbf{R}_z(\Delta\alpha)$	$\mathbf{R}_z(\Delta\alpha \mathrm{C}\beta)\mathbf{R}_x(-\Delta\alpha \mathrm{S}\beta)\mathbf{R}_y(\beta)$	$(\hat{k}\times\hat{j}).\hat{i}=-1$
$\mathbf{R}_y(\beta)\mathbf{R}_x(\Delta\gamma)$	$\mathbf{R}_x(\Delta\gamma \mathrm{C}\beta)\mathbf{R}_z(\Delta\gamma \mathrm{S}\beta)\mathbf{R}_y(\beta)$	$(\hat{i}\times\hat{j}).\hat{k}=1$
$\mathbf{R}_z(\alpha)\mathbf{R}_y(\Delta\beta)$	$\mathbf{R}_y(\Delta\beta \mathrm{C}\alpha)\mathbf{R}_x(\Delta\beta \mathrm{S}\alpha)\mathbf{R}_z(\alpha)$	$(\hat{j}\times\hat{k}).\hat{i}=1$
$\mathbf{R}_z(\alpha)\mathbf{R}_x(\Delta\gamma)$	$\mathbf{R}_x(\Delta\gamma \mathrm{C}\alpha)\mathbf{R}_y(-\Delta\gamma \mathrm{S}\alpha)\mathbf{R}_z(\alpha)$	$(\hat{i}\times\hat{k}).\hat{j}=-1$
$\mathbf{R}_x(\gamma)\mathbf{R}_y(\Delta\beta)$	$\mathbf{R}_y(\Delta\beta \mathrm{C}\gamma)\mathbf{R}_z(-\Delta\beta \mathrm{S}\gamma)\mathbf{R}_x(\gamma)$	$(\hat{j}\times\hat{i}).\hat{k}=-1$
$\mathbf{R}_x(\gamma)\mathbf{R}_z(\Delta\alpha)$	$\mathbf{R}_z(\Delta\alpha \mathrm{C}\gamma)\mathbf{R}_y(\Delta\alpha \mathrm{S}\gamma)\mathbf{R}_x(\gamma)$	$(\hat{k}\times\hat{i}).\hat{j}=1$
Preceded by		
Rotation	Approximation	Sign of middle angle
$\mathbf{R}_z(\Delta\alpha)\mathbf{R}_y(\beta)$	$\mathbf{R}_y(\beta)\mathbf{R}_x(\Delta\alpha \mathrm{S}\beta)\mathbf{R}_z(\Delta\alpha \mathrm{C}\beta)$	$(\hat{j}\times\hat{k}).\hat{i}=1$
$\mathbf{R}_x(\Delta\gamma)\mathbf{R}_y(\beta)$	$\mathbf{R}_y(\beta)\mathbf{R}_z(-\Delta\gamma \mathrm{S}\beta)\mathbf{R}_x(\Delta\gamma \mathrm{C}\beta)$	$(\hat{j}\times\hat{i}).\hat{k}=-1$
$\mathbf{R}_y(\Delta\beta)\mathbf{R}_z(\alpha)$	$\mathbf{R}_z(\alpha)\mathbf{R}_x(-\Delta\beta \mathrm{S}\alpha)\mathbf{R}_y(\Delta\beta \mathrm{C}\alpha)$	$(\hat{k}\times\hat{j}).\hat{i}=-1$
$\mathbf{R}_x(\Delta\gamma)\mathbf{R}_z(\alpha)$	$\mathbf{R}_z(\alpha)\mathbf{R}_y(\Delta\gamma \mathrm{S}\alpha)\mathbf{R}_x(\Delta\gamma \mathrm{C}\alpha)$	$(\hat{k}\times\hat{i}).\hat{j}=1$
$\mathbf{R}_y(\Delta\beta)\mathbf{R}_x(\gamma)$	$\mathbf{R}_x(\gamma)\mathbf{R}_z(\Delta\beta \mathrm{S}\gamma)\mathbf{R}_y(\Delta\beta \mathrm{C}\gamma)$	$(\hat{i}\times\hat{j}).\hat{k}=1$
$\mathbf{R}_z(\Delta\alpha)\mathbf{R}_x(\gamma)$	$\mathbf{R}_x(\gamma)\mathbf{R}_y(-\Delta\alpha \mathrm{S}\gamma)\mathbf{R}_z(\Delta\alpha \mathrm{C}\gamma)$	$(\hat{i}\times\hat{k}).\hat{j}=-1$

axes $\{\omega_1, \omega_2, \omega_3\}$ be denoted by $\{\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3\}$. Then, from Table I, it is to verify that:

