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Constructive Lyapunov Stabilization with Approximate Optimality for A Class of Nonlinear Systems

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***Abstract*— This paper presents a recursive constructing approach to Lyapunov function with optimality for a class of nonlinear systems. The targeted systems are formulated as cascaded system with triangular structure. For this class of nonlinear systems, stabilization problem has been a typical issue and solved by recursively constructing Lyapunov function using so-called back-stepping process. However, as is well known, this constructive design of feedback stabilizing control law is usually lacking time response performance due to the attention of controller design focuses stability only. The presented design approach in this paper puts an optimality into the recursive design process by targeting an approximate solution of Hamiltonian equality. It has been shown that at each stage of the recursive design a Lyapunov function that guarantees optimality can be obtained approximately by policy iteration. Finally, numerical examples are shown to demonstrate the design process.**

I. INTRODUCTION

Recursive Lyapunov function constructing approach is a well-known stabilizing control design method for such kind of nonlinear systems that have triangular structure with cascaded integrators [1]–[3]. Thanks to the contribution of geometrical differential theory to nonlinear system theory, this recursive design method has been used in the last two decades for a more general class of systems that has the triangular structure as nonlinear canonical form [4]. A benefit of this design method is to construct a Lyapunov function for the whole system in a systematic step-by-step procedure, so-called back-stepping design, and a stabilizing control law is obtained at the final step. However, the back-stepping design method just guarantees the stability of the closed-loop system, but it does not consider transient performance of states. At the present stage, to the best of our knowledge, there is a lack of relevant studies on back-stepping design with optimality to improve time response performance.

On the other hand, recently, an iterative constructing strategy is proposed by [5] to solve approximately the dynamical programming. This new contribution in optimal control theory was inspired from the relaxation method used in approximate dynamic programming for Markov decision process with finite state space [6]–[8], and benefit from the formulation of a sum of squares (SOS) program [9]–[11]. This paper targeted also on the learning and adaptive design problem, but a fundamental design methodology for recursively constructing a value function of optimization problem is presented. This motivates an idea for putting optimality into each step of a recursive design procedure.

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In this paper, we propose a recursive constructive design approach to Lyapunov function with optimality of each step. It will be shown that at each design step of the recursive process, a Lyapunov function which guarantees not only stability but also a Hamiltonian equation is constructed by iterative approximation process. Hence, the proposed design method consists of recursive constructing the Lyapunov function with iterative loop. Only a class of nonlinear systems with cascaded structure is considered in this paper, however, to extend the approach to higher dimension nonlinear system with specified structural condition is trivial.

The rest of this paper is organized as follows. Section II presents the problem formulation. Section III gives main result. In section IV, numerical simulations are given for verification.

II. PROBLEM FORMULATION

Consider the following nonlinear system

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = f_2(x_1,x_2) + g_2(x_1,x_2)u \end{cases} \tag{1}$$

where (x_1, x_2) are system states with $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$, $u \in \mathbb{R}$ is its control input, $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R}^2 \to \mathbb{R}$ are polynomial mappings with $f_1(0) = 0$ and $f_2(0, 0) = 0$, $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R}^2 \to \mathbb{R}$ are also polynomial mappings.

The focus of this paper is on designing a recursive constructing approach to Lyapunov function with optimality for system (1). For the development of a control policy, the following assumptions are needed.

Assumption 2.1: There exists a constant $k_1 > 0$ such that

$$|f_1(x_1)| \le k_1|x_1|, \quad \forall x_1 \in \mathbb{R}. \tag{2}$$

Assumption 2.2: There exists a constant $\epsilon > 0$ such that

$$g_1^2(x_1) \ge \epsilon, \quad \forall x_1 \in \mathbb{R}. \tag{3}$$

Remark 2.1: The second-order nonlinear system (1) is a typical system in application of back-stepping design. In addition, such polynomial condition is mild and can be verified by many examples, including the Lorenz system and Van der Pol oscillator, name a few. Assumption 2.2 is reasonable since $g_i(\cdot)$, $i = 1, 2$ being always from zeros are controllable condition.

III. MAIN RESULT

In this section, we elaborate a recursive constructive design approach to stabilize system (1). In particular, we design a control law in two steps and at each step, we aim to solve an optimal control problem.

