2018 26th Mediterranean Conference on Control and Automation (MED)
Zadar, Croatia, June 19-22, 2018

A Case Study on Determining Stability Boundaries of Parameter Uncertain Systems

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Abstract— **Guaranteeing the stability is one of the fundamental problems of control engineering. In most of the dynamical systems, parameter uncertainties can not be avoided. Thus, it is crucial from a practical point of view to propose generic methods for analyzing the stability of uncertain parameter systems. In this study, the extension of previously proposed Lyapunov Equation based stability mapping approach to the case of parameter uncertain systems is presented. Using the present method, it becomes possible to determine the explicit stability boundaries of the uncertain parameters along with the free controller parameters. Unlike most of the conventional approaches, the current method does not include any restrictions related with the number of the uncertain parameters and the way that the uncertain parameter(s) show themselves in the problem formulation. In order to demonstrate the efficiency of the proposed method, two benchmark case studies are discussed in detail. It is shown that the proposed approach is capable of increasing the accuracy of the previous results in specific cases while ensuring a flexible and easily applicable stability analysis environment for such systems.**

I. INTRODUCTION

Uncertainties are unavoidable in dynamic control systems and ignoring them may affect performance and stability characteristics of the closed-loop system in an adverse manner. In the literature, uncertainties are classified under two main groups which are unstructured and parametric uncertainties [1]. Inaccurate representation of system component characteristics, torn-and-worn effects, equipment aging, the effect of environmental conditions on system parameters may lead to perturbations in most of the industrial control systems [2] and these kinds of uncertainties can be represented as variations in certain parameters of the system. Such type of uncertainty representation is named parametric uncertainty.

Within the scope of this study, the stability of parameter uncertain systems is discussed in detail from a Lyapunov Equation point of view. At this point, it should be mentioned that the stability of parameter uncertain systems can be discussed from two different main perspectives. In the first one, it is assumed that the uncertainty bounds of the parameter are known and it is searched that the whole set of polynomials that lie in these intervals are stable or not. Significant progress was achieved in the literature to determine the stability characteristics of systems with the interval, affine-linear and multi-linear type uncertainties ([3], [4], [5], [6]). On the other hand, in some cases, it is not possible to determine strict upper and lower bounds for uncertain parameters. Therefore, it becomes more logical to determine the range of uncertain parameters that make the closed-loop system stable (or unstable). For the given problem formulation there are certain numbers of studies in the literature that discuss the stability of parameter uncertain systems from different perspectives in that sense [7], [8], [9], [10]. Most of them depend on the principles of Parameter Dependent Lyapunov Function (PDLF). However, these methods include structural constraints in general. For instance, some specific methods are only applicable for one or two uncertain parameters. Additionally, the way that uncertain parameters affect the system coefficients is also constrained in general.

In order to overcome such restrictions and difficulties, a based stability mapping approach is presented, which is based on our previous work [11]. We have shown that the proposed method is suitable for determining the bounds of free controller parameters in terms of stability. In the present paper, it is aimed to extend the range of the applicability of the presented approach to the case of parametric uncertainties and to compare it to existing state of the art PDLF based methods. The starting point is the for linear time invariant systems. By using the Kronecker product and the ”vectorization” operator, it is a well-known result that the Lyapunov Equation may be written as a linear set of equations. However, it can be shown that the determinant of the resulting matrix M is closely connected to the stability boundaries of the system and thus a direct mapping into the parameter space is possible. In contrast to other algebraic stability criteria, as for example Hurwitz, one must solve at most two nonlinear equations instead of n coupled nonlinear inequalities.

Compared to the currently existing methods dealing with uncertain robust control, the presented method provides a range of advantages. For instance, it is independent of the number of uncertain parameters and the way that these enter the system. Therefore, the proposed method is directly applicable to a broad range of uncertain systems, unlike the current existing approaches. Moreover, it can also be applied to discrete-time systems with only minor modifications since the problem is formulated in the time domain using the Lyapunov Equation. Finally, it can be very easily implemented and applied which we believe is of great interest for the

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practical usability.