$$\mathbf{R}_{\omega_1}(\theta_1)\mathbf{R}_{\omega_2}(\Delta\theta_2) \approx \mathbf{R}_{\omega_2}(\Delta\theta_2\cos\theta_1)\mathbf{R}_{\omega_3}\left(((\hat{\omega}_2\times\hat{\omega}_1).\hat{\omega}_3)\Delta\theta_2\sin\theta_1\right)\mathbf{R}_{\omega_1}(\theta_1), \tag{15}$$

and

$$\mathbf{R}_{\omega_1}(\Delta\theta_1)\mathbf{R}_{\omega_2}(\theta_2) \approx \mathbf{R}_{\omega_2}(\theta_2)\mathbf{R}_{\omega_3}\left(((\hat{\omega}_2\times\hat{\omega}_1).\hat{\omega}_3)\Delta\theta_1\sin\theta_1\right)\mathbf{R}_{\omega_1}(\Delta\theta_1\cos\theta_2). \tag{16}$$

The quantity $((\hat{\omega}_2\times\hat{\omega}_1).\hat{\omega}_3)$ in the above equations is used to determine the sign of the infinitesimal angle of rotation around the third axis that can also be verified from Table I. In this quantity, the signs $(\times)$ and $(.)$ denote the cross and dot products of vectors, respectively. Other description could probably be obtained, but this one suffices.

III. MANY SUCCESSIVE INFINITESIMAL ROTATIONS

Although the proposed formulae in (15) and (16) were derived from basic definitions, we show analytically that they are correct by constructing the inverting transformation.

A. From Infinitesimal to Large

Generally speaking, the order of rotations cannot be reversed. For example:

$$\mathbf{R}_y(\beta)\mathbf{R}_z(\alpha) \neq \mathbf{R}_z(\alpha)\mathbf{R}_y(\beta). \tag{17}$$

However, should a transformation matrix (let us, from here on, call it the *inverting* transformation denoted by $\mathbf{V}$) be required to reverse the order of rotations, such that:

$$\mathbf{R}_y(\beta)\mathbf{R}_z(\alpha) = \mathbf{V}\mathbf{R}_y(\beta), \tag{18}$$

then $\mathbf{V}$ can be found for this case from the above equation to be:

$$\begin{aligned}\mathbf{V} =& \mathbf{R}_y(\beta)\mathbf{R}_z(\alpha)\mathbf{R}_y^{-1}(\beta)\\ =& \begin{bmatrix} \mathrm{S}^2\beta + \mathrm{C}\alpha\mathrm{C}^2\beta & \mathrm{S}\alpha\mathrm{C}\beta & \frac{1}{2}\mathrm{S}(2\beta)(\mathrm{C}\alpha-1) \\ -\mathrm{S}\alpha\mathrm{C}\beta & \mathrm{C}\alpha & -\mathrm{S}\alpha\mathrm{S}\beta \\ \frac{1}{2}\mathrm{S}(2\beta)(\mathrm{C}\alpha-1) & \mathrm{S}\alpha\mathrm{S}\beta & \mathrm{C}^2\beta+\mathrm{C}\alpha\mathrm{S}^2\beta \end{bmatrix}.\end{aligned} \tag{19}$$

The rotation $\mathbf{R}_z(\alpha)$ can be thought of as many successive infinitesimal rotations each of $\mathbf{R}_z(\Delta\alpha_k)$. Thus:

$$\mathbf{R}_y(\beta)\mathbf{R}_z(\alpha) = \mathbf{R}_y(\beta)\prod_{k=1}^{K}\mathbf{R}_z(\Delta\alpha_k), \tag{20}$$

where K is the number of the successive infinitesimal rotations. A naive attempt could be to rewrite the above equation, by using the rough approximation in (7), as:

$$\mathbf{R}_y(\beta)\mathbf{R}_z(\alpha) \approx \prod_{k=1}^{K}\mathbf{R}_z(\Delta\alpha_k)\mathbf{R}_y(\beta). \tag{21}$$

Let us consider equal infinitesimal angles $\Delta\alpha_k = \frac{\alpha}{K}$, and take the limit as $K\to\infty$. Then:

$$\lim_{K\to\infty}\prod_{k=1}^{K}\mathbf{R}_z(\Delta\alpha_k) = \lim_{K\to\infty}(\mathbf{R}_z(\frac{\alpha}{K}))^K = \mathbf{R}_z(\alpha), \tag{22}$$

which makes (21) incorrect because of (17).

On the other hand, if we can show that many successive rotations with the described approximation in (9) and (11) will result in the inverting transformation $\mathbf{V}$ in (19), then the effectiveness of the proposed formulae will be proved.