A. Optimal Control Design for the first-order system

Consider the subsystem x_1 of system (1)

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\phi_1 \tag{4}$$

where $x_1 \in \mathbb{R}$ is a state and $\phi_1 \in \mathbb{R}$ is a input. The object is to find a state-feedback control policy $\phi_1(x_1)$ such that the following performance index

$$J_1(x_{10}, \phi_1) = \int_0^\infty q_1 x_1^2(t) + r_1 \phi_1^2(t) \mathrm{d}t, \quad x_1(0) = x_{10} \tag{5}$$

is minimized.

Assumption 3.1: There exists a continuously differentiable, positive definite function $V_1^o(x_1)$, such that the Hamilton-Jacobi-Bellman equation satisfies

$$\mathcal{H}_1(V_1^o) = 0, \tag{6}$$

where

$$\mathcal{H}_1(V) = \frac{\partial V}{\partial x_1} f_1(x_1) + q_1 x_1^2 - \frac{g_1^2}{4r_1}\Big(\frac{\partial V}{\partial x_1}\Big)^2. \tag{7}$$

Remark 3.1: According to [5], V_1^o is the unique solution to equation (6) among the set of all continuously differentiable, positive definite functions. It is clear that the closed-loop system composed of (4) and

$$\phi_1^o(x_1) = -\frac{1}{2r_1} g_1(x_1) \frac{\partial V_1^o}{\partial x_1} \tag{8}$$

is globally asymptotically stable at $x_1 = 0$ with V_1^o being the corresponding Lyapunov function. Furthermore, $\phi_1^o(x_1)$ is the optimal control policy, and V_1^o is the optimal cost on the initial condition, *i.e.*,

$$V_1^o(x_{10}) = \min_{\phi_1} J_1(x_{10}, \phi_1) = J(x_{10}, \phi_1^o), \quad \forall x_{10} \in \mathbb{R}. \tag{9}$$

In the spirit of [5], the optimal control problem of subsystem (4) can be solved by sum of squares (SOS) program, if we can find an initial control policy satisfying conditions as shown in the following lemma.

Lemma 3.1: Consider system (4), under Assumptions 2.1 and 2.2, control policy

$$\phi_1^1(x_1) = -\frac{\rho_1 g_1}{2r_1} x_1, \tag{10}$$

and the corresponding function

$$V_1^0(x_1) = \frac{\rho_1}{2} x_1^2 \tag{11}$$

satisfy the condition as follows:

$$-L_1\big(V_1^0(x_1), \phi_1^1(x_1)\big) \text{ is SOS}, \quad \forall x_1 \in \mathbb{R} \tag{12}$$

where

$$L_1(V_1^0, \phi_1^1) = \frac{\partial V_1^0}{\partial x_1}(f_1 + g_1\phi_1^1) + q_1 x_1^2 + r_1(\phi_1^1)^2 \tag{13}$$

and ρ_1 satisfies

$$\frac{\rho_1^2 \epsilon}{4r_1} - \rho_1 k_1 - q_1 \geq 0. \tag{14}$$

Proof: The time derivative of V_1^0 along the trajectory of system composed of (4) and (10) is given by

$$\dot{V}_1^0 = \frac{\partial V_1^0}{\partial x_1}\big(f_1 + g_1 \phi_1^1\big). \tag{15}$$

Substituting (11) and (15) into (13), we have

$$\begin{aligned}
&L_1(V_1^0, \phi_1^1) \\
=&\dot{V}_0^1 + q_1 x_1^2 + r_1(\phi_1^1)^2 \\
=&\rho_1 x_1 f_1(x_1) + \rho_1 g_1(x_1) x_1 \phi_1^1 + q_1 x_1^2 + r_1(\phi_1^1)^2 \\
=&(\sqrt{r_1}\phi_1^1 + \frac{\rho_1 g_1(x_1)}{2\sqrt{r_1}} x_1)^2 - x_1^2(\frac{\rho_1^2 g_1^2}{4r_1} - \frac{\rho_1 f_1}{x_1} - q_1).
\end{aligned}$$

Substituting (10), the above equation is transformed into

$$L_1(V_1^0, \phi_1^1) = -x_1^2(\frac{\rho_1^2 g_1^2}{4r_1} - \frac{\rho_1 f_1}{x_1} - q_1). \tag{16}$$

