2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Design of High-Gain Reduced-Order Observers for Nonlinear Sampled-Data Strict-Feedback Systems with Model Uncertainty

Hitoshi Katayama

***Abstract*— Design of high-gain reduced-order observers is considered for the nonlinear sampled-data strict-feedback system with model uncertainty. First high-gain reduced-order observers are designed by introducing a small positive parameter to the reduced-order observers designed for the sampled-data strict-feedback system without model uncertainty. Then it is shown that there exists a small positive parameter such that the designed observers are semiglobal and practical in sampling period for the exact model of the sampled-data strict-feedback system with model uncertainty. A numerical example is given to show the efficiency of the designed observers.**

I. Introduction

For nonlinear systems, many design methods of controllers such as backstepping, feedback linearization, and passivity-based design of controllers assume that all state variables are available for control ([9], [13], [16]). But all state variables are not usually measured directly from a practical and economical point of view. Hence the observer problem that is to design an observer that estimates unmeasured state variables is important. For linear systems, the observer problem has been well-studied and many design methods of observers have been given ([26], [28]). But the observer problem for nonlinear systems is solved only for specific classes such as state affine systems and uniformly observable systems. Several transformation to state affine systems from the general nonlinear system have been also discussed. Many open problems about nonlinear observer design and its related topics remain ([6], [9], [25], and the references therein).

It has been well-known that high-gain observers play an important role in the control design of nonlinear systems. In the design of high-gain observers, a new technique for a robust design of observers and an avoidance of the peaking phenomena by using globally bounded state feedback controllers has been proposed. A nonlinear separation principle for the stabilization of nonlinear systems by using this technique has been also proven ([13], [15]). Moreover, several modifications of classical high-gain observers have been also discussed to improve their performance ([2]-[4]).

The modern and practical control systems use digital computers, as discrete-time controllers, to control continuous-time plants. Such control systems are referred as sampled-data systems that are continuous-time systems with sampled observations and control inputs realized through zero-order holds [7]. For nonlinear sampled-data systems, the design frameworks of controllers and observers on the basis of discrete-time approximate models such as the Euler model have been recently given ([5], [23], [24]). In the proposed design frameworks, several design methods of state feedback controllers and observers have been given ([1], [17]-[22], [27]). In [10] and [11], we then have designed reduced-order observers that estimate $z_c(kT)$ of the nonlinear sampled-data strict-feedback system

$$\begin{aligned} \dot{x}_c &= f_1(x_c) + g(x_c)z_c, \\ \dot{z}_c &= f_2(x_c, z_c, u_c), \\ y(k) &= x_c(kT) \end{aligned} \tag{1}$$

where $x_c \in \mathbf{R}^{n_x}$, $z_c \in \mathbf{R}^{n_z}$ are the states, $u_c \in \mathbf{R}^m$ is the control input realized through a zero-order hold, i.e., $u_c(t) = u_c(kT) =: u(k)$ for any $t \in [kT, (k+1)T)$, $y \in \mathbf{R}^{n_x}$ is the sampled observation, $T > 0$ is a sampling period, and $k \in \mathbf{N}_0 := \{0, 1, 2, \ldots\}$. The system (1) is the general form of the model of computer-controlled vehicles such as automobiles and ships, where x_c and z_c are the position and the velocity of the vehicle, respectively, and the 1st and 2nd differential equations in (1) correspond to the kinematic and dynamic equations, respectively. We have used the properties of the Euler model of the system (1) to give a simple design method of reduced-order observers that estimate $z_c(kT)$. The proposed design method of reduced-order observers has been also applied to straight-line tracking control of sampled-data underactuated ships [12].

For practical control problems of vehicles, we must consider the influence of the unknown model uncertainty in the dynamic equation and then we extend the system (1) to the following nonlinear sampled-data strict-feedback system with model uncertainty:

$$\begin{aligned} \dot{x}_c &= f_1(x_c) + g(x_c)z_c, \\ \dot{z}_c &= f_2(x_c, z_c, u_c) + \delta(x_c, z_c, u_c), \\ y(k) &= x_c(kT) \end{aligned} \tag{2}$$

where $\delta(x_c, z_c, u_c)$ is the unknown model uncertainty. In this paper, we design high-gain reduced-order observers that estimate $z_c(kT)$ of the system (2) by modifying reduced-order observers designed for the system (1). We first introduce a small parameter $\epsilon > 0$ in the observer gain to design high-gain reduced-order observers. Then we show that there exists a sufficiently small $\epsilon > 0$ such that the designed high-gain observers are semiglobal and practical in T for the exact models of the system (1) and (2). We also give a numerical example to show the efficiency of the proposed design of high-gain reduced-order observers.

H. Katayama is with Department of Electrical and Electronic Engineering, Shizuoka University, Hamamatsu 432 8561, Japan katayama.hitoshi@shizuoka.ac.jp

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In [8] and [14], high-gain observers and high-gain observer-based output feedback control for nonlinear sampled-data systems have been considered. When high-gain observers and output feedback controllers are designed in continuous time, their sampled-data implementation and performance recovery of continuous-time controllers by sampled-data controllers for a sufficiently small sampling period are discussed. The design method of a high-gain reduced-order observer in this paper is based on the Euler model of sampled-data strict-feedback system and the structure of the designed observers is different from that of sampled-data implementation of continuous-time high-gain observers given in [8] and [14].

Notation: Let $\|\cdot\|$ denote the Euclidean norm of a vector and the norm of a matrix and $\mathrm{diag}\{a(i)\}_n = \mathrm{diag}\{a(1),\ldots,a(n)\}$. A function $\alpha : \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$ is of class $\mathcal{K}$ (denoted by $\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A function β: $\mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for any fixed $t \geq 0$, $\beta(\cdot, t) \in \mathcal{K}$ and for each fixed $s \geq 0$, $\beta(s,\cdot)$ is decreasing to zero as its argument tends to infinity [13].

