2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Lyapunov stability results for the parabolic p-Laplace equation

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***Abstract*— Lyapunov stability of the parabolic p-Laplace equation is investigated. The nominal equation is shown to be asymptotically stable, while the stronger property of exponential stability is guaranteed by the presence of lower-order terms satisfying a suitable growth condition. Numerical simulations are provided to support and illustrate the theoretical results.**

I. Introduction

In recent years, partial differential equations have attracted the attention of the control community (see [1], [2], [3] [4] among several others) since many plant models are described by infinite-dimensional systems and hence involve PDEs or systems of PDEs: examples can be found in robotics (haptic controllers and flexible manipulators), in industrial processes (manufacturing, reactors and heat transfer plants) as well as in biomedical applications (tissue engineering).

Due to the high complexity of such models, it could be necessary to handle several sources of uncertainty, this enforcing the interest in the analysis and synthesis of robust control strategies. Sliding-mode is a well established robust control technique having the advantage of constraining the state of the controlled system in a region which results to be invariant with respect to external disturbances. Sliding-mode controllers have also been proposed as possible solution to the problem of robust control for PDEs [5], [6], [7]. All such results pertain classes of linear or quasilinear parabolic and hyperbolic equations. Tackling nonlinear equations is usually much harder, and requires to carefully address some delicate points, which are instead straightforward in the linear case. Among nonlinear problems, equations involving degenerate operators, such as the p-Laplace operator, are particularly challenging. The p-Laplace equation, which is the natural extension of the classical Laplace equation onto the space L^p, and the corresponding solutions, called p-harmonic functions, are used for modeling physical phenomena arising from glaciology, radiation of heat, or plastic moulding [8]. Existence, boundedness, estimates and regularity of solutions have been largely investigated over the last two decades, see for instance [9], [10], [11], [12], [13], [14], [15], [16]. The classical theory of parabolic equations is not sufficient to guarantee that the time-derivative of the solution of the parabolic p-Laplace equation does exists as a function, even though the space-derivatives are Hölder continuous functions. A regularity theory for the time-derivative of the solution has been developed only in recent years [17], [18]. Asymptotic convergence of the solution of the p-parabolic equation to the stationary one has proved in [19] in the case of time-independent data, while large-time geometric properties of solutions have been exploited in [20]. Interesting connections with stability are also provided by the theory of principal eigenvalues [21], [22]. However from a control theoretic perspective, to the best of author's knowledge, no significant result is available yet. As a first step towards that goal, in this paper the Lyapunov stability of the stationary solution is proved, this complementing the result of [19] where convergence to zero is proved using parabolic estimates. In particular, the p-parabolic equations is shown to be asymptotically stable in L^2 as well as in L^{p^*}, where p^* is the so-called Sobolev conjugate [23]. Moreover, the presence of lower order terms with a "good" growth has a regularizing effect on the solution and yields exponential stability also. The proposed results extend straightforwardly to a general class of degenerate parabolic problems, whose inspiration model is the parabolic p-Laplace equation. Future developments on this subject will be focused on investigating the validity of a ISS-like property [24] for the p-parabolic equation, and then using such feature to design robust controllers with the aim of rejecting disturbances and unknown inputs.

A simulation study has been performed to illustrate with numerical examples the different convergence properties of the considered class of equations.

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II. Parabolic p-Laplace equation

The p-Laplace operator Δ_p is a generalization of the classical Laplace operator, and for $p \in [1, \infty)$ is defined as

$$\Delta_p v := \operatorname{div}(|\nabla v|^{p-2} \nabla v).$$

Clearly, such operator is singular for $1 \le p < 2$ and degenerate for $p > 2$, i.e. the modulus of ellipticity vanishes for $|\nabla v| = 0$. The associated parabolic equation is defined in a natural way as

$$u_t = \operatorname{div}(|\nabla u|^{p-2} \nabla u). \tag{1}$$

In the following, we will refer to (1) as the *parabolic p-Laplace equation* or, shortly, as the *p-parabolic equation*.

