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Quadratic Control of Linear Discrete-time Positive Systems

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Abstract— **The linear matrix inequalities approach is proposed to discrete-time linear positive system control design, mounting linear quadratic constraints into design conditions. Coupling together the set of linear matrix inequalities warranting the system positive structure, and the additional inequality guaranteeing asymptotic stability of the controlled system, the design conditions are outlined to construct the positive closed-loop scheme with a state-feedback positive control law gain. Some related properties are deduced to come towards a solution for forced-mode control policy in discrete-time linear positive system using the static decoupling principle. The proposed approaches are numerically illustrated.**

I. Introduction

Positive systems indicate the processes whose variables represent quantities that do not have meaning unless they are nonnegative [18]. The mathematical theory of Metzler matrices has a close relationship to the theory of positive linear continuous-time dynamical systems [7] since, in the relevant description, the state-space system matrix of a positive systems is Metzler and the system input and output matrices are nonnegative matrices [11], [16]. The problem of Metzlerian system stabilization means to design a positive gain of control law in such a way that the Metzler closed-loop system matrix is Hurwitz [4]. The synthesis of state-feedback controllers, guaranteeing the closed-loop system to be asymptotically stable and positive is investigated by linear programming approach in [2]. Consequently, most of the well-tried general techniques designed for linear systems cannot be straightly nominated generally to positive linear systems [9], [20].

In order to reduce the number of boundaries entering the solution in linear programming methods, a synthesis procedure based on linear matrix inequalities (LMI) is proposed for positive discrete-time linear systems in [13], in which the parametric boundaries are defined by n LMIs. Because the solution of such defined base set of LMIs only assures the positivity of the closed-loop system matrix, the design provisions are complemented by another set of LMIs that impose a stable solution in the sense of H_∞ or H_2/H_∞ formulation. An applicable method for state estimation of positive linear discrete-time systems, maintaining the positivity of Luenberger observers is presented in [14]. In addition to these papers, others details and references concerning the discrete-time linear positive systems can be found, e.g., in [3], [21].

The main motivation issues of this paper are the design conditions formulated for control of linear positive discrete-time systems with linear quadratic constraints by using the state-feedback control law. Since, at defined boundaries on elements of a strictly positive system matrix structure, this task cannot be bound explicitly to the solution of the Riccati equation, and the linear quadratic constraints are used to extend the LMI reflecting the Lyapunov stability condition for overall completion of the LMIs set. Because the principle of static decoupling can not be used without substantial modifications for positive linear discrete-time systems, related properties are deduced to come towards a solution for forced-mode control policy.

Used notations are conventional so that $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denotes transpose of the vector $\boldsymbol{x}$ and matrix $\boldsymbol{X}$, respectively, $\boldsymbol{x}_+$, $\boldsymbol{X}_+$ indicates a nonnegative vector and a nonnegative matrix, $\boldsymbol{X} \succ 0$ means that $\boldsymbol{X}$ is a symmetric positive definite matrix, $\rho(\boldsymbol{X})$ reports the eigenvalue spectrum of the square matrix $\boldsymbol{X}$, the symbol $\boldsymbol{I}_n$ marks the n-th order unit matrix, diag$[\,\cdot\,]$ enters up a diagonal matrix, $I\!R^n_n$, $I\!R^{n\times r}_+$ signifies the set of all n-dimensional real non-negative vectors and $n \times r$ real non-negative matrices, respectively, and $\mathcal{Z}_+$ is the set of all positive integers.

II. Linear Discrete-time Positive Systems

To analyze properties of linear discrete-time positive systems, and concisely define these structures, it is preferred the state-space description defined as

$$\boldsymbol{q}(i+1) = \boldsymbol{F}\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{u}(i) \tag{1}$$

$$\boldsymbol{y}(i) = \boldsymbol{C}\boldsymbol{q}(i) \tag{2}$$

The equations (1), (2) represent a linear discrete-time positive system if $\boldsymbol{F} \in I\!R^{n\times n}_+$, $\boldsymbol{G} \in I\!R^{n\times r}_+$, $\boldsymbol{C} \in I\!R^{m\times n}_+$ and $\boldsymbol{q}(i) \in I\!R^n_+$, $\boldsymbol{u}(i) \in I\!R^r_+$, $\boldsymbol{y}(i) \in I\!R^m_+$ for all $i \in \mathcal{Z}_+$. It is considered in the following that the matrix $\boldsymbol{F} \in I\!R^{n\times n}_+$ is strictly positive (all its elements are positive).

Note, $\boldsymbol{F} \in I\!R^{n\times n}_+$ is stable, strictly positive matrix if it is Schur and all its elements are positive [2].

