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On Optimal Anisotropy-Based Control Problem for Discrete-Time Descriptor Systems*

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***Abstract*— This paper is dedicated to optimal state-feedback control problem for discrete-time descriptor systems in presence of "colored" noise with known mean anisotropy level. Here "colored" noise stands for a stationary Gaussian sequence, generated by a linear shaping filter from the Gaussian white noise sequence. The control goal is to find a state feedback control law which makes the closed-loop system admissible and minimizes its a-anisotropic norm (mean anisotropy level a is known).**

I. INTRODUCTION

A descriptor system framework for mathematical modeling and control design has been extensively developed in the last decades. As a rule, such systems describe dynamics of a plant not in abstract state variables, but in physical variables [1]–[3], [5]. This approach substantially simplifies design of mathematical models, simulation of the system's dynamics is more illustrative. Some state variables are redundant, this fact allows to have more freedom while designing control laws [6]. Finally, descriptor systems are a general case of standard state-space systems: a descriptor system without algebraic equations can be easily transformed into the standard state-space system. Despite the obvious advantages, in discrete-time case, a specific behavior such as noncausality may occur while solving the system's equations, due to the presence of algebraic constraints in the system's model. Motivated by this fact, many efforts have been made towards developing methods to solve a number of control problems.

Problems of sensitivity reduction or external disturbance attenuation are well-known in modern control theory. The most studied ones are LQG/$\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems. In LQG/$\mathcal{H}_2$ optimal theory, the Gaussian white noise sequence is considered as the input disturbance. In discrete-time $\mathcal{H}_\infty$ control approach input disturbances are considered as sequences with limited power, i.e. the sequences are square summable. The discrete-time LQG/$\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems were successfully generalized on the class of descriptor systems. The discrete-time LQG/$\mathcal{H}_2$ optimal control problem was solved in [7], [8].

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Anisotropy-based approach deals with the stationary random Gaussian signals with known mean anisotropy level $a \geqslant 0$, which has a sense of "spectral color" of the signal [9], [10], [12]. Similar to $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, anisotropic norm defines a performance index of the system from the input to output. When $a = 0$ the input signal coincides with the Gaussian white noise and anisotropic norm of the system equals the scaled $\mathcal{H}_2$ norm. When $a \to \infty$ the anisotropic norm tends to $\mathcal{H}_\infty$ norm. Hence, both LQG/$\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems are particular cases of anisotropy-based control problem. Also, this approach allows to describe the input disturbance more precisely (the model of "colored" noise is closer to real process), and anisotropic norm minimization gives an opportunity of better tuning the controllers by mathematical methods.

This paper deals with the problem of disturbance attenuation in presence of "colored" noises. The paper represents the solution of optimal state-feedback anisotropy-based control in terms of generalized algebraic Riccati equations.

II. PRELIMINARIES

A. Descriptor systems

The state-space representation of descriptor systems is

$$Ex(k+1) = Ax(k) + Bf(k), \quad (1)$$
$$y(k) = Cx(k) + Df(k) \quad (2)$$

where $x(k) \in \mathbb{R}^n$ is the state, $f(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$ are the input and output signals, respectively, $k \in \mathbb{Z}$, $k \geqslant 0$. E, A, B, C, D are constant real matrices of appropriate dimensions, $\operatorname{rank} E = r < n$. Initial conditions are supposed to be consistent [2], [3].
The following definitions are taken from [4]. The transfer function of the system (1)–(2) is defined by $P(z) = C(zE - A)^{-1}B + D,\ z \in \mathbb{C}$. We also use the following denotation for system (1)–(2):

$$P = \left[\ E,\quad \begin{array}{c|c} A & B \\ \hline C & D \end{array}\ \right]. \quad (3)$$

$\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of system (1)–(2) are defined as follows

$$\|P\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Trace}\left(P^*(e^{i\omega})P(e^{i\omega})\right)d\omega\right)^{\frac{1}{2}},$$

$$\|P\|_\infty = \sup_{\omega\in[0,2\pi]} \sigma_{max}\left(P(e^{i\omega})\right).$$

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Here, $P^*(e^{i\omega}) = P^{\mathrm{T}}(e^{-i\omega})$ is a conjugate system, $\sigma_{max}(X) = \sqrt{\max_j |\lambda_j(X^{\mathrm{T}}X)|}$ is a maximal singular value of the matrix X.

