2018 26th Mediterranean Conference on Control and Automation (MED)
Zadar, Croatia, June 19-22, 2018

H_∞ model reduction for two-dimensional discrete systems in finite frequency ranges

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***Abstract*— This paper examines the design problem H_∞ of the reduced order model for two-dimensional (2D) discrete systems described by the Roesser model with a control input assumed to operate in a finite frequency (FF) domain. Given an asymptotically stable system; our goal is to find a stable reduced order system so that the error of the transfer functions between the original system and the reduced order is limited to a range FF. Using the well-known generalized lemma of Kalman Yakubovich Popov (gKYP) and the Finsler's lemma, sufficient conditions for the existence of the reduction of the H_∞ model for different FF ranges are proposed and then unified in terms of solving a set of linear matrix inequalities (LMIs). An illustrative example is provided to show the utility and potential of the proposed results.**

***Index Terms*— Multidimensional Systems; Roesser Models; Finite Frequency; H_∞ Model Reduction.**

I. Introduction

Over the past decades, extensive researches on two-dimensional (2D) systems have been conducted due to their significance in both theory and practical applications such as multi-dimensional digital filtering, linear image processing, repetitive process control, seismographic data processing, water stream heating, thermal processes, signal processing as well as process control [1], [2], etc.

The two commonly used state-space models for 2D systems are Roesser model and Fornasini-Marchesini local state-space model [3], [4]. Several authors have been interested in these two models with different different objectives in mind. Thus, the controllability and the observability problems are studied in [5]. The stability analysis problem can be found in [6], [7] and the filter have been investigated in [24], [25], etc.

On the other hand, the model reduction is one of the fundamental problems in the field of system analysis and control theory, which has been extensively investigated in the past several decades [8], [9]. The model reduction problem can be formulated as follows: For a given full-order model of a dynamic system, find a reduced-order model such that these two models are close in some sense, such as H_∞, L_∞, etc. As for one-dimensional (1D) linear systems, various effective approaches, such as the optimal Hankel-norm approximation method [10], the aggregation method [11], the balanced truncation method [12]. However, Model reduction thus plays an important role in analysis and design of 2D systems. There are several model reduction methods developed for 2D systems, such as the balanced truncation method in [13], and LMI-based methods in [14] and [15].

In addition, for several engineering systems, the inputs operate in a limited frequency domain. Many of the existing physical systems are affected by external disturbances with known frequency ranges. So, it is well known that to get less conservative conditions, it should be very useful to take into account the frequency domain of the system inputs in the design. There are significant results in the literature reported on FF model reduction for one-dimensional (1D) linear and nonlinear systems (see, e.g., [16], [17], [21], [27] and the references therein) and for two-dimensional (2D) linear systems [18], [26].

In this paper, we present a new approach to solve the problem of FF H_∞ model reduction for two-dimensional (2D) discrete systems represented by the Roesser model. By the aid of the gKYP lemma [19], we derive a new set of sufficient conditions for the H_∞ performance analysis of the error system. To linearize and relax the obtained conditions, we apply the Finsler's lemma and introduce slack matrices. The proposed reduced-order model is more reasonable and can achieve better performance when the frequency range of the control inputs is known in advance. The theoretical results are given in the form of LMIs, which can be solved by standard digital software, thus providing a simple methodology. By comparing with the existing full frequency methods and the FF approach in [18], the FF method proposed in this paper achieves better results in cases where the frequency ranges of the control inputs are known. Finally, we demonstrate by numerical example that the proposed method can achieve a much smaller approximation error than some recent existing results. The rest of the article is organized as follows. Section 2 gives the statement of the problem and preliminaries. Section 3 presents our main result in detail. In section 4, we provide an illustrative example. Finally, the conclusion is given in section 5.

Notations Superscript "T" stands for matrix transposition. In symmetric block matrices or long matrix expressions, we use an asterisk " $*$ " to represent a term that is induced by symmetry. Notation $W > 0$ means that matrix W is positive. I denotes an identity matrix with appropriate dimension. Generally, $sym\{A\}$

This work was not supported by any organization

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978-1-5386-7891-6/18/$31.00 ©2018 IEEE

denotes $A + A^T$, diag{..} stands for block diagonal matrix. $sup_{\sigma_{max}(G)}$ denotes the maximum singular value of transfer matric G. The l_2 norm for a 2D signal $u(i,j)$ is given by

$$\| u \|_2 = \sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u^T(i,j)u(i,j)}$$

where $u(i,j)$ is said to be in the space $l_2\{[0,\infty),[0,\infty)\}$ or l_2, for simplicity, if $\| u \|_2 < \infty$. A 2D signal $u(i,j)$ in the l_2 space is an energy-bounded signal.

