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Distributed Evolutionary Games Reaching Power Indexes: Navigability in a Social Network of Smart Objects

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Abstract— **In the context of coalitional games, a power index allows to determine the magnitude of contributions associated to each player, i.e., a power index provides information about how influential or relevant a player is in a cooperative interaction. Nevertheless, if the number of involved players is big, then the computation of a power index might become intractable. In this paper, we show how to construct a full-potential game whose Nash equilibrium coincides with the Shapley or Banzhaf power index for a family of characteristic functions. Therefore, distributed non-cooperative algorithms can be used for cooperative-game purposes. As a consequence, both the computational time and the information requirements are reduced, allowing the use of power indexes in large-scale systems. As an illustrative example, we present a large-scale social network of smart objects where it is desired to enhance the navigability by means of local decisions.**

Index Terms— **Power indexes, Nash equilibrium, distributed algorithms, large-scale systems**

I. Introduction

The concept of power indexes in the context of coalitional/cooperative games allows to establish the magnitude of players' contributions to the possible coalitions. Therefore, the power indexes provide information about how influential or relevant players are in a cooperative interaction. This concept is quite important in many engineering applications, specially in those scenarios where the number of involved players is big. However, the computation of the power indexes is normally associated to a high computational burden and also with centralized communication requirements, becoming an inconvenient for distributed applications. This issue has been coped with algorithms that compute an estimation of the power index. Some works have proposed to use power indexes in the engineering field. For instance, in [1], it has been shown that the computational burden of the computation of the Shapley power index can be considerably reduced when the coalition's costs are given by an average, and a water system application has been presented. In [2], a distributed bargain protocol with random characteristic function, which allows to obtain a solution that lies in the core, has been discussed. Other works propose the use of an approximated power index as in [3], where a distribution payment mechanism is designed by means of the Shapley power index. Moreover, some control applications have been developed by using a coalitional-game approach, e.g., in [4] and [5], distributed model predictive controllers are designed by associating a power index with communication links, and by using the Shapley and Banzhaf power index, respectively.

Therefore, there is still a relevant interest on applying the concepts of coalitional games in the control design because of the power indexes' properties. Then, it is required the development of novel distributed algorithms able to compute power indexes within short time. Hence, there are still several approaches that have been studied in other fields and that may be applied in the engineering field, e.g., the Banzhaf-Owen coalitional value [6], or the variations on the Shapley value [7].

The contribution of this paper is threefold. As a first contribution, we extend the result regarding the Shapley power index presented in [1] for a family of characteristic functions, which also work for the problem reported in [8], [1] or in [3], among others. Moreover, an extension to the Banzhaf power index is also presented. As a second contribution, we design a full-potential population game such that its Nash equilibrium corresponds to a desired power index, i.e., either the Shapley or the Banzhaf power index. Therefore, we show that these power indexes can be computed in a completely distributed fashion by seeking the aforementioned corresponding Nash equilibrium. It is worth to clarify that the distributed approach presented in this paper is different from the one presented in [1], where it is proposed to capture the required information in a distributed fashion to compute, afterwards, the Shapley power index. As a third contribution, we discuss the case study presented in [8], where an approximation of the Shapley power index is used in order to enhance the navigability in social networks. Different from the work in [8], we compute the exact Shapley power index within reduced time and under a distributed structure. Indeed, we consider a large-scale scenario.

This paper is organized as follows. Section II presents the problem statement. Section III introduces preliminary concepts of coalitional games and presents one of the contributions of the paper consisting in the computation of both the Shapley and Banzhaf power indexes. Section IV shows the design of a non-cooperative game by using the result presented in Section III. Over the end of this paper, Section V presents three different scenarios for a large-scale social network of smart objects case study, involving a large number of players in the game, i.e., 300, 400 and 500 players. Finally, concluding remarks are drawn in Section VI.

Notation: Let $\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$ denote the set of real, positive real, and non-negative real numbers, respectively. Similarly, $\mathbb{Z}_{>0}$ denotes the set of positive integer numbers. Scalar variables are denoted with an associated sub-index, e.g.,

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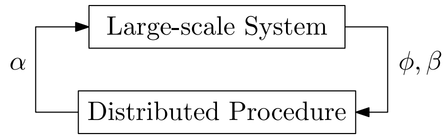

Fig. 1. Closed-loop illustrating a distributed procedure that affects the power index, either the Shapley power index ϕ or the Banzhaf power index β, by applying an action denoted by α.

