A family of extremum seeking laws for a unicycle model with a moving target: theoretical and experimental studies*

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Abstract— In this paper, we propose and practically evaluate a class of gradient-free control functions ensuring the motion of a unicycle-type system towards the extremum point of a time-varying cost function. We prove that the unicycle is able to track the extremum point, and illustrate our results by numerical simulations and experiments that show that the proposed control functions exhibit an improved tracking performance in comparison to standard extremum seeking laws based on Lie bracket approximations.

I. INTRODUCTION

Extremum seeking typically refers to the problem of constructing a gradient-free control law that ensures the motion of a dynamical system to the minimum (or maximum) of a partially or completely unknown and possibly time-varying cost or performance function. Over the past decades, significant advances in the theory and applications of extremum seeking have been made, see, e.g., [1], [2]. Today, there exist many ways to design and analyze extremum seeking laws exploiting, e.g., averaging and singular perturbation techniques, Lie bracket approximation techniques, least squares estimation approaches, stochastic and hybrid approaches, see, e.g., [3]–[12]. Many extremum seeking schemes use control functions depending on the current value of the cost function modulated by time-periodic oscillating excitation (or dither, learning) signals in order to explore and extract sufficient information from the dynamical system and/or from the unknown cost function to solve the extremum seeking problem.

The choice of the control function as well as the excitation signals plays an important role for the performance of the extremum seeking scheme [13]–[16]. In the recent paper [17], a broad family of control functions for extremum seeking schemes based on Lie bracket approximations was presented for systems with single-integrator dynamics and for time-invariant cost functions. This class of controls has several favorable properties including the possibility of adapting and constraining the amplitude of the excitation signal. Moreover, in the paper [18] the extremum seeking problem for time-varying cost functions has been considered in the framework of Lie bracket approximations, but again

with single-integrator dynamics and with standard control functions as used in [5]. The first contribution of this paper is a whole family of control functions which enables a system with unicycle-type dynamics to approximate the gradient-like flow of a time-varying cost functions. This result justify the use of gradient-free controllers presented in [17] in time-varying extremum seeking problems. For the sake of simplicity, we consider a distance-like time-varying cost function. Such problems arise, for example, when a robot has to follow a moving target (tracking problem) and only the distance (but not the relative position) to the moving target can be measured. Although gradient-free control laws for the tracking of a moving target have been previously considered (see, e.g., [19]–[28]), the main advantage of the proposed family of extremum seeking laws is, on the one hand, the high flexibility in designing the control functions such that they meet further specifications like input constraints, and, on the other hand, the family of control functions ensures rigorous stability and tracking properties.

As it will be shown, some important control strategies in the proposed class are not continuously differentiable. In view of this, the second contribution of this paper is the relaxation of the "C^2-requirement" for the Lie bracket approximation approach [5], [18]. Instead, we will require the continuity of Lie derivatives. This result will allow us to exploit a much wider class of admissible extremum seeking laws. Extremum seeking systems with non-C^2 vector fields were also considered in [17], [29] for time-invariant cost functions. However, the results of the above papers are not directly applicable for time-varying extremum seeking problems.

As the third contribution of this paper, we show by numerical simulations and experiments with a mobile robot that the high flexibility of the proposed control functions can be utilized to significantly improve the tracking behavior in comparison to standard extremum seeking approaches considered, for example, in [5].

The rest of this paper is organized as follows. In Section II, we formulate the extremum seeking problem and recall some results on the Lie bracket approximation approach. Section III presents a class of extremum seeking laws for a unicycle-type system and stability results for time-varying cost functions. The numerical simulations and experiments for several extremum seeking laws with different qualitative properties are discussed in Section IV.

II. PROBLEM STATEMENT AND PRELIMINARIES

A. Problem statement

Consider a unicycle model

$$\dot{x}_1 = u\cos(\Omega t),$$
$$\dot{x}_2 = u\sin(\Omega t),$$

$$(1)$$

where $x = (x_1, x_2)^T \in \mathbb{R}^2$ is the state, $u \in \mathbb{R}$ is the control, and $\Omega > 0$ is constant angular velocity. These equations correspond to the standard unicycle model where the equation for the angular velocity $\dot{\theta} = \Omega$, $\theta(0) = 0$, has been eliminated (see, e.g., [24], [30] for details). Note that the model with non-constant angular velocity can also be considered using, e.g., singular perturbations techniques [4].

