2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Exploiting the Superposition Property of Wireless Communication For Average Consensus Problems in Multi-Agent Systems

Fabio Molinari, Sławomir Stańczak, Jörg Raisch

Abstract— **This paper studies system stability and performance of multi-agent systems in the context of consensus problems over wireless multiple-access channels (MAC). We propose a consensus algorithm that exploits the broadcast property of the wireless channel. Therefore, the algorithm is expected to exhibit fast convergence and high efficiency in terms of the usage of scarce wireless resources. The designed algorithm shows robustness against variations in the channel and consensus is always reached. However the consensus value will be depending on these variations.**

I. Introduction

Achieving consensus is an essential task in many distributed control scenarios where a number of control units ("agents") interact to achieve a common aim. Consensus problems in multi-agent systems require the agents to reach an agreement over a certain real-valued scalar or vector, e.g., [1], [2], [3]. Each agent has a local guess of this entity, called the agent's information state, which has to be updated according to some rule, typically a function of the information states of neighbouring agents. Consensus is achieved if all the information states converge to the same value. Consensus-based approaches have been proven to be valuable choices in a wide set of problems, as, for example, the rendez-vous problem [4], control of vehicle formation [5], or the so-called flocking problem [6]. Classical approaches consider *communication* and *computation* as two distinct tasks. Indeed, as communication strategies are usually designed to reliably deliver each information state to a subset of agents by creating independent communication channels, agents have knowledge of other agents' information states. In general, however, each agent is only interested in a function of other agents' information states, which carries less information (in the information-theoretic sense) than the knowledge of individual information states. This opens the door to significant performance gains. Inspired by [7], the authors of [8] proposed an approach that merges communication and computation of nonlinear functions and is based on the nomographic representation of functions. As pointed out in [8], Buck proved in [9] that every real-valued multivariate function is representable in its nomographic form as a function of a finite sum of univariate functions. Based on this deep insight, the authors of [8] concluded that the superposition property of the wireless channel (also called the broadcast property) can be used to approximate an arbitrary function of transmitted signals. According to this, each agent simultaneously broadcasts a suitably chosen function of its information state. Then, each agent postprocesses the received signal, which is a noisy superposition of the locally preprocessed information states transmitted by its neighbours, to estimate the desired function value. If the goal is to achieve average consensus, the employed consensus function is typically linear in the neighbouring agents' information states. In this case, the consensus function is already expressed in its nomographic representation with both pre- and postprocessing functions continuous in the set of real numbers [8]. Using the superposition property of wireless channels then allows for significantly faster convergence when compared to standard communication protocols, but introduces distortions (namely, the unknown channel coefficients) proportional to the transmitted signals, which, if not properly addressed, will cause undesired behaviour. Existing approaches to exploit superposition neglected the influence of channel coefficients, by considering ideal MACs (wireless multiple access channels) [8], [10]. We will relax these assumptions and assume, in a realistic way, channel coefficients with no constraints apart from positivity.

An outline of this paper is as follows. In Section II, consensus problems on graphs are presented; channel superposition and usage of interference for consensus problems over wireless networks are then explained. In Section III, a consensus algorithm exploiting superposition is proposed. The influence of its parameters on convergence rate and consensus value is addressed in Section IV and illustrated via simulations in Section V. Finally, in Section VI, concluding remarks are stated.

Nomenclature

We use $\mathbb{N}$ and $\mathbb{R}$ to denote, respectively, the set of positive integers and the set of real numbers. The set of positive real numbers and nonnegative real numbers are denoted, respectively, by $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$. Given a scalar a, its absolute value is denoted by $|a|$. The closed unit interval is $\mathbb{E} := [0,1] \subset \mathbb{R}$. The $n \times m$ zero matrix is denoted by $\mathbf{0}_{n\times m}$. Given a matrix A, its transpose is A', while its conjugate transpose is A^*. The trace of a matrix A is denoted by $tr(A)$. The element in position (i,j) of A is referred to as

F. Molinari is with the Control Systems Group - Technische Universität Berlin, Germany.
S. Stańczak is with the Network Information Theory Group - Technische Universität Berlin & Fraunhofer Heinrich Hertz Institute, Germany.
J. Raisch is with the Control Systems Group - Technische Universität Berlin, Germany & Max-Planck-Institut für Dynamik Komplexer Technischer Systeme, Germany.
molinari@control.tu-berlin.de, slawomir.stanczak@hhi.fraunhofer.de, raisch@control.tu-berlin.de

This work was funded by the German Research Foundation (DFG) within their priority programme SPP 1914 "Cyber-Physical Networking (CPN)".

$[A]_{ij}$. The $n\times m$ matrix A is positive (nonnegative), denoted by $A > 0$ ($A \geq 0$), if $\forall i, j : 1 \leq i \leq n,\ 1 \leq j \leq m$, $[A]_{ij} > 0$ ($[A]_{ij} \geq 0$). $A \geq 0$ is row-stochastic if $A\mathbf{1} = \mathbf{1}$ where $\mathbf{1}$ is the column vector with all ones. Two $n \times n$ nonnegative matrices A and B are of the same type (denoted by $A \sim B$) if they have zero entries in the same locations. A is double-stochastic if A and A' are both row-stochastic. A nonnegative square matrix A is said to be primitive if there exists $k \in \mathbb{N}$ such that $A^k > 0$. Eigenvalues of the $n \times n$ matrix A are denoted by $\lambda_i(A)$, $1 \leq i \leq n$, and assumed without loss of generality to be ordered as follows: $|\lambda_1(A)| \leq |\lambda_2(A)| \leq \cdots \leq |\lambda_n(A)|$. The identity matrix of dimension $n \times n$ is denoted by $\mathbb{I}_n$; usually, in the event that the context is clear, subscripts are neglected.

