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$\mathcal{H}_\infty$ Observer-Based Controller for Lipschitz Nonlinear Discrete-Time Systems

Noussaiba GASMI, Mohamed BOUTAYEB Assem THABET and Mohamed AOUN

***Abstract*— Within the paper, a relevant $\mathcal{H}_\infty$ observer-based controller design for a class of Lipschitz nonlinear discrete-time systems is proposed. Usually, Bilinear Matrix Inequaities (BMIs) are obtained from the resolution of the observer-based stabilization design problem for this class of systems. Since, the resolution of a BMI is a hard task, then it is interesting to search for a convenient way to linearize the obtained conditions. Therefore, the objective of this paper is to present new Linear Matrix Inequality (LMI) conditions ensuring the convergence of the observer-based controller in a noisy context. Thanks to the introduction of a slack variable the presented LMI conditions are more general and less conservative than the existence ones. Indeed, reformulations of the Lipschitz property and Young's relation in a convenient way lead to a more relaxed new LMI. A numerical example is implemented to show high performances of the proposed design methodology with respect to some existing results.**

I. INTRODUCTION AND PRELIMINARIES

A. Introduction

Special attention has been given to the observation and control of nonlinear systems [1], [2], [3]. It has been known that in many practical systems, it may not be feasible to measure all the state variables, which makes the real control problem more complicated. Observer is a potent tool used to overcome the lack of information and reconstruct the state of dynamic systems [4], [5], [6], [7], [8]. Designing observer for continuous and discrete time nonlinear systems received considerable interest in research on the control design strategies [9], [10], [11]. This is motivated by the fact that state estimates provided by the observer can be used not only in fault diagnosis and system supervising but also in control design. To this fact, observer-based controller design is a particular challenge, specially for nonlinear systems. Lot of design approaches have been proposed to deal with Lipschitzian non-linearities [12], [13], [14]. These approaches still restrictive due to the use of a particular form of the Lypaunov function [15] or of the slack variable [16].

This paper focuses on the observer-based controller design for Lipschitz nonlinear discrete time systems in the presence of bounded disturbances. This work is motivated by the work of [17] where the authors present a useful design procedure to synthesize a decentralized observer-based stabilization for nonlinear interconnected systems. Therefore, the main contribution lies in the introduction of a symmetric lyapunov function that will be associated to a slack variable [18] which is more general than that used in [17], in order to linearize the obtained constraint. Then, thanks to the use of reformulations of the Lipschitz property and Young's relation, less conservative LMI conditions are obtained. Hence, the controller and the observer gains can be computed simultaneously through a less restrictive constraint.

This contribution is organized as follows: In the next section, the problem formulation is introduced. Then, the synthesis procedure is detailed in the third section. In the last section, a numerical example and interesting comparison with some existing results that deal with the same type of non-linearity are considered to prove the superiority of the proposed design methodology.

Notation. The following notation will be used throughout this paper:

- $e_s(i) = \Big[\underbrace{0,...,0,\overbrace{1}^{i^{th}},0,...,0}_{s-components}\Big]^T \in \mathbb{R}^s$, $s \geqslant 1$, is a vector of the canonical basis of $\mathbb{R}^s$;
- $\|A\| = \sqrt{A^TA}$ is the Euclidean vector norm;
- $(\star)$ is used for the blocks induced by symmetry.

B. Preliminaries

Definition 1 ([19]): Consider the vectors U and V described by

$$U = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n \quad \text{and} \quad V = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n \ .$$

For all $i = 0, \ldots, n$, an auxiliary vector $U^{V_i} \in \mathbb{R}^n$ corresponding to U and V is defined as :

$$\begin{cases} U^{V_i} = \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ u_{i+1} \\ \vdots \\ u_n \end{bmatrix} & \text{for} \quad i = 1, \ldots, n \\ U^{V_0} = U \end{cases} \tag{1}$$

□

Lemma 1: Consider a nonlinear function $f : \mathbb{R}^n \to \mathbb{R}^n$. Then the following two items are equivalent [20]:

- f is globally Lipschitz with respect to its argument, i.e.:

$$\| f(U) - f(V) \| \leqslant \gamma_h \| U - V \|, \quad \forall\, U,\, V \in \mathbb{R}^n. \tag{2}$$

Noussaiba GASMI and Mohamed BOUTAYEB are with Centre de Recherche en Automatique de Nancy, CRAN-CNRS UMR 7039, University of Lorraine, FRANCE (e-mail: noussaiba.gasmi@univ-lorraine.fr).

