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Constrained transient optimization using B-splines functions as control inputs

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***Abstract*— The purpose of this paper is to propose a new method for the optimal output transition problem under saturation constraints on the control effort. The recently proposed preview based pseudo inversion approach is here adopted to determine a feedforward action optimizing the set-point following response of a given stable closed-loop system $\Sigma_{f,c}$. The optimal external input reference $r(t)$ forcing $\Sigma_{f,c}$ is assumed to be given by a B-spline function, a second B-spline $\hat{u}(t)$ is used to optimally approximate the actual control effort $u(t)$ produced by $r(t)$. The control points which define the convex hull containing $\hat{u}(t)$ are chosen in such a way to respect the saturation constraints on $u(t)$. In the first part of the transient interval the actual control input $u(t)$ is replaced by $\hat{u}(t)$, after this initial period has elapsed the control law is definitely switched to $u(t)$. Application to a benchmark problem shows the effectiveness of the method.**

***Index Terms*— Model pseudo-inversion, feed-forward control, Constrained control, B-splines.**

I. INTRODUCTION

Achieving an adequate set point following performance under saturation constraints is an important, long standing problem which has been investigating for many years from different points of view. A fundamental approach is based on anti-windup compensation: after a standard linear controller has been designed using classical methods, an anti-windup compensator is introduced to take into account the presence of saturation constraints. See e.g. the survey ([1]) and references therein. An optimum criterion is introduced by the Model Predictive Control (MPC) (see e.g. [2], [3] and references therein): at each time instant a constrained optimal control problem is solved, minimizing a quadratic cost functional over a receding finite horizon. The main inconvenience of MPC is the very large real time computational effort. The problem of moving off line the on line computational burden has been considered in [4]. An alternative to MPC is the Reference Governor (RG) (see e.g. [5]-[8] and references therein), where closed-loop stabilization is obtained through a conventional controller, and constraints are respected through a feed-forward filter varying on line the desired set point. Both MPC and RG approaches have been developed with reference to the discrete-time case, extension of RG to continuous time systems has been recently considered in [9].

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Possible infeasibility arising in MPC and RG can be dealt with by constraints softening methods [10],[11]. These methods are based on additional slack decision variables and penalty functions. A related problem is the difficulty of deriving a lower bound on the constraint violation weight. More recently, [12], [13], proposed the introduction of non smooth penalties into the cost functional, avoiding the use of slack variables altogether. To facilitate the application of a gradient-based optimizer a smooth approximation of the penalty function is then provided through the Kreisselmeier-Steinhauser [14] function.

This paper situates in the recently proposed preview based pseudo-inversion approach [15]-[19]: given a desired trajectory $y_d(t)$ for the output of an asymptotically stable closed loop system, find a corresponding input $r(t)$ yielding an actual output $y(t)$ which is the best approximation of $y_d(t)$. A B-splines based approximation of an optimal control law is here proposed as an alternative method for the hardly constrained optimal output transition problem. Optimality is here intended as the achievement of an actual output which is the best approximation (in the least square sense) of a pre-specified desired output transient trajectory between two given set-points.

The advantage of using B-splines derives from being universal approximators belonging to the convex hull defined by the relative control points, and from their natural continuity at control points [20], [21].

The actual external reference $r(t)$ driving the asymptotically stable closed-loop system whose output has to be controlled is here assumed to be given by a B-spline, a second B-spline $\hat{u}(t)$ is used to optimally approximate the actual control effort $u(t)$ produced by $r(t)$. The control points of $\hat{u}(t)$ are chosen in such a way to respect the saturation constraints on $u(t)$. The idea is to replace $u(t)$ with $\hat{u}(t)$ over the first part of the transient interval following any set point reset. This is the critical period T_c where $u(t)$ may exceed the saturation constraints. Once T_c has elapsed, $\hat{u}(t)$ is discharged and replaced by $u(t)$. Continuity and approximation properties of B-splines allow this switching control law to not produce any appreciable bump on the plant.

In view of the above, it can be said that with respect the previously cited methods, the present approach yields advantages in terms of simplicity and ease of implementation as well as the noteworthy possibility of application both to discrete and continuous time control problems with only minor modifications. Moreover, the method developed here is computationally efficient because it is based on the parsimonious representation of B-splines. This facilitates its implementation over a possibly reduced preview-interval on the new desired set-point.

The paper is organized in the following way. Some mathematical preliminaries are briefly recalled in Section II, the control problem is formulated in Section III and its solution is reported in Section IV, a numerical simulation and concluding remarks are given in Sections V and VI respectively.

II. PRELIMINARIES

A. B-spline functions [20]

B-splines are piecewise polynomial functions derived from a slight adjustment of Bezier's curves in order to obtain polynomial curves that automatically tie together smoothly.

