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Stability analysis for switched linear systems with dwell time and delay in the active mode detection

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Abstract— **In this paper stability analysis for switched systems with delay in the active mode detection is investigated. First the switched system under consideration is recast as a hybrid system. Based on this hybrid formalism, general sufficient stability conditions are then provided. These results are used to establish tractable conditions for stability analysis using sum of squares polynomial and Linear Matrix inequalities. To complete the analysis, relations between the Linear Matrix inequalities feasibility and structural properties of the systems are given. Finally some simulations show the effectiveness of the methods.**

I. Introduction

Networked control systems are becoming more and more present in modern industrial application. A good motivation for their development is that they can be cheaper to set in place, are scalable and enables lower maintenance cost. In turns those architecture impose constraint on the information that can be transmitted among components both in terms of frequency and in terms of regularity. Therefore, when compared with more classical control architecture networked control systems can be harder to predict and need careful investigation. To study networked system different classes of models can be considered [7]. Two approaches are of particular importance, Impulsive systems [4] and Switched systems [8] in this paper the second approach will be considered.

The study of switched systems has been the subject of a growing amount of interest in the past decade and is by now a well studied topic [5], [8], [9]. Switched systems are a subclass of hybrid systems where different dynamical equations describe different operating modes. The transition from one mode to the other is given by a switching rule. The motivation for considering such a class of system is vast and encompasses robust stabilization or modeling and control of systems subject to sampling uncertainty, model singularly perturbed system. An important problem arising when considering switched system is the stability problem. Indeed given a switching rule, stability of every mode does not guaranty stability of the overall switched system while instability of every mode does not guarenty instability of the overall system [9]. An important amount of results exists on stability of switched systems in different settings, see for instance [4], [1], [16], [13], and [10] for a survey on the topic.

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The switching rule from one operating mode to the other may be *a priori unknown* with no hypothesis on the next active mode [5]. This fact begs for identification of the active mode. In practice the identification of this mode may takes some time introducing a delay [12], [15]. Hence when considering stability issue close attention as to be paid to mismatch between estimated active mode and current active mode.

In this paper new stability conditions will be presented for system with delay in the active mode detection. Similar to the setting considered in [14], [16], [17] we consider switched system with minimal dwell time in the active mode and a maximal dwell time of active mode estimation error. While [14], [16], [17] consider Lyapunov-like functions that can increase when the true active mode is not properly estimated, using the hybrid approach we are able to construct strong (hybrid) Lyapunov functions.

In order to prove our results, the switched system under consideration is reformulated as a hybrid system, then tools from hybrid system theory (in the formalism of [6]) are used to find sufficient stability conditions. First general conditions are provided based on this hybrid formalism. Then, this result is used to provide tractable conditions for stability analysis using sum of squares polynomials (SOSs) and Linear Matrix inequalities (LMIs). Relations between Linear Matrix inequalities feasibility and structural properties of the systems are also given.

Notations

For a vector or a matrix v, $v^\top$ denotes its transpose. We define for a matrix A, $He(A) := A + A^\top$. $\mathbb{R}_{\geq 0}$ corresponds to non-negative real numbers. $\mathbb{S}$ denotes the set of symmetric matrices while $\mathbb{S}^+$ denotes the set of positive definite symmetric matrices. For a $M \in \mathbb{S}$, $\lambda_{min}(M)$ and $\lambda_{max}(M)$ denote its smallest and largest eigenvalues, respectively. For two positive definite matrices (resp positive semidefinite) P and Q we write $P > Q$ if $P - Q$ is positive definite (resp $P \geq Q$ if $P - Q$ is positive semidefinite).

II. Problem statement

Considering the following switched system

$$\dot{x}(t) = A_{(\sigma(t),\hat{\sigma}(t))}x(t), t \neq t^{\sigma}_{k+1}, t \neq t^{\hat{\sigma}}_{k+1} \tag{1}$$

With $x(0) = x_0$ given and $\forall s \in (t^{\sigma}_k)_{k\in\mathbb{N}} \cup (t^{\hat{\sigma}}_k)_{k\in\mathbb{N}}$, $x(s) = \lim_{t\to s, t<s} x(t)$.; $\forall t \in \mathbb{R}_{\geq 0}$ $x(t) \in \mathbb{R}^n, \sigma(t), \hat{\sigma}(t) \in \mathcal{D} \subset \mathbb{N}$ a finite set. The sequence of time giving the changes of values for σ is denoted $(t^{\sigma}_k)_{k\in\mathbb{N}}$ and verifies

$$t^{\sigma}_0 = 0,\ t^{\sigma}_{k+1} - t^{\sigma}_k \geq \underline{\tau}.$$

The sequence of time giving the changes of values for $\hat{\sigma}$ is denoted $(t_k^{\hat{\sigma}})_{k\in\mathbb{N}}$ and verifies

$$0 \leq t_k^{\hat{\sigma}} - t_k^{\sigma} \leq \overline{\delta},$$

with furthermore $\overline{\delta} < \underline{\tau}$.

Remark 1: System (1) can be used to express system of the form

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}K_{\hat{\sigma}(t)}x(t),$$

or observer based controller of the form

$$\dot{x} = \begin{pmatrix} \dot{z} \\ \dot{\hat{z}} \end{pmatrix} = \begin{pmatrix} A_{\sigma} & B_{\sigma}K_{\hat{\sigma}} \\ L_{\hat{\sigma}}C_{\sigma} & A_{\hat{\sigma}} + B_{\hat{\sigma}}K_{\hat{\sigma}} - L_{\hat{\sigma}}C_{\hat{\sigma}} \end{pmatrix} \begin{pmatrix} z \\ \hat{z} \end{pmatrix}$$

A. Generalities on hybrid systems

Next, we will embed[1] (1) into the hybrid framework described in [6]. Consider a hybrid system

$$\mathcal{H} : \begin{cases} \xi \in C_{\mathcal{H}}, & \dot{\xi} \in F_{\mathcal{H}}(\xi), \\ \xi \in D_{\mathcal{H}}, & \xi^+ \in G_{\mathcal{H}}(\xi). \end{cases} \tag{2}$$

Roughly speaking, while ξ belongs to $C_{\mathcal{H}}$, the state *flows* according to a differential inclusion characterized by a set-valued mapping $F_{\mathcal{H}}$. When ξ belongs to $D_{\mathcal{H}}$, the state *jumps* according to a discrete dynamic defined by $G_{\mathcal{H}}$. In what follows we will use the concepts and notations from [6]. The most important are recalled below.

