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Regional stability analysis of nonlinear sampled-data control systems: a quasi-LPV approach

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***Abstract*— This paper addresses the stability analysis of sampled-data control for a class of continuous-time nonlinear systems. The proposed approach is based on a local quasi-LPV model for the nonlinear system and the use of a parameter dependent looped-functional to deal with the aperiodic sampling effects. From these ingredients, LMI conditions are proposed to assess local stability. These conditions are then incorporated in convex optimization problems aiming at obtaining maximized estimates of the region of attraction of the origin or maximizing the inter-sampling time for which the stability is ensured.**
Keywords: Sampled-data control, quasi-LPV control, nonlinear systems, Linear Matrix Inequality (LMI).

I. Introduction

Motivated by the development of networked and embedded control, the interest in sampled-data systems has increased in the last decade [1]. The problem is particularly relevant in the context of nonlinear systems. Differently from the linear ones, exact discretization in general cannot be achieved and discretized models obtained by numerical approximations can lead to unexpected behaviors (such as instability) when used to design sampled-data control laws [2]. Hence, it is of paramount importance to take into account the continuous-time behavior of the plant and the discrete-time update of the control signal. To tackle this problem, many approaches can be found in the literature. We can cite for instance the ones based on a hybrid system framework [3], where the sampling phenomena is modeled through a jump system, and the approaches based on a time-delay system framework, where Lyapunov-Krasovskii [4], [5] and looped-functionals [6], [7] are applied for stability assessment.

In particular, some nonlinear systems can be written in the so-called quasi-LPV (linear parameter varying) form. The quasi-LPV approach is based on the possibility of rewriting the plant in a form where nonlinear terms can be replaced by time-varying parameters that depend on the system state. In this case, techniques of LPV control (see for instance [8], [9] and references therein) can be applied to analysis and synthesis of the closed-loop systems. In particular, the controller can be designed as an LPV one, i.e. a time-varying linear control law scheduled by the state dependent parameter. One advantage in this case is that the controller synthesis and the analysis (stability, performance, robustness, etc) of the closed-loop system can be performed in a framework based on LMI constraints and convex optimization.

Many works in the literature consider a continuous-time implementation of LPV control laws, which presuppose that the controller (state feedback or output feedback) is continuously updated to cope with the parameter variation. Nevertheless, in practical digital implementation, the hypothesis of continuous measurement is not fulfilled, since measurements occur only at discrete instants of time. Then, for the discrete implementation of the controller, a discretization based on numerical approximations (e.g. Euler or Tustin methods) is applied. In general, if the sampling interval and/or the parameter variation along the sampling interval are sufficiently small, the behavior of the closed-loop system will be very similar to the one predicted by a true continuous-time implementation. However, no guarantee of stability is provided in this case and unexpected behaviors can occur if sampling period is not conveniently chosen or if the sampling is not periodic. Other works consider a discrete-time LPV model and a discrete-time framework for analysis and design. In this case, two major issues arise: how to obtain a discrete-time model from the continuous one and the fact that the parameter is kept constant between two sampling instants, which may be non realistic for quasi-LPV systems.

Considering the LPV framework, few papers formally address the sampled-data control problem. In [10], under the conservative assumption that the scheduling parameters do not change in the interval between two samples, a lifting technique is applied to derive stability conditions. A time-delay approach is considered in [11] and the proposed stability conditions are in the form of infinite dimensional LMIs. In [12] a model predictive control scheme is considered. In this case, a set of LMIs must be solved at each sampling instant to determine the control law to be applied. On the other hand, there is a lack of papers in the literature considering the sampled-data control for quasi-LPV systems. In this context, we can mainly find works dealing with fuzzy Takagi-Sugeno (T-S) models, which can be seen as a particular form of quasi-LPV modeling. For instance, using a Lyapunov-Krasovskii functional approach, LMI conditions to compute a sampled-data T-S fuzzy controller to guarantee exponential stabilization of the closed-loop system are proposed in [13]. In [14], LMI conditions are derived to compute a T-S state feedback or output feedback to

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asymptotic stabilization and $\mathscr{H}_\infty$ disturbance attenuation for vehicle suspension systems. However, in most cases the quasi-LPV or T-S fuzzy models represent the nonlinear system only regionally [15]. Unfortunately, this fact is ignored in the majority of the papers dealing with this kind of modeling. If a region of validity of the model is not considered, the stability guarantee obtained with the LPV system in a global context will fail for the nonlinear closed-loop system, which may be only locally stable.

In this paper we propose conditions to assess the regional stability of nonlinear systems represented locally by a quasi-LPV system with sampled-data state-feedback quasi-LPV control laws. It is explicitly assumed that the quasi-LPV controller is updated only at the sampling instants and that the control signal is kept constant between two consecutive samples, while the plant evolves continuously. The stability analysis is based on a local polytopic modeling of the quasi-LPV system and the use of a parameter dependent looped-functional to take into account the sampling effects. Both periodic and aperiodic sampling cases can be handled by the proposed approach. Based on the developed conditions, LMI-based optimization problems can be formulated to obtain maximized estimates of the region of attraction of the origin. Moreover, for a given set of admissible initial conditions, an optimization problem aiming at maximizing the upper bound on the intersampling interval is also proposed.

