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Extended balanced truncation for continuous time LTI systems

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Abstract— **In this paper we develop a continuous time counterpart of the balancing approach based on the extended Gramians for discrete time LTI systems as given in [14]. The extended Gramians are solutions to LMI's and can be seen as an extension/generalization of generalized Gramians which are solutions of the standard Lyapunov inequalities. These LMI's are rather different from the discrete time case, and are coupled to dissipativity arguments related to storage functions that are instrumental for obtaining an a priori error bound if balanced truncation is applied based on these extended Gramians. The approach provides flexibility in choosing the Gramians based on which the model is truncated, in order to get a low error bound, and possibly preserve a desired structure such as a network structure.**

I. INTRODUCTION

Balancing is a well-known method for reducing the order of control systems. It relies on realization theory, observability and controllability Gramians and is directly related to the Hankel operator of a system. For stable linear systems, balancing is introduced in [13], and an extensive exposition can be found in [1]. A brief tutorial overview can also be found in [15], which is set up to provide a basis for extending the results to nonlinear systems [7].

Model reduction based on standard balancing for asymptotically stable systems preserves the asymptotic stability, the balanced structure and observability and controllability of the system, i.e., the reduction is based on removing the badly observable and badly controllable parts of the system. However, any other type of structure, such as a port-Hamiltonian system structure, or a network structure is not necessarily preserved. Also, model reduction based on balancing is useful if there is a clear cut between the small and large Hankel singular values. If that is not the case, or if all Hankel singular values are rather large, it means that the error bound that is well known for balanced truncation, [8], [17], [16], is also large. It can be useful to study a class of balancing methods where the error bound can be designed to be small, and where certain internal, physical or network structure can potentially be preserved when applying model reduction based on these methods.

An extension of the above mentioned balancing method is based on the generalized Gramians as is introduced in [10], see e.g., [6], which are solutions to the Lyapunov inequalities, as opposed to equalities. These inqualities do not have a unique solution, and thus offer some freedom for balancing based on solutions to the controllability and observability Lyapunov inequality. It can be proven that model reduction based on balancing generalized Gramians also preserves properties like stability, and results in a similar error bound as for standard balancing, i.e., the sum of the generalized singular values corresponding to the truncated states provide the bound. Furthermore, model reduction based on balancing generalized Gramians may be used to obtain a smaller error bound than for standard balancing, e.g., [6], and it can be used to preserve certain structure, such as a network structure, see [2], [3].

For discrete time systems there is even a further extension available to the above mentioned balancing methods, namely extended balancing, developed in [14]. This method provides more possibilities to impose a certain structure on the reduced order model by considering solutions to specific LMI's. These LMI's have solutions if and only if there exist solutions to the Lyapunov inequalities related to generalized balancing. The discrete time LMI's used in [14] appear first in [4], [5], where solutions are used for robust control design. In [14] the extended Gramians are used for balancing based model order reduction, and similar to other balancing methods an a priori error bound is obtained. These error bounds are not obtained through the usual transfer function approach as in [8], [17], but by using a dissipativity type of property of the error system and of an augmented system, as is (to the best of our knowledge) first presented in [16].

In this paper we extend the discrete time extended balancing approach to continuous time systems. At first sight this may seem a trivial step to take. However, the LMI's as they are presented in [4], [5] have no continuous time counter part. Therefore, we aim at developing a continuous time counter part based on the dissipativity property that is used for determining the error bound in the discrete time case. The LMI's that are developed and have extended Gramians as solutions can be proven to have relations with the generalized Gramians that are solutions to the generalized Lyapunov equations. Furthermore, we show that these new Gramians can be used for balancing.

The outline of the paper is as follows. In Section II a very brief recap of the standard balancing method is presented, after which the problem setting is explained. Then, in Section III, the extended Gramians as solutions to particular LMI's are given, as well as the proof that there exist solutions to these LMI's if and only if there are (positive) solutions to the Lyapunov inequalities. In Section IV first a model comparison is presented that is instrumental for the development of error bounds for the extended balanced truncation method that is presented next. Finally in Section V we present some conclusions.

