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Optimal Synthesis in the Goddard Problem on a Constrained Time Interval

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***Abstract*— In this paper, we consider a problem on the rocket flight final height maximization under the control and endpoint constraints. The problem is considered on a constrained time interval, filling the gap between problems on free and fixed time interval. Using Pontryagin Maximum Principle, we obtain all types of optimal trajectories and then construct optimal synthesis. In addition, we investigate evolution of obtained synthesis w.r.t problem parameters.**

I. PROBLEM STATEMENT

Consider the following optimal control problem (a variant of a classical Goddard problem, see [1], [2], [4], [5], [6], [7], [8], [9], [10], [11], [16], [17]) on the constrained time interval $[0, T_0]$.

Problem A:

$$\begin{cases} \dot{s} = x, & s(0) = 0, & s(T) \to \max, \\ \dot{x} = u - \varphi(x) - g, & x(0) = 0, & x(T) \text{ is free}, \\ \dot{m} = -u, & m(0) = m_0\,, & m(T) \geqslant m_T\,, \\ 0 \leqslant u \leqslant 1, & T \leqslant T_0\,, & \end{cases} \qquad (1)$$

Here, $s(t)$ and $x(t)$ are one-dimensional position and velocity of a vehicle (see [12], [13]), $m(t)$ describes the total mass of vehicle's body and fuel, $u(t)$ is the rate of fuel expenditure, g is a constant gravity force and the function $\varphi(x)$ describes the "friction" (media resistance) depending on the velocity. We assume that (see [14], [15]) $\varphi(0) = 0$, $\varphi'(x) \geqslant 0$ for all x,$\varphi(x)$ is twice smooth for $x \neq 0$, $\varphi''(x) < 0$ for all $x < 0$ and $\varphi''(x) > 0$ for all $x > 0$, which, in particular, implies that $\varphi(x)$ works on decreasing the absolute value of the speed $|x|$.

II. PRELIMINARIES

Note first that since the admissible control set in problem (1) is a convex, the classical Filippov theorem guarantees that a solution (an optimal trajectory) always exists.

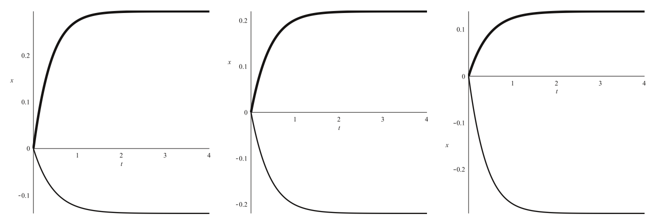
Let us establish some properties of the following control system:

$$\begin{cases} \dot{x} = u - \varphi(x) - g, \\ 0 \leqslant u \leqslant 1. \end{cases} \qquad (2)$$

Define $x_{min} < 0$, $x_{max} > 0$ from the conditions $-\varphi(x_{min}) - g = 0$, $1 - \varphi(x_{max}) - g = 0$, respectively. Due to the properties of the function $\varphi(x)$, x_{min} and x_{max} are unique. Since $\varphi(x)$ is monotonous, the following proposition takes place:

Lemma 1: If $x_0 \in (x_{min}, x_{max})$, then $x(t) \in (x_{min}, x_{max})$ $\forall t \geqslant 0$ for any trajectory of system (2) (see Figure 1).

Moreover, if $u = 1$ $(u = 0)$ on a time interval (t_1, t_2), then $\dot{x}(t) > 0$ (< 0, respectively) on (t_1, t_2).



a: $g < 0.5$ b: $g = 0.5$ c: $g > 0.5$

Fig. 1. "Work region" for $x(t)$

This lemma is proved in [15].

III. MAXIMUM PRINCIPLE

Let $s(t), x(t), m(t), u(t)$, $t \in [0, T]$ be an optimal process. The Pontryagin Maximum Principle (MP) [3]says that there exist numbers $(\alpha_0, \alpha, \alpha_1, \beta_t, \beta_s, \beta_x, \beta_m)$, not all zero, and Lipschitz continuous functions $\psi_s(t), \psi_x(t), \psi_m(t), \psi_t(t)$, that generate *the endpoint Lagrange function*

$$l = -\alpha_0 s(T) - \alpha(m(T) - m_T) + \alpha_1\,(t(T) - T_0) + \\ + \beta_t t(0) + \beta_s s(0) + \beta_x x(0) + \beta_m (m(0) - m_0) \qquad (3)$$

and *the Pontryagin function*

$$H(s, x, m, u) = \psi_s x + \psi_x\,(u - \varphi(x) - g) - \psi_m\, u, \qquad (4)$$

such that the following conditions are satisfied:

