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$\mathcal{H}_\infty$ Model Order Reduction of Uncertain Linear Systems Using generalized KYP Lemma

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***Abstract*— This article presents a new LMI-based approach for the $\mathcal{H}_\infty$ model order reduction problem of continuous-time uncertain linear systems over frequency ranges. Differently from the previous methods in the literature, the approach is specially developed to treat uncertain systems, being able not only to reduce the order of the system, but to remove uncertain parameters. By means of the generalized Kalman-Yakubovich-Popov (gKYP) Lemma design conditions are provided in terms of LMIs relaxations associated to scalar searches and iterative procedures to assess low-frequency specifications. Numerical examples from the literature are used to illustrate the potentialities of the proposed approach when compared to existing methods.**

***Index Terms*— Model reduction, uncertain linear systems, finite frequency, KYP, $\mathcal{H}_\infty$ norm, LMI relaxations**

I. INTRODUCTION

IN TODAY'S world, physical and artificial systems are predominantly described by mathematical models. These models can be used to assess, simulate and/or control the behavior of the system in question. As examples, one can mention climate change prediction and very large scale integration (VLSI), the former physical and the latter artificial. In this framework, there is an ever-growing demand for improved accuracy, naturally leading to models of higher complexity, that considers for instance, high orders, parametric uncertainties and structural constraints. The increased complexity scenario suggests the development of a reduced-order model that reproduces the most important features of the original one. From this appealing proposal two main properties, in general, are desired: a simplified model as close as possible to the original one in terms of some performance criterion and, just as importantly, a model that fits best in the available synthesis conditions. The successful achievement of these requirements increases the chances of obtaining controllers and filters with improved performance, reduced computational complexity, easier implementation and lower costs.

Technically speaking, the Model Order Reduction (MOR) problem is, usually, stated as follows: given an n-th order model $G(s)$, find an r-th lower order model $G_r(s)$, such that $G(s)$ and $G_r(s)$ are as close as possible in terms of some properties of interest. Over the years many methods for solving the MOR problem were proposed [1]–[14]. When dealing with only precisely known systems, the classical methods are: balanced reduction [1] and optimal Hankel norm approximation [2], both applied over balanced models. After a decade, as an alternative solution expressed in terms of convex optimization, Helmersson [3] presented the possibility of using Linear Matrix Inequalities (LMIs) to determine an upper bound for the $\mathcal{H}_\infty$ norm of the approximation error. Right after, LMI-based approaches, using $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms as performance criteria, became quite popular [4]–[6], and did not take longer to appear methods able to handle systems affected by polytopic and affine uncertainties [7]–[11]. Particularly, in the context of $\mathcal{H}_\infty$ model reduction, the mentioned techniques approximate the full-order model over all frequencies. However, in many applications one is not interested in the entire frequency range but rather in a given frequency interval of relevance. Concerning the finite frequency $\mathcal{H}_\infty$ norm MOR problem, there are, generally speaking, two classes of methods based on the extension of the balanced truncation: frequency-weighted balanced truncation [12], [13] and frequency Gramian-based balanced truncation [13], [14]. More recently, LMI-based methods have been developed [15], [16] to cope with the $\mathcal{H}_\infty$ norm in frequency ranges. These papers make use of the so-called generalized Kalman-Yakubovic-Popov (gKYP) Lemma [17], [18] as main tool to take into account the frequency ranges.

This paper follows a similar approach, aiming to improve the results in terms of smaller approximations errors, while treating dynamic systems represented by models with uncertain parameters. As mentioned before, these models present an increasing practical interest because they can provide better approximations for realistic plants. Hence, the main objective of this work is to provide a model reduction technique specially constructed to treat linear models with uncertain parameters. Besides reducing the order of the system, which is standard in

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the literature of order reduction, the proposed method is also capable of removing some uncertain parameters, or changing the polynomial dependency of the system's matrices. These abilities can be particularly useful, since some parameters might have little or no affect at all in the approximation error in a specific frequency range. In this work the $\mathcal{H}_\infty$ norm of the approximation error in a pre-specified frequency range is adopted as performance criterion. The synthesis conditions are expressed in terms of LMI relaxations, and scalar searches and an iterative procedure can be used to improve the accuracy of the solutions. Numerical examples, including a model of a physical system, are presented to illustrate the superior performance of the proposed method.

Notation: The imaginary number is given by $j = \sqrt{-1}$. The space of complex (real) rectangular matrices is represented by $\mathbb{C}^{m\times n}$ ($\mathbb{R}^{m\times n}$); Given a matrix $X \in \mathbb{R}^{m\times n}$, $X < 0$ means X is negative definite; X' denotes the transpose of X. The complex conjugate transpose and a basis for the null space of X (a full column rank matrix such that $XX_\perp = 0$ and $[X^*\ X_\perp]$ has column rank equal to n), are denoted by X^* and $X_\perp$, respectively; The set of complex Hermitian $n \times n$ matrices is denoted by $\mathbf{H}_n$; $\mathrm{He}(X)$ is a short-hand notation for $X + X^*$; For a matrix $X \in \mathbb{C}^{m\times n}$, its maximum singular value is denoted by $\sigma_{max}(X)$. The symbol $\otimes$ indicates the Kronecker product and $\star$ denotes blocks induced by symmetry in Hermitian matrices.