II. Preliminaries and problem setting

We consider a stable finite dimensional linear continuous time-invariant system Σ defined by the triple (A,B,C) as follows:

$$\Sigma : \begin{cases} \dot{x} &= Ax + B\ u \\ y &= Cx \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$.

It is well known that balancing for LTI systems, [13], relies on transforming the system into coordinates $z = Tx$ such that the controllability and observability Gramians, W and M, are simultaneously diagonalized. The eigenvalues of MW are the squared Hankel singular values σ_i^2, $i = 1,\ldots,n$, and $\sigma_1 \geq \ldots \geq \sigma_n$. Furthermore, M and W are solutions of Lyapunov equations. Model reduction based on balancing is usually done by truncating the states corresponding to the small Hankel singular values, i.e., if $\sigma_k >> \sigma_{k+1}$, then we set $z_{k+1} = \cdots = z_n = 0$, which corresponds to truncating the almost non-controllable and non-observable states.

The error that is made in terms of both the Hankel and the H_∞ norm is given by the sum of the truncated Hankel singular values, [8], i.e.,

$$\|\Sigma - \hat{\Sigma}\|_\infty \leq 2 \sum_{i=k+1}^{n} \sigma_i .$$

See e.g., [17] for an more elaborate exposition of balancing and the corresponding reduced order model properties.

Problem setting

Similar to the problem setting of [14] for discrete time linear systems, we aim to develop a method where Σ is approximated by a reduced order model $\hat{\Sigma}$ of order $k < n$ that is obtained by truncation such that $\|\Sigma - \hat{\Sigma}\|_\infty$ is small. We use the partitioning

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}, \tag{2}$$

with $A_{11} \in \mathbb{R}^{k\times k}$, $B_1 \in \mathbb{R}^{k\times m}$, and $C_1 \in \mathbb{R}^{p\times k}$. The triple (A_{11}, B_1, C_1) defines the reduced order system $\hat{\Sigma}$.

Clearly, balanced order reduction is a possible solution to the problem. However, even though the various variations of balanced truncation such as standard balancing, [13], but also e.g., LQG, comprime, H_∞, positive real, and bounded real balancing, [1], [15] and the references therein, preserve the balanced form and the stability of the system, other types of structure (such as a complex specific network structure) are not preserved. Therefore, we consider a generalization of [14] to the continuous time case. As mentioned in the introduction, at first sight, this may seem straightforward. However, because the discrete time LMI's in [14] that play an important role in the general setting of the problem have no extension to the continuous time case, different arguments have to be used.

III. Extended Gramians and dissipativity

In order to provide a continuous time generalization of [14], we have to go back to the dissipativity arguments that eventually lead to a priori error bounds for a reduced order model obtained by truncation. For discrete time systems, useful LMI's related to robust control have been obtained in [4], [5]. These so-called generalized Lyapunov inequalities are used by [14] for the development of the model reduction procedure that solves the problem above for discrete time systems. Because these inequalities directly connect to robust control problems, dissipativity arguments, and solutions to the generalized controllability and observability Gramians are related to the LMI's, these LMI's can be used to develop a model reduction method with a small H_∞ bound.

Here, we study different LMI's, building on dissipativity type of arguments similar to [14]. First, note that it is well known that the observability and controllability Lyapunov inequalities

$$QA + A^T Q + C^T C \leq 0 \tag{3}$$
$$A\tilde{P} + \tilde{P}A^T + BB^T \leq 0 \tag{4}$$

have solutions $Q \geq 0$ and $\tilde{P} \geq 0$, $Q, \tilde{P} \in \mathbb{R}^{n\times n}$ if system (1) is asymptotically stable. Q and $\tilde{P}$ are the generalized observability and controllability Gramians, respectively. Existence of strictly positive solutions is guaranteed if the system is observable and controllable, respectively. However, this is only a sufficient condition for strict positivity in the generalized case, and thus may be a too conservative assumption for our developments. In the LMI developments we do rely on the strict positivity of the solutions, because we need invertibility. Hence, we assume throughout that $Q > 0$ and $\tilde{P} > 0$. Inequality (4) can be rewritten into a Riccati inequality given by

$$PA + A^T P + PBB^T P \leq 0 \tag{5}$$

with $\tilde{P}^{-1} = P$.

