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Discrete-time IDA-PBC for underactuated mechanical systems with input-delay and matched disturbances

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***Abstract*— This work investigates the control problem of discrete-time underactuated mechanical systems with fixed input-delay and matched disturbances. A new control strategy is proposed, which builds upon a discrete-time implementation of the interconnection-and-damping-assignment passivity-based control (IDA-PBC) and extends it in two ways: the disturbances are estimated adaptively; the input-delay is compensated with a recursive algorithm. The resulting control law is constructed from IDA-PBC without solving any additional partial-differential-equation (PDE). Stability conditions are discussed and compared to alternative designs. Numerical simulations for the ball-on-beam system and for the Acrobot system demonstrate the effectiveness of the proposed approach.**

Keywords*— *Underactuated mechanical systems; Input-delay; Disturbance rejection; Discrete-time systems.

I. Introduction

Motivated by the current trend in remote control of robotic systems through wireless communication channels, recent works have investigated the control problem of underactuated mechanical systems with input-delay: a backstepping approach was employed for the stabilization of the cart-pole system in [1]; an adaptive-fuzzy tracking-control approach for multi-input-multi-output (MIMO) systems with dead-band was presented in [2]. While most research has been focusing on continuous-time systems, some results [3], [4] indicate that discrete-time formulations are more appropriate for a digital implementation. Although interconnection-and-damping-assignment passivity-based-control (IDA-PBC) is an effective strategy for equilibrium stabilization of underactuated mechanical systems [5], only a limited number of works have proposed direct discrete-time IDA-PBC designs so far [3], [6] and delay-free systems without disturbances were typically considered for simplicity. Conversely, new continuous-time IDA-PBC designs with enhanced robustness have recently been developed for underactuated systems with matched disturbances (i.e. affecting the actuated part of the state), either constant or bounded [7]–[9]. In conclusion, the extension of IDA-PBC to discrete-time underactuated mechanical systems with input-delay and disturbances remains an open question.

This work presents a new control strategy that builds upon the discrete-time IDA-PBC [3] including a disturbance-compensation term and employing a recursive algorithm [10] in order to compensate the effects of fixed input-delay. A rigorous stability analysis is conducted employing Lyapunov-type functions and sufficient conditions are provided. The resulting control law is constructed from the traditional IDA-PBC without having to solve additional partial-differential-equations (PDE) and only introduces a limited number of design parameters. Differently from other robust IDA-PBC designs, the controller is also applicable in case the disturbances are time-varying and in case the inertia matrix and the input matrix are not constant. Differently from previous works on underactuated systems with input-delay, the proposed approach is not specific to a particular system, does not require model linearization, and does not rely on the previous knowledge of the disturbance bounds or of its structure. Finally, the effectiveness of the control strategy is demonstrated with numerical simulations on the ball-on-beam system and on the Acrobot system.

The rest of the paper is organized as follows: Section II briefly outlines the problem formulation; Section III illustrates the main result; Section IV presents simulation results for the ball-on-beam system and for the Acrobot system; Section V contains the concluding remarks.

II. Problem Formulation

We consider an underactuated mechanical system with generalized position $q \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$ and input matrix $G(q) \in \mathbb{R}^{n\times m}$, where $\mathrm{rank}(G) = m < n$. We define the Hamiltonian $H = \Gamma(q,p) + V(q)$, where $\Gamma(q,p) = \frac{1}{2}p^T M^{-1} p$ is the kinetic energy, $M(q) \in \mathbb{R}^{n\times n}$ is the positive definite and invertible inertia matrix, $p = M\dot{q} \in \mathbb{R}^n$ are the momenta, and $V(q)$ is the open-loop potential energy. Employing the Euler approximation, the discrete-time open-loop dynamics in port-controlled Hamiltonian (PCH) form in the presence of a lumped matched disturbance $\delta \in \mathbb{R}^m$ and of a fixed input-delay $\tau \in \mathbb{N}$ is:

$$\begin{bmatrix} q(k+1) \\ p(k+1) \end{bmatrix} = \begin{bmatrix} q(k) \\ p(k) \end{bmatrix} + T\begin{bmatrix} 0 & I^n \\ -I^n & 0 \end{bmatrix}\begin{bmatrix} \nabla_q H(k) \\ \nabla_p H(k) \end{bmatrix} + T\begin{bmatrix} 0 \\ G \end{bmatrix}(u(k-\tau) - \delta) \tag{1}$$

where $T \in \mathbb{R}^+$ is the sampling interval and $k \in \mathbb{N}$. Finally, I^n is the $n \times n$ identity matrix and the symbol $\nabla_q(\cdot)$ represents the continuous gradient in q. The control aim consists in stabilizing the equilibrium $q = q^*, p = 0$ in closed-loop. In the absence of disturbances and input-delay ($\delta = 0$; $\tau = 0$) this objective is achieved by the discrete-time IDA-PBC [3] according to the following result which is stated for completeness.

