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On robustly stabilizing PID controllers for systems with a certain class of multilinear parameter dependency

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***Abstract*— PID controllers are the most common controller type in industry. However, methods for optimal controller parameter tuning are still under development, especially in the case of robust control. One starting point for controller tuning is the usage of D-decomposition techniques, in order to identify the space of stabilizing controller parameters. In this paper, a novel approach for exact calculation of all stabilizing controller parameters which explicitly considers parameter uncertainties is introduced. Based on previous studies, the focus is expanded to multilinear parameter uncertainties. The usage of the proposed method is demonstrated in the prominent case study of robust control.**

I. Introduction

Stabilization is the basic requirement in most controller design problems. While there exist efficient analysis tools to verify the robustness property of control systems with e.g. the construction of complex value sets and the use of Mikhailov's theorem [12], the synthesis of robust controller is still an active research area. Calculating all stabilizing control parameters is not an easy task. This problem can be traced back to Vyshnegradsky [9]. In this context, the well-known term D-decomposition was first dubbed by Neimark [14]. At the end of the 1940s he proposed an algorithm for calculating stable areas by computing a particular decomposition of the parameter space. The decomposed regions have the property that the number of unstable characteristic roots are invariant in each region. Furthermore, for any point on a boundary the corresponding characteristic equation has at least one root on the imaginary axis. This method is known as the D-subdivision method. Based on the continuity of the roots, the parameter space is divided into several regions with a constant number of stable and unstable roots in the s-plane. These approaches are completely based on frequency sweeping and are limited to the case of two parameters. A way to reduce the problem of frequency sweeping is the introduction of singular frequencies. Such approaches are able to handle low number of control parameters, as appear in PID control.

The parameter space approach is a generic method for mapping a metric like stability into the controller parameter space. Such an approach is followed, e.g. by the singular frequency method [3] and [7], in the case of PID control. In this method, the stability crossing boundaries are straight lines (singular lines) in the K_I-K_D plane. Each singular line corresponds to one singular frequency. The singular lines and singular frequencies are computed from two equations which are obtained by implementing mathematical operations on the real and imaginary parts of the characteristic equation.

Nowadays, there exist various parameter space approaches to calculate the stabilizing control parameter space for systems without uncertainties [2], [7], [3], [5], [6]. Currently, an open question is how the stability boundaries are influenced by uncertainties in the plant parameters. First ideas regarding this question are presented for example in [2]. Based on this, an approach which creates the stabilizing PID parameter space with explicit consideration of the system parameter uncertainties was presented in [11]. However, this study was restricted to a limited class of system uncertainties, containing interval polynomials and polynomials with an affine parameter dependency in order to combine powerful methods like the Kharithonov theorem with the Parameter Space Approach. The current study presents an expansion of these concepts. By utilizing more advanced extremal set theory, it is possible to exactly calculate the robust stabilizing PID controller parameter space for systems with a certain multilinear parameter uncertainty structure by again taking into account the boundary kinematics. Thereby the need of guaranteeing robustness by over bounding the value sets with convex hulls as stated by the mapping theorem of Desoer [15] is omitted, which typically leads to conservative control designs and limits the achievable performance.

The remainder of the present article is organized as follows. In the next section the basic concept of the Parameter Space Approach is briefly summarized. In Section III extremal subsets for multilinear polynomials are presented. Based on this, in Section IV an expansion of the Parameter Space Approach is introduced, which is able to map the previously stated extremal sets. Section V finally illustrates the presented approach with en example system.

II. The Parameter Space Approach

Consider the proper linear time invariant system given by

$$G_S(s,\boldsymbol{q}) = \frac{A(s,\boldsymbol{q})}{R(s,\boldsymbol{q})} = \frac{\sum_{i=0}^{m} a_i(\boldsymbol{q})s^i}{\sum_{i=0}^{f-1} r_i(\boldsymbol{q})s^i}, \tag{1}$$

with an unknown parametric uncertainty $\boldsymbol{q} \in \mathcal{Q} \subset \mathbb{R}^l$ and the degrees $\deg(A) = m$, $\deg(R) = f - 1$. The set of

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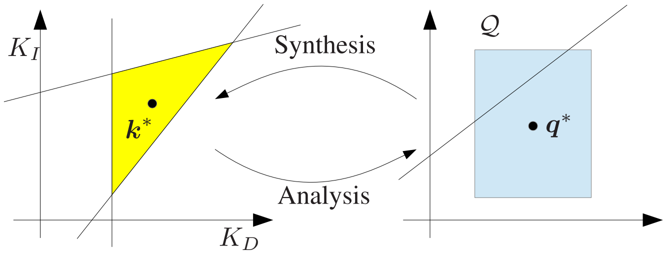

Fig. 1. Synthesis and analysis steps during robust control design.

occurrent parameters $\mathcal{Q}$ is known and is assumed to be an hyper rectangle, i.e. $q_i \in [q_i^-, q_i^+]$.

