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An internal model-based discrete-time dynamic average consensus estimator

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***Abstract*— The dynamic average consensus problem for a group of agents is considered. Each agent is supposed to estimate the average of inputs applied to all agents in a distributed manner. A new structure for the distributed average estimator which can embed the internal model of inputs is proposed. Constructive design procedures are also given for the cases with constant inputs and time-varying inputs. The proposed estimator is validated via numerical simulations.**

I. INTRODUCTION

This paper studies the dynamic average consensus (DAC) problem. It is to construct estimators running on a group of agents connected through communication network so that the each agents' estimator computes the average of all input signals applied to the agents in a distributed manner. A solution to this problem can serve as an essential part of various network applications such as formation control [1]–[3], cooperative control [4], distributed optimization [5]–[7], distributed mapping [8], [9], sensor fusion [10]–[12], and distributed Kalman filtering [13], [14].

Since the DAC problem for the constant input signals had first been introduced [15], many efforts to the problem have been devoted. In [16], Proportional-Integral (PI) type DAC estimator is proposed for the constant input signals. To solve the DAC problem for the time-varying input signals, PI-type estimator is generalized using the internal model principle in [17]. In [18], an adaptive internal model DAC is proposed. In [19], [20], nonlinear DAC algorithms that use derivative information of the inputs are proposed. The algorithms have input-to-state stability properties and achieve DAC of bounded inputs with errors dependent on the derivatives of the inputs. A DAC algorithm based on the singular perturbation theory is given in [21]. In [22], a discrete time DAC estimator that can track the average of the time-varying inputs within a prescribed steady state error is presented. In addition, several discrete-time DAC algorithms are proposed in [23]–[26].

This paper focuses on the internal model-based DAC estimator [17] that generalizes the PI type estimator given in [16]. Although the key parameters of the internal model [27] of the input, such as the order of polynomials or the frequency of the sinusoidal, should be known, the main advantage of the internal model-based DAC solution is that the estimation error converges to zero asymptotically. Another important advantage is that it is not affected by the initial conditions and does not require the bounds of inputs and their time derivatives. Despite these advantages, the main drawback is that a completely constructive design procedure has not been established.

In this paper, we propose a new internal model based dynamic average consensus estimator by reinterpreting the internal model-based PI-type estimator [17] for the discrete-time system. Compared to [16], [17], we propose a simpler estimator in the structural point of view, and the estimator solves the DAC problem with less information exchange. Reducing the amount of information exchange is one of the important issues in multi-agent system problems including the DAC problems because the network throughput is limited practically. Additionally, we propose a concrete design procedure to select the parameters guaranteeing convergence to the average of all inputs. For the constant input, we present a design procedure with the discrete-time root locus method. For the time-varying input, we present a design procedure exploiting the disc margin of the discrete-time LQR.

The subsequent sections are organized as follows. We formulate the problem in Section II. In Section III, we propose new DAC estimators for constant inputs and time-varying inputs, and propose constructive design procedures for the estimators. Moreover, we present an example for constant inputs and time-varying inputs. Concluding remarks are given in Section IV.

Notation: 0_k stands for the zero vector in $\mathbb{R}^k$, $1_k \in \mathbb{R}^k$ a vector with all components being 1. I_k represents the identity matrix in $\mathbb{R}^{k\times k}$. $A \otimes B$ denotes the Kronecker product of matrices A and B. $\bar{\sigma}(M)$ denotes the maximum singular value of matrix M. For two column vectors (or scalars) a and b, $[a; b] := [a^T, b^T]^T$, A block diagonal matrix consisting of two matrices A and B is denoted by $\mathrm{diag}\{A, B\}$. The z-trans form of $f(k)$ is denoted by $\mathcal{Z}(f(k)) := F(z)$. The normal distribution is denoted by $\mathcal{N}(\mu, \sigma^2)$, where μ is the mean and σ^2 is the variance.