To this end, we can use the described approximation in (9) with the parameters in (11) to approximate (20) by:

$$\mathbf{R}_y(\beta)\mathbf{R}_z(\alpha) \approx \prod_{k=1}^{K} \mathbf{R}_z(\Delta\alpha_k^*)\mathbf{R}_x(\Delta\gamma_k^*)\mathbf{R}_y(\beta). \tag{23}$$

Let us again consider equal infinitesimal angles $\Delta\alpha_k^* = \Delta\alpha^*$ and $\Delta\gamma_k^* = \Delta\gamma^*$, and take the limit as $K \to \infty$, as follows:

$$\lim_{K\to\infty} \prod_{k=1}^{K} \mathbf{R}_z(\Delta\alpha_k^*)\mathbf{R}_x(\Delta\gamma_k^*) = \lim_{K\to\infty} \left(\mathbf{R}_z(\Delta\alpha^*)\mathbf{R}_x(\Delta\gamma^*)\right)^K. \tag{24}$$

In order to calculate the limit in (24), we need to use the theory of the *exponential matrix*.

Definition III.1 (Exponential Matrix [9])**.** *The exponential matrix $e^{\mathbf{A}}$, for any arbitrary square matrix $\mathbf{A}$, is defined by:*

$$e^{\mathbf{A}} = \sum_{j=0}^{\infty} \frac{\mathbf{A}^j}{j!}, \tag{25}$$

with $\mathbf{A}^0 = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

Remark 1. *From definition III.1, one can note that:*

$$e^{\mathbf{A}+\mathbf{B}} \neq e^{\mathbf{A}}e^{\mathbf{B}}, \tag{26}$$

unless $\mathbf{A}$ and $\mathbf{B}$ commute, i.e. $\mathbf{AB} = \mathbf{BA}$. Moreover, to compensate for the difference between the two sides when $\mathbf{A}$ and $\mathbf{B}$ do not commute, Baker-Campbell-Hausdorff formula is used. The reader is advised to see [9] and [10] for more details.

For further discussions, it is important to write the following definition.

Definition III.2 (Matrix Logarithm [9])**.** *For any square matrix $\mathbf{A}$, the matrix logarithm is defined by:*

$$\log \mathbf{A} = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(\mathbf{A}-\mathbf{I})^m}{m}, \tag{27}$$

whenever the series converges.

One important result that can be proved directly from definition III.2 can be stated as follows.

Proposition III.1 (Proposition 2.8 in [9])**.** *For any square matrix $\mathbf{A}$ with $||\mathbf{A}|| < \frac{1}{2}$,*

$$\log(\mathbf{I}+\mathbf{A}) = \mathbf{A} + \mathcal{O}(||\mathbf{A}||^2) \tag{28}$$

Proof: For the proof, see Proposition 2.8, chapter 2 in [9]. □

Based on the above, one can easily prove the following result.

Proposition III.2. *For any square matrices $\mathbf{A}$ and $\mathbf{B}$,*

$$\lim_{K\to\infty} \left(\mathbf{I} + \frac{\mathbf{A}}{K} + \frac{\mathbf{B}}{K^2}\right)^K = e^{\mathbf{A}}. \tag{29}$$

Further,

$$\lim_{K\to\infty} (\mathbf{I} + \frac{\mathbf{A}}{K})^K = e^{\mathbf{A}}, \tag{30}$$

Proof: Obviously, the quantity $\frac{\mathbf{A}}{K} + \frac{\mathbf{B}}{K^2}$ is a matrix whose norm decreases as $K \to \infty$. Thus, we can always find sufficiently large K such that the hypothesis of proposition III.1 is satisfied. Taking the logarithm, we get:

$$\log\left(\mathbf{I} + \frac{\mathbf{A}}{K} + \frac{\mathbf{B}}{K^2}\right) = \frac{\mathbf{A}}{K} + \frac{\mathbf{B}}{K^2} + \mathcal{O}(||\frac{\mathbf{A}}{K} + \frac{\mathbf{B}}{K^2}||^2), \tag{31}$$

which can be written, by taking the exponent of both sides, as:

$$\mathbf{I} + \frac{\mathbf{A}}{K} + \frac{\mathbf{B}}{K^2} = \exp(\frac{\mathbf{A}}{K} + \frac{\mathbf{B}}{K^2} + \mathcal{O}(||\frac{\mathbf{A}}{K} + \frac{\mathbf{B}}{K^2}||^2), \tag{32}$$