The Parameter Space Approach (PSA) [2] allows to determine robust stabilizing PID controller parameters $\boldsymbol{k} = (K_P, K_I, K_D) \in \mathcal{K}$ in an iterative fashion. In general it is divided in a synthesis and an analysis step, see Fig. 1. First a candidate $\boldsymbol{q}^* \in \mathcal{Q}$ is chosen. For this particular parameters the stability boundaries in the controller space are calculated during the synthesis step and a controller candidate $\boldsymbol{k}^*$, guaranteeing stability of $G_S(s, \boldsymbol{q}^*)$, is picked. In the following analysis step the stability boundaries are mapped back to the parameter space. If $\mathcal{Q}$ lies completely within the stable area, a robust stabilizing controller has been found. Otherwise another controller candidate or parameter candidate has to be chosen and the procedure is repeated, which leads to an iterative search for suitable controller candidates. Our goal is to omit the analysis step and directly determine the set of control parameters $\mathcal{K}_{\mathcal{Q}}$ which can guarantee robust stability of (1) over complete $\mathcal{Q}$. We will make use of the synthesis step in the Parameter Space Approach which we will therefore briefly summarize in the following.

Let $p(s, \boldsymbol{q}, \boldsymbol{k})$ be the closed loop characteristic polynomial given by:

$$p(s, \boldsymbol{q}, \boldsymbol{k}) = A(s, \boldsymbol{q}) \underbrace{\left(K_D s^2 + K_P s + K_I\right)}_{K(s)} + \underbrace{R(s, \boldsymbol{q}) s}_{:=B(s,\boldsymbol{q})}. \tag{2}$$

Then the set $\mathcal{K}_{\mathcal{Q}}$, we want to determine, follows to be:

$$\mathcal{K}_{\mathcal{Q}} := \left\{ \boldsymbol{k} \in \mathbb{R}^3 \,\middle|\, p(s, \boldsymbol{q}, \boldsymbol{k}) \text{ is Hurwitz } \forall \boldsymbol{q} \in \mathcal{Q} \right\}. \tag{3}$$

In order to determine stability boundaries, we are interested in the case that roots change from the left to the right complex plane via the imaginary axis. It is well known, that the roots of a polynomial are continuous functions of its parameters. The following theorem from [2] is the basis of the PSA.

Theorem 1 (Boundary crossing theorem): The polynomial family $P(s, \mathcal{Q}) = \{p(s, \boldsymbol{q}) | \boldsymbol{q} \in \mathcal{Q}\}$ is robust stable iff:

1) there exists a stable polynomial $p(s, \boldsymbol{q}) \in P(s, \mathcal{Q})$ and
2) $j\omega \notin \text{roots}\left[P(s, \mathcal{Q})\right], \forall \omega \geq 0$.

Three cases can be distinguished how the imaginary axis is crossed:

1) Real Root Boundary (RRB)
A root is crossing the imaginary axis through the origin. This case is described by $p(s = 0, \boldsymbol{q}) = 0$.
2) Complex Root Boundary (CRB)
A root is crossing the imaginary axis. This case is described by $p(s = j\omega, \boldsymbol{q}) = 0$.
3) Infinite Root Boundary (IRB)
A root is changing the half-planes in infinity, iff the highest coefficient of the polynomial vanishes.

In the following we will analyze the resulting stability boundaries in the controller space given by the root boundaries.

A. RRB and IRB

Inserting the RRB condition $s = 0$ into the closed loop polynomial for a fixed $\boldsymbol{q}^*$ leads to the stability boundary $K_I = 0$, if $a_0 \neq 0$, i.e. the system has no pure differentiator. Clearly $K_I > 0$ characterizes the stable side of this stability boundary. An IRB does only exist if the highest coefficient of the closed loop polynomial vanishes. This is exactly the case if either the rank condition $f = m+2$ holds, leading to $K_D = -b_f/a_m$, or if $f = m+1$ holds and then the trivial result $K_D = 0$ is a stability boundary. Note that f can not be smaller than $m+1$ due to the proper transfer function in (1). Both the IRB and the CRB are planes in the controller space which are normal to the corresponding axes.