Definition 1 (Domain of a set-valued mapping): Given a set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^{n_\xi}$, the domain of M is the set $dom\, M = \{x \in \mathbb{R}^m : M(x) \neq \emptyset\}$.

Definition 2 (Data of a hybrid system): The data of a hybrid system $\mathcal{H}$ in $\mathbb{R}^{n_\xi}$ consists of four elements:

- a set $C_{\mathcal{H}} \subset \mathbb{R}^{n_\xi}$, called the *flow set*;
- a set-valued mapping $F_{\mathcal{H}} : \mathbb{R}^{n_\xi} \rightrightarrows \mathbb{R}^{n_\xi}$ with $C_{\mathcal{H}} \subset dom\, F_{\mathcal{H}}$, called the *flow map*;
- a set $D_{\mathcal{H}} \subset \mathbb{R}^{n_\xi}$, called the *jump set*;
- a set-valued mapping $G_{\mathcal{H}} : \mathbb{R}^{n_\xi} \rightrightarrows \mathbb{R}^{n_\xi}$ with $D_{\mathcal{H}} \subset dom\, G_{\mathcal{H}}$, called the *jump map*.

Definition 3 (Hybrid time domains): A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if

$$E = \cup_{j=0}^{J-1}([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_J$. It is a hybrid time domain if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid domain.

Definition 4 (Hybrid arc): A function $\xi : E \to \mathbb{R}^{n_\xi}$ is a hybrid arc if E is a hybrid time domain and if for each $j \in \mathbb{N}$, the function $t \to \xi(t, j)$ is locally absolutely continuous[2] on the interval $I_j = \{t : (t, j) \in E\}$.

Definition 5 (Solution to a hybrid system): A hybrid arc ξ is a solution to the hybrid system $\mathcal{H}$ if $\xi(0,0) \in C_{\mathcal{H}} \cup D_{\mathcal{H}}$, and (S_1) for all $j \in \mathbb{N}$ such that $I_j := \{t : (t, j) \in dom\, \xi\}$ has nonempty interior, $\xi(t, j) \in C_{\mathcal{H}}$ for all $t \in int(I_j)$, $\dot{\xi}(t, j) \in F_{\mathcal{H}}(\xi(t, j))$ for almost all $t \in I_j$
(S_2) for all $(t, j) \in dom\, \xi$ such that $(t, j+1) \in dom\, \xi$,

$$\xi(t, j) \in D_{\mathcal{H}},\ \xi(t, j+1) \in G_{\mathcal{H}}(\xi(t, j)).$$

Definition 6 (Maximal solutions): A solution ξ to $\mathcal{H}$ is maximal if there does not exist another solution ψ to $\mathcal{H}$ such that $dom\, \xi$ is a proper subset of $dom\, \psi$ and $\xi(t, j) = \psi(t, j)$ for all $(t, j) \in dom\, \xi$.

A solution is called complete if it is maximal and defined on an unbounded hybrid time domain. We define as follows the concept of Uniform Global pre-Asymptotic stability (UGpAS) that will be used in the rest of the article.

Definition 7 (UGpAS): Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^{n_\xi}$. Let $\mathcal{A} \subset \mathbb{R}^{n_\xi}$ be closed. The set $\mathcal{A}$ is said to be

- uniformly globally stable for $\mathcal{H}$ if there exists a class-$\mathcal{K}_\infty$ function α such that any solution ξ to $\mathcal{H}$ satisfies $|\xi(t, j)|_{\mathcal{A}} \leq \alpha(|\xi(0, 0)|_{\mathcal{A}})$ for all $(t, j) \in dom\, \xi$,
- uniformly globally pre-attractive for $\mathcal{H}$ if for each $\varepsilon > 0$ and $r > 0$ there exists $T > 0$ such that, for any solution ξ to $\mathcal{H}$ with $|\xi(0, 0)|_{\mathcal{A}} \leq r$, $(t, j) \in dom\, \xi$ and $t+j \geq T$ imply $|\xi(t, j)|_{\mathcal{A}} \leq \varepsilon$,
- uniformly globally pre-asymptotically stable (UGpAS) for $\mathcal{H}$ if it is both uniformly globally stable and uniformly globally pre-attractive.

If furthermore all the maximal solutions of (2) are complete then we say that $\mathcal{A}$ is Uniformly Globally Asymptotically stable (UGAS)

Definition 8 (Candidate Lyapunov function): A function $V : dom\, V \to \mathbb{R}$ is said to be a candidate Lyapunov function for the hybrid system $\mathcal{H}$ if the following conditions hold:
1. $\overline{C_{\mathcal{H}}} \cup D_{\mathcal{H}} \cup G_{\mathcal{H}}(D_{\mathcal{H}}) \subset dom\, V$;
2. V is continuously differentiable on an open set containing $\overline{C_{\mathcal{H}}}$, where $\overline{C_{\mathcal{H}}}$ denotes the closure of $C_{\mathcal{H}}$.