$x_i \in \mathbb{R}$, whereas vectors are denoted without sub-index, e.g., $x \in \mathbb{R}^n$. Moreover, $\mathbb{1}_n$ denotes the vector with n unitary entries, i.e., $\mathbb{1}_n = [1 \quad \dots \quad 1]^\top \in \mathbb{R}^n$. The cardinality of a set $\mathcal{S}$ is denoted by $|\mathcal{S}|$, and we also use the operator $[\cdot]_+ = \max(0, \cdot)$. Throughout the paper, we use the symbol ϕ (β) to associate variables with the Shapley (Banzhaf) power index, e.g., the variable θ^ϕ (θ^β) is associated to the Shapley (Banzhaf) power index.

II. Problem Statement

In a coalitional game, where there are several players interacting to each other, the power indexes allow to determine the magnitude with which a participant is contributing to coalitions, e.g., the Shapley power index ϕ introduced in [9], or the Banzhaf power index β presented in [10]. In other words, the power indexes can provide information about the most influential or relevant participant, or the player that is mostly contributing to the benefit production in the coalitions. In the engineering context, let us consider a large-scale system composed by a finite number of components and/or sub-systems. Moreover, there exists a distributed communication network that constraints how the aforementioned components and/or sub-systems can interact to each other, i.e., each component/sub-system has only partial/limited information about the entire system.

Let us suppose that the power index is associated to the relevance of each component to make the entire system perform as desired. In this regard, it is possible to identify the most critical component for the appropriate operation of the system. Hence, if the suitable operation of the system relies on a unique component (or on a reduced number of components), then the system might be more susceptible to suffer from issues such as faults and/or attacks, i.e., the desired performance of the system depends on few components of the network. An alternative option to avoid this issue is to design systems in which there are equitable responsibilities, i.e., a scenario in which all the components are equally powerful. It is desired that all the components have the same relevance such that the entire system becomes less susceptible to any issue, e.g., an attack. Notice that it should be taken into account that there is a distributed communication network impeding that the power index could be computed in a centralized manner. The objective is to design a distributed procedure that allows to modify the power indexes by applying an action denoted by α.

Figure 1 shows the desired closed-loop in which a distributed procedure is performed in order to manipulate the power indexes. It is quite relevant to emphasize four features:

1) the system is of large-scale nature, i.e., the corresponding coalitional game involves a large number of players,
2) it is required to compute the power indexes (either ϕ or β) within a short time,
3) there is not complete available information, i.e., there is not a central entity with knowledge about the entire system neither the corresponding values/costs associated to the players,
4) any action α that is applied to the system should depend only on partial and local interactions.

III. Power Indexes Preliminaries and Analytical Results

A cooperative game (also known as coalitional game) with transferable utilities is given by a situation where there is a finite number of players, which can obtain a certain payoff by cooperating to each other. The cooperative game is denoted by $G(\mathcal{P}, v)$, where $\mathcal{P} = \{1, \dots, n\}$ is the set of finite players, and $v : 2^{\mathcal{P}} \to \mathbb{R}$, with $v(\emptyset) = 0$ and $2^{\mathcal{P}}$ being the power set of $\mathcal{P}$ (the set of all subset of $\mathcal{P}$ including the empty set), is a characteristic function corresponding to a payoff that the members of the coalition, any subset $\mathcal{C} \subseteq \mathcal{P}$, can distribute among themselves. Therefore, $v(\mathcal{C})$ denotes the payoff that the coalition $\mathcal{C} \subseteq \mathcal{P}$ can achieve. Based on the individual contributions to coalitions, the objective is to determine a *fair* distribution of payoffs for the players. For instance, a possible distribution can be computed by considering the following axioms. A payoff $y = [y_1 \quad \dots \quad y_n]^\top$ satisfies group rationality (or efficiency) if $y^\top \mathbb{1}_n = v(\mathcal{P})$, and the payoff satisfies individual rationality if $y_i \geq v(\{i\})$, for all $i \in \mathcal{P}$, e.g., the Shapley power index considers these axioms. Throughout this paper, we present results regarding both the Shapley and Banzhaf power indexes.