Let $(c_i)_{i=1}^{\ell}$ be a set of ℓ control points for a spline curve $s(v)$ of degree d, with non decreasing knots $(\hat{v}_i)_{i=1}^{\ell+d+1}$

$$s(v) = \sum_{i=1}^{\ell} c_i B_{i,d}(v) \qquad v \in \mathcal{V} \triangleq [\hat{v}_1, \hat{v}_{\ell+d+1}] \qquad (1)$$

where $B_{i,d}(v)$, $d > 1$, is given by the Cox-de Boor recursion formula

$$B_{i,d}(v) = \frac{v - \hat{v}_i}{\hat{v}_{i+d} - \hat{v}_i} B_{i,d-1}(v) + \frac{\hat{v}_{i+1+d} - v}{\hat{v}_{i+1+d} - \hat{v}_{i+1}} B_{i+1,d-1}(v), \qquad (2)$$

and

$$B_{i,0}(v) = \begin{cases} 1, & \hat{v}_i \leq v < \hat{v}_{i+1} \\ 0, & otherwise. \end{cases} \qquad (3)$$

Property 1 (Convex Hull Property). Any value assumed by $s(v)$, $\forall v \in \mathcal{V}$, lies in the convex hull of its $\ell + 1$ control points $(c_i)_{i=1}^{\ell}$.

Property 2. Suppose that the number $\hat{v}_{i+1}$ occurs m times among the knots $(\hat{v}_j)_{j=i-d}^{m+d}$ with m some integer bounded $1 \leq m \leq d+1$, e.g $\hat{v}_i < \hat{v}_{i+1} = \cdots = \hat{v}_{i+m} < \hat{v}_{i+m+1}$, then the spline function $s(v)$ has continuous derivative up to order $d - m$ at knot $\hat{v}_{i+1}$.

This property implies that the spline smoothness can be changed using multiple knot points. It is common choice to set $m = d+1$ multiple knot points for the initial and the last knot points. In this way (1) assumes the first and the final control points as initial and final values.

B. Constrained Least Squares

The continuous-time least square problem has the general form

$$\min_f J(e) = \min_f \|Q^{1/2} e\|_2^2 = \min_f \int_0^T e(t)^T Q(t) e(t) dt, \qquad (4)$$

$$e(t) \triangleq b(t) - D(t) f \qquad (5)$$

where $e(t)$ is the residual, $b(t)$ is the observation vector, $D(t)$ is the design matrix, $Q(t)$ is a positive definite weight matrix for each fixed $t \in [0, T]$ and f is the vector of model parameters. The norm functional $J(e)$ defines the squared weighted L_2 norm of $e(t)$, $t \in [0, T]$. The unique unconstrained solution is known to be given by

$$\hat{f} = \left(\int_0^T D(t)^T Q D(t) dt \right)^{-1} \int_0^T D(t)^T Q b(t) dt, \qquad (6)$$

provided the inverse exists,[22].

Box constraints on the parameters vector to be estimated are represented by the following inequalities

$$\underline{f} \leq f \leq \bar{f} \ , \qquad (7)$$

where $\underline{f}$ ($\bar{f}$) is the lower (upper) bound of the parameter vector f.

The constrained least square problem given by (4),(5) and (7) does not admit a closed form solution, nevertheless a numerical solution can be obtained reformulating such problem in discrete form by considering the discretized residual vector $e(t_k)$, $t_k = k t'_s$, $k = 1, 2, \cdots, N$, where t'_s is the discretization step. Defining the following vectors

$$\bar{e} \triangleq \begin{bmatrix} \bar{e}_1^T & \bar{e}_2^T & \cdots & \bar{e}_N^T \end{bmatrix}^T \qquad (8)$$

$$\bar{b} \triangleq \begin{bmatrix} \bar{b}_1^T & \bar{b}_2^T & \cdots & \bar{b}_N^T \end{bmatrix}^T \qquad (9)$$

$$\bar{D} \triangleq \begin{bmatrix} \bar{D}_1^T & \bar{D}_2^T & \cdots & \bar{D}_N^T \end{bmatrix}^T \qquad (10)$$

where $\bar{e}_k \triangleq e(t_k) = b(t_k) - D(t_k) f$, $\bar{b}_k \triangleq b(t_k)$ and $\bar{D}_k \triangleq D(t_k)$ for $k = 1, \cdots, N$, the following discrete version of the original problem (4),(5) and (7) is obtained

$$\min_f \bar{J}(\bar{e}) = \min_f \|\bar{Q}^{1/2} \bar{e}\|_2^2, \qquad (11)$$

$$\bar{e} \triangleq \bar{b} - \bar{D} f \qquad (12)$$

$$\underline{f} \leq f \leq \bar{f} \qquad (13)$$

where $\bar{Q}$ is the diagonal matrix obtained with the samples $Q(t_k)$ for $k = 1, \cdots, N$.