The focus of the paper is on the Cauchy-Dirichlet problem for (1), and in particular we will deal with

$$\begin{cases} u_t = \operatorname{div}(|\nabla u|^{p-2} \nabla u) + f \\ u(t,x) = 0 \quad t \in [0,\infty),\ x \in \partial\Omega \\ u(0,x) = u_0(x) \quad x \in \Omega \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^n$, $n \ge 2$, is a bounded domain with a smooth boundary, and u_0, f are prescribed data whose regularity will be specified later on. Given a finite time horizon $[0, T]$, a weak solution of (2) on the cylinder $Q_T := [0, T] \times \Omega$ is a function u in a local parabolic Sobolev space that satisfies the identity

$$\int_0^T\int_\Omega (u\varphi_t + |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi)dxdt = \int_0^T\int_\Omega f\varphi dxdt \quad (3)$$

for any function $\varphi \in C_0^1(\bar{Q}_T)$. The definition can be readily extended to the infinite cylinder $[0,\infty)\times\Omega$ by considering test functions φ with compact support in $[0,\infty]\times\bar{\Omega}$.

A. Existence and regularity of solutions

Assume[1] $p \geq 2$ and introduce the functional space

$$V^p(0,T;\Omega) = C(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega)).$$

In formulae, a function $v(t,x)$ belongs to $V^p(0,T;\Omega)$ if the mapping $t \mapsto ||v(t,\cdot)||_2$ is continuous, and the integral $\int_0^T\int_\Omega |\nabla v(t,x)|^p dxdt$ is finite. Following [10], [14], for any fixed T the existence of a local bounded weak solution $u \in V^p(0,T;\Omega)$ can be proved under the assumptions:

A1) The function $f = f(t,x)$ verifies $f(\cdot,x) \in L^\infty(0,\infty)$ for any $x \in \Omega$ and $f(t,\cdot) \in L^{p'}(\Omega)$ for any $t \in [0,\infty)$, where the number p' is the Lebesgue conjugate of p, i.e. $1/p + 1/p' = 1$.

A2) The function $u_0(x) \in W_0^{1,p}(\Omega)$.

Without loss of generality, it will be therefore assumed that for any $T > 0$ a solution $u \in L^\infty(Q_T)$ exists. In fact, solutions can be proved to be Hölder continuous. Moreover, unicity of the weak solutions is provided by the following useful algebraic inequality:

$$(|\zeta|^{p-2}\zeta - |z|^{p-2}z)\cdot(\zeta - z) \geq 2^{1-p}|\zeta - z|^p \quad (4)$$

for any $\zeta, z \in \mathbb{R}^n$. Putting all the pieces together, we can infer that a unique solution $u \in V_{loc}^p(0,\infty;\Omega)\cap L_{loc}^\infty(Q_\infty)$ exists, where $Q_\infty := [0,\infty)\times\Omega$.

Remark 2.1: Thanks to the regularity of solutions, the equality (3) extends to any function $\varphi \in W_0^{1,p}(Q_T)$.

III. Lyapunov stability in L^2

Let us begin our analysis by considering the homogeneous problem, i.e. $f \equiv 0$. The zero function $u^* \equiv 0$ is clearly a weak (stationary) solution of the Cauchy-Dirichlet problem (2) when $u_0(x) = 0$. Let us prove that such stationary solution is an asymptotically stable equilibrium for the homogeneous p-parabolic equation. To this end, let us consider the Lyapunov function candidate

$$V(t) := \frac{1}{2}\int_\Omega |u(t,x)|^2 dx. \quad (5)$$

Recall that for a domain Ω with $|\Omega| < \infty$ the Lebesgue spaces are encapsulated with $L^q(\Omega) \subset L^r(\Omega)$ for $r < q$. Since $p \geq 2$, the solution $u(t,x)$ is in $L^p(\Omega) \subseteq L^2(\Omega)$ for any fixed t, and hence the function $V(t)$ is well-defined. The aim is to evaluate the derivative of $V(t)$ along the solution; however, one has to be careful while performing such operation, as in principle u_t only exists as a distribution and thus differentiation under the integral sign might be not allowed. On the other hand, some recent results pertaining the regularity of time derivatives may be invoked [18], [17].

Proposition 3.1: Let $u \in V_{loc}^p(0,\infty;\Omega)\cap L^\infty(Q_\infty)$ be the weak solution of (2); assume in addition that assumptions A1 *and* A2 *hold. Then the time derivative u_t satisfies:*

i) $u_t \in L_{loc}^2(0,\infty;L_{loc}^q(\Omega))\ \forall q \in \left[1, \frac{2n}{n-\frac{1}{2}}\right]$;

ii) $u_t \in L_{loc}^\infty(0,\infty;L^2(\Omega))$.

