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Stochastic Discrete-time Systems with Delay - Robust Vertex-dependent H_∞ State-feedback Control

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Abstract— **The theory of stochastic state-multiplicative state-feedback control of retarded discrete-time systems is extended to the robust polytopic case, where a vertex-dependent approach is applied. The latter problem is solved by first deriving a stability condition for uncertain systems based on the application of the Finsler lemma to the studied systems, where a special Lyapunov function is applied to each vertex of the uncertainty polytope. Compared to the stability condition which is obtained by the quadratic approach where a single Lyapunov function is applied to the whole uncertainty polytope, the vertex-dependent approach achieves a significatively less-conservative result than the former approach. Based on the vertex-dependent stability condition, a robust bounded real lemma is derived and solved, leading to the solution of the state-feedback control for uncertain systems.**

Keywords : state-feedback H_∞ control, retarded discrete-time systems, input-output

I. Introduction

In the last four decades, the stability analysis, control design and filtering for systems with stochastic state-multiplicative uncertainties have received much attention (see [1] and the references therein), where both continuous-time [2] - [4] and discrete-time [5]-[7] delay-free systems have been considered. The latter problems were extended to retarded systems of various kinds more than two decades ago, and a great body of work has been carried out in this filed including various solutions that ensure a worst case performance bound in the H_∞ sense (see [8] and the references therein, [9] - [13] for continuous-time case and [14] - [17] for the discrete-time case).

In some practical situations, including delayed systems, where state-feedback control is applied, the systems are prone to relatively large parameter uncertainties which can not be compensated via simple control (whether applied to delay-free or retarded systems). Hence, the relative importance of the robust design.

The theory of robust stochastic H_∞ state-feedback control for uncertain delayed systems has been concentrated on both the norm-bounded approach and the quadratic polytopic type description of the uncertain parameters (see [8] and the references therein).

Modeling of uncertainties as norm-bounded may lead to simple control and filtering solutions with considerable over-design whereas considering the uncertainty to be of the polytopic-type a more realistic description of the system is obtained which enables the application of convex optimization method [18], [19]. To the best of our knowledge, the problems of discrete-time state-feedback control and estimation of systems with multiplicative stochastic noise and polytopic-type uncertain parameters were both solved only via the **vertex-independent** Lyapunov approach where a single lyapunov function is uniformly assigned to all the vertices of the uncertainty polytope. This approach yields a solution, usually called the 'quadratic solution', which is very conservative similar to what has been obtained in the control and estimation of deterministic uncertain systems. It is, therefore, instructive to compare the results that were achieved by the traditional 'quadratic approach' to those of the newly applied vertex-dependent approach, where in the latter approach a unique Lyapunov function is applied to each of the vertices of the uncertainty polytope.

In the literature, one can find solutions for additional control problems of retarded stochastic systems, however all of these solutions do not apply the vertex-dependent approach. For example, the full order dynamic output-feedback problem for continuous-time stochastic retarded systems was solved in [13], for nominal systems and norm-bounded uncertain ones.

In the stochastic discrete-time delayed counterpart setting, which has been addressed to a lesser extent comparing to the continuous-time case, the mean square exponential stability and the control and filtering problems of these systems were treated by several groups [14]-[17]. In [14], the state-feedback control problem solution is solved for norm-bounded uncertain systems, for the restrictive case where the same multiplicative noise sequence multiplies both the states and the input of the system. The solution there is delay-dependent. In [20] the static output feedback control of discrete-time retarded systems was solved for both the nominal and the uncertain case. In the uncertain case of the latter solution, a single Lyapunv function was assigned to all the uncertainty polytope. It is instructive to note that the vertex dependent approach can never be inferred from the quadratic ones, hence the need for the new approach.

Here, we continue to adopt the input-output approach, applied in [8], for the solution of the robust stochastic discrete-time state-feedback control problem. This approach is based on the representation of the system's delay action by linear operators, with no delay, which in turn allows one to replace the underlying system with an equivalent one which possesses a norm-bounded uncertainty, and therefore may be treated by the theory of norm bounded uncertain, non-

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retarded systems with state-multiplicative noise. Similarly to the systems treated in [8], in our systems we allow for a time-varying delay where the uncertain stochastic parameters multiply both the delayed and the non delayed states in the state space model of the systems.

