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Asymptotic Stability and Attractivity for 2D Linear Systems*

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Abstract— **Asymptotic stability by definition encapsulates two well-known properties usually called Lyapunov stability and attractivity. In the realm of the linear 1D systems however, it is a well-known property that attractivity is equivalent to asymptotic stability. This result has been assumed to be true when working with multidimensional systems but a definitive proof is missing in the literature. In this paper, we present for the first time a complete proof showing that for a linear discrete 2D model (whether a Fornasini-Marchesini or a Roesser model), asymptotic stability is indeed equivalent to attractivity.**

Index Terms— **multidimensional systems, discrete systems, asymptotic stability, attractivity, Roesser model, Fornasini-Marchesini model**

I. Introduction to nD systems

If we consider a problem of gas filtration, one can either model the problem as partial derivative equations where the control is usually a hard problem, or as a 2D Roesser model [1] where some control tools might be easier to develop as detailed in [2]. For multiple applications, these particular models, called nD models or multidimensional systems, have been used with some success in, for instance, image and signal processing [3], network realizability [4], or interconnected systems [5]. For a general overview on the field of multidimensional systems, one can refer to [6], [7], [8], and [9].

Hence, multidimensional systems have been extensively studied during the past 30 years. Some recent works have tried to extend stability and stabilization results to the case of nonlinear nD systems [10], [11], [12], [13], [14] but the field is mainly concerned with nD discrete linear models. Since the introduction of the two widely used models, the Fornasini-Marchesini model [15] and the Roesser model [1], several concepts of stability have been developed similar to what has been done in the 1D case. Asymptotic stability in particular have been in most studies reduced to stability (starting by its first introduction in [15]) mainly because the field is mostly focusing on linear systems. Note that the papers dealing with nonlinear systems quoted above make the distinction.

Recently, new definitions of stability have been introduced studying the relationships of different notions of stability in the field of multidimensional systems ; notably structural stability ([16] and references therein) and exponential stability (for different form of systems, discrete [17], [11], [18], continuous [14], [19] or discrete/continuous [20], [21], [22]).

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We have, in previous works, investigated the links between these definitions in the case of Fornasini model [23], [24] and Roesser model [25]. We have shown that for both models, exponential and structural stability are equivalent. A question of interest was however left over in these work. Indeed, contrary to the 1D case, a proof showing that asymptotic stability and attractivity are equivalent in the 2D linear case have never been completed as far as we know. Although some results are easily extended, it is well-known that the 2D case (or nD) can be surprising and certain results that are true in the 1D case are false in the 2D case. For instance, the Euclidian division is not anymore available in the 2D case and, we recently proved that asymptotic stability is not equivalent to structural stability contrary to the 1D case [24]. These happen for several reasons, including the change introduced by the boundary conditions, which are infinite sequences of vectors instead of a single vector (see for instance [26], [11]).

The paper is therefore short and follows a very straightforward path. First, we introduce in the next section the two linear models, the stability definitions that are going to be investigated and two lemmas that will be used during the main proof. Then, in Section III, we will introduce the main result showing that asymptotic stability is equivalent to attractivity in the case of 2D linear models. We will take time to detail the proof in this section. Finally we will conclude the paper with remaining open questions.

II. Asymptotic stability for 2D system and boundary conditions

In this section, we introduce the two models, the definition of asymptotic stability for these models and two straightforward lemmas.

The first model considered is the Fornasini-Marchesini model [15] already quoted in the introduction:

$$x(i+1,j+1) = A\,x(i,j+1) + B\,x(i+1,j), \qquad (1)$$

where i and j are two indexes that are usually taken in $\mathbb{N}$, the set of positive integers, x is a vector in $\mathbb{R}^n$ and A and B are square matrices of dimension n. Similarly, the boundary conditions are given by $x(0,j) = \psi(j)$ and $x(i,0) = \varphi(i)$.

The second model is the Roesser model [1]:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \qquad (2)$$

and again i and j are two indexes taken in $\mathbb{N}$, x^h and x^v are vectors in $\mathbb{R}^h$ and $\mathbb{R}^v$ and A_1, A_2, A_3 and, A_4 are matrices of appropriate dimension. The boundary conditions are given by $x^h(0,j) = \psi(j)$ and $x^v(i,0) = \varphi(i)$.