Assem THABET and Mohamed AOUN are with MACS Laboratory : Modeling, Analysis and Control of Systems, National Engineering School of Gabes (ENIG), University of Gabes, TUNISIA.

- for all $i, j = 1, \dots, n$, there exist functions

$$\mathfrak{f}_{ij} \; : \; \mathbb{R}^n \; \times \; \mathbb{R}^n \longrightarrow \mathbb{R} \tag{3}$$

and constant $\underline{\gamma}_{\mathfrak{f}_{ij}}$ and $\overline{\gamma}_{\mathfrak{f}_{ij}}$, so that $\forall\; U, V \in \mathbb{R}^n$

$$f(U) - f(V) = \sum_{i,j=1}^{n,n} \mathfrak{f}_{ij} \mathtt{H}_{ij}(U - V) \tag{4}$$

and

$$\underline{\gamma}_{\mathfrak{f}_{ij}} \leqslant \mathfrak{f}_{ij} \leqslant \overline{\gamma}_{\mathfrak{f}_{ij}} \tag{5}$$

where
$\mathfrak{f}_{ij} \triangleq \mathfrak{f}_{ij}(U^{V_{j-1}}, U^{V_j})$ and $\mathtt{H}_{ij} = e_n(i) e_n^T(j)$.

□

Lemma 2 (Reformulation of Young's relation [21]): Given two matrices U and V of appropriate dimensions, then the following inequality holds for any symmetric positive definite matrix S of appropriate dimensions:

$$U^T V + V^T U \leqslant \frac{1}{2} \left[U + SV\right]^T S^{-1} \left[U + SV\right] \tag{6}$$

□

II. Problem statement

Consider the class of nonlinear discrete-time systems described by:

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k + Df(x_k) + E\omega_k \\ y_k &= Cx_k + F\omega_k \end{cases} \tag{7}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^q$ and $\omega_k \in \mathbb{R}^r$ are the state, the input, the output and the disturbance vectors, respectively. A, B, C, D, E and F are constant matrices of adequate dimensions. $f : \mathbb{R}^n \to \mathbb{R}^p$ is the nonlinear function which is assumed to be globally Lipschitz. The nonlinear function $f(.)$ has the following more detailed form:

$$f(x_k) = \begin{bmatrix} f_1(R_1 x_k) \\ f_2(R_2 x_k) \\ \vdots \\ f_p(R_p x_k) \end{bmatrix}$$

The pairs (A, B) and (A, C) are assumed to be stabilizable and detectable, respectively.

Consider the following Luenberger observer for the nonlinear system (7):

$$\begin{cases} \hat{x}_{k+1} &= A\hat{x}_k + Bu_k + Df(\hat{x}_k) + L(y_k - \hat{y}_k) \\ \hat{y}_k &= C\hat{x}_k \end{cases} \tag{8}$$

coupled with a state estimated feedback controller

$$u_k = K\hat{x}_k \tag{9}$$

where $\hat{x}_k$ is the estimate of the system state x_k. The observer gain matrix L and the control gain matrix K are unknown matrices to be determined such that $\hat{x}_k$ converges asymptotically to x_k.

Define $e_k = x_k - \hat{x}_k$, the error between x_k and $\hat{x}_k$. Then the dynamic of the estimation error e_{k+1} is given by

$$e_{k+1} = x_{k+1} - \hat{x}_{k+1}. \tag{10}$$

Using equations (7), (8) and (9), $\hat{x}_{k+1}$ and e_{k+1} can be rewriting as follows:

$$\hat{x}_{k+1} = (A + BK)\hat{x}_k + LCe_k + Df(\hat{x}_k) + LF\omega_k \tag{11}$$

$$e_{k+1} = (A - LC)e_k + D\left(f(x_k) - f(\hat{x}_k)\right) + (E - LF)\omega_k \tag{12}$$

