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Improved Robust Decentralized MPC

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***Abstract*— Systems become increasingly complex, grow in size and involve many subsystems. For such systems it is often desired to design local controllers for the subsystems, which do not exchange information. Reasons for this are multifold, spanning from limited communication, the desire for scalability, safety, up to flexibility. This work considers decentralized robust model predictive control for constrained linear, time-invariant systems, i.e. every subsystem operates a local controller and no communication between the controller takes place. For the design of the local controllers the proposed approach considers also the dynamics of the other subsystems, which allows to efficiently account for the unknown influence of the other subsystems. The approach guarantees closed loop robust constraint satisfaction and input-to-state stability and is illustrated by simulations.**

I. Introduction

Processes are becoming increasingly connected forming large systems of interconnected subsystems. Examples span from energy networks, to transportation systems and water distribution networks [1]–[3]. While the design of a central, global controller achieves the optimal performance, it is often not desired or possible: Communication between the subsystems might be very limited, not desired for safety and security reasons or impossible. The design of distributed or decentralized controllers, i.e. controllers which only exploit limited or no information about the subsystems while guaranteeing constraint satisfaction, robust operation, as well as performance is thus of significant importance.

Model predictive control (MPC), which is based on the repeated solution of an optimal control problem, see e.g. [4], is a flexible control approach that allows direct consideration of constraints, preview information, as well as systems with multiple inputs and outputs. By now multiple MPC approaches for interconnected systems exist [3], [5]–[7].

In *centralized MPC* a central controller is used: At each sampling instant the subsystems send their state information to the central controller, which computes the feedback solving a single, often large optimization problem. The feedback is sent back to the subsystems and implemented.

In *distributed MPC* each subsystem is equipped with a controller determining the subsystem's input based on the information available locally and from other controllers by solving smaller optimization problems. Various variants for distributed MPC exists, see e.g. [5]–[10], for example, based on cooperative, distributed optimization [8], [10]–[12] or based on robust control approaches, see [13]–[15].

Decentralized MPC denotes approaches without any communication between the subsystems [6], [16]–[18]. The local controllers do not exchange information, so they need to be robust to the unknown interactions between the subsystem. This can, for example, be achieved by tube-based MPC concepts [16], [17] or relying on input-to-state stability [18]. Decentralized MPC usually leads to decreased performance, as only limited information about the neighbors are used, but they do not require communication at all.

This work considers the control of uncertain, time-invariant, linear, constrained systems utilizing for each subsystem a tube-based MPC controller [5], [19]. The local controllers predict the future subsystem behavior based on local models and the local state. So, the influence of the other subsystems leads to a mismatch between the predictions and the real behavior. This mismatch is bounded in form of sets (so-called tubes), which are computed offline.

Existing tube based decentralized control approaches [16], [17] neglect that the subsystem interaction are governed by dynamics, i.e., cannot realize arbitrary trajectories. This simplification is conservative and decrease the performance.

The contribution of this work is an approach to compute tubes which considers directly the dynamics of the other subsystems. The calculated tubes are less restrictive, enabling the design of decentralized controllers with an improved performance. We also discuss closed loop properties.

The remainder of this paper is structured as follows: Section II introduces the problem setup. Section III outlines the decentralized, predictive controller. Section IV reviews existing approaches and presents the novel approach to compute the required tubes. Section V discusses stability properties. Finally, a summary and outlook is provided.

The notation is as follows: For two sets A, B $A \oplus B$, $A \ominus B$, $A \times B$ denote the Minkowski sum, the Minkowski difference and the Cartesian product, see e.g. [20]. For a positive definite matrix M $\|x\|_M^2 = x^T M x$.

II. System and Decentralized Control

A. Interconnected subsystems and global system

We consider the control of M *interconnected subsystems*. Subsystem i is given by

$$x_{k+1}^i = A^{i,i} x_k^i + B^{i,i} u_k^i + w_k^i + d_k^i, \tag{1a}$$

$$d_k^i = \sum_{j=1, i \neq j}^{M} A^{i,j} x_k^j + B^{i,j} u_k^j. \tag{1b}$$

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Here $x_i \in \mathbb{R}^{n_i}$, $n_i \geq 1$ and $u_i \in \mathbb{R}^{p_i}$, $p_i \geq 1$ are the subsystem's state and the control input, respectively. The local disturbance w_i is an unknown, but bounded by

$$w^i \in \mathbb{W}^i. \tag{2}$$

The dynamics of each subsystem i has two parts: the local dynamics $(A^{i,i}x^i_k + B^{i,i}u^i_k + w^i_k)$ and the influence d^i_k of the other subsystems. The local state x^i and the local input u^i need to satisfied at all times the constraints

$$x^i \in \mathbb{X}^i, \qquad u^i \in \mathbb{U}^i. \tag{3}$$

$\mathbb{X}^i$ and $\mathbb{U}^i$ are convex, compact polytopes containing the origin in the interior.

All subsystems together form the *global system* with the state x, the input u and the disturbance w:

$$x = \begin{pmatrix} x^1 \\ \vdots \\ x^M \end{pmatrix}, \quad u = \begin{pmatrix} u^1 \\ \vdots \\ u^M \end{pmatrix}, \quad w = \begin{pmatrix} w^1 \\ \vdots \\ w^M \end{pmatrix}. \tag{4}$$

The dynamics and the constraints of the global system are

$$x_{k+1} = Ax_k + Bu_k + w_k,\ x_k \in \mathbb{X},\ u_k \in \mathbb{U},\ w_k \in \mathbb{W}. \tag{5}$$

B. Decentralized Control

The desired controllers should only utilize local information, i.e. be of the form:

$$u^i_k = \kappa^i_k(x^i_k, z^i_k) \tag{6a}$$
$$z^i_{k+1} = \lambda^i_k(x^i_k, z^i_k), \tag{6b}$$

where z^i is the controller's state. The controllers should achieve the following closed loop property.