Theorem 3.2 [3]: Consider the Euler model (1) with Hamiltonian $H = \Gamma(q,p) + V(q)$ and the following discrete-time controller, where $\kappa, K_v > 0$ are design parameters, $G^\dagger = (G^T G)^{-1} G^T$, and $J_2 = -J_2^T$ a free parameter matrix [5]:

$$
\begin{gathered}
u = \bar{u}_{es} + u_{di} \\
\bar{u}_{es} = G^\dagger\left(\nabla_q H - M_d M^{-1}\left(\overline{\nabla_q H_d}\right) + J_2 M_d^{-1} p\right) \\
u_{di} = -K_v G^T M_d^{-1} p
\end{gathered} \tag{2}
$$

Define $\overline{\nabla_q H_d} = \nabla_q H_d + T\kappa L_V M^{-1} p$, with the closed-loop Hamiltonian $H_d = \frac{1}{2} p^T M_d^{-1} p + V_d(q)$ and the matrix $L_V(q) \in \mathbb{R}^{n \times n}$ so that $P = (M_d^{-1} G G^\dagger M_d M^{-1} L_v M^{-1})$ is positive semi-definite. Then, the equilibrium $(q,p) = (q^*, 0)$, with $q^* = \mathrm{argmin}(V_d)$ is semi-globally practically asymptotically stable (SPAS) for sufficiently small sampling intervals [11] if the output $y = G^T M_d^{-1} p$ is detectable □

The discrete-time closed-loop dynamics (1),(2) becomes:

$$
\begin{bmatrix} q(k+1) \\ p(k+1) \end{bmatrix} = \begin{bmatrix} q(k) \\ p(k) \end{bmatrix} +
+T \begin{bmatrix} 0 & M^{-1} M_d \\ -M_d M^{-1} & J_2 - G k_v G^T \end{bmatrix} \begin{bmatrix} \overline{\nabla_q H_d} \\ \nabla_p H_d \end{bmatrix} \tag{3}
$$

As in the continuous-time design, H_d, M_d, J_2 should satisfy the following matching conditions, where $G^\perp$ is the left annihilator of G (i.e. $G^\perp G = 0$):

$$
G^\perp\left(\nabla_q H - M_d M^{-1} \nabla_q H_d + J_2 M_d^{-1} p\right) = 0 \tag{4}
$$

Computing the increment of the Lyapunov function candidate H_d over one sampling interval with the Euler method gives:

$$
\begin{gathered}
H_d(k+1) - H_d(k) = -T \nabla_p H_d^T G K_v G^T \nabla_p H_d \\
-\kappa T^2 p^T M^{-1} L_V M^{-1} p + O(T^3) \leq 0
\end{gathered} \tag{5}
$$

Using the Euler approximation, which is not Hamiltonian conserving, is appropriate in the context of IDA-PBC [3], since its aim is not to preserve the open-loop Hamiltonian H but to reshape it into H_d through the matching conditions (4).

III. Main Result

Investigating the control problem for the discrete-time system (1) with input-delay τ and disturbance δ is appealing for the following two reasons. Firstly, the input-delay for discrete-time systems can be treated in the same way as a measurement delay [12]. Secondly, the lumped disturbance can be considered constant within each sampling interval. The results in this section rely on the following assumptions:

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Assumption 1: the variation of the lumped disturbance δ during a sampling interval T is bounded, therefore $\exists \varepsilon \in \mathbb{R}^+$ so that $\forall k, |\delta(k+1) - \delta(k)| \leq \varepsilon$.

Assumption 2: there exists a sufficiently small sampling interval T for which the equilibrium $(q^*, 0)$ is SPAS for the closed-loop system (1),(2).

A. Delay-free system with disturbances

We initially consider system (1) without input-delay ($\tau = 0$) and study the control problem in the presence of matched disturbances. Since δ is unknown, we define the estimate $\tilde{\delta}(k) = \left(\hat{\delta}(k) + \beta(k-1)p\right)$ and the following adaptation law according to the discrete-time Immersion & Invariance [13], where $\alpha \in \mathbb{R}^{m \times m}$ is a diagonal matrix of constant parameters:

$$
\begin{gathered}
\hat{\delta}(k+1) = \hat{\delta}(k) + \beta(k-1)p - \beta(k)p \\
-T\beta(k)\left(G\left(u_0' - \tilde{\delta}(k)\right) - \nabla_q H\right) \\
\beta(k) = -\alpha G^T M^{-1}
\end{gathered} \tag{6}
$$