A. Shapley Power Index

The Shapley power index of the player $i \in \mathcal{P}$ is given by [9]

$$\phi_i(\mathcal{P}, v) = \sum_{\mathcal{C} \subseteq \mathcal{P} \setminus \{i\}} h(\mathcal{C}) \left(v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \right), \qquad (1)$$
$$h(\mathcal{C}) = \frac{|\mathcal{C}|!(n - |\mathcal{C}| - 1)!}{n!}.$$

Therefore, the Shapley power index of the game $G = (\mathcal{P}, v)$ is given by $\phi(\mathcal{P}, v) = [\phi_1(\mathcal{P}, v) \quad \dots \quad \phi_n(\mathcal{P}, v)]^\top$. In addition, we also present the normalized Shapley power index as

$$\bar{\phi}(\mathcal{P}, v) = \frac{\phi(\mathcal{P}, v)}{\mathbb{1}_n^\top \phi(\mathcal{P}, v)},$$

such that $\mathbb{1}_n^\top \bar{\phi}(\mathcal{P}, v) = 1$. Therefore, the magnitude order is preserved with the normalized Shapley power index, i.e., if $\phi_i > \phi_j$, then $\bar{\phi}_i > \bar{\phi}_j$. It is relevant to point out that the computation of the Shapley power index requires complete information since it is necessary to evaluate the marginal contributions of each player to each possible coalition to which it can become member. Consequently, besides the computational burden issue, the information requirements are also high. Next, in Proposition 1, we show that the computational burden can be considerably reduced for a family of characteristic functions.

Proposition 1: Let the characteristic function be of the form $v(\mathcal{C}) = a + b(|\mathcal{C}|)\sum_{j\in\mathcal{C}} c(\{j\})$ (with $v(\emptyset) = 0$), where the parameter $a \in \mathbb{R}$, the function $b : \mathbb{Z}_{>0} \to \mathbb{R}$, and $c(\{i\})$ corresponds to the individual cost of the player $i \in \mathcal{P}$. Hence, let

$$\theta^{\phi}(n) = \sum_{\ell=0}^{n-2} \left\{ \frac{(n-2)!}{\ell!(n-2-\ell)!} \psi^{\phi}(\ell) b(\ell+1) \right\}, \qquad (2)$$
$$\psi^{\phi}(\ell) = \frac{(\ell!(n-\ell-1) + (\ell+1)!(n-\ell-2)!)}{(n!)}.$$

Then, the Shapley power index relationship among players is given by

$$\phi_i(\mathcal{P}, v) = \phi_j(\mathcal{P}, v) + (c(\{i\}) - c(\{j\}))\theta^{\phi}(n), \qquad (3)$$

for all the players $i, j \in \mathcal{P}$. ♢

Proof: The proof of this proposition is made following the same reasoning as in [1, Proposition 3], where the Shapley power index for a unique characteristic function has been discussed. Consider two arbitrary players $i, j \in \mathcal{P}$. Then, the corresponding Shapley power index is

$$\begin{aligned}\phi_i(\mathcal{P}, v) = &\sum_{\mathcal{C}\subseteq\mathcal{P}\setminus\{i,j\}} h(\mathcal{C})\,(v(\mathcal{C}\cup\{i\}) - v(\mathcal{C})) \\ &+ \sum_{\mathcal{C}\subseteq\mathcal{P}\setminus\{i,j\}} h(\mathcal{C}\cup\{j\})\,(v(\mathcal{C}\cup\{i,j\}) - v(\mathcal{C}\cup\{j\})),\end{aligned}$$

considering the variable $\psi^{\phi}(|\mathcal{C}|) = h(\mathcal{C}) + h(\mathcal{C}\cup\{i\})$, and the difference $\phi_{i-j}(\mathcal{P}, v) = \phi_i(\mathcal{P}, v) - \phi_j(\mathcal{P}, v)$

$$\phi_{i-j}(\mathcal{P}, v) = \sum_{\mathcal{C}\subseteq\mathcal{P}\setminus\{i,j\}} \psi^{\phi}(|\mathcal{C}|)\,(v(\mathcal{C}\cup\{i\}) - v(\mathcal{C}\cup\{j\})). \qquad (4)$$

Notice that

$$v(\mathcal{C}\cup\{i\}) = \left(a + b(|\mathcal{C}|+1)\left(\sum_{\ell\in\mathcal{C}} c(\{\ell\}) + c(\{i\})\right)\right).$$

Therefore, the difference of the Shapley power indexes in (4) is

$$\phi_{i-j}(\mathcal{P}, v) = (c(\{i\}) - c(\{j\})) \underbrace{\sum_{\mathcal{C}\subseteq\mathcal{P}\setminus\{i,j\}} \psi^{\phi}(|\mathcal{C}|)\, b(|\mathcal{C}|+1)}_{\theta^{\phi}(n)},$$

where $\theta^{\phi}(n)$ is the one presented in (2), obtaining the relation presented in (3). ■

In addition to the result in Proposition 1, we point out in Corollary 1 the monotone behavior of the Shapley power indexes with respect to the costs in the coalitional game. This fact becomes important for some engineering applications in which it is only needed to organize the Shapley power indexes in an incremental/decremental order, since it might be verified using directly the costs, e.g., see the application in [8].

