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Finite-time convergence results in Model Predictive Control

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***Abstract*— Asymptotic stability (convergence and ϵ-δ stability) of invariant sets under model predictive control (MPC) strategies have been extensively studied in the last decades. Lyapunov theory is in some sense the common denominator of the different forms to achieve such results. However, the meaningful problem of the finite-time convergence (for a given fixed control horizon) has not received much attention in the literature (with some remarkable exceptions). In this work a novel set-based MPC that ensures finite-time convergence in a natural way is presented. The contractivity and non-empty interior conditions of the target set, the consideration of an appropriate input set and the continuity of the dynamic model are the main hypothesis to be made. An upper bound for the convergence time is also provided.**

I. INTRODUCTION

Model Predictive Control (MPC) is probably the most employed advanced control technique in process industries, mainly because of (i) the explicit consideration of the model, (ii) the optimal character of the calculated control moves, and (iii) its ability to handle, easily and effectively, hard constraints on control and states.

MPC theoretical background has been widely investigated in the last two decades, showing that MPC is a control technique capable to provide stability, robustness, constraint satisfaction and tractable computation for linear and nonlinear systems [1]. Researchers achieved a consensus on that Lyapunov theory [2] is a suitable framework to provide asymptotic stability of a system controlled by an MPC [3]. To this aim, different stabilizing formulations appeared in literature: *MPC with terminal equality constraint* [4], where stability is guaranteed by imposing a terminal constraint of the form $x(k+N|k)=0$; *MPC with terminal cost* [5], where stability is achieved by incorporating into the cost function a term that penalizes the terminal state; or *MPC with terminal inequality constraint* [6], which extends the terminal constraint to a neighborhood of the origin, on the form $x(k+N|k) \in \Omega$ where Ω is a positive invariant terminal set.

The concept of Positive Invariance is closely related to Lyapunov theory [7], [8]. Set invariance theory focuses the attention on the analysis of stabilizable regions, generalizing this way the stability of a system at the origin or equilibrium point. This theory turned out to be very useful for the analysis of dynamical systems subject to constraints.

However, ensuring asymptotic stability of the closed-loop system may be not enough, given that some specific application may require finite-time convergence. For instance, in [9] a MPC suitable for a closed-loop re-identification is proposed, in which the system is excited (for identification purpose) once it enters a suitable invariant set; in [10] this approach is applied to a polymerization reactor with the intention of re-identifying the model; in [11] the conservatism of the last method is considerably reduced by proposing probabilistic invariant sets as final sets. However, in each of these works, the re-identification begins once the system enters the excitation set, so, ensuring asymptotic convergence is not enough to guarantee that the re-identification of the model actually begins. Other meaningful applications where finite-time convergence is needed, are those requiring two sequential control stages, in which the second one starts once the first reaches a spatial objective.

The majority of the results on the stability of MPC that can be found in literature, however, are mainly devoted to prove asymptotic (or exponential) stability, while it is difficult to find results on finite-time convergence of predictive controllers, when the control horizon is given (and fixed) as it is usually the case. There are few remarkable exceptions: in [12] it is shown that, if the MPC stage cost function is bounded from below by a $\mathscr{K}$-function, then finite-time convergence to a certain terminal set can be ensured. In [13] - in the context of min-max MPC- the authors propose an unconventional cost function based on the idea of distance to a set, and show that finite-time convergence can be ensured by means of an assumption similar to the one of [12].

The main goal of this work is to present a set-based MPC formulation, following the idea of [9], which however is capable to guarantee finite-time convergence to the invariant terminal set, under mild assumption on this set, such as contractivity and non-empty interior.

The note is organized as follows. Section II states the problem and presents some preliminary definitions. The proposed MPC scheme and finite-time convergence analysis are presented in Section III and Section IV, respectively. Finally, some conclusions of this work are given in Section V.

A. Notation

The natural set is defined by $\mathbb{N} = \{1, 2, 3, \ldots\}$, the integers set is represented by $\mathbb{I}$, and the integers between N and M, by $\mathbb{I}_{N:M} \doteq \{N, N+1, \ldots, M\}$. Given any real number $x \in \mathbb{R}$, the floor of x is defined by $\lfloor x \rfloor \doteq$

