2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

A suboptimal LMI formulation for the $\mathcal{H}_2$ static output feedback control of hidden Markov jump linear systems

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***Abstract*— In this work we study the synthesis of $\mathcal{H}_2$ static output feedback controllers for Markov jump linear systems in a context of partial observation of the Markov chain. We consider a hidden Markov model in which the controller has no access to the Markov chain $\theta(k)$, but only to an observable process represented by $\hat{\theta}(k)$. We present a suboptimal condition based on the so-called two-stage procedure in order to design static output feedback controllers that depend only on $\hat{\theta}(k)$ such that the closed-loop system is stochastically stable and its $\mathcal{H}_2$ norm is bounded by some given scalar γ. Our results are given in terms of linear matrix inequalities and are illustrated by a numerical example in the context of systems subject to failures.**

I. Introduction

The study of systems subject to abrupt changes is of special concern in the literature, and therefore it has been subject to a lot of effort in the last decades. This interest resides on the possible situations in which those changes may arise, for instance, in the occurrence of failures in critical applications such as aircraft and nuclear power operation, see, for instance, [1]. In this context, the use of switched systems theory for modeling and controlling those occurrences constitutes a natural approach and, whenever those changes are stochastic, the use of Markov jump linear systems (MJLS) becomes appealing. The MJLS theory is mainly consolidated by now, see for instance the books [2], [3], [4], [5]; and its potential application on the fields of Active Fault-tolerant Control Systems (AFTCS) and Networked Control Systems (NCS) can be seen, e.g., in the works [6], [7], [8].

Considering the control theory for MJLS, the availability of the Markov chain $\theta(k)$ is an important aspect that must be taken into account in the project. In the literature, the most known settings regarding the availability of the Markov mode are the complete observation, cluster, and mode-independent cases. In the former it is considered that the Markov mode $\theta(k)$ is available to the mode-dependent controller as in the dynamic output feedback controllers in [9] and [10]. Alternatively, in the cluster framework it is considered that the Markov modes are unavailable, but instead they can be grouped into disjoint sets called *clusters of observation*, e.g., in the work [11]. Finally, for the mode-independent case, the controller would have no information on $\theta(k)$, and so it would remain constant throughout the time, see, for instance, the paper [12]. More recently it was studied in [13] and [14] the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control in the so-called detector, or hidden MJLS, approach, in which it is considered that the Markov chain is hidden, and the controller has access to a detector $\hat{\theta}(k)$ that provides the only information on the mode $\theta(k)$. This framework is closely related to the AFTCS theory (as $\hat{\theta}(k)$ could be viewed as a failure detector), and also generalizes the aforementioned complete observation, the cluster, and the mode-independent cases.

However, there are fewer results for the more difficult output feedback problem for MJLS. In this context, the work [15] studied mode-dependent dynamic output feedback controllers with partly known transition probabilities; the papers [16], [17] were able to deal with the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ static output feedback control in the cluster and mode-independent cases by means of the so-called two-stage procedure of [18], [19]; and alternatively the work [20] tackled the design of reduced order controllers by means of an iterative procedure that requires a mode-dependent full order controller for starting the algorithm. Under a different hypothesis on the availability of the Markov mode, the work [21] studied the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ static output feedback control for continuous-time MJLS, where it is considered that the observation variable is a distinct Markov process that is also conditioned on the switching rule of the plant. As evidenced in the previous discussion, the problem of finding output feedback controllers that depend only on $\hat{\theta}(k)$ is a quite challenging problem.

We study in this work the $\mathcal{H}_2$ static output feedback problem in the detector approach of [13] and [14]. We present a new suboptimal condition for the design of static output feedback controllers that *depend only on the observation process* $\hat{\theta}(k)$ such that the closed-loop system is stochastically stable, while also guaranteeing that its $\mathcal{H}_2$ norm is bounded by some given constant γ. Our conditions are given in terms of linear matrix inequalities (LMIs) and are obtained by means of the so-called two-stage procedure that was originally presented in [18], [19] for the uncertain polytopic problem and applied to MJLS with uncertain transition probabilities in [17]. In order to illustrate our results and its potential application on AFTCS, we present a numerical example in the context of systems subject to failures.

*Research supported in part by the São Paulo Research Foundation - FAPESP, grants FAPESP-2015/09912-8 and FAPESP-2017/06358-5 for the first author; and the National Council for Scientific and Technological Development - CNPq, grant CNPq-304091/2014-6, the FAPESP/BG Brasil through the Research Center for Gas Innovation, grant FAPESP/BG-2014/50279-4, and the project INCT, grants FAPESP/INCT-2014/50851-0 and CNPq/INCT-465755/2014-3, for the second author.

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This work is organized as follows. The notation is introduced in Section II and the preliminary discussions are performed in Section III by introducing the definition of the system, the class of static controllers, the definitions of stochastic stability and $\mathcal{H}_2$ norm, and the problem formulation. In Section IV, we present the main result that consists of new sufficient LMI conditions that are based on the two-stage procedure of [18], [19]. This method is illustrated by Algorithm 1 and provides $\hat{\theta}(k)$-static output feedback controllers such that the closed-loop system is stochastically stable and its $\mathcal{H}_2$ norm is bounded. The numerical example is shown in Section V and our final remarks are stated in Section VI.

