2018 26th Mediterranean Conference on Control and Automation (MED)
Zadar, Croatia, June 19-22, 2018

Steering-based controllers for stabilizing lean angles in two-wheeled vehicles

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***Abstract*— This study considers the lean angle stability problems in steering controlled tilting vehicles. Despite the fact that the considered nonlinear nonminimum phase model ignores some dynamic characteristics of the real system (we ignore wheel sleeping and wheel inertia), yet it captures important properties of the vehicle dynamics. We establish a stabilizing control strategy for the lean angle of the vehicle. To eliminate impulse behavior (due to the appearance of the derivative of the steering angle) we propose using Lyapunov stability theory a continuous feedback loop for the steering input signal that ensures exponential stability of a desired lean angle. The process of stabilizing a desired lean angle does not imply trajectory tracking, however the current approach can be extended to yield further results regarding trajectory tracking for the complicated model of a two-wheeled vehicle system like bicycles and motorcycles.**

***Keywords*- two-wheeled vehicle, nonlinear nonminimum phase system, steering-based controller, stability, lean angle.**

I. INTRODUCTION

The dynamics and control analysis for bicycles and motorcycles have been investigated by many researchers since the seventies. The technology associated with vehicle dynamics has been applied to motorcycles successfully in various aspects. In [7] the equation of motion and simulation model for studying motorcycle dynamics and control has been demonstrated. A contribution to modelling the steering behavior of motorcycles has been presented in [8]. An analysis of rider behavior and vehicle dynamics rider/cycle system theory with application to handling performance have been considered in [9]. The book [5] offers a wealth of information regarding the behavior and performance and provides useful insights regarding the motorcycle dynamics. A feedback control law is suggested by [3] for a nonlinear, nonholonomic riderless powered two-wheeled bicycle to track a smooth trajectories. A controller that causes a model of a riderless bicycle to recover its balance from a near fall and to converge to a time parameterized path in the ground plane has been considered in [4]. An approach based on model predictive control for trajectory tracking for motorcycles is presented in [2].

The considered two-wheeled vehicle models are inherently unstable in the absence of active control of the steering. In modeling for control synthesis there is a trade off between capturing the relevant dynamical effects while keeping the model as simple as possible [2]. The current research deals with the problem of stabilizing a two-wheeled vehicle by steering control. Our control approach is based on the vehicle model that appears in [6, Chap. 7]. The considered nonlinear model does not include effects of wheel inertia, wheel side-sleep and suspensions, tires profile etc., which play an important role in real situations. However yet, the applied control algorithms is quite complicated and captures some essential properties of the vehicle regarding stability with respect to a desired lean angle. The current model contains quadratic term of the input function and its derivative, which are inherent features of the considered system. We suggest a continuous feedback for the steering input and show (using the approach in [1]) by applying a Lyapunov candidate function that under some mild conditions tracking of a desired lean angle is achievable. It should be emphasized that the proposed control law intends to ensure stability of a desired lean angle of the vehicle rather then tracking a time-parametrized path. However we believe that the paper results and the presented tools might be a point of departure for further analytical study regarding trajectory tracking in relevant vehicles like bicycles and motorcycles. This observation motivates our ongoing work and future challenges in the control framework of single-track vehicles.

The rest of the paper is organized as follows: the next section presents the considered mathematical model of the vehicle. Then we present a Lyapunov-based controller for stabilizing the vehicle in a vertical position, and later expands the controller's operation to stabilizing a desired reference angle. Next, the paper is demonstrating some numerical and simulation results, and finally we discuss the conclusions.

II. MODELING

We consider stability control problems in the nonlinear model of a two-wheeled tilting vehicle. The wheels of the vehicle are considered to have negligible inertial moments and mass, and we assume roll without side or longitudinal-slip. In considering the vehicle model and the associated notations we refer to [6, Chap. 7], however here we are focusing on stability issues in the obtained *nonlinear nonminimum phase system*. The current approach does not suggest a solution to the trajectory tracking control problem, but rather to set-point control, i.e., tracking a desired constant lean. However, the obtained results will be useful for tackling the trajectory tracking control problem in the considered complex system.

Consider Fig. 1 which demonstrates schematically the geometry for a two-wheeled vehicle in a turn. The steering angle is δ and U and V are the components of the velocity in the marked directions of the center of mass projection on the ground plane. The coordinates of the inertial frame are x and y, the angle ϕ represents the rotation of the vehicle with respect to the y-axis and the angular rotation rate is $r = \dot{\phi}$. The instantaneous turn radius is R. It is assumed that R is very large with respect to the wheelbase $a+b$. As indicated above in the current control analysis we neglect the slip angles. (According to [6, Chap. 7] we assume that slip effects are not important as long as the tires do not skid.)

In Fig. 2 we present a rigid body diagram of the two-wheeled vehicle. The lean angle is θ. When $\theta = 0$ the distance of the center of mass from the ground is h. The principal moments of inertia are I_1, I_2, and I_3. The principal axes of the rigid body (we neglect the wheels mass and inertia moments) are assumed to be *parallel to the axis* 1, 2, and 3, as shown in the figure. The 1-axis lies along the longitudinal axis of the vehicle in the xy plane, the 3-axis is aligned with the vehicle axis that is vertical when the lean angle is zero, and the 2-axis is perpendicular the to the other two axes.

Observing Figs. 1 and 2 and consider the approximations

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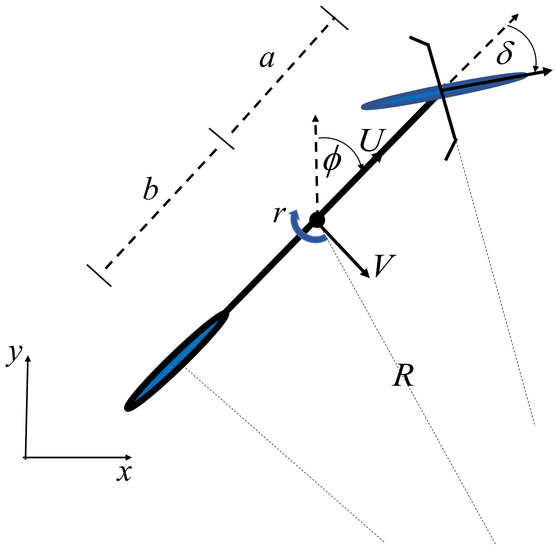

Fig. 1. A two-wheeled vehicle in a turn.

