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Towards a Nonlinear Model Predictive Control using the Extended Modal Series Method

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Abstract— **This paper addresses the Nonlinear Model Predictive Control of input-affine systems. Based on the Extended Modal Series Method, an explicit suboptimal control law is deduced that solves the two point boundary problem given by Pontryagin's Maximum Principle in the related Nonlinear Optimal Control Problem. The deduction of the control law is presented and proven. Then the proposed controller is tested in simulation in the attitude control of a quadrotor, a highly nonlinear and unstable system. The simulation results indicate good performance and near optimal behavior. This work is a background for future development of control techniques for input-affine systems.**

I. INTRODUCTION

Model Predictive Control (MPC) is an advanced control technique that consists on the implementation of an optimal controller in closed loop. This means that at each sampling instant given a model of the system and the initial conditions, an Optimal Control Problem (OCP) is solved minimizing some performance index constrained to the plant dynamics. The solution of the OCP at each instant is what gives MPC advantages over other control techniques, such as good overall performance (optimal for some cost function), the capability to handle Multiple Input Multiple Output systems and incorporate constraints. However, it also provides a great limitation in terms of speed, due to the fact that depending on the type of system, the OCP solution could take significant time.

This problem is presented mainly in nonlinear systems which often require non convex optimization. This is the reason why Nonlinear MPC (NMPC) is primarily used in slow plants. However, because of the highly desirable characteristics of the NMPC, a lot of research has been done recently to expand the applicability of this control method.

In general, there is no analytical solution of a NMPC and many existing approaches differ mainly in the algorithm used to solve the nonlinear optimization problem: Recursive Neural Networks [1] Differential Evolution [2], Sequential Quadratic Programming [3], Particle Swarm Optimization [3], Dynamic Particle Swarm Optimization [4], Optimistic Optimization [5], Neighboring External Updates [6], among others. Nevertheless, most of these approaches are restricted to systems with slow dynamics constraining the applicability of the control technique. Recently a new NMPC strategy has been proposed for input-affine systems [7]. This approach is based on the Modal Series Method, which is technique used in nonlinear analysis [8][9].

As a tool for analyzing nonlinear systems, this method was extended to solve OCP [10][11], becoming the Extended Modal Series Method (EMSM). This technique consists mainly in the application of the modal series method to the two point boundary problem given by Pontryagin's Maximum Principle (PMP) to input-affine systems [12]. The solution of this system of partial differential equations is approximated by an infinite series, and each term of the series can be calculated solving a system of ordinary differential equations. Although it is a recent method, it already has been shown to perform very well and properties such as convergence and stability have been proven [7].

This paper further elaborates on this method starting with a closed expression for these systems of differential equations, which lead to an explicit optimal control law that solves the OPC for input-affine systems and effectively reduces the NMPC problem to matrix multiplication and polynomial integration. The proposed controller is applied to a quadrotor attitude control. This system was chosen because is highly nonlinear and has very fast dynamics, making it an ideal example of the kind of systems the method is aimed at. The results show that not only the controller performs very well but also optimality measures indicate that the control is almost optimal, validating the proposed controller.

The main contribution of this paper is an analytic solution for the systems of ordinary differential equations used in the EMSM. This gives an explicit formula for the optimal states, costates and control law, solving the original OCP for input-affine systems. The main advantage being the ease of implementation and overcoming of the nonlinear optimization problem, reducing the time required to solve the OCP in each iteration allowing NMPC to be applied to highly nonlinear fast dynamic systems.

This paper is organized as follows: in section II the EMSM is introduced and the deduction of the systems of ordinary differential equations shown. In section III the main result is presented: the optimal control law deduction. In section IV the control is applied to the attitude dynamic model of a quadrotor, simulations and results are presented. Finally conclusions and further work are discussed in section V.

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II. Extended Modal Series Method for Input Affine Systems

A. Method Derivation

This subsection is based on [7]. However significant changes have been made particularly because a fix final state is used.

Consider the following input-affine dynamic system:

$$\dot{x} = F(x) + G(x)u \tag{1}$$

Where $F:\mathbb{R}^n \to \mathbb{R}^n$ and $G:\mathbb{R}^n \to \mathbb{R}^{n\times m}$ are analytic vector functions of the state $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ is the system input.

In order to implement NMPC, the optimal control problem is stated as follows:

$$\min_{u\in U} \left\{ J(x,u) = \frac{1}{2}\int_{t_o}^{t_f} \left(x^T Qx + u^T Ru\right) dt \right\} \tag{2}$$

$$s.t. \quad \dot{x} = F(x) + G(x)u \quad x(t_o) = x_o \quad x(t_f) = x_f$$

Where U is the set of admissible inputs $u(t)$, Q is a positive definite matrix and R is a positive semi definite matrix.

The necessary condition for optimality is given by the Pontryagin's Maximum Principle in the form of the following system of partial differential equations:

$$\dot{x} = F - GR^{-1}G^T\lambda \tag{3}$$

$$\dot{\lambda} = -Qx - \partial_x F^T \lambda + \begin{bmatrix} \lambda^T \partial_{x_1} GR^{-1}G^T\lambda \\ \vdots \\ \lambda^T \partial_{x_n} GR^{-1}G^T\lambda \end{bmatrix} \tag{4}$$

Subject to the boundary conditions:

$$x(t_o) = x_o \qquad x(t_f) = x_f \tag{5}$$

Where $\lambda = \lambda(t) \in \mathbb{R}^n$ are known as the costates, and the optimal control input is given by:

$$u^* = -R^{-1}G^T(x^*)\lambda \tag{6}$$

The solution to this system of nonlinear differential equations gives the optimal control as well as the state trajectory. However, it often cannot be found analytically and approximations are required.