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Thus, directly, a sequence of $\boldsymbol{q}(i)$ generated (1) excited by $\boldsymbol{u}(i) \in \mathbb{R}_+^r$ and $\boldsymbol{q}(0) \in \mathbb{R}_+^n$ is asymptotically stable and positive if $\boldsymbol{F}$ is a positive Schur matrix and $\boldsymbol{G} \in \mathbb{R}_+^{n\times r}$ is a non-negative matrix. The discrete-time linear system (1), (2) is asymptotically stable and externally positive if $\boldsymbol{F}$ is a positive Schur matrix and $\boldsymbol{G} \in \mathbb{R}_+^{n\times r}$, $\boldsymbol{C} \in \mathbb{R}_+^{m\times n}$ are non-negative matrices [8]. This concretely illustrates the distinctive features of the system external positiveness and stability.

III. Parameter Constraints

Since a permutation matrix is applied to define the enumeration writing of the system matrix elements constraints for positive linear discrete-time systems, the indexing similar to the addition modulo n of two integers can be used to minimize the number of the n-dimensional composite constraint vectors.

Definition 1: [5], [19] Let n be a fixed positive integer. Two integers j and h are congruent modulo n if they differ by an integral multiple of the integer n. If j and h are congruent modulo n, the expression $(j = h)_{mod\ n}$ is called a congruence, and the number n is called the modulus of the congruence. If $S = \{0, 1, 2, \ldots, n-1\}$ is the complete set of residues for given integer n, the addition modulo n on the set S is $(j + h)_{mod\ n} = r$, where r is the element of S to which the result of the usual sum of integers j and h is congruent modulo n.

Since rows and columns of a square matrix of dimension $n \times n$ are generally denoted from 1 to n, the complete set of residues $S = \{0, 1, 2, \ldots, n\}$ for any positive integer $n+1$ has to be used. Then, in the following, the addition modulo $n+1$ on S is defined as $(j+h)_{mod\ n+1} = r+1$, where r is the element of S to which the result of the usual sum of integers j and k is congruent modulo $n+1$. The used shorthand symbolical notation for $(j+h)_{mod\ n+1} = r+1$ is $(j+h)_{(1\leftrightarrow n)/n} = r+1$.

Considering, for simplicity, a SISO (single input, single output) linear discrete-time strictly positive system, (1), (2) and the control law characterized by the relation

$$u(i) = -\boldsymbol{k}^T \boldsymbol{q}(i) \tag{3}$$

then (1), (3) imply the algebraic inequalities, corresponding to the closed-loop system matrix strictly positiveness,

$$f(h, j) = g(h)k(j) > 0 \text{ for all } h, j \in \langle 1, \ldots, n\rangle \tag{4}$$

where

$$\boldsymbol{F} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ & & \vdots & \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix},\ \boldsymbol{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix},\ \boldsymbol{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \tag{5}$$

Generalizing for MIMO (multiple input, multiple output), the following lema yields.

Lemma 1: [13] Applying the state control

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}(i) = -\begin{bmatrix} \boldsymbol{k}_1^T \\ \vdots \\ \boldsymbol{k}_r^T \end{bmatrix} \boldsymbol{q}(i) \tag{6}$$

on the linear discrete-time strictly positive system (1), (2), while $\boldsymbol{K} \in \mathbb{R}_+^{r\times n}$ is selected to give the closed-loop system matrix

$$\boldsymbol{F}_c = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{K} \tag{7}$$

then $\boldsymbol{F}_c$ is strictly positive, if for given strictly positive matrix $\boldsymbol{F} \in \mathbb{R}_+^{n\times n}$ and non-negative matrix $\boldsymbol{G} \in \mathbb{R}_+^{n\times r}$ there exist positive definite diagonal matrices $\boldsymbol{P}, \boldsymbol{R}_k \in \mathbb{R}^{n\times n}$ such that for $h = 0, 1, 2, \ldots n-1$, $k = 1, 2, \ldots r$

$$\boldsymbol{P} = \boldsymbol{P}^T \succ 0 \tag{8}$$

$$\boldsymbol{T}^h \boldsymbol{F}(j, j+h)_{(1\leftrightarrow n)/n} \boldsymbol{T}^{hT} \boldsymbol{P} - \sum_{k=1}^{r} \boldsymbol{T}^h \boldsymbol{G}_{dk} \boldsymbol{T}^{hT} \boldsymbol{R}_k \succ 0 \tag{9}$$

where

$$\boldsymbol{T} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad \boldsymbol{T}^{-1} = \boldsymbol{T}^T \tag{10}$$

$$\begin{aligned} &\boldsymbol{F}(j, j+h)_{(1\leftrightarrow n)/n} = \\ &= \text{diag}\left[\, f_{1,1+h} \ \cdots \ f_{n-h,n} \ f_{n-h+1,1} \ \cdots \ f_{n,h} \,\right] \end{aligned} \tag{11}$$