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$\max\{n \in \mathbb{I} : n \leq x\}$. The euclidean distance between to points $x,\ y$ on $\mathbb{R}^n$ is represented by $d(x,y)$. The open ball with center in $x \in \mathbb{R}^n$ and radius $\varepsilon > 0$ is defined as $\mathscr{B}_\varepsilon(x) \doteq \{y \in \mathbb{R}^n : d(x,y) < \varepsilon\}$. Let be a constant $\gamma \in \mathbb{R}$ and a set $\Omega \subset \mathbb{R}^n$; the scaled set is defined as $\gamma\Omega \doteq \{\gamma x : x \in \Omega\}$. x is an interior point of Ω if the there exist $\varepsilon > 0$ such that the open ball $\mathscr{B}_\varepsilon(x) \subset \Omega$. The interior of Ω is the set of all interior points and it is denoted by Ω°. If Ω is closed, the boundary of Ω is denoted by $\partial\Omega$, and it is defined as $\partial\Omega \doteq \Omega \setminus \Omega^\circ$. The distance from $x \in \mathbb{R}^n$ to (the fix set) Ω is defined as $d_\Omega(x) \doteq \inf\{d(x,y) :\ y \in \Omega\}$. $d_\Omega(\cdot)$ is a convex and continuous function, and $d_\Omega(x) \geq 0$ for all $x \in \mathbb{R}^n$, while $d_\Omega(x) = 0$ if and only if $x \in \Omega$.

II. Problem statement and preliminary definitions

Consider a discrete time system described by a time-invariant model

$$x(k+1) = f(x(k), u(k)), \quad x(0) = x_0 \tag{1}$$

where $x(k) \in X \subset \mathbb{R}^n$ is the system state at the k–th sample time, x_0 is the initial state and $u(k) \in U \subset \mathbb{R}^m$ is the current control input. The set X is closed, the set U is compact and both contain the origin in their interior. The function $f : X \times U \to X$ is continuous on $Z = X \times U$ and $f(0,0) = 0$.

The next definitions and properties will be referred to system (1), and the corresponding state and input constraints.

Definition 1: **(One-step controllable set)** Given two sets $\Omega \subset X$ and $\Delta \subset U$, the one step controllable set to Ω, $\mathscr{Q}(\Omega,\Delta)$, corresponding to Δ, is the set of all $x \in X$ for which there exists an $u \in \Delta$ such that $f(x,u) \in \Omega$, i.e.,

$$\mathscr{Q}(\Omega,\Delta) \doteq \{x \in X : \exists\ u \in \Delta \text{ such that } f(x,u) \in \Omega\}.$$

This is the set of states in X for which an admissible control does exist in Δ, such that the system can be steered to Ω in one time step. Furthermore, the concept can be generalized to the N-step controllable set $\mathscr{Q}^N(\Omega,\Delta)$, for any $N \in \mathbb{N}$, by applying the above definition iteratively, i.e., $\mathscr{Q}^n(\Omega,\Delta) \doteq \mathscr{Q}(\mathscr{Q}^{n-1}(\Omega,\Delta),\Delta)$, for $n = 1,\cdots,N$, and $\mathscr{Q}^0(\Omega,\Delta) \doteq \Omega$.

Definition 2: **(γ-control invariant set, γ-CIS)** Given $\gamma \in (0,1]$, $\Omega \subseteq X$ is a γ-control invariant set if $x \in \Omega$ implies that $f(x,u) \in \gamma\Omega$ for some $u \in U$. Associated to Ω, is the corresponding input set $\Pi(\Omega) \doteq \{u \in U :\ \exists\ x \in \Omega \text{ such that } f(x,u) \in \gamma\Omega\}$.

This set is such that once the system enters it, then there exists an admissible control input (in $\Pi(\Omega)$) that is able to keep the system inside the set. When $\gamma = 1$, Ω is simply a control invariant set.

It is known from [14] that every $\gamma-$CIS, Ω, is such that $\Omega \subseteq \mathscr{Q}(\Omega,U)$. For our own interest, let us consider the following extension.

Assumption 1: Let $\Omega \subset \mathbb{R}^n$ be a closed and convex γ-CIS, with $\gamma \in (0,1]$, with the corresponding input set $\Pi(\Omega) \subseteq U$. Then, $\Omega \subseteq \mathscr{Q}(\Omega,\Pi(\Omega))^\circ$, where $\mathscr{Q}(\Omega,\Pi(\Omega))^\circ$ is the interior of the one step controllable set to Ω corresponding to $\Pi(\Omega)$.

Property 2 on Appendix shows sufficient conditions for which Assumption 1 is fulfilled.