II. Design of Reduced-order Observers

For the systems (1) and (2), we assume:
A1: f_1, f_2, and g are smooth functions over the compact domain of interest, $f_1(0) = 0$, and $f_2(0,0,0) = 0$,
A2: The $n_z \times n_z$ matrix $\Phi = g^T g$ is nonsingular and its inverse is bounded over the compact domain of interest,
A3: δ is bounded over the compact domain of interest and $\delta(0,0,0) = 0$.
The assumptions **A2** and **A3** are satisfied for some control problems of vehicles (for details see Section III). Since $u_c(t) = u(k)$ for any $t \in [kT,(k+1)T)$, the difference equations corresponding to the exact model and the Euler model of the system (1) are given by

$$\begin{aligned} x(k+1) &= F^e_{1T}(x,z,u)(k) \\ &= x(k) + \int_{kT}^{(k+1)T} [f_1(x_c) + g(x_c)z_c](s)ds, \\ z(k+1) &= F^e_{2T}(x,z,u)(k) \\ &= z(k) + \int_{kT}^{(k+1)T} f_2(x_c(s), z_c(s), u(k))ds, \\ y(k) &= x(k) \end{aligned} \tag{3}$$

and

$$\begin{aligned} x(k+1) &= F^a_{1T}(x,z)(k) \\ &= x(k) + T[f_1(x) + g(x)z](k), \\ z(k+1) &= F^a_{2T}(x,z,u)(k) \\ &= z(k) + Tf_2(x,z,u)(k), \\ y(k) &= x(k), \end{aligned} \tag{4}$$

respectively. Similarly, the difference equations corresponding to the exact model and the Euler model of the system (2) are given by

$$\begin{aligned} x(k+1) &= F^e_{1T}(x,z,u)(k), \\ z(k+1) &= \tilde{F}^e_{2T}(x,z,u)(k), \\ y(k) &= x(k) \end{aligned} \tag{5}$$

and

$$\begin{aligned} x(k+1) &= F^a_{1T}(x,z)(k), \\ z(k+1) &= \tilde{F}^a_{2T}(x,z,u)(k), \\ y(k) &= x(k), \end{aligned} \tag{6}$$

respectively, where $\tilde{F}^e_{2T}(x,z,u)(k) = F^e_{2T}(x,z,u)(k) + \int_{kT}^{(k+1)T} \delta(x_c(s), z_c(s), u(k))ds$ and $\tilde{F}^a_{2T}(x,z,u) = F^a_{2T}(x,z,u) + T\delta(x,z,u)$. Note that $(x,z)(k) = (x_c,z_c)(kT)$ for the exact models. Here we assume that the sampling period is a design parameter and can be assigned arbitrarily. Let $F^i_T = [\,(F^i_{1T})^T \quad (F^i_{2T})^T\,]^T$ and $\tilde{F}^i_T = [\,(F^i_{1T})^T \quad (\tilde{F}^i_{2T})^T\,]^T$ for $i = a$, e. By [23], $\tilde{F}^a_T$ (F^a_T) is one-step consistent with $\tilde{F}^e_T$ (F^e_T), i.e., for each compact set $\Omega \subset \mathbf{R}^{n_x} \times \mathbf{R}^{n_z} \times \mathbf{R}^m$, there exist $\gamma \in \mathcal{K}$ and $T^* > 0$ such that

$$\|\tilde{F}^e_T(x,z,u) - \tilde{F}^a_T(x,z,u)\| \leq T\gamma(T) \tag{7}$$

($\|F^e_T(x,z,u) - F^a_T(x,z,u)\| \leq T\gamma(T)$) for all $(x,z,u) \in \Omega$ and $T \in (0, T^*]$.

Since $y(k) = x(k)$, we want to design a reduced-order observer that estimates $z(k)$ of the exact models (3) and (5). Since the exact model can not be analytically computable in general, it is not used for the design purpose and we must use the Euler model (4) to design a reduced-order observer for the exact models (3) and (5).

A. Preliminary Results

Consider the exact model (3) and the Euler model (4). Note that $z(k)$ of the Euler models (4) and (6) can be calculated directly by

$$\begin{aligned} z(k) &= \Phi^{-1}(y)g^T(y)\left\{\frac{\rho y - y}{T} - f_1(y)\right\}(k) \\ &= \Psi_T(y, \rho y)(k) \end{aligned}$$

at time $k+1$ where ρ is the shift operator, i.e., $(\rho y)(k) = y(k+1)$ ([10], [11], [17]). Then as a candidate of a reduced-order observer, we can consider

$$\begin{aligned} \hat{z}(k+1) &= \hat{z}(k) + Tf_2(y, \Psi_T(y,\rho y), u)(k) \\ &\quad + TH[\Psi_T(y,\rho y) - \hat{z}](k) \\ &= (I - TH)\hat{z}(k) + T\Pi_T(y,\rho y,u)(k) \\ &=: O_T(\hat{z}, y, \rho y, u)(k) \end{aligned} \tag{8}$$

where $H = \mathrm{diag}\{h(i)\}_{n_z}$ is an observer gain and

$$\Pi_T(y,\rho y,u) = H\Psi_T(y,\rho y) + f_2(y, \Psi_T(y,\rho y), u).$$

Let $e = z - \hat{z}$. Then we have $e(k+1) = (I - TH)e(k)$ for the Euler model (4) and we obtain the following results. Proofs are given in [10] and [11].