II. PROBLEM FORMULATION

A. Problem Statement

Consider a discrete-time multi-agent system consisting of N estimators which take possibly different inputs or signals. The signal to the i-th estimator is denoted by u^i. The communication among estimators is modeled by a graph $\mathcal{G}$ and it is assumed that the graph is undirected. Let $\mathcal{A} = \begin{bmatrix} a_{ij} \end{bmatrix} \in \mathbb{R}^{N\times N}$ be an adjacency matrix associated to $\mathcal{G}$ where a_{ij} is the weight of connection. If agent i and j can communicate each other $a_{ij} = a_{ji} > 0$, if they do not

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$a_{ij} = a_{ji} = 0$. The diagonal elements of $\mathcal{A}$ are set to be zero, i.e., $a_{ii} = 0$. The Laplacian of the graph $\mathcal{G}$, denoted by L, is an $N \times N$ matrix whose components are set as $l_{ij} = -a_{ij}$, if $i \neq j$, and $l_{ii} = \sum_{j=1, j\neq i}^{N} a_{ij}$.

The objective of DAC problem is to design a local estimator calculating the average of all input signals in a distributed manner. Formally, we say that the estimator

$$\begin{aligned} \chi^i(k+1) &= F(\chi^i(k), v^i(k), u^i(k)) \\ v^i(k) &= \sum_{i=1}^{N} a_{ij}(\chi^j(k) - \chi^i(k)) \\ \hat{u}^i_{\mathsf{av}}(k) &= G(\chi^i(k), v^i(k)) \end{aligned} \tag{1}$$

where $\hat{u}^i_{\mathsf{av}}(k)$ is an average estimate of i-th estimator, solves the dynamic average consensus problem provided that χ^i for $i = 1, \cdots, N$ are bounded for all $k \geq 0$, and it holds that

$$\lim_{k\to\infty} \left(\hat{u}_{\mathsf{av}}(k) - \frac{1}{N} 1_N 1_N^T u(k) \right) = 0 \tag{2}$$

where $\hat{u}_{\mathsf{av}}(k) := [\hat{u}^1_{\mathsf{av}}(k), \cdots, \hat{u}^N_{\mathsf{av}}(k)]^T$ and $u(k) := [u^1(k), \cdots, u^N(k)]^T$.

B. A Solution to the DAC Problem

One of the well-known DAC estimators is the PI-type estimator reported in [16], [17] which is given by

$$\begin{aligned} \xi^i(k+1) &= A_\xi \xi^i(k) + B_\xi \Big(u^i(k) \\ &- k_P \sum_{j=1}^{N} a_{ij} \big(\hat{u}^i_{\mathsf{av}}(k) - \hat{u}^j_{\mathsf{av}}(k)\big) - k_I \sum_{j=1}^{N} a_{ij} \big(v^i(k) - v^j(k)\big) \Big) \\ \hat{u}^i_{\mathsf{av}}(k+1) &= C_\xi \xi^i(k) + D_\xi \Big(u^i(k) \\ &- k_P \sum_{j=1}^{N} a_{ij} \big(\hat{u}^i_{\mathsf{av}}(k) - \hat{u}^j_{\mathsf{av}}(k)\big) - k_I \sum_{j=1}^{N} a_{ij} \big(v^i(k) - v^j(k)\big) \Big) \\ \eta^i(k+1) &= A_\eta \eta^i(k) + B_\eta \Big(k_I \sum_{j=1}^{N} a_{ij} \big(\hat{u}^i_{\mathsf{av}}(k) - \hat{u}^j_{\mathsf{av}}(k)\big) \Big) \\ v^i(k+1) &= C_\eta \eta^i(k) + D_\eta \Big(k_I \sum_{j=1}^{N} a_{ij} \big(\hat{u}^i_{\mathsf{av}}(k) - \hat{u}^j_{\mathsf{av}}(k)\big) \Big) \end{aligned} \tag{3}$$

where k_P and k_I are positive gains. $\hat{u}^i_{\mathsf{av}}$ is the estimate of the average for the i-th agent, ξ^i, η^i are the internal states.

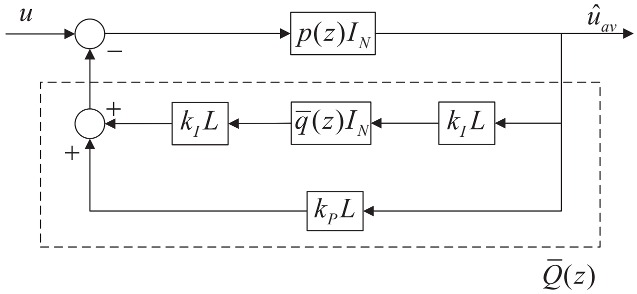


Fig. 1: Dynamic average estimator proposed in [16], [17].