1) nonnegativity condition:

$$\alpha_0 \geq 0, \qquad \alpha \geq 0, \qquad \alpha_1 \geq 0,$$

2) nontriviality condition:

$$(\alpha_0, \alpha, \alpha_1, \beta_t, \beta_s, \beta_x, \beta_m) \neq (0, 0, 0, 0, 0, 0, 0),$$

3) complementarity slackness conditions:

$$\begin{cases} \alpha\,(m(T) - m_T) = 0, \\ \alpha_1\,(t(T) - T_0) = 0, \end{cases} \qquad (5)$$

4) costate (adjoint) equations

$$\begin{cases} -\dot{\psi}_s = H_s = 0, \\ -\dot{\psi}_x = H_x = \psi_s - \psi_x \varphi'(x), \\ -\dot{\psi}_m = H_m = 0, \\ -\dot{\psi}_t = H_t = 0, \end{cases} \qquad (6)$$

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5) transversality conditions:

$$\begin{cases} \psi_s(0) = \beta_s\,, & \psi_s(T) = \alpha_0\,, \\ \psi_x(0) = \beta_x\,, & \psi_x(T) = 0\,, \\ \psi_m(0) = \beta_m\,, & \psi_m(T) = \alpha\,, \\ \psi_t(0) = \beta_t\,, & \psi_t(T) = -\alpha_1\,, \end{cases} \tag{7}$$

6) the "energy conservation law":

$$H(t,s,x,m,u) + \psi_t \equiv 0,$$

reads

$$H(t,s,x,m,u) \equiv \alpha_1 \geq 0, \tag{8}$$

7) and the maximality condition: for almost all t

$$\max_{0\leqslant u'\leqslant 1} H(s(t),x(t),m(t),u') \;=\; H(s(t),x(t),m(t),u(t)). \tag{9}$$

According to (6)–(7), in order to simplify further notation we set $\psi_t = \alpha_1, \psi_s \equiv \alpha_0$, $\psi_m \equiv \alpha$, and write $\psi(t)$ instead of $\psi_x(t)$. Then the maximality condition (9) gives the optimal control in the form

$$u(t) \in \operatorname{Sign}^+(\psi - \alpha), \tag{10}$$

where Sign^+ is the set-valued function

$$\operatorname{Sign}^+(z) = \begin{cases} \{1\}\,, & z > 0, \\ [0,1], & z = 0, \\ \{0\}\,, & z < 0, \end{cases}$$

and the costate $\psi(t)$ is determined by the equation

$$\dot\psi = \; -\alpha_0 + \psi\,\varphi'(x) \tag{11}$$

with the terminal condition $\psi(T) = 0$.

By the assumptions, $\gamma := m_0 - m_T > 0$. If $\gamma \geqslant T$, then the optimal control is obvious: $u \equiv 1$ (full thrust). So, in further considerations we assume that

$$0 < \gamma < T. \tag{12}$$

IV. OPTIMALITY CONDITIONS ANALYSIS

First, we show that the abnormal case $\alpha_0 = 0$ is impossible. Suppose that $\alpha_0 = 0$. Then also $\beta_s = 0$ by (6)–(7), hence equation (11) for ψ reduces to a homogeneous one, and condition $\psi(T) = 0$ yields $\psi(t) \equiv 0$. Hence $\beta_x = 0$ by (7), and nontriviality condition gives $\alpha = \beta_m > 0$. Then (10) yields $u(t) \equiv 0$, and from equations (1) we have $m(T) = m_0$, which contradicts complementarity slackness condition (5), since $m_T < m_0$. Hence $\alpha_0 > 0$, and we may take $\alpha_0 = 1$. Then also $\psi_s \equiv 1$, and equation (11) reads

$$\dot\psi = \; -1 + \psi\,\varphi'(x). \tag{13}$$

This in particular implies that $\dot\psi(t)$ is a continuous function. The Pontryagin function now is

$$H(x,u) = \; x - \,\psi\,(\varphi(x) + g) \,+\, (\psi - \alpha)\,u.$$

Proposition 1: $\psi(t) > 0$ for all $t < T$.

Proof: Since $\psi(T) = 0$, equation (13) yields $\dot\psi(T) = -1$. Then $\psi(t) > 0$ in a left neighborhood of T. Suppose there exists $t' < T$ such that $\psi(t') = 0$ and $\psi(t) > 0$ on (t',T). From (13) we again have $\dot\psi(t') = -1$, which contradicts the previous inequality. ■

Proposition 2: $\alpha > 0$.