Definition 1: System (1)–(2) is called admissible if it is regular $(\exists\lambda : \det(\lambda E - A) \neq 0)$, causal $(\deg\det(\lambda E - A) = \operatorname{rank} E)$, and stable $\Big(\rho(E, A) = \max|\lambda|_{\lambda\in z|\{\det(zE-A)=0\}} < 1\Big)$.
Hereinafter, we suppose that the main concepts for descriptor systems can be defined in terms of system (1), or in terms of the pair (E, A). For more information, see [1], [2]. For regular system (1)–(2) there exist two nonsingular matrices $\overline{W}$ and $\overline{V}$ such that:

$$\overline{W}E\overline{V} = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix}, \overline{W}A\overline{V} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \quad \overline{W}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \; C\overline{V} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \qquad (4)$$

where N is nilpotent of index h, i.e. $N^i \neq 0$ for $i = 1 : (h-1)$, and $N^h = 0$.

Consider an input signal in the following form:

$$f(k) = F_c x(k) + h(k) \qquad (5)$$

where $F_c \in \mathbb{R}^{m\times n}$ is a constant real matrix, $h(k)$ is a new input signal. Equation (1) turns to

$$Ex(k+1) = (A + BF_c)x(k) + Bh(k). \qquad (6)$$

Definition 2: System (1) is called stabilizable if there exists a state feedback control in the form $f(k) = F_{st}x(k)$ such that the pair $(E, A + BF_{st})$ is stable.

Definition 3: System (1) is called causal controllable if there exists a state feedback control in the form (5) such that closed-loop system (6) is causal.

Causality is an important feature of discrete-time descriptor systems. Noncausal behavior means that the system's state depends not only on the current values of the input signal but also on the future values. Noncausal behavior does not allow to implement conversion from descriptor system representation to a standard state-space system. Noncausality is one of the main difficulties in the generalization of existed methods on a class of discrete-time descriptor systems. The impossibility of equivalent transformation between descriptor systems and standard state-space ones is a major motivation in developing new control methods and algorithms. For more information, see [1].

The following results will be used for further investigation.

Theorem 1: [3] Regular discrete-time descriptor system (1)–(2) is admissible if there exists a solution $X = X^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ of the generalized Lyapunov equation

$$A^{\mathrm{T}}XA - E^{\mathrm{T}}XE + Q = 0, \qquad (7)$$

satisfying the condition $E^{\mathrm{T}}XE \geqslant 0$ and $Q > 0$.

Let system (1)–(2) be stable and $D = 0$. Then $\mathcal{H}_2$ norm can be found as [3]:

$$\|P(z)\|_2 = \mathrm{Trace}\,(C\mathcal{G}_{dc}C^{\mathrm{T}})$$

where the controllability Gramian $\mathcal{G}_{dc}$ is a unique symmetric positive semidefinite solution of the projected generalized Lyapunov equation [13]

$$\begin{aligned} A\mathcal{G}_{dc}A^{\mathrm{T}} - E\mathcal{G}_{dc}E^{\mathrm{T}} + P_lBB^{\mathrm{T}}P_l^{\mathrm{T}} & - \\ -(I-P_l)BB^{\mathrm{T}}(I-P_l)^{\mathrm{T}} &= 0, \qquad (8) \\ \mathcal{G}_{dc} - (I-P_r)\mathcal{G}_{dc}(I-P_r)^{\mathrm{T}} &= 0 \qquad (9) \end{aligned}$$

where

$$P_l = \overline{W}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\overline{W}^{-1}, \quad P_r = \overline{V}^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\overline{V}.$$

In the case of $D \neq 0$ it can be shown that

$$\|P\|_2 = \mathrm{Trace}\,(C\mathcal{G}_{dc}C^{\mathrm{T}} + D^{\mathrm{T}}D).$$

B. Mean anisotropy of the signal and anisotropy-based analysis

Now, we provide some background material on anisotropy-based analysis of linear discrete systems. The concepts of mean anisotropy of Gaussian random sequences and anisotropic norm of linear systems are introduced in [10]. An extended exposition of mean anisotropy and anisotropic norm can be found in [11], [14].