Notation. $\mathbb{S}^n$ denotes the set of symmetric matrices of $\mathbb{R}^{n\times n}$. For a given positive scalar T, define $\mathscr{C}^n_{[0,T]}$ as the set of continuous functions from an interval $[0,\ T]$ to $\mathbb{R}^n$ and the union set of continuous functions with support in a certain range defined as $\mathbb{K}^n_{[\underline{T},\overline{T}]} = \cup_{T\in[\underline{T},\overline{T}]}\{\mathscr{C}^n_{[0,T]}\}$. $\|\cdot\|$ stands for the Euclidean norm of a vector. The notation $P>0$ means that P is positive definite. $\mathrm{He}\{A\}>0$ refer to $A+A'>0$, symbols I and 0 represent the identity and the zero matrices of appropriate dimension. $A_{(i)}$ or $x_{(i)}$ represent the i-th line of the matrix A or i-th element of the vector x. $Co\{\cdot\}$ denotes a convex hull and $Ver(\mathscr{B}_\sigma)$ represents the vertices of polytope $\mathscr{B}_\sigma$. $\mathscr{E}(P,c)$ denotes the set $\mathscr{E}(P,c)=\{x\in\mathbb{R}^n; x'Px\le c\}$, with $P=P'>0$, $c>0$. $\partial\mathscr{S}$ defines the boundary of the set $\mathscr{S}$. For a vector $v\in\mathbb{R}^N$, $\Lambda(v)$ is a shortcut for the kronecker product $v\otimes I$.

II. Problem Formulation

Consider a class of continuous-time nonlinear systems with sampled-data control law given by

$$\dot{x}(t) = f(x(t))x(t)+g(x(t))u(t), \tag{1}$$

$$u(t) = l(x(t_k)),\ \forall t\in[t_k,\ t_{k+1}), \tag{2}$$

where $f(\cdot):\mathbb{R}^n\to\mathbb{R}^{n\times n}$, $g(\cdot):\mathbb{R}^n\to\mathbb{R}^{n\times m}$ and $l(\cdot):\mathbb{R}^n\to\mathbb{R}^{n\times m}$ are continuous and (locally) Lipschitz nonlinear functions, with $f(0)=g(0)=l(0)=0$. $l(\cdot):\mathbb{R}^n\to\mathbb{R}^{n\times m}$ represents a nonlinear static state feedback. The state and the control input vector are represented by $x\in\mathbb{R}^n$ and $u\in\mathbb{R}^m$, respectively.

Assume that the nonlinear system (1) can be represented by the following quasi-LPV system

$$\dot{x}(t) = A(\sigma(x(t)))x(t)+B(\sigma(x(t)))u(t), \tag{3}$$

where $\sigma(x(t))$ is a vector in $\mathbb{R}^N$, $\sigma_{(i)}(x(t))$ are functions of the states, for $i=1,\dots,N$. Moreover, we consider that the control law (2) can also be written in a quasi-LPV form, i.e.:

$$u(t) = K(\sigma(x(t_k)))x(t_k),\ \forall t\in[t_k,t_{k+1}),\ \forall k\in\mathbb{N}. \tag{4}$$

The values of $x(t)$ are assumed to be measured and therefore available for feedback only at the sampling instant t_k, with $k\ge 0$ and t_k being an increasing sequence of positive scalars such that $\bigcup[t_k,t_{k+1})=[0,+\infty)$. In the sampling interval $[t_k,t_{k+1})$, the values of $x(t)$ and, consequently, of $\sigma(x(t))$ are kept constant in (4) by means of a zero order holder (ZOH). The difference between two successive sampling instants is denoted by $T_k=t_{k+1}-t_k$ and it satisfies $0\le\underline{T}\le T_k\le\overline{T}$ and $T_k\neq 0$. Note that the particular case where $T_k=\underline{T}=\overline{T}>0$ $\forall k\ge 0$ corresponds to a periodic sampling strategy.

In this paper, we consider the following assumption.

Assumption 1: $A(\sigma)$, $B(\sigma)$ and $K(\sigma)$ depend affinely on σ.

From Assumption 1 and denoting $\sigma(x(t))$ by $\sigma(t)$ for simplicity, $A(\sigma(t))$, $B(\sigma(t))$ and $K(\sigma(t_k))$ can be generically represented as follows:

$$A(\sigma(t)) = A_0+\sum_{i=1}^{N}\sigma_i(t)A_i,\ B(\sigma(t)) = B_0+\sum_{i=1}^{N}\sigma_i(t)B_i,$$
$$K(\sigma(t_k)) = K_0+\sum_{i=1}^{N}\sigma_i(t_k)K_i.$$

Assuming that (2) is a stabilizing control law, the set of all initial conditions ($x(0)\in\mathbb{R}^n$) such that the corresponding trajectories of the closed-loop system (1)-(2) converge asymptotically to the origin corresponds to the so-called region of attraction of the origin ($\mathscr{R}_a$) [16]. Due to the difficulty to analytically determine $\mathscr{R}_a$, a problem of interest is to compute an estimate of the $\mathscr{R}_a$ denoted by $\mathscr{X}_0$ such that $\mathscr{X}_0\subset\mathscr{R}_a\subseteq\mathbb{R}^n$. Then, we can state the following problems:

Problem 1: Considering that $T_k\in[\underline{T},\overline{T}]$ find an estimate $\mathscr{X}_0$ of the region $\mathscr{R}_a$.

Problem 2: Given $\underline{T}$ and a set of admissible initial conditions $\mathscr{X}_0$, compute the maximum allowable sampling interval $\overline{T}$ such that $\forall x(0)\in\mathscr{X}_0$ the trajectories of system (1) under the sampled-data control law (2) converge asymptotically to the origin with $T_k\in[\underline{T},\overline{T}]$.

