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Economic model predictive control for robust periodic operation with guaranteed closed-loop performance

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Abstract— **We present an economic model predictive control scheme for general nonlinear systems based on a terminal cost and a terminal constraint set. We study in particular systems which are optimally operated at some periodic orbit. Besides recursive feasibility of the control scheme, we provide an asymptotic average performance bound which is no worse than the performance value of the system's optimal periodic orbit. By means of a certain (strict) dissipativity assumption, asymptotic convergence to the optimal periodic orbit is shown. Using a tube-based approach, we extend our method to become applicable in the presence of unknown but bounded disturbances. In addition, we propose the concept of robust optimal periodic operation and show how it can essentially improve the closed-loop performance using a simple supply chain network example.**

I. Introduction

Economic model predictive control (EMPC) schemes differ from classical stabilizing model predictive control in terms of their general performance objective function which does not need to be chosen such that the controller stabilizes the system state with respect to an a-priori given reference point or trajectory. However, since the cost can be chosen arbitrarily, the closed-loop system shows potentially complex behavior. Despite the important special case of optimal steady-state operation, as usually considered in economic MPC analysis [1], [2], [3], in nature, economics, and engineering applications, *periodic operation* plays a central role (evolutionary processes, sleeping rhythms, human walk, engines, or supply chain networks, see e.g. [4]). To this end, we present an economic model predictive control scheme for optimal periodic operation (Sec. III) with guarentees in terms of recursive feasibility, asymptotic average performance and convergence to the optimal periodic orbit.

In [5], [6], [7] criteria in form of dissipativity inequalities are provided in order to determine whether or not periodic operation at a certain periodic orbit is the system's possibly best (unique) operational strategy. Given such an orbit, in [8] an EMPC scheme without terminal constraints is presented. The authors provide a bound on sub-optimality w.r.t. closed-loop asymptotic average performance and practical convergence guarantees to the optimal periodic orbit in terms of the online planning horizon. As the planning horizon tends towards infinity, the error term vanishes and optimal asymptotic average performance is recovered.

In [7] a method is presented, which is closely related to the approach presented in this paper. It is based on a terminal region and terminal cost. Compared to the well-known steady-state case, the main drawback of the approach is a (strict) dissipativity assumption for optimality and stability w.r.t. a certain periodic orbit that might be difficult to verify, even in a linear, periodic varying setting. We overcome this problem by relying on dissipativity assumptions that are related to the case in which steady-state operation is optimal with respect to a so called P-step system (Sec. III). In particular, we pose a slightly different dissipativity assumption, which was introduced in [8]. The different assumption originates from strong duality of the optimal steady-state optimization problem [9] which can be easily verified in a linear setting. Transferring this idea to periodic operation allows for efficient analysis of asymptotic convergence to the optimal periodic orbit in the case of piecewise-linear stage cost functions and linear periodic varying dynamics [10].

In applications, external disturbances can have significant impact on the system dynamics. This potentially leads to poor closed-loop performance and a loss of recursive feasibility which can cause safety risks. In [11], a robust economic control scheme was developed for optimal steady-state operation in order to overcome these problems. Using our EMPC approach and the respective assumptions, it is possible to easily combine the concepts of tube based EMPC and periodic EMPC yielding our second contribution, a tube-based robust EMPC scheme (Sec. IV) for robustly optimal periodic operation and for convergence to a neighborhood of the robust optimal periodic orbit. We illustrate our methods using a simple supply chain network (Sec. V).

Notation: Let $\mathcal{I}_{[a,b]}$ denote the integers in the interval $[a,b] \subset \mathbb{R}$ and $\mathcal{I}_{\geq a} = \mathcal{I}_{[a,\infty)} \cup \{\infty\}$, the set of integers greater than or equal to a. Define $\lfloor a \rfloor$ with $a \in \mathbb{R}$ as the largest integer smaller than or equal to a. Let the distance between a point $x \in \mathbb{R}^n$ and a set $\mathcal{A} \subseteq \mathbb{R}^n$ be defined as $|x|_{\mathcal{A}} = \inf_{a \in \mathcal{A}} |x - a|$. We denote $\mathcal{A} \times ... \times \mathcal{A}$ as $\mathcal{A}^P$.

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II. Problem Setup

We consider systems with initial condition $x(0) = x$,

$$x(k+1) = f(x(k),\ u(k),\ w(k)),\ k \in \mathcal{I}_{\geq 0}, \tag{1}$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^n$ s.t. $x(k) \in \mathbb{X} \subseteq \mathbb{R}^n$ and $u(k) \in \mathbb{U} \subseteq \mathbb{R}^m$, $\mathbb{X}$ and $\mathbb{U}$ compact, and unknown but bounded disturbances $w(k) \in \mathbb{W} \subset \mathbb{R}^q$. Given a control sequence $\boldsymbol{u} = \{u(0), ..., u(K)\} \in \mathbb{U}^{K+1}$, and a disturbance sequence $\boldsymbol{w} = \{w(0), ..., w(K)\} \in \mathbb{W}^{K+1}$ we denote the corresponding solution of (1) by $x_{\boldsymbol{u}} = \{x_{\boldsymbol{u}}(0, x), ..., x_{\boldsymbol{u}}(K+1, x)\} \in \mathbb{X}^{K+2}$ with initial condition $x_{\boldsymbol{u}}(0, x) = x$. Consider a constant disturbance $w(k) = 0$ for all $k \in \mathcal{I}_{\geq 0}$. Then for a given $x \in \mathbb{X}$ the set of all feasible control sequences of length $N \in \mathcal{I}_{\geq 0}$, denoted by $\mathbb{U}^N(x)$, is such that $u(k) \in \mathbb{U}$ for all $k \in \mathcal{I}_{[0,N-1]}$ and $x_{\boldsymbol{u}}(k, x) \in \mathbb{X}$ for all $k \in \mathcal{I}_{[0,N]}$. System (1) is equipped with a stage cost function

$$\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R} \tag{2}$$

that is assumed to be continuous but arbitrary otherwise (economic). W.l.o.g. assume $0 \leq \inf_{x\in\mathbb{X},u\in\mathbb{U}} \ell(x, u)$. The overall control goal is to find a feasible input sequence such that it minimizes the asymptotic average cost $\limsup_{T\to\infty} \frac{\sum_{t=0}^{T-1} \ell(x_{\boldsymbol{u}}(t,x),u(t))}{T}$ of the system.

In this paper, we study the case in which the optimal operating behavior of system (1) is not necessarily given by some equilibrium point but by some periodic orbit with period $P \in \mathcal{I}_{\geq 1}$. Therefore we introduce the following notation.