The convex hull $\mathcal{C}(\mathbf{S})$ of a set $\mathbf{S} = \{\mathbf{v}_i \in \mathbb{R}^n,\ 1 \leq i \leq m,\ m \in \mathbb{N},\ n \in \mathbb{N}\}$ is the intersection of all convex sets containing $\mathbf{S}$. So we have $\mathcal{C}(\mathbf{S}) = \{\sum_{j=1}^{m} \lambda_j \mathbf{v}_j : \lambda_j \geq 0\ \ \forall j,\ \sum_{j=1}^{m} \lambda_j = 1\}$.

Given a discrete-time signal $p(k) : \mathbb{N} \to \mathbb{R}$, its zeta-tranform is denoted by $P(z) = \mathcal{Z}(p(k))$.

Finally, given a finite set $\mathcal{V}$, its cardinality is denoted by $|\mathcal{V}|$.

Definition 1 (Directed graph)**.** A directed graph (or digraph) is a pair $(\mathcal{N}, \mathcal{A})$, where $\mathcal{N}$ represents a finite set of nodes and $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of arcs.

In the following, we always assume that $(i, i) \notin \mathcal{A}$, $\forall i \in \mathcal{N}$.

Definition 2 (Neighbors)**.** Given a directed graph $(\mathcal{N}, \mathcal{A})$, the set of neighbours of a node $i \in \mathcal{N}$, denoted by N_i, is the set of those nodes $l \in \mathcal{N}$ for which $(l, i) \in \mathcal{A}$.

By the assumption above, $i \notin N_i$.

Definition 3 (Weighted directed graph)**.** A weighted directed graph is a triple $(\mathcal{N}, \mathcal{A}, w)$, where $(\mathcal{N}, \mathcal{A})$ is a digraph and $w : \mathcal{A} \to \mathbb{R}_{>0}$ associates each arc $(j, i) \in \mathcal{A}$ with a positive weight w_{ij}.

The digraph is balanced if $\forall i \in \mathcal{N}$, $\sum_{j\in N_i} w_{ij} = \sum_{\{j:\ i\in N_j\}} w_{ji}$, i.e. if, for each node, the sum of the weights of all incoming arcs equals the sum of the weights of all outgoing arcs. A directed path in a digraph is a sequence of nodes in which there is an arc pointing from each node in the sequence to its successor in the sequence. The digraph is called strongly connected if there exists a directed path between any two distinct nodes. The digraph is called fully connected (or complete) if there exists an arc between any two distinct nodes.

II. System model and problem statement

A. Consensus for weighted digraphs

We consider a time-varying network described by a sequence of weighted directed graphs

$$\Gamma = \{\Gamma_k : \Gamma_k = (\mathcal{N}, \mathcal{A}, w(k)),\ k \in \mathbb{N}\} \tag{1}$$

with $n = |\mathcal{N}|$ communicating agents (nodes) and with a strongly connected topology. Each agent has the following discrete-time integrator dynamics:

$$x_i(k+1) = x_i(k) + u_i(k),\ i \in \{1, \ldots, n\}. \tag{2}$$

$x_i : \mathbb{N} \to \mathbb{R}$ is the agent's state and $u_i : \mathbb{N} \to \mathbb{R}$ its input. The system (2) can be also expressed compactly as

$$\mathbf{x}(k+1) = \mathbf{x}(k) + \mathbf{u}(k), \tag{3}$$

where $\mathbf{x}(k) = [x_1(k), \ldots, x_n(k)]'$ and $\mathbf{u}(k) = [u_1(k), \ldots, u_n(k)]'$, $\forall k \in \mathbb{N}$.

Definition 4 (Perron matrix)**.** Let a graph $\Gamma_k = (\mathcal{N}, \mathcal{A}, w(k))$, consisting of n communicating agents with dynamics (2), and a parameter $\epsilon_k \in (0, \Delta_k)$ with

$$\Delta_k = \frac{1}{\max_i(\sum_{j\in N_i} w_{ij}(k))} \tag{4}$$

be given. The *Perron matrix* of Γ_k with parameter ϵ_k is the matrix $D_n(k)$ defined to be

$$D_n(k) := \mathbb{I}_n - \epsilon_k \mathcal{L}(\Gamma_k), \tag{5}$$

where $\mathcal{L}(\Gamma_k)$ is the Laplacian of Γ_k [2]. The entries of $D_n(k)$ are $[D_n(k)]_{ii} = 1 - \epsilon_k \sum_{j\in N_i} w_{ij}(k) > 0$, $\forall i \in \mathcal{N}$, $[D_n(k)]_{ij} = \epsilon_k w_{ij}(k)$, $\forall (j, i) \in \mathcal{A}$, and $[D_n(k)]_{ij} = 0$, $\forall i \neq j$ with $(j, i) \notin \mathcal{A}$. We refer to Lemma 3 in [3] for the properties of the *Perron matrix*, which is row-stochastic by construction, and primitive if Γ_k is strongly connected.