Let $U(z) := \frac{n(z)}{d(z)}$, $p(z) := \frac{n_p(z)}{d_p(z)} = C_\xi(zI - A_\xi)^{-1}B_\xi$, $\bar{q}(z) := \frac{n_{\bar{q}}(z)}{d_{\bar{q}}(z)} = C_\eta(zI - A_\eta)^{-1}B_\eta$. The structure of N estimators given by (3) is represented in Fig. 1. The estimator has been generalized in [17] and the convergence of $\hat{u}_{\mathsf{av}}(k)$ to $u_{\mathsf{av}}(k)$ is guaranteed if $p(z)\bar{q}(z)$ is strictly proper as well as $p(z)$ and $\bar{q}(z)$ satisfy the following conditions.

1) All roots of $d_p(z)$ are inside the unit circle, and $n_p(z) - d_p(z)$ includes all roots of $d(z)$.
2) $d_{\bar{q}}(z)$ includes all roots of $d(z)$.
3) the roots of $d_p(z)d_{\bar{q}}(z) + n_p(z)n_{\bar{q}}(z)k_I^2\lambda_i^2 + n_p(z)d_{\bar{q}}(z)k_P\lambda_i$ inside the unit circle $\forall i = 2, \cdots, N$.

It is noted that the first and second condition are easy to be satisfied, but the third condition is not, because the characteristic equation includes not only λ_i, also λ_i^2, which is the main obstacle to derive a design procedure to fulfill the conditions. In addition, this estimator requires that the outputs of ξ dynamics as well as those of η dynamics should be exchanged among agents, which can result in high communication load as the number of agents increases.

III. Proposed Dynamic Average Consensus Protocol

In this section, we propose a simple solution and and provide a constructive design procedure to solve the DAC problem for constant inputs. Moreover, we propose a simple solution and design a procedure for time-varying inputs.

A. Dynamic Average Consensus with Constant Inputs

When it comes to the case that $u(k)$ is a constant vector, the dynamic average consensus estimator is given by

$$\begin{aligned} \xi^i(k+1) &= A_\xi \xi^i(k) + B_\xi \left(u^i(k) - w^i(k) \right) \\ \eta^i(k+1) &= A_\eta \eta^i(k) + B_\eta \hat{u}^i_{\mathsf{av}}(k) \\ \hat{u}^i_{\mathsf{av}}(k) &= C_\xi \xi^i(k) \\ v^i(k) &= C_\eta \eta^i(k) \\ w^i(k) &= \sum_{j=1}^{N} a_{ij}(v^i(k) - v^j(k)), \end{aligned} \tag{4}$$

where

$$A_\xi = \begin{bmatrix} 0 & 1 \\ a_{\xi,0} & a_{\xi,1} \end{bmatrix}, \quad B_\xi = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_\xi = \begin{bmatrix} b_{\xi,0} & b_{\xi,1} \end{bmatrix},$$
$$A_\eta = \begin{bmatrix} 0 & 1 \\ a_{\eta,0} & a_{\eta,1} \end{bmatrix}, \quad B_\eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_\eta = \begin{bmatrix} b_{\eta,0} & b_{\eta,1} \end{bmatrix}.$$

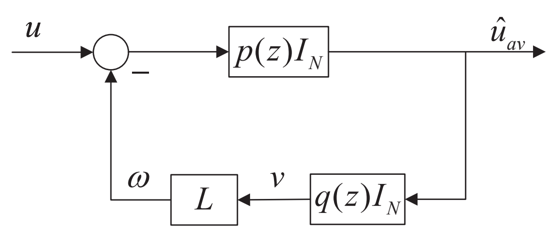


Fig. 2: The structure of proposed dynamic average consensus estimator (4). $p(z)$, $q(z)$ are realized as (4)

Theorem 1: Suppose that the network topology is fixed and $u(k)$ is a constant for all $k \geq 0$. The estimator (4) solves the DAC problem if the following conditions are satisfied.

C.1) A_ξ is stable and $p(1) = 1$ where $p(z)$ is a transfer function w.r.t. $\xi(k)$, i.e., $p(z) = C_\xi(zI - A_\xi)^{-1}B_\xi$.

C.2) $q(z)$ has at least one pole at z=1 where $q(z)$ is a transfer function w.r.t. $\eta(k)$, i.e., $q(z) = C_\eta(zI - A_\eta)^{-1}B_\eta$.

C.3) $1 + \lambda_i p(z)q(z)$ is stable for $i = 2, \cdots, N$ where λ_i is an eigenvalue of L.