This paper is organized as follows: We first bring the newly robust vertex-dependent stability result for uncertain systems. This result, brought in Sections 3, is needed for the derivation, in Section 4, of the less conservative vertex-dependent BRL problem for the latter systems. Based on the robust vertex-dependent BRL result, the solution of the vertex-dependent robust state-feedback control problem is achieved in Section 5.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, $\mathcal{N}$ is the set of natural numbers and the notation $P>0$, (respectively, $P\geq 0$) for $P\in\mathcal{R}^{n\times n}$ means that P is symmetric and positive definite (respectively, semi-definite). We denote by $L^2(\Omega,\mathcal{R}^n)$ the space of square-integrable $\mathcal{R}^n-$ valued functions on the probability space $(\Omega,\mathcal{F},\mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ is a σ algebra of a subset of Ω called events and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. By $(\mathcal{F}_k)_{k\in\mathcal{N}}$ we denote an increasing family of σ-algebras $\mathcal{F}_k\subset\mathcal{F}$. We also denote by $\tilde{l}^2(\mathcal{N};\mathcal{R}^n)$ the n-dimensional space of nonanticipative stochastic processes $\{f_k\}_{k\in\mathcal{N}}$ with respect to $(\mathcal{F}_k)_{k\in\mathcal{N}}$ where $f_k\in L^2(\Omega,\mathcal{R}^n)$. On the latter space the following l^2-norm is defined:

$$||\{f_k\}||^2_{\tilde{l}_2}=E\{\textstyle\sum_0^\infty ||f_k||^2\}=\sum_0^\infty E\{||f_k||^2\}<\infty,\quad \{f_k\}\in\tilde{l}_2(\mathcal{N};\mathcal{R}^n), \tag{1}$$

where $||\cdot||$ is the standard Euclidean norm. We denote by Tr$\{\cdot\}$ the trace of a matrix and by δ_{ij} the Kronecker delta function. Throughout the manuscript we refer to the notation of exponential l^2 stability, or internal stability, in the sense of [6] (see Definition 2.1, page 927, there).

II. Problem Formulation

We consider the following linear retarded system:

$$\begin{aligned}&x_{k+1}=(A_0+D\nu_k)x_k+(A_1+F\mu_k)x_{k-\tau(k)}+B_1w_k+\\&(B_2+G\zeta_k)u_k,\quad x_l=0,\ l\leq 0\\&z_k=C_1x_k+D_{12}u_k\end{aligned} \tag{2a,c}$$

where $x_k\in\mathcal{R}^n$ is the system state vector, $w_k\in\mathcal{R}^q$ is the exogenous disturbance signal and $z_k\in\mathcal{R}^r$ is the state combination (objective function signal) to be regulated and where the time delay is denoted by the integer τ_k and it is assumed that $0\leq\tau_k\leq h,\ \forall k$. The variables $\{\nu_k\},\ \{\mu_k\}$ and $\{\zeta_k\}$ are zero-mean uncorrelated real scalar white-noise sequences that satisfy:

$$E\{\nu_k\nu_j\}=\delta_{kj},E\{\mu_k\mu_j\}=\delta_{kj},E\{\zeta_k\zeta_j\}=\delta_{kj},\forall k,j\ \geq 0.$$

The matrices in (2a-c) are constant matrices of appropriate dimensions.

We seek a constant state-feedback controller

$$u_k=Kx_k \tag{3}$$

that achieves a certain performance requirement. We treat the following two problems:

i) H_∞ State-feedback control :

We consider the system of (2a-c) and the following performance index:

$$J_{SF}\stackrel{\Delta}{=}||\bar{z}_k||^2_{\tilde{l}_2}-\gamma^2||w_k||^2_{\tilde{l}_2}. \tag{4}$$

Our objective is to find a controller of the type of (3) such that J_{SF} given by (4) is negative for all nonzero w_k where $w_k\in\tilde{l}^2_{\mathcal{F}_k}([0,\infty);\mathcal{R}^q)$.

ii) Robust H_∞ State-feedback control :

