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Computation of regular friends for output-nulling and reachability subspaces of linear time-invariant descriptor systems

Christina Kazantzidou, Lorenzo Ntogramatzidis and Tristan Perez

Abstract— **In this paper, we employ the Rosenbrock system matrix pencil to find and parameterise regular friends of output-nulling and reachability subspaces of linear, time-invariant descriptor systems, which yield nonzero characteristic polynomial and assign the corresponding closed-loop finite eigenstructure. In particular, we consider impulse controllable descriptor systems and the computation of these friends is done in such a way to also eliminate impulsive behaviour.**

I. Introduction

Descriptor systems have found applications in several contexts of systems and control theory, for example in areas such as circuit theory, large-scale systems, constrained mechanical systems, robotics, aircraft modeling, biological systems, see e.g. [7], [13], [15], [17], [22] and the references cited therein. Descriptor systems arise also in a variety of contexts where the system at hand is explicit. For example, it is a well-known fact that the discrete-time counterpart of the Hamiltonian system arising in the classical theory of the linear quadratic (LQ) regulator is a set of equations (usually referred to as the *extended symplectic system*), which can be written in descriptor form. Another important example is the modeling of systems where the independent variable is not temporal but, say, it is representative of spatial coordinates: in such cases, causality appears to be an artificial assumption, and a descriptor model appears to be more appropriate.

The fundamental feature of a descriptor system is the fact that the update state equation is written in an implicit form, via the premultiplication of the derivative of the state by a matrix, typically denoted by E, which is allowed to be singular (or even, in some situations, nonsquare). In this way, the resulting mathematical model can capture a wider range of systems, but results in a substantial increase in the mathematical difficulty. Indeed, the introduction of matrix E in the update state equation may cause the system not to have a solution unless the class of allowable functions is extended to include distributions, i.e., the Dirac delta and its distributional derivatives. From a practical point of view, a first, obvious fundamental task of a control strategy is to ensure that impulsive behaviours in the resulting state trajectory are avoided.

Several control and estimation techniques have been generalised to descriptor systems in the past forty years. A mathematical framework that, from the very beginning, has been found to lead to insightful results in this area is the so-called geometric approach to control theory. Several structural system-theoretic properties, such as reachability/controllability and observability/reconstructibility, appear to have a much richer characterisation in this implicit framework than in the standard case, and have been successfully studied for descriptor systems using geometric techniques. An important control problem that has been generalised to descriptor systems is the eigenstructure assignment via state feedback, see e.g. [1], [4], [8], [14], [19], [20], [23].

Several results have been achieved in the past thirty years in the development of a geometric theory for descriptor systems. Concepts such as controlled and conditioned invariant subspaces, controllability/observability subspaces, output-nulling and input containing subspaces, stabilisability and detectability subspaces have been extended to descriptor systems, see e.g. [9]-[11], [13], [14], [19], [20]. These subspaces can be considered as the building blocks of the classical geometric approach to control theory, and are used to express the solvability conditions of a wide range of control and estimation problems, including disturbance decoupling, model matching, unknown-input observation, non-interacting control, fault detection and LQ-H_2-optimal control.

Much less attention has been devoted to the computation of the so-called *friends* of the aforementioned subspaces, which are the matrices that render these subspaces invariant under the closed-loop matrix. However, the friend is typically the main ingredient that is used to turn the solvability condition of any of the problems mentioned above into a strategy for the design of the controller or filter. The problem of determining feedback friends is particularly delicate in the descriptor case, because of the additional requirement of removing impulses from the set of obtainable trajectories of the system. This paper addresses this issue. In particular, we focus on the construction of *regular friends* for the supremal output-nulling reachability subspace $\mathcal{R}^\star$, which is a friend that ensures that the resulting closed-loop system is regular and impulse-free [20].

The key idea is to build a basis for $\mathcal{R}^\star$ by exploiting the null-spaces of the Rosenbrock system matrix pencil, evaluated at the values of the indeterminate that we want to assign as closed-loop finite generalised eigenvalues. This method is the generalisation to descriptor systems of the Moore-Laub technique of [16]. Its main feature is the fact that it leads to a complete parameterisation of regular friends of $\mathcal{R}^\star$ that achieve a desired closed-loop finite spectrum. The degrees of freedom associated with the computation of these friends can be exploited for the formulation of

C. Kazantzidou and T. Perez are with the Institute for Future Environments (IFE) and the School of Electrical Engineering and Computer Science, Queensland University of Technology (QUT), Brisbane, Australia. {christina.kazantzidou, tristan.perez}@qut.edu.au.

L. Ntogramatzidis is with the Department of Mathematics and Statistics, Curtin University, Perth, Australia. L.Ntogramatzidis@curtin.edu.au.

optimisation problems, whose goal is to address objectives such as minimum gain or improved robustness of the closed-loop eigenstructure in the same spirit of [18].

Notation. The origin of a vector space is denoted by $\{0\}$. The image and the kernel of a matrix A are denoted by $\mathrm{im}A$ and $\ker A$, respectively. The spectrum of a square matrix A is denoted by $\sigma(A)$. Given a linear map $A : \mathcal{X} \longrightarrow \mathcal{Y}$ and a subspace $\mathcal{S}$ of $\mathcal{Y}$, $A^{-1}\mathcal{S}$ represents the inverse image of $\mathcal{S}$ with respect to the linear map A, i.e. $A^{-1}\mathcal{S} = \{x \in \mathcal{X} \,|\, A\,x \in \mathcal{S}\}$. If $\mathcal{J} \subseteq \mathcal{X}$, the restriction of the map A to $\mathcal{J}$ will be denoted by $A\,|\,\mathcal{J}$. If $\mathcal{X} = \mathcal{Y}$ and $\mathcal{J}$ is A-invariant, the finite spectrum of A restricted to $\mathcal{J}$ is denoted by $\sigma\,(A\,|\,\mathcal{J})$. The finite spectrum of a square pair (E, A) of a descriptor system is denoted by $\sigma(E, A)$ and the finite spectrum restricted to a subspace $\mathcal{J}$ is denoted by $\sigma(E, A\,|\,\mathcal{J})$. The symbol $\oplus$ will stand for the direct sum of subspaces. The symbol i represents the imaginary unit, i.e., $\mathrm{i} = \sqrt{-1}$, while the symbol $\overline{\alpha}$ represents the complex conjugate of $\alpha \in \mathbb{C}$. Given a matrix M, we denote by M^i its i-th column. The Moore-Penrose pseudo-inverse of M is denoted by $M^\dagger$. For any matrix M with $n + m$ rows, we define the matrices $\overline{\pi}\{M\}$ and $\underline{\pi}\{M\}$ obtained by taking the upper n and lower m rows of M, respectively. Finally, given a self-conjugate set of h complex numbers $\mathcal{L} = \{\lambda_1, \ldots, \lambda_h\}$ containing exactly s complex conjugate pairs, $\mathcal{L}$ is called s-conformably ordered if $2s \leq h$ and the first $2s$ values of $\mathcal{L}$ are complex, while the remaining ones are real and for all odd $k<2s$ there holds $\lambda_{k+1} = \overline{\lambda}_k$ [18].