Proof: We define $v(k) := \left[v^1(k), \cdots, v^N(k)\right]^T$ and $\hat{u}_{\mathsf{av}}(k)$, $w(k)$, similarly. The estimator (4) is rewritten in z-domain as

$$\begin{aligned}\hat{U}^i_{\mathsf{av}}(z) &= p(z)(U^i(z) - \omega^i(z))\\ &= p(z)(U^i(z) - L_i V(z))\\ V^i(z) &= q(z)\hat{U}^i_{\mathsf{av}}(z)\end{aligned}$$

where L_i is i-th row of L and $\omega^i(z) = \mathcal{Z}(w^i(k))$. By stacking above results for all estimators, $\hat{U}_{\mathsf{av}}$ is compactly written as

$$\begin{aligned}\hat{U}_{\mathsf{av}}(z) &= p(z)(U(z) - LV(z))\\ V(z) &= q(z)\hat{U}_{\mathsf{av}}(z)\\ \hat{U}_{\mathsf{av}}(z) &= \left(I_N + Lp(z)q(z)\right)^{-1}p(z)U(z)\\ &:= T(z)U(z).\end{aligned}$$

With V such that $V^TLV = \Delta$, one can obtain

$$T(z) = VV^TT(z)VV^T = V\begin{bmatrix} p(z) & 0^T_{N-1}\\ 0_{N-1} & \bar{T}(z)\end{bmatrix}V^T, \quad (5)$$

where

$$\bar{T}(z) = \mathrm{diag}\left\{\frac{p(z)}{1 + \lambda_2 p(z)q(z)}, \cdots, \frac{p(z)}{1 + \lambda_N p(z)q(z)}\right\}.$$

Since $T(z)$ is stable by the conditions C.1 and C.3, a steady-state value of $\hat{u}_{\mathsf{av}}(k)$ exists and that is independent to the initial value of ξ and η. Recalling that L has a simple zero eigenvalue with respect to the eigenvector $c1_N$ ($c \neq 0$), we decompose V as $V = \left[\frac{1}{\sqrt{N}}1_N \quad W\right]$. Applying the final value theorem to $\hat{U}_{\mathsf{av}}(z)$, one has

$$\begin{aligned}\hat{u}_{\mathsf{av,ss}} &= \lim_{z\to 1}(z-1)T(z)\frac{1}{z-1}\bar{u} = T(1)\bar{u}\\ &= \left[\frac{1}{\sqrt{N}}1_N \quad W\right]\begin{bmatrix} p(1) & 0^T_{N-1}\\ 0_{N-1} & \bar{T}(1)\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{N}}1^T_N\\ W^T\end{bmatrix}\bar{u}\\ &= \frac{1}{N}1_N1^T_Np(1)\bar{u} + W\bar{T}(1)W^T\bar{u}\end{aligned}$$

where $\hat{u}_{\mathsf{av,ss}}$ is a steady-state value of $\hat{u}_{\mathsf{av}}(k)$ and $\bar{u}$ is a vector such that $U(z) = \frac{1}{z-1}\bar{u}$.

From the condition C.3, $\lim_{z\to 1}\frac{p(z)}{1+\lambda_i p(z)q(z)} = 0$ for $i = 2, \cdots, N$, which results in $W^T\bar{T}(1)W\bar{u} = 0$. Thus, $\hat{u}_{\mathsf{av}}(k)$ converges to $\frac{1}{N}1_N1^T_N\bar{u}$ with $p(1) = 1$ in C.1. ∎

Now we provide a design procedure for the case of constant inputs.

Estimator Design Procedure for constant inputs

1) Choose stable polynomials $a_\xi(z) = z^2 + a_{\xi,1}z + a_{\xi,0}$ and $\bar{b}_\xi(z) = z + \bar{b}_{\xi,0}$, after that find $b_{\xi,1}$ such that $b_{\xi,1}\left(1 + \bar{b}_{\xi,0}\right) = a_{\xi,1} + a_{\xi,0} + 1$ where $p(z) = \frac{b_{\xi,1}\bar{b}_\xi(z)}{a_\xi(z)}$.
2) Choose the stable polynomial $\bar{a}_\eta(z) = z + \bar{a}_{\eta,0}$ and $\bar{b}_\eta(z) = z + \bar{b}_{\eta,0}$.
3) Consider the root locus of $1 + kL(z) = 0$ where $L(z) = \frac{\bar{b}_\xi(z)\bar{b}_\eta(z)}{a_\xi(z)(z-1)\bar{a}_\eta(z)}$ and select a proper $k^* > 0$ s.t. all roots of $1 + kL(z) = 0$ are inside the unit circle for any $0 < k < k^*$.
4) Taking $b_{\eta,1} \leq \frac{k^*}{\lambda_{\max}b_{\xi,1}}$, and $q(z) = \frac{b_{\eta,1}\bar{b}_\eta(z)}{(z-1)\bar{a}_\eta(z)}$.