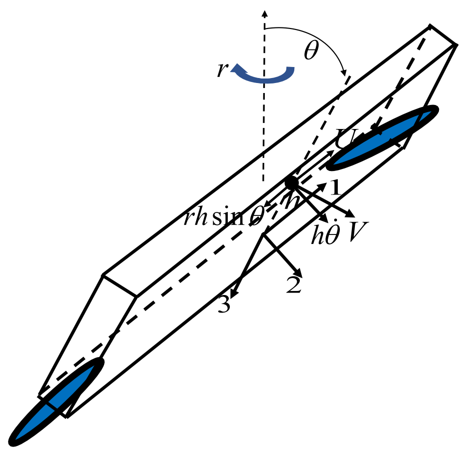

Fig. 2. The vehicle body is tilting at a lean angle θ.

$$r \approx U/R \approx U\delta/(a+b)\,;V \approx Ub\delta/(a+b)\,;R \approx (a+b)/\delta \tag{1}$$

we have using Lagrange formulation the equation for $\theta(t)$ [6, Chap. 7]:

$$\begin{aligned}&(I_1+mh^2)\ddot{\theta}+(I_3-I_2-mh^2)r^2\cos\theta\sin\theta\\&-mhg\sin\theta=-mh\cos\theta(\dot{V}+rU)\end{aligned} \tag{2}$$

where the velocity U is assumed to be constant.

III. Stabilizing the Vertical Position

Considering (1)-(2) we have

$$\begin{aligned}&(I_1+mh^2)\ddot{\theta}+(I_3-I_2-mh^2)\frac{U^2\delta^2\cos\theta\sin\theta}{(a+b)^2}\\&-mgh\sin\theta=-\frac{mh}{a+b}\cos\theta(Ub\dot{\delta}+U^2\delta)\end{aligned} \tag{3}$$

The current control objective is to stabilize the point $\left\{\theta,\dot{\theta}\right\}=\{0,0\}$ (the vehicle vertical position). In the rest of the paper we shall be using the following definitions (see (3))

$$\begin{aligned}I &\doteq I_1+mh^2;\ \alpha\doteq\left(I_3-I_2-mh^2\right)/(I\,(a+b)^2);\\ \beta &\doteq mh/\left(I\,(a+b)\right);\sigma\doteq mgh/I.\end{aligned} \tag{4}$$

Note that all the presented parameters are strictly positive except α which, depending on the system design, might be a positive, zero or negative number. In fact it appears that the sign of α is a factor influencing the controller design and thus, we shall consider separately the case $\alpha > 0$ and $\alpha \le 0$.

A. The case $\alpha > 0$

Using (4) equation (3) becomes:

$$\ddot{\theta}=-\alpha U^2\delta^2\cos\theta\sin\theta+\sigma\sin\theta-\beta\cos\theta\left(b\dot{\delta}U+\delta U^2\right). \tag{5}$$

With $x_1\doteq\theta$, $x_2\doteq\dot{\theta}$ and $x=[x_1,x_2]^T$ we may write (5) for $t\ge 0$ as follows

$$\begin{aligned}\dot{x} &= \left[\begin{array}{c}x_2\\ -\beta\cos x_1(b\dot{\delta}U+\delta U^2)-\alpha U^2\delta^2\cos x_1\sin x_1\end{array}\right]\\ &\quad+\left[\begin{array}{c}0\\ \sigma\sin x_1\end{array}\right]\end{aligned} \tag{6}$$

We assume that $\delta(0,x_1)=0$. To prevent discontinuity in the feedback loop, we suggest a two-phase control signal $\delta(t,x)$:

$$\delta\left(t,x_1\right)=\left\{\begin{array}{l}\frac{t}{\tau}\frac{kx_1(t)}{U^2\cos x_1(t)},\ t\in(0,\tau]\\ \frac{kx_1(t)}{U^2\cos x_1(t)},t\in(\tau,\infty)\end{array}\right. \tag{7}$$

where $\tau,k>0$ are constants, yet to be determined. Note that by definition $\delta\left(t,x_1\right)$ is continuous in t. (Discontinuity in the steering angle δ is physically not seem reasonable. In particular in such case $\dot{\delta}$ displays impulse behavior.) Following (7)

$$\dot{\delta}(t,x_1)=\left\{\begin{array}{l}\frac{1}{\tau}\frac{kx_1}{U^2\cos x_1}+\frac{t}{\tau}\frac{kx_2(\cos x_1+x_1\sin x_1)}{U^2\cos^2 x_1},t\in(0,\tau]\\ \frac{kx_2(\cos x_1+x_1\sin x_1)}{U^2\cos^2 x_1},t\in(\tau,\infty)\end{array}\right. \tag{8}$$

We will consider first the system (6) with δ at the time interval $t\in(\tau,\infty)$ and assume that at the beginning of the second mode (at $t=\tau$, see (7)) the state $x\left(\tau\right)$ satisfies $x_1\left(\tau\right)\in(-\pi/2,\pi/2)$. In considering the first mode that takes place during $t\in[0,\tau]$ we shall state conditions which ensure that indeed $x_1\left(\tau\right)\in(-\pi/2,\pi/2)$. For $\tau>0$ the closed-loop system becomes

$$\begin{aligned}\dot{x} &= \left[\begin{array}{c}x_2\\ -k\beta x_1-\frac{k\beta b(1+x_1\tan x_1)}{U}x_2-\frac{k^2\alpha x_1\tan x_1}{U^2}x_1\end{array}\right]\\ &\quad+\left[\begin{array}{c}0\\ \sigma\sin x_1\end{array}\right]\end{aligned} \tag{9a}$$

and clearly $x=0$ is an equilibrium point of (9a).