Definition II.1 (Cf. [5])**.** Consider $\mathbb{W} = \{0\}$. A set of state and input pairs $\Pi = \{(x_0^p, u_0^p), ..., (x_{P-1}^p, u_{P-1}^p)\}$ with $P \in \mathcal{I}_{\geq 1}$ is called a nominal feasible P-periodic orbit of system (1) if $x_k^p \in \mathbb{X}$, $u_k^p \in \mathbb{U}$, and $x_{k+1}^p = f(x_k^p, u_k^p, 0)$ for all $k \in \mathcal{I}_{[0,P-2]}$ and $x_0^p = f(x_{P-1}^p, u_{P-1}^p, 0)$. It is called a minimal P-periodic orbit if $x_{k_1}^p \neq x_{k_2}^p$ with $k_1 \neq k_2$, $0 \leq k_1, k_2 \leq P-1$. The projection of Π on $\mathbb{X}$ is denoted by $\Pi_{\mathbb{X}} = \{x_0^p, ..., x_{P-1}^p\}$ and on $\mathbb{U}$ by $\Pi_{\mathbb{U}} = \{u_0^p, ..., u_{P-1}^p\}$ respectively. The set of all nominal feasible P-periodic orbits is S_Π^P.

Definition II.2 (Cf. [6])**.** The P-step system of system (1) is defined with $\tilde{x} = (x_0, ..., x_{P-1}) \in \mathbb{X}^P$, $\tilde{u} = (u_0, ..., u_{P-1}) \in \mathbb{U}^P$, $\tilde{w} = (w_0, ..., w_{P-1}) \in \mathbb{W}^P$ and $\tilde{x}(k+P) = f^P(\tilde{x}(k), \tilde{u}(k), \tilde{w}(k))$ with

$$\begin{aligned} &f^P(\tilde{x}, \tilde{u}, \tilde{w}) \\ &= (f(x_{P-1}, u_0, w_0), f(f(x_{P-1}, u_0, w_0), u_1, w_1), ..) \end{aligned} \tag{3}$$

and initial condition $x_{P-1}(0) = x \in \mathbb{X}$ [1]. Given an initial state x, a control and disturbance sequence $\boldsymbol{u} \in \mathbb{U}^{PK}$ and $\boldsymbol{w} \in \mathbb{W}^{PK}$, $K \in \mathcal{I}_{\geq 1}$, the corresponding solution is denoted by

$$\tilde{x}_{\boldsymbol{u}}(k, x) = (x_{\boldsymbol{u}}(k-P+1, x), .., x_{\boldsymbol{u}}(k-1, x), x_{\boldsymbol{u}}(k, x))$$

(with partially undefined states in case $k < P-1$), which implies $\tilde{x}_{\boldsymbol{u}}(k+P, x) = f^P(\tilde{x}_{\boldsymbol{u}}(k, x), \tilde{u}, \tilde{w})$ and $\tilde{x}_{\boldsymbol{u}}(k+1, x) \neq f^P(\tilde{x}_{\boldsymbol{u}}(k, x), \tilde{u}, \tilde{w})$. The stage cost function for the P-step system is given by

$$\tilde{\ell}(\tilde{x}, \tilde{u}) = \sum_{i=0}^{P-1} \ell(x_{\tilde{u}}(i, x_{P-1}), u_i). \tag{4}$$

Definition II.3 (Cf. [6])**.** The distance between a state and input pair $(\tilde{x}, \tilde{u})$ and a nominal P-periodic orbit Π of system (1) is defined as

$$|(\tilde{x}, \tilde{u})|_\Pi = \sum_{i=0}^{P-1} |(x_{\tilde{u}}(i, x_{P-1}), u_i)|_\Pi\,, \tag{5}$$

$$\text{and } |\tilde{x}|_{\Pi_{\mathbb{X}}} = \sum_{i=0}^{P-1} |x_{\tilde{u}}(i, x_{P-1})|_{\Pi_{\mathbb{X}}}. \tag{6}$$

Remark II.4. *Any nominal periodic orbit $\Pi \in S_\Pi^P$ of system (1) is an equilibrium point of the P-step system.*

III. Economic MPC For Periodic Operation

In this section we present an EMPC scheme and provide an asymptotic performance guarantee (Thm. III.6) as well as conditions for asymptotic convergence to the optimal periodic orbit (Thm. III.12).

Consider system (1) *without* disturbances, that is, $w(k) = 0$ for all $k \in \mathcal{I}_{\geq 0}$, and S_Π^P non-empty. Given a state $\tilde{x}$, the operator $(.)_{P-1}$ picks the last element of $\tilde{x}$, i.e. $(\tilde{x})_{P-1} = x_{P-1}$.

Assumption III.1. *Let $(x_i^p, u_i^p) \in \Pi$ for $i \in \mathcal{I}_{[0,P-1]}$. There exists a compact set $\mathbb{X}_f \subseteq \mathbb{X}$ s.t. $\Pi_{\mathbb{X}} \subseteq \mathbb{X}_f$, a feedback law $\tilde{\kappa}_f : \mathbb{X}^P \to \mathbb{U}^P$, and a continuous terminal cost $V_f : \mathbb{X}_f \to \mathbb{R}$ s.t. for all $\tilde{x}$ with $x_{P-1} \in \mathbb{X}_f$ it holds:*

1) $\tilde{\kappa}_f(\tilde{x}) \in \mathbb{U}^P$*;*
2) $f^P(\tilde{x}, \tilde{\kappa}_f(\tilde{x}), 0) \in \mathbb{X}_f^P$*;*
3) $\begin{aligned} &V_f((f^P(\tilde{x}, \tilde{\kappa}_f(\tilde{x}), 0))_{P-1}) - V_f((\tilde{x})_{P-1}) \\ &\quad \leq -\tilde{\ell}(\tilde{x}, \tilde{\kappa}_f(\tilde{x})) + \Sigma_{i=0}^{P-1} \ell(x_i^p, u_i^p). \end{aligned}$

Without loss of generality, let $V_f(x) \geq 0\ \forall x \in \mathbb{X}_f$.

Since a P-periodic orbit becomes a steady-state for the P-step system, the terminal configuration is conceptually related to the steady-state case. This differs from designing a family of P one-step terminal controllers and P terminal cost functions as proposed in [7, Ass. 5.5]. By considering a P-step controller, we only require a decrease in the cost once a period is completed, compare Ass. $III.1$ 3).