The Extended Modal Series Method can be used to transform this problem into a series of linear problems that have analytic solutions [7]. Define the functions

$$\psi(x,\lambda) \triangleq F(x) - G(x)R^{-1}G(x)^T\lambda \tag{7}$$

$$\phi(x,\lambda) \triangleq -Qx - \partial_x F(x)^T\lambda + \begin{bmatrix} \lambda^T \partial_{x_1} G(x)R^{-1}G^T\lambda \\ \vdots \\ \lambda^T \partial_{x_n} G(x)R^{-1}G^T\lambda \end{bmatrix} \tag{8}$$

where $\psi:\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ and $\phi:\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ are nonlinear analytic vector functions (since F and G are assumed to be analytic).

With these definitions, the system of nonlinear differential equations can be written as:

$$\dot{x} = \psi(x,\lambda) \qquad \dot{\lambda} = \phi(x,\lambda) \tag{9}$$

Expanding ψ and ϕ in a Taylor Series around the point $(x,\lambda) = (0,0)$ (which is assumed to be an equilibrium point)

$$\begin{aligned}\dot{x} = &\left(\frac{\partial\psi}{\partial x}\bigg|_{\substack{x=0\\ \lambda=0}}\right)x + \left(\frac{\partial\psi}{\partial\lambda}\bigg|_{\substack{x=0\\ \lambda=0}}\right)\lambda + \frac{1}{2}\begin{bmatrix} x^T\left(\frac{\partial^2\psi_1}{\partial x^2}\big|_{\substack{x=0\\ \lambda=0}}\right)x \\ \vdots \\ x^T\left(\frac{\partial^2\psi_n}{\partial x^2}\big|_{\substack{x=0\\ \lambda=0}}\right)x \end{bmatrix} \\ &+ \begin{bmatrix} x^T\left(\frac{\partial^2\psi_1}{\partial x\partial\lambda}\big|_{\substack{x=0\\ \lambda=0}}\right)\lambda \\ \vdots \\ x^T\left(\frac{\partial^2\psi_n}{\partial x\partial\lambda}\big|_{\substack{x=0\\ \lambda=0}}\right)\lambda \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \lambda^T\left(\frac{\partial^2\psi_1}{\partial\lambda^2}\big|_{\substack{x=0\\ \lambda=0}}\right)\lambda \\ \vdots \\ \lambda^T\left(\frac{\partial^2\psi_n}{\partial\lambda^2}\big|_{\substack{x=0\\ \lambda=0}}\right)\lambda \end{bmatrix} + \dots\end{aligned} \tag{10}$$

$$\begin{aligned}\dot{\lambda} = &\left(\frac{\partial\phi}{\partial x}\bigg|_{\substack{x=0\\ \lambda=0}}\right)x + \left(\frac{\partial\phi}{\partial\lambda}\bigg|_{\substack{x=0\\ \lambda=0}}\right)\lambda + \frac{1}{2}\begin{bmatrix} x^T\left(\frac{\partial^2\phi_1}{\partial x^2}\big|_{\substack{x=0\\ \lambda=0}}\right)x \\ \vdots \\ x^T\left(\frac{\partial^2\phi_n}{\partial x^2}\big|_{\substack{x=0\\ \lambda=0}}\right)x \end{bmatrix} \\ &+ \begin{bmatrix} x^T\left(\frac{\partial^2\phi_1}{\partial x\partial\lambda}\big|_{\substack{x=0\\ \lambda=0}}\right)\lambda \\ \vdots \\ x^T\left(\frac{\partial^2\phi_n}{\partial x\partial\lambda}\big|_{\substack{x=0\\ \lambda=0}}\right)\lambda \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \lambda^T\left(\frac{\partial^2\phi_1}{\partial\lambda^2}\big|_{\substack{x=0\\ \lambda=0}}\right)\lambda \\ \vdots \\ \lambda^T\left(\frac{\partial^2\phi_n}{\partial\lambda^2}\big|_{\substack{x=0\\ \lambda=0}}\right)\lambda \end{bmatrix} + \dots\end{aligned} \tag{11}$$

Assume that the solution has the form:

$$x(t) = \sum_{i=1}^{\infty} g_i(t) \qquad \lambda(t) = \sum_{i=1}^{\infty} h_i(t) \tag{12}$$

Since the trajectory of the states and the costates depends on the time and the initial conditions, it is possible to define the functions Λ and Γ as:

$$x(t) = \Lambda(x_o,t) \qquad \lambda(t) = \Gamma(x_o,t) \tag{13}$$

Now multiplying by a factor ε the initial condition x_o, replacing in $x(t) = \Lambda(x_o,t)$ and $\lambda(t) = \Gamma(x_o,t)$ and expanding in terms of ε, it is obtained (see [7] for details):

$$x_\varepsilon(t) = \Lambda(\varepsilon x_o,t) = \varepsilon g_1(t) + \varepsilon^2 g_2(t) + \cdots = \sum_{i=1}^{\infty} \varepsilon^i g_i(t) \tag{14}$$

$$\lambda_\varepsilon(t) = \Gamma(\varepsilon\lambda_o,t) = \varepsilon h_1(t) + \varepsilon^2 h_2(t) + \cdots = \sum_{i=1}^{\infty} \varepsilon^i h_i(t) \tag{15}$$