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We will use the following notation:

- Ψ will represent the boundary conditions of the model, i.e. $\Psi = (\varphi, \psi)$ where φ is the boundary conditions on the horizontal axis ($j = 0$) and ψ is the boundary conditions on the vertical axis ($i = 0$).
- Given boundary conditions Ψ as above, and given an integer $N \in \mathbb{N}$, we will truncate Ψ at N to obtain boundary conditions $\Psi^N = (\varphi^N, \psi^N)$ as follows. For $i + j \leq N$, $\varphi^N(i) = \varphi(i)$ and $\psi^N(j) = \psi(j)$, and for $i + j > N$, $\varphi^N(i) = 0$ and $\psi^N(j) = 0$. Note that this trivially implies that $\lim_{i\to\infty} \varphi^N(i) = \lim_{j\to\infty} \psi^N(j) = 0$.
- $x(i, j, \Psi)$ is the trajectory of system (1) or (2) with boundary condition Ψ at the point (i, j). If there is no possible confusion, we will simply write $x(i, j)$ instead of $x(i, j, \Psi)$.

With these clarifications in mind, let us now state two lemmas which will be useful later. The first one is obvious:

Lemma 2.1: Given $(p, q) \in \mathbb{N}^2$ and a boundary condition $\Psi = (\varphi, \psi)$, the value of $x(p, q, \Psi)$ depends only on the values of $\varphi(i)$ and $\psi(j)$ for $i \leq p$ and $j \leq q$ respectively. In other words, if two boundary conditions $\Psi = (\varphi, \psi)$ and $\Psi' = (\varphi', \psi')$ are such that $\varphi(i) = \varphi'(i)$ for all $i \leq p$ and $\psi(j) = \psi'(j)$ for all $j \leq q$, then $x(p, q, \Psi) = x(p, q, \Psi')$.

Remark 1: In particular, a direct consequence of Lemma 2.1 is that for a given boundary condition Ψ and integers $i + j = N$, $x(i, j, \Psi) = x(i, j, \Psi^N)$.

As for the second lemma, we have

Lemma 2.2: For any integer $N \in \mathbb{N}$, there is a constant $C_N > 0$ such that for all boundary conditions $\Psi = (\varphi, \psi)$, we have: forall $(i, j) \in \mathbb{N}^2$ satisfying $i + j \leq N$, $\|x(i, j, \Psi)\| \leq C_N \max_{i+j\leq N}(\|\varphi(i)\|, \|\psi(j)\|)$.

Proof: We deal first with the Fornasini-Marchesini model. Consider vectors $\alpha(1), \ldots, \alpha(N)$ and $\beta(1), \ldots \beta(N)$ in $\mathbb{R}^n$. Choose boundary conditions $\Psi = (\varphi, \psi)$ such that $\varphi(i) = \alpha(i)$ for all $i \leq N$ and $\psi(j) = \beta(j)$ for all $j \leq N$. Then by Lemma 2.1, the value of $x(i, j, \Psi)$ for $i + j \leq N$ does not depend on the choice of Ψ. Therefore, the map which to each $(\alpha(1), \ldots, \alpha(N), \beta(1), \ldots \beta(N)) \in (\mathbb{R}^n)^N \times (\mathbb{R}^n)^N$ assigns the vector $(x(i, j, \Psi))_{i+j\leq N} \in (\mathbb{R}^n)^{N(N+1)/2}$ is well-defined. It is moreover a linear map because the system is linear. As the vector space on which it acts is finite dimensional, it is a bounded map. This implies the conclusion of our lemma.

For the Roesser model, the proof is the same, except that we have to change the space on which the boundary conditions are taken. ■

Let us now give the definitions of stability used in this paper.

Definition 2.3 ($\epsilon - \delta$ stability): The point $x_e = 0$ is said to be stable (in the sense of Lyapunov) if for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that if $\|\varphi(i)\| < \delta$ and $\|\psi(j)\| < \delta$ for all $i, j > 0$, then $\|x(i, j)\| < \epsilon$ for all $i, j > 0$.

