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An Approach to linear state Signal Shaping by quadratic Model Predictive Control

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Abstract— **A new approach for using a standard quadratic model predictive control (MPC) for state signal shaping is proposed. The control strategy embeds a property through a linear difference equation into the standard MPC structure. An application example for calculating the reference current of an active power filter device to achieve load compensation on a distribution grid integrated with renewable energy is given.**

I. INTRODUCTION

If a model of the plant is available, model predictive control (MPC) algorithms are known to be well suited for the solution of reference following control problems. A quadratic cost function together with a linear model results in a numerically efficient solvable convex optimization problem, also for cases with linear constraints on states and inputs, [1]. But there are applications, where an exact following of a given reference in time is not essential, as long as the behaviour does not escape a certain shape in state space at any time. Examples vary from robotics, e.g. the coordinated tracking problem, where a coordinated control scheme is used for multiple robots to maintain a formation, [2], process engineering to electrical energy grids. Using MPC algorithms with classical parameter settings, i.e. diagonal weighting matrices, for these kind of problems leads to solutions with exact timing which might be relaxed without making the desired performance worse, see e.g. applications for power converters, [3].

Problems of this kind are well known in the area of signal processing, often referred to as signal shaping, see e.g. [4]. Signal shaping problems can be solved by various approaches from phased-locked loops to iterative learning techniques, [5].

The paper shows that signal shaping problems using a representation of the desired shape by linear difference equations in terms of the state variables are equivalent to standard MPC problems with quadratic cost functions. These are given generically by non-diagonal weighting matrices with a custom factorization a priori known from the structure of the desired signal shape. This opens the door to efficient implementations for various applications, one of these is outlined in detail.

This paper is organized as follows. Section II describes the most relevant aspects of the MPC theory framework. In section III, a concept of classification of signal shapes is given. Section IV integrates the main features and capabilities of the proposed control strategy into the MPC structure. In section V an application example in the field of power quality compensation for renewable energy grid integration is presented. Finally, section VI gives a summary of the proposed control strategy and draws conclusions.

II. MODEL PREDICTIVE CONTROL

The behaviour of a linear system with fixed sampling time t_s and samples taken at times $t = kt_s$ with $k = 0, 1, 2, \ldots$ can be modelled by a discrete state space model

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ refers to the state vector, $\mathbf{u} \in \mathbb{R}^m$ is the input vector, $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the system matrix, and $\mathbf{B} \in \mathbb{R}^{n\times m}$ is the input matrix. Throughout the paper, it is assumed that the state variables could be accessed by the controller, either by direct measurements or good approximations from an observer, [1].

With the state reference $\mathbf{r} \in \mathbb{R}^n$, prediction horizon $H_p \in \mathbb{N}$ and the input horizon $H_u \in \mathbb{N}$, the vectors

$$\mathbf{X}(k) = \begin{pmatrix} \mathbf{x}(k+1) \\ \mathbf{x}(k+2) \\ \vdots \\ \mathbf{x}(k+H_p) \end{pmatrix} \in \mathbb{R}^{H_p n}, \tag{2}$$

$$\Xi(k) = \begin{pmatrix} \mathbf{r}(k+1) \\ \mathbf{r}(k+2) \\ \vdots \\ \mathbf{r}(k+H_p) \end{pmatrix} \in \mathbb{R}^{H_p n}, \tag{3}$$

$$\Delta\mathbf{U}(k) = \begin{pmatrix} \mathbf{u}(k) & - & \mathbf{u}(k-1) \\ \mathbf{u}(k+1) & - & \mathbf{u}(k) \\ & \vdots & \\ \mathbf{u}(k+H_u) & - & \mathbf{u}(k+H_u-1) \end{pmatrix} \in \mathbb{R}^{(H_u+1)m}, \tag{4}$$