We recall now the Theorem 3.18 of [6]:

Theorem 1 (Sufficient conditions for UGpAS): Let $\mathcal{H}$ be a hybrid system and let $\mathcal{A} \subset \mathbb{R}^n$ be closed. If V is a Lyapunov function candidate for $\mathcal{H}$ and there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and a continuous $\rho \in \mathcal{PD}$ such that:

$$\forall \xi \in C_{\mathcal{H}} \cup D_{\mathcal{H}} \cup G_{\mathcal{H}}(D_{\mathcal{H}})$$
$$\alpha_1(|\xi|_{\mathcal{A}}) \leq V(\xi) \leq \alpha_2(|\xi|_{\mathcal{A}}), \tag{3a}$$
$$\forall \xi \in C_{\mathcal{H}}, f \in F_{\mathcal{H}}(\xi), \langle \nabla V(\xi), f \rangle \leq -\rho(|\xi|_{\mathcal{A}}), \tag{3b}$$
$$\forall \xi \in D_{\mathcal{H}}, g \in G_{\mathcal{H}}(\xi), V(g) - V(\xi) \leq \rho(|\xi|_{\mathcal{A}}). \tag{3c}$$

then $\mathcal{A}$ is UGpAs for $\mathcal{H}$.

B. Hybrid system representation

Next we will embed system (1) into the proposed hybrid formalism.

- x the state of the dynamical system
- σ, resp. $\hat{\sigma}$ the real resp. estimated active mode
- τ a timer since the last change of real active mode
- s a logical variable indicating the ability of the system to switch (either the real or the estimated mode)
- $(x^\top, \sigma, \hat{\sigma}, \tau, s)^\top \in \mathcal{E} := \mathbb{R}^n \times \mathcal{D}^2 \times \mathbb{R}_{\geq 0} \times \{0, 1\}$

[1] In this context embed means that the system (1) will be studied trough a hybrid dynamical system containing more solutions. It is however reduired that to any solution of (1) corresponds a solution to the hybrid system.

[2] Local absolute continuity means that $t \to \xi(t, j)$ is differentiable almost everywhere on each I_j with non empty interiors - see for example [6] p. 28.

$$\left.\begin{array}{l}\dot{x}=A_{i,j}x\\ \dot{\sigma}=0\\ \dot{\hat{\sigma}}=0\\ \dot{\tau}=1\\ \dot{s}=0\end{array}\right\}\begin{pmatrix}x\\ \sigma\\ \hat{\sigma}\\ \tau\\ s\end{pmatrix}\in C_{\{i\neq j\}}\quad\left.\begin{array}{l}x^{+}=x\\ \sigma^{+}=\sigma\\ \hat{\sigma}^{+}=\sigma\\ \tau^{+}=\tau\\ s^{+}=0\end{array}\right\}\begin{pmatrix}x\\ \sigma\\ \hat{\sigma}\\ \tau\\ s\end{pmatrix}\in D_{\{i\neq j\}}.$$

$$\left.\begin{array}{l}\dot{x}=A_{i,i}x\\ \dot{\sigma}=0\\ \dot{\hat{\sigma}}=0\\ \dot{\tau}=1\\ \dot{s}=0\end{array}\right\}\begin{pmatrix}x\\ \sigma\\ \hat{\sigma}\\ \tau\\ s\end{pmatrix}\in C^{0}_{\{i\}}\quad\left.\begin{array}{l}x^{+}=x\\ \sigma^{+}=\sigma\\ \hat{\sigma}^{+}=\hat{\sigma}\\ \tau^{+}=\tau\\ s^{+}=1\end{array}\right\}\begin{pmatrix}x\\ \sigma\\ \hat{\sigma}\\ \tau\\ s\end{pmatrix}\in D^{0}_{\{i\}}.$$

$$\left.\begin{array}{l}\dot{x}=A_{i,i}x\\ \dot{\sigma}=0\\ \dot{\hat{\sigma}}=0\\ \dot{\tau}=1\\ \dot{s}=0\end{array}\right\}\begin{pmatrix}x\\ \sigma\\ \hat{\sigma}\\ \tau\\ s\end{pmatrix}\in C^{1}_{\{i\}}\quad\left.\begin{array}{l}x^{+}=x\\ \sigma^{+}\in\mathcal{D}\backslash\{i\}\\ \hat{\sigma}^{+}=\hat{\sigma}\\ \tau^{+}=0\\ s^{+}=1\end{array}\right\}\begin{pmatrix}x\\ \sigma\\ \hat{\sigma}\\ \tau\\ s\end{pmatrix}\in D^{1}_{\{i\}}.$$

$$C_{\{i\neq j\}}=\left\{\xi\in\mathcal{E}|s=1,\tau\in[0,\overline{\delta}]\right\};$$
$$C^{0}_{\{i\}}=\left\{\xi\in\mathcal{E}|s=0,\tau\in[0,\underline{\tau}]\right\};$$
$$C^{1}_{\{i\}}=\left\{\xi\in\mathcal{E}|s=1,\tau\in[\underline{\tau},+\infty)\right\};$$

the flow maps for all $i,j\in\mathcal{D}$, $s\in\{0,1\}$

$$F_{\{i,s=1\}}=F_{\{i,s=0\}}=F_{\{i\neq j\}}(\xi)=\begin{pmatrix}A_{i,j}x\\ 0\\ 0\\ 1\\ 0\end{pmatrix}\tag{4}$$

for ξ in $C_{\mathcal{H}}$ and $F_{\mathcal{H}}(\xi)=\emptyset$ elsewhere; the jump set

$$D_{\{i\neq j\}}=\left\{\xi\in\mathcal{E}|s=1,\tau\in[0,\overline{\delta}]\right\};$$
$$D^{0}_{\{i\}}=\left\{\xi\in\mathcal{E}|s=0,\tau=\underline{\tau}\right\};$$
$$D^{1}_{\{i\}}=\left\{\xi\in\mathcal{E}|s=1,\tau\in[\underline{\tau},+\infty)\right\};$$

and the jump maps

$$G_{\{i\neq j\}}(\xi)=(x^{\top},\sigma,\sigma,\tau,0)^{\top},G^{0}_{\{i\}}(\xi)=(x^{\top}\sigma,\hat{\sigma},\tau,1)^{\top}\tag{5}$$

$$G^{1}_{\{i\}}(\xi)=(x^{\top},\mathcal{D}\backslash\{i\},\hat{\sigma},0,1)^{\top}.\tag{6}$$