Proof: Suppose that $\alpha = 0$. Then from (10) and Proposition (1) we obtain $u \equiv 1$ for a.a. t. Hence $\gamma = T$, which contradicts (12). ■

This property, in view of (5), gives $m(T) = m_T$, and hence

$$\int_0^T u\,dt \;=\; \gamma > 0. \tag{14}$$

Moreover, since $\alpha > 0$, there exists such $t_2 \in (0,T)$ that $\psi(t) < \alpha$ (and then $u = 0$ by (10)) for all $t > t_2$.

Proposition 3: If $T = T_0$, then $x(T) \geq 0$. If $T < T_0$, then $x(T) = 0$.

Proof: Consider first $T = t(T) < T_0$. Thus, from (5) we get $\alpha_1 =,0$ and (8) implies $H \equiv 0$. Since $u = 0$ on the interval (t_2,T), the goal statement immediately follows from $\psi(T) = 0$.

Consider now $T = T_0$. Thus, from (5) we get $\alpha_1 \geq 0$ and the goal statement immediatelly follows from $H \geq 0$, $u = 0$ on the interval (t_2,T) and $\psi(T) = 0$. ■

Lemma 2: There cannot exist $t' < t''$ such that $x(t) < 0$ and $\psi(t) < \alpha$ on (t',t'').

Proof: In this case $u = 0$ and $H = x + \psi\dot x$ on (t',t''). Since $x(t) < 0$ and $\dot x < 0$ (see Lemma 1), we get $H < 0$ on (t',t''), which contradicts (8). ■

Definition 4.1: To define conveniently the control function, we will use the notation

$$u = (u_1, u_2, \ldots) \quad \text{on} \quad (\Delta_1, \Delta_2, \ldots),$$

where Δ_1, Δ_2, etc., are some intervals, if $u(t) = u_1$ on Δ_1, $u(t) = u_2$ on Δ_2, etc.

Now, define the set $M = \{t:\; \psi(t) = \alpha\}$. Obviously, M is closed. Moreover, it is not empty (otherwise $\psi < \alpha$ on $]0,T[$, hence $u \equiv 0$, which contradicts (14)).

A. Some properties of $x(t)$ and $\psi(t)$ related to the set M

Lemma 3: Let some $t' < t'' < T$ be such that $\psi(t') = \psi(t'') = \alpha$ and $\psi(t) > \alpha$ on $]t',t''[$. Then $x(t') < 0$ and $0 < x(t'') \leqslant -x(t')$.

Proof: From the conditions it follows that $\dot\psi(t') \geqslant 0$, $\dot\psi(t'') \leqslant 0$, hence from equation (13) we obtain $\varphi'(x(t'')) \leqslant \varphi'(x(t'))$. On the other hand, since $u = 1$, we have $x(t'') > x(t')$ by Lemma 1. Then from the properties of $\varphi'(x)$ it follows that $x(t') < 0$ and $x(t'') \leqslant -x(t')$.

Further, according to (8), we have $H(t''-0) = x(t'') - \alpha(\varphi(x(t'')) + g) \geq 0$. Since $\alpha > 0$ and always $\varphi(x) + g > 0$, we obtain $x(t'') > 0$. ■

Lemma 4: There cannot exist $t' < t''$ such that $\psi(t') = \psi(t'') = \alpha$ and $\psi(t) < \alpha$ on $]t',t''[$.

Proof: In this case $\dot\psi(t') \leqslant 0$, $\dot\psi(t'') \geqslant 0$, and from equation (13) we obtain $\varphi'(x(t'')) \geqslant \varphi'(x(t'))$. On the other hand, since $u = 0$, by Lemma 1 we have $\dot x < 0$, hence $x(t'') < x(t')$. If $x(t'') \geqslant 0$ then $x(t') > 0$, and from the properties of $\varphi'(x)$ it follows that $\varphi'(x') > \varphi'(x'')$, a contradiction. Therefore, $x(t'') < 0$, and so $x'' < 0$, $\psi < \alpha$ in a left neighborhood of t'', which contradicts Lemma 2. ■

Lemma 5: The following two cases are impossible:

1) There exists $t' < T$ such that $x(t') < 0$, $\psi(t') = \alpha$, $\dot\psi(t') = 0$ and $\psi(t) > \alpha$ (see Fig. 2a) or $\psi(t) < \alpha$ (Fig. 2b) in a right neighborhood of t'.