In literature (see e.g. [11]), the linear consensus protocol

$$u_i(k) = \epsilon_k \sum_{j\in N_i} w_{ij}(k)(x_j(k) - x_i(k)) \tag{6}$$

is widely used and can be expressed in matrix form as

$$\mathbf{u}(k) = -\epsilon_k \mathcal{L}(\Gamma_k)\mathbf{x}(k). \tag{7}$$

By applying (7) to the system (3), the closed loop dynamics becomes

$$\mathbf{x}(k+1) = D_n(k)\mathbf{x}(k). \tag{8}$$

As mentioned in Definition 4, $D_n(k)$ is primitive, therefore it has a unique real eigenvalue that strictly dominates the moduli of all other eigenvalues, which is $\rho(D_n(k)) = \lambda_n(D_n(k)) = 1$ since $D_n(k)$ is also row-stochastic. By the *Perron-Frobenius theorem*, in case of a time-invariant problem (i.e. $\forall k \in \mathbb{N}$, $w(k) = w$) with graph Γ_k unbalanced, the consensus value will be $x^* = \mathbf{w}'\mathbf{x}(0)$, where $\mathbf{w}'D_n = \mathbf{w}'$ and $\mathbf{w}'\mathbf{1} = 1$. Accordingly, $x^* \in \mathcal{C}(\mathbf{x}(0))$. In case of Γ_k balanced, D_n is double stochastic and consequently $x^* = \frac{1}{n}\mathbf{1}'\mathbf{x}(0)$, which is (linear) average consensus.

In the general case, some convergence results for time-variant multi-agent systems are presented in [12]. In the case considered in this paper, Γ is a sequence of weighted digraphs with the same topology but with different positive weights. By Definition 4, for a strongly connected topology, $D_n(k)$ will be a sequence of row-stochastic primitive matrices of the same type and with positive diagonal entries. Their product is characterized by the following result.

Proposition 1. Given two nonnegative $n \times n$ primitive matrices A, B with positive diagonal entries, then AB will be primitive with positive diagonal entries.

Proof: By [13, p. 3], if A is primitive, any nonnegative matrix $\tilde{A}$ of the same type as A is primitive. If $\tilde{A}$ is primitive and C is nonnegative, $(\tilde{A} + C)$ is primitive. As

$$[AB]_{ij} = \sum_{k=1}^{n} [A]_{ik}[B]_{kj} \geq [A]_{ij}[B]_{jj}, \tag{9}$$

and $[B]_{jj} > 0$, $[AB]_{ij}$ is positive whenever $[A]_{ij} > 0$. We can therefore write

$$AB = \tilde{A} + C, \tag{10}$$

where $\tilde{A}$ is nonnegative and of the same type as A, and C is nonnegative. Hence, with the above argument, AB is primitive. Positivity of its diagonal elements is straightforward as $[AB]_{ii} = \sum_{k=1}^{n} [A]_{ik}[B]_{ki} \geq [A]_{ii}[B]_{ii} > 0$. □

By Proposition 1, $\forall k, h \in \mathbb{N}$, the product $D_n(k + h)D_n(k + h - 1) \ldots D_n(k)$ results in a primitive and row-stochastic matrix, therefore, according to [14], (8) achieves consensus. In addition, the agreement value lies in the convex hull of the initial states. Some examples of average-consensus for time-variant systems can be found in [15]. Next we present a result that is used later in this paper.

Proposition 2. Given a row-stochastic matrix P of dimension n and a scalar $\epsilon \in \mathbb{R}_{>0}$, there exists a directed graph Γ_P, such that P is the Perron matrix of Γ_P with parameter ϵ. Moreover, if P is positive, Γ_P is fully connected. If P is symmetric, Γ_P is undirected.

Proof: By (5) we need to show that, given P and $\epsilon \in \mathbb{R}_{>0}$, there is an adjacency matrix $A \in \mathbb{R}_{\geq 0}^{n \times n}$ of a graph with $tr(A) = 0$, such that $P = I - \epsilon(D - A)$, where D is the degree matrix of A as defined in [16], with $[D]_{ii} = \sum_{j=1}^{n} [A]_{ij}$, $1 \leq i \leq n$. This condition can be written as

$$\begin{cases} [P]_{ii} = 1 - \epsilon \sum_{i \neq j} a_{ij} & \forall i \\ [P]_{ij} = \epsilon a_{ij} & \forall i \neq j \\ \sum_j [P]_{ij} = 1 & \forall i \end{cases}. \tag{11}$$

By (11), A is an adjacency matrix of a weighted digraph, whose elements are $a_{ij} = \frac{[P]_{ij}}{\epsilon}$, $\forall i \neq j$, $1 \leq i \leq n$, $1 \leq j \leq n$, and $a_{ii} = 0$, $1 \leq i \leq n$. Whenever P is positive, then, $a_{ij} > 0$, $\forall i \neq j$, in which case Γ_P is fully connected. If P is symmetric, we have $a_{ij} = \frac{[P]_{ij}}{\epsilon} = \frac{[P]_{ji}}{\epsilon} = a_{ji}$, $\forall i \neq j$, $1 \leq i \leq n$, $1 \leq j \leq n$, so that Γ is undirected, since its adjacency matrix is symmetric. This completes the proof. □

B. Exploiting channel superposition for average consensus

To introduce the underlying idea, we first review some existing results. By [9], we know that each multivariate function $f : \mathbb{E}^n \to \mathbb{R}$ has a nomographic representation:

$$f(x_1, \ldots, x_n) = \psi(\sum_{j=1}^{n} \phi_j(x_j)), \tag{12}$$

for some $\psi : \mathbb{R} \to \mathbb{R}$ and $\phi_j : \mathbb{E} \to \mathbb{R}$. We are interested in nomographic representations since they allow us to exploit the superposition (interference) property of the wireless channel for function computation over a multi-agent wireless network.

We consider a wireless network represented by a directed graph $(\mathcal{N}, \mathcal{A})$, where $\mathcal{N} = \{1, \ldots, n\}$ is the set of nodes and $(i, j) \in \mathcal{A} \subset \mathcal{N} \times \mathcal{N}$ if and only if information transmission from $i \in \mathcal{N}$ to $j \in \mathcal{N}$ is established.