Combining (2), (3), (14) and (16), we have

$$\begin{aligned}
&L_1(V_1^0, \phi_1^1) \\
\leq& - x_1^2(\frac{\rho_1^2 \epsilon}{4r_1} - \rho_1 k_1 - q_1) \\
\leq& 0.
\end{aligned}$$

Since L_1 is a polynomial of x_1, we can get $-L_1(V_1^0, \phi_1^1)$ is SOS. Hence, the proof is complete. ■

Remark 3.2: The equivalent express of the above consequence is

$$\dot{V}_1^0 \leq -q_1 x_1^2 - r_1\big(\phi_1^1(x_1)\big)^2. \tag{17}$$

It is immediately concluded that the closed-loop system composed of (4) and (10) is globally asymptotically stable at the origin, with the corresponding Lyapunov function V_1^0.

Then, we apply SOS-programming-based policy iteration with an initial control policy satisfying conditions of Lemma 3.1.

1) Policy evaluation: For $i = 1, 2, \ldots$, solve for an optimal solution ρ_1^i to the following optimization program:

$$\min_{\rho_1} \int_\Omega V_1(x) \mathrm{d}x \tag{18}$$

$$\textit{s.t.} \quad -L_1(V_1, \phi_1^i) \text{ is SOS} \tag{19}$$

$$V_1^{i-1} - V_1 \text{ is SOS} \tag{20}$$

where $\Omega \subset \mathbb{R}$ is an arbitrary compact set containing the origin and $V_1^i = \frac{\rho_1^i}{2} x_1^2$.

2) Policy improvement: Update the control policy by

$$\phi_1^{i+1} = -\frac{\rho_1^i g_1}{2r_1} x_1.$$

Lemma 3.2: Under Assumptions 2.1, 2.2, and 3.1, a control policy

$$\phi_1^*(x_1) = -\frac{\rho_1^* g_1}{2r_1} x_1 \tag{21}$$

and a Lyapunov function

$$V_1^* = \frac{\rho_1^*}{2} x_1^2 \tag{22}$$

satisfy the following conditions:

(1) $\lim_{i\to\infty} V_1^i(x_{10}) = V_1^*(x_{10})$, $\forall x_{10} \in \mathbb{R}$.

(2) The closed-loop system composed of (4) and (21) is globally asymptotically stable at the origin.

(3) Along the trajectory of the above system, the following inequality is satisfied:

$$0 \le V_1^*(x_{10}) - V_1^o(x_{10}) \le -\int_0^\infty \mathcal{H}_1(V_1^*(x_1))\mathrm{d}t. \tag{23}$$

Proof:

(1) The control policy ϕ_1^i and the correspond function V_1^i satisfy (19), *i.e.*,

$$\begin{aligned} L(V_1^i, \phi_1^i) = &\frac{\partial V_1^i}{\partial x_1}(f_1 + g_1\phi_1^i) + q_1 x_1^2 + r_1(\phi_1^i)^2 \\ \le& 0. \end{aligned} \tag{24}$$

By replacing ϕ_1^{i+1} in (19), we obtain

$$\begin{aligned} &L(V_1^i, \phi_1^{i+1}) \\ =&\frac{\partial V_1^i}{\partial x_1}(f_1 + g_1\phi_1^{i+1}) + q_1x_1^2 + r_1(\phi_1^{i+1})^2 \\ =&\frac{\partial V_1^i}{\partial x_1}(f_1 + g_1\phi_1^{i}) + q_1x_1^2 + r_1(\phi_1^{i})^2 \\ +&\frac{\partial V_1^i}{\partial x_1}g_1(\phi_1^{i+1} - \phi_1^i) - r_1(\phi_1^i)^2 + r_1(\phi_1^{i+1})^2 \\ =&L(V_1^i, \phi_1^i) - r_1(\phi_1^{i+1} - \phi_1^i)^2 \\ \le& - r_1(\phi_1^{i+1} - \phi_1^i)^2 \\ \le& 0. \end{aligned}$$

Then, $L(V_1^i, \phi_1^{i+1})$ is SOS. Hence, V_1^i is a feasible solution at step $i+1$. According to the consequences of Lemma 3.1, it is easy to see that ϕ_1^i is globally stabilizing. Integrating both sides of equation (24) along the trajectory of the closed-loop system composed of (4) and ϕ_1^i on the interval $[0,\infty)$, we obtain

$$V_1^i(x_{10}) \ge \int_0^\infty \big(q_1x_1^2 + r_1(\phi_1^i)^2\big)\mathrm{d}t, \quad \forall x_{10} \ne 0. \tag{25}$$

