2018 26th Mediterranean Conference on Control and Automation (MED)
Zadar, Croatia, June 19-22, 2018

Parameter Space Approach for Performance Mapping using Lyapunov Stability

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Abstract— **Calculating the complete controller parameter space, which guarantees specified closed loop performance requirements of a given linear system, is non-trivial. In this paper, a new approach to solve this problem is presented using Lyapunov stability formulations. This method has several advantages in comparison to the existing parameter space approach methods: Currently, the parameter space calculation methods are only applicable for a very restricted system class. They rely on frequency sweeping and the stabilizing parameter space may only be calculated through means of discretization. The proposed method avoids this while reducing the computational complexity and increasing the practicality of the method at the same time. An extensive analysis of the presented method is shown on a practical application example: the longitudinal vehicle guidance (ACC).**

I. Introduction

Calculating all stabilizing control parameters for a given system is the basis of design and tuning in control theory. The problem of symbolically computing all stabilizing controller parameters can be traced back to Vyshnegradsky [16] and has been extensively studied during the last decades. Several methods are available to compute the stable regions in the space of the controller coefficients. The D-decomposition method [6], [11] is based on the division of the parameter space into several regions, such that the number of unstable roots of the characteristic equation in each region is the same for all points in that region.

In practical applications is it often not sufficient to achieve stability, additionally there are performance claims. First approaches to shrink the stabilizing parameter space to guarantee predefined performance requirements started in the early 1960s [14]. Today, a variety of different methods to calculate parameter spaces for this issue are available: A method for mapping gain and phase requirements into the controller parameter space for first order controllers is present in [15]. The θ-stability mapping method is used to map frequency domain performance requirements into the parameter space, as discussed in [3]. A similar frequency based method is called B-stability mapping technique. It is introduced in [12] and can be used to guarantee a specified damping behavior by introducing an upper bound of the amplitude response. In parallel to the frequency based methods, some mapping approaches based on requirements of the closed loop eigenvalues are also available. D. D. Šiljak had presented an Γ-stability mapping approach based on the generalization of Mitrović's method [14]. Similarly, Ackermann presented an overview of common mapping relations in [1]. An expansion of some of this classical Γ-stability mapping approaches for time delay systems are presented in [7], [10].

All the previously mentioned performance mapping methods are based on classical parameter space approach techniques. Thus, they are only applicable for a very restricted system class, as they incorporate frequency sweeping and the parameter space can often only be calculated, if the parameters are discretized. Accordingly, these approaches are not generally applicable and the accuracy of the results is highly influenced by the resolution of the discretization. These problems have been solved for the stability boundary mapping by the recent Lyapunov stability formulation of the authors in [13], as this method bypasses parameter sweeping and may be used for many different control structures. But it still remains to adapt the advantageous Lyapunov stability approach in such a way, that the performant parameter space can be calculated as well. This step is performed in this paper, as the Lyapunov stability approach is extended for performance mapping. This offers a more general approach to the problem, circumventing some of the well-known pitfalls (frequency sweeping and discretization) of the classical parameter space approaches.

The structure of the paper is as follows. First, parameter space calculation methods using Lyapunov formulations are briefly reviewed as they build the basis for the following study. Thereafter in section III, the new performance mapping method is introduced and derived in detail. Here, both cases of settling time as well as damping are investigated. Finally in section IV, a practical application example (ACC) is presented to illustrate the proposed method and clarifies its practicality. Thus, the contribution of this paper lies at the one hand in the extension of the Lyapunov stability approach to performance mapping and on the other hand in the proof of practicality of the proposed approach by using a real world example.

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II. Lyapunov Stabilty Mapping

The aim of the stability mapping approach is to compute the set of all stabilizing parameters. As initially shown by the authors in [13] and already mentioned in section I, this can also be done by using Lyapunov stability analysis. In the present publication, this approach is expanded to performance mapping in section III. Therefore, the basics of the Lyapunov stability mapping approach is revisited in this section. For details see [13].