II. Preliminaries

Let Σ be a linear time-invariant (LTI), multi-input multi-output (MIMO), continuous-time descriptor system described by differential-algebraic equations (DAEs) of the form

$$E\,\dot{x}(t) = A\,x(t) + B\,u(t), \tag{1a}$$

$$y(t) = C\,x(t) + D\,u(t) \tag{1b}$$

and identified with the quintuple (E, A, B, C, D), where $E,\ A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, $\mathrm{rank}\,E \triangleq \ell \leq n$, $x(t) \in \mathcal{X} = \mathbb{R}^n$ is the descriptor variable, $u(t) \in \mathcal{U} = \mathbb{R}^m$ is the control input and $y(t) \in \mathcal{Y} = \mathbb{R}^p$ is the output.

The matrix pencil $\lambda E - A$ is called *regular* if $\det(\lambda E - A)$ is not identically zero, see e.g. [24]. The solution of a descriptor system exists and is unique given $x(0^-)$ and $u(t)$ if its matrix pencil $\lambda E - A$ is regular, see e.g. [24], [14]. The *finite generalised eigenvalues* of a regular matrix pencil $\lambda E - A$ are the roots of $\det(\lambda E - A)$ and can be at most ℓ, see e.g. [14], [8], [24], [1], [7, Ch.3]. If the descriptor system Σ has ℓ finite generalised eigenavlues, then it is called *impulse-free*, see e.g. [23], [7, Ch.7].

Impulsive modes are typically not desired, as they may compromise the integrity of physical systems, for example by damaging actuators or affecting the material properties in mechanical systems, see e.g. [5]-[7], [14]. Therefore, the assumption of impulse controllability for a descriptor system is essential. Impulse controllability of a descriptor system guarantees that ℓ arbitrary finite generalised eigenvalues can be assigned by a state feedback control law $u(t) = H\,x(t) + v(t)$, see e.g. [7, Ch.7], so that the closed-loop system is impulse-free. The descriptor system Σ is impulse controllable if and only if $\mathrm{rank}\,[\begin{array}{ccc} E & A\,E_\infty & B \end{array}] = n$, where E_∞ is a basis matrix for $\ker E$, see e.g. [10], [5], [7, Ch.4].

In this paper, the following standing assumptions are made:

1) the columns of $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ and the rows of $[\begin{array}{cc} C & D \end{array}]$ are linearly independent,[1]
2) the descriptor system Σ is impulse-controllable.

Regularity is not assumed, since impulse controllability implies regularisability, ensuring that there exists a state feedback control such that the closed-loop system is regular, see [11], [7, Ch.4].

We assume with no loss of generality that the descriptor system (1) is in the *dynamics decomposition form*

$$Q\,E\,P = \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix},\ Q\,A\,P = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ Q\,B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$
$$C\,P = [\begin{array}{cc} C_1 & C_2 \end{array}],\ P^{-1}\,x(t) = \begin{bmatrix} \tilde{x}(t) \\ z(t) \end{bmatrix}, \tag{2}$$

i.e., we assume that $Q = P = I_n$ and Σ is described by

$$\begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \tilde{x}(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t), \tag{3a}$$

$$y(t) = [\begin{array}{cc} C_1 & C_2 \end{array}]\begin{bmatrix} \tilde{x}(t) \\ z(t) \end{bmatrix} + D\,u(t), \tag{3b}$$

see e.g. [7, Ch.2]. Under the assumption of impulse controllability, there exists a state feedback control $u(t) = H_1\,\tilde{x}(t) + H_2\,z(t) + v(t)$, such that $\det(A_{22} + B_2\,H_2) \neq 0$, see e.g. [7, Ch.7]. It is clear from this consideration that, with no loss of generality, H_1 can be chosen to be equal to zero. The closed-loop system $\hat{\Sigma}$ under the state feedback $u(t) = H_2\,z(t) + v(t)$ is described by

$$E\,\dot{x}(t) = \hat{A}\,x(t) + B\,v(t), \tag{4a}$$

$$y(t) = \hat{C}\,x(t) + D\,v(t), \tag{4b}$$

where $E = \left[\begin{smallmatrix} I_\ell & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $\hat{A} \triangleq \left[\begin{smallmatrix} A_{11} & \hat{A}_{12} \\ A_{21} & \hat{A}_{22} \end{smallmatrix}\right]$, $\hat{C} \triangleq [\begin{array}{cc} C_1 & \hat{C}_2 \end{array}]$, $\hat{A}_{12} \triangleq A_{12} + B_1\,H_2$, $\hat{A}_{22} \triangleq A_{22} + B_2\,H_2$, $\hat{C}_2 \triangleq C_2 + D\,H_2$. The closed-loop system $\hat{\Sigma}$ is impulse-free and an equivalent form is given by

$$\tilde{Q}\,E\,\tilde{P} = \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix},\ \tilde{Q}\,\hat{A}\,\tilde{P} = \begin{bmatrix} \tilde{A} & 0 \\ 0 & I_{n-\ell} \end{bmatrix},\ \tilde{Q}\,B = \begin{bmatrix} \tilde{B} \\ B_2 \end{bmatrix},$$
$$\hat{C}\,\tilde{P} = [\begin{array}{cc} \tilde{C} & \tilde{C}_2 \end{array}],\ \tilde{P}^{-1}\begin{bmatrix} \tilde{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \tilde{x}(t) \\ \tilde{z}(t) \end{bmatrix},$$

