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Realization of homogeneous multi-agent networks

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***Abstract*— A class of large-scale systems with decentralized information structures, such as multi-agent systems, can be represented by a linear system with a generalized frequency variable. In these models agents are modelled through a strictly proper SISO state space model while the supervisory structure, representing the information exchange among the agents, is represented via a linear state-space model. The starting point of the paper is that the agent $h(s)$ and the overall system $\mathcal{G}(s)$ are known through their Markov parameters. Based on these data a condition is given that characterizes compatibility, i.e., the existence of a transfer function $G(s)$ that describes the network and leads to the relation $\mathcal{G}(s) = G(\frac{1}{h(s)})$. If compatibility holds, the paper also presents an algorithm to compute the Markov parameters of the unknown transfer function $G(s)$. Then, a minimal state space representation of this transfer function can be computed through the Ho-Kalman algorithm.**

I. Introduction and Motivation

Modern engineering systems in the areas of manufacturing, transportation, and telecommunications can be effectively represented as a network of agents that mutually interact and exchange information. Dynamical interactions among agents, and the intrinsic complexity of the physical networks make the analysis and control of multi-agent network systems quite a challenging task.

In order to make the analysis computationally tractable, the simplifying assumption that the agents can be described the same transfer function is often introduced. Then, the overall dynamics can be represented as the interconnection of a scalar transfer matrix and of a feedback control block, that represents the communication exchange among the agents. Under these assumptions, Hara and co-authors have been able to describe the homogeneous multi-agent system dynamics as a linear system with generalized frequency variable, [1]. A series of powerful results were derived regarding controllability, stability and stabilizability , H_2 and H_∞-norm computation of the overall system, see [2], [3], [4]. This class of system descriptions has a potential to provide a theoretical foundation for analyzing and designing large-scale dynamical systems in a variety of areas.

In this paper we investigate the problem of the reconstruction of the transfer function that represents the communication level based on the knowledge of the agent and of the overall behaviour of the system.

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A. Problem statement

Let us consider the following model for hierarchical multi-agent dynamical systems: the system consists of N identical SISO agents whose state space realization is expressed as

$$\begin{aligned} \dot{x}_i &= A_h x_i + b_h u_i \\ y_i &= c_h x_i \end{aligned} \tag{1}$$

and the transfer function is given by

$$h(s) = c_h(sI_{n_h} - A_h)^{-1} b_h, \tag{2}$$

where $c_h^T, b_h \in \mathbb{R}^{n_h}$ and $A_h \in \mathbb{R}^{n_h \times n_h}$.

The agents are connected to each other through the input and output according to the following rule:

$$\begin{aligned} \alpha &= A\beta + Bu \\ y &= C\beta + Du, \\ \beta &= (I_N \otimes h(s))\alpha, \end{aligned} \tag{3}$$

where $u \in \mathbb{R}^m, y \in \mathbb{R}^p, \alpha, \beta \in \mathbb{R}^N$ and A, B, C are real matrices of corresponding dimensions. If the connection is well-defined, the overall system will be given by the upper linear fractional transformation (LFT), see Figure 1:

$$\mathcal{G}(s) = \mathfrak{F}_u(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, h(s)). \tag{4}$$

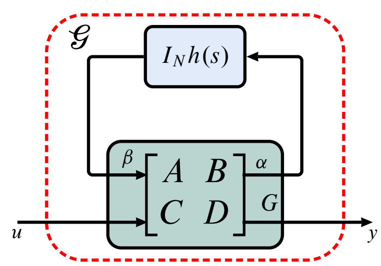

Fig. 1. Homogeneous multi-agent network

Note that if $G(s)$ is interpreted as a standard transfer function, expressed as

$$G(s) = D + C(sI_N - A)^{-1}B, \tag{5}$$

then $\mathcal{G}(s)$ can be rewritten as

$$\mathcal{G}(s) = G(\Phi(s)), \quad \Phi(s) = \frac{1}{h(s)}.$$

A short computation reveals that the state-space realization $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ of $\mathcal{G}(s)$ can be expressed as

$$\mathcal{A} = I_N \otimes A_h + A \otimes (b_h c_h), \tag{6}$$

$$\mathcal{B} = B \otimes b_h, \quad \mathcal{C} = C \otimes c_h, \quad \mathcal{D} = D, \tag{7}$$

or

$$\mathcal{A} = A_h \otimes I_N + (b_h c_h) \otimes A, \tag{8}$$

$$\mathcal{B} = b_h \otimes B, \quad \mathcal{C} = c_h \otimes C, \quad \mathcal{D} = D. \tag{9}$$

Example 1: Let us consider a very simple example of formation control: there are N identical agents moving between the walls placed at l_1 and l_2 in the one-dimensional space. The position of the ith agent is represented as y_i. We assume that the agents are collision-free and all the agents share the common dynamics $h(s)$.