We can therefore define an hybrid system in the formalism of [6] with

$$C_{\mathcal{H}}=(\bigcup_{i,j,\in\mathcal{D},i\neq j}C_{\{i\neq j\}})\cup(\bigcup_{i\in\mathcal{D},s\in\{0,1\}}C^{s}_{\{i\}})\tag{7}$$

$$F(\xi)_{\mathcal{H}}=\begin{pmatrix}A_{i,j}x\\ 0\\ 0\\ 1\\ 0\end{pmatrix}\tag{8}$$

when ξ in $C_{\mathcal{H}}$ and $F_{\mathcal{H}}(\xi)=\emptyset$ elsewhere.

$$D_{\mathcal{H}}=(\bigcup_{i,j,\in\mathcal{D},i\neq j}D_{\{i\neq j\}})\cup(\bigcup_{i\in\mathcal{D},s\in\{0,1\}}D^{s}_{\{i\}})\tag{9}$$

$$G_{\mathcal{H}}(\xi)=(\bigcup_{i,j,\in\mathcal{D},i\neq j}G_{\{i\neq j\}}(\xi))\cup(\bigcup_{i\in\mathcal{D},s\in\{0,1\}}G^{s}_{\{i\}}(\xi))\tag{10}$$

when ξ in $D_{\mathcal{H}}$ and $G_{\mathcal{H}}(\xi)=\emptyset$ elsewhere[3].

Remark 2: Consider a solution $x(t)$ of system (1) with x satisfying (1) for a given switching sequence $(t^{\sigma}_{k})_{k\in\mathbb{N}},(t^{\hat{\sigma}}_{k})_{k\in\mathbb{N}}$.

$$\text{Given } E=\bigcup_{k=0}^{\infty}\Big(([t^{\sigma}_{k},t^{\hat{\sigma}}_{k}],3k)\cup([t^{\hat{\sigma}}_{k},t^{\sigma}_{k}+\underline{\tau}],3k+1)\cup([t^{\sigma}_{k}+\underline{\tau},t^{\sigma}_{k+1}],3k+2)\Big)\tag{11}$$

define the hybrid arc $\xi:E\to\mathcal{E}$ as follows

$$\xi(t,3k)=\Big(x(t)^{\top},\sigma(t),\hat{\sigma}(t),t-t^{\sigma}_{k},1\Big)^{\top},t\in[t^{\sigma}_{k},t^{\hat{\sigma}}_{k}),$$
$$\xi(t,3k+1)=\Big(x(t)^{\top},\sigma(t),\hat{\sigma}(t),t-t^{\sigma}_{k},0\Big)^{\top},t\in[t^{\hat{\sigma}}_{k},t^{\sigma}_{k}+\underline{\tau}),$$
$$\xi(t,3k+2)=\Big(x(t)^{\top},\sigma(t),\hat{\sigma}(t),t-t^{\sigma}_{k},0\Big)^{\top},t\in[t^{\sigma}_{k}+\underline{\tau},t^{\sigma}_{k+1}),$$

$$\xi(t^{\hat{\sigma}}_{k},3k)=\lim_{t\to t^{\sigma}_{k},t<t^{\hat{\sigma}}_{k}}\Big(x(t)^{\top},\sigma(t),\hat{\sigma}(t),t-t^{\sigma}_{k},1\Big)^{\top}.$$
$$\xi(t^{\sigma}_{k}+\underline{\tau},3k+1)=\lim_{t\to t^{\sigma}_{k}+\underline{\tau},t<t^{\sigma}_{k}+\underline{\tau}}\Big(x(t)^{\top},\sigma(t),\hat{\sigma}(t),t-t^{\sigma}_{k},0\Big)^{\top}.$$
$$\xi(t^{\sigma}_{k+1},3k+2)=\lim_{t\to t^{\sigma}_{k+1},t<t^{\sigma}_{k+1}}\Big(x(t)^{\top},\sigma(t),\hat{\sigma}(t),t-t^{\sigma}_{k},0\Big)^{\top}.$$

Using the description (2), (7)-(10), one can see that ξ is a complete solution to the hybrid system (2) with data $C_{\mathcal{H}},F_{\mathcal{H}},D_{\mathcal{H}},G_{\mathcal{H}}$ as in (7)-(10).

In what follows for system (2), (7)-(10) we consider UGpAS with respect to the set

$$\mathcal{A}:=\{\xi\in C_{\mathcal{H}}\cup D_{\mathcal{H}}|x=0\}.\tag{12}$$

III. Main results

A. General stability conditions

For simplicity in the presentation throughout this paper , the following notation is introduced:

$$\eta^{i,j}=\begin{cases}\overline{\delta} & \text{if}\quad i\neq j\\ \underline{\tau} & \text{if}\quad i=j\end{cases}$$

Theorem 2: Consider the hybrid system (2), (7)-(10). If there exist continuous functions $V^{i,j}(x,\tau),V^{i}(x,\tau),i,j\in\mathcal{D}$ and class $\mathcal{K}_{\infty}$ functions $\underline{\sigma}_{i,j},\overline{\sigma}_{i,j}(|x|),\underline{\alpha}_{i},\overline{\alpha}_{i},\rho(|x|)$ such that the following inequalities are verified

$$\forall i,j\in\mathcal{D}$$
$$\forall\tau\in[0,\eta^{i,j}],\ \underline{\sigma}_{i,j}(|x|)\leq V^{i,j}(x,\tau)\leq\overline{\sigma}_{i,j}(|x|)\tag{13a}$$
$$\forall\tau\in[\underline{\tau},\infty),\ \underline{\alpha}_{i}(|x|)\leq V^{i}(x,\tau)\leq\overline{\alpha}_{i}(|x|)\tag{13b}$$

