2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Adaptive Optimal Tracking via Cone Estimation for Discrete-time System under Lipschitz Uncertainty*

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Abstract— **This paper addresses a problem of adaptive optimal tracking for discrete-time plant under bounded nonparametric Lipschitz uncertainty and bounded external disturbance. Solution of the problem is based on closed loop estimation of two different models in parallel. The plant model itself is estimated and used for stabilization. The second model is based on the Lipschitz condition of the uncertainty and is used for closed loop estimation and rejection of the uncertainty. Both models are estimated with the use of cone estimates, which allow to treat the control criterion as an identification criterion to ensure asymptotically optimal tracking with a prescribed accuracy.**

I. INTRODUCTION

Robust control aims to meet a control objective via *a priori* synthesis of controller without estimation of system uncertainty. The optimality of synthesis is typically associated with the maximization of the set of admissible uncertainties [1], [2] or the minimization of a certain control criterion for a norm-bounded set of uncertainties [3]. Note that robust stability and robust performance conditions can be the same for uncertainties of different type. In particular, this is the case in the robust control theory in the ℓ_1 setting where these conditions are equivalent for linear time-varying, nonlinear time-invariant, and nonlinear time-varying uncertainties [4], [5].

Adaptive control aims to improve control performance via closed loop estimation of some unknown values required for control synthesis. In contrast to robust control, adaptive control may look for a possibility to reject a nonparametric time-invariant uncertainty (note that the rejection of uncertainty in discrete-time systems is impossible without its estimation and can not be achieved via strong feedback as it can be done for some continious-time systems). Such a possibility was first demonstrated in [6] for a simplest discrete-time dynamical model under nonparametric nonlinear time-invariant uncertainty of the Lipschitz type. Asymptotically optimal tracking was achieved via online estimation and rejection of the uncertainty and a critical value of $3/2+\sqrt{2}$ for the Lipschitz constant was established for stabilizability of the closed loop system by a feedback. In [7], [8], an adaptive optimal tracking was extended to the model with an additional unknown parameter in the model equation. Maximum capabilities of feedback in stabilizability of more general minimum-phase systems under parametric uncertainty and nonparametric Lipschitz uncertainty were established in [9]–[11].

In this paper, we return to the problem of adaptive asymptotically optimal tracking under the additional assumption that the Lipschitz uncertainty in linear time-invariant minimum-phase SISO plant is bounded but the upper bounds of the uncertainty and the external bounded disturbance are unknown. Under this assumption the plant is adaptively stabilazable under uncertainty with arbitrary Lipshitz constant and the main problem is in the estimation and rejection of the uncertainty in the situation of non-identifiabilty of the plant. Solution of the problem is based on the method of recurrent objective inequalities [12], [13] and a method of two models proposed in [14]. The first model is the plant model itself and is used for stabilization of closed loop system. The second model is based on the Lipschitz condition of uncertainty and is used for closed loop estimation and rejection of the uncertainty. The optimality of adaptive tracking in the situation of non-identifiabilty of the plant is achieved by treating the control criterion as the identification criterion that was first proposed in [15], [16]. The optimality of adaptive control requires the use of set estimates of unknown parameters consistent with measurement data and was achieved in [14] with the use of polyhedral estimates, which are computationally demanding and can be used in the case of relatively small number of estimated parameters. In this paper, we use the cone estimation algorithm proposed in [17]. The cone algorithm has the minimal complexity among polyhedral estimates and can be used in the case of a much larger number of estimated parameters. In a certain sense, the cone algorithm can be considered as an analogue of the recurrent least squares algorithm for the problem of the minimization of the maximal residual in the linear regression model (the optimal estimation problem in this paper differs slightly from the maximal residual problem). In this paper, we describe a more general version of the cone algorithm and prove its finite convergence under clearer and less restrictive additional assumption compared with [17].

