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On Synchronization in FitzHugh-Nagumo Networks with Small Delays

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***Abstract*— This article studies the influence of the small delays on the FitzHugh-Nagumo network synchronization. It is widely known that high delays in signal propagation between the nodes make synchronization difficult or even impossible. The sufficient conditions of the linearized network synchronization for the case of the small delay are obtained. This problem is successfully reduced to the feasibility of the LMIs. The simulation results confirm the efficiency of the obtained conditions. We suppose that the similar conditions can be applicable even to the nonlinear FitzHugh-Nagumo network.**

I. INTRODUCTION

In the recent years an interest in synchronization problems of coupled oscillators has increased [1], [2]. These systems are used in various fields of applied mathematics such as dynamics of nonlinear systems, graph and network theory, and mechanics and have applications in physics, biology and engineering [3], [4].

Synchronization is one of the crucial research problems in coupled oscillators dynamics. Synchronization phenomenon occurs in various science fields [5]. One of such fields is a neural network dynamics. The synchronization depends on different network parameters, one of which is a time delay in a signal propagation between the nodes. Time delays are always present in real physical systems, therefore for the development of the adequate realistic models for dynamical networks we need to include the transmission delays for proper analysis and control design of their dynamics. In neural networks, time delays may induce different rhythmic spatiotemporal patterns [6], [7], modify the stability of existing patterns [8], and play a crucial role in the synchronization behavior [9], [10]. Even the small delays can change the stability of the systems [11], therefore study of the delay influence on the network synchronization is an important problem. This problem is still opened up to the authors' knowledge.

Here we investigate the network synchronization of one of the simplest neuron models, namely FitzHugh-Nagumo model [12], [13]. The rest of the paper is organized as follows: Section II introduces the model. Section III discusses the problem about synchronization of two delay-coupled FitzHugh-Nagumo systems and derives the synchronization conditions. In Sec. IV, these synchronization conditions are generalized to the networks of several nodes. Section V carries out the simulation analysis based on the obtained conditions. Finally, we conclude with Sec. VI.

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II. MODEL DESCRIPTION

The network consists of n identical nodes, the local dynamics of which is described by the FitzHugh-Nagumo (FHN) differential equations [12], [13]. This neuron-like model is a result of simplification of the four-dimensional Hodgkin-Huxley [14] model and its reduction to the two-dimensional oscillator form with the cubic nonlinearity, one fast and one slow variables. The FHN model can be used for the neuron dynamics description, as well as in the context of other systems ranging from electronic circuits [15] to cardiovascular tissues [16], [17] and the climate system [18]. The dynamics of the ith node of the network is described by

$$\begin{aligned} \varepsilon \dot{u}_i(t) &= u_i(t) - \frac{u_i^3(t)}{3} - v_i(t) \\ &\quad + \sum_{j=1}^{n} c_{ij}[u_j(t-\varepsilon h_{ji}) - u_i(t-\varepsilon h_{ij})], \\ \dot{v}_i(t) &= u_i(t) + a, \quad i = 1, \dots, n, \end{aligned} \tag{1}$$

where u_i and v_i represent the fast activation variable (membrane potential) and the slow recovery variable of neuron i, respectively; ε is the time scale parameter which separates the fast and slow dynamics; in the uncoupled system the threshold a affects the system dynamics: the FHN model is excitable having a stable equilibrium point for $|a| > 1$, while it is oscillatory having a stable limit cycle for $|a| < 1$. This is due to a supercritical Andronov-Hopf bifurcation at $|a| = 1$.

The couplings between the nodes are determined by the connectivity graph $G = (V, E)$, where V is a set of vertices and E is a set of edges. Here we assume that the graph G is weighted, connected and undirected, i.e., its adjacency matrix $C = \{c_{ij}\}$ is symmetric and given by

$$\begin{cases} c_{ij} > 0 & \text{if } (i,j) \in E, \\ c_{ij} = 0 & \text{otherwise.} \end{cases}$$

εh_{ij} are the small constant delays, where $h_{ij} \in (0, h_0]$ and $h_{ij} = h_{ji}\ \forall\ i, j = 1, \dots, n$.