II. Problem statement and preliminaries

Consider a 2-D discrete system described by the following Roesser model [3]:

$$\begin{aligned} \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} &= A \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Bu(i,j) \\ y(i,j) &= C \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Du(i,j) \qquad (1) \\ x^h(0,j) &= \varphi(j)\ \forall j \ and\ x^v(i,0) = \phi(i)\ \forall i \end{aligned}$$

where $x^h(i,j) \in \mathbb{R}^{n_h}$, $x^v(i,j) \in \mathbb{R}^{n_v}$ with $n = n_h + n_v$ are horizontal and vertical states, respectively; $y(i,j) \in \mathbb{R}^{n_y}$ is the measured output; A, B, C, D are system matrices with appropriate dimensions and $u(i,j) \in \mathbb{R}^{n_u}$ is the exogenous input with energy bounded and horizontal frequency u_h and vertical frequency u_v of $u(i,j)$ satisfy $(u_h, u_v) \in \Omega_h \times \Omega_v$, where

$$\begin{aligned} \Omega_h &\triangleq \{\mu_h \in \mathbb{R} | \mu_{h_1} \le \mu_h \le \mu_{h_2};\ \mu_{h_1}, \mu_{h_2} \in [-\pi,\pi]\} \\ \Omega_v &\triangleq \{\mu_v \in \mathbb{R} | \mu_{v_1} \le \mu_v \le \mu_{v_2};\ \mu_{v_1}, \mu_{v_2} \in [-\pi,\pi]\} \end{aligned} \qquad (2)$$

with μ_{h_1}, μ_{h_2}, μ_{v_1} and μ_{v_2} being known scalars.

The boundary condition of the system is assumed to satisfy

$$\lim_{n\to\infty} \sum_{k=1}^{n} (|x^h(0,k)|^2 + |x^v(0,k)|^2) < \infty \qquad (3)$$

In this paper, we are interested in approximating the 2D system (1) by a stable $\hat{n}$th-order ($\hat{n} < n$) model described by

$$\begin{aligned} \begin{bmatrix} \hat{x}^h(i+1,j) \\ \hat{x}^v(i,j+1) \end{bmatrix} &= A_r \begin{bmatrix} \hat{x}^h(i,j) \\ \hat{x}^v(i,j) \end{bmatrix} + B_r u(i,j) \\ \hat{y}(i,j) &= C_r \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + D_r u(i,j) \quad (4) \\ \hat{x}^h(0,k) &= 0 \qquad \hat{x}^v(0,k) = 0; \quad \forall\ k \end{aligned}$$

where $\hat{x}^h(i,j) \in \mathbb{R}^{\hat{n}_h}$ and $\hat{x}^v(i,j) \in \mathbb{R}^{\hat{n}_v}$ with $\hat{n} = \hat{n}_h + \hat{n}_v$ are the horizontal and vertical states of the filter respectively, and $\hat{y}(i,j) \in \mathbb{R}^{\hat{n}_y}$ is the estimate of $y(i,j)$. A_r, B_r, C_r and D_r are matrices to be determined.

Augmenting system (1), to include the states of the system (4), we can obtain the following approximation error system

$$\begin{aligned} \begin{bmatrix} \bar{x}^h(i+1,j) \\ \bar{x}^v(i,j+1) \end{bmatrix} &= \bar{A} \begin{bmatrix} \bar{x}^h(i,j) \\ \bar{x}^v(i,j) \end{bmatrix} + \bar{B}u(i,j) \\ \bar{y}(i,j) &= \bar{C} \begin{bmatrix} \bar{x}^h(i,j) \\ \bar{x}^v(i,j) \end{bmatrix} + \bar{D}u(i,j) \quad (5) \end{aligned}$$

