

A Systems Theory Approach to the Synthesis of Minimum Noise Phase-Insensitive Quantum Amplifiers

Ian R. Petersen, Matthew R. James, Valery Ugrinovskii and Naoki Yamamoto

Abstract— We present a systems theory approach to the proof of a result bounding the required level of added quantum noise in a phase-insensitive quantum amplifier. We also present a synthesis procedure for constructing a quantum optical phase-insensitive quantum amplifier which adds the minimum level of quantum noise and achieves a required gain and bandwidth. This synthesis procedure is based on a singularly perturbed quantum system and leads to an amplifier involving two squeezers and two beamsplitters.

I. INTRODUCTION

In the theory of quantum linear systems [1]–[5], quantum optical signals always have two quadratures. These quadratures can be represented either by annihilation and creation operators, or position and momentum operators; e.g., see [4], [5]. The relative size of the two quadratures in a quantum optical signal determines the optical phase of the signal. In designing an amplifier for a quantum optical signal, is often desired to preserve the optical phase of the amplified signal. Such quantum amplifiers are referred to as phase-insensitive amplifiers or phase-preserving amplifiers. In the paper, [6] (see also [7]), Caves recognized the importance of phase-insensitive amplifiers and showed that Heisenberg's uncertainty principle implies that any phase-insensitive amplifier must also introduce an amount of quantum noise which is related to the level of amplification required. The use of phase-insensitive quantum amplifiers plays a key role in areas of quantum technology such as quantum communication and weak signal detection; e.g., see [5], [8]–[12]. Phase-insensitive quantum amplifiers can be implemented using non-degenerate optical parametric amplifiers (NOPAs) [13]; squeezers, beamsplitters and measurement feedforward [14]; or using feedback optical systems [5].

In this paper, we re-derive the noise bound of [6] for phase-insensitive quantum amplifiers using the quantum linear systems notion of physical realizability and in particular the physical realizability of a transfer function matrix; e.g.,

see [1]–[4], [15], [16]. We also present a systematic procedure for synthesizing a quantum optical phase-insensitive quantum amplifier with a given gain and bandwidth, using a pair of degenerate optical parametric amplifiers (squeezers) and a pair of beamsplitters. This approach is based on the singular perturbation of quantum systems [17], [18] to achieve the required DC gain and bandwidth. Compared with the NOPA approach such as described in [5], our approach uses squeezers for which it is typically easier to obtain a higher level of squeezing (and hence amplifier gain). Also, compared to the approach of [14], our approach does not require quantum measurement. In addition, compared to the feedback approach of [5], [13], our approach always achieves the minimum amount of required quantum noise and only requires a fixed level of squeezing for a given amplification. Our proposed quantum optical phase-insensitive quantum amplifier synthesis procedure may be useful in on-chip quantum optical technologies such as described in [19].

In this conference version of the paper all proofs have been removed along with an example due to page limitations. A full version of the paper can be found on arXiv [20].

Notation: I denotes the identity matrix, $J := \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

For a matrix X of operators, X^T and $X^\#$ respectively denote the matrices (of operators) obtained by taking transpose and component-wise adjoint. $X^\dagger := (X^\#)^T$. Also, if X is a complex matrix, then X^T denotes the usual transpose and $X^\#$ denotes the matrix obtained by component-wise complex conjugation. For a single operator (resp. complex scalar) g , we use g^* to denote its adjoint (resp. complex conjugate). If x, y are column vectors (of same length) of operators, then we define the commutator $[x, y^T] := xy^T - (yx^T)^T$. Consequently, $[x, y^\dagger] = [x, (y^\#)^T] = xy^\dagger - (y^\#x^T)^T$.

Matrices of the form $\begin{bmatrix} R_1 & R_2 \\ R_2^\# & R_1^\# \end{bmatrix}$ are denoted by $\Delta(R_1, R_2)$; see also [1], [2].

