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Compositional construction of abstractions via relaxed small-gain conditions Part I: continuous case

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Abstract— **In this paper, we introduce a notion of so-called finite-step simulation functions for discrete-time control systems. In contrast to the existing notions of simulation functions, a finite-step simulation function does not need decay at each time step but after some finite numbers of steps. We show that the existence of such a function guarantees that the mismatch between output trajectories of the concrete and abstract systems lies within an appropriate bound. Using this relaxation, we develop a new type of small-gain conditions which are less conservative than those previously used for compositional construction of approximate abstractions of interconnected control systems. In particular, using finite-step simulation functions, it is possible to construct approximate abstractions, where stabilizability of each subsystem is not necessarily required. The effectiveness of our results is verified by an illustrative example.**

I. Introduction

Classical control theory often focuses on a specific set of qualitative properties of system behavior, such as stabilizing a system around an equilibrium point or ensuring that the system does not enter an unsafe operating condition. In contrast, in several applications such as traffic networks or autonomous transport, there is a wide spectrum of novel requirements, such as those expressed as temporal logic formulae [1], that are difficult to enforce by means of classical control design paradigms; see [2]–[4] for various examples of such requirements. A promising direction to address these sophisticated specifications is the use of formal methods, which were originally developed for specifying and verifying the correct behavior of software and hardware systems [5].

One of the major limitations of current techniques in formal methods is that in practice they can only be applied to control systems with small state space dimension. This is because the computational complexity of constructing discrete representations (known as *discrete* abstractions) usually grows exponentially with the state space dimension. Most formal synthesis techniques rely on the availability of such discrete abstractions. This issue can be addressed by decomposing the overall system into several lower-dimensional subsystems for which individual abstractions are computed. The methodology to obtain abstractions for the overall system via the interconnection of abstractions of the subsystems is called the compositional approach. Given a continuous-space system, which is already “structured” as in [6], such a “compositional” construction typically proceeds into two steps. In the first step, an abstraction itself a continuous system, *but* possibly with a lower dimension, is obtained for each subsystem (such an abstraction is referred to as a *continuous* abstraction). In the second step, a discrete abstraction for each continuous abstraction is computed. In both steps, certain conditions guarantee that the interconnection of the abstractions of the subsystems yields an abstraction of the overall system.

Both of the above steps are crucial to reduce the computational complexity. Therefore, in recent years this area has attracted particular attention; *e.g.* [7]–[12]. The current paper presents a compositional approach for the construction of *continuous* abstractions of a network of discrete-time systems. In the companion paper [13], we address discrete abstractions of the network. Therefore, the results of this paper provides a first pre-processing step to reduce the dimensionality of the network, before the construction of the discrete abstraction.

Among various techniques for computing abstractions, the notion of *approximate* (bi)simulation relation and its variants [5], [14] have received great attention as it relaxes the notion of exact (bi)simulation relation [15], [16] by allowing for the mismatch between outputs of concrete and abstract systems to be below an acceptable bound instead of being strictly zero. Approximate (bi)simulation relations can be quantitatively characterized by Lyapunov-like functions which are called simulation functions [14].

In this paper, a new notion of simulation functions, called finite-step simulation functions, is introduced. In contrast to the existing notions of simulation functions [8], [9], [14], a finite-step simulation function does not have to decay at each time step but after some finite numbers of steps. We show that the existence of such a function guarantees that the mismatch between output trajectories of the concrete and abstract systems lies within an appropriate bound. This relaxation leads to *less* conservative version of small-gain conditions which can be used to compositionally construct abstractions of a network of systems. In particular, in contrast with existing approaches [8], [9], [17] in which stability or stabilizability of each subsystem of the network is required, the relaxed small-gain conditions, proposed in this paper, do *not* rely on this assumption. We verify the effectiveness of our results via an illustrative example.

This paper is organized as follows: First, relevant notation is recalled in Section II. Then the problem formulation is stated in Section III. The relaxed small-gain conditions are made precise in Section IV. The compositional construction of abstractions for networks of linear systems is discussed in Section V. Section VI concludes the paper.

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II. Notation

In this paper, $\mathbb{R}_{\geq 0}(\mathbb{R}_{>0})$ and $\mathbb{N}_0(\mathbb{N})$ denote the nonnegative (positive) real numbers and the nonnegative (positive) integers, respectively. The vector space of real column vectors of length n is denoted by $\mathbb{R}^n$. The ith component of $v \in \mathbb{R}^n$ is denoted by v_i. For any $x \in \mathbb{R}^n$, $x^\top$ denotes its transpose. We write (x, y) to represent $[x^\top, y^\top]^\top$ for $x \in \mathbb{R}^n, y \in \mathbb{R}^p$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $|x|$. Given a real symmetric matrix $N \in \mathbb{R}^{n\times n}$, $N > 0$ ($N \geq 0$) denotes the property that $x^\top N x > 0$ ($x^\top N x \geq 0$) for all $x \neq 0$.

A function $\alpha\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, strictly increasing with $\alpha(0) = 0$. It is of class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if $\alpha \in \mathcal{K}$ and also $\alpha(s) \to \infty$ as $s \to \infty$. A continuous function $\beta\colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$), if for each $s \geq 0$, $\beta(\cdot, s) \in \mathcal{K}$, and for each $r > 0$, $\beta(r, \cdot)$ is decreasing with $\beta(r, s) \to 0$ as $s \to \infty$. Given a function $\varphi\colon \mathbb{N}_0 \to \mathbb{R}^m$, its sup-norm (possibly infinite) is denoted by $\|\varphi\| = \sup\{|\varphi(k)| : k \in \mathbb{N}_0\} \leq \infty$. The identity function is denoted by id. Composition of functions is denoted by the symbol $\circ$ and the i-times repeated composition of a function γ by γ^i. For functions $\alpha, \gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ we write $\alpha < \gamma$ if $\alpha(s) < \gamma(s)$ for all $s > 0$.