The paper is organized as follows. A problem statement is given in section II. Models for stabilizing the closed loop system and for optimal tracking are described in sections III and IV, respectively. The cone algorithm is presented in section V. The final section VI describes the adaptive optimal tracking control.

Notattion:
$\mathbb{N}=\{0,1,2,\ldots\}$ – natural numbers;
$|\varphi|$ – the euclidean norm of the vector $\varphi\in\mathbb{R}^n$;
φ^T – the transpose of the vector ϕ;
$x_p^q = (x_p, x_{p+1}, \ldots, x_q)$ for a real sequence $x = (x_0, x_1, x_2, \ldots)$;

*This work was supported by Russian Scientific Foundation (grant 14-29-00142) in IPME RAS.

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ℓ_∞ – the normed space of bounded real sequences,
$\|x\| = \sup_t |x_t|$ for $x \in \ell_\infty$;
$\|G(\lambda)\| = \sum_{k=0}^{+\infty} |g_k|$ – the induced norm of linear time-invariant system $G : \ell_\infty \to \ell_\infty$ with the stable transfer function $G(\lambda) = \sum_{k=0}^{+\infty} g_k \lambda^k$.

II. PROBLEM STATEMENT

Let a controlled plant be described by a model

$$y_{t+1} = a(q^{-1})y_t + b(q^{-1})u_t + f(y_t) + w_{t+1} \quad \forall t \in \mathbb{N}\,, \quad (1)$$

where y_t, u_t, and w_t are the output, control and external disturbance at the time instatnt t, $f(y_t)$ is an uncertainty, $a(q^{-1})$ and $b(q^{-1})$ are polynomials in the backward shift operator q^{-1},

$$a(q^{-1})y_t = a_0 y_t + \cdots + a_n y_{t-n}\,,$$
$$b(q^{-1})u_t = b_0 u_t + \cdots + b_m u_{t-m}\,,\ b_0 \neq 0\,.$$

A priori information about model (1) is as follows. An unknown coefficient vector

$$\xi = (a_0, \ldots, a_n, b_0, \ldots, b_m)^T$$

belongs to a known *a priori* set Ξ,

$$\xi \in \Xi \subset \mathbb{R}^{n+m+2}\,,$$

and the polynomials $b(\lambda)$ are stable for all $\xi \in \Xi$. The set Ξ is assumed to be polyhedral, that is, Ξ is described by a system of linear inequalities.

The uncertainty $f : \mathbb{R} \to \mathbb{R}$ satisfies the generalized Lipschitz condition

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2| + \gamma \quad \forall x_1, x_2 \in \mathbb{R}\,, \quad (2)$$

and is bounded

$$F = \sup_{x \in \mathbb{R}} |f(x)| < +\infty\,.$$

The unknown external disturbance w is also bounded

$$W = \sup_t |w_t| < +\infty\,,$$

and the constants L, γ, F, and W are unknown.

The problem is to synthesize a feedback $u_t = U_t(y_0^t, u_0^{t-1}, y_t^*)$ that ensures, with a prescribed tolerance, an inequality

$$\limsup_{t \to +\infty} |y_t - y_t^*| \leq \gamma + 2W =: \Gamma \quad (3)$$

for any given bounded reference signal $y^* = (y_0^*, y_1^*, \ldots)$.

It will be clear in the further presentation that the upper bound $\Gamma = \gamma + 2W$ in (3) is minimal. A solution of the stated problem requires an estimation of the uncertainty f for its rejection, wich in turn requires an estimation of the coefficient vector ξ. Despite the non-identifiabilty of the vector ξ, a solution will be obtained with the use of two models in parallel. The first one is used for stabilization of closed loop system while the second model ensures asymptotic optimality of adaptive tracking.

