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Stability and Stabilization of 2D Discrete Stochastic Fornasini-Marchesini Second Model

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Abstract— **This paper deals with the problem of stability and stabilization of two-dimensional (2D) discrete stochastic Fornasini-Marchesini (FM) second model. The proposed results are presented in a Linear Matrix Inequality (LMI) framework. A mean square asymptotic stablilty condition is elaborated through the use of the Leibniz-Newton formula with additional free weighting matrices. Moreover, a sufficient condition is established for the design of a state feedback controller that ensures the mean square stability of the closed loop system. In order to illustrate the effectiveness of the proposed approach, numerical examples have been given.**

keywords: 2D discrete systems, multiplicative noise, stability, robust stability, LMI conditions.

I. INTRODUCTION

Multidimensional (nD) systems have become the subject of interest of recent applied researches [1], [2]. Because the two dimensional systems have a variety of applications including: water stream heating, seismographic data processing, thermal processes, multidimensional digital filtering, process control, image and signal processing [3]–[7]; more focus is paid to the study of this class of systems among multidimensional (nD) systems.

The most commonly useful models in 2D systems are the Roesser model [8] and the Fornasini-Marchesini (FM) model [9].

Based on these state space models, the analysis and synthesis of 2D systems in discrete and continuous frameworks have been the focus of researchers in the area of 2D systems [10]–[12]. For instance, the stability of 2D systems was explored in [13]–[17]. Moreover, much interest has been also devoted to the problem of stabilization of 2D systems [18]–[21].

On the other hand, stochastic systems which are time-dependent processes controlled by brownian motion have attracted significant attention from researchers, because they allow to take into account the random or erratic evolution of the state of these processes [22]–[26]. The stability analysis and control design for 2D discrete stochastic systems have also been broadly studied. For example, the stability and stabilization conditions of uncertain 2D discrete systems with stochastic perturbation was established in [27]. Furthermore, the state estimation of 2D stochastic systems was investigated in [28]. The problem of H_∞ control for 2D stochastic systems has been reported in [29]–[31] which was the extension of the work carried out for 1D stochastic systems [32]–[34]. The stability and stabilization of 2D discrete stochastic Fornasini-Marchesini (FM) second model using the Leibniz-Newton (L-N) formula with free weighting matrices is the focus of the present paper. This approach has been extensively exploited for systems with time-varying delays, see [35]–[38] and recently, for 2D stochastic systems [39]. The main objective in the use of the L-N formula is the resulting flexibility that allows to obtain efficient LMI condition. Unlike the work in [27] where a directe application of the Lyapunov approach results in an LMI condition which was transformed by the use of a Schur complement operation in order to cancel any nonlinear term in the system parameters. Hence, we were motivated to go back to the idea behind the use of the L-N formula which aims to take into account the system equation through the L-N formula instead of its substitution in the Lyapunov function increment. Resulting directely in a formulation without non linear term of the systems parameters. The L-N formula used in this paper contains the stochastic terms. Thus, we propose a new stability and stabilization conditions by employing the L-N formula with free weighting matrices to obtain less conservative results.

This paper is structured as follows. In section II, we will present the problem formulation. In section III, we will elaborate the mean square asymptotic stability of 2D discrete Fornasini-Marchesini (FM) model with multiplicative noise by applying Lyapunov approach combined with the free weighting matrix technique. In section IV, we will deal with the state feedback controller. The stability and stabilization conditions are expressed in terms of Linear Matrix Inequality (LMI). Finally, in order to illustrate the efficiency of the proposed approach, numerical examples are given.

Notations

In this paper, we will use the following notations:

• $sym\{A\}$ stands for the addition of a matrix and its symmetric:

$$sym\{A\} = A^T + A.$$

• $(*)$ designates an asterisk for symmetry-induced terms in a symmetric block matrix.
• $E\{.\}$ represents the expectation operator.
• $diag\{.\}$ stands for a block-diagonal matrix.
• $l_2\{[0,\infty),[0,\infty)\}$ designates the space of square summable sequences on $\{[0,\infty),[0,\infty)\}$.