Note that $b_{\eta,1} > 0$ always exists. Since all zeros and poles of $L(z)$ are inside of the unit circle except for the one at $z = 1$, the existence of k^* is guaranteed by the boundedness of the locus to the unit circle.

B. Design Example for Constant Inputs

Consider a group of four agents that estimate the average of a following constant vector $u(k)$.

$$u(k) = \begin{bmatrix}1 & 3 & 5 & 7\end{bmatrix}^T \quad (6)$$

The Laplacian matrix L of the four agents is given by

$$L = \begin{bmatrix} 2 & 0 & -1 & -1\\ 0 & 4 & -2 & -2\\ -1 & -2 & 3 & 0\\ -1 & -2 & 0 & 3\end{bmatrix} \quad (7)$$

From the Estimator Design Procedure for constant inputs, we choose the parameters of A_ξ, C_ξ, A_η, and C_η to have

$$\begin{aligned} A_\xi &= \begin{bmatrix} 0 & 1\\ -0.34 & 1\end{bmatrix}, \quad C_\xi = \begin{bmatrix}0.74 & -0.4\end{bmatrix},\\ A_\eta &= \begin{bmatrix} 0 & 1\\ -0.3 & 1.3\end{bmatrix}, \quad C_\eta = \begin{bmatrix}0.43 & -1.32\end{bmatrix}.\end{aligned} \quad (8)$$

Fig. 3 shows that each agent's estimate $\hat{u}^i_{\mathsf{av}}$ with the proposed DAC estimator converges to the actual average u_{av} of all input.

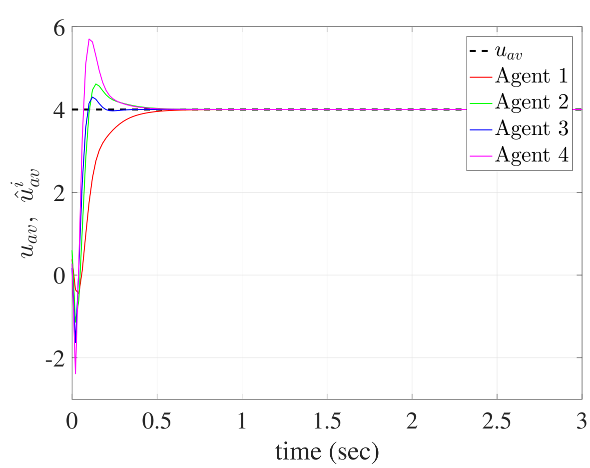


Fig. 3: Average estimate $\hat{u}^i_{\mathsf{av}}$ and the average of u.

C. Average Estimator for Time-varying Inputs

For time-varying inputs, we propose a DAC protocol

$$\begin{aligned}
\xi^i(k+1) &= A_\xi \xi^i(k) + B_\xi \left(u^i(k) - w^i(k)\right) \\
\eta^i(k+1) &= A_\eta \eta^i(k) + B_\eta \hat{u}^i_{\mathsf{av}}(k) \\
\hat{u}^i_{\mathsf{av}}(k) &= C_\xi \xi^i(k) \\
w^i(k) &= \sum_{j=1}^{N} a_{ij} \big\{ \left(K_\xi \xi^i(k) + K_\eta \eta^i(k)\right) \\
&\qquad - \left(K_\xi \xi^j(k) + K_\eta \eta^j(k)\right) \big\}
\end{aligned} \tag{9}$$

where

$$A_\xi = \left[\begin{array}{c|c} 0_{n-1} & I_{n-1} \\ \hline -a_{\xi,0} & -a_{\xi,1} \ \cdots - a_{\xi,n-1} \end{array}\right], \quad B_\xi = \begin{bmatrix} 0_{n-1} \\ 1 \end{bmatrix}$$

$$C_\xi = \begin{bmatrix} b_{\xi,0} & b_{\xi,1} & \cdots & b_{\xi,n-1} \end{bmatrix}$$

$$A_\eta = \left[\begin{array}{c|c} 0_{m-1} & I_{m-1} \\ \hline -a_{\eta,0} & -a_{\eta,1} \ \cdots - a_{\eta,m-1} \end{array}\right], \quad B_\eta = \begin{bmatrix} 0_{m-1} \\ 1 \end{bmatrix}$$

$$C_\eta = \begin{bmatrix} b_{\eta,0} & b_{\eta,1} & \cdots & b_{\eta,m-1} \end{bmatrix}.$$