Thanks to the latter result, u_t does exist as a function, and verifies the local estimate

$$\|u_t(t,\cdot)\|_2 \leq M \quad \forall t \in [t_0-\epsilon, t_0+\epsilon],$$

where $M > 0$ only depends on t_0 and ϵ. Indeed the derivative has even more regularity, as established in the following claim, whose proof is given in Appendix B.

Claim 1: Let $\eta > 0$ and set

$$Q_{\eta,\infty} := (\eta,\infty)\times\Omega.$$

The solution $u(t,x)$ is differentiable almost everywhere in $Q_{\eta,\infty}$ for any $\eta > 0$, and $u_t(t,x)$ equals the strong derivative a.e. in $Q_{\eta,\infty}$.

In view of such property, and based on condition ii) of Proposition 3.1, it is reasonable to introduce the following class of functions and, accordingly, make an assumption on the initial datum $u_0(x)$.

Definition 3.1: For $p \geq 1$ the class of L^p-pointwise bounded functions $B(a,b;L^p(\Omega))$ is defined as

$$B(a,b;L^p(\Omega)) := \{f(t,x) : |f(t,x)| \leq g_f(x) \in L^p(\Omega)\ \forall \text{ a.e. } x \in \Omega,\ \forall t \in (a,b)\}$$

Remark 3.1: It is worth noticing that the following inclusion holds

$$B(a,b;L^p(\Omega)) \supset L^\infty(a,b;L^p(\Omega)).$$

Let us introduce the following set of initial conditions:

$$\mathcal{B}_0 := \left\{ u_0(x) \in W_0^{1,p} : \begin{array}{c}\text{the solution } u(t,x) \text{ is such that} \\ u_t(t,x) \in B(0,\infty;L^1(\Omega))\end{array} \right\}$$

Recalling that $u(t,x)$ is bounded and differentiable a.e., the assumption $u_0(x) \in \mathcal{B}_0$ yields the differentiability of $V(t)$ with

$$\dot{V}(t) = \int_\Omega u(t,x)u_t(t,x)dx. \quad (6)$$

Due to this nice property, the following stability result can be established.

Theorem 3.1: Assume $p \geq 2$ and $f \equiv 0$. The stationary solution u^* is an L^2-asymptotically stable equilibrium with a region of attraction $\mathcal{R}_{u^*} \supseteq \mathcal{B}_0$, i.e. $V(t)$ satisfies a Lyapunov inequality of the type

$$\dot{V}(t) \leq -c[V(t)]^{\frac{p}{2}}, \quad c > 0$$

for any initial condition $u_0(x) \in \mathcal{B}_0$.

Proof: As mentioned, the derivative of $V(t)$ can be written as in (6). On the other hand, thanks to Remark 2.1, the solution u can be used indeed as a test function,

[1] We notice that the given existence and regularity results still hold in the larger range $p \geq 2n/(n-1)$. However, in this case the inequality (4) is slightly different [10]. For $p < 2n/(n-1)$ the local boundedness of solutions is instead no longer guaranteed.

and hence (by implicitly applying the *divergence lemma* and canceling out the boundary terms) one has

$$\dot{V}(t) = -\int_\Omega |\nabla u(t,x)|^p dx \le -C_p \int_\Omega |u(t,x)|^p dx,$$

with $C_p > 0$, where the last estimate follows from Poincaré's inequality. Finally, using Jensen's inequality (see Appendix A), one gets

$$\dot{V}(t) \le -C_p \int_\Omega |u(t,x)|^p dx \le -c\left(\frac{1}{2}\int_\Omega |u(t,x)|^2\right)^{\frac{p}{2}}$$

where $c = 2^{\frac{p}{2}}|\Omega|^{1-\frac{p}{2}}C_p$. Now, by a standard comparison argument [25], the latter chain of inequalities implies asymptotic stability. Indeed, one has

$$\frac{1}{2}\|u(t,\cdot)\|_2^2 = V(t) \le \left(\frac{1}{V(0)^{1-\frac{p}{2}} + c(\frac{p}{2}-1)t}\right)^{\frac{2}{p-2}}$$

