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Stability Analysis on the Networked Multi-Agent System

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***Abstract*— In this paper, a new stability analysis method on the networked multi-agent system is proposed. The main idea is to model the networked control system as a multi-output-multi-input transfer function and to apply the Nyquist criterion for the closed-loop stability. This paper provides two application examples: formation control of first- and second-order agents. The stability conditions are derived with the proposed method, showing the effect of network connectivity with multi-hop communication. The numerical simulations are conducted to verify the analysis. This work is applicable for general agent dynamics, controllers, and communication characteristics.**

I. Introduction

The distributed control of the networked multi-agent systems is getting more attention for its versatile applications: unmanned aerial vehicles (UAVs), mobile sensor networks, automated highway systems, etc. [1] However, there have been not many works considering the communication constraints on the control despite their influence on the stability of the networked control system. The communication is limited in its capacity, delayed, and often incorporates incorrect information, but its stochastic and finite-field dynamics makes analyzing its effects on the stability almost impossible.

Previous studies have coped with this issue by defining the difference between the real and transmitted states as the unknown perturbation. When the perturbed error is bounded, the necessary and sufficient conditions for the closed-loop stability were obtained [2]. In order to specify the communication constraints, Lin *et al.* [3] proposed that the stability is mainly influenced by the time delay and the major source of the delay is the accessing delay, which is constant in some protocols such as time division multiple access (TDMA) protocols [4]. Assuming the constant delay, the stability analysis is more detailed. By augmenting the state-space representation with the delayed states, either the location of the closed-loop poles [5] or the Schur stability [3], [6], [7] of the system matrix was evaluated. Schwager *et al.* [8], [9] provided the trivial but analytic solution of the stability conditions for a first- and second-order dynamics, and experimented the multi-agent network with quad-rotor UAVs.

This paper proposes a new analysis method on the networked multi-agent system that can consider general dynamics and communication characteristics including the time delay. The delay of the communication is modelled as a transfer function and the closed-loop stability of the networked system is evaluated using either Nyquist criterion or root locus. A similar approach has been taken in the consensus problem [10], which is the networked system with a first-order dynamics and uniform delays across the network. In this paper, the stability condition is derived for the general agent dynamics.

The effect of control gain, time delay, and communication graph on the stability margin is investigated, which shows different results from the previous work. In [10], the stability conditions exist for the networked first-order system, and the margin decreases when the agents are more connected with each other. In practice, however, considering that the accessing delay is the main source of the delay and multi-hop communication is possible, it is proved that the networked first-order system is always stable and the connectivity of the network increases the delay margin. The analysis is validated with numerical simulations for the two examples of the networked first- and second-order systems.

The rest of the paper is composed as follows: the formulation of the networked system into a multi-input-multi-output (MIMO) transfer function is described in section II. In section III, the stability conditions of the networked system are given, and the two case studies are analyzed. The analysis is verified through numerical simulations in section IV. The conclusions and future works are suggested in section V.

II. Problem Formulation

The block diagram of the networked system with N agents is shown in Fig. 1. The system dynamics of the physical plant, including the interconnections between N agents, is denoted as $G(s) \in \mathbb{R}^{n\times m}$. The difference between the reference input $r(t) \in \mathbb{R}^n$ and the system state $x(t) \in \mathbb{R}^n$, denoted as $e(t) \in \mathbb{R}^n$, is observed by i-th agent with the transfer function $C_i(s) \in \mathbb{R}^{l\times n}$. The observed states $y_j(t) \in \mathbb{R}^l$ are transmitted from j-th to i-th agent with the transfer function $H_{ij}(s) \in \mathbb{R}^{l\times l}$. Each agent computes its control input $u_i(t) \in \mathbb{R}^{m/N}$ based on the received outputs with the transfer function $K_i(s) \in \mathbb{R}^{m/N\times lN}$. Formulation of each transfer function and the closed-loop dynamics are discussed in this section.

