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Optimal control with singular solution for SIR epidemic systems*

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***Abstract*— Mathematical modeling represents a useful instrument to study the evolution of an epidemic spread and to determine the best control strategy to reduce the number of infected subjects. The computation of the singular solution for a SIR epidemic system with vaccination control is performed; a constructive algorithm for the computation of a bang-singular-bang optimal solution is proposed and, for the specific choice of the parameters typical for a SIR model, the two switching instants, as well as the singular profile, are determined. Numerical simulations are reported to show the results of the described procedure.**

I. INTRODUCTION

Mathematical epidemic modeling allows to study the evolution of a specific epidemic spread and the effects of possible control strategies such as vaccination and/or quarantine and/or treatment. Moreover, it could suggest suitable scheduling of such strategies determing the best action, according to some criterion. Epidemic modeling usually makes use of compartmental description, that is to group subjects into homogeneous categories and model the interactions between the different classes. The most common epidemic model is the SIR one, where S stands for the susceptible individuals, I for the infected patients and R for the removed subjects [1], [2], [3], [4], [5]. Depending on the specificity of the spread to be faced, more complex descriptions coul be proposed, including subjects in the quarantine class, or infected but not infectious patients, or infected people that acquire only partial immunity, and so on, [6], [7], [8], [9]. The mathematical modeling and control of an epidemic spread has been increasing in the last years also for the developments of simulation tools able to face efficiently with nonlinear system equations and to the non availability of the solution is closed form, [10]. Optimal control theory is the natural framework in which study the control action to be applied to eradicate an epidemic spread, once the goal has been defined, as well as the constraints on the available resources. In particular, the applications of the minimum principle allows the determination of the optimal control that, depending on the cost index and on the modeling, could be a bang-bang or even a bang-singular-bang solution, [11], [12]. In fact if the model is linear in the control, as well as the cost index, it is possible to determine the conditions in which such kind of solution exists. In the bang-bang solution the control assumes only the extreme values, whereas the singular one is obtained if the Hamiltonian does not depend on the control in an interval of positive measure. In [1] and [4] the structure of singular control has been deeply investigated in presence of the double control, vaccination and medical treatment, showing that the latter can't be singular, whereas a singular regimen is expected for the optimal vaccination strategy. While the theoretical derivation of bang-singular-bang solutions may not be difficult, its implementation is usually rather elaborate. In fact the optimal control requires the solution of a non linear differential equations system in the state variables with initial conditions, and a non linear differential equations system in the costate variables with final conditions. Moreover, in general it is not easy the determination of the best control sequence and the number of switching points, [13], [14]. In this paper a SIR model is considered, in which, differently from the one analysed in [4], a recovered subject could neither become susceptible, nor infected again. Therefore, it is possible to study the singular surface, that is the manifold over which the state variables move with the singular control if it exists.

A constructive procedure to determine the best sequence of bang-singular-bang control is proposed in Section IV. Preliminarly, in Section II the mathematical model adopted is briefly presented along with the classical optimal control problem formulation for a fixed final time t_f and in Section III the conditions for the simple bang–bang solution are discussed.

The effectiveness of the procedure presented is evidenced by means of some numerical results in Section V.

II. THE MODEL AND THE OPTIMAL CONTROL PROBLEM

Denoting with $x_1(t)$ the number of susceptible subjects S, with $x_2(t)$ the number of infected patients I and with $x_3(t)$ the number of removed individuals R, the classical mathematical model describing the SIR epidemic spread is

$$\begin{aligned} \dot{x}_1 &= -\beta x_1 x_2 - x_1 u + \mu && (1)\\ \dot{x}_2 &= \beta x_1 x_2 - \gamma x_2 && (2)\\ \dot{x}_3 &= \gamma x_2 && (3) \end{aligned}$$

with given initial conditions

$$x_1(t_0) = x_{1,0} \qquad x_2(t_0) = x_{2,0} \qquad x_3(t_0) = x_{3,0} \qquad (4)$$

and box constraints:

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$$u(t) \in [0, U_M] \tag{5}$$

The parameter β depends on the rate of infectious contact, μ takes into account incoming subjects in the susceptible class, γ is related with the recovering rate. The optimal control problem aims at finding the control law $u(t)$ which minimizes the cost function

$$J(x(t), u(t)) = \int_{t_0}^{t_f} (\alpha x_2(t) + cu(t))\, dt \tag{6}$$

for a fixed t_f and under constraints (1)–(5). The aim is to minimize the number of infected subjects in the fixed time interval $[0, t_f]$ by using as less resources as possible; the positive parameters c and α represent the weights of these two contrasting requirements.

