2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Surrounding formation of star frameworks using bearing-only measurements

Quoc Van Tran, Minh Hoang Trinh, and Hyo-Sung Ahn

Abstract— **In this paper, the novel surrounding formation protocol around a beacon is proposed using bearing-only formation. In the surrounding formation problem, all agents surround the beacon such that all the relative angles between some pairs of agents and the beacon satisfy predefined relative angles. Moreover, each agent senses and controls only its direction to the beacon in its local coordinate frame without knowledge of common global coordinate system. We propose an orientation estimator by adding an auxiliary variable which is assumed to be communicated by agents. A bearing-only based control law is proposed using the auxiliary variables. We show that the desired surrounding formation is almost globally asymptotically stable based on input-to-state stability theory. Further, under the proposed control law, the surrounding formation is collision-free under an assumption of initial positions. Numerical simulations are provided to support the proposed control method.**

I. Introduction

Distributed formation control of multi-autonomous agents has captured a lot of research interests during the last decades due to its distributed properties and the simplicity of sensory systems. In distributed formation control, a system of multi-agents has to achieve a desired geometric formation with the condition that agents can only access to local information such as relative distances and relative orientations from a set of their neighbors. Among them, distance-based formation control has been extensively studied in recent years [3]. In distance-based formation control, each agent senses and controls neighboring relative distances. The multi-agent system converges to desired formation shape up to rotation and translation supposed that the interaction graph is rigid or persistent.

Displacement-based and bearing-only based formation control are two other types of formation control in which inter-agent displacements and bearing angles, respectively, are actively controlled by agents. However, in these two control scenarios, a common orientation is required for each agent. Therefore, they are not fully-distributed under this sense. Several methods have been proposed to make aforementioned control problems distributed by aligning or estimating the agents' orientations [1], [2], [4].

We consider a surrounding control problem of star frameworks in which multiple agents sense and control their local bearing angles with regard to a stationary control station. The control station acts as a beacon. The local bearing means that the bearing angle is measured in local coordinate frame. A desired geometric formation of star frameworks is defined by a set of relative bearing angles between some pairs of agents to the control station. The agents achieve the desired relative angles between the agents and the beacon-agent without knowledge of common orientation given that they are able to measure the relative orientations to their neighbors. In this sense, the control problem is fully-distributed. Typical applications of surrounding formation are protection of central station, surveillance, commanding and information transferring from the central control station.

A similar problem is the circular formation (or enclosing a target) in which mobile agents orbit along a common circle and maintain desired spacing along the circle. A number of works have investigated the problem of enclosing a target for holonomic mobile robots [6], [7], and nonholonomic mobile robots [8]–[10]. Most of the works require relative position measurements amongst the robots which increase complexity of sensor systems. Therefore, bearing-only based formation control has been utilized for circular formation of mobile robots in [11], given that only relative bearing measurements are available. The works in [12], [13] proposed projective-like control laws for flocking on circle of autonomous mobile agents. However, global orientation and orientations of all agents are required in such circular formation control schemes. A balanced circular formation control strategy using angle of arrival information without sense of global orientation was proposed in [14]. Under the control strategy, agents are evenly distributed on a circular trajectory. The circular formation differs from the surrounding formation defined in this work; the circular formation requires agents to stay in a circular trajectory while the surrounding formation does not require this, that is, uniform distance distribution around a target is not required in surrounding formation.

This work seeks to investigate the surrounding formation control problem of aforementioned star frameworks. We propose an orientation estimator by adding an auxiliary variable which is assumed to be communicated by agents. A similar estimation strategy using complex number for estimating orientation angles was proposed in [4]. Here, orientations of agents are embedded into estimated direction vectors which are equivalent to bearing vectors if normalized. A bearing-only based formation control law is proposed for each agent, expressed in local coordinate frame, using the estimated direction vectors. Under the control law, agents can be placed with arbitrary relative angles around the control station. Further, the proposed bearing-only based formation control law guarantees almost global asymptotical stability of desired surrounding formation up to a rotation. In addition,

The authors are with School of Mechanical Engineering, Gwangju Institute of Science and Technology, Gwangju, Korea e-mail: {tranvanquoc, trinhhoangminh, hyosung}@gist.ac.kr.

collision avoidance of star frameworks is also analyzed.

The rest of this paper is organized as follows. Background in algebraic graph theory and problem formulation are provided in Section II. Section III analyses bearing rigidity which describes infinitesimal motions of star frameworks. The orientation estimation and proposed control law, and main analyses are presented in Section IV. Section V provides numerical simulations for alignment and surrounding formation of a group of agents and one beacon under star framework. Finally, concluding remarks are given in Section VI.

II. Background and Problem Formulation

A. Background in Graph Theory

Given n dynamic agents in the plane whose interaction topology is characterized by a directed weighted graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V}$ and $\mathcal{E}$ denote the set of vertices and edges, respectively, and $\mathcal{A} = [a_{ij}]$ is a weighted adjacency matrix with $a_{ij} > 0$ for all $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. Each node is labelled by $i \in \mathcal{I} := \{1, \ldots, n\}$. Each edge is denoted by $e_k = (i, j) \in \mathcal{V} \times \mathcal{V}$. We refer to i and j as the *tail* and *head* of the edge (i, j), respectively. The set of neighbors of node i is denoted by $\mathcal{N}_i = \{j : (i, j) \in \mathcal{E}\}$. The degree matrix of $\mathcal{G}$ is denoted by $\Delta = diag(\Delta_{ii})$ where $\Delta_{ii} = deg_{out}(v_i) = \sum_{j \in \mathcal{N}_i} a_{ij}$. The Laplacian matrix $\mathbf{L} = [l_{ij}]$ associated with $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ is defined as $\mathbf{L} = \Delta - \mathcal{A}$. Let $n = |\mathcal{V}|$ and $m = |\mathcal{E}|$, the incidence matrix $\mathbf{H} = [h_{ik}] \in \mathbb{R}^{n \times m}$ is defined by

$$h_{ik} = \begin{cases} +1 & \text{if } i \text{ is the head of the edge } e_k, \\ -1, & \text{if } i \text{ is the tail of the edge } e_k, \\ 0 & \text{otherwise.} \end{cases}$$