B. CRB

In order to determine the stability boundaries in the controller space which correspond to the CRB case, separate the following polynomials in their real and imaginary parts:

$$A(j\omega) = R_A(\omega) + jI_A(\omega),\ B(j\omega) = R_B(\omega) + jI_B(\omega)$$
$$p(j\omega, \boldsymbol{k}) = R_p(\omega, \boldsymbol{k}) + jI_p(\omega, \boldsymbol{k}).$$

Herein $R_{(\cdot)}$ is the real and $I_{(\cdot)}$ the imaginary part of the corresponding polynomials. The condition of the CRB, $p(s = j\omega, \boldsymbol{q}) = 0$, is also separated and yields:

$$\begin{pmatrix} R_A & -\omega^2 R_A \\ I_A & -\omega^2 I_A \end{pmatrix} \begin{pmatrix} K_I \\ K_D \end{pmatrix} + \begin{pmatrix} R_B - K_P \omega I_A \\ I_B + K_P \omega R_A \end{pmatrix} = \boldsymbol{0}. \tag{4}$$

The above matrix is singular and therefore (4) possesses only a solution iff

$$\det \begin{pmatrix} R_A & R_B - K_P \omega I_A \\ I_A & I_B + K_P \omega R_A \end{pmatrix} = 0 \tag{5}$$

is fulfilled. By rearranging we get to:

$$K_P(\omega) = \frac{I_A R_B - R_A I_B}{\omega \left(R_A^2 + I_A^2\right)}. \tag{6}$$

For a fixed $K_P = K_P^*$ this equation characterizes the so called *singular frequencies* ω_{g_i} and the equation itself is called the *singular frequencies generator*. It is important to note that for a given K_P^* a root can only pass the imaginary axis at these particular frequencies. Therefore methods based on frequency grinding may not be reliable if these particular frequencies do not lie on the grinding pattern. By inserting (6) in (4) one obtains the stability boundary of the CRB in the $\mathcal{K}$-space:

$$K_I = \omega_{g_i}^2 K_D + K_I^0, \tag{7}$$

and the K_I-axis intercept of this affine-linear function in K_D is:

$$K_I^0(\omega_g) = -\frac{R_A R_B + I_A I_B}{R_A^2 + I_A^2}. \quad (8)$$

The CRB is therefore given by a line in the K_D-K_I-space with strictly positive slope. By introducing the concept of singular frequencies the problem has been decoupled for a given K_P^*. For each CRB a stable and an unstable side can be assigned, depending on the change of the real part of the crossing root σ. According to [7] the following relation holds

$$\operatorname{sign}\left(\left.\frac{d\sigma}{dK_I}\right|_{\omega_g}\right) = \operatorname{sign}\left(\left.\frac{dK_P(\omega)}{d\omega}\right|_{\omega_g}\right), \quad (9)$$

and therefore the stable side can be directly determined by the derivatives of (6). The controller parameter space is hence partitioned by the RRB, IRB and CRB in convex polygons of stable and unstable areas.

The choice of suitable values for K_P for such stable areas to exist is not trivial. The following necessary condition on the number of singular frequencies Z is given by [3] in the case that $A(s)$ does not have roots on the imaginary axis:

$$Z \geq E\left(\frac{n - m + 2z - 1}{2}\right), \quad (10)$$

with $E(\cdot)$ being the floor function, n the order of polynomial (2), m the order of $A(s)$ and z the number of right half-plane zeros of $A(s)$. As (6) usually has several extrema, the choice of K_P directly influences the number of singular frequencies. More details about finding suitable stabilizing K_P intervals can be found in [8].

III. Extremal Subsets for multilinear polynomials

Checking polynomial families like $P(s, \mathcal{Q})$ for robust stability for all $\boldsymbol{q} \in \mathcal{Q}$ can be very cumbersome, even for low dimensions of $\mathcal{Q}$. Extremal subsets or test sets allow to lessen this effort, by reducing the dimension of the unknown parameter space. Specifically extremal subsets $\mathcal{Q}_T \subset \mathcal{Q}$ are defined by the fact that stability of $P(s, \mathcal{Q}_T)$ implies stability of $P(s, \mathcal{Q})$. Depending on the kind how the unknown parameters influence the coefficients of the characteristic closed loop polynomial, different extremal subsets are stated in the literature.

