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# Experimentally verified multi-objective iterative learning control design with frequency domain specifications

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***Abstract*— This paper considers the design of multi-objective iterative learning control (ILC) schemes for discrete linear systems. In particular, a two step design is developed where the feedback and learning controllers are designed separately. The design procedures developed are able to impose performance specifications over selected frequency ranges to obtain the desired shapes of the sensitivity functions relevant to a particular example. This makes it possible to reject disturbances in specific frequency ranges and hence the desired performance specifications for both the transient response and error convergence speed can be achieved. All design computations required can be completed using linear matrix inequalities (LMIs). Preliminary experimental results demonstrate the effectiveness of the new design are also given..**

## I. Introduction

In many applications, e.g. pick and place robots and batch chemical processes, the system has to follow the same trajectory (reference signal) repeatedly. If feedback control only is applied and the reference signal is a dominant disturbance, the tracking error is (approximately) the same for each repetition of a given task. Iterative Learning Control (ILC) can be introduced into the control system to reduce the repetitive part of the tracking error during subsequent repetitions using a feedforwad signal update. Specifically, ILC is a feedforward control scheme for improving the tracking response of systems that repeat a given task or operation defined over a finite duration. Each repetition is known as a trial, or pass, and when a trial is complete, the system resets to the same initial conditions and the next trial can begin, either immediately after the resetting is complete or after a further period of time has elapsed.

The core advantage of this control structure is the use information from the previous trial to update the control input applied on the next trial and thereby improve the trial-to-trial performance. This feature is a major reason why ILC has been extensively employed in high precision control systems and in other areas, see, [1], [2] as one starting point for the literature.

In application, the objective of ILC is to construct the control input signal such that the output tracks the reference as accurately as possible. Let $y_k(p)$ denote the scalar or vector valued output where $k \geq 0$ denotes the trial number and $0 \leq p \leq \alpha - 1$, where for discrete dynamics $\alpha$ is the number of trials along a pass, where $\alpha$ times the sampling period gives the trial length. Suppose that $y_d(p)$ denotes the specified reference or reference vector. Then the error on trial $k$ is $e_k(p) = y_d(p) - y_k(p)$ and let $\{e_k\}_k$ denote the error sequence generated over the trials.

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Then the basic ILC problem is to control law design to ensure that $\{e_k\}_k$ converges in $k$. A critical property of ILC is that all data generated when it is complete is available for use in designing the next trial input. Hence at sample instant $p$ on trial $k+1$ information from sample $p+\lambda$, $\lambda > 0$ on trial $k$. An ILC with this feature is term phase-lead and has been widely used in applications reported so far.

Another common ILC laws that includes previous trial information is

$$u_{k+1}(p) = u_k(p) + Le_k(p),$$

where $u_k(p)$ denotes the input on trial $k$ and time instant $p$, $L$ acts on the error at the previous trial and is also often termed the learning filter in the literature. Frequently, an ILC law is augmented with a feedback controller that stabilizes the system and suppresses unknown disturbances. This construction results in a so-called current trial ILC scheme. The ILC law is designed to guarantee convergence in the trial domain and in many cases its construction is based on the inverse of the plant dynamics (see, e.g., the relevant references in [2]) resulting from the design and application of the feedback loop.

The design of the feedback and learning controllers must include some requirements on transient dynamics and trial-to-trial error convergence. Often these requirements are defined over the complete frequency spectrum. This is a very strict condition since design requirements and specifications are mostly defined for different frequency ranges of relevance. For example, a closed-loop feedback control system should have small sensitivity in a low frequency range and small complementary sensitivity in the high frequency range. Moreover, since the bandwidth of the reference signal has the strongest influence on the convergence rate, learning over this frequency range should only occur when the ILC scheme is applied.

The contribution of this paper is to provide new insights into the currently known two step ILC design procedures. Specifically, systematic guidelines for design of ILC laws over limited frequency range are proposed. The generalized version of Kalman-Yakubovich-Popov (KYP) lemma [3] is extensively used to permit control law design over selected frequency ranges. These controller design procedures can also include multiple design specifications (e.g., reject disturbances at specific frequencies), whereas the vast majority of currently known designs cannot impose many relevant additional performance specifications. In particular, the developed results allows a designer to specify and/or maximize, frequency ranges where the error convergence condition has to be satisfied. Moreover, the design is based on using control law parametrization in a finite impulse response (FIR) filter form. This allows design procedures over convex sets and therefore they are amenable to effective algorithmic solution in terms of LMIs.