Let $W = \{w_k\}_{-\infty<k<\infty}$ be a stationary sequence of square summable vectors with values in $\mathbb{R}^m$ which is interpreted as a discrete-time random signal. Assembling the elements of W, associated with a time interval $[0, N]$, into a random vector

$$W_{0:N} = \begin{bmatrix} w_0 \\ \vdots \\ w_N \end{bmatrix},$$

we assume that $W_{0:N}$ is absolutely continuously distributed for any $N \geq 0$. The anisotropy A(W) is defined as the minimal value of the relative entropy with respect to the Gaussian distributions in $\mathbb{R}^m$ with zero mean and scalar covariance matrix described by:

$$A(W) = \frac{m}{2}\ln\left(\frac{2\pi e}{m}\mathbf{E}(|W|^2) - h(W)\right),$$

where

$$h(W) = \mathbf{E}\ln f(W) = -\int_{\mathbb{R}^m} f(w)\ln f(w)dw.$$

The mean anisotropy of the sequence W is defined by

$$\overline{\mathbf{A}}(W) = \lim_{N\to+\infty}\frac{\mathbf{A}(W_{0:N})}{N}.$$

It is shown in [14] that

$$\overline{\mathbf{A}}(W) = \mathbf{A}(w_0) + \mathbf{I}\,(w_0; \{w_k\}_{k<0})$$

where $\mathbf{I}\,(w_0; \{w_k\}_{k<0}) = \lim_{s\to-\infty}\mathbf{I}\,(w_0; W_{s:-1})$ is the Shannon mutual information [15] between w_0 and the past history $\{w_k\}_{k<0}$ of the sequence W.

For the Gaussian stationary random sequence W the Shannon mutual information between w_0 and the past history $\{w_k\}_{k<0}$ is defined by

$$\mathbf{I}\,(w_0; \{w_k\}_{k<0}) = \frac{1}{2}\ln\det(\mathbf{cov}(w_0)\mathbf{cov}(\tilde{w}_0)^{-1}),$$

where

$$\tilde{w}_0 = w_0 - \mathbf{E}(w_0|\{w_k\}_{k<0}) \tag{10}$$

is the error of the mean-square optimal prediction of w_0 by the past history $(w_k)_{k<0}$, provided by the conditional expectation.

The random sequence W can be generated from white Gaussian noise V by a shaping filter $G(z) \in \mathcal{H}_2{}^{m\times m}$. Then

$$\|G\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Trace}\, S(\omega) d\omega\right)^{1/2}$$

where $S(\omega) = \widehat{G}(\omega)\widehat{G}^*(\omega)$, $(-\pi \leqslant \omega \leqslant \pi)$, $\widehat{G}(\omega) = \lim_{l\to 1} G(l\mathrm{e}^{i\omega})$ is a boundary value of the transfer function $G(z)$.

The covariance matrix of prediction error (10) and the spectral density $S(\omega)$ are related by the Szegö-Kolmogorov formula:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det S(\omega) d\omega = \ln\det \mathbf{cov}(\tilde{w}_0). \tag{11}$$

Mean anisotropy of the sequence may be defined by the filter's parameters, using the expression

$$\overline{\mathbf{A}}(W) = \overline{\mathbf{A}}(G) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det \frac{mS(\omega)}{\|G\|_2^2} d\omega. \tag{12}$$

Let $Y = PW$ be an output of the linear discrete-time (normal or descriptor) system $P \in \mathcal{H}_\infty{}^{p\times m}$ with a transfer function $P(z)$, which is analytic in the identity circle $|z| < 1$, $P(z)$ has a finite $\mathcal{H}_\infty$-norm.

Definition 4: For a given constant value $a \geqslant 0$ a-anisotropic norm of the system P is defined as

$$|\!|\!| P |\!|\!|_a = \sup\left\{\frac{\|PG\|_2}{\|G\|_2} : \ G \in \mathbf{G}_a\right\}, \tag{13}$$

i.e. the maximum value of the system's gain with respect to the class of shaping filters

$$\mathbf{G}_a = \left\{G \in \mathcal{H}_2{}^{m\times m} : \ \overline{\mathbf{A}}(G) \leqslant a\right\}.$$

The fraction on the right-hand side of (13) can also be interpreted as the ratio of the power norms of the output $Y = \{y(k)\}$ and the input W against the class of shaping filters $\mathbb{G}_a$.

So a-anisotropic norm $|\!|\!| P |\!|\!|_a$ describes a "stochastic gain" of the system $P(z)$ with respect to W. If $\overline{\mathbf{A}}(W) = 0$, then the input signal is the Gaussian white noise sequence. In this case, the shaping filter $G(z)$ can be represented, for example, by an identity $m\times m$-matrix, i.e. $G(z) = I_m$. Therefore, $P(z)G(z) = P(z)$ and $\|G\|_2 = \sqrt{m}$. In this case $|\!|\!| P |\!|\!|_a = \frac{\|P\|_2}{\sqrt{m}}$.

If $\overline{\mathbf{A}}(W) \to \infty$, then $\lim_{a\to\infty} |\!|\!| P |\!|\!|_a = \|P\|_\infty$. For more information, see [14].