A. Classification of the characteristic polynomial

The closed loop characteristic polynomial has coefficients which depend on the unknown parameters:

$$p(s, \boldsymbol{q}) = p_0(\boldsymbol{q}) + p_1(\boldsymbol{q})s + p_2(\boldsymbol{q})s^2 + \cdots + p_n(\boldsymbol{q})s^n. \quad (11)$$

Corresponding to this dependency the following characterization of the polynomials can be stated [13]:

1) Interval polynomials:
Each coefficient is independent of the others and has known lower and upper bounds: $p_i \in \left[p_i^-; p_i^+\right]$.
2) Polynomials with affine dependency:
The coefficients are affine combinations of the unknown parameters $\boldsymbol{q}$.
3) Polynomials with multilinear dependency:
The coefficients posses additionally product terms of the unknown parameters $\boldsymbol{q}$. Example:
$p(s, \boldsymbol{q}) = \underbrace{(5 + 2q_1q_2 + q_3)}_{p_2} s^2 + \underbrace{q_2}_{p_1} s + \underbrace{3 + q_1}_{p_0}$.

In the first case Kharitonov's theorem [2] shows that the stability of only four fixed polynomials guarantees the robust stability of the complete polynomial family. These polynomials are called Kharitonov polynomials and are given by:

$$\begin{aligned} p^{--}(s) &= p_0^- + p_1^- s + p_2^+ s^2 + p_3^+ s^3 + p_4^- s^4 + \dots \\ p^{-+}(s) &= p_0^- + p_1^+ s + p_2^+ s^2 + p_3^- s^3 + p_4^- s^4 + \dots \\ p^{++}(s) &= p_0^+ + p_1^+ s + p_2^- s^2 + p_3^- s^3 + p_4^+ s^4 + \dots \\ p^{+-}(s) &= p_0^+ + p_1^- s + p_2^- s^2 + p_3^+ s^3 + p_4^+ s^4 + \dots \end{aligned} \quad (12)$$

The case of an affine parameter dependency leads to a simple geometric test set which is given by the edges of $\mathcal{Q}$ [4].

The focus of the present paper lies on the case that the parameters exhibit multilinear dependency, which is discussed in detail in the following.

B. Multilinear parameter dependency

For multilinear polynomial dependencies it can be shown that no simple geometric test sets of $\mathcal{Q}$ do exist [1]. Instead, the test set depends on the characteristic polynomial itself. With the Jacobian

$$\boldsymbol{J} = \begin{pmatrix} \dfrac{\partial \operatorname{Re}(p(j\omega, \boldsymbol{q}))}{\partial q_1} & \dots & \dfrac{\partial \operatorname{Re}(p(j\omega, \boldsymbol{q}))}{\partial q_l} \\ \dfrac{\partial \operatorname{Im}(p(j\omega, \boldsymbol{q}))}{\partial q_1} & \dots & \dfrac{\partial \operatorname{Im}(p(j\omega, \boldsymbol{q}))}{\partial q_l} \end{pmatrix} \in \mathbb{R}^{2\times l},$$

we can state the following Theorem from [10]:

Theorem 2: A multilinear polynomial family is stable iff:

1) The surfaces of $\mathcal{Q}$ are stable and
2) the system of equations

$$\begin{aligned} \operatorname{Re} p(j\omega, \boldsymbol{q}) &= 0 \\ \operatorname{Im} p(j\omega, \boldsymbol{q}) &= 0 \\ \operatorname{rank} \boldsymbol{J} &< 2 \end{aligned} \quad (13)$$

does not posses any real solutions on $\mathcal{Q}$.

With the gramian

$$\boldsymbol{G} := \boldsymbol{J}\boldsymbol{J}^T \in \mathbb{R}^{2\times 2} \quad (14)$$

the rank condition from theorem 2 can be verified with the condition

$$\det \boldsymbol{G} = 0. \quad (15)$$

In the case that only two uncertain parameter exist, the simpler condition $\det \boldsymbol{J} = 0$ may be used.