The notation used in this paper is as follows. The null and identity matrices with compatible dimensions are denoted by 0 and $I$ respectively. The notation $X \succ Y$ (respectively $X \prec Y$) means that the symmetric matrix $X - Y$ is positive definite (respectively negative definite). The symbol $(\star)$ denotes block entries in symmetric matrices and $\rho(\cdot)$ and $\overline{\sigma}(\cdot)$ denote the spectral radius and maximum singular value of their matrix arguments, respectively. Finally, the superscript $*$ denotes the complex conjugate transpose of a matrix and $\otimes$ the matrix Kronecker product.

The analysis in the remainder of this paper makes use of the following result, known as the generalized KYP lemma.

*Lemma 1:* [3] For a given linear discrete time-invariant system with the transfer-function matrix $M(z)$ and frequency response matrix $M(e^{j\theta}) = C(e^{j\theta}I - A)^{-1}B + D$, the following inequalities are equivalent

(i) $\begin{bmatrix} M(e^{j\theta}) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} M(e^{j\theta}) \\ I \end{bmatrix} \prec 0, \quad \forall \theta \in \Theta,$
where $\Pi$ is a given real symmetric matrix and $\Theta$ denotes the frequency ranges (see Table I).

(ii)
$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T (\Phi \otimes P + \Psi \otimes R) \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \prec 0, \tag{1}$$
where $R \succ 0$, $P$ is a symmetric matrix, $\Phi = \mathrm{diag}\{-1, 1\}$ and $\Psi$ is specified with the reference to the chosen frequency range - see the below Table I.

TABLE I
FREQUENCY RANGES OF INTEREST

| | LF (low freq.) | MF (middle freq.) | HF (high freq.) |
|---|---|---|---|
| $\Theta$ | $\lvert\theta\rvert \leq \varpi_l$ | $\varpi_1 \leq \theta \leq \varpi_2$ | $\lvert\theta\rvert \geq \varpi_h$ |
| $\Psi$ | $\begin{bmatrix} 0 & 1 \\ 1 & -2\cos(\varpi_l) \end{bmatrix}$ | $\begin{bmatrix} 0 & e^{j\varpi_a} \\ e^{-j\varpi_a} & -2\cos(\varpi_b) \end{bmatrix}$ | $\begin{bmatrix} 0 & -1 \\ -1 & 2\cos(\varpi_h) \end{bmatrix}$ |

where $\varpi_a = \frac{\varpi_1 + \varpi_2}{2}$ and $\varpi_b = \frac{\varpi_2 - \varpi_1}{2}$.

## II. Problem Formulation

This paper addresses the case when the plant dynamics to be controlled can be modeled by the following discrete linear time-invariant state-space model over $0 \leq p \leq \alpha - 1$, $k \geq 0$ as written in the ILC setting as

$$\begin{aligned} x_k(p+1) &= Ax_k(p) + Bu_k(p), \\ y_k(p) &= Cx_k(p), \end{aligned} \tag{2}$$

where $\alpha < \infty$ denotes the number of samples over the finite trial length, $x_k(p) \in \mathbb{R}^n$, $y_k(p) \in \mathbb{R}^m$ and $u_k(p) \in \mathbb{R}^l$ are the state, output and input vectors on trial $k$.

The block diagram of the ILC configuration considered in this paper is shown in Fig. 1. It consists of a unity negative feedback control loop with the feedback controller $C(z)$ applied on the current trial $k$ to ensure stability and/or the required along the trial dynamics. The memory block

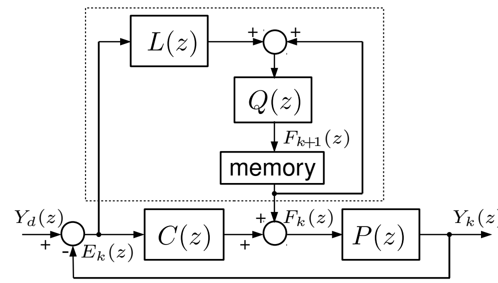


Fig. 1. ILC block diagram representation.

stores the previous trial information which is used for the computation of the next trial control input and $Y_d(z)$ denotes the reference. The block $L(z)$ denotes the learning controller or filter and $Q(z)$ is the robustness filter. All computations within the dashed box are completed off-line during the time elapsed between the end of one trial and the beginning of the next one.