The effectiveness and the efficiency of the approach are discussed over two benchmark case studies which were used in the literature. In the first case study, it is shown that accuracy of the results could also be increased using the proposed technique. Additionally, a free controller parameter is added to the original problem formulation to demonstrate that it is also possible to derive stability conditions on free controller parameters. In the second case study, it is shown that the Lyapunov Equation based approach is directly applicable to the systems that include multiple uncertain parameters without any modifications. For this reason, it can be proposed that the presented Lyapunov Equation based stability mapping technique provides a flexible analysis environment for parameter uncertain systems even in the case of multiple uncertain parameters.

II. Parameter Dependent Lyapunov Functions

In this section, it is aimed to point out the main approaches proposed in the literature from the PDLF point of view. In these methods, it is aimed to determine the ranges of uncertain parameters (q_i s) that make the following closed loop system stable

$$\dot{x} = A\left(q_1, ..., q_f\right) x \tag{1}$$

There are various methods proposed in that sense to determine the bounds of uncertain parameters [10], [12], [13], [14]. However, it can be said that, most of the studies focus on the cases single or double parameter dependencies. For instance, in [15], a technique that is named as guardian maps was proposed for the systems which are in the form:

$$\dot{x} = A\left(q\right) x, \quad A\left(q\right) = A_0 + qA_1 + q^2 A_2 + ... + q^m A_m \tag{2}$$

and

$$\dot{x} = A\left(q_1, q_2\right) x, \quad A\left(q_1, q_2\right) = \sum_{i_1, i_2 = 0}^{i_1 + i_2 = m} q_1^{i_1} q_2^{i_2} A_{i_1, i_2} \tag{3}$$

Using the guardian map approach proposed in [15], it is possible to determine necessary and sufficient conditions for the given uncertainty domains. Here, it should be pointed out that the proposed Lyapunov approach in this paper can also be applied to the type of systems defined in (2). In [16], derived results were extended for the system class that is expressed as:

$$\begin{gathered} \dot{x} = A\left(q_1, q_2, ...q_m\right) x \\ A\left(q_1, q_2, ...q_m\right) = A_0 + \sum_{i=1}^{m} q_i A_i \end{gathered} \tag{4}$$

However, in that method it is only possible to derive sufficient conditions as indicated in [17]. The proposed approaches, given in [15], [16] focus on determining the stability characteristic of the system for given uncertainty bounds. Nonetheless, parameter dependent Lyapunov function method can also be used to derive exact bounds of uncertain parameters. First, single parameter dependency will be discussed. For this purpose, consider the LTI system

$$\dot{x} = A\left(q\right) x, \quad A\left(q\right) = A_0 + qA_1 \quad q \in \mathbb{R} \tag{5}$$

Using the main Lyapunov methodology, it can be stated that the following conditions should be satisfied for the system that is given in terms of stability

$$P(q) > 0 \tag{6}$$

$$A(q)^T P(q) + P(q) A(q) < 0 \tag{7}$$

Equation (7) can be written as matrix equality as:

$$A(q)^T P(q) + P(q) A(q) = -Q(q) \tag{8}$$

where $Q(q) \in \mathbb{R}^{n \times n}$ is any positive definite matrix for all values of uncertain parameters. The solution $P(q)$ of (8) can be written as [18]:

$$P(q) = \int_0^{\infty} e^{tA(q)^T} Q(q) e^{tA(q)} \, \mathrm{dt} \tag{9}$$

When $Q(q)$ is analytic in q, it can be directly concluded that $P(q)$ is also analytic in q. As a result, the solution can be expressed as the sum of infinite power series as:

$$P(q) = P_0 + qP_1 + q^2 P_2 + ... = \sum_{i=0}^{\infty} q^i P_i \tag{10}$$

It was shown in [19] that using the uniform convergence of the integral that is given in (9), infinite power series can be truncated and Lyapunov matrix can be expressed in the following form:

$$P(q) = P_0 + qP_1 + q^2 P_2 + ... + q^m P_m = \sum_{i=0}^{m} q^i P_i \tag{11}$$