[1] If $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ has non-trivial kernel, a subspace $\mathcal{U}_0$ of the input space exists that does not influence the local state dynamics. By performing a suitable (orthogonal) change of basis in the input space, we may eliminate $\mathcal{U}_0$ and obtain an equivalent system for which this condition is satisfied. Likewise, if $[\begin{array}{cc} C & D \end{array}]$ is not surjective, there are some outputs that result as linear combinations of the remaining ones, and these can be eliminated using a dual argument by performing a change of coordinates in the output space.

where $\tilde{Q} = \begin{bmatrix} I_\ell & -\hat{A}_{12}\hat{A}_{22}^{-1} \\ 0 & I_{n-\ell} \end{bmatrix}$, $\tilde{P} = \begin{bmatrix} I_\ell & 0 \\ -\hat{A}_{22}^{-1} A_{21} & \hat{A}_{22}^{-1} \end{bmatrix}$, $\tilde{P}^{-1} = \begin{bmatrix} I_\ell & 0 \\ A_{21} & \hat{A}_{22} \end{bmatrix}$, and

$$\tilde{A} \triangleq A_{11} - \hat{A}_{12}\hat{A}_{22}^{-1} A_{21}, \tag{5a}$$
$$\tilde{B} \triangleq B_1 - \hat{A}_{12}\hat{A}_{22}^{-1} B_2, \tag{5b}$$
$$\tilde{C} \triangleq C_1 - \hat{C}_2\hat{A}_{22}^{-1} A_{21}, \tag{5c}$$

and $\tilde{C}_2 \triangleq \hat{C}_2\hat{A}_{22}^{-1}$ [6], so that the restricted equivalent system (3) can be written as

$$\dot{\tilde{x}}(t) = \tilde{A}\,\tilde{x}(t) + \tilde{B}\,v(t), \tag{6a}$$
$$0 = \tilde{z}(t) + B_2\,v(t), \tag{6b}$$
$$y(t) = \tilde{C}\,\tilde{x}(t) + \tilde{C}_2\,\tilde{z}(t) + D\,v(t). \tag{6c}$$

The characteristic polynomial of $\lambda E - \hat{A}$ is $\det(\lambda E - \hat{A}) = \det(\lambda I_\ell - \tilde{A}) \det I_{n-\ell} = \det(\lambda I_\ell - \tilde{A})$.

Replacing $\tilde{z}(t) = -B_2\,v(t)$ from (6b) to (6c), we obtain the standard system $\tilde{\Sigma}$

$$\dot{\tilde{x}}(t) = \tilde{A}\,\tilde{x}(t) + \tilde{B}\,v(t), \tag{7a}$$
$$y(t) = \tilde{C}\,\tilde{x}(t) + \tilde{D}\,v(t), \tag{7b}$$

where

$$\tilde{D} \triangleq D - \tilde{C}_2 B_2 = D - \hat{C}_2\hat{A}_{22}^{-1} B_2, \tag{8}$$

see also [24].

The *Rosenbrock system matrix pencil* of a descriptor system $\hat{\Sigma}$ is defined as

$$P_{\hat{\Sigma}}(\lambda) \triangleq \begin{bmatrix} \hat{A} - \lambda E & B \\ \hat{C} & D \end{bmatrix} = \left[\begin{array}{cc|c} A_{11} - \lambda I_\ell & \hat{A}_{12} & B_1 \\ A_{21} & \hat{A}_{22} & B_2 \\ \hline C_1 & \hat{C}_2 & D \end{array}\right], \tag{9}$$

see e.g. [21], [9]. The *invariant zeros* of $\hat{\Sigma}$ are the values of $\lambda \in \mathbb{C}$ for which $\operatorname{rank} P_{\hat{\Sigma}}(\lambda) < n + \operatorname{normrank} G_{\hat{\Sigma}}(\lambda)$, where $G_{\hat{\Sigma}}(\lambda) = \hat{C}\,(\lambda E - \hat{A})^{-1} B + D$, see e.g. [2, Ch.6]. For regular systems, $n + \operatorname{normrank} G_{\hat{\Sigma}}(\lambda) = \operatorname{normrank} P_{\hat{\Sigma}}(\lambda)$. The Rosenbrock system matrix pencil of the associated standard system $\tilde{\Sigma}$ described by the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ in (5) and (8) is $P_{\tilde{\Sigma}}(\lambda) = \begin{bmatrix} \tilde{A} - \lambda I_\ell & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}$.

The following lemma shows the connection of the Rosenbrock system matrix pencil of $\hat{\Sigma}$ with the Rosenbrock system matrix pencil of the associated standard system $\tilde{\Sigma}$, [11].

Lemma 2.1: The kernel of the Rosenbrock system matrix pencil of an impulse-free descriptor system $\hat{\Sigma}$ is $\ker P_{\hat{\Sigma}}(\lambda) = \left\{ \begin{bmatrix} \tilde{v} \\ -\hat{A}_{22}^{-1}(A_{21}\,\tilde{v} + B_2\,\tilde{w}) \\ \tilde{w} \end{bmatrix} : \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix} \in \ker P_{\tilde{\Sigma}}(\lambda) \right\}$.

III. Geometric Background for Descriptor Systems

We now recall some concepts of classical geometric control theory for descriptor systems. An *output-nulling* subspace $\mathcal{V}$ for a descriptor system Σ is a subspace of $\mathcal{X}$ which satisfies the inclusion

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subseteq (E\,\mathcal{V} \oplus \{0\}) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix}, \tag{10}$$

see e.g. [14]. From (10) it follows that $\mathcal{V}$, with basis matrix V, is an output-nulling subspace of Σ if and only if there exist Λ, W of suitable dimensions such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} E\,V \\ 0 \end{bmatrix} \Lambda \tag{11}$$

or, equivalently, if and only if there exist F, Λ of suitable dimensions such that

$$\begin{bmatrix} A + B\,F \\ C + D\,F \end{bmatrix} V = \begin{bmatrix} E\,V \\ 0 \end{bmatrix} \Lambda. \tag{12}$$