In this paper, we address the following problem:

Problem 1: For a cost function $\hat{J} \in C^2(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{R})$, $\hat{J} = \hat{J}(x, \gamma)$, with an (unknown) function $\gamma : [0, +\infty) \rightarrow \mathbb{R}^2$, $\gamma = (\gamma_1(t), \gamma_2(t))^T$ and a unique (possibly time-varying) minimum $x^*(\gamma(t))$, the goal is to construct a control function $u = u(t, \hat{J}(x(t), \gamma(t)))$ that asymptotically steers system (1) to an arbitrary small neighborhood of $x^*(\gamma(t))$ as $t \rightarrow +\infty$.

In particular, if

$$\hat{J}(x, \gamma) = J(x - \gamma) = \kappa\|x - \gamma\|^2 \text{ with some } \kappa > 0, \quad (2)$$

then the above extremum seeking problem leads to steering the control system (1) to an arbitrary small neighborhood of the curve $x^*(t) = \gamma(t)$.

B. Notations

For $f_i, f_j : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $x \in \mathbb{R}^n$, we denote the Lie derivative $L_{f_i}f_j(t, x) = \lim_{y \to 0} \frac{1}{y}(f_j(t, x + yf_i(t, x)) - f_j(t, x))$ and the Lie bracket $[f_i, f_j](t, x) = L_{f_i}f_j(t, x) - L_{f_j}f_i(t, x)$; for a function $h \in C^1(\mathbb{R}^n \times \mathbb{R}^k; \mathbb{R})$ and $\xi \in \mathbb{R}^n$, the gradient of h with respect to ξ is denoted as $\nabla_\xi h(\xi, \eta) = \frac{\partial h(\xi, \eta)}{\partial \xi}^T$; for a one-parameter family of non-empty sets $\mathscr{L}_t \subset \mathbb{R}^n$, $t \in \mathbb{R}^+$, with $\mathbb{R}^+$ denoting the set of all non-negative real numbers. $B_\delta(\mathscr{L}_t) = \cup_{y \in \mathscr{L}_t}\{x \in \mathbb{R}^n : \|x - y\| < \delta\}$ is a δ-neighborhood of the set $\mathscr{L}_t$ at time t, and $\bar{B}_\delta(\mathscr{L}_{\lambda,t})$ denotes the closure of $B_\delta(\mathscr{L}_{\lambda,t})$. For $m, n \in \mathbb{Z}$, the notation $i = \overline{m, n}$ means that i takes all possible values from the set $\{m, m+1, \ldots, n\}$.

C. Preliminaries

In this section, we summarize some basic facts about Lie bracket approximation techniques. Consider a control-affine system

$$\dot{x} = f_0(t, x) + \sum_{j=1}^{\ell} f_j(t, x)\sqrt{\omega}u_j(\omega t), \quad (3)$$

where $x = (x_1(t), \ldots, x_n(t))^T \in D \subseteq \mathbb{R}^n$, $x(t_0) = x^0 \in D$, $t_0 \in \mathbb{R}^+$, $\omega > 0$. Let the following assumptions be satisfied:

(A1) $f_i \in C^2(\mathbb{R}^+ \times D; \mathbb{R}^n)$, $i = \overline{0, \ell}$;

(A2) the functions $\|f_i(t, x)\|, \|\frac{\partial f_i(t, x)}{\partial t}\|, \|\frac{\partial f_i(t, x)}{\partial x}\|, \|\frac{\partial^2 f_j(t, x)}{\partial t \partial x}\|,$ $\left\|\frac{\partial [f_j, f_k](t, x)}{\partial t}\right\|, \left\|\frac{\partial [f_j, f_k](t, x)}{\partial x}\right\|$ are bounded on each compact set $x \in \mathscr{X} \subseteq D$ uniformly in $t \geq 0$, for $i = \overline{0, \ell}$, $j = \overline{1, \ell}$, $k = \overline{j, \ell}$.

(A3) the functions u_j are Lipschitz continuous and T-periodic with some $T > 0$, and $\int_0^T u_j(\tau)d\tau = 0$, $j = \overline{1, \ell}$.

Since in time-varying extremum seeking problems it is often necessary to investigate the stability of a family of sets (instead of a single set), we will make use of the following definitions which can be found, e.g., in [18].