II. Notation

The notation used throughout is standard. As usual, the real n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. The space of $n \times m$ real matrices is represented by $\mathbb{B}(\mathbb{R}^m, \mathbb{R}^n)$. The superscript $'$ indicates the transpose of a matrix, the identity operator of order n is represented by I_n, the null operator, by $0_{n\times m}$, and the block diagonal matrix is denoted by $\mathbf{diag}(\cdot)$. For N and M positive integers, the sets $\mathbb{N}$ and $\mathbb{M}$ are defined as $\mathbb{N} := \{1, 2, 3..., N\}$ and $\mathbb{M} := \{1, 2, 3, ..., M\}$, respectively. Furthermore, the set $\mathbb{H}^{n,m}$ represents the linear space of all N-sequence of real matrices $V = \{V_1, V_2, ..., V_N\}$, $V_i \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, $i \in \mathbb{N}$ and we adopt, for simplicity, $\mathbb{H}^n := \mathbb{H}^{n,n}$ and $\mathbb{H}^{n+} := \{V \in \mathbb{H}^n; V_i \geq 0, i = 1, \ldots, N\}$. For $P, V \in \mathbb{H}^{n+}$, we write that $P > V$ if $P_i > V_i$ for each $i = 1, \ldots, N$, and set $\mathrm{Her}(G) := G + G'$ for $G \in \mathbb{B}(\mathbb{H}^n)$. On the probabilistic space $(\Omega, \mathcal{F}, P)$, $\mathbf{E}(\cdot)$ represents the expected value operator. The notation $l_2^r(\Omega, \mathcal{F}, \{\mathcal{F}_k\}, P)$ represents the space of all discrete-time signals $\mathcal{F}_k$-adapted processes such that

$$\|z\|_2 := \sqrt{\sum_{k=0}^{\infty} \mathbf{E}(\|z(k)\|^2)} < \infty.$$

III. Preliminaries

We consider the following Markov jump linear system (MJLS) on a probabilistic space $(\Omega, \mathcal{F}, P)$

$$\mathcal{G} : \begin{cases} x(k+1) &= A_{\theta(k)}x(k) + B_{\theta(k)}u(k) + J_{\theta(k)}w(k), \\ y(k) &= L_{\theta(k)}x(k) + H_{\theta(k)}w(k), \\ z(k) &= C_{\theta(k)}x(k) + D_{\theta(k)}u(k) + E_{\theta(k)}w(k), \\ x(0) &= x_0, \theta(0) = \theta_0, \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state variable, $u(k) \in \mathbb{R}^m$ is the control input, $w(k) \in \mathbb{R}^r$ is the exogenous input, $y(k) \in \mathbb{R}^p$ is the measured output, and $z(k) \in \mathbb{R}^q$ is the controlled output. The process $\theta(k)$ is a discrete-time Markov chain with transition probability matrix given by $\mathbb{P} = [p_{ij}]$ and initial probability distribution $\mu_i = P(\theta_0 = i)$.

In this work, we assume that $\theta(k)$ is not observable, but only an output $\hat{\theta}(k)$ related to the Markov chain and available to the controller. This modeling is called the hidden Markov chain (HMC), or the detector approach, for instance, we can refer to the book [22] for the HMCs and [13] for the detector approach applied to MJLS. Thus, denoting the set of all possible outcomes of $\hat{\theta}(k)$ by $\mathbb{M} = \{1, \ldots, M\}$ and $\hat{\mathcal{F}}_k = \{x(0), w(0), \theta(0), \hat{\theta}(0), \ldots, x(k), w(k), \theta(k)\}$, we assume that

$$P(\hat{\theta}(k) = l \mid \hat{\mathcal{F}}_k) = P(\hat{\theta}(k) = l \mid \theta(k)) = \alpha_{\theta(k)l},$$

for all $l \in \mathbb{M}$, and thus $\sum_{l\in\mathbb{M}} \alpha_{il} = 1$ for all $i \in \mathbb{N}$. The set of all possible outcomes of the detector for a given Markov mode $\theta(k)$ is denoted by $\mathbb{M}_{\theta(k)}$, and so it is clear that $\bigcup_{i\in\mathbb{N}} \mathbb{M}_i = \mathbb{M}$. We define the *detection probability matrix* by $\Upsilon = [\alpha_{il}]$.

Remark 1: The joint process $(\theta(k), \hat{\theta}(k))$ is called a hidden Markov chain, whose theory is well established in the literature, e.g., see [22]. Particularly this partial observation setting applied to Markov jump systems was studied in details, for instance, in the work [13], where it is shown that the complete observation, the cluster, and mode-independent cases for MJLS are generalized by the detector approach.

We want to synthesize static output feedback controllers depending only on the observable part $\hat{\theta}(k)$,

$$u(k) = K_{\hat{\theta}(k)}y(k), \tag{2}$$

and so we get the following closed-loop system

$$\mathcal{G}_K : \begin{cases} x(k+1) &= A_{\theta(k)\hat{\theta}(k)}x(k) + J_{\theta(k)\hat{\theta}(k)}w(k), \\ z(k) &= C_{\theta(k)\hat{\theta}(k)}x(k) + E_{\theta(k)\hat{\theta}(k)}w(k), \\ x(0) &= x_0,\ \theta_0,\ \hat{\theta}_0, \end{cases} \tag{3}$$

where A_{il}, J_{il}, C_{il}, and E_{il}, are given by

$$\left[\begin{array}{c|c} A_{il} & J_{il} \\ \hline C_{il} & E_{il} \end{array}\right] := \left[\begin{array}{c|c} A_i + B_iK_lL_i & J_i + B_iK_lH_i \\ \hline C_i + D_iK_lL_i & E_i + D_iK_lH_i \end{array}\right], \tag{4}$$

for $i \in \mathbb{N}$, $l \in \mathbb{M}_i$.