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{g}_1 & \boldsymbol{g}_2 & \cdots & \boldsymbol{g}_r \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1r} \\ g_{21} & g_{22} & \cdots & g_{2r} \\ & & \vdots & \\ g_{n1} & g_{n2} & \cdots & g_{nr} \end{bmatrix} \tag{12}$$

$$\boldsymbol{G}_{dk} = \text{diag}\left[\ g_{1k} \quad g_{2k} \quad \cdots \quad g_{nk}\ \right] \tag{13}$$

$$\boldsymbol{K}_{dk} = \boldsymbol{R}_k \boldsymbol{P}^{-1}, \quad \boldsymbol{k}_k^T = \boldsymbol{l}^T \boldsymbol{K}_{dk} \tag{14}$$

$$\boldsymbol{l}^T = [1\ 1\ \cdots\ 1] \tag{15}$$

$\boldsymbol{R}_k$ is assumed to be a structured matrix variable of the following form

$$\boldsymbol{R}_k = \text{diag}\left[\ r_{k1} \quad r_{k2} \quad \cdots \quad r_{kn}\ \right] \succ 0 \tag{16}$$

$$\boldsymbol{r}_k^T = \left[\ r_{k1} \quad r_{k2} \quad \cdots \quad r_{kn}\ \right] = \boldsymbol{l}^T \boldsymbol{R}_k \tag{17}$$

and $\boldsymbol{F}(j, j+h)_{(1\leftrightarrow n)/n} \in \mathbb{R}_+^{n\times n}$.

As it is seen from the presented procedure, the resulting conditions prescribe the structure of $\boldsymbol{F}_c \in \mathbb{R}_+^{n\times n}$ but do not guarantee in general that $\boldsymbol{F}_c$ is Schur matrix, since no Lyapunov function is incorporated in the set of LMIs.

IV. Linear Quadratic Control

Next theorem solves the state-feedback control problem for the linear discrete-time positive system (1), (2), given by the controllable pair $(\boldsymbol{F}, \boldsymbol{G})$ and the quadratic weighting matrices pair $(\boldsymbol{Q}, \boldsymbol{U})$ of appropriate dimension, where $\boldsymbol{Q} \succ 0$, $\boldsymbol{R} \succ 0$, $\boldsymbol{F} \in I\!R_{+}^{n\times n}$ is a strictly positive matrix and $\boldsymbol{G} \in I\!R_{+}^{n\times r}$.

Theorem 1: The control (6) stabilizes the linear discrete-time positive system (1), (2) if for given positive definite diagonal matrices $\boldsymbol{Q} \in I\!R^{n\times n}$, $\boldsymbol{U} \in I\!R^{r\times r}$ and strictly positive matrix $\boldsymbol{F} \in I\!R_{+}^{n\times n}$ there exist positive definite diagonal matrices $\boldsymbol{P}, \boldsymbol{R}_k \in I\!R^{n\times n}$ such that for $h = 0, 1, 2, \dots n-1$, $k = 1, 2, \dots r$,

$$\boldsymbol{P} = \boldsymbol{P}^T \succ 0 \tag{18}$$

$$\begin{bmatrix} -\boldsymbol{P} & * & * & * \\ \boldsymbol{F}\boldsymbol{P} - \sum_{k=1}^{r} \boldsymbol{g}_k \boldsymbol{r}_k^T & -\boldsymbol{P} & * & * \\ \sum_{k=1}^{r} \boldsymbol{h}_k \boldsymbol{r}_k^T & \boldsymbol{0} & -\boldsymbol{U}^{-1} & * \\ \boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{Q}^{-1} \end{bmatrix} \prec 0 \tag{19}$$

$$\boldsymbol{T}^h \boldsymbol{F}(j, j+h)_{(1\leftrightarrow n)/n} \boldsymbol{T}^{hT} \boldsymbol{P} - \sum_{k=1}^{r} \boldsymbol{T}^h \boldsymbol{G}_{dk} \boldsymbol{T}^{hT} \boldsymbol{R}_k \succ 0 \tag{20}$$

where $\boldsymbol{T}$, $\boldsymbol{F}(j, j+h)_{(1\leftrightarrow n)/n}$, $\boldsymbol{G}_{dk}$, $\boldsymbol{g}_k$, $\boldsymbol{R}_k$, $\boldsymbol{r}_k^T$, $\boldsymbol{l}^T$ are introduced in (10)-(17) and

$$\boldsymbol{h}_k^T = \begin{bmatrix} 0 & \cdots & 0 & 1_k & 0 & \cdots & 0 \end{bmatrix} \tag{21}$$

When the above conditions hold, the control gain matrix $\boldsymbol{K} \in I\!R_{+}^{r\times n}$ is given as

$$\boldsymbol{K}_{dk} = \boldsymbol{R}_k \boldsymbol{P}^{-1}, \quad \boldsymbol{k}_k^T = \boldsymbol{l}^T \boldsymbol{K}_{dk}, \quad \boldsymbol{K} = \begin{bmatrix} \boldsymbol{k}_1^T \\ \vdots \\ \boldsymbol{k}_r^T \end{bmatrix} \tag{22}$$

Hereafter, $*$ labels the symmetric item in a symmetric matrix.