Remark III.2. *By reformulating the terminal assumption (Ass. III.1) such that the last P system states (instead of just using x_{P-1}) are contained in a terminal set $\tilde{\mathbb{X}}_f \subset \mathbb{X}^P$ together with a corresponding terminal cost function $\tilde{V}_f : \tilde{\mathbb{X}}_f \to \mathbb{R}$, the subsequent results still apply. In this modified setting it would be straight-forward to use the results from the steady-state case [2] in order to fulfill Ass. III.1 with respect to a specific phase of the optimal periodic orbit. However, this modification comes at the cost of constraining the last P system states w.r.t. $\tilde{X}_f$.*

[1] Initial conditions of other states are not relevant for the solution. Furthermore, the time indexing $\tilde{x}(k+P) = f^P(\tilde{x}(k), \tilde{u}(k), \tilde{w}(k))$ simplifies the change between the P-step system notation and the original system notation.

Algorithm 1 Economic MPC for optimal periodic operation

procedure EMPC-P(initial state $x = x(0)$)
  for $k_1 = 0, 1, \ldots$ **do**
    solve $(P_{\mathrm{EMPC-P}})$ with $x = x_{\mathrm{cl}}(k_1 P, x)$
    apply the first P inputs of $\boldsymbol{u}^*(k_1 P)$ to system (1)

Let $N = N_1 P$ with $N_1 \in \mathcal{I}_{>0}$. Define the open loop optimization problem

$$(P_{\mathrm{EMPC-P}}) \begin{cases} \min_{\boldsymbol{u}\in\mathbb{U}^N} J_{\mathrm{MPC}}(x,\boldsymbol{u}) \\ \text{s.t. for all } k \in \mathcal{I}_{[0,N-1]}: \\ \qquad x_{\boldsymbol{u}}(k+1,x) = f(x_{\boldsymbol{u}}(k,x),u(k),0) \\ \qquad x_{\boldsymbol{u}}(k,x) \in \mathbb{X} \\ \qquad u(k) \in \mathbb{U} \\ \qquad x_{\boldsymbol{u}}(N,x) \in \mathbb{X}_f \\ \qquad x_{\boldsymbol{u}}(0,x) = x \end{cases}$$

with finite time open loop cost functional

$$J_{\mathrm{MPC}}(x,\boldsymbol{u}) = \sum_{k=0}^{N-1} \ell(x_{\boldsymbol{u}}(k,x),u(k)) + V_f(x_{\boldsymbol{u}}(N,x)) \quad (7)$$

which can be rewritten as $\sum_{k=0}^{N/P-1} \tilde{\ell}(\tilde{x}_{\boldsymbol{u}}(kP,x),\tilde{u}(kP)) + V_f(x_{\boldsymbol{u}}(N,x))$ and will be solved for $\boldsymbol{u} = (u(0),u(1),..,u(N-1))$ every P-time steps $k_1 P \in \mathcal{I}_{\geq 0}$ using the current system state $x = x(k_1 P)$, see Alg. 1. Starting from an initial state $x = x(0)$, denote the closed-loop system inputs and states by $u_{\mathrm{cl}}(t)$ and $x_{\mathrm{cl}}(t,x)$ for $t \in \mathcal{I}_{\geq 0}$. In terms of the corresponding P-step system we write $\tilde{u}_{\mathrm{cl}}(t) = (u_{\mathrm{cl}}(t),...,u_{\mathrm{cl}}(t+P-1))$ and $\tilde{x}_{\mathrm{cl}}(t,x) = (x_{\mathrm{cl}}(t-P+1,x),..,x_{\mathrm{cl}}(t-1,x),x_{\mathrm{cl}}(t,x))$.

Remark III.3. *Alg. 1 defines a P-step EMPC scheme. Since it operates in open loop for P time steps, robustness with respect to perturbations reduces compared to e.g. [7]. We can overcome this problem as proposed in [8, Rem. 6], by periodically time varying the prediction horizon. Therefore solve $(P_{\mathrm{EMPC-P}})$ at each time step t with N replaced by $N - (t \mod N)$, in which t indicates the real systems time, and apply only the first input to the system each time step. By the dynamic programming principle, this does not influence the subsequent results, but will in general lead to an improvement in robustness against uncertainties and disturbances, see [12].*

Assumption III.4. *The optimization problem $(P_{\mathrm{EMPC-P}})$ is feasible at time $t = 0$ for $x = x(0)$.*

Let the optimal input sequence of $(P_{\mathrm{EMPC-P}})$ with respect to x be denoted by $\boldsymbol{u}^*(x) = (u^*(0,x),..,u^*(N-1,x))$ with corresponding optimal states $x_{\boldsymbol{u}^*(x)}(k,x)$ for $k \in \mathcal{I}_{[0,N]}$ and $x_{\boldsymbol{u}^*(x)}(0,x) = x$.

Theorem III.5. *If Ass. III.1 and Ass. III.4 hold, then Alg. 1 is recursively feasible.*

Proof. Based on the optimal solution $\boldsymbol{u}^*(x(t))$ (Ass. III.4) calculated at time t, define the candidate P-step system input sequence at time $\tau = \tau_1 P \in \mathcal{I}_{\geq t}$ as

$$\tilde{\bar{u}}^t(\tau) = \begin{cases} (u^*(\tau - t, x(t)),...,u^*(\tau-t+P-1,x(t))), \\ \qquad\qquad\qquad\qquad \tau \in \mathcal{I}_{[t,t+N-P]} \\ \tilde{\kappa}_f\left(\tilde{x}_{\bar{u}}(\tau,x)\right), \qquad \text{else}, \end{cases} \quad (8)$$

where we denote the system states, resulting from applying (8), by $x_{\bar{u}}(k,x)$, $k \in \mathcal{I}_{\geq t}$, with $x_{\bar{u}}(0,x(t)) = x(t)$ and corresponding P-step state $\tilde{x}_{\bar{u}}(t,x(t)) = (x_{\bar{u}}(t-P+1,x(t)),..,x_{\bar{u}}(t-1,x(t)),x_{\bar{u}}(t,x(t)))$. Consider

$$\bar{\boldsymbol{u}}^t(k) = (\tilde{\bar{u}}^t(k),\tilde{\bar{u}}^t(k+P),...,\tilde{\bar{u}}^t(k+N-P)) \quad (9)$$

as candidate input trajectory for $k \in \mathcal{I}_{\geq t}$ based on the optimal solution at time t. It constitutes a recursively feasible, suboptimal solution to $(P_{\mathrm{EMPC-P}})$, because by Ass. III.1 for $\tilde{x}$ with $(\tilde{x})_{P-1} \in \mathbb{X}_f$ we have $\tilde{\kappa}_f(\tilde{x}) \in \mathbb{U}^P$ and positive invariance with respect to $\mathbb{X}_f^P \subseteq \mathbb{X}^P$. □

Under our different terminal assumption (Ass. III.1), compared to [7] we are able to state the same performance guarantees as given in [7, Rem. 5.8].