Lemma 2.1: Assume **A1**, **A2**, and
A4: $|1 - Th(i)| < 1$ for given $\hat{T} > 0$, any $T \in (0, \hat{T}]$ and

$i = 1\ldots, n_z$.
Then the system (8) is a reduced-order observer of the Euler model (4). ∎

Lemma 2.2: Assume **A1**, **A2**, and **A4**. Then the reduced-order observer (8) is semiglobal and practical in T for the exact model (3), i.e., there exists $\beta \in \mathcal{KL}$ such that for any pair of strictly positive real numbers (D, d) and a compact set $\Omega \subset \mathbf{R}^{n_x} \times \mathbf{R}^{n_z} \times \mathbf{R}^m$, there exists $T^* > 0$ such that $\|z(0) - \hat{z}(0)\| \le D$ and $(x, z, u)(k) \in \Omega$ for any $k \in \mathbf{N}_0$ imply $\|z(k) - \hat{z}(k)\| \le \beta(\|z(0) - \hat{z}(0)\|, kT) + d$ for any $T \in (0, T^*]$ and $k \in \mathbf{N}_0$ where $(x, z)(k) = (x_c, z_c)(kT)$. ∎

B. High-gain Reduced-order Observers

Consider the exact models (3), (5) and the Euler models (4), (6), and the reduced-order observer (8). Since the eigenvalues of $I-TH$ are close to 1 for small $T > 0$ and $h(i) > 0$, we use the observer gains $H = \text{diag}\{h(i)\}_{n_z}$ with large $h(i) > 0$ to achieve fast convergence of $z(k) - \hat{z}(k)$ within an allowable level. Let $\epsilon > 0$ be a sufficiently small and fixed constant, $0 < \bar{h}(i) < 1$, and $h(i) = \bar{h}(i)/\epsilon$. Then we apply the reduced-order observer (8) to the exact model (5) and the Euler model (6). Note that we first fix a sufficiently small $\epsilon > 0$ and then we discuss the asymptotic behavior as $T = T(\epsilon) > 0$ tends to zero and the robustness of the reduced-order observer (8).

Since $|1 - T(\bar{h}(i)/\epsilon)| < 1$ is equivalent to $0 < T < 2\epsilon/\bar{h}(i)$, we assume the following.
A5: For given $\epsilon > 0$ and $\bar{h}(i) \in (0, 1)$, we choose $T \in (0, \hat{T}]$ where $\hat{T}$ is chosen such that $\hat{T} < 2\epsilon/\bar{h}_{max}$ where $\bar{h}_{max} = \max_{i=1,\ldots,n_z} \bar{h}(i)$.
In this case the assumption **A4** is satisfied.

Remark 2.1: 1) By **A5**, it is necessary to choose a sufficient small $\hat{T} > 0$ when we apply a small $\epsilon > 0$, or equivalently a large $h(i) > 0$, $i = 1, \ldots, n_z$.
2) Consider the Euler model (4) and the reduced-order observer (8) with $h(i) = \bar{h}(i)/\epsilon$. For simplicity of notation, let $F_T^i = F_T^i(y, z, u)$, $F_{jT}^i = F_{jT}^i(y, z, u)$, $\tilde{F}_T^i = \tilde{F}_T^i(y, z, u)$, $\tilde{F}_{2T}^i = \tilde{F}_{2T}^i(y, z, u)$, and $O_T^i = O_T(\hat{z}, y, F_{1T}^i, u)$ for $i = e, a$ and $j = 1, 2$. Note that $\rho y = F_{1T}^a(y, z)$ for the Euler models (4) and (6) and then we have $e(k+1) = F_{2T}^a(k) - O_T^a(k) = (I-TH)e(k)$. By **A4** and **A5**, there exists $P_T > 0$ satisfying

$$(I - TH)^T P_T (I - TH) - P_T = -2TI \tag{9}$$

for any fixed $T \in (0, \hat{T}]$ and fixed $\epsilon > 0$. The solution of (9) is given by $P_T = \text{diag}\{2\epsilon^2(\bar{h}(i)[2\epsilon - T\bar{h}(i)])^{-1}\}_{n_z}$. Let $V_T(z, \hat{z}) = e^T P_T e$. Then we have

$$\alpha_1(\|e\|) \le V_T(z, \hat{z}) \le \alpha_2(\|e\|) \tag{10}$$

$$V_T(F_{2T}^a, O_T^a) - V_T(z, \hat{z}) \le -2T\alpha_3(\|e\|) \tag{11}$$

where $\alpha_1(s) = 2\epsilon q_1 s^2$, $\alpha_2(s) = 2\epsilon^2 q_2(\epsilon) s^2$, $\alpha_3(s) = s^2$, $q_1 = (2\bar{h}_{max})^{-1}$, $q_2(\epsilon) = [\bar{h}_{min}(2\epsilon - \hat{T}\bar{h}_{max})]^{-1}$, and $\bar{h}_{min} = \min_{i=1,\ldots,n_z} \bar{h}(i)$. Let Ω_z, $\hat{\Omega}_z \subset \mathbf{R}^{n_z}$ be compact sets. Then there exists $L_V > 0$ such that

$$|V_T(\eta_1, \hat{\eta}_1) - V_T(\eta_2, \hat{\eta}_2)| \le L_V(\|\eta_1 - \eta_2\| + \|\hat{\eta}_1 - \hat{\eta}_2\|) \tag{12}$$

for any η_1, $\eta_2 \in \Omega_z$ and $\hat{\eta}_1$, $\hat{\eta}_2 \in \hat{\Omega}_z$. ∎