The concept of stochastic stability is presented below.

Definition 1 (Stochastic stability, [13]): System (3) with $w \equiv 0$ is said to be stochastically stable (SS) if $\|x\|_2^2 = \sum_{k=0}^{\infty} \mathbf{E}(\|x(k)\|^2) < \infty$, for every θ_0 and every finite second moment x_0.

Considering the structure in (2), we define the set $\mathbb{K}$ of admissible static output feedback controllers as

$$\mathbb{K} := \{K = \{K_1, \ldots, K_M\} : \text{(3) is SS}\}.$$

In order to study the stochastic stability of (3), we introduce the following operators for $V \in \mathbb{H}^n$:

$$\mathcal{E}_i(V) := \sum_{j\in\mathbb{N}} p_{ij}V_j,$$
$$\mathcal{L}_i(V) := \sum_{l\in\mathbb{M}_i} \alpha_{il}A'_{il}\mathcal{E}_i(V)A_{il},$$
$$\mathcal{T}_j(V) := \sum_{i\in\mathbb{N}} \sum_{l\in\mathbb{M}_i} p_{ij}\alpha_{il}A_{il}V_iA'_{il},$$

for all $i, j \in \mathbb{N}$, and $\mathcal{E}, \mathcal{L}, \mathcal{T} \in \mathbb{B}(\mathbb{H}^n)$, where A_{il} is given in (4) (it depends implicitly on the controller matrices). Also, for $U = \{U_{il} \in \mathbb{B}(\mathbb{R}^n), i \in \mathbb{N}, l \in \mathbb{M}_i\}$, we define

$$\mathcal{D}_j(U) := \sum_{i\in\mathbb{N}} \sum_{l\in\mathbb{M}_i} p_{ij}\alpha_{il}U_{il},$$

for all $j \in \mathbb{N}$. The following result taken from [13] states necessary and sufficient conditions for the stability of (3).

Theorem 1 ([13]): The following assertions are equivalent for a given K:

1) System (3) is SS.
2) There exists $0 < P \in \mathbb{H}^n$, such that $P - \mathcal{L}(P) > 0$.
3) There exists $0 < Q \in \mathbb{H}^n$, such that $Q - \mathcal{T}(Q) > 0$.

Proof: See [13]. ■

Definition 2 (The $\mathcal{H}_2$ norm for MJLS, [13]): Consider that $K \in \mathbb{K}$ and $x_0 = 0$. Let z_s be the controlled output of (3) for the exogenous input $w(k)$ defined as follows:

$$w(k) = \begin{cases} w_s, & k = 0, \\ 0, & k > 0, \end{cases} \tag{5}$$

where w_s is the s-th standard basis of $\mathbb{R}^r$. Then, the $\mathcal{H}_2$ norm of (3) is defined as follows:

$$\|\mathcal{G}_K\|_2^2 := \sum_{s=1}^{r} \|z_s\|_2^2, \quad \|z_s\|_2^2 = \sum_{k=0}^{\infty} \mathbf{E}\left(\|z_s(k)\|^2\right).$$

It is clear that the $\mathcal{H}_2$ norm for MJLS given in Definition 2 becomes equivalent to the deterministic case for $N = 1$. Furthermore, recalling that $\mu_i = P(\theta_0 = i)$, we introduce the following inequalities for a given controller structure K,

$$P - \mathcal{L}(P) > \mathbf{C}, \tag{6}$$

where $\mathbf{C} \in \mathbb{H}^{n+}$ is given by $\mathbf{C}_i := \sum_{l \in \mathbb{M}_i} \alpha_{il} C_{il}' C_{il}$. We have the following proposition adapted from [13] and [23].

Proposition 1 ([13], [23]): If (6) hold for $P > 0$ and a given K, then $K \in \mathbb{K}$, and also

$$\|\mathcal{G}_K\|_2 < \sum_{i \in \mathbb{N}} \sum_{l \in \mathbb{M}_i} \mu_i \alpha_{il} \operatorname{Tr}(J_{il}' \mathcal{E}_i(P) J_{il} + E_{il}' E_{il}). \tag{7}$$

Proof: See, for instance, [13] and [23]. ■

Remark 2: An alternative definition of the $\mathcal{H}_2$ norm given in [23] can be characterized as follows: consider that $w(k)$ is a white noise sequence with covariance matrix given by I_r and independent of the process $(\theta(k), \hat{\theta}(k))$ and the initial condition x_0, and also that the Markov chain is ergodic, for more information, see [22]. If $\nu = \mu$, where $\nu_i = \lim_{k \to \infty} P(\theta(k) = i)$, we have that $\|\mathcal{G}_K\|_2^2 = \lim_{k \to \infty} \mathbf{E}\left(\|z(k)\|^2\right)$. This interpretation will be used in the Monte Carlo simulation of the example in Section V.

Given the previous definitions, we state the main goal of this work as follows: finding $K \in \mathbb{K}$ such that $\|\mathcal{G}_K\|_2 < \gamma$, for the minimum γ possible. A first non-convex formulation for this problem is

$$\inf_{K \in \mathbb{K}} \{\gamma^2 \; ; \; (6) - (7)\}. \tag{8}$$

For obtaining $K \in \mathbb{K}$, we adopt the two-stage procedure introduced in [18] and extended to discrete-time polytopic systems in [19] in order to have an approximation for the non-convex optimization problem in (8).