Let

$$\dot{x} = A(k)\,x \quad ; \quad A \in \mathbb{R}^{n\times n}, \quad x \in \mathbb{R}^n, \quad k \in \mathbb{R}^p \tag{1}$$

be a general linear time invariant (LTI) state-space system in continuous time with the state vector x. The system (1) can either describe an open loop system, where k denotes unknown or uncertain parameters, or a closed loop system with uncertain or control parameters k. It is well-known that the stability of system (1) can be determined using the Lyapunov equation

$$A^{\mathrm{T}}(k)P + PA(k) = -Q \tag{2}$$

as in [2]. For a given positive definite matrix $Q \in \mathbb{R}^{n\times n}$ the system is considered asymptotically (exponential) stable if a positive definite matrix $P \in \mathbb{R}^{n\times n}$ exists, which fulfils (2). Therefore the stabilizing parameter space is given by all parameters k for which (2) can be solved with a positive definite P. If Q is symmetric, the matrix P will also be symmetric. Equation (2) can be rewritten in vector form (see e.g. [2])

$$\left(I \otimes A^{\mathrm{T}}(k) + A^{\mathrm{T}}(k) \otimes I\right)\operatorname{vec}(P) = -\operatorname{vec}(Q), \tag{3}$$

where I is the $n\times n$ identity matrix, $\otimes$ denotes the Kronecker product and $\operatorname{vec}(\cdot)$ arranges matrices into column vectors by rearranging them column-wise. Solving for P in (3) is now a simple matter of solving a linear system of equations

$$\operatorname{vec}(P) = -M^{-1}\operatorname{vec}(Q)$$

where

$$M(k) = I \otimes A^{\mathrm{T}}(k) + A^{\mathrm{T}}(k) \otimes I \tag{4}$$

holds. The inverse M^{-1} and therefore P as well, have the determinant $|M(k)|$ in their denominators. Thereby, the eigenvalues λ_i of $A(k)$ have the following relation to $|M(k)|$ [5]:

$$|M(k)| = \prod_{i=1}^{n}\prod_{j=1}^{n}(\lambda_i + \lambda_j). \tag{5}$$

Thus, the system $A(k)$ is on a stability boundary, if $|M(k)| = 0$ holds. These boundaries restrict the stabilizing parameter space and can be calculated symbolically.

In (5) some duplicated products like $(\lambda_1 + \lambda_2)$ and $(\lambda_2 + \lambda_1)$ are included, which leads to a dimension of $M(k)$ of $n^2 \times n^2$. But there are at most $n(n+1)/2$ unique elements in a $n \times n$ symmetric matrix P. Consequently, the repeated elements can be eliminated using elimination and duplication matrices [8]. As a result, (3) can be reformulated to

$$T^{\dagger}M(k)T\overline{\operatorname{vec}}(P(k)) = -\overline{\operatorname{vec}}(Q), \tag{6}$$

where T is the full column rank duplication matrix and the elimination matrix $T^{\dagger}$ is its pseudo inverse. The duplication matrix T and the elimination matrix $T^{\dagger}$ do not depend on the free parameters k. Additionally, $\overline{\operatorname{vec}}(P(k))$ only includes the unique elements of the original P matrix as

$$\overline{\operatorname{vec}}(P(k)) = \begin{bmatrix} P_{11} & \dots & P_{n1} & P_{22} & \dots \end{bmatrix}^T. \tag{7}$$

Using this approach, all unique elements of the original P matrix are determined by

$$\overline{\operatorname{vec}}(P(k)) = M_{\mathrm{T}}^{-1}(k)\overline{\operatorname{vec}}(-Q) \tag{8}$$

with

$$M_{\mathrm{T}}(k) = T^{\dagger}M(k)T. \tag{9}$$

The corresponding relation to equation (5) is now

$$|M_{\mathrm{T}}(k)| = \prod_{i=1}^{n}\prod_{j\geq i}^{n}(\lambda_i + \lambda_j). \tag{10}$$

Thus, the duplicated eigenvalues are eliminated in this case. As a result, the dimension of $M_T(k)$ is only $(n(n+1)/2) \times (n(n+1)/2)$ compared to the $n^2 \times n^2$ matrix $M(k)$.

III. Performance Mapping

A. General Idea

With the boundary mapping based on the Lyapunov approach explained in section II, systems described by $A(k)$ can be checked for stability analytically. In practical applications it is often necessary or desirable to not only guarantee stability but a certain performance. Here, the settling time and the damping are of particular importance. Both performance indicators can be identified with a certain region for the eigenvalues of $A(k)$ in the complex plane as depicted in Fig. 1. The settling time is connected to α, whereas the damping is related to the angle φ.