Problem (11)-(13) can by solved by the numerical algorithm given in [23] and implemented by the MATLAB® function *lsqlin*.

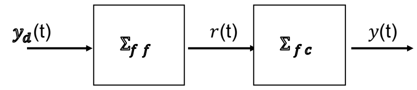

Fig. 1. The two degrees of freedom control system Σ

III. PROBLEM STATEMENT

Let Σ denote the two degrees of freedom control system given by the series connection of a feedforward filter Σ_{ff} with a square closed loop system $(\Sigma_{f,c}, x)$ consisting of the feedback connection of a possibly non-minimum phase and/or non hyperbolic LTI plant (Σ_p, x_p) with a dynamic LTI stabilizing controller (Σ_c, x_c) also providing the internal model of constant signals for robustness issues. The state space representation of $\Sigma_{f,c}$ is

$$\dot{x}(t) = Ax(t) + Br(t), \qquad (14)$$

$$y(t) = Cx(t) \qquad (15)$$

where: $x(t) = [x_c(t)^T, x_p(t)^T]^T \in \mathbb{R}^n$, $r(t) \in \mathbb{R}^q$, $y(t) \in \mathbb{R}^q$. The output $u(t) \in \mathbb{R}^p$ ($p \geq q$) of Σ_c is given by $u(t) = F_c x_c(t) + D_c r(t)$, where D_c is the (possibly null) direct transition matrix of Σ_c.

It is assumed that: A1) $\Sigma_{f,c}$ is asymptotically internally stable, A2) $\Sigma_{f,c}$ has no transmission zero at $s = 0$. The transfer matrices of $\Sigma_{f,c}$ and of the system $\Sigma_{f,u}$ having $r(t)$ and $u(t)$ as input and output respectively, are denoted by $W_{r,y}(s)$ and $W_{r,u}(s)$ respectively.

A preliminary informal definition of the considered problem states that the output $y(t)$ is required to optimally approximate a pre-specified desired trajectory $y_d(t)$ between two given set-points, the external reference $r(t)$ is required to be uniformly bounded and the control input $u_p(t)$ forcing Σ_p is required to satisfy the following saturation constraints

$$\underline{u} \leq u_p(t) \leq \bar{u}, \; t \geq 0. \tag{16}$$

Remark 1 It should be noted that, for technical considerations, which will be explained later, the output $u(t)$ of Σ_c is not the same signal $u_p(t)$ really forcing Σ_p. $\triangle$

As shown in Fig. 1, the purpose of Σ_{ff} is to pre-filter the given $y_d(t)$ to yield the input signal $r(t)$ allowing the fulfillment of the above requirements.

To formally state the above problem, the following notations and preliminary considerations are given.

The internal stability assumption A1 allows the partition of $y_d(t)$ and $r(t)$ in transient and steady state components:

$$y_d(t) = \begin{cases} y_{d,t}(t), \; t \in [\,0, t_s\,] \stackrel{\triangle}{=} T_s \\ \tilde{y}_d, \; t \in (\,t_s, \infty\,) \end{cases} \tag{17}$$

with $y_{d,t}(t_s) = \tilde{y}_d$ and

$$r(t) = \begin{cases} r_t(t), \; t \in [\,0, t_t\,] \stackrel{\triangle}{=} T_t, \;\; t_t > t_s \\ \tilde{r}, \;\; t \in (\,t_t, \infty\,) \end{cases} \tag{18}$$

where T_s is the time interval over which the transient desired output $y_{d,t}(t)$ is required to converge towards the steady state value $\tilde{y}_d$.

By A2) and recalling that $\Sigma_{f,c}$ contains an internal model of constant signals one has $\tilde{r} = W_{r,y}(0)^{-1}\tilde{y}_d = \tilde{y}_d$.

Also the actual control effort $u(t)$ yielded by Σ_c and the actual output $y(t)$ of $\Sigma_{f,c}$ are partitioned in

$$u(t) = \begin{cases} u_t(t), \; t \in T_t \\ \tilde{u}, \; t \in (\,t_t, \infty\,) \end{cases} \tag{19}$$

$$y(t) = \begin{cases} y_t(t), \; t \in T_t \\ \tilde{y}, \; t \in (\,t_t, \infty\,) \end{cases} \tag{20}$$

Choosing t_t sufficiently larger than t_s, the signals $r_t(t)$, $u_t(t)$ and $y_t(t)$ at time t_t will be practically coinciding with the respective steady state values $\tilde{r}$, $\tilde{u}$, $\tilde{y}$. In practice, it is enough to choose t_t as $t_t >> 4\tau_m$, where τ_m is the maximum time constant of $\Sigma_{f,c}$.

Recalling that $\tilde{r} = \tilde{y}_d$, the steady-state value of $u(t)$ is given by

$$\tilde{u} = W_{r,u}(0)\tilde{y}_d. \tag{21}$$

The set-point value $\tilde{y}_d$ must be chosen according to (21) in such a way that the corresponding steady state value $\tilde{u}$ satisfy the saturation constraints (16).