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II. Problem Formulation

In this section, we will consider the following 2D discrete Fornasini-Marchesini second model with multiplicative noise:

$$\begin{aligned} x(i+1,j+1) &= (A_1+\zeta(i,j)\bar{A}_1)x(i,j+1) \\ &+ (A_2+\zeta(i,j)\bar{A}_2)x(i+1,j) \\ &+ (B_1+\bar{\zeta}(i,j)\bar{B}_1)u(i,j+1) \\ &+ (B_2+\bar{\zeta}(i,j)\bar{B}_2)u(i+1,j), \end{aligned} \quad (1)$$

where $x(i,j)\in\mathbb{R}^n$ is the state vector, $u(i,j)\in\mathbb{R}^m$ is the input vector belonging to $l_2\{[0,\infty),[0,\infty)\}$, and $i,j\in\mathbb{Z}$; A_1, $\bar{A}_1$, A_2, $\bar{A}_2$, B_1, $\bar{B}_1$, B_2, $\bar{B}_2$ are known matrices with appropriate dimensions and $\zeta(i,j)$, $\bar{\zeta}(i,j)$ are independent 2D random variables with zero mean satisfying:

$$\left\{ \begin{array}{lllll} E\{\zeta(i,j)\zeta(m,n)\} &=& 1 & for & (i,j)=(m,n) \\ E\{\zeta(i,j)\zeta(m,n)\} &=& 0 & for & (i,j)\neq(m,n) \end{array}\right. \quad (2)$$

The same applies for $\bar{\zeta}(i,j)$.
The boundary conditions are given by:

$$\begin{array}{c} x(0,0)=x_0 \\ x(-i,j)=0 \\ x(j,-i)=0 \end{array} \qquad \forall\ (j,i)\in\mathbb{N}\times\mathbb{N}^* \quad (3)$$

The problem to be addressed in this paper is the mean square stability analysis as well as the design of a state feedback controller that ensures the mean square stability of the closed loop.
The mean square stability is defined as follows:

Definition 1: [27] For $u(i,j)=0$ and for every initial condition, satisfying $E\{\|x(0,0)\|^2\}<\infty$, the 2D discrete stochastic system (1) is mean square asymptotically stable if

$$\underset{i+j\longrightarrow+\infty}{Lim} E\{\|x(i,j)\|^2\}=0.$$

III. Stability of two dimensional discrete stochastic Fornasini-Marchesini second model

The main objective of this section is to establish efficient stability conditions for two dimensional discrete stochastic FM systems. The result of stability analysis is given by Theorem 1.

Theorem 1: [39] The two dimensional discrete stochastic system is mean square asymptotically stable if there exist symmetric positive definite matrices P_1, P_2, and some free weighting matrices E_1, E_2, E_3 that satisfy

$$\mathcal{B} \;=\; \operatorname{diag}\{-P_1,-P_2,P_1+P_2\}+sym\{T\}<0, \quad (4)$$

where:

$$T \;=\; \left[\begin{array}{ccc} T_{11} & T_{12} & -E_1^T \\ T_{21} & T_{22} & -E_2^T \\ E_3^TA_1 & E_3^TA_2 & -E_3^T \end{array}\right],$$

where the elements of the matrix T are as follows:

$$T_{ij} \;=\; E_i^TA_j+\bar{A}_i^TE_3^T\bar{A}_j, \quad i=1,2,\ j=1,2.$$

Proof: To establish the stability condition (4) for the two dimensional discrete stochastic Fornasini-Marchesini second model, we assume that $u(i,j)=0$.

For simplicity reasons, system (1) is written in the following compact form

$$e_3^T\Theta=\{\left(A_1+\zeta(i,j)\bar{A}_1\right)e_1^T+\left(A_2+\zeta(i,j)\bar{A}_2\right)e_2^T\}\Theta, \quad (5)$$

where:

$$\Theta=\begin{bmatrix} x^\top(i,j+1) & x^\top(i+1,j) & x^\top(i+1,j+1)\end{bmatrix}^\top \quad (6)$$

and $e_1=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top$, $e_2=\begin{bmatrix}0 & 1 & 0\end{bmatrix}^\top$, $e_3=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^\top$. As a consequence, we get:

$$\begin{aligned} x(i,j+1)=e_1^T\Theta,\ x(i+1,j)=e_2^T\Theta, \\ x(i+1,j+1)=e_3^T\Theta \end{aligned}$$

