2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Distributed sampled-data control of Kuramoto-Sivashinsky equation under the point measurements

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***Abstract*— We consider sampled-data distributed control of nonlinear PDE system governed by Kuramoto-Sivashinsky equation under point measurements and distributed in space shape functions. It is assumed that the sampling intervals in time and in space are bounded. We derive sufficient conditions ensuring local exponential stability of the closed-loop system in terms of Linear Matrix Inequalities (LMIs) by using Lyapunov-Krasovskii method. Moreover, we give a bound on the domain of attraction. As it happened in the case of heat equation, the time-delay approach to sampled-data control and the descriptor method appeared to be efficient tools for the stability analysis of the sampled-data Kuramoto-Sivashinsky equation.**

I. INTRODUCTION

The Kuramoto-Sivashinsky equation (KSE) is a model which describes a variety of physical and chemical phenomena, such as flame front propagation and flowing thin films on inclined surface (see e.g. [1], [2], [3]). There are several works that study the KSE with distributed control (see e.g. [4], [5], [6], [7]) and boundary control (see e.g. [3], [8]). In [1], a distributed finite-dimensional feedback control algorithm for stabilization of KSE was proposed. In [3], for small values of anti-diffusion parameter, global exponential stability of the closed-loop system in the L^2-sense and asymptotic stability of the closed-loop system in the H^2-sense were proved. In [4], [5], a finite-dimensional controller was designed on the basis of a finite-dimensional system that captures the dominant (slow) dynamics of the infinite-dimensional system. Model decomposition techniques were used for sampled-data case in [7]. The latter approach is a qualitative one without giving an upper bound of the domain of attraction and decay rate. In [8], based on backstepping method, the local rapid stabilization problem for the KSE was studied.

In recent years, a considerable amount of attention has been paid to stability and distributed control of systems described by partial differential equations (PDEs). Distributed sampled-data control of semilinear diffusion equations with globally Lipschitz nonlinearity was studied in [9], [10], [11]. However, the problem of feasible sampled-data controller design for other classes of nonlinear parabolic systems has not been studied yet, and this problem is rather challenging. The objective of this paper is to design a stabilizing sampled-data controller for 1-D KSE by using spatially point measurements available in the discrete-time. In this work we present the LMI-based results on regional stability by using appropriate Lyapunov-Krasovskii functional via the Halanay inequality (see e.g. [12]). Moreover, we find a bound on the domain of attraction of the system.

The paper is organized as follows. In Section II, we formulate the problem. In Section III, a sampled-data controller under the point state measurements is constructed to stabilize the system. A numerical example illustrates the main results in Section IV. The conclusions are finally stated in Section V.

Supported by Israel Science Foundation (grant No. 1128/14)

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Notations and preliminaries. Throughout the paper, $L^2(0,l)$ stands for the Hilbert space of square integrable scalar functions $z(x)$ on $(0,l)$ with the corresponding norm $\|z\|_{L^2} = [\int_0^l z^2(x)dx]^{\frac{1}{2}}$. The Sobolev space $H^k(0,l)$ is defined as $H^k(0,l) = \{z : D^\alpha z \in L^2(0,l),\ \forall\ 0 \le |\alpha| \le k\}$ with norm $\|z\|_{H^k} = \{\sum_{0\le|\alpha|\le k} \|D^\alpha z\|^2_{L^2}\}^{\frac{1}{2}}$. Moreover, $H_0^k(0,l) = \{z \in H^k(0,l) : z(0) = Dz(0) = \cdots = D^{k-1}z(0) = 0, z(l) = Dz(l) = \cdots = D^{k-1}z(l) = 0\}$.

Lemma 1.1: (Halanay's inequality [12] or pp. 138 of [18]) Let $0 < \delta_1 < 2\delta$ and let $V_1 : [t_0 - h, \infty) \to [0,\infty)$ be an absolutely continuous function that satisfies

$$\dot{V}_1(t) \le -2\delta V_1(t) + \delta_1 \sup_{-h\le\theta\le 0} V_1(t+\theta),\ t \ge t_0.$$

Then $V_1(t) \le e^{-2\alpha(t-t_0)} \sup_{-h\le\theta\le 0} V_1(t_0+\theta),\ t \ge t_0$, where α is a unique positive solution of

$$\alpha = \delta - \frac{\delta_1}{2} e^{2\alpha h}. \tag{1}$$

Lemma 1.2: (Wirtinger inequality and its Generalization [19]): Let $g \in H_0^1(0,l)$. Then the following inequality holds:

$$\int_0^l g^2(x)dx \le \frac{l^2}{\pi^2}\int_0^l \left[\frac{dg}{dx}(x)\right]^2 dx.$$