2) There exists $t'' > 0$ such that $x(t'') < 0$, $\psi(t'') = \alpha$, $\dot\psi(t'') = 0$ and $\psi(t) > \alpha$ (see Fig. 3a) or $\psi(t) < \alpha$ (Fig. 3b) in a left neighborhood of t''.

a: $x(t') < 0$ b: $x(t') \leqslant 0$

Fig. 2. Starting from the level $\psi(t') = \alpha$ with $\dot\psi(t') = 0$

a: $x(t'') \leqslant 0$ b: $x(t'') < 0$

Fig. 3. Falling to the level $\psi(t'') = \alpha$ with $\dot\psi(t'') = 0$

Proof: Using equation (13), we write the second time derivative of ψ:

$$\ddot\psi = \dot\psi\varphi'(x) + \psi\varphi''(x)\dot x. \tag{15}$$

Consider the case 1a, where $\psi > \alpha$ in a right neighborhood of t'. Then $u = 1$ there, and Lemma 1 gives $\dot x > 0$. Moreover, $\dot x(t'+0) = 1 - \varphi(x(t')) - g > 0$. Since $x(t') < 0$, we have $\varphi''(x(t')) < 0$, and taking into account that $\dot\psi$ is continuous and $\dot\psi(t') = 0$, we get from (15) that

$$\ddot\psi(t'+0) = \alpha\varphi''(x(t'))\dot x(t'+0) < 0,$$

so $\ddot\psi < 0$ in a right neighborhood of t'. From the condition $\dot\psi(t') = 0$ we obtain $\psi(t) < \alpha$ in a right neighborhood of t', a contradiction.

The case 1b (where $\psi(t) < \alpha$ in a right neighborhood of t'), as well as corresponding cases 2a and 2b are analyzed similarly. Also, the cases 1b and 2b are easily analyzed using Lemma 2. ■

Lemma 6: If $\psi = \alpha$ on an interval $]t', t''[$, then $x = const$ and $u = \varphi(x) + g$ on this interval.

Proof: Indeed, from (13) we get $\dot\psi = -1 + \alpha\varphi'(x) = 0$, so $\varphi'(x(t)) = const$, therefore $x(t) = const$, and hence $u(t) = \varphi(x(t)) + g$. ■

A key qualitative fact is given by the following

Lemma 7: The set M is connected.

Proof: Supposing the opposite, we have such $t' < t''$ that $\psi(t') = \psi(t'') = \alpha$, and in view of Lemma 4, $\psi(t) > \alpha$ on $\omega = (t', t'')$. By Lemma 3 $x(t') < 0$. Since $\dot\psi(t') \neq 0$ in view of Lemma 5, and the function $\dot\psi(t)$ is continuous, we get $\dot\psi(t') > 0$ and $\psi(t) < \alpha$ in a left neighborhood of t'. Taking into account that also $x(t) < 0$ there, we get a contradiction with Lemma 2. ■

Thus, M is a segment $[t_1, t_2]$ with possible $t_1 = t_2$.

Lemma 8: $M \subset]0, T[$, i.e. $t = 0$ and $t = T$ do not belong to M.

Proof: Since $\psi(T) = 0 < \alpha$, the right end $T \notin M$ and we just need to show that $0 \notin M$. Taking into account that M is a segment $[t_1, t_2]$, suppose first that $M = \{0\}$. Then $\psi(t) < \alpha$ on $]0, T[$, which yields $u \equiv 0$ for a.a. $t < T$ and contradicts (14). Now suppose $M = [0, t_2]$, where $0 < t_2 < T$. By Lemma 6, on $[0, t_2]$ we have $x = const = x(0) = 0$, while on $]t_2, T[$ we have $u = 0$ and $x(t) < x(t_2) = 0$, which is impossible by Lemma 2. ■

Lemma 9: $\psi(t) > \alpha$ on $]0, t_1[$.

Proof: Suppose this is wrong. Since $\psi(t) \neq \alpha$ on $]0, t_1[$, we then have $\psi(t) < \alpha$, hence $u = 0$ and $x(t) < x(0) = 0$ there, which contradicts Lemma 2. ■

Now, if $M = \{\hat t\}$ is a point, then, obviously, $u = (1, 0)$ on $(]0, \hat t[,]\hat t, T[)$, and $\hat t = \gamma$ by (14), while T is determined from equation $x(T) = 0$ according to Proposition 3. We say that the corresponding trajectory is of type I.

If M is a segment $[t_1, t_2]$, then by Lemma 6 $x(t) = x_{sing} = const$,

$$u_{sing}(t) = \varphi(x_{sing}) + g \qquad \text{on } [t_1, t_2], \tag{16}$$

so that $u = (1, u_{sing}, 0)$ on $(]0, t_1[,]t_1, t_2[,]t_2, T[)$, where

$$t_2 = t_1 + \frac{\gamma - t_1}{\varphi(x(t_1)) + g}, \tag{17}$$

The values of $x_{sing} = x(t_1)$, u_{sing} and t_1 are to be determined. The corresponding trajectory is of type II.