If, after the implementation of a general consensus protocol, each agent evolves according to

$$x_i(k + 1) = f_i(x_1(k), \ldots, x_n(k)), \; i \in \mathcal{N}, \tag{13}$$

it does not need to reconstruct the individual information states of other agents. By using a nomographic representation of f_i and the interference property of the wireless channel, one use of the noiseless channel is sufficient to compute the function. For this, each node $i \in \mathcal{N}$ will broadcast simultaneously with all the other nodes at instant $k \in \mathbb{N}$ its pre-processed information state $\phi_i(x_i(k))$. By describing the communication with the standard affine model of a wireless multiple-access channel (MAC) [8], the real-valued signal received at node $i \in \mathcal{N}$ becomes

$$Y_i(k) = \sum_{j \in N_i} h_{ij}(k)\phi_j(x_j(k)) + v_i(k), \tag{14}$$

where $h_{ij}(k) \in \mathbb{R}_{>0}$, $\forall j \in N_i$ (otherwise 0), denotes a channel coefficient from node j (transmitter) to node i (receiver) and $v_i \in \mathbb{R}$ is the corresponding receiver noise. Ideally, $h_{ij}(k) = 1$, $\forall j \in N_i$, and $v_i(k) = 0$, $\forall i \in \mathcal{N}$. In this ideal case, every node $i \in \mathcal{N}$ computes $x_i(k + 1) = \psi_i(\phi_i(x_i(k)) + Y_i(k)) = f_i(x_1(k), \ldots, x_n(k))$.

To achieve average-consensus, each node $i \in \mathcal{N}$ may compute at iteration k

$$x_i(k + 1) = f_i(x_1(k), \ldots, x_n(k)) = \frac{\sum_{j \in N_i \cup \{i\}} x_j(k)}{|N_i| + 1}. \tag{15}$$

This function is nomographic with $\phi_j(y_j) = y_j$ and $\psi_i(y) = \frac{y}{|N_i|+1}$, both trivially continuous and differentiable in $\mathbb{R}$. If the receiver noise can be neglected (i.e. $\forall k \in \mathbb{N}$, $\forall i \in \mathcal{N}$, $v_i(k) = 0$), then, for the average-consensus problem, (14) becomes

$$Y_i(k) = \sum_{j \in N_i} h_{ij}(k)x_j(k). \tag{16}$$

Each agent $i \in \mathcal{N}$ then computes $x_i(k+1) = \frac{1}{|N_i|+1}(x_i(k) + Y_i(k))$. Therefore

$$\mathbf{x}(k + 1) = D(k)\mathbf{x}(k) \tag{17}$$

with

$$D(k) = \begin{bmatrix} \frac{1}{|N_1|+1} & \frac{h_{12}(k)}{|N_1|+1} & \cdots & \frac{h_{1n}(k)}{|N_1|+1} \\ \frac{h_{21}(k)}{|N_2|+1} & \frac{1}{|N_2|+1} & \cdots & \frac{h_{2n}(k)}{|N_2|+1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{h_{n1}(k)}{|N_n|+1} & \frac{h_{n2}(k)}{|N_n|+1} & \cdots & \frac{1}{|N_n|+1} \end{bmatrix}. \tag{18}$$

$D(k)$ is a sequence of nonnegative square matrices (according to the definition of channel coefficients). In what follows, the following assumption is made:

Assumption 1. The communication topology $(\mathcal{N}, \mathcal{A})$ is strongly connected.

However, $D(k)$ is in general not row-stochastic. Hence, we cannot expect consensus.

III. Controller Design

In the following, we discuss a control strategy that achieves consensus despite the a priori unknown channel coefficients.

A. Protocol design

Under the assumption of a noiseless channel, each agent $i \in \mathcal{N}$ broadcasts two orthogonal signals, $\tau_i(k) = x_i(k)$ and $\tau_i'(k) = 1$ (by using a MAC of order 2 [17]). Due to the superposition property, each agent $i \in \mathcal{N}$ receives from neighbouring agents two orthogonal real-valued signals, $Y_i(k) = \sum_{j \in N_i} h_{ij}\tau_j(k)$, which is equal to (16), and

$$Y_i'(k) = \sum_{j \in N_i} h_{ij}(k)\tau_j'(k) = \sum_{j \in N_i} h_{ij}(k). \tag{19}$$

B. Controller design

A controller can then be defined by

$$u_i(k) = \sigma_i \left(\frac{Y_i(k)}{Y_i'(k)} - x_i(k) \right) \tag{20}$$

resulting in

$$x_i(k+1) = (1-\sigma_i)x_i(k) + \sigma_i \left[\frac{Y_i(k)}{Y_i'(k)} \right], \tag{21}$$

where $\sigma_i \in (0,1)$, $1 \leq i \leq n$. As the quantity $\frac{Y_i(k)}{Y_i'(k)}$ is a weighted average of neighbours' information states,

$$\frac{Y_i(k)}{Y_i'(k)} = \frac{\sum_{j \in N_i} h_{ij}(k)x_j(k)}{\sum_{j \in N_i} h_{ij}(k)}, \tag{22}$$

the system can be written in matrix form as

$$\mathbf{x}(k+1) = D_n^\sigma(k)\mathbf{x}(k), \tag{23}$$

where

$$D_n^\sigma(k) = \begin{bmatrix} (1-\sigma_1) & \frac{\sigma_1 h_{12}(k)}{\sum_{j \in N_1} h_{1j}(k)} & \cdots & \frac{\sigma_1 h_{1n}(k)}{\sum_{j \in N_1} h_{1j}(k)} \\ \frac{\sigma_2 h_{21}(k)}{\sum_{j \in N_2} h_{2j}(k)} & (1-\sigma_2) & \cdots & \frac{\sigma_2 h_{2n}(k)}{\sum_{j \in N_2} h_{2j}(k)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\sigma_n h_{n1}(k)}{\sum_{j \in N_n} h_{nj}(k)} & \frac{\sigma_n h_{n2}(k)}{\sum_{j \in N_n} h_{nj}(k)} & \cdots & (1-\sigma_n) \end{bmatrix}. \tag{24}$$

The sequence $D_n^\sigma(k)$ is composed of row-stochastic matrices $\forall k \in \mathbb{N}$. We show next that, $\forall \sigma_i \in (0,1)$, $1 \leq i \leq n$, $D_n^\sigma(k)$ is a sequence of primitive matrices of the same type.