The increment of the Lyapunov functional ΔV is given by

$$\Delta V(i+1,j+1)=\Theta^T\left\{e_3\left[\begin{array}{cc} I & I\end{array}\right]P\left[\begin{array}{c} I \\ I\end{array}\right]e_3^T\right\}\Theta- \left[\begin{array}{c} e_1^T\Theta \\ e_2^T\Theta\end{array}\right]^TP\left[\begin{array}{c} e_1^T\Theta \\ e_2^T\Theta\end{array}\right]$$

where:

$$P=\operatorname{diag}\{P_1,P_2\}$$

Or equivalently

$$\Delta V(i+1,j+1)=\Theta^T\left\{e_3\left(P_1+P_2\right)e_3^T-e_1P_1e_1^T- e_2P_2e_2^T\right\}\Theta$$

From equation (5), one can write

$$\begin{aligned} -e_3^T\Theta+\left(A_1+\zeta(i,j)\bar{A}_1\right)e_1^T\Theta+ \\ \left(A_2+\zeta(i,j)\bar{A}_2\right)e_2^T\Theta=0 \end{aligned}$$

The Leibniz Newton formula is applied for system (5)

$$\begin{aligned} E\Big\{2\Theta^T\left[E_1e_1^T+E_2e_2^T+E_3e_3^T\right]^T\times \\ \left[-e_3^T+\left(A_1+\zeta(i,j)\bar{A}_1\right)e_1^T+\right. \\ \left.\left(A_2+\zeta(i,j)\bar{A}_2\right)e_2^T\right]\Theta\Big\}=0 \end{aligned}$$

where E_1, E_2 and E_3 stand for any matrices of appropriate dimensions.
To ensure the stability of the system under consideration, we have to establish some efficient conditions to ensure that $E\{\Delta V\}<0$ holds.
Roughly speaking, there are two methods to establish such conditions, either replacing $e_3^T\Theta$ by its expression in the increment of the Lyapunov function, or introducing the model equation solely through the Leibniz-Newton formula which is the case of this paper. Indeed, the increment of the Lyapunov function doesn't change if we add null term corresponding to the Leibniz-Newton formula, that is:

$$E\{\Delta V\}=E\{\Delta V\}+\mathcal{D}$$

with:

$$\begin{aligned} \mathcal{D}=E\Big\{2\Theta^T\left[E_1e_1^T+E_2e_2^T+E_3e_3^T\right]^T\times \\ \left[-e_3^T+\left(A_1+\zeta(i,j)\bar{A}_1\right)e_1^T+\left(A_2+\zeta(i,j)\bar{A}_2\right)e_2^T\right]\Theta\Big\} \end{aligned}$$

First we focus on the term $\mathcal{D}$, which we split into 3 terms as follows:

$$\mathcal{D} = \mathcal{D}_{11} + \underbrace{E\left\{2\Theta^T\left[e_1E_1^T + e_2E_2^T\right] \times \left[\zeta(i,j)\left(\bar{A}_1e_1^T + \bar{A}_2e_2^T\right)\right]\Theta\right\}}_{\mathcal{D}_{12}} + \underbrace{E\left\{2\Theta^T\left(e_3E_3^T\right) \times \left[\zeta(i,j)\left(\bar{A}_1e_1^T + \bar{A}_2e_2^T\right)\right]\Theta\right\}}_{\mathcal{D}_{13}}$$

With:

$$\mathcal{D}_{11} = E\left\{2\Theta^T\left[e_1E_1^T + e_2E_2^T + e_3E_3^T\right] \times \left[-e_3^T + A_1e_1^T + A_2e_2^T\right]\Theta\right\}$$

$\mathcal{D}_{12}$ is equal to zero since the terms involved in this expression, that is $x(i+1,j)$ and $x(i,j+1)$, are independent from the stochastic term which is a zero-mean stochastic process. The term $\mathcal{D}_{13}$ contains $x(i+1,j+1) = \Theta^Te_3$ which is correlated with the stochastic term and as a consequence could not be zero.