III. PROBLEM STATEMENT

Consider a discrete-time descriptor system P in the following form:

$$\begin{aligned} Ex(k+1) &= Ax(k) + B_w w(k) + B_u u(k), \quad (14)\\ z(k) &= Cx(k) + D_w w(k) + D_u u(k) \quad (15)\end{aligned}$$

where $w(k) \in \mathbb{R}^{m_1}$ and $z(k) \in \mathbb{R}^p$ are the input and output signals, respectively, $u(k) \in \mathbb{R}^{m_2}$ is the control vector. E, A, B_w, B_u, C, D_w, D_u are constant real matrices of appropriate dimensions. The system is assumed to be causally controllable and stabilizable. The input signal is a "colored" Gaussian disturbance with known mean anisotropy level ($\overline{\mathbf{A}}(W) = a \geqslant 0$). In addition, $\mathrm{rank}\, E = \mathrm{rank}\,[\,E \quad B\,] = \mathrm{rank}\,[\,E \quad C^{\mathrm{T}}\,]$.

Problem 1: Optimal state feedback control problem. Anisotropy-based control problem for descriptor systems is similar to the normal ones [9] and can be formulated as follows. For a given system (14)–(15) find an admissible state feedback control

$$u(k) = Kx(k),$$

that minimizes a-anisotropic norm of the closed-loop system:

$$|\!|\!| P_{cl} |\!|\!|_a = \sup_{G(z)\in\mathbb{G}_a} \frac{\|P_{cl}G\|_2}{\|G\|_2} \to \inf_K. \tag{16}$$

IV. MAIN RESULT

Substituting the control law $u(k) = Kx(k)$ into equations (14)–(15), we can write the closed-loop system P_{cl} as

$$\begin{aligned} Ex(k+1) &= (A + B_u K)x(k) + B_w w(k), \quad (17)\\ z(k) &= (C + D_u K)x(k) + D_w w(k). \quad (18)\end{aligned}$$

The idea of the solution of the optimal control problem is based on a saddle point condition that can be formulated as follows. For any admissible shaping filter $G \in \mathbb{G}_a$ and any stabilizing control law $K \in \mathbb{K}(P)$ we introduce the following sets:

$$\mathbb{K}_a^\diamond(G) \doteq Arg \min_{K\in\mathbb{K}(P)} \|P_{cl}\|_2, \ \ G \in \mathbb{G}_a,$$

$$\mathbb{G}_a^\diamond(K) \doteq Arg \max_{G\in\mathbb{G}_a} \frac{\|P_{cl}\|_2}{\|G\|_2}, \ \ K \in \mathbb{K}(P).$$

The set $\mathbb{K}_a^\diamond(G)$ consists of the control laws, which are the solution of the weighted $\mathcal{H}_2$-optimization problem. Here the input signal $W = GV$ is supposed to be "coloured" (correlated). Any control law, given in the form $K \in \mathbb{K}(P)$, minimizes the output dispersion Z of the input signal $W = GV$. The set $\mathbb{G}_a^\diamond(K)$ consists of the shaping filters, which generate Gaussian input disturbances with the worst spectral density of the closed-loop system.

Lemma 1: [16] If the control law K is the saddle point of the mapping $\mathbb{K}_a^\diamond \circ \mathbb{G}_a^\diamond$, then it is the solution to the problem (16).

Hence, the solution is composed of two steps. The first step is to find the worst-case shaping filter $G(z) \in \mathbb{G}_a$ with mean anisotropy level $\overline{\mathbf{A}}(G) \leqslant a$. The second step deals with weighted $\mathcal{H}_2$-control problem solution.

In order to define the state-space representation of the worst-case shaping filter, we give some definitions from the theory of dynamical systems. Consider a normal system $\mathcal{F}(z)$ in the state-space representation

$$x(k+1) = \mathcal{A}x(k) + \mathcal{B}f(k), \quad (19)$$
$$y(k) = \mathcal{C}x(k) + \mathcal{D}f(k). \quad (20)$$

Here, $x(k) \in \mathbb{R}^n$, $f(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$.

The transfer function of system (19)–(20) is given by

$$\mathcal{F}(z) = \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}.$$

Definition 5: The system with transfer function $\mathcal{F}(z)$, satisfying the condition $\mathcal{F}^*(z)\Psi\mathcal{F}(z) = \overline{\Psi}$ for a nonzero matrix $\Psi = \Psi^{\mathrm{T}} \in \mathbb{R}^{p\times p}$ and a nonsingular matrix $\overline{\Psi} = \overline{\Psi}^{\mathrm{T}} \in \mathbb{R}^{m\times m}$, is called the weighted all-pass system. Here $m \leqslant p$. The system, satisfying the condition $\mathcal{F}^*\mathcal{F} = I_m$, is called the all-pass system. For more information, see [17].