To address Problems 1 and 2, we will use the LPV model (3)-(4), considering that Assumption 1 is satisfied.

III. Preliminaries

A. Local Polytopic Modeling

For a set $\mathscr{R}_H\in\mathbb{R}^n$, we consider the following assumption regarding the parameter $\sigma(x(t))$ in system (3)-(4).

Assumption 2: For $x(t)\in\mathscr{R}_H$, it follows that $\sigma(x(t))\in\mathscr{B}_\sigma$, with

$$\mathscr{B}_\sigma = \{\sigma\in\mathbb{R}^N;\ \underline{\sigma}_{(i)}\le\sigma_{(i)}\le\overline{\sigma}_{(i)},\ i=1,\cdots,N\}. \tag{5}$$

In particular, we consider that $\mathscr{R}_H$ is a polyhedral set described as follows:

$$\mathscr{R}_H = \{x \in \mathbb{R}^n; \left|H_{(r)}x\right| \leq \eta_{(r)},\ r = 1,\ldots,r_c\}, \tag{6}$$

with $\eta_{(r)} > 0$, $H_{(r)} \in \mathbb{R}^{1\times n}$.

From the bounds on $\sigma(t)$ defined in (5), it follows that $\sigma(t)$ and $\sigma(t_k)$ belong to convex polytopes in $\mathbb{R}^N$ with 2^N vertices, i.e. $\sigma \in \mathscr{B}_\sigma = Co\{v_1, v_2, \ldots, v_{2^N}\}$. In other words, provided that $x(t) \in \mathscr{R}_H$, $\sigma(t)$ can be obtained as a convex combination of the vertices of $\mathscr{B}_\sigma$, i.e.

$$\sigma(t) = \sum_{g=1}^{2^N} \gamma_g(t) v_g, \text{ with } \gamma_g(t) \geq 0 \text{ and } \sum_{g=1}^{2^N} \gamma_g(t) = 1. \tag{7}$$

Considering now that $\Lambda(v)$ denotes the Kronecker product between a vector $v \in \mathbb{R}^N$ and the identity matrix I, i.e. $\Lambda(v) = v \otimes I = \begin{bmatrix} v_{(1)}I & v_{(2)}I & \ldots & v_{(N)}I \end{bmatrix}'$, and defining $\mathscr{A} = [A_1\ A_2\ \cdots\ A_N]$, $\mathscr{B} = [B_1\ B_2\ \cdots\ B_N]$ and $\mathscr{K} = [K_1\ K_2\ \cdots\ K_N]$, under Assumptions 1 and 2 it is therefore possible to represent $A(\sigma(t))$, $B(\sigma(t))$ and $K(\sigma(t_k))$ as follows:

$$\begin{aligned} A(\sigma(t)) &= A_0 + \mathscr{A}\Lambda(\sigma(t)) = \sum_{g=1}^{2^N} \gamma_g(t) A(v_g), \\ B(\sigma(t)) &= B_0 + \mathscr{B}\Lambda(\sigma(t)) = \sum_{g=1}^{2^N} \gamma_g(t) B(v_g), \\ K(\sigma(t_k)) &= K_0 + \mathscr{K}\Lambda(\sigma(t_k)) = \sum_{g=1}^{2^N} \gamma_g(t_k) K(v_g), \end{aligned} \tag{8}$$

with

$$\begin{aligned} A(v_g) = A_0 + \mathscr{A}\Lambda(v_g),\quad B(v_g) = B_0 + \mathscr{B}\Lambda(v_g), \\ K(v_g) = K_0 + \mathscr{K}\Lambda(v_g). \end{aligned} \tag{9}$$

Hence, provided that $x(t) \in \mathscr{R}_H$, $\forall t \geq 0$, the behavior of closed-loop nonlinear system (1)-(2) can be represented by the following polytopic differential inclusion (PDI):

$$\begin{aligned} \dot{x}(t) &= \left(\sum_{g=1}^{2^N} \gamma_g(t) A(v_g)\right) x(t) + \left(\sum_{g=1}^{2^N} \gamma_g(t) B(v_g)\right) u(t) \\ u(t) &= \left(\sum_{g=1}^{2^N} \gamma_g(t_k) K(v_g)\right) x(t_k),\ t \in [t_k,\ t_{k+1}), \end{aligned} \tag{10}$$

Actually the PDI (10) can represent the dynamics of the nonlinear system (1)-(2) only inside the domain $\mathscr{R}_H$, i.e. $\mathscr{R}_H$ is a region of validity of the model [15].

Since in the sequel analyzed based on the quasi-LPV representation of system (1)-(2) and its polytopic representation (10), which is valid in the set $\mathscr{R}_H$, then, for the estimate $\mathscr{X}_0$ of $\mathscr{R}_a$ to be valid, we must consider that $\mathscr{X}_0 \subseteq \mathscr{R}_H$. In addition, $\mathscr{X}_0$ must be an invariant set.