As stated previously, $f(.)$ is globally Lipschitz. Then, by applying *Lemma 1* to this nonlinear function, we obtain

$$f(\hat{x}_k) = \sum_{i,j=1}^{p,n_i} \varphi_{ij} \mathtt{H}_{ij} R_i \hat{x}_k = \Sigma_1 \hat{x}_k \tag{13}$$

and

$$f(x_k) - f(\hat{x}_k) = \sum_{i,j=1}^{p,n_i} \psi_{ij} \mathtt{H}_{ij} R_i e_k = \Sigma_2 e_k \tag{14}$$

with

$$\begin{gathered} \underline{f}_{ij} \leqslant \varphi_{ij} \leqslant \overline{f}_{ij} \\ \underline{f}_{ij} \leqslant \psi_{ij} \leqslant \overline{f}_{ij} \\ \varphi_{ij} \triangleq \varphi_{ij}(\hat{\varrho}_i^{0_{i,j-1}}, \hat{\varrho}_i^{0_{i,j}}),\; \psi_{ij} \triangleq \psi_{ij}(\varrho_i^{\hat{\varrho}_{i,j-1}}, \varrho_i^{\hat{\varrho}_{i,j}}) \\ \varrho_i = R_i x_k,\; \hat{\varrho}_i = R_i \hat{x}_k \\ \mathtt{H}_{ij} = e_p(i) e_{n_i}^T(j) \end{gathered}$$

We can assume that $\underline{f}_{ij} = 0$ without loss of generality. For more explanation, the reader can refer to [22].
Hence, we can define an augmented system described by the following structure:

$$\bar{x}_{k+1} = \bar{A}\bar{x}_k + \bar{E}\omega_k \tag{15}$$

with

$$\bar{x}_k = \begin{bmatrix} \hat{x}_k \\ e_k \end{bmatrix},\; \bar{A} = \begin{bmatrix} A + BK + D\Sigma_1 & LC \\ 0 & A - LC + D\Sigma_2 \end{bmatrix},$$
$$\bar{E} = \begin{bmatrix} LF \\ E - LF \end{bmatrix}.$$

To synthesize the $\mathcal{H}_\infty$ observer-based controller, we have to find the observer gain L and the controller gain K that guarantee the convergence of the vector $\bar{x}$ asymptotically toward zero. i.e, we must find the parameters L and K such that:

$$\parallel e_k \parallel_{l_2} \leq \lambda \parallel \omega_k \parallel_{l_2} \quad , \quad for \quad \bar{x}_0 = 0 \tag{16}$$

with $\lambda > 0$ is the disturbance attenuation level that will be minimized.
The resolution of this problem return to search a Lyapunov function V so that

$$\Lambda = \Delta V + e_k^T e_k - \lambda^2 \omega_k^T \omega_k < 0. \tag{17}$$

Then, let us consider the following candidate Lyapunov function for system (7)

$$V_k = \bar{x}_k^T P \bar{x}_k \tag{18}$$

with $P = P^T > 0$.
The matrix P has the following form:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \tag{19}$$

Define $\Delta V_k = V_{k+1} - V_k$. Then, along the solution of the augmented system (15) we have:

$$\Lambda = \begin{bmatrix} \bar{x}_k \\ \omega_k \end{bmatrix}^T \Pi \begin{bmatrix} \bar{x}_k \\ \omega_k \end{bmatrix} \quad (20)$$

where

$$\Pi = \begin{bmatrix} \bar{A}^T P \bar{A} - P + \mathcal{I} & \bar{A}^T P \bar{E} \\ (\star) & \bar{E}^T P \bar{E} - \lambda^2 I_r \end{bmatrix} \quad (21)$$

and $\mathcal{I} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$.

Note that $\Lambda < 0$ is satisfied if $\Pi < 0$ which is equivalent to

$$\begin{bmatrix} -P + \mathcal{I} & 0 \\ 0 & -\lambda^2 I_r \end{bmatrix} + \begin{bmatrix} \bar{A}^T \\ \bar{E}^T \end{bmatrix} P \begin{bmatrix} \bar{A} & \bar{E} \end{bmatrix} < 0 \quad (22)$$

Applying the Schur lemma on (22), we get

$$\begin{bmatrix} -P + \mathcal{I} & 0 & \bar{A}^T \\ (\star) & -\lambda^2 I_r & \bar{E}^T \\ (\star) & (\star) & -P^{-1} \end{bmatrix} < 0 \quad (23)$$

In the next section, a useful design methodology is proposed to guarantee the convergence of the estimation error.