Fron Fig. 1 the ILC law can be represented as:

$$F_{k+1}(z) = Q(z)\left(F_k(z) + L(z)E_k(z)\right),$$

and hence the previous trial error feedforward contribution (assuming $Y_d(z) = 0$) to the current trial error is

$$E_k(z) = -\left[(I + P(z)C(z))^{-1}P(z)\right] F_k(z) = -S_P(z)F_k(z),$$

where $S_P(z) = (I + P(z)C(z))^{-1}P(z)$ denotes the process sensitivity function and the propagation of the error from trial-to-trial is given by

$$E_{k+1}(z) = Q(z)\left(I - S_P(z)L(z)\right)E_k(z).$$

Introducing

$$M(z) = Q(z)\left(I - S_P(z)L(z)\right), \tag{3}$$

it follows that the condition for trial-to-trial error convergence can been formulated in $\mathcal{H}_\infty$ control terms as [2]:

$$\|M(z)\|_\infty \triangleq \sup_{\theta \in [-\pi, \pi]} \overline{\sigma}(M(e^{j\theta})) < 1, \tag{4}$$

and minimizing $\|M(z)\|_\infty$ increases the trial-to-trial convergence rate.

Given (3) and (4), it follows that fast trial-to-trail error convergence will occur if $L(z) \approx S_P^{-1}(z)$ for the entire frequency range. However, if $S_P(z)$ is strictly proper its exact inverse is improper. Hence, attention is limited to the frequency range where $L(z)$ can be a good approximation to $S_P^{-1}(z)$. Undesirable frequencies should be cut-off by the $Q(z)$ because the inverse of $S_P(z)$ is not sufficiently matched at higher frequencies. The next sections consider the separate design of $C(z)$ and $L(z)$ to satisfy the error convergence condition over a finite frequency range only and then a suitable cut-off frequency of $Q(z)$ is specified.

## III. Parametrization of a feedback controller

Having obtained conditions for ILC convergence, there is a need for a framework to impose specifications beyond the trial-to-trial error convergence only. In particular, it is required to attenuate the influence of some exogenous signals (such as disturbances or noise) to meet the control performance specifications. Hence, this section starts with the following more general model than (2) of a process to be controlled over $0 \le p \le \alpha-1,\ k \ge 0$,

$$\begin{aligned} x_k(p+1) &= Ax_k(p)+B_{x_1}w_k^1(p)+B_{x_2}w_k^2(p)+Bu_k(p)\\ z_k^1(p) &= C_{z_1}x_k(p)+D_{z_1}w_k^1(p)+D_1u_k(p)\\ z_k^2(p) &= C_{z_2}x_k(p)+D_{z_2}w_k^2(p)+D_2u_k(p)\\ y_k(p) &= Cx_k(p)+D_{y_1}w_k^1(p)+D_{y_2}w_k^2(p) \end{aligned} \tag{5}$$

where $w^i \in \mathbb{R}^r$ is the disturbance input, $z^i \in \mathbb{R}^{q_i}$ is the controlled output and $y \in \mathbb{R}^m$ is the measured output. Also, let a state-space representation of the feedback controller $C(z)$ be given by

$$\begin{aligned} x_k^c(p+1) &= A_cx_k^c(p)+B_cy_k(p),\\ u_k(p) &= C_cx^c(p)+D_cy(p), \end{aligned} \tag{6}$$

where $x^c \in \mathbb{R}^{n_c}$ is the state vector of the controller, which can also be described by $C(z) = D_c + C_c(zI-A_c)^{-1}B_c$.

It is known that multi-objective control problems are hard to solve when the controller structure (6) is directly applied. The crucial problem is that the closed-loop transfer function matrix, on which design specifications are imposed, is nonlinearly dependent on the controller $C(z)$. Hence the resulting design conditions are nonconvex and no computationally effective method exists for checking them.

A multi-objective control problem can be reformulated to one that is convex by applying a particular set of transformations as given in [4]. To proceed, the controller (6) is transformed to the observer-based structure by defining

$$\hat{x}_k(p+1)=(A+BD_cC-BK-LC)\hat{x}_k(p)+Ly_k(p)+Bv_k(p), \tag{7}$$

where $\hat{x}_k(p)$ in an estimate of the state vector and $v_k(p)$ is an additional signal to be used later, $L$ is the observer gain and $K$ is the state feedback gain matrix. Hence the control law takes the form

$$u_k(p) = -K\hat{x}_k(p)+v_k(p). \tag{8}$$

The matrices $K$ and $L$ can be computed using standard techniques such as LQG. Define the state estimation error as $e_k(p) = y_k(p) - C\hat{x}_k(p)$. Using the observer (7) and the control law (8), the system (5) and the controller (6) give the closed-loop system

$$\begin{bmatrix} z_k^i(p)\\ e_k(p)\end{bmatrix} = \left[\begin{array}{c:c} \mathcal{A}_i & \mathcal{B}_i\\ \hdashline \mathcal{C}_i & \mathcal{D}_i\end{array}\right]\begin{bmatrix} w_k^i(p)\\ v_k(p)\end{bmatrix}, \tag{9}$$