V. SINGULAR ARC CONDITIONS

Denote by $\hat x(t)$ the trajectory of system (2) corresponding to the bang-bang control $u = (1, 0)$ on the consecutive intervals $]0, \gamma[,]\gamma, T[$, and set $x_\gamma = \hat x(\gamma)$.

To determine whether the optimal trajectory of problem (1) contains a singular arc, consider the following Cauchy problem for a system of ODEs on $]t_2, T]$, where t_2 is the right bound of M:

$$\begin{cases} \dot x = -\varphi(x) - g, & x(T) = x_T, \\ \dot\psi = -1 + \psi\varphi'(x), & \psi(T) = 0. \end{cases} \tag{18}$$

Since this system is autonomous, we can rewrite it in the reversed time $\tau = T - t$:

$$\begin{cases} \dfrac{dx}{d\tau} = \varphi(x) + g, & x(0) = x_T, \\ \dfrac{d\psi}{d\tau} = 1 - \psi\varphi'(x), & \psi(0) = 0. \end{cases} \tag{19}$$

Since $x(\tau)$ monotone increases in τ (moreover, $\dot x(\tau) > 0$), we can take x as an independent variable, thus obtaining the equation

$$\frac{d\psi}{dx} = \frac{1 - \psi\varphi'(x)}{\varphi(x) + g}, \qquad \psi(x_T) = 0. \tag{20}$$

A nice fact is that this equation can be explicitly solved. Denoting temporarily $\tilde{\varphi}(x) = \varphi(x) + g$, we obtain $\psi'(x)\,\tilde{\varphi}(x) = 1 - \psi(x)\,\tilde{\varphi}'(x)$, so $(\psi(x)\,\tilde{\varphi}(x))' = 1$, whence $\psi(x)\,\tilde{\varphi}(x) = x + c$ and $\psi(x) = (x+c)/\tilde{\varphi}(x)$. Since $\psi(x_T) = 0$, we finally obtain

$$\psi(x) = \frac{x - x_T}{\varphi(x) + g}\,, \tag{21}$$

and so,

$$\psi'(x) = \frac{\varphi(x) + g - (x - x_T)\,\varphi'(x)}{(\varphi(x) + g)^2}\,. \tag{22}$$

Denote the numerator of this fraction by $p(x)$.

If the optimal trajectory does not contain a singular subarc, we have

$$\psi'(x) > 0 \qquad \text{for} \qquad x_T < x < x_\gamma\,.$$

Otherwise, if the optimal trajectory contains a singular subarc $[t_1,t_2]$, we get $x(t) = x_{sing}$ on $[t_1,t_2]$, where the value x_{sing} is determined by the relations:

$$\psi'(x_{sing}) = 0, \qquad x_T < x_{sing} < x_\gamma\,, \tag{23}$$

which in view of (22) reads

$$p(x_{sing}) = varphi(x_{sing}) + g - x_{sing}\,\varphi'(x_{sing}) = 0, \quad x_T < x_{sing} < x_\gamma\,. \tag{24}$$

Note that $p'(x) = -x\,\varphi''(x) < 0$ for $x > 0$, so this equation can have no more than one solution. Since $p(0) = g > 0$, the existence of solution depends on the value $p(x_\gamma)$. If $p(x_\gamma) > 0$, there are no solution, if $p(x_\gamma) < 0$, equation (24) has a unique solution in the above interval, and if $p(x_\gamma) = 0$, the only solution is $x_{sing} = x_\gamma$.

VI. CLASSIFICATION OF EXTREMALS

As follows from the optimality conditions, if $(t, s(t), x(t), u(t))$, $t \in [0,T]$, $T \le T_0$ is an optimal process of (1), then only one of the following cases realises:

1) $T < T_0$ and $x(T) = 0$,
2) $T = T_0$ and $x(T) = 0$ ("treshold" case)
3) $T = T_0$ and $x(T) > 0$,

During the following considerations, we will use repserentaion (21) for $\psi(x)$. Obviously, for given $x(T) = x_T \ge 0$, we get

$$\psi(x) = \frac{x - x_T}{\varphi(x) + g} \text{ for } x \in [x_T, x_\gamma] \tag{25}$$

Singular value x_* is determined by the equation

$$p_q(x_*) = 0, \tag{26}$$

where

$$p_q(x) = \varphi(x) + g - (x - q)\varphi'(x) \text{ for } x \in [q, x_\gamma]$$

Since $p_q'(x) = -(x-q)\varphi''(x) < 0$, for given g we get unique solution x_*.