III. LMI DESIGN METHODOLOGY

The main contribution of this paper is introduced in the following theorem.

Theorem 1: For a disturbance attenuation level $\lambda > 0$, the $\mathcal{H}_\infty$ observer-based controller design problem corresponding to the system (7) and the observer (8) is solvable if there exist matrices $\hat{P}_{11}$, $\hat{P}_{12}$, P_{22}, $\hat{G}_{11}$, $G_{22} \in \mathbb{R}^{n\times n}$, $S_{ij}, \tilde{S}_{ij}, \bar{S}_{ij} \in \mathbb{R}^{n_i \times n_i}$, $\hat{L} \in \mathbb{R}^{q\times n}$, $\hat{K} \in \mathbb{R}^{m\times n}$ of appropriate dimensions such that the following LMI is feasible

$$\min \ \lambda \ \text{subject to}$$

$$\begin{bmatrix} \begin{bmatrix} \Xi_1 & \Xi_2 & 0 & \Xi_3 & 0 \\ (\star) & \Xi_4 & 0 & \Xi_5 & \Xi_6 \\ (\star) & (\star) & \Xi_7 & \Xi_8 & \Xi_9 \\ (\star) & (\star) & (\star) & \Xi_{10} & \Xi_{11} \\ (\star) & (\star) & (\star) & (\star) & \Xi_{12} \end{bmatrix} & \begin{bmatrix} \Delta_1 & \cdots & \Delta_n \end{bmatrix} \\ (\star) & -\Omega\mathcal{Z} \end{bmatrix} < 0 \quad (24)$$

where

$$\Xi_1 = -\hat{P}_{11}$$
$$\Xi_2 = -\hat{P}_{12}$$
$$\Xi_3 = \hat{G}_{11} A^T + \hat{K} B^T$$
$$\Xi_4 = I_n - P_{22}$$
$$\Xi_5 = A^T (I_n - G_{22})^T + C^T \hat{L}^T$$
$$\Xi_6 = A^T G_{22}^T - C^T \hat{L}^T$$
$$\Xi_7 = -\lambda^2 I_r$$
$$\Xi_8 = E^T (I_n - G_{22})^T + F^T \hat{L}^T$$
$$\Xi_9 = E^T G_{22}^T - F^T \hat{L}^T$$
$$\Xi_{10} = \hat{P}_{11} - \hat{G}_{11} - \hat{G}_{11}^T$$
$$\Xi_{11} = \hat{P}_{12} - I_n + G_{22}$$
$$\Xi_{12} = P_{22} - G_{22} - G_{22}^T$$
$$\Delta_i = \begin{bmatrix} \mathcal{M}_{i1} & \cdots & \mathcal{M}_{in_i} \end{bmatrix}$$

$$\mathcal{M}_{ij} = \begin{bmatrix} \hat{G}_{11} E_i^T & 0 & 0 \\ 0 & E_i^T \tilde{S}_{ij}^T & E_i^T \bar{S}_{ij}^T \\ 0 & 0 & 0 \\ D\mathtt{H}_{ij} S_{ij}^T & (I_n - G_{22}) D\mathtt{H}_{ij} & 0 \\ 0 & 0 & G_{22} D\mathtt{H}_{ij} \end{bmatrix}$$

$$\Omega = block - diag(\Omega_1, \cdots, \Omega_p)$$
$$\Omega_i = block - diag(\Omega_{i1}, \cdots, \Omega_{in_i})$$
$$\Omega_{ii} = block - diag(\frac{2}{\bar{h}_{ij}} I_{n_i}, \cdots, \frac{2}{\bar{h}_{ij}} I_{n_i})$$
$$\mathcal{Z} = block - diag(\mathcal{Z}_1, \cdots, \mathcal{Z}_p)$$
$$\mathcal{Z}_i = block - diag(\mathcal{Z}_{i1}, \cdots, \mathcal{Z}_{in_i})$$
$$\mathcal{Z}_{ij} = block - diag(S_{ij}, \tilde{S}_{ij}, \bar{S}_{ij})$$