The control law (2) is modified as follows:

$$
\begin{gathered}
u_0' = \bar{u}_{es} + u_{di} + u_\delta \\
u_\delta = G^\dagger\left(\hat{\delta}(k) - \alpha G_{(k-1)} M_{(k-1)}^{-1} p\right)
\end{gathered} \tag{7}
$$

Proposition 3.1
Consider the closed-loop system (1),(7) under *Assumptions 1-2,* without input-delay and with lumped disturbance $\delta \in \mathbb{R}^m$ estimated according to (6). Define the matrix $L_V(q) \in \mathbb{R}^{n \times n}$ so that $P = (M_d^{-1} G G^\dagger M_d M^{-1} L_v M^{-1})$ is positive semi-definite. Define $\alpha, K_v, \kappa > 0$ so that $\alpha T (G^T M^{-1} G) > 0$ and $\lambda_{min}\{K_v\} \left|\nabla_p H_d^T G\right|^2 \geq \left|\nabla_p H_d^T\right| |G\varepsilon / T\alpha G^T M^{-1} G|$, where $\lambda_{min}\{K_v\}$ is the minimum eigenvalue of K_v. Then the estimation error z is bounded, and the equilibrium $(q^*, 0)$ with $q^* = \mathrm{argmin}(V_d)$ is (locally) stable.

Proof
To prove the first claim we define the vector of estimation errors $z(k) = \tilde{\delta}(k) - \delta(k)$, where $z, \tilde{\delta} \in \mathbb{R}^m$, and the Lyapunov function candidate $W = |z|$ corresponding to the Euclidean norm [14]. Computing z at the next time step and substituting the momenta from (1) we obtain:

$$
\begin{gathered}
z(k+1) = \hat{\delta}(k+1) - \delta(k+1) + \beta(k)p + \\
+T\beta(k)\left(G\left(u_0' - \tilde{\delta}(k) + z(k)\right) - \nabla_q H\right)
\end{gathered} \tag{8}
$$

Computing the increment of W, substituting (6) and (8), and recalling that $|\delta(k+1) - \delta(k)| \leq \varepsilon$ gives:

$$
\begin{gathered}
W(k+1) - W(k) \leq \\
|(1 - T\alpha G^T M^{-1} G)z \pm \varepsilon| - |z|
\end{gathered} \tag{9}
$$

If the lumped disturbance is constant, then $\varepsilon = 0$ and $W(k+1) \leq W(k)$ hence z converges to zero asymptotically. Conversely, z is bounded and converges to $\varepsilon / |T\alpha G^T M^{-1} G|$, where ε typically increases with T.

To prove the second claim we employ the Lyapunov function candidate H_d and compute its increment as in (5). Substituting (1),(7) and considering that $|z|$ converges to $\varepsilon / |T\alpha G^T M^{-1} G|$ we obtain for a sufficiently small T:

$$H_d(k+1) - H_d(k) = -T\nabla_p H_d^T G K_v G^T \nabla_p H_d + \mathcal{O}(T^3) \\ -\kappa T^2 p^T M^{-1} L_V M^{-1} p + T\nabla_p H_d^T G\varepsilon / |T\alpha G^T M^{-1} G| \tag{10}$$

Employing a similar approach to Proposition 2.1 in [9], we can rewrite (10) as

$$H_d(k+1) - H_d(k) \le -T\lambda_{min}\{K_v\}\left|\nabla_p H_d^T G\right|^2 \\ +T\left|\nabla_p H_d^T\right| |G\varepsilon / T\alpha G^T M^{-1} G| + \mathcal{O}(T^2) \tag{11}$$

If $\lambda_{min}\{K_v\}\left|\nabla_p H_d^T G\right|^2 \ge \left|\nabla_p H_d^T\right| |G\varepsilon / T\alpha G^T M^{-1} G|$ then $H_d(k+1) \le H_d(k)$ proving the second claim □

Remark 1: For comparison purposes, considering the IDA-PBC (2) in closed-loop with system (1) and computing the corresponding Lyapunov increment (10) gives:

$$H_d(k+1) - H_d(k) \le -T\lambda_{min}\{K_v\}\left|\nabla_p H_d^T G\right|^2 \\ +T\left|\nabla_p H_d^T\right| |G\delta| + \mathcal{O}(T^2) \tag{12}$$

In this case $H_d(k+1) \le H_d(k)$ and the equilibrium $(q^*, 0)$ is stable only if $\lambda_{min}\{K_v\}\left|\nabla_p H_d^T G\right|^2 \ge \left|\nabla_p H_d^T\right| |G\delta|$. This sufficient-condition is analogous to the one expressed in [9] and is typically more stringent compared to *Proposition 3.1* since it depends on the magnitude of the lumped disturbance rather than on its variation during the sampling interval T.

