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Compositional construction of abstractions via relaxed small-gain conditions Part II: discrete case

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Abstract— **This paper investigates the compositional construction of approximately bisimilar abstractions for incrementally input-to-state stable (δ-ISS) networks of systems. Toward this end, we develop small-gain conditions which can be applied to networks whose subsystems are not necessarily individually δ-ISS. In contrast to existing results, our method for the compositional construction of the abstractions does *not* assume that each subsystem is individually δ-ISS. The effectiveness of our results is illustrated via an example.**

I. Introduction

Modern control systems are becoming larger in scope and more complex in structure. Modern applications demand sophisticated control objectives, which go well beyond standard objectives in classic control theory such as stability or invariance. For example, a complex objective is to automatically synthesize a provably correct control software adjusting the traffic lights such that the congestion is eventually eliminated and freeway throughput is ensured to remain above a minimum threshold. All these requirements have driven the need for bringing ideas from computer science to control theory. In particular, *symbolic methods* have attracted a lot of attention in the past decade by providing a rigorous framework to efficiently address the above issues [1].

In general, symbolic techniques require a discrete representation, called *symbolic model*, of the concrete system representing (approximately) behaviors of the system. A discrete abstraction models the states and dynamics of the original system with a finite set of states and transitions, respectively, which capture all the phenomena of interest. The major issue in the construction of discrete abstractions is that the computation of abstractions does not scale well with the number of state variables in the concrete system. Hence, it is almost infeasible to compute abstractions for large-scale systems. A promising direction to resolve this issue is to decompose the overall system into several subsystems and then compute the abstraction of each subsystem. The methodology to obtain abstractions for the overall system via the interconnection of abstractions of the subsystems is called the *compositional* approach. Existing results on compositional construction of symbolic models include [2]–[8].

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This paper extends the methodology proposed in [9] from continuous abstractions of nonlinear discrete-time networks to discrete abstractions of such systems. In particular, here we provide a compositional scheme for the construction of symbolic models which are approximately *bisimilar* [1] to a network of discrete-time systems. Note that the existence of an approximate bisimulation relation between a concrete system and its discrete abstraction guarantees that a controller exists for the former one enforcing some complex logic properties if and only if a controller exists for the latter one enforcing the same properties [10], [11]. As shown in [10], [11], the existence of such an approximately bisimilar symbolic model is ensured if the concrete system is incrementally input-to-state stable (δ-ISS) [12].

Small-gain theory is used to develop compositional construction of approximately bisimilar discrete abstractions for interconnected nonlinear systems. In particular, discrete abstractions for linear [7] and nonlinear networks [6] were proposed using the classic small-gain theory, where each subsystem is assumed to be individually δ-ISS. However, this requirement can be dispelled if one uses non-conservative small-gain conditions, which are not only sufficient but also necessary to guarantee the desired stability property for the interconnected systems [13]–[15]. This paper proposes non-conservative small-gain conditions ensuring δ-ISS for the interconnected systems. Using such small-gain conditions, we develop compositional constructions of approximately bisimilar symbolic models, where each subsystem is not necessarily required to be δ-ISS. This is in contrast with [6], [7], where each subsystem is required to be δ-ISS. As illustrated via an example, our approach is applicable to networks with subsystems which are *individually* unstable.

Toward the compositional construction of abstractions, we introduce a new notion of approximate bisimulation relation, which is called M*-step* approximate bisimulation relation. This notion establishes the closeness of trajectories of the original system and its abstraction, where the trajectories are close to each other every M time-step. We establish the approximate bisimilarity between the original network and its abstraction, where subsystems are only required to have M-step approximately bisimilar abstractions. This property is guaranteed by the relaxed small-gain conditions proposed in this paper. When $M = 1$, our results recover those in [6], [7].

This paper is organized as follows: First, relevant notation is recalled in Section II. Then the problem formulation is stated in Section III. The relaxed small-gain condition and the compositional construction method are presented in Section IV. The numerical verification of the results is given in Section V. Section VI concludes the paper.

II. Notation

In this paper, $\mathbb{R}_{\geq 0}(\mathbb{R}_{>0})$ and $\mathbb{N}_0(\mathbb{N})$ denote the nonnegative (positive) real numbers and the nonnegative (positive) integers, respectively. The vector space of real column vectors of length n is denoted by $\mathbb{R}^n$. The ith component of $v \in \mathbb{R}^n$ is denoted by v_i. Given $\mathcal{S} \subseteq \mathbb{R}^n$, the Cartesian power of $\mathcal{S}$ is defined by $\mathcal{S}^\ell := \{(s_1, s_2, \ldots, s_\ell) : s_i \in \mathcal{S} \ \forall\ i = 1, \ldots, \ell\}$. By $\mathbb{R}^n_{>0}$ we denote $(\mathbb{R}_{>0})^n = \mathbb{R}_{>0} \times \cdots \times \mathbb{R}_{>0}$. For any $x \in \mathbb{R}^n$, $x^\top$ denotes its transpose. We write (x, y) to represent $[x^\top, y^\top]^\top$ for $x \in \mathbb{R}^n, y \in \mathbb{R}^p$. Given a pair of sets $\mathcal{X}$ and $\mathcal{Y}$ and a relation $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$, $\mathcal{R}^{-1}$ denotes the inverse relation of $\mathcal{R}$, *i.e.* $\mathcal{R}^{-1} = \{(y, x) \in \mathcal{Y} \times \mathcal{X} : (x, y) \in \mathcal{R}\}$. Moreover, $\mathcal{R}(\mathcal{X}) = \{b \in \mathcal{Y} : \exists a \in \mathcal{X} \text{ s.t. } (a, b) \in \mathcal{R}\}$ and $\mathcal{R}^{-1}(\mathcal{Y}) = \{a \in \mathcal{X} : \exists b \in \mathcal{Y} \text{ s.t. } (a, b) \in \mathcal{R}\}$. The infinity norm is denoted by $|x|$ for $x \in \mathbb{R}^n$. Given $\eta \in \mathbb{R}_{>0}$ and $\mathcal{X} \subseteq \mathbb{R}^n$, we denote $[\mathcal{X}]_\eta := (\eta\mathbb{Z}^n) \cap \mathcal{X}$. We note that if $\mathcal{X}$ is convex and with non-empty interior there always exists $\eta \in \mathbb{R}_{>0}$ such that for any $x \in \mathcal{X}$ there exists $y \in [\mathcal{X}]_\eta$ such that $|x - y| \leq \eta$. Order the set $[\mathcal{X}]_\eta$ lexicographically (with respect to the components of the vectors) and define $Y := \{y \in [\mathcal{X}]_\eta : |x - y| \leq \eta\}$ for some given $x \in \mathcal{X}$. We denote the first component of the set Y (in the lexicographical order) by $[x]_\eta$.