Nevertheless, an upper bound for m was not proposed in [19]. Whereas, it was shown in [18] that it is necessary and sufficient to select m as:

$$m \leq \min\left\{ \frac{1}{2}\left(2nr - r^2 + r\right), \left(\frac{1}{2} n(n+1) - 1\right) \right\} \tag{12}$$

for the stability of whole uncertain parameters $q \in \Phi$ where Φ refers to the set of the uncertain parameter. In (12), r represents the rank of A_1. Using these results, it becomes possible to propose the stability range of uncertain parameter in the sense of Lyapunov. Details of the approach can be found in [18]. Several PDLF based approaches were discussed in these sections. It is possible to derive results, if the pre-determined assumptions on the system class and the number of uncertain parameters hold. As a result, it can be seen that most of the previously proposed approaches are case specific. Within the scope of this study, it is aimed to propose an alternative method in order to determine the bounds on uncertain and/or free system parameters. The proposed Lyapunov Equation based approach will be discussed in detail in the following section.

III. Lyapunov Equation Based Stability Mapping Approach

In this section, a Lyapunov Equation based stability mapping approach is proposed to determine the stabilizing parameter space of a given parameter uncertain system. Most of the classical parameter space approaches require frequency sweeping [20] which increases the required computational effort. Additionally, in such approaches, the accuracy of the stability boundaries depends on the step size of the frequency sweeping. However, in the present approach frequency sweeping is eliminated since the problem is directly defined in the time domain. A parameter uncertain LTI closed loop system can be expressed as

$$\dot{x} = A(k,q)x, \quad x \in \mathbb{R}^n \tag{13}$$

where $k \in \mathbb{R}^p$ represents the controller parameters and $q \in \mathbb{R}^m$ stands for the uncertain parameters. It is clear that the dimensions of k and q depend on the controller type and the number of uncertain parameters. Here, it is aimed to determine for which values of the given free and uncertain parameter(s) (k,q), the closed loop system remains stable. For this purpose, the can be reformulated as:

$$A(k,q)^T P(k,q) + P(k,q)A(k,q) = -Q \tag{14}$$

Actually, the matrix equation that is given in (14) is a special case of Sylvester Equation which can be represented as

$$AX + XB = C \tag{15}$$

As indicated in [21], for the existence and uniqueness of the solution for a Sylvester equation, A and $-B$ should not have any common eigenvalues. As a result, it can be proposed that, in the Lyapunov Equation case, the matrices $A(k,q)$ and $-A(k,q)^T$ should not have common eigenvalues in terms of the existence and uniqueness of the solution. Since a matrix A and its transpose A^T have the same eigenvalues, this also corresponds to the case where A has symmetric eigenvalues with respect to the imaginary axis in the Lyapunov Equation. The Lyapunov Equation represented in (14) is a special matrix equation and it is possible to transform it to a set of linear equations using the Kronecker product and vectorization operator. The Kronecker product is defined as:

$$A \otimes B := \left[a_{ij}B\right] = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \tag{16}$$

On the other hand, the vectorization operator vec($\cdot$) transforms an $n \times m$ matrix to an $nm \times 1$ vector by rearranging the matrix entries column after column. As a result, vectorization operator is defined as follows:

$$\mathrm{vec}(B) := \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^{nm\times 1} \tag{17}$$

where b_i represents the i-th column vector of the original B matrix. Using the Kronecker product and vectorization operator, the can be rewritten as:

$$\left(I \otimes A^T(k,q) + A^T(k,q) \otimes I\right) \mathrm{vec}(P(k,q)) = -\mathrm{vec}(Q) \tag{18}$$

where I is the $n \times n$ identity matrix. The equation, which is given in (18), is in the linear set of equations representation. Defining the new $M(k,q)$ matrix as