B. Parameter Dependent Quadratic Function

Consider a parameter dependent quadratic function (PDQF) $V : \mathbb{R}^n \times \mathscr{B}_\sigma \to \mathbb{R}^+$, described as

$$V(x,\sigma) = x'P(\sigma)x, \tag{11}$$

with $0 < P(\sigma) \in \mathbb{S}^n$, $\forall \sigma \in \mathscr{B}_\sigma$. In particular, we suppose that $P(\sigma)$ depends affinely on σ, that is $P(\sigma) = P_0 + \mathscr{P}\Lambda(\sigma)$, with $\mathscr{P} = [P_1\ P_2\ \cdots\ P_N]$. Thus, defining for each $v_g \in Ver(\mathscr{B}_\sigma)$

$$P(v_g) = P_0 + \mathscr{P}\Lambda(v_g), \tag{12}$$

it follows that $P(\sigma)$ can be written as a convex combination of matrices $P(v_g)$, i.e.

$$P(\sigma) = \sum_{g=1}^{2^N} \gamma_g P(v_g), \text{ with } \gamma_g \geq 0 \text{ and } \sum_{g=1}^{2^N} \gamma_g = 1. \tag{13}$$

For a given $c > 0$, the level set associated to the PDQF (11) is therefore given by

$$\mathscr{L}_V(c) = \{x \in \mathbb{R}^n; V(x,\sigma) \leq c, \forall \sigma \in \mathscr{B}_\sigma\} = \bigcap_{\sigma \in \mathscr{B}_\sigma} \mathscr{E}(P(\sigma), c).$$

Inspired by the results of [17], the following lemma can be stated.

Lemma 1: $x \in \mathscr{L}_V(c)$ if and only if $x \in \bigcap_{g \in \{1,\ldots,2^N\}} \mathscr{E}(P(v_g), c)$.

Lemma 2: Define $\Delta V(k) = x'(t_{k+1})P(\sigma(t_{k+1}))x(t_{k+1}) - x'(t_k)P(\sigma(t_k))x(t_k)$ and a domain $\mathscr{D} \subset \mathbb{R}^n$. If

$$\Delta V(k) < -\mu ||x(t_k)||^2,\ \forall x(t_k) \in \mathscr{D} - \{0\},$$

along the trajectories of system (3)-(4), where μ is a positive scalar, then, for any initial condition $x(t_0) = x(0) \in \mathscr{L}_V(c) \subset \mathscr{D}$, it follows that

(i) $\mathscr{L}_V(c)$ is an invariant and contractive set with respect to the discrete-time trajectories (i.e. at sampling instants) of system (3)-(4).

(ii) $x(t_k) \to 0$, for $k \to \infty$.

C. Looped-functional Approach

In order to derive stability conditions for system (3) under the control law (4), we will focus on the behavior of the system in the inter-sampling interval $[t_k, t_{k+1})$. As in [6] and [7], define $x_k(\tau) = x(t_k + \tau)$ and $\sigma_k(\tau) = \sigma(x(t_k + \tau))$, with $\tau \in [0, T_k]$. Hence, for $\tau \in [0, T_k]$, the closed-loop system (3) and (4) can be represented by the following quasi-LPV model:

$$\dot{x}_k(\tau) = A(\sigma_k(\tau))x_k(\tau) + B(\sigma_k(\tau))K(\sigma_k(0))x_k(0). \tag{14}$$

Inspired by the results in [6] and [7], the following theorem provides the basis of the looped-functional approach to address Problems 1 and 2. It can also be seen as a parameter dependent version of the one proposed in [18].

Theorem 1: Consider a PDQF V defined as in (11) and a parameter dependent looped-functional (PDLF) $\mathscr{V}_0 : [0, \overline{T}] \times \mathbb{K}^n_{[\underline{T},\overline{T}]} \times \mathbb{R}^N \times [\underline{T}, \overline{T}] \to \mathbb{R}$, such that

$$\begin{aligned} \mathscr{V}_0(T_k, x_k, \sigma_k, T_k) = \mathscr{V}_0(0, x_k, \sigma_k, T_k) = 0, \\ \mathscr{V}_0(\tau, x_k, \sigma_k, T_k) > 0,\ \forall \tau \in (0, T_k),\ \forall T_k \in [\underline{T}, \overline{T}]. \end{aligned} \tag{15}$$

Define

$$\mathscr{W}(\tau, x_k, \sigma_k, T_k) = V(x_k(\tau), \sigma_k(0)) + \mathscr{V}_0(\tau, x_k, \sigma_k, T_k) \tag{16}$$

and let $\dot{\mathscr{W}}(\tau,x_k,\sigma_k,T_k)$ be the time-derivative of $\mathscr{W}(\tau,x_k,\sigma_k,T_k)$ with respect to τ. If the inequality

$$\dot{\mathscr{W}}(\tau,x_k,\sigma_k,T_k) \leq -\beta \|x_k(0)\|^2 \tag{17}$$

is satisfied along the trajectories of (14), for a positive scalar β and $\forall x_k \in \mathbb{K}^n_{[\underline{T},\overline{T}]}$, such that $x_k(\tau) \in \mathscr{D} \subset \mathbb{R}^n$, $\forall \tau \in [0,T_k]$, $T_k \in [\underline{T},\overline{T}]$, $\forall k \geq 0$, then for any initial condition $x(0) = x_0(0)$ in the set $\mathscr{L}_V(c) \subset \mathscr{D}$, it follows that:

(i) $\Delta V(k) = V(x_{k+1}(0),\sigma_{k+1}(0)) - V(x_k(0),\sigma_k(0)) \leq -\beta\underline{T}\|x_k(0)\|^2$, $\forall k \geq 0$;

(ii) the corresponding trajectories of the closed-loop system (3)-(4), with sampling intervals $T_k \in [\underline{T},\overline{T}]$, never leave $\mathscr{L}_V(c)$ and converge asymptotically to the origin.