Remark 2: Alternatively to (6), the lumped disturbance can be estimated from the previous values of the system state and of the control input with time-delay-control method [15] as:

$$G\tilde{\delta}(k) \cong G_{(k-1)}\delta(k-1) = G_{(k-1)} u_0'(k-1) \\ -\nabla_q H(k-1) + \big(p(k-1) - p(k)\big)/T \tag{13}$$

In this case, the corresponding estimation error remains constant at $z = |\delta(k) - \delta(k-1)| \le \varepsilon$. Defining the control u_0' with $u_\delta = G^\dagger \tilde{\delta}(k)$ the Lyapunov increment (10) becomes:

$$H_d(k+1) - H_d(k) \le -T\lambda_{min}\{K_v\}\left|\nabla_p H_d^T G\right|^2 \\ +T\left|\nabla_p H_d^T\right| |G\varepsilon| + \mathcal{O}(T^2) \tag{14}$$

Compared to TDC (13), the adaptation law (6) allows further design freedom through the parameter α which effectively scales down the constant term ε.

Corollary 3.1

If the disturbance δ is constant and the output $y = G^T M_d^{-1} p$ is detectable, then the equilibrium $(q^*, 0)$ is SPAS for (1),(7) if $\alpha T(G^T M^{-1} G) > 0$ and $\lambda_{min}\{K_v\ \alpha G^T M^{-1} G\} > 1/8$.

Proof

Since the lumped disturbance is constant, $\varepsilon = 0$ and z converges to zero asymptotically (ref. *Proposition 3.1*). Employing the Lyapunov function candidate $W' = H_d + z^T z$, computing its increment and substituting (6),(7),(8) gives:

$$W'(k+1) - W'(k) \le -T\nabla_p H_d^T G K_v G^T \nabla_p H_d \\ +T\nabla_p H_d^T G z + |1 - T\alpha G^T M^{-1} G|^2 z^T z - z^T z + \mathcal{O}(T^2) \tag{15}$$

Omitting terms $\mathcal{O}(T^2)$ for a sufficiently small T and employing a Schur complement argument, we can rewrite (15) as:

$$W'(k+1) - W'(k) \le \\ -T\begin{bmatrix} \nabla_p H_d^T G & z^T \end{bmatrix} \begin{bmatrix} K_v & -I^n/2 \\ -I^n/2 & 2\alpha G^T M^{-1} G \end{bmatrix} \begin{bmatrix} G^T \nabla_p H_d \\ z \end{bmatrix} \tag{16}$$

Finally, $W'(k+1) \le W'(k)$ if $\lambda_{min}\{K_v\ \alpha G^T M^{-1} G\} > 1/8$, concluding the proof □

B. System with input-delay and disturbances

Employing a similar approach to [12], system (1) with fixed input-delay τ and lumped disturbance δ is rewritten as follows, where the terms $y_i = u'(k - \tau + i - 1);\ 1 \le i \le \tau$ represent the values of the control input at previous instants:

$$\begin{bmatrix} q(k+1) \\ p(k+1) \end{bmatrix} = \begin{bmatrix} q(k) \\ p(k) \end{bmatrix} + T\begin{bmatrix} 0 & I^n \\ -I^n & 0 \end{bmatrix}\begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + T\begin{bmatrix} 0 \\ G \end{bmatrix}(y_1 - \delta) \\ y_1(k+1) = y_2 \\ y_i(k+1) = y_{i+1} \\ y_\tau(k+1) = u' \tag{18}$$

We define the error $\zeta_i = u_0'\,(q(k+i-1), p(k+i-1)) - y_i$ where u_0' is the control input (7) of the delay-free system. Employing a similar approach to [10] the control law for system (18) is defined as:

$$u_0'(q(k), p(k)) = \bar{u}_{es} + u_{di} + u_\delta \\ u'(k) = u_0'\big(q(k+\tau), p(k+\tau)\big) + \gamma\zeta_1 \tag{19}$$

In particular, (19) consists of the predictive part $u_0'\big(q(k+\tau), p(k+\tau)\big)$ where the system state at successive instants is computed recursively from (18), and of a corrective part $\gamma\zeta_1$, which accounts for the difference from the delay-free control (7), where γ is a design parameter.

Proposition 3.2

Given the delay-free closed-loop system (1),(7) with stable equilibrium $(q^*, 0)$ according to *Proposition 3.1*. Then control (19) with $|\gamma| < 1$ ensures that the error terms $\zeta = [\zeta_1 \quad \zeta_i \quad \zeta_\tau\,]$ are bounded and converge to zero, while the closed-loop system (18),(19) has a (locally) stable equilibrium in $(q^*, 0)$.