In the robust stochastic H_∞ control case, we assume that the system parameters lie within the following polytope

$$\bar{\Omega}\stackrel{\Delta}{=}\begin{bmatrix} A_0 & A_1 & B_1 & B_2 & C_1 & D_{12} & D & F & G\end{bmatrix}, \tag{5}$$

which is described by the vertices:

$$\bar{\Omega}=\mathcal{C}o\{\bar{\Omega}_1,\bar{\Omega}_2,...,\bar{\Omega}_N\}, \tag{6}$$

where $\bar{\Omega}_i\stackrel{\Delta}{=}$

$$\begin{bmatrix} A_0^{(i)} & A_1^{(i)} & B_1^{(i)} & B_2^{(i)} & C_1^{(i)} & D_{12}^{(i)} & D^{(i)} & F^{(i)} & G^{(i)}\end{bmatrix} \tag{7}$$

and where N is the number of vertices. In other words:

$$\bar{\Omega}=\sum_{i=1}^N\bar{\Omega}_i f_i,\quad \sum_{i=1}^N f_i=1\quad ,f_i\geq 0. \tag{8}$$

Similarly to the nominal case, our objective is to find a controller of the type of (3) such that J_{SF} given by (4) is negative for all nonzero w_k where $w_k\in\tilde{l}^2_{\mathcal{F}_k}([0,\infty);\mathcal{R}^q)$.

III. Robust Mean-square exponential stability

In order to solve the above two problems we bring first the stability result for the retarded, stochastic, discrete-time system that was already derived in [8]. Considering the system of (2a) with $B_1=0,\ B_2=0,\ G=0$, we obtain the following lemma, which is brought here for the sake of completeness :

Lemma 1: [8] The exponential stability in the mean square sense of the system (2a) with $B_1=0,\ B_2=0,\ G=0$, is guaranteed if there exist $n\times n$ matrices $Q>0$, $R_1>0$, $R_2>0$ and M that satisfy the following inequality:

$$\bar{\Gamma}\stackrel{\Delta}{=}\begin{bmatrix}\bar{\Gamma}_{1,1} & \bar{\Gamma}_{1,2} & 0 & 0 & \bar{\Gamma}_{1,5}\\ * & -Q & Q(A_1-M) & QM & 0\\ * & * & \bar{\Gamma}_{3,3} & 0 & \bar{\Gamma}_{3,5}\\ * & * & * & -R_2 & -hM^TR_2\\ * & * & * & * & -R_2\end{bmatrix}<0.$$

where

$$\begin{aligned}
\bar{\Gamma}_{1,1} &= -Q + D^T(Q+h^2R_2)D + R_1,\\
\bar{\Gamma}_{1,2} &= (A_0+M)^TQ,\\
\bar{\Gamma}_{1,5} &= h(A_0^T+M^T)R_2 - R_2h,\\
\bar{\Gamma}_{3,3} &= -R_1 + F^T(Q+h^2R_2)F,\\
\bar{\Gamma}_{3,5} &= h(A_1^T-M^T)R_2.
\end{aligned} \tag{9a-e}$$

We note that inequality (9a) is bilinear in the decision variables because of the terms QM and R_2M. In order to remain in the linear domain, we can define $Q_M = QM$ and choose $R_2 = \epsilon Q$ where ϵ is a positive tuning scalar. The resulting LMI can be found in [8].

In the polytopic uncertain case we obtain two results, the first of which is the quadratic solution which appears in [8] and is based on assigning the same Lyapunov function over the all uncertainty polytope. A new result is obtained here by applying a vertex-dependent Lyapunov function, based on the Finsler lemma [19]. Using Schur's complement, (9a) can be written as:

$$\Psi + \Phi Q \Phi^T < 0, \tag{10}$$

with

$$\Psi \triangleq \begin{bmatrix} \Psi_{1,1} & 0 & 0 & h(A_0^T+M^T)R_2 - R_2h \\ * & \Psi_{2,2} & 0 & h(A_1^T-M^T)R_2 \\ * & * & -R_2 & -hM^TR_2 \\ * & * & * & -R_2 \end{bmatrix},$$

$$\Phi \triangleq \begin{bmatrix} A_0^T+M^T \\ A_1^T-M^T \\ M^T \\ 0 \end{bmatrix}, \Psi_{1,1} = -Q + D^T(Q+h^2R_2)D + R_1$$

and

$$\Psi_{2,2} = -R_1 + F^T(Q+h^2R_2)F. \tag{11a-d}$$

The following new result is thus obtained:

Lemma 2: Inequality (10) is satisfied iff there exist matrices: $0 < Q \in \mathcal{R}^{n\times n}$, $G \in \mathcal{R}^{n\times 4n}$, $M \in \mathcal{R}^{n\times n}$ and $H \in \mathcal{R}^{n\times n}$ that satisfy the following inequality

$$\Omega \triangleq \begin{bmatrix} \Psi + G^T\Phi^T + \Phi G & -G^T + \Phi H \\ -G + H^T\Phi^T & -H - H^T + Q \end{bmatrix} < 0. \tag{12}$$

Proof: Substituting $G = 0$ and $H = Q$ in (12), inequality (10) is obtained. To show that (12) leads to (10) we consider

$$\begin{bmatrix} I & \Phi \\ 0 & I \end{bmatrix} \Omega \begin{bmatrix} I & 0 \\ \phi^T & I \end{bmatrix} = \begin{bmatrix} \Psi + \Phi Q \Phi^T & -G^T - \Phi H^T + \Phi Q \\ -G - H\Phi^T + Q\Phi^T & -H - H^T + Q \end{bmatrix}.$$

Inequality (10) thus follows from the fact that the (1,1) matrix block of the latter matrix is the left side of (10).

Taking $H = G[I_n \; 0 \; 0 \; 0]^T$, $R_2 = \epsilon_r H$ where $\epsilon_r > 0$ is a scalar tuning parameter and denoting $M_H = H^TM$, we note that in (12) the system matrices, excluding D and F, do not multiply Q. It is thus possible to choose vertex dependent $Q^{(i)}$ while keeping H and G constant. We thus arrive at the following result:

Theorem 1: The exponential stability in the mean square sense of the system (2a) where $B_1 = 0$ and where the system matrices lie within the polytope $\bar{\Omega}$ of (5) is guaranteed if there exist matrices $0 < Q_j \in \mathcal{R}^{n\times n}$, $\forall j = 1, ...N$, $0 < R_1 \in \mathcal{R}^{n\times n}$, $M_H \in \mathcal{R}^{n\times n}$, $G \in \mathcal{R}^{n\times 4n}$ and a tuning scalar $\epsilon_r > 0$ that satisfy the following set of inequalities:

$$\Omega_j = \begin{bmatrix} \Psi_j + G^T\Phi^{j,T} + \Phi_j G & -G^T + \Phi_j H \\ -G + H^T\Phi^{j,T} & -H - H^T + Q_j \end{bmatrix} < 0, \tag{13}$$

$\forall j,\ j = 1, 2, ..., N$, where $H \in \mathcal{R}^{n\times n} = G[I_n \; 0 \; 0 \; 0]^T$

$$\Psi_j \triangleq \begin{bmatrix} \Psi_{j,1,1} & 0 & 0 & \Psi_{j,1,4} \\ * & \Psi_{j,2,2} & 0 & h\epsilon_r(A_1^{j,T}H - M_H^T) \\ * & * & -\epsilon_r H & -h\epsilon_r M_H^T \\ * & * & * & -\epsilon_r H \end{bmatrix},$$

$$\Phi_j H = \begin{bmatrix} A_0^{j,T}H + M_H^T \\ A_1^{j,T}H - M_H^T \\ M_H^T \\ 0 \end{bmatrix}, \tag{14}$$

and where

$$\begin{aligned}
\Psi_{j,1,1} &= -Q_j + D^T(Q_j + h^2\epsilon_r H)D + R_1,\\
\Psi_{j,1,4} &= h\epsilon_r(A_0^{j,T}H + M_H^T) - \epsilon_r hH,\\
\Psi_{j,2,2} &= -R_1 + F^T(Q_j + h^2\epsilon_r H)F.
\end{aligned}$$