We need first to consider a special case; if σ_i in (21) is

$$\sigma_i = \bar{\sigma}_i := \epsilon_k \sum_{j \in N_i} h_{ij}(k) \qquad \forall i \in \mathcal{N}, \tag{25}$$

where ϵ_k is chosen as

$$0 < \epsilon_k < \frac{1}{\max_i(\sum_{j \in N_i} h_{ij}(k))}, \tag{26}$$

then $D_n^\sigma(k)$ is the Perron matrix with parameter ϵ_k of a directed weighted graph $\Gamma_{\sigma,k}$ (Proposition 2), and is denoted by $\bar{D}_n^\sigma(k)$. To be precise, we have

$$\Gamma_{\sigma,k} = (\mathcal{N}, \mathcal{A}, w(k)), \ k \in \mathbb{N}, \tag{27}$$

where the weights are determined by the channel coefficients $w_{ij}(k) = h_{ij}(k)$, $\forall (j,i) \in \mathcal{A}$. By Assumption 1, by Definition 4, and since $h_{ij}(k) > 0$ if $(j,i) \in \mathcal{A}$, $\bar{D}_n^\sigma(k)$ is a sequence of primitive row-stochastic matrices.

In the general case, $\forall \sigma_i \in (0,1)$, $1 \leq i \leq n$, the resulting $D_n^\sigma(k)$ is a sequence of nonnegative matrices of the same type as $\bar{D}_n^\sigma(k)$, therefore primitive.

As presented in Section II, since $D_n^\sigma(k)$ is a sequence of row-stochastic primitive matrices of the same type, (23) achieves consensus, i.e. $\mathbf{x}(k) \to \mathbf{x}^* = \mathbf{1}x^*$ where $x^* \in \mathcal{C}(\mathbf{x}(0))$. However, $D_n(k)$ is non-symmetric in general; therefore (23) converges to a weighted average consensus, i.e., in general $x^* \neq \frac{1}{n}\sum_{i=1}^n x_i(0)$.

IV. Influence of σ, n, and h_{ij}

In the following, we assume $\sigma_i = \sigma$ in (21). Thus, the closed loop dynamics (23) is characterized by the matrix

$$D_n^\sigma(k) = \begin{bmatrix} (1-\sigma) & \frac{\sigma h_{12}(k)}{\sum\limits_{j \in N_1} h_{1j}(k)} & \cdots & \frac{\sigma h_{1n}(k)}{\sum\limits_{j \in N_1} h_{1j}(k)} \\ \frac{\sigma h_{21}(k)}{\sum\limits_{j \in N_2} h_{2j}(k)} & (1-\sigma) & \cdots & \frac{\sigma h_{2n}(k)}{\sum\limits_{j \in N_2} h_{2j}(k)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\sigma h_{n1}(k)}{\sum\limits_{j \in N_n} h_{nj}(k)} & \frac{\sigma h_{n2}(k)}{\sum\limits_{j \in N_n} h_{nj}(k)} & \cdots & (1-\sigma) \end{bmatrix}. \tag{28}$$

The parameter $\sigma \in (0,1)$ represents a stubbornness index: for small values of σ, each agent relies more on its current information state than on those of its neighbours, intuitively leading to a slower convergence.

In the following, we analyse the time-variant and the time-invariant cases separately.

A. Time-invariant system

If the channel coefficients do not depend on time,

$$D_n^\sigma(k) = D_n^\sigma, \ \forall k \in \mathbb{N}. \tag{29}$$

Then

$$\mathbf{x}(k) = (D_n^\sigma)^k \mathbf{x}(0). \tag{30}$$

The following proposition shows how the limit $\mathbf{x}^* = \lim_{k \to \infty} \mathbf{x}(k)$ depends on the parameter σ and the channel coefficients.

Proposition 3. Let $\mathbf{w}$ be the left-eigenvector of D_n^σ corresponding to the eigenvalue $\lambda_n = 1$. Then, the limit point $\mathbf{x}^* = \mathbf{1}x^*$, with $x^* = \mathbf{w}'\mathbf{x}(0)$, is independent of σ, while depending on the channel coefficients as follows

$$\begin{cases} x^* = \sum_{i=1}^n \mathbf{w}_i x_i(0) \\ \mathbf{w}_i = \sum_{j \in N_i} \frac{\mathbf{w}_j h_{ji}}{\sum_{l \in N_j} h_{jl}} \end{cases}. \tag{31}$$