C. Multilinear interval polynomials

A special case occurs when the multilinear dependency is given by the multiplication of two interval polynomials and therefore the characteristic polynomial is given in the form

$$p(s) = F_1(s)I_{11}(s)I_{12}(s) + F_2(s)I_{21}(s)I_{22}(s), \quad (16)$$

with the fixed polynomials $F_i(s)$ and the independent interval polynomials $I_{ij}(s)$. A simple extremal subset is given in [5]. For each interval polynomial $I_{ij}(s)$ four Kharitonov polynomials $\mathcal{K}_{ij}(s)$ can be assigned:

$$\mathcal{K}_{ij} = \{p_{ij}^{--}(s), p_{ij}^{-+}(s), p_{ij}^{+-}(s), p_{ij}^{++}(s)\}. \tag{17}$$

Additionally, for each $I_{ij}(s)$ four line segments which connect the Kharitonov polynomials can be defined:

$$\mathcal{S}_{ij} = \Big\{ [p_{ij}^{--}(s); p_{ij}^{-+}(s)], [p_{ij}^{--}(s); p_{ij}^{+-}(s)], \\ [p_{ij}^{-+}(s); p_{ij}^{++}(s)], [p_{ij}^{+-}(s); p_{ij}^{++}(s)] \Big\} \tag{18}$$

With the two sets

$$\begin{aligned} \mathcal{P}_T^1 &= F_1(s)\mathcal{S}_{11}(s)\mathcal{S}_{12}(s) + F_2(s)\mathcal{K}_{21}(s)\mathcal{K}_{22}(s) \\ \mathcal{P}_T^2 &= F_1(s)\mathcal{K}_{11}(s)\mathcal{K}_{12}(s) + F_2(s)\mathcal{S}_{21}(s)\mathcal{S}_{22}(s), \end{aligned} \tag{19}$$

the extremal subset is given by [5]:

$$\mathcal{P}_T = \mathcal{P}_T^1 \bigcup \mathcal{P}_T^2 \tag{20}$$

The set $\mathcal{P}_T$ contains, if all Kharitonov polynomials exist or respectively are different, 512 two dimensional continua which have to be verified for stability.

D. Forming a test set consisting of one dimensional continua

As we would like to apply to the given test sets a modified Parameter Space Approach which takes into account the kinematics of the root boundaries depending on a parameter, we have to further shrink the given test sets to one dimensional continua. In order to achieve this we make the following assumptions:

Assumption 1: The system to be stabilized shows a multilinear dependency of only two unknown parameters.

Assumption 2: These unknown parameters either influence $A(s)$ or $B(s)$, but not both simultaneously.

While these conditions are very restrictive in the general case, please notice that the test set (20) from section III-C satisfies these assumptions. With $F_1(s)$ being e.g. $K(s)$, the closed loop characteristic polynomial with multilinear dependency of two unknown parameters can have one of the two following forms:

$$\begin{aligned} p_I(s) &= A(s,\lambda_1,\lambda_2)K(s) + B(s) \\ p_{II}(s) &= A(s)K(s) + B(s,\lambda_1,\lambda_2), \end{aligned} \tag{21}$$

with some generic parameter $\lambda_i \in [\lambda_i^-, \lambda_i^+] \subset \mathbb{R}, i \in \{1,2\}$. In the case that only two unknown parameters exist, λ_i corresponds to q_i, otherwise λ_i is the variable of one Kharitonov segment, see (18). As the dependence on the unknown parameter is supposed to be multilinear, $A(s,\lambda_1,\lambda_2)$ for example can be written in the form $A(s,\lambda_1,\lambda_2) = A_0(s) + \lambda_1 A_1(s) + \lambda_2 A_2(s) + \lambda_1\lambda_2 A_3(s)$. If we apply Theorem 2 to the above polynomials, the edges of $[\lambda_1^-, \lambda_1^+] \times [\lambda_2^-, \lambda_2^+]$ and the rank condition have to be tested for robust stability. Of course the edges already represent one dimensional continua so we have to focus on the rank condition. Due to Assumption 1 only the matrix $\boldsymbol{J}$ has to considered. This condition can be interpreted as a collinearity test in the complex plane for the polynomials $\partial p/\partial\lambda_1$ and $\partial p/\partial\lambda_2$.

