2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Modeling wireless power transfer in a network of smart devices: a compartmental system approach*

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Abstract— **Wireless power transfer technology provides a possible sustainable and cost-effective way to prolong indefinitely the lifetime of networks of smart devices needed in future Internet-of-Things, while equipping them with batteries of limited capacity. In this paper we show that the theory of compartmental systems, positive interconnected systems exchanging 'mass' (here power) and ruled by mass conservation laws, provides a suitable framework to describe wireless power transfer networks. In particular we show that sustainability of the network of smart devices corresponds to each of them being alimented, directly or indirectly, by nodes having an external source of power, condition known as inflow connectivity in the compartmental systems literature. The framework allows to compute the topology which is optimal in terms of maximizing the overall efficiency of the power transfer.**

I. Introduction

Future Internet-of-Things (IoT) scenarios are based on principles such as high density deployment of sensing, interfacing and computing devices. Ubiquitous monitoring and communication, however, requires that IoT devices are equipped with sufficient and reliable power resources, maintainable over time. In many IoT scenarios, the power constraint is currently one of the key obstacles to a massive large-scale deployment of networks of smart devices. Due to space and cost constraints, in fact, it is asked that the device batteries have a limited capacity, meaning that a reasonable operational lifespan for the network cannot be achieved without periodic replacement or recharging. Connecting all of them to the electric grid means opting for static configurations, and does not apply to portable devices or to hazardous environments. Similarly, battery replacement can be costly and unpractical (think for instance of sensors buried under the asphalt in traffic networks [3]).

In order to overcome such power constraints, a solution that has attracted attention recently is to recharge the device batteries by harvesting energy from the environment [15], [27]. Sources that can be used for this scope can vary from solar to indoor light, from wind to vibration, from thermal to ambient electromagnetic radiation. However, also equipping all nodes with energy harvesting technology could be unfeasible or excessively expensive. An alternative, or complementary, strategy is to share energy wirelessly among the devices in a controlled manner. Methods for wireless power transfer can be based on inductive coupling, electromagnetic radiation and magnetic resonant coupling [28]. Although most of these technologies are still at a research and development stage, some applications (e.g. wireless charging of mobile phones) are starting to appear in commercial products.

A typical scenario, investigated for instance in [19], [24], [29] is that some nodes are serving as wireless chargers for the other surrounding nodes. Such "power base stations" could be nodes equipped with energy harvesting technology as mentioned earlier, or nodes connected with the electric grid, or even mobile nodes (e.g. vehicles) equipped with high volume batteries and transmitters, delivering energy wirelessly to receiving nodes. Depending on the scenario, such power-supplying stations could serve all nodes requiring recharge, or it could be necessary for some of the nodes to act as energy relays, alimenting other nodes. This may happen for instance when power transfer is performed through electromagnetic radiation and not all nodes have line-of-sight to the charging stations [16].

The main contribution of this paper is to propose a system-oriented modeling framework for wireless power transfer in a network of IoT devices. Given that power is a nonnegative quantity, a natural framework is that of positive systems [8]. In particular, identifying the node batteries (where the power "accumulates") with compartments, a power transfer between a pair of nodes can be thought of as a "mass transfer" between two compartments. A suitable framework seems therefore to us that of compartmental systems [13]. Such models have been used extensively in many fields such as physiology, medicine, [2], ecology [17], queueing networks [10], traffic flow [5], etc. In the context of wireless power transfer networks, energy supplies (energy harvested from the environment or from the electric grid) represent inflows of the compartmental system, while power lost in the wireless transfer or consumed at the nodes for their routine operation (or for the energy transmission) represents outflows. The dynamical properties of compartmental models are also well-known. For instance asymptotic stability is associated to outflow connectivity, i.e., to all nodes being connected to a node with losses. Given that no energy transfer has efficiency 1, such property is normally automatically satisfied when the power transfer network is connected. A network has a prolonged lifespan if there exists a stable equilibrium point which is strictly positive, corresponding to positive charge in all batteries at steady state. In compartmental models, existence of a strictly positive equilibrium point which is a global attractor for the dynamics is associated to inflow connectivity, i.e., to all nodes being connected to nodes with

*Work supported in part by a grant from the Swedish Research Council (grant n. 2015-04390 to C.A.).