where $i = 1, 2$ and

$$\mathcal{A}_i = \begin{bmatrix} A+BD_cC-BK & BK\\ 0 & A+BD_cC-LC\end{bmatrix},$$

$$\mathcal{B}_i = \begin{bmatrix} B_{x_i}+BD_cD_{y_i} & B\\ B_{x_i}+BD_cD_{y_i}-LD_{y_i} & 0\end{bmatrix},$$

$$\mathcal{C}_i = \begin{bmatrix} C_{z_i}+D_iD_cC-D_iK & D_iK\\ 0 & C\end{bmatrix}, \mathcal{D}_i = \begin{bmatrix} D_{z_i}+D_iD_cD_{y_i} & D_i\\ D_{y_i} & 0\end{bmatrix}.$$

In the operator domain, (9) takes the form:

$$\begin{bmatrix} Z_k^i(z)\\ E_k(z)\end{bmatrix} = \begin{bmatrix} G_{i1}(z) & G_{i2}(z)\\ G_{i3}(z) & 0\end{bmatrix}\begin{bmatrix} W_k^i(z)\\ V_k(z)\end{bmatrix}, \tag{10}$$

where $G_{i1}(z)$ is the open-loop transfer function matrix from $w_i$ to $z_i$, $G_{i2}(z)$ is the open-loop transfer function matrix from $v$ to $z_i$ and $G_{i3}(z)$ represents the open-loop transfer function matrix from $w_i$ to $e$. The next step is to develop an effective method for computing the feedback controller matrices (6) in the presence of performance specifications, which the Youla parametrization is used.

Assume the signal $v$ in (8) is

$$v_k(p) = He_k(p),$$

where $H$ represents the Youla parameter. The Youla parameter is used to form a closed-loop transfer function which represents a mapping from $w^i$ to $z^i$. Specifically, for each channel

$$G_i(z) = G_{i1}(z) + G_{i2}(z)H(z)G_{i3}(z), \tag{11}$$

and Youla parameter $H(z)$ is selected in the form of a FIR filter

$$H(z) = h_0 + h_1z^{-1} + \cdots + h_nz^{-n} \tag{12}$$

with state-space model

$$H(z) = \left[\begin{array}{c:c} A_h & B_h\\ \hdashline C_h & D_h\end{array}\right] = \left[\begin{array}{cccccc:c} 0 & 1 & 0 & \cdots & 0 & & 0\\ 0 & 0 & 1 & \cdots & 0 & & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1 & & 0\\ 0 & 0 & 0 & \cdots & 0 & & 1\\ \hdashline h_n & h_{n-1} & \ldots & h_2 & h_1 & & h_0\end{array}\right]. \tag{13}$$

Using (9) and (10), the state space model matrices representing $G_{i1}$, $G_{i2}$ and $G_{i3}$ can be easily found. Due to space limitations, the structure of these matrices is not given but they can be found, e.g., in [5]. For single input single output systems, $G_i$ in (11) can be obtained by first forming the series connection of $G_{i2}$ and $G_{i3}$, the result connected in series with $H(z)$ and finally a parallel connection with $G_{i1}$. The resulting $G_i$ is the packed matrix

$$G_i = \left[\begin{array}{c:c} \mathbb{A}_i & \mathbb{B}_i\\ \hdashline \mathbb{C}_i & \mathbb{D}_i\end{array}\right], \tag{14}$$

where again matrices $\mathbb{A}_i$, $\mathbb{B}_i$, $\mathbb{C}_i$, $\mathbb{D}_i$ are not detailed as they follow from routine matrix transformations.

## A. Design specification in the frequency domain

To define frequency domain requirements for the control system, the principal gains approach can be used. These gains (singular values) can be used to assess the performance of the system. The control synthesis problem is to design the control system represented by the closed-loop transfer function matrices $G_i(z)$ in such a way as to satisfy mixed frequency specifications

$$\bar{\sigma}(G_i(e^{j\theta})) \leq \gamma_i, \quad \forall \theta \in \Theta_i, \tag{15}$$

where $\gamma_i$ is the performance bound and $\Theta_i$ denotes the frequency ranges of relevance as in Table I. Also select the matrix $\Pi$ as

$$\Pi = \begin{bmatrix} I & 0 \\ 0 & -\gamma_i^2 I \end{bmatrix}, \tag{16}$$

where $\gamma_i$ is a given scalar that satisfies $0 < \gamma_i \leq 1$. Then

$$\overline{\sigma}(G_i(e^{j\theta})) < \gamma_i \;\Leftrightarrow\; \|G_i(e^{j\theta})\|_\infty < \gamma_i, \; \forall \theta \in \Theta_i.$$

Next, to meet the finite frequency specifications, following [3], the block entries of the matrix $\Psi$ in (1) depend on the chosen frequency range and are defined in Lemma 1. Then straightforward adaptation of Lemma 1 lead to the following result.