give the future state sequence, reference trajectory and input changes, which are used together with the positive semi-definite matrix $\mathbf{Q} \in \mathbb{R}^{H_p n\times H_p n}$ and the positive definite matrix $\mathbf{R} \in \mathbb{R}^{(H_u+1)m\times(H_u+1)m}$ to formulate the standard quadratic cost function, [1]

$$J(k) = \|\mathbf{X}(k) - \Xi(k)\|^2_{\mathbf{Q}} + \|\Delta\mathbf{U}(k)\|^2_{\mathbf{R}} \tag{5}$$

of the optimization problem

$$\min_{\Delta\mathbf{U}(k)} J(k) \tag{6}$$

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leading to an optimal control input change sequence $\Delta\mathbf{U}^{\star}(k)$ as solution. This defines the input $\mathbf{u}(k)$ that is given to the plant, as shown in the standard MPC control loop of Fig. 1.

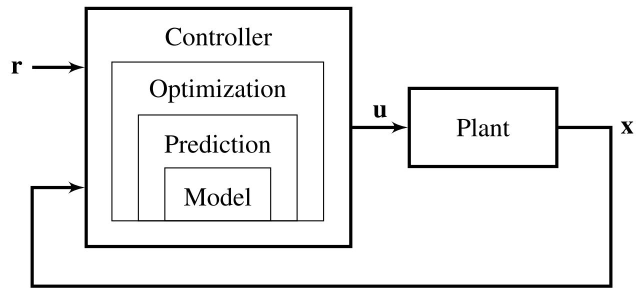

Fig. 1. Model Predictive Controller: Classical Scheme

For most applications, the weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ are chosen as diagonal matrices with positive entries, which stand for costs of an error in an individual state or the costs of an input change, resp. Moreover, it is often not necessary to use the option to vary these costs for each time step, which reduces the number of parameters used for tuning to one for each state and input, i.e. in sum $n+m$, which trade off between tracking error and control effort.

With decompositions

$$\mathbf{Q} = \mathbf{S_Q}'\mathbf{S_Q}\,, \tag{7}$$

$$\mathbf{R} = \mathbf{S_R}'\mathbf{S_R}\,, \tag{8}$$

of the weighting matrices, such as Cholesky factorization, the cost function (5) can be written as, [1]

$$J(k) = \left\| \begin{pmatrix} \mathbf{S_Q}(\mathbf{X}(k) - \Xi(k)) \\ \mathbf{S_R}\Delta\mathbf{U}(k) \end{pmatrix} \right\|^2 . \tag{9}$$

With the lifted system matrix

$$\Theta = \begin{pmatrix} \mathbf{B} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{AB}+\mathbf{B} & \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{A}^2\mathbf{B}+\mathbf{AB}+\mathbf{B} & \mathbf{AB}+\mathbf{B} & \mathbf{B} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^{H_u-1}\mathbf{A}^i\mathbf{B} & \cdots & \cdots & \cdots & \mathbf{B} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{i=0}^{H_p-1}\mathbf{A}^i\mathbf{B} & \sum_{i=0}^{H_p-2}\mathbf{A}^i\mathbf{B} & \cdots & \cdots & \sum_{i=0}^{H_p-H_u}\mathbf{A}^i\mathbf{B} \end{pmatrix},$$

a condition for the solution of the optimization problem (6) is given in terms of a linear system of equations

$$\begin{pmatrix} \mathbf{S_Q}\Theta \\ \mathbf{S_R} \end{pmatrix} \Delta\mathbf{U}(k) = \begin{pmatrix} \mathbf{S_Q}\mathbf{E}(k) \\ 0 \end{pmatrix}, \tag{10}$$

where the difference

$$\mathbf{E}(k) = \Xi(k) - \begin{pmatrix} \mathbf{A} \\ \vdots \\ \mathbf{A}^{H_p} \end{pmatrix} \mathbf{x}(k) - \begin{pmatrix} \mathbf{B} \\ \vdots \\ \sum_{i=0}^{H_p-1}\mathbf{A}^i\mathbf{B} \end{pmatrix} \mathbf{u}(k-1) \tag{11}$$

between the reference trajectory Ξ and the free response of the system at time step k depends on the current state $\mathbf{x}(k)$ and the past input $\mathbf{u}(k-1)$.