III. Problem Formulation

Consider the following nonlinear system

$$\Sigma : \begin{cases} x(k+1) &= g(x(k), u(k)), \\ y(k) &= h(x(k)), \end{cases} \tag{1}$$

where the state $x \in \mathcal{X} \subseteq \mathbb{R}^n$, the (external) input $u \in \mathcal{U} \subseteq \mathbb{R}^m$, and the output $y \in \mathcal{Y} \subseteq \mathbb{R}^q$. We assume that $g\colon \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ and $h\colon \mathcal{X} \to \mathcal{Y}$ are continuous. The symbol U denotes the set of sequences $u : \mathbb{N}_0 \to \mathcal{U}$. For any initial value $\xi \in \mathcal{X}$ and any input $u \in U$, $x(\cdot, \xi, u)$ denotes the corresponding solution to (1).

We refer to system (1) as the concrete system, that is the (complex) system that we actually want to control. On the other hand, controllers will be synthesized with the help of an abstract system, that is a system with a simpler, though less precise, description of the concrete system. This abstract system is given by

$$\Sigma^a : \begin{cases} \hat{x}(k+1) &= \hat{g}(\hat{x}(k), \hat{u}(k)), \\ \hat{y}(k) &= \hat{h}(\hat{x}(k)) \end{cases} \tag{2}$$

with $\hat{x} \in \hat{\mathcal{X}} \subseteq \mathbb{R}^{\hat{n}}$, $\hat{u} \in \hat{\mathcal{U}} \subseteq \mathbb{R}^{\hat{m}}$, and $\hat{y} \in \hat{\mathcal{Y}} \subseteq \mathbb{R}^{\hat{q}}$. We assume that $\hat{g}\colon \hat{\mathcal{X}} \times \hat{\mathcal{U}} \to \hat{\mathcal{X}}$ and $\hat{h}\colon \hat{\mathcal{X}} \to \hat{\mathcal{Y}}$ are continuous. The set of sequences $\hat{u} : \mathbb{N}_0 \to \hat{\mathcal{U}}$ is denoted by $\hat{U}$.

Simulation functions have been introduced in [18] as a quantitative generalization of the notion of simulation relations. Roughly speaking, a simulation function from (2) to (1) is a Lyapunov-like function defined over the Cartesian product of the state spaces explaining how a state trajectory of the abstract system can be related to a state trajectory of the concrete one such that the mismatch between their associated output trajectories remains within some computable bounds. Inspired by the notion of finite-step Lyapunov functions [19], here, we introduce a notion of so-called finite-step simulation function.

We require some notation to state the definition. Consider two systems Σ, Σ^a and a map $q : \mathcal{X} \times \hat{\mathcal{X}} \times \hat{\mathcal{U}} \to \mathcal{U}$. Given $\xi \in \mathcal{X}, \hat{\xi} \in \hat{\mathcal{X}}, \hat{u}(\cdot) \in U$, the iterative evaluation of $q(x(k), \hat{x}(k), \hat{u}(k)) \in \mathcal{U}$ determines a trajectory of Σ as follows. We let $u_q(0) := q(\xi, \hat{\xi}, \hat{u}(0))$ and define $x(1) := g(\xi, u_q(0))$, $\hat{x}(1) := \hat{g}(\hat{\xi}, \hat{u}(0))$. Then $u_q(k) := q(x(k), \hat{x}(k), \hat{u}(k))$, $x(k+1) := g(x(k), u_q(k))$, $\hat{x}(k+1) := \hat{g}(\hat{x}(k), \hat{u}(k))$, $k = 1, 2, \ldots$. In the sequel we abbreviate

$$x(\cdot, \xi, u_q) =: x(\cdot, \xi, q)$$

with the tacit understanding that the term on the right only makes sense, if already a trajectory of Σ^a is specified.

Definition 1: Consider the systems Σ, Σ^a. A pair of continuous functions $V : \hat{\mathcal{X}} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ and $u_V : \mathcal{X} \times \hat{\mathcal{X}} \times \hat{\mathcal{U}} \to \mathcal{U}$ is called finite-step simulation pair, if there exist $M \in \mathbb{N}$, $\underline{\alpha} \in \mathcal{K}_\infty$, $\alpha \in \mathcal{K}_\infty$ with $\alpha < \mathrm{id}$ and $\gamma \in \mathcal{K} \cup \{0\}$ such that for every $\xi \in \mathcal{X}$, $\hat{\xi} \in \hat{\mathcal{X}}$, $\hat{u}(\cdot) \in \hat{U}$ the following conditions hold

$$\underline{\alpha}(|\hat{h}(\hat{\xi}) - h(\xi)|) \leq V(\hat{\xi}, \xi), \tag{3a}$$

$$V(\hat{x}(M, \hat{\xi}, \hat{u}), x(M, \xi, u_V)) \leq \max\left\{\alpha(V(\hat{\xi}, \xi)), \gamma(\|\hat{u}\|)\right\}. \tag{3b}$$

In this case the function V is called a finite-step simulation function from Σ^a to Σ associated with the interface function u_V. For the case $M = 1$ we drop the term finite-step and instead speak of a classic simulation function. □