[3]A similar construction was used in [2] for hybrid observer synthesis

$$\forall i,j \in \mathcal{D}$$

$$\forall \tau \in [0,\eta^{i,j}],\ \langle \frac{\partial V^{i,j}}{\partial x}, A_{i,j}x\rangle + \frac{\partial V^{i,j}}{\partial \tau} \leq -\rho(|x|) \quad (13c)$$

$$\forall \tau \in [\underline{\tau},+\infty),\ \langle \frac{\partial V^{i}}{\partial x}, A_{i,i}x\rangle + \frac{\partial V^{i}}{\partial \tau} \leq -\rho(|x|) \quad (13d)$$

$$\forall i \neq j \in \mathcal{D}$$

$$\forall \tau \in [0,\delta],\ V^{i,i}(x,\tau) - V^{i,j}(x,\tau) \leq -\rho(|x|) \quad (13e)$$

$$V^{i}(x,\underline{\tau}) - V^{i,i}(x,\underline{\tau}) \leq -\rho(|x|) \quad (13f)$$

$$\forall \tau \in [\underline{\tau},+\infty),\ V^{i,j}(x,\tau) - V^{j}(x,\tau) \leq -\rho(|x|) \quad (13g)$$

then the set $\mathcal{A}$ defined in (12) is UGpAS and solution of (2), (7)-(10) does not posses zeno solutions.

Proof: Here, we show that the set $\mathcal{A}$ is UGpAS using the Theorem 1.

Define a hybrid Lyapunov candidate function

$$\mathcal{V}(\xi) : \begin{cases} V^{\sigma,\hat{\sigma}}(x,\tau), \text{ if } \sigma \neq \hat{\sigma}, s = 1 \\ V^{\sigma,\sigma}(x,\tau), \text{ if } \sigma = \hat{\sigma}, s = 0 \\ V^{\sigma}(x,\tau), \text{ if } \sigma = \hat{\sigma}, s = 1 \end{cases} \quad (14)$$

First $\mathcal{V}$ is a candidate hybrid Lyapunov function in the sense of Definition 8 since $\overline{C_{\mathcal{H}}} \cup D_{\mathcal{H}} \cup G_{\mathcal{H}}(D_{\mathcal{H}}) \subset dom\,\mathcal{V}$ and $\mathcal{V}$ is continuously differentiable on an open set containing $\overline{C_{\mathcal{H}}}$.

(ii) Positive definiteness of the hybrid Lyapunov function: By definition of $\mathcal{V}(\xi)$, by virtue of (3a-b) one has $\forall \xi \in C_{\mathcal{H}} \cup D_{\mathcal{H}} \cup G_{\mathcal{H}}(D_{\mathcal{H}})$

$$\min_{i,j} \{\underline{\sigma}_{i,j}(|\xi|_{\mathcal{A}}), \underline{\sigma}_{i}(|\xi|_{\mathcal{A}})\} \leq \mathcal{V}(\xi)$$

$$\mathcal{V}(\xi) \leq \max_{i,j} \{\overline{\sigma}_{i,j}(|\xi|_{\mathcal{A}}), \overline{\sigma}_{i}(|\xi|_{\mathcal{A}})\}.$$

Therefore (3a) is satisfied.

(ii) Conditions during flow ($\xi \in C_{\mathcal{H}}$):
We will distinguish between 3 cases:

Case 1: $\xi \in C_{\{i\neq j\}}$ for any $\sigma = i \neq j = \hat{\sigma}$
$\langle \nabla\mathcal{V}(\xi), f\rangle = \langle \frac{\partial V^{i,j}}{\partial x}, A_{i,j}x\rangle + \frac{\partial V^{i,j}}{\partial \tau}$ with furthermore by definition of $C_{\{i\neq j\}}$, $\tau \in [0,\overline{\delta}]$. In virtue of (13c), $\forall \xi \in C_{\{i\neq j\}}$ it holds that $\langle \nabla\mathcal{V}(\xi), f\rangle \leq -\rho(|\xi|_{\mathcal{A}})$.

Case 2: $\xi \in C^0_{\{i\}}$ for any $\sigma = i = \hat{\sigma}, s = 0$ $\langle \nabla\mathcal{V}(\xi), f\rangle = \langle \frac{\partial V^{i,i}}{\partial x}, A_{i,j}x\rangle + \frac{\partial V^{i,i}}{\partial \tau}$

By definition of $C^0_{\{i\}}$, $\tau \in [0,\underline{\tau}]$. In virtue of (13c), $\forall \xi \in C^0_{\{i\}}$ it holds that $\langle \nabla\mathcal{V}(\xi), f\rangle \leq -\rho(|\xi|_{\mathcal{A}})$.

Case 3: $\xi \in C^1_{\{i\}}$ for any $\sigma = i = \hat{\sigma}, s = 1$
$\langle \nabla\mathcal{V}(\xi), f\rangle = \langle \frac{\partial V^{i}}{\partial x}, A_{i,j}x\rangle + \frac{\partial V^{i}}{\partial \tau}$

By definition of $C^1_{\{i\}}$, $\tau \in [\underline{\tau},+\infty)$. In virtue of (13d), $\forall \xi \in C^0_{\{i\}}$ it holds that $\langle \nabla\mathcal{V}(\xi), f\rangle \leq -\rho(|\xi|_{\mathcal{A}})$.