where

$$\begin{aligned} \bar{A} &= \Psi \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix} \Psi^T;\ \bar{x}^h(i,j) = \begin{bmatrix} x^h(i,j) \\ \hat{x}^h(i,j) \end{bmatrix}; \\ \bar{B} &= \Psi \begin{bmatrix} B \\ B_r \end{bmatrix};\quad \bar{x}^v(i,j) = \begin{bmatrix} x^v(i,j) \\ \hat{x}^v(i,j) \end{bmatrix}; \\ \bar{C} &= \begin{bmatrix} C & -C_r \end{bmatrix} \Psi^T;\ \bar{y}(i,j) = y(i,j) - \hat{y}(i,j); \\ \bar{D} &= D - D_r;\ \bar{n}_h = n_h + \hat{n}_h \quad \bar{n}_v = n_v + \hat{n}_v. \end{aligned}$$

and

$$\Psi = \begin{bmatrix} I_{n_h} & 0 & 0 & 0 \\ 0 & 0 & I_{\hat{n}_h} & 0 \\ 0 & I_{n_v} & 0 & 0 \\ 0 & 0 & 0 & I_{\hat{n}_v} \end{bmatrix} \qquad (6)$$

The matrix transfer function of error system (5) is then given by

$$\bar{G}(z_1,z_2) = \bar{C}[diag\{z_1 I_{\bar{n}_h}, z_2 I_{\bar{n}_v}\} - \bar{A}]^{-1}\bar{B} + \bar{D} \qquad (7)$$

where $z_1 = e^{j\mu_h};\ z_2 = e^{j\mu_v}$.

Problem $\mathbf{FF - H_\infty - MR}$ (Finite frequency H_∞ model reduction): Find an admissible reduced-order system (4) for system (1) such that the following two requirements are satisfied:

- The filtering error system in (5) is robustly asymptotically stable.
- Letting $\gamma > 0$ be a given constant, under zero-initial conditions, the following FF index holds:

$$sup\ \sigma_{max}[\bar{G}(\mu_h,\mu_v)] < \gamma \quad \forall (\mu_h,\mu_v) \in \Omega_h \times \Omega_v \qquad (8)$$

We state the following Finsler's lemma which will be used in the paper.

Lemma 1 *[20] Let $\xi \in \mathbb{R}^n$, $\mathcal{B} \in \mathbb{R}^{n\times n}$ and $\mathcal{Q} \in \mathbb{R}^{m\times n}$ with rank $(\mathcal{Q}) = r < n$ and $\mathcal{Q}^\perp \in \mathbb{R}^{n\times(n-r)}$ be full-column-rank matrix satisfying $\mathcal{Q}\mathcal{Q}^\perp = 0$. Then, the following conditions are equivalent:*

(i) $\xi^*\mathcal{B}\xi < 0, \forall \xi \neq 0 : \mathcal{Q}\xi = 0$
(ii) $\mathcal{Q}^{\perp *}\mathcal{B}\mathcal{Q}^\perp < 0$
(iii) $\exists \tau \in \mathbb{R} : \mathcal{B} - \tau\mathcal{Q}^*\mathcal{Q} < 0$
(vi) $\exists \mathcal{Y} \in \mathbb{R}^{n\times m} : \mathcal{B} + \mathcal{Y}\mathcal{Q} + \mathcal{Q}^*\mathcal{Y}^* < 0$

III. Main results

A. Finite Frequency H_∞ Model Reduction analysis

This subsection is devoted to FF H_∞ model reduction examination. Before proceeding further, we first propose the following Lemma that gives a sufficient condition for the error system (5) with FF specifications in (8).

Lemma 2 *[19] Let $\gamma > 0$ be a given scalar. For error system (5) is asymptotically stable, FF H_∞ performance (8) is satisfied, if there exist Hermitian $S = diag\{S_h,\ S_v\} \in \mathbb{H}_n$, $0 < R = diag\{R_h,\ R_v\} \in \mathbb{H}_n$ such that*

$$\begin{bmatrix} \bar{A} & \bar{B} \\ I & 0 \end{bmatrix}^T \begin{bmatrix} S & \Gamma^* R \\ R\Gamma & -S - \Lambda Q \end{bmatrix} \begin{bmatrix} \bar{A} & \bar{B} \\ I & 0 \end{bmatrix} + \begin{bmatrix} \bar{C} & \bar{D} \\ 0 & I \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \bar{C} & \bar{D} \\ 0 & I \end{bmatrix} < 0 \qquad (9)$$

where

$$\begin{aligned}
\Gamma &= \begin{bmatrix} e^{-j\mu_h^s} I_{\bar{n}_h} & 0 \\ 0 & e^{-j\mu_v^s} I_{\bar{n}_v} \end{bmatrix}; \\
\Lambda &= \begin{bmatrix} 2cos\mu_h^a I_{\bar{n}_h} & 0 \\ 0 & 2cos\mu_v^a I_{\bar{n}_v} \end{bmatrix}; \\
\mu_h^s &= \frac{\mu_{h_1}+\mu_{h_2}}{2}, \quad \mu_v^s = \frac{\mu_{v_1}+\mu_{v_2}}{2}; \\
\mu_h^a &= \frac{\mu_{h_2}-\mu_{h_1}}{2}, \quad \mu_v^a = \frac{\mu_{v_2}-\mu_{v_1}}{2}.
\end{aligned} \tag{10}$$