Proof: Let $n \in \mathbb{N}$ and $\sigma \in (0,1)$ be arbitrary but fixed. Since D_n^σ is primitive, the Perron-Frobenius theorem states that (Theorem 1.2 in [13] p. 9), as $k \to \infty$,

$$(D_n^\sigma)^k = \lambda_n^k \mathbf{v}\mathbf{w}' + 0(k^{m-1}|\lambda_{n-1}^k|), \tag{32}$$

where λ_n and λ_{n-1} are, respectively, the largest and the second largest eigenvalues of D_n^σ and m is the multiplicity of λ_{n-1}. Additionally, $\mathbf{v}'\mathbf{w} = 1$, where $\mathbf{v} > 0$ and $\mathbf{w} > 0$ are, respectively, the right and left eigenvectors of D_n^σ, associated with its largest eigenvalue λ_n. Since D_n^σ is row-stochastic, we have $\lambda_n = 1$, $|\lambda_{n-1}| < 1$, and $\mathbf{v} = \mathbf{1}$. Hence, we have $\mathbf{x}^* = \lim_{k\to\infty} [(D_n^\sigma)^k \mathbf{x}(0)] = \mathbf{v}\mathbf{w}'\mathbf{x}(0) = \mathbf{1}\mathbf{w}'\mathbf{x}(0)$, from which we conclude the first part of (31).

By the definition of the left eigenvector, we have $\mathbf{w}' D_n^\sigma = \mathbf{w}'$. So $\sum_{j=1}^n \mathbf{w}_j [D_n^\sigma]_{ji} = \mathbf{w}_i$, $\forall i \in \{1,\dots,n\}$. This, by using the entries of D_n^σ, becomes

$$\mathbf{w}_i(1-\sigma) + \sum_{j\in N_i} \mathbf{w}_j \sigma \frac{h_{ji}}{\sum_{l\in N_j} h_{jl}} = \mathbf{w}_i, \tag{33}$$

from which the second part of (31) immediately follows. □

B. Time-variant system

Note that in the time-variant case, the equation for agent i can be rewritten as

$$x_i(k+1) = (1-\sigma)x_i(k) + \frac{\sigma}{|N_i|}\sum_{j\in N_i} x_j(k) + \frac{\sigma}{|N_i|}\frac{\sum_{j\in N_i}\sum_{l\in N_i}(h_{ij}(k) - h_{il}(k))x_j(k)}{\sum_{j\in N_i} h_{ij}(k)}, \tag{34}$$

where the last term on the right hand side of (34) represents the impact of the channel at time k. Let $\nu_{ij}(k)$ be

$$\nu_{ij}(k) = \frac{\sum_{l\in N_i} h_{ij}(k) - h_{il}(k)}{\sum_{l\in N_i} h_{il}(k)} x_j(k), \tag{35}$$

$\forall (j,i) \in \mathcal{A}$, $\forall k \in \mathbb{N}$, and $\nu_{ij}(k) = 0$, $\forall (j,i) \notin \mathcal{A}$, $\forall k \in \mathbb{N}$. If the channel coefficients are realizations of a stochastic process, independent and identically distributed, we can prove that the expected value of $\nu_{ij}(k)$ is 0, $\forall k \in \mathbb{N}$, $\forall (j,i) \in \mathcal{A}$.

Putting (35) into (34) yields

$$x_i(k+1) = (1-\sigma)x_i(k) + \frac{\sigma}{|N_i|}\sum_{j\in N_i} x_j(k) + \frac{\sigma}{|N_i|}\sum_{j\in N_i} \nu_{ij}(k), \tag{36}$$

$\forall i \in \mathcal{N}$, which can be written in matrix form as

$$\mathbf{x}(k+1) = D_A^\sigma \mathbf{x}(k) + D_B^\sigma \nu(k), \tag{37}$$

such that the state vector $\mathbf{x}(k) \in \mathbb{R}^n$ and the state disturbance vector $\nu(k) \in \mathbb{R}^{n^2}$, with $\nu(k) = [\nu_{11}(k),\dots,\nu_{1n}(k),\nu_{21}(k),\dots,\nu_{2n}(k),\dots,\nu_{nn}(k)]'$. The dynamics matrix D_A^σ is a row-stochastic matrix whose elements are $[D_A^\sigma]_{ii} = (1-\sigma)$, $\forall i \in \mathcal{N}$, $[D_A^\sigma]_{ij} = \frac{\sigma}{|N_i|}$, $\forall (j,i) \in \mathcal{A}$, and 0 elsewhere. The matrix $D_B^\sigma \in \mathbb{R}^{n\times n^2}$ is a block-diagonal matrix that can be written as

$$D_B^\sigma = \begin{bmatrix} D_{B,1}^\sigma & \mathbf{0}_{1\times n} & \dots & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \\ \mathbf{0}_{1\times n} & D_{B,2}^\sigma & \dots & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \dots & D_{B,n-1}^\sigma & \mathbf{0}_{1\times n} \\ \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \dots & \mathbf{0}_{1\times n} & D_{B,n}^\sigma \end{bmatrix}, \tag{38}$$

where $\forall i \in \mathcal{N}$, $D_{B,i}^\sigma$ is a row-vector of dimension n, whose elements are $[D_{B,i}^\sigma]_j = \frac{\sigma}{|N_i|}$, $\forall j \in \{j \mid (j,i) \in \mathcal{A}\}$, and 0 elsewhere.

We define $X(z)$ and $\mathcal{V}(z)$ as the Zeta-transforms of their respective time-domain signals, i.e. $X(z) = \mathcal{Z}(\mathbf{x}(k))$ and $\mathcal{V}(z) = \mathcal{Z}(\nu(k))$. In a complex frequency domain representation, (37) becomes

$$X(z) = F_A(z)\mathbf{x}(0) + F_B(z)\mathcal{V}(z), \tag{39}$$

where $F_A(z) = z(z\mathbb{I}_n - D_A^\sigma)^{-1}$ and $F_B(z) = (z\mathbb{I}_n - D_A^\sigma)^{-1}D_B^\sigma$, respectively a n-dimensional square matrix and a $(n \times n^2)$ matrix in the complex frequency domain. In the following, we assume that Γ is a sequence of fully connected Γ_k, then, $\forall k \in \mathbb{N}$, $\forall i \in \mathcal{N}$, $|N_i| = n-1$.