The above generalized observability and controllability Gramians can be used for a balancing procedure that in some cases provide better approximations than based on the standard observability and controllability Gramians (the solutions to the equalities), [10], [11], [9], [6], [14]. Here, we aim to find a larger class of Gramians that we can use for a balanced truncation type of approach with an a priori error bound. Our further developments aim at using a storage function

$$(x - x_r)^T Q (x - x_r) + (x + x_r)^T P (x + x_r), \tag{6}$$

where $x_r \in \mathbb{R}^n$ is the state of a reduced order system $\hat{x} \in \mathbb{R}^k$ supplemented with zeros. This storage function can be used to develop a balancing method with a priori error bounds, which is inspired by the proof for the standard balancing error bound as presented for the first time in [16]. Note that this is different from the standard way to determine the error bound for balancing which is based on transfer functions arguments, [8], [17].

As becomes apparent from the developments in Section IV, the storage function (6) gives rise to LMI characterizations, in a similar manner as for the discrete time case in [14]. We first present the LMI's and discuss some relevant properties, and then study the relation with the storage function (6) and model reduction. For the extension of the generalized Gramians, consider the following two linear matrix inequalities (LMI's).

$$\begin{pmatrix} -QA - A^TQ - C^TC & Q - \alpha S - A^TS \\ Q - \alpha S^T - S^TA & S + S^T \end{pmatrix} \geq 0 \qquad (7)$$

and

$$\begin{pmatrix} -PA - A^TP & -P + \beta R + A^TR & -2PB \\ -P + \beta R^T + R^TA & R + R^T & 2R^TB \\ -2B^TP & 2B^TR & 4I \end{pmatrix} \geq 0 \qquad (8)$$

with $R, S \in \mathbb{R}^{n\times n}$, and $\alpha, \beta > 0$ large enough scalars. We call (7) and (8) the extended observability and controllability LMI's with extended observability Gramian (Q, S, α) and extended inverse controllability Gramian (P, R, β), respectively.

Remark 1: It should be noted that we use a similar development as in [14]. However, the LMI's presented above have a very different form than the ones presented in [4], [5], [14] for discrete time systems, and cannot be obtained in a similar manner as given in [4], [5]. Furthermore, in contrast to the discrete time case we need the additional α and β constants for our developments. Our extended Gramians along with the storage function (6) are instrumental to obtain an a priori error bound for a balanced truncation procedure based on the eigenvalues of RS. $\triangleleft$

Now we can formulate the relation between the generalized observability Gramian and the extended observability Gramian.

Theorem 1: (observability Gramians)
The inequality (3) has a solution $Q > 0$ if and only if the LMI (7) has a solution (Q, S, α) with $Q = Q^T > 0$, and $\alpha \in \mathbb{R}$, $\alpha > 0$ large enough.

Proof: *Necessity*: Assume that (7) has a solution (Q, S, α) with $Q > 0$, then it follows directly from multiplying (7) with $(I_n \ O_n)$ from the left and with $(I_n \ O_n)^T$ from the right, that (3) has a solution $Q > 0$.
Sufficiency: Now assume (3) has a solution $Q > 0$, and define $\tilde{A} = \alpha I_n + A$, and $X = -QA - A^TQ - C^TC$. If we take $S = \tilde{A}^{-T}Q$, with α not an eigenvalue of A, then the off-diagonal terms of (7) are zero. It follows that

$$S + S^T = \tilde{A}^{-T}\left(2\alpha Q - X - C^TC\right)\tilde{A}^{-1}$$

Since $Q > 0$, is given, we can find an α such that $2\alpha Q \geq C^TC + X$, which proofs the theorem. ■