II. PRELIMINARIES

A core result in the field of modern system and control theory is the Kalman-Yakupovich-Popov (KYP) Lemma [19]. which states that, given matrices A, B, and a Hermitian matrix Θ, the FDI

$$\begin{bmatrix}(j\omega \mathrm{I} - A)^{-1}B\\ \mathrm{I}\end{bmatrix}^* \Theta \begin{bmatrix}(j\omega \mathrm{I} - A)^{-1}B\\ \mathrm{I}\end{bmatrix} < 0 \qquad (1)$$

holds for all $\omega \in \mathbb{R} \cup \{\infty\}$ if and only if the LMI

$$\begin{bmatrix}A & B\\ \mathrm{I} & 0\end{bmatrix}^* \begin{bmatrix}0 & P\\ P & 0\end{bmatrix} \begin{bmatrix}A & B\\ \mathrm{I} & 0\end{bmatrix} + \Theta < 0 \qquad (2)$$

admits a Hermitian solution P. This equivalence allow us to examine the infinitely many inequalities parametrized by ω given in (1) by solving the finite-dimensional convex feasibility problem (2).

In the last decade, extensions of the KYP Lemma were proposed to deal with finite-frequency specifications [18], [20]. In this context, Iwasaki and Hara version of the so-called generalized KYP Lemma, which deal with specifications in low, middle, and high frequencies, states the equivalence between the FDI

$$\begin{bmatrix}(j\omega \mathrm{I} - A)^{-1}B\\ \mathrm{I}\end{bmatrix}^* \Theta \begin{bmatrix}(j\omega \mathrm{I} - A)^{-1}B\\ \mathrm{I}\end{bmatrix} < 0, \quad \forall \omega \in \Lambda(\Phi, \Psi)$$

and the LMI

$$\begin{bmatrix}A & B\\ \mathrm{I} & 0\end{bmatrix}^* (\Phi \otimes P + \Psi \otimes Q) \begin{bmatrix}A & B\\ \mathrm{I} & 0\end{bmatrix} + \Theta < 0$$

where $Q = Q^* > 0$ and $P = P^*$. The set $\Lambda(\Phi, \Psi)$ represents the frequency interval in which the FDI holds, and it is defined by

$$\Lambda(\Phi, \Psi) = \{\lambda \in \mathbb{C} : \sigma(\lambda, \Phi) = 0,\ \sigma(\lambda, \Psi) \geq 0\}$$

where $\sigma(\lambda, \Pi) = [\lambda^*\ 1]\Pi[\lambda^*\ 1]^*$. The definition of Φ and Ψ is given in [18].

Consider the uncertain robustly stable continuous-time LTI system, of order n given by

$$G(s,\alpha) \triangleq \begin{cases}\dot{x} = A(\alpha)x + B(\alpha)w,\\ y = C(\alpha)x + D(\alpha)w\end{cases} \qquad (3)$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are the state, input and output vectors, respectively. The system matrices $A(\alpha) \in \mathbb{R}^{n\times n}$, $B(\alpha) \in \mathbb{R}^{n\times m}$, $C(\alpha) \in \mathbb{R}^{p\times n}$ and $D(\alpha) \in \mathbb{R}^{p\times m}$ depend affinely on a vector of uncertain parameters $\alpha \in \Omega$, where Ω represents a *multi-simplex* [21], i.e., a Cartesian product $\Omega_{N_1} \times \ldots \times \Omega_{N_m}$ of finite number of unit simplexes $\Omega_{N_1}, \ldots, \Omega_{N_m}$. The dimension of Ω is defined as the index $N = (N_1 \ldots N_m)$ and a unit simplex Ω_m is defined by

$$\Omega_m \triangleq \left\{\delta \in \mathbb{R}^m :\ \sum_{i=1}^{m} \delta_i = 1,\ \delta_i \geq 0,\ i = 1, \ldots, m\right\}$$

where m is the dimension (number of parameters) of the simplex. The element α of Ω is decomposed as $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ according to the structure of Ω and, subsequently, each α_i (being in Ω_i) is decomposed in the the form $(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{iN_m})$. Hence, considering an multi-affine dependence on all uncertain parameters, the system matrices can be generically described by

$$Z(\alpha) = \sum_{i_1=1}^{N_1} \cdots \sum_{i_m=1}^{N_m} \alpha_{1i_1} \cdots \alpha_{mi_m} (Z_{i_1} \cdots Z_{i_m})$$

where $Z(\alpha)$ is any matrix of the system.

This representation is quite general in the sense of representing two classes of uncertainties extensively used in the literature. For instance, if $m = 1$, then $Z(\alpha)$ is a polytopic matrix. Moreover, if the system matrices are originally expressed in the form $Z(\theta) = Z_0 + \sum_{k=1}^{m} \theta_k Z_k$ with $\theta_k \in [-1, 1]$, that is, in the affine form (or hypercubic), they can be automatically converted to $Z(\alpha)$ employing the change of variables proposed in [22].