The graph $\mathcal{G}$ contains a rooted-out branch if there exists at least one node having directed paths to any other nodes. A continuous-time consensus protocol is given as [15]:

$$\dot{\mathbf{x}}(t) = -(\mathbf{L} \otimes \mathbf{I}_d)\mathbf{x}(t) \tag{1}$$

where $\mathbf{x} = [\mathbf{x}_1^T, \ldots, \mathbf{x}_n^T]^T \in \mathbb{R}^{nd}$, and $\mathbf{L}$ is the Laplacian matrix of a weighted directed graph $\mathcal{G}$. The following result is borrowed from [16]:

Theorem 1: The equilibrium set $\mathcal{E}_x = \{\mathbf{x} \in \mathbb{R}^{nd} : \mathbf{x}_i = \mathbf{x}_j,\ \forall i, j \in \mathcal{V}\}$ of (1) is globally exponentially stable if and only if $\mathcal{G}$ has a rooted-out branch. Moreover, there exists $\mathbf{x}^\infty \in \mathbb{R}^d$ such that $\mathbf{x}(t)$ exponentially convergences to $(\mathbf{1}_n \otimes \mathbf{I}_d)\mathbf{x}^\infty$ as $t \to \infty$.

B. Problem Formulation

Consider a group of n single-integrator modeled agents:

$$\dot{\mathbf{p}}_i^i = \mathbf{u}_i^i \tag{2}$$

where $\mathbf{p}_i^i \in \mathbb{R}^2$ and $\mathbf{u}_i^i \in \mathbb{R}^2$ denote the position and control input of agent i, respectively, expressed in its local coordinate frame ${}^i\Sigma$. The orientation misalignment between the local coordinate system ${}^i\Sigma$ and the global coordinate system ${}^g\Sigma$ is denoted by an angle θ_i, $0 \leq \theta_i \leq 2\pi$, as illustrated in Fig.

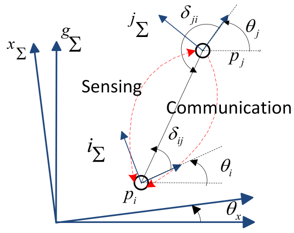


Fig. 1: The sensing and communication protocol between agent i and its neighbor j.

1. Each agent maintains its local coordinate frame and does not have common sense of the global reference frame ${}^g\Sigma$.

There is a stationary control station acting as a beacon x positioned at $\mathbf{p}_x \in \mathbb{R}^2$ such that all agents can sense the direction with respect to it. The direction measurement of agent i is given by an unitary bearing vector, expressed in its local coordinate frame

$$\mathbf{b}_i^i = \frac{\mathbf{p}_x^i - \mathbf{p}_i^i}{\|\mathbf{p}_x^i - \mathbf{p}_i^i\|}.$$

The control station can be a dynamic agent. In this case, it is assumed that the system of n agents- and a control station satisfies the velocity matching condition by additional flocking control scheme [17], [18]. For simplicity, in this paper, we assume that the control station is stationary. Let $\mathbf{b}_i \in \mathbb{R}^2$ be the corresponding measured bearing vector of agent i with regard to the beacon x expressed in ${}^g\Sigma$ and let $\mathbf{b} = \left[\mathbf{b}_1^T, \ldots \mathbf{b}_n^T\right]^T \in \mathbb{R}^{2n}$. Then, the relative angle between agent i, the beacon x, and agent j is the counterclockwise angle $\widehat{\mathbf{b}_i, \mathbf{b}_j}$. It is desired that the agents surround the control station such that all the relative angles satisfy $\alpha_{ij} = \alpha_{ij}^*$, for all $i, j \in \{1, \ldots, n\},\ i \neq j$. We are now ready to define the concept of surrounding formation, which is used repeatedly in this paper:

Definition 2.1 (Surrounding Formation): The star framework of n dynamic agents and a central control station is said to reach surrounding formation for a feasible set of relative angles $\{\alpha_{ij}^*\}_{i,j \in \mathcal{V},\ i \neq j}$ if α_{ij} converges to α_{ij}^* as $t \to \infty$, for all $i, j \in \{1, \ldots, n\},\ i \neq j$.

An example of a star framework of six-agents and a central beacon is illustrated in Fig. 2. Each agent senses and actively controls its bearing vector to the central station located at $\mathbf{p}_x$. In Fig. 2, agent 1 measures and controls the bearing vector $\mathbf{b}_1^1$, expressed in ${}^1\Sigma$. Define the set of desired bearing vectors $\mathbf{b}^* = \left[\mathbf{b}_1^{*T}, \ldots, \mathbf{b}_n^{*T}\right]^T \in \mathbb{R}^{2n}$ given in ${}^g\Sigma$ such that $\alpha_{ij}^* = \widehat{\mathbf{b}_i^*, \mathbf{b}_j^*}$, for all $i \in \{1, \ldots, n-1\}, j = i + 1$. For the star framework in Fig. 2, the surrounding formation can be defined by set of desired relative angles $\alpha^* = \{\alpha_{12}^*, \alpha_{23}^*, \alpha_{34}^*, \alpha_{45}^*, \alpha_{56}^*\}$. Note that the group objective is equivalent to achieving $\mathbf{b} = \bar{\mathbf{Q}}\mathbf{b}^*$, where $\bar{\mathbf{Q}}(\beta) = \mathbf{I}_n \otimes \mathbf{Q}(\beta)$ and $\mathbf{Q} \in \mathbb{R}^{2 \times 2}$ is a rotation matrix. Note that we use $\mathbf{Q}_i$ as a shorthand for $\mathbf{Q}(\theta_i)$-the rotation matrix of agent i.