Based on lemma 2, we give the following theorem which can guarantee the asymptotical stability and the FF H_∞ performance of error system (5).

Theorem 1 *Consider 2D discrete system (1). For given $\gamma > 0$, a reduced-order model of form (4) exists such that the error system in (5) is asymptotically stable and satisfies the specification in (8). If there exist Hermitian matrices $S = diag\{S_h,\ S_v\} > 0$, $R = diag\{R_h, R_v\}$ and symmetric matrices $W = diag\{W_h, W_v\} > 0$ and general matrices M, N, H, satisfying*

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & M\bar{B}-H^T & 0 \\ * & \Phi_{22} & \Phi_{23} & \bar{C}^T \\ * & * & \Phi_{33} & \bar{D}^T \\ * & * & * & -I \end{bmatrix} < 0 \tag{11}$$

and

$$\Pi = \begin{bmatrix} W-M-M^T & -N^T+M\bar{A} \\ * & -W+\bar{A}N+N^T\bar{A}^T \end{bmatrix} < 0 \tag{12}$$

where

$$\begin{aligned}
\Phi_{11} &= P-M-M^T; \\
\Phi_{12} &= \Lambda^* Q - S^T + M\bar{A}^T; \\
\Phi_{22} &= -WQ-P+S\bar{A}+\bar{A}^T S^T; \\
\Phi_{23} &= S\bar{B}+\bar{A}^T R^T; \\
\Phi_{33} &= -\gamma^2 I + R\bar{B}+\bar{B}^T R^T;
\end{aligned}$$

and Λ and Γ defined in (10).

Proof First, we prove that (9) is equivalent to (11). Suppose that (11) holds, denote

$$\begin{aligned}
\mathcal{B} &= \begin{bmatrix} P & \Lambda^* Q & 0 \\ Q\Lambda & -WQ-P+\bar{C}^T\bar{C} & \bar{C}^T\bar{D} \\ 0 & \bar{D}^T\bar{C} & -\gamma^2 I+\bar{D}^T\bar{D} \end{bmatrix}; \\
\mathcal{Y} &= \begin{bmatrix} M \\ N \\ H \end{bmatrix}; \quad \mathcal{Q} = \begin{bmatrix} -I & \bar{A} & \bar{B} \end{bmatrix}.
\end{aligned} \tag{13}$$

By Shur complement, (11) is equivalent to

$$\mathcal{B} + \mathcal{Y}\mathcal{B} + \mathcal{B}^T\mathcal{Y}^T < 0 \tag{14}$$

under condition (iv) of Lemma 1, with

$$\mathcal{Q}^\perp = \begin{bmatrix} \bar{A} & \bar{B} \\ I & 0 \\ 0 & I \end{bmatrix}$$

which, using condition (ii) of Lemma 1, given (9).

In addition, let us construct a Lyapunov function inequality, $\bar{A}$ is stable if and only if there exists $W = \begin{bmatrix} W_h & 0 \\ 0 & W_v \end{bmatrix} > 0$ such that

$$W - \bar{A}^T W \bar{A} > 0 \tag{15}$$

which is rewritten in the form

$$\begin{bmatrix} \bar{A} \\ I \end{bmatrix}^T \begin{bmatrix} W & 0 \\ 0 & -W \end{bmatrix} \begin{bmatrix} \bar{A} \\ I \end{bmatrix} < 0 \tag{16}$$

Define

$$\begin{aligned}
\mathcal{B} &= \begin{bmatrix} W & 0 \\ 0 & -W \end{bmatrix}; \ \mathcal{Y} = \begin{bmatrix} M \\ N \end{bmatrix}; \\
\mathcal{Q} &= \begin{bmatrix} -I & \bar{A} \end{bmatrix}; \ \mathcal{Q}^\perp = \begin{bmatrix} \bar{A} \\ I \end{bmatrix}.
\end{aligned} \tag{17}$$