III. MPC scheme for finite-time convergence with fixed horizon control

In this Section, a general MPC formulation is presented. For a given (fixed) horizon $N \in \mathbb{N}$, and a (compact and convex) Ω that contains the origin in its interior, consider the following (set dependent) cost function:

$$V_N(x,\Omega;\mathbf{u}) \doteq \sum_{j=0}^{N-1} L(x(j),u(j);\Omega) \tag{2}$$

where $L(\cdot) \geq 0$ is a general stage cost that depends in some sense on Ω (it will assume different forms according to each particular MPC formulation) and $\mathbf{u} \doteq \{u(0),\cdots,u(N-1)\}$. Set Ω is called target set and traditionally it is desired that it be a stable and attractive set of the closed-loop system, but in this work is the set where the finite-time convergence wants to be ensured to.

Let $\mathscr{U}_N(x)$ denote the set of admissible control sequences $\mathbf{u}$ satisfying the state and control constraints and the terminal constraints $x(N) \in \Omega$, when the initial state is x. That is, $\mathscr{U}_N(x) =$

$$\{\mathbf{u} \in U^N \mid x(j) \in X, u(j) \in U,\ j \in \mathbb{I}_{0:N-1},\ x(N) \in \Omega\}$$

By Definition 1, for all $i \geq 0$, $\mathscr{Q}^i(\Omega,U)$ denotes the set of states x such that $\mathscr{U}_i(x) \neq \emptyset$. At each time instant k, the MPC control law is derived from the solution of the following optimization problem:

$$\begin{aligned}&\mathscr{P}_N(x,\Omega) : \\ &\qquad V_N^0(x,\Omega) \doteq\ \min\{V_N(x,\Omega;\mathbf{u}) \mid \mathbf{u} \in \mathscr{U}_N(x)\}\end{aligned} \tag{3}$$

where Ω and the initial state $x \in X$ are the optimization parameters and the sequence $\mathbf{u}$ is the optimization variable. The optimal control sequence is given by $\mathbf{u}^0 = \{u^0(0;x), u^0(1;x), \ldots, u^0(N-1;x)\}$, while the corresponding optimal state trajectory is given by

$$\mathbf{x}^0(x) = \{x^0(0;x), x^0(1;x), \ldots, x^0(N;x)\}. \tag{4}$$

where $x^0(0;x) = x$.

By definition of $\mathscr{P}_N(x,\Omega)$, $x^0(N;x) \in \Omega$. The control law, derived from the application of a receding horizon policy, is given by $\kappa_{MPC}(x) = u^0(0;x)$, where $u^0(0;x)$ is the first element of the solution sequence $\mathbf{u}^0(x)$. This way, the closed-loop system under the MPC control law is described as:

$$x(k+1) = f(x(k), \kappa_{MPC}(x(k))). \tag{5}$$

and the optimal cost function is given by:

$$V_N^0(x,\Omega) \doteq V_N(x,\Omega;\mathbf{u}^0(x)) \tag{6}$$

IV. Convergence analysis

A. Some previous results

In [12] and [15] the finite-time convergence to Ω is ensured by imposing the following conditions to the stage cost: (i) there exists a $\mathscr{K}$-function $\ell(\cdot)$ such that $L(x,u;\Omega) \geq \ell(\|(x,u)\|)$ for all $x \notin \Omega$ and for all $u \in U$; and (ii) $L(x,h_L(x);\Omega) = 0$ for all $x \in \Omega$, where $h_L(\cdot)$ is a local control law imposed once the state enters the target set Ω. Furthermore, Ω is assumed to be an invariant set for $x(k+1) = f(x(k), h_L(x(k)))$, which establishes an undesired dependence of Ω on the arbitrary control law $h_L(x)$.

However, given that Ω is assumed to contain the origin in its (non empty) interior, assumptions (i) and (ii) imply that function $L(\cdot)$ is discontinuous on the boundary of Ω, which is a strong assumption that may produce some problems.

The way this formulation ensures finite-time convergence is summarized as follows. By usual procedures in MPC stability theory, the optimal cost function, $V_N^0(x,\Omega)$, is shown to satisfy

$$V_N^0(x(k+1),\Omega) - V_N^0(x(k),\Omega) \leq -L(x(k),\kappa_{MPC}(x(k));\Omega), \quad \forall x(k) \notin \Omega,$$

Given that for all $x \notin \Omega$ there exists an $r > 0$ such that $\|x\| > r$, by assumption (i), it follows that

$$L(x,u;\Omega) \geq \ell(\|(x,u)\|) \geq \ell(\|x\|) \geq \ell(r), \ \forall x \notin \Omega,$$

and then

$$V_N^0(x(k+1),\Omega) - V_N^0(x(k),\Omega) \leq -\ell(r). \tag{7}$$

This means that at each step there is a cost decay of at least $\ell(r)$, for $x \notin \Omega$, which is much stronger than the usual cost decay depending on the distance to the set Ω. So, the finite-time convergence to Ω is achieved.