Proof

To prove the convergence of ζ to zero we define the following Lyapunov function candidate:

$$W'' = \frac{1}{2}\zeta^T\zeta = \frac{1}{2}(\zeta_1^2 + \zeta_2^2 + .. + \zeta_\tau^2) \tag{20}$$

The values of $\zeta_i(k+1)$ at the next time step are:

$$\zeta_i(k+1) = u_0\left(q(k+i), p(k+i)\right) - y_{i+1} = \zeta_{i+1}(k) \\ \zeta_\tau(k+1) = u_0\left(q(k+\tau), p(k+\tau)\right) - u'(k) = -\gamma\zeta_1 \tag{21}$$

Computing the increment of W'' over one sampling interval and substituting (21) gives:

$$W''(k+1) - W''(k) = \\ \frac{1}{2}(\zeta_2^2 + .. + \zeta_\tau^2 + \gamma^2\zeta_1^2) - \frac{1}{2}(\zeta_1^2 + \zeta_2^2 + .. + \zeta_\tau^2) \le 0 \tag{22}$$

which holds true if $|\gamma| < 1$ and proves the first claim. Incidentally, at the equilibrium $(q^*, 0)$ the control input remains constant and $\zeta \equiv 0$ at all successive instants.

In order to prove the second claim we introduce the extended energy function $H_d^* = H_d + W''$ and compute its increment substituting (18),(19):

$$
\begin{aligned}
H_d^*(k+1) - H_d^*(k) &= -T\nabla_p H_d^T G K_v G^T \nabla_p H_d \\
&+ T\nabla_p H_d^T G(\varepsilon/|T\alpha G^T M^{-1} G| + \gamma\zeta_1(k-\tau)) \\
+\frac{1}{2}(\zeta_2^2 + .. + \zeta_\tau^2 + \gamma^2\zeta_1^2) &- \frac{1}{2}(\zeta_1^2 + .. + \zeta_\tau^2) + \mathcal{O}(T^2)
\end{aligned} \tag{23}
$$

Imposing $|\gamma| \propto T < 1$ in (23) and disregarding $\mathcal{O}(T^2)$ recovers (11) concluding the proof □

Remark 3: For comparison purposes, we consider the case $\gamma = 0$ corresponding to the following predictive control law:

$$
\begin{aligned}
u_0'(q(k), p(k)) &= \bar{u}_{es} + u_{di} + u_\delta \\
u'(k) &= u_0'\big(q(k+\tau), p(k+\tau)\big)
\end{aligned} \tag{24}
$$

In this case the errors $\zeta_i(k+1)$ change to:

$$
\begin{aligned}
\zeta_i(k+1) &= u_0'\big(q(k+i), p(k+i)\big) - y_{i+1} = \zeta_{i+1}(k) \\
\zeta_\tau(k+1) &= u_0'\big(q(k+\tau), p(k+\tau)\big) - u'(k) = 0
\end{aligned} \tag{25}
$$

Computing the Lyapunov increment (22) gives:

$$
\begin{aligned}
W'(k+1) - W'(k) &= \\
\frac{1}{2}(\zeta_2^2 + .. + \zeta_\tau^2) - \frac{1}{2}(\zeta_1^2 + .. + \zeta_\tau^2) &\le 0
\end{aligned} \tag{26}
$$

As a result, according to *Proposition 3.2* the closed-loop system (18),(24) also has a stable equilibrium in $(q^*, 0)$. Comparing (22) with (26) and simplifying common terms reveals that the control law (19) results in a more gradual convergence of the error ζ than its alternative (24).

Remark 4: The closed-loop system (18),(19) with $|\gamma| \propto T < 1$ results in the port-controlled Hamiltonian dynamics (3) with vanishing perturbation ζ. In particular, the dynamics of ζ remains decoupled from that of the system state. For instance considering $\tau = 2$ we have:

$$
\begin{bmatrix} q(k+1) \\ p(k+1) \\ \zeta_1(k+1) \\ \zeta_2(k+1) \end{bmatrix} = \begin{bmatrix} q(k) \\ p(k) \\ \zeta_1(k) \\ \zeta_2(k) \end{bmatrix} + \mathcal{O}(T^2) +
T\begin{bmatrix} 0 & M^{-1}M_d & 0 & 0 \\ -M_d M^{-1} & J_2 - G k_v G^T & 0 & 0 \\ 0 & 0 & -1/T & 1/T \\ 0 & 0 & -\gamma/T & -1/T \end{bmatrix}\begin{bmatrix} \nabla_q H_d^* \\ \nabla_p H_d^* \\ \nabla_{\zeta_1} H_d^* \\ \nabla_{\zeta_2} H_d^* \end{bmatrix} \tag{27}
$$

Remark 5: While *Theorem 3.2* [3] was used as staring point for the proposed control, the stability conditions provided in this section do not depend on the matrix $L_V(q)$ introduced in (2). In this respect, (7), (19) can alternatively be constructed from the emulation controller, which is also SPAS. For details on the advantages of (2) over the corresponding emulation controller, the reader is referred to [3].