IV. ROBUST BOUNDED REAL LEMMA

Based on the stability result of Corollary 1, the following result is readily obtained where we consider the system (2a) with $z_k = C_1x_k$ and the following index of performance:

$$J_B = E\{x_{k+1}^TQx_{k+1}\} - x_k^TQx_k + z_k^Tz_k - \gamma^2 w_k^Tw_k.$$

Lemma 3 [8] Consider the system (2a-c). The system is exponentially stable in the mean square sense and, for a prescribed scalar $\gamma > 0$ and a given scalar tuning parameter $\epsilon_b > 0$, the requirement of $J_B < 0$ is achieves for all nonzero $w \in \tilde{l}^2_{\mathcal{F}_k}([0,\infty); \mathcal{R}^q)$, if there exist $n \times n$ matrices $Q > 0$, $R_1 > 0$ and a $n \times n$ matrix Q_m that satisfy $\tilde{\Gamma} < 0$ where $\tilde{\Gamma} =$

$$\begin{bmatrix} \tilde{\Gamma}_{11} & \tilde{\Gamma}_{12} & 0 & 0 & \tilde{\Gamma}_{15} & 0 & C_1^T \\ * & -Q & \tilde{\Gamma}_{23} & Q_m & 0 & QB_1 & 0 \\ * & * & \tilde{\Gamma}_{33} & 0 & \tilde{\Gamma}_{35} & 0 & 0 \\ * & * & * & -\epsilon_bQ & -h\epsilon_bQ_m^T & 0 & 0 \\ * & * & * & * & -\epsilon_bQ & \epsilon_bhQB_1 & 0 \\ * & * & * & * & * & -\gamma^2I_q & 0 \\ * & * & * & * & * & * & -I_r \end{bmatrix} \tag{15}$$

where

$$\begin{aligned}
\tilde{\Gamma}_{11} &= -Q + D^TQ[1+\epsilon_bh^2]D + R_1,\\
\tilde{\Gamma}_{12} &= A_0^TQ + Q_m^T,\\
\tilde{\Gamma}_{15} &= \epsilon_bh[A_0^TQ + Q_m^T] - \epsilon_bhQ,\\
\tilde{\Gamma}_{23} &= QA_1 - Q_m,\\
\tilde{\Gamma}_{33} &= -R_1 + (1+\epsilon_bh^2)F^TQF,\\
\tilde{\Gamma}_{35} &= \epsilon_bh[A_1^TQ - Q_m^T].
\end{aligned}$$

Similarly to the stability condition for the uncertain case of Section 3 we obtain two results for the robust BRL solution. The first one, which is referred to as the quadratic solution, is simply obtained by assigning the same Lyapunov function over the all uncertainty polytope and thus is solved similarly to the robust quadratic condition . A new, possibly less conservative condition is obtained by applying the following vertex-dependent Lyapunov function:

$$\begin{bmatrix} \Psi & \begin{bmatrix} 0 & C_1^T \\ 0 & 0 \\ 0 & 0 \\ hR_2B_1 & 0 \end{bmatrix} \\ \begin{bmatrix} * & * & * & * \\ * & * & * & * \end{bmatrix} & \begin{bmatrix} -\gamma^2 I_q & 0 \\ * & I_r \end{bmatrix} \end{bmatrix}$$

$$+ \begin{bmatrix} \Phi \\ B_1^T \\ 0 \end{bmatrix} Q \begin{bmatrix} \Phi^T & B_1 & 0 \end{bmatrix} \triangleq \hat{\Psi} + \hat{\Phi} Q \hat{\Phi}^T < 0,$$

where Ψ and Φ are given in (11a,b).

Following the derivation of the LMI of Theorem 1, the following result is readily derived:

$$\begin{bmatrix} \hat{\Psi} + \hat{G}^T\hat{\Phi}^T + \hat{\Phi}\hat{G} & -\hat{G}^T + \hat{\Phi}H \\ -\hat{G} + H^T\hat{\Phi}^T & -H - H^T + Q \end{bmatrix} < 0, \quad (16)$$

where now $\hat{G} \in \mathcal{R}^{n\times 4n+q+r}$ and $H \in \mathcal{R}^{n\times n}$. We thus arrive at the following result for the uncertain case, taking $H = \hat{G}[I_n \ 0 \ 0 \ 0]^T$, $R_2 = \epsilon_r H$, $M_H = H^T M$:

Theorem 2 Consider the system (2a-c) where the system matrices lie within the polytope $\bar{\Omega}$ of (5). The system is exponentially stable in the mean square sense and, for a prescribed $\gamma > 0$ and given tuning parameter ϵ_r, the requirement of $J_B < 0$ is achieved for all nonzero $w \in \tilde{l}^2_{\mathcal{F}_k}([0,\infty);\mathcal{R}^q)$, if there exist $0 < Q \in \mathcal{R}^{n\times n}$, $0 < R_1 \in \mathcal{R}^{n\times n}$, $M_H \in \mathcal{R}^{n\times n}$, and $\hat{G} \in \mathcal{R}^{n\times 4n+q+r}$, that satisfy the following set of LMIs:

$$\begin{bmatrix} \hat{\Psi}_j + \hat{G}^T\hat{\Phi}^{j,T} + \hat{\Phi}_j\hat{G} & -\hat{G}^T + \hat{\Phi}_j H \\ -\hat{G} + H^T\hat{\Phi}^{j,T} & -H - H^T + Q_j \end{bmatrix} < 0, \quad (17)$$

where $H \in \mathcal{R}^{n\times n} = \hat{G}[I_n \ 0 \ 0 \ 0]^T$

$$\hat{\Psi}_j = \begin{bmatrix} \Psi_j & \begin{bmatrix} 0 & C_1^{j,T} \\ 0 & 0 \\ 0 & 0 \\ h\epsilon_r B_1^j & 0 \end{bmatrix} \\ \begin{bmatrix} * & * & * & * \\ * & * & * & * \end{bmatrix} & \begin{bmatrix} -\gamma^2 I_q & 0 \\ * & I \end{bmatrix} \end{bmatrix},$$

$$\hat{\Phi}_j H = \begin{bmatrix} \Phi_j H \\ B_1^{j,T} H \\ 0 \end{bmatrix}, \forall j, \ j = 1, 2, ..., N,$$

where Ψ_j and $\Phi_j H$ are given in (14).

V. Robust State-feedback control

In this section we address the problem of finding the following state-feedback control law

$$u_k = Kx_k, \quad (18)$$

that stabilizes the uncertain system and achieves a prescribed level of attenuation. We consider the system of (2a-c) where A_0 is replaced by $(A_0 + B_2 K)$ and C_1 is replaced by $C_1 + D_{12}K$, the following result was obtained in [8]:

Lemma 4 Consider the system (2a-c). For a prescribed scalar $\gamma > 0$, and positive tuning scalar $\epsilon_b > 0$, there exists a state-feedback gain that achieves negative J_E for all nonzero $w \in \tilde{l}^2_{\mathcal{F}_k}([0,\infty);\mathcal{R}^q)$, if there exist $n \times n$ matrices $P > 0$, $\bar{R}_1 > 0$, $n \times n$ matrix M_P and a $l \times n$ matrix K_P that satisfy the following LMI:

$$\Upsilon \triangleq \begin{bmatrix} -P + \bar{R}_1 & \Upsilon_{12} & 0 & 0 & \Upsilon_{15} & 0 & \Upsilon_{17} & \bar{\epsilon}PD^T & 0 & \bar{\epsilon}K_P^T G^T \\ * & -P & A_1P - M_P & M_P & 0 & B_1 & 0 & 0 & 0 & 0 \\ * & * & -\bar{R}_1 & 0 & \Upsilon_{35} & 0 & 0 & 0 & \bar{\epsilon}PF^T & 0 \\ * & * & * & -\epsilon_b P & -h\epsilon_b M_P^T 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\epsilon_b P & h\epsilon_b B_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -I_r & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -P & 0 & 0 \\ * & * & * & * & * & * & * & * & -P & 0 \\ * & * & * & * & * & * & * & * & * & -P \end{bmatrix} < 0, \quad (19)$$

where

$\Upsilon_{12} = PA_0^T + K_P^T B_2^T + M_P^T$,

$\Upsilon_{15} = \epsilon_b h[PA_0^T + K_P^T B_2^T + M_P^T] - \epsilon_b hP$,

$\Upsilon_{17} = PC_1^T + K_P^T D_{12}^T, \quad \Upsilon_{35} = \epsilon_b h[PA_1^T - M_P^T]$,

$\bar{\epsilon}^2 = 1 + \epsilon_b h^2$.