The main drawbacks of the proposal in [12] and [15] is clearly the strong assumption (i), which leads to the discontinuity of the stage cost. In [13], it is stated that it is not convenient to use such a discontinuous stage cost, because it is a major obstacle for implementation, when using standard solvers for linear, quadratic, semi-definite or other smooth, convex nonlinear programming problems. To overcome this drawback, [13] proposes - in the context of robust linear MPC - the following stage cost:

$$L(x,u;\Omega) \doteq d_\Omega(x) + d(x,h_L(x)),$$

where $d_\Omega(x) = \min_{y\in\Omega} \|Q(x-y)\|_p$ and $d(x,h_L(x)) = \|R(u-h_L(x))\|_p$, with $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ being weighting matrices and $\|\|_p$ being a particular norm. The system is given by $x(k+1) = Ax(k) + Bu(k) + w(k)$ with $w \in W$, and the local control law $h_L(x)$ is a fixed linear feedback gain $K \in \mathbb{R}^{m\times n}$. The set Ω is here an invariant set for $x(k+1) = Ax(k) + BKx(k) + w(k)$ and for all possible disturbance $w(k) \in W$, and again, it depends on K, as it is usual in the dual MPC context.

This cost is continuous, and in fact, it does not meet condition (i). However, the way this formulation ensures finite-time convergence is by assuming the following control law:

$$\begin{cases} \kappa_{MPC}(x) & \text{if } x \in X \setminus \mathscr{O}_\infty, \\ h_L(x) & \text{if } x \in \mathscr{O}_\infty, \end{cases}$$

where $\mathscr{O}_\infty$ is the the maximal disturbance invariant set (MDIS). If it is assumed that $\Omega \subset \mathscr{O}_\infty^\circ$, the minimal disturbance invariant set (mDIS) $\mathscr{F}_\infty \subset \mathscr{O}_\infty$ is asymptotically stable for the system in closed loop with the control law defined above. In addition, if $\mathscr{F}_\infty \subset \Omega^\circ$, there is a finite time convergence to the set Ω. This way, the real target set is given by $\mathscr{F}_\infty$, and so, the finite-time convergence to Ω is trivially achieved by the asymptotic convergence to $\mathscr{F}_\infty$ and the assumption $\mathscr{F}_\infty \subset \Omega^\circ$.

Remark 1: As a particular case, if the nominal scenario ($W = \{0\}$) is considered, then set mDIS is given by $\mathscr{F}_\infty = \{0\}$, and classical attractivity of the origin is achieved. So, any set including the origin in its interior - as the proposed Ω - will trivially be finite-time attractive.

Other works devoted to achieve finite-time convergence in the context of dual-mode model predictive control are [6] and [16]. However, that is a different case since it considers a variable horizon MPC, which constitutes a different scenario to the one considered in this work.

The idea in the next subsection is to show that if set Ω is selected to be a $\gamma-$CIS for the open-loop system and its associated input set $\Pi(\Omega)$ is also considered in the stage cost, then a finite-time convergence to Ω can be ensured, with no further assumption on the stage cost. This result is potentially useful in many applications where two sequential optimization stages naturally arise.

B. Main result

Consider the following set-based stage cost:

$$L(x,u;\Omega) \doteq d_\Omega(x) + d_{\Pi(\Omega)}(u) \tag{8}$$

where the function $d_\Omega : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the distance from the state to the set Ω, with a metric on $\mathbb{R}^n$ and $d_{\Pi(\Omega)} : \mathbb{R}^m \longrightarrow \mathbb{R}$ is the distance from the control to the set $\Pi(\Omega)$, with a metric on $\mathbb{R}^m$. Set Ω is assumed to be a $\gamma-$CIS for the open-loop system (1), which fulfills Assumption 1, while $\Pi(\Omega)$ is the corresponding input set.

Remark 2: Note that according to this formulation, the control objective is considered to be reached once the system enters Ω, and no further implicit objectives are considered. However, the interesting point is that the system will not be in open loop, since the controller will not allow the state to jump outside Ω.

The next Lemma establishes the asymptotic convergence of the closed-loop derived from Problem (3), to Ω, when the stage cost is given by (8)[1].

[1]As it is known, convergence is the main condition to ensure stability by means of Lyapunov classical methods.