Such a matrix F is called a *friend* of $\mathcal{V}$. The set of all friends F of an output-nulling subspace $\mathcal{V} \subseteq \mathcal{X}$ is denoted by $\mathfrak{F}(\mathcal{V})$. If $\mathcal{V} \cap \ker E \neq \{0\}$, regularity is not preserved after the state feedback, which makes it essential to introduce the notion of regular friends of $\mathcal{V}$, [20]. A linear map $F : \mathcal{X} \to \mathcal{U}$ is called a *regular friend* of $\mathcal{V}$ if $F \in \mathfrak{F}(\mathcal{V}_s)$ for some $\mathcal{V}_s$ that satisfies $\mathcal{V} = \mathcal{V}_s \oplus (\mathcal{V} \cap \ker E)$, [20]. The set of all regular friends F of $\mathcal{V} \subseteq \mathcal{X}$ is denoted by $\mathfrak{F}_r(\mathcal{V})$. It is clear from the definition that $\mathfrak{F}_r(\mathcal{V}) = \mathfrak{F}(\mathcal{V}_s)$. The set of output-nulling subspaces is closed under subspace addition. Therefore, there exists a maximum element, which is denoted by $\mathcal{V}^\star$ and can be computed using the monotonically non-increasing sequence of subspaces $\mathcal{V}_0 = \mathcal{X}$, $\mathcal{V}_i = \begin{bmatrix} A \\ C \end{bmatrix}^{-1}\Big((E\,\mathcal{V}_{i-1} \oplus \{0\}) + \operatorname{im}\begin{bmatrix} B \\ D \end{bmatrix}\Big)$, $i \in \{1, 2, \ldots, n-1\}$. This sequence converges to $\mathcal{V}^\star$ in at most $n-1$ steps, i.e., $\mathcal{V}^\star = \mathcal{V}_k$ where $k \leq n-1$ is such that $\mathcal{V}_{k+1} = \mathcal{V}_k$. We can write $\mathcal{V}^\star = \mathcal{V}_s^\star \oplus (\mathcal{V}^\star \cap \ker E)$ for some $\mathcal{V}_s^\star$ and then we can also write $\mathcal{V}^\star + \ker E = \mathcal{V}_s^\star \oplus \ker E$. Consequently, a regular friend of $\mathcal{V}^\star$ is a friend of $\mathcal{V}_s^\star$, which is also a regular friend of $\mathcal{V}^\star + \ker E$. We denote by v the dimension of $\mathcal{V}_s^\star$.

An *input-containing* subspace $\mathcal{S}$ for Σ is a subspace of $\mathcal{X}$ which satisfies

$$E^{-1}\,[\,A \;\; B\,]\,((\mathcal{S} \oplus \mathcal{U}) \cap \ker [\,C \;\; D\,]) \subseteq \mathcal{S}. \tag{13}$$

The set of input-containing subspaces is closed under subspace intersection, so there exists a minimum element, which is denoted by $\mathcal{S}^\star$ and can be computed using the monotonically non-decreasing sequence of subspaces $\mathcal{S}_0 = \ker E$, $\mathcal{S}_i = E^{-1}\,([\,A \;\; B\,]\,((\mathcal{S}_{i-1} \oplus \mathcal{U}) \cap \ker [\,C \;\; D\,]))$, $i \in \{1, 2, \ldots, \ell - 1\}$. There holds $\mathcal{S}^\star = \mathcal{S}_k$, where $k \leq \ell - 1$ is such that $\mathcal{S}_{k+1} = \mathcal{S}_k$.

The *output-nulling reachability* subspace $\mathcal{R}^\star$ represents the set of initial states for which there exists an impulsive input and a trajectory from the origin such that $y = 0$ and $E\,x(0) = E\,x_0$, [9]. The subspace $\mathcal{R}^\star$ can be computed by $\mathcal{R}^\star = (\mathcal{V}^\star + \ker E) \cap \mathcal{S}^\star$, [9]. Since $\ker E \subseteq \mathcal{R}^\star$, we can write $\mathcal{R}^\star = \mathcal{R}_s^\star \oplus \ker E$ for some $\mathcal{R}_s^\star$. Consequently, a regular friend of $\mathcal{R}^\star$ is a friend of $\mathcal{R}_s^\star$. Denoting by r the dimension of $\mathcal{R}_s^\star$, the dimension of $\mathcal{R}^\star$ is $\dim \mathcal{R}^\star = r + \dim(\ker E) = r + n - \ell$.

Let $F \in \mathfrak{F}_r(\mathcal{V}^\star)$. Then $\sigma(E, A + B\,F \,|\, \mathcal{V}^\star)$ represents the *inner finite spectrum* of the closed loop with respect to $\mathcal{V}^\star$ and consists of the finite generalised eigenvalues of $(E, A + B\,F)$ restricted to $\mathcal{R}^\star$, i.e., $\sigma(E, A + B\,F \,|\, \mathcal{R}^\star)$, which are all freely assignable[2] with a suitable choice of F in $\mathfrak{F}_r(\mathcal{V}^\star)$ and the finite generalised eigenvalues of $(E, A + B F)$ restricted to $\mathcal{V}^\star/\mathcal{R}^\star$, i.e., $\mathcal{Z} \triangleq \sigma(E, A + B F \,|\, \mathcal{V}^\star/\mathcal{R}^\star)$, which are fixed for all the choices of F in $\mathfrak{F}_r(\mathcal{V}^\star)$. Thus, the set $\mathcal{Z}$ does not depend on the choice of the regular friend F of $\mathcal{V}^\star$. The elements of $\mathcal{Z}$ are the invariant zeros of Σ.

IV. Computation of Regular Friends of Reachability and Output-Nulling Subspaces

Consider now a descriptor system Σ in the equivalent form (3). Let $u(t) = H\,x(t) + v(t) = H_2\,z(t) + v(t)$ be a preliminary state feedback applied to Σ, so that the closed-loop system $\hat{\Sigma}$ is impulse-free. A regular friend of $\mathcal{R}^\star$ for $\hat{\Sigma}$ is denoted by $\hat{F}$. Applying the input $v(t) = \hat{F}\,x(t)$, we obtain

$$\begin{aligned} E\,\dot{x}(t) &= (A + B\,(\hat{F} + H))\,x(t), \\ y(t) &= (C + D\,(\hat{F} + H))\,x(t), \end{aligned}$$

which shows that a regular friend of $\mathcal{R}^\star$ for Σ is $F = \hat{F} + H$.[3] The main theorem presents a procedure for the parameterisation of all regular friends $\hat{F}$ of $\mathcal{R}^\star$ for $\hat{\Sigma}$ that place the closed-loop finite generalised eigenvalues restricted to $\mathcal{R}^\star$ at arbitrary locations.