The problem to be addressed in this paper is the design of a reduced model for (3), of order r, $r \leq n$, with state-space representation given by

$$G_r(s,\alpha) \triangleq \begin{cases}\dot{x}_r = A_r(\alpha)x_r + B_r(\alpha)w,\\ y_r = C_r(\alpha)x_r + D_r(\alpha)w\end{cases} \qquad (4)$$

where $x_r \in \mathbb{R}^r$, $w \in \mathbb{R}^m$, $y_r \in \mathbb{R}^p$ are the state, input and output vectors, respectively, and $A_r(\alpha) \in \mathbb{R}^{r\times r}$, $B_r(\alpha) \in \mathbb{R}^{r\times m}$, $C_r(\alpha) \in \mathbb{R}^{p\times r}$ and $D_r(\alpha) \in \mathbb{R}^{p\times m}$ are the uncertain matrices of the reduced model to be determined such that the dynamics of (3) and (4) are evenly matched as possible. Combining the dynamics of both systems and creating a signal error $e = y - y_r$, called *approximation error*, one has the augmented system

$$\begin{aligned} \dot{z} &= \bar{A}(\alpha)z + \bar{B}(\alpha)w, \\ e &= \bar{C}(\alpha)z + \bar{D}(\alpha)w \end{aligned} \tag{5}$$

where $\bar{C}(\alpha) = \begin{bmatrix} C(\alpha) & -C_r(\alpha) \end{bmatrix}$, $\bar{D}(\alpha) = D(\alpha) - D_r(\alpha)$,

$$z = \begin{bmatrix} x \\ x_r \end{bmatrix},\ \bar{A}(\alpha) = \begin{bmatrix} A(\alpha) & 0_{n\times n_r} \\ 0_{n_r\times n} & A_r(\alpha) \end{bmatrix},\ \bar{B}(\alpha) = \begin{bmatrix} B(\alpha) \\ B_r(\alpha) \end{bmatrix}.$$

As performance criterion, it is adopted the $\mathcal{H}_\infty$ norm associated to the transfer matrix obtained from input w to output e of system (5), that is,

$$\| \bar{C}(\alpha)(sI - \bar{A}(\alpha))^{-1}\bar{B}(\alpha) + \bar{D}(\alpha) \|_\infty \leq \gamma \tag{6}$$

where γ is a guaranteed cost for the approximation error. The following lemma provides a specialization of the gKYP Lemma to compute an upper bound for the $\mathcal{H}_\infty$-norm on a low-frequency interval for system (5).

Lemma 1 ($\mathcal{H}_\infty$ Low Frequency): Let matrices $\overline{A}(\alpha) \in \mathbb{R}^{n+r\times n+r}$, $\overline{B}(\alpha) \in \mathbb{R}^{n+r\times m}$, $\overline{C}(\alpha) \in \mathbb{R}^{p\times n+r}$ and $\overline{D}(\alpha) \in \mathbb{R}^{p\times m}$, and scalar ω_l be given. Then, the following statements are equivalent.

i) $\sigma_{max}(G(s,\alpha)) < \gamma$, $\forall s = j\omega$, $\omega \leq |\omega_l|$ and $\forall \alpha \in \Omega$, where $G(s,\alpha) = \overline{C}(\alpha)(s\mathrm{I} - \overline{A}(\alpha))^{-1}\overline{B}(\alpha) + \overline{D}(\alpha)$

ii)$\exists P(\alpha)$, $0 < Q(\alpha) \in \mathbf{H}_n$ such that the following inequality holds $\forall \alpha \in \Omega$

$$\begin{bmatrix} \overline{A}(\alpha) & \overline{B}(\alpha) \\ \mathrm{I} & 0 \end{bmatrix}^* \begin{bmatrix} -Q(\alpha) & P(\alpha) \\ P(\alpha) & \omega_l^2 Q(\alpha) \end{bmatrix} \begin{bmatrix} \overline{A}(\alpha) & \overline{B}(\alpha) \\ \mathrm{I} & 0 \end{bmatrix} + \begin{bmatrix} \overline{C}(\alpha) & \overline{D}(\alpha) \\ 0 & \mathrm{I} \end{bmatrix}^* \begin{bmatrix} \mathrm{I} & 0 \\ 0 & -\gamma^2\mathrm{I} \end{bmatrix} \begin{bmatrix} \overline{C}(\alpha) & \overline{D}(\alpha) \\ 0 & \mathrm{I} \end{bmatrix} < 0 \tag{7}$$

Proof: See [18] or [17]. ■

Remark 1: The conditions of Lemma 1 could be extended to deal with the case of middle- and high-frequency interval by appropriate choices of Ψ and Φ (see [17] for details).

Remark 2: In [23] it is demonstrated that variables $P(\alpha)$ and $Q(\alpha)$ can be made real symmetric matrices without introducing conservatism.

III. MAIN RESULTS

As a starting point of the proposed procedure, it is presented an equivalent condition to Lemma 1, where the system matrices do not appear multiplying the variables $P(\alpha)$ and $Q(\alpha)$. This task is accomplished with the help of the Finsler's Lemma [24, Lem. 2-iv], that introduces the so-called *slack variables*.

Lemma 2: Let matrices $\overline{A}(\alpha) \in \mathbb{R}^{n+r\times n+r}$, $\overline{B}(\alpha) \in \mathbb{R}^{n+r\times m}$, $\overline{C}(\alpha) \in \mathbb{R}^{p\times n+r}$ and $\overline{D}(\alpha) \in \mathbb{R}^{p\times m}$, and scalar ω_l be given. There exist $P(\alpha)$ and $Q(\alpha)$ satisfying (7) if and only if there exist matrices $P(\alpha)$, $0 < Q(\alpha) \in \mathbf{H}_{n+r}$, and $K(\alpha)$, $E(\alpha) \in \mathbb{R}^{n+r\times n+r}$, $Y(\alpha) \in \mathbb{R}^{p\times n+r}$, $F(\alpha) \in \mathbb{R}^{m\times n+r}$, $L(\alpha)$ and $U(\alpha) \in \mathbb{R}^{n+r\times p}$, $H(\alpha) \in \mathbb{R}^{p\times p}$ and $M(\alpha) \in \mathbb{R}^{m\times p}$ such that