The corresponding Hamiltonian is

$$\begin{aligned} H(x, \lambda, u) &= (\alpha x_2 + cu) + \lambda_1 (-\beta x_1 x_2 - x_1 u + \mu) + \\ &\quad + \lambda_2 (\beta x_1 x_2 - \gamma x_2) + \lambda_3 \gamma x_2 = \\ &= (\alpha x_2 - \lambda_1 \beta x_1 x_2 + \lambda_1 \mu + \lambda_2 \beta x_1 x_2 + \\ &\quad - \gamma \lambda_2 x_2 + \lambda_3 \gamma x_2) + (c - \lambda_1 x_1)\, u = \\ &= F(x, \lambda) + G(x, \lambda) u \end{aligned} \tag{7}$$

where the linear dependence from the control $u(t)$ has been evidenced, and $\lambda = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}^T$ denotes the vector of the costate functions. The condition

$$H(t) = K \quad \forall t \in [t_0, t_f] \tag{8}$$

holds for a certain unknown $K \in R$, since t_f is fixed.

The equations for the costate λ are

$$\dot{\lambda}_1 = \beta \lambda_1 x_2 + \lambda_1 u - \beta \lambda_2 x_2 \tag{9}$$

$$\dot{\lambda}_2 = -\alpha + \lambda_1 \beta x_1 - \lambda_2 \beta x_1 + \lambda_2 \gamma - \lambda_3 \gamma \tag{10}$$

$$\dot{\lambda}_3 = 0 \tag{11}$$

with final conditions

$$\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = 0 \tag{12}$$

being the state variables not fixed.

From (11) and (12) and the continuity of λ, one has

$$\lambda_3(t) = 0 \qquad \forall t \in [t_0, t_f] \tag{13}$$

so simplifying (7) and (10). Therefore, from now on only the costate variables $\lambda_1(t)$ and $\lambda_2(t)$ will be considered, as well as only the state variables $x_1(t)$ and $x_2(t)$ will be taken into account, being $x_3(t)$ dependent only from $x_2(t)$ and fully determined once $x_2(t)$ is known.

Then, from now on, the notations $x = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T$ and $\lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix}^T$ are used.

The minimum principle condition is:

$$G(x, \lambda) u \leq G(x, \lambda) \omega \qquad \forall \omega \in [0, U_M] \tag{14}$$

Condition (8), evaluated in t_f, together with (12), implies

$$\alpha x_2(t_f) + cu(t_f) = K \tag{15}$$

The minimum principle (14) yields the optimal control

$$u^o(t) = \begin{cases} U_M & G(x, \lambda) < 0 \\ u_S(t) & G(x, \lambda) = 0 \\ 0 & G(x, \lambda) > 0 \end{cases} \tag{16}$$

A switching between different expressions of $u^o(t)$ occurs at any time t_s in which the condition

$$G(x(t_s), \lambda(t_s)) = 0 \tag{17}$$

holds, that is, for the case here considered, when

$$c - \lambda_1(t_s) x_1(t_s) = 0 \tag{18}$$

Clearly, condition (18) can be satisfied only if neither $\lambda_1(t_s)$ nor $x(t_s)$ are equal to zero, being $c > 0$.

Moreover, since

$$G(x(t_f), \lambda(t_f)) = c - \lambda_1(t_f) x_1(t_f) = c > 0 \tag{19}$$

conditions (16) applied to (15) give the final admissible value for $x_2(t)$

$$x_2(t_f) = \frac{K}{\alpha} \tag{20}$$

The expression (16) of the optimal control suggests different scenarios depending on the evolution of the switching function $G(x, \lambda)$. It is worth to be noted that the number of switching points does not influence the value of the cost index in the proposed formulation; the choice of the best strategy is determined comparing the values of the cost index in the different configurations.

