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Helmholtzian Strategy to Stay in Focus with Binocular Vision

Justin Ruths[1] and Bijoy K. Ghosh[2]

***Abstract*— We study problems that can be applied to controlling the rotational motion of a pair of human eyes. Eyes move to acquire a point target and the control task is to direct the eye-pair towards the general target direction and, if the target is close by, to focus on the target. Roughly speaking, the former task is accomplished by versional eye movements and the latter task of pinpointing the eyes on a specific point is accomplished by vergence eye movements. We assume that the versional movement rotates the eye-pair identically whereas the vergence movements are specific to each eye with the goal of focusing. Although it is commonly believed and evidenced by collected data that 'versional eye movements satisfy Listing's Law', we show in this paper that Listing's eye movements do not maintain focus. Perhaps surprisingly, we show that if the eye controller maintains a Donders' surface originally proposed by Helmholtz, eye movements away from points on the Transverse Plane maintains focus. For points above or below the Transverse Plane, we show by simulation that rotations satisfying the Helmholtz condition maintain focus as well. Recorded data from human eye movement satisfying Listing's law supports the observation presented here.**

I. Introduction

Eye Movement Scientists have been interested in modeling and control of a single human eye since 1845 with notable studies conducted by Listing [1], Donders [2] and Helmholtz [3]. Specifically, it was observed by Donders [2] that the oculomotor system chooses just one angle of ocular torsion for any one gaze direction. Several studies have focused on three dimensional eye movements [4], [5], [6]. Assuming the human eye to be a rigid sphere, the oculomotor system can be viewed as a mechanical control system and one can apply geometric theory with Lagrangian and Hamiltonian viewpoints [7], [8]. This paper extends our earlier studies [9], [10], [11] from monocular control to controlling a pair of eyes optimally (see earlier work [12], [13], [14]).

Any eye orientation can be reached, starting from one specific orientation called the primary orientation (typically chosen as the orientation when the eyes are looking straight), by rotation about a single axis. Listings law states that, starting from a frontal gaze, any other gaze direction is obtained by a rotation matrix whose axis of rotation is constrained to lie on a plane, called the Listings plane. Consequently, the set of all orientations the eye can assume is a submanifold [15] of $\mathbf{SO(3)}$ called **LIST**. Listing had shown, subsequently verified by others [16], [17], that with

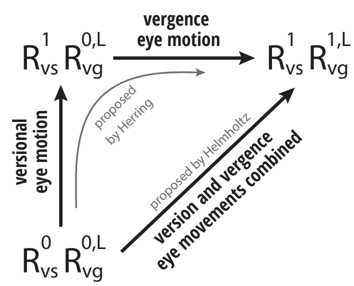

Fig. 1. In this figure only the left eye movement has been shown. The versional eye movement for the left and the right eyes are the same.

targets located at optical infinity and keeping the head fixed, eye orientations are restricted to the submanifold **LIST** [9].

In binocular vision, Listings law is not valid for fixation of nearby targets, which is the main point of this paper. It has been observed [18] (see also [19]), that when a pair of human eyes fixate on a nearby point target, the axes of rotations of the two eyes are not located on the Listings plane. The eye rotations are not independently controlled (as proposed by Helmholtz [3] in (1866)), but can be viewed as a concatenation of **version** followed by **vergence** as in Fig. 1 (in the spirit of what was originally proposed by Hering [20] in 1868). The version component produces identical eye movement in both the eyes, and is used to follow a target located far away by following the general direction of the target. In order to focus on a closer target, generally the eyes rotate in opposite directions, using the vergence component.

The version component, identical for both the eyes, is assumed to satisfy a suitable Donders' constraint [21]. On the other hand, for the vergence components of each of the two eyes, we assume, following [18] that the rotation vectors are restricted to the mid-sagittal plane, with eye orientations restricted to a different submanifold **MS** of SO(3) (see [12]). The main question we address in this paper is the following: If the goal of the binocular, versional eye movement is to maintain focus, as closely as possible, what is the best choice of the Donders' constraint that the versional system should satisfy? Quite surprisingly, we find in this paper that such a choice is the **Helmholtz gimbal** [22] and not the Listing's constraint. Specifically, under the Listing's constraint, versional eye movement does not maintain focus (illustrated in Fig. 2). Hence the focus has to be constantly adjusted by vergence eye movements following version.