Proof: Since system (1), (2) is linear in $\boldsymbol{q}(i)$, the quadratic Lyapunov function can be chosen as

$$v(\boldsymbol{q}(i)) = \boldsymbol{q}^T(i) \boldsymbol{S} \boldsymbol{q}(i) \tag{23}$$

where $\boldsymbol{S}$ is defined in analogy with (8) in diagonal positive definite structure.

Considering positive definite diagonal matrices $\boldsymbol{Q} \in I\!R^{n\times n}$, $\boldsymbol{U} \in I\!R^{r\times r}$ and exploiting the Krasovskii theorem [10], the difference of the Lyapunov function can be prescribed as

$$\begin{aligned} \Delta v(\boldsymbol{q}(i)) = \boldsymbol{q}^T(i+1)\boldsymbol{S}\boldsymbol{q}(i+1) - \boldsymbol{q}^T(i)\boldsymbol{S}\boldsymbol{q}(i) \leq \\ \leq -(\boldsymbol{q}^T(i)\boldsymbol{Q}\boldsymbol{q}(i) + \boldsymbol{u}^T(i)\boldsymbol{U}\boldsymbol{u}(i)) < 0 \end{aligned} \tag{24}$$

Since with (6), (7) evidently (1) takes the form

$$\boldsymbol{q}(i+1) = \boldsymbol{F}_c \boldsymbol{q}(i) \tag{25}$$

then, substituting (25) and (6) into (24) gives

$$\boldsymbol{q}^T(i)(\boldsymbol{F}_c^T \boldsymbol{S} \boldsymbol{F}_c - \boldsymbol{S} + \boldsymbol{Q} + \boldsymbol{K}^T \boldsymbol{U} \boldsymbol{K})\boldsymbol{q}(i) < 0 \tag{26}$$

$$\boldsymbol{F}_c^T \boldsymbol{S} \boldsymbol{F}_c - \boldsymbol{S} + \boldsymbol{Q} + \boldsymbol{K}^T \boldsymbol{U} \boldsymbol{K} \prec 0 \tag{27}$$

respectively. Premultiplying the left side and postmultiplying the right side by the matrix $\boldsymbol{P} = \boldsymbol{S}^{-1}$ it can be observed immediately that the inequality (27) implies

$$\boldsymbol{P}\boldsymbol{F}_c^T \boldsymbol{P}^{-1} \boldsymbol{F}_c \boldsymbol{P} - \boldsymbol{P} + \boldsymbol{P}\boldsymbol{Q}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{K}^T \boldsymbol{U} \boldsymbol{K} \boldsymbol{P} \prec 0 \tag{28}$$

and applying the Schur complement property, the equivalent form of (28) is

$$\begin{bmatrix} -\boldsymbol{P} + \boldsymbol{P}\boldsymbol{Q}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{K}^T\boldsymbol{U}\boldsymbol{K}\boldsymbol{P} & \boldsymbol{P}\boldsymbol{F}_c^T \\ \boldsymbol{F}_c\boldsymbol{P} & -\boldsymbol{P} \end{bmatrix} \prec 0 \tag{29}$$

Rewriting (29) as follows

$$\begin{bmatrix} -\boldsymbol{P}+\boldsymbol{P}\boldsymbol{Q}\boldsymbol{P} & \boldsymbol{P}\boldsymbol{F}_c^T \\ \boldsymbol{F}_c\boldsymbol{P} & -\boldsymbol{P} \end{bmatrix} + \begin{bmatrix} \boldsymbol{P}\boldsymbol{K}^T \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{U} \begin{bmatrix} \boldsymbol{K}\boldsymbol{P} & \boldsymbol{0} \end{bmatrix} \prec 0 \tag{30}$$

it yields, evidently,

$$\begin{bmatrix} -\boldsymbol{P}+\boldsymbol{P}\boldsymbol{Q}\boldsymbol{P} & \boldsymbol{P}\boldsymbol{F}_c^T & \boldsymbol{P}\boldsymbol{K}^T \\ \boldsymbol{F}_c\boldsymbol{P} & -\boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{K}\boldsymbol{P} & \boldsymbol{0} & -\boldsymbol{U}^{-1} \end{bmatrix} \prec 0 \tag{31}$$