Remark 2: The sufficiency proof of Theorem 1 is rather conservative in picking α. Define $Y = Q - \alpha S - A^TS$. For the case that $X > 0$, we can find a tighter bound for α by Schur complements and completion of the squares arguments. In particular, note that with the right multiplications (7) is equivalent to

$$\begin{pmatrix} X & 0 \\ 0 & -Y^TX^{-1}Y + S + S^T \geq 0 \end{pmatrix}.$$

This can be worked out further to obtain a less conservative bound on α, i.e., pick α such that $2\alpha Q \geq C^TC$ $\triangleleft$

The results about the generalized and extended observability Gramians have a controllability version as follows.

Theorem 2: (controllability Gramians)
The inequality (5) has a solution $P > 0$ (and thus (4) has a solution $\tilde{P} > 0$ if and only if the LMI (8) has a solution (P, R, β) with $P > 0$, and $\beta \in \mathbb{R}$, $\beta > 0$ large enough.

Proof: First note that solving the the LMI in (8) is equivalent to solving the following LMI in (9).
Necessity: This goes along the same lines as the necessity proof of Theorem 1.
Sufficiency: Similar to the proof of Theorem 1, choose $R = \tilde{A}^{-1}P$ with $\tilde{A} = A + BB^TP + \beta I$, and $-\beta$ not an eigenvalue of $A + BB^TP$. Furthermore, define $W = -PA - A^TP - PBB^TP$. Then it follows after some derivations that if we pick β such that $2\beta P \geq W$, the theorem is proven. ■

These results are instrumental for the model comparison in the next section. In order to ease the further analysis, it is also important to realize that the extended Gramians transform in a standard manner under a coordinate transformation, i.e.,

Proposition 1: Assume we apply an invertible state transformation $x = W\bar{x}$. Then the extended Gramians transform as $\bar{Q} = W^TQW$, $\bar{P} = W^TPW$, $\bar{S} = W^TSW$, $\bar{R} = W^TRW$, $\bar{\alpha} = \alpha$, and $\bar{\beta} = \beta$.

Proof: Substitute $A = W\bar{A}W^{-1}$, $B = W\bar{B}$, and $C = \bar{C}W^{-1}$ into (7) and (9). Multiply both inequalities from the right with $\mathrm{diag}(W, W)$ and from the left with its transposed, then the result is obtained straightforwardly. ■

IV. Model comparison and extended balanced truncation

In this section we use the LMI's to establish inequality relations for comparison of models with the ultimate goal to use this for determining the error made by truncating the original model based on the extended Gramians. For that, we first compare the following two models

$$\Sigma : \begin{cases} \dot{x} &= Ax + Bu, \quad x(0) = x_0 \\ y &= Cx + Du \end{cases} \qquad (10)$$

$$\begin{pmatrix} -PA-A^TP-PBB^TP & -P+\beta R+A^TR+PBB^TR \\ -P+\beta R^T+R^TA+R^TBB^TP & R+R^T-R^TBB^TR \end{pmatrix} \geq 0 \tag{9}$$

$$\Sigma_r : \begin{cases} \dot{x}_r &= Ax_r+Bu+v, \quad x_r(0)=x_{r0} \\ y_r &= Cx_r+Du. \end{cases} \tag{11}$$

As in [14], Σ and Σ_r are identical apart from a perturbation signal $v(t)\in\mathbb{R}^n$, and a potentially different initial state. If the same input u is applied, we can use the extended Gramians to quantify the differences between Σ and Σ_r. In particular, (7) implies that

$$\begin{aligned}
&\begin{pmatrix} x-x_r \\ v \end{pmatrix}^T \begin{pmatrix} -QA-A^TQ-C^TC & Q-\alpha S-A^TS \\ Q-\alpha S^T-S^TA & S+S^T \end{pmatrix} \\
&\times \begin{pmatrix} x-x_r \\ v \end{pmatrix} \geq 0 \\
&\Leftrightarrow \\
&-2(x-x_r)^TQ(\dot{x}-\dot{x}_r)-2\alpha(x-x_r)^TSv-2(\dot{x}-\dot{x}_r)^TSv \\
&\geq (y-y_r)^T(y-y_r) \\
&\Leftrightarrow \\
&\frac{d}{dt}\left((x-x_r)^TQ(x-x_r)^T\right) \\
&\leq -|y-y_r|^2-2\alpha(x-x_r)^TSv-2(\dot{x}-\dot{x}_r)^TSv
\end{aligned}$$