III. TWO COUPLED FITZHUGH-NAGUMO SYSTEMS

We start our consideration with the simplest case of the network, namely two coupled FHN systems. Some results regarding the behavior analysis of two delay-coupled FHN systems are obtained in [10], [19]. The ith FHN system has the following form:

$$\begin{aligned} \varepsilon \dot{u_i}(t) &= u_i(t) - \frac{u_i^3(t)}{3} - v_i(t) \\ &\quad + c[u_j(t-\varepsilon h) - u_i(t-\varepsilon h)], \\ \dot{v_i}(t) &= u_i(t) + a, \end{aligned} \tag{2}$$

The system (2) has the unique equilibrium point $x^* = \mathrm{col}(u_1^*, v_1^*, u_2^*, v_2^*)$, where $u_i^* = -a$ and $v_i^* = -a + a^3/3$, $i = 1, 2$. Linearizing (2) around the equilibrium point x^* we obtain

$$\begin{aligned} \varepsilon \dot{u_i}(t) &= (1-a^2)u_i(t) - v_i(t) \\ &\quad + c[u_j(t-\varepsilon h) - u_i(t-\varepsilon h)], \\ \dot{v_i}(t) &= u_i(t), \quad i = 1, 2. \end{aligned} \tag{3}$$

Subtracting the first equation of the system 1 from the first one of the system 2 (3), and the second one from the second one, respectively, and making the following substitution

$$\delta_1 = v_1 - v_2, \qquad \delta_2 = u_1 - u_2,$$

we get

$$\begin{aligned} &\dot{\delta_1}(t) = \delta_2(t), \\ &\varepsilon \dot{\delta_2}(t) = -\delta_1(t) + (1-a^2)\delta_2(t) - 2c\delta_2(t-\varepsilon h), \end{aligned} \tag{4}$$

which can be presented in a matrix form

$$E_\varepsilon \dot{z}(t) = Az(t) + Hz(t-\varepsilon h), \tag{5}$$

where

$$z = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}, \quad E_\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix},$$
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1-a^2 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 \\ 0 & -2c \end{bmatrix}.$$

To study synchronization of two FHN systems (2) we find the conditions of the error system (5) stability. Consider the "fast" system

$$\dot{\delta_2}(t) = (1-a^2)\delta_2(t) - 2c\delta_2(t-h). \tag{6}$$

To ensure the exponential stability of (5) for all $h \in (0, h_0]$ and small enough ε the following assumptions should be fulfilled [20]:

A1) The value $\xi \equiv 1 - a^2 - 2c$ is negative.

A2) The "fast" system (6) is exponentially stable for all $h \in [0, h_0]$.

To study stabilizability of the "fast" system (6) consider its characteristic equation for $h = h_0$

$$\lambda - 1 + a^2 + 2c\mathrm{e}^{-\lambda \mathrm{h_0}} = 0. \tag{7}$$

Let $a = 0.9$ meaning that the uncoupled system is in oscillatory regime. The areas of paramaters c and h, for which the characteristic equation (7) has roots with negative real part, are marked by blue color in Fig. 1. The maximum value of the delay h_0, for which the "fast" system (6) is stable, equals to 5.23 and is achieved for $c = 0.096$.

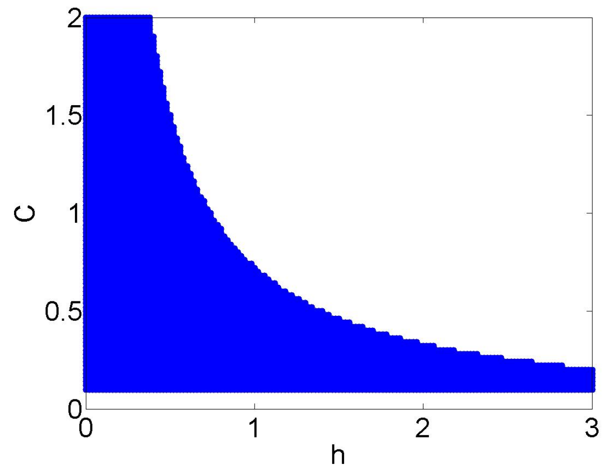

Fig. 1. The areas of paramaters c and h with fixed $a = 0.9$, for which the "fast" system (6) is exponentially stable.