IV. Main results

We adopt the two-stage procedure for obtaining the static output feedback gain (2) that was proposed in [18], and used in [16] for obtaining output feedback controllers for MJLS.

Consider the following LMIs for a given $\gamma > 0$, $F = \{F_l, l \in \mathbb{M}\}$, $i \in \mathbb{N}, l \in \mathbb{M}_i$, and system matrices in (4),

$$\sum_{i \in \mathbb{N}} \sum_{l \in \mathbb{M}_i} \mu_i \alpha_{il} \operatorname{Tr}(W_{il}) < \gamma^2, \tag{9}$$

$$\begin{bmatrix} W_{il} & \bullet & \bullet & \bullet \\ Z_{il} J_i & \operatorname{Her}(Z_{il}) - \mathcal{E}_i(P) & \bullet & \bullet \\ E_i & 0 & I & \bullet \\ Y_l H_i & B_i' Z_{il}' & D_i' & -\operatorname{Her}(X_l) \end{bmatrix} > 0, \tag{10}$$

$$P_i > \sum_{l \in \mathbb{M}_i} \alpha_{il} M_{il}, \tag{11}$$

$$\begin{bmatrix} M_{il} & \bullet & \bullet & \bullet \\ G_{il}(A_i + B_i F_l) & \Xi_{il} & \bullet & \bullet \\ C_i + D_i F_l & 0 & I & \bullet \\ -X_l F_l + Y_l L_i & B_i' G_{il}' & D_i' & -\operatorname{Her}(X_l) \end{bmatrix} > 0, \tag{12}$$

where $\Xi_{il} = \operatorname{Her}(G_{il}) - \mathcal{E}_i(P)$. We define the set of variables of (9)-(12) as follows,

$$\psi := \{P_i, W_{il}, M_{il}, G_{il}, Z_{il}, X_l, Y_l\} \cup \phi,$$

where $\phi = \emptyset$ if γ is not considered as a decision variable, and $\phi = \{\gamma_a\}, \gamma_a = \gamma^2$, otherwise. The set of all solutions of (10)-(12) is represented by

$$\Psi(F) := \{\psi; \; (9)\text{-}(12) \text{ hold}\}.$$

We have the following theorem whose goal is to design a controller in the form given in (2), inspired in [16], [19].

Theorem 2: Suppose that for a given F, we have that $\psi \in \Psi(F)$. Then by setting the control gain (2) as $K_l = X_l^{-1} Y_l$ for all $l \in \mathbb{M}$, we have that $K \in \mathbb{K}$ and $\|\mathcal{G}_K\| < \gamma^2$.

Proof: If (9)-(12) holds, we have that $-\operatorname{Her}(X_l) > 0$, that implies that X_l is non-singular. In this case, we set $K_l = X_l^{-1} Y_l$ and rewrite (12) as $\Phi_{il}^{(1)} + \operatorname{Her}(U' X_l V_{il}^{(1)}) > 0$, where

$$\Phi_{il}^{(1)} := \begin{bmatrix} M_{il} & \bullet & \bullet & \bullet \\ G_{il} \tilde{A}_{il} & \operatorname{Her}(G_{il}) - \mathcal{E}_i(P) & \bullet & \bullet \\ \tilde{C}_{il} & 0 & I & \bullet \\ 0 & B_i' G_{il}' & D_i' & 0 \end{bmatrix}, \tag{13}$$

$U := \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix}$, $V_{il}^{(1)} := \begin{bmatrix} S_{il} & 0 & 0 & -I \end{bmatrix}$, $S_{il} := K_l L_i - F_l$, as well as $\tilde{A}_{il} := A_i + B_i F_l$ and $\tilde{C}_{il} := C_i + D_i F_l$. By means of the Elimination Lemma (see [19], [24]), we take the orthogonal complements of $V_{il}^{(1)}$ and U by

$$\mathcal{N}_{v_1} := \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ S_{il} & 0 & 0 \end{bmatrix}, \mathcal{N}_u := \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix},$$

respectively, and so by multiplying (13) to the left hand side by $\mathcal{N}_{v_1}'$ and to the right by its transpose, we get that (12) holds if and only if

$$\begin{bmatrix} M_{il} & \bullet & \bullet \\ G_{il}(\tilde{A}_{il} + B_i S_{il}) & \operatorname{Her}(G_{il}) - \mathcal{E}_i(P) & \bullet \\ \tilde{C}_{il} + D_i S_{il} & 0 & I \end{bmatrix} > 0, \tag{14}$$

for all $i \in \mathbb{N}$, $l \in \mathbb{M}_i$. Thus we have that $\tilde{A}_{il} + B_i S_{il} = A_i + B_i K_l L_i = A_{il}$, as well as $\tilde{C}_{il} + D_i S_{il} = C_i + D_i K_l L_i =$