In the next Sections III and IV, two basic possible strategies will be discussed, starting from the simple non singular bang-bang solution (Section III) up to the bang-singular-bang one (Section IV). All the possible scenarios are composed of combinations of these two possible solutions.

III. The Bang-Bang solution

Consider the switching function $G(x, \lambda)$. In this Section it is discussed the non singular solution; it occurs when the switching function is zero only on isolated points called switching instants.

From condition (19) and (16), it results that the control in the last subinterval $[t_s, t_f]$, for a suitable t_s, must be equal to 0.

Two situations are possible:

1. $u(t) = 0 \quad t \in [t_0, t_f]$;
2. $u(t) = U_M \quad t \in [t_0, t_s]$ and then $u(t) = 0 \quad t \in (t_s, t_f]$

In case 1, the system (1)–(2), with initial conditions in (4), evolves without control up to the final instant t_f; once $x(t)$ is known, the evolution of the costate dynamics (9)–(10), with final conditions (12), can be obtained. The solution is acceptable if $x(t)$ and $\lambda(t)$ satisfy condition (19) over the whole time interval $[0, t_f]$; in this case, the determination of K follows from conditions (20).

Then, the value of the cost index (6) is simply equal to the integral of the $\alpha x_2(t)$ function.

In case 2, assume the initial control effort $u(t) = U_M$; substituting this expression into the system (1)–(2), the evolution of the state is completely determined up to the switching point t_s, that is the one in which (18) is satisfied. From that instant on, the system evolves with null control, from the initial condition $x(t_s)$, up to the final instant t_f, where (20) is satisfied. Also the evolution of the costate functions depends on t_s: for $t \geq t_s$, $\lambda_1(t) = \lambda_1(t_s;t)$ and $\lambda_2(t) = \lambda_2(t_s;t)$. Note that two parameters are unknown: K and the switching instant t_s; their values can be determined by the following conditions:

$$\lambda_1(t_s;t_f) = 0 \tag{21}$$

$$\lambda_2(t_s;t_f) = 0 \tag{22}$$

$$x_2(t_s;t_f) = \frac{K}{\alpha} \tag{23}$$

$$\lambda_1(t_s) = \frac{c}{x_1(t_s)} \tag{24}$$

If equations (21)-(24) are compatible (and therefore K and t_s may be determined), the solution proposed in case 2 is acceptable if in the interval $[0,t_s]$ condition $G(x(t_f),\lambda(t_f)) = c - \lambda_1(t_f)x_1(t_f) = c < 0$ is satisfied, as well as condition (19) is verified in $[t_s,t_f]$.

The number of switching points depends on the number of sign changes of the function $G(x,\lambda)$. From the final condition (12), as already noted, it results that the sequence of optimal control must end with the interval in which $u(t) = 0$. Therefore, if the first control in the optimal sequence is the maximum value allowed $u(t) = u_M$, the number of switches must be odd, whereas if the initial control is null the number of switching instants must be even.

IV. The singular solution

A singular solution occurs when $G(x,\lambda) = 0$ over a finite time interval $[t_1,t_2]$, $t_0 < t_1 < t_2 < t_f$. This implies that also all its time derivatives are equal to zero in the same interval. Be $G^{(k)}(x,\lambda)$ the k-th time derivative of $G(x,\lambda)$. Note that $G(x,\lambda) = 0$ implies, for (8), that $F(x,\lambda) = K$ at the same time instants.