Therefore, we replace Θ^Te_3 by its expression which gives the following equation:

$$\begin{aligned}\mathcal{D}_{13} &= E\Big\{2\Theta^T\Big[e_1\left(A_1 + \zeta(i,j)\bar{A}_1\right)^T + e_2\left(A_2 + \zeta(i,j)\bar{A}_2\right)^T\Big] \times E_3^T \times \left[\zeta(i,j)\left(\bar{A}_1e_1^T + \bar{A}_2e_2^T\right)\right]\Theta\Big\}\\ &= E\left\{2\left(\Theta^Te_1\bar{A}_1^TE_3^T\bar{A}_1e_1^T\Theta\right)\right\}E\left\{\zeta^T(i,j)\zeta(i,j)\right\} + E\left\{2\left(\Theta^Te_2\bar{A}_2^TE_3^T\bar{A}_1e_1^T\Theta\right)\right\}E\left\{\zeta^T(i,j)\zeta(i,j)\right\} +\\ &\quad E\left\{2\left(\Theta^Te_1\bar{A}_1^TE_3^T\bar{A}_2e_2^T\Theta\right)\right\}E\left\{\zeta^T(i,j)\zeta(i,j)\right\} + E\left\{2\left(\Theta^Te_2\bar{A}_2^TE_3^T\bar{A}_2e_2^T\Theta\right)\right\}E\left\{\zeta^T(i,j)\zeta(i,j)\right\}\end{aligned}$$

Taking into account (2), $\mathcal{D}_{13}$ is rewritten as follows:

$$\begin{aligned}\mathcal{D}_{13} = {} & E\left\{2\left(\Theta^Te_1\bar{A}_1^TE_3^T\bar{A}_1e_1^T\Theta\right)\right\} + E\left\{2\left(\Theta^Te_2\bar{A}_2^TE_3^T\bar{A}_1e_1^T\Theta\right)\right\} +\\ & E\left\{2\left(\Theta^Te_1\bar{A}_1^TE_3^T\bar{A}_2e_2^T\Theta\right)\right\} + E\left\{2\left(\Theta^Te_2\bar{A}_2^TE_3^T\bar{A}_2e_2^T\Theta\right)\right\}\end{aligned}$$

which implies that

$$\begin{aligned}\mathcal{D} = E\{\Theta^T\{sym\{&-e_1E_1^Te_3^T - e_2E_2^Te_3^T - e_3E_3^Te_3^T + e_1E_1^TA_1e_1^T + e_2E_2^TA_1e_1^T + e_3E_3^TA_1e_1^T + e_1E_1^TA_2e_2^T +\\ & e_2E_2^TA_2e_2^T + e_3E_3^TA_2e_2^T + e_1\bar{A}_1^TE_3^T\bar{A}_1e_1^T + e_2\bar{A}_2^TE_3^T\bar{A}_1e_1^T + e_1\bar{A}_1^TE_3^T\bar{A}_2e_2^T + e_2\bar{A}_2^TE_3^T\bar{A}_2e_2^T\}\Theta\}\end{aligned}$$

The increment of the Lyapunov function is then

$$\Delta V = E\{\Theta^T\mathcal{B}\Theta\}$$

where $\mathcal{B}$ is given by equation (4) in Theorem 1.

This ends the proof of the first part of Theorem 1, that is to establish an efficient condition to ensure that the increment of the Lyapunov function is negative. In the second part of the proof, we will show that $E\{\Delta V\} < 0$ implies effectively the mean square asymptotic stability of the system, that is $\lim\limits_{i+j\longrightarrow+\infty} E\{\|x(i,j)\|^2\} = 0$. We just proved that:

$$E\{\Delta V(i+1,j+1)\} = E\{\Theta^T\mathcal{B}\Theta\} < 0,$$

which implies that the increment of the Lyapunov function satisfies:

$$E\{V(i+1,j+1)\} < E\{x^T(i,j+1)P_1x(i,j+1) + x^T(i+1,j)P_2x(i+1,j)\}$$

Given any nonnegative integer θ and taking into account the inequality above, we get easily:

$$\sum_{i=0}^{\theta+1} E\left\{V(i,\theta+1-i)\right\} < \sum_{i=0}^{\theta+1} E\left\{\|x(i,\theta-i)\|_{P_2}\right\} + \sum_{i=0}^{\theta+1} E\left\{\|x(i-1,\theta-i+1)\|_{P_1}\right\}. \quad (7)$$