Numerical advantages result from the constant left hand side of (10), i.e. at each instant k only the difference $\mathbf{E}(k)$ is computed from (11) as right hand side to solve (10) in the least-square sense, e.g. using a QR algorithm, [1].

Implementations of MPC controllers use the moving horizon principle to set an update time strategy for the controlled input signal. Only the first element of the computed optimal input vector sequence $\Delta\mathbf{U}^{\star}(k)$ is given to the system, then the state values are measured or observed, resp., and fed back to the controller. This whole process is executed in each time step k, meaning that the MPC is generating a full new optimal input sequence in each sample time, to only process the first value of the sequence and discard the rest, [1].

III. LINEAR SIGNAL SHAPING

This section is dedicated to introduce a concept for classification of signal shapes, which is given by the next Definition.

Definition 1. The set

$$\mathscr{X}_{\mathbf{V}} = \{\mathbf{x}(1), \mathbf{x}(2), \ldots \,|\, \mathbf{V} \begin{pmatrix} \mathbf{x}(k+1) \\ \vdots \\ \mathbf{x}(k+T) \end{pmatrix} = \mathbf{0}\ \forall k = 0, 1, \ldots\} \tag{12}$$

is called a *linear shape class* of a discrete-time vector signal defined by the matrix $\mathbf{V} \in \mathbb{R}^{s\times nT}$.

Example 1: The shape classes of the exponential scalar signals $x(k) = x_0\lambda^k$ can be given, considering $n = 1$, $T = 2$, and $s = 1$, by the vector

$$\begin{pmatrix} \lambda & -1 \end{pmatrix} \in \mathbb{R}^{1\times 2},$$

which leads to signals of decreasing amplitude for $|\lambda| < 1$ and constant for $\lambda = 1$.

Example 2: A two-element vector signal with the property that x_1 is the finite difference of x_2 and its previous value could be represented with $n = 2$, $T = 2$, and $s = 1$ by the vector

$$\begin{pmatrix} 0 & -1 & -1 & 1 \end{pmatrix} \in \mathbb{R}^{1\times 4}.$$

Example 3: A two-element vector signal with the properties that x_1 is exponentially decreasing with $\lambda = \frac{1}{2}$ and constant x_2 could be represented with $n = 2$, $T = 2$, and $s = 2$ by the matrix

$$\mathbf{V} = \begin{pmatrix} \frac{1}{2} & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \in \mathbb{R}^{2\times 4},$$

which can be seen by inserting in the definition

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_1(k+2) \\ x_2(k+2) \end{pmatrix} = \mathbf{0}$$

$$\begin{pmatrix} x_1(k+1) - 2x_1(k+2) \\ x_2(k+1) - x_2(k+2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is equivalent to the autonomous state space model

$$x_1(k+1) = 0.5 x_1(k) \,,$$
$$x_2(k+1) = x_2(k) \,.$$

Representing a desired behaviour of a system by a class of state signal shapes given by a matrix $\mathbf{V}$ has advantages for the solution of the corresponding control problem, which will be shown in the next section.

IV. STATE SHAPING BY MPC

If a desired behaviour of a dynamical system lies in a special signal shape class given by its kernel representation (12), it is useful to investigate control algorithms which ensure that state signals are driven to this class. It is shown in this section first, to which optimization problems this corresponds and next, that these belong to the class of linear MPC problems.