The above considerations show that the main question about the investigated optimization problem concerns the transient state. To this purpose the transient reference $r_t(t)$ is assumed to be given by a B-spline function and the actual $u_t(t)$ output of Σ_c is optimally approximated by a B-spline $\hat{u}_t(t)$, which is required to satisfy

$$\underline{u} \leq \hat{u}_t(t) \leq \bar{u}, \; t \in T_t. \tag{22}$$

The B-spline $\hat{u}_t(t)$, is used instead of $u_t(t)$ as the forcing input of Σ_p, for $t \in [0, t_c] \stackrel{\triangle}{=} T_c$, for some sufficiently large $t_c < t_t$. Hence the control input $u_p(t)$, really forcing Σ_p, $\forall t \geq 0$, is composed of previously defined signals:

$$u_p(t) = \begin{cases} \hat{u}_t(t), \; 0 \leq t \leq t_c \\ u_t(t), \; t_c < t \leq t_t \\ \tilde{u}, \qquad t > t_t. \end{cases} \tag{23}$$

For $t > t_c$, $\hat{u}_t(t)$ is discharged and replaced by the actual control effort $u(t)$ produced by Σ_c. Continuity and approximation properties of B-splines allow this switching control law to not produce any appreciable bump on the plant for sufficiently large t_c.

The interval T_c is defined as the critical part of T_t, namely the compact interval $T_c \subseteq T_t$ where $u_t(t)$ can violate the saturation constraints, while no violation occurs for $t \geq t_c$. This definition is based on the practical consideration that the control effort reaches its maximum in the first part of the transient interval due to: 1) the initial state of $\Sigma_{f,c}$ is too far from the new steady-state value to be reached, 2) a desired transient trajectory $y_{d,t}(t)$ characterized by a very short t_s has been chosen.

From a practical point of view, T_c is determined choosing t_c as the minimum time instant t' such that

$$|\hat{u}_{t,i}(t)| < M, \quad t' \leq t \leq t_t, \; i = 1, \cdots, p, \tag{24}$$

with $M << \min\{|\underline{u}_i|, \bar{u}_i\}$, $i = 1, ..., p$.

In other words, the approximating signal $\hat{u}_t(t)$ will force Σ_p as long as the modulus of the signal itself will not enter definitely into the hysteresis band bounded by M. As the B-spline $\hat{u}_t(t)$ satisfying (22) is an optimal approximation of $u_t(t)$, one has that also $u_p(t)$, given by (23), exactly satisfies (16).

Remark 2 It should be noted that replacing the stabilizing control law $u(t)$ produced by Σ_c with the saturated $u_p(t)$ given by (23) does not compromise the internal stability of $\Sigma_{f,c}$. In fact the control signals $u(t)$ and $u_p(t)$ only differs over the bounded time interval $T_c \subseteq T_t$, and for any internal state $x(t_c)$ of $\Sigma_{f,c}$ reached through $\hat{u}_t(t)$, $t \in T_c$, the stabilizing $u(t)$ drives $x(t)$ to a stable equilibrium point for $t \to \infty$. $\triangle$

Defining $e(t)$ over T_t as $e(t) \stackrel{\triangle}{=} [e_1^T(t), e_2^T(t)]^T \stackrel{\triangle}{=} [y_{d,t}^T(t) - y_t^T(t), \hat{u}_t^T(t) - u_t^T(t)]^T$, the forgoing considerations lead to the following formal problem definition:

Formal problem definition *It is required to find the parameter vector composed of all the control points defining the* *two B-spline functions $r_t(t)$ and $\hat{u}_t(t)$, $t \in T_t$ with $r_t(t)$ and $\hat{u}_t(t)$ smoothly converging to $\tilde{r} = \tilde{y}_d$, and $\tilde{u} = W_{r,u}(0)\tilde{y}_d$, respectively, such that the following functional is minimized:*

$$J(e) = \int_0^{t_t} e^T(t)Q(t)e(t)dt =$$
$$= \int_0^{t_t} (e_1^T(t)Q_1(t)e_1(t) + e_2^T(t)Q_2(t)e_2(t))dt \qquad (25)$$

subject to (22), and where $Q(t) = diag\{Q_1(t), Q_2(t)\}$ is a suitably chosen positive definite weight matrix.

Remark 3 The following should be noted: 1) In practice t_c results to be a bit larger than t_s, nevertheless the exact value of t_c is not "a priori" exactly known, the only information being $t_c < t_t$. In the light of (24), this implies the necessity of computing $\hat{u}_t(t)$ for $t \in T_t$ and explains why the minimization of the weighted L_2 norm of $e_2(t)$ is performed over the whole T_t, and is not limited to T_c. 2) Forcing Σ_p with the saturated input $u_p(t)$ (and not with the actual input $u_t(t)$) prevents the functional (25) from attaining its absolute minimum, which corresponds to the particular case of an exact overlapping of $u_t(t)$ with $\hat{u}_t(t)$ over T_c.