The distributed estimator (9) has two internal state vectors, ξ^i and η^i, and they driven by the external input u^i and the weighted sum of differences between itself $K_\xi \xi^i + K_\eta \eta^i$ and those of neighbors. The dimensions of ξ^i and η^i depend on the internal model of u^i.

From the structural point of view, the proposed estimator has a simpler structure than the estimator proposed in [16], [17]. In addition, and more importantly, the estimator (9) exchanges only a scalar value $K_\xi \xi^i + K_\eta \eta^i$, which means that the communication load is minimum.

Theorem 2: Let $U(z) := \frac{b_{\mathsf{u}}(z)}{a_{\mathsf{u}}(z)}$, $p(z) := \frac{b_\xi(z)}{a_\xi(z)} = C_\xi(zI - A_\xi)^{-1}B_\xi$, $q(z) := \frac{b_\eta(z)}{a_\eta(z)} := C_\eta(zI - A_\eta)^{-1}B_\eta$. Suppose that the network topology is fixed. The estimator (9) solves the DAC problem if the following conditions are satisfied.

C.4) All roots of $a_\xi(z)$ are inside the unit circle, and $b_\xi(z) - a_\xi(z)$ includes all roots of $a_{\mathsf{u}}(z)$.

C.5) $a_\eta(z)$ includes all roots of $a_{\mathsf{u}}(z)$.

C.6) $A_\psi - \lambda_i(L) B_\psi K_\psi$, $i = 2, \cdots, N$, are all Schur where

$$A_\psi = \begin{bmatrix} A_\xi & 0_{n\times m} \\ B_\eta C_\xi & A_\eta \end{bmatrix}, \quad B_\psi = \begin{bmatrix} B_\xi \\ 0_m \end{bmatrix}$$
$$C_\psi = \begin{bmatrix} C_\xi & 0_m^T \end{bmatrix}, \quad K_\psi = \begin{bmatrix} K_\xi & K_\eta \end{bmatrix}.$$

The conditions C.4 and C.5 ensure that the distributed estimator (9) contains the internal model of the input $u(k)$ so that the estimators can track any signal that (including u^i's and their average) can be generated by the internal model. Although these two conditions come from the same idea of [17], the condition C.6 is closely related to the structure of the average estimator and thus quite different from [17]. Thanks to the simple structure, the condition C.6 is linear in $\lambda_i(L)$ and this allows us to use well developed tools for a consensus of linear systems. Combining the internal model embedding technique for conditions C.4 and C.5, and a robust design tool exploiting the gain margin property of optimal control for condition C.6, we provide a constructive and concrete design procedure to select proper parameters of the proposed estimator.

Estimator Design Procedure

1) Choose n and m (the orders of $a_\xi(z)$ and $a_\eta(z)$, respectively) such that $n \geq l$ and $m \geq l$ (l is the order of $a_{\mathsf{u}}(z)$).
2) Choose the coefficients of $a_\xi(z)$ such that A_ξ is a Schur.
3) Choose an $(m-l)$-th order polynomial $\gamma(z)$ such that all the roots of $\gamma(z)$ are different from those of $a_\xi(z)$, and take $a_\eta(z) = a_{\mathsf{u}}(z)\gamma(z)$.
4) Choose an $(n-l)$-th order polynomial $\bar{\gamma}(z)$ of the form $\bar{\gamma}(z) = -(z^{n-l} + \bar{\gamma}_{n-l-1} z^{n-l-1} + \cdots + \bar{\gamma}_0)$, and take $b_\xi(z) = a_\xi(z) + a_{\mathsf{u}}(z)\bar{\gamma}(z)$.
5) Find K_ψ by choosing sufficiently small $\epsilon > 0$ such that $\left[\frac{\lambda_{\max}}{\lambda_{\min}} - \left(1 + \frac{R}{\delta_\psi}\right)\right]^2 < \frac{R(R+\delta_\psi)}{\delta_\psi^2}$ where $\delta_\psi = \sigma_{\max}(B_\psi^T P_\psi B_\psi)$.