Proposition 1. All elements of the shape class $\mathbf{V} \in \mathbb{R}^{s \times nT}$ will be for $T = H_p$ minimizers of the optimization problem

$$\min_{\mathbf{X}(k)} \sum (\mathbf{V}\,\mathbf{X}(k))^2 \,. \tag{13}$$

Proof. Equation (12) ensures that $\mathbf{V}\,\mathbf{X}(k) = \mathbf{0}$, thus the expression $(\mathbf{V}\,\mathbf{X}(k))^2 = \mathbf{0}$ and the sum over a zero vector structurally gives zero as value of the cost function. This result is the minimum because the cost function is a sum of squares (SOS). □

Definition 2. For any integer $H_p \geq T$, let $\mathbf{V}_j = \mathbf{V}_{:,(j-1)n+1:jn}$, the band matrix

$$\mathbf{P}_V = \begin{pmatrix} \mathbf{V}_1 & \mathbf{V}_2 & \cdots & \mathbf{V}_T & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_1 & \mathbf{V}_2 & \cdots & \mathbf{V}_T & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{V}_1 & \mathbf{V}_2 & \cdots & \mathbf{V}_T \end{pmatrix} \in \mathbb{R}^{p_1 \times p_2}, \tag{14}$$

with $p_1 = s(H_p - T + 1)$ and $p_2 = nH_p$, can be computed by shifting the shape class matrix $\mathbf{V}$ of (12) for integer multiples of n columns with the help of matrices of zeros $\mathbf{0} = \{0\}^{s \times n}$.

Remark: For $H_p = T$, the matrix $\mathbf{P}_V = \mathbf{V}$ is trivial.

Proposition 2. If a MPC optimization problem (6) with prediction horizon $H_p \geq T$, reference $\Xi(k) = \mathbf{0}$, weighting matrices $\mathbf{R} = \mathbf{0}$, and $\mathbf{Q} = \mathbf{P}_V'\mathbf{P}_V$ has zero cost $J(k) = 0$, the state sequence $\mathbf{x}(k+1), \mathbf{x}(k+2), \ldots$ is in the shape class given by the matrix $\mathbf{V}$.

Proof. For the cost function (5) of (6), the following holds:

$$J(k) = \|\mathbf{X}(k) - \Xi(k)\|^2_{\mathbf{Q}} + \|\Delta\mathbf{U}(k)\|^2_{\mathbf{R}} = \|\mathbf{X}(k)\|^2_{\mathbf{P}_V'\mathbf{P}_V} =$$
$$= \mathbf{X}'(k)\mathbf{P}_V'\mathbf{P}_V\mathbf{X}(k) = (\mathbf{P}_V\mathbf{X}(k))^2 = \sum_{i=0}^{H_p-T-2} (\mathbf{V}\mathbf{X}(k+i))^2$$

Zero cost function value can only be reached if each of the elements $\mathbf{V}\mathbf{X}(k+i) = 0$, for $i = 0, \ldots, H_p - T - 2$, which ensures that $\mathbf{X}(k)$ is in the shape class $\mathbf{V}$ given by equation (12). □

Remark: For $\mathbf{R} \neq \mathbf{0}$, having a zero cost $J(k) = 0$ will be practically impossible. This would mean that with no input changes (i.e. constant input signal for the whole H_u) the shape class given by $\mathbf{V}$ has to be followed perfectly, because both parts of the cost function are SOS. Thus the minimum $J(k)$ will rely on a tradeoff between the cost of control effort introduced by $\mathbf{R}$ and the original cost of control error of not being in the shape class. Note that the lower the control error the closer it is to the shape class, since it is a sum of least squares. In this scenario, the matrix $\mathbf{R}$ tuning needs to reflect the input restrictions appropriately while prioritizing the control error.

Example 4: Continuing Example 3 with shape class matrix

$$\mathbf{V} = \begin{pmatrix} \frac{1}{2} & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \in \mathbb{R}^{2 \times 4} \,,$$

for a prediction horizon $H_p = 3 > T = 2$ the band matrix

$$\mathbf{P}_V = \begin{pmatrix} \frac{1}{2} & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \in \mathbb{R}^{4 \times 6} \,,$$

gives the symmetric but non-diagonal weighting matrix

$$\mathbf{Q} = \mathbf{P}_V'\mathbf{P}_V = \begin{pmatrix} \frac{1}{4} & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{5}{4} & 0 & -\frac{1}{2} & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} \,.$$

Proposition 2 gives the case of ideal following the shape, whereas for a controllable pair $(\mathbf{A}, \mathbf{B})$, the MPC controller in general only forces the state signals to be close to the shape class given by $\mathbf{V}$. For $\Theta'\mathbf{Q}\Theta + \mathbf{R} > 0$, the solution of the optimization problem, i.e. the optimal input sequence, is unique, [1].