Derivating $x_\varepsilon(t)$ and $\lambda_\varepsilon(t)$ with respect to time in (14) and (15) to obtain $\dot{x}_\varepsilon(t) = \psi(x_\varepsilon(t),\lambda_\varepsilon(t))$ and $\dot{\lambda}_\varepsilon(t) = \phi(x_\varepsilon(t),\lambda_\varepsilon(t))$ gives the following system (using (10) and (11)):

$$\begin{aligned}\varepsilon\dot{g}_1(t) + \varepsilon^2\dot{g}_2(t) + \cdots = &\ \varepsilon\left[\left(\frac{\partial\psi}{\partial x}\right)g_1(t) + \left(\frac{\partial\psi}{\partial\lambda}\right)h_1(t)\right] \\ &+ \varepsilon^2\left[\left(\frac{\partial\psi}{\partial x}\right)g_2(t) + \left(\frac{\partial\psi}{\partial\lambda}\right)h_2(t) + \frac{1}{2}\begin{bmatrix} g_1^T(t)\left(\frac{\partial^2\psi_1}{\partial x^2}\right)g_1(t) \\ \vdots \\ g_1^T(t)\left(\frac{\partial^2\psi_n}{\partial x^2}\right)g_1(t) \end{bmatrix}\right. \\ &\left. + \begin{bmatrix} g_1^T(t)\left(\frac{\partial^2\psi_1}{\partial x\partial\lambda}\right)h_1(t) \\ \vdots \\ g_1^T(t)\left(\frac{\partial^2\psi_n}{\partial x\partial\lambda}\right)h_1(t) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} h_1^T(t)\left(\frac{\partial^2\psi_1}{\partial\lambda^2}\right)h_1(t) \\ \vdots \\ h_1^T(t)\left(\frac{\partial^2\psi_n}{\partial\lambda^2}\right)h_1(t) \end{bmatrix}\right] + \dots\end{aligned} \tag{16}$$

$$\varepsilon\dot{h_1}(t)+\varepsilon^2\dot{h_2}(t)+\cdots=\varepsilon\left[\left(\frac{\partial\phi}{\partial x}\right)g_1(t)+\left(\frac{\partial\phi}{\partial\lambda}\right)h_1(t)\right]$$
$$+\varepsilon^2\left[\left(\frac{\partial\phi}{\partial x}\right)g_2(t)+\left(\frac{\partial\phi}{\partial\lambda}\right)h_2(t)+\frac{1}{2}\begin{bmatrix}g_1^T(t)\left(\frac{\partial^2\phi_1}{\partial x^2}\right)g_1(t)\\ \vdots\\ g_1^T(t)\left(\frac{\partial^2\phi_n}{\partial x^2}\right)g_1(t)\end{bmatrix}\right.$$
$$\left.+\begin{bmatrix}g_1^T(t)\left(\frac{\partial^2\phi_1}{\partial x\partial\lambda}\right)h_1(t)\\ \vdots\\ g_1^T(t)\left(\frac{\partial^2\phi_n}{\partial x\partial\lambda}\right)h_1(t)\end{bmatrix}+\frac{1}{2}\begin{bmatrix}h_1^T(t)\left(\frac{\partial^2\phi_1}{\partial\lambda^2}\right)h_1(t)\\ \vdots\\ h_1^T(t)\left(\frac{\partial^2\phi_n}{\partial\lambda^2}\right)h_1(t)\end{bmatrix}\right]+\ldots \tag{17}$$

This holds for arbitrary ε in a neighborhood of the origin, thus the coefficients that correspond to each power of ε must be equal. Generating infinite systems of ordinary differential equations, each corresponding to a power of ε and of $2n$ equations (n states and n costates):

$$\varepsilon := \begin{cases}\dot{g_1}=\left(\frac{\partial\psi}{\partial x}\right)g_1+\left(\frac{\partial\psi}{\partial\lambda}\right)h_1\\ \dot{h_1}=\left(\frac{\partial\phi}{\partial x}\right)g_1+\left(\frac{\partial\phi}{\partial\lambda}\right)h_1\end{cases} \tag{18}$$

$$\varepsilon^2 := \begin{cases}\dot{g_2}=\left(\frac{\partial\psi}{\partial x}\right)g_2+\left(\frac{\partial\psi}{\partial\lambda}\right)h_2+\frac{1}{2}\begin{bmatrix}g_1^T\left(\frac{\partial^2\psi_1}{\partial x^2}\right)g_1\\ \vdots\\ g_1^T\left(\frac{\partial^2\psi_n}{\partial x^2}\right)g_1\end{bmatrix}\\ \qquad+\begin{bmatrix}g_1^T\left(\frac{\partial^2\psi_1}{\partial x\partial\lambda}\right)h_1\\ \vdots\\ g_1^T\left(\frac{\partial^2\psi_n}{\partial x\partial\lambda}\right)h_1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}h_1^T\left(\frac{\partial^2\psi_1}{\partial\lambda^2}\right)h_1\\ \vdots\\ h_1^T\left(\frac{\partial^2\psi_n}{\partial\lambda^2}\right)h_1\end{bmatrix}\\ \dot{h_2}=\left(\frac{\partial\phi}{\partial x}\right)g_2+\left(\frac{\partial\phi}{\partial\lambda}\right)h_2+\frac{1}{2}\begin{bmatrix}g_1^T\left(\frac{\partial^2\phi_1}{\partial x^2}\right)g_1\\ \vdots\\ g_1^T\left(\frac{\partial^2\phi_n}{\partial x^2}\right)g_1\end{bmatrix}\\ \qquad+\begin{bmatrix}g_1^T\left(\frac{\partial^2\phi_1}{\partial x\partial\lambda}\right)h_1\\ \vdots\\ g_1^T\left(\frac{\partial^2\phi_n}{\partial x\partial\lambda}\right)h_1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}h_1^T\left(\frac{\partial^2\phi_1}{\partial\lambda^2}\right)h_1\\ \vdots\\ h_1^T\left(\frac{\partial^2\phi_n}{\partial\lambda^2}\right)h_1\end{bmatrix}\end{cases} \tag{19}$$