Lemma 2: [17] For given matrices Ψ and $\overline{\Psi}$ system $\mathcal{F} \in \mathcal{H}_\infty{}^{p\times m}$ is the weighted all-pass system if there exists a matrix $\mathcal{R} = \mathcal{R}^{\mathrm{T}}$, which satisfies the following generalized algebraic Riccati equation:

$$\mathcal{R} = \mathcal{A}^{\mathrm{T}}\mathcal{R}\mathcal{A} + \mathcal{C}^{\mathrm{T}}\Psi\mathcal{C}, \quad (21)$$
$$0 = \mathcal{B}^{\mathrm{T}}\mathcal{R}\mathcal{A} + \mathcal{D}^{\mathrm{T}}\Psi\mathcal{C}, \quad (22)$$
$$\overline{\Psi} = \mathcal{B}^{\mathrm{T}}\mathcal{R}\mathcal{B} + \mathcal{D}^{\mathrm{T}}\Psi\mathcal{D}. \quad (23)$$

Now we formulate the conditions of all-pass systems for descriptor system (1)–(2), supposing that the following rank assumptions take place:

$$\operatorname{rank} E = \operatorname{rank}\begin{bmatrix} E & B \end{bmatrix} = \operatorname{rank}\begin{bmatrix} E & C^{\mathrm{T}} \end{bmatrix}. \quad (24)$$

Lemma 3: Admissible system (1)–(2) is the all-pass system if there exists a matrix $\tilde{R} = \tilde{R}^{\mathrm{T}}$, satisfying the condition $E^{\mathrm{T}}\tilde{R}E \geqslant 0$, such that

$$B^{\mathrm{T}}\tilde{R}B + D^{\mathrm{T}}D = I, \quad (25)$$
$$B^{\mathrm{T}}\tilde{R}A + D^{\mathrm{T}}C = 0, \quad (26)$$
$$A^{\mathrm{T}}\tilde{R}A + C^{\mathrm{T}}C - E^{\mathrm{T}}\tilde{R}E = 0. \quad (27)$$

Proof: For the admissible system there exist two nonsingular matrices $\overline{W}$ and $\overline{V}$, which transform initial system (1)–(2) to the form (4).

Then rank assumption (24) is equivalent to

$$\operatorname{rank}\left(\overline{W}E\overline{V}\right) = \operatorname{rank}\begin{bmatrix} \overline{W}E\overline{V} & \overline{W}B \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \overline{W}E\overline{V} & \overline{V}^{\mathrm{T}}C^{\mathrm{T}} \end{bmatrix}.$$

It means that

$$\operatorname{rank}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} I & 0 & B_1 \\ 0 & 0 & B_2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} I & 0 & C_1^{\mathrm{T}} \\ 0 & 0 & C_2^{\mathrm{T}} \end{bmatrix},$$

consequently,

$$B_2 = 0, \;\; C_2 = 0. \quad (28)$$

Introduce the matrix $\tilde{R}$ in the following way

$$\tilde{R} = \overline{W}^{\mathrm{T}}R\overline{W} = \overline{W}^{\mathrm{T}}\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}\overline{W}. \quad (29)$$

Consider equation (27) and rewrite it, taking into account (29),

$$A^{\mathrm{T}}\overline{W}^{\mathrm{T}}R\overline{W}A + C^{\mathrm{T}}C - E^{\mathrm{T}}\overline{W}^{\mathrm{T}}R\overline{W}E = 0. \quad (30)$$

Left and right multiplying on nonsingular matrices $\overline{V}^{\mathrm{T}}$ and $\overline{V}$ gives

$$\overline{V}^{\mathrm{T}}A^{\mathrm{T}}\overline{W}^{\mathrm{T}}R\overline{W}A\overline{V} + \overline{V}^{\mathrm{T}}C^{\mathrm{T}}C\overline{V} - \overline{V}^{\mathrm{T}}E^{\mathrm{T}}\overline{W}^{\mathrm{T}}R\overline{W}E\overline{V} = 0. \quad (31)$$