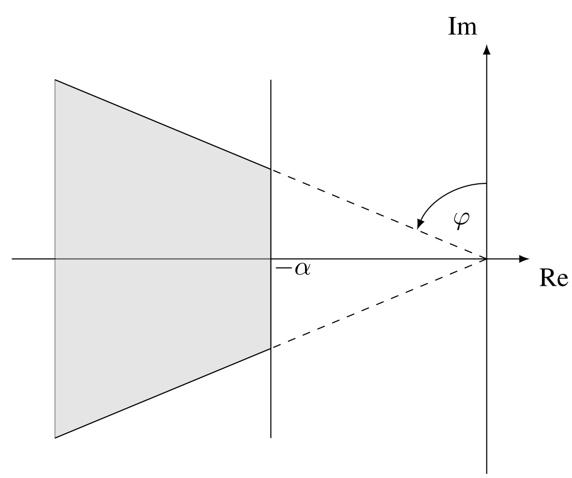

Fig. 1: Performance area in the complex plane

The idea of this paper is now, to extend the Lyapunov approach for performance mapping by finding a surrogate-matrix $\mathcal{A}(k)$, which has eigenvalues on the stability boundary, if the matrix $A(k)$ has eigenvalues on the performance boundaries from Fig. 1. The surrogate-matrix $\mathcal{A}(k)$ should be as simple as $A(k)$ to not increase the computational complexity. If $\mathcal{A}(k)$ is at hand, the classical Lyapunov stability approach can be applied to $\mathcal{A}(k)$ in order to derive the performance mapping for the original matrix $A(k)$. The construction of the surrogate-matrix $\mathcal{A}(k)$ will be presented for settling time and damping separately, but the combined area can also be computed easily: Given the area which satisfies the performance criterion regarding the settling time and the area satisfying the damping requirements. Then every parameter from the union of both areas obviously fulfills both requirements.

Further note, that in the case of $\mathcal{A}(k)$ being on a stability boundary, if and only if $A(k)$ is on its performance boundary, no additional fictitious boundaries arise. In other cases, a stability boundary of $\mathcal{A}(k)$ may not match a performance boundary. Nevertheless, the problem of fictitious boundaries is well-known for the parameter space approach and will be addressed along subsection III-C. All ideas are based on the well-known fact, that eigenvalues are continuous functions of the coefficients of the characteristic polynomial.

B. Mapping of the Settling Time

To check, whether the system $A(k)$ fulfils a particular settling time, one has to investigate, if all eigenvalues λ of $A(k)$ are on the left hand side of the the axis $-\alpha \pm j\omega$ with $\omega \in \mathbb{R}^+$. Examine the matrix

$$\mathcal{A}(k) = A(k) + \alpha I \tag{11}$$

and let v_i be an eigenvector of $A(k)$ with the eigenvalue λ_i. Then it holds

$$\mathcal{A}(k)\, v_i = A(k)\, v_i + \alpha I\, v_i = (\lambda_i + \alpha)\, v_i. \tag{12}$$

If now $A(k)$ has eigenvalues on the performance boundary $-\alpha \pm j\omega$, $\mathcal{A}(k)$ has eigenvalues at $\pm j\omega$. Thus, the surrogate-matrix $\mathcal{A}(k)$ is on its stability boundary, if and only if $A(k)$ is on its performance boundary. The settling time can be investigated by applying the established parameter space approach using Lyapunov stability on the matrix $\mathcal{A}(k)$ and no additional fictitious boundaries arise. Here, the parameter α can either be fixed or be added to the vector k as unknown parameter. Further note, that all calculations are of similar complexity by using A or $\mathcal{A}$, as the dimensions are the same and the order of the polynomials in k is not changed. This also holds for the additional unknown α, as α is not multiplied by $A(k)$. Likewise, for $\alpha = 0$ the same equations as for the classical Lyapunov approach arise, which emphasizes the consistency of the approach.

C. Mapping of the Damping

Considering that the eigenvalue λ of $A(k)$ is on the performance boundary in the upper left complex half plane - e.g. $\varphi \in [0, \frac{\pi}{2})$. As shown in Fig. 2, lies the value $\lambda(1 - j\tan(\varphi))$ on the imaginary axis.