Note that $\mathcal{O}(||\frac{\mathbf{A}}{K} + \frac{\mathbf{B}}{K^2}||^2)$ can be written as $\frac{c}{K^2}$, where c is an arbitrary constant because $||\frac{\mathbf{A}}{K} + \frac{\mathbf{B}}{K^2}||^2 \leq ||\frac{\mathbf{A}}{K}||^2 + ||\frac{\mathbf{B}}{K^2}||^2 \leq \frac{c}{K^2}$. Thus:

$$\mathbf{I} + \frac{\mathbf{A}}{K} + \frac{\mathbf{B}}{K^2} = \exp(\frac{\mathbf{A}}{K} + \frac{\mathbf{B}}{K^2} + \frac{c}{K^2}), \tag{33}$$

and

$$\left(\mathbf{I} + \frac{\mathbf{A}}{K} + \frac{\mathbf{B}}{K^2}\right)^K = \exp(\mathbf{A} + \frac{\mathbf{B}}{K} + \frac{c}{K}). \tag{34}$$

The quantity $\frac{\mathbf{B}+c}{K} \to 0$ as $K \to \infty$, and since the exponential function is continuous, the result in (29) is true. Equation (30) is a special case of (29). □

Let us now go back to (24). Notice that the matrices $\mathbf{R}_z(\Delta\alpha^*)$ and $\mathbf{R}_x(\Delta\gamma^*)$ can be written from (4) as $\mathbf{I} + \Delta\alpha^*\mathbf{D}_z$ and $\mathbf{I} + \Delta\gamma^*\mathbf{D}_x$, respectively, where:

$$\mathbf{D}_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{D}_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \tag{35}$$

In group theory, the matrices $\mathbf{D}_z$ and $\mathbf{D}_x$ are usually called *infinitesimal rotation generators*, and they are used to define the *Lie Algebra*, [6] and [9].

By taking $\Delta\alpha^* = \frac{\alpha^*}{K}$ and $\Delta\gamma^* = \frac{\gamma^*}{K}$, one gets:

$$\begin{aligned} \left(\mathbf{R}_z(\Delta\alpha^*)\mathbf{R}_x(\Delta\gamma^*)\right)^K &= \left((\mathbf{I} + \Delta\alpha^*\mathbf{D}_z)(\mathbf{I} + \Delta\gamma^*\mathbf{D}_x)\right)^K \\ &= \left((\mathbf{I} + \frac{\alpha^*}{K}\mathbf{D}_z)(\mathbf{I} + \frac{\gamma^*}{K}\mathbf{D}_x)\right)^K \\ &= \left(\mathbf{I} + \frac{\alpha^*\mathbf{D}_z + \gamma^*\mathbf{D}_x}{K} + \frac{\alpha^*\gamma^*}{K^2}\mathbf{D}_z\mathbf{D}_x\right)^K \\ &= \left(\mathbf{I} + \frac{\overline{\mathbf{D}}}{K} + \frac{\alpha^*\gamma^*}{K^2}\mathbf{D}_z\mathbf{D}_x\right)^K, \end{aligned} \tag{36}$$

where

$$\overline{\mathbf{D}} = \alpha^*\mathbf{D}_z + \gamma^*\mathbf{D}_x = \begin{bmatrix} 0 & \alpha^* & 0 \\ -\alpha^* & 0 & \gamma^* \\ 0 & -\gamma^* & 0 \end{bmatrix}. \tag{37}$$

By using (29) in proposition III.2, the limit in (24) will be:

$$\lim_{K\to\infty} \left(\mathbf{R}_z(\Delta\alpha^*)\mathbf{R}_x(\Delta\gamma^*)\right)^K = e^{\overline{\mathbf{D}}}. \tag{38}$$

Inserting (11) in (37) yields:

$$\overline{\mathbf{D}} = \begin{bmatrix} 0 & \alpha\cos\beta & 0 \\ -\alpha\cos\beta & 0 & -\alpha\sin\beta \\ 0 & \alpha\sin\beta & 0 \end{bmatrix}. \tag{39}$$

Direct calculations show that $\overline{\mathbf{D}}$ to the jth power is given by:

$$\overline{\mathbf{D}}^j = (-1)^{\frac{j-1}{2}} \begin{bmatrix} 0 & \alpha^j C\beta & 0 \\ (-\alpha)^j C\beta & 0 & (-\alpha)^j S\beta \\ 0 & \alpha^j S\beta & 0 \end{bmatrix} \tag{40}$$

for odd j,

$$\overline{\mathbf{D}}^j = (-1)^{\frac{j}{2}} \begin{bmatrix} \alpha^j C^2\beta & 0 & \alpha^j C\beta S\beta \\ 0 & \alpha^j & 0 \\ \alpha^j C\beta S\beta & 0 & \alpha^j C^2\beta \end{bmatrix} \tag{41}$$

for even j, and

$$\overline{\mathbf{D}}^j = \mathbf{I} \;\text{ for }\; j = 0 \tag{42}$$

Finally, after expanding the exponential term in (38) by using the definition in (30) and the expressions for $\overline{\mathbf{D}}^j$ in (40), (41) and (42), and by using the well known series expansions of the sines and cosines, one can deduce that:

$$\lim_{K\to\infty} \Big(\mathbf{R}_z(\Delta\alpha^*) \mathbf{R}_x(\Delta\gamma^*) \Big)^K = \mathbf{V}. \tag{43}$$

Thus, the formulae of the approximation in (9) and (11) are correct.