Moreover, if $g \in H_0^2(0,l)$, then

$$\int_0^l \left[\frac{dg}{dx}(x)\right]^2 dx \le \frac{l^2}{\pi^2}\int_0^l \left[\frac{d^2 g}{dx^2}(x)\right]^2 dx.$$

II. PROBLEM FORMULATION

In this paper, we consider the problem of stabilization of one dimensional KSE with homogeneous Dirichlet and Neumann boundary conditions[3]:

$$\begin{cases} z_t(x,t) + z_{xx}(x,t) + \nu z_{xxxx}(x,t) + z(x,t)z_x(x,t) \\ = \sum_{j=1}^N b_j(x)u_j(t),\ 0 < x < l,\ t \ge 0, \\ z(0,t) = z(l,t) = 0, \\ z_x(0,t) = z_x(l,t) = 0, \end{cases} \tag{2}$$

where ν is a positive constant, $z(x,t)$ describes the height of the film fluctuations, and $u_j(t) \in \mathbb{R}$, $(j = 1,2,\cdots,N)$ are the control inputs. The open-loop system (2) (subject to $u_j(t) \equiv 0$) may become unstable if ν is small enough (see the example below).

As in [1], [9], [10], [13], consider the points $0 = x_0 < x_1 < \cdots < x_N = l$ that divide $[0,l]$ into N sampling intervals $\Omega_j = [x_{j-1}, x_j)$. Let $0 = t_0 < t_1 < \cdots < t_k \cdots$ with $\lim_{k\to\infty} t_k = \infty$ be sampling time instants. The sampling intervals in time and in space may be variable but bounded,

$$0 \le t_{k+1} - t_k \le h,\ 0 < x_j - x_{j-1} = \Delta_j \le \Delta.$$

The control inputs $u_j(t)$ enter (2) through the shape functions

$$\begin{cases} b_j(x) = 1,\ x \in \Omega_j, \\ b_j(x) = 0,\ x \notin \Omega_j, \end{cases} \quad j = 1,\ldots,N. \tag{3}$$

Sensors provide point measurements of the state:

$$y_{jk} = z(\bar{x}_j, t_k), \bar{x}_j = \frac{x_{j-1}+x_j}{2}, \quad \begin{array}{l} j = 1,\ldots,N, \\ k = 0,1,2\ldots \end{array} \tag{4}$$

The aim is to design for (2) an exponentially stabilizing sampled-data controller which can be implemented by zero-order hold devices:

$$u_j(t) = -\mu y_{jk},\ j = 1,\ldots,N,\ t \in [t_k, t_{k+1}),\ k = 0,1\ldots \tag{5}$$

where μ is a positive constant to be determined later, and y_{jk} is given by (4).

III. Sampled-data control under point measurements

By selecting the controller (5) subject to (4), one arrives at the closed-loop system

$$\begin{cases} z_t(x,t) + z_{xx}(x,t) + \nu z_{xxxx}(x,t) + z(x,t)z_x(x,t) \\ = -\mu \sum_{j=1}^{N} b_j(x) z(\bar{x}_j, t_k),\ t \in [t_k, t_{k+1}),\ k = 0,1,\ldots \\ z(0,t) = z(l,t) = 0, \\ z_x(0,t) = z_x(l,t) = 0. \end{cases} \tag{6}$$

A. Well-posedness and stability of (6)

We start with the well-posedness of the sampled-data closed-loop system (6) initialized with $z_0(x) = z(x,0)$. We will use the step method for solution of time-delay systems [18].

Let $H = L^2(0,l)$. We define an unbounded linear operator $A : D(A) \subset H \to H$ as follows:

$$\begin{cases} Af = -\nu f'''',\ \forall f \in D(A), \\ D(A) = H^4(0,l) \cap H_0^2(0,l). \end{cases} \tag{7}$$

It is well-known that A is a dissipative operator, and A generates an analytic semigroup. The domain $D(A)$ is dense in H. Operator $-A$ is positive, which implies its square root $(-A)^{\frac{1}{2}}$ is also positive. Then $D((-A)^{\frac{1}{2}}) = H_0^2(0,l)$. The norm of $H_0^2(0,l)$ is endowed by the induced inner product:

$$\langle f, g\rangle_{H_0^2(0,l)} = \langle (-A)^{\frac{1}{2}} f, (-A)^{\frac{1}{2}} g\rangle.$$

Then, $\|f\|_{H_0^2(0,l)} = \nu^{\frac{1}{2}} [\int_0^l |f''(x)|^2 dx]^{\frac{1}{2}}$. Moreover, $H_0^2(0,l)$ norm is equivalent to the inherent norm $\|\cdot\|_{H^2}$ of Sobolev space $H^2(0,l)$. All relevant material on fractional operator degrees can be found in [23] (see pp. 81-83).

