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Robust Linear Quadratic Regulator for Uncertain Linear Discrete-Time Systems with Delay in the States: an augmented system approach

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***Abstract*— In this paper we deal with the regulation problem for a class of uncertain discrete-time systems with known constant delays in the states. Uncertainties are assumed norm-bounded and affect all parametric matrices of the system. Applying the lifting method, the delayed system is transformed into an augmented delay-free system. Then, the control law is obtained from combination of penalty functions and robust regularized least-squares problem, when there exist uncertainties in the data. The solution provided is given in terms of augmented Riccati equations presented in a framework given by an array of matrices.**

***Index Terms*— Delay systems, uncertain discrete-time system, Riccati equation, robust LQR, state feedback.**

I. Introduction

State Time-Delay Systems (STDS) constitute a class of systems in which future dynamics depend on the past and current states. This class of systems can be found in a vast category of applications: in communication network control systems, where there exist problems of information transfer delay and risk of data loss [1] and in unmanned aerial vehicle systems [2]. The delay may constitute an integral characteristic of the controlled plant, as is the case of logistic system with perishable goods [3].

The inclusion of delays in a dynamic model can provide a better representation of real phenomena. However, their presence can affect the dynamic characteristics of the systems and increase the difficulty of the stability analysis and the controller design. As a consequence, control of delayed systems has attracted the attention of many researchers in different fields of science [see, e.g., 4,5-8].

In the literature, a classical approach called *lifting method* or *augmented approach* [see, e.g., 5,6,9] is used to design the control of delayed systems through introducing additional state variables, transforming the state space with delay into a one without delay. In this way, the new system with augmented state space can be analyzed according to standard methods for delay-free systems.

Xia et al. [6] used the lifting method to deal with the stability and stabilization problems of STDS subject to parametric uncertainties and with both constant and time-varying delays. The state feedback control was designed in terms of linear matrix inequalities (LMIs). Based on switched Lyapunov function approach, some stability conditions have been obtained for systems with time-varying delays, given by [9]. One of these conditions was used to design a state feedback controller with varying gains through LMIs. Solutions expressed in the form of LMIs, without applying the lifting method, can be seen in [7,8].

On the other hand, solutions based on recursive Riccati equations have been obtained for delayed systems. In particular, [5, Chap. 6] applied the augmented approach in STDS with constant delay and designed a control law via an augmented Riccati equation associated with the standard Linear Quadratic Regulator (LQR) problem. Moreover, the augmented approach was extended to the multi-delay case in the input by [10] which investigated the infinite-horizon linear-quadratic optimal control problem, developing a strategy control by appropriate treatment of the Riccati equation. Other approaches based on Riccati equations can be found in [11,12].

In this paper, we deal with the recursive solutions of the Robust Linear Quadratic Regulator (RLQR) for STDS subject to constant delay and parametric uncertainties. Motivated by recent results obtained by [5,10], we used the lifting method rewriting the STDS as an uncertain augmented system. Then, the controller is designed in terms of Riccati equations through robust regularized least-squares approach combine with penalty functions, see [13,14] and references therein. The main feature of the RLQR is the recursiveness of the algorithm given in terms of an augmented Riccati equation.

This paper is organized as follows: Section II presents the problem formulation and the definition of the augmented system; Section III introduces the penalty function and the robust regularized least-square approach; Section IV presents the deduction and the algorithm of RLQR for STDS via augmented system; Section V shows an application of the RLQR to the control of an industrial electric heater [15,16] and its effectiveness is compared with the controller developed by [5,6]. The paper ends with a conclusion in Section VI where the main contributions are discussed.

Notations: Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^n$ denotes the set of n-dimensional vectors whose elements are in $\mathbb{R}$ and $\mathbb{R}^{m\times n}$ is the set of $m\times n$ real matrices. I_n indicates the identity matrix of dimension $n\times n$. The null matrix of appropriate dimension is denoted by O. For real matrix P, $P\succ 0$ ($P\succeq 0$) stands a symmetric and positive (semi)definite matrix. The matrix $P^{\frac{1}{2}}$ is the square root of $P\succeq 0$. The superscript T stands for matrix transposition. The matrix $diag(A_1,\dots,A_s)$

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represents a diagonal block matrix consisting of $\{A_1,\ldots,A_s\}$. The notation $\|x\|$ is adopted for the Euclidean norm of x and $\|x\|_P$ the weighted norm of x defined by $(x^T P x)^{\frac{1}{2}}$. For convenience, $Y^T X(\bullet)$ represents $Y^T XY$.