In order to achieve the goal, we assume that each agent i ($i \in \mathcal{V}$) can sense and exchange information about relative orientation with its neighboring agents. Figure 1 describes

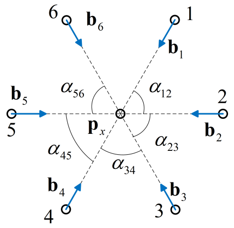


Fig. 2: A star framework of six agents and a control station.

such sensing and communication topology, the sensing is bidirectional in which agent i senses δ_{ij} and its neighbor j senses δ_{ji}. However, the communication is unidirectional; the neighbor j sends δ_{ji} back to i. From the sensing and communication information, agent i can obtain the relative orientation between i and j as

$$\theta_{ij} = PV(\theta_j - \theta_i) = PV(\delta_{ji} - \delta_{ij} + \pi), \tag{3}$$

where $PV(\theta_j - \theta_i) := [\theta_j - \theta_i \ mod \ 2\pi] - \pi$ [2].

Assumption 1: In surrounding formation of star frameworks, the control station assigns desired relative angles and commands desired bearings to agents via communication. An example of bearing assignments can be as follows:

- $\mathbf{b}_1^* = \mathbf{Q}(\alpha_0^*)[1, \ 0]^T$, for any $\alpha_0^* \in \mathbb{R}$
- $\mathbf{b}_{i+1}^* = \mathbf{Q}(\alpha_{ij}^*)\mathbf{b}_i^*, \ i = 1, \ldots, n-1$.

Note that the desired surrounding formation defined by $\alpha^* = \{\alpha_0^*, \alpha_{12}^*, \ldots, \alpha_{(n-1)n}^*\}$ can be designed and assigned beforehand to each agent. However, Assumption 1 allows control station to arbitrary assign relative angles and change the surrounding formation configuration of star frameworks in real time.

Under the Assumptions 1, we can now state the main problem of this paper.

Problem 1: Given a set of desired relative angles $\{\alpha_{ij}^* | \ j = i+1, \ i \in \{1, \ldots, n-1\}\}$, design a distributed control law for each agent using only bearing information such that $\alpha_{ij} \to \alpha_{ij}^*$ exponentially, for all $i \in \{1, \ldots, n-1\}, \ j = i+1$.

III. Bearing Rigidity Theory for Star Frameworks

Since the target-surrounding configuration satisfying the group objective is nonunique, this section studies all these configurations. Because the focus of this paper is maintaining some relative angles, the main tool used in this section is bearing rigidity theory [1]. Let $\mathbf{p} = [\mathbf{p}_1^T, \ldots, \mathbf{p}_n^T, \mathbf{p}_x^T]^T \in \mathbb{R}^{2n+2}$ be the position stacked vector of the system of n agents and the beacon x in the global coordinate frame. Denote the incidence matrix as $\mathbf{H} \in \mathbb{R}^{n \times (n+1)}$ and $\bar{\mathbf{H}} = \mathbf{H} \otimes \mathbf{I}_2$. Let $\mathbf{z}_i = \mathbf{p}_x - \mathbf{p}_i$ be the displacement between agent i and the beacon, and $\mathbf{z} = [\mathbf{z}_1^T, \ldots, \mathbf{z}_n^T]^T \in \mathbb{R}^{2n}$; then we have $\mathbf{z} = \bar{\mathbf{H}}\mathbf{p}$. The stacked bearing vector written in the global coordinate frame is given by $\mathbf{b} = [\mathbf{b}_1^T, \ldots, \mathbf{b}_n^T]^T \in \mathbb{R}^{2n}$. The bearing rigidity matrix is defined as

$$\mathbf{R}(\mathbf{p}) = \partial \mathbf{b} / \partial \mathbf{p} = diag\left(\frac{\mathbf{P}_{b_i}}{\|\mathbf{z}_i\|}\right) \bar{\mathbf{H}} \tag{4}$$

where $\mathbf{P}_{b_i} := \mathbf{I}_2 - \mathbf{b}_i \mathbf{b}_i^T \in \mathbb{R}^{2 \times 2}$ is the orthogonal projection matrix. Figure 3a gives an example of a target-centric system of six agents. The graph is actually a star graph. The incidence matrix in this example is explicitly written as follows:

$$\mathbf{H} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

The infinitesimally bearing rigid motions are motions of the star framework that preserve all relative angles $\alpha_{ij}, \ \forall i, j \in \{1, \ldots, n\}$. For star graphs, infinitesimally bearing rigid motions include:

- Translations: these motions are spanned by $\mathbf{v}_1 = \mathbf{1}_{n+1} \otimes [1 \ 0]^T = [1 \ 0 \ 1 \ 0 \ldots \ 1 \ 0]^T$ and $\mathbf{v}_1 = \mathbf{1}_{n+1} \otimes [0 \ 1]^T = [0 \ 1 \ 0 \ 1 \ldots \ 0 \ 1]^T$ (see Fig. 3b).
- Stretches and Shrinks: these motions are spanned by $\mathbf{v}_{2+i} = [0 \ldots 0 \ r_i \mathbf{b}_i^T \ 0 \ldots 0]^T, \ \forall i \in \mathcal{V}$. Intuitively, the angles $\alpha_{ij}, \forall j \in \mathcal{V} \setminus \{i\}$ are invariant if vertex i slightly moves along its bearing direction $\mathbf{b}_i$. Note that $r_i > 0$ associates with a stretch and $r_i < 0$ associates with a shrink as illustrated in Fig. 3c.
- Co-rotations: these infinitesimal motions are not in $Null(\mathbf{R})$.[1] Figure 3d depicts an example when the whole framework rotates around the beacon. These motions for star frameworks are spanned by $\left[(\mathbf{b}_1^\perp)^T \|\mathbf{z}_1\|, \ldots, (\mathbf{b}_n^\perp)^T \|\mathbf{z}_n\|, \mathbf{0}^T\right]^T = (\mathbf{I}_{n+1} \otimes \mathbf{J})[\mathbf{z}_1^T, \ldots, \mathbf{z}_n^T, \mathbf{0}^T]^T$. Here, $\mathbf{b}_i^\perp = \mathbf{J}\mathbf{b}_i, \ \forall i = 1, \ldots, n$. The matrix $\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is the perpendicular operator in $\mathbb{R}^2$ and it can be checked that $\mathbf{b}_i^T \mathbf{b}_i^\perp = \mathbf{b}_i^T \mathbf{J} \mathbf{b}_i = 0, \forall i = 1, \ldots, n$.