$$\begin{bmatrix} -Q(\alpha) & P(\alpha) & 0 & 0 \\ P(\alpha) & \omega_l^2 Q(\alpha) & 0 & 0 \\ 0 & 0 & \mathrm{I}_p & 0 \\ 0 & 0 & 0 & -\gamma^2\mathrm{I}_m \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} K(\alpha) & L(\alpha) \\ E(\alpha) & U(\alpha) \\ Y(\alpha) & H(\alpha) \\ F(\alpha) & M(\alpha) \end{bmatrix} \begin{bmatrix} -\mathrm{I} & \overline{A}(\alpha) & 0 & \overline{B}(\alpha) \\ 0 & \overline{C}(\alpha) & -\mathrm{I} & \overline{D}(\alpha) \end{bmatrix} \right\} < 0 \tag{8}$$

holds for all $\alpha \in \Omega$.

Proof: The dependence on α is omitted to shorten the formulas. The equivalence between (7) and (8) is established using Finsler's Lemma. Rewriting (7) as

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \mathrm{I} & 0 \\ \overline{C} & \overline{D} \\ 0 & \mathrm{I} \end{bmatrix}^* \begin{bmatrix} -Q & P & 0 & 0 \\ P & \omega_l^2 Q & 0 & 0 \\ 0 & 0 & \mathrm{I}_p & 0 \\ 0 & 0 & 0 & -\gamma^2\mathrm{I}_m \end{bmatrix} \begin{bmatrix} \overline{A} & \overline{B} \\ \mathrm{I} & 0 \\ \overline{C} & \overline{D} \\ 0 & \mathrm{I} \end{bmatrix} < 0 \tag{9}$$

then computing the orthogonal matrix of

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \mathrm{I} & 0 \\ \overline{C} & \overline{D} \\ 0 & \mathrm{I} \end{bmatrix} \Longrightarrow \begin{bmatrix} \overline{A} & \overline{B} \\ \mathrm{I} & 0 \\ \overline{C} & \overline{D} \\ 0 & \mathrm{I} \end{bmatrix}_\perp = \begin{bmatrix} -\mathrm{I} & \overline{A} & 0 & \overline{B} \\ 0 & \overline{C} & -\mathrm{I} & \overline{D} \end{bmatrix}$$

and applying Finsler's Lemma yields (8) ■

As noticed, if the reduced systems matrices $A_r(\alpha)$, $B_r(\alpha)$, $C_r(\alpha)$ and $D_r(\alpha)$ are decision variables, (8) is actually a Parameter-Dependent (PD) Bilinear Matrix Inequality (BMI). Aiming a convex approximation for (8), i.e., a PD-LMI, structural constraints are imposed to the slack variables as $L(\alpha) = U(\alpha) = M(\alpha) = 0$ and

$$K(\alpha) = \begin{bmatrix} K_{11}(\alpha) & \beta_1 J_\sigma(\hat{K}) \\ K_{21}(\alpha) & \beta_2 \hat{K} \end{bmatrix},\quad E(\alpha) = \begin{bmatrix} E_{11}(\alpha) & J_\sigma(\hat{K}) \\ E_{21}(\alpha) & \hat{K} \end{bmatrix},$$
$$Y(\alpha) = \begin{bmatrix} Y_1(\alpha) & 0_{p\times r} \end{bmatrix},\quad F(\alpha) = \begin{bmatrix} F_1(\alpha) & 0_{m\times r} \end{bmatrix}$$

and the following change of variables are performed $M_{A_r}(\alpha) = \hat{K}A_r(\alpha)$, $M_{B_r}(\alpha) = \hat{K}B_r(\alpha)$, $M_{C_r}(\alpha) = \hat{H}C_r(\alpha)$, and $M_{D_r}(\alpha) = \hat{H}D_r(\alpha)$, giving rise to the main contribution of the paper. For convenience, matrices $Q(\alpha)$ and $P(\alpha)$ were also partitioned as

$$Q(\alpha) = \begin{bmatrix} Q_{11}(\alpha) & Q_{12}(\alpha) \\ Q_{12}(\alpha)^* & Q_{22}(\alpha) \end{bmatrix},\quad P(\alpha) = \begin{bmatrix} P_{11}(\alpha) & P_{12}(\alpha) \\ P_{12}(\alpha)^* & P_{22}(\alpha) \end{bmatrix}$$