We can rewrite (31) as

$$\begin{bmatrix} A_1^{\mathrm{T}} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}\begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} C_1^{\mathrm{T}} \\ C_2^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} C_1 & C_2 \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = 0, \quad (32)$$

or

$$\begin{bmatrix} A_1^{\mathrm{T}}R_{11}A_1 & A_1^{\mathrm{T}}R_{12} \\ R_{21}A_1 & R_{22} \end{bmatrix} + \begin{bmatrix} C_1^{\mathrm{T}}C_1 & C_1^{\mathrm{T}}C_2 \\ C_2^{\mathrm{T}}C_1 & C_2^{\mathrm{T}}C_2 \end{bmatrix} - \begin{bmatrix} R_{11} & 0 \\ 0 & 0 \end{bmatrix} = 0. \quad (33)$$

It leads to

$$A_1^{\mathrm{T}}R_{11}A_1 + C_1^{\mathrm{T}}C_1 - R_{11} = 0,\; A_1^{\mathrm{T}}R_{12} + C_1^{\mathrm{T}}C_2 = 0,$$
$$R_{21}A_1 + C_2^{\mathrm{T}}C_1 = 0,\; R_{22} + C_2^{\mathrm{T}}C_2 = 0. \quad (34)$$

Now consider equation (26): $B^{\mathrm{T}}\overline{W}^{\mathrm{T}}R\overline{W}A + D^{\mathrm{T}}C = 0.$ By right multiplication on $\overline{V}$ we have

$$B_1^{\mathrm{T}}R_{11}A_1 + D^{\mathrm{T}}C_1 = 0, \quad (35)$$
$$B_1^{\mathrm{T}}R_{12} + D^{\mathrm{T}}C_2 = 0. \quad (36)$$

For equation (25) we have

$$B_1^{\mathrm{T}}R_{11}B_1 + B_1^{\mathrm{T}}R_{12}B_2 + B_2^{\mathrm{T}}R_{22}B_2 + D^{\mathrm{T}}D = I. \quad (37)$$

Taking into account (28), equations (34)–(37) are equivalent to

$$A_1^{\mathrm{T}}R_{11}A_1 + C_1^{\mathrm{T}}C_1 - R_{11} = 0, \quad (38)$$
$$B_1^{\mathrm{T}}R_{11}A_1 + D^{\mathrm{T}}C_1 = 0, \quad (39)$$
$$B_1^{\mathrm{T}}R_{11}B_1 + D^{\mathrm{T}}D = I. \quad (40)$$

Evidently, the initial descriptor system is equivalent to the following normal one:

$$x_1(k+1) = A_1x_1(k) + B_1u(k),$$
$$y(k) = C_1x_1(k) + Du(k),$$

for which conditions (34), (35), (37) coincide with the conditions of Lemma 2 if $\Psi = I_p$, $\overline{\Psi} = I_m$. ■

Lemma 4: Let the system with state-space representation (1)–(2) be admissible. Let the following assumption hold

$$\operatorname{rank} E = \operatorname{rank} [\, E \quad B\,]. \tag{41}$$

Consider a generalized Lyapunov equation

$$A\tilde{\mathcal{G}}A^{\mathrm{T}} - E\tilde{\mathcal{G}}E^{\mathrm{T}} + BB^{\mathrm{T}} = 0. \tag{42}$$

Then $\mathcal{H}_2$ norm of the system can be computed as

$$\|P\|_2 = \operatorname{Trace}(C\tilde{\mathcal{G}}C^{\mathrm{T}} + D^{\mathrm{T}}D). \tag{43}$$

Proof:

In [13] it is shown that the solution of the generalized projected Lyapunov equation (8)–(9) coincides with the solution of generalized Lyapunov equation (42) if $B_1B_2^{\mathrm{T}} = 0$.

Under rank assumption (41) $B_2 = 0$ and, therefore, the condition $B_1B_2^{\mathrm{T}} = 0$ holds. Hence, the solution of generalized Lyapunov equation (42) defines a controllability Gramian of system (1)–(2). Using definition of $\mathcal{H}_2$ norm we get (43). ■

Rank restrictions (24) are required for uniqueness and existence of solution of the equation (25)–(27). This equation is required for worst-case shaping filter design $G \in \mathbb{G}_a^{\diamond}(K)$. This filter is also used for a-anisotropic norm calculation of the admissible system. From saddle-point condition, we first suppose that there exists an optimal controller.

The following theorem gives conditions on the parameters of the worst-case shaping filter with a bounded mean anisotropy level $\overline{\mathbf{A}}(G) \leqslant a$.