Remark 2. *As mentioned before, the order of the infinitesimal rotations can be reversed as in* (5)*, to very good accuracy. However, if many successive infinitesimal rotations are reversed, the result will be different.*

To elucidate, let us again consider the rotation $\mathbf{R}_y(\beta)\mathbf{R}_z(\alpha)$ divided into many successive infinitesimal rotations as in (20). Let us also use the described approximation as in (23). One can think of reversing the order of the infinitesimal rotations in (24) by collecting all rotations around z-axis together, as:

$$\lim_{K\to\infty} \prod_{k=1}^{K} \mathbf{R}_z(\Delta\alpha_k^*) \prod_{k=1}^{K} \mathbf{R}_x(\Delta\gamma_k^*) = \lim_{K\to\infty} \Big(\mathbf{R}_z(\Delta\alpha^*) \Big)^K \Big(\mathbf{R}_x(\Delta\gamma^*) \Big)^K \tag{44}$$

The limit above can be determined as:

$$\begin{aligned} &\lim_{K\to\infty} \Big(\mathbf{R}_z(\Delta\alpha^*) \Big)^K \Big(\mathbf{R}_x(\Delta\gamma^*) \Big)^K \\ &\quad = \lim_{K\to\infty} \Big(\mathbf{R}_z(\Delta\alpha^*) \Big)^K \lim_{K\to\infty} \Big(\mathbf{R}_x(\Delta\gamma^*) \Big)^K \\ &\quad = \lim_{K\to\infty} \Big(\mathbf{I} + \Delta\alpha^* \mathbf{D}_z \Big)^K \lim_{K\to\infty} \Big(\mathbf{I} + \Delta\gamma^* \mathbf{D}_x \Big)^K \\ &\quad = \lim_{K\to\infty} \Big(\mathbf{I} + \frac{\alpha^*}{K} \mathbf{D}_z \Big)^K \lim_{K\to\infty} \Big(\mathbf{I} + \frac{\gamma^*}{K} \mathbf{D}_x \Big)^K \\ &\quad = e^{\alpha^* \mathbf{D}_z} e^{\gamma^* \mathbf{D}_x}. \end{aligned} \tag{45}$$

Now, by substituting $\alpha^* = \alpha\cos\beta$ and $\gamma^* = -\alpha\sin\beta$ in (45), and by using the expansion in (30), one can obtain:

$$\begin{aligned} &\lim_{K\to\infty} \Big(\mathbf{R}_z(\Delta\alpha^*) \Big)^K \Big(\mathbf{R}_x(\Delta\gamma^*) \Big)^K \\ &= \begin{bmatrix} C(\alpha C\beta) & S(\alpha C\beta)C(\alpha S\beta) & -S(\alpha C\beta)S(\alpha S\beta) \\ -S(\alpha C\beta) & C(\alpha C\beta)C(\alpha S\beta) & -S(\alpha S\beta)C(\alpha C\beta) \\ 0 & S(\alpha S\beta) & C(\alpha S\beta) \end{bmatrix} \\ &\neq \mathbf{V} \end{aligned} \tag{46}$$

This shows that the larger the number of the successive infinitesimal rotations is, the more significant the error resulting from reversing the order of rotations will be.

B. Physical Interpretation

The analysis above provides a good physical interpretation of the successive infinitesimal rotations. To elaborate, rotating a reference frame around the z-axis by angle α and then around y-axis by angle β, is not equivalent to rotating it around y-axis by angle β, then around x-axis by angle $\gamma^* = -\alpha\sin\beta$, and then around z-axis by angle $\alpha^* = \alpha\cos\beta$, i.e.:

$$\begin{aligned} \mathbf{R}_y(\beta)\mathbf{R}_z(\alpha) &\neq \lim_{K\to\infty} \Big(\mathbf{R}_z(\Delta\alpha^*) \Big)^K \Big(\mathbf{R}_x(\Delta\gamma^*) \Big)^K \mathbf{R}_y(\beta) \\ &= \mathbf{R}_z(\alpha^*)\mathbf{R}_x(\gamma^*)\mathbf{R}_y(\beta). \end{aligned} \tag{47}$$