To obtain the sufficient conditions of the "fast" system (6) stability let $h = h_0$ and use the following relation

$$\delta_2(t-h_0) = \delta_2(t) - \int_{t-h_0}^{t} \dot{\delta_2}(s)\mathrm{d}s,$$

via the descriptor model transformation introduced in [21] to present the system (6) in the following form

$$\dot{\delta_2}(t) = \xi\delta_2(t) + 2c \int_{t-h_0}^{t} \dot{\delta_2}(s)\mathrm{d}s.$$

Now introduce the following Lyapunov-Krasokvskii functional

$$V(\delta_2(t), \dot{\delta_2}(t+s)) = p\delta_2^2(t) + \int_{t-h_0}^{t} (h_0 + s - t)r\dot{\delta_2}(s)\mathrm{d}s,$$

where $p > 0$, $r > 0$. To get the LMI conditions we use the descriptor method which is described in [22], [23]

$$\begin{aligned} \dot{V}(\delta_2(t), \dot{\delta_2}(t+s)) &\leq 2p\delta_2(t)\dot{\delta_2}(t) + h_0 r\dot{\delta}_2^2(t) \\ &- rh_0 \left(\frac{1}{h_0} \int_{t-h_0}^{t} \dot{\delta_2}(s)\mathrm{d}s \right)^2 + 2[p_2\delta_2(t) + p_3\dot{\delta_2}(t)] \\ &\times \left[\xi\delta_2(t) + 2c \int_{t-h_0}^{t} \dot{\delta_2}(s)\mathrm{d}s - \dot{\delta_2}(t) \right], \end{aligned}$$

where p_2 and p_3 are "slack variables". Then we get the following LMI

$$\begin{bmatrix} 2\xi p_2 & p - p_2 + \xi p_3 & 2h_0 c p_2 \\ * & -2p_3 + h_0 r & 2h_0 c p_3 \\ * & * & -h_0 r \end{bmatrix} < 0. \tag{8}$$

The feasibility of LMIs (8), $p > 0$, $r > 0$ with respect to p, r, p_2, p_3 leads to the system (6) exponential stability.

Thus, the following theorem holds:

Theorem 1: If the system (2) parameters satisfy the condition $1 - a^2 - 2c < 0$ and the LMIs (8), $p > 0$, $r > 0$ are feasible, then two linearized FHN systems (3) exponentially synchronize for all $h \in (0; h_0]$, i.e., the error system (4) is exponentially stable.

Note that in the case of synchronization absence we can achieve system (2) synrchonization by using the control algorithms [24], [25].

IV. NETWORK OF FITZHUGH-NAGUMO SYSTEMS

Now we consider the network of n FHN nodes (1). The system (1) has the unique equilibrium point $x^* = \mathrm{col}(u_1^*, v_1^*, \dots, u_n^*, v_n^*)$, where $u_i^* = -a$ and $v_i^* = -a + a^3/3$, $i = 1, \dots, n$. Linearizing (1) around the equilibrium point x^* we obtain

$$\begin{aligned} \varepsilon \dot{u}_i(t) &= (1-a^2)u_i(t) - v_i(t) \\ &\quad + \sum_{j=1}^{n} c_{ij}[u_j(t-\varepsilon h_{ij}) - u_i(t-\varepsilon h_{ij})], \\ \dot{v}_i(t) &= u_i(t). \end{aligned} \tag{9}$$

By the averaging over all nodes we get the averaged trajectory described by

$$\begin{aligned} \varepsilon \dot{\bar{u}}(t) &= (1-a^2)\bar{u}(t) - \bar{v}(t), \\ \dot{\bar{v}}(t) &= \bar{u}(t), \end{aligned} \tag{10}$$

because the connectivity graph G of the network (9) is undirected, where

$$\bar{u} = \frac{1}{n}\sum_{j=1}^{n} u_j, \quad \bar{v} = \frac{1}{n}\sum_{j=1}^{n} v_j. \tag{11}$$

Subtracting the first equation of (10) from the first one of (9), and the second one from the second one, respectively, we obtain

$$\begin{aligned} &\varepsilon[\dot{u}_i(t) - \dot{\bar{u}}(t)] = (1-a^2)[u_i(t) - \bar{u}(t)] \\ &-[v_i(t) - \bar{v}(t)] + \sum_{j=1}^{n} c_{ij}\Big[u_j(t-\varepsilon h_{ij}) - \bar{u}(t-\varepsilon h_{ij}) \\ &\qquad + \bar{u}(t-\varepsilon h_{ij}) - u_i(t-\varepsilon h_{ij})\Big], \\ &\dot{v}_i(t) - \dot{\bar{v}}(t) = u_i(t) - \bar{u}(t). \end{aligned} \tag{12}$$