Definition 1: (Decentralized robust constraint satisfaction) *The control laws* (6) *guarantee* decentralized robust constraint satisfaction *for the global system* (5) *for sets* $\mathbb{X}^i_0$, $0 \in \mathrm{int}(\mathbb{X}^i_0)$*, if for any initial conditions* $x^i_0 \in \mathbb{X}^i_0$ *and any disturbance sequences* $\{w^i_k\}$, $w^i_k \in \mathbb{W}^i$ *the constraints* $u^i_k \in \mathbb{U}^i$ *and* $x^i_k \in \mathbb{X}^i$ *are satisfied.*

The set $\mathbb{X}^i_0$ is introduced as it might not be possible to guarantee constraint satisfaction for all states inside $\mathbb{X}^i$. One objective of this work is to increase the size of $\mathbb{X}^i_0$ without increasing the online computational complexity.

We furthermore desire to achieve input-to-state-stability:

Definition 2: (Input-to-state stability (ISS)) *The closed loop* (5), (6) *is said to be input-to-state stable, if it achieves decentralized robust constraint satisfaction and if there exists a class* $\mathcal{KL}$ *function* α *and a class* $\mathcal{K}$ *function* β *[21] such that for* $x^i_0 \in \mathbb{X}^i_0$ *for* $i = 1, \ldots, M$ *and admissible disturbance sequence* $\{w_k\}$, $w_k \in \mathbb{W}$*:*

$$\|x_{k+m}\| \leq \alpha(\|x_k\|, m) + \max_{0 \leq l \leq m-1} \beta(\|w_{k+l}\|). \tag{7}$$

This bound on the global state x_k (and convergence to the origin if $w_k \equiv 0$) implies a bound on each subsystems state x^i_k (convergence of each subsystem states to zero in absence of disturbances, since $\|x^i_{k+m}\| \leq \|x_{k+m}\|$ for any i.

III. Proposed decentralized MPC

We first outline the structure of the local controllers. In the second part the resulting MPC formulations are presented. The structure of the local controllers is similar to [16], [17].

A. Subsystem controllers

The controller for subsystem i consists of two components: the first component is a MPC control law based on the so-called nominal model:

$$\hat{x}^i_{k+1} = A^{i,i}\hat{x}^i_k + B^{i,i}\hat{u}^i_k, \tag{8}$$

which ignores the interactions (d^i_k) and the disturbance (w^i_k), compare (1). Consequently, the resulting predictive control law computes an input $\hat{u}^i_k$ based on the nominal state $\hat{x}^i_k$:

$$\hat{u}^i_k = \kappa^i_{k,MPC}(\hat{x}^i_k), \tag{9}$$

as outlined in the following subsection. The second component is a feedback $K^i \Delta x^i_k$ depending on the difference Δx^i_k between the real state x^i_k and nominal state $\hat{x}^i_k$:

$$\Delta x^i_k = x^i_k - \hat{x}^i_k, \tag{10}$$

This feedback counteracts and bounds the prediction mismatch Δx^i_k arising from neglecting the interaction term d^i_k and the disturbance w^i_k in the nominal model (8).

The overall feedback to subsystem i is given by

$$u^i_k = K^i \Delta x^i_k + \hat{u}^i_k = K^i(x^i_k - \hat{x}^i_k) + \kappa^i_{k,MPC}(\hat{x}^i_k). \tag{11}$$

Note that this feedback depends only on information available locally, the nominal state $\hat{x}^i_k$ and the real state x^i_k, it is thus a decentralized feedback of the form (6) (where $z^i_k = \hat{x}^i_k$). The question arises under which conditions the control law (11) guarantees decentralized constraint satisfaction and input-to-state stability (Definitions 1, 2).

For the gains K^i in the control laws (11) we assume:

Assumption 1: (Condition on controller gains K^i*)* *The control gains* K^i *are such that* $A^{i,i} + B^{i,i}K^i$ *and* $A + BK$ *are asymptotic stable, where* $K \in \mathbb{R}^{p \times n}$ *is*

$$K = \begin{pmatrix} K^1 & 0 & \cdots \\ 0 & K^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{12}$$

This Assumption requires that one can design control gains K^i stabilizing simultaneously the global dynamics and each subsystems dynamics, see e.g. [13], [15]–[17], [22].

B. Local MPC control laws

Subsystem i uses a model predictive control law to compute the nominal input $\hat{u}^i_k$, which is described in the following. The controller determines an input sequence $\hat{\mathbf{u}}^i$ and a predicted state sequence $\hat{\mathbf{x}}^i$

$$\hat{\mathbf{u}}^i = \{\hat{u}^i_{k|k}, \ldots, \hat{u}^i_{k+N-1|k}\}, \tag{13a}$$
$$\hat{\mathbf{x}}^i = \{\hat{x}^i_{k|k}, \ldots, \hat{x}^i_{k+N|k}\}, \tag{13b}$$

minimizing a convex, quadratic cost function $J^i(\hat{\mathbf{x}}^i, \hat{\mathbf{u}}^i)$:

$$J^i = \|\hat{x}^i_{k+N|k}\|^2_{P^i} + \sum_{j=k}^{k+N-1} \|\hat{x}^i_{j|k}\|^2_{Q^i} + \|\hat{u}^i_{j|k}\|^2_{R^i}. \quad (14)$$