C_{il}, that are the matrices A_{il} and C_{il} in (4). Similarly, the other inequality set from the Elimination Lemma would be obtained by applying $\mathcal{N}_u$ in (12), yielding (14) with $S_{il}=0$. Considering that $G_{il}\mathcal{E}_i(P)^{-1}G'_{il} \geq \text{Her}(G_{il}) - \mathcal{E}_i(P)$ (see [25]) in the previous inequality set, applying the congruence transformation $\mathbf{diag}(I, G_{il}^{-1}, I)$ and the Schur complement, multiplying the resulting inequalities by α_{il}, summing them up for all $l \in \mathbb{M}_i$, and considering (11) we get (6). Similarly, we can rewrite (10) as $\Phi_{il}^{(2)} + \text{Her}(U'X_l V_{il}^{(2)}) > 0$, where $V_{il}^{(2)} := \begin{bmatrix} T_{il} & 0 & 0 & -I \end{bmatrix}$, $T_{il} := K_l H_i$, and U as previously defined, as well as

$$\Phi_{il}^{(2)} := \begin{bmatrix} W_{il} & \bullet & \bullet & \bullet \\ Z_{il}J_{il} & \text{Her}(Z_{il}) - \mathcal{E}_i(P) & \bullet & \bullet \\ E_{il} & 0 & I & \bullet \\ 0 & B'_i Z'_{il} & D'_i & 0 \end{bmatrix}. \tag{15}$$

Defining the orthogonal complement of $V_{il}^{(2)}$ by

$$\mathcal{N}_{v_2} := \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ T_{il} & 0 & 0 \end{bmatrix}$$

and multiplying (15) to the left hand side by $\mathcal{N}'_{v_2}$ and to the right by its transpose, we get that (10) holds if and only if

$$\begin{bmatrix} W_{il} & \bullet & \bullet \\ Z_{il}(J_i + B_i T_{il}) & \text{Her}(Z_{il}) - \mathcal{E}_i(P) & \bullet \\ E_i + D_i T_{il} & 0 & I \end{bmatrix} > 0, \tag{16}$$

for all $i \in \mathbb{N}$, $l \in \mathbb{M}_i$. Recalling that $T_{il} = K_l H_i$, we get $J_{il} = J_i + B_i T_{il}$ and $E_{il} = E_i + D_i T_{il}$, that are the matrices J_{il} and E_{il} in (4). Similarly, the other inequality set from the Elimination Lemma is obtained by applying $\mathcal{N}_u$ in (10), yielding (16) with $T_{il} = 0$. Finally, considering the similar reasoning previously applied to (14), Equation (16) implies that $W_{il} > J'_{il}\mathcal{E}_i(P)J_{il} + E'_{il}E_{il}$ also holds, and so by applying the trace operator, multiplying them by $\mu_i\alpha_{il}$, summing them up for all $l \in \mathbb{M}_i$ and $i \in \mathbb{N}$, we get the upper bound in (7). Thus by Proposition 1 and (9), we get that $K \in \mathbb{K}$ and $\|\mathcal{G}_K\|_2 < \gamma$, and so the claim follows. ∎

Specifically on the nature of F, it is clear that, by setting $S_{il} = 0$ on (14) we get precisely the other inequality set of the Elimination Lemma as discussed in the proof of Theorem 2, and so we have that a necessary condition for (12) to hold is that system (3) controlled by $u(k) = F_{\hat{\theta}(k)}x(k)$ is SS (see also [19]). Considering this fact, in the next theorem we provide a possible form of calculating F. For that, we present the following LMI set taken from [13], for all $i \in \mathbb{N}$, $l \in \mathbb{M}_i$,

$$\begin{bmatrix} M_{il} & A_i G_l + B_i Y_l \\ \bullet & \text{Her}(G_l) - \mathcal{D}_i(M) \end{bmatrix} > 0. \tag{17}$$

We have the following theorem.

Theorem 3 ([13]): If (17) holds for $M_{il} > 0, i \in \mathbb{N}, l \in \mathbb{M}_i$, G_l and Y_l, $l \in \mathbb{M}$, then system (1) controlled by a state feedback control law $u(k) = F_{\hat{\theta}(k)}x(k)$ with gains given by $F_l = Y_l G_l^{-1}$, $l \in \mathbb{M}$, is SS.

Proof: See [13] and Theorem 1. ∎

Given the result of Theorem 2 and F (calculated, for instance, by Theorem 3), we can obtain the best upper bound of the $\mathcal{H}_2$ norm of (3) by the following optimization problem,

$$\inf_{\psi \in \Psi} \gamma_a, \tag{18}$$

for $\gamma_a = \gamma^2$, where (18) is an approximation of (8).

The two-stage procedure is summarized in Algorithm 1. We stress that the result of Theorem 2 *depends explicitly on the existence of a given set of matrices F* in order to provide a solution for (9)-(12), thus the optimization problem (18) will depend on this choice of F.

Algorithm 1 The two-stage procedure, based on [18]

1: Calculate the stochastic stabilizing state feedback gains F through Theorem 3;
2: Use F as an input in (18) and calculate the static output feedback gains $K \in \mathbb{K}$.

Remark 3: In the case that (9)-(12) do not hold for the controller set F calculated by means of Theorem 3, some alternative state feedback design conditions can be found, for instance, in the works [13], [14], [26].