II. LINEAR QUANTUM SYSTEMS

We consider a class of linear quantum systems described by the quantum stochastic differential equations (QSDEs), (e.g., see [1], [3], [4]):

$$\begin{aligned} \begin{bmatrix} da(t) \\ da(t)^\# \end{bmatrix} &= A \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt + B \begin{bmatrix} du(t) \\ du(t)^\# \end{bmatrix}; \\ \begin{bmatrix} dy(t) \\ dy(t)^\# \end{bmatrix} &= C \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt + D \begin{bmatrix} du(t) \\ du(t)^\# \end{bmatrix}, \end{aligned} \quad (1)$$

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where

$$\begin{aligned} A &= \Delta(A_1, A_2), & B &= \Delta(B_1, B_2), \\ C &= \Delta(C_1, C_2), & D &= \Delta(D_1, D_2). \end{aligned} \quad (2)$$

Here, $A_1 \in \mathbb{C}^{n \times n}$, $A_2 \in \mathbb{C}^{n \times n}$, $B_1 \in \mathbb{C}^{n \times m}$, $B_2 \in \mathbb{C}^{n \times m}$, $C_1 \in \mathbb{C}^{m \times n}$, $C_2 \in \mathbb{C}^{m \times n}$, $D_1 \in \mathbb{C}^{m \times m}$ and $D_2 \in \mathbb{C}^{m \times m}$. Also, $a(t) = [a_1(t) \cdots a_n(t)]^T$ is a vector of (linear combinations of) annihilation operators. The vector u represents the input signals and is assumed to admit the decomposition:

$$du(t) = \beta_u(t)dt + d\tilde{u}(t)$$

where $\tilde{u}(t)$ is the noise part of $u(t)$ and $\beta_u(t)$ is an adapted process (see [21], [22] and [23]). The noise $u(t)$ is a vector of quantum noises. The noise processes can be represented as operators on an appropriate Fock space (for more details see [24] and [22]). The process $\beta_u(t)$ represents variables of other systems which may be passed to the system (1) via an interaction. More details concerning this class of quantum systems can be found in the references [1], [3], [4], [25]).

Definition 1: (See [3], [4], [25].) A complex linear quantum system of the form (1), (2) is said to be *physically realizable* if there exists a complex commutation matrix $\Theta = \Theta^\dagger$, a complex Hamiltonian matrix $M = M^\dagger$, and a coupling matrix N such that

$$\Theta = TJT^\dagger \quad (3)$$

where $T = \Delta(T_1, T_2)$ is non-singular, M and N are of the form

$$M = \Delta(M_1, M_2), N = \Delta(N_1, N_2) \quad (4)$$

and

$$\begin{aligned} A &= -\iota\Theta M - \frac{1}{2}\Theta N^\dagger JN; \\ B &= -\Theta N^\dagger J; \\ C &= N; \\ D &= I. \end{aligned} \quad (5)$$

In this definition, if the system (1) is physically realizable, then the matrices M and N define a complex open harmonic oscillator with scattering matrix $S = I$, coupling operator vector

$$L = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix}$$

and Hamiltonian operator

$$\mathcal{H} = \frac{1}{2} \begin{bmatrix} a^\dagger & a^T \end{bmatrix} M \begin{bmatrix} a \\ a^\# \end{bmatrix};$$

e.g., see [26], [22], [21], [27], [25] and [28].

Theorem 1 (See [25], [29].): The linear quantum system (1), (2) is physically realizable if and only if there exists a complex matrix $\Theta = \Theta^\dagger$ such that Θ is of the form in (3), and

$$\begin{aligned} A\Theta + \Theta A^\dagger + BJB^\dagger &= 0; \\ B &= -\Theta C^\dagger J; \\ D &= I. \end{aligned} \quad (6)$$

The complex transfer function matrix corresponding to the system (1) is given by

$$G(s) = C(sI - A)^{-1}B + D.$$

Definition 2: A complex transfer function matrix $G(s)$ is said to be physically realizable if it is the transfer function of a physically realizable linear quantum system.

A physically realizable transfer function matrix corresponds to a linear quantum system which satisfies the laws of quantum mechanics and can be implemented using physical components such as arising in quantum optics; e.g., see [5], [30]–[35].

III. PROBLEM FORMULATION

A phase-insensitive quantum amplifier is a two-input two-output physically realizable quantum linear system with transfer function $\bar{G}(s)$ as illustrated in Figure 1. In this diagram, the first input channel and the first output channel are the signal input and output channels respectively. Also, the second input channel and the second output channel are noise input and output channels. The noise output channel is not used in the operation of the amplifier but is included for consistency with the physical realizability theory for quantum linear systems; e.g., see [3], [16], [25]. As with any quantum linear system, each input and output channel consists of two quadratures; e.g., see [3]–[5]. Hence, the transfer function matrix $\bar{G}(s)$ is a four-by-four transfer function matrix. In order to define a phase-insensitive quantum amplifier, a physically realizable transfer function matrix $\bar{G}(s)$ should satisfy certain gain and phase-insensitivity properties over a specified frequency range. These properties will be formally defined below.