A. Agent Dynamics

Consider a state-space representation for the networked agent dynamics as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where the system matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are assumed to be constant and controllable. Then, the transfer function of the MIMO system is given as:

$$G(s) = (sI_n - A)^{-1}B \tag{2}$$

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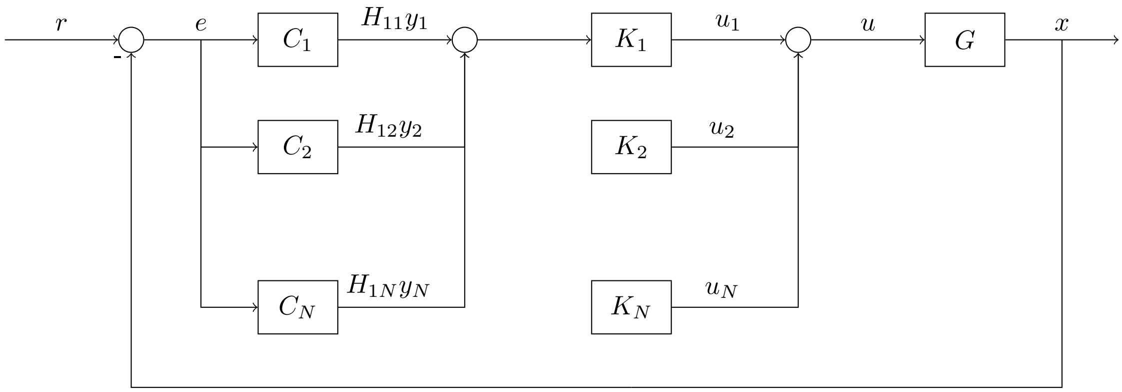


Fig. 1: Block diagram of the networked system

where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. The control law for each vehicle i is assumed to be linear as:

$$U_i(s) = -K_i(s)\sum_{j=1}^{N} H_{ij}(s)Y_j(s) \tag{3}$$

where the capital letters $U_i(s)$ and $Y_j(s)$ denote the frequency domain of $u_i(t)$ and $y_j(t)$, respectively. The observed states $Y_j(s)$ are obtained with the observers as:

$$Y_i(s) = C_i(s)X(s) \tag{4}$$

where $X(s)$ is the frequency domain of $x(t)$.

B. Communication Dynamics

The network protocol using the flooding algorithm [4] is considered for its physical applicability. The main idea of the flooding algorithm is that each agent copies the outdated outputs. Each agent takes turn to broadcast its output, while the others receive it to avoid the interference. The time slot is allocated equally by time T, similar to TDMA protocol. After all the agents take their turn, the networked agents have the outdated outputs of the connected agents with different time delays. The flooding algorithm thus enables the multi-hop communication with constant delays proportional to the time slot T. Defining the number of turns to transmit the data from j-th to i-th agent as τ_{ij}, the delayed communication is modelled as

$$H_{ij}(s) = e^{-\tau_{ij}Ts}I_l \tag{5}$$

The exponential function complicates the characteristic equation, and the closed-loop transfer function has a non-causality issue. The delay is approximated with Pade approximation to analyze the stability as

$$H_{ij}(s) \simeq \frac{1-\tau_{ij}Ts/2}{1+\tau_{ij}Ts/2}I_l \tag{6}$$

The difference between the approximated and real function exists in the time response, showing the undershoot due to the non-minimum phase zero. However, the frequency domain is well approximated and stability condition remains the same.

C. Networked Dynamics

The open-loop transfer function of the whole networked system is derived from Fig. 1 as

$$G_{OL}(s) = G(s)K(s)H(s)C(s) \tag{7}$$

where the augmented matrices are defined as

$$\begin{aligned} K(s) &= [K_1^T(s), \cdots, K_N^T(s)]^T \in \mathbb{R}^{m\times lN} \\ H(s) &= \begin{bmatrix} H_{11}(s) & \cdots & H_{1N}(s) \\ \vdots & \ddots & \vdots \\ H_{N1}(s) & \cdots & H_{NN}(s) \end{bmatrix} \in \mathbb{R}^{lN\times lN} \\ C(s) &= [C_1^T(s), \cdots, C_N^T(s)]^T \in \mathbb{R}^{lN\times n} \end{aligned} \tag{8}$$