The matrices Q^i, R^i and P^i are positive definite. The input sequence $\hat{\mathbf{u}}^i$ and a state sequence $\hat{\mathbf{x}}^i$ need to satisfy the nominal (decoupled) dynamics (8):

$$\hat{x}^i_{j+1|k} = A^{i,i}\hat{x}^i_{j|k} + B^{i,i}\hat{u}^i_{j|k}, \quad (15a)$$

and are subject to constraints of the form

$$\hat{x}^i_{j|k} \in \tilde{\mathbb{X}}^i_k,\ \hat{u}^i_{j|k} \in \tilde{\mathbb{U}}^i_k,\ j = k, \ldots, k+N-1, \quad (15b)$$

as well as a terminal constraint

$$\hat{x}^i_{k+N|k} \in \mathbb{T}^i_k. \quad (15c)$$

Furthermore, there are constraints on the first part of the predicted state sequence, which are for $k = 0$ given by:

$$x^i_0 - \hat{x}^i_{0|0} \in \Xi^i \quad (16)$$

or for $k > 0$ by:

$$\hat{x}^i_{k|k} = \hat{x}^i_k. \quad (17)$$

Note that $\tilde{\mathbb{X}}^i_k$ and $\tilde{\mathbb{U}}^i_k$ are tightened versions of (3). These sets as well as the terminal constraint set $\mathbb{T}^i_k$ and initial state constraint set Ξ^i are convex, polytopic sets containing the origin, details are provided in the next section. They guarantee constraint satisfaction despite the mismatch of the nominal system and the real system. Furthermore, motivated by [23], we assume that the initial nominal states are either free and optimized ($k = 0$) or are fixed to the nominal dynamics (8). This allows an easier consideration of the resulting interactions, as outlined in the next section.

In summary, for $k > 0$, the optimization problem $\mathcal{O}^i_k(\hat{x}^i_k)$ for subsystem i becomes

$$\min_{\hat{\mathbf{x}}^i, \hat{\mathbf{u}}^i} J(\hat{\mathbf{x}}^i, \hat{\mathbf{u}}^i) \text{ s.t. (15), (17).} \quad (18)$$

The first part of the optimal input is the nominal input for subsystem i:

$$\hat{u}^i_k = \kappa_{k,MPC}(\hat{x}^i_k) = \hat{u}^{i,\star}_{k|k}. \quad (19)$$

For $k = 0$, the nominal state $\hat{x}^i_0$ is an optimization variable, i.e. the resulting formulation $\mathcal{O}^i_0(x^i_0)$ becomes

$$\min_{\hat{\mathbf{x}}^i, \hat{\mathbf{u}}^i} J^i(\hat{\mathbf{x}}^i, \hat{\mathbf{u}}^i) \text{ s.t. (15), (16)} \quad (20)$$

and its solution provides the initial nominal state $\hat{x}^i_0 = \hat{x}^{i,\star}_0$ and the nominal input $\hat{u}^i_0 = \kappa_{0,MPC}(\hat{x}^i_0)$.

Note that the resulting optimization problem (18), (20) are convex quadratic programs, for which by now several efficient solution strategies exist, see e.g. [3], [24]–[26].

IV. Design of Suitable Tubes

We use the idea of tubes to guarantee that the real system satisfies the constraints (3) although the controllers' decisions are based on the nominal model, ignoring the interactions and disturbances. Loosely speaking, tubes are bounds in form of sets on the worst case difference Δx^i between the real state x^i and its nominal prediction $\hat{x}^i$. The tubes are sets $\Delta\mathbb{X}^i_k$, computed offline and are used to setup the constraints (15b) and to design the terminal sets (15c).

The mismatch Δx^i_k (10) is governed by the dynamics

$$\Delta x^i_{k+1} = (A^{i,i} + B^{i,i}K^i)\Delta x^i_k + d^i_k + w^i_k. \quad (21)$$

In contrast to centralized robust MPC, where only a disturbance w^i_k is present, here also a coupling term d^i_k depending on the other subsystems (1b) needs to be considered. So, one cannot directly use centralized results to compute the tubes.

However, with suitable tubes one can guarantee rather straightforwardly decentralized robust constraint satisfaction. In detail, we consider that the tubes satisfy

Condition 1: (Required properties of $\Delta\mathbb{X}^i_k$) *The sets $\Delta\mathbb{X}^i_k$, $k > 0$ guarantee that $\Delta x^i_k \in \Delta\mathbb{X}^i_k$ given by* (5), (11), (21), *if $\Delta x^i_0 \in \Delta\mathbb{X}^i_0$ and $x_m \in \mathbb{X}$, $u_m \in \mathbb{U}$, $m < k$.*
This assumption allows us to derive the following general result, which is independent of the actual approach used to compute the tube. Note that $\Delta\mathbb{X}^i_0$ is a design/tuning variable.

Proposition 1: (Decentr. robust constraint satisfaction) *Let Assumptions 1 hold and let $\Delta\mathbb{X}^i_k$ satisfy Condition 1. Furthermore, assume that the tightened constraints* (15b)*, the terminal sets $\mathbb{T}^i_k$* (15c) *and the initial sets Ξ^i* (16) *satisfy:*

$$\tilde{\mathbb{X}}^i_k = \mathbb{X}^i \ominus \Delta\mathbb{X}^i_k,\ \tilde{\mathbb{U}}^i_k = \mathbb{U}^i \ominus K\Delta\mathbb{X}^i_k, \quad (22a)$$

$$\mathbb{T}^i_k \subseteq \tilde{\mathbb{X}}^i_k,\ K\mathbb{T}^i_k \subseteq \tilde{\mathbb{U}}^i_k,\ (A^{i,i} + B^{i,i}K^i)\mathbb{T}^i_k \subseteq \mathbb{T}^i_{k+1}, \quad (22b)$$

$$\Xi^i = \Delta\mathbb{X}^i_0. \quad (22c)$$

If $\mathcal{O}^i_0(x_0)$, $i = 1, \ldots, M$ are feasible, then the closed loop achieves decentralized robust constraint satisfaction.
The proof can be found in the appendix.