Definition 1: A family of non-empty sets $\mathscr{L}_t \subseteq D$, $t \in \mathbb{R}^+$, is said to be *locally practically uniformly asymptotically stable* for (3) if it is

• *practically uniformly stable*: for any $\varepsilon > 0$ there exist $\delta, \omega_0 > 0$ such that, for all $t_0 \in \mathbb{R}^+$ and $\omega > \omega_0$, the following property holds for the solutions of (3):
$x^0 \in B_\delta(\mathscr{L}_{t_0}) \Rightarrow x(t) \in B_\varepsilon(\mathscr{L}_t)$ for all $t \geq t_0$;

• $\hat{\delta}$-*practically uniformly attractive with some* $\hat{\delta} > 0$: for every $\varepsilon > 0$ there exist $t_1 \geq 0$ and $\omega_0 > 0$ such that, for all $t_0 \in \mathbb{R}^+$ and $\omega > \omega_0$, the following property holds for the solutions of (3):
$x^0 \in B_{\hat{\delta}}(\mathscr{L}_{t_0}) \Rightarrow x(t) \in B_\varepsilon(\mathscr{L}_t)$ for all $t \geq t_0 + t_1$;

• the solutions of (3) are *practically uniformly bounded*: for each $\delta > 0$ there are $\varepsilon > 0$ and $\omega_0 > 0$ such that, for all $t_0 \in \mathbb{R}^+$ and $\omega > \omega_0$, $x^0 \in B_\delta(\mathscr{L}_{t_0}) \Rightarrow x(t) \in B_\varepsilon(\mathscr{L}_t)$ for all $t \geq t_0$.

If the attractivity property holds for every $\hat{\delta} > 0$, then the family of sets $\mathscr{L}_t$ is called *semi-globally practically uniformly asymptotically stable* for (3). For systems independent of ω we omit the terms *practically* and *semi*.

The following result from [18] allows to establish practical asymptotic stability properties of (3) from asymptotic stability properties of the so-called Lie bracket system.

Theorem 1: Let (A1)–(A3) hold. Suppose that a family of sets $\mathscr{L}_t \subset D$ is locally (globally) uniformly asymptotically stable for the Lie bracket system

$$\dot{\bar{x}} = f_0(t, \bar{x}) + \frac{1}{T}\sum_{i=1}^{n}\sum_{\substack{j=2 \\ j>i}}^{n}[f_i, f_j](t, \bar{x})\int_0^T\int_0^\theta u_j(\theta)u_i(\tau)d\tau d\theta,$$

$$(4)$$

and suppose that there exists a compact set $S \subset \mathbb{R}^n$ such that $\mathscr{L}_t \subseteq S$ for all $t \in \mathbb{R}^+$. Then $\mathscr{L}_t$ is locally (semi-globally) practically uniformly asymptotically stable for (3).

In order to characterize the stability and tracking behavior of (1) with respect to time-varying cost functions of the form (2), we will consider the following family of level sets of the cost function $\hat{J}$:

$$\mathscr{L}_{\lambda,t} = \{x \in \mathbb{R}^2 : \hat{J}(x, \gamma(t)) - \hat{J}(x^*(\gamma(t)), \gamma(t)) \leq \lambda\}, \lambda, t \geq 0.$$

Let us emphasize that most of the results presented in the following apply to general time-varying cost functions not necessarily restricted to the form (2). However, in target tracking applications, the cost function often takes the form (2) with a possibly unknown γ.

III. MAIN RESULTS

A. Family of extremum seeking controls

In [17], we have introduced a novel family of extremum seeking controls for systems with integrator dynamics and time-invariant cost functions. In this section, we show that

a similar result can be obtained for system (1) with time-varying cost functions.

Theorem 2: Consider the control function

$$u^{\omega}(t,\hat{J}(x,\gamma)) = \sqrt{\vartheta \alpha \omega}\big(F_1(\hat{J}(x,\gamma))\cos(\omega t) \\ + F_2(\hat{J}(x,\gamma))\sin(\omega t)\big), \quad (5)$$

where $\omega = k\Omega$ *with some* $k \in \mathbb{N}$, $k > 1$, $\alpha = 4(1 - k^{-2})$, $\vartheta > 0$, *and the functions* F_1, F_2 *are such that* $F_s \in C(\mathbb{R};\mathbb{R})$, $F_s \circ \hat{J} \in C^1(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R})$ *(s = 1,2), and*

$$F_2(z) = -F_1(z)\int \frac{dz}{F_1(z)^2}. \quad (6)$$

Then the Lie bracket system corresponding to the closed-loop system (1) *with control function* (5) *has the form*

$$\dot{\bar{x}} = -\vartheta \nabla_{\bar{x}}\hat{J}(\bar{x},\gamma) + \Phi(\bar{x},\gamma), \quad (7)$$

with $\Phi(\bar{x},\gamma) = (-\varphi_2(\bar{x},\gamma), \varphi_1(\bar{x},\gamma))^T$,

$$\varphi_s(\bar{x},\gamma) = \frac{1}{2k}\nabla_{\bar{x}_s}\Big(F_1^2(\hat{J}(\bar{x},\gamma)) + F_2^2(\hat{J}(\bar{x},\gamma))\Big), \; s = 1,2.$$