(iii) Conditions during jump ($\xi \in D_{\mathcal{H}}$):
Again, we will distinguish between 3 cases:

Case 1: $\xi \in D_{\{i\neq j\}}$ for any $\sigma = i \neq j = \hat{\sigma}$

$$\forall \xi \in D_{\{i\neq j\}}, g \in G_{\mathcal{H}}(\xi), \mathcal{V}(g) - \mathcal{V}(\xi) = V^{i,i}(x,\tau) - V^{i,j}(x,\tau).$$

With furthermore by definition of $D_{\{i\neq j\}}$, $\tau \in [0,\overline{\delta}]$. In virtue of (13e), $\forall \xi \in D_{\{i\neq j\}}$ it holds that

$$\forall \xi \in D_{\{i\neq j\}}, g \in G_{\mathcal{H}}(\xi), \mathcal{V}(g) - \mathcal{V}(\xi) \leq -\rho(|\xi|_{\mathcal{A}})$$

Case 2: $\xi \in D^0_{\{i\}}$ for any $\sigma = i, s = 0$

$$\forall \xi \in D^0_{\{i\}}, g \in G_{\mathcal{H}}(\xi), \mathcal{V}(g) - \mathcal{V}(\xi) = V^{i}(x,\tau) - V^{i,i}(x,\tau).$$

With furthermore by definition of $D^0_{\{i\}}$, $\tau = \underline{\tau}$. In virtue of (13f), $\forall \xi \in D^0_{\{i\}}$ it holds that

$$\forall \xi \in D^0_{\{i\}}, g \in G_{\mathcal{H}}(\xi), \mathcal{V}(g) - \mathcal{V}(\xi) \leq -\rho(|\xi|_{\mathcal{A}})$$

Case 3: $\xi \in D^1_{\{i\}}$ for any $\sigma = i, s = 1$

$$\forall \xi \in D^1_{\{i\}}, g \in G_{\mathcal{H}}(\xi), \mathcal{V}(g) - \mathcal{V}(\xi) = V^{i,j}(x,\tau) - V^{j}(x,\tau).$$

for some $\tau \in [\underline{\tau},+\infty), \sigma = j, \hat{\sigma} = i$ In virtue of (13g), it holds that

$$\forall \xi \in D^1_{\{i\}}, g \in G_{\mathcal{H}}(\xi), \mathcal{V}(g) - \mathcal{V}(\xi) \leq -\rho(|\xi|_{\mathcal{A}})$$

All the condition of Theorem 3 are verified thus the set $\mathcal{A}$ is UGpAS. ∎

This theorem can be seen as a reformulation of (2) for system (2), (7), (10). However the proposed theorem does not allow for numerically tractable conditions to ensure stability of system (1). In what follows two specific structure will be imposed on the functions $V^{i,j}, V^{i}$ in order to derive tractable conditions for stability analysis.

B. Sum of Squares Polynomial

In what follows we will consider $V^{i,j}, V^{i}$ to be sum of square polynomial. The growing interest for investigating this class of Lyapunov function stems from that fact the while it is in general difficult to test for positive definitness of a given polynomial, it is however possible to recast the problem of sum of square decomposition as a convex optimization problem (see for example [11]). A polynomial $p(x) := p(x_1, \cdots, x_n)$ is a sum of squares (later named SOS) if

$$p(x) = \sum_{i}^{M} f_i(x)^2$$

for a finite M and f_i polynomials.

Theorem 3: Consider the hybrid system (2), (7)-(10). If there exist SOS-polynomials $V^{i,j}(x,\tau), V^{i}(x), a^{i,j}(x,\tau), a^{i}(x,\tau), a^{i,j}_{\mathcal{F}}(x,\tau), a^{i}_{\mathcal{F}}(x,\tau), a^{i,j}_{\mathcal{J}}(x,\tau), a^{i}_{\mathcal{J}}(x,\tau), \forall i,j \in \mathcal{D}$ and $\varepsilon > 0$ such that the sum of squares constraint given in (15) are verified

$$\forall i,j \in \mathcal{D}$$

$$V^{i,j}(x,\tau) - a^{i,j}(x,\tau)g_{\eta^{i,j}}(\tau) - \varepsilon|x|^2 \text{ is SOS} \quad (15a)$$

$$V^{i}(x) - \varepsilon|x|^2 \text{ is SOS} \quad (15b)$$

$$-\langle \frac{\partial V^{i,j}}{\partial x}, A_{i,j}x\rangle - \frac{\partial V^{i,j}}{\partial \tau} - a^{i,j}_{\mathcal{F}}(x,\tau)g_{\eta^{i,j}}(\tau) - \varepsilon|x|^2 \text{ is SOS} \quad (15c)$$

$$-\langle \frac{\partial V^{i}}{\partial x}, A_{i,i}x\rangle - \varepsilon|x|^2 \text{ is SOS} \quad (15d)$$

$$\forall i \neq j \in \mathcal{D}$$

$$V^{i,j}(x,\tau) - V^{i,i}(x,\tau) - a^{i,j}_{\mathcal{J}}(x,\tau) g_{\overline{\delta}}(\tau) - \varepsilon|x|^2 \text{ is SOS} \quad (15e)$$

$$V^{i,i}(x,\underline{\tau}) - V^{i}(x) - \varepsilon|x|^2 \text{ is SOS} \quad (15f)$$

$$V^{j}(x,\tau) - V^{i,j}(x,\tau) - a^{j}_{\mathcal{F}}(x,\tau)\tau - \varepsilon|x|^2 \text{ is SOS} \quad (15g)$$

Where $g_{\eta^{i,j}}(\tau) := \tau(\eta^{i,j} - \tau)$. Then the set $\mathcal{A}$ defined in (12) is UGpAS.

Proof: Note that in virtue of (15a) $\forall i,j \in \mathcal{D}$, $\tau \in [0, \eta^{i,j}]$

$$\varepsilon|x|^2 \leq V^{i,j}(x,\tau)$$

$$\leq \max_{\tau \in [0\eta^{i,j}]} \{V^{i,j}(x,\tau) + a^{i,j}(x,\tau) g_{\eta^{i,j}}(\tau)\} \leq \gamma(x)$$

where $\gamma(x)$ is a class $\mathcal{K}$ function

Note that in virtue of (15b), $\forall \tau \in [\underline{\tau}, \infty]$

$$\varepsilon|x|^2 \leq V^{i}(x) \leq \gamma(x)$$

where $\gamma(x)$ is a class $\mathcal{K}$ function

From equations (15c), (15d) it is clear that the conditions (3c), (3d) are verified. Furthermore, from (15e), (15f), (15g) it is clear that the conditions (3e), (3f), (3g) are verified with $\rho(x) = \varepsilon|x|^2$.