Corollary 1: The Shapley power index $\phi_i(\mathcal{P}, v) > \phi_j(\mathcal{P}, v)$ if and only if $c(\{i\}) > c(\{j\})$ for positive $\theta^{\phi}(n)$ in (2), and $\phi_i(\mathcal{P}, v) < \phi_j(\mathcal{P}, v)$ in case $\theta^{\phi}(n)$ is negative.

Proof: This fact immediately follows from (3). ■

Alternatively, there are other concepts of power indexes [11], which can be computed within reduced computational time and under distributed information structures, e.g., the Banzhaf power index. Therefore, in the following section we briefly discuss the computation of the Banzhaf power index.

B. Banzhaf Power Index

The Banzhaf power index for the $i \in \mathcal{P}$ player is given by [10]

$$\beta_i(\mathcal{P}, v) = \frac{1}{2^{n-1}} \sum_{\mathcal{C}\subseteq\mathcal{P}\setminus\{i\}} (v(\mathcal{C}\cup\{i\}) - v(\mathcal{C})). \qquad (5)$$

Therefore, the Banzhaf power index of the game $G = (\mathcal{P}, v)$ is given by $\beta(\mathcal{P}, v) = [\beta_1(\mathcal{P}, v) \quad \ldots \quad \beta_n(\mathcal{P}, v)]^{\top}$. In addition, we also present the normalized Banzhaf power index as

$$\bar{\beta}(\mathcal{P}, v) = \frac{\beta(\mathcal{P}, v)}{\mathbb{1}_n^{\top}\beta(\mathcal{P}, v)},$$

such that $\mathbb{1}_n^{\top}\bar{\beta}(\mathcal{P}, v) = 1$. The relevance order is preserved with the normalized Shapley power index, i.e., if $\beta_i > \beta_j$, then $\bar{\beta}_i > \bar{\beta}_j$.

Likewise the computation of the Shapley power index in (1), the Banzhaf computation in (5) requires complete information in order to evaluate all the possible coalition to which each player can join. Indeed, the computational burden is also high with a large number of players. Then, Corollary 2 below shows an alternative computation for the Banzhaf power index.

Corollary 2: Let the characteristic function be of the form $v(\mathcal{C}) = a + b(|\mathcal{C}|)\sum_{j\in\mathcal{C}} c(\{j\})$ (with $v(\emptyset) = 0$), where the parameter $a \in \mathbb{R}$, the function $b : \mathbb{Z}_{>0} \to \mathbb{R}$, and $c(\{i\})$ corresponds to the individual cost of the player $i \in \mathcal{P}$. Hence, let

$$\theta^{\beta}(n) = \sum_{\ell=0}^{n-2} \left\{ \frac{(n-2)!}{\ell!(n-2-\ell)!} \psi^{\beta}(\ell) b(\ell+1) \right\}, \qquad (6)$$
$$\psi^{\beta}(n) = \frac{2}{2^{n-1}}.$$

Then, the Banzhaf power index relationship among players is given by

$$\beta_i(\mathcal{P}, v) = \beta_j(\mathcal{P}, v) + (c(\{i\}) - c(\{j\}))\theta^{\beta}(n), \qquad (7)$$

for all the players $i, j \in \mathcal{P}$. ♢

Proof: Consider two arbitrary players $i, j \in \mathcal{P}$. Then, following the same reasoning as in Proposition 1, the corresponding Banzhaf power index is

$$\begin{aligned}\beta_i(\mathcal{P}, v) = &\sum_{\mathcal{C}\subseteq\mathcal{P}\setminus\{i,j\}} \frac{1}{2^{n-1}}\,(v(\mathcal{C}\cup\{i\}) - v(\mathcal{C})) \\ &+ \sum_{\mathcal{C}\subseteq\mathcal{P}\setminus\{i,j\}} \frac{1}{2^{n-1}}\,(v(\mathcal{C}\cup\{i,j\}) - v(\mathcal{C}\cup\{j\})),\end{aligned}$$

it follows that the difference of the Banzhaf power indexes $\beta_{i-j}(\mathcal{P},v)=\beta_i(\mathcal{P},v)-\beta_j(\mathcal{P},v)$ is

$$\beta_{i-j}(\mathcal{P},v)=\sum_{\mathcal{C}\subseteq\mathcal{P}\setminus\{i,j\}}\psi^\beta(n)\left(v(\mathcal{C}\cup\{i\})-v(\mathcal{C}\cup\{j\})\right),\quad (8)$$

where $\psi^\beta(n)=\frac{2}{2^{n-1}}$. Therefore, replacing the characteristic function yields

$$\beta_{i-j}(\mathcal{P},v)=(c(\{i\})-c(\{j\}))\underbrace{\sum_{\mathcal{C}\subseteq\mathcal{P}\setminus\{i,j\}}\psi^\beta(n)\,b(|\mathcal{C}|+1)}_{\theta^\beta(n)},$$

where $\theta^\beta(n)$ is the one presented in (6), obtaining the desired relation presented in (7). ■

Similar to the Shapley power index (Corollary 1), Corollary 3 shows the monotone behavior of the Banzhaf power indexes with respect to the costs.