Thus, considering the notation (14), it can write using (7) that

$$\boldsymbol{F}_c\boldsymbol{P} = \boldsymbol{F}\boldsymbol{P} - \sum_{k=1}^{r} \boldsymbol{g}_k \boldsymbol{k}_k^T \boldsymbol{P} = \boldsymbol{F}\boldsymbol{P} - \sum_{k=1}^{r} \boldsymbol{g}_k \boldsymbol{r}_k^T \tag{32}$$

$$\boldsymbol{K}\boldsymbol{P} = \sum_{k=1}^{r} \boldsymbol{h}_k \boldsymbol{k}_k^T \boldsymbol{P} = \sum_{k=1}^{r} \boldsymbol{h}_k \boldsymbol{r}_k^T \tag{33}$$

where

$$\boldsymbol{r}_k^T = \boldsymbol{k}_k^T \boldsymbol{P} \tag{34}$$

and $\boldsymbol{h}_k$ is a column vector with zero elements except one on the k-th position as is defined in (21).

Inserting (32), (33) then (31) gives

$$\begin{bmatrix} -\boldsymbol{P}+\boldsymbol{P}\boldsymbol{Q}\boldsymbol{P} & * & * \\ \boldsymbol{F}\boldsymbol{P} - \sum_{k=1}^{r} \boldsymbol{g}_k \boldsymbol{r}_k^T & -\boldsymbol{P} & * \\ \sum_{k=1}^{r} \boldsymbol{h}_k \boldsymbol{r}_k^T & \boldsymbol{0} & -\boldsymbol{U}^{-1} \end{bmatrix} \prec 0 \tag{35}$$

Extracting as follows

$$\begin{bmatrix} \boldsymbol{P}\boldsymbol{Q}\boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{Q} \begin{bmatrix} \boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \tag{36}$$

then (35) obtains its equivalent form

$$\begin{bmatrix} -\boldsymbol{P} & * & * & * \\ \boldsymbol{F}\boldsymbol{P} - \sum_{k=1}^{r} \boldsymbol{g}_k \boldsymbol{r}_k^T & -\boldsymbol{P} & * & * \\ \sum_{k=1}^{r} \boldsymbol{h}_k \boldsymbol{r}_k^T & \boldsymbol{0} & -\boldsymbol{U}^{-1} & * \\ \boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{Q}^{-1} \end{bmatrix} \prec 0 \tag{37}$$

Therefore, combining (36) with (8)-(17) this concludes the proof. ■

Corollary 1: The structures of (30), (36) also imply

$$\begin{bmatrix} PK^TUKP & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} PK^TU \\ 0 \end{bmatrix} U^{-1} \begin{bmatrix} UKP & 0 \end{bmatrix} \quad (38)$$

$$\begin{bmatrix} PQP & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} PQ \\ 0 \\ 0 \end{bmatrix} Q^{-1} \begin{bmatrix} QP & 0 & 0 \end{bmatrix} \quad (39)$$

and, consequently, (19) can be reformulated as

$$\begin{bmatrix} -P & * & * & * \\ FP - \sum_{k=1}^{r} g_k r_k^T & -P & * & * \\ \sum_{k=1}^{r} h_k r_k^T U & 0 & -U & * \\ PQ & 0 & 0 & -Q \end{bmatrix} \prec 0 \quad (40)$$

which eliminates matrix inverse from LMI. Because the matrices Q and U are not LMI variables, the inequalities (19) and (40) are equivalent.

Since in [6] is suggested to choose diagonal weighting matrices Q, U, with diagonal elements equal to inverse of desired maximum squared values of $q(i)$, $u(i)$, respectively, the acceptance of the matrix inverses in (19) is quite natural. In this sense, the use of positively defined diagonal matrices Q and U is also justifiable because requiring a state or input variable of the process to fix permanently the zero value is unusual. Possibly indefinite LQ problem can be formulated using the results presented in [1].

Corollary 2: The linear quadratic (LQ) control on finite time horizon for a linear discrete-time system resolves the input $u(i)$ defined on a finite interval $\langle 0, s-1\rangle$, $s > 0$, such that $q(i)$ is driven to the equilibrium minimizing the performance index

$$J_s = q^T(s)\Xi_s q(s) + \sum_{i=0}^{s-1} r\big(q(i), u(i)\big) \quad (41)$$

$$r\big(q(i), u(i)\big) = q^T(i)Qq(i) + u^T(i)Uu(i) \quad (42)$$

where s is finite, $Q, \Xi \in \mathbb{R}^{n\times n}$ are positive semi-definite symmetric matrices and $U \in \mathbb{R}^{m\times m}$ is a positive definite symmetric matrix.