IV. Problem solution

Thanks to the B-spline convex hull property, it is possible to satisfy constraints (22) on $\hat{u}_t(t)$ through a suitable definition of control points. To this purpose, next section explains how the B-spline function can be expressed in a parametric form.

A. B-spline representation of signals

Identifying the parameter v of (1) with the time variable $t \in [0, t_t]$, the corresponding B-spline curve $s(t)$ can be used to represent a scalar time function defined over the transient interval T_t.
Let ℓ and d be the number of control points and degree respectively of a generic scalar B-spline $s(t)$, $t \in [0, t_t]$, $t_t < \infty$. Any control point c_i, $i = 1, \cdots, \ell$, of $s(t)$ belongs to $\mathbb{R}^2$ and is completely defined by its (c_x, c_y) coordinates

$$c_i \triangleq [c_{x_i}, c_{y_i}]^T \qquad (26)$$

The coordinate c_{x_i} indicates the time instant $t_i \in [0, t_t]$ at which the c_y-coordinate of the control point c_i assumes the corresponding value c_{y_i}. To preserve the causality of the signal, each t_i is a priori assigned partitioning the interval $[0, t_t]$ in $\ell - 1$ equi-spaced intervals.

Choosing a knot vector $(\hat{t}_i)_{i=1}^{\ell+d+1}$ with $\hat{t}_1 = \cdots = \hat{t}_{d+1} = 0$ and $\hat{t}_{\ell+1} = \cdots = \hat{t}_{\ell+d+1} = t_t$, Property 2 of B-splines implies that $s(t)$ starts from the c_y-coordinate of the initial control point c_1 at time $t = 0$ and ends at the c_y-coordinate of final control point c_ℓ at time instant $t = t_t$. The remaining knot points $\hat{t}_j$, $j = d+2, \cdots, \ell$ are chosen uniformly equispaced within $[0, t_t]$. Defining the vectors

$$\mathbf{c}_s \triangleq \begin{bmatrix} c_{y_1} & c_{y_2} & \cdots & c_{y_\ell} \end{bmatrix}^T \qquad (27)$$

and

$$\mathbf{B}_d(t) \triangleq \begin{bmatrix} B_{1,d}(t) & B_{2,d}(t) & \cdots & B_{\ell,d}(t) \end{bmatrix} \qquad (28)$$

it is possible to rewrite (1) as

$$s(t) = \sum_{i=1}^{\ell} c_{y_i} B_{i,d}(t) = \mathbf{B}_d(t)\mathbf{c}_s. \qquad (29)$$

Equation (29) is now used to define a compact B-spline representation of the $q \times 1$ external reference vector $r_t(t)$ and of the $p \times 1$ control effort vector $\hat{u}_t(t)$, $t \in T_t$. To this purpose the following vectors and matrix are defined:

$$\mathbf{c}_{r_t} \triangleq \left[\mathbf{c}^1{}_{r_t}^T, \cdots, \mathbf{c}^q{}_{r_t}^T \right]^T, \qquad \bar{\mathbf{B}}_{rd}(t) \triangleq \text{diag}\left[\mathbf{B}_d(t)\right]. \qquad (30)$$

Each $\mathbf{c}_{r_t}^i \triangleq \left[c_{r_t}^{i,1}, \cdots, c_{r_t}^{i,\ell}\right]^T$, $i = 1, \cdots, q$, is defined as in (27). The dimensions of $\mathbf{c}_{r_t}$ are $(q\ell \times 1)$. The dimensions of the block diagonal matrix $\bar{\mathbf{B}}_{rd}(t)$ are $(q \times q\ell)$.

The above notation and (29) allow $r_t(t)$ to be rewritten as

$$r_t(t) = \bar{\mathbf{B}}_{rd}(t)\mathbf{c}_{r_t}, \quad t \in T_t. \qquad (31)$$

The same procedure can be applied to the B-spline $\hat{u}_t(t)$ approximating the actual control effort $u_t(t)$, one has

$$\hat{u}_t(t) = \bar{\mathbf{B}}_{ud}(t)\mathbf{c}_{u_t}, \quad t \in T_t, \qquad (32)$$

where $\mathbf{c}_{u_t}$ is defined as $\mathbf{c}_{r_t}$, but has dimensions $(p\ell \times 1)$, and $\bar{\mathbf{B}}_{ud}(t)$ is defined analogously to $\bar{\mathbf{B}}_{rd}(t)$, but has dimensions $(p \times p\ell)$.