We solve $F_A(z)$ and $F_B(z)$ as functions of the complex variable z and of parameters n and σ; then, the final value theorem for time-discrete systems gives

$$\lim_{k\to\infty} \mathbf{x}(k) = \lim_{z\to 1}(z-1)\left(F_A(z)\mathbf{x}(0) + F_B(z)\mathcal{V}(z)\right) \tag{40}$$

$$= \lim_{z\to 1} \frac{1}{n}\mathbf{1}\mathbf{1}'\mathbf{x}(0) + \frac{\sigma}{n(n-1)}\Xi\mathcal{V}(z), \tag{41}$$

$n \in \mathbb{N}$, $\sigma \in (0,1)$ and Ξ a $n \times n^2$ matrix in the form

$$\Xi = \mathbf{1}_n \xi', \tag{42}$$

where $\xi \in \mathbb{R}^{n^2}$ is a column vector whose elements are $[\xi]_{hn+1} = 0$, $\forall h = 0\dots n-1$, and 1 elsewhere.

By (41), stubborn systems (small values of σ) reduce the impact of time-varying channel coefficients on the agreement value more than systems with higher values of σ. However, as argued in the beginning of this section, we may expect the convergence rate to decrease if $\sigma > 0$ becomes smaller.

A smaller impact of time-varying channel coefficients is also a benefit of a larger network (where n is large).

The formal analysis of the transient behaviour (function of time-varying channel coefficients), together with the relaxation of the assumption of a fully connected topology, will be the subject of future work.

V. Numerical example

Let us consider the balanced communication topology in Figure 1. Let the initial information states $x_i(0)$ be randomly generated out of an uniform distribution between 0 and 2π, i.e. $x_i(0) \sim \mathcal{U}(0, 2\pi)$, $\forall i \in \mathcal{N}$. First, consider the time-invariant system, where channel coefficients are distributed like $h_{ij} \sim \mathcal{U}(0,10)$, $\forall (j,i) \in \mathcal{A}$. The system in Figure 2a, with $\sigma_i = \sigma = 0.2$, achieves weighted average consensus. Larger values of σ give a higher rate of convergence, resulting in a faster system, as in Figure 2b, where the initial

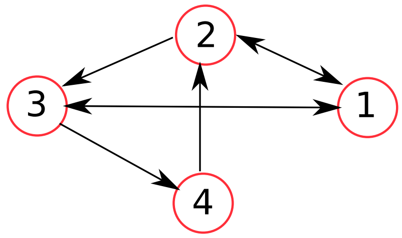
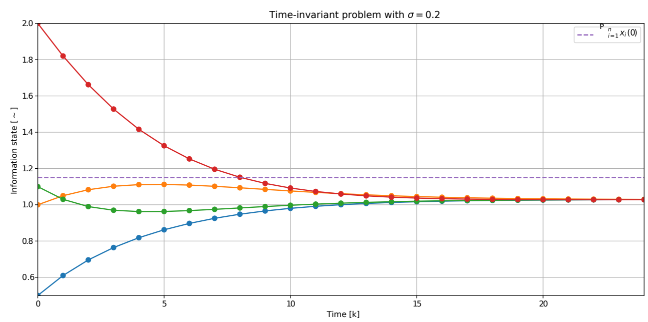
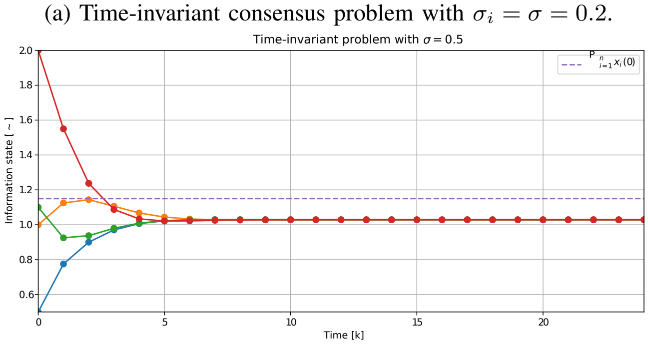


Fig. 1: Communication digraph with topology $(\mathcal{N}, \mathcal{A})$.

state vector and the realizations of channel coefficients are the same as in Figure 2a, but $\sigma = 0.5$. In this case, as already shown, x^* is independent of σ, which affects therefore only the convergence rate.



(a) Time-invariant consensus problem with $\sigma_i = \sigma = 0.2$.



(b) Time-invariant consensus problem with $\sigma_i = \sigma = 0.5$.

Fig. 2: Consensus problem with constant channel coefficients.

We will now consider the time-variant case for $\sigma = 0.5$ in Figure 3. Channel coefficients are generated at each step under the assumption that they are independent and identically distributed. For the same σ, the convergence rate stays roughly the same as the one of the time-invariant case in Figure 2b. Also, as already proven, $x^* \in \mathcal{C}(\mathbf{x}(0))$.

According to Section IV, for a fully connected network, in case of smaller σ (more stubborn system), the rate of convergence is expected to be much slower, but the consensus value is closer to the linear average of initial information states, as in Figure 4a.

By increasing the network size to $n = 30$ and setting σ to 0.8, with a fully connected topology, agents achieve consensus (Figure 4b), converging to an agreement value closer to the linear average consensus.