The corresponding MIMO closed-loop transfer function from the reference input to the system states is

$$G_{CL}(s) = (I_n + G_{OL}(s))^{-1}G_{OL}(s) \tag{9}$$

III. Stability Analysis

A. General Stability Condition

The stability condition of a multivariable feedback system is derived using the Nyquist criterion. Let $\mathfrak{N}(0, I_n + G_{OL}(s), \Omega_R)$ be the number of clockwise encirclements of the point 0 by the locus of $I_n + G_{OL}(s)$ as s traverses the closed contour Ω_R in the complex plane in a clockwise sense. The Nyquist criterion states that a closed-loop system is asymptotically stable if and only if the following equation is satisfied:

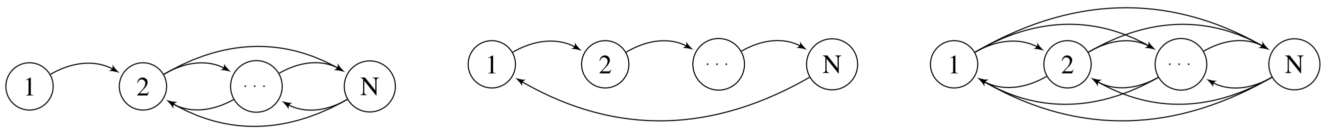
$$\mathfrak{N}(0, I_n + G_{OL}(s), \Omega_R) = -P \tag{10}$$

where P is the number of CRHP zeros of $G_{OL}(s)$, and the Nyquist contour Ω_R avoids imaginary zeros of $G_{OL}(s)$ by indentations of radius $1/R$.

The marginal stability is obtained when

$$\begin{aligned} Re(\det(I_n + G_{OL}(j\omega))) &= 0, \\ Im(\det(I_n + G_{OL}(j\omega))) &= 0. \end{aligned} \tag{11}$$

In the root locus, marginal stability is also satisfied when the closed-loop poles are on the imaginary axis. Substituting



(a) (b) (c)

Fig. 2: Communication graph

$s = j\omega$, the characteristic equation of the closed-loop transfer function is zero, resulting in the same marginal stability condition as Eqn. (11).

The gain margin is obtained by structuring the controller as $K(s) = kK_{struct}(s)$ where k is a scalar and $K_{struct}(s)$ is a scaled transfer function of the controller $K(s)$. When the networked system is open-loop stable, the system is closed-loop stable with the gain $k = 0$. Suppose that the closed-loop system is marginal stability with the gain k_c. Then, the closed-loop system is asymptotically stable for $k < k_c$. The delay margin for T is obtained such that $k_c > 0$.

B. Stability Condition for Uncoupled System

For the uncoupled linear systems, the networked system dynamics can be separated into each agent's individual dynamics as:

$$\begin{aligned} G(s) &= I_N \otimes G_i(s) \\ C(s) &= \mathbf{1}_{N\times 1} \otimes C_i \end{aligned} \tag{12}$$

where $G_i(s) = (sI - A_i)^{-1} B_i$ is the individual dynamics, and $\otimes$ is the Kronecker product.

Assuming that the number of output and control input is the same, i.e. $l = m$, the matrices $K_s(s) \in \mathbb{R}^{m\times l}$ and $H_s(s) \in \mathbb{R}^{l\times l}$ exist such that

$$K(s)H(s) = K_s(s) \circ H_s(s) \tag{13}$$

where $\circ$ is the Hadamard product.

Then, the open-loop transfer function can be simplified from the properties of the Kronecker product as:

$$G_{OL}(s) = (K_s(s) \circ H_s(s)) \otimes (G_i(s)C_i(s)) \tag{14}$$

When the rank of the matrices B_i and C_i is 1, the characteristic equation of the close-loop transfer function is simplified to

$$\det\left(I_N + \frac{b_i K_s(s) \circ H_s(s)}{\det(sI - A)}\right) = 0 \tag{15}$$

where b_i is the non-zero element of the matrix B_i.

C. Case Study I: Formation Control of First-Order System

The formation control is a representative application of the networked but dynamically uncoupled system. The relationship between the control gain and time delay of the formation control is analyzed by implementing a simple P-gain controller as

$$K_{s,ij} = \begin{cases} k, & \text{if } i \neq j \\ -N_i k, & \text{if } j \in \mathcal{N}_i \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

where $\mathcal{N}_i$ is a set of agents that are transmitting their outputs to the agent i, and N_i is the cardinality of the set $\mathcal{N}_i$. For instance, the communication graph (a) in Fig. 2 is a topology for the leader-follower formation control where $N_1 = 0$. The graphs (b) and (c) are examples of fully spanned networks, i.e. $N_i > 0$ for all i.