III. MODEL FOR STABILIZING THE CLOSED LOOP SYSTEM

In this section, a model for stabilizing the closed loop system is described. Model equation (1) and the upper bounds on the uncertainty f and the disturbance w imply an inequality

$$|y_{t+1} - a(q^{-1})y_t - b(q^{-1})u_t| \leq F + W =: D \quad \forall\, t \in \mathbb{N}$$

with the unknown upper bound D in the right hand side. With notation

$$\varphi_t = (y_t, \ldots, y_{t-n}, u_t, \ldots, u_{t-m})^T$$

we have

$$a(q^{-1})y_t + b(q^{-1})u_t = \phi_t^T \xi\,,$$

and

$$|y_{t+1} - \varphi_t^T \xi| \leq D\,. \quad (4)$$

Inequalities (4) play the role of *stabilization model* (that is, a model for stabilizing the closed loop system). A purpose of the estimation algorithm for stabilization is to compute in a finite time an estimate $\theta_\infty^s = (\xi_\infty^{sT}, D_\infty^s)^T$ of the unknown vector $\theta^s = (\xi^T, D)^T$ that satisfies inequalities

$$|y_{t+1} - \varphi_t^T \xi_\infty^s| \leq D_\infty$$

with a prescribed tolerance for all sufficiently large t and, in addition, ensures the inequality $D_\infty^s \leq D$.

IV. MODEL FOR OPTIMAL TRACKING

In this section, a model for adaptive optimal tracking is described. To construct a finite memory feedback, choose a small number $\varepsilon > 0$ and consider a partition of the real axis

$$\mathbb{R} = \bigcup_{k \in \mathbb{Z}} [k\varepsilon, (k+1)\varepsilon)\,.$$

Let $(y_1, y_2, y_3, \ldots)$ be a sequence of outputs generated by some feedback. We will store in computer memory some outputs according to the following algorithm. For any $t > 0$, the current output $y_t \in [k\varepsilon, (k+1)\varepsilon)$ is stored in memory if and only if no previously stored outputs fall into the same interval $[k\varepsilon, (k+1)\varepsilon)$. Let $(y_{t_1}, y_{t_2}, y_{t_3}, \ldots)$ be the subsequence of all stored outputs with $y_{t_1} = y_1$. Note that the subsequence $(y_{t_1}, y_{t_2}, y_{t_3}, \ldots)$ may be finite and this is definitely the case if the sequence $(y_1, y_2, y_3, \ldots)$ is bounded.

Denote by i_t the number of the nearest to y_t neighbor among the previous stored outputs:

$$|y_t - y_{i_t}| = \min_{\{j \ \mid\ t_j < t\}} |y_t - y_{t_j}|\,.$$

Let ξ_t^o be an estimate of the unknown vector ξ provided by an estimation algorithm for optimal tracking at the time instant t. The estimate of the unknown $f(y_t)$ is computed at the time instant t as follows.

$$\hat{f}_t(y_t) = y_{i_t+1} - \varphi_{i_t}^T \xi_t\,. \quad (5)$$

Formula (5) is motivated by equality

$$y_{i_t+1} - \varphi_{i_t}^T \xi = f(y_{i_t}) + w_{i_t+1}$$

and by observation that for the Lipschitz uncertainty f the value of $f(y_{i_t})$ is the best available estimate of $f(y_t)$ as y_{i_t} is the nearest to y_t stored output.

It follows from (1), (2) and the boundedness of w that

$$\begin{array}{l} |y_{i+1} - \varphi_i^T \xi - (y_{j+1} - \varphi_j^T \xi)| = \\ |f(y_i) + w_{i+1} - (f(y_j) + w_{j+1})| \le \\ \le L|y_i - y_j| + \Gamma\,, \quad \Gamma = \gamma + 2W \end{array} \tag{6}$$

for all $0 \le j < i \le t$.