While being viewed over the time segment $t \in [t_0, t_1]$, with the operator A at hand, the system (6) can be written as an evolution equation in H:

$$\begin{cases} \frac{d}{dt} z(\cdot,t) = Az(\cdot,t) + F(z(\cdot,t)), \\ z(\cdot,0) = z_0(\cdot). \end{cases} \tag{8}$$

Here, the nonlinear term $F : H^2(0,l) \to L^2(0,l)$ is defined on functions $z(\cdot,t)$ according to

$$\begin{aligned} F(z(\cdot,t)) &= -z(x,t)z_x(x,t) - z_{xx}(x,t) \\ &\quad -\mu \sum_{j=1}^{N} b_j(x) z_0(\bar{x}_j),\ t \in [t_0, t_1). \end{aligned} \tag{9}$$

Definition 3.1: A function $z \in C([0,T]; H_0^2(0,l)) \cap L^2([0,T]; D(A))$ such that $\dot{z} \in L^2([0,T]; L^2(0,l))$ is called a strong solution of (8) if (8) holds for almost everywhere on $[0,T]$.

Our definition of strong solution follows [24] (see pp. 207).

Lemma 3.1: Given any $z_0 \in H_0^2(0,l)$ and $F(z(\cdot,t))$ defined by (9), where $t \in [t_0, t_1]$. Then
(i) F is Lipschitz continuous locally in z, that is, there exists a positive constant $L(R)$ such that the following inequality

$$\|F(z_1) - F(z_2)\|_{L^2} \le L(R) \|(z_1 - z_2)\|_{H_0^2(0,l)} \tag{10}$$

holds for $z_1,\ z_2 \in H_0^2(0,l)$ with $\|z_1\|_{H_0^2(0,l)} \le R$, $\|z_2\|_{H_0^2(0,l)} \le R$.
(ii) There exists a unique strong solution of (8) on some interval $[0,T] \subset [0,t_1]$ (here $T > 0$ depends on z_0). Moreover, if this solution admits a priori estimate

$$\|z\|_{H_0^2(0,l)} \le C(z_0),$$

where $C(z_0)$ is a constant depending on initial state z_0, then the solution exists on the entire interval $[0,t_1]$.

Proof: The proof of (i) can be easily established. Here we only prove (ii). The existence and uniqueness of local strong solution of (8) on some interval $[0,T] \subset [0,t_1]$ follow now from Theorem 3.3.3 of [22]. Continuation of this solution under a priori bound to entire interval $[0,t_1]$ follows from Theorem 6.23.5 of [28]. ■

For the stability analysis, we present (6) as

$$\begin{cases} z_t(x,t) + z_{xx}(x,t) + \nu z_{xxxx}(x,t) + z(x,t)z_x(x,t) \\ = -\mu z(x,t_k) + \mu \sum_{j=1}^{N} b_j(x) \int_{\bar{x}_j}^{x} z_\xi(\xi,t_k) d\xi, \\ \qquad t \in [t_k, t_{k+1}),\ k = 0,1,\cdots \\ z(0,t) = z(l,t) = 0, \\ z_x(0,t) = z_x(l,t) = 0. \end{cases} \tag{11}$$

We will use an input delay approach to sampled-data control ([26], [27]), where the sampling time t_k is presented as delayed time $t - \tau(t)$ with $\tau(t) = t - t_k$ for $t \in [t_k, t_{k+1})$.

In order to derive the stability conditions for (11), we define a Lypunov-Krasovskii functional

$$
\begin{aligned}
V_1(t) \quad &= p_1 \int_0^l z^2(x,t)dx + p_3\nu \int_0^l z_{xx}^2(x,t)dx \\
&+ r(t_{k+1}-t) \int_0^l \int_{t_k}^t e^{2\delta(s-t)} z_s^2(x,s)dsdx, \\
&t \in [t_k, t_{k+1}),\ p_1 > 0,\ p_3 > 0,\ r > 0.
\end{aligned} \tag{12}
$$

Here the first two terms are a combination of the usual terms of the canonical energy of the solution of KSE, whereas r-term treats sampled-data control as suggested in [16]. In the time-derivative of r-term we will have a positive term $r(t_{k+1}-t)\int_0^l z_t^2(x,t)dx$ (see (20) below). To compensate such a term in $\dot{V}_1$, we choose the p_3-term that guarantees convergence in H^2-norm (and not in H^1-norm like [16] for the case of diffusion-reaction equation).