Remark 2: i) In [18] the decay condition is given in the so-called implication form, while here (3b) is formulated in the max form;
ii) a finite-step simulation function does not have to decay at each step, but after a *fixed* finite number of steps. Therefore, every classic simulation function is a finite-step simulation function, but the converse does not necessarily hold;
iii) a finite-step simulation function is a classic simulation function for M-sampled systems that are obtained by evaluation of the solutions of Σ and Σ^a at the times jM, $j \in \mathbb{N}_0$. The dynamics of the sampled system, which we call $\mathcal{L}^M(\Sigma)$, can be described as follows. Given $u \in U$, $j \in \mathbb{N}_0$ define $u^M(j) := \big(u(j), u(j+1), \ldots, u(j+M-1)\big) \in \mathcal{U}^M$. Then for $\xi \in \mathcal{X}, u \in \mathcal{U}^M$, we define $g^M(\xi, u)$ iteratively by setting $g^1(\xi, u^1) := g(\xi, u_1)$ and $g^{k+1}(\xi, u^{k+1}) := g\big(g^k(x, u^k), u_{k+1}\big)$, $k = 1, \ldots, M-1$. In that way, we obtain the system $\mathcal{L}^M(\Sigma)$ defined by

$$\mathcal{L}^M(\Sigma) : \begin{cases} x^M(k+1) &= g^M(x^M(k), u^M(k)), \\ y^M(k) &= \tilde{h}(x^M(k)) \end{cases} \tag{4}$$

where $x^M \in \mathcal{X}$, $u^M \in \mathcal{U}^M$, $y^M \in \mathcal{Y}$. With the same arguments as in [20, Remark 4.2], one can show that a function V is a finite-step simulation function from Σ^a to Σ if and only if it is a classic simulation function from $\mathcal{L}^M(\Sigma^a)$ to $\mathcal{L}^M(\tilde{\Sigma})$. □

Before proceeding to our first result, we define the concept of $\mathcal{K}$-boundedness, which is required in this work.

Definition 3: Consider systems Σ and Σ^a. Let $W : \mathcal{X} \times \hat{\mathcal{X}} \to \mathbb{R}_{\geq 0}$ be a continuous and nonnegative function. Also, let $q : \mathcal{X} \times \hat{\mathcal{X}} \times \hat{\mathcal{U}} \to \mathcal{U}$ be given. The pair $(g, \hat{g})$ is said to be $\mathcal{K}$-bounded on $(\mathcal{X}, \hat{\mathcal{X}}, \hat{\mathcal{U}})$ with respect to $(W, q, |\cdot|)$ if there exist $\kappa_1, \kappa_2 \in \mathcal{K}$ such that for all $\xi \in \mathcal{X}$, $\hat{\xi} \in \hat{\mathcal{X}}$, and

$\hat{\mu} \in \hat{\mathcal{U}}$

$$W\Big(g\big(\xi, q(\xi, \hat{\xi}, \hat{\mu})\big), \hat{g}(\hat{\xi}, \hat{\mu})\Big) \leq \kappa_1\big(W(\xi, \hat{\xi})\big) + \kappa_2(|\hat{\mu}|). \quad (5)$$

$\square$

Theorem 4: Consider systems Σ and Σ^a. Let (V, u_V) be a finite-step simulation pair from Σ^a to Σ. Assume that the pair $(g, \hat{g})$ is $\mathcal{K}$-bounded on $(\mathcal{X}, \hat{\mathcal{X}}, \hat{\mathcal{U}})$ with respect to $(V, u_V, |\cdot|)$. Then there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $\xi \in \mathcal{X}$, all $\hat{\xi} \in \hat{\mathcal{X}}$, all $\hat{u} \in \hat{\mathcal{U}}$ and all $k \in \mathbb{N}_0$ we have

$$\big|\hat{y}(k, \hat{\xi}, \hat{u}) - y(k, \xi, u_V)\big| \leq \max\{\beta(V(\hat{\xi}, \xi), k), \gamma(\|\hat{u}\|)\}. \quad (6)$$

$\square$

The proof of Theorem 4 follows arguments similar to those in [21, Proposition 1]. It is presented for the sake of completeness. The strategy is to first show that the existence of a max-form finite-step simulation pair (V, u_V) implies the existence of a dissipative-form finite-step simulation function associated with the same interface function u_V. This fact is the content of the following lemma whose proof is not given due to space reasons.

Lemma 5: Let (V, u_V) be a finite-step simulation pair from Σ^a to Σ. Then there exist a continuous map $V_d : \hat{\mathcal{X}} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ and $\underline{\alpha}_d, \alpha_d \in \mathcal{K}_\infty$ with $\alpha_d < \mathrm{id}$ and $\gamma_d \in \mathcal{K} \cup \{0\}$ such that for every $\xi \in \mathcal{X}$, $\hat{\xi} \in \hat{\mathcal{X}}$, $\hat{u} \in \hat{U}$, the following conditions hold

$$\underline{\alpha}_d(|\hat{h}(\hat{\xi}) - h(\xi)|) \leq V_d(\hat{\xi}, \xi), \quad (7a)$$

$$V_d(\hat{x}(M, \hat{\xi}, \hat{u}), x(M, \xi, u_V)) - V_d(\hat{\xi}, \xi) \leq -\alpha_d(V_d(\hat{\xi}, \xi)) + \gamma_d(\|\hat{u}\|). \quad (7b)$$

$\square$

Proof: (of Theorem 4) Let the functions V_d, $\underline{\alpha}_d$, α_d and γ_d be as given by Lemma 5. For ease of notation, we denote $V := V_d$, $\underline{\alpha} := \underline{\alpha}_d$, $\alpha := \alpha_d$ and $\gamma := \gamma_d$. Choose any $\rho \in \mathcal{K}_\infty$ with $\rho < \mathrm{id}$. We aim to show that V and γ satisfy he claim and to this end we need to construct appropriate β. Without loss of generality we can assume that $\gamma \in \mathcal{K}_\infty$ (if not we can just enlarge it). If $\hat{u}$ is unbounded the claim in (6) is then true for any $\mathcal{KL}$ function β so that we may from now on assume that we only consider finite sequences in $\hat{U}$.