where $V(0) = \frac{1}{2}\|u_0(x)\|_2^2$. ■

Let us investigate the problem further, and work towards obtaining a more general stability result. To this end, let us consider now problem (2) with a datum f satisfying assumption A1 and independent of t, i.e. $f(t,x) = f(x)$. Accordingly, let us denote by u_f^* the solution of the corresponding elliptic problem or, equivalently, the stationary solution of (2), that is

$$-\mathrm{div}(|\nabla u_f^*|^{p-2}\nabla u_f^*) = f \tag{7}$$

Focusing on such stationary solution, a natural question arises: how is u_f^* related to the solution of the original p-parabolic equation? It has been proved in [19] that, under very mild conditions (e.g. even irregular data), the solution $u(t,x)$ converges to $u_f^*(x)$ in L^1 as $t \to \infty$. We will prove that a stronger property holds under assumptions A1-A2, namely u_f^* is an asymptotically stable equilibrium in L^2.

In this regard, consider the modified Lyapunov function candidate

$$W(t) = \frac{1}{2}\int_\Omega |u(t,x) - u_f^*(t,x)|^2 dx. \tag{8}$$

The following statement constitutes the main stability result of the paper, and generalizes the assessment of Theorem 3.1.

Theorem 3.2: Assume $p \ge 2$ and let $f(t,x) = f(x)$ be such that A1 *is fulfilled. The stationary solution u_f^* is an L^2-asymptotically stable equilibrium with a region of attraction $\mathcal{R}_{u_f^*} \supseteq \mathcal{B}_0$, i.e. the function $W(t)$ satisfies a Lyapunov inequality of the type*

$$\dot{W}(t) \le -c[W(t)]^{\frac{p}{2}}, \quad c > 0 \tag{9}$$

for any initial condition $u_0(x) \in \mathcal{B}_0$.

Proof: Proceeding as in the proof of Theorem 3.1, and using the regularity of the time derivative $u_t - u_{f,t}^*$, with a slight abuse of notation one has

$$\begin{aligned}\dot{W}(t) &= \int_\Omega (u-u_f^*)(u_t - u_{f,t}^*)dx \\ &= \int_\Omega (u-u_f^*)(\Delta_p u - \Delta_p u_f^*)dx \\ &= \underbrace{\int_{\partial\Omega} \langle (u-u_f^*)(|\nabla u|^{p-2}\nabla u - |\nabla u_f^*|^{p-2}\nabla u_f^*), \mathbf{n}\rangle d\sigma}_{=0} \\ &\quad - \int_\Omega (\nabla u - \nabla u_f^*)\cdot(|\nabla u|^{p-2}\nabla u - |\nabla u_f^*|^{p-2}\nabla u_f^*)dx\end{aligned}$$

where the dependency on (t,x) has been omitted and $\mathbf{n}$ stands for the outer normal to $\partial\Omega$. Applying inequality (4) yields

$$\dot{W}(t) \le -2^{1-p}\int_\Omega |\nabla u - \nabla u_f^*|^p dx,$$

and hence, mimicking the steps of the proof of Theorem 3.1, it is straightforward to attain the desired estimate (9) with $c = (2|\Omega|)^{1-\frac{p}{2}}C_p$. ■

IV. Stability in L^{p^*}, lower-order terms and generalizations.

For the sake of simplicity we limit to consider the homogeneous case $f \equiv 0$ only: the extension to the general case is straightforward.

A. Lyapunov stability in L^{p^*}

A stronger stability result can be proved indeed under the same conditions. Let $2 \le p < n$ and consider the Lyapunov function candidate

$$V^*(t) := \frac{1}{p^*}\int_\Omega |u(t,x)|^{p^*} dx$$

where p^* is the Sobolev conjugate. Adapting the steps of the proof of Theorem 1, one gets the inequality

$$\dot{V}^*(t) \le -\int_\Omega |u|^{p^*-1}|\nabla u|^p dx$$

Since $u(t,x) \in L^\infty(Q_\infty)$ by construction, the function $v(t,x) = \frac{1}{\gamma}|u(t,x)|^\gamma \in W_0^{1,p}$ for any $\gamma \ge 1$, with