The dynamics of the first-order system can be generalized as

$$A_i = -a, \quad B_i = b, \quad C_i = 1 \tag{17}$$

where a and b are positive real constants. Note that the state is position in case of formation control, but the consensus problem is a special case of the networked first-order agents with $a = 0$ and $b = 1$. The stability of the consensus algorithm can be thus analyzed the same.

Theorem 1: The networked first-order system is closed-loop stable for all k and T regardless of the communication topology.

Proof: For the given first-order system, the characteristic equation for closed-loop transfer function is obtained from Eqn. (15) as

$$\det\left(I_N + \frac{bK_s(s) \circ H_s(s)}{s + a}\right) = 0 \tag{18}$$

The Nyquist plot of the transfer function Eqn. (18) is shown in Fig. 3. The figure is plotted with the communication graph for three agents in Fig. 2. The Nyquist plot of the communication graph (a) is distinct from the other two, (b) and (c), and the stability condition corresponds with the network with two agents as previously expected. For all the communication graphs, as there is no unstable pole in the open-loop transfer function and the Nyquist contour does not encircle 0, the system is closed-loop stable.

Also, the analytic solution of the closed-loop poles is derived from the characteristic equation as

$$\begin{aligned} s = &- \frac{f(k,T)}{c_3 T} \\ &\pm \frac{\sqrt{f(k,T)^2 - c_4 abkT^2 - c_5 bkT - c_6 T}}{c_3 T} \end{aligned} \tag{19}$$

where $f(k,T) = aT + c_1 bkT + c_2$ and c_i's are non-negative real constants which vary depending on the communication

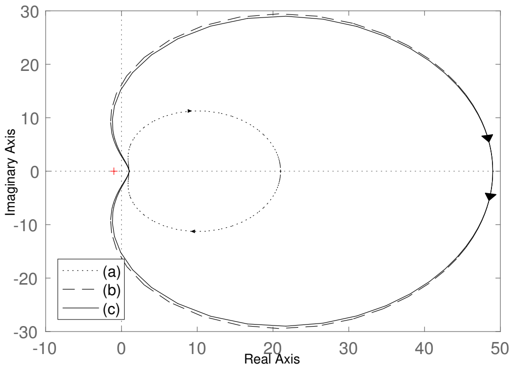


Fig. 3: Nyquist plot of the networked first-order system with the network topology (a), (b), and (c) in Fig. 2

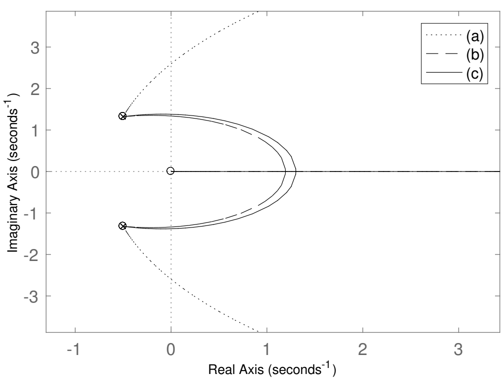


Fig. 4: Root locus of the networked second-order system with the network topology (a), (b), and (c) in Fig. 2

graph. Therefore, the closed-loop poles stay on the left-hand-plane regardless of the communication topology. ■

Theorem 1 is different from [10], where the delay exists uniformly even for the agent itself. Under the assumption that the accessing delay of the communication is the main source of the delay, the delay from the output to the control input of the agent itself is negligible, and the formation control of the first-order system is always stable.

D. Case Study II: Formation Control of Second-Order System

The stability of the formation control of second-order system is analyzed. Defining the state vector as its position and velocity, the dynamics of the second-order system is generalized as

$$A_i = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ c \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{20}$$

where b and c are positive real constants.

Theorem 2: Suppose that the network is fully spanned, i.e. $N_i > 0$ for all i. The gain and delay margin of the networked second-order system increases as the communication graph is more connected.