Let be given $\varphi(x), g, T_0$ and γ increases from 0.

1) Define $\hat{x}(t)$ for $t \in [0,\gamma]$ from the initial value problem
$$\dot{\hat{x}} = 1 - \varphi(\hat{x}) - g, \qquad \hat{x}(0) = 0, \tag{27}$$
set $\hat{x}_\gamma = \hat{x}(\gamma)$.
2) Define T from the initial value problem:
$$\dot{\hat{x}} = -\varphi(\hat{x}) - g, \quad \hat{x}(\gamma) = \hat{x}_\gamma, \qquad T = \int_0^{\hat{x}_\gamma} \frac{dx}{\varphi(x) + g} \tag{28}$$
3) Define $\psi(x) = \dfrac{x}{\varphi(x) + g}$ for $x \in [0, \hat{x}_\gamma]$, find $\psi'(x) = \dfrac{\varphi(x) + g - x\varphi'(x)}{(\varphi(x) + g)^2}$ and check $p(\hat{x}_\gamma)$, where $p(x) = \varphi(x) + g - x\varphi'(x)$.
   a) If $p(x) > 0$ then trajectory of type I with $x(T) = 0$, $T < T_0$.

   Note that $p'(x) = -x\varphi'' < 0$, i.e. $p(x)$ decreases while $\hat{x}_\gamma$ increases.
   Thus, there exists γ_* and corresponding x_* such that $p(x_*) = 0$, and (28) gives T_*.
   Here, we consider $T_0 > T_*$. If $T_* \ge T_0$, we have degenerate synthesis consisting only of bang-bang trajectories.

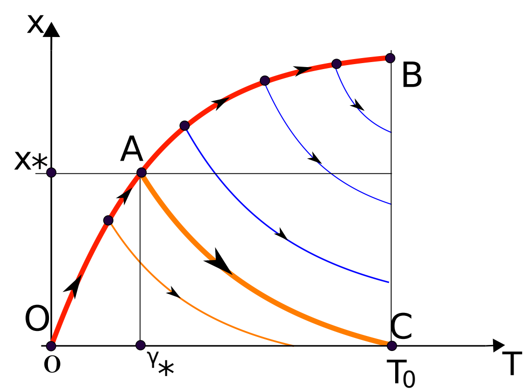

Fig. 4. Degenerate synthesis for $T_* \ge T_0$

4) Let $\gamma > \gamma_*$. Thus, we get $p(x_*) = 0$ and the trajectory is of type II with $x(T) = 0$ and singular arc x_*.
   We finish with the value γ_0 corresponding to trajectory of type II with $x(T_0) = 0$ and singular arc x_*.
5) Let $\gamma > \gamma_0$. Thus, we get
$$\psi = \frac{x - x_T}{\varphi(x) + g}$$
and
$$\psi' = \frac{\varphi + g - (x - x_T)\varphi'(x))}{(\varphi(x) + g)^2}.$$
Singular value x_* is determined by the equation $p(x_*) = 0$, where $p(x) = \varphi + g - (x - x_T)\varphi'$. As before, we get $p' = -(x - x_T)\varphi'' < 0$.

Lemma 10: $\dfrac{dx_*}{dx_T} > 0$.

Proof: According to (26),

$$p_q(x) = \varphi(x) + g - (x - q)\,\varphi'(x).$$

where $q = x_T$. Setting $q_1 = q$, we get singular value x_1 determined by equation

$$\varphi(x_1) + g - (x_1 - q)\,\varphi'(x_1), \tag{29}$$

Setting $q_2 = q + \bar{q}$, $\bar{q} > 0$, we get singular value $x_2 = x_1 + \bar{x}$ determined by equation

$$\varphi(x_2) + g - (x_2 - q - \bar{q})\,\varphi'(x_2). \tag{30}$$

Subtsrakting (30) from (29), we get

$$x_1\varphi'(x_1) - q\varphi'(x_1) - x_2\varphi'(x_2)q\varphi'(x_2) + \\ +\bar{q}\varphi'(x_2) - \varphi(x_1) + \varphi(x_2) = 0,$$

i.e.

$$-(x\varphi'(x))'(x_1)\bar{x} + q\varphi''(x_1)\bar{x} + \bar{q}\varphi'(x_2) + \varphi'(x_1)\bar{x} = 0,$$

hence

$$\bar{x} = \bar{q}\frac{\varphi'(x_2)}{\varphi''(x_1)(x_1 - q)}.$$

Thus,

$$\frac{dx_*}{dx_T} := \frac{dx_*}{dq} = \frac{\bar{x}}{\bar{q}} = \frac{\varphi'(x_2)}{\varphi''(x_*)(x_* - q)} > 0$$

by definition. ■

Thus, increasing x_T, we obtain x_* increasing too. We get bang-singular-bang trajectories. We finish with the value $\gamma = \gamma^* : x_* = \widehat{x}_{\gamma^*}$.