$$|\nabla v|^p = |u|^{p(\gamma-1)}|\nabla u|^p$$

Let us denote by γ^* the solution to the algebraic equation

$$p(\gamma^*-1) = p^*-1,$$

that is $\gamma^* = 1 + \frac{p^*}{p} - \frac{1}{p} \ge 1$. Applying the Gagliardo-Nirenberg-Sobolev inequality, one gets

$$\begin{aligned}\dot{V}^*(t) &\le -\int_\Omega |u|^{p^*-1}|\nabla u|^p dx \\ &\le -\left(\frac{S_p}{\gamma^*}\right)^p \left(\int_\Omega |u|^{\gamma^* p^*} dx\right)^{\frac{p}{p^*}}\end{aligned} \tag{10}$$

where S_p is the Sobolev constant. Finally, by Jensen's inequality, the integral in the latter term can be bounded as

$$-\left(\int_\Omega |u|^{\gamma^* p^*} dx\right)^{\frac{p}{p^*}} \leq -|\Omega|^{1-\gamma^*}\left(\int_\Omega |u|^{p^*} dx\right)^{\frac{\gamma^* p}{p^*}} \tag{11}$$

where $\frac{\gamma^* p}{p^*} = 1 + \frac{(p-1)}{p^*} > 1$. The reasoning above leads to the following enhanced stability result.

Theorem 4.1: Assume $p \geq 2$ and $f \equiv 0$. The stationary solution u^* is an L^{p^*}-asymptotically stable equilibrium with a region of attraction $\mathcal{R}_{u^*} \supseteq \mathcal{B}_0$, i.e. $V^*(t)$ satisfies a Lyapunov inequality of the type

$$\dot{V}^*(t) \leq -c[V^*(t)]^{\frac{\gamma^* p}{p^*}}, \quad c = p^{*\frac{\gamma^* p}{p^*}} \left(\frac{S_p}{\gamma^*}\right)^p |\Omega|^{1-\gamma^*} > 0$$

for any initial condition $u_0(x) \in \mathcal{B}_0$.

Proof: The proof follows immediately by merging conditions (10)-(11), and observing that the integral in the right-hand side of (11) equals $p^* V^*(t)$. ■

B. Lower-order terms: exponential stability

Let us consider now a variation of the original differential problem, namely

$$\begin{cases} u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + g(u) + f \\ u(t,x) = 0 \quad t \in [0,\infty), \ x \in \partial\Omega \\ u(0,x) = u_0(x) \quad x \in \Omega \end{cases} \tag{12}$$

Assumption 4.1: The lower-order term $g(\cdot)$ verifies suitable growth conditions:

$$-\overline{\lambda}|w|^2 - \overline{k}(w) \leq g(w)w \leq -\underline{\lambda}|w|^2 - \underline{k}(w) \quad \forall w \in \mathbb{R}$$

$$(g(w) - g(v))(w - v) \leq -\alpha|w - v|^2 - \beta(w, v) \ \forall w, v \in \mathbb{R}$$

where $\overline{\lambda} \geq \underline{\lambda} \geq 0$, $\alpha > 0$ and $\overline{k}(\cdot), \underline{k}(\cdot), \beta(\cdot,\cdot) \geq 0$ are smooth non-negative functions with $\overline{k}(0) = \underline{k}(0) = 0$.

Remark 4.1: We notice that a simple and natural example of function $g(\cdot)$ fulfilling the latter conditions is

$$g(w) = -\lambda w - \mu|w|^{p-2}w, \quad \mu \geq 0.$$

As mentioned, to simplify the presentation, it will be assumed $f \equiv 0$. We will prove that the presence of a lower-order term satisfying Assumption 4.1 with $\underline{\lambda} > 0$ guarantees an improvement of the stability condition, i.e. it provides exponential stability of the equilibrium. To this end, let us consider once again the Lyapunov function candidate $V(t)$ defined in (5).

Theorem 4.2: Assume $p \geq 2$, $f \equiv 0$ and suppose that Assumption 4.1 is satisfied with $\underline{\lambda} > 0$. The stationary solution u^* of the problem (12) is an L^2-exponentially stable equilibrium with a region of attraction $\mathcal{R}_{u^*} \supseteq \mathcal{B}_0$, i.e. $V(t)$ satisfies a Lyapunov inequality of the type

$$\dot{V}(t) \leq -2\underline{\lambda}V(t)$$

for any initial condition $u_0(x) \in \mathcal{B}_0$.