IV. Proposed Control Law and Main Analysis

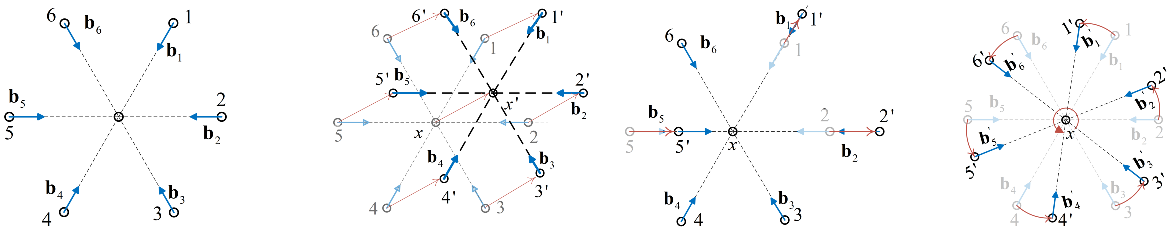
A. Bearing estimation and bearing-only formation control law

The orientation of agent can be expressed in terms of orientation angle [4] or rotation matrix [1]. If the orientation angle is used, it might need to run several orientation estimators simultaneously when applied to high dimensions. However, orientation angles represented in higher than two-dimensional space are highly coupled; this limits the application of angle estimator. Rotation matrix can be used to represent orientations which can be applied to arbitrary dimensions [1]. In this paper, for the consistent uses of directional vectors (bearing vector is a normalized direction vector) we express orientations in terms of directional vectors, which provides a simple way of expressing orientations. Define an auxiliary variable $\hat{\mathbf{z}} = [\hat{\mathbf{z}}_1^T, \ldots, \hat{\mathbf{z}}_n^T]^T \in \mathbb{R}^{2n}$; then the update law and bearing-only formation control law are proposed as

$$\dot{\hat{\mathbf{z}}}_{\mathbf{i}}(t) = \textstyle\sum_{j \in \mathcal{N}_i} a_{ij} (\mathbf{Q}_{ij}^T \hat{\mathbf{z}}_j(t) - \hat{\mathbf{z}}_i)(t), \tag{5a}$$

$$\mathbf{u}_i^i(t) = \dot{\mathbf{p}}_i^i(t) = \mathbf{P}_{\mathbf{b}_i^i} [\hat{\mathbf{z}}_i]_\times^T \mathbf{b}_i^*, \tag{5b}$$

[1]See also the bearing rigidity theory in $SE(2)$ [5].



(a) The original shape (b) Translation (c) Shrinking and Stretching (d) Rotation

Fig. 3: Infinitesimal motions that preserves the set of bearing vectors in star frameworks.

where, $\mathbf{P}_{\mathbf{b}_i^i} = \mathbf{I}_2 - \mathbf{b}_i^i \mathbf{b}_i^{iT}$ is the projection matrix, $\mathbf{Q}_{ij} = \mathbf{Q}_i^T \mathbf{Q}_j$ denotes the rotation of θ_{ij}, a_{ij} is the element with index (i,j) of adjacency matrix $\mathbf{A}[a_{ij}] \in \mathbb{R}^{n\times n}$ of the orientation sensing and communication graph (see Fig. 1), and the skew matrix form of $\hat{\mathbf{z}} = [x\ y]^T$ is denoted as $[\hat{\mathbf{z}}_i]_\times := \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$. In the estimation law (5a), the relative orientations $\theta_{ij}, j \in \mathcal{N}_i$, are measured and communicated between neighboring agents, while the states $\hat{\mathbf{z}}_j, j \in \mathcal{N}_i$, are sent to agent i by communication.

B. Analysis

The dynamics (5a) can be rewritten as

$$\begin{aligned} \dot{\hat{\mathbf{z}}}_\mathbf{i} &= \textstyle\sum_{j\in\mathcal{N}_i} a_{ij}(\mathbf{Q}_{ij}\hat{\mathbf{z}}_j - \hat{\mathbf{z}}_i) \\ \mathbf{Q}_i^T\dot{\hat{\mathbf{z}}}_\mathbf{i} &= \textstyle\sum_{j\in\mathcal{N}_i} a_{ij}(\mathbf{Q}_j^T\hat{\mathbf{z}}_j - \mathbf{Q}_i^T\hat{\mathbf{z}}_i) \\ blkdiag\{\mathbf{Q}_i^T\}\dot{\hat{\mathbf{z}}} &= -(\mathbf{L}\otimes\mathbf{I}_2)blkdiag\{\mathbf{Q}_i^T\}\hat{\mathbf{z}}, \end{aligned} \tag{6}$$

where $blkdiag\{.\}$ denotes the block diagonal matrix, $\hat{\mathbf{z}} = [\hat{\mathbf{z}}_1^T, \dots \hat{\mathbf{z}}_n^T]^T \in \mathbb{R}^{2n}$ denotes the stacked vector of auxiliary variables. Then, by using linear transformation $\mathbf{q} := blkdiag\{\mathbf{Q}_i^T\}\hat{\mathbf{z}}$ where $blkdiag\{\mathbf{Q}_i^T\}$ is nonsingular since $det(blkdiag\{\mathbf{Q}_i^T\}) = \prod_{i=1}^n det(\mathbf{Q}_i) = 1$, equation (6) can be further rewritten as

$$\dot{\mathbf{q}} = -(\mathbf{L}\otimes\mathbf{I}_2)\mathbf{q}. \tag{7}$$

It is a globally exponentially stable system, similar to (1), if and only if the interaction graph has a rooted-out branch. By Theorem 1, the following condition is required for the globally exponential convergence of the estimation law (5a).

Assumption 2: The directed interaction graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ of the n agents has a rooted out-branch [20].