Lemma 1: Let $x = x(0)$ in the $N-$step controllable set to Ω, $\mathscr{Q}^N(\Omega, U)$, corresponding to U. Consider the MPC formulation $\mathscr{P}_N(x, \Omega)$, (3), with the stage cost (8). Then, the closed-loop system $x(k+1) = f(x(k), \kappa_{MPC}(x(k)))$ satisfies

$$V_N^0(x(k{+}1), \Omega) - V_N^0(x(k), \Omega) \leq -d_\Omega(x(k)) - d_{\Pi(\Omega)}(u(k)),$$

for all $k \geq 0$, and so $\lim_{k\to\infty} d_\Omega(x(k)) = 0$ and $\lim_{k\to\infty} d_{\Pi(\Omega)}(u(k)) = 0$.

Proof: Let $x \in \mathscr{Q}^N(\Omega, U)$, at a given time k. Suppose that the optimal cost function is given by

$$V_N^0(x, \Omega) = \sum_{j=0}^{N-1} d_\Omega(x^0(j; x)) + d_{\Pi(\Omega)}(u^0(j; x)), \tag{9}$$

where $u^0(j; x)$, $j \in \mathbb{I}_{0:N-1}$, is the optimal solution and $x^0(j; x)$, $j \in \mathbb{I}_{0:N-1}$, the corresponding state trajectory.

Let the successor state of x under the closed-loop system be given by $x^+ = x^0(1; x)$. One feasible solution to Problem $\mathscr{P}_N(x^+, \Omega)$ at time $k+1$ is given by the sequence $\hat{\mathbf{u}} = \{u^0(1; x), \cdots, u^0(N-1; x), \hat{u}\}$, where $\hat{u}$ is a control action in $\Pi(\Omega)$, such that $\hat{x} \doteq f(x^0(N), \hat{u}) \in \Omega$ (this input does exist because of the control invariant condition of Ω). Therefore, the corresponding feasible cost function of Problem $\mathscr{P}_N(x^+, \Omega)$ can be written as

$$\begin{aligned} V_N(x^+, \Omega; \hat{\mathbf{u}}) &= V_N^0(x, \Omega) + d_\Omega(\hat{x}) + d_{\Pi(\Omega)}(\hat{u}) \\ &- d_\Omega(x) - d_{\Pi(\Omega)}(u^0(0; x)), \end{aligned}$$

where $d_\Omega(\hat{x}) + d_{\Pi(\Omega)}(\hat{u}) = 0$. Furthermore, by optimality it is derived that $V_N^0(x^+, \Omega) \leq V_N(x^+, \Omega; \hat{\mathbf{u}})$, which implies that

$$V_N^0(x^+, \Omega) - V_N^0(x, \Omega) \quad \leq \quad -d_\Omega(x) - d_{\Pi(\Omega)}(u^0(0; x)).$$

In other words, for any time $k \in \mathbb{N}$, the optimal cost function satisfies

$$V_N^0(x(k+1), \Omega) - V_N^0(x(k), \Omega) \leq -d_\Omega(x(k)) - d_{\Pi(\Omega)}(u(k)),$$

and given that $V_N^0(\cdot)$ is positive definite, this implies that $\lim_{k\to\infty} d_\Omega(x(k)) = 0$ and $\lim_{k\to\infty} d_{\Pi(\Omega)}(u(k)) = 0$. ■

Before presenting the main result of the work, let us introduce the following lemma that is fulfilled by the proposed set-based stage cost (8), but it is not necessarily true for a different stage cost.

Lemma 2: Consider the MPC formulation $\mathscr{P}_N(x, \Omega)$, (3), with the stage cost (8). Consider also $\mathscr{Q}(\Omega, \Pi(\Omega))$, which is the one step controllable set to Ω, corresponding to the input set $\Pi(\Omega)$. Then, if $x(0) \in \mathscr{Q}(\Omega, \Pi(\Omega))$, $x(1) \doteq f(x(0), \kappa_{MPC}(x(0))) \in \Omega$.