Proof: Repeating the first part of the proof of Theorem 3.1 one gets

$$\dot{V}(t) \leq -\int_\Omega |\nabla u|^p dx + \int_\Omega u g(u) dx,$$

and hence Assumption yields

$$\begin{aligned} \dot{V}(t) &\leq -\int_\Omega |\nabla u|^p dx - \int_\Omega (\underline{\lambda}|u|^2 + \underline{k}(u)) dx \\ &\leq -\underline{\lambda}\int_\Omega |u|^2 dx = -2\underline{\lambda}V(t), \end{aligned}$$

that is the desired condition. In particular, exponential stability is guaranteed thanks to

$$V(t) \leq e^{-2\underline{\lambda}t}V(0) = \frac{1}{2}e^{-2\underline{\lambda}t}||u_0||_2^2$$

■

C. Degenerate parabolic equations in general form

We present here a wider class of equations for which the stability theory developed so far is still valid. In this regard, let us emphasize that rather than the explicit form of the p-Laplace operator, its growth conditions have been used to attain the desired estimates.

Consider the following differential problem

$$\begin{cases} u_t = \operatorname{div}(a(x,t,\nabla u)) + g(x,t,u) + f \\ u(t,x) = 0 \quad t \in [0,\infty), \ x \in \partial\Omega \\ u(0,x) = u_0(x) \quad x \in \Omega \end{cases} \tag{13}$$

and, following the setting given in [10], let us make some preliminary assumptions.

- C1) The mapping $(\xi,\sigma) \mapsto a(\xi,\sigma,z)$ is measurable for all $z \in \mathbb{R}^n$, and the mapping $z \mapsto a(\xi,\sigma,z)$ is continuous for a.e. $(\xi,\sigma) \in \mathbb{R}^n \times \mathbb{R}$.
- C2) There exist constants $c_2 \geq c_1 > 0$ with
$$c_1|z|^p \leq a(\xi,\sigma,z)\cdot z \leq c_2|z|^p$$
for all $z \in \mathbb{R}^n$ and for a.e. $(\xi,\sigma) \in \mathbb{R}^n \times \mathbb{R}$.
- C3) For a.e. $(\xi,\sigma) \in \mathbb{R}^n \times \mathbb{R}$
$$(a(\xi,\sigma,z) - a(\xi,\sigma,\zeta))\cdot(z - \zeta) > 0$$
for all $z, \zeta \in \mathbb{R}^n$ with $z \neq \zeta$.
- C3′) For a.e. $(\xi,\sigma) \in \mathbb{R}^n \times \mathbb{R}$
$$(a(\xi,\sigma,z) - a(\xi,\sigma,\zeta))\cdot(z - \zeta) \geq \theta|z - \zeta|^p, \ \theta > 0$$
for all $z, \zeta \in \mathbb{R}^n$.
- C4) The lower-order term $g(\xi,\sigma,w)$ fulfills Assumption 4.1 uniformly in (ξ,σ).

The following general result can be then established with the same technique that has been used to prove Theorem 3.1 and Theorem 3.2.

Proposition 4.1: Consider the degenerate parabolic problem (13) and assume that conditions C1, C2 and C4 hold true with $p \geq 2$. The achievable[2] stability results are summarized in Table I.

[2]We notice that, when $\underline{\lambda} = 0$, the condition C3 is sufficient to guarantee asymptotic stability of the equilibrium only in the homogeneous case $f \equiv 0$

TABLE I

STABILITY RESULTS FOR THE DEGENERATE PARABOLIC EQUATION

Type of stability	C3+$\{\underline{\lambda} > 0\}$	C3+$\{\underline{\lambda} = 0\}$	C3$'$
L^2-asymptotic	$\surd$	–	$\surd$
L^{p^*}-asymptotic	$\surd$	–	$\surd$
L^2-exponential	$\surd$	–	–

V. NUMERICAL SIMULATIONS

Let us illustrate the proposed stability results by means of a series of numerical examples corresponding to different scenarios.