V. Numerical Example

All calculations were performed using MATLAB R2015 along with LMILAB, see, for instance, the manual [27]. We consider a modified version of the system taken from [9] that consists of a classical two mass system coupled with a spring and a damper. The position and the velocity of the first mass are measured by an imperfect sensor that may fail according to a Markov chain considering three possibilities: for $\theta(k) = 1$, the sensor succeeds in transmitting the signal with a nominal noise level; for $\theta(k) = 2$ the transmitted signal is attenuated and the noise level is higher; finally, given that $\theta(k) = 3$ the sensor may fail completely and so the signal is lost. The transition probability matrix and the initial probability distribution are given by

$$\left[\begin{array}{c|c} \mathbb{P} & \mu' \end{array}\right] = \left[\begin{array}{ccc|c} 0.7000 & 0.3000 & 0 & 0.1892 \\ 0.1000 & 0.6000 & 0.3000 & 0.5676 \\ 0 & 0.7000 & 0.3000 & 0.2432 \end{array}\right].$$

In order to get a discrete-time formulation, a zero order hold of sampling period 0.6 s is used. In this setting, we want to control the position and velocity of the second mass by choosing, for all $i \in \mathbb{N}$, $E_i = 0_{3\times 2}$, and

$$\left[\begin{array}{c|c} C_i & D_i \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 50.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.8000 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 \end{array}\right].$$

The behavior of the sensor is modeled by

$$\left[\begin{array}{c|c} L_1 & H_1 \\ \hline L_2 & H_2 \\ \hline L_3 & H_3 \end{array}\right] = \left[\begin{array}{cccc|cc} 1.0000 & 0 & 0 & 0 & 0 & 0.1000 \\ 0 & 0 & 1.0000 & 0 & 0 & 0.1250 \\ \hline 0.1000 & 0 & 0 & 0 & 0 & 0.1500 \\ 0 & 0 & 0.1000 & 0 & 0 & 0.1750 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The remaining system matrices (A_i, B_i, J_i), $i \in \mathbb{N}$, can be calculated from [9].

We consider a detector of $\theta(k)$ with the following behavior: the nominal and attenuated states, that is, $\theta(k) = 1$ and $\theta(k) = 2$, cannot be perfectly distinguished, however the detector will surely know whenever a transmission is lost. For modeling this behavior, we consider the following *detection probability matrix*,

$$\Upsilon = \begin{bmatrix} \rho & 1-\rho & 0 \\ 1-\rho & \rho & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{19}$$

where $\rho = P(\hat{\theta}(k) = i \mid \theta(k) = i)$, $i \in \{1, 2\}$, is the probability of correct detection. We want to calculate static output feedback controllers, their respective upper bounds γ and actual $\mathcal{H}_2$ norms of the closed-loop system, represented by $\gamma_K := \|\mathcal{G}_K\|_2$.

In order to illustrate the two-stage procedure presented in Algorithm 1, we set $\rho = 0.7$ in (19) and calculate the following state feedback gains by means of Theorem 3 in the first step:

$$\begin{bmatrix} F_1 \\ \hline F_2 \\ \hline F_3 \end{bmatrix} = \begin{bmatrix} -1.4028 & 2.6006 & 0.2422 & -1.4010 \\ \hline -1.5552 & 2.5486 & 0.2355 & -1.4414 \\ \hline -2.3013 & 2.5311 & 0.2775 & -1.5053 \end{bmatrix}.$$

This gain set F is used in the second step of Algorithm 1, resulting in $\gamma = 68.5866$ and $\gamma_K = 56.4138$, and the following static output feedback gains that *switch according to the observation process*, $\hat{\theta}(k)$:

$$\begin{bmatrix} K_1 \\ \hline K_2 \\ \hline K_3 \end{bmatrix} = \begin{bmatrix} 4.3613 & -1.4381 \\ \hline 5.6084 & -2.5923 \\ \hline 0 & 0 \end{bmatrix}.$$

Figure 1 shows the mean value curve of $\|z(k)\|^2$ and its standard deviation, along with γ and γ_K, in function of k for a Monte Carlo simulation of 2000 rounds by considering $w(k)$ as a white noise sequence whose covariance matrix is given by I_2 (see Remark 2). The curves of Figure 1 illustrate two

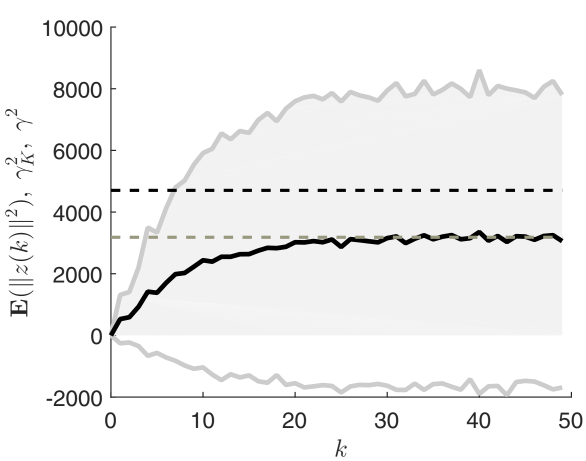

Fig. 1. $\mathbf{E}(\|z(k)\|^2) \pm \sigma$ (full black line), γ_K^2 (dashed grey line) and γ^2 (dashed black line) in function of k.

aspects of both the $\mathcal{H}_2$ control and our formulation. The first point concerns the stochastic version of the $\mathcal{H}_2$ control for MJLS, that is discussed in Remark 2 and, in more details, in [2] and [23]. In this setting, we have that $\mathbf{E}(\|z(k)\|^2) \to \gamma_K^2$ as $k \to \infty$ given that $w(k)$ is the aforementioned white noise sequence. Besides, the conservatism of our conditions, along with the fact that we are dealing with a partial observation case, is illustrated by means of the gap between the actual $\mathcal{H}_2$ norm γ_K and the guaranteed cost value γ represented by the dashed black line in Figure 1.