Combining with the constraint (20) of SOS program, it can be concluded that

$$V_1^o(x_{10}) \le V_1^{i-1}(x_{10}) \le V_1^i(x_{10}), \quad \forall x_{10} \in \mathbb{R} \tag{26}$$

As a result, the sequence $\{V_1^i\}$ has a limit, *i.e.*, $\lim_{i\to\infty} V_1^i = V_1^*$. Since $V_1^i = \frac{\rho_1^i}{2}x_1^2$, we have $\lim_{i\to\infty}\rho_1^i = \rho_1^*$, $V_1^* = \frac{\rho_1^*}{2}x_1^2$.

(2) By the above consequences, we have

$$L_1(V_1^*, \phi_1^*) \le 0. \tag{27}$$

Hence, ϕ_1^* is globally stabilizing with V_1^* being the corresponding Lyapunov function.

(3) It is clear that

$$\mathcal{H}_1(V_1^*) = L_1(V_1^*, \phi_1^*), \tag{28}$$

and we have

$$V_1^*(x_{10}) \ge V_1^o(x_{10}). \tag{29}$$

Then, the first inequality in (23) holds.

On the other hand,

$$\begin{aligned} &\mathcal{H}_1(V_1^*) \\ =&\mathcal{H}_1(V_1^*) - \mathcal{H}_1(V_1^o) \\ =&\frac{\partial V_1^*}{\partial x_1}(f_1 + g_1\phi_1^*) + r_1(\phi_1^*)^2 - \frac{\partial V_1^o}{\partial x_1}(f_1 + g_1\phi_1^o) - r_1(\phi_1^o)^2 \\ =&(\frac{\partial V_1^*}{\partial x_1} - \frac{\partial V_1^o}{\partial x_1})(f + g_1\phi_1^o) - r_1(\phi_1^* - \phi_1^o)^2 \\ \le&(\frac{\partial V_1^*}{\partial x_1} - \frac{\partial V_1^o}{\partial x_1})(f + g_1\phi_1^o). \end{aligned}$$

Integrating both sides of the above equation along the trajectory of the closed-loop system composed of (4) and (8), we obtain

$$V_1^*(x_{10}) - V_1^o(x_{10}) \le -\int_0^\infty \mathcal{H}_1(V_1^*(x_1))\mathrm{d}t. \tag{30}$$

Hence, the proof is complete. ■

B. Optimal Control Design for the second-order system

In the last part, we have designed a suboptimal control policy for the first-order subsystem (4). Now, we continue to construct Lyapunov function with optimality for the second-order system.

To proceed, adding and subtracting $g_1(x_1)\phi_1^*(x_1)$ on the right-hand side of (4), we have

$$\begin{cases} \dot{x}_1 = [f_1 + g_1\phi_1^*] + g_1(x_2 - \phi_1^*) \\ \dot{x}_2 = f_2 + g_2 u. \end{cases} \tag{31}$$

Then, let $\tilde{x}_2 = x_2 - \phi_1^*(x_1)$ and $\phi_2 = f_2(x_1, x_2) - \dot{\phi}_1^* + g_2(x_1, x_2)u$, we obtain the following system

$$\begin{cases} \dot{x}_1 = [f_1 + g_1\phi_1^*] + g_1\tilde{x}_2 \\ \dot{\tilde{x}}_2 = \phi_2 \end{cases} \tag{32}$$

where $(x_1, \tilde{x}_2)$ are system states, ϕ_2 is its control input.

Consider system (32), the objective is to find a control policy ϕ_2, such that the following performance index

$$\begin{aligned} &J_2(x_{10}, \tilde{x}_{20}, \phi_2) \\ =&\int_0^\infty q_1x_1^2(t) + q_2\tilde{x}_2^2(t) + r_2(\phi_2(t))^2\mathrm{d}t \end{aligned} \tag{33}$$

is minimized, where $x_1(0) = x_{10}, \tilde{x}_2(0) = \tilde{x}_{20}$.