- Example 1: parabolic p-Laplace equation with zero data, i.e. $f \equiv 0$.
- Example 2: parabolic p-Laplace equation with $f \equiv 0$ and a lower-order term

$$g(u) = -2u - 3|u|^{p-2}u.$$

In each simulated scenario the space dimension is $n = 2$ and the exponent $p = 3$. The low dimension $n = 2$ has been chosen only for allowing to describe graphically the behavior of solutions. For representation purposes, the domain Ω has been assumed to be the square $[-4, 4]^2$, even though the boundary is not smooth at the vertices, this slightly weakening the theoretical assumptions used to establish the formal results. The initial condition has been set as

$$u_0(x) = \sin\left(\frac{\pi}{4}x\right)(e^{-64} - e^{-y^2}).$$

The behavior of the solution of Example 1 is depicted of Figures 1-2, corresponding to evaluation at $t = 10$ and $t = 1000$ respectively. The asymptotic convergence to the stationary null solution, though quiet slow, can be easily deduced from the plots.

On the other hand, as proved in Section IV-B, the presence of a lower-order term provides additional regularity and guarantees exponential stability. This fact is shown in Figures 3-4, where the solution of Example 2 evaluated at $t = 5$ and $t = 100$ is reported: as expected, the solution is characterized by a remarkably higher convergence rate.

As a further comparison, the time histories of the L^2 norm of the solution of the p-parabolic equation with and without lower-order term are depicted in Figure 5.