Remark 1: In formula (6), we assume that $F_1(z) \neq 0$ except for at most a countable set of isolated zeros $Z^* = \{z_k^*\}$. We treat the function $\Psi_1(z) := \int \frac{dz}{F_1(z)^2}$ as an antiderivative of $\frac{1}{F_1(z)^2}$ defined on the open set $\mathbb{R} \setminus Z^*$, so that (6) holds as an identity with continuous functions in a neighborhood of each point $z \notin Z^*$. As the functions F_1 and F_2 are assumed to be globally continuous, formula (6) is treated in the sense that $F_2(z_k^*) = -\lim_{z \to z_k^*} F_1(z)\Psi_1(z)$ at each $z_k^* \in Z$.

Proof of Theorem 2: Substituting the controls (5) into system (1), we obtain a system of the form (3) given by

$$\dot{x} = \sum_{j=1}^{4} f_j(t,x)\sqrt{\omega}v_j(\omega t), \quad (8)$$

where v_j are the new $\frac{2\pi k}{\omega}$-periodic inputs,

$$v_1(\omega t) = \cos(\omega t)\cos\left(\frac{\omega t}{k}\right), \; v_2(\omega t) = \sin(\omega t)\cos\left(\frac{\omega t}{k}\right),$$

$$v_3(\omega t) = \cos(\omega t)\sin\left(\frac{\omega t}{k}\right), \; v_4(\omega t) = \sin(\omega t)\sin\left(\frac{\omega t}{k}\right),$$

and the new vector fields are $f_i(t,x) = (F_i(\hat{J}(x,\gamma(t))),0)^T$, $f_{i+2}(t,x) = (0,F_i(\hat{J}(x,\gamma(t))))^T$, $i = 1,2$. Direct construction of system (4) with the use of (6) completes the proof. ∎

Notice that the above control laws leave a lot of freedom for tuning by choosing the functions F_1 and F_2 appropriately.

B. Stability conditions

If the functions F_1 and F_2 are of class C^2, then the stability properties of the unicycle model (1) controlled by (5) can be deduced from the stability properties of the corresponding Lie bracket system (7). This directly follows from Theorem 1:

Corollary 1: Let the functions $F_s(\hat{J}(\cdot,\cdot))$ satisfy (A1)–(A2), and suppose that a one-parameter family of sets $\mathscr{L}_{\lambda,t}$

is locally (globally) uniformly asymptotically stable for (7) with some $\lambda > 0$, and there exists a compact set $S \subset \mathbb{R}^n$ such that $\mathscr{L}_{\lambda,t} \subseteq S$ for all $t \in \mathbb{R}^+$, then $\mathscr{L}_{\lambda,t}$ is locally (semi-globally) practically uniformly asymptotically stable for (1) with controls (5).

In combination with asymptotic stability conditions of families of sets for the Lie bracket system, the above result describes a solution to Problem 1 with a wide class of time-varying cost functions $\hat{J}$, provided that λ is small enough. Although, unlike [18], the Lie bracket system for (1) is not the exact gradient flow of the cost function, it admits the same asymptotic stability conditions for $\mathscr{L}_{\lambda,t}$ as proposed in [18] because of the property $(\Phi(\bar{x},\gamma), \nabla_{\bar{x}}\hat{J}(\bar{x},\gamma)) \equiv 0$. However, many functions F_1, F_2 described by (6) fail to satisfy the C^2-condition at the origin (so that Corollary 1 and the results of [18] are not applicable), but exhibit much better performance in comparison with systems with smooth vector fields (see [17], [29], [31] and Section IV for some examples). To overcome such limitation, we will present stability results under relaxed assumptions. Note that, although extremum seeking problems for systems with non-C^2 vector fields were previously considered, e.g., in [17], [29], the results of the above papers are not applicable because of several reasons. First, it is easy to see that the time-varying function $x^*(\gamma(t))$ is not a solution of system (7), therefore, the considered problem cannot be reduced to control design in a neighborhood of an admissible trajectory. Second, the approximation result in [29] has been proved under the assumption that the Lie bracket system possesses C^2-vector fields: $[f_i, f_j] \in C^2(\mathbb{R}^+ \times D)$. However, the function φ_1, φ_2 in Theorem 2 do not necessary satisfy this requirement. The following result establishes the stability of the unicycle model (1) controlled by (5) under relaxed assumptions. For clarity of presentation and because of space limitations, our next theorem and its proof will be stated for system (1) with time-varying cost functions of the form (2). It is expected that similar results can be obtained for a wide class of time-varying cost functions (but with more involved conditions). We leave the general case for future studies.