Theorem III.6. *Let Ass. III.1 and Ass. III.4 hold, then under application of Alg. 1 the closed-loop system has an average performance which is no worse than that of the optimal periodic orbit Π, i.e.*

$$\frac{1}{P}\sum_{k=0}^{P-1} \ell(x_k^p,u_k^p) \geq \limsup_{T\to\infty} \frac{\sum_{k=0}^{T-1} \ell(x_{\mathrm{cl}}(k,x(0)),u_{\mathrm{cl}}(k))}{T}$$

with $(x_i^p,u_i^p) \in \Pi$ for $i \in \mathcal{I}_{[0,P-1]}$.

Proof. The proof follows along the lines of [2, Thm. 18]. □

A. Convergence to the optimal periodic orbit

Assumption III.7. *There exists a continuous storage function $\tilde{\lambda}: \mathbb{R}^{nP} \to \mathbb{R}$ and a $\mathcal{K}_\infty$ function α s.t. system (3) is strictly dissipative in the sense of [5, Def. 1], i.e.*

$$\tilde{\lambda}(f^P(\tilde{x},\tilde{u},0)) - \tilde{\lambda}(\tilde{x}) \leq s(\tilde{x},\tilde{u}) - \alpha(|(\tilde{x},\tilde{u})|_\Pi) \quad (10)$$

with supply rate $s(\tilde{x},\tilde{u}) = \tilde{\ell}(\tilde{x},\tilde{u}) - \sum_{k=0}^{P-1}\ell(x_k^p,u_k^p)$.

Note, that Ass. III.7 differs from strict dissipativity as defined in [7, Def. 3.3]. We consider dissipativity with respect to the 'average' over P elements of the periodic orbit Π in the supply rate $s(\tilde{x},\tilde{u})$, rather than dissipativity with respect to P decoupled supply rates for each element of Π. This enables us to note the following.

Remark III.8. *Consider systems of the type*

$$x(k+1) = A(k)x(k) + B(k)u(k) \quad (11)$$
$$A_x x(k) \leq b_x, \quad \forall k \in \mathcal{I}_{\geq 0} \quad (12)$$
$$A_u u(k) \leq b_u, \quad \forall k \in \mathcal{I}_{\geq 0} \quad (13)$$

with time-varying, P-periodic matrices $A(k) = A(k+P), A(k) \in \mathbb{R}^{n\times n}$, $B(k) = B(k+P), B(k) \in \mathbb{R}^{n\times m}$ and a continuous, convex, piece-wise linear stage cost function

$\ell(x,u)$. *According to Rem. II.4, a P-periodic optimal orbit is obtained by solving*

$$(P_{\text{orbit}})\begin{cases}\min_{\tilde{x}\in\mathbb{X}^P,\tilde{u}\in\mathbb{U}^P}\tilde{\ell}(\tilde{x},\tilde{u})\\ \text{s.t.}\quad \tilde{x}=f^P(\tilde{x},\tilde{u},0).\end{cases}$$

The corresponding Lagrangian reads

$$\tilde{\nu}^T(\tilde{x}-f^P(\tilde{x},\tilde{u},0))+\tilde{\ell}(\tilde{x},\tilde{u})-\sum_{k=0}^{P-1}\ell(x_k^p,u_k^p).$$

It can be easily verified that the optimal dual variables $\tilde{\nu}^*$ define a valid, linear storage function $\tilde{\lambda}(\tilde{x})=\tilde{\nu}^{*\top}\tilde{x}$. Further, it can be shown that if and only if (P_{orbit}) has a unique solution (can be verified via LP [13]), strict dissipativity holds [10].

For the purpose of analysis, we define the *rotated stage cost* $\tilde{L}(\tilde{x},\tilde{u})=s(\tilde{x},\tilde{u})+\tilde{\lambda}(\tilde{x})-\tilde{\lambda}(f^P(\tilde{x},\tilde{u},0))$, the *rotated P-step terminal cost* $\tilde{V}_f(\tilde{x})=V_f((\tilde{x})_{P-1})+\tilde{\lambda}(\tilde{x})$ with $(\tilde{x})_{P-1}=x_{P-1}$, and the *auxiliary objective* $J_{\text{aux}}(x,\boldsymbol{u})=\sum_{k=0}^{N/P-1}\tilde{L}(\tilde{x}_{\boldsymbol{u}}(kP,x),\tilde{u}(kP))+\tilde{V}_f(\tilde{x}_{\boldsymbol{u}}(N,x))$.

Lemma III.9. *Solving (P_{EMPC}) using the rotated objective J_{aux} yields the same minimizer as in the case of J_{MPC}.*

Proof. Analogous to [2] it can be shown that $J_{\text{aux}}(x,\boldsymbol{u})=J_{\text{MPC}}(x,\boldsymbol{u})+c$ for some $c\in\mathbb{R}$. ☐

Lemma III.10. *For all $\tilde{x}\in\{\tilde{x}\in\mathbb{X}^P|x_{P-1}\in\mathbb{X}_f\}$ it holds $\tilde{V}_f(f^P(\tilde{x},\tilde{\kappa}_f(\tilde{x}),0))-\tilde{V}_f(\tilde{x})\leq-\tilde{L}(\tilde{x},\tilde{\kappa}_f(\tilde{x}))$.*

Proof. By Ass. III.1 for $\tilde{x}\in\{\tilde{x}\in\mathbb{X}^P|x_{P-1}\in\mathbb{X}_f\}$ it holds $V_f((f^P(\tilde{x},\tilde{\kappa}_f(\tilde{x}),0))_{P-1})-V_f((\tilde{x})_{P-1})\leq-\tilde{\ell}(\tilde{x},\tilde{\kappa}_f(\tilde{x}))+\sum_{k=0}^{P-1}\ell(x_k^p,u_k^p)$. Adding $-\tilde{\lambda}(\tilde{x})+\tilde{\lambda}(f^P(\tilde{x},\tilde{\kappa}_f(\tilde{x}),0))$ on both sides yields $V_f((f^P(\tilde{x},\tilde{\kappa}_f(\tilde{x}),0))_{P-1})-V_f((\tilde{x})_{P-1})-\tilde{\lambda}(\tilde{x})+\tilde{\lambda}(f^P(\tilde{x},\tilde{\kappa}_f(\tilde{x}),0))\leq-\tilde{L}(\tilde{x},\tilde{\kappa}_f(\tilde{x}))$, where the left hand side equals $\tilde{V}_f(f^P(\tilde{x},\tilde{\kappa}_f(\tilde{x}),0))-\tilde{V}_f(\tilde{x})$. ☐

Consider in the following $X_N=\{x\in\mathbb{X}|\exists\boldsymbol{u}\in\mathbb{U}^N(x)\text{ s.t. }x_{\boldsymbol{u}}(N,x)\in\mathbb{X}_f\}$, i.e. the set of states for which $(P_{\text{EMPC-P}})$ is feasible.