Adding and subtracting $\rho \circ \alpha(V(\hat{\xi}, \xi))$ on the right hand-side of (7b) yields for all $\xi \in \mathcal{X}$, $\hat{\xi} \in \hat{\mathcal{X}}$, $\hat{u} \in \hat{U}$ that

$$V(\hat{x}(M, \hat{\xi}, \hat{u}), x(M, \xi, u_V)) - V(\hat{\xi}, \xi) \leq -(\mathrm{id} - \rho) \circ \alpha(V(\hat{\xi}, \xi)) + \gamma(\|\hat{u}\|) - \rho \circ \alpha(V(\hat{\xi}, \xi)). \quad (8)$$

Fix $\hat{u} \in \hat{U}$, let $b := \alpha^{-1} \circ \rho^{-1} \circ \gamma(\|\hat{u}\|)$, and define $S := \big\{(\hat{\xi}, \xi) \in (\hat{\mathcal{X}}, \mathcal{X}) \colon V(\hat{\xi}, \xi) > b\big\}$. Also, denote $S^c := (\hat{\mathcal{X}} \times \mathcal{X}) \backslash S$. It follows from (8) that for all $(\hat{\xi}, \xi) \in S$

$$V(\hat{x}(M, \hat{\xi}, \hat{u}), x(M, \xi, u_V)) - V(\hat{\xi}, \xi) \leq -(\mathrm{id} - \rho) \circ \alpha(V(\hat{\xi}, \xi)).$$

For $(\hat{\xi}, \xi) \in S$ let $k_0 := k_0(\hat{\xi}, \xi, u) := \min\big\{k \in \mathbb{N}_0 \colon (\hat{x}(k, \hat{\xi}, \hat{u}), x(k, \xi, u)) \in S^c\big\} \leq \infty$. By [21, Lemma 3], there exist $\beta \in \mathcal{KL}$ and a real number $P > 1$ such that for all $(\hat{\xi}, \xi) \in S$

$$V(\hat{x}(k, \hat{\xi}, \hat{u}), x(k, \xi, u_V)) \leq \max_{i \in \{0, \ldots, M-1\}} P^i \beta\big(V(\hat{x}(i, \hat{\xi}, \hat{u}), x(i, \xi, u_V)), k\big) \quad (9)$$

for all $k \in \{0, \ldots, k_0 - 1\}$. Recalling the $\mathcal{K}$-boundedness condition (5)[1], for each $k < k_0$ we get

$$\begin{aligned}
&V\big(\hat{x}(k+1, \hat{\xi}, \hat{u}), x(k+1, \xi, u_V)\big) \leq \\
&\quad \kappa_1\Big(V\big(\hat{x}(k, \hat{\xi}, \hat{u}), x(k, \xi, u_V)\big)\Big) + \kappa_2(\|\hat{u}\|) \\
&= \kappa_1\Big(V\big(\hat{x}(k, \hat{\xi}, \hat{u}), x(k, \xi, u_V)\big)\Big) + \kappa_2 \circ \gamma^{-1} \circ \rho \circ \alpha(b) \\
&\leq \big(\kappa_1 + \kappa_2 \circ \gamma^{-1} \circ \rho \circ \alpha\big)\big(V(\hat{x}(k, \hat{\xi}, \hat{u}), x(k, \xi, u_V))\big) \\
&=: \tilde{\kappa}\big(V(\hat{x}(k, \hat{\xi}, \hat{u}), x(k, \xi, u_V))\big). \qquad (10)
\end{aligned}$$

By repeatedly applying (9) to (10), there exists $\overline{\beta} \in \mathcal{KL}$ such that for all $k \in \{0, \ldots, k_0 - 1\}$

$$V(\hat{x}(k, \hat{\xi}, \hat{u}), x(k, \xi, u_V)) \leq \overline{\beta}(V(\hat{\xi}, \xi), k). \quad (11)$$

Now we claim that there exists $\tilde{\sigma} \in \mathcal{K}$ such that for all $k \geq k_0$

$$V(\hat{x}(k, \hat{\xi}, \hat{u}), x(k, \xi, u_V)) \leq \tilde{\sigma}(b). \quad (12)$$

To establish this claim, note that

$$\begin{aligned}
&V(\hat{x}(k_0 + M, \hat{\xi}, \hat{u}), x(k_0 + M, \xi, u_V)) \\
&\quad \leq -(\mathrm{id} - \rho) \circ \alpha\big(V(\hat{x}(k_0, \hat{\xi}, \hat{u}), x(k_0, \xi, u_V))\big) + \gamma(\|\hat{u}\|) \\
&\qquad + (\mathrm{id} - \rho \circ \alpha)\big(V(\hat{x}(k_0, \hat{\xi}, \hat{u}), x(k_0, \xi, u_V))\big) \\
&\quad \leq -(\mathrm{id} - \rho) \circ \alpha\big(V(\hat{x}(k_0, \hat{\xi}, \hat{u}), x(k_0, \xi, u_V))\big) + \gamma(\|\hat{u}\|) \\
&\qquad + (\mathrm{id} - \rho \circ \alpha)(b) \\
&\quad = -(\mathrm{id} - \rho) \circ \alpha\big(V(\hat{x}(k_0, \hat{\xi}, \hat{u}), x(k_0, \xi, u_V))\big) + b \\
&\quad \leq b. \qquad (13)
\end{aligned}$$