V. APPLICATION EXAMPLE

Renewable energy generation is non-dispatchable, distributed, and intermittent with high fluctuations by nature. Therefore, the current increasing share of renewable energy sources leads to voltage and frequency fluctuations at the interconnected distribution grid that affect the power quality, [6].

Active power filters (APF) come as a solution that can compensate unbalanced, harmonic and reactive components of the load currents under various supply conditions. In this context, selecting an appropriate control strategy for these devices is critical to get the desired compensation characteristics, [7].

The most common power control theories for APF implementation are: Instantaneous Symmetrical Component (ISC), Instantaneous Reactive Power (IRP) and Synchronous Reference Frame (SRF), [8]. All these strategies involve several transformations on each iteration, and even trigonometric calculations, which increases the computational effort, [8]. Moreover, to control either voltages or currents, the controller needs to alternate between different operation modes, [9]. Note that the introduction of this example is meant as an application reference of the theory developed.

A. Control principle

The aim of an APF is to compensate the distortions on the system signals, i.e. the voltage and current signals are perfect fundamental harmonic functions. Solving the initial value problem of the homogeneous ordinary second order differential equation (ODE)

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}+(2\pi f)^2x(t)=0, \tag{15}$$

leads to a sinusoidal state signal $x(t)$ with a specific frequency f, zero offset, and an arbitrary amplitude, [10]. We will show that the ODE (15) can also be reversely used to define the characteristics of harmonic oscillation needed as reference signals for the APF control problem.

B. Linear difference equation

In order to transform the property (15) into a linear shape class of the form of (12), it needs to be translated into discrete-time form. Using forward numerical differentiation with a step size equal to the sample time t_s and an accuracy of $O(t_s^2)$, [11], the second derivative of the differential equation has the discrete time approximation

$$\ddot{x}(k)\approx\frac{2x(k)-5x(k+1)+4x(k+2)-x(k+3)}{t_s^2}, \tag{16}$$

that when replaced in (15) starting at $k+1$ (first future state value), leads to a linear shape class with $s=1$, $n=1$, and $T=4$, defined by the row vector

$$\mathbf{v}=\frac{1}{t_2^2}\begin{pmatrix} 2+4\pi^2f^2t_s^2 & 5 & 4 & -1 \end{pmatrix}\in\mathbb{R}^{1\times 4}. \tag{17}$$

C. State space model representation

In order to analyze the control principle derived from (15), a simplified two state model of a three node single-phase system (supply, load, and compensator) is introduced, as shown in Fig. 2.

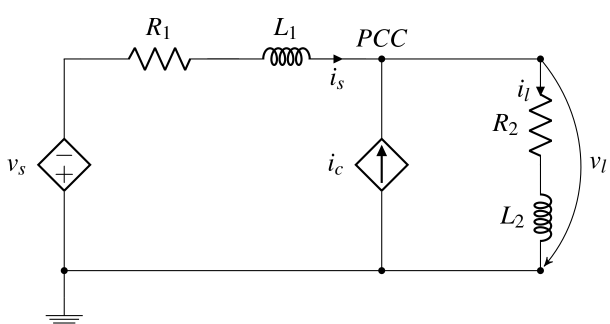

Fig. 2. Equivalent circuit of a three node single-phase grid with an ideal compensator

In Fig. 2, the compensator is an ideal controlled current source i_c, steered by the control input. The supply voltage v_s is an ideal controlled voltage source, capable of introducing distorted voltages. The resistance R_1 and inductance L_1 correspond to the transmission between supply voltage and the point of common coupling PCC. The resistance R_2 and inductance L_2 are modeling the load with voltage v_l. Currents i_s and i_l correspond to the supply and load respectively. The focus of the compensation will be the load states v_l and i_l.