IV. Simulation Results

A. Ball-on-beam System

The ball-on-beam system consists of a ball with point mass that is free to move along an actuated beam hinged in the middle and subject to a torque u. The continuous-time open-loop dynamics with disturbance δ and input-delay τ is:

$$
\begin{aligned}
\ddot{q}_1 + g\sin(q_2) - q_1\dot{q}_2^2 &= 0 \\
(L^2 + q_1^2)\ddot{q}_2 + gq_1\cos(q_2) + 2q_1\dot{q}_1\dot{q}_2 &= u(k-\tau) - \delta
\end{aligned} \tag{28}
$$

The corresponding discrete-time port-controlled Hamiltonian model (18) is obtained defining the inertia matrix $M = \begin{bmatrix} 1 & 0 \\ 0 & L^2 + q_1^2 \end{bmatrix}$, the input matrix $G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the open-loop potential energy $V = gq_1\sin(q_2)$. The parameter L is half the length of the beam, $q_1 \in (-L; L)$ is the position of the ball from the midpoint of the beam, $q_2 \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ is the inclination of the beam from the horizontal, and g is the gravity constant. The control aim is to stabilize the beam in the horizontal position with the ball at its midpoint ($q_1 = 0; q_2 = 0$). The discrete-time IDA-PBC [3] is employed here as baseline:

$$
\begin{aligned}
\bar{u}_{es} &= T\kappa\left(p_1\sqrt{L^2 + q_1^2} - p_2\sqrt{2}\right) + \\
&\frac{q_1}{\sqrt{2}(L^2 + q_1^2)}\left(-p_1^2\sqrt{L^2 + q_1^2} + p_1 p_2\sqrt{2} + \frac{p_2^2}{\sqrt{L^2 + q_1^2}}\right) \\
&+ gq_1\cos(q_2) - g\sqrt{2(L^2 + q_1^2)}\sin(q_2) - \\
&-k_p\sqrt{\frac{L^2 + q_1^2}{2}}\left(q_2 - 1/\sqrt{2}\,\mathrm{arcsinh}\left(\frac{q_1}{L}\right)\right) \\
u_{di} &= \frac{K_v}{L^2 + q_1^2}\left(p_1 - p_2\sqrt{\frac{2}{L^2 + q_1^2}}\right)
\end{aligned} \tag{29}
$$

The design parameters are $k_p, K_v, \kappa > 0$, the closed-loop inertia matrix M_d and potential energy V_d are defined as in [5]. The following values were employed in the numerical simulation: $L = 0.5; g = 9.81; T = 0.01; k_p = 1; K_v = 1; \kappa = 0.8; \alpha = 5; \gamma = 0.9$ and $\gamma = 0$ with input-delay $\tau = 5$. The disturbance was defined as $\delta = -0.1 + 0.02\mathrm{sign}(\dot{q}_2)$. The initial conditions were set to: $q(0) = (0.2; -0.1)$; $p(0) = (0; 0)$. Fig. 1 represents the time history of the position with control (19): the position settles at $q = (0.005; 0.005)$ and the ball remains within the length of the beam, while a higher overshoot corresponds to $\gamma = 0$. Conversely, with the baseline IDA-PBC (29) the system position settles around $q = (-0.195; -0.001)$. Notably, the robust IDA-PBC [7] is not applicable here since the inertia matrix M_d is not constant.

B. Acrobot System

The Acrobot system consists of an articulated pendulum with a single actuator at the elbow joint (q_2) and an unactuated shoulder joint (q_1). The open-loop system dynamics is:

$$
\dot{p} = -\nabla_q\left(\frac{1}{2}p^T M^{-1} p + V\right) + G(u(k-\tau) - \delta) \tag{30}
$$

The open-loop potential energy is $V = g(c_4\cos(q_1) + c_5\cos(q_1 + q_2))$, the input matrix is $G^T = [0 \quad 1]$, while $M = \begin{bmatrix} c_1 + c_2 + 2c_3\cos(q_2) & c_2 + c_3\cos(q_2) \\ c_2 + c_3\cos(q_2) & c_2 \end{bmatrix}$ is the open-loop inertia matrix with determinant $\Delta = \det(M) > 0$.