II. Eye Movement under Version and Vergence

Before we begin any discussion, it would be important to agree upon one inertial frame with respect to which all

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[1]Justin Ruths is with the Departments of Mechanical and Systems Engineering, Richardson, TX, USA. `jruths at utdallas.edu`

[2]Bijoy K. Ghosh is with the Department of Mathematics and Statistics, Texas Tech University, Broadway and Boston, Lubbock, TX 79409-1042, USA. `bijoy.ghosh at ttu.edu`

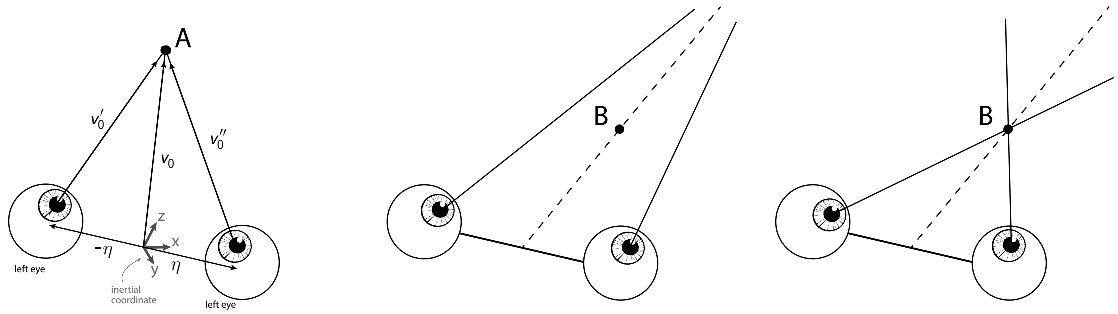


Fig. 2. Version and vergence eye movements. The initial target point is at A. v_0 is the target vector with respect to the inertial coordinate (x,y,z). Initially the two eyes were on the x axis; z is the straight gaze direction; and y is the head top vector perpendicular to the floor of a standing human. The head shown in the figure has rotated so that the eyes are not on the x-axis.

vectors and rotation matrices will be represented. We assume that the head and the two eyes are looking straight (along the z-axis) and that the eyes are focused at infinity. The inertial frame we consider is aligned with the center of the two eyes as the x-coordinate; the y-coordinate is assumed to be perpendicular to the floor; and the z-coordinate is perpendicular to x and y and points towards the frontal direction as in Fig. 2. Note that the centers of the two eyes can subsequently move away from the x-axis.

Let us now consider a point target **A** (see Fig. 2), with v_0 being the corresponding target vector. Likewise we have another point target **B**, with v_1 being the corresponding target vector. Finally we assume that the vectors η and $-\eta$ are the centers of the two eyes. Thus we have vectors v_0, v_1 and η all in $\mathbb{R}^3$. We shall call the vector v_0 the initial versional vector and the vector v_1 the final versional vector. The vector η is currently arbitrary, and to be chosen later. Using a suitable Donders' constraint we calculate a unique (see [23]) rotational matrix R^0_{vs} that rotates the frontal gaze $\mathbf{e}_3 = [0,0,1]^T$ to the direction of v_0,

$$R^0_{vs}\mathbf{e}_3 = \tilde{v}_0. \tag{1}$$

Throughout, we assume the notation that $\tilde{v} = \frac{v}{\|v\|}$ (the norm in the denominator is the standard Euclidean norm). Let us now define vectors $v'_0 = v_0 + \eta$ and $v''_0 = v_0 - \eta$. Using the **MS** constraint (defined formally below in (16)) we now calculate[1] the rotational matrix $R^{0,L}_{vg}$ such that

$$R^{0,L}_{vg}\mathbf{e}_3 = \widetilde{[R^0_{vs}]^{-1}v'_0}. \tag{2}$$

We also calculate the rotational matrix $R^{0,R}_{vg}$ such that

$$R^{0,R}_{vg}\mathbf{e}_3 = \widetilde{[R^0_{vs}]^{-1}v''_0}. \tag{3}$$

We now use the final versional vector v_1, and use the same Donders' constraint to calculate the unique rotational matrix R^1_{vs} such that

$$R^1_{vs}\mathbf{e}_3 = \tilde{v}_1.$$

$R^{1,L}_{vg}$ and $R^{1,R}_{vg}$ can be computed likewise under **MS** constraint.

For the left eye, the initial rotation matrix can be written as $R^{0,L} = R^0_{vs}\, R^{0,L}_{vg}$ (see Fig. 1). Likewise, for the right eye, the initial rotation matrix can be written as $R^{0,R} = R^0_{vs}\, R^{0,R}_{vg}$. Using the already chosen Donders' constraint, we compute the **versional optimal control** (see Section V) and the corresponding versional rotation matrices $R_{vs}(t)$, subject to the initial and the final conditions

$$R_{vs}(0) = R^0_{vs}, \quad R_{vs}(1) = R^1_{vs}.$$

The left and the right rotation matrices are now given by

$$R^L(t) = R_{vs}(t)\, R^{0,L}_{vg} = R_{vs}(t)\, [R_{vs}(0)]^{-1}\, R^{0,L},$$
$$R^R(t) = R_{vs}(t)\, R^{0,R}_{vg} = R_{vs}(t)\, [R_{vs}(0)]^{-1}\, R^{0,R}$$

respectively. Additionally, we note that

$$R^L(t)\mathbf{e}_3 = R_{vs}(t)\, [R_{vs}(0)]^{-1}\, \widetilde{v'_0}, \tag{4}$$
$$R^R(t)\mathbf{e}_3 = R_{vs}(t)\, [R_{vs}(0)]^{-1}\, \widetilde{v''_0}. \tag{5}$$