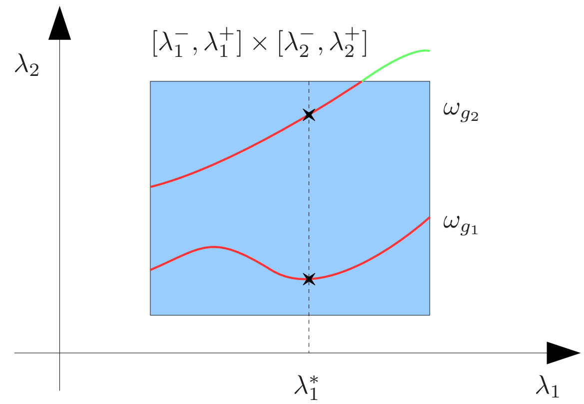

Fig. 2. Construction of one dimensional test sets from the rank condition $\det \boldsymbol{J} = 0$.

Assumption 2 guarantees that either both derivatives do not depend on $K(s)$ in the case of p_{II} or both polynomials are multiplied by $K(s)$ in the case of p_I. Therefore the resulting equation will not depend on the control parameters but only on the frequency ω and the two unknown parameters λ_1 and λ_2.

With the real and imaginary parts of A_i or B_i, depending if we consider p_I or p_{II}, the rank condition $\det \boldsymbol{J} = 0$ yields:

$$(R_1 + \lambda_2 R_3)(I_2 + \lambda_1 I_3) - (R_2 + \lambda_1 R_3)(I_1 + \lambda_2 I_3) = 0.$$

Solving this condition for λ_2 gives:

$$\lambda_2 = \underbrace{\frac{R_3 I_1 - R_1 I_3}{R_3 I_2 - R_2 I_3}}_{\xi(j\omega)} \lambda_1 + \underbrace{\frac{R_2 I_1 - R_1 I_2}{R_3 I_2 - R_2 I_3}}_{\eta(j\omega)}. \tag{22}$$

This equation represents a line in the q_2-q_1-plane for a fixed frequency and is therefore called the Jacobi-line. Of course it is also possible to solve for λ_1 instead. A practical implementation should choose between both in the case that singularities occur. In the following we will exemplary deal with the first case. Substituting λ_2 with the above expression in (6) offers an implicit connection between the unknown parameter λ_1 and its corresponding singular frequencies ω_{g_i}. By solving for theses frequencies and inserting them into (22) we can determine the corresponding λ_{2_i} which belong to the given λ_1. Due to the possible existence of several singular frequencies this relationship is not unique. The resulting parameter combinations are part of the test set of the closed loop polynomial together with edges of the two dimensional continua. Therefore by screening $\lambda_1 \in [\lambda_1^-, \lambda_1^+]$ we may determine several one dimensional continua over the varying parameter q_1, which will have to be mapped by the PSA into the control parameter space. Figure 2 illustrates the proposed approach. Of course only the parts of the one dimensional continua have to be considered which lie in the set $[\lambda_1^-, \lambda_1^+] \times [\lambda_2^-, \lambda_2^+]$. For a given λ_1^* the resulting equations are highly non-linear and have therefore to be solved numerically. Note that while procedures based on grinding may be in general unreliable in the context of robust control, we precisely grind the points which are critical from a stability point of view.

IV. Modification of the Parameter Space Approach

The previous thoughts have shown that if only two unknown parameters exist in a system, one dimensional continua have to be checked for robust stability, i.e. the edges of $\mathcal{Q}$ and the rank condition. If on the other hand the multilinear dependency is formed by the multiplication of two interval polynomials, the extremal subset contains two dimensional continua, which in turn can be reduced to one dimensional continua with the results of III-D. Therefore we will extend the Parameter Space Approach to be able to deal with these cases. Consider a closed loop polynomial given by $p(j\omega, \lambda)$, where $\lambda \in \mathbb{R}$ is a generic variable.

A. RRB and IRB

If the rank condition $f = m + 2$ is fulfilled, the IRB condition yields

$$K_D = -\frac{b_f(\lambda)}{a_m(\lambda)}, \tag{23}$$

which in this case parameterizes a band parallel to the K_I-axis. The limits have to be identified by a extremum search by calculating $dK_D/d\lambda$. In the case that the coefficients show an affine dependency it can be easily shown that no extrema exists [11] due to monotony.

If $\exists\lambda \in [\lambda^-, \lambda^+], a_0(\lambda) \neq 0$, the RRB is still given by $K_I = 0$, otherwise no RRB exists.