$$M(k,q) = \left(I \otimes A^T(k,q) + A^T(k,q) \otimes I\right) \tag{19}$$

all entries of $P(k,q)$ can be determined from

$$\mathrm{vec}(P(k,q)) = M^{-1}(k,q)\mathrm{vec}(-Q) \tag{20}$$

With respect to the Lyapunov Theorem, $P(k,q)$ must be a positive definite matrix for guaranteeing the stability of (13). All leading principal minors of $P(k,q)$ must be positive in terms of stability. Since all entries of $P(k,q)$ can be determined from (20), it becomes possible to determine all the leading principal minors of $P(k,q)$. However, parametric calculation of the leading principal minors requires high computational effort.

Considering the numerators and denominators of these leading principal minors, it can be proposed that $2n$ symbolic equations need to be solved in order to determine stability boundaries. However, this computational complexity can be reduced by analysing equation (20) in detail. All the denominator elements of the $P(k,q)$ matrix that is given in (20) are equal to the determinant of $M(k,q)$. As a result, the denominators of the leading principal minors of $P(k,q)$ only include the determinant of $M(k,q)$ and its increasing powers. So that, it can be proposed that it is sufficient to solve only the determinant of $M(k,q)$ to check the denominators of the leading principal minors of $P(k,q)$. The required number of equations that should be solved in order to determine stabilizing parameters are reduced to $n+1$ by this analysis. However, solving $n+1$ symbolic equations still needs a high computational effort. On the other hand, significant reductions on the computational complexity may occur if the relations between the $A(k,q)$, $P(k,q)$ and $M(k,q)$ matrices are analyzed in detail. Using (19), another important relation between the eigenvalues of $A(k,q)$ and the determinant of $M(k,q)$ can be given as [22]:

$$|M(k,q)| = \prod_{i=1}^{n}\prod_{j=1}^{n}(\lambda_i + \lambda_j) \tag{21}$$

where $\lambda_1, ..., \lambda_n$ are the eigenvalues of the $A(k,q)$.

A stable continuous time LTI system may become unstable in three different ways as indicated by the crossing of the three different types of stability boundaries in the Parameter Space Approach (PSA) [20]. The PSA method is using the characteristic equation for the stability analysis. The stability boundaries, which bound the stabilizing paramter space are named as: Real Root Boundary (RRB $s = 0$), Complex Root Boundary (CRB $s = \pm j\omega$ and Infinite Root Boundary (IRB $s \to \infty$). Focusing on the Lyapunov Equation based mapping method, according to [23], it can be proposed that it is necessary and sufficient to determine the parameter values that make

$$|M(k,q)| = 0 \quad \text{and} \quad |M(k,q)| \to \infty \tag{22}$$

under the condition that $A(k,q)$ does not have any symmetric eigenvalues with respect to the imaginary axis. In general $|M(k,q)|$ can be expressed in the rational form as:

$$|M(k,q)| = \frac{m_{num}(k,q)}{m_{den}(k,q)} \tag{23}$$

where $m_{num}(k,q)$ and $m_{den}(k,q)$ are the numerator and denominator polynomials respectively. It is clear that when $|M(k,q)|$ is a pure polynomial then $m_{den}(k,q)$ can be taken as 1. The solutions of (22) respectively leads to $m_{num}(k,q) = 0$ and $m_{den}(k,q) = 0$. The intersection points of the IRB and RRB or IRB and CRB lead to zero by zero division and the value of $|M(k,q)|$ is undefined in these cases. It can be interpreted that some eigenvalues may traverse to the right or left half plane by crossing the origin or over the infinity in such cases.

Solutions of (22) divide the whole parameter space into several subregions in terms of stability. A parameter pair can be selected from each subregion and the stability characteristics of each region can be determined.