For convenience we define

$$
\|z(\cdot,t)\|_V^2 = p_1 \int_0^l z^2(x,t)dx + p_3\nu \int_0^l z_{xx}^2(x,t)dx, \tag{13}
$$

where p_1 and p_3 are positive constants, and $z(\cdot,t) \in H_0^2(0,l)$. The choice of such norm is motivated by the Lyapunov-Krasovskii functional (12).

Remark 3.1: To find the domain of attraction for system (11), we will use positive invariance principle: we derive stability conditions in terms of matrix inequalities that guarantee $V_1(t) \leq V_1(0)$ for all $t \geq 0$. These matrix inequalities (matrices Θ_1 and Θ_2 in (4.7)) are affine in $z_x(x,t)$. Our objective is to guarantee that $\max\limits_{x\in[0,l]} |z_x(x,t)|^2 < C^2$ for all $t \geq 0$. This allows to verify the matrix inequalities in the vertices $z_x = \pm C$. Therefore, if the initial condition satisfies $\|z_0\|_V < \sqrt{\frac{p_3\nu}{l}}C$, then from the Sobolev inequality we obtain the desired bound on z_x:

$$
\begin{aligned}
&\max_{x\in[0,l]} |z_x(x,t)|^2 \leq l\|z_{xx}(\cdot,t)\|_{L^2}^2 \leq \frac{l}{p_3\nu} V_1(t) \\
&\leq \frac{l}{p_3\nu} V_1(0) = \frac{l}{p_3\nu} \|z_0\|_V^2 < C^2.
\end{aligned}
$$

Now we are in a position to formulate our main result:

Theorem 3.1: Consider the closed-loop system (11). Given positive scalars C, R, h, μ, Δ and $\delta_1 < 2\delta$, let there exist $r > 0$, $\lambda \geq 0$, $p_i > 0$ $(i = 1,2,3)$ that satisfy the linear matrix inequalities (LMIs):

$$
\Theta_{i|_{z_x=C}} < 0,\ \Theta_{i|_{z_x=-C}} < 0,\ i = 1,2, \tag{14}
$$

and

$$
\bar{\Theta} = \begin{bmatrix} -\delta_1 p_1 & \frac{\mu}{2}\frac{\Delta}{\pi} R^{-1}(p_2+p_3) \\ * & -\delta_1 p_3 \nu \end{bmatrix} < 0, \tag{15}
$$

where

$$
\Theta_1 = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ * & rh + \theta_{22} & \theta_{23} \\ * & * & \theta_{33} \end{bmatrix}, \tag{16}
$$

$$
\Theta_2 = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \mu p_2 h \\ * & \theta_{22} & \theta_{23} & \mu p_3 h \\ * & * & \theta_{33} & 0 \\ * & * & * & -re^{-2\delta h}h \end{bmatrix}, \tag{17}
$$

$$
\begin{aligned}
&\theta_{11} = 2\delta p_1 + \frac{\Delta}{\pi}\mu R p_2 - 2\mu p_2 - \lambda\frac{\pi^4}{l^4}, \\
&\theta_{12} = p_1 - p_2 - \mu p_3 - p_3 z_x, \\
&\theta_{13} = -p_2,\ \theta_{22} = -2p_3 + \frac{\Delta}{\pi}\mu R p_3, \\
&\theta_{23} = -p_3,\ \theta_{33} = 2\delta p_3\nu - 2p_2\nu + \lambda.
\end{aligned} \tag{18}
$$

Then for any initial function $z_0 \in H_0^2(0,l)$ satisfying $\|z_0\|_V < \sqrt{\frac{p_3\nu}{l}}C$, the following properties hold:
(i) [Well-posedness] There exists a strong solution of (11) for all $t \geq 0$.
(ii) [Stability] The strong solution of (11) satisfies

$$
\begin{aligned}
&p_1 \int_0^l z^2(x,t)dx + p_3\nu \int_0^l z_{xx}^2(x,t)dx \\
&\leq e^{-2\alpha t}\left[p_1 \int_0^l z^2(x,0)dx + p_3\nu \int_0^l z_{xx}^2(x,0)dx\right]
\end{aligned} \tag{19}
$$

for all $t \geq 0$, where α is a unique positive solution of (1). Furthermore, if the strong inequalities (14) and (15) are feasible for $\delta = \frac{\delta_1}{2} > 0$, then the Dirichlet boundary value problem (11) initialized with $z_0 \in H_0^2(0,l)$ such that $\|z_0\|_V \leq \sqrt{\frac{p_3\nu}{l}}C$, is exponentially stable with a small enough decay rate.