The control objective for the ith agent is to control its position by collecting its relative position with respect to the other agents and the walls. In particular, the target (reference) position r_i of the ith agent is given by

$$r_i = F_i y + b_i, \tag{10}$$

where F_i characterizes the weighted relative position with respect to the other agents and b_i takes a nonzero value when the information of the wall positions is known and zero when no information is available for the agent i. Note that $F_{ij} \neq 0$ if j is an agent that the ith agent can sense, and zero otherwise.

In this paper it is assumed that $h(s)$ and $\mathcal{G}(s)$ are known. More precisely, it is supposed that both systems are given through their Markov parameters, i.e., the sets $m = \{m_k = c_h A_h^k b_h \in \mathbb{R},\ k \geq 0\}$ and $\mathcal{M} = \{\mathcal{M}_k = \mathcal{C}\mathcal{A}^k\mathcal{B} \in \mathbb{R}^{p\times m},\ k \geq 0\}$, respectively. Let $M = \{M_k = CA^kB \in \mathbb{R}^{p\times m},\ k \geq 0\}$ be the set of Markov parameters that corresponds to $G(s)$. Then, we call the given data compatible if there exists a G such that (4) holds. In the same manner we can use the terminology for h and $\mathcal{G}$: they are compatible if there is a G such that (4) holds.

The paper provides a characterization of the compatible sets. We also present a method to determine the transfer function $G(s)$. The difficulty of the problem lies in the fact that the available data, i.e., the state space realizations of the transfer functions are not necessarily related through (6) or (8), which explains the role of the Markov parameters (as invariants for the particular realization) in the reconstruction process.

The proposed algorithm provides a method to compute the Markov parameters of the unknown transfer function $G(s)$. Then, a minimal state space representation of this transfer function can be computed through the well-known Ho-Kalman algorithm, see [5].

Section II introduces the notion of compatible realization and explains its relevance to the state space representation of homogeneous multi-agent dynamical systems. Section III recalls the controllability and observability results related to this class of systems. Finally, Section IV presents the main result of the paper: it provides the compatibility condition and describes the proposed realization algorithm. With illustrative purposes the paper also contains some small examples.

II. Compatible realizations

As a first step, let us relate the state space transformations $\mathcal{T}$ of the realization $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ to the realizations of $h(s)$ and $G(s)$, respectively.

Lemma 1: Let us consider the state space transformations T_i on $h(s)$ and T on $G(s)$, i.e.,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \sim \begin{bmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{bmatrix},$$

$$\begin{bmatrix} A_h & b_h \\ c_h & 0 \end{bmatrix} \sim \begin{bmatrix} T_i A_h T_i^{-1} & T_i B \\ c_h T_i^{-1} & 0 \end{bmatrix}.$$

Then $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \sim (\mathcal{T}\mathcal{A}\mathcal{T}^{-1}, \mathcal{T}\mathcal{B}, \mathcal{C}\mathcal{T}^{-1}, \mathcal{D})$ with

$$\mathcal{T} = T_h(T \otimes I_N), \quad T_h = \operatorname{diag}(T_i)_{i=1,N}. \tag{11}$$

Given (1) one can immediately check that for the aggregate state $\xi = \begin{bmatrix} x_1^T, & \cdots, & x_N^T \end{bmatrix}^T$ we have

$$\dot{\xi} = (I_N \otimes A_h)\xi + (I_N \otimes b_h)\alpha, \tag{12}$$

$$\beta = (I_N \otimes c_h)\xi. \tag{13}$$

Then, from (3) we have

$$\dot{\xi} = (I_N \otimes A_h + A \otimes (b_h c_h))\xi + (B \otimes b_h)u,$$
$$y = (C \otimes c_h)\xi.$$

Note, that the loop (4) is always well-defined. Rewriting these equations with $\tilde{x}_i = T_i x_i$, $\tilde{\alpha} = T\alpha$ and $\tilde{\beta} = T\beta$ we obtain

$$\dot{\zeta} = \mathcal{T}(I_N \otimes A_h + A \otimes (b_h c_h))\mathcal{T}^{-1}\zeta + \mathcal{T}(B \otimes b_h)u,$$
$$y = (C \otimes c_h)\mathcal{T}^{-1}\zeta,$$

with $\zeta = \mathcal{T}\xi$, as claimed.