Inequalities (6) contain information on unknown parameters ξ, L, γ, W associated with the Lipschitz property of the uncertainty. It turns out that the use of only one of these equalities at each time instant is sufficient for synthesis of adaptive optimal control. Let

$$\theta_t^o = (\xi_t^{oT}, L_t^o, \Gamma_t^o)^T$$

be an estimate of the unknown vector $\theta^o = (\xi^T, L, \Gamma)^T$ computed at the time instant t. Then the objective inequality for the estimate θ_t^o at the time instant $t+1$ is as follows

$$|y_{t+1} - \varphi_t^T \xi_t^o - (y_{i_t+1} - \varphi_{i_t}^T \xi_t^o)| \le L_t^o |y_t - y_{i_t}| + \Gamma_t^o\,. \tag{7}$$

A purpose of the estimation algorithm for optimal tracking is to compute in a finite time an estimate $\theta_\infty^o = (\xi_\infty^{oT}, L_\infty^o, \Gamma_\infty^o)^T$ of the unknown vector $\theta = (\xi^T, L, \Gamma)^T$ that satisfies inequalities

$$|y_{t+1} - \varphi_t^T \xi_\infty^o - (y_{i_t+1} - \varphi_{i_t}^T \xi_\infty^o)| \le L_\infty^o |y_t - y_{i_t}| + \Gamma_\infty^o \tag{8}$$

with a prescribed tolerance for all sufficiently large t and, in addition, ensures inequality

$$L_\infty^o \varepsilon + \Gamma_\infty^o \le L\varepsilon + \Gamma = L\varepsilon + \gamma + 2W\,. \tag{9}$$

Note that the right hand side in (9) converges to the desired upper bound in (3) as $\varepsilon \to 0$. Inequalities (7) play the role of auxiliary *optimal tracking model*. It will be clear in the further presentation that this auxiliary model together with adaptive controller represent a model of closed loop system consistent with measurement data for all sufficiently large t.

V. Cone algorithm for asymptotic optimal solution of infinite system of linear inequalities

In this section, the cone algorithm for asymptotic optimal solution of infinite system of linear inequalities is formulated and its properties relevant to adaptive control are proven.

Consider an infinite system of linear inequalities in the Euclidean space $\mathbb{R}^l$ with respect to the vector θ

$$\psi_t^T \theta \ge \zeta_t\,, \quad \zeta_t \in \mathbb{R}\,, \quad |\psi_t| = 1\,, \quad t = 1, 2, \ldots. \tag{10}$$

Let *a priori* set Θ of admissible values of θ is described by a finite system of linear inequalities

$$\Theta = \{\ \theta \in \mathbb{R}^l \ \big|\ P\theta \ge p\ \}\,, \quad P \in \mathbb{R}^{N \times l}\,, \quad p \in \mathbb{R}^N\,. \tag{11}$$

It is assumed that there exists a vector $\theta \in \Theta$ satisfying (10). The inequality $\psi_t^T \theta \ge \zeta_t$ becomes known at the time instant t. Let $c^T\theta$ be an *objective function* with a given $c \in R^n$ and $\delta > 0$ be a given number characterizing a desired accuracy of solution. The problem is to construct a sequence of estimates θ_t of the unknown solution θ such that

$$\exists t_* \ \forall t \ge t_* \quad \theta_t = \theta_\infty\,, \quad \psi_t^T \theta_\infty \ge \zeta_t - \delta\,, \tag{12}$$

and

$$c^T \theta_\infty \le c^T \theta\,. \tag{13}$$

Inequality (13) is a requirement for optimality of estimated solution θ_∞ and can not be guaranteed by any estimation algorithm of projection type because all of them are based on decreasing $|\theta_t - \theta|$ and can not provide the minimization of the objective function $c^T\theta_t$. The simplest set estimates for solution of the problem are cones in $\mathbb{R}^l$ of the form

$$C_t = \{\ \hat\theta \ \big|\ \hat\theta = \theta_t + \sum_{k=1}^{l} \lambda_k e_k^t\,, \forall \lambda_k \ge 0,\ |e_k^t| = 1\ \},$$

where θ_t is the vertex of the cone and $e_1^t, \ldots, e_l^t$ are its edges. The vertices θ_t will play the role of estimates of the vectors of unknown parameters for two models described in the previous section.