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external supply of energy. This last property gives a useful criterion for deciding the topology of the network of energy transfers. A question that is often posed in the context of wireless power sharing is in fact to design the transmission routes that lead to the most efficient overall transmission while guaranteeing energetic survival of all nodes. Several methods have appeared in the literature [6], [14], [30], [31]. We show in the paper that in our approach such question can be formulated in terms of computing a maximal directed spanning forest rooted at the nodes that are energy sources.

II. Background material

In this section we review material on positive and compartmental systems. Most of it can be found in textbooks such as [2], [8], [11] or papers such as [13].

A. Linear algebraic notions

A matrix $A \in \mathbb{R}^{n\times n}$ is said Hurwitz stable if all its eigenvalues $\lambda_i(A)$, $i = 1, \ldots, n$, have $\mathrm{Re}[\lambda_i(A)] < 0$. It is said marginally stable if $\mathrm{Re}[\lambda_i(A)] \leqslant 0$, $i = 1, \ldots, n$, and $\lambda_i(A)$ such that $\mathrm{Re}[\lambda_i(A)] = 0$ have an associated Jordan block of order one. A is said *irreducible* if there does not exist a permutation matrix Π such that $\Pi^T A \Pi$ is block triangular. Denoting $\mathcal{G}(A)$ the graph whose adjacency matrix is A, then A irreducible iff $\mathcal{G}(A)$ is strongly connected. A matrix A is said (column, omitted hereafter) *diagonally dominant* if

$$|A_{ii}| \geqslant \sum_{i\neq j} |A_{ij}|, \quad \forall\, i = 1, \ldots, n. \tag{1}$$

It is said *strictly diagonally dominant* when all inequalities of (1) are strict, and *weakly diagonally dominant* when at least one (but not all) of the inequalities (1) is strict. The following theorem characterizes nonsingularity and stability in terms of diagonal dominance.

Theorem 1 *[23] If A is strictly diagonally dominant or weakly diagonally dominant and irreducible then it is nonsingular. If in addition $a_{ii} < 0\ \forall\ i = 1, \ldots, n$, then A is Hurwitz stable.*

B. Metzler matrices and compartmental matrices

A matrix is said (elementwise) nonnegative (denoted $A \geq 0$) if all its entries are nonnegative, i.e., $a_{ij} \geq 0$. It is said positive (denoted $A > 0$) if $a_{ij} > 0\ \forall\, i, j$. Denote $\rho(A)$ the *spectral radius* of A, i.e., the smallest positive real number such that $\rho(A) \geq |\lambda_i(A)|$, $i = 1, \ldots, n$, and $\mu(A)$ its *spectral abscissa*: $\max\{\mathrm{Re}[\lambda_i(A)], i = 1, \ldots, n\}$. A matrix is said *Metzler* if $a_{ij} \geq 0\ \forall\, i \neq j$. A Metzler matrix can always be written as $A = B - \alpha I$, with $B \geq 0$ and $\alpha > 0$, scalar. A Metzler matrix is Hurwitz if $\mu(A) < 0$, i.e., if $A = B - \alpha I$ and $\alpha > \rho(B)$. Notice that when $\mu(A) = 0$ (i.e., $\alpha = \rho(B)$) the Metzler matrix is singular. From Theorem 1 and the Perron-Frobenius theorem ([12], Thm. 8.4.4), irreducible Metzler matrices with $\mu(A) = 0$ are marginally (but not asymptotically) stable, i..e, $\lambda_i(A) = 0$ for some i, and $\lambda_i(A)$ algebraically simple. A Metzler matrix is called a *compartmental matrix* if $\sum_{i=1}^n a_{ij} \leq 0$ $\forall\, j = 1, \ldots, n$. Denote $\mathbf{1} = \begin{bmatrix}1 & \ldots 1\end{bmatrix}^T$. The condition, which in matrix form reads $\mathbf{1}^T A \leq 0$, corresponds to diagonal dominance, which from Theorem 1 leads to marginal stability or asymptotic stability if A is also irreducible, or at least “outflow connected”. In the compartmental systems literature, in fact, the j-th diagonal element is expressed as $a_{jj} = -(a_{0j} + \sum_{i\neq j} a_{ij})$, and $-a_{0j} \leq 0$ is called the outflow rate coefficient from node j because it represents an outflow of mass from the j-th node. By construction, then, $\mathbf{1}^T A = -a_0 = -\begin{bmatrix}a_{01} & \ldots & a_{0n}\end{bmatrix} \leq 0$, i.e., $-a_0$ provides the amount of diagonal dominance of A.