Let ${}^x\Sigma$ denote an arbitrary fixed reference frame located at the beacon position $\mathbf{p}_x$ with its orientation angle θ_x such that all auxiliary variables $\hat{\mathbf{z}}_i$ are expressed in the common coordinate frame ${}^x\Sigma$ (see Fig. 4). Further, let $\mathcal{S}_x(\phi)$ denote the set of directional vector from x to a point i, i.e., $\mathbf{p}_{ix}$, such that $\mathbf{p}_{ix} = -\mathbf{z}_i = \mathbf{p}_i - \mathbf{p}_x$ makes with the horizontal axis of ${}^x\Sigma$ an angle ϕ. Denote $\mathbf{L}$ and $\mathbf{Q}^T := blkdiag\{\mathbf{Q}_i^T\}$ as the Laplacian matrix of the interaction graph of orientation sensing and communication, and the block diagonal matrix of rotation matrices, respectively. The convergence characteristics of the dynamics (5a) is shown by the following theorem:

Theorem 2: Under the update law (5a), there exists a finite vector $\hat{\mathbf{z}}^\infty \in \mathcal{S}_x(\theta_\infty)$ such that $\hat{\mathbf{z}}$ globally exponentially converges to $\hat{\mathbf{z}}^* = (\hat{\mathbf{z}}_i^*, \dots, \hat{\mathbf{z}}_n^*) \in S_x(\theta_1+\theta_\infty)\times\dots\times S_x(\theta_n+\theta_\infty)$ if and only if $\mathcal{G}$ has a spanning tree. Further, $\hat{\mathbf{z}}^*$ is nontrivial equilibrium if and only if an initial value $\mathbf{Q}^T\hat{\mathbf{z}}(t_0)$ is not in $Null(\mathbf{L}^T\otimes\mathbf{I}_2)^\perp$.

Proof: Under the Assumption 2, the dynamic system (7) reaches a consensus. Specifically, the $\mathbf{q}_i$ converges to $\hat{\mathbf{z}}^\infty \in \mathbb{R}^2$, i.e., $\mathbf{q}_i^* = \mathbf{q}_j^* = \hat{\mathbf{z}}^\infty$, $\forall i,j \in \mathcal{V}$. Let θ^∞ be the orientation angle of $\hat{\mathbf{z}}^\infty$ with respect to the beacon's reference frame ${}^x\Sigma$ then $\hat{\mathbf{z}}^\infty \in \mathcal{S}_x(\theta^\infty)$. Thus, the convergence of the axillary variable is $\hat{\mathbf{z}}_i \to \hat{\mathbf{z}}_i^* = \mathbf{Q}_i\mathbf{q}_i^* = \mathbf{Q}_i\hat{\mathbf{z}}^\infty \in \mathcal{S}_x(\theta_i+\theta^\infty)$, for all $i = (1,\dots,n)$. It is worthy to note that $\hat{\mathbf{z}}_i^*$ is obtained by rotating $\hat{\mathbf{z}}^\infty$ an angle of θ_i about x this implies that all $\hat{\mathbf{z}}_i^*$ lie in the circle $(x, \|\hat{\mathbf{z}}^\infty\|)$ as illustrated in Fig. 5.

To prove the second part of the Theorem 2 considering a solution of (7) as $\mathbf{q}(t) = e^{-(\mathbf{L}\otimes\mathbf{I_2})(t-t_0)}\mathbf{q}(t_0)$. Let $\mathbf{J}$ be the Jordan form associated with $\mathbf{L}$, i.e., $\mathbf{L} = \mathbf{PJP}^{-1}$. Here, column vectors of $\mathbf{P} = [\mathbf{w}_{r1}\dots\mathbf{w}_{rn}]$ are right eigenvectors of $\mathbf{L}$, with the first eigenvector associated with the zero eigenvalue of $\mathbf{L}$. Then, $\mathbf{P}^{-1} = [\mathbf{w}_{l1}^T\dots\mathbf{w}_{ln}^T]^T$ contains left eigenvectors of $\mathbf{L}$. Since every nonzero eigenvalue of $\mathbf{L}$ has positive real part, we have $exp(-(\mathbf{L}\otimes\mathbf{I}_2)(t-t_0)) = (\mathbf{P}\otimes\mathbf{I}_2)exp(-(\mathbf{J}\otimes\mathbf{I}_2)(t-t_0))(\mathbf{P}^{-1}\otimes\mathbf{I}_2)$. As $t\to\infty$ the steady state solution

$$\begin{aligned} \lim_{t\to\infty}\mathbf{q}(t) &= \lim_{t\to\infty}\mathbf{P}e^{-\mathbf{J}(t-t_0)}\mathbf{P}^{-1}\mathbf{q}(t_0) \\ &= (\mathbf{w}_{r1}\otimes\mathbf{I}_2)(\mathbf{w}_{l1}\otimes\mathbf{I}_2)\mathbf{q}(t_0) \end{aligned}$$

this indicates that $\mathbf{q}^*$ is trivial solution if and only if $\mathbf{q}(t_0)$ is perpendicular to $\mathbf{w}_{l1}\otimes\mathbf{I}_2$ which is the left eigenvector of $\mathbf{L}$ corresponding to the zero eigenvalue. Thus, it follows that $(\mathbf{w}_{l1}\otimes\mathbf{I}_2)\mathbf{q}(t_0) = 0$ if and only if $\mathbf{q}(t_0) = \mathbf{Q}^T\hat{\mathbf{z}}(t_0)$ is in $Null(\mathbf{L}^T\times\mathbf{I}_2)^\perp$, which completes the proof. ■

Suppose that the agent i moves under the control law

$$\dot{\mathbf{p}}_i(t) = -\mathbf{P}_{\mathbf{b}_i}\mathbf{b}_i^*. \tag{8}$$

Under this control law, there are two equilibria $\mathbf{p}_i = \mathbf{p}_{ia}^*$ where $\mathbf{b}_i = \mathbf{b}_i^*$ and $\mathbf{p}_i = \mathbf{p}_{ib}^*$ where $\mathbf{b}_i = -\mathbf{b}_i^*$. The following result is borrowed from [1], [21]:

Lemma 1: Under control law (8): (a) $d_i(t) = \|\mathbf{p}_x - \mathbf{p}_i(t)\|$ is invariant; (b) the equilibria $\mathbf{p}_i = \mathbf{p}_{ia}^*$ is almost globally exponentially stable while the equilibrium $\mathbf{p}_i = \mathbf{p}_{ib}^*$ is exponentially unstable.