*Theorem 1:* Let $\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^* & \Psi_{22} \end{bmatrix}$ be given. For the state-space realization of $G_i$ given by (14), suppose that there exist matrices $R_i \succ 0$ and a symmetric $P_i$ such that the following LMIs

$$\begin{bmatrix} -\mathbb{A}_i^T P_i \mathbb{A}_i + \Psi_{11}\mathbb{A}_i^T R_i \mathbb{A}_i + \Psi_{12}\mathbb{A}_i^T R_i + (\Psi_{12}\mathbb{A}_i^T R_i)^* + P_i + \Psi_{22} R_i & -\mathbb{A}_i^T P_i \mathbb{B}_i + \Psi_{11}\mathbb{A}_i^T R_i \mathbb{B}_i + \Psi_{12}^* R_i \mathbb{B}_i & -\mathbb{C}_i^T \\ -\mathbb{B}_i^T P_i \mathbb{A}_i + \Psi_{11}\mathbb{B}_i^T R_i \mathbb{A}_i + \Psi_{12}\mathbb{B}_i^T R_i & -\mathbb{B}_i^T P_i \mathbb{B}_i + \Psi_{11}\mathbb{B}_i^T R_i \mathbb{B}_i - \gamma_i^2 I & \mathbb{D}_i^T \\ \mathbb{C}_i & \mathbb{D}_i & -I \end{bmatrix} \prec 0, \tag{17}$$

are feasible. Then the closed-loop system satisfies the required finite frequency specifications (15).

*Proof:* The LMI (17) can be obtained by employing the same procedure as given in [5] and hence the detials omitted. ∎

It is important to stress that (17) can be adopted for the case when design specifications must hold for the entire frequency range. The difference is that in this case when $\Theta_i = [0, \pi]$ in (15), it is possible to set $\Psi = 0$ in (1).

*Theorem 2:* For the state-space realization of $G_i$ in the form (14), suppose that exist $P_i$ such that the following LMIs

$$\begin{bmatrix} \mathbb{A}_i^T P_i \mathbb{A}_i & \mathbb{A}_i^T P_i \mathbb{B}_i & \mathbb{C}_i^T \\ \mathbb{B}_i^T P_i \mathbb{A}_i & \mathbb{B}_i^T P_i \mathbb{B}_i - \gamma_i I & \mathbb{D}_i^T \\ \mathbb{C}_i & \mathbb{D}_i & -I \end{bmatrix} \prec 0, \tag{18}$$

are feasible. Then the closed-loop system satisfies the required entire frequency specifications.

*Proof:* This result foloows as an obvious consequence of Theorem 1 when $\Psi = 0$. ∎

Given $H(z) : (A_h, B_h, C_h, D_h)$ the feedback controller matrices in (6) on completing the design are

$$\begin{aligned} A_c &= \begin{bmatrix} A + BD_cC - BK - LC - BD_hC & BC_h \\ -B_hC & A_h \end{bmatrix}, \\ B_c &= \begin{bmatrix} L + BD_h \\ B_h \end{bmatrix}, C_c = \begin{bmatrix} -K + D_hC & C_h \end{bmatrix}, D_c = D_h. \end{aligned} \tag{19}$$

## IV. Design of a Learning Controller

Based on the results obtained in the previous section, i.e. by considering that there exist stabilizing feedback controller $C(z)$ satisfying multi-objective performance requirements and hence the sensitivity function $S_P(z)$ of (3) has been determined, an effective procedure for designing the learning controller $L(z)$. The goal is to compute matrices $A_L$, $B_L$, $C_L$ and $D_L$ that define its minimal state-space realization of $L(z)$. Furthermore, since the bandwidth of the reference signal has the most effect on the trial-to-trial error convergence, it is relevant to impose performance specifications over this specific frequency range.

To proceed, it is routine to show that (4) limited to such a frequency range is equivalent to

$$\begin{bmatrix} M(e^{j\theta}) \\ I \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} M(e^{j\theta}) \\ I \end{bmatrix} \prec 0, \; \forall \theta \in \Theta. \tag{20}$$

Moreover, choosing $\Pi = \mathrm{diag}\{I, -\gamma^2 I\}$, where $\gamma$ satisfies $0 < \gamma \leq 1$, and using Lemma 1 gives that (20) is equivalent to (1)(see also [6] for more details on finite frequency rages for ILC design). However, (1) is not convex due to the product of the $L$-filter parameters and the matrices $P$ and $R$.