Proof: Step 1: From (ii) of Lemma 3.1, it follows that there exists a strong solution of (8) initialized with $z_0 \in H_0^2(0,l)$ on some interval $[0,T] \subset [0,t_1]$. We will prove in Step 3 that if the LMIs (14), (15) are feasible, the solution of (8) starting from $\|z_0\|_V < \sqrt{\frac{p_3\nu}{l}}C$ admits a priori bound, which guarantees the existence of the strong solution of (8) on the entire interval $[0,t_1]$. Then, by applying the same arguments step-by-step for $[t_k,t_{k+1}], k = 1,2,\cdots$ we conclude that the strong solution exists for all $t \geq 0$.

Step 2. Assume formally that strong solutions of (11) starting from $\|z_0\|_V < \sqrt{\frac{p_3\nu}{l}}C$ exist for all $t \geq 0$.

Calculating the derivative of the functional (12), we get

$$
\begin{aligned}
&\dot{V}_1(t) + 2\delta V_1(t) = 2p_1 \int_0^l z(x,t)z_t(x,t)dx \\
&+2p_3\nu \int_0^l z_{xx}(x,t)z_{xxt}(x,t)dx \\
&-r \int_0^l \int_{t_k}^t e^{2\delta(s-t)} z_s^2(x,s)dsdx \\
&+r(t_{k+1}-t) \int_0^l z_t^2(x,t)dx + 2\delta p_1 \int_0^l z^2(x,t)dx \\
&+2\delta p_3\nu \int_0^l z_{xx}^2(x,t)dx,\quad t \in [t_k, t_{k+1})
\end{aligned} \tag{20}
$$

Denote

$$
\rho(x,t) \triangleq \frac{1}{t-t_k} \int_{t_k}^t z_s(x,s)ds. \tag{21}
$$

Here we understand $\lim_{t\to t_k^+}\rho(x,t)=z_t(x,t_k)$ and obtain

$$z(x,t)=z(x,t_k)+(t-t_k)\rho(x,t). \tag{22}$$

Then Jensen's inequality yields

$$\begin{aligned}
&-r\int_0^l\int_{t_k}^t e^{2\delta(s-t)}z_s^2(x,s)dsdx\\
&\le -re^{-2\delta h}\int_0^l\frac{1}{t-t_k}\left[\int_{t_k}^t z_s(x,s)ds\right]^2dx \\
&=-re^{-2\delta h}(t-t_k)\int_0^l\rho^2(x,t)dx.
\end{aligned}\tag{23}$$

We apply the descriptor method ([15], [17], [18]), where z_t is not substituted from the differential equation by

$$\begin{aligned}
z_t\ &=-z_{xx}(x,t)-\nu z_{xxxx}(x,t)-z(x,t)z_x(x,t)\\
&-\mu z(x,t_k)+\mu\sum_{j=1}^N b_j(x)\int_{\bar{x}_j}^x z_\xi(\xi,t_k)d\xi,
\end{aligned}$$

but is treated as an additional state. The latter can be done by employing the following equality:

$$\begin{aligned}
&2\int_0^l[p_2z(x,t)+p_3z_t(x,t)][-z_t(x,t)-z_{xx}(x,t)\\
&-\nu z_{xxxx}(x,t)-z(x,t)z_x(x,t)-\mu z(x,t_k)]dx\\
&+2\mu\sum_{j=1}^N\int_{x_{j-1}}^{x_j}[p_2z(x,t)+p_3z_t(x,t)]\int_{\bar{x}_j}^x z_\xi(\xi,t_k)d\xi dx=0,
\end{aligned}\tag{24}$$

where $p_2>0$, $p_3>0$ are some scalars.

Integration by parts and substitution of the boundary conditions of (11) lead to

$$\int_0^l z(x,t)[z(x,t)z_x(x,t)]dx=0, \tag{25}$$

$$\int_0^l z(x,t)z_{xxxx}(x,t)dx=\int_0^l z_{xx}^2(x,t)dx, \tag{26}$$

and

$$\int_0^l z_t(x,t)z_{xxxx}(x,t)dx=\int_0^l z_{xx}(x,t)z_{xxt}(x,t)dx. \tag{27}$$