Taking the integral from 0 to τ, for all $\tau>0$, yields

$$\begin{aligned}
&\|y-y_r\|_2^2 \leq (x_0-x_{r0})^TQ(x_0-x_{r0}) \\
&-2\int_0^\infty \left(\alpha(x(t)-x_r(t))^TSv(t)+(\dot{x}(t)-\dot{x}_r(t))^TSv(t)\right)dt.
\end{aligned} \tag{12}$$

Note that the extended observability Gramian components α and S only appear if $v\neq 0$.

We can do something similar, but a bit more involved, in the controllability case, i.e., (8) implies that

$$\begin{aligned}
&\begin{pmatrix} x+x_r \\ v \\ u \end{pmatrix}^T \begin{pmatrix} -PA-A^TP & -P+\beta R+A^TR & -2PB \\ -P+\beta R^T+R^TA & R+R^T & 2R^TB \\ -2B^TP & 2B^TR & 4I \end{pmatrix} \\
&\times \begin{pmatrix} x+x_r \\ v \\ u \end{pmatrix} \geq 0 \\
&\Leftrightarrow \\
&-2(x+x_r)^TPA(x+x_r)-4(x-x_r)^TPBu-2(x-x_r)^TPv \\
&+2\beta(x+x_r)^TRv+2(x+x_r)^TA^TRv+4u^TB^TRv+2v^TRv \\
&+4u^Tu \geq 0 \\
&\Leftrightarrow \\
&\frac{d}{dt}\left((x+x_r)^TP(x+x_r)\right) \\
&\leq 4|u|^2+2\beta(x+x_r)^TRv+2(\dot{x}+\dot{x}_r)^TRv
\end{aligned}$$

Taking the integral from 0 to τ, for all $\tau>0$ yields

$$\begin{aligned}
&4\|u\|_2^2 \geq -(x_0+x_{r0})^TP(x_0+x_{r0}) \\
&-2\int_0^\infty \left(\beta(x(t)+x_r(t))^TRv(t)+(\dot{x}(t)+\dot{x}_r)(t)^TRv(t)\right)dt
\end{aligned} \tag{13}$$

Now we can use a particular choice of the perturbation signal v to establish an error bound for balanced truncation. For that, we first compare the model Σ_r and the truncated model for such signal. The following result follows straightforwardly.

Lemma 1: Consider the reduced order system $\hat{\Sigma}$ given by the triple (A_{11},B_1,C_1) of (2), with state $\hat{x}$, and output $\hat{y}$. Now consider system Σ_r given in (11). Partition x_r accordingly in $x_r=(x_{r1}^T,x_{r2}^T)^T$ Choose

$$v(t)=\begin{pmatrix} 0 \\ -A_{21}x_{r1}(t)-B_2u(t) \end{pmatrix}$$

and $x_{r0}=0$, $\hat{x}(0)=0$, then $y_r(t)=\hat{y}(t)$, and $x_{r2}(t)=0$ for $t\geq 0$. $\lhd$

Now assume that S and R are symmetric, then they can transform just like Gramians (note that P is the inverse of the generalized controllability Gramian, and consequently R is treated similarly), i.e., the eigenvalues $\lambda_i(R^{-1}S)$ are invariant under coordinate transformation (see Proposition 1). Define $\sigma_i=\sqrt{\lambda_i}$, $i=1,\ldots,n$, and order them as $\sigma_1\geq\sigma_2\ldots\sigma_n>0$. Next we assume that $x_{r2}\in\mathbb{R}$, and thus we can find coordinates such that S and R are in a block diagonal form as follows:

$$S=\begin{pmatrix} S_1 & 0 \\ 0 & \sigma_n \end{pmatrix}, \quad R=\begin{pmatrix} R_1 & 0 \\ 0 & \sigma_n^{-1} \end{pmatrix} \tag{14}$$