Definition 1 *A compartmental matrix is said* outflow connected *if all nodes of $\mathcal{G}(A)$ are connected via a directed path to nodes with nonzero outflow rate a_{0j}.*

The outflow connectivity condition is also called “no trap condition” in the compartmental systems literature [13], and algebraically corresponds to A having no eigenvalue in the origin [2], [1].

The following proposition and corollary assemble the stability properties of compartmental matrices, which can be found in [13] and [2] (Theorem 13.2 and par. 13B) (or deduced from Theorem 1).

Proposition 1 *A compartmental matrix A is nonsingular and therefore Hurwitz iff it is outflow connected. When this holds, then*

1) *$a_{jj} < 0\ \forall\, j$;*
2) *$(-A)^{-1} \geq 0$.*

For the special case of A irreducible, a single node having outflow is enough to guarantee asymptotic stability.

Corollary 1 *An irreducible compartmental matrix A is nonsingular and therefore Hurwitz iff $\mathbf{1}^T A \leq 0$, $\mathbf{1}^T A \neq 0$. Furthermore, $(-A)^{-1} > 0$.*

Proof. For irreducible compartmental matrices, the condition $\mathbf{1}^T A \leq 0$, $\mathbf{1}^T A \neq 0$ corresponds to outflow connectivity. The final claim can be found e.g. in [9]. ■

C. Positive linear systems and positive equilibria

The system

$$\dot{x} = Ax + Bu, \tag{2}$$

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, is said *(internally) positive* if $x(0) \geq 0$ and $u \geq 0$ imply $x(t) \geq 0\ \forall\, t \geq 0$. Necessary and sufficient conditions for positivity are that A is Metzler and $B \geq 0$ [8].

Definition 2 *The system* (2) *is said* inflow connected *from the input if its inputs influence, directly or indirectly, the entire state vector x.*

Inflow connectivity is sometimes referred to as *excitability* [18], [20]. For positive systems, inflow connectivity implies that each state variable can be made positive by a nonnegative input. In particular, we are interested in conditions which guarantee that a (strictly) positive asymptotically stable equilibrium point exists, i.e., $\exists\, x^* > 0$ such that $Ax^* + B\bar{u} = 0$ for some constant $\bar{u} \geq 0$.

From [20], we have the following.

Proposition 2 *In the positive linear system* (2)*, any two of the following conditions imply the third:*

1) *For any constant $\bar{u} \geq 0$, $\bar{u} \neq 0$, $\exists$ equilibrium point $x^* > 0$;*
2) *A is Hurwitz;*
3) *The system is inflow connected.*

For the special case of A irreducible we have that any input will suffice to achieve inflow connectivity, from [8].

Corollary 2 *If A is Metzler and irreducible, then any $B \geq 0$, $B \neq 0$, is such that the system* (2) *is inflow connected.*

Putting together all these conditions we have the following result.

Theorem 2 *Consider the positive linear system* (2)*. If A is compartmental, outflow connected and inflow connected, then for each constant $\bar{u} \geq 0$, $\bar{u} \neq 0$, there exists a unique strictly positive asymptotically stable equilibrium point $x^* = -A^{-1}B\bar{u} > 0$ with domain of attraction $\mathbb{R}^n_+$.*

Proof. From Proposition 1, A compartmental and outflow connected implies A Hurwitz and hence $-A^{-1} \geq 0$ nonsingular, while from Proposition 2, A Hurwitz plus inflow connectivity guarantee the existence of $x^* > 0$ for (2). Uniqueness follows from the invertibility of A. Its asymptotic stability follows from the Hurwitz property of A, which also guarantees that convergence occurs on the entire state space, here $\mathbb{R}^n_+$. ■

The result can be strengthened when A irreducible.