By adding to $\dot{V}_1+2\delta V_1$ the equality (24), and using (22), (23), (25), (26), (27), we obtain

$$\begin{aligned}
&\dot{V}_1(t)+2\delta V_1(t)=2p_1\int_0^l z(x,t)z_t(x,t)dx\\
&-2p_2\nu\int_0^l z_{xx}^2(x,t)dx-re^{-2\delta h}(t-t_k)\int_0^l\rho^2(x,t)dx\\
&+r(t_{k+1}-t)\int_0^l z_t^2(x,t)dx\\
&+2\delta p_1\int_0^l z^2(x,t)dx+2\delta p_3\nu\int_0^l z_{xx}^2(x,t)dx\\
&+2\int_0^l[p_2z(x,t)+p_3z_t(x,t)][-z_t(x,t)-z_{xx}(x,t)\\
&-\mu z(x,t)+\mu(t-t_k)\rho(x,t)]dx\\
&+2\mu\sum_{j=1}^N\int_{x_{j-1}}^{x_j}[p_2z(x,t)+p_3z_t(x,t)]\int_{\bar{x}_j}^x z_\xi(\xi,t_k)d\xi dx\\
&-2p_3\int_0^l z_t(x,t)z(x,t)z_x(x,t)dx.
\end{aligned}\tag{28}$$

From Young's and Wirtinger's inequalities, we have

$$\begin{aligned}
&2\mu\sum_{j=1}^N\int_{x_{j-1}}^{x_j}[p_2z(x,t)+p_3z_t(x,t)]\int_{\bar{x}_j}^x z_\xi(\xi,t_k)d\xi dx\\
&\le\mu\bar{R}\int_0^l[p_2z^2(x,t)+p_3z_t^2(x,t)]dx\\
&+\mu\bar{R}^{-1}(p_2+p_3)\frac{\Delta^2}{\pi^2}\int_0^l z_x^2(x,t_k)dx,\ \forall\bar{R}>0.
\end{aligned}\tag{29}$$

Due to the Dirichlet boundary conditions, Lemma 1.2 implies

$$\lambda\int_0^l\left[z_{xx}^2(x,t)-\frac{\pi^4}{l^4}z^2(x,t)\right]dx\ge 0, \tag{30}$$

where $\lambda\ge 0$.

Substituting (29) into (28), and using Halanay's inequalities, we obtain

$$\begin{aligned}
&\dot{V}_1(t)+2\delta V_1(t)-\delta_1\sup_{\theta\in[-h,0]}V_1(t+\theta)\\
&\le\dot{V}_1(t)+2\delta V_1(t)-\delta_1V_1(t_k)\\
&\le(2p_1-2p_2-2\mu p_3)\int_0^l z(x,t)z_t(x,t)dx\\
&+(2\delta p_3\nu-2p_2\nu+\lambda)\int_0^l z_{xx}^2(x,t)dx\\
&-2\int_0^l[p_2z(x,t)+p_3z_t(x,t)]z_{xx}(x,t)dx\\
&+\left[2\delta p_1+\mu(\bar{R}-2)p_2-\lambda\frac{\pi^4}{l^4}\right]\int_0^l z^2(x,t)dx\\
&+\left[r(t_{k+1}-t)-2p_3+\mu\bar{R}p_3\right]\int_0^l z_t^2(x,t)dx\\
&-2p_3\int_0^l z_t(x,t)z(x,t)z_x(x,t)dx\\
&-re^{-2\delta h}(t-t_k)\int_0^l\rho^2(x,t)dx\\
&+\mu(p_2+p_3)\frac{\Delta^2}{\pi^2}\bar{R}^{-1}\|z(\cdot,t_k)\|_{L^2}\|z_{xx}(\cdot,t_k)\|_{L^2}\\
&+2\mu\int_0^l[p_2z(x,t)+p_3z_t(x,t)][(t-t_k)\rho(x,t)]dx\\
&-\delta_1p_1\int_0^l z^2(x,t_k)dx-\delta_1p_3\nu\int_0^l z_{xx}^2(x,t_k)dx.
\end{aligned}\tag{31}$$

Here we use the fact that

$$\|z_x(\cdot,t_k)\|_{L^2}^2\le\|z(\cdot,t_k)\|_{L^2}\|z_{xx}(\cdot,t_k)\|_{L^2}.$$

Set

$$\begin{aligned}
&\eta=\mathrm{col}\{z(x,t),z_t(x,t),z_{xx}(x,t),\rho(x,t)\},\\
&\eta_0=\mathrm{col}\{z(x,t),z_t(x,t),z_{xx}(x,t)\},\\
&\bar{\eta}=\mathrm{col}\{\|z(\cdot,t_k)\|_{L^2},\|z_{xx}(\cdot,t_k)\|_{L^2}\},
\end{aligned}\tag{32}$$

and choose next $R=\frac{\pi}{\Delta}\bar{R}$. Since $0\le t_{k+1}-t_k\le h$, from (31) it follows that

$$\begin{aligned}
&\dot{V}_1(t)+2\delta V_1(t)-\delta_1\sup_{\theta\in[-h,0]}V_1(t+\theta)\\
&\le\int_0^L\frac{h-t+t_k}{h}\eta_0^T\Theta_1\eta_0+\frac{t-t_k}{h}\eta^T\Theta_2\eta+\bar{\eta}^T\bar{\Theta}\bar{\eta}dx,\\
&\qquad\qquad\forall t\in[t_k,t_{k+1}),
\end{aligned}\tag{33}$$

where $\bar{\Theta}$, Θ_1 and Θ_2 are given by (15), (16), (17) respectively.