Assumption 3.2: There exists a continuously differentiable, positive and definite function $V_2^o(x_1, \tilde{x}_2)$, such that the HJB equation satisfies

$$\mathcal{H}_2(V_2^o) = 0, \tag{34}$$

where

$$\begin{aligned} \mathcal{H}_2(V) =&\frac{\partial V}{\partial x_1}(f_1 + g_1\phi_1^* + g_1\tilde{x}_2) \\ &+q_1x_1^2 + q_2\tilde{x}_2^2 - \frac{1}{4r_2}\Big(\frac{\partial V}{\partial \tilde{x}_2}\Big)^2. \end{aligned} \tag{35}$$

Remark 3.3: Similar to Assumption 3.1, it is easy to see that the globally stabilizing control policy

$$\phi_2^o(\tilde{x}_2) = -\frac{1}{2r_2}\frac{\partial V_2^o}{\partial \tilde{x}_2} \tag{36}$$

is the unique optimal control policy, and V_2^o is the optimal cost on the initial condition $col(x_{10}, \tilde{x}_{20})$,*i.e.*,

$$\begin{aligned} &V_2^o(x_{10}, \tilde{x}_{20}) \\ =&\min_{\phi_2} J_2(x_{10}, \tilde{x}_{20}, \phi_2) \\ =&J_2(x_{10}, \tilde{x}_{20}, \phi_2^o), \quad \forall col(x_{10}, \tilde{x}_{20}) \in \mathbb{R}^2. \end{aligned} \tag{37}$$

First, we find an initial control policy satisfying conditions as shown in the following lemma.

Lemma 3.3: Under Assumptions 2.1, 2.2, 3.1 and 3.2, a controller in the form of

$$\phi_2^1 = -\sqrt{\frac{r_1 + q_2}{r_2}}\tilde{x}_2 \tag{38}$$

and a corresponding function

$$V_2^0 = \frac{\rho_1^*}{2}x_1^2 + \frac{\rho_2}{2}\tilde{x}_2^2 \tag{39}$$

satisfy

$$-L_2(V_2^0, \phi_2^1) \text{ is SOS}, \quad \forall x_1 \in \mathbb{R}, \tilde{x}_2 \in \mathbb{R}, \tag{40}$$

where

$$\begin{aligned} L_2(V_2^0, \phi_2^1) =&\frac{\partial V_2^0}{\partial x_1}(f_1 + g_1\phi_1^* + g_1\tilde{x}_2) \\ &+ \frac{\partial V_2^0}{\partial \tilde{x}_2}\phi_2^1 + q_1x_1^2 + q_2\tilde{x}_2^2 + r_2(\phi_2^1)^2, \end{aligned} \tag{41}$$

and ρ_2 satisfies

$$\rho_2 = 2\sqrt{(r_1 + q_2)r_2}. \tag{42}$$

Proof: The time derivative of V_2^0 along the trajectory of closed-loop system composed of (32) and (38) is given by

$$\dot{V}_2^0 = \frac{\partial V_2^0}{\partial x_1}(f_1 + g_1\phi_1^* + g_1\tilde{x}_2) + \frac{\partial V_2^0}{\partial \tilde{x}_2}\phi_2^1. \tag{43}$$

Substituting (39) and (43) into (41), we have

$$\begin{aligned} &L_2(V_2^0, \phi_2^1) \\ =&\dot{V}_2 + q_1x_1^2 + q_2\tilde{x}_2^2 + r_2(\phi_2^1)^2 \\ =&\rho_1^*f_1x_1 + \rho_1^*g_1x_1\phi_1^* + \rho_1^*g_1x_1\tilde{x}_2 + \rho_2\tilde{x}_2\phi_2^1 \\ &+ q_1x_1^2 + q_2\tilde{x}_2^2 + r_2(\phi_2^1)^2 \\ \leq& - r_1{\phi_1^*}^2 + \rho_1^*g_1x_1\tilde{x}_2 + \rho_2\tilde{x}_2\phi_2^1 + q_2\tilde{x}_2^2 + r_2(\phi_2^1)^2 \\ =& - (\frac{\rho_1^2 g_1}{2\sqrt{n_1}}x_1 - \sqrt{r_1}\tilde{x}_2)^2 + (\sqrt{q_2 + r_1}\tilde{x}_2 + \sqrt{r_2}\phi_2^1)^2. \end{aligned}$$

Substituting (38), the above equation is transformed into

$$\begin{aligned} L_2(V_2^0, \phi_2^1) =& -(\frac{\rho_1^2 g_1}{2\sqrt{n_1}}x_1 - \sqrt{r_1}\tilde{x}_2)^2 \\ \leq& \, 0. \end{aligned}$$

Since L_2 is a polynomial of $(x_1, \tilde{x}_2)$, $-L_2(V_2^0, \phi_2^1)$ is SOS. Hence, the proof is complete. ■

Then, we apply SOS-programming-based policy iteration with an initial control policy ϕ_2^1 satisfying conditions of Lemma 3.3.