Corollary 3 *Consider the system* (2)*. If A is compartmental, irreducible with $\mathbf{1}^T A \leq 0$, $\mathbf{1}^T A \neq 0$, then for any $B \geq 0$ ($B \neq 0$) and for each constant $\bar{u} \geq 0$, $\bar{u} \neq 0$, there exists a unique strictly positive asymptotically stable equilibrium point $x^* = -A^{-1}B\bar{u} > 0$ with domain of attraction $\mathbb{R}^n_+$.*

Proof. Corollary 1 implies that the system is outflow connected, while from Corollary 2 the condition $B \geq 0$, $B \neq 0$ implies that the system is inflow connected. Hence Thereom 2 applies. ■

III. Compartmental models for wireless power transfer

Consider a network of n agents (IoT devices), each equipped with its own battery. Denote x_i the amount of power stored in the i-th battery. Clearly $x_i \geq 0$. The i-th battery dynamics is given by the following ODE:

$$\dot{x}_i = -\omega_i(x_i) - \sum_{j=1}^{n} \phi_{ji}(x_i) + \sum_{j=1}^{n} \eta_{ij}\phi_{ij}(x_j) + h_i \tag{3}$$

where h_i is the energy supplied externally to the i-th node (harvested from the environment or provided by the electrical grid), $\phi_{ij}(x_j)$ is the energy sent by the j-th agent to the i-th agent, $\eta_{ij} \in [0,\, 1]$ is the efficiency of the $j \to i$ energy transfer, $\omega_i(x_i)$ is the energy depleted by the i-th agent while operating (other than the shared energy, e.g. power needed by the transmitter, receiver or by all other functions, such as for wireless data transmission of a sensor to a fusion center, etc.). Notice that nodes that have external supplies of energy (i.e, having $h_i > 0$) are treated in the same way as nodes for which $h_i = 0$.

Natural constraints on the terms of (3) are: $h_i \geq 0$ (power supply cannot be negative), $\omega_i(x_i) \geq 0$ (any operation that the agent does consumes power), $\phi_{ij}(x_j) \geq 0$ (transferred energy is nonnegative) and $\phi_{ii}(x_i) = 0$ ($i \to i$ energy transfers are meaningless). Notice that we have assumed that both terms ω_i and ϕ_{ij} are functions only of the battery power of the "donor" agent, which, in the graph that represents the energy exchanges among the nodes, is the tail of each directed edge.

A. Linear models for wireless power transfer

Let us focus on a linear model, by assuming that $\omega_i(x_i) = \delta_i x_i$ and $\phi_{ij}(x_j) = s_{ij}x_j$, where δ_i is a constant rate of power used at node i, and the rate coefficients $0 \leqslant s_{ij} \leqslant 1$ represent the energy transfer routes as well as the policy adopted by the agent j towards its neighboring nodes. Denoting $S = [s_{ij}]$, by construction $S \geqslant 0$ and

$$\sum_i s_{ij} = \begin{cases} 1 & \text{if } j \text{ transmits power} \\ 0 & \text{if } j \text{ does not.} \end{cases}$$

If for instance the agent j sends energy to k neighbors, all with the same priority (e.g. transmitting with a radial symmetry), then we can assume $s_{ij} = 1/k$ for any neighbor i of j. If instead directional beacons or resonant couplings are used then it could be $s_{kj} \neq s_{ij}$ for two neighbors k and i of j. Clearly $s_{ii} = 0\ \forall i$ in all cases. With these choices, the model (3) becomes in vector form

$$\dot{x} = (-\Delta + \Gamma)x + h = Ax + h \tag{4}$$

where

$$\Delta = \begin{bmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{bmatrix}$$

and

$$\Gamma = \begin{bmatrix} -\sum_{i=1}^n s_{i1} & \eta_{12}s_{12} & \dots & \eta_{1n}s_{1n} \\ \eta_{21}s_{21} & -\sum_{i=1}^n s_{i2} & & \eta_{2n}s_{2n} \\ \vdots & & \ddots & \vdots \\ \eta_{n1}s_{n1} & \dots & & -\sum_{i=1}^n s_{in} \end{bmatrix}$$