Solving the system due to ε, it gives the functions $g_1(t)$ and $h_1(t)$, then substituting in the system generated by ε^2 it becomes a system of linear time-invariant non-homogeneous ordinary differential equations with solutions $g_2(t)$ and $h_2(t)$. This process can be repeated recursively in order to obtain the functions $g_i(t)$ and $h_i(t)$ [7].

III. Explicit Control Law

Equations (16) and (17) give a constructive algorithm to form the systems of differential equations for the ω-th power of ε. However, this algorithm is inefficient and does not give insight into the method. A closed form for these systems not only reduces computational burden but also enables an analytic solution which in turn allows for the explicit optimal control law to be calculated. This analytic solution is the main result of the article:

Theorem 1: The system of ordinary differential equations for each iteration ω can be written in the following compact form:

$$\begin{bmatrix}\dot{g}_\omega\\ \dot{h}_\omega\end{bmatrix}=\begin{bmatrix}\frac{\partial}{\partial x}\psi & \frac{\partial}{\partial\lambda}\psi\\ \frac{\partial}{\partial x}\phi & \frac{\partial}{\partial\lambda}\phi\end{bmatrix}\Bigg|_{\substack{x=0\\ \lambda=0}}\begin{bmatrix}g_\omega\\ h_\omega\end{bmatrix}+\begin{bmatrix}\chi_\omega^\psi\\ \chi_\omega^\phi\end{bmatrix} \tag{20}$$

Whose solution is:

$$\begin{bmatrix}g_\omega(t)\\ h_\omega(t)\end{bmatrix}=e^{(t-t_o)A}\begin{bmatrix}g_\omega(t_o)\\ h_\omega(t_o)\end{bmatrix}+\int_{t_o}^{t}e^{(t-s)A}\begin{bmatrix}\chi_\omega^\psi(s)\\ \chi_\omega^\phi(s)\end{bmatrix}ds \tag{21}$$

Where

$$\chi_{\omega,l}^\psi=\sum_{|(\alpha,\beta)|=2}^{\omega}\left\{\frac{D^{(\alpha,\beta)}}{(\alpha,\beta)!}\psi_l(0)\Omega(\alpha,\beta,\omega)\right\} \tag{22}$$

$$\chi_{\omega,l}^\phi=\sum_{|(\alpha,\beta)|=2}^{\omega}\left\{\frac{D^{(\alpha,\beta)}}{(\alpha,\beta)!}\phi_l(0)\Omega(\alpha,\beta,\omega)\right\} \tag{23}$$

$$\Omega(\alpha,\beta,\omega)=\sum_{|(\theta,\alpha,\beta)|=\omega}\left\{\prod_{q=1}^{n}\left(\sum_{\substack{u+v=\alpha_q+\beta_q+\theta_q\\ \alpha_q\le u\le\omega\alpha_q\\ \beta_q\le v\le\omega\beta_q}}GH\right)\right\} \tag{24}$$

$$G=G(\alpha_q,u)=\sum_{\substack{\sigma(a)=u\\ |a|=\alpha_q}}\left(\frac{\alpha_q!}{a_1!a_2!\ldots a_\omega!}\right)\left(\prod_{t=1}^{\omega}g_{t,q}^{a_t}\right) \tag{25}$$

$$H=H(\beta_q,v)=\sum_{\substack{\sigma(b)=v\\ |b|=\beta_q}}\left(\frac{\beta_q!}{b_1!b_2!\ldots b_\omega!}\right)\left(\prod_{t=1}^{\omega}h_{t,q}^{b_t}\right) \tag{26}$$

$$A=\begin{bmatrix}\frac{\partial}{\partial x}\psi & \frac{\partial}{\partial\lambda}\psi\\ \frac{\partial}{\partial x}\phi & \frac{\partial}{\partial\lambda}\phi\end{bmatrix}\Bigg|_{\substack{x=0\\ \lambda=0}}\qquad \sigma(x)=\sum_{t=1}^{\omega}tx_t \tag{27}$$

With boundary conditions $x(t_o)=x_o$ and $x(t_f)=x_f$.

Before the proof of the theorem the following lemma is required:

Lemma 1.1: Let $\xi_{\omega,l}^\psi$ be the function corresponding to $\dot{g}_{\omega,l}$, that is, the component l of the function that multiplies ε^ω expanded from ψ. Likewise let $\xi_{\omega,l}^\phi$ be the function corresponding to $\dot{h}_{\omega,l}$. Then, they can be calculated as follows:

$$\xi_{\omega,l}^\psi=\sum_{|(\alpha,\beta)|=1}^{\omega}\left\{\frac{D^{(\alpha,\beta)}}{(\alpha,\beta)!}\psi_l(0)\Omega(\alpha,\beta,\omega)\right\} \tag{28}$$

$$\xi_{\omega,l}^\phi=\sum_{|(\alpha,\beta)|=1}^{\omega}\left\{\frac{D^{(\alpha,\beta)}}{(\alpha,\beta)!}\phi_l(0)\Omega(\alpha,\beta,\omega)\right\} \tag{29}$$

With $\Omega(\alpha,\beta,\omega)$ defined as in (24).