B. CRB

The CRBs will change both their slope and their K_I-axis intercept depending on λ. Applying the method developed in [11] we can analyze the kinematic movement of the CRB by calculating its instantaneous center of rotation and its trajectory over λ. The condition

$$\frac{dK_I}{d\lambda} = 2\omega_g \frac{d\omega_g}{d\lambda} K_D^M + \frac{dK_I^0}{d\lambda} = 0, \tag{24}$$

gives the instantaneous center of rotation M for a given λ. Solving the above equation for the K_D coordinate gives:

$$K_D^M = -\frac{1}{2\omega_g}\frac{dK_I^0}{d\lambda} \cdot \frac{d\lambda}{d\omega_g}. \tag{25}$$

Inserting K_D^M into the CRB equation (7) yields the K_I coordinate:

$$K_I^M = -\frac{1}{2}\omega_g \frac{dK_I^0}{d\omega_g} + K_I^0. \tag{26}$$

Please note that the expression $dK_I^0/d\omega_g$ is a total differential and has to be computed as:

$$\frac{dK_I^0}{d\omega_g} = \frac{\partial K_I^0}{\partial \omega_g} + \frac{\partial K_I^0}{\partial \lambda}\frac{d\lambda}{d\omega_g}. \tag{27}$$

In order to determine the rate of the singular frequency ω_g with respect to λ, the implicit function F, for a fixed K_P^*, is introduced from (6)

$$F(\omega, \lambda) := K_P^* \omega \left(R_A^2 + I_A^2\right) + R_A I_B - I_A R_B = 0, \tag{28}$$

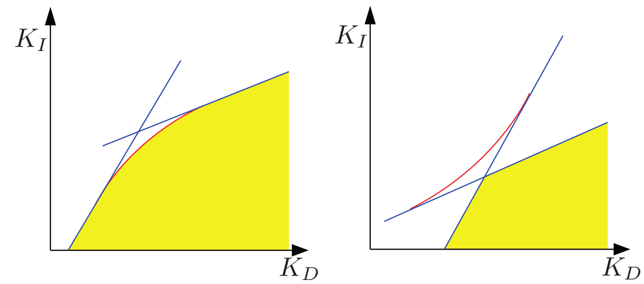

Fig. 3. Non-Convex (left) and convex (right) movement of a CRB with yellow stability side.

and implicit differentiation gives:

$$\frac{d\omega_g}{d\lambda} = -\frac{\partial F/\partial\lambda}{\partial F/\partial\omega}. \tag{29}$$

The parametrized curve of the center of rotation $M(\lambda)$ has the form:

$$\begin{pmatrix} K_D^M(\lambda, \omega_g(\lambda)) \\ K_I^M(\lambda, \omega_g(\lambda)) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2\omega_g}\frac{dK_I^0}{d\omega_g} \\ -\frac{1}{2}\omega_g \frac{dK_I^0}{d\omega_g} + K_I^0 \end{pmatrix}. \tag{30}$$

It can be easily shown that the following relation holds and hence each CRB represents a tangent to this curve:

$$\frac{dK_I^M}{dK_D^M} = \frac{dK_I^M/d\lambda}{dK_D^M/d\lambda} = \omega_g^2. \tag{31}$$

Depending on the movement of the CRB the trajectory of the instantaneous center of rotation is a stability boundary or not, see figure 3. If the stability of the interval $[\lambda^-, \lambda^+]$ is included by the CRBs of λ^- and λ^+ we call the movement convex, otherwise non-convex. A condition for a convex movement can be given as [11]:

$$\operatorname{sign}\left(\left.\frac{\partial K_P(\omega)}{\partial\omega}\right|_{\omega_g} \cdot \frac{dK_D^M}{d\omega_g}\right) = 1, \forall\lambda \in [0, 1]. \tag{32}$$

Only trajectories with non-convex movements have to be considered as stability boundaries in the K_D-K_I-plane. Please note that a single movement can contain convex and non-convex parts which should be treated separately.

V. Example

Consider the following system inspired by the well known example from [1] which shows an multilinear parameter dependency and which is used to demonstrate that in this case no general geometric test sets do exist:

$$\begin{aligned} G(s, q_1, q_2) &= \frac{0.01}{s^3 + a_2 s^2 + a_1 s + a_0}, \text{ with} \\ a_0 &= 2 + r^2 + 6(q_1 + q_2) + 2q_1 q_2 \\ a_1 &= a_2 = 2 + q_1 + q_2. \end{aligned} \tag{33}$$

Herein $q_1 \in [0, 2]$ and $q_2 \in [0, 2.5]$ denote the unknown parameter of the system, while $r = 0.5$ is a fixed parameter. It can be shown that while the edges of the square $\mathcal{Q} = [0, 2] \times [0, 2.5]$ are stable, the system is not robustly stable