II. PROBLEM FORMULATION

Consider a class of state time-delay systems subject to uncertainties described by

$$\begin{aligned} x_{k+1} &= \mathcal{F}_k^\delta x_k + \mathcal{F}_{d,k}^\delta x_{k-d} + \mathcal{G}_k^\delta u_k,\ \forall k \geq 0, \\ x_k &= \varphi_0(k), \quad k \in [-d,0] \end{aligned} \tag{1}$$

where

$$\mathcal{F}_k^\delta \leftarrow F_k + \delta F_k, \quad \mathcal{F}_{d,k}^\delta \leftarrow F_{d,k} + \delta F_{d,k} \\ \text{and} \quad \mathcal{G}_k^\delta \leftarrow G_k + \delta G_k,$$

for $k = 0,\ldots,N-1$, where $x_k \in \mathbb{R}^n$ is the state at instant k, $x_{k-d} \in \mathbb{R}^n$ is the vector of delayed states of d samples, $u_k \in \mathbb{R}^m$ is the control input. The delay d is a constant positive integer and $\varphi_0(k)$ denotes the initial condition for $k = -d, -d+1, \ldots, 0$. Matrices $F_k, F_{d,k} \in \mathbb{R}^{n\times n}$ and $G_k \in \mathbb{R}^{n\times m}$ are known and the uncertain matrices $\delta F_k, \delta F_{d,k} \in \mathbb{R}^{n\times n}$ and $\delta G_k \in \mathbb{R}^{n\times m}$ are assumed to be of the usual norm-bounded type as

$$\begin{bmatrix} \delta F_k & \delta F_{d,k} & \delta G_k \end{bmatrix} = H_k \ \triangle_k \ \begin{bmatrix} E_{F_k} & E_{F_{d,k}} & E_{G_k} \end{bmatrix} \tag{2}$$

where $H_k \in \mathbb{R}^{n\times p}$ is a nonzero matrix, $E_{F_k}, E_{F_{d,k}} \in \mathbb{R}^{q\times n}$ and $E_{G_k} \in \mathbb{R}^{q\times m}$ are known matrices and $\triangle_k \in \mathbb{R}^{p\times q}$ is a contraction matrix with $\| \triangle_k \| \leq 1$. Associated with STDS (1) we consider the following N-stage cost function

$$J_N(x,u) = x_N^T P_N x_N + \sum_{k=0}^{N-1} \left(x_k^T Q_k x_k + u_k^T R_k u_k \right) \tag{3}$$

with P_N, $Q_k \succeq 0$ and $R_k \succ 0$ are weighting matrices with appropriate dimensions. The goal dealt within this paper consists in designing the best state-feedback control sequence $\mathcal{U} = \{u_0^*, \ldots, u_{N-1}^*\}$ that regulates the STDS (1) subject to parametric uncertainties defined in (2). To this purpose, the discrete-time delay system will be transformed into a non-delay systems by state space augmentation based on *lifting method* [5,6]. Thus, the STDS (1) can be written by the following system:

$$z_{k+1} = \mathcal{A}_k^\delta z_k + \mathcal{B}_k^\delta v_k,\ \forall k \geq 0$$

with $\mathcal{A}_k^\delta := A_k + \delta A_k \in \mathbb{R}^{n_d,n_d}$, $\mathcal{B}_k^\delta := B_k + \delta B_k \in \mathbb{R}^{n_d,m}$, $n_d = (d+1)n$, and the uncertain parameter matrices are modeled as

$$\begin{bmatrix} \delta\mathcal{A}_k & \delta\mathcal{B}_k \end{bmatrix} = \mathcal{H}_k \ \Delta_k \ \begin{bmatrix} E_{\mathcal{A}_k} & E_{\mathcal{B}_k} \end{bmatrix}, \quad \|\Delta_k\| \leq 1$$

where

$$z_k := \begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-d+1} \\ x_{k-d} \end{bmatrix},\ z_0 := \begin{bmatrix} \varphi_0(0) \\ \varphi_0(-1) \\ \vdots \\ \varphi_0(-d+1) \\ \varphi_0(-d) \end{bmatrix},\ v_k := u_k,$$

$$\mathcal{A}_k^\delta := \begin{bmatrix} \mathcal{F}_k^\delta & \ldots & O & \mathcal{F}_{d,k}^\delta \\ I_n & \ldots & O & O \\ \vdots & \ddots & \vdots & \vdots \\ O & \ldots & I_n & O \end{bmatrix},\ \mathcal{B}_k^\delta := \begin{bmatrix} \mathcal{G}_k^\delta \\ O \\ \vdots \\ O \end{bmatrix},$$

$$\mathcal{H}_k := \begin{bmatrix} H_k^T & O & \cdots & O \end{bmatrix}^T \in \mathbb{R}^{p,n_d}, \quad \Delta_k := \begin{bmatrix} \triangle_k & O \end{bmatrix} \in \mathbb{R}^{p,\ell},$$

$$E_{\mathcal{A}_k} := \begin{bmatrix} E_{F_k} & \ldots & O & E_{F_{d,k}} \end{bmatrix} \in \mathbb{R}^{\ell,n_d} \ \text{ and } \ E_{\mathcal{B}_k} := E_{G_k} \in \mathbb{R}^{\ell,m}.$$