The projection matrix $\mathbf{P}_{\mathbf{b}_i^i}$ in (5b) can be rewritten as $\mathbf{P}_{\mathbf{b}_i^i} = \mathbf{I}_2 - \mathbf{b}_i^i(\mathbf{b}_i^i)^T = \mathbf{I}_2 - \mathbf{Q}_i^T\mathbf{b}_i\mathbf{b}_i^T\mathbf{Q}_i = \mathbf{Q}_i^T\left(\mathbf{I}_2 - \mathbf{b}_i\mathbf{b}_i^T\right)\mathbf{Q}_i = \mathbf{Q}_i^T\mathbf{P}_{\mathbf{b}_i}\mathbf{Q}_i$.

Under the control law (5b), the dynamics of agent i in global reference frame is given as

$$\begin{aligned}\dot{\mathbf{p}}_i(t) &= -\mathbf{Q}_i\mathbf{Q}_i^T\mathbf{P}_{\mathbf{b}_i}\mathbf{Q}_i[\hat{\mathbf{z}}_i]_\times^T\mathbf{b}_i^* \\ &= -\mathbf{P}_{\mathbf{b}_i}\mathbf{Q}_i\|\hat{\mathbf{z}}_i\|\mathbf{Q}(\theta_i)\mathbf{Q}^T(\hat{\theta}_i+\theta_x+\theta^\infty)\mathbf{b}_i^* \\ &= -\mathbf{P}_{\mathbf{b}_i}\|\hat{\mathbf{z}}_i\|\mathbf{Q}^T(\Delta\theta_i+\theta_x+\theta^\infty)\mathbf{b}_i^* \\ &= -\mathbf{P}_{\mathbf{b}_i}\|\hat{\mathbf{z}}_i\|\mathbf{Q}^T(\Delta\theta_i)\mathbf{Q}^T(\theta_x+\theta^\infty)\mathbf{b}_i^* \\ &= -\mathbf{P}_{\mathbf{b}_i}\|\hat{\mathbf{z}}_i\|\mathbf{Q}^T(\theta_x+\theta^\infty)\mathbf{b}_i^* \\ &\quad + \mathbf{P}_{\mathbf{b}_i}\|\hat{\mathbf{z}}_i\|(\mathbf{I}_2-\mathbf{Q}^T(\Delta\theta_i))\mathbf{Q}^T(\theta_x+\theta^\infty)\mathbf{b}_i^* \\ &= -\|\hat{\mathbf{z}}_i\|\mathbf{P}_{\mathbf{b}_i}\mathbf{b}_i^{c*}+\mathbf{h}(t) \qquad (9)\end{aligned}$$

where $\mathbf{b}_i^{c*} = \mathbf{Q}^T(\theta_x+\theta^\infty)\mathbf{b}_i^*$, $\|\hat{\mathbf{z}}_i\| > 0$ is control gain, and $\mathbf{h}(t) = \mathbf{P}_{\mathbf{b}_i}\|\hat{\mathbf{z}}_i\|(\mathbf{I}_2-\mathbf{Q}^T(\Delta\theta_i))\mathbf{Q}^T(\theta_x+\theta^\infty)\mathbf{b}_i^*$ can be considered as an input to the system:

$$\dot{\mathbf{p}}_i(t) = -\|\hat{\mathbf{z}}_i\|\mathbf{P}_{\mathbf{b}_i}\mathbf{b}_i^{c*}. \qquad (10)$$

Lemma 2: The equilibrium $\mathbf{p}_i = \mathbf{p}_i^{c*}$ corresponding to $\mathbf{b}_i = \mathbf{b}_i^{c*}$ of the unforced system (10) is almost globally exponentially stable, while the equilibrium $\mathbf{p}_i = \mathbf{p}_i^{c'}$ corresponding to $\mathbf{b}_i = -\mathbf{b}_i^{c*}$ of the unforced system (10) is exponentially unstable.

Proof: The proof follows directly from Lemma 1 ■

Lemma 3: The input $\mathbf{h}(t)$ is bounded. Moreover, $\mathbf{h}(t) \to \mathbf{0}$ exponentially.

Proof: Since

$$\begin{aligned}\|\mathbf{h}(t)\| \le \|\mathbf{P}_{\mathbf{b}_i}\|\|\hat{\mathbf{z}}_i\|\|(\mathbf{I}_2-\mathbf{Q}^T(\Delta\theta_i))\| \\ \times\|\mathbf{Q}^T(\theta_x+\theta^\infty)\|\|\mathbf{b}_i^*\|\end{aligned} \qquad (11)$$

and each component on the right hand side of (11) is bounded, thus, $\|\mathbf{h}(t)\|$ is also bounded. Since $\Delta\theta_i \to 0$ exponentially as $t \to 0$ based on Theorem 2, $\mathbf{Q}^T(\Delta\theta_i) \to \mathbf{I}_2$ exponentially, and thus $\|\mathbf{h}(t)\| \to 0$ exponentially. ■

Theorem 3 (Input-to-state stability): The equilibrium $\mathbf{p}_i = \mathbf{p}_i^{c*}$ corresponding to $\mathbf{b}_i = \mathbf{b}_i^{c*}$ of the system (9) is almost globally asymptotically stable.

Proof: Under Assumption 2, (5a) is (almost) globally asymptotically stable (Theorem 2), the unforced system (10) is (almost) globaly asymptotically stable (Lemma 2), and input $\mathbf{h}(t)$ is bounded (Lemma 3). Using the application of almost global input-to-state stability theorem [1], [19], the equilibrium $\mathbf{p}_i = \mathbf{p}_i^{c*}$ corresponding to $\mathbf{b}_i = \mathbf{b}_i^{c*}$ of the system (9) is almost globally asymptotically stable. ■

It is worth noting that from Theorem 3, the agents achieve a desired surrounding formation up to a infinitesimal rotation of $\mathbf{Q}^T(\theta_x+\theta^\infty) = \mathbf{Q}(-\theta_x-\theta^\infty)$. Further, the equilibrium $\mathbf{p}_i = \mathbf{p}_i^{c*}$ corresponding to $\mathbf{b}_i = \mathbf{b}_i^{c*}$ of the system (9) is locally exponentially stable if the initially $\mathbf{b}_i(0) \neq -\mathbf{b}_i^{c*}$, $\forall i \in \{1,\ldots,n\}$ (Lemma 4, [21]).