Proposition 1 requires that suitable tubes are given. In the remainder of this section we outline existing approaches and present a new, less conservative approach that takes the interactions between the systems into account.

A. Existing approaches to obtain suitable tubes

One way to compute bounds on Δx^i_k is to treat the interaction term d^i_k as an additional, additive "disturbance" d^i_k, similar as in [16], [17]. It is assumed that this disturbance satisfies $d^i_k \in \mathbb{D}^i$, where

$$\mathbb{D}^i = \bigoplus_{j=1, i\neq j}^{M} A^{i,j}\mathbb{X}^j \oplus B^{i,j}\mathbb{U}^j, \quad (23)$$

assuming that the other subsystems $j = 1, \ldots, M$, $j \neq i$ satisfy their own constraints (3) at time k.

Note that beyond the constraints little information about the other subsystems for the dynamics of d^i_{k+m}, $m > 0$ is used. This simplification allows to utilize standard methods to determine a tube for Δx_k^i (21) by a robust positive invariant set $\Delta\mathbb{X}_{ex.}^i$:

$$\Delta\mathbb{X}_{ex.}^i \supseteq (A^{i,i}+B^{i,i}K^{i,i})\Delta\mathbb{X}_{ex.}^i \oplus \mathbb{D}^i \oplus \mathbb{W}^i. \tag{24}$$

If $\Delta x_k^i \in \Delta\mathbb{X}_{ex.}^i$, then for any $x_k \in \mathbb{X}$, $u_k \in \mathbb{U}$ and $w_k^i \in \mathbb{W}^i$, $\Delta x_{k+1}^i \in \Delta\mathbb{X}_{ex.}^i$ holds. Thus, the robust invariance of the set $\Delta\mathbb{X}_{ex.}^i$ is guaranteed as long as the constraints (3) are satisfied and as long as $w_k^i \in \mathbb{W}^i$, which leads to the following proposition

Proposition 2: (Tubes as robust invariant sets)
The sets $\Delta\mathbb{X}_k^i = \Delta\mathbb{X}_{ex.}^i$ (24) *satisfy Condition 1.*
The proof follows similar ideas as in [16], [17], only the additionally disturbances w_k^i need to be considered.

This approach allows to compute suitable tubes bounding the effect of the disturbances and the subsystems' interactions. However, these tubes are conservative, as the dynamics of other subsystems is not taken into account.

Remark 1: (Choice/computation of the sets)
The terminal sets $\mathbb{T}_k^i$ as well as the tightened constraints $\tilde{\mathbb{X}}_k^i$, $\tilde{\mathbb{U}}_k^i$ can be time-invariant, since the conditions (22) *are the same for every k. To compute these sets and the robust positive invariant sets $\Delta\mathbb{X}_{ex.}^i$ one can use efficient software tools, e.g. [27], [28].*

B. Tube calculation taking the interaction dynamics into account (Main result)

This section proposes an approach to bound Δx_k^i, $k \geq 0$, $i = 1,\dots,M$ taking the interaction dynamics of the other subsystems into account. This enables less conservative tightened constraints $\tilde{\mathbb{X}}_k^i$, $\tilde{\mathbb{U}}_k^i$ and terminal sets $\mathbb{T}_k^i$.

As above the bounds on the worst case values of Δx^i (21) given by the tubes $\overline{\Delta\mathbb{X}}_k^i$ are valid if the system (5) is controlled by the (yet to be designed) control law (11).

We aim to determine sets (tubes) $\overline{\Delta\mathbb{X}}_k^i$ bounding the admissible Δx_k^i exploiting the knowledge of the dynamics and constraints of subsystems i and the other subsystems. The basic idea is to find such tubes by a convex "forward" prediction, as described in the following:

The set $\overline{\Delta\mathbb{X}}_k^i$ consists of all $\Delta\overline{x}_k^i$ such that there exists a state sequences $\{\overline{x}_m\}$, an input sequences $\{\overline{u}_m\}$, disturbance sequences $\{\overline{w}_m\}$ and mismatch sequences $\{\Delta\overline{x}_m^i\}$, where $m = 0,\dots,k-1$, such that

$$\Delta\overline{x}_{m+1}^i = (A^{i,i}+B^{i,i}K^{i,i})\Delta\overline{x}_m^i + \overline{w}_m^i + \sum_{l=1,i\neq l}^{M} A^{i,l}\overline{x}_m^l + B^{i,l}\overline{u}_m^l, \tag{25a}$$

$$\Delta\overline{x}_0^i \in \Delta\mathbb{X}_0^i,\ \overline{w}_m \in \mathbb{W}, \tag{25b}$$

$$\overline{x}_{m+1} = A\overline{x}_m + B\overline{u}_m + \overline{w}_m, \tag{25c}$$

$$\overline{x}_m \in \mathbb{X},\ \overline{u}_m \in \mathbb{U}, \tag{25d}$$

where $\Delta\mathbb{X}_0^i$ is a given design parameter. Note that the set $\overline{\Delta\mathbb{X}}_k^i$ can be written in short form as

$$\overline{\Delta\mathbb{X}}_k^i = \{\Delta\overline{x}_k^i : \exists\{\overline{x}_m,\overline{u}_m,\overline{w}_m,\Delta\overline{x}_m^i\},\ \text{s.t. (25) holds}\}. \tag{26}$$

The sets $\overline{\Delta\mathbb{X}}_k^i$ have the desired properties, i.e, are tubes:

Proposition 3: (Time varying tubes)
The sets $\Delta\mathbb{X}_k^i = \overline{\Delta\mathbb{X}}_k^i$ (26) *satisfy Condition 1.*
The proof is located in the appendix.