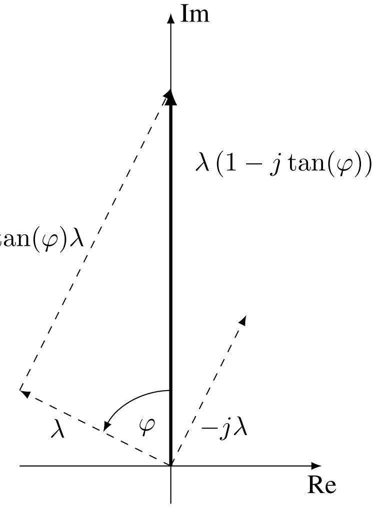

Fig. 2: Eigenvalues of $\mathcal{A}(k)$ for the case of damping

Keeping this fact in mind, one can define the matrix

$$\mathcal{A}(k) = A(k) - j\tan(\varphi)A(k) \quad \text{with } \varphi \in [0, \frac{\pi}{2}). \tag{13}$$

All eigenvalues of this surrogate-matrix $\mathcal{A}(k)$ are given by

$$\begin{aligned}\mathcal{A}(k)\, v_i &= A(k)\, v_i - j\tan(\varphi)A(k)\, v_i \\ &= \lambda(1 - j\tan(\varphi))\, v_i.\end{aligned} \tag{14}$$

Thus, if $A(k)$ is on its performance boundary in the upper left complex half plane, the matrix $\mathcal{A}(k)$ has an eigenvalue on the imaginary axis and therefore on the stability boundary. As $A(k)$ is real, the eigenvalues of $A(k)$ are complex conjugated. Hence, this statement also holds for the performance boundary in the lower left half plane.

As mentioned in III-A, fictitious boundaries arise for eigenvalues of $A(k)$, as the surrogate-matrix $\mathcal{A}(k)$ is on its stability boundary. These fictitious boundaries are no performance boundaries. The corresponding eigenvalues of $A(k)$ are located in the lower right complex half plane and thus in the unstable area. Hence, the additional boundaries do not contribute to the parameter space. Thus, the following statement holds: The eigenvalues of $A(k)$ are in the angle-sector $(-\pi/2 - \varphi, \pi/2 + \varphi)$ for $\varphi \in [0, \frac{\pi}{2})$, if and only if the eigenvalues of the surrogate matrix $\mathcal{A}(k)$ are in the open left half plane. Further note, that this also holds true for the matrix $\mathcal{A}(k) = A(k) + j\tan(\varphi)A(k)$, with the situation as in Fig.2 mirrored at the real axis.

Even though $\mathcal{A}(k)$ from (13) is a candidate of choice, this surrogate-matrix is complex. Thus, the Lyapunov stability boundary mapping has to be extended for complex matrices, which has not been done in literature yet. Let

$$\dot{x} = \mathcal{A}(k)\, x \quad ; \quad \mathcal{A} \in \mathbb{C}^{n \times n}, \quad x \in \mathbb{C}^n, \quad k \in \mathbb{R}^p \tag{15}$$

be a system as in (1) with complex A and x. The Lyapunov equation for this given system is

$$A^*(k)P + PA(k) = -Q, \qquad (16)$$

as in [2], where A^* denotes the complex conjugate transposed of A and P is hermitian. The system is considered stable, if a positive definite P exists, which solves (16) for positive definite Q. Again (16) can be rewritten in vector form

$$\left(I \otimes A^{\mathrm{T}}(k) + A^*(k) \otimes I\right) \operatorname{vec}(P) = -\operatorname{vec}(Q). \qquad (17)$$

Solving for P leads to a matrix $M(k)$ with the determinant

$$|M(k)| = \prod_{i=1}^{n} \prod_{j=1}^{n} (\lambda_i + \bar{\lambda}_j). \qquad (18)$$

If now the eigenvalue λ_i of $A(k)$ is on the imaginary axis (and thus on a stability boundary), then it holds $\lambda_i + \bar{\lambda}_i = 0$ and thus $|M(k)| = 0$. Hence, the Lyapunov stability boundary mapping is also applicable for complex matrices A by adjusting the corresponding equations for calculating P. Moreover, also the methods for saving computational time using the matrix T as explained in section II can be applied (with slight modifications) as a hermitian matrix is also uniquely described by $\frac{n^2+n}{2}$ elements. Further note, that for real $A(k)$ the same equations and stability boundaries as for the common Lyapunov approach are derived.