Denote $\mathcal{G}(S)$ the energy sharing graph, whose adjacency matrix is S. Notice that if self-loops are disregarded, $\mathcal{G}(A)$ and $\mathcal{G}(S)$ have the same topology by construction (at least as long as all $\eta_{ij} > 0$), but different weights.

In the following, we show that the linear system (4) is a compartmental system, hence the conditions discussed in Sections II-B and II-C concerning existence of positive equilibria and their stability can be applied. Notice first that the outflow rate coefficients for the system (4) are $a_{0j} = \delta_j + \sum_{i\neq j}(1-\eta_{ij})s_{ij}$, i.e., the energy losses due to energy transfer are counted as outflows at the donor node. With this convention, the coefficient a_{0j} assumes the meaning of overall energy loss rate at node j.

A few preliminary facts, themselves of independent interest, are formulated in the following lemma. Denote $\mathcal{J} = \{j = 1,\ldots,n, \text{ s.t. } a_{0j} > 0\}$ and $\mathcal{H} = \{j = 1,\ldots,n, \text{ s.t. } h_j > 0\}$.

Lemma 1 *Consider the linear wireless power transfer system* (4).

1) *The matrix A is Metzler;*
2) *The matrix A is compartmental;*
3) *The matrix A is outflow connected (and hence nonsingular) iff for each $i = 1,\ldots,n$, either $i \in \mathcal{J}$ or $\exists$ a directed path in $\mathcal{G}(A)$ connecting i to some $j \in \mathcal{J}$.*
4) *If $h = 0$ and A nonsingular, then $x^* = 0$ is asymptotically stable.*
5) *The matrix A is inflow connected iff for each $i = 1,\ldots,n$, either $i \in \mathcal{H}$ or $\exists$ a directed path in $\mathcal{G}(A)$ connecting some $j \in \mathcal{H}$ to i.*

Proof.

1) Follows from $\eta_{ij}s_{ij} \geq 0 \;\forall\, i,j$.
2) Computing explicitly, $\forall\, j = 1,\ldots,n$

$$\begin{aligned}\sum_{i=1}^{n} a_{ij} &= -\delta_j - \sum_{i=1}^{n} s_{ij} + \sum_{i=1}^{n} \eta_{ij}s_{ij} \\ &= -\delta_j - \sum_{i=1}^{n}(1-\eta_{ij})s_{ij} = -a_{0j} \leq 0.\end{aligned} \tag{5}$$

3) By definition, outflow connectivity corresponds to directed connectivity to a node in $\mathcal{J}$. From Proposition 1, outflow connectivity corresponds to nonsingularity.
4) A nonsingular compartmental matrix is Hurwitz.
5) By definition, inflow connectivity corresponds to each node being reachable from the nonzero components of the input (i.e., h_i) through a directed path on $\mathcal{G}(A)$.

■

Existence of a positive equilibrium point can then be investigated using Theorem 2.

Theorem 3 *The linear wireless power transfer system* (4) *admits a unique strictly positive equilibrium point $x^* = -A^{-1}h$ for a constant $h \neq 0$ iff $\mathcal{G}(S)$ is outflow connected and inflow connected from h. The domain of attraction of x^* is $\mathbb{R}^n_+$.*

Proof. From Lemma 1, the system (4) is a compartmental system which is also outflow connected and inflow connected from a given h. Hence Theorem 2 applies and the conclusion follows. ■

When S (and hence A, if $\eta_{ij} > 0 \;\forall\, i,j$) is irreducible, then a single node supplying energy is enough to guarantee that the system (4) has a unique strictly positive equilibrium point, provided that some energy losses are present.