Let $\Omega \subset \mathbf{R}^{n_x} \times \mathbf{R}^{n_z} \times \mathbf{R}^m$ be a compact set. By **A1-A3**, there are strictly positive real numbers L_2, l_1, l_2, and l_δ such that $\|f_2(\eta_1, \eta_2, \tau) - f_2(\bar{\eta}_1, \bar{\eta}_2, \bar{\tau})\| \le L_2(\|\eta_1 - \bar{\eta}_1\| + \|\eta_2 - \bar{\eta}_2\| + \|\tau - \bar{\tau}\|)$, $\|g(\eta_1)\| \le l_1$, $\|\Phi(\eta_1)^{-1}\| \le l_2$, and $\|\delta(\eta_1, \eta_2, \tau)\| \le l_\delta$ for any (η_1, η_2, τ), $(\bar{\eta}_1, \bar{\eta}_2, \bar{\tau}) \in \Omega$.

Consider the Euler model (6) and the reduced-order observer (8) with $h(i) = \bar{h}(i)/\epsilon$. Then we have

$$\begin{aligned}
& V_T(\tilde{F}_{2T}^a, O_T^a) - V_T(z, \hat{z}) \\
= \; & -2T\|e\|^2 + 2T\delta^T P_T(I - TH)e + T^2\delta^T P_T\delta \\
\le \; & -T\alpha_3(\|e\|) + T^2\delta^T P_T\delta \\
& \quad + T\delta^T P_T(I - TH)^2 P_T\delta
\end{aligned}$$

where $\delta = \delta(x, z, u)$. Here we have $\delta^T P_T\delta \le 2\epsilon^2 q_2(\epsilon)\|\delta\|^2$ and

$$P_T(I - TH)^2 P_T = 4\epsilon^2 \text{diag}\{\frac{[\epsilon - T\bar{h}(i)]^2}{\bar{h}^2(i)[2\epsilon - T\bar{h}(i)]^2}\}_{n_z}.$$

Since

$$\begin{aligned}
\frac{[\epsilon - T\bar{h}(i)]^2}{\bar{h}^2(i)[2\epsilon - T\bar{h}(i)]^2} \; & \le \; \max\{\frac{[\epsilon - T\bar{h}_{min}]^2}{\bar{h}_{min}^2[2\epsilon - T\bar{h}_{max}]^2}|_{T=0}, \\
& \qquad \frac{[\epsilon - T\bar{h}_{min}]^2}{\bar{h}_{min}^2[2\epsilon - T\bar{h}_{max}]^2}|_{T=\hat{T}}\} \\
& = \; \frac{1}{\bar{h}_{min}^2}\max\{\frac{1}{4}, \frac{[\epsilon - \hat{T}\bar{h}_{min}]^2}{[2\epsilon - \hat{T}\bar{h}_{max}]^2}\} \\
& =: \; \theta(\epsilon),
\end{aligned}$$

we have $\delta^T P_T(I-TH)^2 P_T\delta \le 4\epsilon^2\theta(\epsilon)\|\delta\|^2$. Since $\|\delta\| \le l_\delta$ for any $(y, z, u) \in \Omega$, we obtain

$$\begin{aligned}
V_T(\tilde{F}_{2T}^a, O_T^a) - V_T(z, \hat{z}) \le & -T\alpha_3(\|e\|) \\
& + T[4\epsilon^2\theta(\epsilon) + 2T\epsilon^2 q_2(\epsilon)]l_\delta^2
\end{aligned} \tag{13}$$

for any $(y, z, u) \in \Omega$.

Remark 2.2: The function $V_T(z, \hat{z}) = e^T P_T e$ satisfying (10), (12), and (13) corresponds to an ISS-Lyapunov function [21] for the Euler model $e(k + 1) = \tilde{F}_T^a(k) - O_T^a(k)$. ∎

In the following let $\Psi_T^i = \Psi_T(y, F_{1T}^i)$ and $f_2^i = f_2(y, \Psi_T^i, u)$ for $i = e, a$. Then we have the following results.

Lemma 2.3: Assume **A1-A3** and **A5**. For any strictly positive real numbers $(\Delta_x, \Delta_z, \hat{\Delta}_z, \Delta_u, \nu)$, there exist $\epsilon > 0$ and $T^* \in (0, \hat{T}]$ such that

$$V_T(\tilde{F}_{2T}^e, O_T^e) - V_T(z, \hat{z}) \le -T\alpha_3(\|e\|) + T\nu \tag{14}$$

for all y, z, $\hat{z}$, u, and T satisfying $\|y\| \le \Delta_x$, $\|z\| \le \Delta_z$, $\|\hat{z}\| \le \hat{\Delta}_z$, $\|u\| \le \Delta_u$, and $T \in (0, T^*]$. ∎

Proof. Let $(\Delta_x, \Delta_z, \hat{\Delta}_z, \Delta_u, \nu)$ be given and $\Omega = \mathbf{B}_{\Delta_x} \times \mathbf{B}_{\Delta_z} \times \mathbf{B}_{\Delta_u}$. Then we can find $T_1^* > 0$ and $\gamma \in \mathcal{K}$ satisfying (7). Let

$$\Delta_L = \sup_{(y,z,u)\in\Omega, \|\hat{z}\|\le\hat{\Delta}_z} \max\{\|\tilde{F}_T^e\|, \|\tilde{F}_T^a\|, \|O_T^e\|, \|O_T^a\|, \Delta_x, \Delta_z, \hat{\Delta}_z\}$$

and $\mathcal{Z} = \hat{\mathcal{Z}} = \mathbf{B}_{\Delta_L}$. Then there exist $L_V > 0$ and $T_2^* > 0$ satisfying (10), (12), and (13). Let $\epsilon > 0$ and $T_3^* > 0$ be such that