Lemma III.11. *For all $x\in X_N$, $k\in\mathcal{I}_{\geq0}$ it holds that*

$$\begin{aligned}&J_{\text{aux}}(x_{\text{cl}}(kP+P,x),\bar{\boldsymbol{u}}^{kP}(kP+P))\\&\qquad-J_{\text{aux}}(x_{\text{cl}}(kP,x),\boldsymbol{u}^*(x_{\text{cl}}(kP,x)))\\&\leq-\alpha\left(|\tilde{x}_{\text{cl}}(kP,x),\tilde{u}_{\text{cl}}(kP)|_{\Pi}\right).\end{aligned}$$

Proof. Using the auxiliary objective we have by Lem. III.10 that for all $k\in\mathcal{I}_{\geq0}$ it holds $J_{\text{aux}}(x_{\text{cl}}(kP+P,x),\bar{\boldsymbol{u}}^{kP}(kP+P))-J_{\text{aux}}(x_{\text{cl}}(kP,x),\boldsymbol{u}^*(x_{\text{cl}}(kP,x)))\leq-\tilde{L}(\tilde{x}_{\text{cl}}(kP,x),\tilde{u}_{\text{cl}}(kp))$. Using the strict dissipativity inequality from Ass. III.7 completes the proof. ☐

Theorem III.12. *Let Ass. III.1, III.4, and III.7 be satisfied. If $x(0)\in X_N$ then the closed loop system resulting from application of the P-step MPC controller (Alg. 1) asymptotically converges to the optimal periodic orbit Π.*

Proof. Under application of Alg. 1 we have along the closed loop system by Lem. III.11

$$\begin{aligned}&J_{\text{aux}}(x_{\text{cl}}(t+P,x),\boldsymbol{u}^*(x_{\text{cl}}(t+P,x)))\\&\qquad-J_{\text{aux}}(x_{\text{cl}}(t,x),\boldsymbol{u}^*(x_{\text{cl}}(t,x)))\\&\leq J_{\text{aux}}(x_{\text{cl}}(t+P,x),\bar{\boldsymbol{u}}^t(t+P))\\&\qquad-J_{\text{aux}}(x_{\text{cl}}(t,x),\boldsymbol{u}^*(x_{\text{cl}}(t,x)))\\&\leq-\alpha(|\tilde{x}_{\text{cl}}(t,x),\tilde{u}_{\text{cl}}(t)|_{\Pi}),\end{aligned}\tag{14}$$

i.e. that the sequence $J_{\text{aux}}(x_{\text{cl}}(t,x),\boldsymbol{u}^*(x_{\text{cl}}(t,x)))$ is nonincreasing with t. It is bounded from below, since the stage cost function $L(.,.)$ in (2) and terminal cost function $V_f(.)$ in Ass. III.1 are continuous and both, X_N and $\mathbb{X}_f$ are compact. Therefore the sequence converges, which implies by (14) that $\alpha(|\tilde{x}_{\text{cl}}(t,x),\tilde{u}_{\text{cl}}(t)|_{\Pi})\to0$ as $t\to\infty$. Using (5) this implies also that the system follows exactly the optimal periodic orbit Π for $t\to\infty$ which completes the proof. ☐

IV. Robust EMPC for Periodic Operation

Based on [11] we extend the EMPC scheme from the previous section to be applicable under the presence of disturbances $w(k)\in\mathbb{W}$, $\mathbb{W}$ compact, using a tube-based approach, see e.g. [14]. Define the nominal system as

$$z(t+1)=f(z(t),v(t),0),\ z(0)=z\tag{15}$$

and let the error between the real, disturbed system state $x(t)$ and the nominal system state $z(t)$ be $e(t)=x(t)-z(t)$. Further let

$$u(t)=\phi(v(t),x(t),z(t))\tag{16}$$

be an error feedback in order to keep the real system state $x(t)$ close to the nominal system state $z(t)$. The error dynamics is then defined as

$$e^+=f(x,\phi(v,x,z),w)-f(z,v,0).\tag{17}$$

Definition IV.1 (Cf. [11])**.** A set $\Omega\subseteq\mathbb{R}^n$ is robust control invariant (RCI) for the error dynamics (17) if there exists a feedback law (16) such that for all $x(t),z(t)\in\mathbb{R}^n$ with $e(t)\in\Omega$ and $x(t)\in\mathbb{X}$, $\phi(v(t),x(t),z(t))\in\mathbb{U}$ and for all $w\in\mathbb{W}$ it holds that $e(t+1)\in\Omega$.

The concept of tube-based robust model predictive control is to perform the open-loop optimization for the nominal system and then apply the input according to (16) to the real system. This way we guarantee that the real, disturbed system state $x(t)$ will always stay within a compact RCI set Ω around the nominal, calculated (predicted) states $z(t)$. In order to guarantee that $(x,u)\in\mathbb{X}\times\mathbb{U}$ under application of (16), we must tighten the state and input constraints $\mathbb{X}$ and $\mathbb{U}$ of (15) as in [11] to $\bar{\mathbb{Z}}=\{(z,v)\in\mathbb{X}\times\mathbb{U}|(x,\phi(v,x,z))\in\mathbb{X}\times\mathbb{U}\text{ for all }x\in\{z\}\oplus\Omega\}$. In the following we denote the projection of $\bar{\mathbb{Z}}$ on $\mathbb{X}$ as $\bar{\mathbb{X}}$ and $\bar{\mathbb{Z}}$ on $\mathbb{U}$ as $\bar{\mathbb{U}}$ respectively.

Assumption IV.2. *There exists a function $\phi:\bar{\mathbb{U}}\times\mathbb{X}\times\bar{\mathbb{X}}\to\mathbb{U}$ and an RCI set Ω according to Def. IV.1.*

A. Robust optimal periodic operation

Using the concept of an integrated stage cost function [11]

$$\ell^{\text{int}}(z,v)=\int_{x\in\{z\}\oplus\Omega}\ell(x,\phi(v,x,z))dx, \tag{18}$$

we define the robust optimal periodic orbit.