Similarly, the augmented version of the quadratic cost function (3) associated with STDS is given by:

$$\mathcal{J}_N(z,v) = z_{N+1}^T \mathcal{P}_N z_{N+1} + \sum_{k=0}^{N-1} \left(z_k^T \mathcal{Q}_k z_k + v_k^T \mathcal{R}_k v_k \right) \tag{4}$$

with

$$\mathcal{P}_N := diag(P_N, O, \ldots, O), \quad \mathcal{R}_k := R_k \\ \text{and} \quad \mathcal{Q}_k := diag(Q_k, O, \ldots, O).$$

Therefore, the STDS (1) is redefined as an augmented system which belongs to a class of discrete-time state-space systems. For this class of systems, the recursive Riccati equation-based control was extended for systems subject to parametric uncertainties by [13,14]. In this approach, a min-max optimization problem is solved where the worst influence of uncertainties is maximized and the criterion $\mathcal{J}_N$ along with the trajectory z_k^* is minimized. Thereby, we consider the following robust control problem: given the initial state $z_0 \in \mathbb{R}^{n_d}$, solve the min-max optimization problem

$$\begin{aligned} &\min_{v\in\mathcal{U}} \ \max_{\delta\mathcal{A}_k, \delta\mathcal{B}_k} \ \mathcal{J}_N(z,v) \\ &\text{s.t.} \quad z_{k+1} = \mathcal{A}_k^\delta z_k + \mathcal{B}_k^\delta v_k, \end{aligned} \tag{5}$$

with the state-feedback control law in the form

$$v_k = \mathcal{K}_k z_k$$

where the matrix $\mathcal{K}_k \in \mathbb{R}^{m\times n_d}$ is the gain to be determined. In next sections, the solution of the RLQR is presented, the optimality of the recursive regulator is ensured by the equivalence with the standard LQR.

III. FUNDAMENTAL CONCEPTS

The preliminary results presented in this section deal with constrained optimization problems. Firstly, the penalty function method is revisited [17]. It aims to convert a constrained problem into an unconstrained problem. Then, robust regularized least-squares approach [18] is defined for systems subject to uncertainties in the data and the solution is obtained by combination of penalty function method.

A. Penalty Function

Consider the constrained optimization problem

$$\begin{aligned} \min_{x} \;\; & f(x) \\ \text{s.t.} \;\; & h(x) = 0. \end{aligned} \tag{6}$$

Suppose that this problem can be rewritten as an unconstrained optimization problem

$$\begin{aligned} \min_{x} \;\; & f(x) + \mu \, \|h(x)\|^2 \\ \text{s.t.} \;\; & x \in \mathbb{R}^n, \end{aligned} \tag{7}$$

where $\mu > 0$ is the *penalty parameter* and the term $\mu \|h(x)\|^2$ is called *penalty function*. The penalty method is a procedure for approximation of constrained problems (6) by unconstrained problems (7). The approximation consists in adding to the objective function the penalty parameter which prescribes a high cost for any violation of the constraints. This approach defines a sequence of solutions for the problem (7) whose limit point is an optimal solution to the original problem.

B. The Robust Regularized Least-Squares

Consider the following constrained optimization problem

$$\begin{aligned} \min_{z} \; \max_{\delta N, \delta y} \;\; & F(z) = \|z\|_U^2 + \|Mz - w\|_V^2 \\ \text{s.t.} \;\; & (N + \delta N) z = y + \delta y, \end{aligned} \tag{8}$$

with $M \in \mathbb{R}^{r \times s}$ and $N \in \mathbb{R}^{l \times s}$ are known matrices, $w \in \mathbb{R}^r$ and $y \in \mathbb{R}^l$ are a measurement vector, $z \in \mathbb{R}^s$ is an unknown vector, $U \succ 0$ and $V \succ 0$ are weighting matrices, and $\{\delta N, \delta y\}$ are perturbations modeled by

$$\begin{bmatrix} \delta N & \delta y \end{bmatrix} = L \, \triangle \, \begin{bmatrix} E_N & E_y \end{bmatrix}$$

where L is a nonzero matrix, $\{E_N, E_y\}$ are known matrices and $\triangle$ is an arbitrary contraction with $\|\triangle\| \leq 1$. Associated with (8), we have for each penalty parameter $\mu > 0$ the following unconstrained optimization problem

$$\begin{gathered} \min_{x_\mu} \; \max_{\delta A, \delta b} \; \mathcal{F}(x_\mu) = \left\| x_\mu \right\|_Q^2 + \left\| (A + \delta A) x_\mu - (b + \delta b) \right\|_{W(\mu)}^2 \\ \text{with} \quad \begin{bmatrix} \delta A & \delta b \end{bmatrix} = H \, \Delta \, \begin{bmatrix} E_A & E_b \end{bmatrix} \end{gathered} \tag{9}$$