Proof: The characteristic equation for closed-loop transfer function is

$$\det\left(I_N + \frac{cK_s(s) \circ H_s(s)}{s^2 + bs + c}\right) = 0 \tag{21}$$

The root locus of the transfer function Eqn. (21) is shown in Fig. 4. The figure is plotted with the communication graph in Fig. 2 with three agents. The effect of delay is shown as the non-minimum phase zero and the perturbation in the poles. As the gain k increases, two poles cross the imaginary axis, implying that there exists a single lower bound, k_c, for the gain k.

The marginal stability conditions are derived with respect to different communication graphs in Fig. 2. When the network is fully connected as the graph (c), the matrix in Eqn. (21), of which the determinant is the characteristic equation, is symmetric with the diagonal terms $s^2 + bs + c + ck(N-1)$ and the off-diagonal terms $-ck(1 - sT/2)/(1 + sT/2)$. For the determinant to be zero, the following equation is obtained.

$$\begin{aligned} &(s^2 + bs + c + ck(N-1))(1 + sT/2) \\ &= -ck(1 - sT/2) \text{ or } (N-1)ck(1 - sT/2) \end{aligned} \tag{22}$$

Substituting $s = j\omega$, the closed-loop poles are on the imaginary axis if either of the following equations are satisfied.

$$\begin{aligned} &\begin{cases} -T\omega^3 + (ck(N-2)T + cT + 2b)\omega = 0 \\ -(bT+2)\omega^2 + 2ckN + 2c = 0 \end{cases} \\ \text{or } &\begin{cases} -T\omega^3 + (2ck(N-1)T + cT + 2b)\omega = 0 \\ -(bT+2)\omega^2 + 2c = 0 \end{cases} \end{aligned} \tag{23}$$

Considering that both the time slot T and the control gain k are positive, the condition for the closed-loop stability is obtained as

$$k < \frac{b}{c} \cdot \frac{(2/T)^2 + b(2/T) + c}{2(2/T) - (N-2)b} \triangleq k_c, \quad T < \frac{4}{(N-2)b} \tag{24}$$

When the network is less connected as the graph (b), the matrix in Eqn. (21) has larger delays in the off-diagonal terms. Considering that the off-diagonals are proportional to the gain k, it is evident that the critical gain k_c decreases. However, when one of the agents is not dependent of any other agents and the others are fully connected as the graph (a), the stability margin is the same with the fully connected graph (c). Therefore, under the assumption that the whole network is spanned by the communication graph, it can be concluded that the more the control network is connected, the more robust the networked system is to the time delay. ■

For the networked system with single-hop communication with uniform delay, the bound is known to be decreased in the order of $1/\lambda(L)$, where $\lambda(\cdot)$ is the eigenvalue of a matrix and L is the Laplacian matrix of a communication

Variable	Value
a	0.5
b	1
c	2
Δt (sec)	0.01

TABLE I: Formation control specification

graph [10]. The tendency is reversed in Theorem 2 as multi-hop communication is used and the network is designed as $N_i > 0$.

Note that the resultant stability condition contradicts with the previous work [8]. For the same second-order system, the trivial solution from the previous work is periodic as:

$$T = \frac{\pi m}{\omega}, \quad \omega = \frac{1}{2}\sqrt{-b^2 + 4(1 + N_i k)c} \tag{25}$$

where m is a positive integer. This research provides more intuitive solution with lower degree of characteristic equation, simplifying the analysis.

IV. Numerical Simulations

A. Simulation Setup

The numerical simulations for the case studies are conducted to verify the theoretic stability analysis and demonstrate the system performance. The values for the agent dynamics are specified in Table I. The reference input for the desired formation is given as a constant, $r(t) = [0, 0.5, -1]^T$ m. The communication graph is set with three agents as (a) in Fig. 2.

B. Simulation Results

The distance between the agents is shown in Fig. 5 and Fig. 6 with different control gains and time delays. The distance between the agent 1 and 2 is plotted in the solid line, and that between the agent 1 and 3 is shown in the dashed line.

For the networked first-order system, the response is stable regardless of the control gain and time delay, as theoretically proven in the previous section. For all time slot T and control gain k, the formation is stable. The tracking performance, however, varies with respect to the control gain due to its effect on the steady-state error.

The result also proves that the stability of networked second-order system depends on the control gain. When the gain k is small, the steady-state error occurs for the damping effect of c in the system dynamics, but the distance between each agents is stable. Increasing the gain up to the critical value k_c, the response shows divergence. The critical value for the control gain differs from the theoretic value for its discrete-time simulation, but provides an approximation to the real value.