Defining

$$\varrho(x_1)\doteq x_1\tan x_1,\varrho_1(x_1)\doteq\frac{k\varrho(x_1)}{U^2},\varrho_2(x_1)\doteq\frac{1+\varrho(x_1)}{U} \tag{10}$$

and rewrite (9a) as

$$\dot{x}=\left[\begin{array}{cc}0 & 1\\ -k\beta-k\alpha\varrho_1 & -k\beta b\varrho_2\end{array}\right]x+\left[\begin{array}{c}0\\ \sigma\sin x_1\end{array}\right] \tag{11}$$

for $t>\tau$. Note that for $k>0$

$$\varrho\left(x_1\right),\varrho_1\left(x_1\right),\varrho_2\left(x_1\right)>0,\forall x_1\in\left(-\pi/2,\pi/2\right),x_1\neq 0. \tag{12}$$

Consider the function $\Omega(x)$ with

$$\Omega\left(x\right)=\frac{1}{2}x^T\left[\begin{array}{cc}\mu & \nu\\ \nu & 1\end{array}\right]x\doteq\frac{1}{2}x^TPx \tag{13}$$

where μ, ν are constants that will be determined shortly. Clearly $\Omega = \mu x_1^2/2 + \nu x_1 x_2 + x_2^2/2$ and thus the derivative of Ω along the solution of (11) is

$$\begin{aligned}\dot{\Omega} &= \mu x_1 \dot{x}_1 + \nu x_2 \dot{x}_1 + \nu x_1 \dot{x}_2 + x_2 \dot{x}_2 \\ &= \mu x_1 x_2 + \nu x_2^2 - \nu k \left(\beta + \alpha \varrho_1\right) x_1^2 \\ &\quad -\nu k \beta b \varrho_2 x_1 x_2 - \nu k \left(\beta + \alpha \varrho_1\right) x_1 x_2 \\ &\quad -k \beta b \varrho_2 x_2^2 + \sigma \sin x_1 \left(\nu x_1 + x_2\right) \end{aligned} \tag{14}$$

We rewrite (14) as

$$\begin{aligned}\dot{\Omega} &= -x^T \begin{bmatrix} k\nu(\beta + \alpha\varrho_1) & \frac{-\mu + k\nu(\beta + \alpha\varrho_1 + \beta b \varrho_2)}{2} \\ \frac{-\mu + k\nu(\beta + \alpha\varrho_1 + \beta b \varrho_2)}{2} & k\beta b \varrho_2 - \nu \end{bmatrix} x \\ &\quad + \left(\nu x_1 + x_2\right) \sigma \sin x_1 \\ &\doteq -x^T Q(x_1) x + \left(\nu x_1 + x_2\right) \sigma \sin x_1 \end{aligned} \tag{15a}$$

Let us take

$$\mu = k\nu\beta \left(1 + b/U\right). \tag{16}$$

Then, by definition (see (10)) equation (15a) becomes

$$\begin{aligned}&\dot{\Omega}(x) \\ &= -x^T \begin{bmatrix} k\nu(\beta + \alpha\varrho_1) & \frac{\nu k \varrho (k\alpha/U^2 + \beta b/U)}{2} \\ \frac{\nu k \varrho (k\alpha/U^2 + \beta b/U)}{2} & k\beta b \varrho_2 - \nu \end{bmatrix} x \\ &\quad + \left(\nu x_1 + x_2\right) \sigma \sin x_1 \\ &\doteq -x^T Q(x_1) x + \left(\nu x_1 + x_2\right) \sigma \sin x_1 \end{aligned} \tag{17}$$

Recalling the terms $\varrho, \varrho_1, \varrho_2$ in (10) the matrix $Q(x_1)$ in (17) can be rewritten as

$$\begin{aligned}Q(x_1) &= \begin{bmatrix} k\nu\beta & 0 \\ 0 & k\beta b/U - \nu \end{bmatrix} \\ &\quad + \begin{bmatrix} k^2 \nu \alpha \varrho / U^2 & \frac{\nu k \varrho (k\alpha/U^2 + \beta b/U)}{2} \\ \frac{\nu k \varrho (k\alpha/U^2 + \beta b/U)}{2} & k\beta b \varrho / U \end{bmatrix} \\ &= \begin{bmatrix} k\nu\beta & 0 \\ 0 & k\beta b/U - \nu \end{bmatrix} \\ &\quad + k\varrho(x_1) \begin{bmatrix} k\nu\alpha/U^2 & \frac{\nu (k\alpha/U^2 + \beta b/U)}{2} \\ \frac{\nu (k\alpha/U^2 + \beta b/U)}{2} & \beta b/U \end{bmatrix} \\ &\doteq R + S(x_1) \end{aligned} \tag{18}$$

where $\varrho(x_1) = x_1 \tan x_1 > 0$ for nonzero x_1 in $(-\pi/2, \pi/2)$. Recalling that $\alpha > 0$ we will chose $k, \nu > 0$ such that $R = R^T > 0$ and $S(x_1) = S^T(x_1) \geq 0$ for all $x_1 \in (-\pi/2, \pi/2)$. It is clear that for

$$0 < k_1 \doteq \nu U/(\beta b) < k \tag{19}$$

$R > 0$. Next

$$\det S = k^2 \varrho^2 [(k\nu\alpha/U^2)\beta b/U - (\nu(k\alpha/U^2 + \beta b/U)/2)^2] \tag{20}$$

where $\varrho(x_1) = x_1 \tan x_1$. Hence, for $x_1 \in (-\pi/2, \pi/2)$ and

$$0 < \nu \leq \nu_1 \doteq \frac{4k\alpha U b \beta}{(k\alpha + U b \beta)^2} \tag{21}$$

$S(x_1) \geq 0$ for all $k > 0$. (In the domain $(-\pi/2, \pi/2)$, $S(x_1) = 0$ if and only if $x_1 = 0$). We also note that for a fixed $k > 0$ one can select v such that both (19) and (21) hold and furthermore, observing (16) and (19)

$$\begin{aligned}\mu &= k\nu\beta \left(1 + b/U\right) > \nu\beta \left(1 + b/U\right) \nu U/(\beta b) \\ &= \nu^2 \left(1 + U/b\right) > \nu^2 \end{aligned} \tag{22}$$

which ensures that $P = P^T > 0$ in (13) and therefore $Q(x_1) = Q^T(x_1) > 0$ in (17) for all $x_1 \in (-\pi/2, \pi/2)$.