1) Policy evaluation: For $i = 1, 2, \ldots$, solve for an optimal solution (ρ_1^i, ρ_2^i) to the following optimization program:

$$\min_{(\rho_1, \rho_2)} \int_\Omega V_2(x)\mathrm{d}x \tag{44}$$

$$\textit{s.t.} \quad - L_2(V_2, \phi_2^i) \text{ is SOS} \tag{45}$$

$$V_2^{i-1} - V_2 \text{ is SOS} \tag{46}$$

2) Policy improvement: Update the control policy by

$$\phi_2^{i+1} = -\frac{\rho_2^i}{2r_2}\tilde{x}_2. \tag{47}$$

Lemma 3.4: Under Assumptions 2.1, 2.2, 3.1, and 3.2, there exists a function

$$V_2^* = \frac{\tilde{\rho}_1^*}{2}x_1^2 + \frac{\rho_2^*}{2}\tilde{x}_2^2 \tag{48}$$

and a controller of the form

$$\phi_2^* = -\frac{\rho_2^*}{2r_2}\tilde{x}_2 \tag{49}$$

satisfying the following conditions:

(1) $\lim_{i\to\infty} V_2^i(x_{10}, x_{20}) = V_2^*(x_{10}, x_{20}), \forall \mathrm{col}(x_{10}, x_{20}) \in \mathbb{R}^2$.

(2) The closed-loop system composed of (32) and (49) is globally asymptotically stable at the origin.

(3) Along the trajectory of the above system, the following inequalities are satisfied:

$$\begin{aligned} &0 \leq V_2^*(x_{10}, x_{20}) - V_2^o(x_{10}, x_{20}) \\ &\leq -\textstyle\int_0^\infty \mathcal{H}_2(V_2^*(x_1(t), x_2(t)))\mathrm{d}t. \end{aligned}$$

The proof is similar to Lemma 3.2.

C. Recursive Constructing Approach

At this point, we designed a control law in two steps, each of which consists of constructing the Lyapunov function with iterative loop. We summarize the recursive constructing approach here:

step 1: Consider the first-order system

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\phi_1 \tag{50}$$

with the performance index

$$J_1 = \int_0^\infty q_1x_1^2 + r_1\phi_1^2\mathrm{d}t, \quad x_1(0) = x_{10}, \tag{51}$$

find an initial condition satisfying (12). Then, using SOS-programming-based policy iteration, we have state-feedback control ϕ_1^* and the Lyapunov function V_1^* satisfying Lemma 3.2.

step 2: Consider the second-order system

$$\begin{cases} \dot{x}_1 = [f_1 + g_1\phi_1^*] + g_1\tilde{x}_2 \\ \dot{\tilde{x}}_2 = \phi_2 \end{cases} \tag{52}$$

with the performance index

$$J_2 = \int_0^\infty q_1x_1^2 + q_2\tilde{x}_2^2 + r_2(\phi_2)^2\mathrm{d}t, \quad x_2(0) = x_{20}, \tag{53}$$

find an initial condition satisfying (40). Then, using SOS-programming-based policy iteration, we have state-feedback control ϕ_2^* and the Lyapunov function V_2^* satisfying Lemma 3.4.