With this results for complex matrices, the performance mapping for the damping goes as in the case for the settling time. First, the matrix $\mathcal{A}(k)$ is setup as in (13). Here, the parameter $\psi = \tan(\varphi)$ can be taken as fixed or as uncertain parameter to be investigated. The stability of $\mathcal{A}(k)$ is then checked with the new modified Lyapunov approach for complex matrices in (16). Note, that for $\varphi = 0$ the same equations as for the classical Lyapunov approach arise.

The calculation time for the performance mapping of the damping is in the same range as for just checking stability, if the damping is fixed: The dimensions are all the same and all techniques for dimension reduction of M known for the real case are also applicable for complex valued matrices. But, if $\psi = \tan(\varphi)$ is taken as unknown parameter, the complexity is slightly increased. This is caused by the fact, that ψ is multiplied with $A(k)$ in the calculation of $\mathcal{A}(k)$. Thus, the order of all polynomials in k (including ψ) is increased by one. Although, the rise in computational costs is marginal.

D. Influence of the Infinity Root Boundary

In the derivation of the performance mapping only the cases have been considered, where the stability is influenced by eigenvalues, which cross the imaginary axis. These cases are denoted as complex root boundary (for the pairwise crossing of two complex conjugated eigenvalues) and real root boundary (one pole crosses through the origin) in literature [1]. In addition, also the crossing of poles through infinity is possible, which is called infinity root boundary (IRB) [1]. In the previous discussion the IRB has been disregarded. The IRB occurs, if the order of the characteristic polynomials changes, e.g. the highest coefficient of the characteristic equation becomes zero. For $\varphi \in [0, \frac{\pi}{2})$, the highest coefficient is neither influenced by α nor by φ. Thus, the IRB is not affected by the performance mapping.

Fig. 3: Photo of the Test Vehicle

IV. Application Example

Developments regarding driver assistant systems, in the scope of highly automated driving, has gained significant process during the last years. Currently, this is one of the most active fields of research and development in the automotive industry. IAV GmbH has worked in this domain since decades. During the last months, IAV GmbH has driven more than 150.000 km with their highly automated vehicle prototypes, which are used for the evaluation of new automation concepts. A photo of the test vehicle (VW Golf), which is used in the current study, is shown in Fig. 3.

The software framework, which is used to realize the highly automated driving functionality can handle functionalities in high-speed scenarios (like ACC, lane keeping and active lane change) as well as low-speed scenarios (like paring assist). The framework consist of three layers. The first layer realizes the environment perception. Therefore, the main focus is on object detection and prediction as well as on lane detection. Moreover, a free space calculation is realized. The second layer consist of the motion planning module. Here, a strategy module which planes the next movements of the vehicle as well as a trajectory planning module is implemented. The last layer consists of a follow-up controller. The follow-up control concept in order to realize a high performant lateral and longitudinal vehicle dynamic will be focus in this study.

More in detail, the particular task is to optimize the parameter tuning of the longitudinal controller, see Fig. 4. A proper tuning of the controller is important as a slow system response influences the driver comfort in a negative way. Additionally, a low damping of the system response can lead to rear-end collisions. The longitudinal controller is a meshed control-loop, which consists of an inner loop for velocity control (output is the velocity set point $v_{v,ego}$) of the vehicle and an outer loop for the distance control (output is the velocity set point $v_{d,ego}$), see Fig. 4. Here, v_{Lead} is the