Definition 2.4 (attractivity): The point $x_e = 0$ is said to be attractive when, for all boundary conditions $\Psi = (\varphi, \psi)$, if $\lim_{j\to\infty} \psi(j) = 0$ and $\lim_{i\to\infty} \varphi(i) = 0$, then $\lim_{i+j\to\infty} x(i, j) = 0$. We will call this property *attractivity* as in the 1D case.

Remark 2: Here, $\lim_{i+j\to\infty} x(i, j) = 0$ means that for all $\epsilon > 0$, there exists an integer $N \in \mathbb{N}$ such that for all $(i, j) \in \mathbb{N}^2$, if $i + j \geq N$, then $\|x(i, j)\| < \epsilon$.

Definition 2.5 (asymptotic stability): The point $x_e = 0$ is said to be asymptotically stable if $x_e = 0$ is stable and attractive.

Remark 3: It is important to observe that the notion of attractivity is slightly different than the one commonly used for 1D systems. Indeed, in the 1D case, one only asks that $\lim_{i\to\infty} x(i) = 0$ without constraints on the boundary conditions. In the 2D case, we ask x to reach the origin in all possible directions i or j. But, for instance, the direction $i = 0$ and $j \to \infty$ is exactly matching the boundary condition $\psi(j)$. Therefore, in the 2D case, attractivity is only checked for a particular type of boundary conditions ($\lim_{j\to\infty} \psi(j) = 0$ and $\lim_{i\to\infty} \varphi(i) = 0$). This has been discussed in [11].

III. Main result: asymptotic stability is equivalent to attractivity

In this section, we introduce the main result of this paper: in the case of a linear 2D model, the notion of asymptotic stability is reduced to attractivity similar to the 1D case. This result mainly uses the linearity of the system and therefore is not dependent on the choice of the linear 2D model. Thus, it is true for both the discrete Roesser and Fornasini-Marchesini models. Note that the explicit form of these models is used mainly for these reasons: to determine the correct space of boundary conditions, the correct notion of attractivity, and for Lemma 2.1.

Theorem 3.1: If the equilibrium point of a linear 2D model is attractive, it is also asymptotically stable.

Proof: We argue by contradiction and assume that there is a system for which 0 is an attractive equilibrium point which is not asymptotically stable, hence is not stable in the sense of Lyapunov. We are going to construct boundary conditions converging to zero at infinity such that the associated trajectory does not converge to zero, contradicting our attractivity assumption.

To construct such a trajectory, the first step is to prove the following:

Claim. Assume that we have a 2D linear system which is not stable in the sense of Lyapunov but for which 0 is an attractive equilibrium point. Then there exist a constant $\epsilon > 0$, and for each $k \in \mathbb{N}$, a small constant $\eta_k > 0$, an integer $N_k \in \mathbb{N}$, a couple of integers $(i_k, j_k) \in \mathbb{N}^2$ and an boundary condition Ψ_k such that

(1) $i_k + j_k < N_k < i_{k+1} + j_{k+1}$.
(2) $\eta_k \leq 1/(k + 1)^2$.
(3) $\|\Psi_k\| \leq \eta_k$.
(4) $\|x(i_k, j_k, \Psi_k)\| \geq \epsilon$.
(5) Set $\epsilon_k = \frac{3}{\pi^2(k+1)^2}\epsilon$. If $k \geq 1$, then for $i + j \leq N_{k-1}$ we have $\|x(i, j, \Psi_k)\| \leq \epsilon_k$.
(6) Denoting by $\Psi_k^{i_k+j_k}$ the truncation of Ψ_k at $i_k + j_k$ (see the previous section), then for $i + j \geq N_k$ we have

$$\left\| x(i,j,\Psi_k^{i_k+j_k}) \right\| \le \epsilon_k.$$

This claim is graphically illustrated in Figure 1.

Remark 4: Note that we cannot take η_k to be $1/(k+1)^2$, as η_k has also to be small enough so that condition (i) (displayed after Figure 1) holds.