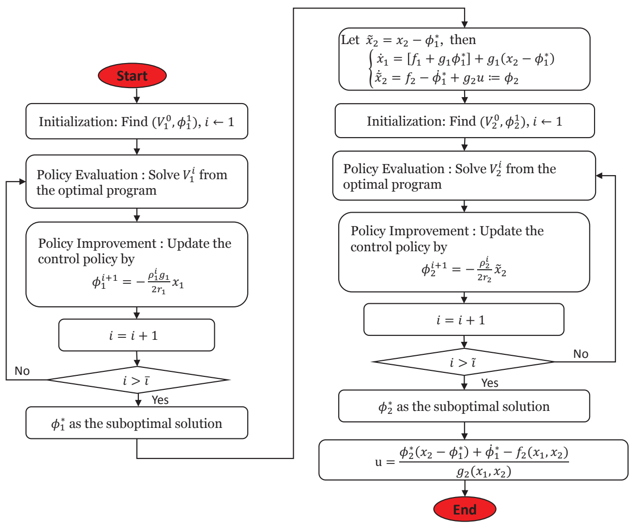

Fig. 1: Flowchart of Recursive Constructing Approach.

Theorem 3.1: Under Assumptions 2.1, 2.2, 3.1, and 3.2, the system (1) can be globally stabilized by a state-feedback controller of the form

$$u = \frac{\phi_2^*(x_2 - \phi_1^*) + \dot{\phi}_1^* - f_2(x_1, x_2)}{g_2(x_1, x_2)}, \tag{54}$$

and at each design step of the recursive process, along the trajectory of the subsystem with ϕ_i^*, $i = 1, 2$, the following inequalities hold:

$$0 \leq V_i^*(x_0) - V_i^o(x_0) \leq -\int_0^\infty \mathcal{H}_i(V_i^*(x(t)))\mathrm{d}t. \tag{55}$$

Proof: Clearly, if we can construct a controller to stabilize the system (32), then the stabilization problem of the original system (1) is solved. According to the consequences of Lemma 3.2 and Lemma 3.4, it is immediately concluded that the closed-loop system composed (1) and (54) is globally asymptotically stable at the origin and inequalities (55) are satisfied. ■

IV. Simulation

In this section, we will present two examples to elaborate our results.

Example 4.1: Consider the following second-order nonlinear system:

$$\begin{cases} \dot{x}_1 = x_1 + (1 + x_1^2)x_2 \\ \dot{x}_2 = x_1 x_2 (1 + x_2^2) + u \end{cases} \tag{56}$$

where (x_1, x_2) is the state and u is the control input. Clearly, Assumptions 2.1 and 2.2 are verifiable.

We choose the initial conditions $(x_1, x_2) = (2, -1)$, and parameters $q_1 = 1$, $q_2 = 0.25$, $r_1 = 1$, $r_2 = 4$. Then we can get $\rho_1 = 9$ satisfying inequality (14) and $\rho_2 = 4.4721$ satisfying equality (42). Simulation results are shown in Fig. 2 for the profile of sates of system (56), Fig. 3 for the comparison of the value functions in step 1 and Fig. 4 for the comparison of the value functions in step 2.

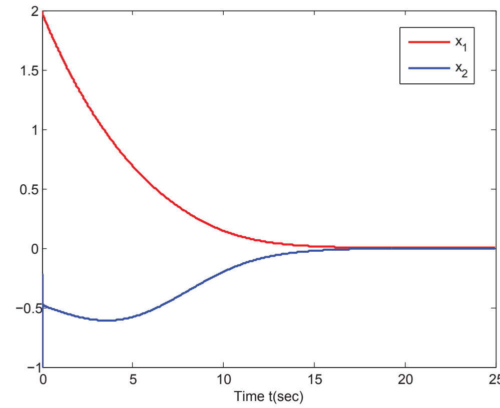

Fig. 2: The profile of states of system (56).

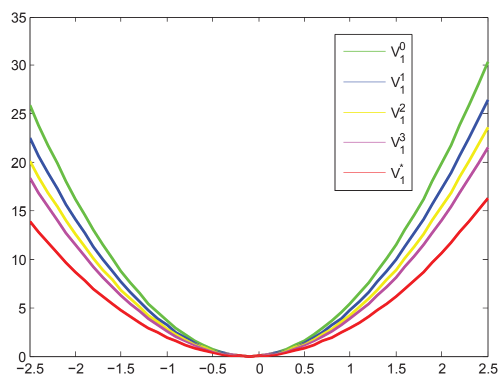

Fig. 3: Comparison of the value functions in step 1.