V. Conclusions

In this paper, a new stability analysis method for the networked multi-agent system has been proposed. The main idea is to model the networked system including the communication dynamics into a MIMO transfer function and to apply the Nyquist criterion for the closed-loop stability. This paper suggested two case studies with the formation control of the first- and second-order system. Applying the proposed method, it was that the first-order networked system is stable in all time delays and control gains, while the second-order networked system has a stability margin, which increases when increasing the connectivity of the communication graph. Numerical simulations supported the theoretical analysis.

The strength of the proposed method lies on that it can analyze any agent dynamics, controllers, and communication characteristics. For the future work, stability of the dynamically coupled system, e.g. slung-load system [11], will be analyzed. Also, for compensating the uncertainty and nonlinear dynamics, more complicated controllers including nonlinear controllers will be considered. Approximation of the stochastic packet dynamics into the deterministic transfer function will extend the feasibility of the proposed analysis.

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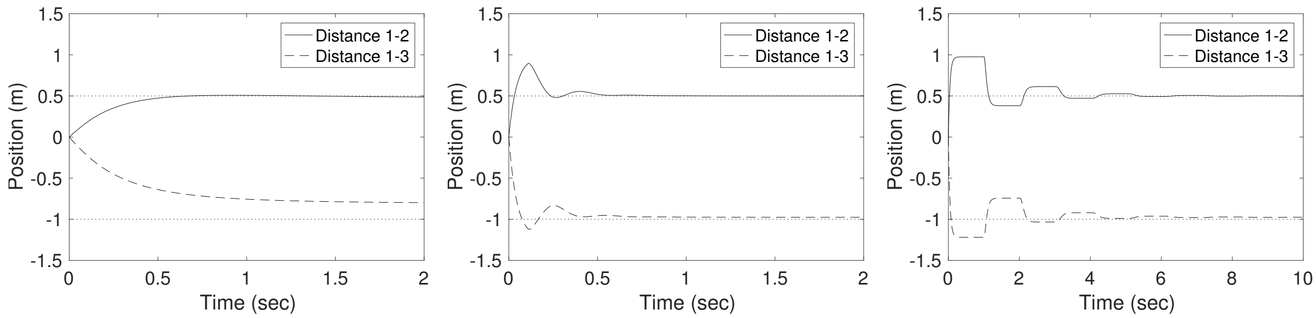
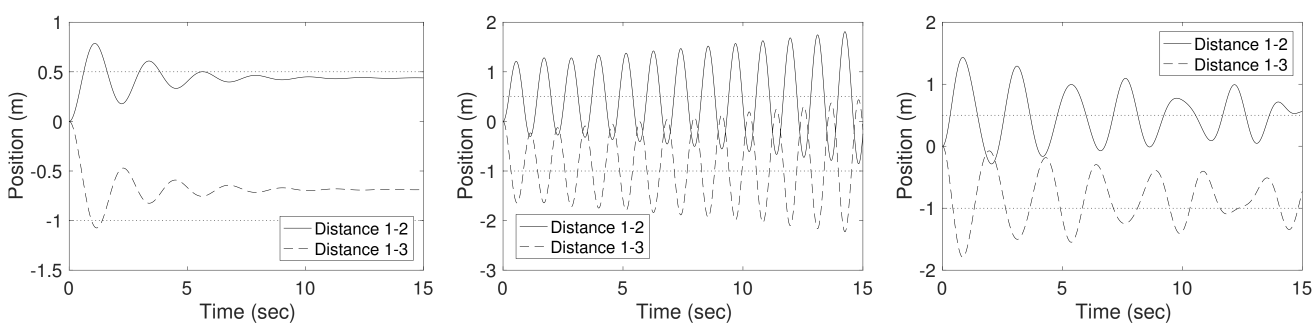


(a) $k = 1$, $T = 0.1$sec (b) $k = 10$, $T = 0.1$sec (c) $k = 10$, $T = 1$sec

Fig. 5: Formation control of the first-order system



(a) $k = 1$, $T = 0.1$sec (b) $k = k_c$, $T = 0.1$sec (c) $k = k_c$, $T = 1$sec

Fig. 6: Formation control of the second-order system