On the other hand, we want also to investigate the behavior of the two-stage procedure as we vary the entries of the detection probability matrix given in (19). Intuitively, we get some cases of interest, for instance, by setting $\rho = 1.0$, that yields the complete observation case (recalling that $\rho = P(\hat{\theta}(k) = i \mid \theta(k) = i)$, $i \in \{1, 2\}$). Similarly, due to the structure of the *detection probability matrix* in (19), we would also have this perfect observation scenario by setting $\rho = 0$: if we know for sure that $\hat{\theta}(k)$ is wrong for only two given possible outcomes, we can always assume that the other one is correct. Finally, by taking $\rho = 0.5$, we would have a cluster case scenario (see [23]), as the detector would be unable to tell the difference between $\theta(k) = 1$ and $\theta(k) = 2$. In order to effectively illustrate this result, we find F through Theorem 3 and the final controllers K by means of (18) for $\rho \in [0, 1]$. The curves of guaranteed costs γ and actual $\mathcal{H}_2$ norms are shown in Figure 2. We

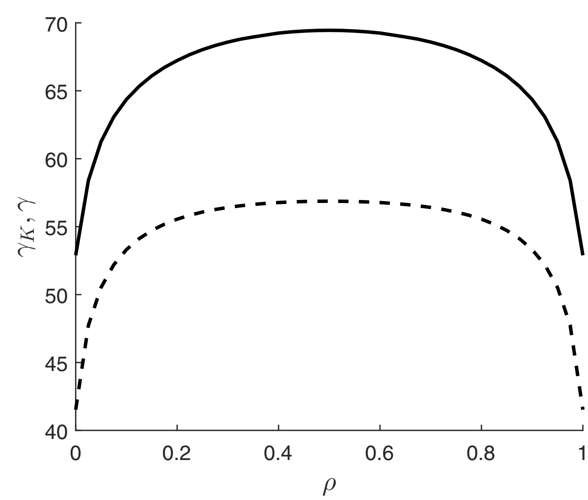

Fig. 2. γ (full line) and γ_K (dashed line) in function of the probability of correct detection ρ.

point out that even in the complete observation case, that is $\rho = 1.0$, we have that $\gamma_K \neq \gamma$. This illustrates the suboptimal nature of problem (18) that can not retrieve the complete observation cost, instead yielding conservative results. On the other hand, considering the behavior of the costs in Figure 2 in relation to ρ, we have a symmetry with respect to $\rho = 0.5$, that configures the worst cost situation in this example. The controller matrices depending on $\hat{\theta}(k)$ for $\rho = 1.0$ (the complete observation gains represented by "K_l") are given by

$$\begin{bmatrix} K_1 \\ \hline K_2 \\ \hline K_3 \end{bmatrix} = \begin{bmatrix} 3.8961 & -1.0638 \\ \hline 18.1876 & -14.4088 \\ \hline 0 & 0 \end{bmatrix}$$

with $\gamma = 52.8922$ and $\gamma_K = 41.5258$, as well as for $\rho = 0.5$ (the cluster gains "$K_l^{(c)}$")

$$\begin{bmatrix} K_1^{(c)} \\ \hline K_2^{(c)} \\ \hline K_3^{(c)} \end{bmatrix} = \begin{bmatrix} 4.8454 & -1.8493 \\ \hline 4.8454 & -1.8493 \\ \hline 0 & 0 \end{bmatrix},$$

for $\gamma^{(c)} = 69.4509$ and $\gamma_K^{(c)} = 56.8727$. We point out that for both settings, the gain in $\hat{\theta}(k) = 3$ is zero due to the failure of the sensor in this mode of operation. Moreover, in the cluster case, we have precisely the same gain for both $\hat{\theta}(k) = 1$ and $\hat{\theta}(k) = 2$ caused by the aforementioned effect of the detector being unable to tell the difference between those two modes of operation.

VI. Conclusion

In this work we studied the $\mathcal{H}_2$ static output feedback control of MJLS considering that the Markov mode $\theta(k)$ is not available for the controller, where the only information of the underlying Markov chain is given by some detector $\hat{\theta}(k)$ in the spirit of the so-called hidden Markov models in [22]. Sufficient LMI conditions for the synthesis of static output feedback controllers that *depend only on the available process* $\hat{\theta}(k)$ are given in order to ensure the stochastic stability of the closed-loop system, as well as guaranteeing that its $\mathcal{H}_2$ norm is bounded by some constant γ. The technique used in this work relies on the so-called two-stage procedure of [18], [19] that was also applied to MJLS in the works [16], [17] for the case of uncertain transition probabilities. A numerical example concerning a system whose sensor is subject to noise and failures is given in order to illustrate our results.

Considering the extension of this work, we want to reduce the conservatism of our result by improving the given LMI conditions through the use of slack variables, and also by the use of iterative algorithms. Furthermore, the design of $\mathcal{H}_\infty$ static output feedback, as well of higher order controllers by means of the two-stage procedure, is also a topic of interest. For the latter, we believe that there are some variations that were not yet explored in the literature, mainly concerning dual operators in the MJLS theory.

References

[1] I. Hwang, S. Kim, Y. Kim, and C. Seah, "A survey of fault detection, isolation, and reconfiguration methods," *IEEE Transactions on Control Systems Technology*, vol. 18, no. 3, pp. 636–653, May 2010.