Thus, we get trajectory of type II with $x(T_0) = x_T$, value x_* determined by $p(x_*) = 0$ and singulart subarc bound times

$$t_1 := \int_0^{x_*} \frac{dx}{1 - \varphi(x) - g}, \tag{31}$$

and

$$t_2 := T_0 - \int_{x_T}^{x_*} \frac{dx}{\varphi(x) + g}, \tag{32}$$

bounded by value

$$t_2 := T_0 - \int_0^{x_*} \frac{dx}{\varphi(x) + g}, \tag{33}$$

where

$$\varphi(x_*) + g = x_*\varphi'(x_*) \tag{34}$$

6) Let $\gamma > \gamma^*$. Consider $\psi_1 = \dfrac{x - x_T^1}{\varphi(x) + g}$ and $\psi_2 = \dfrac{x - x_T^2}{\varphi(x) + g}$, where $x_T^2 > x_T^1$. For $x > x_T^2$, we get

$$\psi_2 = \psi_1 - \frac{x_T^2 - x_T^1}{\phi(x) + g} < \psi_1,$$

$$\psi_2' = \psi_1' + \varphi'(x)\frac{x_T^2 - x_T^1}{(\varphi(x) + g)^2} > \psi_1'$$

Thus, if $\psi_1'(x) \geq 0$, then $\psi_2'(x) > 0$. Since for $\gamma = \gamma^*$ we got $\psi'(\widehat{x}_{\gamma^*}) = 0$ then for $\gamma > \gamma^*$ we get $\psi'(\widehat{x}_\gamma) > 0$.

VII. CONSTRUCTION OF OPTIMAL SYNTHESIS

Finally, we get the following synthesis of optimal trajectories depending on value $\gamma := m_0 - m_T$:

1) $\gamma \in (0, \gamma_*]$ – the optimal trajectory is bang-bang with $x(T) = 0$, $T < T_0$
2) $\gamma \in (\gamma_*, \gamma_0]$ –the optimal trajectory is bang-singular with $x(T) = 0$, $T \leq T_0$
3) $\gamma \in (\gamma_0, \gamma^*]$ –the optimal trajectory is bang-singular with $x(T_0) > 0$,
4) $\gamma \in (\gamma^*, T_0]$ – the optimal trajectory is bang-bang with $x(T_0) > 0$.

The typical synthesis is shown on the following picture:

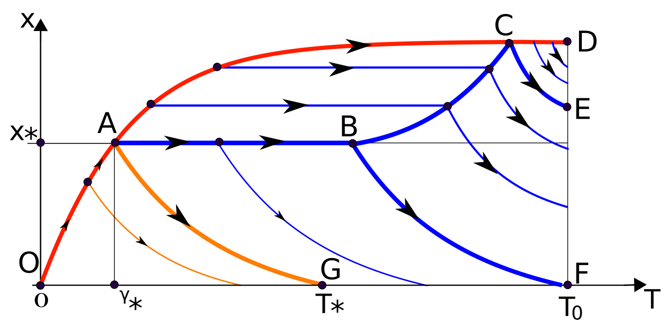

Fig. 5. Optimal synthesis in plane xT for $g > 0$. Set OAG is filled with bang-bang trajectories with $x(T) = 0$, set $ABFG$ is filled with bang-singular-bang trajectories with $x(T) = 0$, set $ACEFB$ is filled with bang-singular-bang trajectories with $x(T_0) \geq 0$, set CDE is filled with bang-bang trajectories with $x(T_0) > 0$.

Note that the set of parameters corresponding to bang-singular trajectories lies "between" sets corresponding to bang-bang trajectories.

VIII. EVOLUTION OF SYNTHESIS W.R.T. GRAVITY PARAMETER

Note that (34) yields that if g decreased then x_* decreased too. Our goal is to obtain information about t_2 dependance on g, which gives us information on B point position evolution.