Definition IV.3. The robust optimal periodic orbit Π^* with optimal period length P^* of system (1), (2) is defined as

$$\underset{P\in\mathcal{I}_{\geq 1},\Pi\in S_\Pi^P}{\operatorname{argmin}}\sum_{i=0}^{P-1}\left(\int_{x\in\{z_i^p\}\oplus\Omega}\ell(x,\phi(v_i^p,x,z_i^p))dx\right) \tag{19}$$

with $(z_i^p,v_i^p)\in\Pi^*$, Ω an RCI set, and minimal P^*.

Note, that the RCI set can be seen as an outer approximation of the real system's behavior for all possible disturbances in $\mathbb{W}$. Thus, the stage cost function is integrated over the RCI set Ω, centered at the nominal state in order to take the influence of the disturbance into account. This can be also seen as averaging over all possible disturbances. Next, we define under which condition such a periodic orbit is the system's possibly best operation.

Definition IV.4. System (1) is said to be robustly optimally operated at the periodic orbit Π with respect to the stage cost (2) and the constraints $x\in\mathbb{X}$ and $u\in\mathbb{U}$ if for any feasible nominal input sequence $\boldsymbol{v}$ and its associated nominal state sequence $z_{\boldsymbol{v}}$ it holds that

$$\liminf_{T\to\infty}\frac{\sum_{t=0}^{T}\ell^{\text{int}}(z_{\boldsymbol{v}}(t,z(0)),v(t))}{T}\geq\frac{1}{P}\sum_{k=0}^{P-1}\ell^{\text{int}}(z_k^p,v_k^p),$$

with $(z_k^p,v_k^p)\in\Pi$.

We can state a similar sufficient condition as in [5, Cor. 14], which implies that a robust optimal periodic orbit Π is indeed the systems possibly best operation w.r.t. Def. IV.4.

Assumption IV.5. *There exists a continuous storage function $\tilde{\lambda}:\mathbb{R}^{nP}\to\mathbb{R}$ s.t. the nominal system* (15) *is dissipative with respect to the periodic orbit Π and the integrated stage cost function* (18)*, i.e.*

$$\tilde{\lambda}(f^P(\tilde{z},\tilde{v},0))-\tilde{\lambda}(\tilde{z})\leq s(\tilde{z},\tilde{v}) \tag{20}$$

with supply rate $s(\tilde{z},\tilde{v})=\tilde{\ell}^{\text{int}}(\tilde{z},\tilde{v})-\sum_{k=0}^{P-1}\ell^{\text{int}}(z_k^p,v_k^p)$.

Theorem IV.6. *If Ass. IV.5 holds, then system* (1) *is robustly optimally operated at Π according to Def. IV.4.*

Proof. The statement follows directly from [5, Cor. 13]. □

B. Tube-based robust economic MPC for periodic operation

We modify Ass. III.1 such that we can transfer the ideas from Sec. III to the robust setting.

Assumption IV.7. *Let $(z_i^p,v_i^p)\in\Pi$ for $i\in\mathcal{I}_{[0,P-1]}$. There exists a compact set $\bar{\mathbb{X}}_f\subseteq\bar{\mathbb{X}}$ s.t. $\Pi_{\mathbb{X}}\subseteq\bar{\mathbb{X}}_f$, a feedback law $\tilde{\kappa}_f:\bar{\mathbb{X}}^P\to\bar{\mathbb{U}}^P$, and a continuous terminal cost $\bar{V}_f:\bar{\mathbb{X}}_f\to\mathbb{R}$ s.t. for all $\tilde{z}$ with $z_{P-1}\in\bar{\mathbb{X}}_f$ it holds:*

1) $\tilde{\kappa}_f(\tilde{z})\in\bar{\mathbb{U}}^P$;
2) $f^P(\tilde{z},\tilde{\kappa}_f(\tilde{z}),0)\in\bar{\mathbb{X}}_f^P$;
3) $V_f((f^P(\tilde{z},\tilde{\kappa}_f(\tilde{z}),0))_{P-1})-V_f((\tilde{z})_{P-1})$ $\leq-\tilde{\ell}^{\text{int}}(\tilde{z},\tilde{\kappa}_f(\tilde{z}))+\Sigma_{i=0}^{P-1}\ell^{\text{int}}(z_i^p,v_i^p)$.

Without loss of generality, let $\bar{V}_f(z)\geq 0\ \forall z\in\bar{\mathbb{X}}_f$.

Let $N=N_1P$, $N_1\in\mathcal{I}_{>0}$ and define the nominal open loop optimization problem with integrated stage cost as

$$(P_{\text{REMPC-P}})\begin{cases}\min_{\boldsymbol{v}\in\bar{\mathbb{U}}^N}J_{\text{MPC}}^{\text{int}}(z,\boldsymbol{v})\\ \text{s.t. for all } k\in\mathcal{I}_{[0,N-1]}:\\ \qquad z_{\boldsymbol{v}}(k+1,z)=f(z_{\boldsymbol{v}}(k,z),v(k),0)\\ \qquad (z_{\boldsymbol{v}}(k,z),v(k))\in\bar{\mathbb{Z}}\\ \qquad z_{\boldsymbol{v}}(N,z)\in\bar{\mathbb{X}}_f\\ \qquad z_{\boldsymbol{v}(t)}(0,z)=z\end{cases}$$

with finite time open loop cost functional

$$J_{\text{MPC}}^{\text{int}}(z,\boldsymbol{v})=\sum_{k=0}^{N-1}\ell^{\text{int}}(z_{\boldsymbol{v}}(k,z),v(k))+\bar{V}_f(z_{\boldsymbol{v}}(N,z)).$$

Assumption IV.8. *The optimization problem $(P_{\text{REMPC}-\text{P}})$ is feasible at time $t=0$ for $z=x(0)$.*

Theorem IV.9. *If Ass. IV.2, IV.7 and IV.8 hold, then Alg. 2 is recursively feasible and the closed-loop system has an asymptotic average performance which is no worse than that of the robust optimal periodic orbit Π, i.e.*

$$\frac{1}{P}\sum_{k=0}^{P-1}\ell^{\text{int}}(z_k^p,v_k^p)\geq\limsup_{T\to\infty}\frac{\sum_{k=0}^{T-1}\ell^{\text{int}}(z_{\text{cl}}(k,z(0)),v_{\text{cl}}(k))}{T}$$

with $(z_i^p,v_i^p)\in\Pi$ for $i\in\mathcal{I}_{[0,P-1]}$.