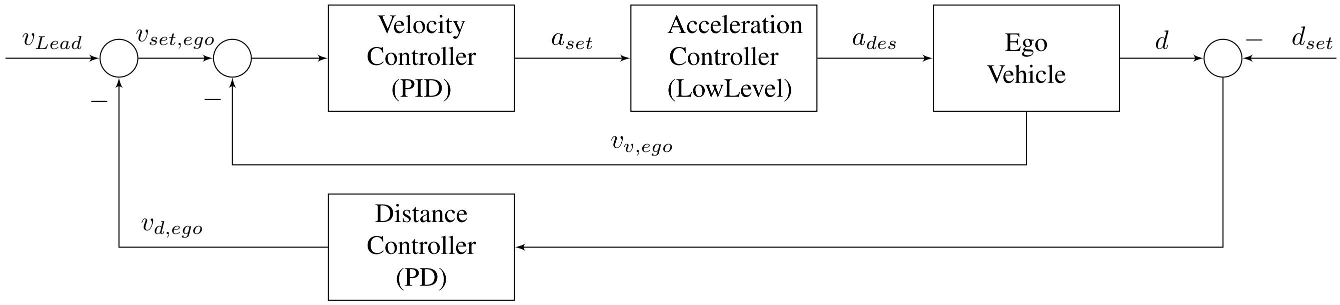

Fig. 4: Architecture for Longitudinal Control

velocity of the vehicle in front, d is the actual and d_{set} the desired distance between both vehicles. The system model describing the longitudinal vehicle dynamic (input: set point a_{set}; output: measured acceleration a_{des}) can be written as

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{T_l} & -\frac{2\cdot D_l}{T_l} \end{bmatrix} \ , \ B = \begin{bmatrix} 0 \\ \frac{K_l}{T_l^2} \end{bmatrix} \ , \ C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (19)$$

with $T_l = 0.52$, $D_l = 0.845$, $K_l = 1$. Several test drives with different vehicle velocities were made and a classical parameter fitting was done to create this linear model. Aim of the parameter fitting was to minimize the quadratic fitting error with focus on highway scenarios (velocity operation point between 80 and 120 km/h). The dynamics of the closed-loop system using PID-control is then given by

$$A(K_I, K_P, K_D) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{K_l}{T_l^2}K_I & -\frac{1}{T_l} - K_P\frac{K_l}{T_l^2} & -\frac{2\cdot D_l}{T_l} - K_D\frac{K_l}{T_l^2} \end{bmatrix} . \quad (20)$$

A PID-type controller is chosen to be able to compare the results with the results from classical methods, which are only applicable for PID-type controller. Nevertheless, the presented approach can be applied for non-PID controllers. Further note, that the parameters T_L and D_l also could be considered as uncertain in this approach if aiming for a robust pole placement. Also, the free parameters k don't need to influence A in an affine way as in the example.

Fig. 5 shows, how the stabilizing controller gain K_P is influenced by the performance index α for fixed K_D and K_I. The boundaries are depicted with black lines and the performant area is shaded grey. It is obvious, that the range of stabilizing parameters shrinks by increasing the requirements. However, it is quite hard to predict the exact dependency of K_P and α without such a detailed analysis.

In Fig. 6 the stabilizing K_D and K_I parameter space is visualized for varying values of α. The results correspond to Fig. 5 with $K_P = 5$. Here, a similar effect can be seen: the range of stabilizing parameters shrinks by increasing the performance requirements. The shape of the resulting stability regions is quite unexpected. The stability regions for $\alpha = 0$ are linear boundaries. This fits perfectly to

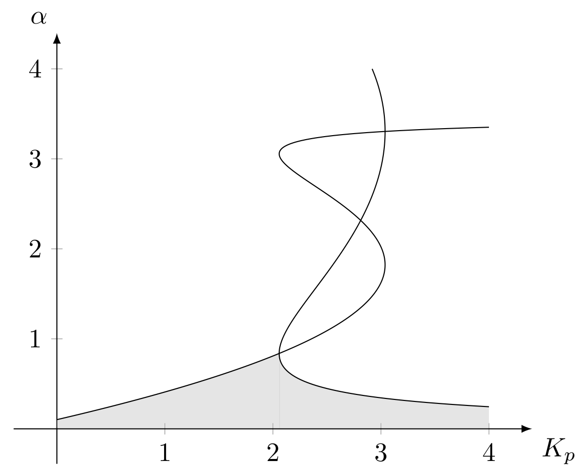

Fig. 5: K_P over α with $K_D = 1$ and $K_I = 1$

the classical observations of the stability boundaries in Ackermann [1]. For increasing values of α, the shape of the boundaries changes significantly and some of the boundaries become nonlinear. Consequently, such boundaries could not be created by the classical methods in [1], which shows, that these methods are not applicable in the presented case.