A function $\alpha \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, strictly increasing with $\alpha(0) = 0$. It is of class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if $\alpha \in \mathcal{K}$ and also $\alpha(s) \to \infty$ as $s \to \infty$. A continuous function $\beta \colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$), if for each $s \geq 0$, $\beta(\cdot, s) \in \mathcal{K}$, and for each $r > 0$, $\beta(r, \cdot)$ is decreasing with $\beta(r, s) \to 0$ as $s \to \infty$. Given a function $\varphi \colon \mathbb{N}_0 \to \mathbb{R}^m$, its sup-norm (possibly infinite) is denoted by $\|\varphi\| = \sup\{|\varphi(k)| : k \in \mathbb{N}_0\} \leq \infty$. The identity function is denoted by id. Composition of functions is denoted by the symbol $\circ$ and the i-times repeated composition of a function γ by γ^i. For functions $\alpha, \gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ we write $\alpha < \gamma$ if $\alpha(s) < \gamma(s)$ for all $s > 0$.

III. Problem Formulation

Consider the following discrete-time control system

$$\Sigma\colon \quad x(k+1) = g(x(k), u(k)), \tag{1}$$

where the state $x \in \mathcal{X} \subset \mathbb{R}^n$ and the input $u \in \mathcal{U} \subset \mathbb{R}^m$. The sets $\mathcal{X}$ and $\mathcal{U}$ are assumed to be convex, bounded and with non-empty interior. We assume that $g \colon \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ is continuous.

The set of all inputs $u : \mathbb{N}_0 \to \mathcal{U}$ is denoted by U. Note that, by a slight abuse of notation, we will generally use $u \in \mathcal{U}$ and $u \in U$ where u being a vector or function, respectively, will be clear from context. Given any input $u \in U$ and any initial value $\xi \in \mathcal{X}$, the corresponding solution to (1) is denoted by $x(\cdot, \xi, u)$.

Definition 1: Given $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$, we call the function g in (1) *$\mathcal{K}$-bounded* on $(\mathcal{X}, \mathcal{U})$ if there exist $\kappa_1, \kappa_2 \in \mathcal{K}$ such that

$$\left|g(\xi, \mu) - g(\hat{\xi}, \hat{\mu})\right| \leq \kappa_1(\left|\xi - \hat{\xi}\right|) + \kappa_2(|\mu - \hat{\mu}|) \tag{2}$$

for all $\xi, \hat{\xi} \in \mathcal{X}$ and all $\mu, \hat{\mu} \in \mathcal{U}$. □

This definition is a variant of the $\mathcal{K}$-boundedness definition from [13]. The importance of the $\mathcal{K}$-boundedness will become obvious in the subsequent results.

We assume that system (1) can be decomposed into ℓ subsystems

$$\Sigma_i\colon \quad x_i(k+1) = g_i(x_1(k), \ldots, x_\ell(k), u(k)), \tag{3}$$

where each $g_i \colon \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_n \times \mathcal{U} \to \mathcal{X}_i$ is continuous and the state $x_i \in \mathcal{X}_i \subset \mathbb{R}^{n_i}$ for each $i \in \{1, \ldots, \ell\}$, and the input $u \in \mathcal{U}$. The sets $\mathcal{X}_i$ for $i = 1, \ldots, \ell$ are assumed to be convex and with non-empty interior. From the decomposition, we have that $n = n_1 + \ldots + n_\ell$, $x = (x_1, \ldots, x_\ell)$ and $g = (g_1, \ldots, g_\ell)$.

The ultimate aim of this paper is to construct a discretized representation, called a symbolic model, of the concrete system Σ in the form of a discrete abstraction for the system's behavior. A discrete abstraction models the states and dynamics of the system with a finite set of symbols and transitions which capture all the phenomena of interest. To this end, we need to describe both the original system as well as its symbolic models in a unified way. The following definitions are required toward such a unified treatment.

Definition 2: [1] A system Σ is a quintuple $\Sigma = (\mathcal{X}, \mathcal{U}, g, \mathcal{Y}, H)$, consisting of a set of states $\mathcal{X}$, a set of inputs $\mathcal{U}$, a transition map $g : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$, a set of outputs $\mathcal{Y}$ and an output function $H : \mathcal{X} \to \mathcal{Y}$. If $\mathcal{Y} = \mathcal{X}$ and the output function $H(x) = x$ for any $x \in \mathcal{X}$, then the notation reduces $\Sigma = (\mathcal{X}, \mathcal{U}, g)$. □

System Σ is said to be symbolic if $\mathcal{X}$ and $\mathcal{U}$ are *finite* sets and metric if the output set $\mathcal{Y}$ is equipped with a metric $\mathbf{d} : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_{\geq 0}$. Similarly, a network of systems can be described as follows.

The original version of Definition 2 in [1] considers the codomain of the transition map g as the power set of $\mathcal{X}$. Here we only reduce the codomain of g to $\mathcal{X}$ because of the so-called determinism [1] of the system, which indeed improves the readability of the paper.