Assume for the moment that the claim is true. Then the obvious inequality $\left\|\Psi_k^{i_k+j_k}\right\| \le \|\Psi_k\|$ implies, by using (2) and (3), that the boundary condition $\Psi_\infty := \sum_{k=1}^{\infty} \Psi_k^{i_k+j_k}$ is well defined[1]. Moreover, it is easily checked that, by putting $\Psi_\infty = (\varphi_\infty, \psi_\infty)$, we have that φ_∞ and ψ_∞ both converge to zero. Therefore, by attractivity, $x(i,j,\Psi_\infty) \to 0$ when $i+j \to +\infty$.

To unveil the contradiction, we now show that this trajectory is greater than $\epsilon/2$ at carefully constructed points. Indeed, fix $k \in \mathbb{N}^*$ and let $l \le k$ be any integer. The system is linear so that the map which, to each boundary condition, assigns the corresponding solution is linear. Using this fact and the triangle inequality, we get

$$\left\| x(i_l, j_l, \sum_{1\le m\le k} \Psi_m^{i_m+j_m}) \right\| \ge \left\| x(i_l,j_l,\Psi_l^{i_l+j_l}) \right\| - \sum_{1\le m\le k,\, m\ne l} \left\| x(i_l,j_l,\Psi_m^{i_m+j_m}) \right\|.$$

From Lemma 2.1 and property (4) we deduce that

$$\left\| x(i_l,j_l,\Psi_l^{i_l+j_l}) \right\| \ge \epsilon.$$

For $m < l$, property (1) implies that $i_l + j_l > N_m$, so that by property (6) we get

$$\left\| x(i_l,j_l,\Psi_m^{i_m+j_m}) \right\| \le \epsilon_m.$$

For $m > l$, property (1) implies that $i_l+j_l \le i_{m-1}+j_{m-1} < N_{m-1}$ so that by property (5) and Lemma 2.1 we get

$$\left\| x(i_l,j_l,\Psi_m^{i_m+j_m}) \right\| \le \epsilon_m.$$

It follows that

$$\left\| x(i_l, j_l, \sum_{1\le m\le k} \Psi_m^{i_m+j_m}) \right\| \ge \epsilon - \sum_{1\le m\le k} \epsilon_m.$$

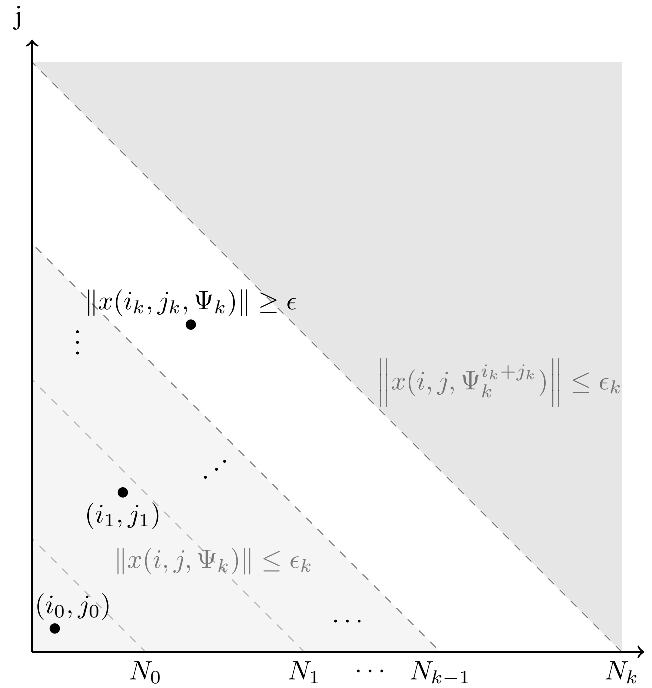
Now, a consequence of lemma 2.2 is that for fixed (i,j), we have $\lim_{k\to+\infty} x(i,j,\sum_{1\le m\le k} \Psi_m^{i_m+j_m}) = x(i,j,\Psi_\infty)$. Therefore, by letting k go to infinity in the inequality above we deduce that for all $l \in \mathbb{N}$, we have

$$\|x(i_l,j_l,\Psi_\infty)\| \ge \epsilon - \sum_{k=1}^{+\infty} \epsilon_k \ge \epsilon/2.$$

As by property (1) we have $i_l + j_l \to +\infty$ when $l \to +\infty$, there follows that $x(i,j,\Psi_\infty)$ does not converge to 0 when $i+j$ goes to $+\infty$ and this conludes the proof of the theorem, provided that we justify our claim.