By Lemma 1, (16) and (17) are equivalent to

$$\begin{bmatrix} W & 0 \\ 0 & -W \end{bmatrix} + \begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} -I & \bar{A} \end{bmatrix} + \begin{bmatrix} -I & \bar{A} \end{bmatrix}^T \begin{bmatrix} M \\ N \end{bmatrix}^T < 0 \tag{18}$$

which is nothing but (12). ∎

B. Finite Frequency H_∞ Model Reduction design

In this section, a methodology is established for designing the FF H_∞ reduced-order model (4). The main objective is to determine the reduced-order matrices such that error system (5) is asymptotically stable with an H_∞-norm bounded by γ.

Theorem 2 *For given $\gamma > 0$,ρ_1, ρ_2 and ρ_3 a reduced-order model in form (4) exists such that the error system in (5) is asymptotically stable and satisfies the FF specification in (8), if there exist matrices $\hat{A}_r$, $\hat{B}_r$, $\hat{C}_r$, $\hat{D}_r$, M_u, N_u, H_1, U, $u = 1,2$, and hermitian matrices $S_s = diag\{S_{sh}, S_{sv}\}$, $R_s = diag\{R_{sh}, R_{sv}\}$, $s = 1,2,3$ and symmetric matrices $W_s = diag\{W_{sh}, W_{sv}\}$, $s = 1,2,3$, satisfying*

$$\begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} > 0, \quad \begin{bmatrix} W_1 & W_2 \\ * & W_3 \end{bmatrix} > 0 \tag{19}$$

$$\bar{\Phi} = \begin{bmatrix} \bar{\Phi}_{11} & \bar{\Phi}_{12} & \bar{\Phi}_{13} & \bar{\Phi}_{14} & \bar{\Phi}_{15} & 0 \\ * & \bar{\Phi}_{22} & \bar{\Phi}_{23} & \bar{\Phi}_{24} & \bar{\Phi}_{25} & 0 \\ * & * & \bar{\Phi}_{33} & \bar{\Phi}_{34} & \bar{\Phi}_{35} & C^T \\ * & * & * & \bar{\Phi}_{44} & \bar{\Phi}_{45} & -\hat{C}_r^T \\ * & * & * & * & \bar{\Phi}_{55} & D^T-D_r^T \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{20}$$

and

$$\bar{\Pi} = \begin{bmatrix} \bar{\Pi}_{11} & \bar{\Pi}_{12} & M_1 A - N_1^T & \bar{\Pi}_{14} \\ * & \bar{\Pi}_{22} & -\rho_1 U^T E^T + M_2 A & \bar{\Pi}_{24} \\ * & * & -W_1 + N_1 A + A^T N_1^T & \bar{\Pi}_{34} \\ * & * & * & \bar{\Pi}_{44} \end{bmatrix} < 0 \tag{21}$$

$$
\begin{aligned}
\bar{\Phi}_{11} &= S_1 - M_1 - M_1^T;\\
\bar{\Phi}_{12} &= S_2 - EU - M_2^T;\\
\bar{\Phi}_{13} &= \Gamma^* R_1 + M_1 A - N_1^T;\\
\bar{\Phi}_{14} &= \Gamma^* R_2 + E\hat{A}_r - N_2^T;\\
\bar{\Phi}_{15} &= -H_1^T + E\hat{B}_r;\\
\bar{\Phi}_{22} &= S_3 - U - U^T;\\
\bar{\Phi}_{23} &= S_2^T\Gamma - \rho_1 U^T E^T + M_2 A;\\
\bar{\Phi}_{24} &= \Gamma^* R_3 - \rho_2 U^T + \hat{A}_r;\\
\bar{\Phi}_{25} &= M_2 B + \hat{B}_r - \rho_3 U^T;\\
\bar{\Phi}_{33} &= -\Lambda R_1 - S_1 + N_1 A + A^T N_1^T;\\
\bar{\Phi}_{34} &= -\Lambda R_2 - S_2 + A^T N_2^T + \rho_1 E\hat{A}_r;\\
\bar{\Phi}_{35} &= S_1 A + A^T R_1^T + \rho_1 E\hat{B}_r;\\
\bar{\Phi}_{44} &= -\Lambda R_3 + \rho_2(\hat{A}_r + \hat{A}_r^T);\\
\bar{\Phi}_{45} &= R_2 B + \rho_2\hat{B}_r + \rho_3\hat{A}_r;\\
\bar{\Phi}_{55} &= -\gamma^2 I + H_1 B + B^T H_1^T + \rho_3(\hat{B}_r + \hat{B}_r^T);\\
\bar{\Pi}_{11} &= W_1 - M_1 - M_1^T;\\
\bar{\Pi}_{12} &= W_2 - EU - M_2^T;\\
\bar{\Pi}_{14} &= -N_2^T + E\hat{A}_r;\\
\bar{\Pi}_{22} &= W_3 - U - U^T;\\
\bar{\Pi}_{24} &= -\rho_2 U^T + \hat{A}_r;\\
\bar{\Pi}_{34} &= -W_2 + A^T N_2^T + \rho_1 E\hat{A}_r;\\
\bar{\Pi}_{44} &= -W_3 + \rho_2(\hat{A}_r + \hat{A}_r^T).
\end{aligned}
$$