Theorem 1: If there exist matrices $K_{11}(\alpha)$ and $E_{11}(\alpha) \in \mathbb{C}^{n\times n}$, $\hat{K}$ and $M_{A_r}(\alpha) \in \mathbb{R}^{r\times r}$, $K_{21}(\alpha)$ and $E_{21}(\alpha) \in \mathbb{C}^{r\times n}$, $P_{12}(\alpha)$ and $Q_{12}(\alpha) \in \mathbb{C}^{n\times r}$, $\hat{H} \in \mathbb{R}^{p\times p}$, $M_{B_r}(\alpha) \in \mathbb{R}^{r\times m}$, $M_{C_r}(\alpha) \in \mathbb{R}^{p\times r}$, $M_{D_r}(\alpha) \in \mathbb{R}^{p\times m}$, $Y_1(\alpha) \in \mathbb{C}^{p\times n}$, $F_1(\alpha) \in \mathbb{C}^{m\times n}$, Hermitian matrices $P_{11}(\alpha)$ and $Q_{11}(\alpha) \in \mathbf{H}_n$ and $P_{22}(\alpha)$ and $Q_{22}(\alpha) \in \mathbf{H}_r$ with $Q(\alpha) > 0$, and given scalars $\sigma \in \{0, 1, \ldots, n-r\}$, ω_l, β_1 and $\beta_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} -Q(\alpha) & P(\alpha) & 0 & 0 \\ P(\alpha) & \omega_l^2 Q(\alpha) & 0 & 0 \\ 0 & 0 & \mathrm{I}_p & 0 \\ 0 & 0 & 0 & -\gamma^2\mathrm{I}_m \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \overline{K}(\alpha) & 0 \\ \overline{E}(\alpha) & 0 \\ Y(\alpha) & \mathrm{I}_p \\ F(\alpha) & 0 \end{bmatrix} \begin{bmatrix} -\mathrm{I} & \overline{A}_L(\alpha) & 0 & \overline{B}_L(\alpha) \\ 0 & \overline{C}_L(\alpha) & -\mathrm{I} & \overline{D}_L(\alpha) \end{bmatrix} \right\} < 0 \tag{10}$$

with

$$\overline{K}(\alpha)=\begin{bmatrix}K_{11}(\alpha) & \beta_1 J_\sigma(\mathrm{I}_r)\\ K_{21}(\alpha) & \beta_2\mathrm{I}_r\end{bmatrix},\quad \overline{E}(\alpha)=\begin{bmatrix}E_{11}(\alpha) & J_\sigma(\mathrm{I}_r)\\ E_{21}(\alpha) & \mathrm{I}_r\end{bmatrix},$$

$$\overline{A}_L(\alpha)=\begin{bmatrix}A(\alpha) & 0_{n\times n_r}\\ 0_{n_r\times n} & M_{A_r}(\alpha)\end{bmatrix},\quad \overline{B}_L(\alpha)=\begin{bmatrix}B(\alpha)\\ M_{B_r}(\alpha)\end{bmatrix},$$

$$\overline{C}_L(\alpha)=\begin{bmatrix}\hat{H}C(\alpha) & -M_{C_r}(\alpha)\end{bmatrix},\ \overline{D}_L(\alpha)=\hat{H}D(\alpha)-M_{D_r}(\alpha)$$

holds for all $\alpha\in\Omega$, then $A_r(\alpha)=\hat{K}^{-1}M_{A_r}(\alpha)$, $B_r(\alpha)=\hat{K}^{-1}M_{B_r}(\alpha)$, $C_r(\alpha)=\hat{H}^{-1}M_{C_r}(\alpha)$ and $D_r(\alpha)=\hat{H}^{-1}M_{D_r}(\alpha)$ are the reduced matrices such that γ is a guaranteed cost for the $\mathcal{H}_\infty$ norm of system (5).

Proof: Considering the proposed structural constraints and change of variables, inequality (10) can be written as

$$\begin{bmatrix}-Q(\alpha) & P(\alpha) & 0 & 0\\ P(\alpha) & \omega_l^2Q(\alpha) & 0 & 0\\ 0 & 0 & \mathrm{I}_p & 0\\ 0 & 0 & 0 & -\gamma^2\mathrm{I}_m\end{bmatrix}+ \mathrm{He}\left\{\begin{bmatrix}K(\alpha) & 0\\ E(\alpha) & 0\\ Y(\alpha) & \hat{H}\\ F(\alpha) & 0\end{bmatrix}\begin{bmatrix}-\mathrm{I} & \overline{A}(\alpha) & 0 & \overline{B}(\alpha)\\ 0 & \overline{C}(\alpha) & -\mathrm{I} & \overline{D}(\alpha)\end{bmatrix}\right\}<0$$

which is a sufficient condition for (8). ∎

Theorem 1 is the first contribution of this paper, providing synthesis condition in terms of PD-LMIs (considering fixed values of β_1, β_2 and σ). $J_\sigma(\hat{K})$ is an operator structured as

$$J_\sigma(\hat{K}_{r\times r})=\begin{bmatrix}0_{r\times\sigma} & \hat{K}_{r\times r} & 0_{r\times n-r-\sigma}\end{bmatrix}'$$

and it is used as an extra degree of freedom in the linearization procedure [11]. Regarding the infinite dimensional nature of the problem, i.e., PD-LMIs that must be checked for all $\alpha\in\Omega$, Section IV discusses in details how to obtain a finite dimensional test by imposing fixed structures (polynomial) for the optimization variables.

With respect to matrices $H(\alpha)$ and $K(\alpha)$ in Lemma 2, $\hat{H}$ and $\hat{K}$ were set fixed in Theorem 1, as parameter-independent, with the purpose of providing reduced matrices with polynomial dependence instead of rational dependence. Although the latter representation would provide a more accurate model (smaller approximation error), the synthesis conditions (for controllers or filters) available in the literature usually consider only matrices with polynomial dependence. Compared to previous results note that, differently from [16], the synthesis conditions of Theorem 1 do not require an initial reduced model, which could be a hard task, specially in the context of uncertain systems. Moreover, with respect to [25], the constraint $\bar{D}(\alpha)\bar{D}(\alpha)'-\gamma^2I$ is not required in the proposed conditions (less conservative).