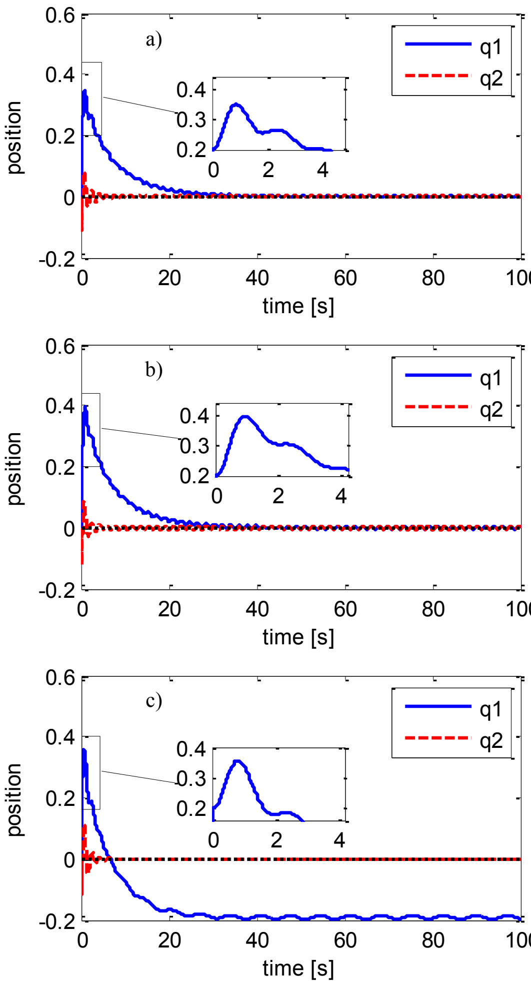

Figure 1. Ball-on-beam system: (a) position with control (19) and $\gamma = 0.9$; (b) with control (19) and $\gamma = 0$; (c) with baseline IDA-PBC (29).

The terms c_1, c_2, c_3, c_4, c_5 are constant system parameters, while g is the gravity constant. The control aim is to stabilize the upright position ($q_1 = q_2 = 0$). The continuous-time IDA-PBC [16] is used as basis for the discrete-time design:

$$\begin{aligned} u_{es} &= \frac{1}{2}\nabla_{q_2}(p^T M^{-1} p) + \nabla_{q_2} V - [k_2 \quad k_3] M^{-1} \nabla_q V_d \\ u_{di} &= \frac{K_v}{\Delta_d}(k_2 p_1 - k_1 p_2) \end{aligned} \tag{31}$$

where $\nabla_q V_d$, M_d are defined as in [16] and $K_v > 0$ (ref. Appendix). Since M_d is constant, the robust IDA-PBC [7] can be employed here in order to compensate the matched disturbances δ:

$$\begin{aligned} u &= \bar{u}_{es} + u_{di} + u_{PID} \\ u_{PID} &= -\left[K_p G^T M_d^{-1} G K_1 G^T M^{-1} + K_1 G^T M^{-1}\right]\nabla_q V_d \\ &\quad -[K_2 K_i (K_2^T + K_3^T G^T M_d^{-1} G K_1) G^T M^{-1}]\nabla_q V_d \\ &\quad -\left[K_i G^T M^{-1} \nabla_q^2 V_d M^{-1} + K_2 K_i K_3^T G^T M_d^{-1}\right] p \\ &\quad -\left(K_p G^T M_d^{-1} G K_2 + K_3\right) K_i \zeta \\ \dot{\zeta} &= (K_2^T G^T M^{-1} + K_3^T G^T M_d^{-1} G K_1 G^T M^{-1}) \nabla_q V_d \\ &\quad + K_3^T G^T M_d^{-1} p \end{aligned} \tag{32}$$

where $K_2 = (G^T M_d^{-1} G)^{-1}$ and K_p, K_i, K_1, K_3 are additional design parameters. The discrete-time counterparts of IDA-PBC (31) and (32) are obtained according to [3] choosing the matrix $L_V = \begin{bmatrix} \cos(q_2)^2 & c_3\cos(q2) \\ c_3\cos(q2) & c_3^2 \end{bmatrix}$.

The following parameters were employed in the numerical simulation: $c_1 = 2.3333$; $c_2 = 5.3333$; $c_3 = 2$; $c_4 = 3$; $c_5 = 2$; $g = 9.81$; $T = 0.01$; $k_1 = 0.3386$; $k_2 = 1$; $k_3 = 5.9073$. The design parameters for IDA-PBC (31) and (19) are: $\mu = -0.6019$; $k_0 = -350$; $k_u = 10$; $K_v = 30$; $\kappa = 0.8$; $\alpha = 2$; $\gamma = 0.9$ and $\gamma = 0$ (ref. Appendix). The parameters of the robust IDA-PBC (32) are: $K_p = K_v$; $K_i = 0.02$; $K_1 = 0.005$; $K_3 = 25$. The disturbance was defined as $\delta = -4 + 2\mathrm{sign}(\dot{q}_2)$. The initial conditions were set to $q(0) = (\pi; 0)$; $p(0) = (0; 0)$, and the input-delay to $\tau = 9$. Fig. 2 represents the time history of the position with control (19) and with the discrete-time version of the robust IDA-PBC (32). In the former case the system position settles at $q = (-0.001; 0.003)$, while with the latter the final position is $q = (6.96, -1.94)$. Although the discrete-time version of the robust IDA-PBC (32) can stabilize the upright position without input-delay, its performance degrades for larger τ.