Proof: Since $x(0) = x \in \mathscr{Q}(\Omega, \Pi(\Omega))$, there is a control action $u(0) \in \Pi(\Omega)$ such that

$$x(1) = f(x(0), u(0)) \in \Omega,$$

From the $\gamma-$invariance of the target set Ω, there exist control actions $u(k) \in \Pi(\Omega)$, $k = 1, \ldots, N-1$, for which $x(k) \in \gamma\Omega \subseteq \Omega$, for $k = 2, \ldots, N-1$. The use of such a control sequence produces the cost

$$\begin{aligned} V_N(x, \Omega; \mathbf{u}) &= d_\Omega(x(0)) + \underbrace{d_{\Pi(\Omega)}(u(0))}_{=0} \\ &+ \underbrace{\sum_{j=1}^{N-1} (d_\Omega(x(j)) + d_{\Pi(\Omega)}(u(j)))}_{=0} \\ &= d_\Omega(x(0)) \end{aligned}$$

While any control action that leaves $x(1)$ outside Ω would produce a cost $d_\Omega(x(1)) > 0$. Thus, the MPC will drive the state to the target set in one step. ■

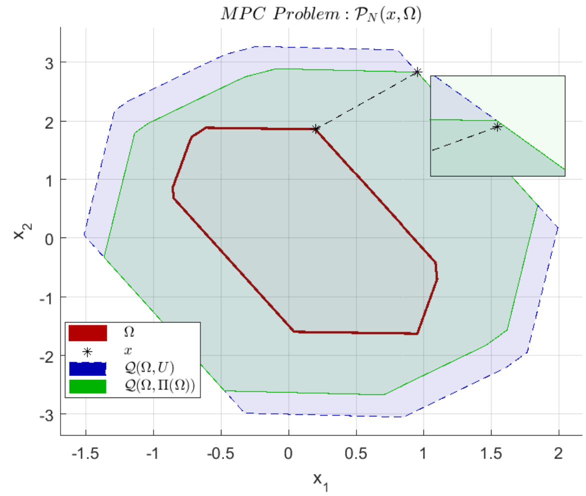

Fig. 1. Closed-loop state evolution, starting at $\mathscr{Q}(\Omega, \Pi(\Omega))$. When the initial state $x(0) \in \mathscr{Q}(\Omega, \Pi(\Omega))$ the controller steers the state inside Ω in one time step.

The result of the above Lemma is basically due to the fact that any $u \in \Pi(\Omega)$ does not add positive cost to the stage cost (8), and it is enough to reach the set Ω from $\mathscr{Q}(\Omega, \Pi(\Omega))$. Furthermore, this is possible due to the continuity of system (1) and the implicit contractivity (Assumption 1) of the target set Ω.

Figure 1 and 2 show the closed-loop evolution corresponding to the MPC Problem $\mathscr{P}(\Omega, x)$, (3), with the stage cost (8), for two given initial states. The controller is stopped when the state enters the target set Ω. Note that when the initial state $x(0) \in \mathscr{Q}(\Omega, \Pi(\Omega))$ (Fig. 1) the controller steers the state inside Ω in one time step, as Lemma 2 claimed. On the other hand, if the initial state is in $x(0) \in \mathscr{Q}(\Omega, U) \setminus \mathscr{Q}(\Omega, \Pi(\Omega))$ - even when it is possible to reach it in only one time step, by the one step set definition - the MPC controller reaches Ω in two time steps (Fig. 2). The model and MPC parameters of this scenario simulations are described in the appendix.

Next, the main theorem of this note is presented, where both, the finite-time convergence and an upper bound for such a time are established.

Theorem 1: Consider the MPC formulation $\mathscr{P}_N(x, \Omega)$, (3), with the stage cost (8). Then, Ω is locally reached in

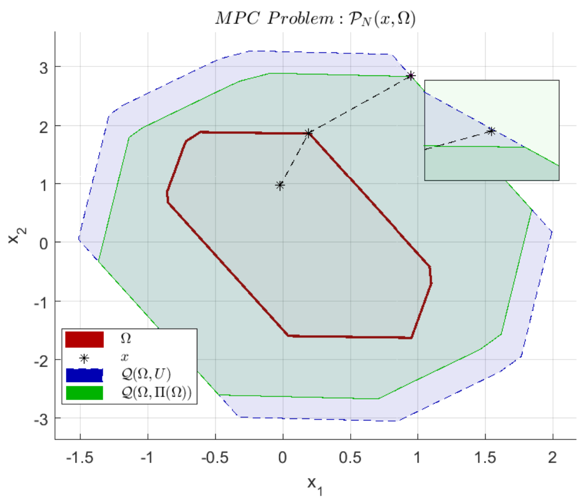

Fig. 2. Closed-loop state evolution, starting at $\mathcal{Q}(\Omega, U)$. The state does not reach the target set Ω in one step, since its initial condition lays outside $\mathcal{Q}(\Omega, \Pi(\Omega))$.