Corollary 1 (Alignment): If $\alpha_{ij}^* = 0$ in (5b), for all $i = 1,\ldots,n-1$, $j = i+1$, the n agents align around the control station.

Theorem 4: Under the control law (5b), all agents preserve their distances to the beacon.

Proof: Consider the agent- beacon distance $d_i = \|\mathbf{p}_i - \mathbf{p}_x\|$, $i \in \mathcal{V}$. The time derivative $\dot{d}_i = \frac{d}{dt}\|\mathbf{p}_i^i - \mathbf{p}_x^i\| = \frac{(\mathbf{p}_i^i-\mathbf{p}_x^i)^T}{\|\mathbf{p}_i^i-\mathbf{p}_x^i\|}\frac{d}{dt}(\mathbf{p}_i^i - \mathbf{p}_x^i) = \frac{(\mathbf{p}_i^i-\mathbf{p}_x^i)^T}{\|\mathbf{p}_i^i-\mathbf{p}_x^i\|}\dot{\mathbf{p}}_i^i$ (since the beacon is stationary $\frac{d}{dt}\mathbf{p}_x^i = 0$). From the definition of bearing vector we have

$$\dot{d}_i = (\mathbf{b}_i^i)^T\dot{\mathbf{p}}_i^i = (\mathbf{b}_i^i)^T\mathbf{P}_{\mathbf{b}_i^i}[\hat{\mathbf{z}}_i]_\times^T\mathbf{b}_i^* = 0,$$

which completes the proof. ■

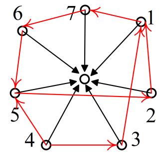
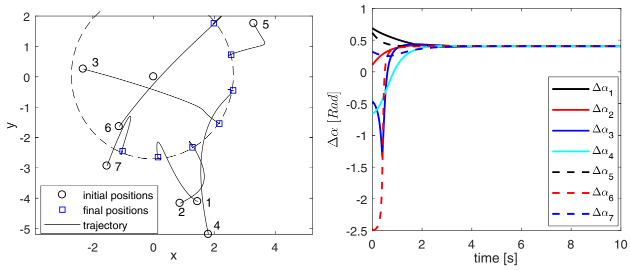


Fig. 4: The information flows: black edges - bearing sensing and control graph, red edges - orientation sensing and communication graph.



(a) Convergence of the variables $\hat{\mathbf{z}}_i$ to a circle.

(b) The orientation estimation errors

Fig. 5: Simulation 1: orientation estimation (5a).

According to Theorem 4, trajectories of all agents are circular arcs centered at the control station. The following result therefore is obtained.

Corollary 2 (Collision Avoidance): The control law (5b) guarantees collision-free formation if there is no pair of two agents located at the same distance to the beacon.

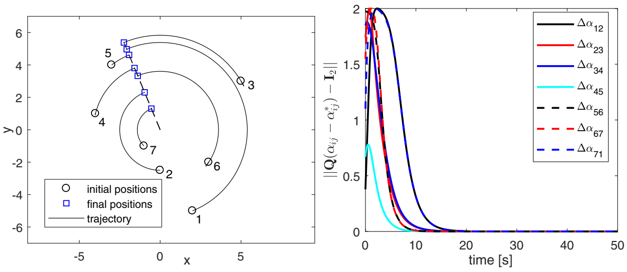
Proof: It follows from Theorem 4 that $d_i(t) = d_i(0)$ for all time $t \ge 0$. For $d_i(0) \neq d_j(0)$, $\forall i,j \in \{1,\ldots,n\}$, there holds $d_i(t) \neq d_j(t)$, $\forall t$, i.e., no collision can happen. ■

In practice, the initial positions of the agents should be chosen such that the agents are located at distinct distances from the stationary beacon.

V. Simulation

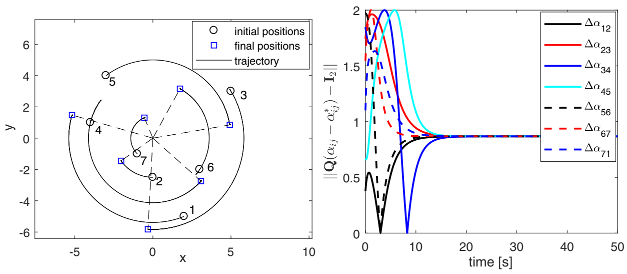
Consider a star framework of seven agents and a beacon. The orientation sensing topology between agents is given by a directed graph as described in Fig. 4. The initial positions are chosen as $\mathbf{p}_x(0) = [0,\ 0]^T$, $\mathbf{p}_1(0) = [2,-5]^T$, $\mathbf{p}_2(0) = [0,-2.5]^T$, $\mathbf{p}_3(0) = [5,3]^T$, $\mathbf{p}_4(0) = [-4,1]^T$, $\mathbf{p}_5(0) = [-3,4]^T$, $\mathbf{p}_6(0) = [3,-2]^T$, and $\mathbf{p}_7(0) = [-1,-1]^T$. Further, the orientation angles of seven agents are given as $\theta_i = \pi * i/7$, $i = 1,\ldots,6$, and $\theta_7 = 0$.

The effectiveness of the estimation law (5a) is shown in Fig. 5b. The bearing estimation error is defined as $\Delta\alpha_i := \widehat{\hat{\mathbf{z}}_i^*, \mathbf{q}_{ave}^*} - \theta_i$, where $\mathbf{q}_{ave}^* := Ave\{\mathbf{Q}_i^T\hat{\mathbf{z}}_i(t_0)\}$ is the average value. The initial values of each coordinate of auxiliary variables are randomly chosen in the interval $(-6,\ 6)$. Figure



(a) Trajectories of the agents (b) Alignment angle errors

Fig. 6: Simulation 2: Seven agents align in a straight line.