When $|M(k,q)|$, which is given in (21), is analysed in detail, it can be observed that duplicated products of eigenvalue pairs are included. For instance, both $(\lambda_1 + \lambda_2)$ and $(\lambda_2 + \lambda_1)$ are included in (21). However, it is sufficient to check only one of them in terms of stability.

Since both $P(k,q)$ and Q are symmetric matrices, these redundant multipliers can be eliminated using transformations. Any given $n \times n$ symmetric S matrix, includes only $n(n+1)/2$ unique elements. $\overline{\mathrm{vec}}(S) \in \mathbb{R}^{n(n+1)/2}$ that only includes these unique elements can be written as:

$$\overline{\mathrm{vec}}(S) = \begin{bmatrix} S_{11} & \dots & S_{n1} & S_{22} & \dots & S_{n2} & \dots & S_{nn} \end{bmatrix}^{\mathrm{T}} \tag{24}$$

It can be asserted that for any given symmetric S matrix, there exists a full column rank transformation $D_n \in \mathbb{R}^{n^2 \times n(n+1)/2}$ such that:

$$\mathrm{vec}(S) = D_n \overline{\mathrm{vec}}(S) \tag{25}$$

In the literature, this transformation matrix D_n is named duplication matrix and as indicated in [17], it is independent of the entries of the S matrix and only depends on the dimension of S. By using the duplication matrix, (18) can be rewritten as follows:

$$M_T(k,q)\overline{\mathrm{vec}}(P(k)) = \overline{\mathrm{vec}}(-Q) \tag{26}$$

where

$$M_T(k,q) = D_n^{+} M(k,q) D_n \tag{27}$$

In (27), D_n^{+} represents the Moore-Penrose inverse of the duplication matrix D_n. In literature, the matrix D_n^{+} is also named as the elimination matrix.

As a result, it can be proposed that all the unique entries of the original $P(k,q)$ matrix can be determined from:

$$\overline{\mathrm{vec}}(P(k,q)) = M_T^{-1}(k,q)\overline{\mathrm{vec}}(-Q) \tag{28}$$

It is also possible to set a relation between the eigenvalues of the $A(k,q)$ and the $|M_T(k,q)|$. Using the Kronecker Sum and elimination matrix properties that is obtained in [24], it can be proposed that the determinant of the new $n(n+1)/2 \times n(n+1)/2$ dimensional $M_T(k)$ matrix can be expressed as:

$$|M_T(k,q)| = \prod_{i=1}^{n} \prod_{j \geq i}^{n} (\lambda_i + \lambda_j) \tag{29}$$

Compared to (21), it can be concluded that redundant multiplications are eliminated in (29). By this further analysis, now it is possible to determine the stability boundaries of a given system by calculating the determinant of a $n(n+1)/2 \times n(n+1)/2$ dimensional $M_T(k,q)$ instead of a $n^2 \times n^2$ dimensional $M(k,q)$ matrix.

IV. Case Studies

In this section, two benchmark case studies are included to verify the correctness and the efficiency of the proposed Lyapunov Equation based stability mapping approach. Derived results are also compared with the previously proposed PDLF based methods. Mathematica 10.3 software was used for the symbolic calculations.

A. Case Study I

In order to demonstrate the derived results, the same system that was used in [18] will be discussed as a case study. The considered system can be represented as:

$$\dot{x} = (A_0 + qA_1)\,x, \tag{30}$$

where

$$\begin{aligned} A_0 &= \begin{bmatrix} 0.7493 & -2.4358 & -1.6503 \\ -2.0590 & -3.3003 & -1.4833 \\ -1.5019 & 1.2149 & -4.8737 \end{bmatrix} \\ A_1 &= \begin{bmatrix} 1.2149 & 1.6640 & -2.2091 \\ 0.7542 & -0.1501 & 0.2109 \\ 2.1990 & 0.6493 & -0.2214 \end{bmatrix} \end{aligned} \tag{31}$$