It is quite obvious, that not all state transformations $\mathcal{T}$ of $\mathcal{G}$ can be obtained in this way. As an example, take $N = 2, n_h = 2$ and consider the following transformations:

$$\mathcal{T} = \begin{bmatrix} a & 0 \\ c & b \end{bmatrix}, T_h = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}, T = \begin{bmatrix} x & y \\ z & w \end{bmatrix},$$

with $a, b, T_1, T_2, T \in \mathbb{R}^{2\times 2}$ nonsingular. Condition $\mathcal{T} = T_h(T \otimes I_N)$ implies that $c = \frac{z}{w}b$, which cannot be satisfied for nonsingular c, e.g., for $c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

This fact motivates to call realizations that are related by transformations of the type $\mathcal{T} = T_h(T \otimes I_N)$ as compatible (to the given hierarchical structure).

Note, that (8) is not a compatible realization: since A_h is square, according to (29) we have

$$(I_N \otimes A_h) = P(A_h \otimes I_N)P^{-1},$$

for some permutation matrix P. One can show that

$$P(I_N \otimes b_h) = (b_h \otimes I_N), \quad (I_N \otimes c_h)P^{-1} = (c_h \otimes I_N).$$

Moreover, $\mathcal{T} = P$. But P does not have a block diagonal structure, in general.

Example 2: As an example let us consider the case

$$h(s) \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 1 & 0 \end{array}\right], \quad G(s) \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \end{array}\right].$$

Applying (6) we have

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix},$$

while (8) gives the (not compatible) realization

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}.$$

The letter is obtained from the compatible realization through the permutation

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

i.e., we have an interblock mixing. Observe that actually this is not the more natural selection: the permutation

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

leads to the (not compatible) realization

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

Interblock mixing means that the states of the realization contains a blending of the components of ξ defined by (12) that corresponds to different blocks. A compatible realization does not mix these components.

III. Controllability and Observability

Up to this point we do not consider the properties of the realizations. It is immediate that the possible uncontrollability (unobservabity) of the realization of $h(s)$ is inherited by the one of $\mathcal{G}(s)$ (supposed that $D = 0$). To make this fact obvious, consider the Kalman decomposition

$$A_h = \begin{bmatrix} A_{r\bar{o}} & A_{12} & A_{13} & A_{14} \\ 0 & A_{ro} & 0 & A_{24} \\ 0 & 0 & A_{\bar{r}\bar{o}} & A_{34} \\ 0 & 0 & 0 & A_{\bar{r}o} \end{bmatrix},$$

$$b_h = \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \\ 0 \\ 0 \end{bmatrix}, \quad c_h = \begin{bmatrix} 0 & C_{ro} & 0 & C_{\bar{r}o} \end{bmatrix},$$

and observe by (8) and the definition of the Kronecker product that $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ inherits this structure. By applying the same argument, one can observe that this is also true for the case when the realization of $G(s)$ is considered to be in the Kalman form.

Thus, in order to have controllability (observability), it is necessary to have controllable (observable) realizations for both $h(s)$ and $G(s)$. The remaining question is wether these conditions are also sufficient.

Example 2 reveals that even $h(s)$ and $G(s)$ have a controllable realization, the resulting realization is not controllable. Take, e.g., $w^T = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$ and $\lambda = 0$ to check that for the compatible realization we have $w^T \mathcal{B} = 0$ with $w^T \mathcal{A} = \lambda w^T$. Observe that the uncontrollability can be revealed only on the global level, in general – the controllability decomposition leads to a not compatible realization. Thus, in light of these facts the following result has a great practical relevance:

Theorem 1 ([1]): If $\operatorname{rank} B = N$ ($\operatorname{rank} C = N$), then $(\mathcal{A}, \mathcal{B})$ is controllable ($(\mathcal{A}, \mathcal{C})$ is observable) if and only if (A_h, b_h) is controllable ((A_h, c_h) is observable).