Proof: Suppose that $x(0) = x_0(0) \in \partial\mathscr{L}_V(c)$ and assume that (17) is satisfied. Then it follows that

$$\mathscr{W}(\rho,x_0,\sigma_0,T_0) < \mathscr{W}(0,x_0,\sigma_0,T_0),\ \forall \rho \in (0,T_0], \tag{18}$$

Since by definition $\mathscr{V}_0(\rho,x_0,\sigma_0,T_0) > 0$ and $\mathscr{V}_0(0,x_0,\sigma_0,T_0) = 0$, from (16) and (18), it follows that $V(x_0(\rho),\sigma_0(0)) < \mathscr{W}(\rho,x_0,\sigma_0,T_0) < \mathscr{W}(0,x_0,\sigma_0,T_0) = V(x_0(0),\sigma_0(0))$, i.e. $x_0(\rho) \in \mathscr{L}_V(c) \subset \mathscr{D}$, $\forall \rho \in (0,T_0]$. Hence integrating (17) over the interval $[0,T_0]$, and taking into account (15), one gets $\Delta V(0) < -\beta T_0\|x_0(0)\|^2 \leq -\beta\underline{T}\|x_0(0)\|^2$, which implies that $x_1(0) \in \mathscr{L}_V(c_1) \subset \mathscr{L}_V(c)$. Repeating now the reasoning for $k = 1,\ldots,\infty$, we conclude that $\Delta V(k) < -\beta T_k\|x_k(0)\|^2 \leq -\beta\underline{T}\|x_k(0)\|^2$, $\forall k \geq 0$, which ensures, from Lemma 2, that $\lim\limits_{k\to\infty} x_k(0) = \lim\limits_{k\to\infty} x(t_k) = 0$. Moreover, it follows that $\mathscr{L}_V(c)$ is a positively invariant set and, since $V(x_k(0),\sigma_k(0)) \to 0$, for $k \to \infty$, we conclude that $V(x_k(\rho),\sigma_k(0)) \to 0$ for $k \to \infty$ and hence $x_k(\rho) \to 0$ for $k \to \infty$, which concludes the proof of item (ii). ∎

IV. Stability Assessment

In this section, from the theoretical results of Theorem 1, we propose conditions in LMI form to provide a solution to Problem 1. For this, consider the PDLF:

$$\begin{aligned}\mathscr{V}_0(\tau,x_k,\sigma_k,T_k) = (T_k-\tau)\Big\{&(x_k(\tau)-x_k(0))'[F(\sigma_k(0))\\ &(x_k(\tau)-x_k(0)) + 2G(\sigma_k(0))x_k(0)] + \tau x_k(0)'X(\sigma_k(0))x_k(0)\\ &+\int_0^\tau \dot{x}_k'(\theta)R\dot{x}_k(\theta)d\theta\Big\},\end{aligned} \tag{19}$$

where $0 < R \in \mathbb{S}^n$ and $F(\sigma_k(0)) = F_0 + \mathscr{F}\Lambda(\sigma_k(0))$, $G(\sigma_k(0)) = G_0 + \mathscr{G}\Lambda(\sigma_k(0))$, $X(\sigma_k(0)) = X_0 + \mathscr{X}\Lambda(\sigma_k(0))$, with $\mathscr{F} = [F_1\ F_2\ \cdots\ F_N]$, $\mathscr{G} = [G_1\ G_2\ \cdots\ G_N]$, $\mathscr{X} = [X_1\ X_2\ \cdots\ X_N]$, F_j, $X_j \in \mathbb{S}^n$ and $G_j \in \mathbb{R}^{n\times n}$, $j = 0,\ldots,N$. By definition, note that $\mathscr{V}_0(T_k,x_k,\sigma_k,T_k) = \mathscr{V}_0(0,x_k,\sigma_k,T_k) = 0$.

From the choices of V and $\mathscr{V}_0$ in (11) and (19), respectively, the following theorem can be stated:

Theorem 2: If there exist symmetric positive definite matrix $R \in \mathbb{S}^n$, symmetric matrices P_j, F_j and $X_j \in \mathbb{S}^n$, matrices G_j, $Q_j \in \mathbb{R}^{3n\times n}$, $K_j \in \mathbb{R}^{m\times n}$, $j = 0,1,\ldots,N$ and Y_1 and $Y_2 \in \mathbb{R}^{n\times n}$, satisfying

$$\Pi_1(v_f,v_g) + T_k\Pi_2(v_f) + T_k\Pi_3(v_f) < 0, \tag{20}$$

$$\begin{bmatrix}\Pi_1(v_f,v_g) - T_k\Pi_3(v_f) & T_kQ(v_f)\\ * & -T_kR\end{bmatrix} < 0, \tag{21}$$

$$\begin{bmatrix}P(v_f) & H'_{(r)}\\ * & \eta^2_{(r)}\end{bmatrix} > 0,\quad r = 1,\ldots,r_c, \tag{22}$$

$$\begin{bmatrix}F(v_f) & G'(v_f)-F(v_f)\\ G(v_f)-F(v_f) & F(v_f)-2G(v_f)\end{bmatrix} > 0, \tag{23}$$

$$X(v_f) > 0, \tag{24}$$

$\forall (v_f,v_g) \in Ver(\mathscr{B}_\sigma)\times Ver(\mathscr{B}_\sigma)$ and $T_k \in \{\underline{T},\overline{T}\}$, with