Furthermore, under these conditions for any fixed $x_1 \in (-\pi/2, \pi/2)$, $\lambda_{\min}(Q(x_1)) \geq \lambda_{\min}(R) = \min\{k\nu\beta, k\beta b/U - \nu\} > 0$ and thus

$$x^T Q(x_1) x \geq \left(\min\{k\beta v, k\beta b/U - \nu\}\right) \|x\|^2 > 0 \tag{23}$$

where $\|\cdot\|$ is the Euclidean norm. Observing (15a) we note that $\sigma |\sin x_1 (\nu x_1 + x_2)| \leq \sigma |x_1| (|\nu x_1| + |x_2|) \leq \sigma(\nu |x_1|^2 + |x_1 x_2|)$ and since $|x_1 x_2| \leq \|x\|^2/2$ one has $\sigma |\sin x_1 (\nu x_1 + x_2)| \leq \sigma(\nu + 1/2)\|x\|^2$. Hence, if under the conditions (19) and (21) k and v could be determined such that

$$\min\{k\beta v, k\beta b/U - \nu\} > \sigma\left(\nu + 1/2\right) \tag{24}$$

then $\dot{\Omega}(x) < 0$ for all $x_1 \in (-\pi/2, \pi/2)$. Hence, provided x_1 remain within the admissible domain (see later discussion) during the second control mode ($t > \tau$) we have that $x(t)$ converges (exponentially) to $x = 0$.

Remark 1. If, for a given pair $\{k, \nu\}$ that satisfies (19) and (21) the condition (24) does not hold one can increase k for a fixed ν such that (19), (21), and (24) are satisfied and $\dot{\Omega}(x) < 0$ so long as $x_1 \in (-\pi/2, \pi/2)$.

Now we can consider the first control mode associated with the input action during the time interval $[0, \tau]$. Suppose the condition (24) holds. We wish to define a set of 'admissible initial conditions' $B_{r_0} = \left\{x : \sqrt{x_1^2 + x_2^2} \leq r_0\right\}$ for the system (9a) and a $\tau > 0$ such that for each $x(0) \in B_{r_0}$ the resulting trajectory (for the selected k) satisfies $x(t) \in \Omega_c$ for all $t \in [0, \tau]$. To this end let us define $r_1 \doteq \inf\{\|\bar{x}\| : \bar{x} \in \partial\Omega_c\}$ where $\partial\Omega_c$ is the boundary of Ω_c. The radius of the ball B_{r_0}, namely, r_0 is taken such that

$$r_0 + k\beta b r_0/U < r_1. \tag{25}$$

We will show that a $\tau > 0$ can be determined such that

$$\|x(\tau)\| = \left\| x(0) + \int_0^\tau F(x, t) dt \right\| < r_1 \tag{26}$$

where (see (6))

$$\begin{aligned}F &= \begin{bmatrix} x_2 \\ -\beta \cos x_1 (b\dot{\delta} U + \delta U^2) - \alpha U^2 \delta^2 \cos x_1 \sin x_1 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ \sigma \sin x_1 \end{bmatrix} \end{aligned} \tag{27}$$

with (see (7)-(8))

$$\delta = t \frac{k}{\tau} \frac{x_1}{U^2 \cos x_1};\ t \in [0, \tau] \tag{28}$$

and

$$\dot{\delta} = \frac{k}{\tau} \frac{x_1}{U^2 \cos x_1} + t \frac{k}{\tau} \frac{x_2 (\cos x_1 + x_1 \sin x_1)}{U^2 \cos^2 x_1};\ t \in [0, \tau]. \tag{29}$$

Observing (26)-(27) *as long as* $\|x(t)\| < r_1$ we have

$$\left| \int_0^\tau x_2(t) dt \right| < \tau r_1. \tag{30}$$

Next, from (26)-(29), *so long as* $\|x(t)\| < r_1$ we have

$$\begin{aligned}
\left|\int_0^\tau (-\beta \cos x_1 b\dot{\delta} U dt\right| &< \int_0^\tau \beta b \frac{k}{\tau}\left|x_1(0) + \int_0^t x_2(\rho) d\rho\right| / U dt \\
&+ \int_0^\tau t\beta b \frac{k}{\tau}(r_1 + r_1^2 \tan r_1)/U dt \\
&< \beta b \frac{k}{\tau}\int_0^\tau (r_0 + r_1 t) dt / U \\
&+ \int_0^\tau t\beta b \frac{k}{\tau}(r_1 + r_1^2 \tan r_1)/U dt \\
&= k\beta b r_0/U + \tau k\beta b r_1/(2U) \\
&+ \tau k\beta b(r_1 + r_1^2 \tan r_1)/(2U). \quad (31)
\end{aligned}$$

while referring to the same equations we observe the following

$$\left|\int_0^\tau (-\beta \cos x_1 \delta U^2 dt\right| = \int_0^\tau \beta t \frac{k}{\tau} dt = \tau k\beta/2 \quad (32)$$

and further

$$\begin{aligned}
&\left|\int_0^\tau -\alpha U^2 \delta^2 \cos x_1 \sin x_1 dt\right| \\
&< \int_0^\tau \alpha \left(t\frac{k}{\tau}\right)^2 r_1 \tan r_1 dt = \tau^2 k\alpha r_1 \tan r_1/3 \quad (33)
\end{aligned}$$

and finally

$$\left|\int_0^\tau \sigma \sin x_1 dt\right| < \int_0^\tau \sigma \sin r_1 dt = \tau\sigma \sin r_1. \quad (34)$$