Theorem 3: Let the cost $J = J(x - \gamma(t))$ be of the form (2), $\rho > 0$, $D = \cup_{t \geq 0}\mathscr{L}_{\rho,t}$, and F_1, F_2 be defined from (6). *Assume that:*

(B1) $F_i \circ J \in C^2(D \setminus \{0\};\mathbb{R})$, $L_{F_j}(F_i \circ J) \in C(D;\mathbb{R})$, and $L_{F_k}L_{F_j}(F_i \circ J) \in C(D;\mathbb{R})$ for all $i,j,k \in \{1,2\}$;

(B2) *the first-order partial derivatives of* $F_i \circ J$ *and of* $L_{F_i}(F_j \circ J)$ *are uniformly bounded in* $D \setminus \{0\}$ *for all* $i,j,k \in \{1,2\}$;

(B3) $\gamma \in C^1(\mathbb{R}^+;\mathbb{R}^2)$, *and there exists a* $\nu > 0$ *such that* $\|\dot{\gamma}(t)\| \leq \nu$ *for all* $t \in \mathbb{R}^+$.

Then, for any $\lambda \in (0,\rho)$, $\delta \in \left(0, \frac{\sqrt{\rho} - \sqrt{\lambda}}{\sqrt{\kappa}}\right)$, *and* $\vartheta > \frac{\nu}{2\sqrt{\kappa\lambda}}$, *the family of sets*

$$\mathscr{L}_{\lambda,t} = \{x \in \mathbb{R}^2 : J(x - \gamma(t)) \leq \lambda\}, \; t \in \mathbb{R}^+ \quad (9)$$

is practically uniformly asymptotically stable for system (1) *with* $x^0 \in B_\delta(\mathscr{L}_{\lambda,t_0})$.

The proof of this result can be found in arXiv: 1802.09285. Note that the proof technique is similar to [17, Theorem 3].
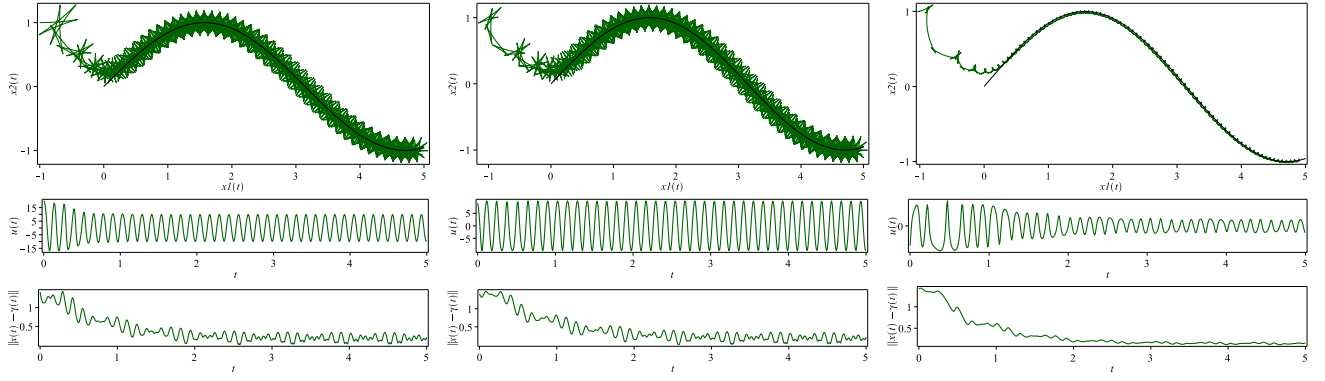
Fig. 1. Trajectory (top), controls and the tracking error $\|x(t) - \gamma(t)\|$ (bottom) for system (1) with controls (11) (left), (12) (center) and (14) (right).