B. Constrained Least Square Spline Identification

Equations (31) and (32) indicate that the B-spline representation of $r_t(t)$ and $\hat{u}_t(t)$ requires the identification of vectors $\mathbf{c}_{r_t}$ and $\mathbf{c}_{u_t}$. These parameter vectors are computed using a constrained least squares procedure as indicated in Section II-B. Some passages, not reported for brevity, show that, in the present case, the residual equation (5) related to functional (25) is defined by the following vectors and matrices

$$e(t) = [e_1(t)^T, e_2(t)^T]^T \qquad (33)$$
$$f \triangleq \begin{bmatrix} \mathbf{c}_{r_t}^T & \mathbf{c}_{u_t}^T \end{bmatrix}^T \qquad (34)$$
$$D(t) \triangleq \begin{bmatrix} G_a(t) & 0 \\ G_b(t) + D_c\bar{\mathbf{B}}_{rd}(t) & -\bar{\mathbf{B}}_{ud}(t) \end{bmatrix} \qquad (35)$$
$$b(t) \triangleq [(y_{d,t}(t) - Ce^{At}x(0))^T \ \ (-F_c e^{At}x(0))^T]^T \qquad (36)$$

with

$$\mathbf{G}_a(t) \triangleq \int_0^t Ce^{A(t-\tau)}B\bar{\mathbf{B}}_d(\tau)d\tau$$

and

$$\mathbf{G}_b(t) \triangleq \int_0^t F_c e^{A(t-\tau)}B\bar{\mathbf{B}}_d(\tau)d\tau.$$

In the present case, the vector component $\mathbf{c}_{u_t}$ must be such that, the corresponding B-spline $\hat{u}_t(t)$ satisfy (22), $\forall t \in T_t$. By Property 1 of B-spline functions, this will be satisfied imposing the following constraints to the control points of $\hat{u}_t(t)$

$$\underline{u}_i \le c_{u_t}^{i,j} \le \bar{u}_i, \quad i = 1, ..., p \quad j = 1, ..., \ell. \qquad (37)$$

The solution $\hat{f} \triangleq \begin{bmatrix} \hat{\mathbf{c}}_{r_t}^T & \hat{\mathbf{c}}_{u_t}^T \end{bmatrix}^T$ of the constrained least square problem (33)-(37) can be numerically computed as explained in the second part of Section II-B: discretization of the continuous constrained least square problem according to (8)-(13) and successive application of the algorithm [23] implemented by the MATLAB® function *lsqlin*. It is easy to see that definition (35) implies that $D(t)$ is full column rank, hence the constrained least square problem admits a unique solution that can be computed in polynomial time. This solution is used to compute $r_t(t)$ and $\hat{u}_t(t)$, $t \in T_t$ according to (31) and (32) respectively, putting $\hat{\mathbf{c}}_{r_t}^T = \mathbf{c}_{r_t}^T$ and $\hat{\mathbf{c}}_{u_t}^T = \mathbf{c}_{u_t}^T$. After computing $\hat{u}_t(t)$, the time instant t_c can be identified.

Remark 4 The presented approach can be easily extended to the case of a switching desired output set-point. At some transition instants t_h, $h \in \mathbb{Z}^+$, the actual set point $\tilde{y}_{d,h}$ is reset to a new value $\tilde{y}_{d,h+1}$, according to the particular application.

To obtain a constrained optimal output transition over each interval $T_h \triangleq [t_h, t_{h+1})$ $(t_0 = 0)$, the proposed method can be still applied provided that $t_{h+1} - t_h > t_{t,h}$, $\forall h \in \mathbb{Z}^+$, where $t_{t,h}$ has the same meaning of t_t. Namely each transient interval $T_{t,h} \triangleq [t_h, t_h + t_{t,h})$ should be long enough to guarantee that the actual output $y(t)$, starting from the previous set point $\tilde{y}_{d,h}$, has practically reached the new set point $\tilde{y}_{d,h+1}$. The next transition instant and the new set-point may be known in advance with a very short preview time. In practice, the preview interval is only determined by the computation time for estimating the spline coefficients, namely the control points, so that the method can be practically implemented on line. △

V. NUMERICAL RESULTS

The method has been tested on the benchmark example proposed in [25] where a two-mass-spring system described by the triplet $\Sigma_p = (C_p, A_p, B_p)$

$$A_p = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & 0 & 0 \\ k/m_2 & -k/m_2 & 0 & 0 \end{bmatrix}, \tag{38}$$

$$B_p = \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix}, \quad C_p = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix},$$

with $m_1 = m_2 = 1$ and $k = 1$ is considered. It is noted that the dynamical matrix A_p has two null eigenvalues.

The performed numerical results refer to the Problem 4 in [25] and only the nominal case is here dealt with. The aim is to design a feedback/feedforward controller for a unit-step output command tracking problem. The following saturation constraint on the control input is assumed: $|u_p(t)| \le 1$, namely $|\underline{u}_1| = \bar{u}_1 = 1$.