$$\begin{aligned}\Pi_1(v_f,v_g) &= \mathrm{He}\{M_1'P(v_f)M_3 - M_{12}'G(v_f)M_2\\ &\quad -Q(v_f)M_{12}\} - M_{12}'F(v_f)M_{12} + \mathrm{He}\{(M_1'Y_1'\\ &\quad +M_3'Y_2')(A(v_g)M_1 + B(v_g)K(v_f)M_2 - IM_3)\},\\ \Pi_2(v_f) &= M_3'RM_3 + \mathrm{He}\{M_3'(F(v_f)M_{12} + G(v_f)M_2)\},\\ \Pi_3(v_f) &= M_2'X(v_f)M_2,\\ Q(v_f) &= Q_0 + \mathscr{Q}\Lambda(v_f),\quad F(v_f) = F_0 + \mathscr{F}\Lambda(v_f),\\ G(v_f) &= G_0 + \mathscr{G}\Lambda(v_f),\quad X(v_f) = X_0 + \mathscr{X}\Lambda(v_f),\end{aligned} \tag{25}$$

being matrices $A(v_g)$, $B(v_g)$, $K(v_f)$ and $P(v_f)$ defined as in (9) and (12), $\mathscr{Q} = [Q_1,\ldots,Q_N]$ and the auxiliary matrices

$$\begin{aligned}M_1 = [I\ 0\ 0],\ M_2 = [0\ I\ 0],\ M_3 = [0\ 0\ I]\\ M_{12} = M_1 - M_2,\end{aligned} \tag{26}$$

then control law (4) with gain matrices K_j, $j = 0,1,\ldots,N$, ensures that for any initial condition $x(0) \in \mathscr{L}_V(1)$ the correspondent trajectories of the sampled-data nonlinear closed-loop system (1)-(2), represented by the quasi-LPV systems (3)-(4) under Assumptions 1 and 2, converge asymptotically to the origin, $\forall T_k \in [\underline{T},\overline{T}]$.

Proof: Define a matrix $Q(\sigma_k(0)) = Q_0 + \mathscr{Q}\Lambda(\sigma_k(0))$ and $\chi(\tau) = [x_k'(\tau)\quad x_k'(0)\quad \dot{x}_k'(\tau)]'$. Since $R > 0$, we have that [18]:

$$\begin{aligned}\textstyle\int_0^\tau \dot{x}_k'(\theta)R\dot{x}_k(\theta)d\theta \geq 2\chi'(\tau)Q(\sigma_k(0))(x_k(\tau)\\ -x_k(0)) - \tau\chi'(\tau)Q(\sigma_k(0))R^{-1}Q'(\sigma_k(0))\chi(\tau).\end{aligned} \tag{27}$$

Consider the PDQF V and the PDLF $\mathscr{V}_0$ defined in (11) and (19), respectively. Hence, based on auxiliary matrices M_1, M_2, M_3 and M_{12} defined in (26), it follows from (27) that

$$\begin{aligned}\dot{\mathscr{W}} \leq \chi'(\tau)[\hat{\Pi}_1(\sigma_k) + (T_k-\tau)\Pi_2(\sigma_k)\\ +(T_k-2\tau)\Pi_3(\sigma_k) + \tau Q(\sigma_k(0))R^{-1}Q'(\sigma_k(0))]\chi(\tau),\end{aligned} \tag{28}$$

with

$$\begin{aligned}\hat{\Pi}_1(\sigma_k) &= \mathrm{He}\{M_3'P(\sigma_k(0))M_1 - Q(\sigma_k(0))M_{12}\\ &\quad -M_{12}'G(\sigma_k(0))M_2\} - M_{12}'F(\sigma_k(0))M_{12},\\ \Pi_2(\sigma_k) &= \mathrm{He}\{M_3'(F(\sigma_k(0))M_{12} + G(\sigma_k(0))M_2)\} + M_3'RM_3\\ \Pi_3(\sigma_k) &= M_2'X(\sigma_k(0))M_2.\end{aligned}$$

From (14), for any matrices Y_1 and Y_2 of appropriate dimensions, we have that equality

$$(Y_1x_k(\tau)+Y_2\dot{x}_k(\tau))'[A(\sigma_k(\tau))x_k(\tau)+(B(\sigma_k(\tau)) \\ K(\sigma_k(0)))x_k(0)-\dot{x}_k(\tau)]=0. \quad (29)$$

holds. Hence, using $\chi(\tau)$ and the auxiliary matrices from (26), one gets that $\chi'(\tau)\Theta(\sigma_k(0),\sigma_k(\tau))\chi(\tau)=0$, with $\Theta(\sigma_k(0),\sigma_k(\tau))=(Y_1M_1+Y_2M_3)'[A(\sigma_k(\tau))M_1+B(\sigma_k(\tau))K(\sigma_k(0))M_2-M_3]$. Then, it is possible to rewrite (28) as

$$\dot{\mathscr{W}}\leq\chi(\tau)'\{\Pi_1(\sigma_k(0),\sigma_k(\tau))+(T_k-\tau)\Pi_2(\sigma_k) \\ +(T_k-2\tau)\Pi_3(\sigma_k)+\tau Q(\sigma_k)R^{-1}Q'(\sigma_k)\}\chi(\tau), \quad (30)$$

with $\Pi_1(\sigma_k(0),\sigma_k(\tau))=\hat{\Pi}_1(\sigma_k)+\text{He}\{\Theta(\sigma_k(0),\sigma_k(\tau))\}$. Hence, by applying Schur's complement, a sufficient condition to verify $\dot{\mathscr{W}}(\tau,x_k,\sigma_k,T_k)\leq-\beta||x_k(0)||^2$ is given by