finite-time by the system $x(t+1) = f(x(t), \kappa_{MPC}(x(t)))$, with $x = x(0)$ in the $N-$step controllable set $\mathcal{Q}^N(\Omega, U)$, corresponding to U. Even more , the system reaches Ω in at most $\lfloor K \rfloor$ steps[2], with

$$K = \frac{V_N^0(x, \Omega)}{\min\limits_{y \in \partial \mathcal{Q}(\Omega, \Pi(\Omega))} d_\Omega(y)} + 1 \tag{10}$$

Proof: Let $x = x(0) \in \mathcal{Q}^N(\Omega, U) \setminus \mathcal{Q}(\Omega, \Pi(\Omega))$, where $\mathcal{Q}(\Omega, \Pi(\Omega))$ is the one step controllable set corresponding to $\Pi(\Omega)$. Take $m \in \mathbb{N}$ such that $m > K - 1 = \frac{V_N^0(x(0), \Omega)}{\min\limits_{y \in \partial \mathcal{Q}(\Omega, \Pi(\Omega))} d_\Omega(y)}$, and suppose that $x(k) \notin \mathcal{Q}(\Omega, \Pi(\Omega))$ for all $k = 1, 2, ..., m$, then

$$-d_\Omega(x(k)) \leq - \min_{y \in \partial \mathcal{Q}(\Omega, \Pi(\Omega))} d_\Omega(y) < 0, \quad k = 0, 1, ..., m.$$

Moreover, from Lemma 1

$$V_N^0(x(k+1), \Omega) - V_N^0(x(k), \Omega) \leq -d_\Omega(x(k)).$$

Then

$$\begin{aligned} V_N^0(x(k+1), \Omega) &- V_N^0(x(k), \Omega) \\ &\leq - \min_{y \in \partial \mathcal{Q}(\Omega, \Pi(\Omega))} d_\Omega(y) \\ &< 0, \end{aligned}$$

for all $k = 0, 1, ..., m$, and it follows that

$$V_N^0(x(m), \Omega) - V_N^0(x(0), \Omega) \leq -m \min_{y \in \partial \mathcal{Q}(\Omega, \Pi(\Omega))} d_\Omega(y).$$

So,

$$\begin{aligned} V_N^0(x(m), \Omega) &\leq -m \min_{y \in \partial \mathcal{Q}(\Omega, \Pi(\Omega))} d_\Omega(y) + V_N^0(x(0), \Omega) \\ &< 0, \end{aligned}$$

[2]Note that K is well defined since by Assumption 1 $\Omega \subset \mathcal{Q}(\Omega, \Pi(\Omega))^\circ$. This means that $\min\limits_{y \in \partial \mathcal{Q}(\Omega, \Pi(\Omega))} d_\Omega(y) \neq 0$.

which is a contradiction. So, $x(k)$ must be inside $\mathcal{Q}(\Omega, \Pi(\Omega))$ for some $k \leq K - 1$. Then, Lemma 2 ensures that $x(k) \in \Omega$ for some $k \leq K$, which concludes the proof. ■

V. Conclusion

In this work, a new and simple MPC formulation has been presented, which ensures finite-time convergence to a given state space region. The necessary conditions to achieve such a convergence are studied in detail and an upper bound for the convergence time is provided. In spite of its simplicity, this result increases the applicability of the two-stage MPC controllers, in which a second stage begins to control the system once it reaches - under the first stage controller - some particular objective region. Given the particular characteristics of the proposed MPC, a future extension to the robust case, i.e., robust finite-time convergence to a robust invariant set, seems to be plausible.

APPENDIX

A. How to accomplish Assumption 1

Before presenting Property 2, which provides a particular case for which Assumption 1 is fulfilled, we need to introduce the next property:

Property 1: ([17]) Let $\Omega \subset \mathbb{R}^n$ a set and $\mathcal{Q}^N(\Omega, U)$ the N step controllable set to Ω for system (1). If Ω is closed then $\mathcal{Q}^N(\Omega)$ is closed.

Proof: Consider a sequence $\{x_k\}_{k \in \mathbb{N}} \subset \mathcal{Q}(\Omega, U)$ converging to $\bar{x}$. Proving that $\mathcal{Q}(\Omega, U)$ is closed implies that $\bar{x} \in \mathcal{Q}(\Omega, U)$. Indeed, by definition of $\mathcal{Q}(\Omega, U)$ there exists a corresponding sequence $\{u_k\} \subset U$ such that

$$f(x_k, u_k) \in \Omega, \tag{11}$$

By the compactness of U, $\{u_k\}$ admits a subsequence $\{u'_k\} \subset U$ which converges to $\bar{u} \in U$. Let us consider the subsequence $\{x'_k\} \subset \mathcal{Q}(\Omega, U)$ corresponding to the subsequence $\{u'_k\}$. Clearly $x'_k \to \bar{x}$.