When the constraint (24) is feasible, the matrices gain are given by $L = G_{22}^{-1} \hat{L}$ and $K = \hat{K}^T \hat{G}_{11}^{-T}$. □

Proof. To linearize the inequality (23), a slack variable G is added. Then, multiplying both sides of inequality (23) by $block - diag(I, I, G)$ and its transpose, then using the inequality $-GP^{-1}G^T \leq P - G - G^T$ yield to

$$\begin{bmatrix} -P + \mathcal{I} & 0 & \bar{A}^T G^T \\ (\star) & -\lambda^2 I_r & \bar{E}^T G^T \\ (\star) & (\star) & P - G - G^T \end{bmatrix} < 0. \quad (25)$$

where P is in the form of (19). Then, we take the following structure of G :

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}. \quad (26)$$

By substituting (19) and (26) in inequality (25) and after mathematical developments, the following inequality is obtained:

$$\begin{bmatrix} -P_{11} & -P_{12} & 0 & \Pi_{14} & \Pi_{15} \\ (\star) & I_n - P_{22} & 0 & \Pi_{24} & \Pi_{25} \\ (\star) & (\star) & -\lambda^2 I_r & \Pi_{34} & \Pi_{35} \\ (\star) & (\star) & (\star) & \Pi_{4,4} & \Pi_{45} \\ (\star) & (\star) & (\star) & (\star) & \Pi_{55} \end{bmatrix} < 0 \quad (27)$$

where

$$\begin{cases} \Pi_{14} = (A + BK + D\Sigma_1)^T G_{11}^T \\ \Pi_{15} = (A + BK + D\Sigma_1)^T G_{21}^T \\ \Pi_{24} = (A + D\Sigma_2)^T G_{12}^T + (LC)^T (G_{11} - G_{12})^T \\ \Pi_{25} = (A + D\Sigma_2)^T G_{22}^T + (LC)^T (G_{21} - G_{22})^T \\ \Pi_{34} = E^T G_{12}^T + (LF)^T (G_{11} - G_{12})^T \\ \Pi_{35} = E^T G_{22}^T + (LF)^T (G_{21} - G_{22})^T \\ \Pi_{44} = P_{11} - G_{11} - G_{11}^T \\ \Pi_{45} = P_{12} - G_{12} - G_{21}^T \\ \Pi_{55} = P_{22} - G_{22} - G_{22}^T \end{cases}$$

Note here that the controller gain L is coupled with both $G_{11} - G_{12}$ and $G_{21} - G_{22}$. To eliminate this bilinear terms, we choose to take $G_{11} - G_{12} = 0$ and $G_{21} - G_{22} = 0$. Likewise, in $(1,5)$, G_{21} multiplies the controller gain K. To overcome this problem, G_{21} should be taken zero. So, we can choose $G = \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix}$ as in [16] or $G = \begin{bmatrix} G_{11} & G_{11} \\ 0 & G_{22} \end{bmatrix}$ as in [17].

In this contribution, we will use a more general form of the matrix G to decrease the conservatism and to get a more relaxed LMI. Therefore, G_{12} is kept, and by using some mathematical tools, a more general form of the matrix G is defined.

So, the following structure of G is taken

$$G = \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix}. \tag{28}$$

Pre and post multiplying (27) by $block-diag(\hat{G}_{11}, I, I, \hat{G}_{11}, I)$ and its transpose with $\hat{G}_{11} = G_{11}^{-1}$ and using the notations $\hat{P}_{12} = \hat{G}_{11}P_{12}$, $\hat{P}_{11} = \hat{G}_{11}P_{11}\hat{G}_{11}^T$ lead to :

$$\begin{bmatrix} -\hat{P}_{11} & -\hat{P}_{12} & 0 & \Pi_{14} & 0 \\ (\star) & I_n - P_{22} & 0 & \Pi_{24} & \Pi_{25} \\ (\star) & (\star) & -\lambda^2 I_r & \Pi_{34} & \Pi_{35} \\ (\star) & (\star) & (\star) & \Pi_{44} & \hat{P}_{12} - \hat{G}_{11}G_{12} \\ (\star) & (\star) & (\star) & (\star) & \Pi_{55} \end{bmatrix} < 0 \tag{29}$$