That being said, the rotation above is equivalent to rotating the reference frames around y-axis by angle β followed by infinitesimal rotations around x-axis by $\Delta\gamma^* = -\Delta\alpha\sin\beta$ then around z-axis by $\Delta\alpha^* = \Delta\alpha\cos\beta$, repeated infinitely many times, i.e.:

$$\mathbf{R}_y(\beta)\mathbf{R}_z(\alpha) = \lim_{K\to\infty} \Big(\mathbf{R}_z(\Delta\alpha^*) \mathbf{R}_x(\Delta\gamma^*) \Big)^K \mathbf{R}_y(\beta). \tag{48}$$

IV. CASE STUDY

As an application of the proposed formulae, we consider the example presented in [3] about the error analysis of the Euler angle transformations. Let us consider a sequence of Euler angles that transform coordinate axes ①, into coordinate axes ④, passing by intermediate coordinate axes frames ②, ③. Let the transformation that describes this sequence of rotations be:

$$\mathbf{T}_1^4 = \mathbf{T}_3^4 \mathbf{T}_2^3 \mathbf{T}_1^2 = \mathbf{R}_x(\gamma)\mathbf{R}_y(\beta)\mathbf{R}_z(\alpha), \tag{49}$$

such that:

$$\mathbf{T}_4^1 = (\mathbf{T}_1^4)^{-1} = \mathbf{T}_2^1 \mathbf{T}_3^2 \mathbf{T}_4^3 = \mathbf{R}_z(-\alpha)\mathbf{R}_y(-\beta)\mathbf{R}_x(-\gamma). \tag{50}$$

The example presented in [3] discusses the error resulting from known perturbations that may affect such transformation, assuming that we can quantify those perturbations. Note that identifying the number and nature of the possible perturbations is not an easy task, even from theoretical point of view. However, since our target is to show how the proposed formulae can be used to solve practical problems, we assume that the reader is familiar with this problem, and for further details the reader is advised to read [3]. The author in [3] showed that the worst case involves 9 different perturbations that would transform coordinate axes ① into coordinate axes ④′ that is different from coordinate axes ④. That is to say:

$$\begin{aligned} \mathbf{T}_1^{4'} = &\mathbf{R}_z(\delta_9)\mathbf{R}_y(\delta_8)\mathbf{R}_x(\gamma+\delta_7)\mathbf{R}_z(\delta_6)\mathbf{R}_y(\beta+\delta_5)\mathbf{R}_x(\delta_4) \\ &\mathbf{R}_z(\alpha+\delta_3)\mathbf{R}_y(\delta_2)\mathbf{R}_x(\delta_1), \end{aligned} \tag{51}$$

where $\delta_1, \ldots, \delta_9$ are presumably known infinitesimal perturbations. The error between coordinate axes frames ④ and ④′ can also be expressed as a series of Euler infinitesimal

$$\mathbf{T}_4^{4'} = \mathbf{T}_1^{4'}\mathbf{T}_4^1 = \mathbf{R}_z(\delta_9)\mathbf{R}_y(\delta_8)\mathbf{R}_x(\gamma+\delta_7)\mathbf{R}_z(\delta_6)\mathbf{R}_y(\beta+\delta_5)\mathbf{R}_x(\delta_4)\overbrace{\mathbf{R}_z(\alpha+\delta_3)\mathbf{R}_y(\delta_2)}^{\text{Approximate}}\underbrace{\mathbf{R}_x(\delta_1)\mathbf{R}_z(-\alpha)}_{\text{Approximate}}\mathbf{R}_y(-\beta)\mathbf{R}_x(-\gamma). \quad (52)$$

transformations, as:

$$\mathbf{T}_4^{4'} = \mathbf{R}_x(\varepsilon_1)\mathbf{R}_y(\varepsilon_2)\mathbf{R}_z(\varepsilon_3), \quad (53)$$

where ε_1, ε_2 and ε_3 are the error angles. Now, the problem is to find ε_1, ε_2 and ε_3 in terms of α, β, γ, and the known perturbations $\delta_1,...,\delta_9$, analytically. It is worth mentioning here, that the techniques presented in [3] to solve this problem are meant to give analytical solutions.

Firstly, one can obtain the transformation from coordinate axes ④ into ④′ as shown in (52) above. In [3], three different methods were presented to solve the described problem:

1) **Matrix Techniques**. First, the rotation matrices $\mathbf{R}_x(\gamma)$, $\mathbf{R}_y(\beta)$ and $\mathbf{R}_z(\alpha)$, and the *perturbed rotation matrices* $[\mathbf{R}_z(\delta_9)\mathbf{R}_y(\delta_8)\mathbf{R}_x(\gamma)]$, $[\mathbf{R}_x(\delta_7)\mathbf{R}_z(\delta_6)\mathbf{R}_y(\beta)]$, $[\mathbf{R}_y(\delta_5)\mathbf{R}_x(\delta_4)\mathbf{R}_z(\alpha)]$, and $[\mathbf{R}_z(\delta_3)\mathbf{R}_y(\delta_2)\mathbf{R}_x(\delta_1)]$ are constructed. Then, by multiplying them by each other, one can find a one-to-one correspondence between the error rotation matrices $\mathbf{R}_x(\varepsilon_1), \mathbf{R}_y(\varepsilon_2)$, and $\mathbf{R}_z(\varepsilon_3)$ and the simplified product of the right-hand side of (52). Finally, by matching the corresponding elements, one can find the errors ε_1, ε_2 and ε_3.
2) **Similarity transformation**. In this technique, the error and perturbed rotation matrices written above can be transformed into vectors exploiting the fact that these matrices are skew-symmetric. Thus, the matrix equation in (52) is converted into vector equation. Then, the errors can be found by using the dot product between vector representing the perturbations and that representing the errors, which is usually easier to solve.
3) **Piograms**. This technique was proposed in [3], and it exploits the Piograms presented in [4] and [5]. A Piogram is a symbolic representation of a rotation or successive rotations of a reference frame. Assuming that we have the vectors corresponding to the perturbed rotation matrices and the errors obtained from the second method above, the Piogram that represents the transformation $\mathbf{T}_1^4$ in (49) is drawn. Then, the dot product between the two vectors is carried out graphically by tracing the paths along the Piogram from the perturbation vector to the error vector.

By using the proposed formulae in (15) and (16), the problem can be solved without the need for matrices operations, vectors construction with dot product, and/or Piograms. Basically, we simplify (52) by using the proposed formulae until we reach (53), and hence the errors can be found. Equation (52) can be attacked in different directions by using the proposed formulae. All directions should lead to the same results because the error angles ε_i are infinitesimal and the order of rotations in this case will not matter as indicated in (5). Before we begin, we need to state some useful tips on how to calculate such expressions: (a)

1) $\cos(\theta+\Delta\theta)$ and $\sin(\theta+\Delta\theta)$ are approximated by $\cos\theta$ and $\sin\theta$, respectively, for any angle θ and any infinitesimally small angle $\Delta\theta$.
2) Reversing the order of several successive infinitesimal rotations as in (5) is valid, as long as their number is not so high as explained in Remark 2.
3) If the successive rotations are around the same axis the angles of rotations can be added up.

Thus, let us start by applying the formulae in (15) and (16) to the quantities indicated by the down and above braces in (52). One possible way to solve the problem is shown in (55) below. Finally, comparing the solution obtained in (55) with (53), we get:

$$\begin{aligned}
\varepsilon_1 =& \delta_1 \mathrm{C}\alpha\mathrm{C}\beta + \delta_2\mathrm{S}\alpha\mathrm{C}\beta - \delta_3\mathrm{S}\beta + \delta_4\mathrm{C}\beta + \delta_7 \\
\varepsilon_2 =& \delta_1(\mathrm{C}\alpha\mathrm{S}\beta\mathrm{S}\gamma - \mathrm{S}\alpha\mathrm{C}\gamma) + \delta_2(\mathrm{S}\alpha\mathrm{S}\beta\mathrm{S}\gamma + \mathrm{C}\alpha\mathrm{C}\gamma) \\
&+ \delta_3\mathrm{C}\beta\mathrm{S}\gamma + \delta_4\mathrm{S}\beta\mathrm{S}\gamma + \delta_5\mathrm{C}\gamma + \delta_6\mathrm{S}\gamma + \delta_8 \\
\varepsilon_3 =& \delta_1(\mathrm{C}\alpha\mathrm{S}\beta\mathrm{C}\gamma + \mathrm{S}\alpha\mathrm{S}\gamma) + \delta_2(\mathrm{S}\alpha\mathrm{S}\beta\mathrm{C}\gamma - \mathrm{C}\alpha\mathrm{S}\gamma) \\
&+ \delta_3\mathrm{C}\beta\mathrm{C}\gamma + \delta_4\mathrm{S}\beta\mathrm{C}\gamma - \delta_5\mathrm{S}\gamma + \delta_6\mathrm{C}\gamma + \delta_9, \quad (54)
\end{aligned}$$

which is identical to the solution presented in [3].

Perhaps, solving the problem above by using the proposed formulae is not the fastest way. However, with the proposed formulae, one does not need matrices construction and multiplication, vectors formulation and dot product, or Piograms to solve the problem.