Definition 3: Given a collection of systems $\Sigma_i = (\mathcal{X}_i, \mathcal{X}_1 \times \cdots \times \mathcal{X}_{i-1} \times \mathcal{X}_{i+1} \times \cdots \times \mathcal{X}_\ell \times \mathcal{U}_i, g_i)$ for $i = 1, \cdots, \ell$, define the network $\Sigma(\{\Sigma_i\}_{i=1}^{\ell}) := (\mathcal{X}, \mathcal{U}, g)$ where $\mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_\ell$, $\mathcal{U} := \mathcal{U}_1 \times \cdots \times \mathcal{U}_\ell$, $g(x, u) := \big(g_1(x, u_1), \ldots, g_\ell(x, u_\ell)\big)$, where $x = (x_1, \ldots, x_\ell)$ and $u = (u_1, \ldots, u_\ell)$. □

Let $\mathbf{u}^k := \{u_i\}_{i=1}^k \in \mathcal{U}^k$ be a sequence of inputs, where $u_i \in \mathcal{U}$ for all $i \in \{1, \cdots, k\}$.

Definition 4: Given $M \in \mathbb{N}$, an input sequence $\mathbf{u}^M \in \mathcal{U}^M$, and state $x \in \mathcal{X}$ and a transition map $g : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$, an *$M$-step* successive transition, denoted by g^M, is defined by

$$g^M(x, \mathbf{u}^M) := g(g^{M-1}(x, \mathbf{u}^{M-1}), u_M),$$

where $g^1(x, \mathbf{u}^1) = g(x, u_1)$. □

Now we introduce a variant of approximate bisimulation relation [1], which is called *M-step* approximate bisimulation relation.

Definition 5: Let $M \in \mathbb{N}$ and $\Sigma_i = (\mathcal{X}_i, \mathcal{U}_i, g_i, \mathcal{Y}_i, H_i)$, $i = 1, 2$, be metric systems with the same output sets $\mathcal{Y}_1 = \mathcal{Y}_2$ and the associated metric $\mathbf{d}_{\mathcal{Y}}$, and let $\varepsilon > 0$ be a given precision. A relation $\mathcal{R} \subseteq \mathcal{X}_1 \times \mathcal{X}_2$ is called an M-step ε-approximate bisimulation relation between Σ_1 and Σ_2 if for all $(x_1, x_2) \in \mathcal{R}$ the following conditions hold: (i) $\mathbf{d}_{\mathcal{Y}}(H_1(x_1), H_2(x_2)) \leq \varepsilon$; (ii) $\forall \mathbf{u}_1^M \in \mathcal{U}_1^M$ $\exists \mathbf{u}_2^M \in \mathcal{U}_2^M$, $\forall z_1 = g_1^M(x_1, \mathbf{u}_1^M)$ $\exists$ $z_2 = g_2^M(x_2, \mathbf{u}_2^M)$ such that $(z_1, z_2) \in \mathcal{R}$; (iii) $\forall \mathbf{u}_2^M \in \mathcal{U}_2^M$ $\exists \mathbf{u}_1^M \in \mathcal{U}_1^M$, $\forall z_2 = g_2^M(x_2, \mathbf{u}_2^M)$ $\exists$ $z_1 = g_1^M(x_1, \mathbf{u}_1^M)$ such that $(z_1, z_2) \in \mathcal{R}$.
Systems Σ_1 and Σ_2 are M-step ε-approximately bisimilar, if there exists an ε-approximate bisimulation relation $\mathcal{R}$ between Σ_1 and Σ_2 such that $\mathcal{R}(\mathcal{X}_1) = \mathcal{X}_2$ and $\mathcal{R}^{-1}(\mathcal{X}_2) = \mathcal{X}_1$. □

When $M = 1$, Definition 5 reduces to the classic notion of ε-approximate bisimulation relation in [1]. Therefore, we drop the term "finite-step" and instead speak of a "classic" ε-approximate bisimulation abstraction when $M = 1$.

As shown in [10], [11], the δ-ISS property [16], as recalled next, guarantees the existence of approximately bisimilar abstractions with arbitrary precision.

Definition 6: System (1) is said to be incrementally input-to-state stable (δ-ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $\xi, \hat{\xi} \in \mathcal{X}$, all $u, \hat{u} \in U$ and all $k \in \mathbb{N}_0$ we have

$$\left|x(k, \xi, u) - x(k, \hat{\xi}, \hat{u})\right| \leq \max\Big\{\beta\big(\big|\xi - \hat{\xi}\big|, k\big), \gamma(\|u - \hat{u}\|)\Big\}. \tag{4}$$

□

Now, we provide sufficient conditions for the δ-ISS property in terms of the existence of a so-called finite-step δ-ISS Lyapunov function.

Definition 7: Let $M \in \mathbb{N}$. A continuous function $V \colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ is called a finite-step δ-ISS Lyapunov function for (1) if there exist $\underline{\alpha}, \overline{\alpha}, \alpha \in \mathcal{K}_\infty$ with $\alpha < \mathrm{id}$ and $\gamma \in \mathcal{K}$ such that for all $\xi, \hat{\xi} \in \mathcal{X}$ and all $u, \hat{u} \in U$, the following holds

$$\underline{\alpha}(\big|\xi - \hat{\xi}\big|) \leq V(\xi, \hat{\xi}) \leq \overline{\alpha}(\big|\xi - \hat{\xi}\big|), \tag{5}$$

$$V(x(M, \xi, u), x(M, \hat{\xi}, \hat{u})) \leq \max\{\alpha(V(\xi, \hat{\xi})), \gamma(\|u - \hat{u}\|)\}. \tag{6}$$

When $M = 1$, we drop the term "finite-step" and instead speak of a "classic" δ-ISS Lyapunov function. □

Proposition 8: Let $g \colon \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ as in (1) be $\mathcal{K}$-bounded on $(\mathcal{X}, \mathcal{U})$. If there exists a finite-step δ-ISS Lyapunov function V for (1), then (1) is δ-ISS. □
The proof follows the same argument as those in the proof of [13, Theorem 7]. Thus it is not presented here.