The conditions of existence of a singular solution can be formulated as

$$\begin{cases} G(x,\lambda) = 0 \\ G^{(1)}(x,\lambda) = 0 \end{cases} \tag{25}$$

whose expressions, for the considered system, are

$$\begin{cases} c - \lambda_1(t)x_1(t) = 0 \\ \beta\lambda_2(t)x_1(t)x_2(t) - \lambda_1(t)\mu = 0 \end{cases} \tag{26}$$

or, equivalently,

$$\begin{pmatrix} x_1(t) & 0 \\ -\mu & \beta x_1(t)x_2(t) \end{pmatrix} \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix} \tag{27}$$

If $x_1(t) \neq 0$ and $x_2(t) \neq 0$, one gets

$$\lambda_1(t) = \frac{c}{x_1(t)} \qquad \lambda_2(t) = \frac{c\mu}{\beta}\frac{1}{x_1^2(t)x_2(t)} \tag{28}$$

Otherwise, no solution can be found, since for $x_1(t) = 0$ the condition $G(x,\lambda) = 0$ is not satisfied, and, for $x_2(t) = 0$ one has

$$\text{rank}\begin{pmatrix} x_1(t) & 0 \\ -\mu & 0 \end{pmatrix} = 1 \quad \text{and} \quad \text{rank}\begin{pmatrix} x_1(t) & 0 & c \\ -\mu & 0 & 0 \end{pmatrix} = 2 \tag{29}$$

Computing $G^{(2)}(x,\lambda)$, the expression

$$\begin{aligned} G^{(2)}(x,\lambda) \quad = \quad & \beta x_1x_2\left(-\alpha + \lambda_1\beta x_1 - \lambda_2\beta x_1 + \lambda_2\gamma\right) + \\ & +\beta x_1\lambda_2\left(\beta x_1x_2 - \gamma x_2\right) + \\ & +\beta\lambda_2x_2\left(-\beta x_1x_2 - x_1u + \mu\right) + \\ & -\mu\left(\beta\lambda_1x_2 + \lambda_1u - \beta\lambda_2x_2\right) = \\ & -\alpha\beta x_1x_2 + \beta^2\lambda_1x_1^2y_1 - \beta^2\lambda_2x_1^2x_2 + \\ & +\beta\gamma\lambda_2x_1x_2 + \beta^2\lambda_2x_1^2x_2 - \beta\gamma\lambda_2x_1x_2 + \\ & -\beta^2\lambda_2x_1x_2^2 - \beta\lambda_2x_1x_2u + \beta\mu\lambda_2x_2 + \\ & -\beta\mu\lambda_1x_2 - \mu\lambda_1u + \beta\mu\lambda_2x_2 = \\ & -\alpha\beta x_1x_2 + \beta^2\lambda_1x_1^2x_2 - \beta^2\lambda_2x_1x_2^2 + \\ & -\beta\lambda_2x_1x_2u + \beta\mu\lambda_2x_2 + \\ & -\beta\mu\lambda_1x_2 - \mu\lambda_1u + \beta\mu\lambda_2x_2 = \\ & \beta x_2\left[-\alpha x_1 + \lambda_1\left(\beta x_1^2 - \mu\right) + \right. \\ & \left.\lambda_2\left(2\mu - \beta x_1x_2\right)\right] - \left(\beta\lambda_2x_1x_2 + \mu\lambda_1\right)u \end{aligned} \tag{30}$$

is found, which allows to obtain the control as function of the state and the costate

$$u_S(x,\lambda) = \beta x_2\frac{-\alpha x_1 + \lambda_1\left(\beta x_1^2 - \mu\right) + \lambda_2\left(2\mu - \beta x_1x_2\right)}{\beta\lambda_2x_1x_2 + \mu\lambda_1} \tag{31}$$

Making use of (28), expression (31) becomes

$$u_S(x) = \beta x_2\frac{\beta cx_1 - \alpha x_1 - \frac{c\mu}{x_1} + \frac{c\mu}{\beta}\frac{1}{x_1^2x_2}\left(2\mu - \beta x_1x_2\right)}{\beta\frac{c\mu}{\beta}\frac{1}{x_1^2x_2}x_1y_2 + \mu\frac{c}{x_1}}$$

which gives the state feedback control law

$$u_S(x) = \beta x_2\left(\frac{\beta c - \alpha}{2c\mu}x_1^2 - 1\right) + \frac{\mu}{x_1} \tag{32}$$