Choosing the quadratic Lyapunov function as follows

$$v(q(i)) = q^T(i)\Xi(i-1)q(i) \quad (43)$$

where $\{\Xi(i) : i \in \langle 0, s-1\rangle\}$, $\Xi(-1) = \Xi(0)$, is a sequence of symmetric positive definite matrices, then

$$\Delta v(q(i)) = v(q(i+1)) - v(q(i)) \quad (44)$$

Defining, at the time instant s, the functional V_s as

$$V_s = \sum_{i=0}^{s-1} \Delta v(q(i)) \quad (45)$$

which, in turn, is equivalent to

$$V_s = q^T(s)\Xi(s-1)q(s) - q^T(0)\Xi(0)q(0) \quad (46)$$

then, adding (46) to (41) as well as subtracting (46) from (41), and setting $\Xi(s-1) = \Xi_s$, this choice ensures that the performance index takes the form

$$J_N = q^T(0)\Xi(0)q(0) + \sum_{i=0}^{s-1} p(q(i), u(i)) \quad (47)$$

$$p(q(i), u(i)) = r(q(i), u(i)) + \Delta v(q(i)) \quad (48)$$

If a negative $\Delta v(q(i))$ is prescribed, the optimal LQ control on the finite interval can ensure that $p(q(i), u(i))$ gets zero at each time instant $i \in \langle 0, s–1\rangle$. Details of the algorithm to achieve these results can be find, e.g., in [12], [17]. For application to an infinite time horizon, (48) so justifies the use of the inequality (24).

Corollary 3: Taking into account the required positive definite diagonal matrix structures of $P, R_k \in \mathbb{R}^{n\times n}$, $k = 1, 2, \dots r$, as are prescribed in Theorem 1, then, instead of the inequality (19), a different matrix inequality, or a set of matrix inequalities, can be used to ensure the stability of the closed-loop. In [13] this role plays the matrix inequalities resulting from H_2, H_∞ and H_2/H_∞ formulation. Evidently, the simplest one reflecting the Lyapunov inequality, takes the following form

$$\begin{bmatrix} -P & * \\ FP - \sum_{k=1}^{r} g_k r_k^T & -P \end{bmatrix} \prec 0 \quad (49)$$

Since (49) can be simply derived from (19), the proof is omitted.

Remark 1: Considering the square strictly positive system (1), (2) and the forced mode control policy

$$u(i) = -Kq(i) + Ww(i) \quad (50)$$

where $W \in \mathbb{R}^{m\times m}$ is the signal gain matrix, $w(i) \in \mathbb{R}^m$ is a desired output signal vector and $r = m$, then (1), (2), (7), (50) imply at a system steady-state

$$q_o = F_c q_o + GWw_o \quad (51)$$

$$y_o = Cq_o \quad (52)$$

while q_o, y_o, w_o are steady-state values vectors of $q(i)$, $y(i)$, $w(i)$, respectively. Therefore,

$$q_o = (I_n - F_c)^{-1}GWw_o \quad (53)$$

$$y_o = C(I_n - F_c)^{-1}GWw_o \quad (54)$$

and, if [22]

$$rank \begin{bmatrix} F & G \\ C & 0 \end{bmatrix} = n + m \quad (55)$$

then, prescribing $y_o = w_o$, it can obtain

$$W = (C(I_n - F_c)^{-1}G)^{-1} \quad (56)$$

Since I_n is non-negative matrix and with (22) the matrix F_c is a strictly positive Schur matrix, the matrix $I_n - F_c$ is, in general, neither a non-negative nor a non-positive matrix [15]. This implies that W is not a non-negative matrix and so closed-loop system with the input $\boldsymbol{w}(i)$ and the output $\boldsymbol{y}(i)$, described as

$$\boldsymbol{q}(i+1) = \boldsymbol{F}_c\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{W}\boldsymbol{w}(i) \tag{57}$$

$$\boldsymbol{y}(i) = \boldsymbol{C}\boldsymbol{q}(i) \tag{58}$$

is not internally positive.

Thus, the state and output variables in the forced mode will only be positive for a suitably chosen initial vector $\boldsymbol{q}(0)$, whose some (all) values are positive and the rest values are equal to zero.

V. Illustrative Example

The system representation (1), (2) is supported by the sampling period $t_s = 0.02s$ and the following matrix parameters

$$\boldsymbol{F} = \begin{bmatrix} 0.9361 & 0.0116 & 0.1219 & 0.1149 \\ 0.0112 & 0.9197 & 0.0375 & 0.0156 \\ 0.0198 & 0.0792 & 0.8784 & 0.1098 \\ 0.0012 & 0.0428 & 0.0035 & 0.9593 \end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix} 0.0081 & 0.0043 \\ 0.0110 & 0.0041 \\ 0.0028 & 0.0063 \\ 0.0025 & 0.0034 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Is can be verified that the matrix $\boldsymbol{F}$ is strictly positive but not Schur.