This shows that (12) holds for all $k = k_0 + lM$, $l \in \mathbb{N}_0$. Moreover, assume that $V\big(\hat{x}(k, \hat{\xi}, \hat{u}), x(k, \xi, u_V)\big) > b$ for all $k \geq k_0 + i + lM$, $i = 1, \ldots, M-1$, $l \in \mathbb{N}_0$.[2] So we have

$$\begin{aligned}
&V(\hat{x}(k_0 + 1 + M, \hat{\xi}, \hat{u}), x(k_0 + 1 + M, \xi, u_V)) \\
&\quad - V\big(\hat{x}(k_0 + 1, \hat{\xi}, \hat{u}), x(k_0 + 1, \xi, u_V)\big) \leq \\
&\qquad -(\mathrm{id} - \rho) \circ \alpha\big(\hat{x}(k_0 + 1, \hat{\xi}, \hat{u}), x(k_0 + 1, \xi, u_V)\big).
\end{aligned}$$

By reapplication of [21, Lemma 3], we have

$$V(\hat{x}(k, \hat{\xi}, \hat{u}), x(k, \xi, u_V)) \leq \quad (14)$$

$$\max_{i \in \{1, \ldots, M-1\}} P^{i-1} \beta\big(V(\hat{x}(k_0 + i, \hat{\xi}, \hat{u}), x(k_0 + i, \xi, u_V)), k - k_0 - 1\big) \quad (15)$$

for all $k \geq k_0 + i + lM$, $i = 1, \ldots, M-1$, and $l \in \mathbb{N}_0$. Take $i = 1$ in (15). Also, let $\beta_0(s) := \beta(s, 0)$ for all $s \in \mathbb{R}_{\geq 0}$. Exploiting (5) and the monotonicity of β, κ_1 and κ_2 yield

$$\begin{aligned}
&V(\hat{x}(k, \hat{\xi}, \hat{u}), x(k, \xi, u_V)) \\
&\leq \beta_0\Big(\kappa_1\big(V\big(\hat{x}(k_0, \hat{\xi}, \hat{u}), x(k_0, \xi, u_V)\big)\big) + \kappa_2(\|\hat{u}\|)\Big) \\
&\leq \beta_0 \circ (2\kappa_1)\big(V\big(\hat{x}(k_0, \hat{\xi}, \hat{u}), x(k_0, \xi, u_V)\big) + \beta_0 \circ (2\kappa_2)(\|\hat{u}\|) \\
&\leq \beta_0 \circ (2\kappa_1)(b) + \beta_0 \circ (2\kappa_2) \circ \gamma^{-1} \circ \rho \circ \alpha(b)
\end{aligned}$$

[1] By possibly rescaling κ_i, for $i = 1, \ldots, 4$. However, for simplicity, we keep the same notation.

[2] This is the worst case. Otherwise, there exists $\hat{k} \in \{k_0 + 1, \ldots, M-1\}$ such that $V(x(k + M, \xi, u)) \leq b$ for all $k = \hat{k} + lM$, $l = 0, 1, 2, \ldots$.

$$\leq \max\left\{\beta_0 \circ (2\kappa_1)(b) + \beta_0 \circ (2\kappa_2) \circ \gamma^{-1} \circ \rho \circ \alpha(b), b\right\}$$

for all $k \geq k_0 + 1 + lM$, $l \in \mathbb{N}_0$. By repeating this procedure for $i \in \{2, \dots, M-1\}$ it follows that there exists $\tilde{\sigma} \in \mathcal{K}$ such that

$$V(\hat{x}(k, \hat{\xi}, \hat{u}), x(k, \xi, u_V)) \leq \overline{\sigma}(b), \tag{16}$$

for all $k \geq k_0 + i + lM$, $i = 1, \dots, M-1$, $l \in \mathbb{N}_0$. Combining (13) and (16) establishes (12) with $\tilde{\sigma} := \max\{\overline{\sigma}, \mathrm{id}\}$.

Given (3a), (11), and (12), we have

$$\underline{\alpha}(|\hat{h}(\hat{x}(k, \hat{\xi}, \hat{u})) - h(x(k, \xi, u_V))|) \leq \max\left\{\overline{\beta}(V(\hat{\xi}, \xi), k), \tilde{\sigma} \circ \rho^{-1} \circ \alpha^{-1} \circ \gamma(\|u\|)\right\}$$

for all $k \in \mathbb{N}_0$. The monotonicity of $\underline{\alpha}$ yields that for all $k \in \mathbb{N}_0$

$$|\hat{y}(k, \hat{\xi}, \hat{u}) - y(k, \xi, u_V)| \leq \max\left\{\tilde{\beta}(V(\hat{\xi}, \xi), k), \sigma(\|u\|)\right\},$$

where $\tilde{\beta}(\cdot, \cdot) := \underline{\alpha}^{-1} \circ \overline{\beta}(\cdot, \cdot)$ and $\sigma(\cdot) := \underline{\alpha}^{-1} \circ \tilde{\sigma} \circ \rho^{-1} \circ \alpha^{-1} \circ \gamma(\cdot)$. This completes the proof. ■

IV. Interconnected Control Systems

Assume that the concrete system (1) can be decomposed into ℓ interconnected subsystems

$$\Sigma_i : \begin{cases} x_i(k+1) = g_i(x_i(k), w_i(k), u_i(k)), \\ y_i(k) = h_i(x_i(k)), \end{cases} \tag{17}$$

with states $x_i \in \mathcal{X}_i \subseteq \mathbb{R}^{n_i}$ and with partitioned inputs and outputs

$$w_i = (w_{i1}, \dots, w_{i(i-1)}, w_{i(i+1)}, \dots, w_{i\ell}) \in \mathbb{R}^{p_i},$$
$$y_i = (y_{i1}, \dots, y_{i\ell}) \in \mathbb{R}^{q_i},$$

with $w_{ij} \in \mathcal{W}_{ij} \subseteq \mathbb{R}^{p_{ij}}$, $y_{ij} \in \mathcal{Y}_{ij} \subseteq \mathbb{R}^{q_{ij}}$, external input $u_i \in \mathcal{U}_i \subseteq \mathbb{R}^{m_i}$, and output function $h_i(x) := (h_{i1}(x_i), \dots, h_{i\ell}(x_i))$.