A. Computation of the singular surface

The singular curve is the state space locus where the system evolution lays when the singular control $u_S(x)$ is applied between the switching instants t_1 and t_2. From (8), when conditions (25) are satisfied,

$$F(x,\lambda) = K$$

holds, yielding to the expression

$$\alpha x_2 - \lambda_1\beta x_1x_2 + \lambda_1\mu + \lambda_2\beta x_1x_2 - \gamma\lambda_2x_2 = K \tag{33}$$

Making use of (28) in (33), one gets the singular curve

$$SC(K;x) = (\alpha - \beta c)x_2 + 2\frac{c\mu}{x_1} - \frac{c\gamma\mu}{\beta x_1^2} - K = 0 \tag{34}$$

function of the state $x(t)$ and depending on the parameter K; due to its structure, it can be easily written as a function $x_2 = x_2(x_1)$:

$$x_2 = \frac{K}{\alpha - \beta c} - \frac{2c\mu}{\alpha - \beta c}\frac{1}{x_1} + \frac{c\gamma\mu}{\beta(\alpha - \beta c)}\frac{1}{x_1^2} \qquad (35)$$

The same results can be obtained observing that the singular solution must satisfy (8), as any other solution, with, in addition, (25). Since (8), when $G(x,\lambda) = 0$ is equivalent to $F(x,\lambda) = K$, the following system of equations can be written

$$\begin{cases} F(x,\lambda) = K \\ G(x,\lambda) = 0 \\ G^{(1)}(x,\lambda) = 0 \end{cases} \qquad (36)$$

corresponding to the compact form

$$\begin{pmatrix} \mu - \beta x_1 x_2 & \beta x_1 x_2 - \gamma x_2 \\ x_1 & 0 \\ -\mu & \beta x_1 x_2 \end{pmatrix} \lambda = \begin{pmatrix} K - \alpha x_2 \\ c \\ 0 \end{pmatrix} \qquad (37)$$

It is known that, as for any three equations - two variables systems, a condition for the existence of a solution $\lambda \in R^2$ is that the three equations are dependent, i.e.

$$\det \begin{pmatrix} \mu - \beta x_1 x_2 & \beta x_1 x_2 - \gamma x_2 & K - \alpha x_2 \\ x_1 & 0 & c \\ -\mu & \beta x_1 x_2 & 0 \end{pmatrix} = 0 \quad (38)$$

Computing the determinant (38), one gets

$$\begin{aligned} &-\beta c\mu x_1 x_2 + c\gamma\mu x_2 + \beta K x_1^2 x_2 - \alpha\beta x_1^2 x_2^2 + \\ &-\beta c\mu x_1 x_2 + \beta^2 c x_1^2 x_2^2 = \\ &= \beta\left[-2c\mu x_1 + K x_1^2 + (\beta c - \alpha)x_1^2 x_2 + \frac{c\gamma\mu}{\beta}\right] x_2 = 0 \end{aligned}$$

and, then,

$$x_2 = 0 \qquad (39)$$

or

$$-2c\mu x_1 + K x_1^2 + (\beta c - \alpha)x_1^2 x_2 + \frac{c\gamma\mu}{\beta} = 0 \qquad (40)$$

Clearly, (40) is the same as (35). The singular curve (34) obviously depends on the unknown parameter K.

The determination of the switching instants t_1 and t_2 may be outlined as follows;:

1. set $u(t) = U_M$.
2. Integrate the x dynamics (1)–(2), getting $x_1(t)$, $x_2(t)$.
3. Define $t = t_1$ as the unknown (first) time instant in which (34) is verified, so that