Now we make the following substitution

$$\delta_{1i} = u_i - \bar{u}, \qquad \delta_{2i} = v_i - \bar{v},$$

to present the system (12) in the following form

$$\begin{aligned} \varepsilon \dot{\delta}_{1i}(t) &= (1-a^2)\delta_{1i}(t) - \delta_{2i}(t) \\ &\quad + \sum_{j=1}^{n} c_{ij}[\delta_{1j}(t-\varepsilon h_{ij}) - \delta_{1i}(t-\varepsilon h_{ij})], \\ \dot{\delta}_{2i}(t) &= \delta_{1i}(t). \end{aligned} \tag{13}$$

Since $\forall i, j = 1, \dots, n$ $h_{ij} \in (0; h_0]$, we can consider the network with identical delay h_0 for all nodes and present it in a matrix form

$$E_\varepsilon \dot{x}(t) = Ax(t) + Hx(t - \varepsilon h_0), \tag{14}$$

where $x = \mathrm{col}(\delta_{11}, \dots, \delta_{1n}, \delta_{21}, \dots, \delta_{2n})$, $x \in \mathbb{R}^{2n}$, $E_\varepsilon, A, H \in \mathbb{R}^{2n \times 2n}$ and

$$E_\varepsilon = \begin{bmatrix} \varepsilon I_n & 0 \\ 0 & I_n \end{bmatrix}, \ H = \begin{bmatrix} -L(G) & 0 \\ 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} (1-a^2)I_n & -I_n \\ I_n & 0 \end{bmatrix},$$

$$L(G) = \begin{bmatrix} d_1 & -c_{12} & \cdots & -c_{1n} \\ -c_{21} & d_2 & \cdots & -c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{n1} & -c_{n2} & \cdots & d_n \end{bmatrix}$$

where $d_i = \sum_{j=1}^{n} c_{ij}$, $L(G) \in \mathbb{R}^{n \times n}$ is the Laplacian matrix of the connectivity graph G.

Remind that the graph G is connected by the assumption, therefore its minimal eigenvalue is equal to 0 and the algebraic connectivity $\lambda_2 > 0$. Let P be the square invertible matrix such that

$$P^{-1}L(G)P = \Lambda = \mathrm{diag}(0, \lambda_2, \dots, \lambda_n).$$

Then one can use the following matrix

$$S = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix},$$

for the system (14) conversion

$$\begin{aligned} S^{-1}E_\varepsilon S &= \begin{bmatrix} I_n & 0 \\ 0 & \varepsilon I_n \end{bmatrix}, \ S^{-1}HS = \begin{bmatrix} 0 & 0 \\ 0 & -\Lambda \end{bmatrix}, \\ S^{-1}AS &= \begin{bmatrix} 0 & I_n \\ -I_n & (1-a^2)I_n \end{bmatrix}. \end{aligned} \tag{15}$$

Let $z = S^{-1}x$, $z = \mathrm{col}(z_1, z_{e1}, z_{n+1}, z_{e2})$, $z_e = \mathrm{col}(z_{e1}, z_{e2}) \in \mathbb{R}^{2n-2}$. Then meaning (15) the system (14) can be presented as

$$\begin{aligned} \dot{z}_1(t) &= z_{n+1}(t), \\ \varepsilon \dot{z}_{n+1}(t) &= -z_1(t) + (1-a^2)z_{n+1}(t), \end{aligned} \tag{16a}$$

$$\tilde{E}_\varepsilon \dot{z}_e(t) = \tilde{A}z_e(t) + \tilde{H}z_e(t - \varepsilon h_0), \tag{16b}$$

where $\tilde{E}_\varepsilon, \tilde{A}, \tilde{H} \in \mathbb{R}^{(2n-2)\times(2n-2)}$ and

$$\tilde{E}_\varepsilon = \begin{bmatrix} I_{n-1} & 0 \\ 0 & \varepsilon I_{n-1} \end{bmatrix}, \ \tilde{H} = \begin{bmatrix} 0 & 0 \\ 0 & -\Lambda_e \end{bmatrix},$$

$$\tilde{A} = \begin{bmatrix} 0 & I_{n-1} \\ -I_{n-1} & (1-a^2)I_{n-1} \end{bmatrix}, \ \Lambda_e = \mathrm{diag}(\lambda_2, \dots, \lambda_n).$$