Observing (30)-(31), (26) holds for every $x(0) \in B_{r_0}$ if

$$\begin{aligned}
&r_0 + k\beta b r_0/U + \tau r_1 + \tau k\beta b r_1/(2U) \\
&+ \tau k\beta b(r_1 + r_1^2 \tan r_1)/(2U) \\
&+ \tau k\beta/2 + \tau^2 k\alpha r_1 \tan r_1/3 + \tau\sigma \sin r_1 < r_1. \quad (35)
\end{aligned}$$

Since $\tau > 0$ can be taken as small as needed and in (35) the only terms which are not multiplied by τ (or τ^2) are $r_0, k\beta b r_0/U$ for any r_0 that satisfies (25) a trajectory $x(t)$ that starts in $x(0) \in B_{r_0}$ belongs to B_{r_1} and $x(\tau) \in \Omega_c$ and $\lim_{t\to\infty} x(t) = 0$.

B. The case $\alpha \le 0$

For the current case we start again by considering the second control mode, namely,

$$\delta(t, x_1) = \frac{k x_1}{U^2 \cos x_1}, \tau < t. \quad (36)$$

Assume momentarily that $k > 0$ is given. Then one can select a ϖ with $0 < \varpi < \beta$ and $\varkappa \in (0, \pi/2)$ such that

$$\begin{aligned}
\beta + \alpha k x_1 \tan x_1/U^2 &= \beta - |\alpha| k x_1 \tan x_1/U^2 \\
&= \beta + \alpha\varrho_1 > \varpi, \forall x_1 \in (-\varkappa, \varkappa) \quad (37)
\end{aligned}$$

Applying the controller $\delta(\cdot,\cdot)$ in (6) we arrive at (9a) and using (10) we have (11). Note that now $-k\beta - k\alpha\varrho_1 = -k\left(\beta - |\alpha| k x_1 \tan x_1/U^2\right) < -k\varpi$. Using the function $\Omega(x)$ in (13) we obtain $\dot{\Omega}(x)$ in (14) and (15a). Let μ be taken as in (16), i.e. $\mu = k\nu\beta(1 + b/U)$. Recalling (37) one can determine $\alpha_1 < \alpha \le 0$ and $\alpha_2 > 0$ such that $\alpha = \alpha_1 + \alpha_2$ and

$$\beta + \alpha_1 k x_1 \tan x_1/U^2 > \varpi, \ \forall x_1 \in (-\varkappa, \varkappa) \quad (38)$$

(If $\alpha = 0$ then $\alpha_1 = -\alpha_2$). With this observation we can write (17) in case $\alpha \le 0$

$$\begin{aligned}
\dot{\Omega} &= -x^T \begin{bmatrix} k\nu(\beta + (\alpha_1 + \alpha_2)\varrho_1) & \frac{\nu k\varrho(k\alpha/U^2 + \beta b/U)}{2} \\ \frac{\nu k\varrho(k\alpha/U^2 + \beta b/U)}{2} & k\beta b\varrho_2 - \nu \end{bmatrix} x \\
&+ (\nu x_1 + x_2)\sigma \sin x_1 \\
&\doteq -x^T Q_1(x_1) x + (\nu x_1 + x_2)\sigma \sin x_1. \quad (39)
\end{aligned}$$

where $\varrho, \varrho_1, \varrho_2$ are given in (10). Hence, we have

$$\begin{aligned}
Q_1(x_1) &= \begin{bmatrix} k\nu(\beta + \alpha_1\varrho_1) & 0 \\ 0 & k\beta b/U - \nu \end{bmatrix} \\
&+ \begin{bmatrix} k^2\nu\alpha_2\varrho/U^2 & \frac{\nu k\varrho(k\alpha/U^2 + \beta b/U)}{2} \\ \frac{\nu k\varrho(k\alpha/U^2 + \beta b/U)}{2} & k\beta b\varrho/U \end{bmatrix} \\
&= \begin{bmatrix} k\nu(\beta + \alpha_1 k\varrho/U^2) & 0 \\ 0 & k\beta b/U - \nu \end{bmatrix} \\
&+ k\varrho \begin{bmatrix} k\nu\alpha_2/U^2 & \frac{\nu(k\alpha/U^2 + \beta b/U)}{2} \\ \frac{\nu(k\alpha/U^2 + \beta b/U)}{2} & \beta b/U \end{bmatrix} \\
&\doteq R_1(x_1) + S_1(x_1) \quad (40)
\end{aligned}$$

where $\varrho = x_1 \tan x_1$ and $\alpha_2 > 0$ and we have now (compare with (20) and note that α_2 replaces α in the first term below)

$$\det S = k^2\varrho^2[(k\nu\alpha_2/U^2)\beta b/U - (\nu(k\alpha/U^2 + \beta b/U)/2)^2]. \quad (41)$$

Hence, for every v that satisfies

$$0 < \nu \le \nu_1 \doteq \frac{4k\alpha_2 U b\beta}{(k\alpha + U b\beta)^2} \quad (42)$$

we have $S_1(x_1) = S_1^T(x_1) \ge 0$ for all $x_1 \in (-\pi/2, \pi/2)$. Regarding the matrix $R_1(x_1)$, since $(\beta + \alpha_1\varrho_1) > 0$ for $x_1 \in (-\varkappa, \varkappa)$ if $k > k_1 \doteq \nu U/\beta b$ then $R_1(x_1) > 0$ for all $x_1 \in (-\varkappa, \varkappa)$. Hence, for any fixed $x_1 \in (-\varkappa, \varkappa)$, $\lambda_{\min}(Q_1(x_1)) \ge \lambda_{\min}(R_1) = \min\{k\nu\varpi, k\beta b/U - \nu\} > 0$ where ϖ is given in (38) and thus for $x \ne 0$

$$x^T Q_1(x_1) x \ge \min\{k\nu\varpi, k\beta b/U - \nu\}\|x\|^2 > 0 \quad (43)$$