It is now easy to see that

$$[R_{vs}(0)]^{-1}v'_0 \;=\; [R_{vs}(0)]^{-1}[v_0+\eta] \;=\; \|v_0\|\mathbf{e}_3 + [R_{vs}(0)]^{-1}\eta,$$
$$[R_{vs}(0)]^{-1}v''_0 \;=\; [R_{vs}(0)]^{-1}[v_0-\eta] \;=\; \|v_0\|\mathbf{e}_3 - [R_{vs}(0)]^{-1}\eta.$$

We propose to choose η such that $[R_{vs}(0)]^{-1}\eta$ is in the transverse plane, in particular, with $\mathbf{e}_1 = [1,0,0]^T$,

$$[R_{vs}(0)]^{-1}\eta \;=\; \varepsilon\mathbf{e}_1. \tag{6}$$

The gaze directions of the left and right eyes now follow

$$R_{vs}(t)\begin{pmatrix}\varepsilon\\0\\\|v_0\|\end{pmatrix}/\sqrt{\varepsilon^2+\|v_0\|^2}, \tag{7}$$

and

$$R_{vs}(t)\begin{pmatrix}-\varepsilon\\0\\\|v_0\|\end{pmatrix}/\sqrt{\varepsilon^2+\|v_0\|^2}, \tag{8}$$

respectively, under the versional eye movement. The question we would like to ask is the following:

Question 2.1: Under what condition on the Donders' surface would the left and the right eye gaze directions in (7)

[1] It follows from the Extended Listing Theorem in [12], page 6477 that the rotation matrix on **MS** is unique as long as the vergence gaze directions (the right hand sides of (2), (3)) are not straight, i.e. $\neq (0,0,1)^T$.

and (8), continue to meet at a point? i.e when do the two eyes continue to stay in focus under version?

We note that at the end of the versional rotation the unnormalized eye gazes are at

$$R^1_{vs}\begin{pmatrix}\varepsilon\\0\\\|v_0\|\end{pmatrix} = \|v_0\|\,\tilde{v}_1+\varepsilon\,R^1_{vs}\mathbf{e}_1$$

and

$$R^1_{vs}\begin{pmatrix}-\varepsilon\\0\\\|v_0\|\end{pmatrix} = \|v_0\|\tilde{v}_1-\varepsilon\,R^1_{vs}\mathbf{e}_1,$$

respectively, for the left and the right eyes. The orientation matrices of the left and the right eyes are at $R^1_{vs}R^{0,L}_{vg}$ and $R^1_{vs}R^{0,R}_{vg}$ respectively. A vergence movement of the left eye rotates from $R^{0,L}_{vg}$ to $R^{1,L}_{vg}$ and the right eye from $R^{0,R}_{vg}$ to $R^{1,R}_{vg}$.

Note from Fig. 2 that at the end of the versional eye movement, a vergence eye movement is required to focus the left eye towards v_1' and likewise for the right eye towards v_1''. For the vergence rotation of the left eye (analogously for the right eye), we would consider **vergence optimal control**[2] $R^L_{vg}(t)$ in **MS** subject to initial and final conditions

$$R^L_{vg}(0)=R^{0,L}_{vg},\quad R^L_{vg}(1)=R^{1,L}_{vg}.$$

The orientation of the left eye during the vergence rotation is at $R^1_{vs}\,R^L_{vg}(t)$, which can also be written as

$$\overline{R^L_{vg}}(t)\,R^1_{vs}, \tag{9}$$

where $\overline{R^L_{vg}}(t)$ is the vergence rotation with respect to the local versional coordinate provided by R^1_{vs}. In fact, we have

$$R^L_{vg}(t) = [R^1_{vs}]^{-1}\,\overline{R^L_{vg}}(t)\,R^1_{vs}.$$

Remark 2.1: The interpretation of the orientation matrix (9) is that the eyes are first rotating versionally followed by vergence in the local versional coordinates (as was originally proposed by Herring). The optimal vergence control is computed separately for the left and the right eye, under the **MS** constraint, in the inertial coordinate (x,y,z).