where

$$\begin{cases} \Pi_{14} = & \hat{G}_{11}(A + BK + D\Sigma_1)^T \\ \Pi_{24} = & (A + D\Sigma_2)^T(\hat{G}_{11}G_{12})^T \\ & +(LC)^T(I_n - \hat{G}_{11}G_{12})^T \\ \Pi_{25} = & (A - LC + D\Sigma_2)^T G_{22}^T \\ \Pi_{34} = & E^T(\hat{G}_{11}G_{12})^T + (LF)^T(I_n - \hat{G}_{11}G_{12})^T \\ \Pi_{35} = & (E - LF)^T G_{22}^T \\ \Pi_{44} = & \hat{P}_{11} - \hat{G}_{11} - \hat{G}_{11}^T \\ \Pi_{55} = & P_{22} - G_{22} - G_{22}^T \end{cases}$$

We can notice tat the observer gain L multiplies G_{22} and $(I_n - \hat{G}_{11}G_{12})$ in the same constraint. To overcome this problem, we take $I_n - \hat{G}_{11}G_{12} = G_{22}$, which amounts to choose $G_{12} = G_{11} - G_{11}G_{22}$. The obtained form of the matrix G is described by

$$G = \begin{bmatrix} G_{11} & G_{11} - G_{11}G_{22} \\ 0 & G_{22} \end{bmatrix}. \tag{30}$$

Then, we obtain

$$\begin{bmatrix} -\hat{P}_{11} & -\hat{P}_{12} & 0 & \Pi_{14} & 0 \\ (\star) & I_n - P_{22} & 0 & \Pi_{24} & \Pi_{25} \\ (\star) & (\star) & -\lambda^2 I_r & \Pi_{34} & \Pi_{35} \\ (\star) & (\star) & (\star) & \Pi_{44} & \hat{P}_{12} - \hat{G}_{11}G_{12} \\ (\star) & (\star) & (\star) & (\star) & \Pi_{55} \end{bmatrix} + \sum_{i,j=1}^{p,n_i} \varphi_{ij}\left(U_i^T V_{ij} + V_{ij}^T U_i\right) + \sum_{i,j=1}^{p,n_i} \psi_{ij}\left(\tilde{U}_i^T \tilde{V}_{ij} + \tilde{V}_{ij}^T \tilde{U}_i\right) + \sum_{i,j=1}^{p,n_i} \psi_{ij}\left(\bar{U}_i^T \bar{V}_{ij} + \bar{V}_{ij}^T \bar{U}_i\right) < 0 \tag{31}$$

where

$$\begin{cases} \Pi_{14} = \hat{G}_{11}(A + BK)^T \\ \Pi_{24} = A^T(I_n - G_{22})^T + (LC)^T G_{22}^T \\ \Pi_{25} = (A - LC)^T G_{22}^T \\ \Pi_{34} = E^T(I_n - G_{22})^T + (LF)^T G_{22}^T \\ \Pi_{35} = (E - LF)^T G_{22}^T \\ \Pi_{44} = \hat{P}_{11} - \hat{G}_{11} - \hat{G}_{11}^T \\ \Pi_{55} = P_{22} - G_{22} - G_{22}^T \end{cases}$$

and

$$U_i^T = \begin{bmatrix} \hat{G}_{11}E_i^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tilde{U}_i^T = \begin{bmatrix} 0 \\ E_i^T \\ 0 \\ 0 \\ 0 \end{bmatrix}, \bar{U}_i^T = \begin{bmatrix} 0 \\ E_i^T \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V_{ij} = \begin{bmatrix} 0 & 0 & 0 & \mathtt{H}_{ij}^T D^T & 0 \end{bmatrix}$$

$$\tilde{V}_{ij} = \begin{bmatrix} 0 & 0 & 0 & \mathtt{H}_{ij}^T D^T (I_n - G_{22})^T & 0 \end{bmatrix}$$

$$\bar{V}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 & \mathtt{H}_{ij}^T D^T G_{22}^T \end{bmatrix}.$$