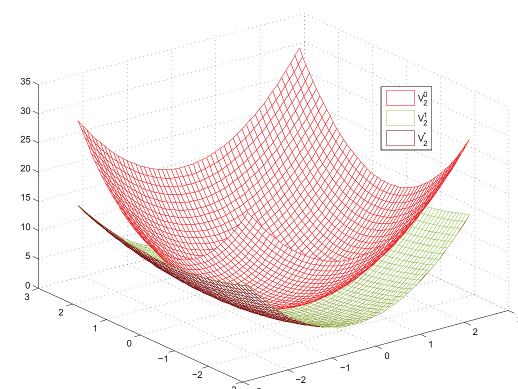

Fig. 4: Comparison of the value functions in step 2.

Example 4.2: Consider a second-order nonlinear system as following:

$$\begin{cases} \dot{x}_1 = 3x_1 + (2 + x_1^2 + 3x_1^4)x_2 \\ \dot{x}_2 = x_1^3 + x_2^2 + x_1^2 x_2 + u \end{cases} \tag{57}$$

where (x_1, x_2) is the state and u is the control input. Clearly, Assumptions 2.1 and 2.2 are verifiable.

We choose the initial conditions $(x_1, x_2) = (1, -0.5)$, and parameters $q_1 = 2$, $q_2 = 1$, $r_1 = 1$, $r_2 = 2$. Then we can get $\rho_1 = 10$ satisfying inequality (14) and $\rho_2 = 4$ satisfying equality (42). Simulation results are shown in Fig. 5 for the profile of sates of system (57), Fig. 6 for the comparison of the value functions in step 1 and Fig. 7 for the comparison of the value functions in step 2.

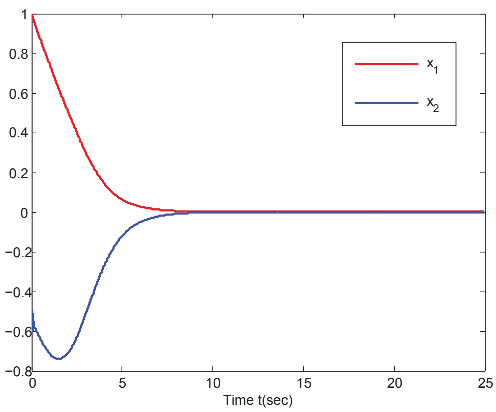

Fig. 5: The profile of sates of system (56).

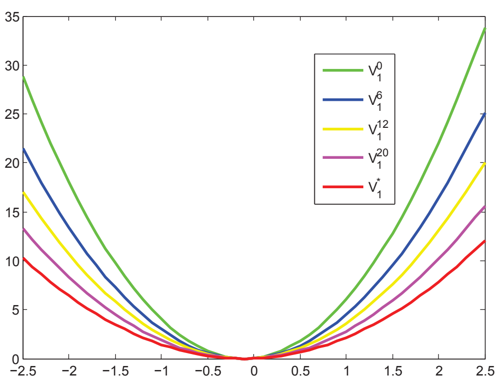

Fig. 6: Comparison of the value functions in step 1.

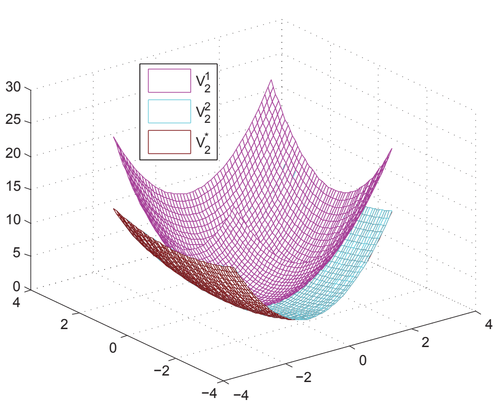

Fig. 7: Comparison of the value functions in step 2.

V. Conclusion

This paper developed a recursive constructing approach to Lyapunov function with optimality for cascaded system with triangular structure. Different from conventional backstepping design, the new method put optimality into each step of recursive design procedure to improve transient performance of closed-loop system. It has been shown that at each design stage of the recursive process, a Lyapunov function which guarantees optimality is constructed by policy iteration. It is noted that SOS-programming-based policy iteration can not only obtain globally stabilizing control policy but also reduce significantly computational burden. In addition, recursive design can construct a Lyapunov function for a whole system in a systematic step-by-step procedure. Hence, to extend the approach to higher dimension nonlinear system with specified structural condition is trivial.

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