If $\operatorname{rank} B \neq N$ ($\operatorname{rank} C \neq N$), then $(\mathcal{A}, \mathcal{B})$ is controllable ($(\mathcal{A}, \mathcal{C})$ is observable) if and only if (A_h, b_h, c_h) is a minimal representation and (A, B) is controllable ((A, C) is observable).

Theorem 2 ([1]): Assume that $h(s)$ is strictly proper. Then the realization $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ of $\mathcal{G}(s)$ is minimal if and only if the realization (A_h, b_h, c_h) of $h(s)$ and the realization (A, B, C, D) of $G(s)$ is a minimal.

Example 3: Let us modify Example 2 to

$$h(s) \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 1 & 0 \end{array}\right], \quad G(s) \sim \left[\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right].$$

Applying (6) we have

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix},$$

i.e., a controllable pair (check, e.g., by the Kalman rank condition).

Considering

$$h(s) \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \end{array}\right], \quad G(s) \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \end{array}\right],$$

we have

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

i.e., a controllable pair, as expected.

IV. Minimal compatible realizations

Theorem 2 guarantees that the compatible realization of $\mathcal{G}(s)$ through (6) is minimal, provided that we start from minimal realizations of $h(s)$ and $G(s)$. In this section we investigate the problem of obtaining a minimal realization of $G(s)$ if minimal realizations of $\mathcal{G}(s)$ and $h(s)$ are given, provided that (4) holds. This problem is closely related to the following assertion:

Theorem 3: For given transfer functions $h(s)$, $G(s)$ and $\mathcal{G}(s)$ relation (4) holds if and only if any minimal realization of $\mathcal{G}(s)$ is equivalent to a compatible one, i.e, for any minimal realization of $h(s)$ and $G(s)$ there is a transformation matrix $\mathcal{T}$ that relates the given $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ to (6).

In view of Theorem 2 and recalling the fact that all minimal realizations of a given plant are related through a suitable state transformation $\mathcal{T}$, the assertion is immediate. Note that Theorem 2 shows that (4) implies $n = Nn_h$, where n, n_h and N are the minimal state dimensions of $\mathcal{G}(s)$, $h(s)$ and $G(s)$, respectively. Thus, the relation $n = Nn_h$, as a basic structural condition, is necessary in order to have (4), i.e., h and $\mathcal{G}$ to be compatible.

It is nontrivial, however, to obtain the $\mathcal{T}$ guaranteed by Theorem 3 even if we suppose that the two transfer functions are compatible. Instead of trying to construct directly such a transformation, in what follows we provide an algorithm to compute a minimal realization of $G(s)$ provided that the ones for $\mathcal{G}(s)$ and $h(s)$ are available, or equivalently, we compute the Markov parameters of G if the Markov parameters of these two transfer functions are available. Recall that a minimal realization of a transfer function starting from the Markov parameters is provided by the Ho-Kalman algorithm, [5]. Having a minimal realization of $G(s)$ one can compute a minimal compatible realization through (6). A possible procedure to obtain $\mathcal{T}$ is to reduce each of the minimal realizations of $\mathcal{G}(s)$ to the canonical form with $\mathcal{T}_1$ and $\mathcal{T}_2$. Then $\mathcal{T} = \mathcal{T}_2^{-1}\mathcal{T}_1$.

A. Computation of M

Recall that we denote the Markov parameters of $\mathcal{G}(s)$ by $\mathcal{M}_k = \mathcal{C}\mathcal{A}^k\mathcal{B} \in \mathbb{R}^{p\times m}$, those of $G(s)$ by $M_k = CA^kB \in \mathbb{R}^{p\times m}$ and those of $h(s)$ by $m_k = c_h A_h^k b_h \in \mathbb{R}$, for $k \geq 0$, respectively.

Starting from (6) one has the following relations:

$$\mathcal{M}_0 = M_0 \otimes m_0 = m_0(M_0 \otimes 1) = m_0 M_0, \tag{14}$$

$$\begin{aligned}\mathcal{C}(\mathcal{A} - I_N \otimes A_h)^k \mathcal{B} &= m_0^{k-1}\mathcal{C}(A^k \otimes b_h c_h)\mathcal{B} = \\ &= m_0^{k+1}(M_k \otimes 1) = m_0^{k+1} M_k,\ k \geq 1.\end{aligned} \tag{15}$$