$$SC(K; x(t_1), y(t_1)) = 0 \qquad (41)$$

   Note the dependency

$$t_1 = t_1(K) \qquad (42)$$

   Moreover, from (28), also $\lambda_1(t_1)$ and $\lambda_2(t_1)$ are known.
4. Set $u(t) = u_S(t)$ and integrate the dynamics (1)–(2) plus (9)-(10), starting from the initial conditions $x_1(t_1)$, $x_2(t_1)$, $\lambda_1(t_1)$ and $\lambda_2(t_1)$, until a certain $t = t_2$, $t_1 < t_2 < t_f$. The evolutions in the time interval $[t_1, t_2]$, say $x_1(t_1(K);t)$, $x_2(t_1(K);t)$, as well as $\lambda_1(t_1(K);t)$ and $\lambda_2(t_1(K);t)$, depend, of course, on the instant t_1. Clearly, $x_1(t_1(K);t)$ and $x_2(t_1(K);t)$ satisfies (34) $\forall t \in [t_1, t_2]$.
   Compute $x_1(t_1(K);t_2)$, $x_2(t_1(K);t_2)$, $\lambda_1(t_1(K);t_2)$ and $\lambda_2(t_1(K);t_2)$
5. Set $u(t) = 0$ and integrate (1)–(2) and (9)–(10) from $x_1(t_1(K);t_2)$, $x_2(t_1(K);t_2)$, $\lambda_1(t_1(K);t_2)$ and $\lambda_2(t_1(K);t_2)$ until the fixed final instant t_f, so obtaining the full evolutions $x_1(t_1(K),t_2;t)$, $x_2(t_1(K),t_2;t)$, $\lambda_1(t_1(K),t_2;t)$ and $\lambda_2(t_1(K),t_2;t)$, in which the dependency from the switching instants t_1 and t_2 has been evidenced.

If a singular solution exists, the following conditions must hold:

$$x_2(t_1(K),t_2;t_f) = \frac{K}{\alpha} \qquad (43)$$

$$\lambda_1(t_1(K),t_2;t_f) = 0 \qquad (44)$$

$$\lambda_2(t_1(K),t_2;t_f) = 0 \qquad (45)$$

From (42), (41), (43), (44) and (45) it could be possible to determine the three unknows t_1, t_2 and K.

V. Simulation results

The computation of the optimal bang–singular–bang control is here performed. Since the explicit solutions in analytic form required for the conditions (43)—(45) are not easy to compute, the control problem is solved using the steps 1–3, 4 and 5 separately. Starting from step 5, the point $x(t_2) \in SC(K;x)$ is found since, from such a point and after a time $T < t_f$, the final conditions in (43)–(45) are satisfied. The point $x(t_2)$ is computed numerically, and the values K and T are obtained. Then, one has that $t_2 = t_f - T$. Moreover, once K is known, steps 1–3 can be performed, finding the first time instant t_1 in which the forced dynamics under input $u(t) = U_M$ intersects the full known singular curve $SC(K;x)$. From t_1 to t_2, the singular control $u_S(x)$ in (32) is used.

In the proposed simulation, the final time has been fixed to $t_f = 20$ and the weights $\alpha = 1$ and $c = 1$ in the cost function (6) are chosen. The model parameters are fixed to $\beta = 0.01$, $\gamma = 0.4$ and $\mu = 10$, while the initial conditions have been chosen as $x_{1,0} = 50$ and $x_{2,0} = 10$.

In Figures 1 and 2 the time evolution of the two state variables $x_1(t)$, the Susceptible individuals, and $x_2(t)$, the Infected ones, are reported. For a comparative purpose, the evolution of the same variables without control action are also depicted, evidencing that the effect of the control is to reduce strongly the number of infected subjects; this corresponds to a long term increment of (uninfected) susceptible individuals. The discontinuities corresponding to the switches bang–singular at $t = t_1 = 5.32$ and singular–bang at $t = t_2 = 15.3$ are well evident in Figure 1.

The optimal control is reported in Figure 3. It is composed by the three time segments $[0, t_1)$, with $u(t) = U_M$, $[t_1, t_2)$, with $u(t) = u_s(t)$, and $[t_2, t_f]$, with $u(t) = 0$, according to