Corollary 3: The Banzhaf power index $\beta_i(\mathcal{P},v)>\beta_j(\mathcal{P},v)$ if and only if $c(\{i\})>c(\{j\})$ for positive $\theta^\beta(n)$ in (6)*, and $\beta_i(\mathcal{P},v)<\beta_j(\mathcal{P},v)$ in case $\theta^\beta(n)$ is negative.*

Proof: This fact immediately follows from (7). ■

IV. Evolutionary Game Dynamics

Once the coalitional-game approach has been introduced, we present the preliminary concepts corresponding to population games. Then, we discuss the design of full-potential games such that a power index can be computed in a distributed fashion by means of a Nash equilibrium

A. Preliminary Concepts

Consider a population where $\mathcal{S}=\{1,\ldots,n\}$ is the set of the n available strategies. Let $x_i\in[0,m]$ be the portion of agents selecting the i^{th} strategy from $\mathcal{S}$, and $m\in\mathbb{R}_{>0}$ be the population mass. In addition, let $x=[x_1\quad\ldots\quad x_n]^\top$ be the population state, and $\Delta=\{x\in\mathbb{R}^n_{\geq 0}:x^\top\mathbb{1}_n=m\}$ be the simplex set representing all the admissible population states. Agents make decisions in order to improve their benefits, which are given by a fitness function $f_i:\Delta\to\mathbb{R}$. Therefore, the population fitness function is denoted by $f=[f_1\quad\ldots\quad f_n]^\top$. The objective in the population is to achieve a Nash equilibrium denoted by $x^\star\in\Delta$ as defined next.

Definition 1: The population state $x^\star\in\Delta$ is a Nash equilibrium if each used strategy entails the maximum benefit for the proportion who is choosing it, i.e., the set $\mathrm{NE}(f)=\{x^\star\in\Delta:\forall\, i\in\mathcal{S},\, x_i^\star>0\Rightarrow f_i(x^\star)\geq f_j(x^\star),\ \forall\, j\in\mathcal{S}\}$ *corresponds to the Nash equilibria.* ◇

Now consider that the agents' interactions within the population are constrained as in [12], [13]. The interaction constraints are described by an undirected graph $\mathcal{G}=(\mathcal{S},\mathcal{E})$, where each node is associated to a strategy and $\mathcal{E}\subseteq\{(i,j):i,j\in\mathcal{S}\}$ is the set of links representing possible interactions, i.e., if $(i,j)\in\mathcal{E}$, then agents selecting the i^{th} strategy can interact with those selecting the j^{th} strategy. Hence, the set with which the portion of agents x_i can interact is given by $\mathcal{N}_i=\{j:(i,j)\in\mathcal{E}\}$. Notice that, under this consideration, the set of Nash equilibria depend on the graph topology as defined next.

Definition 2: The population state $x^\star\in\Delta$ is a Nash equilibrium if, for the given agents interaction $\mathcal{G}=(\mathcal{S},\mathcal{E})$, each used strategy entails the maximum benefit that the proportion who is choosing it can get in its neighborhood, i.e., the set $\mathrm{NEG}(f,\mathcal{G})=\{x^\star\in\Delta:x_i^\star>0\Rightarrow f_i(x^\star)=\max_{j\in\mathcal{N}_i}f_j(x^\star),\ \forall\, i\in\mathcal{S}\}$ *corresponds to the Nash equilibria over graphs.* ◇

In addition, there is a relationship between the sets $\mathrm{NE}(f)$ and $\mathrm{NEG}(f,\mathcal{G})$ as shown in Lemma 1.