In the following we assume that the system (10) is in the form such that S and R are as in (14). Pick $\alpha=\beta$ large enough, then it follows that

$$\begin{aligned}
&\frac{d}{dt}\left((x-x_r)^TQ(x-x_r)+\sigma_n^2(x+x_r)^TP(x+x_r)^T\right) \\
\leq\;& -|y-y_r|^2-2\alpha(x-x_r)^TSv-2(\dot{x}-\dot{x}_r)^TSv \\
&+4\sigma_n^2|u|^2+2\alpha\sigma_n^2(x+x_r)^TRv+2\sigma_n^2(\dot{x}+\dot{x}_r)^TRv \\
=\;& -|y-y_r|^2+4\sigma_n^2|u|^2
\end{aligned} \tag{15}$$

with v as in Lemma 1. Integrating from 0 to τ for all $\tau>0$ yields that $\|y-y_r\|_2\leq 2\sigma_n\|u\|_2$ if the systems Σ and Σ_r are initially at rest. Consequently we have the following lemma

Lemma 2: Suppose Σ has extended Gramians (P,R,α) and (Q,S,α), where R and S are in the form of (14).

If Σ and the truncated system $\hat{\Sigma}$ are initially at rest then $\|\Sigma-\hat{\Sigma}\|_\infty \leq 2\sigma_n$.

Proof: The proof follows directly from the developments above, using the L_2 induced norm. ■

The above truncation result is useful for the further model reduction procedure. It is however also relevant to establish that the extended Gramians are preserved under this truncation, i.e., similar to [12], [14] it follows that

Lemma 3: Suppose P and Q are of the form

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix} \quad Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix}$$

and R and S are as in (14), then (P_{11}, R_{11}, β) and (Q_{11}, S_{11}, α) are the extended controllability and observability Gramian, respectively, of the truncated system $\hat{\Sigma}$. ◁

We are now ready to state our main result on an error bound for model order reduction for continuous time LTI systems based on extended Gramians:

Theorem 3: Suppose that Σ has balanced extended Gramians in the sense that R and S are balanced, i.e.,

$$S = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix}, \quad R = \begin{pmatrix} \sigma_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \sigma_n^{-1} \end{pmatrix}.$$

Assume $\sigma_k > \sigma_{k+1}$, and write $S = diag(S_k, S_{n-k})$, $R = diag(R_k, R_{n-k})$ Then the truncated k^{th} order system $\hat{\Sigma}$ has balanced extended Gramians (P_{11}, R_k, α) and (Q_{11}, S_k, α). Furthermore

$$\|\Sigma-\hat{\Sigma}\|_\infty \leq 2 \sum_{i=k+1}^{n} \sigma_i$$

Proof: The result follows from applying Lemma 2 and Lemma 3 iteratively. ■

In the same manner as in [14] our result can be further specialized by taking multiplicity of the singular values into account.

The extended Gramians can thus be used for a balancing procedure with a priori error bound. The freedom in choosing R and S provides flexibility to impose certain structure to the reduced order system. In particular in the light of the recent interest in networks and synchronization, it may be relevant to exploit this freedom to preserve the network and synchronization structure for the reduced order model.

V. CONCLUSIONS

In this paper we have developed a balancing method based on extended Gramians for continuous time linear systems. A priori error bounds are provided. These results are relevant for preserving certain specific structures in the reduced order model that are not preserved by standard and generalized balancing. In [2], [3] a generalized balancing approach for networked passive systems is presented, and it is shown that the reduced order system can be presented as a network again. However, this can only be proven for a reduced order system that has a complete graph structure. In our ongoing research we study the possibility to preserve more specific graph structures, such as tree or star graphs. We anticipate that the extended balancing approach may be helpful for that problem setting.

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