The system (16a) is stable for $|a| \geq 1$ and unstable for $|a| < 1$. However, the stability of the system (16b) is sufficient condition for the network (9) synchronization. Consider the following linear system:

$$\dot{y}_1 - \dot{\bar{y}} \ = \ a_1(y - \bar{y}), \tag{17a}$$

$$\begin{bmatrix} \dot{y}_2 - \dot{\bar{y}} \\ \vdots \\ \dot{y}_n - \dot{\bar{y}} \end{bmatrix} = \begin{bmatrix} a_2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{bmatrix} \begin{bmatrix} y_2 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{bmatrix}, \tag{17b}$$

where $\bar{y} = n^{-1}\sum_{i=1}^{n} y_i$, $a_1 > 0$ meaning the system (17a) is unstable and $a_i < 0$, $i = 1, \ldots, n$ meaning the system (17b) is stable. Therefore, $y_2 \to \bar{y} = n^{-1}\sum_{i=1}^{n} y_i, \ldots, y_n \to \bar{y} = n^{-1}\sum_{i=1}^{n} y_i$. Then $n\sum_{i=2}^{n} y_i \to (n-1)\sum_{i=1}^{n} y_i$, and $y_i \to \bar{y} \to y_1$, $i = 2, \ldots, n$, that means the synchronization.

Thus, if the system (16b) solution is exponentially stable then the network (9) synchronizes. To study the stability of the system (16b) consider also the "fast" system

$$\dot{z}_{e2}(t) = (1-a^2)I_{n-1}z_{e2}(t) - \Lambda_e z_{e2}(t-h_0). \qquad (18)$$

To ensure the exponential stability of (16b) the following assumptions should be fulfilled [20]:

A1) The "fast" matrix $\Xi \equiv (1-a^2)I_{n-1} - \Lambda_e$ is Hurwitz.
A2) The "slow' matrix Ξ^{-1} is Hurwitz.
A3) The "fast" system (18) is exponentially stable.

Since all eigenvalues of the matrix $(1-a^2)I_{n-1}$ equal $1-a^2$ and the algebraic connectivity $\lambda_2(G)$ is the minimal eigenvalue of the matrix Λ_e, the conditions A1, A2 are fulfilled if $1 - a^2 - \lambda_2(G) < 0$.

To derive the sufficient conditions of the system (18) stability we can follow the descriptor approach like in Sec. III and obtain the following LMI

$$\begin{bmatrix} P_2^{\mathrm{T}}\Xi + \Xi^{\mathrm{T}}P_2 & P - P_2^{\mathrm{T}} + \Xi^{\mathrm{T}}P_3 & h_0 P_2^{\mathrm{T}}\Lambda_e \\ * & -P_3 - P_3^{\mathrm{T}} + h_0 R & h_0 P_3^{\mathrm{T}}\Lambda_e \\ * & * & -h_0 R \end{bmatrix} < 0, \quad (19)$$

where $P, R, P_2, P_3 \in \mathbb{R}^{n-1}$, $P > 0$, $R > 0$. Thus, we reduce the problem to the feasibility of the LMIs (19), $P > 0$, $R > 0$ with respect to $P,\ R,\ P_2,\ P_3$.

The following theorem holds:

Theorem 2: If the network (9) parameters satisfy the condition $1 - a^2 - \lambda_2(G) < 0$ and the LMIs (19), $P > 0$, $R > 0$ are feasible, then the network of linearized FHN systems (9) exponentially synchronize for all $h_{ij} \in (0; h_0]$, i.e., the error system (13) is exponentially stable.

V. SIMULATION

Now consider the FHN network (1) of $n = 10$ nodes with threshold a equal to 0.9 and $\varepsilon = 0.1$. The connectivity graph G is characterized by the symmetric adjacency matrix C, which is a sparse matrix with link density equal to 0.5. This means there are approximately $0.5n^2$ normally distributed nonzero entries. These nonzero entries are drawn from a Gaussian distribution with mean $\mu = 0.3$ and variance $\sigma^2 = 0.1$. The algebraic connectivity of this graph G is equal to 0.2825, i.e., the condition $1 - a^2 - \lambda_2(G) < 0$ from the Theorem 2 is fulfilled. The delays εh_{ij} in signal propagation have uniform distribution on the interval $(0; 0.05]$.

By solving the LMI (19) with given parameters we found that the maximum value of the delay h_0, for which LMI (19) is feasible, equals to 0.5. One can see that $h_{ij} \in (0, 0.5]$, therefore the conditions of the Theorem 2 are fulfilled, then linearized FHN network (9) exponentially synchronizes. Solvability of the LMI was verified in the *Matlab* environment [26] with using of the *Yalmip* packet (Sedumi solver) [27].