Using finite-step δ-ISS Lyapunov functions, small-gain conditions ensuring incremental ISS of the network Σ are developed in the next section, where each subsystem does *not* have to be necessarily δ-ISS. Such small-gain conditions allow for compositional construction of finite abstractions for (1) without assuming that each subsystem is individually δ-ISS.

IV. Main Results

In this section, the compositional construction of approximately bisimilar abstractions for the interconnected system (1) is given. We begin with the construction of abstractions for each subsystem. Then sufficient conditions are proposed, which guarantee the existence of an approximate bisimulation relation between the network Σ and its abstraction.

A. Symbolic models for the subsystems

Following Definition 2, each subsystem Σ_i as in (3) can be denoted by $\Sigma_i = (\mathcal{X}_i, \mathcal{W}_i \times \mathcal{U}_i, g_i)$, where $\mathcal{W}_i := \mathcal{X}_1 \times \cdots \times \mathcal{X}_{i-1} \times \mathcal{X}_{i+1} \times \cdots \times \mathcal{X}_\ell$, $\mathcal{U}_i = \mathcal{U}$. We note that $\mathcal{X}_i \subset \mathbb{R}^{n_i}$ for each $i \in \{1, \ldots, \ell\}$, and $\mathcal{U} \subset \mathbb{R}^m$. System Σ_i is equipped with the metric $\mathbf{d}_i(x_i, \hat{x}_i) = |x_i - \hat{x}_i|$, where $x_i, \hat{x}_i \in \mathcal{X}_i$.

Discrete abstractions of Σ_i's are also defined using Definition 2.

Definition 9: Given $\Sigma_i = (\mathcal{X}_i, \mathcal{W}_i \times \mathcal{U}_i, g_i)$, for $i = 1, \ldots, \ell$, and a state quantization vector $\eta^x \in \mathbb{R}^\ell_{>0}$ and an input quantization constant $\eta^u > 0$, define systems $\Sigma_i^a = (\mathcal{X}_i^a, \mathcal{W}_i^a \times \mathcal{U}_i^a, g_i^a)$ for $i = 1, \ldots, \ell$, where $\mathcal{X}_i^a = [\mathcal{X}_i]_{\eta_i^x}$, $\mathcal{W}_i^a := [\mathcal{X}_1]_{\eta_1^x} \times \cdots \times [\mathcal{X}_{i-1}]_{\eta_{i-1}^x} \times [\mathcal{X}_{i+1}]_{\eta_{i+1}^x} \times \cdots \times [\mathcal{X}_\ell]_{\eta_\ell^x}$, $\mathcal{U}_i^a := [\mathcal{U}_i]_{\eta_u}$, $g_i^a : \mathcal{X}_i^a \times \mathcal{W}_i^a \times \mathcal{U}_i^a \to \mathcal{X}_i^a$ which is described by $(x_i, w_i, u) \mapsto [g_i(x_i, w_i, u)]_{\eta_i^x}$. □

Remark 10: We note that the construction of the sets $[\mathcal{X}]_{\eta_i^x}$ and $[\mathcal{U}]_{\eta^u}$ is a simple task if the sets $\mathcal{X}$ and $\mathcal{U}$ are already given as the union of boxes, *i.e.* the union of Cartesian products of the form $\prod_{i=1}^n [a_i, b_i]$, where $a_i < b_i$ are real numbers. □

The same metric as that for Σ_i is considered for Σ_i^a, *i.e.* $\mathbf{d}_i(x_i^a, \hat{x}_i^a) = |x_i^a - \hat{x}_i^a|$. We note that Σ_i^a is a finite system as the sets $\mathcal{X}_i^a$, $\mathcal{W}_i^a$ and $\mathcal{U}_i^a$ are finite.

We need the following assumptions on subsystems (3) to show the main result of the paper.

Assumption 11: Let $M \in \mathbb{N}$. Suppose that for each subsystem Σ_i, there exists a Lipschitz continuous function $W_i \colon \mathcal{X}_i \times \mathcal{X}_i \to \mathbb{R}_{\geq 0}$ such that the following conditions hold

(i) There exist functions $\overline{\alpha}_i, \underline{\alpha}_i \in \mathcal{K}_\infty$ such that for all $\xi_i, \hat{\xi}_i \in \mathcal{X}_i$

$$\underline{\alpha}_i(\big|\xi_i - \hat{\xi}_i\big|) \leq W_i(\xi_i, \hat{\xi}_i) \leq \overline{\alpha}_i(\big|\xi_i - \hat{\xi}_i\big|). \tag{7}$$

(ii) There exist $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$, $j = 1, \ldots, \ell$, with $\gamma_{ii} < \mathrm{id}$, and $\gamma_{iu} \in \mathcal{K} \cup \{0\}$ such that for all $\xi, \hat{\xi} \in \mathcal{X}$ and all $u, \hat{u} \in U$ the estimate

$$W_i(x_i(M, \xi, u), x_i(M, \hat{\xi}, \hat{u})) \leq \max\left\{\max_{j \in \{1, \ldots, \ell\}}\Big\{\gamma_{ij}(W_j(\xi_j, \hat{\xi}_j))\Big\}, \gamma_{iu}(\|u - \hat{u}\|)\right\} \tag{8}$$

holds, where $x_i(\cdot, \xi, u)$ denotes the ith component of the solution $x(\cdot, \xi, u)$ of Σ, that corresponds to the subsystem Σ_i. □

We note that the function W_i satisfying (7) and (8) is not a finite-step δ-ISS Lyapunov function for the ith subsystem. In fact, it is an estimate for trajectories of the overall system Σ, or rather the ith component of this trajectory.