We interpret the outputs y_{ii} as external outputs, whereas the outputs y_{ij} with $i \neq j$ are internal outputs which are used to define the interconnected systems. In particular, we assume that $w_{ij} = y_{ji}$ for all $i, j \in \{1, \dots, \ell\}$, $i \neq j$. Note that $h_{ij} \equiv 0$ if there is no connection from the ith subsystem to the jth subsystem.

We assume that each transition map $g_i \colon \mathbb{R}^{n_i} \times \mathbb{R}^{p_i} \times \mathbb{R}^{m_i} \to \mathcal{X}_i$ is continuous. Defining $n = n_1 + \ldots + n_\ell$, $x := (x_1, \dots, x_\ell)$, $\mathcal{X} := \mathcal{X}_1 \times \ldots \times X_\ell$, $\mathcal{U} := \mathcal{U}_1 \times \ldots \times U_\ell$, $g := (g_1, \dots, g_\ell) : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ and $h := (h_{11}, h_{22}, \dots, h_{\ell\ell})$, we call Σ the concrete composite system of the subsystems Σ_i.

Given the concrete composite system of the subsystems (17), assume that there exist corresponding abstractions of each subsystem (17) with the following dynamics

$$\Sigma_i^a : \begin{cases} \hat{x}_i(k+1) = \hat{g}_i(\hat{x}_i(k), \hat{w}_i(k), \hat{u}_i(k)), \\ \hat{y}_i(k) = \hat{h}_i(\hat{x}_i(k)), \end{cases} \tag{18}$$

with appropriate dimensions and the similar structure as those in (17).

To verify the distance between the output trajectories h of (17) and $\hat{h}$ of (18), we make the following assumption.

Assumption 6: Let $M \in \mathbb{N}$ be given. Suppose that for each Σ_i with $i \in \{1, \dots, \ell\}$, there exists an M-step simulation pair (W_i, u_{W_i}), where $W_i \colon \hat{\mathcal{X}}_i \times \mathcal{X}_i \to \mathbb{R}_{\geq 0}$ and $u_{W_i} : \mathcal{X} \times \hat{\mathcal{X}} \times \hat{\mathcal{U}} \to \mathcal{U}_i$ such that the following holds

(i) There exist functions $\overline{\alpha}_i, \underline{\alpha}_i \in \mathcal{K}_\infty$ such that for all $(\hat{\xi}_i, \xi_i) \in \hat{\mathcal{X}}_i \times \mathcal{X}_i$

$$\underline{\alpha}_i(|\hat{h}_i(\xi_i) - h_i(\xi_i)|) \leq W_i(\hat{\xi}_i, \xi_i). \tag{19}$$

(ii) For all $\hat{\xi} \in \hat{\mathcal{X}}$, $\xi \in \mathcal{X}$ and $\hat{u} \in \hat{\mathcal{U}}$ there exist $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$, $j = 1, \dots, \ell$, and $\gamma_{iu} \in \mathcal{K} \cup \{0\}$ such that

$$W_i(\hat{x}_i(M, \hat{\xi}, \hat{u}), x_i(M, \xi, u_W)) \leq \max\left\{\left\{\max_{j \in \{1, \dots, \ell\}} \gamma_{ij}(W_j(\hat{\xi}_j, \xi_j))\right\}, \gamma_{iu}(\|\hat{u}\|)\right\}. \tag{20}$$

□

We note that in (20) the notation $\hat{x}_i(M, \hat{\xi}, \hat{u})$ denotes the ith component of the solution of the interconnected abstraction system determined by the initial condition $\hat{\xi} \in \hat{X}$ and the input $\hat{u} \in \hat{U}$. The interpretation of $x_i(M, \xi, u_W)$ is then along the same lines.

Theorem 7: Let Assumption 6 hold. Assume the functions γ_{ij} given in (20) satisfy

$$\gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \cdots \circ \gamma_{i_{r-1} i_r} \circ \gamma_{i_r i_1} < \mathrm{id} \tag{21}$$

for all sequences $(i_1, \dots, i_r) \in \{1, \dots, \ell\}^r$ and $r = 1, \dots, \ell$. Then there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K} \cup \{0\}$ such that for all $\xi \in \mathcal{X}$, all $\hat{\xi} \in \hat{\mathcal{X}}$ and all $\hat{u} \in \hat{\mathcal{U}}$ the output trajectory of the abstract composite system of subsystems Σ_i^a and that of the subsystems Σ_i satisfy (6). □

The proof follows arguments similar to those in [22]. It is not presented for the sake of space constraints.

In contrast to the existing results relying on classic small-gain theory, *e.g.*, [8], [9], the above small-gain theorem does not require that the mismatch between the output trajectories of each subsystem and its corresponding abstraction satisfies the estimate (6) when considered in isolation. In fact, the mismatch between the trajectories of some subsystems and their corresponding abstractions may grow in the open-loop sense, as long as in the local interconnection with other subsystems they satisfy estimates (19) and (20). In other words, this result allows for neighboring subsystems to exercise a stabilizing effect on a given local subsystem. This interesting capability is illustrated by the following example.