Note that the set $\overline{\Delta\mathbb{X}}_k^i$ is given by affine constraints, i.e. it is a convex polytope. Consequently, it is possible to compute $\tilde{\mathbb{X}}_k^i$ and $\tilde{\mathbb{U}}_k^i$ by solving linear programs, which can be done efficiently [24].

Remark 2: (Non global computation)
The definition of $\overline{\Delta\mathbb{X}}_k^i$ (26) *includes the global dynamics* (25c). *If there are a large number of subsystems, then it might be challenging to calculate the tightened constraints $\tilde{\mathbb{X}}_k^i$ and $\tilde{\mathbb{U}}_k^i$ based on the set $\overline{\Delta\mathbb{X}}_k^i$. A rather straightforward extension is to consider only for some subsystem the dynamics directly, e.g. the one with a large influence, and treat the remaining subsystems similar as above/in [16], [17] using simple bounds. This reduces the accuracy, but enables the application of the approach to large-scale systems. A detailed discussion of this aspect is beyond the scope of this work.*

In the following we focus on the case that $\overline{\Delta\mathbb{X}}_0^i$ is chosen based on the existing approach, which allows to derive certain additional results. The next result underpins that the proposed approach enables less conservative results:

Proposition 4: (Decreased conservatism)
If $\overline{\Delta\mathbb{X}}_0^i = \Delta\mathbb{X}_{ex.}^i$ with $\Delta\mathbb{X}_{ex.}^i$ as in (24) *for $i = 0,\dots,M$, then $\overline{\Delta\mathbb{X}}_k^i \subseteq \Delta\mathbb{X}_{ex.}^i$ for $\overline{\Delta\mathbb{X}}_k^i$ given by* (26).
The proof can be found in the appendix.

Proposition 3 requires to compute the sets $\overline{\Delta\mathbb{X}}_k^i$ for $k = 1,2,\dots$, i.e. possibly an infinite number of sets. Fortunately, it is possible to stop the computation of $\overline{\Delta\mathbb{X}}_k^i$ at some (an arbitrary) k and approximate safely the remaining sets. Thus only a finite number of sets need to be computed.

Proposition 5: (Monotonicity of $\overline{\Delta\mathbb{X}}_k^i$)
If for $i = 1,\dots,M$ $\overline{\Delta\mathbb{X}}_0^i = \Delta\mathbb{X}_{ex.}^i$ with $\Delta\mathbb{X}_{ex.}^i$ as in (24), *then $\overline{\Delta\mathbb{X}}_k^i \subseteq \overline{\Delta\mathbb{X}}_{k-m}^i$ for $m < k$ for $\overline{\Delta\mathbb{X}}_k^i$ given by* (26).
The proof is sketched in the Appendix.

In summary, the proposed approach allows to determine the tube with less conservatism, if $\overline{\Delta\mathbb{X}}_0^i$ are chosen based on the existing approach. It also allows to trade computational complexity with accuracy as it is sufficient to compute the sets $\overline{\Delta\mathbb{X}}_k^i$ only for a finite, selectable number of steps.

Remark 3: (Time-invariant tubes)
Above, we focused on the choice of $\overline{\Delta\mathbb{X}}_0^i$ based on the existing approach, which resulted in time-varying tubes. An open question is wheter it is possible to determine robust positive invariant sets for (25).

V. ISS STABILITY

We shortly discuss input-to-state stability of the closed loop system. To guarantee ISS the choice of the penalty matrices in the cost function (14) need to be:

Assumption 2: (Choice of terminal penalties P^i)
For $i = 1,\dots,M$ the weighting matrices P^i are such that $P^i = (A^{i,i}+B^{i,i}K^i)^T Q^i (A^{i,i}+B^{i,i}K^i) + (K^i)^T R^i K^i$.
Now, we can state requirements ford ISS (Definition 2):

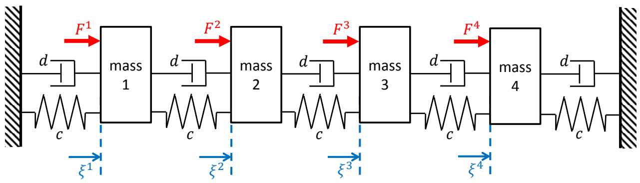

Fig. 1. Example system: Chain of masses connected by springs.

Proposition 6: (ISS of the closed loop)
Let Assumptions 1, 2 hold and let $\Delta\mathbb{X}^i_k$ satisfy Condition 1 and the tightened constraints (15b)*, the terminal sets $\mathbb{T}^i_k$* (15c)*, the initial sets Ξ^i* (16) *be given by* (22)*. If the optimization problems $\mathcal{O}^i_0(x_0)$, $i = 1, \ldots, M$ are feasible, then ISS of the closed loop* (5), (11) *is achieved.*
The corresponding proof is sketched in the appendix.

VI. Illustrative Example

To illustrate the results we consider a system of four masses connected with each other and with walls by spring and dampers as shown in Figure 1. Attached to each mass is an actuator, which can supply a force $F^i \in [-20, 20]$. The position $\xi^i \in [-8, 8]$ and speed $\rho^i \in [-10, 10]$ of each mass are the state of subsystem i, whereas F^i are the subsystems' inputs. The spring constants are $c = 1$, each mass has an inertia 1 and the damping coefficient is $d = 0.1$.