In [18] a PDLF based method was used to determine the stability boundaries, which have been given as:

$$q \in (-18.3861, -1.2729) \cup (2.1538, 3.7973) \tag{32}$$

It is also possible to determine the stabilizing regions using the Lyapunov Equation based stability mapping technique that was proposed in Section III. For the system that is given in (30), the corresponding $|M(q)|$ can be derived as:

$$\begin{aligned} |M(q)| = &-84.8832q^9 - 1035.43q^8 + 8644.17q^7 \\ &-14679.1q^6 + 60519.9q^5 - 212831q^4 + 24827.5q^3 \\ &-319418q^2 + 506036q + 2.08949 \times 10^6 \end{aligned} \tag{33}$$

The real solutions of (33) lead to the following values:

$$\begin{aligned} q_1 &= -18.38565 \qquad q_2 = -1.27289 \\ q_3 &= 2.153729 \qquad q_4 = 3.797347 \end{aligned} \tag{34}$$

The given roots of $|M(q)|$ in (34) divide the whole q space (from $-\infty$ to $+\infty$) into five regions. By selecting a specific

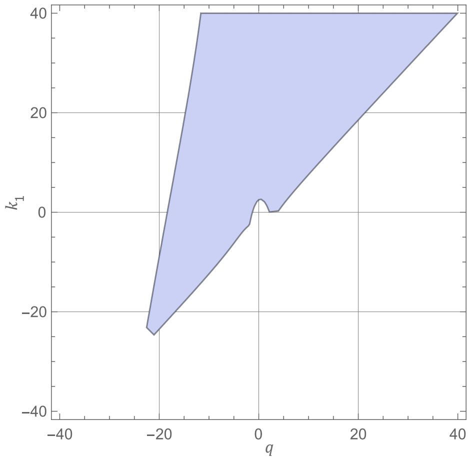

Fig. 1. The mixed controller and uncertain parameter region that make the closed loop uncertain system stable

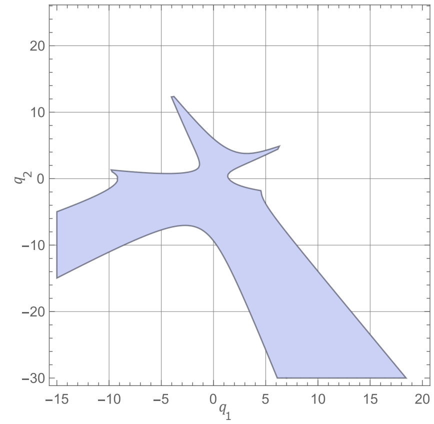

Fig. 2. The region that make the closed loop uncertain system stable.

q value from each region, the stability characteristics of that region can be determined. Hence, the following

$$q \in (-18.38565, -1.27289) \cup (2.153729, 3.797347) \quad (35)$$

stabilizing parameter intervals are determined for this case. The stabilizing parameter intervals found using the proposed approach are slightly different than the intervals given in (32). For instance, the first stability boundary was given as $q = -18.3861$ in (32), while it is determined as $q = -18.38565$. In order to verify the correctness of the results, the uncertain parameter is selected as $q = -18.3860$ and eigenvalues of the closed loop system are determined as:

$$\begin{aligned} \lambda_1 &= -11.4658 + 30.7965i \\ \lambda_2 &= -11.4658 - 30.7965i \\ \lambda_3 &= 0.0000982723 \end{aligned} \quad (36)$$

It is clear that for the given value of q the closed loop system is unstable and the result is consistent with our method. This slight variation in (32) and (35) may possibly depend on the precision levels of the solvers used in [18]. However, it must be pointed out that especially in the case of multi-linear parameter uncertainties, relatively small parameter region sets can occur in terms of stability (isolated stable/unstable point(s)) as indicated in [25]. From this point of view, higher accuracy levels are required for the correct calculation of stability boundaries. As it is shown in this case study, the proposed Lyapunov Equation based approach gives satisfactory results from this perspective.