Proof: See the appendix. ■

Now the proof of Theorem 1.

Proof:

By the preceding lemma for a given ω the system of ordinary differential equations can be written by definition as:

$$\dot{g}_{\omega,l}=\xi_{\omega,l}^\psi \qquad \dot{h}_{\omega,l}=\xi_{\omega,l}^\phi \tag{30}$$

For this system the unknown functions are $g_{\omega,l}$ and $h_{\omega,l}$. Take for instance $g_{\omega,l}$, it can only appear in the sum for the multiindices:

$$\alpha=(0,0,\ldots,\overbrace{1}^{l\text{-th place}},\ldots,0)$$
$$\beta=(0,0,\ldots,0,\ldots,0)$$
$$\theta=(0,0,\ldots,\omega-1,\ldots,0)$$

Thus $g_{\omega,l}$ appears only once in the sum multiplied by $\frac{\partial}{\partial x_l}\psi(0)$. This holds for every $g_{\omega,l}$ and $h_{\omega,l}$.

Writing explicitly all these terms out of the sums and arranging in matrix form yields (20). Note that the functions (22) and (23) are the same as (28) and (29) but starting from 2 because the multiindices those sum is 1 are the ones that were taken out of the sum.

The solution of (20) is a well known fact of linear systems. ■

Theorem 1 gives an explicit solution to the systems of differential equations for g_ω and h_ω, which enables the computation of the optimal trajectories of the states x^* and costates λ^* using:

$$x^*(t) = \sum_{i=1}^{\infty} g_i(t) \qquad \lambda^*(t) = \sum_{i=1}^{\infty} h_i(t) \tag{31}$$

In turn this allows for the calculation of the optimal control law u^* with:

$$u^* = -R^{-1}G^T(x^*)\lambda^* \tag{32}$$

Solving the original Optimal Control Problem for input-affine systems (2).

In practical implementation the series in (31) will be truncated. Also, the solution of (21) can be found approximating $e^{(t-t_o)A}$ with a series expansion. As a result the exponential and the functions g_ω and h_ω will be matrices of polynomials, reducing the method to polynomial integration and multiplication.

IV. APPLICATION EXAMPLE

As an application example the attitude control of a quadrotor is presented.

A. Quadrotor Attitude Dynamic System

The attitude dynamics of a quadrotor are given by the differential equations [13]:

$$\begin{aligned} \dot{\phi} &= p + q\sin\phi\tan\theta + r\cos\phi\tan\theta & \dot{p} &= \left(\frac{I_y - I_z}{I_x}\right)qr + \frac{\tau_\phi}{I_x} \\ \dot{\theta} &= q\cos\phi - r\sin\phi & \dot{q} &= \left(\frac{I_z - I_x}{I_y}\right)pr + \frac{\tau_\theta}{I_y} \\ \dot{\psi} &= \frac{\sin\phi}{\cos\theta}q + \frac{\cos\phi}{\cos\theta}r & \dot{r} &= \left(\frac{I_x - I_y}{I_z}\right)qp + \frac{\tau_\psi}{I_z} \end{aligned} \tag{33}$$

Where:

- p, q and r are the angular rates of change of ϕ, θ and ψ.
- ϕ, θ and ψ are the Euler angles roll, pitch and yaw.
- I_x, I_y and I_z are the moments of inertia in each axis.
- τ_ϕ, τ_θ and τ_ψ are the torque inputs.

In order to apply the EMSM it is necessary to rewrite these equations in an input-affine form: Let $x = (p,q,r,\phi,\theta,\psi)^T$ be the state vector and $u = (\tau_\phi,\tau_\theta,\tau_\psi)^T$ be the input vector. Rearranging (33) yields:

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \underbrace{\begin{bmatrix} \left(\frac{I_y - I_z}{I_x}\right)qr \\ \left(\frac{I_z - I_x}{I_y}\right)pr \\ \left(\frac{I_x - I_y}{I_z}\right)qp \\ p + q\sin\phi\tan\theta + r\cos\phi\tan\theta \\ q\cos\phi - r\sin\phi \\ \frac{\sin\phi}{\cos\theta}q + \frac{\cos\phi}{\cos\theta}r \end{bmatrix}}_{F(x)} + \underbrace{\begin{bmatrix} \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{G(x)} \begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} \tag{34}$$

The vector functions $F(x)$ and $G(x)$ that appear in (1) are indicated in (34).

B. Simulations and Results

In order to obtain meaningful results and validate the proposed controller a complete quadrotor system was simulated (both the translational and angular dynamics).

Given spacial reference points, a translational control loop calculated the angular position of the quadrotor and total thrust for each sample time. These angular positions became the references for the angular control loop in which the proposed NMPC based on the EMSM was implemented.

Two types simulations are performed, the first one with linear displacements and the second one with curves. In both cases the mean error and optimality were measured.

The optimality is measured with the functions $\Delta\psi(t)$ and $\Delta\phi(t)$ defined as:

$$\Delta\psi(t) = \dot{x}(t) - \psi(t) \qquad \Delta\phi(t) = \dot{\lambda}(t) - \phi(t) \tag{35}$$

A completely optimal solution would imply $\Delta\psi(t) = 0$ and $\Delta\phi(t) = 0$, therefore these functions measure the optimality of the solutions.