We can conclude now the above discussion. Going back to the beginning of the current subsection, select $k > 0$ and $\alpha_1 < \alpha \le 0$ and $\alpha_2 > 0$ such that $\alpha = \alpha_1 + \alpha_2$ and $\varkappa \in (0, \pi/2)$ such that (38) for some $\varpi > 0$. Then determine $0 < v < \min\{v_1, k_1\beta b/U\}$. Then $Q_1(x_1) > 0$. If in addition $\min\{k\nu\varpi, k\beta b/U - \nu\} > \sigma(\nu + 1/2)$ we have

$$\dot{\Omega}(x) < 0, \forall x : x_1 \in (-\varkappa, \varkappa) \quad (44)$$

where the function $\Omega(x)$ is given in (13). The last result depends on the system parameters and the velocity U. The computational process can be modelled recursively. In case the condition $\min\{k\nu\varpi, k\beta b/U - \nu\} > \sigma(\nu + 1/2)$ does not hold one can further decrease the term $\varkappa$ such that for all $x_1 \in (-\varkappa, \varkappa) \subset (-\pi/2, \pi/2)$ the symmetric matrix $R_1(x_1) + S_1(x_1)$ is positive definite and $\dot{\Omega}(x) < 0$ (see Remark 1).

As for the application of the first control mode associated with the input action during the time interval $[0, \tau]$, the analysis is almost identical to the one presented in the previous case, ($\alpha > 0$) with one difference: the starting point is the ball $B_r = \left\{x : \sqrt{x_1^2 + x_2^2} \le r < \varkappa < \pi/2\right\}$ in which $\dot{\Omega}(x)$ satisfies (44).

IV. Stability in Tilting

We consider now the stability problem with respect to a point $\left\{\theta, \dot{\theta}\right\} = \{\theta_d, 0\}$ where $\theta_d \in (-\pi/2, \pi/2)$ and $\theta_d \neq 0$. Using $x_1 \doteq \theta$, $x_2 \doteq \dot{\theta}$ and $x = [x_1, x_2]^T$ the reference set-point is now $x_d = [x_{1d}, x_{2d}]^T = [\theta_d, 0]^T$. Observing (6) we wish to find a constant value for the steering angle δ, namely, δ_d in terms of the vector state x_{1d} such that the following holds

$$-\delta_d \beta U^2 \cos x_{1d} - \delta_d^2 \alpha U^2 \cos x_{1d} \sin x_{1d} + \sigma \sin x_{1d} = 0 \quad (45)$$

Note that if δ_d solves the last equation for a given x_{1d} then $-\delta_d$ solves the equation for $-x_{1d}$. If $\alpha = 0$ the solution of the last equation is $\delta_d = \sigma \sin x_{1d} / \left(\beta U^2 \cos x_{1d}\right)$ with $sgn\left\{\delta_d\right\} = sgn\left\{x_{1d}\right\}$. Assume that $\alpha \neq 0$. Since $x_{1d} \neq 0$ we rewrite (45) as

$$\delta_d^2 + \delta_d \beta / \left(\alpha \sin x_{1d}\right) - \sigma / \left(\alpha U^2 \cos x_{1d}\right) = 0 \quad (46)$$

The solution of the last equation for δ_d is

$$\delta_d = \frac{-\beta/(\alpha \sin x_{1d}) \pm \sqrt{\beta^2/(\alpha \sin x_{1d})^2 + 4\sigma/(\alpha U^2 \cos x_{1d})}}{2}. \quad (47)$$

Due to physical reasons we take the solution that satisfies $sgn\left\{\delta_d\right\} = sgn\left\{x_{1d}\right\}$. Assume first that $\alpha > 0$. Then, if $x_{1d} > 0$

$$\delta_d = \frac{-\beta/(\alpha \sin x_{1d}) + \sqrt{\beta^2/(\alpha \sin x_{1d})^2 + 4\sigma/(\alpha U^2 \cos x_{1d})}}{2} \quad (48)$$

and if $x_{1d} < 0$

$$\delta_d = \frac{-\beta/(\alpha \sin x_{1d}) - \sqrt{\beta^2/(\alpha \sin x_{1d})^2 + 4\sigma/(\alpha U^2 \cos x_{1d})}}{2} \quad (49)$$

Assume now that $\alpha < 0$. Then the situation is more complicated and we confine x_{1d} to be in $(-\varkappa, \varkappa) \subset (-\pi/2, \pi/2)$ such that $\beta^2 > 4\left|\alpha\right| \sigma \sin \varkappa \tan \varkappa / U^2$. In this case $\sqrt{\beta^2/(\alpha \sin x_{1d})^2 + 4\sigma/(\alpha U^2 \cos x_{1d})} < \left|\beta/(\alpha \sin x_{1d})\right|$ and therefore analytically the two solutions in (47) satisfies $sgn\left\{\delta_d\right\} = sgn\left\{x_{1d}\right\}$.