$$4\epsilon^2\theta(\epsilon)l_\delta^2 < \nu, \tag{15}$$

$$\begin{aligned}&4\epsilon^2\theta(\epsilon)l_\delta^2 + T_3^*\epsilon^2 q_2(\epsilon)l_\delta^2 \\ &\quad + L_V[1 + l_1 l_2(\frac{\bar{h}_{max}}{\epsilon} + L_2)]\gamma(T_3^*) \le \nu\end{aligned} \tag{16}$$

and $T^* = \min\{T_1^*, T_2^*, T_3^*\}$. By (12), (13), and the one-step consistency between $\tilde{F}_T^e$ and $\tilde{F}_T^a$, we have

$$\begin{aligned}& V_T(\tilde{F}_{2T}^e, O_T^e) - V_T(z, \hat{z}) \\ \le\ & V_T(\tilde{F}_{2T}^a, O_T^a) - V_T(z, \hat{z}) \\ & + |V_T(\tilde{F}_{2T}^e, O_T^e) - V_T(\tilde{F}_{2T}^a, O_T^a)| \\ \le\ & -T\alpha_3(\|e\|) + T[4\epsilon^2\theta(\epsilon) + 2T\epsilon^2 q_2(\epsilon)]l_\delta^2 \\ & + L_V[T\gamma(T) + \|O_T^e - O_T^a\|]\end{aligned}$$

for any $(y, z, u) \in \Omega$ and $T \in (0, T^*]$. By direct calculation, we have $O_T^e - O_T^a = TH(\Psi_T^e - \Psi_T^a) + T(f_2^e - f_2^a)$, $\Psi_T^e - \Psi_T^a = \frac{1}{T}\Phi^{-1}(y)g^T(y)(F_{1T}^e - F_{1T}^a)$, $\|\Psi_T^e - \Psi_T^a\| \le l_1 l_2 \gamma(T)$, $\|O_T^e - O_T^a\| \le T l_1 l_2[(\bar{h}_{max}/\epsilon) + L_2]\gamma(T)$, and we obtain

$$\begin{aligned}& V_T(\tilde{F}_{2T}^e, O_T^e) - V_T(z, \hat{z}) \\ \le\ & -T\alpha_3(\|e\|) + T\{4\epsilon^2\theta(\epsilon)l_\delta^2 + 2T\epsilon^2 q_2(\epsilon)l_\delta^2 \\ & \qquad + L_V[1 + l_1 l_2(\frac{\bar{h}_{max}}{\epsilon} + L_2)]\gamma(T)\}.\end{aligned}$$

Hence by (15) and (16), we have the assertion.

Theorem 2.1: Assume **A1**-**A3** and **A5**. Then there exists $\epsilon > 0$ such that the reduced-order observer (8) with $h(i) = \bar{h}(i)/\epsilon$ is semiglobal and practical in T for the exact model (5). ∎

Proof. Let x and z be the states of the exact model (5) and let $\mathcal{X}$, $\mathcal{Z}$, $\mathcal{U}$ give a compact set $\Omega = \mathcal{X} \times \mathcal{Z} \times \mathcal{U}$ that guarantees the one-step consistency between $\tilde{F}_T^e$ and $\tilde{F}_T^a$. Let $(x, z, u)(k) \in \Omega$ for any $k \in \mathbf{N}_0$ and let $0 < r < R$ be given. Then we first show that if $r \le V_T(z, \hat{z})(k) \le R$, then there exist $\epsilon > 0$ and $T_1^* > 0$ such that

$$\Delta V_T(k) \le -\frac{1}{2}T\alpha_3(\|e(k)\|) \tag{17}$$

for any $T \in (0, T_1^*]$ where $\Delta V_T(k) = V_T(k+1) - V_T(k)$ and $V_T(k) = V_T(z, \hat{z})(k)$ for simplicity of notation. To show (17), let $(\Delta_x, \Delta_z, \hat{\Delta}_z, \Delta_u, \nu)$ be strictly positive real numbers satisfying

$$\begin{aligned}&\Delta_x \ge \sup_{x\in\mathcal{X}}\|x\|,\ \Delta_z \ge \sup_{z\in\mathcal{Z}}\|z\|,\ \Delta_u \ge \sup_{u\in\mathcal{U}}\|u\|, \\ &\quad \hat{\Delta}_z \ge \sup_{z\in\mathcal{Z}}\|z\| + \alpha_1^{-1}(R),\ \nu \le \frac{1}{2}\alpha_3(\alpha_2^{-1}(r)).\end{aligned} \tag{18}$$

By (10) and $V_T(k) \le R$, we have $\|\hat{z}(k)\| \le \|z(k)\| + \alpha_1^{-1}(V_T(k)) \le \|z(k)\| + \alpha_1^{-1}(R) \le \hat{\Delta}_z$. Hence if we choose $T_1^* > 0$ from Lemma 2.3, we have $\Delta V_T(k) \le -T\alpha_3(\|e(k)\|) + T\nu$ for any $T \in (0, T_1^*]$. Moreover $r \le V_T(k)$ and (10) imply $\|e(k)\| \ge \alpha_2^{-1}(V_T(k)) \ge \alpha_2^{-1}(r)$. By the choice of ν in (18), we have $\nu \le \frac{1}{2}\alpha_3(\alpha_2^{-1}(r)) \le \frac{1}{2}\alpha_3(\|e(k)\|)$ and hence (17).