Once a reduced model is obtained by Theorem 1, an iterative procedure could be applied. The strategy frequently employed by the methods in the literature is the so-called alternating direction method. Basically, it consists of using the matrices of the reduced system and the slack variables as inputs, fixing one at a time and determining the other at each iteration. Instead, this work pursues a different approach, inspired by the two step procedure presented in [26], [27], that have provided compelling results for reduced order controller design. By striving for a lower computational effort, certainly an advantageous feature in an iterative scheme, the proposed condition presents only two extra variables (besides the Lyapunov matrix and the KYP variable).

First note that (5) can be equivalently rewritten as

$$\begin{aligned}\dot{z}&=\tilde{A}(\alpha)z+\tilde{B}_1(\alpha)w+\tilde{B}_2u,\\ e&=\tilde{C}_1(\alpha)z+\tilde{D}_1(\alpha)w+\tilde{D}_2u,\\ y_u&=\tilde{C}_2z+\tilde{D}_{1u}w\end{aligned}\tag{11}$$

where

$$\tilde{A}(\alpha)=\begin{bmatrix}A(\alpha) & 0_{n\times n_r}\\ 0_{n_r\times n} & 0_{n_r\times n_r}\end{bmatrix},\quad \tilde{B}_2=\begin{bmatrix}0_{n\times n_r} & 0_{n\times p}\\ \mathrm{I}_{n_r\times n_r} & 0_{n_r\times p}\end{bmatrix},$$

$$\tilde{C}_1(\alpha)=\begin{bmatrix}C(\alpha) & 0_{p\times n+n_r}\end{bmatrix},\quad \tilde{D}_2=\begin{bmatrix}0_{p\times n_r} & -\mathrm{I}_p\end{bmatrix},$$

$$\tilde{B}_1(\alpha)=\begin{bmatrix}B(\alpha)\\ 0_{n_r\times m}\end{bmatrix},\quad \tilde{C}_2=\begin{bmatrix}0_{n_r\times n} & \mathrm{I}_{n_r}\\ 0_{m\times n} & 0_{m\times n_r}\end{bmatrix},$$

$$\tilde{D}_{1u}=\begin{bmatrix}0_{m\times n_r} & \mathrm{I}_m\end{bmatrix}',\quad \tilde{D}_1(\alpha)=D(\alpha)$$

and u is a fictitious input given by $u=\Theta y_u$ where

$$\Theta(\alpha)=\begin{bmatrix}A_r(\alpha) & B_r(\alpha)\\ C_r(\alpha) & D_r(\alpha)\end{bmatrix}\tag{12}$$

In fact, by closing the loop in (11) with the fictitious input results in (5). Moreover, the affine dependency of the closed-loop matrices in (5) on $\Theta(\alpha)$ is expressed as

$$\begin{bmatrix}\overline{A}(\alpha) & \overline{B}(\alpha)\\ \overline{C}(\alpha) & \overline{D}(\alpha)\end{bmatrix}=\begin{bmatrix}\tilde{A}(\alpha) & \tilde{B}_1(\alpha)\\ \tilde{C}_1(\alpha) & \tilde{D}_1(\alpha)\end{bmatrix}+\begin{bmatrix}\tilde{B}_2\\ \tilde{D}_2\end{bmatrix}\Theta\begin{bmatrix}\tilde{C}_2 & \tilde{D}_{1u}\end{bmatrix}$$

Theorem 2 presented in the sequence explores the augmented representation given in (11). It provides a new synthesis condition in the form of PD-LMI that can be used if the reduced order matrices are available (for instance, designed with Theorem 1) and is the second main contribution of the paper.

Theorem 2: Let $\hat{\Theta}$, in the form (12), be the reduced matrices obtained by any MOR method and scalar ω_l be given. If there exist matrices $\hat{K}$ and $M_{A_r}(\alpha)\in\mathbb{R}^{r\times r}$, $P_{12}(\alpha)$ and $Q_{12}(\alpha)\in\mathbb{C}^{n\times r}$, $\hat{H}\in\mathbb{R}^{p\times p}$, $M_{B_r}(\alpha)\in\mathbb{R}^{r\times m}$, $M_{C_r}(\alpha)\in\mathbb{R}^{p\times r}$, $M_{D_r}(\alpha)\in\mathbb{R}^{p\times m}$, hermitian matrices $P_{11}(\alpha)$ and $Q_{11}(\alpha)\in\mathbf{H}_n$ and $P_{22}(\alpha)$ and $Q_{22}(\alpha)\in\mathbf{H}_r$, with $Q(\alpha)>0$, such that (13) holds for all $\alpha\in\Omega$, then $A_r(\alpha)=\hat{K}^{-1}M_{A_r}(\alpha)$, $B_r(\alpha)=\hat{K}^{-1}M_{B_r}(\alpha)$, $C_r(\alpha)=\hat{H}^{-1}M_{C_r}(\alpha)$ and $D_r(\alpha)=\hat{H}^{-1}M_{D_r}(\alpha)$ are the reduced matrices such that γ is a guaranteed cost for the $\mathcal{H}_\infty$ norm of system (5).

$$\begin{bmatrix}\Psi_{11} & \star & \star & \star & \star\\ \Psi_{21} & \Psi_{22} & \star & \star & \star\\ \Psi_{31} & \Psi_{32} & \Psi_{33} & \star & \star\\ \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} & \star\\ -C(\alpha) & \Psi_{52} & \Psi_{53} & 0_{p\times r} & \Psi_{55}\end{bmatrix}<0\tag{13}$$

with $\Psi_{11}=A(\alpha)'Q_{11}(\alpha)A(\alpha)+A(\alpha)'P_{11}(\alpha)+P_{11}(\alpha)A(\alpha)+C(\alpha)'C(\alpha)+\omega_l^2Q_{11}(\alpha)+M_{A_r}(\alpha)'\hat{A}_r(\alpha)+M_{C_r}(\alpha)'\hat{C}_r(\alpha)$; $\Psi_{21}=P_{12}(\alpha)'A(\alpha)+\omega_l^2Q_{12}(\alpha)'$;