Finally, in Fig. 7 the effect of damping requirements to the stabilizing parameter space is analysed. The key results are similar to the previous case study: the range of stabilizing parameters shrinks by increasing the performance requirements and the stability regions for $\psi = 0$ are linear boundaries. This fits perfectly to the results of Ackermann [1] as well. Again, for increasing values of ψ, the shape of the boundaries changes significantly and two nonlinear boundaries arise. These two coincide for $\psi = 0$. Thus, the boundary to the left is the additional fictitious boundary mentioned in section III, which clearly lies in the unstable area for all ψ. The results in Fig. 7 cannot be gathered by present parameter space approach methods. Those need a discretization of the parameter space to perform the frequency sweeping, whereas no discretization of the parameter space is needed in the presented approach.

In Fig. 5 to 7 it can be seen how dramatically the

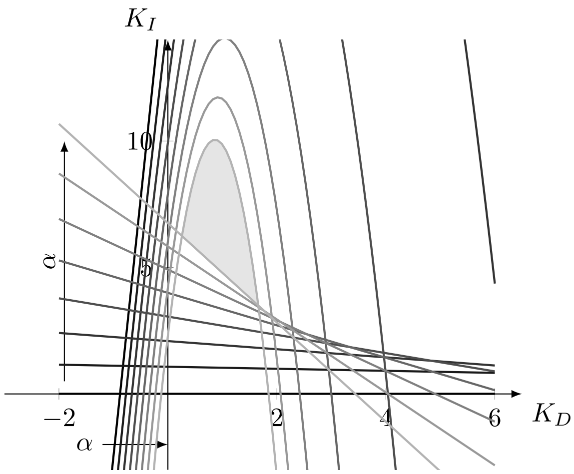

Fig. 6: K_I over K_D for $\alpha = 0 \ldots 1.5$ with $K_P = 5$

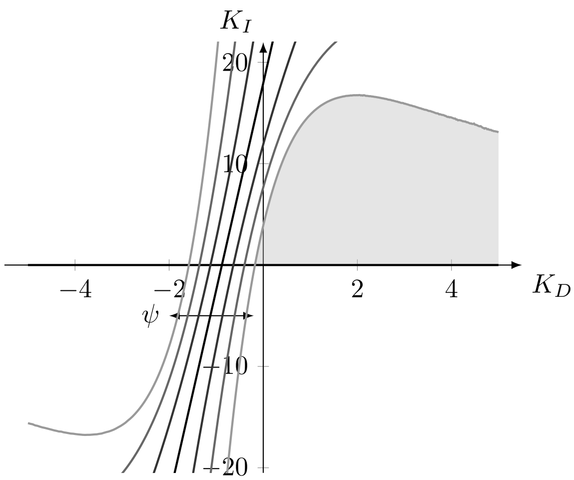

Fig. 7: K_I over K_D for $\psi = 0 \ldots 0.3$ with $K_P = 5$

stabilizing parameter space shrinks by increasing the performance requirements. This shows the practical relevance of such studies, as it would be quite hard to find by trial and error a controller parameter set which fulfills the previously mentioned performance requirements. It is also visible, that the center of gravity of the stabilizing parameter space, which is often suggested as performant choice in literature, leads to poor controller parameters.

Finally, the presented parameter space charts were used to find suitable controller parameters for the velocity controller of the ACC system. By using the presented approach it was easy possible to find a suitable trade-off between a fast settling time and a well damped system. In practice, this can be seen as maximizing the driver comfort and having an optimal safety behaviour of the car in dense traffic situations.

V. Conclusion

Several methods to analytically calculate the performant parameter space of linear systems exists, but rely on frequency sweeping and discretization. In this paper, the advantageous method of the Lyapunov stability boundary mapping has been extended from stability analysis towards performance investigation. With the proposed method settling time and damping can be analysed without increase of computational costs. From a theoretical point of view, the Lyapunov stability boundary mapping has in passing also be extended for complex matrices and perfect agreement with the well-known results for the stability analysis could be achieved. From the point of practicality, the performant parameter space computation was applied to a real world ACC-model, which has been validated with an extensive amount of measurement data. Here, surprising effects regarding the strong nonlinear behaviour between control parameters and performance parameters have been observed.

As future work, the authors aim to extend the methods also for discrete-time systems. Whereas the Lyapunov stability boundary mapping has been extended for such system, the performance mapping has not. This will on the one hand supplement the theory of the parameter space approach and on the other hand increase the practicality, as many controllers in practice are implemented in discrete-time.

VI. Acknowledgments

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