Now, by applying *Lemma* 2 on (31), we obtain

$$U_i^T V_{ij} + V_{ij}^T U_i \leqslant \frac{1}{2}\left[U_i + S_{ij}V_{ij}\right]^T S_{ij}^{-1}\left[U_i + S_{ij}V_{ij}\right]$$

$$\bar{U}_i^T \bar{V}_{ij} + \bar{V}_{ij}^T \bar{U}_i \leqslant \frac{1}{2}\left[\bar{U}_i + \bar{S}_{ij}\bar{V}_{ij}\right]^T \bar{S}_{ij}^{-1}\left[\bar{U}_i + \bar{S}_{ij}\bar{V}_{ij}\right]$$

$$\tilde{U}_i^T \tilde{V}_{ij} + \tilde{V}_{ij}^T \tilde{U}_i \leqslant \frac{1}{2}\left[\tilde{U}_i + \tilde{S}_{ij}\tilde{V}_{ij}\right]^T \tilde{S}_{ij}^{-1}\left[\tilde{U}_i + \tilde{S}_{ij}\tilde{V}_{ij}\right].$$

Finally, using the equation of G given by (30) and simple changes of variables $\hat{K} = \hat{G}_{11}K^T$ and $\hat{L} = G_{22}L$, the LMI described in (24) is obtained. □

IV. Numerical example

Consider the example studied in [15]. The corresponding state-space model is described by (7) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, E = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, F = 0, R = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

and the nonlinear function: $f(x_k) = \alpha \sin(x_{2k})$.

To show the superiority of the proposed design methodology with respect to [23], [24], [15], we search for the α_{max} tolerated by each approach. The results are given in the Table I.

Method	α_{max}
[23]	1
[24]	1630
[15]	31541
LMI (24)	3366944

TABLE I: Different values of α_{max}.

According to the values given in Table I, it is clear that the proposed design methodology presents more degree of freedom then the existing ones.

V. Conclusion

In this paper, new results on $\mathcal{H}_\infty$ observer-based controller design are presented. Less conservative LMI condition is established using some mathematical tools (the reformulation of Young's relation) and useful lemmas (the definition of Lipschitz function). Comparison with some existing results is provided to prove the superiority of the proposed design methodology.

As a future work, we will try to implement this method with an embedded system with real time constaints to show that this approach is a powerful tool for this kind of problems.

References

[1] R. Rajamani, "Observer for lipschitz nonlinear systems," *IEEE Trans. on Autom. Control*, vol. 43, pp. 397–401, 1998.

[2] M. Abbaszadeh and H. J. Marquez, "Nonlinear observer design for one-sided lipschitz systems," in *Proc. American Control Conf.*, Marriott Waterfront, Baltimore, MD, USA, June 30-July 02, 2010, pp. 5284–5289.

[3] P. Pagilla and Y. Zhu, "Controller and observer design for lipschitz nonlinear systems," in *Proc. American Control Conf.*, Boston, Massachusetts,USA, 2004, pp. 2379–2384.

[4] Q. Ha and H. Trinh, "State and input simultaneous estimation for a class of nonlinear systems," *Automatica*, vol. 40, pp. 1779–1785, 2004.

[5] N. Gasmi, A. Thabet, M. Boutayeb, and M. Aoun, "Observer design for a class of nonlinear discrete time systems," in *2015 16th International Conference on Sciences and Techniques of Automatic Control and Computer Engineering (STA)*, Dec 2015, pp. 799–804.

[6] Y. Wang, C. Chan, K. Cheung, and W. Chan, "Fault estimation for a class of nonlinear dynamical systems," in *IEEE Proc. Conf. on Decision and Control*, Phoenix, Arizona USA, 1999, pp. 3128–3129.

[7] G. I. Bara, A. Zemouche, and M. Boutayeb, "Observer synthesis for lipschitz discrete-time systems," in *IEEE International Symposium on Circuits and Systems*, vol. 4, 2005, pp. 3195–3198.

[8] F. Zhu and Z.Han, "A note on observers for lipschitz nonlinear systems," *IEEE Trans. on Autom. Control*, vol. 47, pp. 1751–1754, 2002.