where $\quad Q := U, \quad A := \begin{bmatrix} M \\ N \end{bmatrix}, \quad \delta A := \begin{bmatrix} 0 \\ \delta N \end{bmatrix}, \quad b := \begin{bmatrix} w \\ y \end{bmatrix},$

$$\delta b := \begin{bmatrix} 0 \\ \delta y \end{bmatrix}, \quad W(\mu) := \begin{bmatrix} V & 0 \\ 0 & \mu I_l \end{bmatrix}, \quad H := \begin{bmatrix} 0 \\ L \end{bmatrix},$$

$$\Delta := \triangle, \quad E_A := E_N \quad E_b := E_y \quad \text{and} \quad x_\mu := z.$$

A solution for the optimization problem (9) was presented in [18] and redefined in terms of an array of matrices by [19] according to the next result.

Lemma 3.1: Consider the optimization problems (8) and (9). Suppose that $W(\mu)$ is positive definite and the matrix $\begin{bmatrix} I_s \\ A \\ E_A \end{bmatrix}$ is full column rank. Then, there exists a unique solution x^* for (8-9), which can be obtained according to:

(i) for each $\mu > 0$, the optimal solution $\hat{x}_{\mu,\widehat{\lambda}}$, and minimum value $\mathcal{F}(\hat{x}_{\mu,\widehat{\lambda}})$ for problem (9) are given by

$$\begin{bmatrix} \hat{x}_{\mu,\widehat{\lambda}} \\ \mathcal{F}(\hat{x}_{\mu,\widehat{\lambda}}) \end{bmatrix} = \begin{bmatrix} O & O \\ O & b \\ O & E_b \\ I_s & O \end{bmatrix}^T \begin{bmatrix} Q^{-1} & O & O & I_s \\ O & \widehat{W}^{-1}(\mu,\widehat{\lambda}) & O & A \\ O & O & \widehat{\lambda}^{-1} I & E_A \\ I_s & A^T & E_A^T & O \end{bmatrix}^{-1} \begin{bmatrix} O \\ b \\ E_b \\ O \end{bmatrix} \tag{10}$$

where

$$\begin{gathered} \widehat{W}^{-1}(\mu,\widehat{\lambda}) = diag(V^{-1}, \mu^{-1} I - \widehat{\lambda}^{-1} L L^T) \\ \text{e } \widehat{\lambda} > ||\mu L^T L||. \end{gathered}$$

(ii) whenever the matrix $\left[\begin{smallmatrix} N \\ E_N \end{smallmatrix}\right]$ is full row rank, then the optimal solution $z^* = \lim_{\mu \to +\infty} \; \hat{x}_{\mu,\widehat{\lambda}}$ and the minimum value $F(z^*) = \lim_{\mu \to +\infty} \; \mathcal{F}(\hat{x}_{\mu,\widehat{\lambda}})$ for problem (8) is given by

$$\begin{bmatrix} z^* \\ F(z^*) \end{bmatrix} = \begin{bmatrix} O & O \\ O & w \\ O & y \\ O & E_y \\ I_s & O \end{bmatrix}^T \begin{bmatrix} U^{-1} & O & O & O & I_s \\ O & V^{-1} & O & O & M \\ O & O & O & O & N \\ O & O & O & O & E_N \\ I_s & M^T & N^T & E_N^T & O \end{bmatrix}^{-1} \begin{bmatrix} O \\ w \\ y \\ E_y \\ O \end{bmatrix}.$$

Proof:

(i) It follows from solution obtained by [18] which is rewritten in terms of an array of matrices since the invertibility of the block matrix in (10) is guaranteed according to [20, Lemma 2.1].

(ii) The optimal solution is obtained when $\mu \to \infty$ according to penalty function. Notice that when $\mu \to \infty$ then $\mu \to \widehat{\lambda}$. ■

IV. Robust Linear Quadratic Regulator for STDS via Augmented System

In this section we deal with the RLQR for uncertain linear discrete-time systems with delays in the states redefined as an augmented system. The solution for the augmented system is obtained from robust regularized least-squares approach aforementioned. For each step $k \geq 0$, the robust control problem (5) is solved through the minimization of the control input v_k and states z_{k+1} against the maximization of the uncertainties $\delta \mathcal{A}_k$ and $\delta \mathcal{B}_k$, providing an optimal sequence $\{z_{k+1}^*, v_k^*\}_{k=0}^{N-1}$.