V. CONCLUSION

In this paper the infinitesimal rotation of the reference frame around a certain axis followed or preceded by a large rotation around another axis was discussed. In many cases, such transformations are required to be reversed as in the error analysis of the perturbations affecting coordinate rotations or some motion tracking algorithms. Good approximations are usually obtained by using the dot product of the unit vectors of the coordinate systems before and after the reversal, in general. In this work, we presented direct formulae that describe such approximations. Then, we showed that the proposed formulae of the given approximation can be used to construct the inverse of any transformation comprising successive large rotations. Moreover, we used the proposed formulae to find the error resulting from the perturbations that may affect Euler angle transformations in the problem posed in [3]. Obviously, the proposed formulae can solve the problem without matrices multiplication, vectors dot products, or Piograms.

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\begin{aligned}
\mathbf{T}_4^{4'} =& \mathbf{R}_z(\delta_9)\mathbf{R}_y(\delta_8)\mathbf{R}_x(\gamma+\delta_7)\mathbf{R}_z(\delta_6)\mathbf{R}_y(\beta+\delta_5)\mathbf{R}_x(\delta_4)\mathbf{R}_y(\delta_2\mathrm{C}\alpha)\mathbf{R}_x(\delta_2\mathrm{S}\alpha)\overbrace{\mathbf{R}_z(\alpha+\delta_3)\mathbf{R}_z(-\alpha)}^{\mathbf{R}_z(\delta_3)}\mathbf{R}_y(-\delta_1\mathrm{S}\alpha)\cdots \\
& \mathbf{R}_x(\delta_1\mathrm{C}\alpha)\mathbf{R}_y(-\beta)\mathbf{R}_x(-\gamma) \\
=& \mathbf{R}_z(\delta_9)\mathbf{R}_y(\delta_8)\mathbf{R}_x(\gamma+\delta_7)\mathbf{R}_z(\delta_6)\underbrace{\mathbf{R}_y(\beta+\delta_5)\mathbf{R}_x(\overbrace{\delta_1\mathrm{C}\alpha+\delta_2\mathrm{S}\alpha+\delta_4}^{\sigma_1})}_{\text{Approximate}}\mathbf{R}_y(\overbrace{-\delta_1\mathrm{S}\alpha+\delta_2\mathrm{C}\alpha}^{\sigma_2})\underbrace{\mathbf{R}_z(\delta_3)\mathbf{R}_y(-\beta)}_{\text{Approximate}}\mathbf{R}_x(-\gamma) \\
=& \mathbf{R}_z(\delta_9)\mathbf{R}_y(\delta_8)\mathbf{R}_x(\gamma+\delta_7)\mathbf{R}_z(\delta_6)\mathbf{R}_x(\sigma_1\mathrm{C}\beta)\mathbf{R}_z(\sigma_1\mathrm{S}\beta)\overbrace{\mathbf{R}_y(\beta+\delta_5)\mathbf{R}_y(\sigma_2)\mathbf{R}_y(-\beta)}^{\mathbf{R}_y(\sigma_2+\delta_5)}\mathbf{R}_x(-\delta_3\mathrm{S}\beta)\mathbf{R}_z(\delta_3\mathrm{C}\beta)\mathbf{R}_x(-\gamma) \\
=& \mathbf{R}_z(\delta_9)\mathbf{R}_y(\delta_8)\underbrace{\mathbf{R}_x(\gamma+\sigma_1\mathrm{C}\beta-\delta_3\mathrm{S}\beta+\delta_7)\mathbf{R}_z(\overbrace{\sigma_1\mathrm{S}\beta+\delta_3\mathrm{C}\beta+\delta_6}^{\sigma_3})}_{\text{Approximate}}\underbrace{\mathbf{R}_y(\sigma_2+\delta_5)\mathbf{R}_x(-\gamma)}_{\text{Approximate}} \\
=& \mathbf{R}_z(\delta_9)\mathbf{R}_y(\delta_8)\mathbf{R}_z(\sigma_3\mathrm{C}\gamma)\mathbf{R}_y(\sigma_3\mathrm{S}\gamma)\mathbf{R}_x(\gamma+\sigma_1\mathrm{C}\beta-\delta_3\mathrm{S}\beta+\delta_7)\mathbf{R}_x(-\gamma)\mathbf{R}_z(-(\sigma_2+\delta_5)\mathrm{S}\gamma)\mathbf{R}_y((\sigma_2+\delta_5)\mathrm{C}\gamma) \\
=& \mathbf{R}_x(\sigma_1\mathrm{C}\beta-\delta_3\mathrm{S}\beta+\delta_7)\mathbf{R}_y((\sigma_2+\delta_5)\mathrm{C}\gamma+\sigma_3\mathrm{S}\gamma+\delta_8)\mathbf{R}_z(-(\sigma_2+\delta_5)\mathrm{S}+\gamma\sigma_3\mathrm{C}\gamma+\delta_9)
\end{aligned} \tag{55}
$$

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