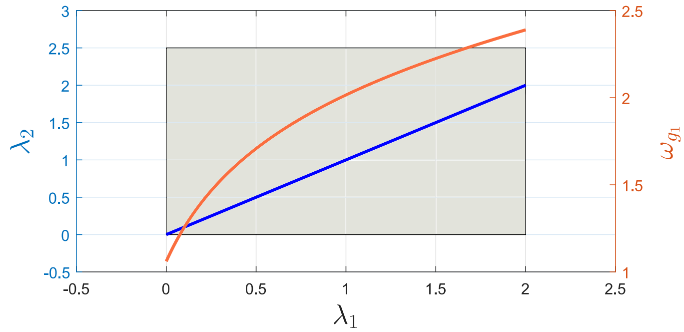

Fig. 4. Additional stability critical continua (blue) and corresponding singular frequency (orange) plotted over λ_1.

over $\mathcal{Q}$. Particularly there is a ball with radius $r = 0.5$ around the point $q_1 = q_2 = 1$ which is unstable.

By closing the control loop with a PID controller we are able to determine the stability critical continua which can be obtained from the Jacobi condition (22) and the frequency decoupling equation (6). Figure 4 shows that in this simple example in addition to the edges one line has to be mapped to the control gain space.

Figure 5 shows the resulting stability area in the K_D-K_I-plane for $K_P = 0$. Herein the green line represents the RRB, the red line represents the stability critical boundaries given by the trajectories of the instantaneous centers of rotation from the edges of $\mathcal{Q}$. The cyan curve is the boundary resulting from the additional Jacobi condition. It can be seen that this curve consists of a convex and a non-convex movement of its corresponding CRB, see figure 3. The black line is the CRB which occurs in the point $q_1 = q_2 = 0$. Consequently, the stable area is the yellow marked one. The gray area is the difference between the stable area which follows from the edge theorem and the real stable area. As expected, one can see, that in the case of the uncontrolled system $K_D = K_P = K_I = 0$, the edges are stable while the whole system is not due to the additional boundary in cyan.

CONCLUSION

The parameter space approach represents an intuitive method for control design. However as the existing mapping methods only deal with the nominal plant without uncertainty, robustness has to be achieved on a cumbersome iterative fashion. In this paper, an extension is proposed which explicitly considers multilinear parameter uncertainties for the controller parameter space by analyzing the root boundary kinematics. Thereby it is for the first time possible to precisely determine the robust stabilizing set of PID control gains for the whole polynomial family. Hence the analysis step is omitted and the control designer can focus on performance criteria without the conservatism of over bounding.

We believe that the discussed approach is a promising extension of the parameter space approach. Further research topics in this area will be the extension to systems with a more general parameter dependency structure. On the other hand the obtained boundaries in parameter space may be actively used in an control parameter optimization which inherently guarantees robust stability.

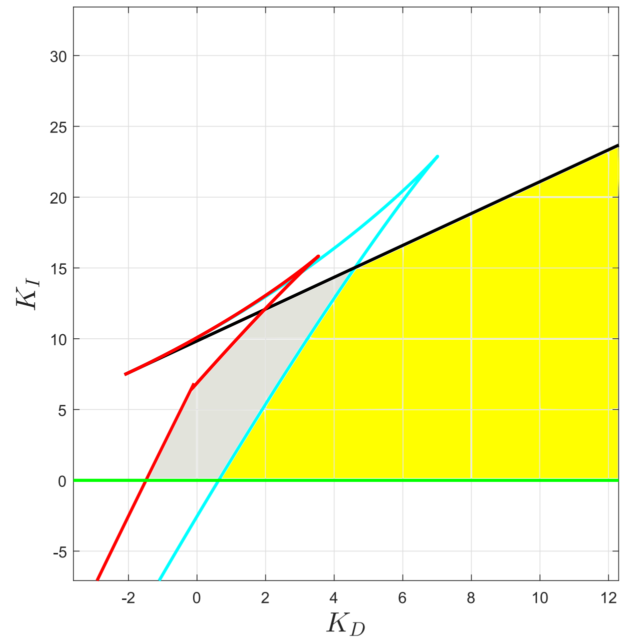

Fig. 5. Robust stable area (yellow), difference to stable area from edges (gray), for $K_P = 0$.

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