We first present a geometric description of updating the cones C_t. Every cone C_t is described by a subsystem of l linear inequalities (10) or inequalities from description (11) of the *a priori* set Θ. The vertex θ_t is the point of the minimum of the objective function on C_t. After obtaining the next inequality $\psi_{t+1}^T \hat\theta \ge \zeta_{t+1}$ that defines a half-space in $\mathbb{R}^l$, the cone C_t is updated only if the distance from the vertex θ_t to this half-space is greater than δ. At first an intermediate estimate $\hat\theta_{t+1}$ that minimizes the objective function $c^T\hat\theta$ at the intersection of the cone with the half-space is calculated. After this, a new inequality is added to the cone description and an inequality from the description of C_t that is satisfied for $\hat\theta_{t+1}$ strictly is discarded. The updated set of inequalities defines an intermediate cone $\hat C_t$. Then the updating is repeated (without taking into account the distance to *a priori* constraints) by successive presentation of inequalities from the description of the *a priori* set Θ until all these inequalities are satisfied. The described algorithm is realized by following formulas.

$$C_{t+1} := C_t, \quad \text{if } \psi_{t+1}^T \theta_t \ge \zeta_{t+1} - \delta\,. \tag{14}$$

Otherwise the edges $\hat e_k^{t+1}$ and the vertex $\hat\theta_{t+1}$ of the updated cone $\hat C_{t+1}$ are computed as follows.

$$\begin{array}{l} k_t := \underset{\{k \mid \psi_{t+1}^T e_k^t > 0\}}{\operatorname{argmin}} \ \dfrac{c^T e_k^t}{\psi_{t+1}^T e_k^t}\,, \\ \hat e_k^{t+1} := \left\{ \begin{array}{ll} e_k^t\,, & k = k_t\,, \\ (e_k^t - \frac{\psi_{t+1}^T e_k^t}{\psi_{t+1}^T e_{k_t}^t} e_{k_t}^t)_{nor}\,, & k \ne k_t\,, \end{array} \right. \\ \hat\theta_{t+1} := \theta_t + \dfrac{\zeta_{t+1} - \psi_{t+1}^T \theta_t}{\psi_{t+1}^T e_{k_t}^t} e_{k_t}^t\,, \end{array} \tag{15}$$

where $(e)_{nor}$ denotes the normalized vector $e/|e|$. If $\hat\theta_{t+1} \notin \Theta$ then the updating continues with successive presentation of inequalities from the description of Θ without taking into account the distance δ to the corresponding half-spaces until the vertex of the cone $\hat{C}_{t+1}$ gets into in Θ. In order to avoid cycling in the case of multiple minima k_t, we specify the choice of k_t by the following lexicographic procedure. Complete the vector c to an orthonormal basis $c, c_2, \ldots, c_l$ in $\mathbb{R}^l$. If the argmin k_t is not unique, choose an index for which the minimum is reached by successive replacing the vector c by $c_2, c_3, \ldots$ until the unique argmin is achieved.

The properties of the cone algorithm relevant to the synthesis of adaptive control are formulated in the following theorem.

Theorem 1. Let the following assumptions be true.

1. There exists $\theta \in \Theta$ such that $\psi_t^T \theta \geq \zeta_t$ for all $t = 0, 1, 2, \ldots$.

2. The orthogonal projection of Θ onto $\{ \ \hat{\theta} \mid c^T \hat{\theta} = 0 \ \}$ is bounded.

4. The initial cone C_0 satisfies $\Theta \subset C_0$ and

$$\theta_0 = \operatorname*{argmin}_{\hat{\theta} \in C_0} c^T \hat{\theta} \,,$$

i.e. $c^T e_k^0 \geq 0$ for all k.