C. Linear Displacements

The first trajectory has a diamond shape that induces simultaneous movements parallel to the x and y axis. The simulation results are presented in Fig. 1, the orange solid lines represent the angular references while the blue dotted lines are the variable trajectories.

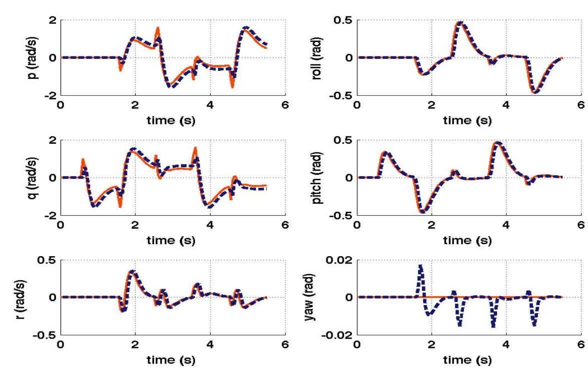

Fig. 1. Angular position. Reference (orange solid line) and trajectory (blue dotted line)

The mean error is shown in table I.

TABLE I
MEAN ERROR IN THE STATE VARIABLES

E_p	0.168 rad/s	E_ϕ	0.022547 rad
E_q	0.21115 rad/s	E_θ	0.027046 rad
E_r	0.026704 rad/s	E_ψ	0.0020553 rad

As shown in table I the angular error is almost negligible, with the maximum being 0.21115 *rad/s* for the angular rates of change and 0.027046 *rad* for the angular position.

Table II contains the mean value of optimality for the states and costates of the angular position.

TABLE II

MEAN VALUES OF OPTIMALITY

$\overline{\Delta\psi_4}$	0.44828	$\overline{\Delta\phi_4}$	4.8278×10^{-5}
$\overline{\Delta\psi_5}$	0.54161	$\overline{\Delta\phi_5}$	5.3557×10^{-5}
$\overline{\Delta\psi_6}$	0.047787	$\overline{\Delta\phi_6}$	5.0051×10^{-21}

Note that the optimality of the costates is greater than those of the states, this may be due to the high degree of nonlinearities in the system model.

D. Curve Displacements

The second trajectory is a circular motion that requires constant changes in the force direction implying constant changes in the angular position as depicted in Fig. 2.

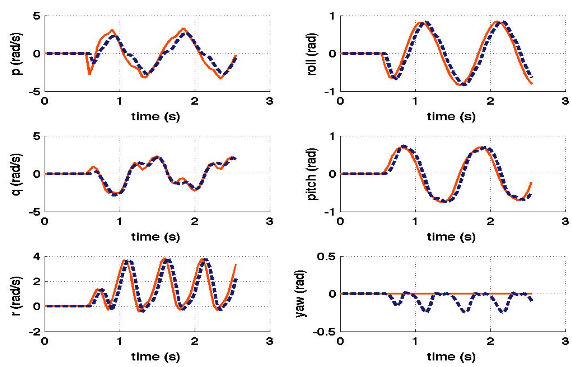

Fig. 2. Angular position, reference (orange solid line) and result (blue dotted line)

The continuous changes are the cause of the slight higher error of the states shown in table III.

TABLE III

MEAN ERROR IN THE STATE VARIABLES

E_p	0.55542 rad/s	E_ϕ	0.14503 rad
E_q	0.23032 rad/s	E_θ	0.095551 rad
E_r	0.60202 rad/s	E_ψ	0.060601 rad

As before the errors remain significantly small, the maximum being 0.60202 *rad/s* for the angular rate of change and 0.14503 *rad* for the angular position. Table IV presents the mean values form the states and costates corresponding to the angular positions.

TABLE IV

MEAN VALUES OF OPTIMALITY

$\overline{\Delta\psi_4}$	1.5446	$\overline{\Delta\phi_4}$	1.8572×10^{-4}
$\overline{\Delta\psi_5}$	1.4174	$\overline{\Delta\phi_5}$	1.9423×10^{-4}
$\overline{\Delta\psi_6}$	0.5591	$\overline{\Delta\phi_6}$	4.8266×10^{-20}

Finally, comparing the time required to compute and solve the systems of ordinary differential equations for each power of ε given by (16) and (17), the explicit formulation is faster than the constructive method. The ratio depends on the power of ε considered. If T_{exp} and T_{con} are the times for the explicit formulation and constructive method respectively, then for $\omega=2$, $T_{\text{con}}/T_{\text{exp}}=2$, for $\omega=5$, $T_{\text{con}}/T_{\text{exp}}=4$ and for $\omega=10$, $T_{\text{con}}/T_{\text{exp}}=14$.

V. CONCLUSIONS AND FUTURE WORK

In this paper a Nonlinear Model Predictive Control (NMPC) problem was addressed using a novel approach based on the Extended Modal Series Method (EMSM). This new method solves the two point boundary problem given by Pontryagin's Maximum Principle for input-affine systems expressing the solution as a series of functions that can be found solving systems of ordinary differential equations.

Until now only a constructive approach for finding these systems is known. In this paper a closed form expression for these systems of ordinary differential equations is deduced. It not only reduces computational burden but also enables the deduction of an explicit optimal control law that solves the original Optimal Control Problem (OCP) for these input-affine systems. As a result the time required to solve the OCP in each iteration of the NMPC is reduced, allowing the implementation of the NMPC to nonlinear systems with fast dynamics.

In order to test this controller a quadrotor system was simulated and the EMSM was employed for the angular control. Simulation results showed that although the dynamic system is highly nonlinear the control presents good performance and gives near optimal results.