The model has the state vector:

$$\mathbf{x}\in\mathbb{R}^n=\begin{bmatrix} v_l & i_l & i_s & \frac{\mathrm{d}i_c}{\mathrm{d}t} & v_s \end{bmatrix}', \tag{18}$$

with the disturbance $d(t)=\frac{\mathrm{d}v_s}{\mathrm{d}t}\in\mathbb{R}^z$, and the control input $u(t)=\frac{\mathrm{d}^2i_c}{\mathrm{d}t^2}\in\mathbb{R}^m$. The model is given as

$$\frac{\mathrm{d}x_1}{\mathrm{d}t}=\frac{R_1R_2L_2-R_2^2L_1}{(L_1+L_2)^2}x_2+\frac{R_1^2L_2-R_1R_2L_1}{(L_1+L_2)^2}x_3+\frac{R_2L_1^2+R_1L_2^2}{(L_1+L_2)^2}x_4 +\frac{R_2L_1-R_1L_2}{(L_1+L_2)^2}x_5+\frac{L_1L_2}{L_1+L_2}u(t)+\frac{L_2}{L_1+L_2}d(t), \tag{19}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t}=\frac{-R_2}{L_1+L_2}x_2+\frac{-R_1}{L_1+L_2}x_3+\frac{L_1}{L_1+L_2}x_4+\frac{1}{L_1+L_2}x_5, \tag{20}$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t}=\frac{-R_2}{L_1+L_2}x_2+\frac{-R_1}{L_1+L_2}x_3+\frac{-L_2}{L_1+L_2}x_4+\frac{1}{L_1+L_2}x_5. \tag{21}$$

$$\frac{\mathrm{d}x_4}{\mathrm{d}t}=u(t), \tag{22}$$

$$\frac{\mathrm{d}x_5}{\mathrm{d}t}=d(t). \tag{23}$$

From Fig. 2, with the use of Kirchhoff's circuit laws, (20) and (21) can be derived. However, to avoid an output with direct feed-through for v_l, additional states in (22) and (23) are declared, so $x_1=v_l$ can be expressed as a state in (19) that can be targeted by the controller. This also leads to the reformulation of the original input i_c and disturbance v_s. Note that for a real implementation the actual input is i_c, which could be calculated as $i_c=i_l-i_s$.

The continuous-time state space model is

$$\dot{\mathbf{x}}(t)=\mathbf{A}_c\mathbf{x}(t)+\mathbf{b}_cu(t)+\mathbf{e}_cd(t), \tag{24}$$

where the elements of $\mathbf{A}_c\in\mathbb{R}^{n\times n},\mathbf{b}_c\in\mathbb{R}^{n\times m},\mathbf{e}_c\in\mathbb{R}^{n\times z}$, correspond to the coefficients from the state equations in (19) to (23). In the model, $m=1$ refers to the number of inputs, $n=5$ is the number of states, and $z=1$ is the number of disturbances. This state space model is discretized temporally using the zero-order-hold method and sampling time t_s, such that $t=kt_s$.

Remark: Since the model described in this section is a disturbed discrete state space model, the MPC needs to consider a prediction for the future values of the disturbance $d(t)$ for the whole H_p, which typically is considered to be the same as the disturbance measured in the previous period, [1].

D. Periodic moving horizon

In the case of the periodic moving horizon principle, it differs from the standard moving horizon from Section II, since it takes steps of the size of one full period $\tau=\frac{1}{f}$ instead of unitary steps t_s. This means that the controller does not only take the first optimal input of the sequence, but rather the whole optimal input sequence and feeds it into the system. The size of the control and prediction horizon matches the size of a period $H_u=H_p=\frac{\tau}{t_s}$,which means that the system can run for a full period without any further influence or processing from the controller, thus reducing the computational effort.