Consider the following optimization problem:

$$\min_{z_{k+1}, v_k} \; \max_{\delta \mathcal{A}_k, \delta \mathcal{B}_k} \; \mathcal{J}_{k,\mu}(z_{k+1}, v_k) \tag{12}$$

for all $k \geq 0$ with the quadratic cost functional $\mathcal{J}_{k,\mu}(z_{k+1}, v_k)$ given by

$$\begin{aligned} \mathcal{J}_{k,\mu}(z_{k+1}, v_k) = & \begin{bmatrix} z_{k+1} \\ v_k \end{bmatrix}^T \begin{bmatrix} \mathcal{P}_{k+1} & O \\ O & \mathcal{R}_k \end{bmatrix} \begin{bmatrix} z_{k+1} \\ v_k \end{bmatrix} + \\ & \left(\begin{bmatrix} O & O \\ I_{n_d} & -\mathcal{B}_k^\delta \end{bmatrix} \begin{bmatrix} z_{k+1} \\ v_k \end{bmatrix} - \begin{bmatrix} -I_{n_d} \\ \mathcal{A}_k^\delta \end{bmatrix} z_k \right)^T \begin{bmatrix} \mathcal{Q}_k & O \\ O & \mu I_{n_d} \end{bmatrix} \left(\bullet \right) \end{aligned}$$

with $\mu > 0$ fixed. Notice that the unconstrained optimization problem (12) is an approximation of the constrained optimization problem (5), in each step k, obtained by penalty functions. Moreover, (12) is a particular case of the problem (9) when are performed the following identifications:

$$x_\mu \leftarrow \begin{bmatrix} z_{k+1} \\ v_k \end{bmatrix}, \quad Q \leftarrow \begin{bmatrix} \mathcal{P}_{k+1} & O \\ O & \mathcal{R}_k \end{bmatrix}, \quad W(\mu) \leftarrow \begin{bmatrix} \mathcal{Q}_k & O \\ O & \mu I_{n_d} \end{bmatrix},$$

$$A \leftarrow \begin{bmatrix} O & O \\ I_{n_d} & -\mathcal{B}_k \end{bmatrix}, \quad \delta A \leftarrow \begin{bmatrix} O & O \\ O & -\delta\mathcal{B}_k \end{bmatrix}, \quad b \leftarrow \begin{bmatrix} -I_r \\ \mathcal{A}_k \end{bmatrix} z_k,$$

$$\delta b \leftarrow \begin{bmatrix} 0 \\ \delta\mathcal{A}_k \end{bmatrix} z_k, \quad H \leftarrow \begin{bmatrix} 0 \\ \mathcal{H}_k \end{bmatrix}, \quad \Delta \leftarrow \Delta_k,$$

$$E_A \leftarrow \begin{bmatrix} 0 & E_{\mathcal{B}_k} \end{bmatrix} \text{ and } E_b \leftarrow E_{\mathcal{A}_k}.$$

The next result provides the solution of the optimization problem (12) in terms of partitioned matrices to compute the optimal state trajectory, input control and cost function.

Lemma 4.1: Consider the optimization problem (12) with $\mu > 0$ fixed. Suppose that $diag(\phi(\mu,\widehat{\lambda}), \widehat{\lambda}^{-1} I_{n_d})$ is positive semidefinite. Then, the recursive robust solution for all $k \geq 0$ is given by

$$\begin{bmatrix} z^*_{k+1,\mu} \\ v^*_{k,\mu} \\ \mathcal{J}^*_{k,\mu} \end{bmatrix} = \begin{bmatrix} I_{n_d} & O & O \\ O & I_m & O \\ O & O & z_k \end{bmatrix}^T \begin{bmatrix} \mathcal{L}_{k,\mu} \\ \mathcal{K}_{k,\mu} \\ \mathcal{P}_{k,\mu} \end{bmatrix} z_k$$

where the closed loop system matrix $\mathcal{L}_{k,\mu}$, the feedback-gain $\mathcal{K}_{k,\mu}$ and the solution of the Riccati equation $\mathcal{P}_{k,\mu}$ are obtained from (11).

Remark 4.1: The parameter $\widehat{\lambda}$ is defined over the interval $(\|\mu\mathcal{H}_k^T\mathcal{H}_k\|, \infty)$ for each $\mu \in (0,\infty)$. In this way, the penalty parameter μ and the variable $\widehat{\lambda}$ can be seen as robustness parameters under which the robust regulation is obtained.

The optimal recursive solution to the RLQR in Lemma 4.1 is obtained when we perform the limit of μ, i.e., assuming $\mu \to \infty$. As consequence, $\widehat{\lambda} \to \infty$ and $\phi(\mu,\widehat{\lambda}) \to 0$. In this case, it should be remarked that the matrix $E_{\mathcal{B}_k}$ must be full row rank to satisfy (11) and the feedback gain $\mathcal{K}_{k,\mu}$ satisfies the following equations:

$$\mathcal{L}_{k,\infty} = \mathcal{A}_k + \mathcal{B}_k \mathcal{K}_{k,\infty} \quad \text{and} \quad E_{\mathcal{A}_k} + E_{\mathcal{B}_k}\mathcal{K}_{k,\infty} = 0.$$

Moreover, the optimal N-stage cost function is given by

$$\mathcal{J}^*_N(z_0, v^*) = z_0^T \mathcal{P}_{0,\infty} z_0.$$

The procedure to calculate the RLQR deduced from standard discrete-time systems subject to parametric uncertainties is presented in Algorithm 1.