■

Next we will give another sufficient stability result in order to recast the stability analysis as an LMI feasibility problem:

Theorem 4: Consider the hybrid system (2), (7)-(10) and assume that $\forall i, i \in \mathcal{D}$, there exists $P_0^{i,j} \in \mathbb{S}^+$, $P_1^{i,j} \in \mathbb{S}$ and $P_i \in \mathbb{S}^+$ such that the following inequalities are verified:

$$\forall i,j \in \mathcal{D},\ P_0^{i,j} + \eta^{i,j} P_1^{i,j} > 0 \quad (16)$$

$$\forall i,j \in \mathcal{D}, \begin{cases} He\Big(A_{i,j}^\top P_0^{i,j}\Big) + P_1^{i,j} < 0 \\ He\Big(A_{i,j}^\top \left(P_0^{i,j} + \eta^{i,j} P_1^{i,j}\right)\Big) + P_1^{i,j} < 0 \\ He\Big(A_{i,i}^\top P_i\Big) < 0 \end{cases} \quad (17)$$

$$\forall i,j \in \mathcal{D}, \quad \begin{cases} P_0^{i,i} - P_0^{i,j} < 0 \\ P_0^{i,i} - P_0^{i,j} + \delta\left(P_1^{i,i} - P_1^{i,j}\right) < 0 \end{cases} \quad (18)$$

$$\forall i \in \mathcal{D}, \quad P_i - (P_0^{i,i} + \underline{\tau} P_1^{i,i}) < 0 \quad (19)$$

$$\forall i,j \in \mathcal{D}, \quad P_0^{i,j} - P_j < 0. \quad (20)$$

Then the set $\mathcal{A}$ defined in (12) is UGpAS and solution of (2), (7)-(10) does not posses zeno solutions.

Proof: Let's consider the continuous functions $V^{i,j}(x,\tau)$ and $V^{i}(x,\tau)$ defined by:

$$V^{i,j}(x,\tau) = x^\top \left(P_0^{i,j} + \tau P_1^{i,j}\right) x,\ V^{i}(x,\tau) = x^\top P_i\, x.$$

For every $x \in \mathbb{R}^n$, one has

$$\forall i,j \in \mathcal{D},\ \forall \tau \in [0,\eta^{i,j}], \quad \lambda_{\min}^{i,j}|x|^2 \leq V^{i,j}(x,\tau) \leq \lambda_{\max}^{i,j}|x|^2$$
$$\forall i \in \mathcal{D},\ \forall \tau \in [\underline{\tau}, +\infty), \quad \lambda_{i,\min}|x|^2 \leq V^{i}(x,\tau) \leq \lambda_{i,\max}|x|^2$$

where $\forall i,j \in \mathcal{D}$,

$$\lambda_{\min}^{i,j} = \min_{\tau \in [0,\eta^{i,j}]} \lambda_{min}\left(P_0^{i,j} + \tau P_1^{i,j}\right)$$

$$\lambda_{\max}^{i,j} = \max_{\tau \in [0,\eta^{i,j}]} \lambda_{max}\left(P_0^{i,j} + \tau P_1^{i,j}\right),$$

$$\lambda_{i,\min} = \min \lambda_{\min}(P_i) \quad ; \quad \lambda_{i,\max} = \max \lambda_{\max}(P_i)$$

Consequently, conditions (13a)–(13b) hold. Now computing the partial derivatives of $V^{i,j}$ and V^i w.r.t. x and τ, one gets:

$$\langle \frac{\partial V^{i,j}}{\partial x}, A_{i,j} x\rangle + \frac{\partial V^{i,j}}{\partial \tau} = x^\top \left(He(A_{i,j}^\top P_0^{i,j} + \tau A_{i,j}^\top P_1^{i,j}) + P_1^{i,j}\right) x$$

and $\langle \frac{\partial V^{i}}{\partial x}, A_{i,i} x\rangle + \frac{\partial V^{i}}{\partial \tau} = x^\top (He(A_{i,i}^\top P_i))\, x$, which by continuity imply the existence of $\varepsilon > 0$ such that according to (17) that

$$\forall \tau \in [0,\delta], \quad \langle \frac{\partial V^{i,j}}{\partial x}, A_{i,j} x\rangle + \frac{\partial V^{i,j}}{\partial \tau} < -\varepsilon|x|^2,$$
$$\forall \tau \in [0,\underline{\tau}], \quad \langle \frac{\partial V^{i,i}}{\partial x}, A_{i,i} x\rangle + \frac{\partial V^{i,i}}{\partial \tau} < -\varepsilon|x|^2,$$
$$\forall \tau \in [\overline{\tau},+\infty], \quad \langle \frac{\partial V^{i}}{\partial x}, A_{i,i} x\rangle + \frac{\partial V^{i}}{\partial \tau} < -\varepsilon|x|^2.$$

Finally, using (18)–(20) and the fact that $-\varepsilon|x|^2 = -\rho(|x|)$, one gets the inequalities (13e)–(13g) and this concludes the proof. ■

C. Relationship between LMIs and structural property of the switched system

Until now only sufficient conditions for stability of the original switched system have been presented. No structural relationship has been given between stability of the original system and feasibility of the proposed LMIs (and S.O.S). This aspect will be briefly discussed.

Proposition 1: Assume there exist $i^\star \in \mathcal{D}$ such that $A_{i^\star,i^\star}$ is unstable. Then the set of LMI's (16)-(20) has no solution.