We first assume that

$$\max_{x\in[0,l]} |z_x(x,t)| < C, \ \forall t \geq 0. \tag{34}$$

Under the assumption (34), from (33) we obtain

$$\dot{V}_1(t) + 2\delta V_1(t) - \delta_1 \sup_{\theta\in[-h,0]} V_1(t+\theta) \leq 0 \tag{35}$$

if $\Theta_1 < 0, \ \Theta_2 < 0, \ \bar{\Theta} < 0$ hold for all $z_x \in (-C, C)$.

Matrices Θ_1 and Θ_2 given by (16), (17) are affine in z_x. Hence, $\Theta_1 < 0$ and $\Theta_2 < 0$ for all $z_x \in (-C, C)$ if these LMIs in the vertices $z_x = \pm C$ are satisfied meaning that (14) holds.

We prove next (34). By the Sobolev inequality we obtain

$$\begin{aligned} \max_{x\in[0,l]} |z_x(x,t)|^2 &\leq l\|z_{xx}(\cdot,t)\|^2_{L^2} \\ &\leq \frac{l}{p_3\nu}\|z(\cdot,t)\|^2_V \leq \frac{l}{p_3\nu}V_1(t). \end{aligned} \tag{36}$$

Therefore, it is sufficient to show that $V_1(t) < \frac{p_3\nu}{l}C^2$. The initial condition $V_1(0) = \|z_0\|^2_V < \frac{p_3\nu}{l}C^2$ implies $\max_{x\in[0,l]} |z_x(x,0)|^2 < C^2$. Let $t^* \in (0,\infty)$ be the smallest time instance such that $V_1(t^*) \geq \frac{p_3\nu}{l}C^2$. Since V_1 is continuous in time, we have $V_1(t^*) = \frac{p_3\nu}{l}C^2$ and $V_1(t) < \frac{p_3\nu}{l}C^2$ for $t \in [0, t^*)$. Together with (36) this implies $\max_{x\in[0,l]} |z_x(x,t)|^2 < C^2$ for $t \in [0,t^*)$ and, therefore, the feasibility of (14) and (15) guarantees that (35) is true for $t \in [0,t^*)$. Hence, $V_1(t) \leq e^{-2\alpha t}V_1(0) < \frac{p_3\nu}{l}C^2$ holds for all $t \in [0,t^*]$, which contradicts to the definition of t^*. Thus, for $t \geq 0$,

$$\|z_0\|_V < \sqrt{\frac{p_3\nu}{l}}C \Rightarrow (34) \Rightarrow (19).$$

Step 3: Now we continue to prove the well-posedness. When $k = 0$, we obtain that if the LMIs conditions (14) and (15) are satisfied, then for any possible solution of (8) initialized with $\|z_0\|_V < \sqrt{\frac{p_3\nu}{l}}C$ admits a priori estimate

$$V_1(t) \leq e^{-2\alpha t}V_1(0), \ \forall t \in [0,T] \subset [0,t_1], \tag{37}$$

where α is the solution of (1). Thus, continuation of this solution of (8) initialized with $z_0 \in H^2_0(0,l)$ subject to $\|z_0\|_V < \sqrt{\frac{p_3\nu}{l}}C$ under a priori bound to entire interval $[0,t_1]$ follows from (ii) of Lemma 3.1. We apply the same arguments step-by-step for $[t_1,t_2]$, $[t_2,t_3]$, $\cdots$. For $t \in [t_1,t_2]$, the system (6) can be also rewritten as an evolution equation (8) initialized with $z(x,t_1)$, where $F(z(\cdot,t))$ is given by (9) with $z(\bar{x}_j,t_1)$ instead of $z_0(\bar{x}_j)$. Thus, for the modified F, (10) holds for $z_1, \ z_2 \in H^2_0(0,l)$ with $\|z_1\|_{H^2_0(0,l)} \leq R$, $\|z_2\|_{H^2_0(0,l)} \leq R$. Therefore, by arguments of Steps 1 and 2 we conclude there exists a strong solution for $t \in [t_1,t_2]$. Then, by step method, the strong solution of (11) exists for all $t \geq 0$. Furthermore, Halanay's inequality implies (19) for all $t \geq 0$, which completes the proof.