In what follows, we show that the left hand side can be computed starting from any, i.e., not necessarily compatible, realizations. Observe that (3) can be written as

$$\alpha = A\beta + Bu \tag{16}$$
$$y = C\beta + Du,$$
$$\dot{x} = \mathbb{A}_h x + \mathbb{B}_h \alpha \tag{17}$$
$$\beta = \mathbb{C}_h x,$$

with $(\mathbb{A}_h, \mathbb{B}_h, \mathbb{C}_h)$ any minimal realization of $I_N \otimes h(s)$. Then the system $\mathcal{G}(s)$ will have the realization

$$\mathcal{A}_o = \mathbb{A}_h + \mathbb{B}_h A \mathbb{C}_h, \quad \mathcal{B}_o = \mathbb{B}_h B, \quad \mathcal{C}_o = C\mathbb{C}_h. \tag{18}$$

Note, that in contrast to the compatible realizations, which are special in that they can be transformed into (6), every minimal realization of $\mathcal{G}(s)$ should be of this form taking a suitable choice for $(\mathbb{A}_h, \mathbb{B}_h, \mathbb{C}_h)$.

Let us denote by $\mathcal{T}$ a state transformation that gives $(\mathcal{T}\mathbb{A}_h\mathcal{T}^{-1}, \mathcal{T}\mathbb{B}_h, \mathbb{C}_h\mathcal{T}^{-1}) = (I_N \otimes A_h, I_N \otimes b_h, I_N \otimes c_h)$. Then, one has $(\mathcal{T}\mathcal{A}_o\mathcal{T}^{-1}, \mathcal{T}\mathcal{B}_o, \mathcal{C}_o\mathcal{T}^{-1}) = (\mathcal{A}, \mathcal{B}, \mathcal{C})$. Thus, we have

$$\begin{aligned}&\mathcal{C}(\mathcal{A} - I_N \otimes A_h)^k \mathcal{B} = \mathcal{C}\mathcal{T}\mathcal{T}^{-1}(\mathcal{A} - I_N \otimes A_h)^k \mathcal{T}\mathcal{T}^{-1}\mathcal{B} = \\ &\mathcal{C}_o(\mathcal{T}^{-1}\mathcal{A}\mathcal{T} - \mathcal{T}^{-1} I_N \otimes A_h \mathcal{T})^k \mathcal{B}_o = \mathcal{C}_o(\mathcal{A}_o - \mathbb{A}_h)^k \mathcal{B}_o = \\ &\mathcal{C}_o(\mathbb{B}_h A \mathbb{C}_h)^k \mathcal{B}_o = m_0^{k+1} M_k,\end{aligned}$$

i.e., $\mathcal{C}_o(\mathcal{A}_o - \mathbb{A}_h)^k \mathcal{B}_o = m_0^{k+1} M_k$.

It remains to prove that the left hand side of the expression is known (it can be computed), i.e., one can compute the Markov parameters M_k of $G(s)$, provided that $m_0 \neq 0$.

For $k = 0$ we have $\mathcal{C}_o\mathcal{B}_o = \mathcal{M}_0$. Thus, from $\mathcal{M}_0 = m_0 M_0$ we can compute M_0. For $k = 1$ we have

$$\begin{aligned}&\mathcal{C}_o(\mathcal{A}_o - \mathbb{A}_h)\mathcal{B}_o = \mathcal{M}_1 - \mathcal{C}_o\mathbb{A}_h\mathcal{B}_o = \\ &= \mathcal{M}_1 - C\mathbb{C}_h\mathbb{A}_h\mathbb{B}_h B = \mathcal{M}_1 - C I_N \otimes m_1 B = \\ &= \mathcal{M}_1 - m_1 M_0.\end{aligned}$$

We have used the fact that $\mathbb{C}_h\mathbb{A}_h\mathbb{B}_h$ is a Markov parameter of $I_N \otimes h(s)$ and that the set of Markov parameters does not depend on the particular realization: thus,

$$\mathbb{C}_h\mathbb{A}_h\mathbb{B}_h = I_N \otimes c_h A_h b_h.$$

Since M_0 is already known, from $\mathcal{M}_1 - m_1 M_0 = m_0^2 M_1$ we can compute M_1.