Theorem 4.1: Let $\hat{\Sigma}$ be an impulse-free descriptor system as in (4). Let $r = \dim\mathcal{R}^\star - n + \ell$ and let $\mathcal{L} = \{\lambda_1, \ldots, \lambda_r\}$ be s-conformably ordered with elements all different from the invariant zeros of the descriptor system. Let $K \triangleq \mathrm{diag}\{k_1, k_2, \ldots, k_r\}$, where $k_i \in \mathbb{C}^d, (d = n + m - \mathrm{normrank}\, P_{\hat{\Sigma}}(\lambda))$ for $i \in \{1, 2, \ldots, 2s\}$ and for all odd $i < 2s$ we have $k_{i+1} = \overline{k}_i$, and $k_i \in \mathbb{R}^d$ for $i \in \{2s+1, \ldots, r\}$. Let $\hat{M}_K$ be an $(n+m) \times r$ complex matrix given by

$$\hat{M}_K = [\; N_{\hat{\Sigma}}(\lambda_1) \;\; N_{\hat{\Sigma}}(\lambda_2) \;\; \ldots \;\; N_{\hat{\Sigma}}(\lambda_r) \;]\, K, \tag{14}$$

where $N_{\hat{\Sigma}}(\lambda_i)$ is a basis for $\ker P_{\hat{\Sigma}}(\lambda_i)$, and let for all $i \in \{1, 2, \ldots, r\}$

$$\hat{m}_{K,i} \triangleq \begin{cases} \mathfrak{Re}\{\hat{M}_K^i\} & \text{if } i < 2\,s \text{ is odd}, \\ \mathfrak{Im}\{\hat{M}_K^i\} & \text{if } i \le 2\,s \text{ is even}, \\ \hat{M}_K^i & \text{if } i > 2\,s. \end{cases} \tag{15}$$

Finally, let

$$X_K \triangleq \overline{\pi}\{[\; \hat{m}_{K,1} \;\; \ldots \;\; \hat{m}_{K,r} \;]\},\; \hat{Y}_K \triangleq \underline{\pi}\{[\; \hat{m}_{K,1} \;\; \ldots \;\; \hat{m}_{K,r} \;]\}. \tag{16}$$

The following statements hold:
- The matrix $E\,X_K$ and consequently the matrix X_K are generically full column rank with respect to the parameter matrix K, i.e., $\mathrm{rank}\, E\,X_K = \mathrm{rank}\, X_K = r$ for every K except possibly for those in a set of Lebesgue measure zero.
- The set of regular friends of $\mathcal{R}^\star$ such that $\sigma(E, \hat{A} + B\,\hat{F} \,|\, \mathcal{R}^\star) = \mathcal{L}$ is parameterised as $\hat{F}_K = \hat{Y}_K\,X_K^\dagger$ and $\hat{F}_K = \hat{Y}_K\,(E\,X_K)^\dagger$, where K is such that $\mathrm{rank}\, E\,X_K = r$.

Proof: We will prove the theorem by using the lemmas and Theorem 3.1 in [18]. Let K be such that $\mathrm{rank}\,\overline{\pi}\{\hat{M}_K\} = r$ and $\hat{M}_K$ be partitioned as $\hat{M}_K = \begin{bmatrix} v'_1 & v'_2 & \ldots & v'_r \\ w'_1 & w'_2 & \ldots & w'_r \end{bmatrix}$, where $v'_i = \begin{bmatrix} \hat{v}'_i \\ z'_i \end{bmatrix}$ for each $i \in \{1, 2, \ldots, r\}$ and there holds

$$\left[\begin{array}{cc|c} A_{11} - \lambda_i\, I_\ell & \hat{A}_{12} & B_1 \\ A_{21} & \hat{A}_{22} & B_2 \\ \hline C_1 & \hat{C}_2 & D \end{array}\right] \begin{bmatrix} \hat{v}'_i \\ z'_i \\ w'_i \end{bmatrix} = 0 \tag{17}$$

or, equivalently,

$$\begin{bmatrix} \tilde{A} - \lambda_i\, I_\ell & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} \tilde{v}'_i \\ \tilde{w}'_i \end{bmatrix} = 0, \tag{18}$$

where $\hat{v}'_i = \tilde{v}'_i$ and $w'_i = \tilde{w}'_i$. The above matrix is the Rosenbrock system matrix pencil of the standard system $\tilde{\Sigma}$ in (7). For the standard system $\tilde{\Sigma}$, it was proved in [18] that $\tilde{X}_K$ is generically full column rank with respect to the parameter matrix K, i.e., $\mathrm{rank}\,\tilde{X}_K = r$, for every K except possibly for those in a set of Lebesgue measure zero. Thus, the matrix $E\,X_K = \begin{bmatrix} \tilde{X}_K \\ 0 \end{bmatrix}$ and consequently the matrix $X_K = \begin{bmatrix} \tilde{X}_K \\ Z_K \end{bmatrix}$, where $Z_K \triangleq -\hat{A}_{22}^{-1}\,(A_{21}\,\tilde{X}_K + B_2\,\hat{Y}_K)$ (see Lemma 2.1), will also have rank equal to r for every K except possibly for those belonging to a set of Lebesgue measure zero.