III. Quaternion Representation of Rotations

Quaternions [24] offer an elegant and numerically stable way of representing rotations. A quaternion is composed of four elements $q=(q_0,q_1,q_2,q_3)$ in which q_0 is called the scalar part and the remaining elements compose the vector part. The rotation encoded in a quaternion is a rotation around the axis given by the vector part of the quaternion by an angle given as a function of the scalar part. For convenience, we define a function $vec: S^3\to\mathbb{R}^3$ that extracts the vector part of the quaternion. A surjective (2-to-1) map $rot: S^3\to SO(3)$ provides the rotation matrix corresponding to a quaternion q,

$$rot(q) := \begin{pmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3 & 2(q_1q_3+q_0q_2)\\ 2(q_1q_2+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1)\\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2\end{pmatrix}. \tag{10}$$

[2]The version and vergence optimal control problems were introduced and studied in [14] (see also [12], [13]).

A rotation of a vector v by a rotation matrix R, $v'=Rv$, is accomplished through **quaternion conjugation**.

$$v'=Rv=rot(q)v=vec[q*q_v*q^{-1}], \tag{11}$$

where $q_v=[0,v]$ is the quaternion with scalar part zero and vector part of v and $*$ is quaternion (Hamilton) product. The inverse quaternion $q^{-1}=[q_0,-q_1,-q_2,-q_3]/\sqrt{q_0^2+q_1^2+q_2^2+q_3^2}$.

When we assume the eye is a perfect sphere and $\omega=(\omega_1,\ \omega_2,\ \omega_3)^T$ to be the angular velocity vector of the eye, the rotational rigid-body dynamics [8] is described by

$$\dot{q}(t)=\frac{1}{2}q_\omega*q, \tag{12}$$

$$\dot{\omega}(t)=T(t), \tag{13}$$

where $T(t)=(T_1,T_2,T_3)^T$ is the external torque vector applied by the muscles of the eye and $q(t)$ is the quaternion defined in the inertial coordinate system as the rotation of the eye from the frontal gaze $(0,0,1)^T$. The quaternion $q_\omega=(0,\omega_1,\omega_2,\omega_3)$. When (12) is expanded the quaternion dynamics become [25]

$$\frac{d}{dt}\begin{pmatrix}q_0\\q_1\\q_2\\q_3\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-\omega_1q_1-\omega_2q_2-\omega_3q_3\\ \omega_1q_0+\omega_2q_3-\omega_3q_2\\ \omega_2q_0+\omega_3q_1-\omega_1q_3\\ \omega_3q_0+\omega_1q_2-\omega_2q_1\end{pmatrix}. \tag{14}$$

While (13), (14) capture the general dynamics of the eye, empirical studies have established that certain eye motions restrict themselves to well-defined manifolds in the space of rotations. In particular, Listing's plane enforces

$$\mathbf{LIST}=\{q\mid q_3=0\}, \tag{15}$$

primarily observed during versional and saccadic eye motions under head fixed condition; and the mid-sagittal plane enforces

$$\mathbf{MS}=\{q\mid q_1=0\}, \tag{16}$$

primarily observed during vergence (focusing) eye motion in binocular vision. Furthermore, in the next section we introduce a third (Helmholtz) constraint that we claim is closely related to the notion of retaining/losing focus during versional motion.

IV. The story of losing focus

In this section we go back to the versional eye movement as described by (7) and (8). We note that the vectors on which the rotation matrix $R_{vs}(t)$ operates, are in the Transverse Plane. The following theorem completely answers question 2.1.

Theorem 4.1: Assume that the vectors ρ_1 and ρ_2 are two independent vectors on the Transverse Plane, $y=0$. A necessary and sufficient condition for the three vectors $rot(q)\rho_1$, $rot(q)\rho_1$ and $e=(1,0,0)^T$ are coplanar is given by the Helmholtz constraint

$$\mathbf{HELM}=\{q\mid q_0q_3-q_1q_2=0\}. \tag{17}$$

Remark 4.1: The Donders' surface described by (17) is followed by the well known Helmholtz Gimbal[3]. By constructing such a gimbal, Helmholtz had studied 'rotation of the binocular eye' and did observe cases where the eyes maintained focus. He also did make extensive comparison with the Fick Gimbal.

Proof of Theorem 4.1:

Let us write $\rho_1 = (a\ 0\ b)^T$, $\rho_2 = (c\ 0\ d)^T$. It follows that

$$rot(q)\,\rho_1 = \begin{pmatrix} (q_0^2+q_1^2-q_2^2-q_3^2)a+2(q_1q_3+q_0q_2)b \\ 2(q_1q_2+q_0q_3)a+2(q_2q_3-q_0q_1)b \\ 2(q_1q_3-q_0q_2)a+(q_0^2+q_3^2-q_1^2-q_2^2)b \end{pmatrix}$$

and

$$rot(q)\,\rho_2 = \begin{pmatrix} (q_0^2+q_1^2-q_2^2-q_3^2)c+2(q_1q_3+q_0q_2)d \\ 2(q_1q_2+q_0q_3)c+2(q_2q_3-q_0q_1)d \\ 2(q_1q_3-q_0q_2)c+(q_0^2+q_3^2-q_1^2-q_2^2)d \end{pmatrix}.$$