Algorithm 1 Robust LQR for STDS via Augmented System

Define the parameters of the augmented system: $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{H}_k$, $E_{\mathcal{A}_k}$, $E_{\mathcal{B}_k}$, $\mathcal{P}_N$, $\mathcal{Q}_k$, $\mathcal{R}_k$ and z_0.

Initial conditions: Set $\mu > 0$ and N.

Compute for each $k = 0, \ldots, N-1$:

$$\mathcal{L}_{k,\mu}, \; \mathcal{K}_{k,\mu} \text{ and } \mathcal{P}_{k,\mu} \text{ via (11).}$$

Obtain:

$$x_{k+1} = \mathcal{F}^\delta_k x_k + \mathcal{F}^\delta_{d,k} x_{k-d} + \mathcal{G}^\delta_k u_k$$
$$\text{with } u_k = v_k = \mathcal{K}_k z_k.$$

The next result shows that solution presented in Lemma 4.1, in terms of an array of matrices, reduces to the form of standard Riccati equations. The necessary and sufficient conditions for the existence of solution for the RLQR are given based on the standard LQR for systems not subject to uncertainties [see, e.g, 21].

Theorem 4.1: Consider the optimization problem (12). The optimal recursive algebraic solution, for $\mu \to \infty$, is given by

$$\mathcal{P}_k = \overline{\mathcal{A}}_k^T \left[\mathcal{P}_{k+1} - \mathcal{P}_{k+1}\overline{\mathcal{B}}_k (I_m + \overline{\mathcal{B}}_k^T \mathcal{P}_{k+1} \overline{\mathcal{B}}_k)^{-1} \overline{\mathcal{B}}_k^T \mathcal{P}_{k+1} \right] \overline{\mathcal{A}}_k + \overline{\mathcal{Q}}_k$$

with $\overline{\mathcal{B}}_k = \mathcal{B}_k \overline{\mathcal{R}}_k^{-\frac{1}{2}}$,

$$\overline{\mathcal{R}}_k = \mathcal{R}_k^{-1} - \mathcal{R}_k^{-1} E^T_{\mathcal{B}_k} (E_{\mathcal{B}_k}\mathcal{R}_k^{-1}E^T_{\mathcal{B}_k})^{-1} E_{\mathcal{B}_k}\mathcal{R}_k^{-1},$$
$$\overline{\mathcal{A}}_k = \mathcal{A}_k - \mathcal{B}_k\mathcal{R}_k^{-1}E^T_{\mathcal{B}_k}(E_{\mathcal{B}_k}\mathcal{R}_k^{-1}E^T_{\mathcal{B}_k})^{-1}E_{\mathcal{A}_k},$$
$$\overline{\mathcal{Q}}_k = \mathcal{Q}_k + E^T_{\mathcal{A}_k}(E_{\mathcal{B}_k}\mathcal{R}_k^{-1}E^T_{\mathcal{B}_k})^{-1}E_{\mathcal{A}_k}.$$

Proof: It follows from algebraic manipulations of expressions presented in (11). ■

V. Illustrative Example

The numerical example considered in this paper is an adaptation from a model of an industrial electric heater studied in [15,16]. The electric heater is divided into five heating zones, each of them is measured by a thermocouple and represented by state variable $x_{i,k}$, $i = \{1, \ldots, 5\}$. The control inputs $u_{i,k}$ are electrical energy signals applied in each zone of the electric heater. The main goal is to control the

$$\begin{bmatrix} \mathcal{L}_{k,\mu} \\ \mathcal{K}_{k,\mu} \\ \mathcal{P}_{k,\mu} \end{bmatrix} = \begin{bmatrix} O & O & O \\ O & O & O \\ O & O & O \\ O & O & -I_{n_d} \\ O & O & O \\ O & O & \mathcal{A}_k \\ O & O & E_{\mathcal{A}_k} \\ I_{n_d} & O & O \\ O & I_m & O \end{bmatrix}^T \begin{bmatrix} O & I_{n_d} & O & O & O & O & O & I_{n_d} & O \\ I_{n_d} & -\mathcal{P}_{k+1,\mu} & O & O & O & O & O & O & O \\ O & O & \mathcal{R}_k^{-1} & O & O & O & O & O & I_m \\ O & O & O & O & I_{n_d} & O & O & O & O \\ O & O & O & I_{n_d} & -\mathcal{Q}_k & O & O & O & O \\ O & O & O & O & O & \phi(\mu,\widehat{\lambda}) & O & I_{n_d} & -\mathcal{B}_k \\ O & O & O & O & O & O & \widehat{\lambda}^{-1} I_{n_d} & O & -E_{\mathcal{B}_k} \\ I_{n_d} & O & O & O & O & I_{n_d} & O & O & O \\ O & O & I_m & O & O & -\mathcal{B}_k^T & -E^T_{\mathcal{B}_k} & O & O \end{bmatrix}^{-1} \begin{bmatrix} O \\ O \\ O \\ -I_{n_d} \\ O \\ \mathcal{A}_k \\ E_{\mathcal{A}_k} \\ O \\ O \end{bmatrix} \tag{11}$$