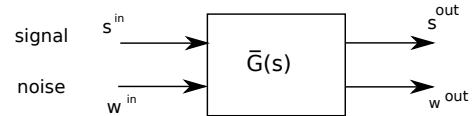


Fig. 1. Phase-insensitive quantum amplifier.

In this paper, we will formally consider the properties of $\bar{G}(s)$ to hold at DC; i.e., at $s = 0$. In addition, we will look at synthesizing amplifiers such that these properties also hold (approximately) out to some bandwidth frequency. However, it would be straightforward to extend out techniques so that the required phase-insensitive amplifier properties hold on any specified frequency interval.

In the pioneering paper [6], Caves showed that phase-insensitive quantum amplification can only be achieved at the expense of adding noise to the signal; see also [7], [13]. We re-derive this result using the quantum linear systems theory notion of physical realizability and then give a systematic synthesis procedure for designing a physical phase-insensitive amplifier achieving a specified gain, which can be implemented using quantum optics.

As in [3], [4], we write the transfer function $\bar{G}(s)$ in “doubled-up” form, specifying both quadratures of each input and output channels as follows:

$$\begin{bmatrix} s^{out} \\ w^{out} \\ s^{out*} \\ w^{out*} \end{bmatrix} = \bar{G} \begin{bmatrix} s^{in} \\ w^{in} \\ s^{in*} \\ w^{in*} \end{bmatrix} = \begin{bmatrix} G & H \\ H^\# & G^\# \end{bmatrix} \begin{bmatrix} s^{in} \\ w^{in} \\ s^{in*} \\ w^{in*} \end{bmatrix}. \quad (7)$$

Here and in the sequel, we will usually drop the dependence of transfer functions on the Laplace variable s for simplicity of notation. Using this notation, the two quadratures of the input signal are denoted by $\begin{bmatrix} s^{in} \\ s^{in*} \end{bmatrix}$ and a similar notation applies to the other input and output signals. Furthermore, we write

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}, \quad H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}. \quad (8)$$

We now present a result on the physical realizability of a transfer function matrix; e.g., see [1], [3], [16], [25].

Lemma 1 ([1], [3], [16]): A transfer function matrix $\bar{G}(s)$ of the form (7) is physically realizable if and only if

$$\bar{G}^\sim(s) J \bar{G}(s) = J$$

for all $s \in \mathbb{C}$ and the matrix $\bar{G}(\infty)$ is of the form $\bar{G}(\infty) = \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix}$ where $S^\dagger S = S S^\dagger = I$. Here, $\bar{G}^\sim(s) = \bar{G}(-s^*)^\dagger$.

Note that it follows from this result that any physically realizable transfer function $\bar{G}(s)$ will satisfy

$$\bar{G}(j\omega)^\dagger J \bar{G}(j\omega) = J \quad (9)$$

for all ω ; see also [1], [15].

Definition 3 (see also [6]): A physically realizable transfer function matrix $\bar{G}(s)$ of the form (7), (8) is said to be *phase-insensitive* at frequency ω if

$$h_{11}(j\omega) = 0. \quad (10)$$

We will be mostly concerned with the phase-insensitive property at DC and hence, we will usually drop the frequency specification. Also, we will be concerned with the corresponding *amplifier gain* squared amplitude

$$g_{11}(j\omega)^* g_{11}(j\omega)$$

and *noise* squared amplitude

$$g_{12}(j\omega)^* g_{12}(j\omega) + h_{12}(j\omega)^* h_{12}(j\omega)$$

at a given frequency ω (usually DC).

IV. MAIN RESULTS

In this section, we first re-derive the main result of [6] in terms of the physical realizability notions given in the previous section. That is, we show that any physically realizable transfer function matrix which is phase-insensitive at a given frequency ω will have the property that the minimum possible value of the noise squared amplitude at that frequency is equal to the *amplifier gain* squared amplitude at that frequency minus one. Also, for the case of $\omega = 0$, (the DC case), we give a method for synthesizing a physically realizable transfer function which achieves this lower bound on the noise squared amplitude. Furthermore, this construction allows these properties to be (approximately) continued out to some arbitrary bandwidth.