For odd $i < 2s$ we have $\lambda_{i+1} = \overline{\lambda}_i$ and $k_{i+1} = \overline{k}_i$. Therefore $v'_{i+1} = \overline{v}'_i$ and $w'_{i+1} = \overline{w}'_i$. Let $[\; v_i \;\; v_{i+1} \;] \triangleq [\; v'_i \;\; v'_{i+1} \;]\, U$, $[\; w_i \;\; w_{i+1} \;] \triangleq [\; w'_i \;\; w'_{i+1} \;]\, U$, for odd $i < 2s$ where $U \triangleq \frac{1}{2}\begin{bmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{bmatrix}$. Then we obtain the real valued vectors

$$\begin{array}{lll} v_i = \frac{1}{2}(v'_i + v'_{i+1}), & w_i = \frac{1}{2}(w'_i + w'_{i+1}), & \text{if } i < 2s \text{ and odd}, \\ v_i = \frac{1}{2\mathrm{i}}(v'_i - v'_{i-1}), & w_i = \frac{1}{2\mathrm{i}}(w'_i - w'_{i-1}), & \text{if } i \le 2s \text{ and even}, \\ v_i = v'_i, & w_i = w'_i, & \text{if } i > 2s, \end{array}$$

from which it follows that the matrices X_K and $\hat{Y}_K$ can be written as $X_K = [\; v_1 \;\; v_2 \;\; \ldots \;\; v_r \;]$ and $\hat{Y}_K = [\; w_1 \;\; w_2 \;\; \ldots \;\; w_r \;]$. Since, for this choice of K, we have $\mathrm{rank}\, X_K = r$, the equation $\hat{F}_K X_K = \hat{Y}_K$ has the solution $\hat{F}_K = \hat{Y}_K\, X_K^\dagger$. Since $\tilde{F}_K\,\tilde{X}_K = \tilde{Y}_K = \hat{Y}_K$, or, equivalently, $[\; \tilde{F}_K \;\; 0 \;]\, X_K = \hat{Y}_K$, another solution of the equation $\hat{F}_K X_K = \hat{Y}_K$ is $\hat{F}_K = [\; \tilde{F}_K \;\; 0 \;] = [\; \hat{Y}_K\,\tilde{X}_K^\dagger \;\; 0 \;] = \hat{Y}_K\,(E\,X_K)^\dagger$. This means that $\hat{F}_K\, v'_i = w'_i,\; i \in \{1, 2, \ldots, r\}$ and

$$\begin{bmatrix} \hat{A} + B\,\hat{F}_K \\ \hat{C} + D\,\hat{F}_K \end{bmatrix} v'_i = \begin{bmatrix} E\,v'_i \\ 0 \end{bmatrix} \lambda_i.$$

For all odd $i \in \{1, 2, \ldots, 2s\}$, we have that $v'_i = v_i - \mathrm{i}\, v_{i+1}$, $v'_{i+1} = v_i + \mathrm{i}\, v_{i+1}$ and $\hat{F}_K\,[\; v_i \;\; v_{i+1} \;] = [\; w_i \;\; w_{i+1} \;]$, so that we obtain

$$\begin{bmatrix} \hat{A} + B\,\hat{F}_K \\ \hat{C} + D\,\hat{F}_K \end{bmatrix} v_i = \begin{bmatrix} E\,(v_i\,\mathfrak{Re}\{\lambda_i\} + v_{i+1}\mathfrak{Im}\,\{\lambda_i\}) \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} \hat{A} + B\,\hat{F}_K \\ \hat{C} + D\,\hat{F}_K \end{bmatrix} v_{i+1} = \begin{bmatrix} E\,(v_{i+1}\,\mathfrak{Re}\{\lambda_i\} - v_i\,\mathfrak{Im}\{\lambda_i\}) \\ 0 \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} \hat{A} + B\,\hat{F}_K \\ \hat{C} + D\,\hat{F}_K \end{bmatrix} [\; v_i \;\; v_{i+1} \;] = \begin{bmatrix} E\,v_i & E\,v_{i+1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{Re}\{\lambda_i\} & -\mathfrak{Im}\{\lambda_i\} \\ \mathfrak{Im}\{\lambda_i\} & \mathfrak{Re}\{\lambda_i\} \end{bmatrix}.$$

Defining $\Lambda_{i,i+1} = \begin{bmatrix} \mathfrak{Re}\{\lambda_i\} & -\mathfrak{Im}\{\lambda_i\} \\ \mathfrak{Im}\{\lambda_i\} & \mathfrak{Re}\{\lambda_i\} \end{bmatrix}$, $i \in \{1, 2, \ldots, 2s-1\}$ and $\Lambda_i = \lambda_i,\; i \in \{2s+1, \ldots, r\}$, we find

$$\begin{bmatrix} \hat{A} + B\,\hat{F}_K \\ \hat{C} + D\,\hat{F}_K \end{bmatrix} X_K = \begin{bmatrix} E\,X_K \\ 0 \end{bmatrix} \Lambda,$$

[2] An assignable set of finite generalised eigenvalues here is always intended to be a set of complex numbers which is mirrored with respect to the real axis.

[3] If Σ is not in the dynamics decomposition form (3), then $F = (\hat{F} + H)\,P^{-1}$.

where $\Lambda = \operatorname{diag}\{\Lambda_{1,2}, \ldots, \Lambda_{2s-1,2s}, \Lambda_{2s+1}, \ldots, \Lambda_r\}$. From (17)-(18), it follows that $\sigma(E, \hat{A} + B\hat{F} \,|\, \mathcal{R}^\star) = \sigma(\tilde{A} + \tilde{B}\tilde{F} \,|\, \tilde{\mathcal{R}}^\star)$.

Finally, we show that this parameterisation is exhaustive, i.e., given $\mathcal{L}$ and a regular friend $\hat{F}$ of $\mathcal{R}^\star$ such that $\sigma(E, \hat{A}+B\hat{F} \,|\, \mathcal{R}^\star) = \mathcal{L}$, we show that there exists K such that, constructing X_K, $\hat{Y}_K$ from $\hat{M}_K$, there holds $\hat{F} = \hat{Y}_K X_K^\dagger$. The set of regular friends $\hat{F}$ of $\mathcal{R}^\star$ such that $\sigma(E, \hat{A} + B\hat{F} \,|\, \mathcal{R}^\star) = \mathcal{L}$ is parameterised as $\hat{F} R = -\Omega$, where Ω satisfies the linear equation

$$\begin{bmatrix} \hat{A} \\ \hat{C} \end{bmatrix} R = \begin{bmatrix} E R \\ 0 \end{bmatrix} \Lambda + \begin{bmatrix} B \\ D \end{bmatrix} \Omega,$$

where R is a basis for some $\mathcal{R}_s^\star$ such that $\mathcal{R}^\star = \mathcal{R}_s^\star \oplus \ker E$, and Λ is such that $\sigma(\Lambda) = \mathcal{L}$ and satisfies

$$\begin{bmatrix} \hat{A} + B\hat{F} \\ \hat{C} + D\hat{F} \end{bmatrix} R = \begin{bmatrix} E R \\ 0 \end{bmatrix} \Lambda.$$