In the latter case the state-feedback gain is given by:

$$K = K_P P^{-1}. \quad (20)$$

In the uncertain case two approaches can be applied. The first one is obtained by assigning a single Lyapumov function over all the uncertainty polytope [the so called 'quadratic' solution]. We obtain the following result already derived in [8]:

Lemma 5 [8] Consider the system (2a-c), where the system matrices lie within the polytope $\bar{\Omega}$ of (5). For a prescribed scalar $\gamma > 0$, and positive tuning scalar $\epsilon_b > 0$, there exists a state-feedback gain that achieves negative J_E for all nonzero $w \in \tilde{l}^2_{\mathcal{F}_k}([0,\infty);\mathcal{R}^q)$, if there exist $n \times n$ matrices $P > 0$, $\bar{R}_1 > 0$, $n \times n$ matrix M_P and a $l \times n$ matrix K_P that satisfy the following set of LMIs:

$$
\begin{bmatrix}
\bar{\Upsilon}^i_{11} & \bar{\Upsilon}^i_{12} & 0 & 0 & \bar{\Upsilon}^i_{15} & 0 & \bar{\Upsilon}^i_{17} & \bar{\Upsilon}^i_{18} & 0 & \bar{\Upsilon}^i_{1,10} \\
* & -P & A^i_1 P - M_P & M_P & 0 & B^i_1 & 0 & 0 & 0 & 0 \\
* & * & -\bar{R}_1 & 0 & \bar{\Upsilon}^i_{35} & 0 & 0 & 0 & \bar{\epsilon} P F^{i,T} & 0 \\
* & * & * & -\epsilon_b P & -h\epsilon_b M_P^T & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\epsilon_b P & h\epsilon_b B^i_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -I_r & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -P & 0 & 0 \\
* & * & * & * & * & * & * & * & -P & 0 \\
* & * & * & * & * & * & * & * & * & -P
\end{bmatrix} < 0, \qquad (21)
$$

$\forall i,\ i = 1, 2, ..., N$, where

$$
\begin{aligned}
&\bar{\Upsilon}^i_{11} = -P + \bar{R}_1, \\
&\bar{\Upsilon}^i_{12} = P A_0^{i,T} + K_P^T B_2^{i,T} + M_P^T, \\
&\bar{\Upsilon}^i_{15} = \epsilon_b h [P A_0^{i,T} + K_P^T B_2^{i,T} + M_P^T] - \epsilon_b h P, \\
&\bar{\Upsilon}^i_{17} = P C_1^{i,T} + K_P^T D_{12}^{i,T}, \\
&\bar{\Upsilon}^i_{18} = \bar{\epsilon} P D^{i,T}, \\
&\bar{\Upsilon}^i_{1,10} = \bar{\epsilon} K_P^T G^{i,T}, \\
&\bar{\Upsilon}^i_{35} = \epsilon_b h [P A_1^{i,T} - M_P^T], \\
&\bar{\epsilon}^2 = 1 + \epsilon_b h^2.
\end{aligned}
$$

In the latter case, the state-feedback gain is given by (20).

The second approach to the robust uncertain case can be achieved by applying the results of the vertex-dependent BRL of Theorem 2. We start with (19) of Lemma 4 and we interchange rows 2 and 3 (and columns 2 and 3). We obtain the following LMI,

$$
\bar{\Upsilon} = \Psi + \hat{P}\Phi + \Phi^T \hat{P} \qquad (22)
$$

where

$$
\Phi^T = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
A_0 + B_2 K & A_1 \\
0 & 0 \\
\epsilon h (A_0 + B_2 K) & \epsilon h A_1 \\
0 & 0 \\
C_1 + D_{12} K & 0 \\
\bar{\epsilon} D & 0 \\
0 & \bar{\epsilon} \bar{F} \\
\bar{\epsilon} G K & 0
\end{bmatrix}, \quad
\hat{P} = \begin{bmatrix}
P & 0 \\
0 & P \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix},
$$

and

$$
\Psi = \begin{bmatrix}
-P + \bar{R}_1 & 0 & M_p^T & 0 & \epsilon h M_p^T & 0 & 0 & 0 & 0 & 0 \\
* & -R_1 & -M_p^T & 0 & -\epsilon h M_p^T & 0 & 0 & 0 & 0 & 0 \\
* & * & -P & M_p & 0 & B_1 & 0 & 0 & 0 & 0 \\
* & * & * & -\epsilon_b P & -h\epsilon_b M_p^T & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\epsilon_b P & h\epsilon_b B_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & *0 & -\gamma^2 I_q & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -I_r & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -P & 0 & 0 \\
* & * & * & * & * & * & * & * & -P & 0 \\
* & * & * & * & * & * & * & * & * & -P
\end{bmatrix}.
$$