Assume $r \le V_T(k) \le R$. Let $s_T(t) = V_T(k) + (\frac{t}{T} - k)\Delta V_T(k)$ for any $t \in [kT, (k+1)T)$ and $k \ge 0$. Then $s_T(kT) = V_T(k)$ and we have $s_T(t) \le V_T(k)$ and $\dot{s}_T \le \frac{1}{T}\Delta V_T(k) \le -\frac{1}{2}\alpha_3(\|e(k)\|)$ for any $t \in [kT, (k+1)T)$. By (10) we obtain

$$\dot{s}_T(t) \le -\frac{1}{2}\alpha_3(\alpha_2^{-1}(V_T(k))) \le -\frac{1}{2}\alpha_3(\alpha_2^{-1}(s_T(t))).$$

Since $\alpha_3(s) = s^2$ and $\alpha_2(s) = 2\epsilon^2 q_2(\epsilon)s^2$, we have

$$\dot{s}_T(t) \le -\frac{1}{4\epsilon^2 q_2(\epsilon)}s_T(t)$$

for any $t \in [kT, (k+1)T)$. Hence we have

$$\begin{aligned}s_T(t) &\le \exp\left(-\frac{1}{4\epsilon^2 q_2(\epsilon)}(t - kT)\right)s_T(kT) \\ &= \exp\left(-\frac{1}{4\epsilon^2 q_2(\epsilon)}(t - kT)\right)V_T(k).\end{aligned}$$

If $V_T(k) \le r$, then by (14), we have $V_T(k+1) \le r + T\nu$. Let $0 < T_2^* \le T_1^*$ be such that $r + T_2^*\nu \le R$ for any $T \in (0, T_2^*]$. Then from the proof of Theorem 1 in [5], we can show that $V_T(0) \le R$ implies

$$V_T(k) \le \max\{\beta_1(V_T(0), kT), r + T\nu\} \tag{19}$$

where

$$\beta_1(s, t) = s\exp\left(-\frac{1}{4\epsilon^2 q_2(\epsilon)}t\right).$$

If $\|e(0)\| \le \alpha_2^{-1}(R)$, then $V_T(0) \le R$. By (10) and (19), we have

$$\begin{aligned}\|e(k)\| &\le \alpha_1^{-1}(\beta_1(V_T(0), kT) + r + T\nu) \\ &\le \beta(\|e(0)\|, kT) + \alpha_1^{-1}(2(r + T\nu))\end{aligned}$$

where

$$\beta(s, t) = \alpha_1^{-1}(2\beta_1(s, t)) = \sqrt{\frac{\epsilon q_2(\epsilon)}{q_1}}\exp\left(-\frac{1}{8\epsilon^2 q_2(\epsilon)}t\right)s.$$

For any $D > d > 0$, let $R = \alpha_2(D)$, $r = \frac{1}{4}\alpha_1(d)$ and choose $0 < T^* \le T_2^*$ such that $\nu T^* \le \frac{1}{4}\alpha_1(d)$. Then we have $\|e(0)\| \le \alpha_2^{-1}(R) = D$ and $\alpha_1^{-1}(2(r + T\nu)) \le \alpha_1^{-1}(2(r + T^*\nu)) \le d$ for any $T \in (0, T^*]$. Hence the proof is complete.

Remark 2.3: 1) From the proof of Theorem 2.1, we have

$$\|e(k)\| \le \sqrt{\frac{\epsilon q_2(\epsilon)}{q_1}}\exp\left(-\frac{kT}{8\epsilon^2 q_2(\epsilon)}\right)\|e(0)\| + d$$

and hence we do not have a peaking phenomena for the designed reduced-order observer (8) with $h(i) = \bar{h}(i)/\epsilon$ and we can achieve fast convergence of $z(k) - \hat{z}(k)$ within an allowable level for sufficiently small $\epsilon > 0$ and $T > 0$.
2) We can include external bounded disturbance signals w_c in the model uncertainty. ∎

III. A Numerical Example

Consider the simplified model of the three degree-of-freedom (3DOF) sampled-data ship with model uncertainty and bounded external disturbance

$$\begin{aligned}\dot{x}_c &= R(x_c)z_c, \\ \dot{z}_c &= -M^{-1}[C(z_c) + D]z_c + q(z_c, u_c) \\ &\quad + \hat{\delta}(x_c, z_c, u_c) + w_c \\ y(k) &= x_c(kT)\end{aligned} \tag{20}$$

where $x_c \in \mathbf{R}^3$ and $z_c \in \mathbf{R}^3$ correspond to the position and the velocity of the ship, respectively, $u_c(t) = u(k) \in \mathbf{R}^2$ for any $t \in [kT, (k+1)T)$ is the control input, $w_c \in \mathbf{R}^3$ is the bounded external disturbance, $R(x) = \begin{bmatrix} \bar{R}(x) & 0 \\ 0 & 1 \end{bmatrix}$ with $\bar{R}(x) = \begin{bmatrix} \cos x_3 & -\sin x_3 \\ \sin x_3 & \cos x_3 \end{bmatrix}$ is the rotation matrix in yaw, $M = \mathrm{diag}\{m_{11}, M_2\} > 0$ with $M_2 = \begin{bmatrix} m_{22} & m_{23} \\ m_{23} & m_{33} \end{bmatrix}$ is the inertia matrix including hydrodynamic added inertia, $D = \mathrm{diag}\{d_{11}, D_2\} > 0$ with $D_2 = \begin{bmatrix} d_{22} & d_{23} \\ d_{32} & d_{33} \end{bmatrix}$ is the linear damping matrix, $C(z) = \begin{bmatrix} 0 & -\bar{c}(z)^T \\ \bar{c}(z) & 0 \end{bmatrix}$ with $\bar{c}(z) = [\, m_{22}z_2 + m_{23}z_3 \quad -m_{11}z_1 \,]$ is the Coriolis-centripetal matrix, and $q(z,u) = [\, u_1 \quad \lambda_{21}(z)u_2 + \lambda_{22}(z) \quad \lambda_{31}(z)u_2 + \lambda_{32}(z) \,]^T$. Since it is hard to identify $M > 0$ and D precisely, we assume $\hat{\delta}(x,z,u) = \bar{\delta}\{-M^{-1}[C(z)+D]z + q(z,u)\}$ with $|\bar{\delta}| \leq 0.5$. The model (20) is of the form (2) with $f_1(x) = 0$, $g(x) = R(x)$, $f_2(x,z,u) = -M^{-1}[C(z)+D]z + q(z,u)$, and $\delta = \hat{\delta} + w$. Since $R^{-1}(x) = R^T(x)$, the assumptions **A1** and **A2** are satisfied. We also have $\Phi^{-1}(y)g^T(y) = R^T(y)$.