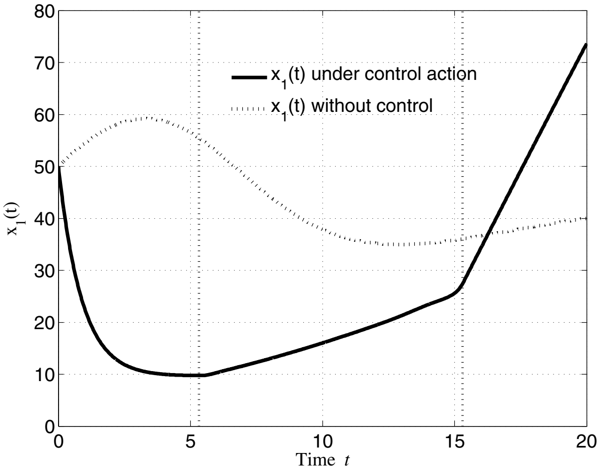

Fig. 1. Time history of the first state component $x_1(t)$, corresponding to the Susceptible individuals. Both the uncontrolled and the controlled cases are considered.

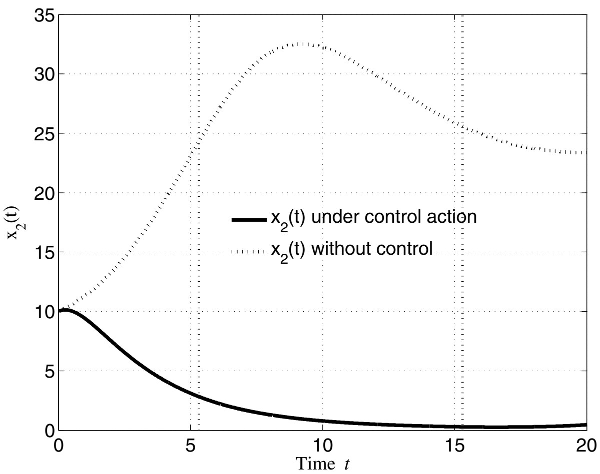

Fig. 2. Time history of the second state component $x_2(t)$, corresponding to the Infected individuals. Both the uncontrolled and the controlled cases are considered.

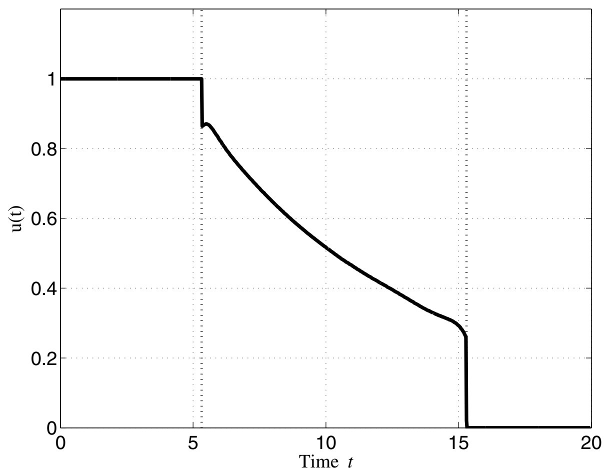

Fig. 3. The Bang–Singular–Bang optimal control $u(t)$.

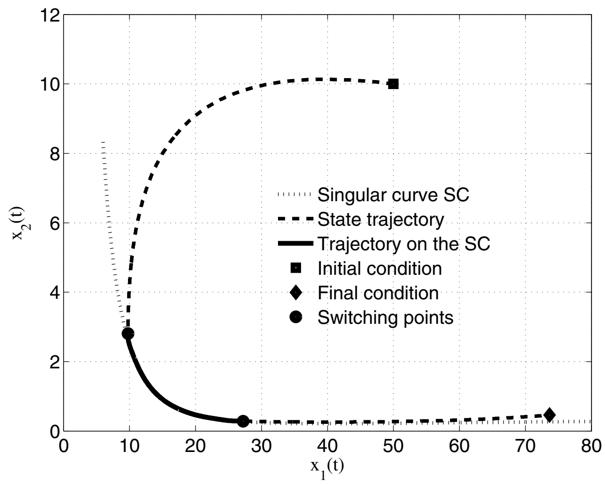

Fig. 4. State trajectory in the x_1–x_2 plane, compared with the singular curve.

expressions in (16). Discontinuities in $t = t_1 = 5.32$ and in $t = t_2 = 15.3$, evidenced by the vertical dotted lines, are present and are compatible with the condition for $u(t)$ to be continuous almost everywhere.