Proposition 12: Suppose that Assumption 11 holds. Let L_i be the Lipschitz constant associated with W_i in $\mathcal{X}_i \times \mathcal{X}_i$. Then, for any desired precision $\varepsilon_i > 0$ and for any quantization parameters $\eta^x \in \mathbb{R}^\ell_{>0}$ and $\eta^u > 0$ satisfying the following inequalities

$$\max\left\{\max_{j\in\{1,\dots,\ell\},j\neq i}\left\{\gamma_{ij}\circ\overline{\alpha}_j(\eta_i^x)\right\},\gamma_{iu}(\eta^u)\right\} + L_i\eta_i^x \leq \rho_i\circ\underline{\alpha}_i(\varepsilon_i), \quad (9)$$

$$\overline{\alpha}_i(\eta_i^x) \leq \underline{\alpha}_i(\varepsilon_i) \qquad (10)$$

with $\rho_i := \mathrm{id} - \gamma_{ii}$, the systems Σ_i and Σ_i^a are M-step ε_i-approximately bisimilar. □

Proof: The proof is inspired by that of [11, Theorem 5.1]. Consider the relation $\mathcal{R}_i := \{(\xi_i,\hat{\xi}_i) \in \mathcal{X}_i \times \mathcal{X}_i^a | W_i(\xi_i,\hat{\xi}_i) \leq \underline{\alpha}(\varepsilon_i)\}$. Take any pair $(\xi_i,\hat{\xi}_i) \in \mathcal{R}_i$. It follows from the first inequality (7) and the monotonicity of $\underline{\alpha}_i$ that item (i) of Definition 5 holds. Now we establish item (ii) of Definition 5. Take any input sequence $\mathbf{u} \in \mathcal{U}^M$ and any $(\xi_1,\dots,\xi_{i-1},\xi_{i+1},\dots,\xi_\ell) \in \mathcal{W}_i$ and the associated M-step successive transition $g_i^M(\xi,\mathbf{u}^M)$, where $\xi = (\xi_1,\xi_2,\dots,\xi_\ell)$. Let $z_i := x_i(M,\xi,\mathbf{u}^M) = g_i^M(\xi,\mathbf{u}^M)$. Also, take any input sequence $\hat{\mathbf{u}} \in (\mathcal{U}^a)^M$ and any $(\hat{\xi}_1,\dots,\hat{\xi}_{i-1},\hat{\xi}_{i+1},\dots,\hat{\xi}_\ell) \in \mathcal{W}_i^a$ such that $\left|\xi_j - \hat{\xi}_j\right| \leq \eta_j^x$ for all $j \in \{1,\dots,\ell\}$ and $\left\|\mathbf{u}^M - \hat{\mathbf{u}}^M\right\| \leq \eta^u$. Let $\hat{z}_i := x_i(M,\hat{\xi},\hat{\mathbf{u}}^M)$, where $\hat{\xi} = (\hat{\xi}_1,\hat{\xi}_2,\dots,\hat{\xi}_\ell)$, and $\tilde{z}_i := [\hat{z}_i]_{\eta_i^x}$ which is generated by the M-step successive transition $(g_i^a)^M(\hat{\xi},\hat{\mathbf{u}}^M)$ of the subsystem Σ_i^a.

From the Lipschitz continuity of W_i, we have that

$$W_i(z_i,\tilde{z}_i) \leq W_i(z_i,\hat{z}_i) + L_i\left|\tilde{z}_i - \hat{z}_i\right|.$$

It follows from (8) that

$$W_i(z_i,\tilde{z}_i) \leq \max\left\{\max_{j\in\{1,\dots,\ell\}}\left\{\gamma_{ij}(W_j(\xi_j,\hat{\xi}_j))\right\}, \gamma_{iu}(\left\|\mathbf{u}^M - \hat{\mathbf{u}}^M\right\|)\right\} + L_i\left|\tilde{z}_i - \hat{z}_i\right|.$$

It follows from the choice of $(\xi_i,\hat{\xi}_i)$ that

$$W_i(z_i,\tilde{z}_i) < \max\left\{\gamma_{ii}\circ\underline{\alpha}_i(\varepsilon_i), \max_{j\in\{1,\dots,\ell\},j\neq i}\left\{\gamma_{ij}(W_j(\xi_j,\hat{\xi}_j))\right\}, \gamma_{iu}(\left\|\mathbf{u}^M - \hat{\mathbf{u}}^M\right\|)\right\} + L_i\left|\tilde{z}_i - \hat{z}_i\right|.$$

By the second inequality of (7), we have

$$W_i(z_i,\tilde{z}_i) \leq \max\left\{\gamma_{ii}\circ\underline{\alpha}_i(\varepsilon_i), \max_{j\in\{1,\dots,\ell\},j\neq i}\left\{\gamma_{ij}\circ\overline{\alpha}_j(\left|\xi_j - \hat{\xi}_j\right|)\right\}, \gamma_{iu}(\left\|\mathbf{u}^M - \hat{\mathbf{u}}^M\right\|)\right\} + L_i\left|\tilde{z}_i - \hat{z}_i\right|.$$

As $\gamma_{ii} < \mathrm{id}$, there exists some $\rho_i < \mathrm{id}$ such that $\gamma_{ii} = \mathrm{id} - \rho_i$. In that way, we have

$$W_i(z_i,\tilde{z}_i) \leq \max\left\{(\mathrm{id} - \rho_i)\circ\underline{\alpha}_i(\varepsilon_i), \max_{j\in\{1,\dots,\ell\},j\neq i}\left\{\gamma_{ij}\circ\overline{\alpha}_j(\left|\xi_j - \hat{\xi}_j\right|)\right\}, \gamma_{iu}(\left\|\mathbf{u}^M - \hat{\mathbf{u}}^M\right\|)\right\} + L_i\left|\tilde{z}_i - \hat{z}_i\right|.$$