Future work divides into two parts. First, the results presented in this paper set a background that enables further theoretical advancements such as error bounds in the optimality of the solutions and better approximations. Second, practical implementation of the explicit suboptimal control laws in order to further validate the proposed control techniques and development of efficient implementation algorithms. This also includes complete implementation for a Model Predictive Control.

APPENDIX

Proof of the lemma

Proof:

First rewrite (10) and (11) using the compact form of Taylor's Formula [14] it is an expansion in $2n$ variables:

$$\dot{x}=\psi(x,\lambda)=\psi(\Lambda,\Gamma)=\sum_{|\alpha|+|\beta|=1}^{\infty}\frac{D^\alpha D^\beta}{\alpha!\beta!}\psi(0)\Lambda^\alpha\Gamma^\beta$$
$$=\sum_{|\alpha|+|\beta|=1}^{\infty}\frac{\partial_{x_1}^{\alpha_1}\dots\partial_{x_n}^{\alpha_n}\partial_{\lambda_1}^{\beta_1}\dots\partial_{\lambda_n}^{\beta_n}}{\alpha_1!\dots\alpha_n!\beta_1!\dots\beta_n!}\psi(0)\Lambda_1^{\alpha_1}\dots\Lambda_n^{\alpha_n}\Gamma_1^{\beta_1}\dots\Gamma_n^{\beta_n} \quad (36)$$

$$\dot{\lambda}=\sum_{|\alpha|+|\beta|=1}^{\infty}\frac{\partial_{x_1}^{\alpha_1}\dots\partial_{x_n}^{\alpha_n}\partial_{\lambda_1}^{\beta_1}\dots\partial_{\lambda_n}^{\beta_n}}{\alpha_1!\dots\alpha_n!\beta_1!\dots\beta_n!}\phi(0)\Lambda_1^{\alpha_1}\dots\Lambda_n^{\alpha_n}\Gamma_1^{\beta_1}\dots\Gamma_n^{\beta_n} \quad (37)$$

Where α and β are indices with n positions each: $\alpha=(\alpha_i)_{i=1}^n$ and $\beta=(\beta_i)_{i=1}^n$ and $|\alpha|=\sum_{i=1}^n\alpha_i$. And

$$\Lambda_k=\Lambda_k(x_o,t)=\sum_{j=1}^{\infty}g_{j,k}(t) \qquad \Gamma_k=\Gamma_k(x_o,t)=\sum_{j=1}^{\infty}h_{j,k}(t) \quad (38)$$

Now, change the initial conditions $x(t_o)=\varepsilon x_o$. We have:

$$\psi(x_\varepsilon,\lambda_\varepsilon)=\psi(\Lambda(\varepsilon x_o,t),\Gamma(\varepsilon x_o,t))$$
$$=\sum_{|\alpha|+|\beta|=1}^{\infty}\frac{D^\alpha D^\beta}{\alpha!\beta!}\psi(0)\Lambda^\alpha(\varepsilon x_o,t)\Gamma^\beta(\varepsilon x_o,t)=\sum_{j=1}^{\infty}\varepsilon^j\dot{g}_j(t) \quad (39)$$

$$\phi(x_\varepsilon,\lambda_\varepsilon)=\sum_{|\alpha|+|\beta|=1}^{\infty}\frac{D^\alpha D^\beta}{\alpha!\beta!}\phi(0)\Lambda^\alpha(\varepsilon x_o,t)\Gamma^\beta(\varepsilon x_o,t)=\sum_{j=1}^{\infty}\varepsilon^j h_j(t) \quad (40)$$

and

$$\Lambda_k(\varepsilon x_o,t)=\sum_{j=1}^{\infty}\varepsilon^j g_{j,k}(t) \qquad \Gamma_k(x_o,t)=\sum_{j=1}^{\infty}\varepsilon^j h_{j,k}(t)$$

Then, the expansion becomes:

$$\psi(x_\varepsilon,\lambda_\varepsilon)=\sum_{|\alpha|+|\beta|=1}^{\infty}\frac{D^\alpha D^\beta}{\alpha!\beta!}\psi(0)\bar{\Omega}(\alpha,\beta)=\sum_{k=1}^{\infty}\varepsilon^k\xi_k^\psi \quad (41)$$

$$\phi(x_\varepsilon,\lambda_\varepsilon)=\sum_{|\alpha|+|\beta|=1}^{\infty}\frac{D^\alpha D^\beta}{\alpha!\beta!}\phi(0)\bar{\Omega}(\alpha,\beta)=\sum_{k=1}^{\infty}\varepsilon^k\xi_k^\phi \quad (42)$$

Where

$$\bar{\Omega}(\alpha,\beta)=\prod_{q=1}^{n}\Big(\sum_{k=1}^{\infty}\varepsilon^k g_{k,q}(t)\Big)^{\alpha_q}\prod_{q=1}^{n}\Big(\sum_{k=1}^{\infty}\varepsilon^k h_{k,q}(t)\Big)^{\beta_q} \quad (43)$$

The purpose is to calculate the coefficient of ε^ω for any ω.