[1]The series $\sum \left\|\Psi_k^{i_k+j_k}\right\|$ is bounded by a convergent series $\sum \frac{1}{(k+1)^2}$.

Fig. 1. Graphical illustration of the claim in Theorem 3.1



In order to prove the claim, we first notice that our system is assumed not to be stable in the sense of Lyapunov, which mathematically is expressed as the following: there exists $\epsilon > 0$ such that for all $\eta > 0$, there is an boundary condition Ψ satisfying $\|\Psi\| \le \eta$ but for which there is a point (i,j) with $\|x(i,j;\Psi)\| \ge \epsilon$.

So, given this $\epsilon > 0$, we can just choose some $0 < \eta_0 \le 1$. By instability, there is an boundary condition Ψ_0 with $\|\Psi_0\| \le \eta_0$ and there is $(i_0,j_0) \in \mathbb{N}^2$ such that $\|x(i_0,j_0,\Psi_0)\| \ge \epsilon$. By attractivity, there is an integer N_0, which we may assume to satisfy $i_0 + j_0 < N_0$, such that for all $i+j \ge N_0$, we have $\left\|x(i,j,\Psi_0^{i_0+j_0})\right\| \le \epsilon_0$.

Using Lemma 2.2 (with $N = N_0$) and using instability, if we choose $0 < \eta_1 \le 1/2^2$ small enough, then there is an boundary condition Ψ_1 satisfying $\|\Psi_1\| \le \eta_1$ such that

(i) for all $(i,j) \in \mathbb{N}^2$ with $i+j \le N_0$, we have $\|x(i,j,\Psi_1)\| \le \epsilon_1$.
(ii) there is $(i_1,j_1) \in \mathbb{N}^2$ such that $\|x(i_1,j_1,\Psi_1)\| \ge \epsilon$.

As $\epsilon_1 < \epsilon$, conditions (i) and (ii) imply the inequality $i_1 + j_1 > N_0$. Finally, by attractivity, there is an integer N_1, which we may assume to satisfy $N_1 > i_1 + j_1$, such that for $i+j \ge N_1$ we have

$$\left\|x(i,j,\Psi_1^{i_1+j_1})\right\| \le \epsilon_1.$$

It is now clear that we can prove the claim by induction on k as the step from k to $k+1$ uses the same principles.

This completes the proof of the theorem. ■

IV. CONCLUSIONS

In this paper, we introduced a result that has never been proven in the literature of 2D linear systems as far as we know. Indeed, it is shown that if the equilibrium point of a linear 2D model is attractive, it is also asymptotically stable.

We believe that the result remains correct in the continuous case, at least for continuous Roesser models, although we have not a complete proof yet. To expand on this a little bit, we note that given Lemmas 2.1 and 2.2, the proof of our main result uses basically only the linearity assumption. Hence, if we could prove the analog of these lemmas for continuous systems, we should be able to prove that attractivity and asymptotic stability are also equivalent for these systems. Now, the analog of Lemma 2.1 for continuous systems seems straightforward, but it is less obvious to prove the analog of Lemma 2.2, as our argument in the discrete case, using finite dimensional vector spaces, breaks down in the continuous case. However, we believe that in the case of a continuous Roesser model, in which only first derivatives appear, we should be able to use a Gronwall type argument to circumvent this difficulty. This idea should not work for a continuous Fornasini-Marchesini model in which second order derivatives also appear.

Further questions have not been addressed by this work, the first one concerns the nonlinear case. Examples exist in the literature of 1D models showing that the equivalence between asymptotic stability and attractivity is not true anymore when dealing with nonlinear systems [27, p. 192]. We believe these examples can be generalized to 2D systems but this is still an open problem.

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