Corollary 4 *Consider the linear wireless power transfer system* (4). *Assume that the power transfer matrix S is irreducible. If $\delta_i > 0$ for some i, or $\eta_{ij} < 1$ for some (i,j) such that $s_{ij} \neq 0$, then for any constant $h \geq 0$, $h \neq 0$, the system* (4) *admits a unique strictly positive equilibrium point $x^* = -A^{-1}h$. The domain of attraction of x^* is $\mathbb{R}^n_+$.*

Proof. Using Lemma 1 and the irreducibility of S, the proof of the theorem follows straightforwardly from Theorem 3. ■

B. Spectral radius interpretation

Let us split Γ of (4) into diagonal and off-diagonal parts: $\Gamma = -\Gamma_d + \Gamma_o$. The following proposition is a reformulation of Theorem 2 of [1] for our case. A proof is given for completeness.

Proposition 3 *Consider the system* (4). *The compartmental matrix A is nonsingular iff $\rho(F) < 1$, where $F = \Gamma_o(\Delta+\Gamma_d)^{-1}$ and $(\Delta+\Gamma_d)_{ii} > 0 \;\forall\, i$.*

Proof. From Proposition 1, A compartmental and nonsingular has $a_{jj} < 0 \;\forall\, j$, hence $\Delta + \Gamma_d$ has all positive diagonal elements (and is invertible). Then (4) can be written as

$$\begin{aligned}\dot{x} = Ax + h &= -(\Delta + \Gamma_d - \Gamma_o)x + h \\ &= -\left(I - \Gamma_o(\Delta+\Gamma_d)^{-1}\right)(\Delta+\Gamma_d)x + h \\ &= -(I-F)(\Delta+\Gamma_d)x + h.\end{aligned} \tag{6}$$

By construction $F \geq 0$, hence for it Perron-Frobenius theorem applies. Computing explicitly:

$$\begin{aligned}F &= \Gamma_o(\Delta+\Gamma_d)^{-1} \\ &= \begin{bmatrix} 0 & \frac{\eta_{12}s_{12}}{\delta_2+\sum_{i\neq 2}s_{i2}} & \cdots & \frac{\eta_{1n}s_{1n}}{\delta_n+\sum_{i\neq n}s_{in}} \\ \frac{\eta_{21}s_{21}}{\delta_1+\sum_{i\neq 1}s_{i1}} & 0 & & \vdots \\ \vdots & & \ddots & \\ \frac{\eta_{n1}s_{n1}}{\delta_1+\sum_{i\neq 1}s_{i1}} & \cdots & & 0 \end{bmatrix}.\end{aligned}$$

Hence, summing over the rows,

$$\mathbf{1}^T F = \begin{bmatrix} \frac{\sum_{i\neq 1}\eta_{i1}s_{i1}}{\delta_1+\sum_{i\neq 1}s_{i1}} & \cdots & \frac{\sum_{i\neq n}\eta_{in}s_{in}}{\delta_n+\sum_{i\neq n}s_{in}} \end{bmatrix} \leqslant \mathbf{1}^T.$$

Since the Perron-Frobenius eigenvalue is extremal over the positive vectors (i.e., $Fx \leqslant \xi x$ for $x > 0$ and $\xi > 0$ implies $\rho(F) \leqslant \xi$), it must be $\rho(F) \leqslant 1$. To show that $\rho(F) < 1$, assume by contradiction that $\rho(F) = 1$, then from the Perron-Frobenius theorem, $\rho(F) = 1$ must be an eigenvalue of F and the matrix $I - F$ must be singular. But this means that also $A = -(I-F)(\Delta+\Gamma_d)$ must be singular which is false. To show the converse, it is enough to say that from $\rho(F) < 1$ it follows that $I - F$ is nonsingular, and hence so must be A, since $\Delta + \Gamma_d$ is nonsingular.

■

Apart from compartmental models, this property is well-known in the literature on power control of wireless networks

[21], where the models used are very similar to (4), although the interactions have a different meaning (interference, not exchanges). See also [25], [26] for related material.