$$\text{with} \quad \phi(\mu,\widehat{\lambda}) = \mu^{-1} I_{n_d} - \widehat{\lambda}^{-1}\mathcal{H}_k\mathcal{H}_k^T \quad \text{and} \quad \widehat{\lambda} > ||\mu\mathcal{H}_k^T\mathcal{H}_k||.$$

temperature of each zone of the electric heater. We assume that dynamic model of the electric heater proposed by [15,16] can suffer influence of parametric uncertainties. Given the STDS (1), consider the nominal matrices obtained by [15,16]

$$F_k = \begin{bmatrix} 0.97421 & 0.15116 & 0.19667 & -0.05870 & 0.07144 \\ -0.01455 & 0.88914 & 0.26953 & 0.11866 & -0.22047 \\ 0.06376 & 0.12056 & 1.00049 & -0.03491 & -0.02766 \\ -0.05084 & 0.09254 & 0.28774 & 0.82569 & 0.02570 \\ 0.01723 & 0.01939 & 0.29285 & 0.03544 & 0.87111 \end{bmatrix},$$

$$F_{d,k} = \begin{bmatrix} -0.01000 & -0.08837 & -0.06989 & 0.18874 & 0.20505 \\ 0.02363 & 0.03384 & 0.05282 & -0.09906 & -0.00191 \\ -0.04468 & -0.00798 & 0.05618 & 0.00157 & 0.03593 \\ -0.04082 & 0.01153 & -0.07116 & 0.16472 & 0.00083 \\ -0.02537 & 0.03878 & -0.04683 & 0.05665 & -0.03130 \end{bmatrix},$$

$$G_k = \begin{bmatrix} 0.53706 & -0.11185 & 0.09978 & 0.04652 & 0.25867 \\ -0.51718 & 0.73519 & 0.57518 & 0.40668 & -0.12472 \\ 0.29469 & 0.31528 & 1.16420 & -0.29922 & 0.23883 \\ -0.20191 & 0.19739 & 0.41686 & 0.66551 & 0.11366 \\ -0.11835 & 0.16287 & 0.20378 & 0.23261 & 0.36525 \end{bmatrix}$$

for all $k \geq 0$ and initial conditions $x_0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$ and $\varphi_0(k) = x_0$. Assume the following parameters to the parametric uncertainties:

$$H_k = \begin{bmatrix} 0.42931 & 0.29140 & 0.37186 & 0.94027 & 0.45495 \end{bmatrix}^T,$$
$$E_{F_k} = \begin{bmatrix} 0.89166 & 0.27856 & 0.68917 & 0.78440 & 0.92555 \end{bmatrix},$$
$$E_{F_{d,k}} = \begin{bmatrix} 0.00702 & 0.12333 & 0.64426 & 0.36737 & 0.08285 \end{bmatrix}$$
and $$E_{G_k} = \begin{bmatrix} 0.69388 & 0.83614 & 0.32470 & 0.14643 & 0.62555 \end{bmatrix}.$$

The values adopted for weighting matrices of the N-stage cost function (3) are:

$$P_N = I_5, \quad Q_k = I_5 \quad \text{and} \quad R_k = I_5, \ \forall k \geq 0.$$

The performance of RLQR is compared with the augmented approach developed by [6, Corollary 3] and the standard LQR applied to the augmented system according to [5, Chap. 6, pg. 259]. All the routines were implemented in the software *MATLAB*®, version 8.4.0.150421 (R2014b) using *LMI Control toolbox* [22] for solving the LMIs. For RLQR, we consider $\mu \to \infty$. As result, we have $\widehat{\lambda}^{-1} = 0$ and $\phi(\mu, \widehat{\lambda}) = 0$.

In order to perform this comparison, we consider two scenarios for the delay in the states: $d = 2$ and $d = 7$. The results obtained are illustrated in Figures 1 to 4. For all graphics were performed 1000 experiments. In each experiment j, the variable $\triangle_k$ is randomly selected from a uniform distribution over the interval $[-1, 1]$ for each instant of time k.