[9] A. Alessandri, "Design of observers for lipschitz nonlinear systems using lmi," in *NOLCOS, IFAC Symposium on Nonlinear Control Systems*, Stuttgart, Germany, 2004.

[10] M. Benallouch, M. Boutayeb, and M. Zasadzinski, "Observers design for one-sided lipschitz discrete-time systems," *Syst. Control Letters*, vol. 61, pp. 879–886, 2012.

[11] D. Ho and G. Lu, "Robust stabilization for a class of discrete-time nonlinear systems via output feedback: The unified lmi approach," *Int. J. of Control*, vol. 76, pp. 105–115, 2003.

[12] S. Ibrir, W. Xie, and C. Su, "Observer-based control of discrete-time lipschitzian non-linear systems: application to one-link flexible joint robot," *International Journal of Control*, vol. 78, no. 6, pp. 385–395, 2005.

[13] X. Mao, W. Liu, L. Hu, Q. Luo, and J. Lu, "Stabilization of hybrid stochastic differential equations by feedback control based on discrete-time state observations," *Syst. Control Letters*, vol. 73, pp. 88–95, 2014.

[14] C. Wang, Z. Zuo, Z. Lin, and Z. Ding, "Consensus control of a class of lipschitz nonlinear systems with input delay," *Circuits and Systems I: Regular Papers, IEEE Transactions on*, vol. 62, no. 11, pp. 2730–2738, Nov 2015.

[15] B. Grandvallet, A. Zemouche, H. Souley-Ali, and M. Boutayeb, "New lmi condition for observer-based $\mathcal{H}_\infty$ stabilization of a class of nonlinear discrete-time systems," *SIAM Journal on Control and Optimization*, vol. 51, 02 2013.

[16] H. Kheloufi, A. Zemouche, F. Bedouhene, and H. Souley-Ali, "Robust $\mathcal{H}_\infty$ observer-based controller for lipschitz nonlinear discrete-time systems with parameter uncertainties," in *53rd IEEE Conference on Decision and Control*, Dec 2014, pp. 4336–4341.

[17] H. Kheloufi, A. Zemouche, F. Bedouhene, C. Bennani, and H. Trinh, "New decentralized control design for interconnected nonlinear discrete-time systems with nonlinear interconnections," in *2016 IEEE 55th Conference on Decision and Control (CDC)*, Dec 2016, pp. 6030–6035.

[18] H. Bibi, F. Bedouhene, A. Zemouche, H. R. Karimi, and H. Kheloufi, "Output feedback stabilization of switching discrete-time linear systems with parameter uncertainties," *Journal of the Franklin Institute*, vol. 354, no. 14, pp. 5895 – 5918, 2017.

[19] A. Zemouche, R. Rajamani, B. Boulkroune, H. Rafaralahy, and M. Zasadzinski, "Convex optimization based dual gain observer design for lipschitz nonlinear systems," in *2016 American Control Conference (ACC)*, July 2016, pp. 125–130.

[20] ——, "$\mathcal{H}_\infty$ circle criterion observer design for lipschitz nonlinear systems with enhanced lmi conditions," in *2016 American Control Conference (ACC)*, July 2016, pp. 131–136.

[21] A. Zemouche, M. Zerrougui, B. Boulkroune, F. Bedouhene, H. Souley-Ali, and M. Zasadzinski, "$\mathcal{H}_\infty$ observer-based stabilization for lipschitz nonlinear systems," in *2016 European Control Conference (ECC)*, June 2016, pp. 2017–2022.

[22] L. Hassan, A. Zemouche, and M. Boutayeb, "$\mathcal{H}_\infty$ observers design for a class of nonlinear time-delay systems in descriptor form," *International Journal of Control*, vol. 84, no. 10, pp. 1653–1663, 2011.

[23] S. Ibrir, "Static output feedback and guaranteed cost control of a class of discrete-time nonlinear systems with partial state measurements," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 7, pp. 1784–1792, 2008.

[24] S. Ibrir and S. Diopt, "Novel lmi conditions for observer-based stabilization of lipschitzian nonlinear systems and uncertain linear systems in discrete-time," *Appl. Math. Comput.*, vol. 206, pp. 579–588, 2008.