In steps 1 to 4, we find the parameters of the estimator satisfying the conditions 1 and 2. Step 5 is a simple method to find a proper gain K_ψ satisfying the condition 3 without checking it for all eigenvalues of L. For the step 5, we employ the result regarding 'disc margin' of discrete-time multi-agent system in [28], which is briefly summarized as follows.

For given $\epsilon > 0$, a discrete-time linear system

$$x(k+1) = Ax(k) + Bu(k) \tag{10}$$

is also stabilized by the control input $u(k) = -cK_\epsilon(k)$ where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, and P_ϵ is a solution to

$$\begin{aligned}
P_\epsilon &= A^T P_\epsilon A - A^T P_\epsilon B (B^T P_\epsilon B + R)^{-1} B^T P_\epsilon A + \epsilon I \\
K_\epsilon &:= (B^T P_\epsilon B + R)^{-1} B^T P_\epsilon A
\end{aligned}$$

and c is an arbitrary complex number included in a set

$$\bar{\Omega}_{i,\epsilon} = \left\{ \sigma + j\omega : \left[\sigma - \left(1 + \frac{r_i}{\delta}\right)\right]^2 + \omega^2 < \frac{r_i(r_i+\delta)}{\delta^2} \right\}.$$

Here, $R = \mathrm{diag}\{r_1, \cdots, r_q\}$, $\delta = \sigma_{\max}(B^T P_\epsilon B)$. Furthermore, as ϵ approaches to zero, the set $\bar{\Omega}_{i,\epsilon}$ becomes the set $H := \{z \in \mathbb{C} : \mathrm{Re}(z) > \frac{1}{2}\}$. Based upon this result, we claim following proposition.

Proposition 1: Suppose that the network is undirected and fixed, and λ_i is the i-th eigenvalue of the Laplacian L. For the sufficiently small $\epsilon > 0$, $A_\psi - \lambda_i B_\psi K_\psi$ for $i = 2, \cdots, N$ are all Schur where $K_\psi = \lambda_{\min}^{-1}(B_\psi^T P_\psi B_\psi + R)^{-1} B_\psi^T P_\psi A_\psi$ and P_ψ is a unique solution to

$$P_\psi = A_\psi^T P_\psi A_\psi - A_\psi^T P_\psi B_\psi (B_\psi^T P_\psi B_\psi + R)^{-1} B_\psi^T P_\psi A_\psi + \epsilon I.$$

Proof: Since the network is undirected (i.e., L is a symmetric real matrix), all eigenvalues of L are positive and lie on the real axis, so one can easily infer that the following relation holds true

$$1 \leq \frac{\lambda_2}{\lambda_{\min}} \leq \cdots \leq \frac{\lambda_N}{\lambda_{\min}} \leq \frac{\lambda_{\max}}{\lambda_{\min}}.$$

Meanwhile, as ϵ approaches zero, P_ψ converges to zero [29], and then $\delta_\psi = \sigma_{\max}(B_\psi^T P_\psi B_\psi)$ also converges to zero. Therefore, if ϵ is sufficiently small, δ_ψ obtained with ϵ satisfies the following equation

$$\left[\frac{\lambda_{\max}}{\lambda_{\min}} - \left(1 + \frac{R}{\delta_\psi}\right)\right]^2 < \frac{R(R+\delta_\psi)}{\delta_\psi^2}. \tag{11}$$

Once above inequality is satisfied with the sufficiently small ϵ, the inequality still holds true for each $\frac{\lambda_{N-1}}{\lambda_{\min}}, \cdots, \frac{\lambda_2}{\lambda_{\min}}$.

Since $\frac{\lambda_i}{\lambda_{\min}}$ for $i = 2, \cdots, N$ are included in the disc margin of K_ψ, and $A_\psi - B_\psi K_\psi$ is obviously Schur, $A_\psi - \lambda_i B_\psi K_\psi$ for $i = 2, \cdots, N$ are all Schur with sufficiently small ϵ. ■

By Proposition 1, the gain K_ψ selected according to step 5 guarantees the condition C.6 in Theorem 2.