Using a sampling time of $T = 0.1$ and a zero order hold the continuous-time dynamics are exactly discretized. With respect to the disturbances, we assume that each mass is affected by a disturbance w^i_k with $\|w^i_k\|_\infty \leq 0.01$. The state/input constraints are given by $\mathbb{X}^i = [-8, 8] \times [-10, 10]$ and $\mathbb{U}^i = [-20, 20]$, respectively.

The controller gains K^i are chosen as

$$K^1 = K^4 = \begin{pmatrix} 0.9224 & -1.9062 \end{pmatrix},$$
$$K^2 = K^3 = \begin{pmatrix} 0.9219 & -1.9065 \end{pmatrix}.$$

Note that for these controller gains Assumption 1 is satisfied.

Sets $\Delta\mathbb{X}^i_k$ and tightened constraints: Figure 2 shows sets $\overline{\Delta\mathbb{X}}^i_k$ (26) for varying k starting with the initial sets $\overline{\Delta\mathbb{X}}^i_0 = \Delta\mathbb{X}^i_{ex.}$ for the second subsystem/mass (due to symmetry the sets for the third subsystem/mass is equal)[1]. Observe that the size of the sets decreases as k increases. Moreover, note that the sets are contained in each other and that in particular they are inside $\Delta\mathbb{X}^i_{ex.}$, i.e. the bounds/tubes of the proposed approach (Section IV-B) are less restrictive than the one of the existing approach (Section IV-A). These observations are in accordance with Propositions 4 and 5.

The tightened state/input constraints are also boxes:

$$\tilde{\mathbb{X}}^i_k = [-\tilde{\xi}^i_k, \tilde{\xi}^i_k] \times [-\tilde{\rho}^i_k \times \tilde{\rho}^i_k], \quad \tilde{\mathbb{U}}^i_k = [-\tilde{F}^i_k, \tilde{F}^i_k]. \quad (27)$$

Figure 3 illustrate the tightened state constraints for the existing/proposed approach using $\overline{\Delta\mathbb{X}}^i_0 = \Delta\mathbb{X}^i_{ex.}$. Observe that for the proposed approach the tightened constraints

[1]The set computations are done in Matlab using YALMIP [29], MPT3 [27] and the PnPMPC toolbox [28].

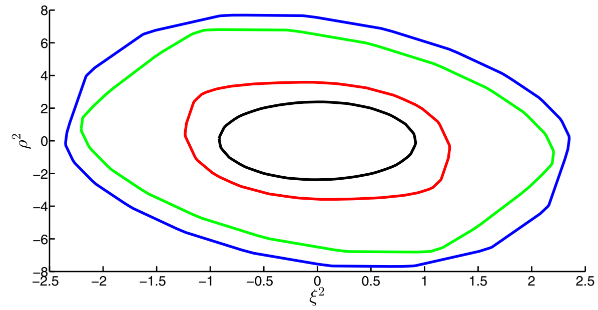

Fig. 2. Boundaries of sets/tube cross-section $\overline{\Delta\mathbb{X}}^2_k$ for $k = 0$ (blue), $k = 4$ (green), $k = 10$ (red) and $k = 40$ (black).

monotonically increase in size (become less conservative) as k increases and are less restrictive as the ones for the existing approach for $k > 0$; as expected (Propositions 4, 5).

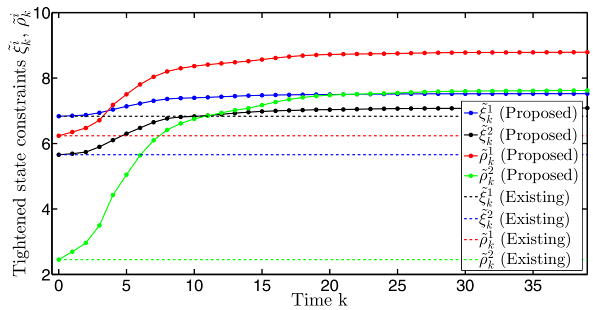

Fig. 3. Tightened state constraints $\tilde{\mathbb{X}}^i_k$, (27).

Simulation results - stabilization: Next, the proposed, decentralized MPC scheme is used for stabilization. Therefore we choose $Q^i = I$, $R^i = 0.1$, a terminal penalty P^i satisfying Assumption 2 and the horizon length as $N = 30$.

Figure 4 shows the closed loop response of the proposed decentralized MPC scheme with randomly generate process noise $w_k \in \mathbb{W}$. We observe that the closed loop system is asymptotic stable and the constraints are satisfied.

Simulation results - set point tracking: Finally, we investigate the decentralized tracking of (unreachable) set points. This is done using a rather straightforward extension of the proposed approach based on [30], compare [31].

The set points are chosen such that $\xi^1 = \xi^4 = -4$ and $\xi^2 = \xi^3 = 4$. As these set points cannot be reached the controller of the first/last subsystem should move the mass as as possible in the negative direction, whereas the second and third mass should be moved in the positive direction. Note that the each subsystem knows only its own set point.

Figure 5 and Figure 6 illustrate the results for the existing approach and for the proposed approach, respectively. We observe that the proposed approach delivers better results, the positions ξ are closer to the desired set points.