Example 8: Consider the following system composed of two subsystems

$$\begin{aligned} x_1(k+1) &= A_1 x_1(k) + D_1 w_1(k) + B_1 u_1(k), \\ x_2(k+1) &= A_2 x_2(k) + D_2 w_2(k) + B_2 u_2(k), \end{aligned} \tag{22}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ and the matrices A_i, B_i, D_i for $i = 1, 2$ have appropriate dimensions. The existing methodologies for compositional abstraction of interconnected linear time-invariant systems [8], [9] need the pair (A_i, B_i) to be stabilizable for each subsystem. If the requirement is not met, the existing approaches are not applicable. The individual stabilizability requirement can be removed by letting neighboring subsystems have a stabilizing effect on a given local system. Once the stabilizing effect is evident, we use small-gain theory to construct the abstraction. This is the idea behind the small-gain condition proposed in this paper.

To verify the stabilizing effect, we need to look at the solution of the system in future times. For simplicity, take

$u_i \equiv 0$ for $i = 1, 2$, $w_1 = x_2$ and $w_2 = x_1$ in (22). Let the matrices A_i and D_i for $i = 1, 2$ be given by

$$A_1 = \begin{bmatrix} 1.01 & 0 & 0.28 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.4 \\ 0.3 \\ -0.01 \end{bmatrix},$$
$$A_2 = -0.23, \quad D_2 = \begin{bmatrix} -0.73 \\ 0.56 \\ -0.19 \end{bmatrix}^\top.$$

Note that the first subsystem is individually unstable. Now by looking at the solution two steps ahead, we have

$$\begin{aligned} x_1(k+2) &= \tilde{A}_1 x_1(k) + \tilde{D}_1 w_1(k), \\ x_2(k+2) &= \tilde{A}_2 x_2(k) + \tilde{D}_2 w_2(k), \end{aligned} \tag{23}$$

where

$$\tilde{A}_1 = \begin{bmatrix} 0.73 & 0.22 & 0.20 \\ -0.22 & 0.16 & -0.05 \\ 0 & 0 & 0 \end{bmatrix}, \tilde{D}_1 = \begin{bmatrix} 0.3 \\ -0.07 \\ 0 \end{bmatrix},$$
$$\tilde{A}_2 = -0.07, \quad \tilde{D}_2 = \begin{bmatrix} -0.56 \\ -0.13 \\ -0.16 \end{bmatrix}^\top.$$

The eigenvalues of both $\tilde{A}_1$ and $\tilde{A}_2$ lie within the unit disk. This motivates us to construct an abstraction of the original system (22) from the auxiliary system (23) using the relaxed small-gain conditions in Theorem 7. □

V. Abstractions for Linear Systems

Motivated by Example 8 and item iii) of Remark 2, this section uses the small-gain conditions given by Theorem 7 to develop a procedure to *compositionally* compute abstractions for a network of linear subsystems. In particular, we focus on the construction of abstractions whose output trajectories are close to those of the concrete system at times jM, $j \in \mathbb{N}_0$ for some $M \in \mathbb{N}$. Toward this end, we provide geometric conditions under which an abstraction for one *single* subsystem is constructed. This can be applied to each subsystem. Then we verify that the interconnection of concrete subsystems and their corresponding abstractions satisfy the small-gain conditions in Theorem 7.

Consider the following network of ℓ linear subsystems

$$\Sigma_i : \begin{cases} x_i(k+1) &= A_i x_i(k) + D_i w_i(k) + B_i u_i(k), \\ y_i(k) &= C_i x_i(k), \end{cases} \tag{24}$$

where $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, $C_i \in \mathbb{R}^{q_i \times n_i}$ and $D_i \in \mathbb{R}^{n_i \times p_i}$. Note that w_i and u_i are partitioned into subvectors as those for (17) and we have $w_{ij} = y_{ji}$ for all $i, j \in \{1, \dots, \ell\}$, $i \neq j$.

Let $M \in \mathbb{N}$ be given. For each i, assume that $u_i(k) = 0$ for all $k \neq jM - 1$, $j \in \mathbb{N}$. For ease of representation, suppose that $w_{ij} = x_j$ for all $i, j \in \{1, \dots, \ell\}$, $i \neq j$. In that way, $\mathcal{L}^M(\Sigma_i)$ is, with abuse of notation, given by

$$\mathcal{L}^M(\Sigma_i) : \begin{cases} \tilde{x}_i(k+1) &= \tilde{A}_i \tilde{x}_i(k) + \tilde{D}_i \tilde{w}_i(k) + \tilde{B}_i \tilde{u}_i(k), \\ \tilde{y}_i(k) &= \tilde{C}_i \tilde{x}_i(k), \end{cases} \tag{25}$$

where $\tilde{x}_i(k) \in \mathbb{R}^{n_i}$ with $\tilde{x}_i(0) = x_i(0)$, $\tilde{u}_i(k) := u_i((k+1)M - 1)$, $\tilde{w}_i(k) := w_i(k)$, $\tilde{B}_i := B_i$ and $\tilde{C}_i := C_i$. Following the same arguments as those for the computation of g^M in (4), the matrices $\tilde{A}_i$ and $\tilde{D}_i$ can be iteratively computed from the A_j's and D_j's, $j \in \{1, \dots, \ell\}$. We use the tuple $\mathcal{L}^M(\Sigma_i) = (\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{D}_i)$ to denote the ith subsystem (25).

While the pairs (A_i, B_i) may not be necessarily stabilizable, we assume that the pairs $(\tilde{A}_i, \tilde{B}_i)$ are stabilizable as discussed in Example 8. Therefore, we can use the geometric conditions developed in [9], [18], [23] to construct abstraction of each subsystem (25), denoted by $\tilde{\Sigma}_i^a = (\hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{D}_i)$. To do so, take the following simulation function candidate from $\tilde{\Sigma}_i^a$ to $\tilde{\Sigma}_i$

$$W_i(\hat{\xi}_i, \tilde{\xi}_i) = (\tilde{\xi}_i - P_i \hat{\xi}_i)^\top N_i (\tilde{\xi}_i - P_i \hat{\xi}_i), \tag{26}$$

where $N_i > 0$, and the associated linear interface $\tilde{u}_{W_i}$ is given by

$$u_{W_i} = K_i(\tilde{\xi}_i - P_i \hat{\xi}_i) + Q_i \hat{\xi}_i + R_i \hat{u}_i + S_i \hat{w}_i \tag{27}$$

where K_i, P_i, Q_i, R_i, and S_i are matrices of appropriate dimensions. Note that W_i is a simulation function from $\mathcal{L}^M(\Sigma)_i^a$ to $\tilde{\Sigma}_i$ while it is a finite-step simulation function from Σ_i^a to Σ_i (see the item iii) of Remark 2).