According to the control scheme described in Section III, for robustness reasons the stabilizing linear observer based controller (Σ_c, x_c) also provides an internal model Σ_m of constant signals. This also holds if, as in this case, the internal model is already provided by the plant. The state space representation (A_m, B_m, x_m) of Σ_m is characterized by $A_m = 0_{1\times 1}$ (null matrix) and $B_m = I_{1\times 1}$ (identity matrix). The state of the observer is denoted by $\xi(t)$.

According to [26], the state space representation of the closed loop control system $\Sigma_{f,c}$ is given by (C, A, B, x) where:

$$A = \begin{bmatrix} A_p & B_p K_m & -B_p K_p \\ -B_m C_p & A_m & 0 \\ L C_p & B_p K_m & A_p - L C_p - B_p K_p \end{bmatrix} \tag{39}$$

$$C = \begin{bmatrix} C_p & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ B_m \\ 0 \end{bmatrix}, x(t) = \begin{bmatrix} x_p(t) \\ x_m(t) \\ \xi(t) \end{bmatrix} \tag{40}$$

with $x_c(t) \triangleq [x_m^T(t), \xi^T(t)]^T$. The control effort is $u(t) \triangleq F_c x(t) + D_c r(t)$ with $F_c \triangleq [0 \quad K_m \quad -K_p]$, and $D_c = 0$. The assigned closed loop eigenvalues are: $[-0.5, -1, -0.6, -1.2, -2, -3, -3.5, -4, -4.5]$. The gain matrices K_m, K_p and L are not reported for brevity.

As $W_{r,u}(0) = 0$, by (21) one has that any chosen $\tilde{y}_d$ is compatible with the saturation constraints. The pre-specified desired trajectory for both output components is the following S-shape function

$$\begin{cases} y(0), & 0 \le t \le a \\ 2\left(\tilde{y}_d - y(0)\right)\left(\frac{t-a}{b-a}\right)^2 + y(0), & a \le t \le \frac{a+b}{2} \\ \left(\tilde{y}_d - y(0)\right)\left(1 - 2\left(\frac{t-b}{b-a}\right)^2\right) + y(0), & \frac{a+b}{2} \le t \le b \\ \tilde{y}_d = 1, & t \ge b \end{cases}$$

with $a = 0$, $b = 5$, $y(0) = 0$, and $\tilde{y}_d = 1$.

Unlike classical methods (see e.g. [24], [27], [28], [29] and references therein), the present approach does not require either a peculiar structure of the desired output function or its derivability. The above S-shape function only represents the particular desired output trajectory for the present example.

To evidence the effects of the proposed method, two different sets of simulations, corresponding to two different approaches have been performed.

The first one does not explicitly consider any saturation constraint.To this purpose: 1) $e(t) = e_1(t)$, $Q(t) = Q_1(t)$, 2) the parameter vector f to be estimated reduces to $f = c_{r_t}$ because $\hat{u}_t(t)$ and the respective constraints (37) are not defined, 3) the control input $u_p(t)$ forcing Σ_p coincides with the control effort $u(t)$ yielded by Σ_c, $\forall t \in T_t$. The solution of the unconstrained least square problem is of the kind (6).

The second set of simulations refers to the direct application of the present approach. The choice $Q_1(t) = Q_2(t) = 1, t \in T_t$, has been adopted to equally weigh both components of both errors $e_1(t)$ and $e_2(t)$.The two sets of simulations are referred to as Unconstrained and Constrained Approach (UA and CA) respectively.

The following choices hold in both cases: 1) x(0) is the null vector, 2) the transient interval has been chosen as $T_t = [0, t_t]$ with $t_t = 15 > 4\tau_m = 8$, where $\tau_m = 2$ is the maximum time constant of $\Sigma_{f,c}$.

The constrained least squares problem corresponding to the CA has been solved using the Matlab function *lsqlin*

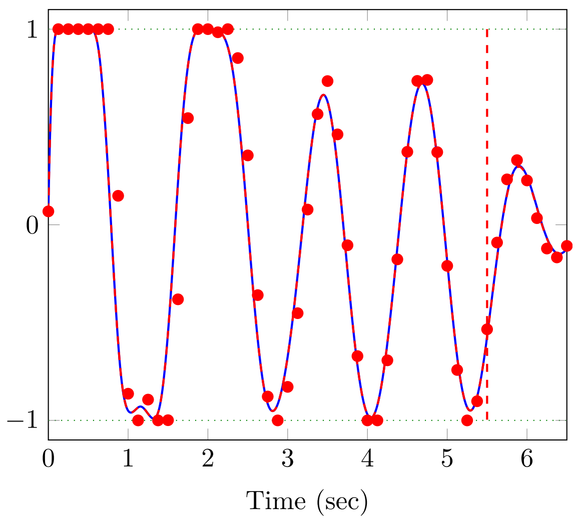

Fig. 2. The trajectory of the actual control input $u_p(t)$ (blue continuous line) and of the computed B-spline $\hat{u}_t(t)$ (red continuous line) with its respective control points (red circles) over the interval $[0,\ 6.5] \subseteq T_t$, respectively.