Furthermore, it is possible to determine the stabilizing parameter space, when there are more than one parameter using our method. However, the method proposed in [18] is valid for only single parameter dependencies. For instance, assume that the first entry of the A_0 matrix ($A_{0_{11}}$) includes a free controller parameter k_1 as:

$$A_{0_{11}} = 0.7493 - k_1 \quad (37)$$

then the stabilizing parameter region can be determined as it is given in Figure 1. It can be interpreted from Figure 1, that for $k_1 = 0$, the stabilizing parameter region is identical with the stabilizing region given in (35).

B. Case Study II

In the previous case study, it was shown that the proposed Lyapunov Equation based stability mapping approach is capable of calculating stability boundaries of single parameter dependent systems with higher precision. If the number of the uncertain parameters increases, generally large modifications for PDLF based methods have to be applied. However, this is not the case for our technique and it can be demonstrated by utilizing a parameter uncertain system that includes two uncertain parameters. In this case study, the system used in [13] will be analyzed to verify the results. For this purpose, the following system can be considered

$$\dot{x} = (A_0 + q_1 A_1 + q_2 A_2)\, x \quad (38)$$

where

$$\begin{aligned} A_0 &= \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix}, \\ A_1 &= \begin{bmatrix} 0.916 & -0.8119 & -0.2168 \\ -0.6863 & -0.1001 & -0.4944 \\ -0.1673 & 0.7383 & -0.2912 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 1.215 & 1.664 & -2.209 \\ 0.7542 & -0.1501 & 0.2109 \\ 2.199 & 0.6493 & -0.2214 \end{bmatrix}. \end{aligned} \quad (39)$$

Using the Lyapunov Equation based stability mapping approach, the uncertain parameter region that correspond to a stable system is given in Figure 2. The derived region is exactly the same as the one given in [13]. As a result, it is shown that the proposed technique is also suitable for the systems that include more than one uncertain parameter. For the case of known uncertainty bounds on uncertain parameters, the proposed method can still be applicable. However, in such a case, the given (or known) uncertainty range should be intersected with the exact stabilizing parameter space in order to propose the robust stability of the given parameter uncertain system.

With the help of this case study, it was shown that the determination of the non-convex shaped stability region can be very challenging in general. Therefore, parameter space studies - like the one presented in the current paper - cannot be neglected, especially from a robust controller design point of view. Moreover, it can be stated that classical parameter space calculation methods like the Parameter Space Approach [20] cannot be easily applied here. The reason is the complicated coupling of the uncertain parameters in the characteristic equation, which makes the frequency sweeping (which is a key component in order to create the stability charts for this classical methods) nearly impossible. Consequently, the presented stability mapping approach is currently one of the most efficient methods to create such advanced, non-convex stability region charts.

V. CONCLUSION

In this study, a benchmark case for uncertain systems is presented - by utilizing the Lyapunov Equation based stability mapping approach. The current technique does not include strict conditions on the system type and the number of uncertain parameters unlike most of the previously proposed methods. Compared to these methods, the proposed approach provides various advantages like providing a flexible analysis approach, applicability to various system classes, eliminating the conditions on uncertain parameter numbers and the way that they enter the system matrix. Using the proposed method, it is also possible to determine the stability boundaries of free controller parameters in addition to uncertain parameters.

In order to verify the effectiveness of the stability mapping method, two benchmark case studies are discussed in detail. In the first case study, it is shown that the exact stability boundaries can be calculated with higher precision compared to the currently existing methods. Additionally, with the help of the second case study, it is verified that the proposed technique can be easily applied to the systems that include more than one uncertain parameter. Furthermore, the importance of parameter space based approaches is pointed out since the resulting stability region has advanced, non-linear shape. Compared to the classical parameter space approaches it can be said that the proposed method presents flexibility and computational advantage since it does not require any frequency sweeping.

As a future work, it is aimed to focus on the possible application areas. In that sense, highly automated driving systems seem suitable as a practical robust controller design problem due to their problem formulation which requires analytical methods.

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