The spectral radius $\rho(F)$ (and hence also $\mu(A)$) corresponds to the slowest mode of the system (4), and therefore determines the convergence speed of the system to x^*. From the expression of F, such convergence speed is inversely proportional to Δ and Γ_d, i.e., to the diagonal coefficients a_{jj} of A.

IV. Wireless power transfer network construction

In this section we consider the problem of designing energy transfer routes so that the network of IoT devices is sustainable, i.e., all the n batteries never discharge completely. In our formulation, this corresponds to choosing $\mathcal{G}(S)$ so that the system (4) admits a strictly positive equilibrium point. The following theorem provides a simple necessary and sufficient topological condition.

Theorem 4 *Consider the system* (4)*. The matrix A satisfies Theorem 3 iff $\mathcal{G}(S)$ contains a spanning forest rooted at $\mathcal{H}$ and with leaves in $\mathcal{J}$.*

Proof. Rooting the spanning forest at $\mathcal{H}$ is equivalent to the compartmental system being inflow connected, while the condition that leaf nodes are in $\mathcal{J}$ implies that it is outflow connected. Hence Theorem 3 applies. ∎

Remark 1 For a leaf node j it is $s_{ij} = 0 \;\forall\, i$, i.e., Γ has an empty column and $a_{0j} = \delta_j$, meaning that in order for A^{-1} to exist it must be $\delta_j > 0$, i.e., a leaf node cannot remain idle (because its battery would "overflow" as $t \to \infty$).

In order to design $\mathcal{G}(S)$, we will make the (realistic) assumption that no energy transfer occurs with efficiency 1, and that all the power depletion rates δ_i are equal (identical IoT devices).

Assumption 1

A1 *The matrix Γ is such that $0 < \eta_{ij} < 1 \;\forall\, i, j = 1, \ldots, n$, $i \neq j$.*

A2 *The matrix Δ is such that $\delta_i = \delta > 0 \;\forall\, i = 1, \ldots, n$.*

Assumption A2 guarantees that the network is always outflow connected, i.e., $\mathcal{J} = \{1, \ldots, n\}$.

Under Assumptions A1-A2, the solution that minimizes the total energy loss rate due to wireless power transfer can be computed explicitly. Call $W(\mathcal{G}(S)) = \sum_{ij}(1 - \eta_{ij})s_{ij}$ the energy loss rate of $\mathcal{G}(S)$. Our problem of minimizing the energy loss due to wireless transfer while maintaining all IoT devices alive can be formulated as the following optimization problem. Denote S_{all} the adjacency matrix of all possible power transfer routes for a given network.

Problem 1 *(Energy loss minimization in wireless power transfer)*

$$\min_{S \subseteq S_{\text{all}}} W(\mathcal{G}(S)) \;\; \textit{subject to} \;\; \mathcal{G}(S) \;\; \textit{inflow connected}$$

In the expression above $S \subseteq S_{\text{all}}$ means that S is a submatrix of S_{all}.

Lemma 2 *Assume A1 and A2 hold. Consider a power transfer network $\mathcal{G}(S)$ which is inflow connected. If $\mathcal{G}(S)$ contains directed cycles then $\exists\, S' \subset S$ such that the subgraph $\mathcal{G}(S')$ of $\mathcal{G}(S)$ is inflow connected and $W(\mathcal{G}(S')) < W(\mathcal{G}(S))$.*

Proof. Consider $\mathcal{G}(S)$ which is inflow connected and with directed cycles. Let $k_1 \to k_2 \to \ldots \to k_\ell \to k_1$ be one such directed cycle. Assume without loss of generality that k_1 is the node of the cycle closer to an energy source (possibly it could also be $k_1 \in \mathcal{H}$). But then if $\mathcal{G}(S') = \mathcal{G}(S) \setminus (k_\ell, k_1)$ is the graph $\mathcal{G}(S)$ without the (k_ℓ, k_1) edge, we have that also $\mathcal{G}(S')$ must be inflow connected, and in addition

$$\begin{aligned} W(\mathcal{G}(S)) &= \sum_{(i,j)\in\mathcal{G}(S)} (1 - \eta_{ij})s_{ij} \\ &= \sum_{(i,j)\in\mathcal{G}(S')} (1 - \eta_{ij})s_{ij} + (1 - \eta_{k_\ell k_1})s_{k_\ell k_1} \\ &> W(\mathcal{G}(S')) \end{aligned}$$

since $(1 - \eta_{k_\ell k_1})s_{k_\ell k_1} > 0$. ∎

The following theorem gives the optimal solution to Problem 1.