where $E = \begin{bmatrix} I & 0 \end{bmatrix}^T$ *and* Γ, Λ *are defined in (10). Moreover, if (19-21) are feasible, the parameters of the reduced-order model can be obtained by*

$$A_r = U^{-1}\hat{A}_r,\ \ B_r = U^{-1}\hat{B}_r,\ \ C_r = \hat{C}_r;\ \ D_r = \hat{D}_r. \tag{22}$$

Proof For the slack matrix matrices in theorem 1, we first structuring them as the following specific block form:

$$
\begin{aligned}
M &= \Psi\begin{bmatrix} M_1 & U \\ M_2 & U \end{bmatrix}\Psi^T,\ N = \Psi\begin{bmatrix} N_1 & \rho_1 U \\ N_2 & \rho_2 U \end{bmatrix}\Psi^T,\\
H &= \begin{bmatrix} H_1 & \rho_3 U \end{bmatrix}\Psi^T
\end{aligned}
\tag{23}
$$

with Ψ in (6). Moreover, for matrix variables S, $R > 0$, $W > 0$ in Theorem 1, we define:

$$
\begin{aligned}
S &= \Psi\begin{bmatrix} S_1 & S_2 \\ * & S_3 \end{bmatrix}\Psi^T,\ R = \Psi\begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix}\Psi^T,\\
W &= \Psi\begin{bmatrix} W_1 & W_2 \\ * & W_3 \end{bmatrix}\Psi^T.
\end{aligned}
\tag{24}
$$

In addition, the following equations are readily obtained:

$$\bar{\Gamma} = \Psi diag\{\Gamma,\ \Gamma\}\Psi^T, \quad \bar{\Lambda} = \Psi diag\{\Lambda,\ \Lambda\}\Psi^T \tag{25}$$

Due to $\Psi^T\Psi = I$, by replacing (5) into (11) and (12) and combining (19-21), we have

$$\bar{\Phi} = \Upsilon\ \Phi\ \Upsilon^T; \qquad \bar{\Pi} = \Xi\ \Pi\ \Xi^T, \tag{26}$$

where

$$
\begin{aligned}
&\Upsilon = diag\{\Psi,\ \Psi,\ I_{n_u},\ I_{n_y}\}; \quad \Xi = diag\{\Psi,\ \Psi\},\\
&\hat{A}_r = UA_r; \quad \hat{B}_r = UB_r; \quad \hat{C}_r = C_r; \quad \hat{D}_r = D_r,
\end{aligned}
$$

and Φ, Π are in Theorem 1, respectively. ∎

Remark 1 *When scalars ρ_1, ρ_2 and ρ_3 of Theorem 2 are constant, the conditions in Theorem 2 are LMIs. To select values for these scalars, numerical optimization techniques could be used (for example fminsearch in MATLAB) to optimize some performance measure (for example disturbance attenuation level γ).*

Remark 2 *If we take $R_s = diag\{R_{sh}\ ,R_{sv}\} = 0$, $s = 1, 2, 3$, we can use Theorem 2 to solve the H_∞ model reduction problem in the entire frequency (EF) domain of two-dimensional (2D) discrete systems described by Roesser model.*

Remark 3 *It is expected that the obtained result provides less conservative than the proposed one in [18], thanks to added slack variables using the Finsler's lemma. Furthermore, it should be mentioned that there is no constraints on matrix variables involved by the same lemma.*