$$\Psi(\sigma_k(0),\sigma_k(\tau))=\begin{bmatrix}\Pi(\sigma_k(0),\sigma_k(\tau)) & \tau Q(\sigma_k)\\ * & -\tau R\end{bmatrix}<0 \quad (31)$$

with $\Pi(\sigma_k(0),\sigma_k(\tau))=\Pi_1(\sigma_k(0),\sigma_k(\tau))+(T_k-\tau)\Pi_2(\sigma_k)+(T_k-2\tau)\Pi_3(\sigma_k)$. Recalling from (7) that $\Lambda(\sigma_k(0))=\sigma_k(0)\otimes I$ and $\Lambda(\sigma_k(\tau))=\sigma_k(\tau)\otimes I$, it follows that (31) is affine on $\sigma_k(0)$ and $\sigma_k(\tau)$. Furthermore, under Assumption 2, provided that $x_k(\tau)\in\mathscr{R}_H$, it follows that $\sigma_k(\tau)$ and $\sigma_k(0)\in\mathscr{B}_\sigma$. In this case, by convexity, a necessary and sufficient condition to satisfy (31) consists in satisfying it at any combination of the vertices of $\mathscr{B}_\sigma$, i.e. we must verify

$$\Psi(v_f,v_g)<0,\quad\forall(v_f,v_g)\in Ver(\mathscr{B}_\sigma)\times Ver(\mathscr{B}_\sigma). \quad (32)$$

Noting now that since (32) is also affine with respect to the variable $\tau\in[0,T_k]$, from convexity arguments, it suffices to ensure that this relation is verified for $\tau=0$ and $\tau=T_k$. In other words, it is satisfied $\forall\tau\in[0,T_k]$ iff

$$\Pi_1(v_f,v_g)+T_k\Pi_2(v_f)+T_k\Pi_3(v_f)<0, \quad (33)$$

and

$$\begin{bmatrix}\Pi_1(v_f,v_g)-T_k\Pi_3(v_f) & T_kQ(v_f)\\ * & -T_kR\end{bmatrix}<0. \quad (34)$$

Furthermore since $T_k\in[\underline{T},\overline{T}]$, applying the same convexity argument, it follows that (20) and (21) verified for $T_k=\underline{T}$ and $T_k=\overline{T}$ (i.e. for $T_k\in\{\underline{T},\overline{T}\}$) are equivalent to (33) and (34). Moreover, also by convexity arguments, if (22)-(24) are verified $\forall v_f\in\mathscr{B}_\sigma$ it follows that $\mathscr{V}_0(\tau,x_k,\sigma_k,T_k)>0$ and $P(\sigma(t_k))>0$, provided $x_k(\tau)\in\mathscr{R}_H$.

We have now to show that if $x_0(0)\in\mathscr{L}_V(1)$, then the trajectory never leaves $\mathscr{R}_H$. To this end, note that (22) ensures that $\mathscr{L}_V(1)\subset\mathscr{R}_H$. Thus, if $x_0(0)\in\mathscr{L}_V(1)$, it follows that (20)-(21) ensure that $\dot{\mathscr{W}}(0,x_0,\sigma_0,T_0)\leq-\beta||x_0(0)||^2$. Hence, following the reasoning in Lemma 2, we conclude that $x_k(\tau)\in\mathscr{L}_V(1)$ $\forall\tau\in[0,T_k]$ and $k\geq0$. Thus, the simultaneous satisfaction of conditions (20)-(22) ensures that conditions of Theorem 1 are fulfilled with $\mathscr{D}=\mathscr{R}_H$ and $\mathscr{L}_V(1)$ for the LPV system (3)-(4), under Assumption 2. Hence, as (3)-(4) represents the nonlinear system (1)-(2) provied $x(t)\in\mathscr{R}_H$ and $\mathscr{L}_V(1)\subset\mathscr{R}_H$, it follows that $\mathscr{L}_V(1)$ is included in the region of attraction of its origin. ■

V. Optimization Problems

A. Maximization of the estimate of the region of attraction

Note that the set $\mathscr{L}_V(1)\subset\mathscr{R}_H$ is, by definition, included in $\mathscr{R}_a$ and can be used as an estimate of it. Then, from Problem 1 (given $T_k\in[\underline{T},\overline{T}]$ and the state feedback gains K_j) the goal is to maximize the set $\mathscr{L}_V(1)$, considering some size criterion such as the maximization of the minor axis of $\mathscr{E}(P(v_f),1)$, $\forall f=1,\ldots,2^N$. This can be accomplished by the solution of the following optimization problem:

$$\begin{array}{l} \textit{min}\ \ \delta \\ \text{subject to} \\ (20),\ (21),\ (22),\ (23),\ (24),\ P(v_f)<\delta I \\ \forall(v_f,v_g)\in Ver(\mathscr{B}_\sigma)\times Ver(\mathscr{B}_\sigma),\ \text{for } T_k\in\{\underline{T},\overline{T}\}. \end{array} \quad (35)$$

B. Maximization of the sampling interval

From Problem 2, we consider a given region of admissible initial conditions for the sampled-data system (3)-(4) in the following form:

$$\mathscr{E}(X_0,1)=\{x\in\mathbb{R}^n;\ x'X_0x\leq1\},\ \text{ with }\ X_0=X_0'>0. \quad (36)$$

Then, for given $\underline{T}$ and state feedback gains K_j, the aim is to maximize the value of $\overline{T}$ for which the closed-loop stability is ensured for $T_k\in[\underline{T},\overline{T}]$. This can be accomplished from the solution of the following optimization problem:

$$\begin{array}{l} \textit{max}\ \ \overline{T} \\ \text{subject to} \\ (20),\ (21),\ (22),\ (23),\ (24),\ P(v_f)\leq X_0 \\ \forall(v_f,v_g)\in Ver(\mathscr{B}_\sigma)\times Ver(\mathscr{B}_\sigma),\ \text{for } T_k\in\{\underline{T},\overline{T}\}. \end{array} \quad (37)$$

Hence, the optimization problem (37) can be solved by iteratively increasing $\overline{T}$ and testing the feasibility of the LMIs.