VII. Summary and Outlook

This work presents a method to design predictive control laws for the robust, decentralized control of uncertain, constrained, linear systems consisting of multiple interacting

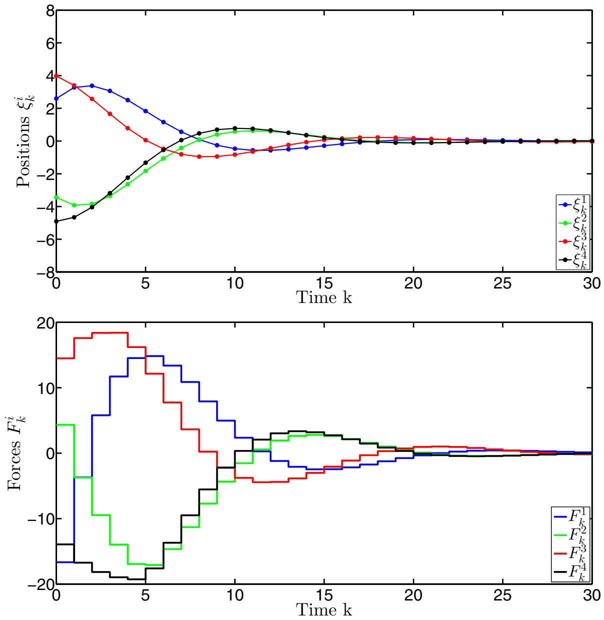

Fig. 4. Simulation results, stabilization

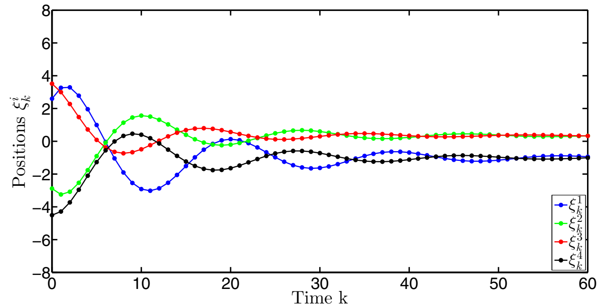

Fig. 5. Simulation results, set point tracking, existing approach

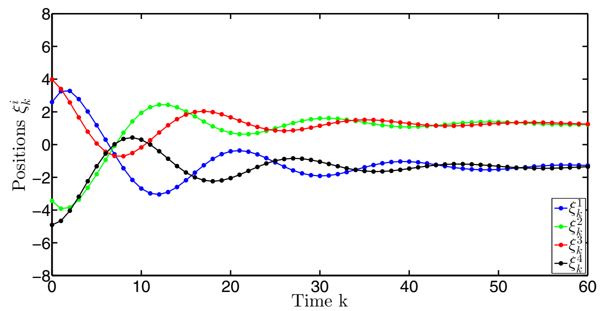

Fig. 6. Simulation results, set point tracking, proposed approach

subsystems. A new approach to compute bounds/tubes on the influence of the subsystem interaction and disturbances is presented. The approach takes the coupled subsystems dynamics directly into account, which leads to less conservative results. It achieves closed loop robust stability and robust constraint satisfaction and is illustrated by simulations.

The results can be extended to the output feedback case, i.e. considering that the local state is unavailable and needs to be estimated from measurements. Moreover, it might be interesting to use the proposed approach to derive resilient distributed control schemes, which also work if the communication is not possible at certain time instants, e.g. due to packet drop outs.

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APPENDIX

A. Proof of Proposition 1

We show that (a) the constraints (3) are satisfied at k, if $\mathcal{O}_0^i(x_0^i)$/$\mathcal{O}_k^i(\hat{x}_k^i)$ is feasible and $\Delta x_k^i \in \overline{\Delta\mathbb{X}}_k^i$, $i = 1, \ldots, M$, if $k > 1$, and (b) feasibility of the optimization problems $\mathcal{O}_k^i(\hat{x}_k^i)$ for $k > 0$.

(a) Constraint satisfaction: Feasibility of $\mathcal{O}_0^i(x_0^i)$ implies $\hat{x}_0^i \in \tilde{\mathbb{X}}_0^i$, $\hat{u}_0^i \in \tilde{\mathbb{U}}_0^i$ and $\Delta x_0^i \in \overline{\Delta\mathbb{X}}_0^i$. Feasibility of $\mathcal{O}_k^i(\hat{x}_k^i)$ implies only that $\hat{x}_k^i \in \tilde{\mathbb{X}}_k^i$ and $\hat{u}_k^i \in \tilde{\mathbb{U}}_k^i$, but we have $\Delta x_k^i \in \overline{\Delta\mathbb{X}}_k^i$ due to Condition 1. Due to the choice of the tightened constraints (22a), it follows that $u_k^i \in \mathbb{U}^i$ and $x_k^i \in \mathbb{X}^i$ using (10), (11) and standard arguments, compare [5].

(b) Recursive feasibility: If $\mathcal{O}_k^i(\hat{x}_k^i)$/$\mathcal{O}_0^i(x_0^i)$ is feasible, then there are at least one $\hat{\mathbf{u}}^i$ and $\hat{\mathbf{x}}^i$ satisfying the nominal dynamics, $\hat{x}_{k+j|k}^i \in \tilde{\mathbb{X}}_{k+j}^i$, $\hat{u}_{k+j|k}^i \in \tilde{\mathbb{U}}_{k+j}^i$ where $j = 1, \ldots, N-1$ and $\hat{x}_{k+N|k}^i \in \mathbb{T}_k^i$. One can verify that the control/input sequence given by $\hat{x}_{k+j|k+1}^i = \hat{x}_{k+j|k}^i$, $\hat{x}_{k+N+1|k+1}^i = (A^{i,i} + B^{i,i}K^i)\hat{x}_{k+N|k} \in \mathbb{T}_{k+1}^i$, $\hat{u}_{k+j|k+1}^i = \hat{u}_{k+j|k}^i$ and $\hat{u}_{k+N+1|k+1}^i = K^i\hat{x}_{k+N|k}$ is a feasible solution for $\mathcal{O}_{k+1}^i(\hat{x}_{k+1}^i)$ due to the assumptions made on the terminal sets and tightened constraints (22).