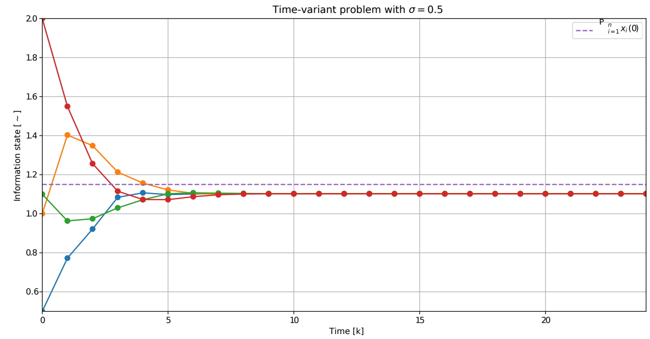
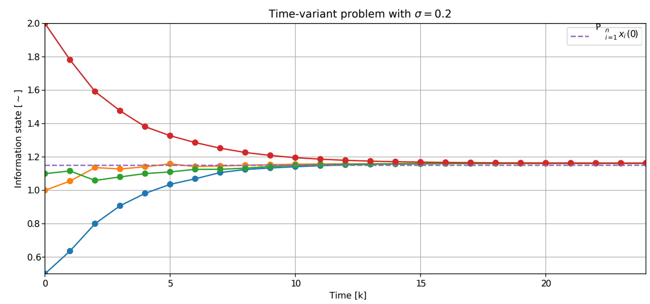
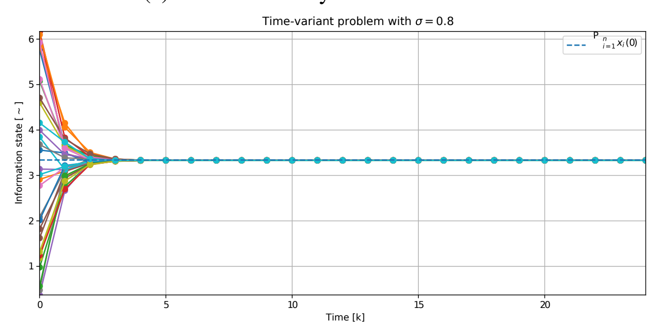


Fig. 3: Time-variant system with $\sigma = 0.5$.



(a) Time-variant system with $\sigma = 0.2$.



(b) Time-variant system with $n = 30$ and $\sigma = 0.8$.

Fig. 4: Time-variant system over a fully connected communication topology.

As a conclusion, the proposed control offers robustness against the variation of positive channel coefficients (under Assumption 1); the system always achieves consensus. However, the agreement value is depending on the realizations of the channel coefficients.

VI. Conclusion

In this paper, we investigated a consensus scheme that exploits the superposition property of wireless communication by relying on broadcast of information. In particular, we took the case of unknown (time-variant and time-invariant) channel coefficients into account. In both cases, the resulting consensus is a weighted average one. We introduced a tuning parameter that, in the time-invariant case, influences the convergence rate, but not the asymptotic consensus. It was also shown, that this tuning parameter affects both convergence rate and value in the time-varying case. Finally, for systems with a large number of agents, the resulting consensus value will be close to the linear average value.

References

[1] M. H. DeGroot, "Reaching a consensus," *Journal of the American Statistical Association*, vol. 69, no. 345, pp. 118–121, 1974.

[2] W. Ren, R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Systems*, vol. 27, no. 2, pp. 71–82, 2007.

[3] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[4] S. Martinez, J. Cortes, and F. Bullo, "On robust rendezvous for mobile autonomous agents," *IFAC Proceedings Volumes*, vol. 38, no. 1, pp. 115–120, 2005.

[5] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE transactions on automatic control*, vol. 49, no. 9, pp. 1465–1476, 2004.

[6] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: Algorithms and theory," *IEEE Transactions on automatic control*, vol. 51, no. 3, pp. 401–420, 2006.

[7] S. Stanczak, M. Wiczanowski, and H. Boche, *Fundamentals of resource allocation in wireless networks: theory and algorithms*. Springer Science & Business Media, 2009, vol. 3.

[8] M. Goldenbaum, H. Boche, and S. Stańczak, "Nomographic gossiping for f-consensus," in *Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks (WiOpt), 2012 10th International Symposium on*. IEEE, 2012, pp. 130–137.

[9] R. C. Buck, "Approximate complexity and functional representation," *Journal of Mathematical Analysis and Applications*, vol. 70, no. 1, pp. 280–298, 1979.

[10] M. Zheng, M. Goldenbaum, S. Stańczak, and H. Yu, "Fast average consensus in clustered wireless sensor networks by superposition gossiping," in *Wireless Communications and Networking Conference (WCNC), 2012 IEEE*. IEEE, 2012, pp. 1982–1987.

[11] W. Ren and R. W. Beard, *Distributed consensus in multi-vehicle cooperative control*. Springer, 2008.

[12] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Transactions on automatic control*, vol. 50, no. 2, pp. 169–182, 2005.

[13] E. Seneta, *Non-negative matrices and Markov chains*. Springer Science & Business Media, 2006.

[14] J. Wolfowitz, "Products of indecomposable, aperiodic, stochastic matrices," *Proceedings of the American Mathematical Society*, vol. 14, no. 5, pp. 733–737, 1963.

[15] Y. G. Sun, L. Wang, and G. Xie, "Average consensus in networks of dynamic agents with switching topologies and multiple time-varying delays," *Systems & Control Letters*, vol. 57, no. 2, pp. 175–183, 2008.

[16] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on automatic control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[17] M. Goldenbaum, H. Boche, and S. Stańczak, "Harnessing interference for analog function computation in wireless sensor networks," *IEEE Transactions on Signal Processing*, vol. 61, no. 20, pp. 4893–4906, 2013.