To obtain a convex formulation of this problem, again assume that $L(z)$ has the form of an $n$-th order FIR filter given by

$$L(z) = \alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \ldots + \alpha_n z^{-n}, \tag{21}$$

and associated minimal state-space realization

$$L(z) = C_L(zI - A_L)^{-1} B_L + D_L, \tag{22}$$

with matrices

$$\begin{aligned} A_L &= \begin{bmatrix} 0 & I_{n-1} \\ 0 & 0 \end{bmatrix}, B_L = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ C_L &= \begin{bmatrix} \alpha_n & \alpha_{n-1} & \cdots & \alpha_2 & \alpha_1 \end{bmatrix}, D_L = \begin{bmatrix} \alpha_0 \end{bmatrix}. \end{aligned} \tag{23}$$

In the above form, the parameters $\alpha_0, \alpha_1, \ldots, \alpha_n$ to be designed are only present in the matrices $C_L$ and $D_L$. This again means that calculation of products of matrices whose entries are filter parameters and hence these can be computed via a constrained convex optimization procedure using LMIs. To see this, introduce the matrices $A_{sp}$, $B_{sp}$, $C_{sp}$ and $D_{sp}$ as the state-space realization of the sensitivity function $S_P(z)$. Then a state-space representation of $L(z)S_P(z)$ in (3) is

$$\begin{aligned} \mathcal{A} &= \begin{bmatrix} A_{sp} & 0 \\ B_L C_{sp} & A_L \end{bmatrix}, \; \mathcal{B} = \begin{bmatrix} B_{sp} \\ B_L D_{sp} \end{bmatrix}, \\ \mathcal{C} &= \begin{bmatrix} D_L C_{sp} & C_L \end{bmatrix}, \; \mathcal{D} = D_L D_{sp}. \end{aligned} \tag{24}$$

In this realization, the filter parameters appear in $\mathcal{C}$ and $\mathcal{D}$ only and the following result gives an LMI design for the $L$-filter.

*Theorem 3:* Let $\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^* & \Psi_{22} \end{bmatrix}$ be given. Then a stable $n$-order filter $L(z)$ of the form (21) can be designed such that the ILC convergence condition (20) holds for a chosen finite frequency range $\Theta$ of Lemma 1 if there exist matrices $P$, $R \succ 0$, $\mathcal{C}$, and $\mathcal{D}$ such that the following LMI is feasible

$$\begin{bmatrix} -\mathcal{A}^T P\mathcal{A} + \Xi_{11}\mathcal{A}^T R\mathcal{A} + \Xi_{12}\mathcal{A}^T R + \Xi_{12}^T R\mathcal{A} + P + \Xi_{22} R & -\mathcal{A}^T P\mathcal{B} + \Xi_{11}\mathcal{A}^T R\mathcal{B} + \Xi_{12}^T R\mathcal{B} & -\mathcal{C}^T \\ -\mathcal{B}^T P\mathcal{A} + \Xi_{11}\mathcal{B}^T R\mathcal{A} + \Xi_{12}\mathcal{B}^T R & -\mathcal{B}^T P\mathcal{B} + \Xi_{11}\mathcal{B}^T R\mathcal{B} - \gamma^2 I & 1 - \mathcal{D}^T \\ -\mathcal{C} & 1 - \mathcal{D} & -1 \end{bmatrix} \prec 0. \tag{25}$$

Moreover, the required $Q(z)$-filter can be selected as a low-pass filter with the cut-off frequency equal to the highest frequency for which the above result is valid.

*Proof:* This is given for the low frequency (LF) range since this choice is often encountered in physical applications and the others follow by identical steps. For this range, the matrix $\Psi$ in (1) can be partitioned as

$$\Psi = \left[\begin{array}{c|c} \Psi_{11} & \Psi_{12} \\ \hline \Psi_{12}^T & \Psi_{22} \end{array}\right] = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & -2\cos(\varpi_l) \end{array}\right]$$

and with the notation introduced in (24), the state-space representation of $M(z)$ in (3) for $Q(z) = 1$ is

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline -\mathcal{C} & 1 - \mathcal{D} \end{array}\right].$$

Direct application of Lemma 1 for the above state-space model matrices gives (25) and the proof is complete. ■

*Remark 1:* The robustness filter $Q(z)$ can be implemented as a zero-phase filter (e.g. by using the `filtfilt` routine in MATLAB) since such filtering is performed off-line using previous trial information and the known reference trajectory.