Lemma 1: If the possible interaction in the population is given by an undirected connected graph $\mathcal{G}$, then the set of equilibria $\mathrm{NE}(f)=\mathrm{NEG}(f,\mathcal{G})$. ◇

Proof: Let $\mathrm{supp}(x)=\{i\in\mathcal{S}:\ x_i>0\}$. Since $x^\star\in\Delta$, then $\mathrm{supp}(x^\star)\neq\emptyset$, i.e., there exists an $i\in\mathcal{S}$ for which $x_i^\star>0$. Now, let $f_i(x^\star)=\max_{j\in\mathcal{N}_i}f_j(x^\star)$. If $x_i^\star=m$, then $x_j^\star=0$, for all $j\in\mathcal{S}\setminus\{i\}$, and it is concluded that $x^\star\in\mathrm{NE}(f)$. If $x_i^\star<m$ then, there is another strategy $j\in\mathcal{S}$ such that $x_j^\star>0$. Moreover, let $f_j(x^\star)=\max_{a\in\mathcal{S}}f_a(x^\star)$. Notice that, since the graph $\mathcal{G}$ is connected, then there exists a path on $\mathcal{G}$ connecting i to j, i.e., a path $\tilde{\mathcal{E}}_{i\to j}\subseteq\mathcal{E}$. It follows that

$$\begin{aligned}f_i(x^\star)&=\max_{a\in\mathcal{N}_i}f_a(x^\star),\\&=\max_{a\in\mathcal{N}_i}\left(\max_{k\in\mathcal{N}_a}f_k(x^\star)\right),\\&\ \ \vdots\\&=\max_{a\in\mathcal{N}_i}\left(\max_{k\in\mathcal{N}_a}\cdots\left(\max_{\ell\in\mathcal{N}_r}f_\ell(x^\star)\right)\right),(\text{covering }\tilde{\mathcal{E}}_{i\to j}),\end{aligned}$$

then $f_i(x^\star)=\max_{a\in\mathcal{S}}f_a(x^\star)$, for which it is concluded that $x^\star\in\mathrm{NE}(f)$. ■

Remark 1: Notice that when associating the population mass m with the total payoff that the coalitional game can generate $v(\mathcal{P})$, and the scalar x_i with the Shapley power index ϕ_i, for all $i=1,\ldots n$ (since the set of strategies in the population $\mathcal{S}$, and the set of finite players in the coalitional game $\mathcal{P}$ are the equivalent), then the simplex set Δ represents the group rationality axiom, i.e., $\phi^\top\mathbb{1}_n=v(\mathcal{P})=m=x^\top\mathbb{1}_n$. ◇

The distributed population dynamics converge to a Nash equilibrium by using a distributed interaction structure [12], [13]. Any of the distributed population dynamics presented in [12] can be used, e.g., the distributed replicator dynamics, the distributed projection dynamics, or the distributed Smith dynamics, which we present next,

$$\dot{x}_i=\sum_{j\in\mathcal{N}_i}\Big(x_j[f_i(x)-f_j(x)]_+-x_i[f_j(x)-f_i(x)]_+\Big).\quad (9)$$

It has been shown in [12], that for a full-potential game f whose corresponding potential function $J(x)$ is strictly concave, then the Nash equilibrium $x^\star\in\mathrm{int}(\Delta)$ is locally asymptotically stable under the distributed Smith dynamics in (9). In the following section, we use this stability result to design a full-potential game whose Nash equilibrium corresponds to a power index in the context of coalitional games.