Using the fact that $\max\{a,b\} \leq a+b$ for any $a,b \geq 0$, we get

$$W_i(z_i,\tilde{z}_i) \leq (\mathrm{id} - \rho_i)\circ\underline{\alpha}_i(\varepsilon_i) + \max\left\{\max_{j\in\{1,\dots,\ell\},j\neq i}\left\{\gamma_{ij}\circ\overline{\alpha}_j(\left|\xi_j - \hat{\xi}_j\right|)\right\}, \gamma_{iu}(\left\|\mathbf{u}^M - \hat{\mathbf{u}}^M\right\|)\right\} + L_i\left|\tilde{z}_i - \hat{z}_i\right|.$$

Recalling the choice of $\tilde{z}_i$ and the condition (9) gives

$$W_i(z_i,\tilde{z}_i) \leq \underline{\alpha}_i(\varepsilon_i).$$

Then, item (ii) of Definition 5 holds. With the similar arguments as those for the proof of item (ii), one can can show item (iii) of Definition 5.

For any $\xi_i \in \mathcal{X}_i$, take $\hat{\xi}_i := [\xi_i]_{\eta_i^x} \in \mathcal{X}_i^a$. It follows from the second inequality of (7) and the choice of $\hat{\xi}_i$ that

$$W_i(\xi_i,\hat{\xi}_i) \leq \overline{\alpha}(\left|\xi_i - \hat{\xi}_i\right|) \leq \overline{\alpha}_i(\eta_i^x).$$

From (10), we have

$$W_i(\xi_i,\hat{\xi}_i) \leq \underline{\alpha}_i(\varepsilon_i).$$

This implies that $\mathcal{R}(\mathcal{X}_i) = \mathcal{X}_i^a$. To show the converse, we note that taking $\xi_i = \hat{\xi}_i$ for any $\hat{\xi}_i \in \mathcal{X}_i^a$ yields that $W_i(\xi_i,\hat{\xi}_i) \leq \underline{\alpha}(\varepsilon_i)$. This implies that $\mathcal{R}^{-1}(\mathcal{X}_i^a) = \mathcal{X}_i$. This completes the proof. ■

B. Symbolic models for the network

Using Definition 3, we define symbolic models for the network $\Sigma(\{\Sigma_i\}_{i=1}^\ell)$ as $\Sigma^a(\{\Sigma_i^a\}_{i=1}^\ell) := (\mathcal{X}^a,\mathcal{U}^a,g^a)$, where $\mathcal{X}^a := \mathcal{X}_1^a\times\cdots\times\mathcal{X}_\ell^a, \mathcal{U}^a := \mathcal{U}_1^a\times\cdots\times\mathcal{U}_\ell^a, g^a(x,u) = \left(g_1^a(x,u_1),\dots,g_\ell^a(x,u_\ell)\right)$, where $x = (x_1,\dots,x_\ell)$ and $u = (u_1,\dots,u_\ell)$. System Σ^a is equipped with the metric $\mathbf{d}(x,\hat{x}) = |x - \hat{x}|$, where $x,\hat{x} \in \mathcal{X}^a$.

Proposition 13: Consider the systems (3) and let Assumption 11 hold. Assume the functions γ_{ij} given in (8) satisfy

$$\gamma_{i_1i_2}\circ\gamma_{i_2i_3}\circ\cdots\circ\gamma_{i_{r-1}i_r}\circ\gamma_{i_ri_1} < \mathrm{id} \qquad (11)$$

for all sequences $(i_1,\dots,i_r) \in \{1,\dots,\ell\}^r$ and $r = 1,\dots,\ell$. Then there exists $\sigma_i \in \mathcal{K}_\infty$, $i = 1,\dots,\ell$ such that function $V : \mathcal{X}\times\mathcal{X} \to \mathbb{R}_{\geq 0}$ defined by

$$V(\xi,\hat{\xi}) := \max_{i\in\{1,\dots,\ell\}}\sigma_i^{-1}(W_i(\xi_i,\hat{\xi}_i)) \qquad (12)$$

is a finite-step δ-ISS Lyapunov function for the network Σ. □

The proof follows similar arguments as those in [13]. Hence, it is not presented here.

By the $\mathcal{K}$-boundedness condition of the dynamics g and Proposition 8, Proposition 13 implies δ-ISS of network (1). In contrast to [6], Proposition 13 does not require that every subsystem Σ_i is δ-ISS. Now we state the main result, which establish the classic approximate bisimilarity ($i.e.$ the approximate bisimilarity as in Definition 5 with $M = 1$) between $\Sigma(\{\Sigma_i\}_{i=1}^{\ell})$ and $\Sigma^a(\{\Sigma_i^a\}_{i=1}^{\ell})$.

Theorem 14: Consider system (1). Let $g : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ as in (1) be $\mathcal{K}$-bounded on $(\mathcal{X},\mathcal{U})$. Let $V : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ come from Proposition 13. There exists a *classic* δ-ISS function $V_c : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ with the corresponding gain functions $\underline{\alpha}_c, \overline{\alpha}_c, \alpha_c \in \mathcal{K}_\infty$ and $\gamma_c \in \mathcal{K}$ as in Definition 7 for the network Σ.
Moreover, assume that V_c is Lipschitz on $\mathcal{X} \times \mathcal{X}$ with the Lipschitz constant L. For any desired precision $\varepsilon > 0$, take the quantization parameters $\eta_c^x, \eta_c^u > 0$ such that the following inequalities hold

$$\begin{aligned} \gamma_c(\eta_c^u) + L\eta_c^x &\leq \rho_c \circ \underline{\alpha}_c(\varepsilon), &(13)\\ \overline{\alpha}_c(\eta_c^x) &\leq \underline{\alpha}_c(\varepsilon), &(14)\end{aligned}$$

where $\rho_c := \mathrm{id} - \alpha_c$. Let $\eta_i^x = \eta_c^x$ for all $i \in \{1, \ldots, \ell\}$ and $\eta^u = \eta_c^u$. Then $\Sigma^a(\{\Sigma_i^a\}_{i=1}^{\ell})$, where each Σ_i^a is constructed by the corresponding η_i^x and η^u, and $\Sigma(\{\Sigma_i\}_{i=1}^{\ell})$ are ε-approximately bisimilar. □

Before proceeding to the proof of Theorem 14, we emphasize that the theorem gives the same quantization parameters for each subsystem Σ_i^a, which can be conservative in practice. As shown in [6], this conservatism can be relaxed by splitting the corresponding graph of the network Σ into strongly connected subgraphs [17] and then applying Theorem 14 to each strongly connected subgraph.