*Remark 2:* To minimize the inaccuracies between the computed $L$ and the known $S_P^{-1}$, the term $\gamma$ in (25) has to be minimized. This can be achieved using the linear objective minimization procedure

$$\min_{R \succ 0, \mu > 0} \mu \quad \text{subject to (25)}$$

substituting $\mu = \gamma^2$.

## V. A COMPLETE DESIGN PROCEDURE

To summarize the developments to this stage, a design procedure for ILC scheme considered is as follows.

Step 1: Compute the matrices $K$ and $L$ using LQG control, e.g., using MATLAB routines `dlqr()` and `dlqe()`.

Step 2: Impose the performance specification in finite and/or entire frequency range based on required shape of the sensitivity function.

Step 3: Solve the LMIs (17)-(18) and compute the feedback controller parameters from (19).

Step 4: Check the frequency content of the reference signal and carry out the finite frequency range design of the learning filter using Theorem 3.

Step 5: Obtain the learning controller parameters from (23).

## VI. DESIGN EXAMPLE WITH AN EXPERIMENTAL VALIDATION

To validate the new controller design, the results of an experimental validation on a laboratory servomechanism system are giveen and discussed in this section. The system consists of a DC motor and an inertial mass (brass cylinder, weight 2.03 [kg], diameter 0.066 [m], length 0.068 [m]), which are connected through a rigid shaft. Rotational motion of the mass is exited by the DC motor, load position is measured by an incremental encoder and the whole system operates with a PC-based digital controller. The system is fully integrated with MATLAB/SIMULINK and operates in real-time.

The block scheme of the servo with the direct connection to the representation (5) is shown in Fig. 2 where $J_m = 11.18 \cdot 10^{-4}[kg \cdot m^2]$ is the moment of inertia, $B_m = 3.5077 \cdot 10^{-6}[N \cdot m \cdot s]$ is the viscous friction coefficient, $K_e = 0.1288[\frac{V}{rad/s}]$ is the electromotive force constant, $K_m = 0.0163[\frac{N \cdot m}{A}]$ is the motor torque constant, $R_m = 2[\Omega]$ is the resistance and $L_m = 0.001[H]$ is the inductance.

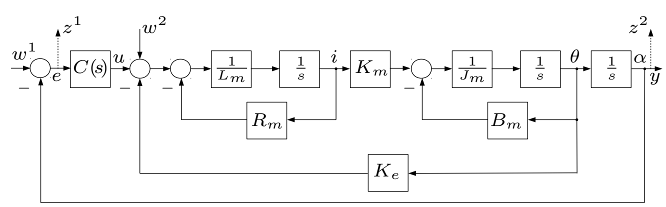


Fig. 2. Block scheme of the modular servo

A continuous-time model (see the block diagram of the system in Fig. 2) has been converted to the discrete state-space form using a sampling time of $T_s = 0.01$ and the resulting matrices re

$$A = \begin{bmatrix} -0.0005 & -0.0639 & 0 \\ 0.0072 & 0.9911 & 0 \\ 0.0001 & 0.01 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0.4958 \\ 0.0690 \\ 0.0003 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 \; 0 \; 24 \end{bmatrix}, \; D = 0.$$

Moreover, by (5), $B_{x_1} = 0$, $B_{x_2} = B$, $C_{z_1} = -C$, $C_{z_2} = C$, $D_{z_1} = 1$, $D_1 = 0$, $D_{z_2} = 0$, $D_2 = 0$, $D_{y_1} = 0$ and $D_{y_2} = 0$. Hence $z^1 = e$, $z^2 = y$, and $y = \alpha$ and the state vector $x = [i \; \omega \; \alpha]^T$. The reference trajectory for the mass position is of duration 6.5 secs as shown in Fig. 3 and consists of harmonic components from 0 to $2[Hz]$ and hence the frequency range from 0 to $2[Hz]$ is of primary importance. The reference signal consists of 6 rotations in the positive direction, a return path, 6 rotations in the negative direction and then a return to the start position. Based on the separation