B. Proof of Proposition 3

The idea to show that $\Delta\overline{x}_k^i \in \overline{\Delta\mathbb{X}}_k^i$ is to basically explicitly define the set of all possible Δx_k^i of the closed loop system and then show that $\overline{\Delta\mathbb{X}}_k^i$ contains this set.

We use an inductive proof: we show for $k \geq 1$ that $\Delta\overline{x}_k^i \in \overline{\Delta\mathbb{X}}_k^i$ holds, if for $m < k$ $\Delta\overline{x}_m^i \in \overline{\Delta\mathbb{X}}_m^i$, the state/input constraints are satisfied for $m < k$ and $\Delta\overline{x}_0^i \in \Delta\mathbb{X}_0^i$.

Let us denote by $\mathcal{X}_0^i$ the set of states x_0^i for which $\mathcal{O}_0^i(x_0^i)$ is feasible. If $x_0^i \in \mathcal{X}_0^i$ for $i = 1, \ldots, M$, then also the optimization problems $\mathcal{O}_m^i(\hat{x}_m^i)$, $1 \leq m < k$ are feasible (compare (b) of the proof above), if Condition 1 holds up to k. Consequently, the set of all possible Δx_k^i is given by

$$\Omega_k^i = \{\Delta x_k^i | \exists\{\Delta x_m^i, w_m, x_m, u_m\} \text{ s. t. (25a)-(25c), (29)}\} \tag{28}$$

where $m < k$ and

$$x_0^l \in \mathcal{X}_0^l, \tag{29a}$$
$$u_m^l = K^l \Delta x_m^l + \kappa_{m,MPC}^l(x_m^l - \Delta x_m^l, x_m^l) \tag{29b}$$

Note that, since $\mathcal{O}_0^i$ is feasible, for $k = 1$, adding to the conditions (25a)-(25c) and (29) the constraints (25d) is non restrictive, i.e. the set Ω_k^i (28) is the same set as

$$\{\Delta x_k^i | \exists\{\Delta x_m^i, w_m, x_m, u_m\} \text{ s. t. (25), (29) hold}\} \tag{30}$$

Similarly for $k > 1$ under the condition that $\Delta x_m^l \in \overline{\Delta\mathbb{X}}_m^l$, the sets Ω_k^i (28) and (30) are equal, since constraint satisfaction holds, compare part (a) of the proof above.

Finally note that we have

$$\overline{\Delta\mathbb{X}}_k^i = \{\Delta x_k^i | \exists\{\Delta x_m^i, w_m, x_m, u_m\} \text{ s. t. (25) hold}\}$$

which is the set Ω_k^i (28) without the constraints (29). Consequently, $\overline{\Delta\mathbb{X}}_k^i \supseteq \Omega_k^i$, i.e. $\overline{\Delta\mathbb{X}}_k^i$ is an outer approximation of the set Ω_k^i (28). This means that since (by definition) $\Delta x_k^i \in \Omega_k^i$, Δx_k^i is also in $\Delta x_k^i \in \overline{\Delta\mathbb{X}}_k^i$.

C. Proof of Proposition 4

Observe that removing the constraints (25c) from (25) yields conditions similar to (24). Thus, if $\overline{\Delta\mathbb{X}}_0^i = \Delta\mathbb{X}_{ex.}^i$ and $\Delta\mathbb{X}_{ex.}^i$ satisfies (24), then the additional constraints in (25c) will yield a $\overline{\Delta\mathbb{X}}_k^i$, which is contained in $\Delta\mathbb{X}_{ex.}^i$.

D. Proof of Proposition 5

Removing (25c) for $m = 0, ..., k-1$ from (25) yields the set $\Delta\mathbb{X}_{k-m}^i$ (if $\Delta\mathbb{X}_{ex.}^i$ satisfies (24) with equality) or a set contained in $\Delta\mathbb{X}_{k-m}^i$. The conditions defining $\Delta\mathbb{X}_{k-m}^i$ are less restrictive than the ones for $\Delta\mathbb{X}_k^i$: $\Delta\mathbb{X}_k^i \subseteq \Delta\mathbb{X}_{k-m}^i$.

E. Proof of Proposition 6

The requirements of Proposition 1 hold, i.e. all optimization problems are feasible. One can show that there is a constant c^i independent of x_0^i such the choice of $\hat{x}_0^i$ in $\mathcal{O}_0^i(x_0^i)$ guarantees $\|\hat{x}_0^i\| \leq c^i\|x_0^i\|$, $\|\Delta x_0^i\| \leq c^i\|x_0^i\|$.

Moreover, with the choice of P^i, Q^i and R^i the nominal dynamics (8) is exponentially stable, compare [5], [21].

The global dynamics of the mismatch Δx_k (21) is

$$\Delta x_{k+1} = (A^d + B^d K)\Delta x_k + w_k + d_k \tag{31}$$

where A^d, B^d are block diagonal matrices with blocks $A^{i,i}$, $B^{i,i}$ (structure as (12)). Using (10), (11) and $\tilde{A} = A + BK$:

$$\Delta x_{k+1} = \tilde{A}\Delta x_k + w_k + (A - A^d)\hat{x}_k + (B - B^d)\hat{u}_k$$

which is ISS in $\hat{x}_k$, $\hat{u}_k$, w_k and Δx_0, since $\tilde{A}$ is asymptotic stable. Using ISS properties and the exponential stability of $\hat{x}_k$, $\hat{u}_k$, we see that Δx_k is ISS in w_k, Δx_0 and $\hat{x}_0$, compare [21]. Finally, using $x_k = \hat{x}_k + \Delta x_k$ and the above bound on Δx_0 and $\hat{x}_0$ results in ISS, see [21].