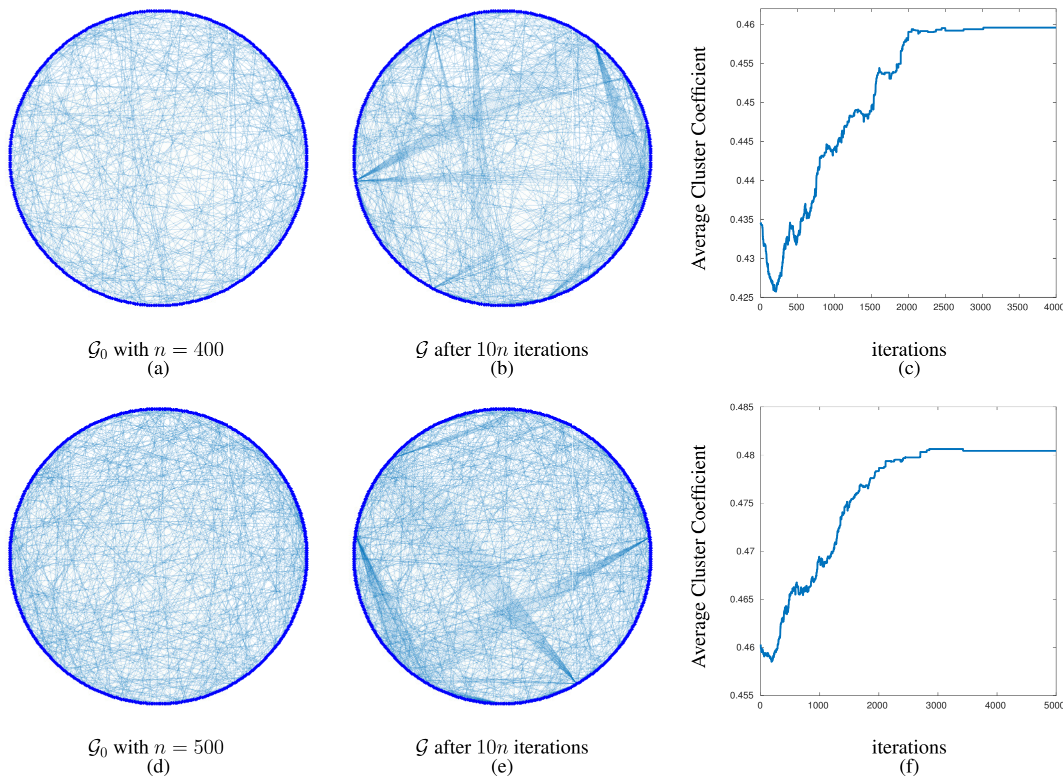

Fig. 2. Evolution of the *Average Cluster Coefficient* under the proposed approach, using fully distributed information, and maintaining constant the number of links. Sub-figures (a) and (d) represent the initial random graph for 400 and 500 players (objects). Sub-figures (b) and (e) represent the initial final graph for 400 and 500 players (objects) when adopting the proposed friendship process. Sub-figures (c) and (f) represent the evolution of the *Average Cluster Coefficient* for 400 and 500 players (objects) showing that it is improved when adopting the proposed approach.

B. Designing Games to Converge Power Indexes

In Section III, an alternative computation of power indexes has been presented. Therefore, the result obtained in Proposition 1 and Corollary 2 are exploited in order to design a non-cooperative game such that its Nash equilibrium corresponds with a power index. Thus, distributed algorithms that converge to Nash equilibria can be used to obtain a desired power index as presented in Proposition 2 below.

Proposition 2: Let $f : \Delta \to \mathbb{R}^n$ be a full-potential population game whose concave continuously differentiable potential function is given by $J : \mathbb{R}^n \to \mathbb{R}$ and whose possible interaction is given by a connected graph $\mathcal{G}$, i.e.,

$$J(x) = \sum_{i \in \mathcal{P}} J_i(x_i) = \sum_{i \in \mathcal{P}} -\frac{1}{2}x_i^2 - r(n)\, c(\{i\})x_i, \qquad (10)$$

then the Nash equilibrium $x^\star \in \mathrm{int}(\Delta)$ corresponds to the Shapley power index if $r(n) = \theta^\phi(n)$ and with population mass $m = v(\mathcal{P})$, and to the Banzhaf power index if $r(n) = \theta^\beta(n)$ and with population mass $m = \beta^\top \mathbb{1}_n$. $\diamondsuit$

Proof: Since the potential function is concave and the Nash equilibrium $x^\star \in \mathrm{int}(\Delta)$ by assumption, then

$$x^\star = \arg \max_{x \in \mathrm{int}(\Delta)} J(x).$$

Hence, since $f(x) = \nabla J(x)$, then $f(x^\star) \in \mathrm{span}\{\mathbb{1}_n\}$. It follows that $f_i(x_i) = -x_i - c(\{i\})r(n)$, for all $i \in \mathcal{P}$. Regarding the equilibrium point

$$-x_i^\star - c(\{i\})r(n) = -x_j^\star - c(\{j\})r(n), \ \forall i, j \in \mathcal{P}. \qquad (11)$$

Arranging the latter equality, it is obtained that

$$x_i^\star = x_j^\star + (c(\{i\}) - c(\{j\}))r(n),$$

which is the condition for the Shapley power index if $r(n) = \theta^\phi(n)$. In addition, notice that $x^{\star\top}\mathbb{1}_n = v(\mathcal{P})$. The same procedure can be followed with $m = \beta^\top \mathbb{1}_n$ obtaining the equivalence with the Banzhaf power index if $r(n) = \theta^\beta(n)$ and making $x^{\star\top}\mathbb{1}_n = \beta^\top \mathbb{1}_n$. ■

Indeed, if the algorithm procedure to adjust the system only requires the magnitude order of the power index values, then the selected population mass does not affect the performance of the procedure. Notice that this claim is made thanks to Corollaries 1 and 3. This is the situation in the application treated in this paper [8]. Nevertheless, in case that the exact power value in needed, players must get to know about the total utility that the cooperation is generating, but it does not imply that all players have to interact to each other.