However, since the results of [17] are proved for the case of time-invariant vector fields $f_i(x)$ and constant x^*, they are not directly applicable. The proof of Theorem 3 requires some extensions of the approach of [17] to control-affine systems with time-varying vector fields and non-vanishing drift term. Furthermore, unlike many other results on the time-varying extremum seeking problems, we do not assume that $\gamma(t)$ is uniformly bounded.

IV. NUMERICAL SIMULATIONS AND EXPERIMENTS

In this section, we illustrate our results with examples and discuss some interesting choices of the functions F_1 and F_2 in the control law (6).

A. Moving target tracking

Let $\Omega = 5$, $\omega = 50$, and $\gamma(t) = (0.1t, \sin(0.1t))^T$, so that the cost function is of the form

$$\hat{J}(x, \gamma) = J(x - \gamma) = (x_1 - 0.1t)^2 + (x_2 - \sin(0.1t))^2. \quad (10)$$

In all simulations, we assume that system (1) is initialized at $x_1(0) = -1, x_2(0) = 1$, the functions F_1, F_2 satisfy (6), and $\vartheta = 1$. For the first case, consider $F_1(J) = J$, $F_2(J) \equiv 1$. Then control (5) takes the form

$$u(t) = \sqrt{\alpha\omega}\big(J(x - \gamma)\cos(\omega t) + \sin(\omega t)\big). \quad (11)$$

Here and in the sequel, we denote $u(t) := u^\omega(t, J(x - \gamma))$. Such type of controls were introduced in [5] and also used in other classical extremum seeking approaches (e.g., [9]), possibly with additional filters. The main advantages of this control are its simple analytical form and applicability for a wide class of cost functions. The corresponding plots are shown in Fig. 1 (left). The control with $F_1(J) = \sin(J)$, $F_2(J) = \cos(J)$ was previously considered in [24]:

$$u(t) = \sqrt{\frac{\alpha\omega}{2}}\big(\sin(J(x - \gamma))\cos(\omega t) + \cos(J(x - \gamma))\sin(\omega t)\big),$$
$$|u(t)| \le \sqrt{\alpha\omega} \quad \text{for any } J \text{ and all } x \in \mathbb{R}^2, t \ge 0. \quad (12)$$

It possesses similar properties and, moreover, has an a priori known bound. The corresponding plots are shown in Fig. 1 (center).

Although the controls (11) and (12) possess several useful properties, they always lead to non-vanishing oscillations of the trajectories of the extremum seeking system. This can be explained, in particular, by the fact that the controls (11) and (12) do not vanish for $J = 0$, i.e., when approaching the target. Requiring $F_1(J), F_2(J) \to 0$ as $J \to 0$, it is possible to construct control laws which reduce the amplitude of oscillations and ensure better convergence properties. In particular, these properties are ensured with the following control proposed in [31]:

$$u(t) = \sqrt{\alpha\omega}\big(\sqrt{|J(x - \gamma)|}\sin(\ln(|J(x - \gamma)|))\cos(\omega t)$$
$$+ \sqrt{|J(x - \gamma)|}\cos(\ln(|J(x - \gamma)|))\sin(\omega t)\big). \quad (13)$$

In order to combine the advantages of controls (13) (i.e., vanishing amplitudes when reaching the target) and (12) (i.e., bounded excitation signals independent of the cost function), the following control function has been proposed in [17]:

$$u(t) = \sqrt{\alpha\omega\phi}\big(\sin(\psi)\cos(\omega t) + \cos(\psi))\sin(\omega t)\big), \quad (14)$$

where $\phi = \frac{1 - e^{-|J(x - \gamma)|}}{1 + e^{J(x - \gamma)}}$, $\psi = e^{|J(x - \gamma)|} + 2\ln(e^{|J(x - \gamma)|} - 1)$ for $J \neq 0$, and $u = 0$ for $J = 0$. Fig. 1 (right) presents the simulation results for system (1) with the controls (14). It can be seen that the controls (13) and (14) exhibit smaller tracking error and the control amplitudes. However, it has to be noticed that both controls (13) and (14) exhibit better behavior of an extremum seeking system only in case of known minimal value of the cost function, and control (14) requires that $x(t_0)$ is close enough to $x^*(\gamma(t_0))$ (under a proper scaling of the cost function) for better convergence properties.