Moreover, by using the whole optimal input sequence, the controller ensures that the controlled state has a sinusoidal shape for the full prediction horizon, which means that the target sine wave of the controller is the same during a whole period, since it belongs to the same processed optimal input sequence. While in the traditional moving horizon principle, in each time step, the controller can aim for a different sine wave with different amplitude or phase, since it processes a new optimal input sequence in each step. Considering that the harmonic distortion targeted here is of periodic steady state nature (no spikes), the periodic moving horizon principle is the strategy of choice for the update of the proposed controller.

E. Simulation and results

Table I shows the main simulation parameters considered for the construction of the state space model.

Parameter	Symbol	Value
Frequency	f	50 Hz
Sampling time	t_s	20 µs
Supply resistance	R_1	1 Ω
Supply inductance	L_1	10 µH
Load resistance	R_2	10 Ω
Load inductance	L_2	0.1 H

TABLE I
SIMULATION PARAMETERS

In Table II the weighting values of the matrix $\mathbf{Q}$ for the individual states are given. In this case the focus is the compensation of v_l and i_l. The matrix $\mathbf{R}$ has a value of 10^4 on each element of its diagonal in order to adjust the cost of control effort associated with the flexibility of the change of input sequence $\Delta\mathbf{U}$ in comparison with the cost of control error of the conditions addressed in the matrix $\mathbf{Q}$, as explained in the *Remark* of Proposition 2.

State	Weight	Value
v_l	α_1^2	10^3
i_l	α_2^2	10^2
i_s	α_3^2	0
$\frac{di_c}{dt}$	α_4^2	0
v_s	α_5^2	0

TABLE II
MATRIX $\mathbf{Q}$ WEIGHTING

Since in this case there are two states being evaluated for the sinusoidal condition, v_l and i_l, the linear shape class is given as matrix $\mathbf{V}$ from the vector (17) but with two conditions, i.e. $s = 2$ and extended for all states $n = 5$. This restructuring can be performed starting from the coefficients in (17), with the help of the Kronecker product and considering the weighting values in Table II, leading to

$$\mathbf{V} = \mathbf{v} \otimes \begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 \end{pmatrix} \tag{25}$$

With $\mathbf{V} \in \mathbb{R}^{2\times 20}$, the pattern band matrix $\mathbf{P}_V$ can be computed as given in (14), which includes the complete state vector $\mathbf{x}$.

By implementing the MPC with the conditions proposed in this paper, a close loop simulation for 80 ms yields the compensation results as shown in Fig. 3 for the load.