Therefore, since $x'_k \to \bar{x}$, and $u'_k \to \bar{u}$, and by continuity of the function f we have that

$$f(x'_k, u'_k) \to f(\bar{x}, \bar{u}),$$

Since the sequence $\{f(x'_k, u'_k)\} \subset \Omega$, and Ω is closed, $f(\bar{x}, \bar{u}) \in \Omega$, which means that $\bar{x} \in \mathcal{Q}(\Omega, U)$, and so $\mathcal{Q}(\Omega, U)$ is closed. The fact that $\mathcal{Q}^N(\Omega, U)$ is closed follows by induction, which concludes the proof. ■

The following property represents a stronger geometric property than the one presented in [14], and also shows a particular case that fulfills Assumption 1.

Property 2: Let $\Omega \subset X^\circ$ be a compact and convex γ-CIS, with $\gamma < 1$ and with the origin as an interior point, with the corresponding input set $\Pi(\Omega) \subseteq U$. Then, $\Omega \subseteq \mathcal{Q}(\Omega, \Pi(\Omega))^\circ$, where $\mathcal{Q}(\Omega, \Pi(\Omega))^\circ$ is the interior of the one step controllable set to Ω, corresponding to $\Pi(\Omega)$.

Proof: By the result in [14] we know that $\Omega \subseteq \mathcal{Q}(\Omega, \Pi(\Omega))$. It remains to show that every point of Ω is an interior point of $\mathcal{Q}(\Omega, \Pi(\Omega))$. Consider an $\bar{x} \in \Omega$. Given that by assumption, Ω is a $\gamma-$CIS, then there exists an $\bar{u} \in \Pi(\Omega)$

such that $f(\bar{x},\bar{u}) \in \gamma\Omega$. Furthermore, since $\gamma < 1$, and Ω contains the origin in its interior, then $\gamma\Omega \subset \Omega^\circ$. Therefore,

$$\varepsilon \doteq \inf\{d(y,z):\ y \in \partial\Omega,\ z \in \gamma\Omega > 0\} \tag{12}$$

is such that $\varepsilon > 0$, being $\partial\Omega$ the boundary of Ω.

Since f is continuous at $\bar{x}$, there is $\delta > 0$ such that for all $x \in \mathscr{B}_\delta(\bar{x})$ it holds

$$d(f(x,\bar{u}), f(\bar{x},\bar{u})) < \varepsilon, \tag{13}$$

Since $f(\bar{x},\bar{u}) \in \gamma\Omega$ and $d(f(x,\bar{u}), f(\bar{x},\bar{u})) < \varepsilon$, (due to (12)), and given that Ω is compact and convex, it follows that $f(x,\bar{u}) \in \Omega$, and then $x \in \mathscr{Q}(\Omega,\Pi(\Omega))$, since $\mathscr{Q}(\Omega,\Pi(\Omega))$ is closed by Property 1. So $\mathscr{B}_\delta(\bar{x}) \subset \mathscr{Q}(\Omega,\Pi(\Omega))$, i.e. $\Omega \subseteq \mathscr{Q}(\Omega,\Pi(\Omega))^\circ$. ■

B. Simulation parameters

For the simulation results plotted in Figure 1 and 2, the simulated system is similar to the one presented in [9]. It is a second order stable linear system,

$$x(k+1) = Ax(k) + Bu(k),$$

with

$$\begin{aligned} A &= \begin{bmatrix} 0.7476 & -0.4984 \\ 0.0356 & 1.0680 \end{bmatrix}, \\ B &= \begin{bmatrix} 0.3 \\ -0.4 \end{bmatrix}. \end{aligned} \tag{14}$$

The constraints of the system are given by $X = \left\{x \in \mathbb{R}^2 : \|x\|_\infty \le 10\right\}$ and $U = \{u \in \mathbb{R} : \|u\|_\infty \le 4\}$. The horizon of the MPC controller $\mathscr{P}(\Omega,x)$, (3), with the stage cost (8), is given by $N = 7$, being Ω a $\gamma-$CIS with $\gamma = 0.65$, and its corresponding input set $\Pi(\Omega) = \{u \in \mathbb{R} : \|u\|_\infty \le 3\}$.

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