A necessary and sufficient condition for the three vectors $rot(q)\rho_1$, $rot(q)\rho_1$, e to be coplanar is that

$$\det\begin{pmatrix} 2(q_1q_2+q_0q_3)a+2(q_2q_3-q_0q_1)b & 2(q_1q_2+q_0q_3)c+2(q_2q_3-q_0q_1)d \\ 2(q_1q_3-q_0q_2)a+(q_0^2+q_3^2-q_1^2-q_2^2)b & 2(q_1q_3-q_0q_2)c+(q_0^2+q_3^2-q_1^2-q_2^2)d \end{pmatrix}$$
$$= 2(ad-bc)(q_0q_3-q_1q_2) = 0.$$

Since ρ_1 and ρ_2 are independent, the condition (17) follows. (Q.E.D.)

V. Optimal constrained eye movement control

We now derive trajectories of the binocular eye system such that initially the eyes are focused at a point v_0 in the visual field. Through the application of external torques (using eye muscles) the gaze is changed such that the two eyes are again focused at a point v_1. As described in Section II and illustrated in Fig. 1, the eyes first execute a versional motion to shift the eye-gaze from an initial direction to a final (possibly not focused) gaze direction, and then a vergence motion to focus the eyes back on the final target point.

Fundamental to this study, we aim to identify if and when the eyes maintain or lose focus throughout this process. We consider separately the constraints **LIST** and **HELM** imposed on the versional motion followed by a **MS**-constrained vergence motion to focus the eyes (note that this final vergence motion may require refocusing at a different depth, even if the eyes are focused at the end of the version motion). To generate the version and vergence trajectories we derive solutions to an optimal control problem that minimizes the power of the external torques,

$$\text{cost:} \quad \int_0^1 T_1^2(t)+T_2^2(t)+T_3^2(t)\ dt. \tag{18}$$

The sequence of execution is as follows:

1) Compute q_{vs}^0 and q_{vs}^1 according to Section II using either **LIST** (15) or **HELM** (17).
2) Solve the optimal control problem corresponding to cost (18), general dynamics (13)-(14), endpoints $q(0) = q_{vs}^0$ and $q(1) = q_{vs}^1$, and the corresponding constraint (either **LIST** or **HELM**). This produces the minimum power version torques $T_{vs}(t)$ and the corresponding trajectory $q_{vs}(t)$.
3) Compute $q_{vg}^{0,L}$ and $q_{vg}^{1,L}$ using the **MS** constraint (16) according to Section II.
4) Solve the optimal control problem corresponding to cost (18), general dynamics (13)-(14), endpoints $q(0) = q_{vg}^{0,L}$ and $q(1) = q_{vg}^{1,L}$, and the **MS** constraint. This produces the minimum power vergence torques $T_{vg}^L(t)$ and corresponding trajectory $q_{vg}^L(t)$.
5) Repeat the previous two steps to produce $T_{vg}^R(t)$ and $q_{vg}^R(t)$.
6) Produce the final torques and quaternion trajectories

$$T^i(t) = \begin{cases} T_{vs}(t), & t \in [0,1] \\ T_{vg}^i(t-1), & t \in (1,2] \end{cases}, \tag{19}$$

$$q^i(t) = \begin{cases} q_{vs}(t) * q_{vg}^i(0), & t \in [0,1] \\ q_{vs}(1) * q_{vg}^i(t-1), & t \in (1,2] \end{cases}, \tag{20}$$

with $i \in \{L,R\}$.

We take durations of unit length for both version and vergence problems (easily generalized) For all of these optimizations, note that the duration of each version and vergence is one. It is easy to generalize this to allow for different durations (vergence motions are typically accepted to be slower than version motions). The version and vergence optimizations are currently separately optimized. Again it would be straightforward to optimize both of these problems simultaneously by creating a much larger optimization problem.

A. Numerical Solution of Optimal Control Problems

While framing goals as optimal control problems is a powerful and systematic way of optimizing and studying the motion of dynamical systems, in order to benefit from this formulation we need equally powerful numerical methods to solve the optimal control problems. The eye dynamics are highly nonlinear and the cost functions and the constraints we use are not necessarily convex. These features of physical processes makes their optimal control formulations very challenging (if not impossible) to solve analytically. In this line of work, we employ a numerical method that efficiently and accurately discretizes the optimal control problem into a nonlinear, nonconvex optimization problem. While the transformed problem is still quite challenging, there are a number of versatile solvers that perform well.