The behaviour of the controlled system is evidenced in Figure 4, in which the trajectory in the x_1–x_2 plane is depicted. It is composed by a first segment, the dashed line from the initial condition (small black square in $(50, 10)$) to the singular curve (dotted line). Then, a segment (solid line) of the trajectory along the singular curve follows, until the trajectory leaves the singular curve and reaches the final condition (small black diamond) with the new dashed segment.

The possibility of the full computation of the state and the input, making use of the conditions (12), allows to find the time evolution of the costate λ. In Figures 5 and 6 they are reported. They can be useful to verify easily the fulfilment of conditions (25) between the two switching instants t_1 and t_2, marked by the vertical dotted lines. It is also interesting to observe that $\lambda_1(t)$ shows a more evident change in the shape at the switching instants, like the susceptible subjects $x_1(t)$, than $\lambda_2(t)$, smother as $x_2(t)$.

VI. CONCLUSIONS

In an optimal control problem, depending on the parameters of the model, the optimal solution could be a bang-bang or a bang-singular-bang one or a combination of both. In general it is not possible to determine in advance the number of switching points and the sequence of the different control actions. It is proposed a constructive algorithm to determine the singular solution, if it exists, in an optimal control problem designed to face the SIR epidemic spread. The first results evidenced the positive effects of the optimal control on the number of infected subjects. The optimal solution has two switching points and consists of three parts: firstly, the maximum effort at the beginning of the control interval; then, a singular arc up to another switching instant

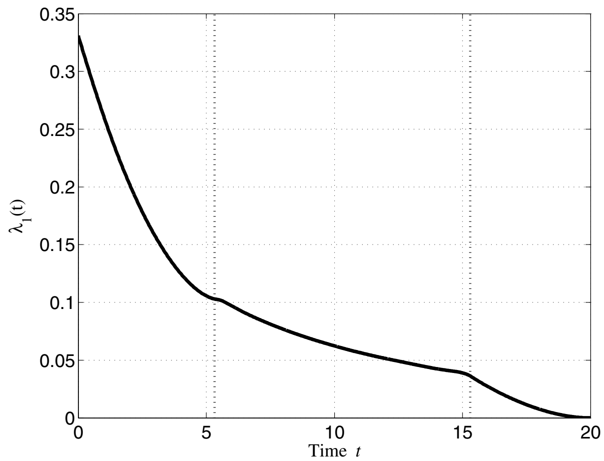

Fig. 5. Time history of the costate variable $\lambda_1(t)$.

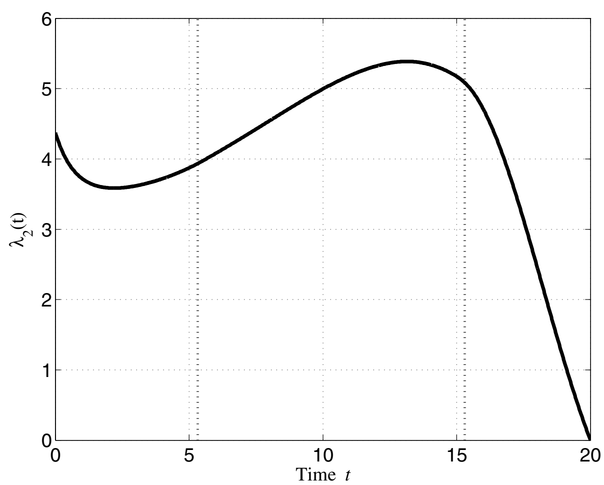

Fig. 6. Time history of th costate variable $\lambda_2(t)$.

from which the best strategy is to adopt the lower limit of the control.

An interesting aspect of the proposed approach is that the computation of the singular control does not require the solution of the differential equations of the costate variables with final conditions, so avoiding the forward (state) and backward (costate) combination of the integration procedure. The best sequence of bang- bang or bang-singular-bang solution is determined making use of the concept of singular surface.

The possibility of obtaining the full behaviours of all the variables involved makes the procedure complete and effective.

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