The matrices $\hat{A}_i$, Q_i, $\hat{D}_i$, S_i, $\hat{C}_i$, N_i and K_i are computed by the following algorithm:

(i) Calculate $\hat{A}_i$ and Q_i satisfying the following equation
$$\tilde{A}_i P_i = P_i \hat{A}_i - \tilde{B}_i Q_i. \tag{28}$$

(ii) Calculate $\hat{D}_i$ and S_i satisfying the following equation
$$\tilde{D}_i = P_i \hat{D}_i - \tilde{B}_i S_i. \tag{29}$$

(iii) Obtain $\hat{C}_i$ satisfying
$$\hat{C}_i = \tilde{C}_i P_i. \tag{30}$$

(iv) Let $\epsilon_i > 0$ and $0 < \lambda_i < \frac{1}{3}$ be given. Find $N_i > 0$ and K_i such that the following inequalities hold
$$\tilde{C}_i^\top \tilde{C}_i \leq N_i \tag{31}$$
$$(1 + \epsilon_i)(\tilde{A}_i + \tilde{B}_i K_i)^\top N_i (\tilde{A}_i + \tilde{B}_i K_i) \leq \lambda_i N_i. \tag{32}$$

Note that there are no condition on $\hat{B}_i$ and R_i. In fact, it can be chosen arbitrarily. For instance, one can choose $\hat{B}_i$ as an *identity* with an appropriate dimension [23], which makes the abstract system fully actuated and, hence, the synthesis problem over it much easier. Also, as suggested in [18], one can choose $R_i = (B_i^\top N_i B_i)^{-1} B_i^\top N_i P_i \hat{B}_i$.

Condition (31) guarantees that W_i in (26) satisfies (19) for some appropriate $\underline{\alpha} \in \mathcal{K}_\infty$. Conditions (32)–(30) ensure the decay rate (20). Necessary and sufficient conditions under which conditions (31)–(30) hold are given in [9], [18].

The following theorem summarizes the results of this section.

Theorem 9: Consider $\mathcal{L}^M(\Sigma_i) = (\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{D}_i)$ and $\tilde{\Sigma}_i^a = (\hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{D}_i)$. Suppose that there exist the matrices N_i, K_i, P_i, Q_i, and S_i satisfying (31)–(30). Then W_i defined by (26) is a simulation function from $\tilde{\Sigma}_i^a$ to $\tilde{\Sigma}_i$. □

To verify the effectiveness of Theorem 9, consider system (22) with matrices A_1, A_2, D_1 and D_2 as in Example 8 and

$$\begin{aligned} B_1 &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\top, & C_1 &= \begin{bmatrix} 0.1 & 0 & 0 \end{bmatrix}, \\ B_2 &= 1, & C_2 &= 0.1. \end{aligned} \tag{33}$$

It is easy to see that the pair (A_1, B_1) is not stabilizable. Taking $M = 2$, $\tilde{\Sigma}_i = (\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{D}_i)$ with $\tilde{A}_i$ and $\tilde{D}_i$ as in Example 8 and $\tilde{B}_i$ and $\tilde{C}_i$ as in (33). In that case, the pairs $(\tilde{A}_i, \tilde{B}_i)$ for $i = 1, 2$ are stabilizable. Pick $K_1 = (-1.59, -0.69, 0)$ and $K_2 = 0.07$. Then we have that

$$N_1 = \begin{bmatrix} 0.96 & -0.13 & 0 \\ -0.13 & 1.22 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}, \quad N_2 = 0.01,$$

which satisfy (32) with $\epsilon_1 = \epsilon_2 = 1$. Choosing the matrices P_1 and P_2 as

$$P_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top, \qquad P_2 = 1,$$

equations (28)–(30) are satisfied by

$$\begin{array}{llll} \hat{A}_1 = 0.95, & \hat{A}_2 = -0.07, & \hat{D}_1 = 0.3, & \hat{D}_2 = -0.1, \\ \hat{C}_1 = 0.1, & \hat{C}_2 = 0.1, & Q_1 = 1.01, & Q_2 = 0, \\ S_1 = 0.37, & S_2 = 0.46. & & \end{array}$$

Finally, by (32) and the fact that $4\left|\sqrt{N_1}D_1\right|\left|\sqrt{N_2}D_2\right| < 1$, one can easily verify that the small-gain conditions (21) hold.

VI. Conclusions

This paper has introduced a notion of so-called finite-step simulation functions. In contrast with a classic simulation function, a finite-step simulation function does not have to decay at each time step but after some finite numbers of steps. We have shown that the existence of such a function guarantees that the mismatch between output trajectories of the concrete and abstract systems lies within appropriate bounds. Using finite-step simulation functions, we have developed relaxed small-gain conditions to compositionally construct abstractions of a network of systems. The main advantage of such small-gain conditions over existing ones in [8], [9] is that no assumption on the stabilizability of the subsystems of the network is required.

The results of this paper can only be viewed as a first step, as necessarily we are interested in discrete abstraction. The question of how to construct discrete abstractions within the framework presented here is the topic of the companion paper [13].

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