Consider a change of coordinates which brings Λ to the Jordan real canonical form such that the blocks are ordered having the s complex conjugate pairs of finite generalised eigenvalues first. The above can then be written as

$$\begin{bmatrix} \hat{A} + B\hat{F} \\ \hat{C} + D\hat{F} \end{bmatrix} R\,T = \begin{bmatrix} E R \\ 0 \end{bmatrix} T\,\Lambda_J,$$

where $\Lambda_J \triangleq T^{-1}\Lambda\,T = \operatorname{diag}\{\Lambda_{1,2}, \ldots, \Lambda_{2s-1,s}, \Lambda_{2s+1}, \ldots, \Lambda_r\}$, $\Lambda_{i,i+1} = \begin{bmatrix} \Re\mathfrak{e}\{\lambda_i\} & -\Im\mathfrak{m}\{\lambda_i\} \\ \Im\mathfrak{m}\{\lambda_i\} & \Re\mathfrak{e}\{\lambda_i\} \end{bmatrix}$, $i \in \{1, \ldots, 2s-1\}$ and $\Lambda_i = \lambda_i$, $i \in \{2s+1, \ldots, r\}$. The matrix $R\,T$ is also a basis for $\mathcal{R}_s^\star$. If we let $X = R\,T$, $\hat{Y} = \hat{F} R\,T$, it follows that

$$\begin{bmatrix} \hat{A} & B \\ \hat{C} & D \end{bmatrix} \begin{bmatrix} X \\ \hat{Y} \end{bmatrix} = \begin{bmatrix} E X \\ 0 \end{bmatrix} \Lambda_J.$$

Denoting by $v_1, \ldots, v_r$ the r columns of X and by $w_1, \ldots, w_r$ the r columns of $\hat{Y}$, we obtain for all odd $i < 2\,s$

$$\begin{bmatrix} \hat{A} & B \\ \hat{C} & D \end{bmatrix} \begin{bmatrix} v_i & v_{i+1} \\ w_i & w_{i+1} \end{bmatrix} = \begin{bmatrix} E\,v_i & E\,v_{i+1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Re\mathfrak{e}\{\lambda_i\} & -\Im\mathfrak{m}\{\lambda_i\} \\ \Im\mathfrak{m}\{\lambda_i\} & \Re\mathfrak{e}\{\lambda_i\} \end{bmatrix},$$

and for $i > 2\,s$

$$\begin{bmatrix} \hat{A} & B \\ \hat{C} & D \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} E\,v_i \\ 0 \end{bmatrix} \lambda_i.$$

Let $v_i' = v_i - \mathrm{i}\,v_{i+1}$, $v_{i+1}' = v_i + \mathrm{i}\,v_{i+1}$, $w_i' = w_i - \mathrm{i}\,w_{i+1}$, $w_{i+1}' = w_i + \mathrm{i}\,w_{i+1}$, $i < 2\,s$ and odd, and $v_i' = v_i$, $w_i' = w_i$, $i > 2\,s$. Then we have for all odd $i < 2\,s$

$$\begin{bmatrix} \hat{A} & B \\ \hat{C} & D \end{bmatrix} \begin{bmatrix} v_i' \\ w_i' \end{bmatrix} = \begin{bmatrix} \lambda_i\,E\,v_i' \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} \hat{A} & B \\ \hat{C} & D \end{bmatrix} \begin{bmatrix} v_{i+1}' \\ w_{i+1}' \end{bmatrix} = \begin{bmatrix} \overline{\lambda}_i\,E\,v_{i+1}' \\ 0 \end{bmatrix},$$

and for $i > 2\,s$

$$\begin{bmatrix} \hat{A} & B \\ \hat{C} & D \end{bmatrix} \begin{bmatrix} v_i' \\ w_i' \end{bmatrix} = \begin{bmatrix} \lambda_i\,E\,v_i' \\ 0 \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} \hat{A} & B \\ \hat{C} & D \end{bmatrix} \begin{bmatrix} v_1' & v_2' & \ldots & v_r' \\ w_1' & w_2' & \ldots & w_r' \end{bmatrix} = \begin{bmatrix} \lambda_1\,E\,v_1' & \lambda_2\,E\,v_2' & \ldots & \lambda_r\,E\,v_r' \\ 0 & 0 & \ldots & 0 \end{bmatrix}.$$

This implies that

$$\begin{bmatrix} v_i' \\ w_i' \end{bmatrix} \in \ker \begin{bmatrix} \hat{A} - \lambda_i\,E & B \\ \hat{C} & D \end{bmatrix}, \quad i \in \{1, \ldots, r\}.$$

Thus, there exists a matrix K for which $X = X_K$, $\hat{Y} = \hat{Y}_K$, where $X_K = \overline{\pi}\{[\ \hat{m}_{K,1} \quad \hat{m}_{K,2} \quad \ldots \quad \hat{m}_{K,r}\ ]\}$, $\hat{Y}_K = \underline{\pi}\{[\ \hat{m}_{K,1} \quad \hat{m}_{K,2} \quad \ldots \quad \hat{m}_{K,r}\ ]\}$, as required.

The proof that the parameterisation $\hat{Y}_K\,(E\,X_K)^\dagger$ is exhaustive follows from the associated standard system (7) and the fact that $E\,R = \left[\begin{smallmatrix} \tilde{R} \\ 0 \end{smallmatrix}\right]$.

■

Corollary 4.1: If the descriptor system Σ in (3) is regular and the dimension of $\mathcal{R}^\star$ is equal to n, then the procedure for the computation of regular friends of $\mathcal{R}^\star$ can be carried out in one step as shown in Theorem 4.1, without the preliminary state feedback H. This follows from the fact that $\ell = \operatorname{rank} E$ finite generalised eigenvalues are assigned. Therefore, another parameterisation of a regular friend of $\mathcal{R}^\star$ is $F_K = Y_K\,X_K^\dagger$, where X_K, Y_K are computed using (14)-(16) in Theorem 4.1 for the descriptor system Σ instead of $\hat{\Sigma}$, see also [12].[4]

Remark 4.1: The same result of Theorem 4.1 holds for the computation of regular friends of $\mathcal{V}^\star$ when we consider distinct $\lambda_1, \lambda_2, \ldots, \lambda_r, \zeta_1, \zeta_2, \ldots, \zeta_{v-r}$, where $\zeta_1, \zeta_2, \ldots, \zeta_{v-r}$ are the invariant zeros of $\hat{\Sigma}$.