B. Experimental results

The above examples show that the proposed new extremum seeking control laws perform very well in numerical simulations. In this section we want to illustrate that the benefits also transfer to an experimental setup. We validated the control on a three-wheeled mobile robot (see Fig. 3) both in a fixed and a moving target tracking scenario.

Due to limitations in the experimental setup we do not directly measure the distance to the target but evaluate it using (x_1, x_2)-position measurements of both the robot and the target obtained from tracking them with a camera.
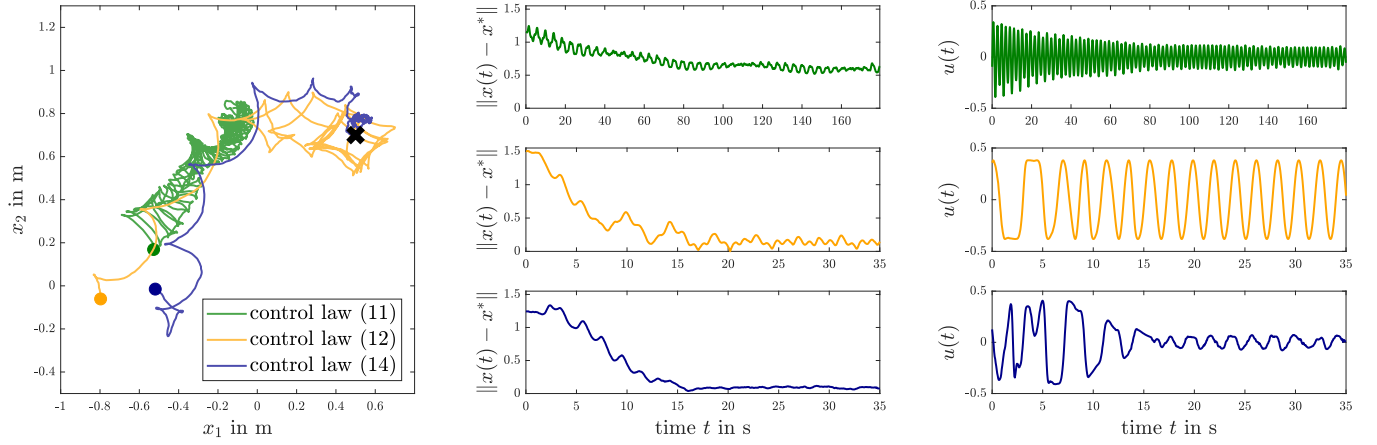
Fig. 2. Experimental results for the fixed target scenario. The plot on the left-hand side shows the trajectories of the robot in the (x_1, x_2)-plane using the three different control laws (11) (green), (12) (orange) and (14) (blue). The target point is indicated by a black cross. The plots in the middle depict the evolution of the distance to the target, again for the control laws (11) (green, upper), (12) (orange, middle) and (14) (blue, lower). The plots on the right-hand side depict the control input $u(t)$.

In the fixed target scenario, we let $\omega = 3$, $\Omega = 1.5$ and assume the cost function to be of the form (2) with $\gamma \equiv x^\star = (0.5, 0.7)^\top$ being the constant position of the target. We compared the control laws (11), (12) and (14) where the parameters α and κ were tuned under the assumption that the input is bounded as $|u(t)| \le 0.4$, see Table I.

The experimental results are depicted in Fig. 2. Control law (11) shows the worst performance and does not converge very close to the target, even in much longer time. The performance of (12) and (14) is comparable in terms of the accumulated squared distance error and the convergence time. However, while control law (12) is non-vanishing and thus the robot circulates around the target in the end, the robot only makes small movements in the end when using control law (14), and the resulting total control effort is drastically reduced in comparison to control law (12). The reason why the control input does not vanish completely is the imperfect rotational motion of the robot when $u(t) = 0$. In the moving target scenario, we let $\omega = 3$, $\Omega = 1$. The goal is to track a target moving along a figure eight curve, i.e., the cost function takes the form (2) with

$$\gamma = \big(0.8\cos(0.025t) + 0.08, 0.8\sin(0.05t) + 0.5\big)^\top. \quad (15)$$

We compared the control laws (12) and (14). Again, the

Fig. 3. The omni wheel robot used in the experiments.