Let $(m_{11}, m_{22}, m_{23}, m_{33}) = (198.8, 206, 15.6, 101.5)$, $(d_{11}, d_{22}, d_{23}, d_{32}, d_{33}) = (56.1, 90.9, -21.8, -26.9, 38.8)$, and the control input $u = [\, u_1 \quad u_2 \,]^T$ is given by the state feedback controller

$$
\begin{aligned}
u_1 &= -\frac{1}{T+0.3}[z_1 - 0.8] - d_1(z), \qquad (21)\\
u_2 &= \frac{1}{\Xi(x,z)}\{[-d_3(z) - \lambda_{32}(z) + \kappa(x,z,\bar{x}_3) \\
&\quad - \frac{0.3}{T+0.3}\tilde{z}_3] - \Gamma(x)u_1 \sin x_3\}
\end{aligned}
$$

where $(x,z)(k) = (x_c, z_c)(kT)$ and $d_1(z)$, $\Xi(x,z)$ and etc. are given in Appendix. The state feedback controller (21) achieves straight-line trajectory tracking and achieves $\lim_{t\to\infty} z_c(t) = [\, 0.8 \quad 0 \quad 0 \,]^T$ for the system (20) with $\bar{\delta} = 0$ and $w_c = 0$ in the continuous-time SPUA stable sense [12]. Then there exists $t^* \geq 0$ that depends on $z_1(0)$ such that $z_1(t) > 0$ for any $t \geq t^*$.

Let $\bar{H} = \mathrm{diag}\{0.1, 0.2, 0.2\}$ and $H = \bar{H}/\epsilon$. Then the high-gain reduced-order observer is given by

$$
\hat{z}(k+1) = (1 - \frac{T}{\epsilon}\bar{H})\hat{z}(k) + T\Pi(y, \rho y, u)(k) \qquad (22)
$$

where $\Pi(y, \rho y, u) = (1/\epsilon)\bar{H}\Psi_T - M^{-1}[C(\Psi_T) + D]\Psi_T + q(\Psi_T, u)$ and $\Psi_T = \Psi_T(y, \rho y) = (1/T)R^T(y)(\rho y - y)$.

Let $x_c(0) = [\, 0 \quad 10 \quad \pi/6 \,]^T$, $z_c(0) = [\, 0.5 \quad -0.5 \quad -0.5 \,]^T$, and $w_c = 0$. In this case we can see $z_1(t) > 0$ for any $t \geq 0$ from Figs 1-3 and hence the assumption **A3** is also satisfied. Figs 1-3 show the simulation results of the time responses of the velocity $z_c(t)$ of the system (20) and its estimate $\hat{z}(k)$ given by the observer (22) that is designed for the system (20) with $\bar{\delta} = 0$ and $w_c = 0$. Fig 1 shows that the performance of the designed observer with $(T, \epsilon) = (0.1, 1)$ becomes worse for the system (20) with $\bar{\delta} = 0.5$. Figs 2 and 3 show the performance of the designed observer for the system (20) with $\bar{\delta} = 0.5$ when $T = 0.1$ (s) and $T = 0.01$ (s). Fig 4 shows the performance of the designed observer for the system (20) with $\bar{\delta} = 0.5$ and $w_c = [\, 0.4 \quad 0.2 \quad 0.1 \,]^T$ when $T = 0.01$ (s). We can see that the designed observer gives a better performance for smaller $\epsilon > 0$. Figs 2-4 indicate that a small parameter $\epsilon > 0$ plays an important role to attenuate an influence of both model uncertainty and bounded external disturbances, and to obtain fast convergence of $z_c(kT) - \hat{z}(k)$ within an allowable level for small sampling periods.

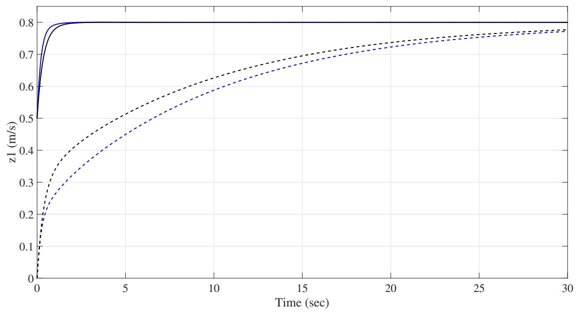

Fig. 1. Simulation result of the time responses of $z_c(t)$ of the system (20) with $\bar{\delta} = 0$ (black line) and $\bar{\delta} = 0.5$ (blue line) and $\hat{z}(k)$ given by the reduced-order high-gain observer (22) (black broken line: $\bar{\delta} = 0$ and blue broken line: $\bar{\delta} = 0.5$) for $(T, \epsilon) = (0.1, 1)$.