Theorem 2: Denote $\hat{A} = A + B_uK$, $\hat{C} = C_1 + D_uK$, and suppose, that the closed-loop system is admissible. Then for any mean anisotropy level $a \geqslant 0$ the worst-case shaping filter with state-space representation

$$G = \left[\begin{array}{c} E, \quad \begin{array}{c|c} \hat{A} + B_wL & B_w\Sigma^{1/2} \\ \hline L & \Sigma^{1/2} \end{array} \end{array}\right] \tag{44}$$

and mean anisotropy $\overline{\mathbf{A}}(G) = a$ is defined from the unique solution (q, R) of the generalized algebraic Riccati equation

$$\begin{aligned} E^{\mathrm{T}}RE &= \hat{A}^{\mathrm{T}}R\hat{A} + q\hat{C}^{\mathrm{T}}\hat{C} + L^{\mathrm{T}}\Sigma^{-1}L, &(45)\\ \Sigma &= (I_{m_1} - qD_w^{\mathrm{T}}D_w - B_w^{\mathrm{T}}RB_w)^{-1}, &(46)\\ L &= \Sigma(B_w^{\mathrm{T}}R\hat{A} + qD_w^{\mathrm{T}}\hat{C}) &(47)\end{aligned}$$

where a scalar parameter q satisfies the condition $q \in [0, \quad \|P_{cl}\|_\infty^{-2})$, and $R = R^{\mathrm{T}}$ with $E^{\mathrm{T}}RE \geqslant 0$. Moreover,

$$-\frac{1}{2}\ln\det\left(\frac{m_1\Sigma}{\operatorname{Trace}(LP_GL^{\mathrm{T}} + \Sigma)}\right) = a \tag{48}$$

where $P_G \in \mathbb{R}^{n\times n}$ is a controllability Gramian for the shaping filter G. It satisfies the projected generalized Lyapunov equation

$$EP_GE^{\mathrm{T}} = (\hat{A} + B_wL)P_G(\hat{A} + B_wL)^{\mathrm{T}} - B_w\Sigma B_w^{\mathrm{T}}. \tag{49}$$

Proof: Using definition of a-anisotropic norm (13) we can construct a Lagrange function

$$\mathfrak{L} = \|P_{cl}G\|_2^2 - \mu\|G\|_2^2 - \lambda\overline{\mathbf{A}}(G). \tag{50}$$

It can be shown that Lagrange function (50) reaches its maximum when

$$q\Lambda(\omega) - I_{m_1} + \sigma S^{-1}(\omega) = 0 \tag{51}$$

where $\Lambda(\omega) = \hat{P}_{cl}^*(\omega)\hat{P}_{cl}(\omega)$, $S(\omega) = \hat{G}^*(\omega)\hat{G}(\omega)$.

It follows from (51) that

$$S(\omega) = \sigma(I_{m_1} - q\Lambda(\omega))^{-1}$$

defines the spectral density of the worst case input disturbance. Without loss of generality we may put $\sigma = 1$ and rewrite the equation (51) as

$$q\hat{P}_{cl}^*(\omega)\hat{P}_{cl}(\omega) + (\hat{G}^{-1})^*(\omega)\hat{G}^{-1}(\omega) = I_{m_1}.$$

Define

$$\Theta = \begin{bmatrix} \sqrt{q}\hat{P}_{cl}(\omega) \\ \hat{G}^{-1}(\omega) \end{bmatrix}.$$

Then the closed-loop system with the worst-case shaping filter must satisfy the following factorization $\Theta^*\Theta = I_{m_1}$. Noting that the shaping filter G is assumed to be invertible. Invertibility of operator G stands for possibility of obtaining $v(k)$ from $w(k)$. Hence, we get

$$G^{-1} = \left[\begin{array}{c} E, \quad \begin{array}{c|c} \hat{A} & B_w \\ \hline -\Sigma^{-1/2}L & \Sigma^{-1/2} \end{array} \end{array}\right].$$

The state-space representation of Θ is

$$\begin{bmatrix} q^{1/2}P_{cl} \\ G^{-1} \end{bmatrix} = \left[\begin{array}{c} E, \quad \begin{array}{c|c} \hat{A} & B_w \\ \hline \Omega & \Delta \end{array} \end{array}\right]$$

where

$$\Omega = \begin{bmatrix} q^{1/2}\hat{C} \\ -\Sigma^{-1/2}L \end{bmatrix}, \quad \Delta = \begin{bmatrix} q^{1/2}D_w \\ \Sigma^{-1/2} \end{bmatrix}.$$