(a) Surrounding formation (b) Bearing angle errors

Fig. 7: Simulation 3: Surrounding around a beacon using bearing-only formation.

5a shows the circle convergences of auxiliary variables. While, all bearing estimation errors reach a consensus angle, that is, bearing angles are estimated up to a rotation (see Fig. 5b).

We provide simulation results to support the proposed control strategy. Figure 6a shows the consensus of the seven agents to a common orientation with regard to the beacon when zero desired relative angles are assigned. Note that the alignment configuration can be altered arbitrary by changing the desired angle α_0^*.

Additionally, the agent surrounds the reference with desired angle $\alpha_{ij}^* = \frac{2\pi}{7}$, $j = i+1$ with the relative angle errors converge to zero exponentially. The simulation result for this surrounding formation is given in Fig. 7a. The relative bearing angle error is defined as $\Delta\alpha_{ij} = \|\mathbf{Q}(\alpha_{ij} - \alpha_{ij}^*) - \mathbf{I}_2\|$ as shown in Fig. 7b.

VI. Conclusions

In this paper, we considered the problem of surrounding a fixed beacon based on bearing-only measurement. This problem naturally leads to the bearing rigidity theory for star-frameworks. We proposed an orientation estimation and a bearing-only control law to drive agents to surround the formation. Under the proposed control strategy, the agents almost globally asymptotically converge to a desired formation specified by some subtended angles around the target. A sufficient condition for collision avoidance between the agents were also provided. Numerical simulations were provided to support the effectiveness of the proposed strategy.

For further studies, we will conduct experimental evaluations with quadcopter systems. Hardware realization will lead to practical issues such as controlling with nonlinear dynamics model, attenuating noises in local measurements, and dealing with delays in communications.

Acknowledgment

This work was supported by the National Research Foundation (NRF) of Korea under the grants NRF-2016M1B3A1A01937575 and NRF-2017R1A2B3007034.

References

[1] S. Zhao and D. Zelazo, *Bearing rigidity and almost global bearing-only formation stabilization*, IEEE Transactions on Automatic Control, vol. 61, no. 5, pp. 1255-1268, 2015.
[2] K.-K. Oh and H.-S. Ahn, *Formation control and network localization via orientation alignment*, IEEE Transactions on Automatic Control, vol. 59, no.2, pp. 540-545, 2014.
[3] K.-K. Oh, M.-C. Park, and H.-S. Ahn, *A survey of multi-agent formation control*, Automatica, vol. 53, pp. 424-440, 2015.
[4] B.-H. Lee and H.-S. Ahn, *Distributed formation control via global orientation estimation*, Automatica, vol. 73, pp. 125-129, 2016.
[5] D. Zelazo, P. R. Giordano, and A. Franchi, *Bearing-only formation control using an $SE(2)$ rigidity theory*. In Proc. IEEE 54th Annual Conference on Decision and Control (CDC), 2015, Osaka, Japan, pp. 6121-6126.
[6] J. Guo, G. Yan, and Z. Lin, *Local control strategy for moving-target-enclosing under dynamically changing network topology*, System & Control Letters, vol. 59, no. 10, pp. 654-661, 2010.
[7] T. Kim and T. Sugie, *Cooperative control for target-capturing task based on a cyclic pursuit strategy*, Automatica, vol. 43, no. 8, pp. 1426-1431, 2007.
[8] N. Ceccarelli, M. Di Marco, A. Garulli, and A. Giannitrapani, *Collective circular motion of multi-vehicle systems*, Automatica, vol. 44, no. 12, pp. 3025-3035, 2008.
[9] Y. Lan, G. Yan, and Z. Lin, *Distributed control of cooperative target enclosing based on reachability and invariance analysis*, System & Control Letters, vol. 59, no. 7, pp. 381-389, 2010.
[10] R. Zheng, Z. Liu, M. Fu, and D. Sun, *Distributed circumnavigation by unicycles with cyclic repelling strategies*, in Proc. of the 9th Asian Control Conference, 2013.
[11] R. Zheng, Y. Liu, and D. Sun, *Enclosing a target by nonholonomic mobile robots with bearing-only measurements*, Automatica, vol. 53, pp. 400-407, 2015.
[12] N. Moshtagh, A. Jadbabaie, and K. Daniilidis, *Distributed coordination of dynamic rigid bodies*, in Proc. of the IEEE Conference on Decision and Control (CDC), 2007, pp. 1480-1485.
[13] E. W. Justh and P. S. Krishnaprasad, *Natural frames and interacting particles in three dimensions*, in Proc. of the IEEE Conference on Decision and Control (CDC), 2005, pp. 2841-2846.
[14] J. Guo, G. Yan, and Z. Lin, *Balance circular formation control based on gossip communication and angle of arrival information*, in Proc. of the 30th Chinese Control Conferences, 2011, pp. 6036-6041.
[15] L. Scardovi and R. Sepulchre, *Synchronization in networks of identical linear system*, Automatica, vol. 45, pp. 2557-2562, 2009.
[16] L. Moreau, *Stability of continuous-time distributed consensus algorithms*, in Proc. of the IEEE Conference on Decision and Control (CDC), 2004, pp. 3998-4003.
[17] C. W. Reynolds, *Flocks, herds, and schools: D distributed behavioural model*, in Computer Graphics(ACM SIGGRAPH'87 Conf. Proc.), vol. 21, 1987, pp. 25-34.
[18] R. Olfati-Saber, *Flocking for multi-agent dynamic system: algorithms and theory*, IEEE Transactions on Automatic Control, vol. 51, no.3, pp. 401-420, 2006.
[19] D. Angeli and L. Praly, *Stability robustness in the presence of exponentially unstable isolated equilibria*, IEEE Transactions on Automatic Control, vol. 7, no. 56, pp. 1582-1592, 2011.
[20] C. Godsil and G. Royle, *Algebraic graph theory*, Springer, 2001.
[21] M.H. Trinh, K.-K. Oh, and H.-S. Ahn, *Bearing-only control of leader first follower formations*, in Proc. of the 14th IFAC Symposium on Large Scale Complex Systems: Theory and Application, 2016, pp. 7-12.