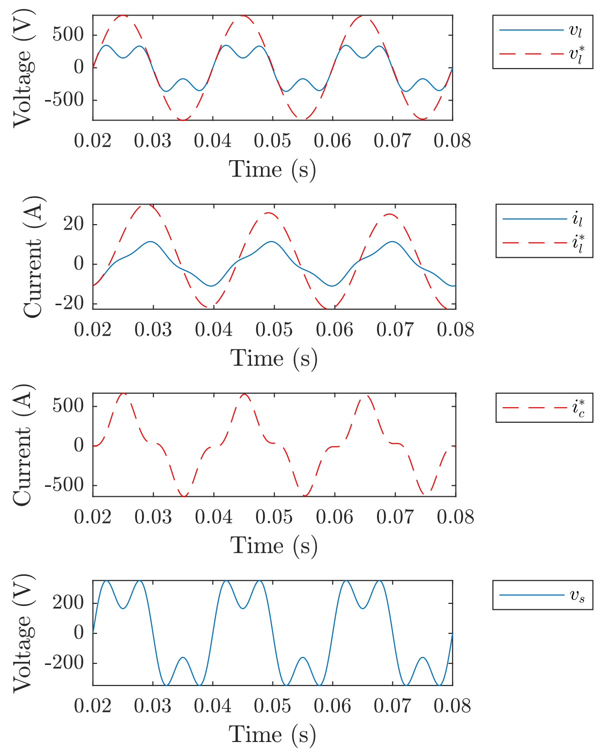

Fig. 3. Load compensation results

The simulation considers a future disturbance equal to the disturbance measured in the previous period, thus the first period is skipped since it is part of the initialization stage of the controller. From top to bottom, the first plot of Fig. 3 shows the controlled dashed state signal v_l^* having a sinusoidal shape in comparison with the original signal v_l. The second plot shows the controlled dashed state signal i_l^* also having a sinusoidal shape in comparison with the original state signal i_l. The third plot shows the compensation signal provided by the controller optimal input sequence through i_c. The bottom plot shows the distorted supply voltage v_s that considers a fundamental signal at 50 Hz of amplitude $230\sqrt{2}$ V with harmonic distortion of third order and 50% of strength in respect to the fundamental amplitude.

In order to analyze the quality of the compensation, the total harmonic distortion (*THD*) is measured as follows

$$THD = \frac{\sqrt{\sum_{n=2}^{\infty} X_n^2}}{X_1}, \tag{26}$$

where THD is the total harmonic distortion compared to the fundamental signal, n is the harmonic order ($n = 1$ refers to the fundamental signal), and X_n is the amplitude or RMS (root mean square) of the n^{th} harmonic, [12].

Fig. 4 shows how the compensation of the signals at the beginning present a spike but immediately start to sink over time; v_l goes down from 50.4% (corresponding to the harmonic distortion strength factor) to around 0.1% and i_l also goes down from 17.5% to around 0.1%. As a reference, according to the European Norm EN 50160, [13], the THD of the voltage, including all harmonics up to the 40^{th} order, shall be less than or equal to 8%. This defines the maximum distortion allowed as standard for power quality. Thus the MPC controller compensates the distortion of the load to values considerably lower than the quality standard limit.

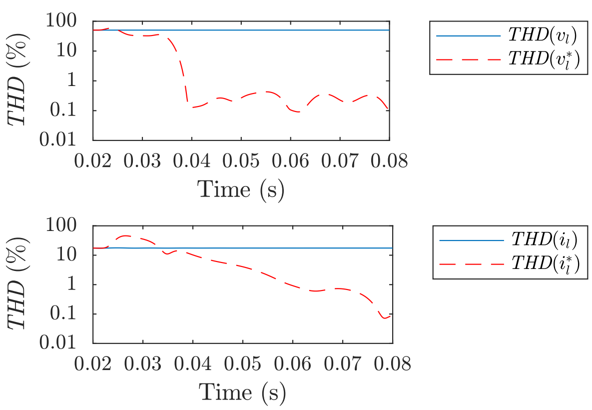

Fig. 4. Total harmonic distortion comparison

VI. CONCLUSIONS

The presented approach to signal shaping uses linear kernels as representations for classes of state signals. These could be given as matrices, as well as linear discrete-time models to represent the plant dynamics. A shape following control problem in contrast to a reference following control problem has no absolute timing restrictions, which could lead to major relaxations.

It is shown, that the control problem of following a shape class is equivalent to a standard linear MPC problem with in general non-diagonal weighting matrices, which gives access to efficient numeric algorithms. Moreover, the decomposition of the Hessian matrix of the underlying optimization problem can be done structurally from the matrix representation of the signal class.

The application example shows the ability of this approach to effectively turn a distorted signal into sinusoidal shape, compensating the harmonic distortion in a smart grid to satisfactory levels. This is achieved on both current and voltage signals simultaneously, without changing operation modes nor specifying a reference signal for each state, all while saving some computational effort by performing only one iteration per period.

For further development, a control over the amplitude can be integrated. One possibility is adding an RMS condition to the cost function that would lead to a non-quadratic cost function, which affords more complex algorithms to solve the optimization problem. Also, this new way of using the MPC for solving ODE conditions, opens several possibilities in other fields that not only involve sine waves, but different ODEs or properties that can be expressed by linear difference equations.

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