Proof: We present a sketch of the proof. We first construct the δ-ISS function V_c from the finite-step δ-ISS Lyapunov function V generated by Proposition 13. Then we show the approximate bisimilarity between Σ and Σ^a.

Define the auxiliary system by

$$\begin{aligned} z_1(k+1) &= g(z_1, \mathrm{sat}_{\mathcal{U}}(d_1 + \varphi(|z_1 - z_2|\, d_2)), &(15a)\\ z_2(k+1) &= g(z_2, \mathrm{sat}_{\mathcal{U}}(d_1 + \varphi(|z_1 - z_2|\, d_2)), &(15b)\end{aligned}$$

where $\varphi \in \mathcal{K}_\infty$, $d_1 : \mathbb{N}_0 \to \mathcal{U}$, $d_2 : \mathbb{N}_0 \to [-1,1]^m$. The set of all possible sequences of d_1 and d_2 are, respectively, denoted by $\mathcal{D}_1$ and $\mathcal{D}_2$. Also, the function $\mathrm{sat}_{\mathcal{U}} : \mathbb{R}^m \to \mathcal{U}$ is defined by

$$\mathrm{sat}_{\mathcal{U}}(u) := \begin{cases} u & \forall u \in \mathcal{U} \\ \arg\min_{\nu \in \mathcal{U}} |u - \nu|_2 & \forall u \notin \mathcal{U} \end{cases},$$

where $|\cdot|_2$ denotes the Euclidean norm in $\mathbb{R}^m$. A solution to (15) starting from the initial values (ξ_1, ξ_2) and the inputs (d_1, d_2) is denoted by $z_\varphi(k, \xi_1, \xi_2, d_1, d_2) = \big(z_{\varphi,1}(k, \xi_1, \xi_2, d_1, d_2), z_{\varphi,2}(k, \xi_1, \xi_2, d_1, d_2)\big)$.

Denote the gain functions associated with V by $\underline{\alpha}, \overline{\alpha}, \alpha \in \mathcal{K}_\infty$ with $\alpha < \mathrm{id}$ and $\gamma \in \mathcal{K}$ as in Definition 7. The function $V_c : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ is defined by

$$V_c(\xi) := \max_{k \in \{0,\ldots,M-1\}} \sup_{d_1 \in \mathcal{D}_1, d_2 \in \mathcal{D}_2} \alpha^{k/M}\Big(V\big(z_\varphi(M-1-k, \xi, d_1, d_2)\big)\Big). \tag{16}$$

From the definition of V_c and the $\mathcal{K}$-boundedness condition, it is not hard to see that V_c satisfies the following inequalities

$$\underline{\alpha}_c(|\xi - \hat{\xi}|) \leq V_c(\xi, \hat{\xi}) \leq \overline{\alpha}_c(|\xi - \hat{\xi}|), \tag{17}$$

where $\underline{\alpha}_c(\cdot) := \underline{\alpha}(\cdot)$, $\overline{\alpha}_c(\cdot) := \max_{k \in \{0,\ldots,M-1\}} \underline{\alpha}^{-1} \circ \overline{\alpha} \circ (\kappa_1(\cdot) + \kappa_2 \circ \varphi(\cdot))$, where κ_1, κ_2 are as in the $\mathcal{K}$-boundedness condition in Definition 1. Following similar arguments as those in proof of [18, Theorem 2.22], we can also show that the following holds

$$V_c(z_\varphi(1,\xi,d_1,d_2)) \leq \max\big\{\alpha^{1/M}(V_c(\xi)), \gamma(\|d_2\|\varphi(|\xi_1 - \xi_2|))\big\}. \tag{18}$$

From the convexity of the set $\mathcal{U}$ and definition of system (15), we have

$$V_c(x(1,\xi,u), x(1,\hat{\xi},\hat{u})) \leq \max\big\{\alpha^{1/M}(V_c(\xi,\hat{\xi})), \gamma(\|u - \hat{u}\|)\big\} \tag{19}$$

when $\|u - \hat{u}\| \leq \varphi(|\xi - \hat{\xi}|)$. By the continuity of the dynamics and V_c, one sees that (19) also holds for all $\xi, \hat{\xi} \in \mathcal{X}$ and $u_1, u_2 \in \mathcal{U}$ (possibly with a larger $\gamma \in \mathcal{K}$), which gives γ_c. This completes the first part of the proof.