V. Numerical Example

Example 5.1: Consider a continuous-time descriptor system Σ described by the matrices

$$E = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad A = \left[\begin{array}{cc|cc} 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right],$$

$$B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 1 & 0 \\ -1 & 1 \end{array}\right], \quad C = [\ 0 \quad 0 \mid 0 \quad 4\ ], \quad D = [\ 0 \quad 1\ ].$$

The descriptor system is impulse controllable since $\operatorname{rank}[\ E \quad A\,E_\infty \quad B\ ] = 4 = n$, and regular since $\det(\lambda\,E - A) = -(\lambda + 1) \neq 0$. Denoting by $\mathbf{e}_i$ the i-th canonical basis vector of $\mathbb{R}^4$, we compute the subspaces $\mathcal{V}^\star = \operatorname{span}\{17\mathbf{e}_1 + \mathbf{e}_3 - 4\mathbf{e}_4, \mathbf{e}_2, 4\mathbf{e}_3 + \mathbf{e}_4\}$, $\ker E = \operatorname{span}\{\mathbf{e}_3, \mathbf{e}_4\}$, $\mathcal{V}^\star + \ker E = \mathcal{X}$, $\mathcal{S}^\star = \mathcal{X}$, $\mathcal{R}^\star = (\mathcal{V}^\star + \ker E) \cap \mathcal{S}^\star = \mathcal{X}$.

We notice that Σ is not impulse-free, because $A_{22} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is singular. Applying the state feedback control law $u(t) = H\,x(t) + v(t)$, where $H = \left[\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right]$, to Σ, we obtain the impulse-free, closed-loop system $\hat{\Sigma} = (E, \hat{A}, B, \hat{C}, D)$, where

$$\hat{A} = A + B\,H = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix},$$

$$\hat{C} = C + D\,H = [\ 0 \quad 0 \quad 0 \quad 5\ ].$$

[4] The upper part of $\hat{M}_K$ is not affected by a preliminary state feedback, hence the notation X_K is maintained, see also [11]. However, the lower part of $\hat{M}_K$ will be different, i.e., the constructed Y_K and $\hat{Y}_K$ are not the same.

Let us choose $\lambda_1 = -1,\ \lambda_2 = -2$. We compute the kernels of the Rosenbrock system matrix pencil $P_{\hat\Sigma}$ for λ_1, λ_2 and choose $K = I_2$. Then we have

$$\hat{M}_K = [\ N_{\hat\Sigma}(-1)\ \ N_{\hat\Sigma}(-2)\ ]\, K = \operatorname{im}\begin{bmatrix} X_K \\ \hat{Y}_K \end{bmatrix},$$

$$X_K = \begin{bmatrix} 4 & 7 \\ -3 & 1 \\ 8 & 15 \\ 1 & 2 \end{bmatrix},\ \hat{Y}_K = \begin{bmatrix} -8 & -15 \\ -5 & -10 \end{bmatrix}.$$

A regular friend of $\mathcal{R}^\star$ is computed by

$$F_{K,1} = \hat{Y}_K X_K^\dagger + H = -\frac{1}{3501}\begin{bmatrix} 1326 & 93 & 2826 & 375 \\ 855 & 390 & 1875 & -3246 \end{bmatrix}$$

and the closed-loop characteristic polynomial is $800(\lambda+1)(\lambda+2)/3501$. Another regular friend of $\mathcal{R}^\star$ can be computed by

$$F_{K,2} = \hat{Y}_K (E\,X_K)^\dagger + H = -\frac{1}{25}\begin{bmatrix} 53 & 4 & 0 & 0 \\ 35 & 5 & 0 & -25 \end{bmatrix},$$

so that the closed-loop characteristic polynomial is $(\lambda+1)(\lambda+2)$.

Since Σ is regular and $\dim \mathcal{R}^\star = n$, we may also compute a regular friend of $\mathcal{R}^\star$ in one step. We compute

$$M_K = [\ N_\Sigma(-1)\ \ N_\Sigma(-2)\ ]\, K = \operatorname{im}\begin{bmatrix} X_K \\ Y_K \end{bmatrix},$$

$$X_K = \begin{bmatrix} 4 & 7 \\ -3 & 1 \\ 8 & 15 \\ 1 & 2 \end{bmatrix},\ Y_K = \begin{bmatrix} -8 & -15 \\ -4 & -8 \end{bmatrix},$$

$$F_{K,3} = Y_K X_K^\dagger = -\frac{1}{3501}\begin{bmatrix} 1326 & 93 & 2826 & 375 \\ 684 & 312 & 1500 & 204 \end{bmatrix}.$$

The closed-loop characteristic polynomial is $175(\lambda+1)(\lambda+2)/3501$.

VI. Conclusions

In this paper, we showed that we can employ the Rosenbrock system matrix pencil to parameterise regular friends of output-nulling and reachability subspaces of square, impulse controllable, LTI, MIMO descriptor systems.

Under the assumption of impulse controllability, a preliminary state feedback H can be applied to the plant, so that the closed-loop system is impulse-free and we can compute the associated regular friends considering the impulse-free system. In the case of impulse-free descriptor systems, we showed that we may compute the associated regular friends $\hat{F}_K$ in two ways. The regular friends for the open-loop descriptor system are then computed by $\hat{F}_K + H$. We also showed that, in the case of regular descriptor systems and if the dimension of the output-nulling reachability subspace is equal to n, then the associated friends may be computed in one step, since $\ell = \operatorname{rank} E$ finite generalised eigenvalues are assigned and regularity is guaranteed. However, it is clear that the parameterisation $F_K = \hat{Y}_K (E\,X_K)^\dagger + H$ is numerically more efficient.

An important direction for future work is to exploit the results to obtain a procedure for the computation of regular friends of these subspaces with complete inner/outer finite spectrum assignment, along the lines of what is achieved in [18] for the standard case.

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