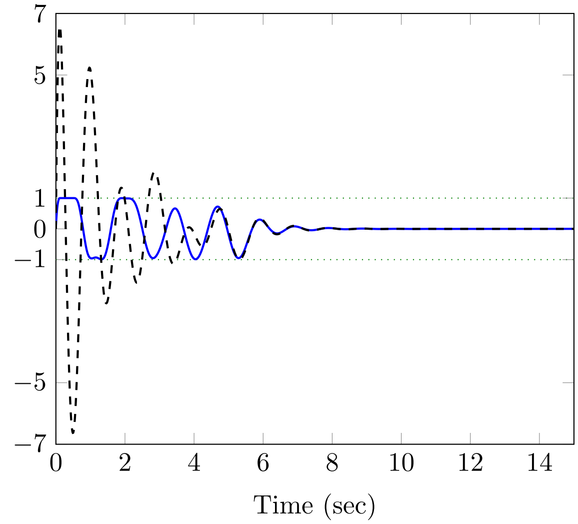

Fig. 4. The trajectory of actual control input forcing Σ_p in the CA (blue continuous line) and UA (black dashed line) respectively.

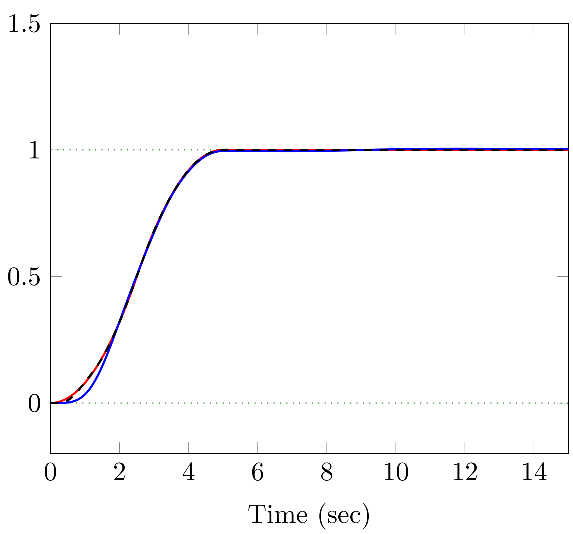

Fig. 3. The actual output $y(t)$ of $\Sigma_{f,c}$ in the CA (blue continuous line) and UA (black dashed line) respectively. The red continuous line represents the desired output trajectory $\tilde{y}_d(t)$.

considering a uniformly discretized residual vector $e(k\,t'_s)$ with $k = 0, 1, \cdots$, and $t'_s = 0.01$ sec.
The approximating B-spline $\hat{u}_t(t)$ has been computed using $\ell = 120$ control points, choosing $M = 0.6$, condition (24) gives a critical interval $T_c = [0 \quad 5.5]$, (see Fig 2). At the critical instant $t_c = 5.5$, $u_p(t)$ switches from the B-spline $\hat{u}_t(t)$ to the control signal $u_t(t)$ yielded by Σ_c. As $\hat{u}_t(t)$ and $u_t(t)$ are practically overlapped for $t \geq t_c$, no perceptible discontinuity is observed in $u_p(t)$ at the switching instant t_c.

Fig 3 shows that the UA yields slightly better tracking performance with respect to the CA. Nevertheless, CA produces a control input satisfying the saturation constraints, while UA does not (see Fig 4). At time $t_t = 15$, the steady state value $\tilde{u} = 0$ has been practically achieved in both cases.

VI. Conclusions

It has been here shown how pseudo-inversion and B-splines can be advantageously used for handling hardly constrained transient optimization problems. The three key points are:1) to use the preview based pseudo-inversion approach to compute the optimal reference $r_t(t)$, 2) to impose a B-spline function structure to $r_t(t)$ and $\hat{u}_t(t)$, 3) to replace $u_t(t)$ with $\hat{u}_t(t)$ over T_c.
Feasibility of the constrained optimization problem formally defined in Section III is a rather direct consequence of the used approach. In fact, modeling $\hat{u}_t(t), t \in T_t$, as a B-spline function implies its membership to the convex hull of its control points, that can be arbitrarily chosen on the basis of the saturation constraints, whatever they are. The feasibility of the corresponding optimization problem $\forall t \in T_t$ directly follows. Also the stability of Σ is a straightforward consequence of the proposed method: internal stability derives by Remark 2 and by the uniform boundedness of $r(t)$ directly implied by the way it is computed.

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