Notice that

$$\sum_{i=0}^{\theta+1} E\left\{\|x(i-1,\theta-i+1)\|_{P_1}\right\} = E\left\{\|x(-1,\theta+1)\|_{P_1}\right\} + \sum_{i=1}^{\theta+1} E\left\{\|x(i-1,\theta-i+1)\|_{P_1}\right\},$$

and since we assume that $x(i,j) = 0$, for every $i < 0$ or $j < 0$, we get

$$\sum_{i=0}^{\theta+1} E\left\{\|x(i-1,\theta-i+1)\|_{P_1}\right\} = \sum_{i=0}^{\theta} E\left\{\|x(i,\theta-i)\|_{P_1}\right\}.$$

Similarly we have

$$\sum_{i=0}^{\theta+1} E\left\{\|x(i,\theta-i)\|_{P_2}\right\} = \sum_{i=0}^{\theta} E\left\{\|x(i,\theta-i)\|_{P_2}\right\}.$$

Taking account of (7) we get finally that

$$\sum_{i=0}^{\theta+1} E\left\{V(i,\theta+1-i)\right\} < \sum_{i=0}^{\theta} E\left\{V(i,\theta-i)\right\}.$$

The condition above implies that

$$\sum_{i=0}^{\theta+1} E\left\{V(i,\theta+1-i)\right\} < \infty$$

and as a consequence

$$E\left\{V(i,j)\right\} \longrightarrow 0, \text{ when } i+j=\theta \longrightarrow +\infty.$$

The proof is completed.

Remark For 2-D systems, the boundary conditions are, in general, taken as follows:

$$x(i,0) = \varphi(i), x(0,i) = \psi(i) \quad \forall\ i \in \mathbb{N}$$

with φ and ψ two bounded functions.

The functions φ and ψ must satisfy one of the following condition:

$$1) \quad \varphi(i) \underset{i\to 0}{\longrightarrow} 0, \qquad \psi(i) \underset{i\to 0}{\longrightarrow} 0 \quad (8)$$

$$2)\ \exists(L,\ M) \in \mathbb{N}\times\mathbb{N} : \psi(i) = 0, \forall i \geqslant L, \varphi(j) = 0, \forall j \geqslant M \quad (9)$$

However, we adopt a third convention to set the boundary condition as in (3).
The reason is related to the convergence proof where we need to show

$$\sum_{i=0}^{\theta} E\left\{V(i,\theta+1-i)\right\} \longrightarrow 0 \text{ when } \theta \longrightarrow +\infty.$$

If the initial conditions are chosen according to (8) or (9) then one has to show that

$$\sum_{i=0}^{\theta} V(i,\theta-i) - V(0,\theta) - V(\theta,0) \underset{\theta\to+\infty}{\longrightarrow} 0$$

In the proof we need to use the model equation to take into account condition (3) whereas the boundary conditions in (8) or (9) do not comply necessarily with the model equation and removing them yields a quantity that vanishes to zero. ■

IV. Stabilization of two dimensional Discrete Stochastic Fornasini-Marchesini second model

In this section, we will deal with the stabilization of system (5) by a state feedback controller. More exactly, we have to find a state feedback $u(i,j) = Kx(i,j)$ such that the closed-loop system of (5) is mean-square asymptotically stable, where $K \in \mathbb{R}^{m\times n}$ is a constant gain matrix to be determined. The following closed-loop system can be given by applying the state feedback controller to system (5).

$$\begin{aligned} e_3^T\Theta = \{&[(A_1+B_1K)+\zeta(i,j)\bar{A}_1+\bar{\zeta}(i,j)\bar{B}_1K]e_1^T \\ &+[(A_2+B_2K)+\zeta(i,j)\bar{A}_2+\bar{\zeta}(i,j)\bar{B}_2K]e_2^T]\}\Theta \end{aligned} \quad (10)$$

Where Θ is defined in (6).
The result on the stabilization is summarized by Theorem 2.