VI. Example

Consider the Lorenz system borrowed from [13]:

$$\begin{array}{l} \dot{x}_1(t)=-ax_1(t)+ax_2(t)+u(t) \\ \dot{x}_2(t)=cx_1(t)-x_2(t)-x_1(t)x_3(t) \\ \dot{x}_3(t)=x_1(t)x_2(t)-bx_3(t) \end{array} \quad (38)$$

with $a=10$, $b=8/3$, $c=28$, $d=25$ and the control law $u(t)=-17.3491x_1(t_k)-13.4530x_2(t_k)-0.1956x_3(t_k)-0.0031x_1(t_k)x_2(t_k)-0.0040x_1^2(t_k)-0.0002x_1(t_k)x_3(t_k)$, $t\in[t_k,\ t_{k+1})$. Assuming $x_1(t)\in[-d,d]$ and $\sigma(x(t))=x_1(t)$, the Lorenz system (38) can be represented in the quasi-LPV form (14) with matrices

$$A_0=\begin{bmatrix}-a & a & 0\\ c & -1 & 0\\ 0 & 0 & -b\end{bmatrix},\ A_1=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix},\ B_0=\begin{bmatrix}1\\0\\0\end{bmatrix},$$

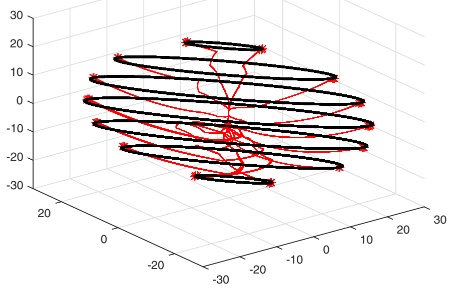

Fig. 1. For $T_k = 0,075$, state trajectories (in red) in $\mathbb{R}^3$ for initial conditions on the boundary of the set $\mathscr{E}(P(v_f),1)$ (in black).

$B_1 = 0$ and the control law (4) is defined with the gain matrices

$$K_0 = \begin{bmatrix} -17.3491 & -13.4530 & -0.1956 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} -0.0040 & -0.0031 & -0.0002 \end{bmatrix}.$$

Let us consider first the case of constant sampling period, i.e. $T_k = \underline{T} = \overline{T}$. Applying the optimization problem (37) with $X_0 = I$, we obtain the maximum T_k feasible of 0.075s. Then, for $T_k = \underline{T} = \overline{T} = 0.075$s, $\forall k$, it is ensured the asymptotic stability for $\mathscr{L}_V(1)$ defined by the following matrices:

$$P_0 = 10^{-2}\begin{bmatrix} 2.14 & 1.66 & 0.12 \\ 1.66 & 1.45 & 0.09 \\ 0.12 & 0.09 & 0.17 \end{bmatrix}, P_1 = 10^{-6}\begin{bmatrix} 5.03 & 3.87 & 0.27 \\ 3.87 & 2.98 & 0.21 \\ 0.27 & 0.21 & 0.01 \end{bmatrix}$$

with $trace(P(v_1)) = 0.0374$ and $trace(P(v_2)) = 0.0378$. In Figure 1, a tri-dimensional approximation of the set $\mathscr{L}_V(1)$ is depicted along with some trajectories starting on its boundary. As expected, the trajectories starting from initial conditions on the boundary of the set $\mathscr{L}_V(1)$ converge to the origin.

From the optimization problem (35) and considering the aperiodic case, Table I shows a trade-off between the size of the sampling interval $T_k = [\underline{T}, \overline{T}]$ and the size of the obtained region $\mathscr{E}(P(v_f),1)$. We consider a fixed value for $\underline{T}$ ($\underline{T} = 0.01$s) and different values for $\overline{T}$. As $\overline{T}$ increases, i.e., larger is the interval $[\underline{T}, \overline{T}]$, larger values for δ are obtained, i.e., smaller is $\mathscr{E}(P(v_f),1)$ (noted by the trace of $P(v_f)$). Observe that the case of constant sampling periods ($\underline{T} = \overline{T}$) leads to a larger set when compared to the cases of aperiodic sampling.

TABLE I

Example 1 - Values of $trace(P(v_f))$, with $\underline{T} = 0.01$ and different values of $\overline{T}$.

trace $\times 10^{-2}$	$\overline{T}$			
	$1 \times \underline{T}$	$3 \times \underline{T}$	$5 \times \underline{T}$	$7 \times \underline{T}$
$P(v_1)$	0.8919	0.9112	1.2007	1.7495
$P(v_2)$	0.8915	0.9116	1.2032	1.7299

VII. Conclusion

This work has addressed the stability analysis problem of sampled-data nonlinear systems considering quasi-LPV models. The control signal has been assumed to be constant between two successive sampling instants and the continuous behavior of the nonlinear plant has been explicitly considered (i.e., no discretization has been performed). Based on a looped-functional and a polytopic approach, LMI conditions to assess the regional stability of the nonlinear closed-loop system has been proposed.

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