D. Design Example for Time-varying Inputs

Consider a group of four agents estimates an average of sinusoidal inputs. The i-th input u^i is given by

$$\begin{aligned} x^i(k+1) &= e^{FT} x^i(k) \\ u^i(k) &= M^i x^i(k) \end{aligned} \tag{12}$$

where $x \in \mathbb{R}^2$ is the state, a step-time of $T = 0.04\pi$, $\omega = 1$, and

$$F = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, M = \begin{bmatrix} 0 & 4 \\ 0 & 7 \\ 0 & 3 \\ 0 & 6 \end{bmatrix}, x(0) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 1 \\ 1 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Here, M^i and $x^i(0)$ are the i-th row of M and $x(0)$, respectively.

The Laplacian matrix L of the four agents is given by

$$L = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 4 & -2 & -2 \\ -1 & -2 & 3 & 0 \\ -1 & -2 & 0 & 3 \end{bmatrix} \tag{13}$$

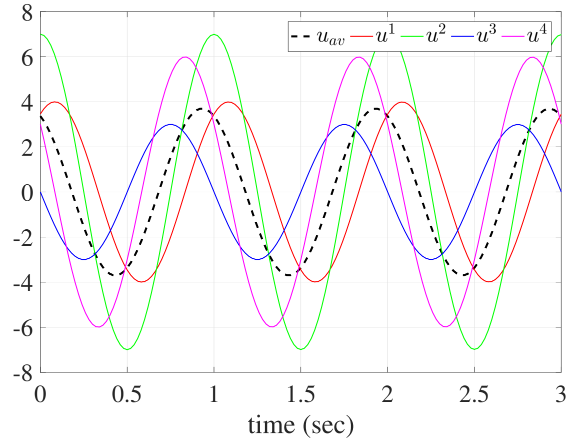
From the Estimator Design Procedure 1) to 4), we choose the parameters of A_ξ, C_ξ, and A_η as follows.

$$\begin{aligned} A_\xi &= \begin{bmatrix} 0 & 1 \\ -0.34 & 1 \end{bmatrix}, \quad A_\eta = \begin{bmatrix} 0 & 1 \\ -1 & 2\cos(h) \end{bmatrix}, \\ C_\xi &= \begin{bmatrix} -0.66 & -1 + 2\cos(h) \end{bmatrix} \end{aligned} \tag{14}$$

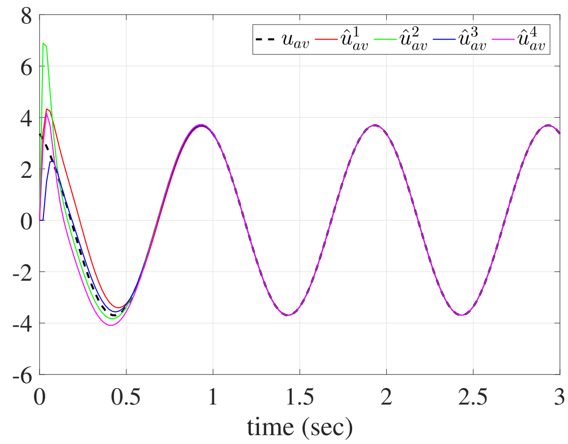
We find $K_\psi = \begin{bmatrix} -0.21 & 0.26 & -0.02 & 0.19 \end{bmatrix}$ with $\epsilon = 0.025$ to satisfy Estimator Design Procedure 5). Fig. 4 shows the simulation results to verify the proposed DAC estimator for time-varying input signals. Fig. 4a shows a sinusoidal input signals u^i of each agent and the average u_{av} of all input signals. Fig. 4b shows that each agent's estimate $\hat{u}^i_{\mathsf{av}}$ converges to the average of all inputs.

IV. Conclusion And Future Work

In this paper, we proposed a new dynamic average consensus estimator based on internal model principle for discrete-time case. The proposed estimator has a simpler structure than PI-Type estimator and the communication load is also reduced. The constructive and concrete design procedures are provided for both constant input case and time-varying case. The proposed estimator was validated by the example for constant and time-varying input case. For the future work, we plan to analyze the convergence of overall system for the Distributed Kalman Filter including the proposed distributed estimator. Also, we plan to conduct a lab-scale experiment of the Distributed Kalman Filter with real mobile robots.



(a) Each input u^i endowed to i-th agent and their average u_{av}



(b) The estimate of each agent $\hat{u}^i_{\mathsf{av}}$

Fig. 4: Simulation result of the proposed DAC estimator for time-varying inputs.

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