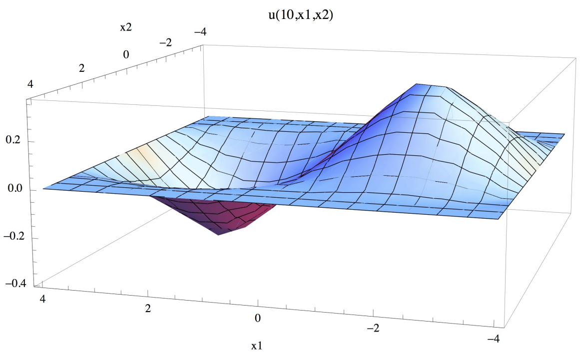

Fig. 1. Example 1: solution $u(t, x_1, x_2)$ evaluated at $t = 10$

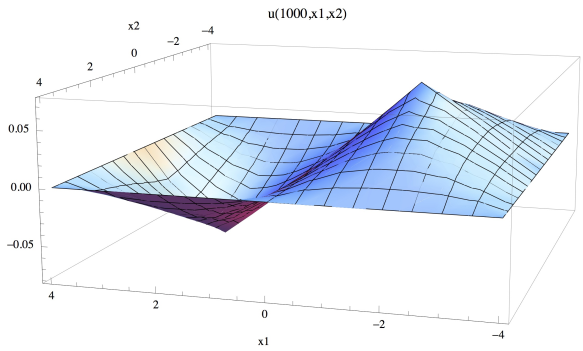

Fig. 2. Example 1: solution $u(t, x_1, x_2)$ evaluated at $t = 1000$

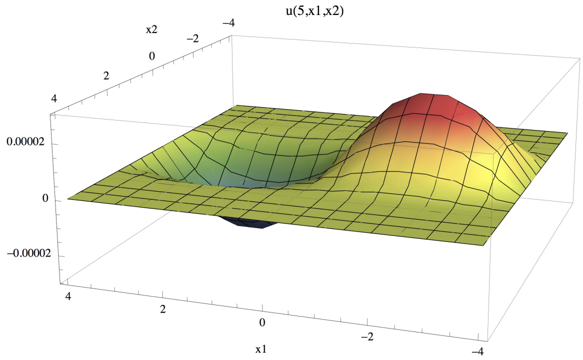

Fig. 3. Example 2: solution $u(t, x_1, x_2)$ evaluated at $t = 5$

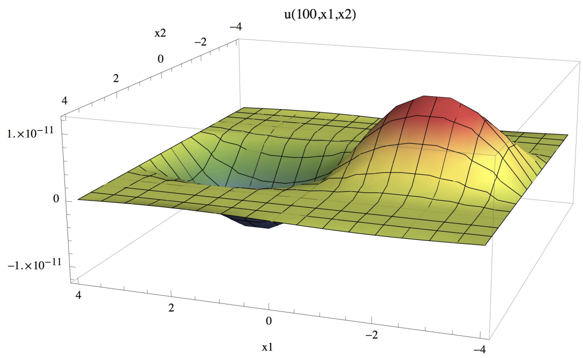

Fig. 4. Example 2: solution $u(t, x_1, x_2)$ evaluated at $t = 100$

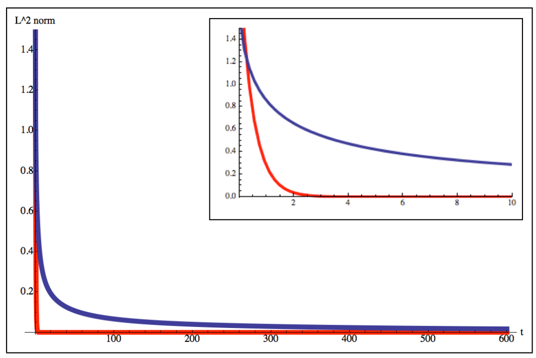

Fig. 5. Time histories of the L^2 norm of solutions in Example 1 (blue) and Example 2 (red).

ACKNOWLEDGMENT

The authors are grateful to Francesco Petitta for the fruitful discussion and his kind suggestions.

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APPENDIX A – SOME USEFUL INEQUALITIES

For the readers' convenience a collection of classical inequalities that have been used in the paper are reported.

Poincaré's inequality. Let $v(x) \in W_0^{1,p}(\Omega)$, $p \geq 1$. There exists a positive constant $C_p > 0$ such that

$$\|v\|_p \leq C_p \|\nabla v\|_p.$$

Jensen's inequality. Let $h(x) \in L^1(\Omega)$ be non-negative with $|\Omega| < \infty$, and consider a continuous, convex and non-negative function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. Then the following estimate holds:

$$\varphi\left(\frac{1}{|\Omega|}\int_\Omega h(x)dx\right) \leq \frac{1}{|\Omega|}\int_\Omega \varphi(h(x))dx.$$

Gagliardo-Nirenberg-Sobolev inequality. Let $v(x) \in W_0^{1,p}(\Omega)$ and $1 \leq p < n$. There exists a positive constant $S_p > 0$ such that

$$\|v\|_{p^*} \leq S_p \|\nabla v\|_p,$$

where $p^* > p$, referred to as the Sobolev conjugate of p, is given by the relationship $1/p^* = 1/p - 1/n$.

For a detailed discussion about these inequalities and their connections with more general properties of Sobolev spaces, one may refer to the classical textbooks [23], [26].

APPENDIX B – PROOF OF CLAIM 1

Let us demonstrate the claim. As proved in [18, Corollary 2.5], the solution $u(t,x)$ verifies

$$u_t(t,x) \in W^{\alpha,2}_{loc}(Q_T) \quad \forall \alpha \in \left(0, \frac{1}{4}\right), \quad \forall T > 0.$$

By the theory of fractional capacities of open sets, the function $u_t(t,x)$ admits a $cap_{\alpha,2}$-quasi continuous representative [27], i.e. $u_t(t,x)$ can be identified with a continuous function for any $(t,x) \in K \setminus \mathcal{E}$ with $cap_{\alpha,2}(\mathcal{E}) = 0$, where K is an arbitrary compact subset $K \subset Q_T$. We recall that the capacity $cap_{s,p}(\mathcal{A})$ of the open set $\mathcal{A}$ is defined as

$$cap_{s,p}(\mathcal{A}) = \inf\left\{ \|\omega\|^p_{W^{s,p}(\mathbb{R}^n)} : \begin{array}{l} \omega \in W^{s,p}(\mathbb{R}^n), \\ \omega \geq 1 \text{ a.e. in } \mathcal{A} \end{array} \right\},$$

and that the following implication holds

$$cap_{s,p}(\mathcal{E}) = 0 \implies |\mathcal{E}| = 0.$$

As a consequence, the function $u_t(t,x)$ admits an a.e.-quasi continuous representative.

Let us select now an arbitrary point of continuity (t_0,x_0) for $u_t(t,x)$; by definition, the function $u_t(t,x)$ is then bounded in a neighbourhood $\mathcal{N}_{(t_0,x_0)}$ of (t_0,x_0), and hence by Rademacher's theorem [28], $u(t,x)$ is differentiable a.e. in $\mathcal{N}_{(t_0,x_0)}$. Since (t_0,x_0) is arbitrary, such a property holds a.e. in Q_T, and therefore the claim has been proved.