Remark 4 *In Theorem 2, μ_{h_1}, μ_{h_2}, μ_{v_1} and μ_{v_2} are frequency bounds, which are given beforehand. Conditions (19), (20) and (21) are LMIs, and thus the above optimization problem is convex and can be easily solved by Yalmip [22] and SeDuMi [23] in MATLAB 7.6.*

IV. Simulation Study

In this section, we use an example to illustrate the effectiveness of the theoretical results developed before. Consider a 2D discrete system described by Roesser model in (1) with $(n_h, n_v) = (1, 4)$ and the following system matrices [18]:

$$
\begin{aligned}
A &= \begin{bmatrix} 0.2 & -1 & 0 & 0 & 0 \\ 0.1 & 0 & 1 & 0 & 0 \\ -0.1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\\
C &= \begin{bmatrix} 0.2 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}
\tag{27}
$$

Assume that the reduced-order model in (4) has an order $(\hat{n}_h, \hat{n}_v) = (1, 1)$.
The purpose here is to design a FF H_∞ reduced-order model in the form of (4) such that the resulting error system in (5) is asymptotically stable with a guaranteed H_∞ norm bound γ.

By solving the LMI conditions given in Theorem 2, the minimum H_∞ performances γ and maximum error are given in the following Tables:

It is clearly shown that Theorem 2 yields less conservative results than the FF method proposed in [18].
The reduced-order parameters are given as follows:

1) **Finite Frequency** : (**Th 2**)*:*

- For FF domain $[\frac{-\pi}{4}, \frac{\pi}{4}] \times [\frac{-\pi}{4}, \frac{\pi}{4}]$, $\rho_1 = 0.0618$, $\rho_2 = 0.0302$ and $\rho_3 = 0.1792$; we obtained $\gamma = 0.1199$ and

Frequency	*Methods*	γ	*Maximum error*
$FF\ ([\frac{-\pi}{4},\frac{\pi}{4}]\times[\frac{-\pi}{4},\frac{\pi}{4}])$	*Theorem 1 in* [18]	0.1271	0.1078
$FF\ ([\frac{-\pi}{4},\frac{\pi}{4}]\times[\frac{-\pi}{4},\frac{\pi}{4}])$	**Theorem 2**	**0.1199**	**0.1057**
$FF\ ([\frac{-\pi}{8},\frac{\pi}{8}]\times[\frac{-\pi}{8},\frac{\pi}{8}])$	*Theorem 1 in* [18]	0.0948	0.0821
$FF\ ([\frac{-\pi}{8},\frac{\pi}{8}]\times[\frac{-\pi}{8},\frac{\pi}{8}])$	**Theorem 2**	**0.0635**	**0.0590**

TABLE I: Comparison of reduced order performance and maximum error obtained in different finite frequency (FF) ranges.

Frequency	*Methods*	γ	*Maximum error*
EF	[18]	*Infeasible*	–
EF	**Theorem 2** ($\mathbf{R_{s=1,2,3}=0}$)	**0.3687**	**0.2926**

TABLE II: Comparison of reduced order performance and maximum error obtained in different entire frequency ranges

reduced-order model

$$\begin{aligned} A_r &= \begin{bmatrix} 0.1817 & -0.8731 \\ -0.0076 & 0.0512 \end{bmatrix}, \\ B_r &= \begin{bmatrix} -1.1442 & 0.0217 \\ -0.0176 & -1.1309 \end{bmatrix}, \\ C_r &= \begin{bmatrix} -0.1480 & 0.8497 \\ -0.9270 & -0.7754 \end{bmatrix}, \\ D_r &= \begin{bmatrix} 0.9978 & -0.0180 \\ -0.0189 & 1.0208 \end{bmatrix}. \end{aligned} \tag{28}$$

- For FF domain $[\frac{-\pi}{8},\frac{\pi}{8}]\times[\frac{-\pi}{8},\frac{\pi}{8}]$, $\rho_1 = 0.0618$, $\rho_2 = 0.0302$ and $\rho_3 = 0.1792$; we obtained $\gamma = 0.1199$ and reduced-order model

$$\begin{aligned} A_r &= \begin{bmatrix} 0.1943 & -0.8930 \\ -0.0115 & 0.0979 \end{bmatrix}, \\ B_r &= \begin{bmatrix} -1.0978 & 0.0319 \\ -0.0113 & -0.9988 \end{bmatrix}, \\ C_r &= \begin{bmatrix} -0.1687 & 0.9044 \\ -0.9210 & -0.8213 \end{bmatrix}, \\ D_r &= \begin{bmatrix} 1.0017 & -0.0141 \\ 0.0128 & 1.0243 \end{bmatrix}. \end{aligned} \tag{29}$$