V. Social Network of Smart Objects Case Study

In order to illustrate the performance of the proposed distributed methodology to compute both the Shapley and Banzhaf power index, we discuss the same case study presented in [8] with a different friendship selection procedure.

Let us consider a graph $\mathcal{G} = (\mathcal{P}, \mathcal{E})$ representing the topology of a social network. The set $\mathcal{P} = \{1, \ldots, n\}$

represents n smart objects and $\mathcal{E} \subseteq \{(i,j) : i,j \in \mathcal{P}\}$ represents the possible interactions among smart objects. Moreover, let $\mathcal{N}_i = \{j : (i,j) \in \mathcal{E}\}$ define the set of friends of the i^{th} smart object, and let $\mathcal{F}_i = \{j \in \cup_{\ell \in \mathcal{N}_i} \mathcal{N}_\ell\}$ be the set of possible smart objects with which the i^{th} object can create a new friendship, i.e., the i^{th} object can establish a new friendship with the object $j \in \mathcal{N}_\ell$ because there exists an object $\ell \in \mathcal{N}_i$ that can introduce it.

The objective is to design an algorithmic distributed procedure in order to enhance the navigability of the social network of smart objects by preserving the total amount of links within the network. Notice that the procedure should perform in a distributed manner, where each object $i \in \mathcal{P}$ can only interact with its neighborhood $\mathcal{N}_i$, and can create a new friendship only with those that are friends of friends $\mathcal{F}_i$. To this end, we consider the *clustering coefficient* [14] as in [8], which is a measure about the degree to which nodes the graph tends to create clusters, e.g., in social network, to which point nodes tend to construct tight groups. In addition, the *cluster coefficient* denoted by ω_i is a local measurement that is computed as follows:

$$\omega_i = \frac{2L_i}{|\mathcal{N}_i|(|\mathcal{N}_i| - 1)},\ \forall i \in \mathcal{P},$$

where $L_i \in \mathbb{Z}_{\geq 0}$ is the number of links between $\mathcal{N}_i$. Since the coefficient ω_i determines to which nodes the network tends to make groups, those nodes with small *clustering coefficient* might get isolated, which would affect the navigability of the network. In [8], it has been proposed to consider the following characteristic function:

$$v(\mathcal{C}) = \begin{cases} \left(1 - \frac{1}{|\mathcal{C}|} \sum_{j \in \mathcal{C}} \omega_j\right) & \forall \mathcal{C} \subseteq \mathcal{P}, \mathcal{C} \neq \emptyset, \\ 0 & \mathcal{C} = \emptyset. \end{cases} \tag{12}$$

Then, more importance is assigned to those players with small *clustering coefficient*. It follows that at each iteration, a player $i \in \mathcal{P}$ receives the opportunity to evaluate whether establishing a new friendship. Since it is desired to preserve the total number of links within the whole network, prior to making the new friendship, the i^{th} player has to drop an existing link. To this end, it is proposed that the i^{th} player breaks the relationship with the one with the smallest power index in the neighborhood $\mathcal{N}_i$. Hence, the i^{th} player establishes a new friendship with the one with highest power index in the set of friend candidates $\mathcal{F}_i$. Notice that the characteristic function in (12) satisfies the required form in Proposition 1 and Corollary 2, and it is possible to compute the respective power index by using (3) and (7). Hence, notice that if the procedure only requires the magnitude order of the power indexes, then it can be performed by using Corollaries 1 and 3, which is the case in [8]. In addition, the result in Proposition 2 can be applied and then the power indexes can be computed in a completely distributed fashion by using the Smith dynamics in (9). In order to evaluate the performance of the distributed algorithm, consider three Watts-Strogatz Small Worlds with $n = 300$, $n = 400$ and $n = 500$ number of nodes. Each node is connected to $\text{round}(\frac{n}{50})$ nodes, and rewiring probability of 15% [14]. These graphs are presented in Figures 2(a), 2(d), and 2(g), respectively. On the other hand, Figures 2(b), 2(e), and 2(h) show the obtained graph after having applied the proposed distributed procedure. Moreover, Figures 2(c), 2(f), and 2(i) present the evolution of the *average clustering coefficient* showing a suitable performance, since it is increased along the time until it achieves a steady state.

VI. Conclusions

We have formally introduced an alternative computation of both the Shapley and Banzhaf power indexes for a family of characteristic functions. Moreover, we have shown how to design a full-potential games whose Nash equilibrium corresponds to a desired power index. Finally, we have presented a large-scale case study in order to illustrate the suitable performance of the proposed methodology.

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