Such a method that transforms the original method is called a direct numerical method for solving optimal control problems (rather than transforming the dual problem). The method used in this work is a pseudospectral method, a direct collocation method, which means that the decision variables of the discretized optimization are simply the states and controls of the optimal control problem at the discretization points [26]. This makes implementing arbitrary dynamics,

[3] In the Helmholtz strategy, a rigid body is first rotated with respect to the horizontal axis and then with respect to the rotated vertical axis.

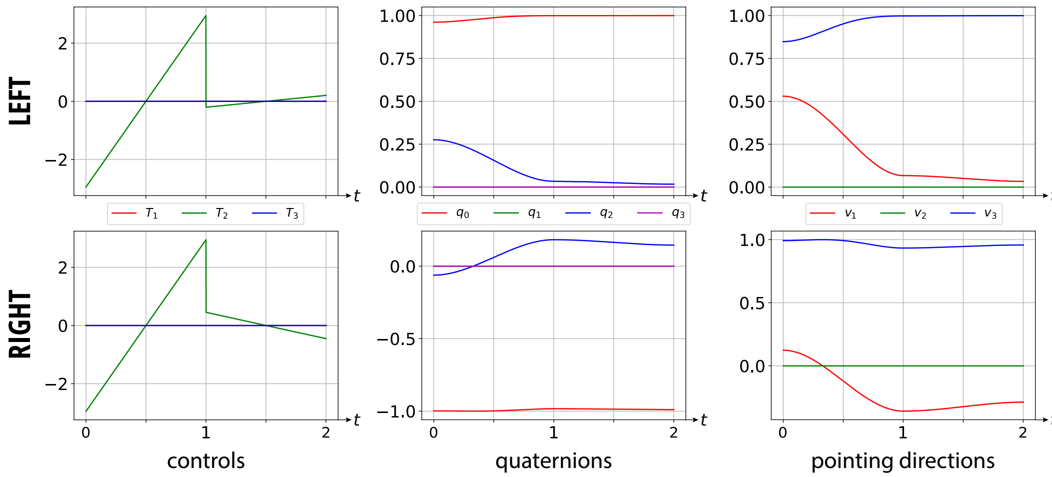


Fig. 3. Case I (both initial ($v_0 = v_a$) and final ($v_1 = v_b$) gaze vectors lie in the transverse plane): The power optimal solution showing the controls, quaternion evolution, and gaze vector evolution for the left and right eyes over the complete version-followed-by-vergence motion. In Case I, the Listing and Helmholtz (17) constraints produce identical solutions that remain in the transverse plane.

v_a	$[0.75,\ 0,\ 2]^T$
v_b	$[-0.4,\ 0,\ 3]^T$
v_A	$[0.75,\ -0.25,\ 2]^T$
v_B	$[-0.4,\ 0.35,\ 3]^T$

TABLE I
ENDPOINT CONSTRAINTS CONSIDERED; v_a AND v_b ARE LOCATED IN THE TRANSVERSE PLANE AND v_A AND v_B ARE NOT.

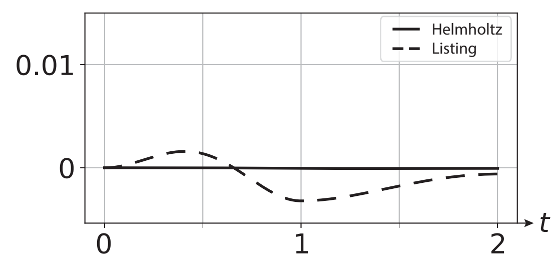


Fig. 4. Case II (initial gaze vector ($v_0 = v_a$) lies in the transverse plane but the final gaze vector ($v_1 = v_B$) does not): The coplanarity - the shortest distance between the lines spanned by the left and right gaze vectors - over the entire motion. The Listing constraint yields trajectories that cause the eyes to lose focus whereas the Helmholtz constraint (17) maintains focus.

cost functions, or constraints very simple - important for this work since we implement several different constraints. In addition, the spectral component of this method provides high levels of accuracy through representation in terms of the orthogonal Legendre polynomials. One of the limitations of this method is its reliance on polynomials (and interpolation), which means that the solutions must be smooth. In the context of eye motion, we expect all trajectories to be smooth and, therefore, this does not limit the generality of our approach. More details on the pseudospectral method can be found in [27], [28]. The only parameter to set is the order of approximation, which is the order of the polynomial/interpolation, and we denote it N. In all of the cases here, we choose $N = 40$. We code these problems in the AMPL modeling language and used the solver KNITRO (with its multistart option enabled to overcome local minima).