Decreasing g from g_1 to $g_2 = g_1 - \bar{g}$ we get new value $x_*^2 = x_*^1 - \bar{x}$, where $\bar{x}$ is determined by the following two equations:

$$\begin{aligned} \varphi(x_{*1}) + g_1 &= x_{*1}\varphi'(x_{*1}), \\ \varphi(x_{*2}) + g_2 &= x_{*2}\varphi'(x_{*2}) \end{aligned} \tag{35}$$

read

$$\begin{aligned} \varphi(x_{*1}) + g_1 &= x_{*1}\varphi'(x_{*1}), \\ \varphi(x_{*1} - \bar{x}) + g_1 - \bar{g} &= (x_{*1} - \bar{x})\varphi'(x_{*1} - \bar{x}) \end{aligned} \tag{36}$$

substracting the second equation from the first one, we get

$$\varphi(x_{*1}) - \varphi(x_{*1} - \bar{x}) + \bar{g} = x_{*1}\varphi'(x_{*1}) - (x_{*1} - \bar{x})\varphi'(x_{*1} - \bar{x}), \tag{37}$$

i.e.

$$\varphi'(x_{*1})\bar{x} + \bar{g} = \big(x_{*1}\varphi'(x_{*1})\big)'\bar{x}), \tag{38}$$

$$\varphi'(x_{*1})\bar{x} + \bar{g} = (x_{*1}\varphi''(x_{*1})\bar{x}) + \varphi'(x_{*1})\bar{x}, \tag{39}$$

thus

$$\bar{x} = \frac{\bar{g}}{x_{*1}\varphi''(x_{*1})} \tag{40}$$

Thus, switching time changes from

$$t_2^1 := T_0 - \int\limits_0^{x_*} \frac{dx}{\varphi(x)+g}, \tag{41}$$

to

$$t_2^2 := T_0 - \int\limits_0^{x_* - \frac{\overline{g}}{x_* \varphi''(x_*)}} \frac{dx}{\varphi(x)+g-\overline{g}}, \tag{42}$$

Thus, we need to determine the sign of

$$t_2^2 - t_2^1 = \int\limits_0^{x_*} \frac{dx}{\varphi(x)+g} - \int\limits_0^{x_* - \frac{\overline{g}}{x_* \varphi''(x_*)}} \frac{dx}{\varphi(x)+g-\overline{g}} =$$
$$= \int\limits_{x_* - \frac{\overline{g}}{x_* \varphi''(x_*)}}^{x_*} \frac{dx}{\varphi(x)+g-\overline{g}} - \overline{g} \int\limits_0^{x_*} \frac{dx}{(\varphi(x)+g)(\varphi(x)+g-\overline{g})} \tag{43}$$

Using expanding in series up to order 1, we can prove that $t_2^2 - t_2^1 < 0$, thus the following fact takes place

Lemma 11: If g decreased, then switching time t_2 is decrased too.

Thus, if g decreased to 0, synthesis shown on Fig.5 transforms to the second typical form shown on Fig. 6:

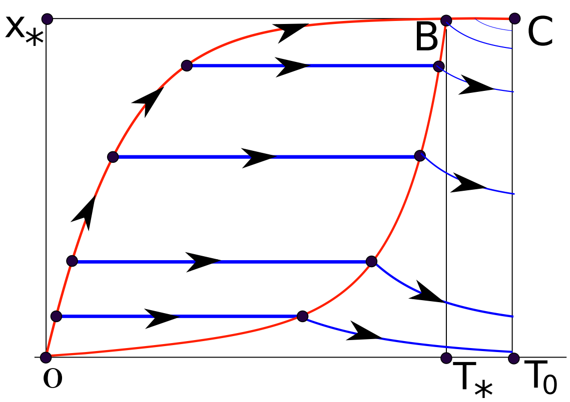

Fig. 6. Optimal synthesis in plane xOT for $g = 0$

Note also that for the linear-quadratic resistance function

$$\varphi(x) = \frac{b}{2}x^2 + cx, b > 0, c > 0$$

we get

$$x_* = \sqrt{\frac{2g}{b}},$$

hence

$$\frac{\varphi'_*}{g}(t_2^2 - t_2^1) = \frac{1}{x_*^2 \varphi''_*} + \frac{1}{\varphi_* + g} - \frac{1}{g} =$$
$$= \frac{1}{2g} + \frac{1}{2g + c\sqrt{\frac{2g}{b}}} - \frac{1}{g} = -\frac{1}{2g} + \frac{1}{2g + c\sqrt{\frac{2g}{b}}} < 0$$

and we get $t_2^2 - t_2^1 < 0$ analytically, without expanding in series.

IX. Conclusions

In this paper, a problem on maximization of final height of rocket flight under the control and endpoint constraints is consodered. In case of constrained time interval, we use optimality conditions in form of Pontryagin Maximum Principle. We obtain all types of optimal trajectories and then construct optimal synthesis using analytical form of adjoint (costate) variable. We construct the synthesis in coordinates (x,t). This allows us to investigate extremal evolution and synthesis evolution w.r.t problem parameters.

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