Using Lemma 3 and substituting Θ into (25)–(27) we get (45)–(47). Under the conditions of lemma 4 $\mathcal{H}_2$ norm of the shaping filter is defined by the formula

$$\|G\|_2 = \operatorname{Trace}(LP_GL^{\mathrm{T}} + \Sigma). \tag{52}$$

Taking into account definition of mean anisotropy (12) and (52), we get equation (48). The proof is completed. ■

Consider an extended system given as

$$\begin{aligned} E_*\hat{x}(k+1) &= A_*\hat{x}(k) + B_{u*}u(k) + B_{v*}v(k), &(53)\\ z(k) &= C_*\hat{x}(k) + D_wv(k) &(54)\end{aligned}$$

where $\hat{x}(k) \in \mathbb{R}^{2n}$, $z(k) \in \mathbb{R}^p$, and $v(k) \in \mathbb{R}^{m_1}$ is a Gaussian white noise sequence. Here $E_* = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}$, $A_* = \begin{bmatrix} A & B_wL \\ 0 & A + B_wL \end{bmatrix}$, $B_{u*} = \begin{bmatrix} B_u \\ 0 \end{bmatrix}$, $C_* = \begin{bmatrix} C & D_wL \end{bmatrix}$. As the shaping filter G is assumed to be invertible, this problem is equivalent to the standard $\mathcal{H}_2$-optimization problem.

Theorem 3: Let system (53)–(54) be stabilizable and causally controllable. The optimal state-space control law, that solves the weighted $\mathcal{H}_2$-optimization problem, can be found in the following form:

$$K = \Gamma_1 + \Gamma_2 \tag{55}$$

where Π and $\Gamma = [\,\Gamma_1, \quad \Gamma_2\,]$ are found from the solution of the following generalized algebraic Riccati equation

$$E_*^{\mathrm{T}} T E_* = A_*^{\mathrm{T}} T A_* + C_*^{\mathrm{T}} C_* + \Gamma^{\mathrm{T}} \Pi \Gamma, \quad (56)$$

$$\Pi = (B_{u*}^{\mathrm{T}} T B_{u*} + D_w^{\mathrm{T}} D_w), \quad (57)$$

$$\Gamma = -\Pi^{-1}(B_{u*}^{\mathrm{T}} T A_* + D_w^{\mathrm{T}} C_*). \quad (58)$$

The proof of theorem 3 can be found in [8] and references therein.

The solution of the optimal control problem consists of solving coupled generalized Riccati equations (45)–(47) and (56)–(58), projected generalized Lyapunov equation (49), and nonlinear special type equation (48).

If $a = 0$ the parameters of worst case shaping filter $G(z)$ are $L = 0$ and $\Sigma = I_{m_1}$. In this case the solution of the anisotropy-based optimal control problem is equivalent to the solution of $\mathcal{H}_2$ optimal control problem.

V. Numerical example

Consider the following system:
$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 1.15 & -0.3 \\ 0.1 & 0.3 \end{bmatrix}$, $B_w = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$, $B_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D_w = 0.2,\ D_u = 0.1$.

It is easy to see, that $\operatorname{rank}(E) = \operatorname{rank}[E \quad B_w] = \operatorname{rank}[E \quad C^{\mathrm{T}}] = 1$. The system is causal, but unstable, $\rho(E, A) = 1.25$.

Now we find a state feedback control $u(k) = Kx(k)$ for the given mean anisotropy level $a = 0.4$, using the techniques from the derived theorems.

The optimal controller is $K^* = \begin{bmatrix} -1.6514 & 0 \end{bmatrix}$. The closed-loop system is admissible. Its spectral radius is $\rho(E, A + B_u K^*) = 0.4014$ and a-anisotropic norm is $|\!|\!| P_{cl}^{SF} |\!|\!|_a = 0.4978$.

VI. Conclusion

In this paper, the problem of anisotropy-based optimal control for descriptor systems based on generalized algebraic Riccati equations is solved. The solution to the optimal control problem consists of solving two generalized Riccati equations (in order to define the parameters of the shaping filter and the parameters of the optimal control law), the projected generalized Lyapunov equation and the special type equation. All these equations are connected. While solving the optimal control problem one has to deal with four matrix variables ($P_G \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times n}$, $T \in \mathbb{R}^{2n\times 2n}$, $K \in \mathbb{R}^{n\times m_2}$) and one scalar variable q. The total amount of unknown variables in this algorithm is $6n^2 + n\times m_2 + 1$. This technique provides a minimum value of the a-anisotropic norm of the closed-loop system. If $E = I_n$, then the derived solution also covers state-feedback optimal anisotropy-based control problem for normal systems.

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