Proof. Recursive feasibility w.r.t. the nominal system (15) can be shown as in Thm. III.5. By standard tube-based MPC arguments, recursive feasibility w.r.t. to the real system (1) follows from Ass. IV.2 and application of (16) to system (1). The second statement follows directly from Thm. III.6. □

Similar to Ass. III.7, we use a *strict* dissipativity condition with respect to the optimal periodic orbit in order to provide asymptotic convergence guarantees for the robust setting.

Assumption IV.10. *There exists a continuous storage function $\tilde{\lambda}:\mathbb{R}^{nP}\to\mathbb{R}$ and a $\mathcal{K}_\infty$ function α s.t. the nominal system* (15) *is strictly dissipative with respect to the periodic orbit Π and the integrated stage cost function* (18)*, i.e.*

$$\tilde{\lambda}(f^P(\tilde{z},\tilde{v},0))-\tilde{\lambda}(\tilde{z})\leq s(\tilde{z},\tilde{v})-\alpha(|(\tilde{z},\tilde{v})|_\Pi) \tag{21}$$

with supply rate $s(\tilde{z},\tilde{v})=\tilde{\ell}^{\text{int}}(\tilde{z},\tilde{v})-\sum_{k=0}^{P-1}\ell^{\text{int}}(z_k^p,v_k^p)$.

Algorithm 2 Robust EMPC for periodic operation

```
procedure REMPC-P(initial state z = x(0))
    for k_1 = 0, P, 2P, .. do
        solve (P_REMPC-P) with z = z_cl(k_1, z)
        for k_2 = k_1, k_1 + 1, .., k_1 + P − 1 do
            u_cl(k_2) = φ(v^{*,k_1}(k_2), x_cl(k_2, x(0)), z_cl(k_2, z))
            v_cl(k_2) = v^{*,k_1}(k_2)
```

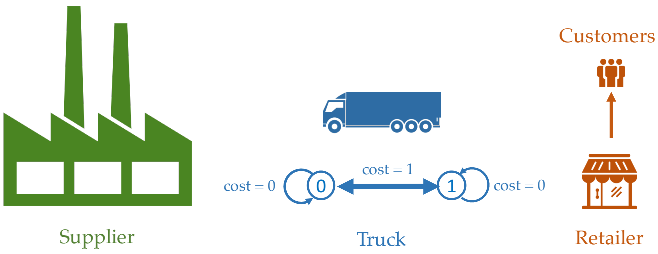

Fig. 2. Illustration of the simple supply chain network example.

Consider $\bar{X}_N = \{z \in \bar{\mathbb{X}} | \exists \boldsymbol{v} \in \bar{\mathbb{U}}^N(z) \text{ s.t. } z_{\boldsymbol{v}}(N,z) \in \bar{\mathbb{X}}_f\}$, i.e. the set of states for which ($P_{\text{REMPC-P}}$) is feasible.

Theorem IV.11. *Let Ass. IV.2, IV.7, IV.8 and IV.10 be satisfied. If $x(0) \in \bar{X}_N$, then the closed loop system resulting from application of the P-step robust MPC controller (Alg. 2) asymptotically converges to the neighborhood $\Pi_{\mathbb{X}} \oplus \Omega$ of the robust optimal periodic orbit Π.*

Proof. For the nominal system (15) it follows as in Thm. III.12 that $(z_{\text{cl}}(t, x(0)), v_{\text{cl}}(t)) \rightarrow \Pi$ for $t \rightarrow \infty$. By definition it holds $x_{\text{cl}}(t, x(0)) = z_{\text{cl}}(t, x(0)) + e_{\text{cl}}(t, 0)$ and by Ass. IV.2 we have that $e_{\text{cl}}(t, 0) \in \Omega$ for all $t \in \mathcal{I}_{\geq 0}$. This shows that $x_{\text{cl}}(t, x(0)) \rightarrow \Pi_{\mathbb{X}} \oplus \Omega$ as $t \rightarrow \infty$. $\square$

V. Example

We aim at controlling the simple supply chain network in Fig. 2 economically, see [10] for a more detailed explanation of the model, which we will describe briefly in the following. The state $x_{S,1}(k) \in \mathbb{R}$ represents the number of goods in the supplier production process, $x_{S,2}(k) \in \mathbb{R}$ in the supplier storage, $x_{T,L}(k) \in \mathbb{R}$ in the truck, $x_R(k) \in \mathbb{R}$ in the retailer storage and $x_{T,P}(k) \in \{0, 1\}$ describes the truck position. Inputs are represented using the truck navigation $u_{T,P} \in \{0, 1\}$ (0:stay; 1:drive), the truck loading of goods $u_{T,L} \in \mathbb{R}$, and the supplier production request $u_S \in \mathbb{R}$ as well as external disturbances $w \in \mathbb{W}$. Let

$$\begin{bmatrix} x_{S,1}(k+1) \\ x_{S,2}(k+1) \\ x_{T,P}(k+1) \\ x_{T,L}(k+1) \\ x_R(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{=:A} \underbrace{\begin{bmatrix} x_{S,1}(k) \\ x_{S,2}(k) \\ x_{T,P}(k) \\ x_{T,L}(k) \\ x_R(k) \end{bmatrix}}_{=:x(k)} +$$

$$\underbrace{\begin{bmatrix} 0 \\ 0 \\ f_{T,P}(x_{T,P}(k), u_{T,P}(k)) \\ 0 \\ 0 \end{bmatrix}}_{=:f_G(x(k),u(k))} + B_{\sigma(k)} \underbrace{\begin{bmatrix} u_S(k) \\ u_{T,L}(k) \end{bmatrix}}_{=:u_B(k)} + w(k)$$

describe the dynamics with uniformly distributed disturbances $w(k) \in \{w \in \mathbb{R}^5 | w_0 = w_1 = w_2 = w_3 = 0, w_4 \in \mathcal{I}_{[-3,-1]}\}$. The function $f_{T,P}$ encodes the dynamics of the position of the truck on the graph, see Fig. 2. The matrices $B_{\sigma(k)} \in \{B_0, B_1\}$ are given as

$$B_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix}^\top, B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}^\top,$$

with the switching policy $\sigma(k) = x_{T,P}$, and constraints $0 \leq x_{S,1} \leq a, 0 \leq x_{S,2} \leq a, 0 \leq x_{T,L} \leq 10, -a \leq x_R \leq a$, with $a \in \mathbb{R}, a > 100$. The stage cost is defined as

$$\ell(x,u) = \begin{cases} x_{S,1} + 0.5x_{S,2} + x_{T,L} + u_{T,P} - 10x_R, x_R < 0 \\ x_{S,1} + 0.5x_{S,2} + x_{T,L} + u_{T,P} + x_R, \ x_R \geq 0. \end{cases}$$

A larger demand than available goods (negative number of goods), means, that customers reach out for the store but they get disappointed, because the product they wish to buy is not available. This results in unhappiness of the customers which makes it more likely that they will go to another store the next time. Therefore we decide on a high penalty for negative number of goods. The optimal periodic orbit under the nominal demand $w_4 = -1$ is

$$P^* = 2, \Pi^* = \left\{ \left(\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right) \right\} \tag{22}$$

with average cost $\frac{1}{2} \sum_{k=0}^{1} \ell(x_k^{p*}, u_k^{p*}) = 3.5$.