■

IV. EXAMPLE

Consider the KSE governed by (2) with $l = 2\pi$ and $\nu = 0.5 < 1$. Figure 1 demonstrates the state $z(x,t)$ for the open-loop system initialized by $z(x,0) = (1-\cos x)\sin x$, $x \in [0,2\pi]$. It is seen that the open-loop system is unstable.

Consider the sampled-data controller under the point measurements. We choose $\Delta = \frac{\pi}{8}$, $\nu = 0.5$, $\mu = 3$. Here for sampled-data control law (5) with the point measurements, by using Yalmip Toolbox of Matlab we verify LMI conditions of Theorem 3.1 with $\delta_1 = 0.5$, $\delta = 0.26$, $C = 1$, $R = 1$. We find that the closed-loop system preserves the exponential stability for $t_{k+1} - t_k \leq h = 0.23$ for any initial values satisfying $\|z(\cdot,0)\|_V < \sqrt{\frac{p_3\nu}{l}}C = 0.5$. For $h = 0.23$, the above controller locally exponentially stabilizes the closed-loop system with a decay rate $\alpha = 0.0089$.

Next, a finite difference method is applied to compute the displacement of the closed-loop system (11) to illustrate the effect of the proposed feedback control law. We choose $\mu = 3$, and initial condition $z(x,0) = 0.05(1-\cos(x))\sin(x)$. Hence, $\|z(\cdot,0)\|_V < 0.5$. The steps of space and time are taken as $\frac{\pi}{16}$ and 0.0001, respectively. Simulations of solutions under the sampled-data in time and in space controller $u_j(t) = -3z(\bar{x}_j,t_k)$ with $x_j - x_{j-1} = \frac{\pi}{8}$, $j = 1,\cdots,16$, $t_{k+1} - t_k = 0.23$, where the spatial domain is divided into sixteen sub-domains, show that the closed-loop system is exponentially stable in Figure 2. Figure 3 illustrates instability for $t_{k+1} - t_k \geq 1$. The simulation of the solution confirms the theoretical result.

V. CONCLUSIONS

In the present paper, sampled-data distributed control of KSE has been introduced under point discrete-time measurements and distributed in space shape functions. By using the time-delay approach to sampled-data control and constructing a suitable Lyapunov-Krasovskii functional, sufficient conditions for the regional exponential stability are derived in terms of LMIs. An interesting, yet technically challenging, open question is extension of the obtained results (eg. [29]-[31]) to the observer-based sampled-data control of nonlinear PDEs or coupled ODE-PDE system, which may be a topic for future research.

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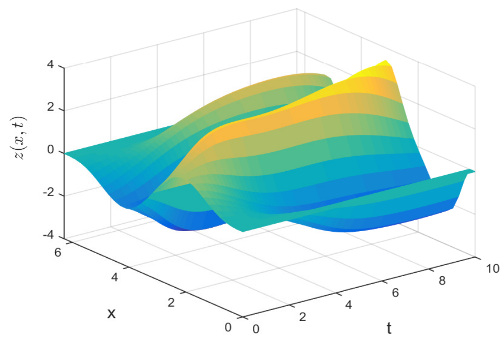

Fig. 1. Open-loop system (without control input)

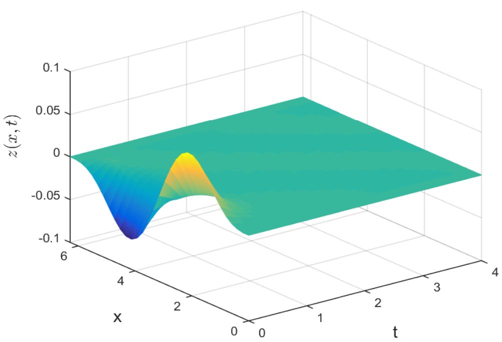

Fig. 2. Closed-loop system (with sampled-data control)

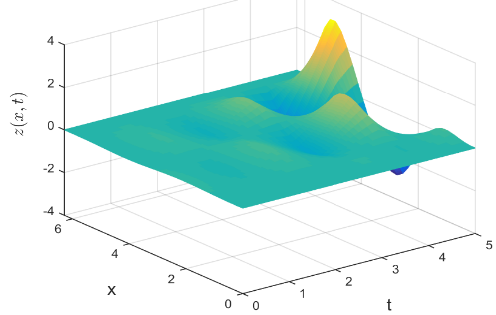

Fig. 3. Closed-loop system is unstable

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