Theorem 2 (see also [6]): At any frequency ω , given a desired phase-insensitive quantum amplifier gain at that frequency g_{11} , then

$$\begin{aligned} & \min [g_{12}^* g_{12} + h_{12}^* h_{12}] \\ & = g_{11}^* g_{11} - 1. \end{aligned} \quad (11)$$

Here the minimum is taken over all transfer function matrices (7), (8) satisfying the physical realizability condition (9), the phase-insensitivity condition (10) and with the given amplifier gain g_{11} . Furthermore, this minimum is achieved by the transfer function matrix defined by

$$\begin{aligned} g_{12} &= 0; g_{21} = \sqrt{\frac{g_{11}^* g_{11} - 1}{g_{11}^* g_{11}}}; g_{22} = \sqrt{1 + g_{11}^* g_{11}}; \\ h_{11} &= 0; h_{12} = \frac{1}{g_{11}^*} \sqrt{g_{11}^* g_{11} (g_{11}^* g_{11} - 1)}; \\ h_{21} &= \sqrt{\frac{(g_{11}^* g_{11})^2 - 1}{g_{11}^* g_{11}}}; h_{22} = 1. \end{aligned} \quad (12)$$

In order to construct a physically realizable quantum system corresponding to a phase-insensitive amplifier whose DC transfer function matrix is derived from the above theorem, we will use the following lemma which is referred to as the Shale decomposition.

Lemma 2 ([36], see also [1], [35], [37], [38].):

Consider a 4×4 complex matrix \bar{G} of the form (7), (8) satisfying the physical realizability condition (9). Then there exists a real diagonal matrix $R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$ and 2×2 unitary matrices S_1 and S_2 such that

$$\bar{G} = \begin{bmatrix} S_1 & 0 \\ 0 & S_1^\# \end{bmatrix} \begin{bmatrix} -\cosh(R) & -\sinh(R) \\ -\sinh(R) & -\cosh(R) \end{bmatrix} \begin{bmatrix} S_2 & 0 \\ 0 & S_2^\# \end{bmatrix}. \quad (13)$$

Note that the decomposition given in this lemma is constructive; e.g., see [37], [38].

This lemma shows that the problem of physically realizing the two channel DC gain transfer function matrix \bar{G} can be reduced to the problem of physically realizing each of the single channel transfer function matrices $\bar{G}_1 = \begin{bmatrix} -\cosh(r_1) & -\sinh(r_1) \\ -\sinh(r_1) & -\cosh(r_1) \end{bmatrix}$, and $\bar{G}_2 =$

$\begin{bmatrix} -\cosh(r_2) & -\sinh(r_2) \\ -\sinh(r_2) & -\cosh(r_2) \end{bmatrix}$. Then, the unitary transfer matrices $\begin{bmatrix} S_1 & 0 \\ 0 & S_1^\# \end{bmatrix}$, and $\begin{bmatrix} S_2 & 0 \\ 0 & S_2^\# \end{bmatrix}$ can be physically implemented using beamsplitters; e.g., see [35], [39]. Indeed, since S_1 and S_2 are both 2×2 matrices, it follows that each of these can be implemented by a single beamsplitter. For example, as in [39] (with the addition of phase shifters on the input and output channels), we can write the input-output relations of a beamsplitter in the form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathcal{R} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where \mathcal{R} is a unitary matrix of the form

$$\mathcal{R} = \begin{bmatrix} e^{j\phi_1} \sin(\theta) & e^{j(\phi_1+\phi_3)} \cos(\theta) \\ e^{j\phi_2} \cos(\theta) & -e^{j(\phi_2+\phi_3)} \sin(\theta) \end{bmatrix} \quad (14)$$

and ϕ_1 , ϕ_2 , ϕ_3 and θ are parameters of the beamsplitter. Furthermore, it is straightforward to verify that any 2×2 unitary matrix S can be represented as a matrix of the form (14).

To realize a single channel DC transfer function matrix

$$\bar{G}_r = \begin{bmatrix} -\cosh(r) & -\sinh(r) \\ -\sinh(r) & -\cosh(r) \end{bmatrix}, \quad (15)$$

we consider a single channel dynamic squeezer following the approach of [40]; see also [1].