B. Case I: Trajectories in the transverse plane

Theorem 4.1 was derived based on the assumption that the vector $\rho_0^L = [R_{vs}(0)]^{-1}\, \widetilde{v_0'}$, for the left eye, on which the versional matrix $R_{vs}(t)$ (as described in (4)) acts, is located on the transverse plane. The same is assumed about ρ_0^R. Here we specify an initial gaze direction $v_0 = v_a$ and final gaze direction $v_1 = v_b$, both located on the transverse plane (see Table I). In Fig. 3 we show the controls $T^L(t)$ and $T^R(t)$, quaternion trajectories $q^L(t)$ and $q^R(t)$, and gaze directions for each eye $v^L(t) = vec(q^L(t))$ and $v^R(t) = vec(q^R(t))$. We observe that the **LIST** and **HELM** version constraints generate identical solutions. While not explicitly shown by Theorem 4.1, we observe some special symmetry/degeneracy when both endpoints lie in the transverse plane. Moreover, we compute the "coplanarity", which is defined as the minimum distance between the lines spanned by the left and right gaze vectors

$$\text{coplanarity}(t) = \left\{ \min\ \|p_L(t) - p_R(t)\| \;\middle|\; \begin{array}{l} p_L(t) \in \text{span}\{v^L(t)\}, \\ p_R(t) \in \text{span}\{v^R(t)\} \end{array} \right\}. \tag{21}$$

For this case (we omit the figure) the coplanarity for both **LIST** and **HELM** are identically zero. We also observe that the trajectories stay in the transverse plane (see $v_2(t) \equiv 0$).

C. Case II: Trajectories that leave the transverse plane

We now consider the scenario when the initial gaze $v_0 = v_a$ lies in the transverse plane, but the final gaze $v_1 = v_B$ lies outside of the transverse plane (see Table I). Our goal is to test the **LIST** and **HELM** constraints as claimed in the proof of Theorem 4.1. We omit the trajectory plots

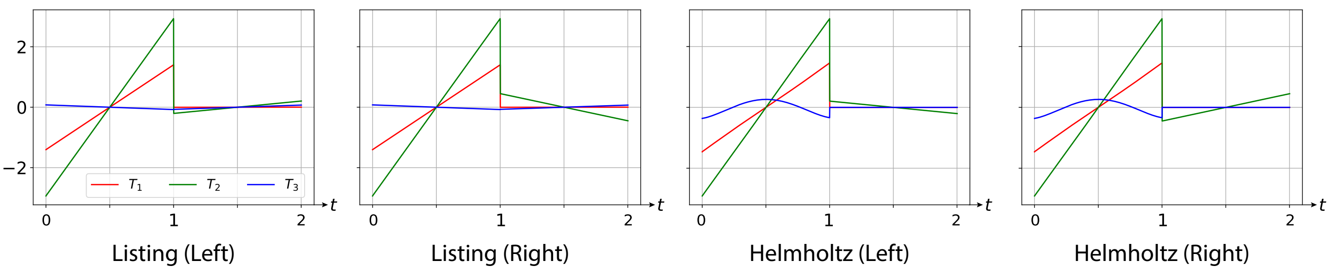


Fig. 5. Case III (both initial ($v_0 = v_A$) and final ($v_1 = v_B$) gaze vectors lie off of the transverse plane): The power optimal solution showing a comparison of the controls following the Listing constraint and following the Helmholtz constraint (17).

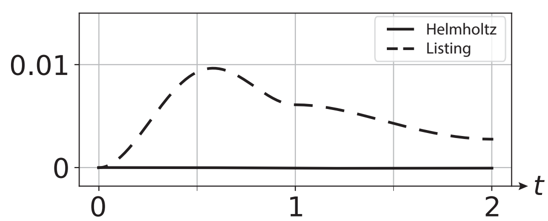


Fig. 6. Case III: The coplanarity - the shortest distance between the lines spanned by the left and right gaze vectors - over the entire motion. The Listing constraint yields trajectories that cause the eyes to lose focus whereas the Helmholtz constraint (17) maintains focus.

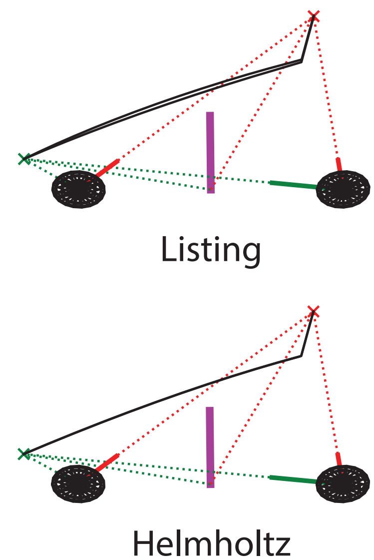


Fig. 7. Case III: Viewed from behind the eyes, the magenta line indicates the "forward" $+z$ direction. Under the Helmholtz constraint the eyes maintain focus as indicated by a single black line of focus over the gaze trajectory. Under the Listing constraint the eyes are not focused and we observe two separate lines over the gaze trajectory. Skewed lines in $\mathbb{R}^3$ (such as those represented by span of the non-focused gaze vectors) have a unique shortest path that separates them. The two curves in this plot are the endpoints of these shortest paths over the trajectory of the system. The length of these shortest paths gives the scalar quantification of coplanarity used in Figs. 4 and 6. The sharp angle close to the final gaze point (red $\times$) demarcates the transition from version to vergence.