4. There exists $\delta_1 > 0$ and an index k_* such that $c^T e_{k_*}^t \geq \delta_1$ for all sufficiently large t.

Then the number of updatings of the cones C_t defined by (14) and (15) is finite and the final estimate θ_∞ satisfies inequalities *(12)* and *(13)*.

Proof: Let us explain why formulas (15) realize the geometric description of the updating algorithm given before these formulas. The lower formula implies $\psi_{t+1}^T \hat{\theta}_{t+1} = \zeta_{t+1}$, i.e. the vector $\hat{\theta}_{t+1}$ lies on the boundary of the inequality $\psi_{t+1}^T \hat{\theta} \geq \zeta_{t+1}$. Since θ_t is updated along the edge $\hat{e}_{k_t}^t$ of the cone C_t, the vector $\hat{\theta}_{t+1}$ remains on the boundary of the $l-$ inequalities from the description of C_t, and the inequality whose boundary does not contain the edge $\hat{e}_{k_t}^t$ is discarded from the description of C_t. Thus, the cone $\hat{C}_{t+1}$ is obtained by replacing one of the inequalities in the description of C_t by the new inequality $\psi_{t+1}^T \hat{\theta} \geq \zeta_{t+1}$.

It follows from the definitions of the index k_t and the edges $\hat{e}_k^{t+1}$ that $c^T \hat{e}_k^{t+1} \geq 0$ for all k and, consequently,

$$c^T \hat{\theta}_{t+1} = \min_{\hat{\theta} \in \hat{C}_{t+1}} c^T \hat{\theta} \,.$$

One can see from the definition of $\hat{\theta}_{t+1}$ and the condition of updating $\zeta_{t+1} - \psi_{t+1}^T \theta_t > \delta$ that $c^T \hat{\theta}_{t+1} \geq c^T \theta_t$ for all t. In view of condition 1 of the theorem we have $\theta \in C_t$ for all t and, consequently, $c^T \hat{\theta}_t \leq c^T \theta$ for all t. Then

$$c^T \hat{\theta}_t \nearrow J_\infty \tag{16}$$

where the sign $\nearrow$ denotes the monotone convergence from below to the limit J_∞.

The finite convergence of the cones C_t is proved by induction on the dimension l. The finite convergence for $l = 2$ and $l = 3$ was proven in [17]. Assume that the finite convergence is proven for the dimension $l - 1$. Let the cone C_t be updated, i.e. $\zeta_{t+1} - \psi_{t+1}^T \theta_t > \delta$ and $k_t = k_*$, where k_* is the index from condition 4 of the theorem. Then

$$c^T \hat{\theta}_{t+1} := c^T \theta_t + \frac{\zeta_{t+1} - \psi_{t+1}^T \theta_t}{\psi_{t+1}^T e_{k_*}^t} c^T e_{k_*}^t \geq c^T \theta_t + \delta \delta_1 \,.$$

Taking into account (16) one can conclude that $k_t \neq k_*$ for all sufficiently large t. It means that all further updatings of θ_t are produced on the boundary of the same inequality, that is, on a hyperplane of the dimension $l-1$. The finite convergence follows now by the inductive hypothesis. Impossibility of cycling was proven in [17]. The theorem is proven. ■

VI. Adaptive suboptimal tracking based on method of two models and cone estimation

In this section, a method of two models and the cone estimation algorithm are applied to solution of the problem stated at section II.

A. Stabilizing estimation algorithm

An estimation algorithm for stabilization of closed loop system is formulated as follows. Recall that the purpose of this estimation algorithm is to compute in a finite time an estimate $\theta_\infty^s = (\xi_\infty^{sT}, D_\infty^s)^T$ of the unknown vector $\theta = (\xi^T, D)^T$ that satisfies (4) with a prescribed tolerance for all sufficiently large t and, in addition, ensures the inequality $D_\infty^s \leq D$. Let

$$\theta_t^s = (\xi_t^{sT}, D_t^s)^T$$

be an estimate of the unknown vector $\theta = (\xi^T, D)^T$ computed at the time instant t and y_{t+1} be the measured output y_{t+1} after some control input u_t. Define

$$s_{t+1} = \operatorname{sign}(y_{t+1} - \varphi_t^T \xi_t^s) \,, \quad \upsilon_{t+1} = (s_{t+1} \varphi_t^T, 1)^T \,.$$