We finally conclude that for a given constant steering angle $\delta = \delta_d$ (i.e. $\dot{\delta} = 0$) that satisfies (depending on $sgn\{\alpha\}$) one of the equations (47)-(49), the selected x_d is an equilibrium point of the equation (16) and in fact

$$\begin{gathered} x_{2d} = 0 \\ -\beta \cos x_{1d} \delta_d U^2 - \alpha U^2 \delta_d^2 \cos x_{1d} \sin x_{1d} + \sigma \sin x_{1d} = 0 \end{gathered} \quad (50)$$

Next we consider a controller stabilizes the point $x_d = [x_{1d}, x_{2d}]^T = [\theta_d, 0]^T$. We assume that $\delta(0, x_1) = \delta_d$. Recalling (7) let the steering input be taken as

$$\delta(t, x_1) = \begin{cases} \frac{t}{\tau} \frac{k(x_1 - x_{1d})}{U^2 \cos x_1} + \delta_d, 0 < t \leq \tau \\ \frac{k(x_1 - x_{1d})}{U^2 \cos x_1} + \delta_d, \tau < t \end{cases} \quad (51)$$

We consider first the second control mode (in $\tau < t$). Defining $\Delta \doteq x_1 - x_{1d}$ we have now

$$\dot{\delta}(t, x_1) = \begin{cases} \frac{1}{\tau} \frac{k\Delta}{U^2 \cos x_1} + \frac{t}{\tau} \frac{k x_2 (\cos x_1 + \Delta \sin x_1)}{U^2 \cos^2 x_1}, t \in (0, \tau] \\ \frac{k x_2 (\cos x_1 + \Delta \sin x_1)}{U^2 \cos^2 x_1}, t \in (t, \infty). \end{cases} \quad (52)$$

Implementing the controller in (6) we have the closed-loop system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} x_2 \\ -k\beta\Delta - \frac{k\beta b(1 + \Delta \tan x_1)}{U} x_2 - \frac{k^2 \alpha \Delta \tan x_1}{U^2} \Delta \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ -\beta \cos x_1 \delta_d U^2 - \alpha U^2 \delta_d^2 \cos x_1 \sin x_1 + \sigma \sin x_1 \end{bmatrix} \end{aligned} \quad (53)$$

We use a change of variables $z = [z_1, z_2]^T = x - x_d$ and we have from (53) (recall that $\Delta = x_1 - x_{1d} = z_1$ and defining $\Lambda \doteq z_1 + x_{1d}$)

$$\begin{aligned} \dot{z} &= \begin{bmatrix} z_2 \\ -k\beta z_1 - \frac{k\beta b(1 + z_1 \tan \Lambda)}{U} z_2 - \frac{k^2 \alpha z_1 \tan \Lambda}{U^2} z_1 \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ -\beta \cos \Lambda \delta_d U^2 - \alpha U^2 \delta_d^2 \cos \Lambda \sin \Lambda + \sigma \sin \Lambda \end{bmatrix}. \end{aligned} \quad (54)$$

Let

$$\bar{\varrho}(z_1) \doteq z_1 \tan(z_1 + x_{1d}), \bar{\varrho}_1(z_1) \doteq \frac{k\bar{\varrho}(z_1)}{U^2}, \bar{\varrho}_2(z_1) \doteq \frac{1 + \bar{\varrho}(z_1)}{U} \quad (55)$$

and hence ($\Lambda = z_1 + x_{1d}$)

$$\begin{aligned} \dot{z} &= \begin{bmatrix} 0 & 1 \\ -k\beta - k\alpha\bar{\varrho}_1 & -k\beta b\bar{\varrho}_2 \end{bmatrix} z \\ &+ \begin{bmatrix} 0 \\ \sigma \sin \Lambda - \beta \cos \Lambda \delta_d U^2 - \alpha U^2 \delta_d^2 \cos \Lambda \sin \Lambda \end{bmatrix} \end{aligned} \quad (56)$$

Consider the quadratic function $\Omega(z)$

$$\Omega(z) = \frac{1}{2} z^T \begin{bmatrix} \mu & \nu \\ \nu & 1 \end{bmatrix} z \doteq \frac{1}{2} z^T P z$$

Comparing $\Omega(z)$ to $\Omega(x)$ in (13) and the state equation (56) to (11) we have now (see (17)-(18) and take $\mu = k\nu\beta(1 + b/U)$ according to (16))

$$\begin{aligned} \dot{\Omega} &= -z^T \begin{bmatrix} k\nu(\beta + \alpha\bar{\varrho}_1) & \frac{\nu k\bar{\varrho}(k\alpha/U^2 + \beta b/U)}{2} \\ \frac{\nu k\bar{\varrho}(k\alpha/U^2 + \beta b/U)}{2} & k\beta b\bar{\varrho}_2 - \nu \end{bmatrix} z \\ &\quad + (\nu z_1 + z_2)\left(\sigma \sin \Lambda - \beta \cos \Lambda \delta_d U^2 \right. \\ &\quad \left. -\alpha U^2 \delta_d^2 \cos \Lambda \sin \Lambda\right) \\ &\doteq -z^T \bar{Q}(z_1) z + (\nu z_1 + z_2)\left(\sigma \sin \Lambda - \beta \cos \Lambda \delta_d U^2 \right. \\ &\quad \left. -\alpha U^2 \delta_d^2 \cos \Lambda \sin \Lambda\right) \end{aligned} \quad (57)$$

Recalling the definition of $\bar{\varrho}, \bar{\varrho}_1, \bar{\varrho}_2$ in (55) the matrix $\bar{Q}(z_1)$ in (57) can be rewritten as (see (18))

$$\begin{aligned} \bar{Q}(z_1) &= \begin{bmatrix} k\nu\beta & 0 \\ 0 & k\beta b/U - \nu \end{bmatrix} \\ &+ k\bar{\varrho}(z_1) \begin{bmatrix} k\nu\alpha/U^2 & \frac{\nu(k\alpha/U^2 + \beta b/U)}{2} \\ \frac{\nu(k\alpha/U^2 + \beta b/U)}{2} & \beta b/U \end{bmatrix} \\ &\doteq \bar{R} + \bar{S}(\bar{z}_1) \end{aligned} \quad (58)$$