[2] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-Time Markov Jump Linear Systems*. Springer, 2005.

[3] E.-K. Boukas, *Stochastic Switching Systems: Analysis and Design*. Birkhäuser, 2006.

[4] O. L. V. Costa, M. D. Fragoso, and M. G. Todorov, *Continuous-Time Markov Jump Linear Systems*. New York, NY, USA: Springer, 2013.

[5] V. Dragan, T. Morozan, and A.-M. Stoica, *Mathematical Methods in Robust Control of Linear Stochastic Systems*. New York, NY, USA: Springer, 2013.

[6] M. Mahmoud, J. Jiang, and Y. Zhang, *Active Fault Tolerant Control Systems - Stochastic Analysis and Synthesis*. Germany: Springer, 2003.

[7] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138–162, 2007.

[8] A. P. C. Gonçalves, A. R. Fioravanti, and J. C. Geromel, "Markov jump linear systems and filtering through network transmitted measurements," *Signal Processing*, vol. 90, no. 10, pp. 2842–2850, 2010.

[9] J. C. Geromel, A. P. C. Gonçalves, and A. R. Fioravanti, "Dynamic output feedback control of discrete-time Markov jump linear systems through linear matrix inequalities," *SIAM Journal on Control and Optimization*, vol. 48, no. 2, pp. 573–593, 2009.

[10] O. L. V. Costa and E. F. Tuesta, "H_2-control and the separation principle for discrete-time Markovian jump linear systems," *Mathematics of Control, Signals and Systems*, vol. 16, no. 4, pp. 320–350, 2004.

[11] J. B. do Val, J. C. Geromel, and A. P. C. Gonçalves, "The H_2-control for jump linear systems: cluster observations of the Markov state," *Automatica*, vol. 38, no. 2, 2002.

[12] A. R. Fioravanti, A. P. C. Gonçalves, and J. C. Geromel, "Optimal H_2 and H_∞ mode-independent control for generalized Bernoulli jump systems," *ASME. Journal Dynamic Systems, Measurement and Control*, vol. 136, no. 4, pp. 011 004–011 004–6, 2013.

[13] O. L. V. Costa, M. D. Fragoso, and M. G. Todorov, "A detector-based approach for the H_2 control of Markov jump linear systems with partial information," *IEEE Transactions on Automatic Control*, vol. 60, no. 5, pp. 1219–1234, 2015.

[14] M. G. Todorov, M. D. Fragoso, and O. L. V. Costa, "A new approach for the H_∞ control of Markov jump linear systems with partial information," in *2015 54th IEEE Conference on Decision and Control (CDC)*, 2015, pp. 3592–3597.

[15] L. Zhang and E.-K. Boukas, "H_∞ control for discrete-time Markovian jump linear systems with partly unknown transition probabilities," *International Journal of Robust and Nonlinear Control*, vol. 19, no. 8, pp. 868–883, 2009.

[16] C. F. Morais, M. F. Braga, R. C. L. F. Oliveira, and P. L. D. Peres, "An LMI approach for H_2 dynamic output feedback control of discrete-time Markov systems with uncertain probabilities," in *2016 IEEE Conference on Computer Aided Control System Design (CACSD)*, Sept 2016, pp. 1–6.

[17] ——, "$\mathcal{H}_\infty$ static output feedback control of discrete-time Markov jump linear systems with uncertain transition probability matrix," in *2014 American Control Conference*, June 2014, pp. 489–494.

[18] D. Peaucelle and D. Arzelier, "An efficient numerical solution for H_2 static output feedback synthesis," in *2001 European Control Conference (ECC)*, Sept 2001, pp. 3800–3805.

[19] D. Mehdi, E. K. Boukas, and O. Bachelier, "Static output feedback design for uncertain linear discrete time systems," *IMA Journal of Mathematical Control and Information*, vol. 21, no. 1, pp. 1–13, 2004.

[20] C. F. Morais, M. F. Braga, R. C. L. F. Oliveira, and P. L. D. Peres, "Reduced-order dynamic output feedback control of uncertain discrete-time Markov jump linear systems," *International Journal of Control*, vol. 90, no. 11, pp. 2368–2383, 2017.

[21] S. Aberkane, J. C. Ponsart, M. Rodrigues, and D. Sauter, "Output feedback control of a class of stochastic hybrid systems," *Automatica*, vol. 44, no. 5, pp. 1325–1332, 2008.

[22] S. M. Ross, *Introduction to Probability Models*, 10th ed. Orlando, FL, USA: Academic Press, Inc., 2010.

[23] A. M. de Oliveira and O. L. V. Costa, "H_2-filtering for discrete-time hidden Markov jump systems," *International Journal of Control*, vol. 90, no. 3, pp. 599–615, 2017.

[24] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. SIAM studies in applied mathematics, 1994.

[25] M. C. de Oliveira, J. Bernussou, and J. C. Geromel, "A new discrete-time robust stability condition," *Systems & Control Letters*, vol. 37, no. 4, pp. 261–265, 1999.

[26] A. M. de Oliveira and O. L. V. Costa, "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ of hidden Markov jump systems," *International Journal of Robust and Nonlinear Control*, vol. 28, no. 4, pp. 1261–1280, 2018.

[27] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox: For Use with MATLAB. User's Guide. Version 1*, ser. Computation, visualization, programming. MathWorks, 1995.