For $k = 2$ we have $\mathcal{C}_o(\mathcal{A}_o - \mathbb{A}_h)^2\mathcal{B}_o =$

$$= \mathcal{C}_o\mathcal{A}_o^2\mathcal{B}_o - \mathcal{C}_o\mathcal{A}_o\mathbb{A}_h\mathcal{B}_o - \mathcal{C}_o\mathbb{A}_h\mathcal{A}_o\mathcal{B}_o + \mathcal{C}_o\mathbb{A}_h^2\mathcal{B}_o.$$

Using (18) we obtain

$$\begin{aligned}&\mathcal{C}_o\mathcal{A}_o\mathbb{A}_h\mathcal{B}_o = \mathcal{C}_o\mathbb{A}_h^2\mathcal{B}_o + \mathcal{C}_o\mathbb{B}_h A\mathbb{C}_h\mathbb{A}_h\mathcal{B}_o = \\ &= m_2 M_0 + m_0 m_1 M_1, \\ &\mathcal{C}_o\mathbb{A}_h\mathcal{A}_o\mathcal{B}_o = \mathcal{C}_o\mathbb{A}_h^2\mathcal{B}_o + \mathcal{C}_o\mathbb{A}_h\mathbb{B}_h A\mathbb{C}_h\mathcal{B}_o = \\ &= m_2 M_0 + m_0 m_1 M_1.\end{aligned}$$

Then, from

$$\begin{aligned}&\mathcal{C}_o(\mathcal{A}_o - \mathbb{A}_h)^2\mathcal{B}_o = \\ &= \mathcal{M}_2 - m_2 M_0 - 2m_0 m_1 M_1 = m_0^3 M_2\end{aligned}$$

we have M_2.

For $k = 3$ we have $\mathcal{C}_o(\mathcal{A}_o - \mathbb{A}_h)^3\mathcal{B}_o =$

$$\begin{aligned}&= \mathcal{C}_o\mathcal{A}_o^3\mathcal{B}_o - \mathcal{C}_o\mathcal{A}_o^2\mathbb{A}_h\mathcal{B}_o - \mathcal{C}_o\mathbb{A}_h\mathcal{A}_o^2\mathcal{B}_o - \\ &- \mathcal{C}_o\mathcal{A}_o\mathbb{A}_h\mathcal{A}_o\mathcal{B}_o + \mathcal{C}_o\mathbb{A}_h\mathcal{A}_o\mathbb{A}_h\mathcal{B}_o + \\ &+ \mathcal{C}_o\mathbb{A}_h^2\mathcal{A}_o\mathcal{B}_o + \mathcal{C}_o\mathcal{A}_o\mathbb{A}_h^2\mathcal{B}_o - \mathcal{C}_o\mathbb{A}_h^3\mathcal{B}_o,\end{aligned}$$

i.e.,

$$\mathcal{C}_o(\mathcal{A}_o - \mathbb{A}_h)^3\mathcal{B}_o = \\ = \mathcal{M}_3 - 2m_0m_2M_1 - m_1^2M_1 - 3m_0^2m_1M_2 - m_3M_0.$$

It turns out that, repeating the process iteratively, $\mathcal{C}_o(\mathcal{A}_o - \mathbb{A}_h)^k\mathcal{B}_o$ is an expression that depends on $\mathcal{M}_k, m_0, \ldots, m_k, M_0, \ldots, M_{k-1}$, i.e., it can be computed based on the information available at step k.

Since the matrix products are not commutative, we cannot provide a closed formula here, but the computations can be performed by a suitable program, if necessary. Details are left out for brevity.

For the general case let l be the relative degree of $h(s)$, i.e., the first integer for which $m_l \neq 0$. Then

$$\mathcal{C}(I_N \otimes A_h)^l\mathcal{B} = \mathcal{C}(I_N \otimes A_h^l)\mathcal{B} = m_lM_0. \quad (19)$$

Let us write (6) as

$$\mathcal{A} - I_N \otimes A_h = A \otimes b_hc_h,$$

i.e., for any integers $k_1, k_2 \geq 0$ we have

$$\mathcal{A}_{k_1,k_2} := (I_N \otimes A_h)^{k_1}(\mathcal{A} - I_N \otimes A_h)(I_N \otimes A_h)^{k_2} = \\ = A \otimes A_h^{k_1}b_hc_hA_h^{k_2}. \quad (20)$$

Then, from (20) it follows that

$$\mathcal{C}\mathcal{A}_{l,l}\mathcal{B} = m_l^2M_1.$$

Let $k_1 + k_2 = l$. Then

$$\mathcal{A}_{l,k_2}\mathcal{A}_{k_2,l} = m_l(A^2 \otimes A_h^lb_hc_hA_h^l),$$

hence

$$\mathcal{C}\mathcal{A}_{l,k_1}\mathcal{A}_{k_2,l}\mathcal{B} = m_l^3M_2.$$

Iterating the process we have

$$\mathcal{C}\mathcal{A}_{l,k_1}\mathcal{A}_{k_2,k_3}...\mathcal{A}_{k_{2v},l}\mathcal{B} = m_l^{2v+1}M_{v+1}, \quad (21)$$

with $k_1 + k_2 = l, \cdots, k_{2v-1} + k_{2v} = l$.