A. EMPC for periodic operation

It can be verified (see [10] for details) that (22) fulfills Ass. III.7 by leveraging strong duality arguments of the optimal periodic orbit with respect to the system dynamics as described in Rem. III.8 by investigating different trajectories of the truck. Note that we were not able to establish a relation between [7, Ass. 5.3] and strong duality, neither could we guess a periodic storage function as required in order to properly apply the EMPC scheme [7].

For Ass. III.1 choose, as explained in [10],

$$\mathbb{X}_f = \mathbb{X} \cap \left\{ \left\{ \begin{matrix} x_{S,1} = 0 \\ x_{S,2} = 0 \\ x_{T,P} = 1 \\ x_{T,L} = 2 - x_R \\ x_R \leq 0 \end{matrix} \right\} \cup \left\{ \begin{matrix} x_{S,1} = 2 \\ x_{S,2} = 0 \\ x_{T,P} = 0 \\ x_{T,L} = 0 \\ x_R = 1 \end{matrix} \right\} \right\},$$

$$\tilde{\kappa}_f(\tilde{x}) = \begin{cases} \left(\begin{bmatrix} 1 \\ x_R - 2 \\ 2 - x_R - x_{T,L} + x_{S,2} \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right), \ x_{T,P} = 1 \\ ([1, 2, 0]^\top, [1, -2, 2]^\top), \ x_{T,P} = 0 \end{cases}$$

with $\tilde{x} = ((*), x)$, $(*) \in \mathbb{R}^5$ and terminal cost $V_f(x) = x_{T,P}(-11x_R)$. Note that $\mathbb{X}_f$ is a manifold, rather than a set of states, contained in the periodic orbit. A particular advantage of our approach is that Ass. III.1 has to be satisfied only over a period, while [7] poses a terminal assumption which has to hold for each time-step. The latter might be less intuitive to construct for the given example at hand. In Fig. 3, a sample closed-loop simulation of Alg. 1 is shown with constant nominal disturbance $w_4 = -1$ and $x(0) = [1, 1, 1, 4, -4]^\top$.

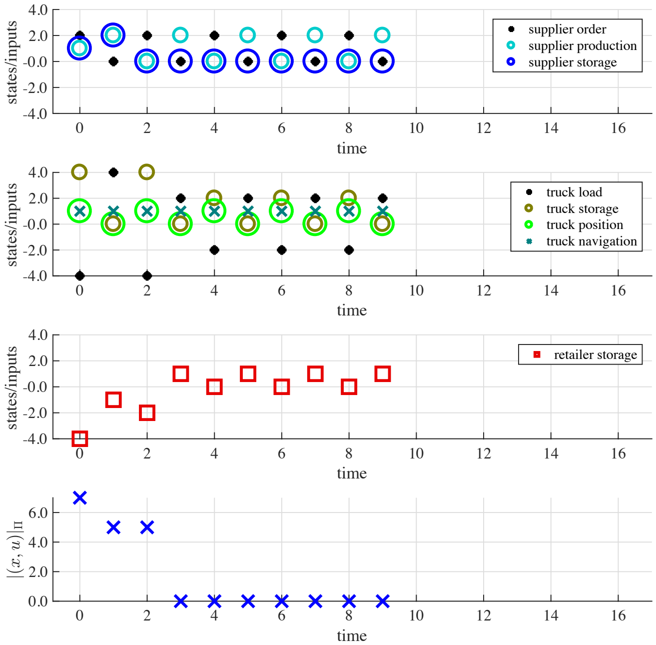

Fig. 3. Closed-loop distance to optimal orbit Alg. 1 without disturbances.

B. Robust EMPC for periodic operation

For constructing the tube (Ass. IV.2) let

$$\phi(v,x,z) = \begin{cases} [v_{T,L} - e_T, v_S]^\top, x_{T,P} = 0 \\ [v_{T,L} + e_R, v_S - e_R]^\top, x_{T,P} = 1, \end{cases} \tag{23}$$

as in [10], which yields the RCI set $\Omega = \{x \in \mathbb{R}^5 | [0,0,0,-4,-4]^\top \leq x \leq [4,0,0,0,0]^\top\}$. Solving (19) using Ω yields the robust optimal periodic orbit

$$P = 2,$$
$$\Pi \approx \left\{ \left(\begin{bmatrix} 2 \\ 0 \\ 0 \\ 4 \\ 4.2727 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right), \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 6 \\ 3.2727 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right) \right\}.$$

It can be shown as in the nominal case that Π fulfills Ass. IV.10. The terminal configuration from the previous section can be easily modified in order to fulfill Ass. IV.7, see [10].

Using a closed-loop simulation under uniformly distributed disturbances we calculated the average performance over 4000 simulation steps in Tab. I. We compare our results to the method proposed in [8]. With Alg. 1 and a 33.3% smaller planning horizon we get already the same performance as with the unconstrained algorithm [8]. In case of the mixed integer problem structure, a larger planning horizon results in a harder optimization problem (more discrete variables) compared to the additional terminal constraints with a shorter horizon. The tube-based approach (Alg. 2) yields an essential performance improvement.

VI. Conclusions

Based on the P-step system concept we developed an EMPC scheme for optimal periodic operation. This allowed us to establish performance guarantees and asymptotic convergence to the optimal periodic orbit. Moreover, for periodically time-varying systems and piece-wise linear stage cost, convergence to the optimal periodic orbit can be verified systematically under our assumptions. Based on the resulting EMPC scheme, we presented a tube-based extension, in order to consider the presence of disturbances explicitly for better performance and strict feasibility. We illustrated our findings using a simple supply chain network.

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TABLE I

Simulational closed-loop validation of Alg. 1 and Alg. 2.

Algorithm	Planning hor. N	Avg. Performance
Alg. 1	4	36.71
Alg. [8]	4	45.92
Alg. [8]	6	36.07
Alg. 2	4	9.72