(they look qualitatively similar to Case III) and present in Fig. 4 the coplanarity observed in each case. From the coplanarity plots, we observe a discrepancy between the **LIST** and **HELM** trajectories. The nonzero coplanarity value at $t = 1$ for the dashed line indicates that the eyes are no longer focused after the version motion following the **LIST** constraint. Contrasting that is the identically zero coplanarity of the **HELM** constraint trajectory.

D. Case III: Beyond the transverse plane

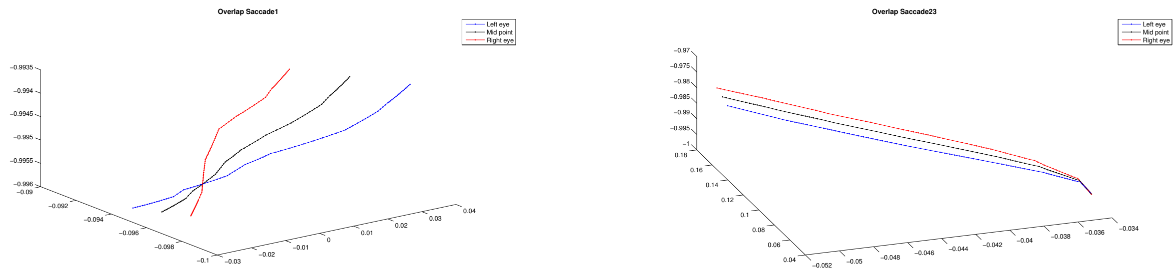
Finally we consider the most general case in which both initial ($v_0 = v_A$) and final ($v_1 = v_B$) gaze directions lie outside of the transverse plane. Here we show in Fig. 5 the way in which the **LIST** and **HELM** constraints shape the torques. Although the difference is somewhat subtle it evidences its effect in the coplanarity (see Fig. 6). The amount the eyes lose focus is large enough to observe in the trajectories of the left and right eye gaze vectors $v^L(t)$ and $v^R(t)$ in Fig. 7. The curves identify the endpoints of the shortest path between the span$\{v^L(t)\}$ and span$\{v^R(t)\}$. This simulation shows that the focusing ability of the Helmholtz constraint compared to the Listing's constraint goes beyond the transverse plane. In this sense, **HELM** as a constraint is more general than what was originally anticipated.

VI. Discussion & Conclusion

Empirical studies on eye movement have indicated that a strategy of version followed by vergence is preferred over an approach when the two eyes are controlled independently. This is perhaps because during version, eyes are rotated identically and therefore versional eye control is easier to implement. In Theorem 4.1 of this paper, we argue that when versional rotation acts on vectors that initiate from the transverse plane (Case I and II in Section V) then the Helmholtz constraint (17) is to be preferred over Listing's constraint (15). The former approach guarantees that the eyes maintain focus throughout the versional gaze change, whereas the latter approach does not. The result from Case III in section V, strongly suggest that the main result of this paper, i.e. preferring **HELM** over **LIST**, applies even for versions not initiated from the transverse plane.

It turns out that **HELM** as a constraint is not typically what is implemented during versional eye movement of human eyes, it is the **LIST** that is typically used[4], which only seems to maintain focus when the entire trajectory is

[4] Eye movement saccade trajectories have been displayed following **LIST** in Fig. 8. Eyes are typically focused initially and gradually lose focus during version following **LIST**.



(a) Eyes showing defocused to focus and back.

(b) Focused eyes are slightly defocused during a saccade.

Fig. 8. The displayed data is obtained from measurements of gaze directions of human eyes as they undergo a saccade. The right and the left gaze trajectories are in red and blue, respectively. The black trajectory is the center of the smallest line segment between the two eyes. The trajectories show how focused the eyes are during a saccade. Typically the end of saccade would signify maximal defocusing, which the data seems to show.

in the transverse plane. This paper suggests an interesting adaptation of binocular vision, using **HELM** as a constraint, to ensure a consistent coherent vision that keeps the versional gazes for the two eyes always in focus. This may find relevance to applications in robotic vision, especially in scenarios that demand high visual acuity at high speeds.

In this paper, the separation vector η between the center of the two eyes is assumed fixed. In future work, we plan to generalize this scenario to derive **HELM**-like constraints that would guarantee versional eye gazes remain in focus even when the robotic eyes translate along with rotation. Our numerical work seems to indicate that the current Helmholtz constraint would generalize.

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