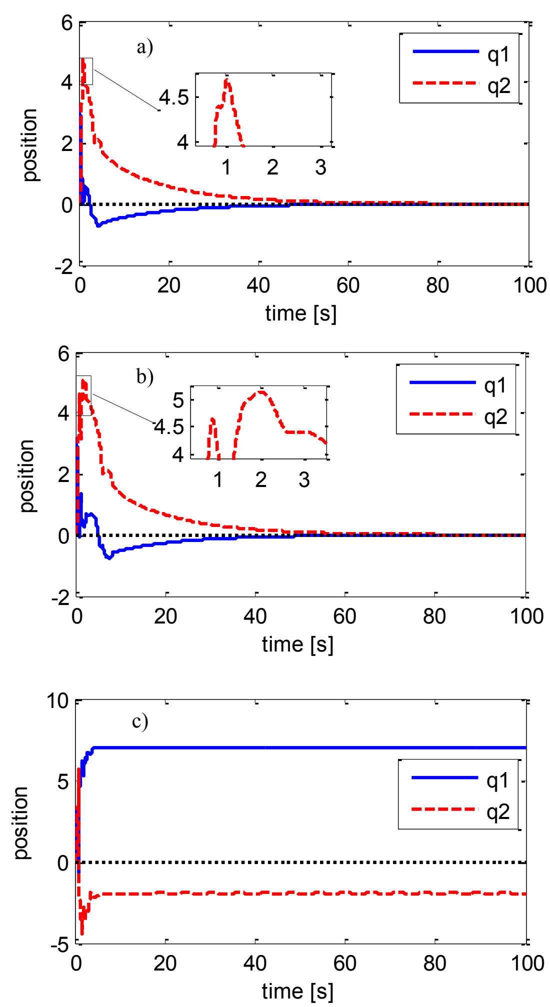

Figure 2. Acrobot system: (a) position with control (19) and $\gamma = 0.9$; (b) with control (19) and $\gamma = 0$; (c) with discrete-time version of (32).

Finally, with the discrete-time version of the baseline IDA-PBC (31) the final position is $q = (5.93; -9.59)$. Similarly to the ball-on-beam system, a higher overshoot is also registered for the Acrobot if $\gamma = 0$ in (19). This observation suggests that employing $\gamma \neq 0$, apart from affecting the dynamics of the error ζ (ref. *Remark 3*) can also be beneficial in terms of transient performance.

V. Conclusions

This work presented a new discrete-time IDA-PBC design for systems with fixed input-delay and variable matched disturbances. In particular, the proposed approach does not require solving additional PDE beyond the conventional matching conditions and is applicable to a large class of underactuated mechanical systems, including those with non-constant inertia matrix. Stability conditions were discussed and related to different IDA-PBC designs. Simulations on two benchmark examples demonstrated the effectiveness of the proposed approach. Further work will aim to extend the results to continuous-time systems and to unmatched disturbances. Finally, the results will be validated experimentally.

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Appendix

Constructive details of IDA-PBC for the Acrobot system [16]:

$$\begin{aligned}
\nabla_{q_1} V_d &= -k_0 \sin(q_1 - \mu q_2) - b_1 \sin(q_1) \\
&\quad -b_2 \sin(q_1 + q_2) - b_3 \sin(q_1 + 2q_2) \\
&\quad -b_4 \sin(q_1 - q_2) + k_u(q_1 - \mu q_2) \\
\nabla_{q_2} V_d &= k_0 \mu \sin(q_1 - \mu q_2) - b_2 \sin(q_1 + q_2) \\
&\quad -2b_3 \sin(q_1 + 2q_2) + b_4 \sin(q_1 - q_2) \\
&\quad -k_u(q_1 - \mu q_2) \\
M_d &= \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}
\end{aligned} \tag{A1}$$

The coefficients μ, k_0, k_u are design parameters, the determinant of M_d is $\Delta_d = k_1 k_3 - k_2^2 > 0$, while b_1, b_2, b_3, b_4 are defined as follows:

$$\begin{aligned}
b_1 &= \frac{g}{2k_2}\left(c_3 c_4 \pm 2c_4\sqrt{c_1 c_2}\right) \\
b_2 &= \frac{g\mu}{2k_2(\mu+1)}\left(c_3 c_4 \pm 2c_5\sqrt{c_1 c_2}\right) \\
b_3 &= \frac{g\mu c_3 c_5}{2k_2(\mu+2)} \\
b_4 &= \frac{g\mu c_3 c_4}{2k_2(\mu-1)}
\end{aligned} \tag{A2}$$

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