An optical cavity consists of a number of mirrors, one of which is partially reflective; e.g., see [26], [41]. If we include a nonlinear optical element inside such a cavity, an optical squeezer can be obtained. By using suitable linearizations and approximations, such an optical squeezer can be described by a quantum stochastic differential equation as follows:

$$\begin{aligned} da &= -\frac{\kappa}{2}adt - \chi a^*dt - \sqrt{\kappa}du; \\ dy &= \sqrt{\kappa}adt + du, \end{aligned} \quad (16)$$

where $\kappa > 0$, χ is a complex number associated with the strength of the nonlinear effect and a is a single annihilation operator associated with the cavity mode; e.g., see [26], [41]. This leads to a linear quantum system of the form (1) as follows:

$$\begin{aligned} \begin{bmatrix} da(t) \\ da(t)^* \end{bmatrix} &= \begin{bmatrix} -\frac{\kappa}{2} & -\chi \\ -\chi^* & -\frac{\kappa}{2} \end{bmatrix} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} dt \\ &\quad - \sqrt{\kappa} \begin{bmatrix} du \\ du^* \end{bmatrix}; \\ \begin{bmatrix} dy \\ dy^* \end{bmatrix} &= \sqrt{\kappa} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} dt + \begin{bmatrix} du \\ du^* \end{bmatrix}. \end{aligned} \quad (17)$$

Note that it is straightforward to verify that this system is stable if and only if $\kappa^2 > 4\chi\chi^*$.

As shown in [40], these QSDEs are physically realizable with the corresponding (S, L, \mathcal{H}) parameters (e.g., see [27] for a discussion of (S, L, \mathcal{H}) parameters) given by

$$S = I; N = \begin{bmatrix} \sqrt{\kappa} & 0 \\ 0 & \sqrt{\kappa} \end{bmatrix}; M = \begin{bmatrix} 0 & -j\chi \\ j\chi^* & 0 \end{bmatrix}.$$

A diagram of a dynamic optical squeezer is shown in Figure 2.

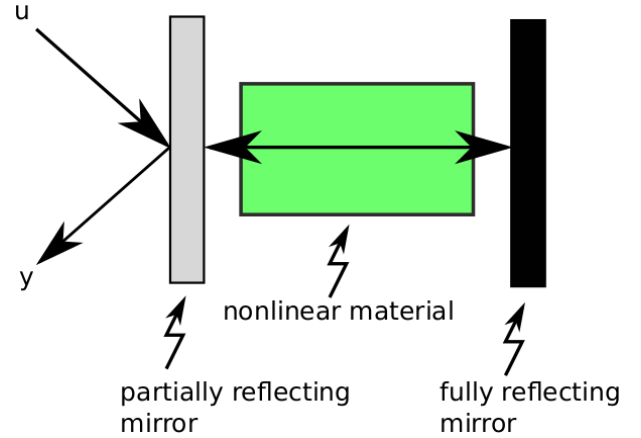


Fig. 2. Optical squeezer.

Now, we choose the parameters κ and χ to be of the form $\kappa = \epsilon\bar{\kappa}$ and $\chi = \epsilon\bar{\chi}$, where $\bar{\kappa} > 0$, $\bar{\chi}$ is chosen to be real and $\epsilon > 0$ is a parameter which will determine the amplifier bandwidth. Introducing the change of variables $\begin{bmatrix} \tilde{a}(t) \\ \tilde{a}(t)^* \end{bmatrix} =$

$$\begin{aligned} \epsilon^{-\frac{1}{2}} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix}, \text{ the QSDEs (17) reduce to} \\ \begin{bmatrix} d\tilde{a}(t) \\ d\tilde{a}(t)^* \end{bmatrix} &= \frac{1}{\epsilon} \begin{bmatrix} -\frac{\bar{\kappa}}{2} & -\bar{\chi} \\ -\bar{\chi} & -\frac{\bar{\kappa}}{2} \end{bmatrix} \begin{bmatrix} \tilde{a}(t) \\ \tilde{a}(t)^* \end{bmatrix} dt \\ &\quad - \frac{\sqrt{\bar{\kappa}}}{\epsilon} \begin{bmatrix} du \\ du^* \end{bmatrix}; \\ \begin{bmatrix} dy \\ dy^* \end{bmatrix} &= \sqrt{\bar{\kappa}} \begin{bmatrix} \tilde{a}(t) \\ \tilde{a}(t)^* \end{bmatrix} dt + \begin{bmatrix} du \\ du^* \end{bmatrix}. \end{aligned} \quad (18)$$