The objective inequality (4) for the estimate θ_t^s is of the form

$$|y_{t+1} - \varphi_t^T \xi_t^s| \leq \hat{D}_t^s$$

and can be rewritten as

$$\upsilon_{t+1}^T \theta_t^s \geq s_{t+1} y_{t+1} \,. \tag{17}$$

Inequalities (17) with respect to θ_t^s are of the form (10) with

$$\psi_t = \upsilon_t / |\upsilon_t| \,, \quad \zeta_t = s_t y_t / |\upsilon_t| \,. \tag{18}$$

The priot set (11) for stabilization model is of the form

$$\Theta^s = \left\{ \ (\hat{\xi}^{sT}, \hat{D}^s)^T \ \middle| \ \hat{\xi}^s \in \Xi \,, \hat{D}^s \geq 0 \ \right\}$$

and we assume that the *a priori set* Ξ is described by a system of linear inequalities. Take an initial cone

$$C_0^s = \{ \ \hat{\theta}^s \ \big| \ \hat{\theta} = \theta_0^s + \sum_{k=1}^{n+m+3} \lambda_k e_k^{s0} \,, \forall \lambda_k \geq 0 \ \} \supset \Xi \times [0, \infty)$$

where $\theta_0^s = (\xi_0^{sT}, 0)^T$ with arbitrary $\xi_0^s \in \Xi$ and $c^T e_k^{s0} \geq 0$ for all k and $c = (0, \ldots, 0, 1)^T$. The estimates θ_t^s will be computed by the cone algorithm applied to the system of inequalities (17) and the objective function with $c = (0, \ldots, 0, 1)^T$.

B. Optimal tracking estimation algorithm

An estimation algorithm for optimal tracking is formulated as follows. Recall that the purpose of this estimation algorithm is to compute in a finite time an estimate $\theta^o_\infty = (\xi^{oT}_\infty, L^o_\infty, \Gamma^o_\infty)^T$ of the unknown vector $\theta = (\xi^T, L, \Gamma)^T$ that satisfies inequalities (8) with a prescribed tolerance for all sufficiently large t and, in addition, ensures (9). Let

$$\theta^o_t = (\xi^{oT}_t, L^o_t, \Gamma^o_t)^T$$

be an estimate of the unknown vector $\theta^o = (\xi^T, L, \Gamma)^T$ computed at the time instant t and y_{t+1} be the measured output y_{t+1} after some control input u_t. Define

$$\begin{aligned} &\eta_{t+1} = \operatorname{sign}[y_{t+1} - \varphi_t^T\xi_t - (y_{i_t+1} - \varphi_{i_t}^T\xi_t)], \\ &\psi_{t+1} = (\eta_{t+1}(\varphi_t - \varphi_{i_t})^T, |y_t - y_{i_t}|, 1)^T, \\ &\zeta_{t+1} = \eta_{t+1}(y_{t+1} - y_{i_t+1}). \end{aligned} \tag{19}$$

With this notation the objective inequalities

$$|y_{t+1} - \varphi_t^T\xi_t^o - (y_{i_t+1} - \varphi_{i_t}^T\xi_t^o)| \le L_t^o|y_t - y_{i_t}| + \Gamma_t^o$$

for the estimate θ^o_t take, after division by $|\psi_{t+1}|$, form (10). The prior set (11) for optimal tracking estimation is of the form

$$\Theta^o = \left\{\ (\xi^o, L^o, \Gamma^o)\ \middle|\ \xi^o \in \Xi\,, L^o \ge 0\,, \Gamma^o \ge 0\ \right\}.$$