$\Psi_{22} = \omega_l^2 Q_{22}(\alpha) + \hat{A}_r(\alpha)' M_{A_r}(\alpha) + \hat{C}_r(\alpha)' M_{C_r}(\alpha)$;
$\Psi_{31} = -B(\alpha)' Q_{11}(\alpha) A(\alpha) + B(\alpha)' P_{11}(\alpha) + D(\alpha)' C(\alpha)$;
$\Psi_{32} = B(\alpha)' P_{12}(\alpha) + \hat{B}_r(\alpha)' M_{A_r}(\alpha) + \hat{D}_r(\alpha)' M_{C_r}(\alpha) + M_{B_r}(\alpha)' \hat{A}_r(\alpha) + M_{D_r}(\alpha)' \hat{C}_r(\alpha)$;
$\Psi_{33} = -B(\alpha)' Q_{11}(\alpha) B(\alpha) + D(\alpha)' D(\alpha) + \hat{B}_r(\alpha)' M_{B_r}(\alpha) + \hat{D}_r(\alpha)' M_{D_r}(\alpha) + M_{B_r}(\alpha)' \hat{B}_r(\alpha) + M_{D_r}(\alpha)' \hat{D}_r(\alpha) - \gamma^2 \mathrm{I}_p$;
$\Psi_{41} = -Q_{12}(\alpha)' A(\alpha) + P_{12}(\alpha)'$;
$\Psi_{42} = P_{22}(\alpha) - M_{A_r}(\alpha) - \hat{K}' \hat{A}_r(\alpha)$;
$\Psi_{43} = -Q_{12}(\alpha)' B(\alpha) - M_{B_r}(\alpha) - \hat{K} \hat{B}_r(\alpha)$;
$\Psi_{44} = -Q_{22}(\alpha) - \hat{K} - \hat{K}'$; $\Psi_{52} = -M_{C_r}(\alpha) - \hat{H}' \hat{C}_r(\alpha)$;
$\Psi_{53} = -D(\alpha) - M_{D_r}(\alpha) - \hat{H} \hat{D}_r(\alpha)$; $\Psi_{55} = -\hat{H} - \hat{H}' - \mathrm{I}_p$;

Proof: The argument α is omitted to shorten the formulas. Starting from (9) written in the form $\mathcal{B}_\perp^* \mathcal{Q} \mathcal{B}_\perp < 0$, one can assume $\mathcal{B}_\perp = \mathcal{R}\mathcal{P}_\perp$ as

$$\begin{bmatrix} \overline{A} & \overline{B} \\ \mathrm{I} & 0 \\ \overline{C} & \overline{D} \\ 0 & \mathrm{I} \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \mathrm{I} & 0 & 0 \\ \tilde{C}_1 & \tilde{D}_1 & \tilde{D}_2 \\ 0 & \mathrm{I} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \\ \Theta\tilde{C}_2 & \Theta\tilde{D}_{1u} \end{bmatrix}$$

and obtain $\mathcal{P}_\perp^* \hat{\mathcal{Q}} \mathcal{P}_\perp < 0$, where $\hat{\mathcal{Q}} = \mathcal{R}^* \mathcal{Q} \mathcal{R}$. Computing $\mathcal{P}$ such that $\mathcal{P}\mathcal{P}_\perp = 0$, one has $\mathcal{P} = \begin{bmatrix} \Theta\tilde{C}_2 & \Theta\tilde{D}_{1u} & -\mathrm{I} \end{bmatrix}$ Applying the Projection Lemma [28] with a constrained slack variable, the following condition is obtained:

$$\hat{\mathcal{Q}} + \mathrm{He}\left\{ \begin{bmatrix} \tilde{C}_2'\hat{\Theta}' \\ \tilde{D}_{1u}'\hat{\Theta}' \\ -\mathrm{I} \end{bmatrix} \begin{bmatrix} \hat{K} & 0 \\ 0 & \hat{H} \end{bmatrix} \begin{bmatrix} \Theta\tilde{C}_2 & \Theta\tilde{D}_{1u} & -\mathrm{I} \end{bmatrix} \right\} < 0$$

that is precisely (14) with the proposed change of variables (13). ∎

Remark 3: Theorem 2 do not reduce the model order but, instead, seeks a finer solution, i.e., with a smaller approximation error.

Note that the reduced matrices provided by Theorem 2 are obtained only in terms of optimization variables. As a consequence, the polynomial degrees associated to these variables defines the degrees of the reduced matrices. For instance, their degrees could be different than the initialization input matrices degrees. Furthermore, the solutions of Theorems 2 could be used as new input parameters, giving rise to an iterative procedure, with nonincreasing γ, which is explored in the numerical experiments. It is also worth mentioning that the reformulation of the model reduction problem in terms of a static output-feedback problem is similar to the strategy given in [16] (actually uses a constrained state-feedback formulation), but the derivation of the proposed results is much simpler.