Theorem 2: The two dimensional stochastic Fornasini-Marchesini second model is stabilizable through state-feedback $u(i,j) = Kx(i,j)$ if there exist a matrix Y and symmetric positive definite matrices X, $\bar{P}_1$, $\bar{P}_2$ such that:

$$\mathcal{B} = \begin{bmatrix} -\bar{P}_1 & 0 & \mathcal{B}_{1,3} & X^T\bar{A}_1^T & Y^T\bar{B}_1^T \\ * & -\bar{P}_2 & \mathcal{B}_{2,3} & X^T\bar{A}_2^T & Y^T\bar{B}_2^T \\ * & * & \mathcal{B}_{3,3} & 0 & 0 \\ * & * & * & -\frac{1}{2}X & 0 \\ * & * & * & * & -\frac{1}{2}X \end{bmatrix} < 0 \quad (11)$$

with:

$$\begin{aligned} \mathcal{B}_{1,3} &= X^TA_1^T + Y^TB_1^T \\ \mathcal{B}_{2,3} &= X^TA_2^T + Y^TB_2^T \\ \mathcal{B}_{3,3} &= -2X + \bar{P}_1 + \bar{P}_2 \end{aligned}$$

where: $Y = KX$, $\bar{P}_1 = X^TP_1X$, $\bar{P}_2 = X^TP_2X$, $X = E_3^{-1}$
If the above condition is satisfied, then a state feedback control law is given by $u(i,j) = Kx(i,j)$ with $K = YX^{-1}$.

Proof: Similary to the stability analysis, we can check the stabilization following similar arguments.
The increment of the Lyapunov functional ΔV is given by

$$\begin{aligned} \Delta V(i+1,j+1) = \Theta^T\{&e_3(P_1+P_2)e_3^T - e_1P_1e_1^T - \\ &e_2P_2e_2^T\}\Theta \end{aligned}$$

From equation (10), one can write:

$$\begin{aligned} &-e_3^T\Theta + [(A_1+B_1K)+\zeta(i,j)\bar{A}_1+\bar{\zeta}(i,j)\bar{B}_1K]e_1^T\Theta \\ &\quad+[(A_2+B_2K)+\zeta(i,j)\bar{A}_2+\bar{\zeta}(i,j)\bar{B}_2K]e_2^T\Theta = 0 \end{aligned}$$

The Leibniz Newton formula is applied for system (10)

$$\begin{aligned} &E\{2\Theta^T[[E_1e_1^T+E_2e_2^T+E_3e_3^T]^T \times \\ &([-e_3^T+(A_1+B_1K)+\zeta(i,j)\bar{A}_1+\bar{\zeta}(i,j)\bar{B}_1K]e_1^T + \\ &[(A_2+B_2K)+\zeta(i,j)\bar{A}_2+\bar{\zeta}(i,j)\bar{B}_2K]e_2^T)]\Theta\} = 0 \end{aligned}$$

Let us compute this formula,

$$\begin{aligned} &\mathcal{F} = E\left\{2\Theta^T\left[E_1e_1^T+E_2e_2^T+E_3e_3^T\right]^T \times\right. \\ &\left[\left(-e_3^T+(A_1+B_1K)+\zeta(i,j)\bar{A}_1+\bar{\zeta}(i,j)\bar{B}_1K\right)e_1^T+\right. \\ &\left.\left.\left((A_2+B_2K)+\zeta(i,j)\bar{A}_2+\bar{\zeta}(i,j)\bar{B}_2K\right)e_2^T\right]\Theta\right\} \\ &= \mathcal{F}_1+\mathcal{F}_2+\mathcal{F}_3 \end{aligned}$$

with:

$$\begin{aligned} \mathcal{F}_1 = E\left\{2\Theta^T\left[e_1E_1^T+e_2E_2^T+e_3E_3^T\right]\times\right. \\ \left.\left[-e_3^T+(A_1+B_1K)e_1^T+(A_2+B_2K)e_2^T\right]\Theta\right\} \end{aligned}$$

$$\begin{aligned} &\mathcal{F}_2 = E\left\{2\Theta^T\left[e_1E_1^T+e_2E_2^T\right]\times\left[\zeta(i,j)\left(\bar{A}_1e_1^T+\right.\right.\right. \\ &\left.\left.\left.\bar{A}_2e_2^T\right)+\bar{\zeta}(i,j)\left(\bar{B}_1Ke_1^T+\bar{B}_2Ke_2^T\right)\right]\Theta\right\} \end{aligned}$$