(23a-c)

Applying similar result to those of the robust BRL of Section 3 we obtain the following set of LMIs for the uncertain case:

$$
\bar{\Gamma} = \begin{bmatrix}
\Psi + \hat{G}^T \Phi + \Phi^T \hat{G} & -\hat{P} + \hat{G}^T - \Phi^T \hat{H} \\
* & -\hat{H} - \hat{H}^T
\end{bmatrix} < 0, \qquad (24)
$$

where

$$
\hat{G}^T = \begin{bmatrix}
\bar{G}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{G}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

The proof of the latter is obtained by substituting $\hat{G} = \hat{P}$ and $\hat{H} = \varepsilon \hat{G}$, $\varepsilon > 0$ in (24). Following some matrix manipulations and letting $\varepsilon \to 0$, the LMI of (22) is recovered. To show that (24) leads to (22) we multiply (24) by $\tilde{\Upsilon} = \begin{bmatrix} I & -\Phi^T \\ 0 & I \end{bmatrix}$ from the left and by $\tilde{\Upsilon}^T$ from the right. Carrying out both multiplications, the LMI of (22) is obtained. The latter means that if (24) is satisfied then also (22) is satisfied. Applying the latter result to the uncertain polytopic case, we arrive to the following theorem:

Theorem 3 Consider the system (2a-c), where the system matrices lie within the polytope $\bar{\Omega}$ of (5). For a prescribed scalar $\gamma > 0$, and positive tuning scalar $\epsilon_g > 0$, there exists a state-feedback gain that achieves negative J_E for all nonzero $w \in \tilde{l}^2_{\mathcal{F}_k}([0,\infty);\mathcal{R}^q)$, if there exist N $n \times n$ matrices $P^{(i)} > 0$, $\bar{R}_1^{(i)} > 0$, $M_P^{(i)}$, $i = 1,2,..N$ and a $n \times n$ matrix $\hat{G}$ and a $l \times n$ matrix K_G that satisfy the following set of LMIs $\bar{\Gamma}^{(j)} =$

$$\begin{bmatrix} \Psi^{(j)}+\hat{G}^T\Phi^{(j)}+\Phi^{T,(j)}\hat{G} & -\hat{P}^{(j)}+\hat{G}^T-\epsilon_g\Phi^{T,(j)}\hat{G} \\ * & -\epsilon_g[\hat{G}+\hat{G}^T] \end{bmatrix} < 0,$$
$$i = 1,2,...N, \tag{25}$$

where $\Psi^{(j)}$ and $\Phi^{(j)}$ are obtained from (23a,c) by replacing A_0, B_1, B_2, C_1 and D_{12}, D, F with $A_0^{(j)}$, $B_1^{(j)}$, $B_2^{(j)}$, $C_1^{(j)}$ and $D_{12}^{(j)}$, $D^{(j)}$, $F^{(j)}$, respectively and by replacing P, $\bar{R}$ and M_p by $P^{(j)}$, $\bar{R}^{(j)}$ and $M_p^{(j)}$ in (23b,c). In the latter case the state-feedback gain is given by

$$K = K_G\hat{G}^{-1}. \tag{26}$$

VI. Conclusions

In this paper the new theory of robust vertex-dependent H_∞ state-feedback control of state-multiplicative noisy systems is developed for discrete-time delayed uncertain polytopic systems. In these systems, the stochastic uncertainties are encountered in both the delayed and the non delayed states in the state space model of the system. The delay is assumed to be unknown and time-varying where only the bound on its size is given. Delay dependent synthesis methods are developed which are originally based on the input-output approach. This approach transforms the delayed system to a non-retarded system with norm-bounded operators.

We note that whereas both the robust state-feedback and the robust static output feedback control of our systems have already been published, they all treat only the quadratic case which can never yield the vertex-dependent result. In order to apply the latter approach, a newly vertex-dependent stability approach was first introduced followed by the robust BRL derivation.

Based on the robust BRL derivation, the robust vertex-dependent state-feedback control problem is formulated and solved where a superior result is obtained compared to the linear quadratic solution case. Some over-design is admitted to our solution due to the use of the bounded operators which enable us to transform the retarded system to a norm-bounded one. Some additional over-design is also admitted in our solution due to the special structure imposed on the decision variable H.

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