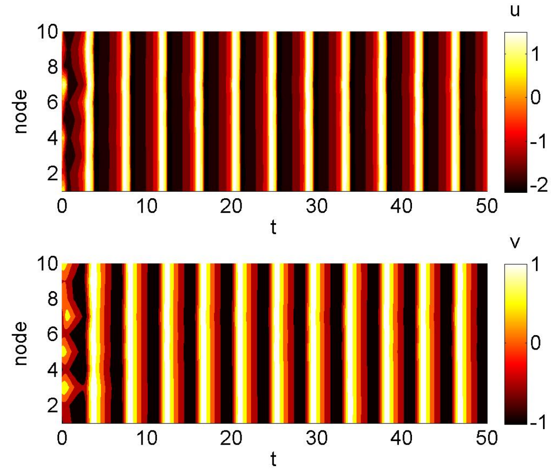

Fig. 2. Dynamics of $n = 10$ FitzHugh-Nagumo systems (1) with a random topology and small delay. (a) and (b): time series of the membrane potential and the recovery variable of all nodes, respectively. Parameters: $n = 10$, $\varepsilon = 0,1$ $a = 0.9$. The nonzero entries are drawn from a Gaussian distribution with mean $\mu = 0.3$ and variance $\sigma^2 = 0.1$. The delays εh_{ij} in signal propagation have uniform distribution on the interval $(0; 0.05]$ Initial conditions: $u_i(t) = u_i^0$, $v_i(t) = v_i^0$, $i = 1, \ldots, n$, for $t \in [-h_0, 0]$, where u_i^0 and v_i^0 have uniform distribution on the interval $[-1; 1]$.

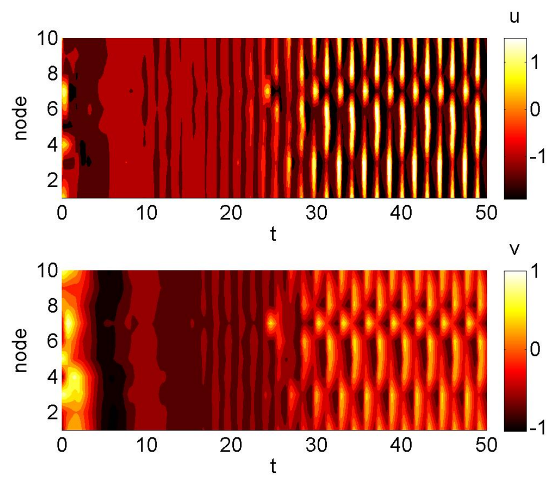

Fig. 3. Dynamics of $n = 10$ FitzHugh-Nagumo systems (1) with a random topology. (a) and (b): time series of the membrane potential and the recovery variable of all nodes, respectively. Parameters: the delays εh_{ij} in signal propagation have uniform distribution on the interval $(0; 0.5]$. Other parameters and initial conditions as in Fig. 2.

Figure 2 presents the results of the simulation of $n = 10$ FHN systems (1) dynamics. X-axis corresponds to the time, while Y-axis corresponds to the number of node. The amplitude of the state dynamics is marked by the color. One can see that there is synchronization between the membrane potential values (see Fig. 2(a)) and recovery variable values (see Fig. 2(b)) of all nodes, respectively.

Now we consider the case of the network with the same parameters but with the delays 10 times greater than in the previous case, i.e., the delays εh_{ij} have uniform distribution on the interval $(0; 0.5]$. One can see that in this case

$h_{ij} \notin (0, 0.5]$, i.e., the conditions of the Theorem 2 are not fulfilled. Figure 3 presents the results of the simulation of such network dynamics: there is no synchronization. Thus, increasing of the delay not proportional to ε may hinder synchronization.

VI. CONCLUSION

The influence of the delay in the signal propagation between the nodes on the FitzHugh-Nagumo network synchronization has been studied. We have started our consideration with the simplest case of the network — two coupled linearized FitzHugh-Nagumo systems. The sufficient conditions of the systems synchronization have been derived. This problem is successfully reduced to the feasibility of the LMIs. Afterwards, we have generalized this result to the case of linearized FitzHugh-Nagumo network. The obtained conditions guarantee the linearized network synchronization for the small delays. The simulation of nonlinear FitzHugh-Nagumo network dynamics corresponds to the obtained conditions. We expect our results to be helpful for the further investigation of the neural networks.

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