By the existence of V_c and following the proof of Proposition 12, we can establish the classic approximate bisimilarity between the network Σ and its abstraction $\Sigma^a(\{\Sigma_i^a\}_{i=1}^{\ell})$. In particular, Consider the relation $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{X}^a$ defined by $(\xi, \hat{\xi}) \in \mathcal{R}$ if and only if $V_c(\xi, \hat{\xi}) \leq \underline{\alpha}(\varepsilon)$. This together with taking $\eta_c^x, \eta_c^u > 0$ satisfying (13) and (14) show that $\mathcal{R}$ is an ε-approximate bisimulation relation. ■

Remark 15: To obtain the quantization parameters η_c^x and η_c^u satisfying (13) and (14), one needs the knowledge of the gain functions associated with V_c, *i.e.* $\underline{\alpha}_c$, $\overline{\alpha}_c$, α_c and γ_c. It follows from the proof of Theorem 14, the latter gain functions are obtained from the function φ in (15) and the gain functions associated with V. In [19], [20], the numerical methods for the computation of the gain functions associated with V are given. On the other hand, the proof of Theorem 14 only requires that φ in (15) to be of class $\mathcal{K}_\infty$. However, it follows from the choice of $\overline{\alpha}_c$ and the choice of γ_c (see (18)) that φ can be appropriately chosen such that (13) and (14) are satisfied. □

V. Numerical Example

Consider the discrete-time system

$$\Sigma : \begin{cases} x_1(k+1) = 1.01x_1(k) + 0.7x_5(k) + 0.01u_1(k), \\ x_2(k+1) = 0.3x_5(k) + 0.01u_2(k), \\ x_3(k+1) = 0.1x_5(k) + 0.01u_3(k), \\ x_4(k+1) = 0.1x_5(k) + 0.01u_4(k), \\ x_5(k+1) = -0.1x_5(k) - 0.73x_1(k) - 0.1x_3(k) \\ \qquad\qquad\quad +0.01u_5(k), \end{cases} \tag{20}$$

where $x_i, u_i \in \mathbb{R}$ for each $i \in \{1, \ldots, 5\}$. Note that the first subsystem is not δ-ISS when it is decoupled from the other subsystem. This implies that the classic small-gain conditions are not applicable to prove δ-ISS for system (20). However, we can use the small-gain conditions given by Proposition 13 to show the δ-ISS property for system (20). To do this, take $W_i(\xi_i, \hat{\xi}_i) = |\xi_i - \hat{\xi}_i|$ for $i \in \{1, \ldots, 5\}$. This choice of W_i satisfies (7) with $\underline{\alpha}_i = \overline{\alpha}_i = \mathrm{id}$. One can also verify that the W_i's satisfy (8) with $M = 2$ and the following gain functions

$$\begin{aligned}
&\gamma_{11}(s) = \gamma_{13}(s) = \gamma_{15}(s) = \gamma_{u1}(s) = 0.95s,\\
&\gamma_{21}(s) = \gamma_{23}(s) = \gamma_{25}(s) = \gamma_{u2}(s) = 0.42s,\\
&\gamma_{31}(s) = \gamma_{33}(s) = \gamma_{35}(s) = \gamma_{u3}(s) = 0.15s,\\
&\gamma_{41}(s) = \gamma_{43}(s) = \gamma_{45}(s) = \gamma_{u4}(s) = 0.15s,\\
&\gamma_{51}(s) = \gamma_{53}(s) = \gamma_{55}(s) = \gamma_{u5}(s) = 0.8s,\\
&\gamma_{ij} \equiv 0, i \in \{1,5\}, j = 2, j = 4, \forall s \geq 0.
\end{aligned}$$

Clearly the above gain functions satisfy the small-gain conditions (11). We choose quantization parameters such that conditions (9) and (10) hold, which by the use of Proposition 12 implies the M-step ε_i-approximate bisimilarity between the subsystems Σ_i and Σ_i^a. Then, by Theorem 14, we can show that Σ and its symbolic models Σ^a are ε-approximately bisimilar. Let Σ_i^a, $i \in \{1,\dots,5\}$, be finite subsystems as in Definition 9. Let $\eta^u = \eta_i^x = 0.02\varepsilon_1$ for all $i \in \{1,\dots,5\}$, where ε_1 is the approximate bisimilarity precision between Σ_1 and Σ_1^a. This choice of η^u and η_i^x satisfies (9) and (10) with $\varepsilon_i \leq \varepsilon_{i+1}$ for all $i \in \{1,\dots,4\}$. Hence, by Proposition 12, the systems Σ_i and Σ_i^a are M-step approximately bisimilar with the precision ε_i, $\forall i \in \{1,\dots,5\}$.

Now we aim to design a controller maintaining all the states in some safe set $\mathcal{S}$. The idea here is to design a local controller for abstraction $\hat{\Sigma}_i$ and then refine it to a controller for system Σ_i. We choose $\varepsilon_i = 0.1$ for all $i \in \{1,\dots,5\}$. Also let $\mathcal{S} = [3,4]^5$, $x_i \in [3,4]$, $u_i \in [-10^3, 10^3], i \in \{1,\dots,5\}$. The states trajectories of the closed-loop system Σ under control inputs u_i are illustrated by Figure 1.

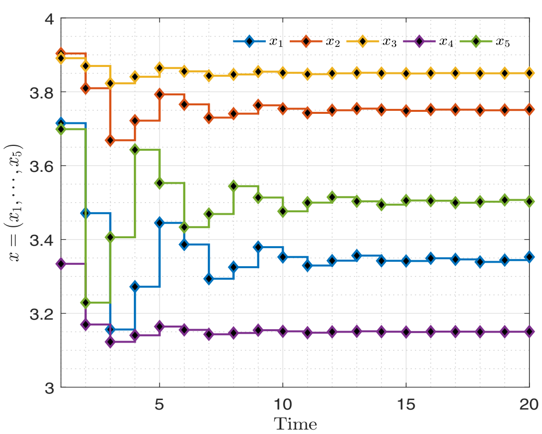

Fig. 1: States trajectories of the closed-loop system Σ.

VI. Conclusions

We have proposed a compositional approach for the construction of approximately bisimilar symbolic models for interconnected systems using the notion of δ-ISS. In particular, our approach can be applicable to networks whose subsystems are not necessarily individually δ-ISS. To do this, we have introduced a notion of so-called M-step approximately bisimilar relation. We established the approximate bisimilarity between the original network and its abstraction, where the subsystems are only required to have M-step approximately bisimilar abstractions. This has been obtained by the use of the relaxed small-gain conditions proposed in this paper.

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