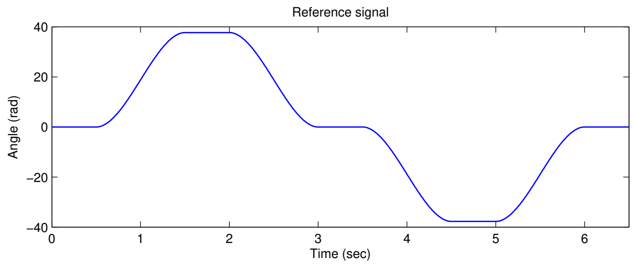


Fig. 3. The reference trajectory

rule and using the simple pole placement technique the observer gain $L = [0.0039 \quad -0.0598 \quad -0.0074]^T$ and the gain matrix $K = [-0.0170 \ -2.3440 \ -21.9580]$ are derived. Furthermore, assuming that the initial controller (6) is a static with $D_c = 0.0001$, and solving the set of LMIs (17) yields a feasible solution. Application of the feedback controller design procedure then gives the corresponding stabilizing controller matrices as

$$A_c = \begin{bmatrix} -0.0089 & -1.2260 & -10.7572 \\ 0.0061 & 0.8294 & -2.9437 \\ 0.0001 & 0.0092 & 0.8155 \end{bmatrix}, B_c = \begin{bmatrix} 0.0054 \\ -0.0595 \\ -0.0074 \end{bmatrix},$$
$$C_c = \begin{bmatrix} -0.0170 & -2.3440 & -21.8811 \end{bmatrix}, \ D_c = \begin{bmatrix} 0.0032 \end{bmatrix}.$$

Given this data, the design procedure for the learning controller over the frequency range from 0 to $2[Hz]$ has been completed and gives the following FIR polynomial coefficients

$$\alpha_0 = 7.7126, \ \alpha_1 = \begin{bmatrix} 3.9546 & -11.1428 \end{bmatrix}. \tag{26}$$

Given (23) and (26) a state-space model of the learning controller is given by the matrices

$$A_L = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_L = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_L = \begin{bmatrix} 3.955 & -11.14 \end{bmatrix}, D_L = 7.713$$

Figure 4 shows that the frequency responses of $L(z)$ and $S_P(z)$ are almost identical in the low frequency range up to $10[Hz]$. Moreover, the test system has a tendency to amplify high frequency signals (noise) and hence $Q(z)$ should be chosen as a low-pass filter with the cut-off frequency below $10[Hz]$. In this example, a sixth order low-pass digital Butterworth filter with cut-off frequency equal to $4[Hz]$ has been used. With the design completed, the resulting ILC scheme was experimentally tested over 20 trials. After each trial the corresponding root mean square (RMS) value of the tracking error was computed using

$$\text{RMS} = \sqrt{\frac{1}{\alpha}\sum_{p=1}^{\alpha} e(p)^2}.$$

In Fig. 5 the RMS value of the tracking error is shown as a function of the trial number and this plot confirms the convergence of the error with increasing $k$.

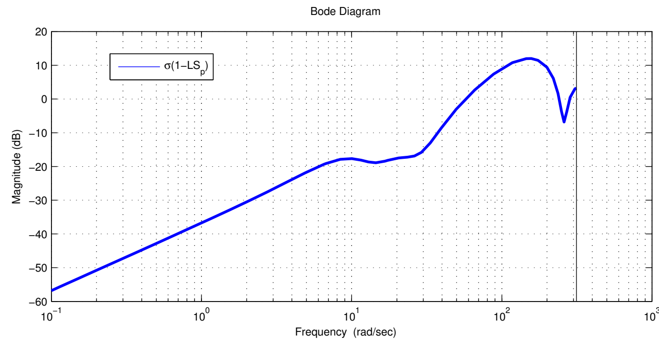


Fig. 4. Magnitude plot of $(1 - LS_P^{-1})$.

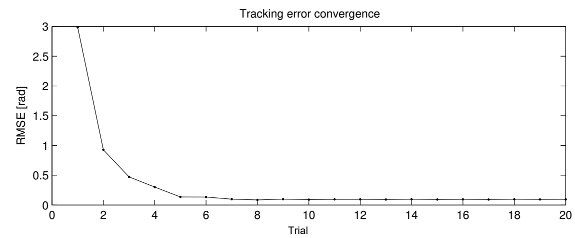


Fig. 5. Convergence of the tracking error.

## VII. Conclusions

In this paper a systematic procedure for the design of both feedback and learning controllers for ILC schemes has been developed. Applying the, design it is possible to impose many control performance constrains over finite and/or semi-finite frequency ranges. It is shown that an FIR form of required controllers leads to a problem formulation in terms of LMIs. Experimental validation of the results on a laboratory setup has also been undertaken to validate the new design.

Area for possible future research include the use of the two-dimensional/repetitive processes setting for ILC design to develop a one step design procedure and hence an optimal combination of the feedback with the learning action could result. This means that interplay between the feedback and feedforward actions have to be considered. Also robust control should be investigated.

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