Analogously with the alrady presented process,

$$\mathcal{C}\mathcal{A}_{l,k_1}\mathcal{A}_{k_2,k_3}...\mathcal{A}_{k_{2v},l}\mathcal{B} = \mathcal{C}_0\mathcal{A}^0_{l,k_1}\mathcal{A}^0_{k_2,k_3}...\mathcal{A}^0_{k_{2v},l}\mathcal{B}_0,$$

with

$$\mathcal{A}^0_{k_1,k_2} = \mathbb{A}_h^{k_1}(\mathcal{A}_0 - \mathbb{A}_h)\mathbb{A}_h^{k_2}.$$

Thus, the left hand side of (21) can be computed by using the already known Markov parameters and M_k results.

Example 4: For illustrative purposes let us consider the systems

$$h(s) \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 1 & 1 \\ \hline 0 & 1 & 0 \end{array}\right], \quad G(s) \sim \left[\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right].$$

For $\mathcal{G}(s)$, that fulfils (4), we have

$$\mathcal{G}(s) \sim \left[\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & 0 \end{array}\right],$$

with

$$\mathcal{A}_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \mathcal{B}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ \mathcal{C}_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}.$$

This generates the data of our problem, i.e., the Markov parameters

$$m_0 = 1,\ m_1 = 1,\ m_2 = 2,\ m_3 = 3,\ m_4 = 5 \\ \mathcal{M}_0 = [0,\ 1],\ \mathcal{M}_1 = [1,\ 1],\ \mathcal{M}_2 = [2,\ 2], \\ \mathcal{M}_3 = [5,\ 3],\ \mathcal{M}_4 = [10,\ 5].$$

Based on this data one can verify that $N = 2$, i.e., the necessary condition for compatibility is satisfied. Applying the steps of the proposed algorithm we obtain

$$M_0 = [0,\ 1],\ M_1 = [1,\ 0],\ M_2 = [0,\ 0], \\ M_3 = [0,\ 0],\ M_4 = [0,\ 0],$$

i.e., the Markov parameters of $G(s)$. One can verify that the order of G is $\hat{N} = 2$.

B. Testing compatibility

One can observe that starting from the given data, i.e., m and $\mathcal{M}$, the proposed algorithm always provides a set of Markov parameters M. It remains to check wether the corresponding G fulfils (4) or not. Let us denote the order of G by $\hat{N}$.

Theorem 4: m and $\mathcal{M}$, i.e., $h(s)$ and $\mathcal{G}(s)$, are compatible if and only if $n = \hat{N}n_h$.

We have already seen the necessity of the condition. For sufficiency let us suppose that G does not fulfil (4). Let us consider then $\hat{\mathcal{G}}(s) = \mathfrak{F}_u(G, h(s))$. By construction its Markov parameters satisfy $\hat{\mathcal{M}} = \mathcal{M}$ which, by the uniqueness of the Markov parameters, contradicts our hypothesis. Thus G proves compatibility.

Example 5: To illustrate a case when $\hat{N} \neq N$ let us consider the same $h(s)$ as in Example 4 and $\mathcal{G}(s)$ defined by

$$\mathcal{A}_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \mathcal{B}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}, \\ \mathcal{C}_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}.$$

Thus we have the set of data

$$m_0 = 1,\ m_1 = 1,\ m_2 = 2,\ m_3 = 3,\ m_4 = 5 \\ \mathcal{M}_0 = [0,\ 1],\ \mathcal{M}_1 = [3,\ 1],\ \mathcal{M}_2 = [5,\ 2], \\ \mathcal{M}_3 = [12,\ 3],\ \mathcal{M}_4 = [23,\ 5].$$

Applying the steps of the proposed algorithm we obtain

$$M_0 = [0,\ 1],\ M_1 = [3,\ 0],\ M_2 = [-1,\ 0], \\ M_3 = [0,\ 0],\ M_4 = [2,\ 0],$$

which leads to $\hat{N} = 3$. Since $N = 2$, it follows that the data is not compatible.