Proof: Assume LMI's (16)-(20) has a solution then there exist $P_{i^\star} > 0$ such that $He\Big(A_{i^\star,i^\star} P_{i^\star}\Big) < 0$, which imply that $A_{i^\star,i^\star}$ is Hurwitz. This contradict the initial assumption. ■

Proposition 2: Assume that $\forall i \in \mathcal{D}, A_{i,i}$ is Hurwitz, then, considering Theorem 4, there exists Δ and T such that $\forall\, \delta \leq \Delta, \forall\, \underline{\tau} \geq T$ the LMIs (16)-(20) are feasible.

Proof: Since by assumption $\forall\, i \in \mathcal{D}$, $A_{i,i}$ is hurwitz, then there exist $\{P_i\}_{i\in\mathcal{D}}$ positive definite matrix such that $A_{i,i}^\top P_i + P_i\, A_{i,i} < 0$. Consider $\varepsilon > 0$ we define $P_0^{i,i} = \varepsilon P_i$ and $P_1^{i,i} = \frac{\varepsilon^2}{\underline{\tau}} P_i$. Hence by construction LMIs (19) are verified. We have therefore:

$$He\Big(A_{i,i}^\top P_0^{i,i}\Big) + P_1^{i,i} = \varepsilon\Big(He(A_{i,i}^\top P_i) + \frac{\varepsilon}{\underline{\tau}} P_i\Big),$$

$$He\Big(A_{i,i}^\top \left(P_0^{i,i} + \underline{\tau} P_1^{i,i}\right)\Big) + P_1^{i,i} = \varepsilon\Big(He(A_{i,i}^\top P_i(1+\varepsilon)) + \frac{\varepsilon}{\underline{\tau}} P_i\Big)$$

From $A_{i,i}^\top P_i + P_i\, A_{i,i} < 0$ and by continuity for ε sufficiently small:

$$He\Big(A_{i,i}^\top P_i\Big) + \frac{\varepsilon}{\underline{\tau}} P_i < 0,\ He\Big(A_{i,i}^\top P_i(1+\varepsilon)\Big) + \frac{\varepsilon}{\underline{\tau}} P_i < 0$$

Define $\forall i \neq j \in \mathcal{D}$: $P_0^{i,j} = (1+\varepsilon)P_i$, $P_1^{i,j} = -I - He\Big(A_{i,j}^\top P_0^{i,j}\Big)$ We have $He\Big(A_{i,j}^\top P_0^{i,j}\Big) + P_1^{i,j} = -I < 0$

$$He\Big(A_{i,j}^\top \left(P_0^{i,j} + \eta^{i,j} P_1^{i,j}\right)\Big) + P_1^{i,j} = He\Big(A_{i,j}^\top\, \delta\, P_1^{i,j}\Big) - I$$

By continuity there exist $\Delta > 0$ sufficiently small such that $\forall \delta < \Delta$

$$He\Big(A_{i,j}^{\top}\, \delta\, P_1^{i,j}\Big) - I < 0$$

And such that $P_0^{i,j} + \delta\Big(- I - He\Big(A_{i,j}^{\top}\, P_0^{i,j}\Big) > 0$. Hence LMIs (16), (17) are verified. Furthermore,

$$P_0^{i,i} - P_0^{i,j} = -\varepsilon^2 P_i < 0$$

Moreover

$$P_0^{i,j} - P_j = \varepsilon(1+\varepsilon)P_i - P_j$$

which implies that $\varepsilon(1+\varepsilon)P_i - P_j < 0$ for ε sufficiently small. Therefore all the LMIs of Theorem 4 are verified. ■

IV. SIMULATIONS

Consider the academic example given in [16], [17]

$$A_1 = \begin{bmatrix} 0.2 & -0.5 \\ 0.5 & -0.3 \end{bmatrix}, B_= \begin{bmatrix} -0.4 \\ 1.8 \end{bmatrix}, K_=[2.1124, -0.9336],$$

$$A_2 = \begin{bmatrix} 0.2 & 0.3 \\ -1 & 0.4 \end{bmatrix}, B_= \begin{bmatrix} 0.1 \\ 1.5 \end{bmatrix}, K_=[-1.3111, -1.3158],$$

Next we set $\underline{\tau} = 2$. Note that the matrix

$$e^{(A_1+B_1K_2)(\delta)} e^{(A_2+B_2K_2)(\tau-\delta)} e^{(A_2+B_2K_1)(\delta)} e^{(A_1+B_1K_1)(\tau-\delta)} \tag{21}$$

is not Hurwitz for $\delta \geq 0.7$

Using the conditions given in Theorem 4 one can ensure stability for $\delta \leq 0.34$, while using the conditions given in Theorem 3 one can ensure stability for $\delta \leq 0.68$. Table I illustrates the effectiveness of our approach. Note that unlike [16] and [17] in this paper we are only considering stability analysis and not control gain computation. Furthermore in [16] average dwell time switching is considered which is a (strictly) bigger class than the minimal switching that is considered here. Only considering the problem of stability analysis one finds out that in [16], [17] for fixed $(\delta, \underline{\tau})$ one is required to solve product of LMI's and unknown parameter (i.e. a BMI) this task can be accomplished performing a line search on the unknown parameter.

V. CONCLUSION

In this paper, switched system with minimal dwell time and active-mode estimation delay are reformulated as hybrid system. Using this hybrid modelling, first general conditions for stability are given. Then, this result is used to provide tractable conditions for stability analysis using sum of squares polynomials (SOSs) and Linear Matrix inequalities (LMIs). Then relations between Linear Matrix inequalities feasibility and structural properties of the systems are given. The effectiveness of the method is illustrated on a simulation.

	[16]	[17]	Th 4	Th 3	(21)
δ	0.35	0.6	0.34	0.68	0.7
Method	BMI's	BMI's	LMI's	SOS's	

TABLE I

MAXIMAL TIME δ OF ACTIVE MODE ESTIMATION DELAY FOR DWELL TIME $\underline{\tau} = 2$.

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