First consider $\bar{\Omega}(\alpha,\beta)$, for now truncate the expansion of x_ε and λ_ε up to the term m. Recalling the multinomial expansion:

$$\Big(\sum_{k=1}^{m}\varepsilon^k g_{k,l}(t)\Big)^{\alpha_l}=\sum_{|k|=\alpha_l}\frac{\alpha_l!}{k_1!\dots k_m!}\Big(\prod_{t=1}^{m}g_{t,l}^{k_t}\Big)\varepsilon^{\sigma(k)} \quad (44)$$

Where $\sigma(x)=\sum_{t=1}^{m}tx_t$

Since $k=(k_1,\dots k_m)$, and $|k|=\alpha_l$, then $\min|\sigma(k)|=\alpha_l$ and $\max|\sigma(k)|=m\alpha_l$. Thus we can rewrite (44) as:

$$\left(\sum_{k=1}^{m}\varepsilon^k g_{k,l}(t)\right)^{\alpha_l}=\sum_{\rho=\alpha_l}^{m\alpha_l}F_l(\rho)\varepsilon^\rho \quad (45)$$

The same can be done for the h's, yielding:

$$\bar{\Omega}(\alpha,\beta)=\prod_{q=1}^{n}\left(\sum_{\rho=\alpha_q}^{m\alpha_q}F_q(\rho)\varepsilon^\rho\right)\prod_{q=1}^{n}\left(\sum_{\rho=\beta_q}^{m\beta_q}G_q(\rho)\varepsilon^\rho\right) \quad (46)$$

Now, note that the minimum power of ε in this expansion is the sum of the minimum powers of each term of the product, i.e. $\sum_{q=1}^{n}(\alpha_q+\beta_q)$, while the maximum power is the sum of each maximum, i.e. $\sum_{q=1}^{n}m(\alpha_q+\beta_q)$.

Then, given α and β, the product $\Lambda^\alpha\Gamma^\beta$ only contributes to ε^ω, where $|\alpha|+|\beta|\le\omega\le m(|\alpha|+|\beta|)$. Recalling that m is the limit of the truncated series if $m\to\infty$ then it is clear that for a given order ω it is unnecessary to consider vectors α and β that satisfy $\omega<|\alpha|+|\beta|$.

This allows to truncate the 'external Taylor series' (equations (36) and (37)) as well as the 'internal Taylor series' (equation (38)) up to ω instead of ∞. Furthermore we have an expression for $\bar{\Omega}$:

$$\prod_{q=1}^{n}\left(\sum_{\rho=\alpha_q}^{\omega\alpha_q}F_q(\rho)\varepsilon^\rho\right)\left(\sum_{\rho=\beta_q}^{\omega\beta_q}G_q(\rho)\varepsilon^\rho\right)=\prod_{q=1}^{n}\left(\sum_{\mu=\alpha_q+\beta_q}^{\omega(\alpha_q+\beta_q)}\chi(\mu)\varepsilon^\mu\right) \quad (47)$$

Where

$$\chi(\mu)=\sum_{\substack{u+v=\mu\\ \alpha_q\le u\le\omega\alpha_q\\ \beta_q\le v\le\omega\beta_q}}F_q(u)G_q(v) \quad (48)$$

Once more, the minimum power of (47) is the sum of the minimum powers of each term, and the maximum is the sum of all the maximum powers:

$$\prod_{q=1}^{n}\left(\sum_{\mu=\alpha_q+\beta_q}^{\omega(\alpha_q+\beta_q)}\chi(\mu)\varepsilon^\mu\right)=\sum_{W=|\alpha|+|\beta|}^{\omega(|\alpha|+|\beta|)}\Theta(W)\varepsilon^W \quad (49)$$

Where

$$\Theta(W)=\sum_{|\theta|=W-(|\alpha|+|\beta|)}\left(\prod_{q=1}^{n}\chi(\alpha_q+\beta_q+\theta_q)\right) \quad (50)$$

Going back to (41) and (42) (which now go to ω) we have:

$$\sum_{|\alpha|+|\beta|=1}^{\omega}\left(\frac{D^\alpha D^\beta}{\alpha!\beta!}\psi(0)\sum_{W=|\alpha|+|\beta|}^{\omega(|\alpha|+|\beta|)}\Theta(W)\varepsilon^W\right)=\sum_{k=1}^{\omega^2}\varepsilon^k\xi_k^\psi \quad (51)$$

$$\sum_{|\alpha|+|\beta|=1}^{\omega}\left(\frac{D^\alpha D^\beta}{\alpha!\beta!}\phi(0)\sum_{W=|\alpha|+|\beta|}^{\omega(|\alpha|+|\beta|)}\Theta(W)\varepsilon^W\right)=\sum_{k=1}^{\omega^2}\varepsilon^k\xi_k^\phi \quad (52)$$

Thus it follows directly that

$$\xi_\omega^\psi=\sum_{|\alpha|+|\beta|=1}^{\omega}\left(\frac{D^\alpha D^\beta}{\alpha!\beta!}\psi(0)\Theta(\omega)\right) \quad \xi_\omega^\phi=\sum_{|\alpha|+|\beta|=1}^{\omega}\left(\frac{D^\alpha D^\beta}{\alpha!\beta!}\phi(0)\Theta(\omega)\right)$$

In order to write explicitly $\Theta(W)$, first consider $F_q(\rho)$ and $G_q(\rho)$, which first appear in (45). Evaluating at $\varepsilon=1$ and using the multinomial theorem it follows that:

$$F_q(u)=\sum_{\substack{\sigma(a)=u\\ |a|=\alpha_q}}\frac{\alpha_q!}{a!}\left(\prod_{t=1}^{\omega}g_{t,q}^{a_t}\right) \quad G_q(v)=\sum_{\substack{\sigma(b)=v\\ |b|=\beta_q}}\frac{\beta_q!}{b!}\left(\prod_{t=1}^{\omega}h_{t,q}^{b_t}\right)$$

Which together with (48) and (50) give the desired result.

■

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