Control law	Fixed target		Moving target	
	α	κ	α	κ
(11)	$2.25 \cdot 10^{-4}$	10	–	–
(12)	$4.84 \cdot 10^{-2}$	4	$5.29 \cdot 10^{-2}$	4
(14)	$3.249 \cdot 10^{-1}$	4	$2.5 \cdot 10^{-1}$	1

TABLE I

PARAMETERS USED IN THE EXPERIMENTS.

parameters α and κ were tuned under the assumption that the input is bounded as $|u(t)| \le 0.4$, see Table I. The experimental results are depicted in Fig. 4 (left) for control (12) and in Fig. 4 (right) for control law (14). Both control laws achieve tracking the moving target, where control law (14) shows a better behavior in terms of the tracking error while requiring only approximately half the control effort.

All in all, the experimental results show that the new extremum seeking control laws can lead to improved performance also in practical implementations. Nevertheless, due to low upper limits for ω and Ω, there is still quite a gap between experimental and simulative results.

CONCLUSIONS

In this paper, a novel family of extremum seeking laws have been introduced for unicycle models. We have proved that the proposed controls can be utilized for tracking an extremum point of a time-varying cost function by extending the theoretical results from [17] and [18]. In particular, we have discussed how the results can be applied to moving target tracking problems. We have illustrated by simulations as well as experiments that the proposed family of extremum seeking laws performs remarkably well for these type of problems. Our next goals are to construct further extensions of the family of control functions (6) for more general classes of cost functions and to evaluate their performance with simulations and experiments.
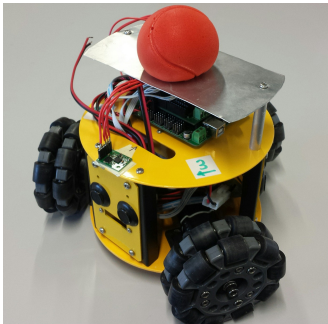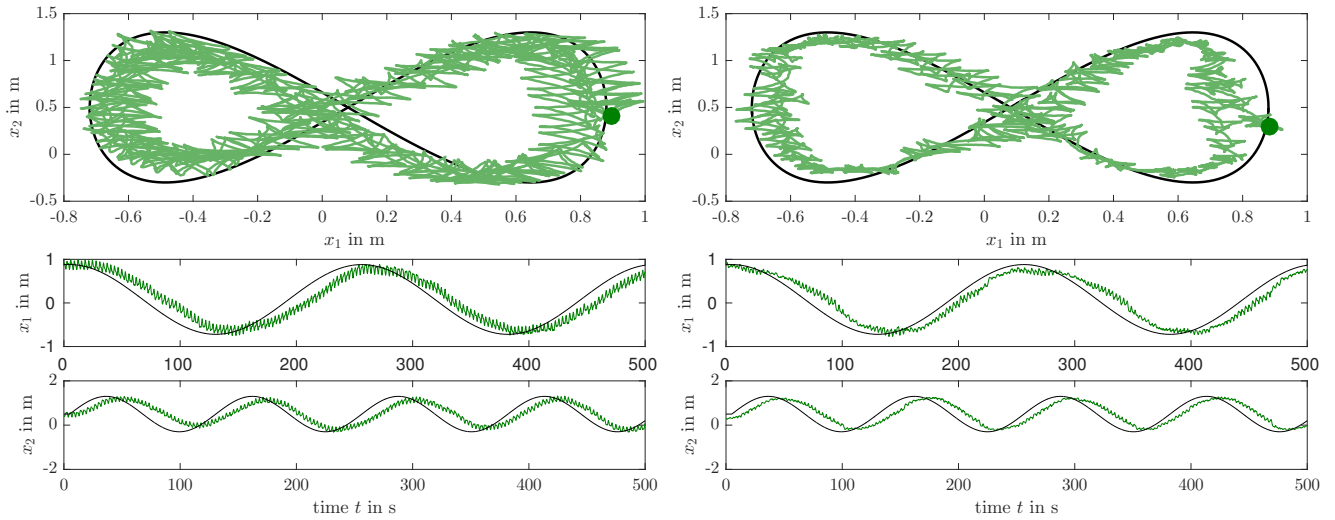
Fig. 4. Experimental results for the moving target scenario using control law (12) (left) and (14) (right). The target trajectory is depicted in black and the robot trajectory is depicted in green. The accumulated tracking error is $\int_0^{500} \|x(t) - \gamma(t)\|^2 \mathrm{d}t \approx 367.7589$ for (12) and ≈ 75.3409 for (14), and the accumulated control effort is $\int_0^{500} |u(t)| \mathrm{d}t \approx 129.2032$ for (12) and ≈ 64.7733 for (14).

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