The trajectories z_k derived from Corollary 3 in [6] were calculated according to the case of the Equation 11 in [6, pg. 20] with uncertainties and the LQR was applied for the uncertain matrices $\mathcal{F}_k^\delta$, $\mathcal{F}_{d,k}^\delta$ and $\mathcal{G}_k^\delta$. For comparison purposes, the N-stages cost function is calculated according with (3).

The behavior of mean of Euclidean norm of the trajectories z_k and the mean of the associated costs J_N^* when $d = 2$ are presented in Figures 1 and 2, respectively. The RLQR response is faster if compared with the other methods.

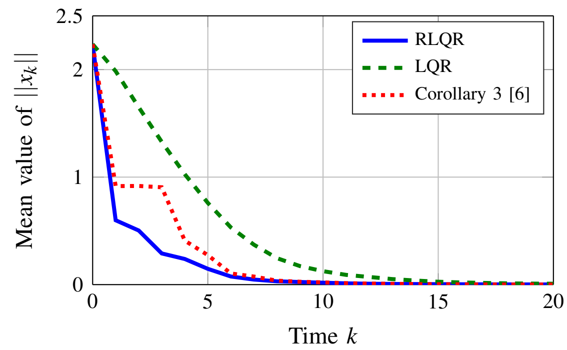

Fig. 1. Comparison of the response of the uncertain system when $d = 2$.

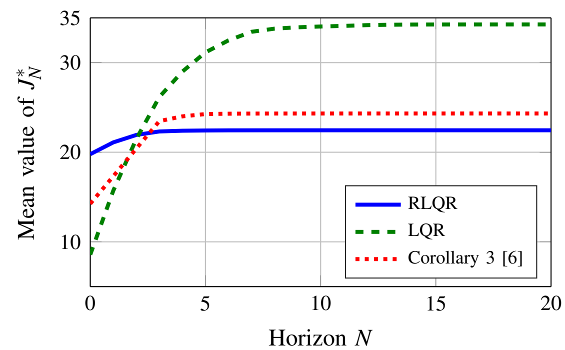

Fig. 2. Comparison of the N-stage cost when $d = 2$.

Figures 3 and 4 correspond to the mean of Euclidean norm of the trajectories z_k and the mean of the associated costs J_N^* when $d = 7$, respectively. Even with the increase in the delay, the performance of the RLQR is faster when compared with the standard LQR and Corollary 3 presented in [6].

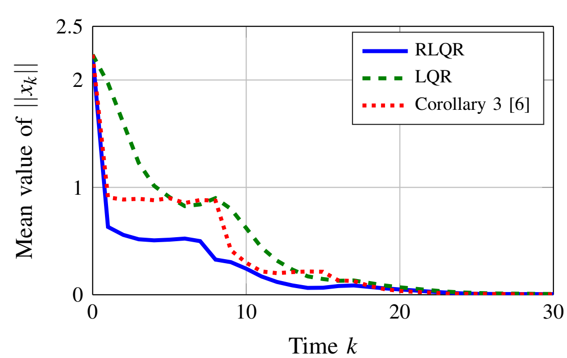

Fig. 3. Comparison of the response of the uncertain system when $d = 7$.

The mean value of optimal cost J_N^* obtained for each method in each scenario considered for the delay is presented in the Table I.

VI. Conclusions

Considering the regulation problem for uncertain discrete-time systems with known constant delays in the states, we proposed a recursive robust solution for RLQR associated

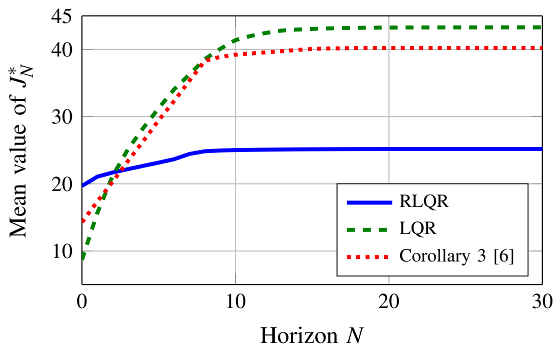

Fig. 4. Comparison of the N-stage cost when $d = 7$.

TABLE I
MEAN VALUES OF OPTIMAL COST J_N^*

Delay	Methods		
	RLQR	Corollary 3 [6]	LQR
$d = 2$	22,45045	24,01046	30,72879
$d = 7$	25.17858	40.21448	43.29083

with Riccati equations. Based on lifting method, the uncertain STDS was reformulated as an augmented system without delay. Then, the controller synthesis problem is solved and the algorithm is presented in form of partitioned matrices. The results of the computational experiments showed the effectiveness of the RLQR in comparison with approaches available in the literature, highlighting its advantage in applications where the delay is small. However, the lifting method leads to a higher state dimension which can deserve more computational efforts. The development of alternative recursive approaches without increase state dimension will be considered in a future work.

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