The transfer function matrix of this system at DC is given by

$$\begin{aligned} G(0) &= I + \begin{bmatrix} -\frac{2\bar{\kappa}^2}{\bar{\kappa}^2 - 4\bar{\chi}^2} & \frac{4\bar{\kappa}\bar{\chi}}{\bar{\kappa}^2 - 4\bar{\chi}^2} \\ \frac{4\bar{\kappa}\bar{\chi}}{\bar{\kappa}^2 - 4\bar{\chi}^2} & -\frac{2\bar{\kappa}^2}{\bar{\kappa}^2 - 4\bar{\chi}^2} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1+\alpha^2}{1-\alpha^2} & \frac{2\alpha}{1-\alpha^2} \\ \frac{2\alpha}{1-\alpha^2} & -\frac{1+\alpha^2}{1-\alpha^2} \end{bmatrix} \end{aligned}$$

where $\alpha = \frac{2\bar{\chi}}{\bar{\kappa}} = \frac{2\chi}{\kappa}$. In order for the system to be stable, we require $\alpha^2 < 1$. In order to construct a physically realizable quantum system with DC transfer function matrix \bar{G}_r defined in (15), we equate $G(0)$ with \bar{G}_r . That is,

$$\begin{bmatrix} -\frac{1+\alpha^2}{1-\alpha^2} & \frac{2\alpha}{1-\alpha^2} \\ \frac{2\alpha}{1-\alpha^2} & -\frac{1+\alpha^2}{1-\alpha^2} \end{bmatrix} = \begin{bmatrix} -\cosh(r) & -\sinh(r) \\ -\sinh(r) & -\cosh(r) \end{bmatrix}.$$

This is equivalent to the equations

$$\begin{aligned} \cosh(r) &= \frac{1+\alpha^2}{1-\alpha^2}; \\ \sinh(r) &= -\frac{2\alpha}{1-\alpha^2} \end{aligned}$$

To see that these equations are consistent, we calculate

$$\cosh^2(r) - \sinh^2(r) = \frac{1 + 2\alpha^2 + \alpha^4}{(1 - \alpha^2)^2} - \frac{4\alpha^2}{(1 - \alpha^2)^2} = 1$$

as required.

Now given r , we construct the corresponding value of α satisfying $\alpha^2 < 1$ such that $\sinh(r) = -\frac{2\alpha}{1-\alpha^2}$. This is equivalent to the equation

$$\alpha^2 - \frac{2\alpha}{\sinh(r)} - 1 = 0.$$

This equation has two solutions:

$$\alpha = \frac{1 + \cosh(r)}{\sinh(r)} = \frac{1}{\tanh(\frac{r}{2})}$$

and

$$\alpha = \frac{1 - \cosh(r)}{\sinh(r)} = -\tanh(\frac{r}{2}).$$

However, $\tanh(\frac{r}{2}) \in (-1, 1)$ and hence only the solution

$$\alpha = -\tanh(\frac{r}{2}) \quad (19)$$

satisfies the condition $\alpha^2 < 1$.

The above discussion leads to the following result.

Lemma 3: Given any matrix G_r of the form (15), there exists a physically realizable quantum system of the form (18) corresponding to a stable single channel dynamic squeezer such that its transfer function matrix $G(s)$ satisfies

$$G(0) = G_r.$$

Here, the ratio $\alpha = \frac{2\bar{x}}{\bar{\kappa}}$ satisfying $\alpha^2 < 1$ is uniquely determined by the design equation (19) and the parameter $\epsilon > 0$ can be chosen to achieve any desired bandwidth.

We now combine Theorem 2 with Lemmas 2 and 3 to obtain the following theorem which is our main result.

Theorem 3: Given any desired quantum phase-insensitive amplifier DC gain g_{11} , there exists a corresponding physically realizable linear quantum system of the form (1) which achieves this DC gain and introduces the minimal amount of DC quantum noise defined by (11). Furthermore, this transfer function matrix satisfies the DC phase-insensitivity condition (10). In addition, the parameters in this linear quantum system can be chosen to achieve a specified bandwidth over which the above conditions will hold approximately. Finally, this system can be constructed from two beamsplitters and two stable dynamic squeezers of the form (17).

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