We can compare now the current situation (stabilizing the vehicle when tilting) with the previous one (stabilizing the vehicle in a vertical position). We clearly have $\bar{R} = R$ where latter matrix is given in (18). However, unlike the positive function $\varrho(x_1) \doteq x_1 \tan x_1$ which ensures in the previous development that $S(x_1) > 0$, the function $\bar{\varrho}(z_1) \doteq z_1 \tan(z_1 + x_{1d})$ is indefinite in sign and therefore the symmetric matrix $\bar{S}(\bar{z}_1)$ is indefinite. Hence, in order to ensure positive definiteness of $\bar{Q}(z_1)$, $|z_1|$ should be relatively small, namely, x_1 is sufficiently close to x_{1d}. Next, the term $(\nu x_1 + x_2)\sigma \sin x_1$ in (17) is replaced now (see (57)) by $(\nu z_1 + z_2)\left(\sigma \sin \Lambda - \beta \cos \Lambda \delta_d U^2 - \alpha U^2 \delta_d^2 \cos \Lambda \sin \Lambda\right)$ where $\Lambda = z_1 + x_{1d}$. However, by definition when z_1 is in a small neighborhood of 0, x_1 is in a small neighborhood of x_{1d}, which means following (56) that $\left|\sigma \sin \Lambda - \beta \cos \Lambda \delta_d U^2 - \alpha U^2 \delta_d^2 \cos \Lambda \sin \Lambda\right|$ closes to zero. In conclusion, using the previous control approach stability with respect to $x_{1d} \neq 0$ can be guaranteed provided the states are perturbed just slightly off the equilibrium point x_d.

V. EXAMPLES

The examples demonstrate the proposed controllers in stabilizing a desired lean angle. Note that the proposed controllers ensure a lean angle setpoint and not trajectory tracking. The latter research topic is currently being conducted and will expand this work.

Example 1: Stabilizing the vertical position. The system physical parameters (in MKS units) are (see Figs. 1 and 2) $a = b = 0.5$, $h = 1$, $m = 120$, $I_1 = 11$, $I_2 = 15$, $I_3 = 12$, the velocity is $U = 10$. The controller constants are (see (7)) $\tau = 0.2$ and $k = 80$. The initial conditions (angles are in degrees) are $x(0) = y(0) = 0$, $\phi(0) = 0$, $\theta(0) = 20[deg.]$. The results appear in Figs. 3 and 4:

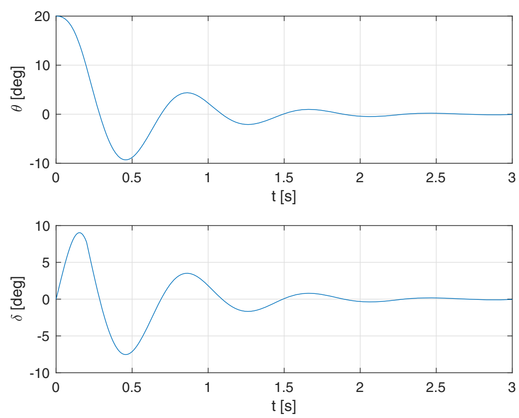

Fig. 3. Time histories of the lean and the steering angles in Example 1.

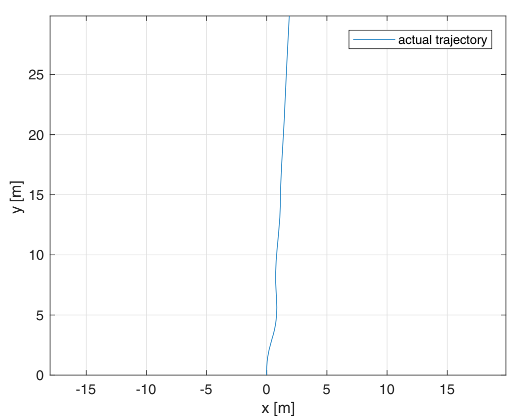

Fig. 4. The resulting path due to the controller action in Example 1.

Example 2: Stabilizing the setpoint $(\theta_d, \dot{\theta}_d) = (10, 0)$. The system parameters, controller constants, and initial conditions are as in Example 1. The results are demonstrated in Figs. 5 and 6:

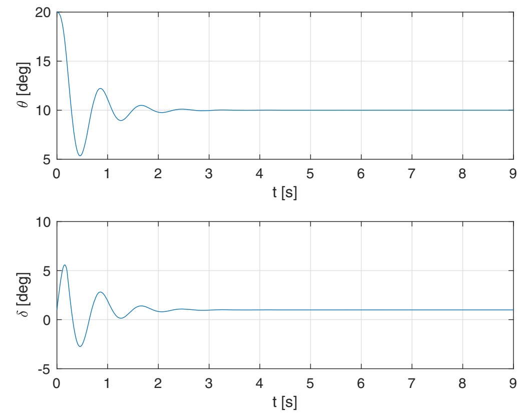

Fig. 5. Time histories of the lean and the steering angles Example 2.

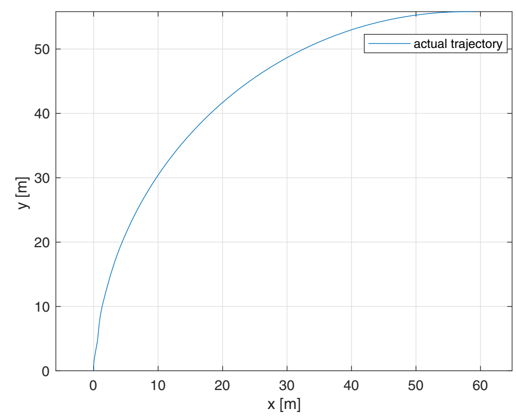

Fig. 6. The resulting path due to the controller action in Example 2.

VI. CONCLUSION

This study proposes a control approach for stabilizing a desired lean angle of the nonlinear nonminimum phase model of a two-wheeled vehicle. Following the Lagrangian formulation the obtained model includes quadratic and derivative terms of the steering input signal. To eliminate impulsive behavior we suggest a continuous feedback for the steering input and establish exponential stability of a desired set-point, provided some listed conditions are satisfied. We believe that further study will lead to less conservative conditions for stability in a more complex system that includes effects like wheel sleeping and inertia of the wheels, and moreover the presented approach will allow us to extend existing control algorithm for tracking a time-parameterized path (a geometric path with an associated timing-law) in the control frameworks of two-wheeled vehicles like bicycle and motorcycles.

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