To solve the design task, the auxiliary parameters are constructed as

$$\boldsymbol{T} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\boldsymbol{F}(i,i)_{(1\leftrightarrow 4)} = \text{diag}\begin{bmatrix} 0.9361 & 0.9197 & 0.8784 & 0.9593 \end{bmatrix}$$

$$\boldsymbol{F}(i,i+1)_{(1\leftrightarrow 4)/4} = \text{diag}\begin{bmatrix} 0.0116 & 0.0375 & 0.1098 & 0.0012 \end{bmatrix}$$

$$\boldsymbol{F}(i,i+2)_{(1\leftrightarrow 4)/4} = \text{diag}\begin{bmatrix} 0.1219 & 0.0156 & 0.0198 & 0.0428 \end{bmatrix}$$

$$\boldsymbol{F}(i,i+3)_{(1\leftrightarrow 4)/4} = \text{diag}\begin{bmatrix} 0.1149 & 0.0112 & 0.0792 & 0.0035 \end{bmatrix}$$

Within these parameters, prescribing $\boldsymbol{Q} = \boldsymbol{I}_4$, $\boldsymbol{U} = \boldsymbol{I}_2$, then solving (18)-(20) with the SeDuMi packet, the stability conditions imply the values of the LMI variables

$$\boldsymbol{P} = \text{diag}\begin{bmatrix} 0.0179 & 0.0011 & 0.0045 & 0.0006 \end{bmatrix}$$

$$\boldsymbol{R}_1 = \text{diag}\begin{bmatrix} 0.0079 & 0.0003 & 0.0060 & 0.0001 \end{bmatrix}$$

$$\boldsymbol{R}_2 = \text{diag}\begin{bmatrix} 0.0003 & 0.0024 & 0.0002 & 0.0021 \end{bmatrix}$$

which give, prescribed by (22), (56), the control law parameters for (50)

$$\boldsymbol{K} = \begin{bmatrix} 0.4421 & 0.2493 & 1.3333 & 0.0912 \\ 0.0166 & 2.1357 & 0.0414 & 3.4377 \end{bmatrix}$$

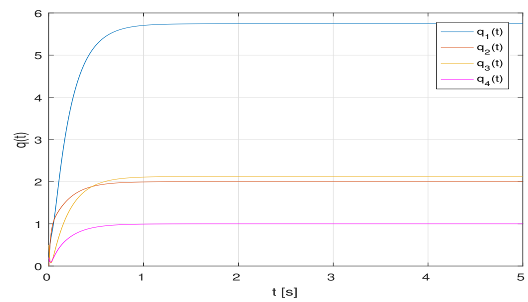

Fig. 1: Closed-loop system state response

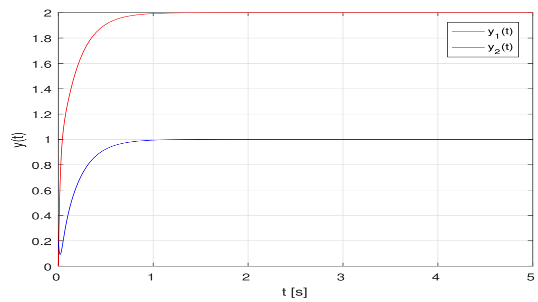

Fig. 2: Closed-loop system output response

$$\boldsymbol{W} = \begin{bmatrix} 16.7431 & -18.6882 \\ -22.2493 & 28.7268 \end{bmatrix}$$

Evidently, the gain matrix $\boldsymbol{K}$ is positive matrix and $\boldsymbol{W}$ is not non-negative.

Interesting can be the matrix $\boldsymbol{F}_c$, defined by the following matrix elements

$$\boldsymbol{F}_c = \begin{bmatrix} 0.9324 & 0.0005 & 0.1110 & 0.0995 \\ 0.0062 & 0.9083 & 0.0227 & 0.0006 \\ 0.0185 & 0.0651 & 0.8745 & 0.0879 \\ 0.0000 & 0.0348 & 0.0001 & 0.9473 \end{bmatrix}$$

and the eigenvalue spectrum

$$\rho(\boldsymbol{F}_c) = \{\ 0.8423 \quad 0.9898 \quad 0.9152 \pm 0.0190\,\mathrm{i}\ \}$$

It is evident that $\boldsymbol{F}_c$ is strictly positive Schur matrix, with dominant diagonal matrix elements. Note, $\boldsymbol{F}_c(4,1)$ is positive real element closest to zero.

The simulation results for control of the system are shown in Fig. 1 and 2. Fig. 1 shows the closed-loop system state response in the forced control mode, in Fig. 2 is shown the output evolution for the same regime. The vector of the desired output values is

$$\boldsymbol{w}(i) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{for all } i$$

To obtain the responses with positive system state and output variables, the system initial state is set as

$$\boldsymbol{q}^T(0) = \begin{bmatrix} 0 & 0 & 0.5 & 0.2 \end{bmatrix}$$

Evidently, the state and output responses of the closed-loop scheme are positive and asymptotically stable.