Take an initial cone

$$C_0^o = \{\ \hat\theta^o \ \big|\ \hat\theta = \theta_0^o + \sum_{k=1}^{n+m+4} \lambda_k e_k^{o0}\,, \forall\lambda_k \ge 0\ \} \supset \Xi\times[0,\infty)$$

where $\theta^o_0 = (\xi_0^{oT}, 0, 0)^T$ with arbitrary $\xi_0^o \in \Xi$ and $c^T e_k^{o0} \ge 0$ for all k. The vector c is associated with the objective function

$$c^T\theta^o := L^o\varepsilon + \Gamma^o \tag{20}$$

(see (9)). Estimates θ^o_t for optimal tracking will be computed by the cone algorithm applied to the system of inequalities (10) and the objective function (20).

C. Adative suboptimal tracking based on two models

Let $y^* = (y_0^*, y_1^*, \ldots)$ be a bounded reference signal, the estimates θ^s_t be computed by the stabilizing estimation algorithm of subsection (VI-A) and the estimates θ^o_t be computed by the optimal tracking estimation algorithm of subsection (VI-B). Adaptive controller is defined by the following formulas.

Adaptive controller: Control u_t is defined as follows.

a) stabilizing controller:

$$\phi_t^T\xi_t^s = y^*_{t+1}, \text{ if } \ |y_t - y_{i_t}| > \varepsilon\,, \tag{21}$$

b) optimal tracking controller:

$$\phi_t^T\xi_t^o = y^*_{t+1} - \hat f_t(y_t), \text{ if } \ |y_t - y_{i_t}| \le \varepsilon\,. \tag{22}$$

Note that the control u_t is defined by (21) and (22) because it is multiplied by estimates b^s_{0t} and b^o_{0t} of the coefficient b_0 and all estimates are nonzero in view of the assumption that $b_0 \ne 0$ for all $\xi \in \Xi$.

The main result on optimal tracking is presented in the next theorem.

Theorem 2. Assume that

$$\left\|\frac{1}{b(\lambda)}\right\| \le C_1,\ \|a(\lambda)\| \le C_2,\ \|b(\lambda)\| \le C_3 \quad \forall\xi\in\Xi$$

with known constants C_1, C_2 and C_3. Then for all sufficiently small $\delta > 0$ the output y of closed loop system (1), (21), and (22) satisfies inequality

$$\limsup_{t\to+\infty} |y_t - y_t^*| \le \Gamma + L\varepsilon + \delta K\,, \tag{23}$$

where $K = K_0 + K_1 D + K_2 L + K_3|y_0| + K_4|f(y_0)| + K_5\|y^*\|$ and the constants K_i, $i = 0, \ldots, 5$ depends on n, m, C_1, C_2, C_3.

The proof of Theorem 2 repeats the proof of Theorem 1 in [14] with a difference in the estimation algorithms.

Comment 1. Inequality (23) means that the adaptive controller ensures the optimal upper bound for the tracking error (3) with the accuracy $L\varepsilon + \delta K$, which can be made arbitrary small by the choice of sufficiently small ε and δ. The smaller is ε, the larger amount of data is necessary to store and the smaller is δ, the slower is the convergence of estimates ξ^s_t and ξ^o_t to their final values.

Comment 2. Let us comment a role of auxiliary model (7). It follows from (5), (7), and the finite convergence of the cone algorithm that adaptive optimal tracking controller (22) ensures inequalities

$$|y_{t+1} - y^*_{t+1}| \le L_t^o|y_t - y_{i_t}| + \Gamma_t^o + \delta|\psi_{t+1}|$$

with unchanged estimates $\theta^o_t = (\xi_t^{oT}, L^o_t, \Gamma^o_t)^T$ for all sufficiently large t. It means that the auxiliary model together with the adaptive controller provide online validation of the model of closed loop system for all sufficiently large t.

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