V. Conclusion

The paper has provided a method to determine the transfer function $G(s)$ starting from the knowledge of $h(s)$ and $\mathcal{G}(s)$. It is supposed that the available data is the set of Markov parameters that corresponds to the known systems.

A condition to test compatibility of the initial data was given. The proposed algorithm provides a method to compute the Markov parameters of the unknown transfer function $G(s)$. Once these Markov parameters are known, a minimal state space representation of the transfer function $G(s)$ can be obtained by applying the classical Ho-Kalman algorithm.

This paper assumes a full knowledge of the systems. In practice, however, the Markov parameters can be obtained through measurements and their set is only partially known. The possibility to apply the presented method in an identification context, and wether a partial realization approach is reliable for this class of problems it is subject to further research. It is also a research topic how to check compatibility in that setting.

Appendix

A. Notations and basic facts

For the matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ their Kronecker product $A\otimes B$ is the block matrix:

$$A\otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}. \tag{22}$$

The Kronecker product has the following properties:

$$A\otimes(B+C) = A\otimes B + A\otimes C, \tag{23}$$
$$(A+B)\otimes C = A\otimes C + B\otimes C, \tag{24}$$
$$(kA)\otimes B = A\otimes(kB) = k(A\otimes B), \tag{25}$$
$$(A\otimes B)\otimes C = A\otimes(B\otimes C). \tag{26}$$

If the matrices are nonsingular, then

$$(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}, \tag{27}$$

while theu have compatible dimensions, then

$$(A\otimes B)(C\otimes D) = (AC)\otimes(BD). \tag{28}$$

In general, the Kronecker product is not commutative. However, there exist permutation matrices P and Q such that

$$A\otimes B = P\,(B\otimes A)\,Q. \tag{29}$$

If A and B are square matrices, then we can take $P = Q^T$.

B. Ho-Kalman algorithm

The classical Ho-Kalman algorithm, see [5], can be summarized as follows. Consider a MIMO system with m inputs and p outputs and denote by H_r and $\tau(H_r)$ the matrices

$$H_r = \begin{bmatrix} M_1 & M_2 & \dots & M_r \\ M_2 & M_3 & \dots & M_{r+1} \\ \dots & & & \\ M_r & M_{r+1} & \dots & M_{2r} \end{bmatrix}$$

and

$$\tau(H_r) = \begin{bmatrix} M_2 & M_3 & \dots & M_{r+1} \\ M_3 & M_4 & \dots & M_{r+2} \\ \dots & & & \\ M_{r+1} & M_{r+2} & \dots & M_{2r+1} \end{bmatrix},$$

respectively. $\{M_k\}$ is the set of Markov parameters of the transfer function G and r is greater than or equal to the MacMillan degree of the system ($r = \max_l \operatorname{rank} H_l$). Then there exist matrices P and Q such that

$$PH_rQ = \begin{bmatrix} \mathbb{I}_s & 0 \\ 0 & 0 \end{bmatrix} = J.$$

Let us consider the matrices $U_s = \begin{bmatrix} \mathbb{I}_s & 0 \end{bmatrix}$ and $E_k = \begin{bmatrix} \mathbb{I}_k & 0_k & \dots & 0_k \end{bmatrix}$, where the dimension of the matrix U_s may vary according to the dimensions of the expressions in which it appears. Then a minimal state space realization (A,B,C) is given by

$$A = U_sJP\tau(H_r)QJU_s^*,$$
$$B = U_sJPH_rE_q^* \quad C = E_pH_rQJU_s^*.$$

Note that the realization algorithm presented above gives the desired result only in the full information case. In the finite information case one has to deal with the partial realization problem, i.e., the problem of finding the minimal order [1] rational function that has as first Markov parameters the given ones. In view of the realization theory, the problem is equivalent to the one of giving an extension sequence such that the Ho–Kalman algorithm gives the unique rational function with the desired property.

In general the set of extension sequences that leads to rational functions with the same degree that has the first Markov parameters identical to the given ones has more then one element, i.e., the solution of the minimal partial realization problem in general is not unique, see, e.g., [6], [7].

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[1] The largest among the numerator and denominator degree, called also MacMillan degree.