VI. Concluding Remarks

The paper is concentrated specifically on effectively computation the full state feedback control gain, to accomplish the closed-loop system matrix be Schur and strictly positive. The primary aims are the algebraic constraints, defined as a set of LMIs, implying from the predefined closed-loop system matrix structure and replenished by the Lyapunov matrix inequality containing prescribed quadratic constrains to guarantee closed-loop system asymptotic stability and dynamics. Whilst the conclusions described here can be obtained only via positive definite and diagonal matrix variables associated with given set of LMIs, progress is made in incorporating such LMI structure in diagonal stabilizability of discrete-time positive systems.

As it is illustrated by given numerical example, the proposed design fulfilment provides numerically effective computational frameworks.

References

[1] M. Ait Rami and J. Moore, "Partial stabilizability and hidden convexity of LQ problem," *Numerical Algebra, Control and Optimization*, vol. 6, no. 3, pp. 221-239, 2016.

[2] M. Ait Rami and F. Tadeo, "Controller synthesis for positive linear systems with bounded controls," *IEEE Trans. Circuits and Systems*, vol. 54, no. 2, pp. 151-155, 2007.

[3] M. Ait Rami, F. Tadeo, and A. Benzaouia, "Control of constrained positive discrete systems, *Proc. 2007 American Control Conference ACC'07*, New York, USA, pp. 5851-5856, 2007.

[4] J. Back and A. Astolfi, "Design of positive linear observers for positive linear systems via coordinate transformations and positive realizations," *SIAM J. Control and Optimization*, vol. 47, no. 1, pp. 345-373, 2008.

[5] G. Birkhoff and S. Mac Lane, *A Survey of Modern Algebra*, Macmillan Publishing, New York, 1977.

[6] A.E. Bryson, *Applied Linear Optimal Control. Examles and Algorithms*, Cambridge University Press, Cambridge, 2002.

[7] J.M. Carnicer, J.M. Pena, and R.A. Zalik, "Strictly totally positive systems," *J. Approximation Theory*, vol. 92, pp. 411-441, 1998.

[8] L. Farina and S. Rinaldi, *Positive Linear Systems. Theory and Applications*, John Wiley & Sons, New York, 2000.

[9] H. Gao, J. Lam, C. Wang, and S. Xu, "Control for stability and positivity. Equivalent conditions and computation," *IEEE Trans. Circuits and Systems*, vol. 52, no. 9, pp. 540-544, 2005.

[10] W.M. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and Control. A Lyapunov-Based Approach*, Princeton University Press, Princeton, 2008.

[11] T. Kaczorek and L. Sajewski, *The Realization Problem for Positive and Fractional Systems*, Springer, Cham, 2014.

[12] D. Krokavec and A. Filasová, "Novel fault detection criteria based on linear quadratic control performances," *Int. J. Applied Mathematics and Computer Science*, vol. 22, no. 4, pp. 929-938, 2012.

[13] D. Krokavec and A. Filasová, "On control of discrete-time LTI positive systems," *Applied Mathematical Sciences*, vol. 11, no. 50, pp. 2459-2476, 2017.

[14] D. Krokavec and A. Filasová, "Stabilization of discrete-time LTI positive systems," *Archives of Control Sciences*, vol. 27, no. 4, pp. 575-594, 2017.

[15] P. De Leenheer and D. Aeyels, "Stabilization of positive linear systems," *Systems & Control Letters*, vol. 44, no. 4. pp. 259-271, 2001.

[16] O. Pastravanu and M.H. Matcovschi, "(M, β)-stability of positive linear systems," *Mathematical Problems in Engineering*, vol. 2016, 11p, 2016.

[17] C.L. Phillips and H.T. Nagle, *Digital Control System Analysis and Design*, Prentice Hall, Englewood Clifs, 1984.

[18] *Positive Systems*, F. Cacace, L. Farina, R. Setola and A. Germani (Eds.), Lecture Notes in Control and Information Sciences, vol. 471, Springer, Cham, 2016.

[19] D.J.S. Robinson, *An Introduction to Abstract Algebra*, Walter de Gruyter, Berlin, 2003.

[20] B. Shafai and A. Oghbaee, "Positive quadratic stabilization of uncertain linear system," in *Proc. 2014 IEEE Multi-Conference on Systems and Control*, Antibes, France, pp. 1412-1417, 2014.

[21] X. Xue and L. Guo, "A kind of nonnegative matrices and its application on the stability of discrete dynamical systems," *J. Mathematical Analysis and Application*, vol. 331, no. 2, pp. 1113-1121, 2007.

[22] Q.G. Wang, *Decoupling Control*, Springer-Verlag, Berlin, 2003.