$$\begin{aligned} &\mathcal{F}_3 = E\left\{2\Theta^T\left(e_3E_3^T\right)\times\left[\zeta(i,j)\left(\bar{A}_1e_1^T+\bar{A}_2e_2^T\right)+\right.\right. \\ &\left.\left.+\ \bar{\zeta}(i,j)\left(\bar{B}_1Ke_1^T+\bar{B}_2Ke_2^T\right)\right]\Theta\right\} \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\mathcal{F}_3 = E\{2\Theta^T[(e_1(A_1+B_1K)^T+e_1\bar{A}_1{}^T\zeta^T(i,j)+ \\ &e_1K^T\bar{B}_1^T\bar{\zeta}^T(i,j))+(e_2(A_2+B_2K)^T+e_2\bar{A}_2{}^T\zeta^T(i,j)+ \\ &e_2K^T\bar{B}_2^T\bar{\zeta}^T(i,j))]\times[\zeta(i,j)(\bar{A}_1e_1^T+\bar{A}_2e_2^T)+ \\ &\bar{\zeta}(i,j)(\bar{B}_1Ke_1^T+\bar{B}_2Ke_2^T)]\Theta\} \end{aligned}$$

The conditions on $\zeta(i,j)$ and $\bar{\zeta}(i,j)$ allow to rewrite $\mathcal{F}_3$. Adding the Leibniz-Newton formula to the increment of Lyapunov functional $E\{\Delta V\}$, and looking for a condition which ensures that $E\{\Delta V\} < 0$ yields the condition $E\{\Theta^\top\mathcal{M}\Theta\} < 0$ with

$$\mathcal{M} = \begin{bmatrix} -P_1 & 0 & 0 \\ 0 & -P_2 & 0 \\ 0 & 0 & P_1+P_2 \end{bmatrix} + sym(R) < 0$$

and

$$\begin{aligned} R = &\begin{bmatrix} E_1^\top \\ E_2^T \\ E_3^\top \end{bmatrix}\begin{bmatrix} (A_1+B_1K) & (A_2+B_2K) & -I \end{bmatrix} + \\ &\begin{bmatrix} \bar{A}_1^\top \\ \bar{A}_2^\top \\ 0 \end{bmatrix} E_3^\top \begin{bmatrix} \bar{A}_1 & \bar{A}_2 & 0 \end{bmatrix} + \\ &\begin{bmatrix} K^T\bar{B}_1^\top \\ K^\top\bar{B}_2^\top \\ 0 \end{bmatrix} E_3^\top \begin{bmatrix} \bar{B}_1K & \bar{B}_2K & 0 \end{bmatrix} \end{aligned}$$

In order to get a solution for the state feedback, one has to set $E_1 = E_2 = E_3$. Unfortunately, this option makes

the condition too conservative and we had to remove any dependence between the pair (E_1, E_2) and E_3. Removing completely E_1 and E_2 gives a nice result and provides a condition with less conservativeness. As a consequence we assume that: $E_1 = E_2 = 0$ and $E_3 = E_3^T > 0$
Taking into account the assumption above for matrix $\mathcal{M}$ and performing a Schur complement, one gets:

$$C = \begin{bmatrix} -P_1 & 0 & C_{1,3} & \bar{A}_1^T & K^T \bar{B}_1^T \\ * & -P_2 & C_{2,3} & \bar{A}_2^T & C_{2,5} \\ * & * & C_{3,3} & 0 & 0 \\ * & * & * & -\frac{1}{2}E_3^{-1} & 0 \\ * & * & * & * & -\frac{1}{2}E_3^{-1} \end{bmatrix} < 0$$

with:

$$\begin{aligned} C_{1,3} &= (A_1 + B_1 K)^T E_3, C_{2,3} = (A_2 + B_2 K)^T E_3 \\ C_{2,5} &= K^T \bar{B}_2^T, C_{3,3} = -2E_3 + P_1 + P_2 \end{aligned}$$

Then, proceeding by pre multiplication by $diag\left(E_3^{-T}, E_3^{-T}, E_3^{-T}, I, I\right)$ and post multiplication by $diag\left(E_3^{-1}, E_3^{-1}, E_3^{-1}, I, I\right)$ yields the matrix $\mathcal{B}$ in Theorem 2, which ends the proof. ■

V. Example

To highlight the effectiveness of the proposed method, numerical examples will be given. Consider 2D discrete stochastic FM system in (1) with the following coefficient matrices:

$$A_1 = \begin{bmatrix} -0.61 & 0.19 \\ 0.14 & 0.49 \end{bmatrix}, A_2 = \begin{bmatrix} -0.8 & 0.39 \\ -0.61 & 0.33 \end{bmatrix},$$

$$\bar{A}_1 = \begin{bmatrix} 0 & 0.3 \\ 0.3 & 0 \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \bar{B}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \bar{B}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Figures 1 and 2 represent the two state variables of the open-loop system using the following boundary condition $x_0 = \begin{bmatrix} 1 & -1 \end{bmatrix}'$. We can deduce that the open-loop system is unstable.