2) **Entire Frequency Case** : (**Th 2** (**with** $\mathbf{R_s = 0}$))*:* For $\rho_1 = 0.2173$, $\rho_2 = 0.2032$ and $\rho_3 = -0.0928$; we obtained $\gamma = 0.3687$ and reduced-order model

$$\begin{aligned} A_r &= \begin{bmatrix} 0.2118 & -1.0567 \\ 0.0872 & -0.0357 \end{bmatrix}, \\ B_r &= \begin{bmatrix} -0.9787 & -0.0034 \\ -0.0045 & -0.9344 \end{bmatrix}, \\ C_r &= \begin{bmatrix} -0.2031 & 1.0947 \\ -1.0193 & -1.0820 \end{bmatrix}, \\ D_r &= \begin{bmatrix} 1.0111 & 0.0010 \\ -0.0245 & 0.9978 \end{bmatrix}. \end{aligned} \tag{30}$$

To illustrate the effectiveness of these designed reduced orders, by respectively connecting (28), (29) and (30) to the system in (27), the frequency responses of the error systems are depicted in figures 1, 2 and 3. These figures clearly demonstrate the effectiveness of the proposed method.

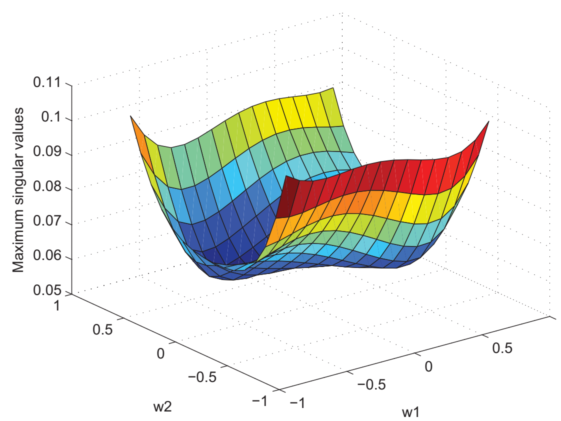

Fig. 1: Frequency response of the systems error system with reduced-order (28).

V. Conclusion

A new approach has been developed to study the FF H_∞ model reduction problem for two-dimensional (2D) discrete systems described by Roesser model. To reduce the error and establish less conservative results, we apply the Finsler's lemma and we introduce slack matrices to provide extra free dimensions in the solution space of the H_∞ optimization. The effectiveness of the developed results has been demonstrated by an illustrative example.

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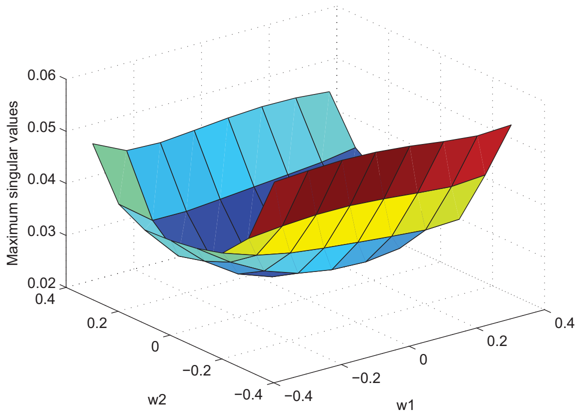

Fig. 2: Frequency response of the systems error system with reduced-order (29).

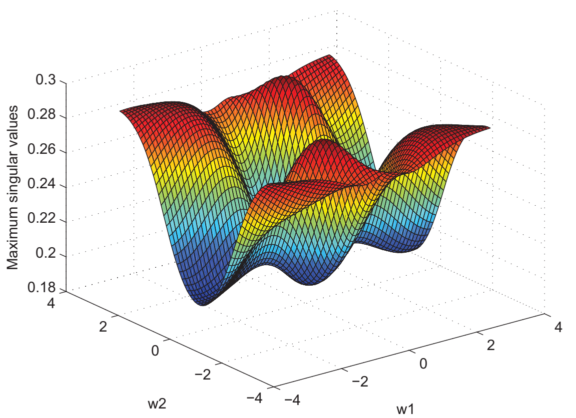

Fig. 3: Frequency response of the systems error system with reduced-order (30).

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