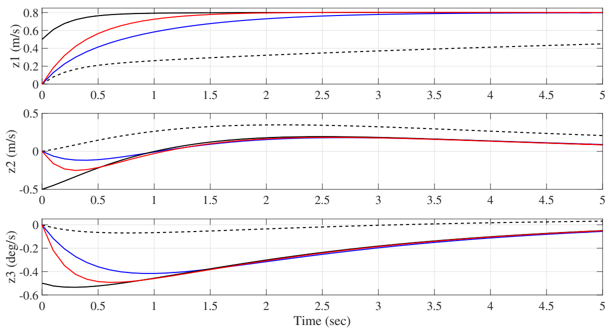

Fig. 2. Simulation result of the time responses of $z_c(t)$ (black line) of the system (20) with $\bar{\delta} = 0.5$ and $\hat{z}(k)$ given by the reduced-order high-gain observer (22) (black broken line for $(T, \epsilon) = (0.1, 1)$, blue line for $(T, \epsilon) = (0.1, 0.1)$, red line for $(T, \epsilon) = (0.1, 0.05)$.

IV. Conclusions

In this paper we have considered the design of high-gain reduced-order observers for the nonlinear sampled-data strict-feedback system with model uncertainty. We have introduced a small parameter $\epsilon > 0$ to the reduced-order observers designed for the sampled-data strict-feedback system without model uncertainty. Then we have shown the existence of $\epsilon > 0$ such that the modified observers are semiglobal and practical in T for the exact model of the sampled-data strict-feedback system with model uncertainty. A numerical example has been also given to illustrate the efficiency of the designed observers. High-gain reduced-order observer-based output feedback control and the related topics such as performance recovery are in the future work.

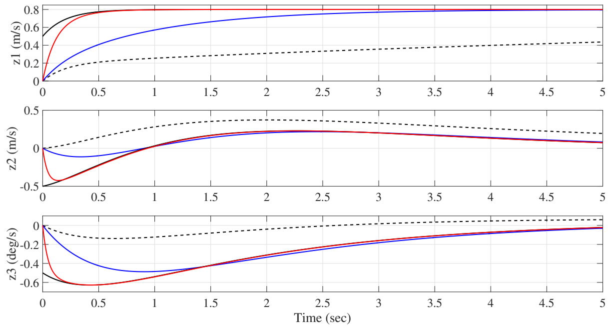

Fig. 3. Simulation result of the time responses of $z_c(t)$ (black line) of the system (20) with $\bar{\delta} = 0.5$ and $\hat{z}(k)$ given by the reduced-order high-gain observer (22) (black broken line for $(T, \epsilon) = (0.01, 1)$, blue line for $(T, \epsilon) = (0.01, 0.1)$, red line for $(T, \epsilon) = (0.01, 0.01)$.

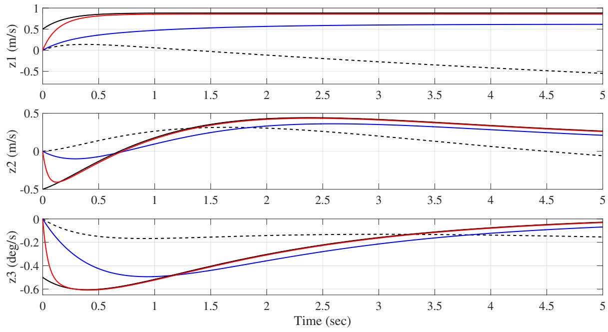

Fig. 4. Simulation result of the time responses of $z_c(t)$ (black line) of the system (20) with $\bar{\delta} = 0.5$ and $w_c = [0.4\ 0.2\ 0.1]^T$, and $\hat{z}(k)$ given by the reduced-order high-gain observer (22) (black broken line for $(T, \epsilon) = (0.01, 1)$, blue line for $(T, \epsilon) = (0.01, 0.1)$, red line for $(T, \epsilon) = (0.01, 0.01)$.

APPENDIX

A. Notation in Section III

$\lambda_{21}(z) = 0.102z_1^2 - 0.0232(z_2 - 1.18z_3)^2$, $\lambda_{22}(z) = -0.0232[z_1^2 + (z_2 - 1.18z_3)^2]\tan^{-1}((z_2 - 1.18z_3)/z_1)$, $\lambda_{31}(z) = 0.287z_1^2 + 0.307(z_2 - 1.18z_3)^2$, $\lambda_{32}(z) = -13.23\lambda_{22}(z)$, $[\, d_1(z) \quad d_2(z) \quad d_3(z)\,]^T = -M^{-1}[C(z) + D]z$, $\Gamma(x) = 6/(36 + x_2^2)$, $\Xi(x, z) = \lambda_{31}(z) + \lambda_{21}(z)\Gamma(x)\cos x_3$, $\bar{x}_3 = \tan^{-1}(-x_2/6)$, $\tilde{z}_3 = z_3 - \bar{z}_3$, $\bar{z}_3 = -0.3\tilde{x}_3/(T+0.3) - \Gamma(x)(z_1 \sin x_3 + z_2 \cos x_3)$, $\tilde{x}_3 = x_3 - \bar{x}_3$, $\kappa(x, z, \bar{x}_3) = -0.3\dot{\tilde{x}}_3/(T + 0.3) + \Gamma(x)^2 x_2(z_1 \sin x_3 + z_2 \cos x_3)^2/3 - \Gamma(x)\{d_1(z)\sin x_3 + [d_2(z) + \lambda_{22}(z)]\cos x_3 + (z_1 \cos x_3 - z_2 \sin x_3)z_3\}$.

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