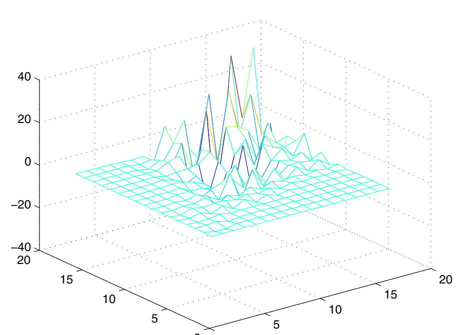

Fig. 1: Evolution of the first state component of open-loop system

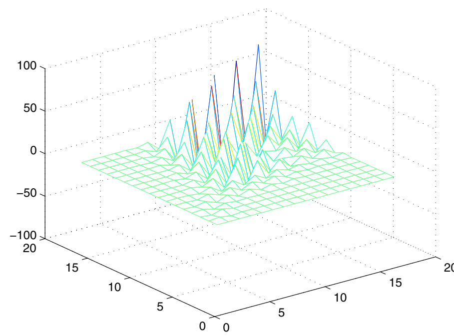

Fig. 2: Evolution of the second state component of open-loop system

To stabilize this system, we have to design a state-feedback control law such that the closed-loop system is mean-square asymptotically stable.
By solving LMI (11), we can find a feasible solution with $X = \begin{bmatrix} 4.3139 & -0.0459 \\ -0.0459 & 5.1165 \end{bmatrix}$, $\bar{P}_1 = \begin{bmatrix} 2.1340 & 0.8761 \\ 0.8761 & 3.8863 \end{bmatrix}$, $\bar{P}_2 = \begin{bmatrix} 3.4237 & -0.9245 \\ -0.9245 & 3.0045 \end{bmatrix}$, $K = \begin{bmatrix} 0.5147 & -0.2694 \end{bmatrix}$.

Figures 3 and 4 represent the two state variables of the closed-loop system which converge to zero under the same boundary condition.

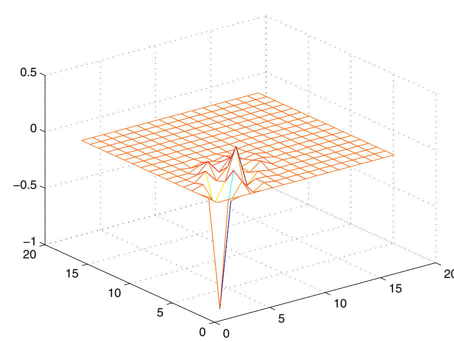

Fig. 3: Evolution of the first state component of closed-loop system

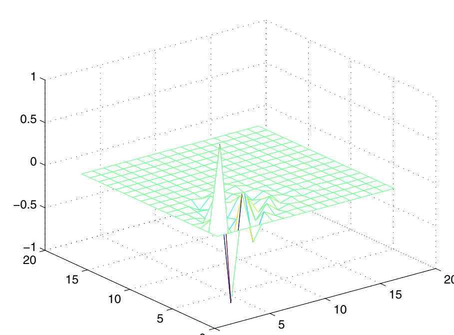

Fig. 4: Evolution of the second state component of closed-loop system

VI. Conclusion

To sum up, the mean square asymptotic stablilty and stabilization of two dimensional Fornasini-Marchesini (FM) second model with multiplicative noise are the focus of this paper. A new mean square stability condition expressed in terms of LMI has been established using Lyapunov approach combined with the Leibniz-Newton formula for the purpose of reducing the conservatism. The obtained condition is then used to derive a controller for mean square stabilizability by state feedback for this class of systems. To highlight the effectiveness of our method, numerical examples are given.

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