Theorem 5 *Assume A1 and A2 hold. Let $\mathcal{G}(S_{\text{all}})$ be inflow connected. Then Problem 1 is solved by the maximum directed spanning forest of $\mathcal{G}(S_{\text{all}})$ rooted at $\mathcal{H}$.*

Proof. From $\mathcal{G}(S_{\text{all}})$ of edge weights $0 < \eta_{ij}s_{ij} < 1$ form $\hat{\mathcal{G}}(S_{\text{all}})$ of edge weights $0 < 1 - \eta_{ij}s_{ij} < 1$. From Lemma 2 we know that the optimal solution of Problem 1 cannot contain any directed cycle, i.e., it must be a Directed Acyclic Graph (DAG). From Theorems 4 and 3, we know that the optimal solution must be inflow connected and outflow connected. The latter property is always guaranteed by Assumption A2. Hence the DAG must be rooted at $\mathcal{H}$. Among all DAGs of $\hat{\mathcal{G}}(S_{\text{all}})$ rooted at $\mathcal{H}$, the minimal directed spanning forest is the one that minimizes the sum of the edge weights $1 - \eta_{ij}s_{ij}$, hence it must be the one solving Problem 1. Since $\hat{\mathcal{G}}(S_{\text{all}})$ and $\mathcal{G}(S_{\text{all}})$ have corresponding edges that sum to 1, this solution corresponds to the maximum directed spanning forest of $\mathcal{G}(S_{\text{all}})$ rooted at $\mathcal{H}$. ∎

An efficient algorithm to compute the maximal directed spanning tree (or forest) of a graph is the well-known Edmonds algorithm [4], [7], [22].

Example 1 In Fig. 1 an example of size $n = 10$ is shown, in which 3 nodes have external energy supplies. The maximum directed spanning tree is computed using the Edmonds algorithm.

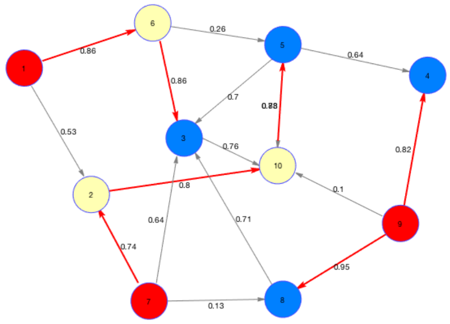

Fig. 1. Example 1. Maximum directed spanning forest for a network of 10 IoT nodes (red edges). Nodes in red have external power supply, while nodes in blue are terminal leaves of the trees. The efficiencies-weighted routing coefficients $\eta_{ij}s_{ij}$ of the energy transfer are reported on the edges.

V. FINAL COMMENTS AND OUTLOOK

Our wireless power transfer model (4) is obtained under a number of simplificative assumptions. For instance the power depleted by a node for its functioning obeys to the linear law $\omega_i(x_i) = \delta_i x_i$ even when x_i is very small. Clearly, a minimum power threshold $x_i^{\min_1}$ should be imposed, below which a node stops functioning and enters into sleep mode. Similarly, a lower threshold $x^{\min_2}$ can be imposed below which energy transmission is refused. Also an upper bound on the power should be considered, i.e., $x_i(t) \leq x_i^{\max}$, representing max battery capacity at a node.

The assumption made here that the energy input h is constant is valid if we consider the electrical grid as source of energy. However, if we think of harvesting energy from the environment, then basically all sources have an intermittent nature, and therefore are better represented by random processes than by constant, deterministic inputs. We intend to explore these aspects in the future.

Acknowledgements. The authors would like to thank Steffi Knorn for introducing them to the problem and for discussions on the topic.

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