Enforcing stability: As a general consequence of the gKYP Lemma, the proposed synthesis conditions do not guarantee that the synthesized reduced model is robustly stable. As the original system is assumed robustly stable as initial hypothesis, the robust stability of the reduced system, associated to the Hurwitz stability of matrix $A_r(\alpha)$, can be enforced by the following condition

$$\begin{bmatrix} A_r(\alpha)' & \mathrm{I} \end{bmatrix} \begin{bmatrix} 0 & W(\alpha) \\ W(\alpha) & 0 \end{bmatrix} \begin{bmatrix} A_r(\alpha) \\ \mathrm{I} \end{bmatrix} < 0, \quad W(\alpha) > 0 \tag{14}$$

Noting that $\begin{bmatrix} A_r(\alpha) \\ \mathrm{I} \end{bmatrix}_\perp = \begin{bmatrix} \mathrm{I} & -A_r(\alpha) \end{bmatrix}$ and applying the Projection Lemma, inequality (14) is equivalent to

$$\begin{bmatrix} 0 & W(\alpha) \\ W(\alpha) & 0 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} X_1(\alpha) \\ X_2(\alpha) \end{bmatrix} \begin{bmatrix} \mathrm{I} & -A_r(\alpha) \end{bmatrix} \right\} < 0$$

Finally, fixing the structures $X_1(\alpha) = \hat{K}$ and $X_2(\alpha) = \xi\hat{K}$, for $\xi > 0$, and employing the change of variables $M_{A_r}(\alpha) = \hat{K} A_r(\alpha)$, yields the following condition

$$\begin{bmatrix} \mathrm{He}(\hat{K}) & W(\alpha) - M_{A_r}(\alpha) + \xi\hat{K}' \\ W(\alpha) - M_{A_r}(\alpha)' + \xi\hat{K} & -\mathrm{He}(\xi M_{A_r}(\alpha)) \end{bmatrix} < 0 \tag{15}$$

which is a PD-LMI condition (for a fixed value of ξ) and, when solved jointly with the conditions of Theorems 1 and 2, assures the robust stability of the reduced system.

Remark 4: Note that $\hat{K}$ and $M_{A_r}(\alpha)$ in (15) must be the same optimization variables used in Theorems 1 and 2 (same structure), and $W(\alpha)$ is a Lyapunov matrix used only to prove the robust stability of $A_r(\alpha)$.

With respect to [15], the stability condition proposed here is much simpler, checking only the stability of $A_r(\alpha)$ ($A(\alpha)$ is assumed robustly stably as hypothesis) and thus requiring a lower computational complexity.

IV. FINITE DIMENSIONAL CONDITIONS

The proposed conditions were presented in terms of PD-LMIs (for fixes values of the scalar parameters), comprising a high level representation which cannot be numerically solved (programmed). The reason for this option is both theoretical and numerical. From the theoretical point of view, it is known for over a decade that PD-LMIs can be approximated by polynomial solutions [29]. Moreover, nowadays there are available software that can perform the trick job: polynomial manipulations and application of relaxations for the resulting polynomial positivity test. The task left to the user is basically the choice of the polynomial degrees associated to the optimization variables. Particularly for the case of multi-simplex parameters used in this paper, a degree vector $g = (g_1, \ldots, g_q)$ must be chosen, where g_i is associated to the i-th simplex inside the multi-simplex [21]. As a general rule, the degrees chosen for the variables that define the reduced order matrices depends on the purpose of the design. For instance, $g = (1, \ldots, 1)$ guarantees that the resulting system is multi-affine or polytopic on the uncertainty. The degree for the other variables are related to the conservativeness of the solution (quality of the polynomial approximation). Note that null components of g guarantee that the associated parameters will be removed from the resulting system. It is important to observe the flexibility of this strategy, that allows the reduced system to be precisely known (degree zero), affine (degree one) or of higher degrees. The programming of the synthesis conditions was realized using the ROLMIP (Robust LMI Parser) toolbox [30], that works jointly with Yalmip [31]. In this paper MOSEK ApS solver was used as the SDP solver [32].

V. NUMERICAL EXAMPLES

Since the proposed synthesis conditions are only sufficient, i.e. the resulting γ are only guaranteed costs, the worst case $\mathcal{H}_\infty$ norm of the closed-loop system is calculated using the analysis condition given in (7) (the guaranteed cost is denoted as $\gamma_{analysis}$). The examples presented in the sequence were computed with MATLAB ($R2016b$) 64bits, in a computer with Intel® Core$^{\text{TM}}$ i7-3610QM 2.3GHz, 12Gb RAM and Windows 10 64 bits.

Example 1: Although the proposed methods are specially developed to treat uncertain systems, where the singular values cannot be determined, the purpose of this example is to evaluate the conservativeness of the approach with respect to other methods from the literature in the case of precisely known systems and using the $\mathcal{H}_\infty$ norm as performance criterion. This example is borrowed from [15], [16]. Consider the following fourth-order strictly proper system: $C = \begin{bmatrix} -0.877 & -1.252 & 0.774 & -0.111 \end{bmatrix}$,

$$A = \begin{bmatrix} -2.218 & -1.974 & 0.306 & -1.287 \\ -1.529 & -2.809 & 0.909 & -1.141 \\ -1.049 & -0.050 & -2.144 & -1.210 \\ -1.299 & 1.745 & 0.345 & -2.141 \end{bmatrix}, \quad B = \begin{bmatrix} -1.712 \\ -0.089 \\ -0.671 \\ -1.800 \end{bmatrix}.$$

For this example, the pre-specified frequency range is $|\omega| \leq 2$ rad/s and a second-order reduced model is desired. Applying Theorem 1 with $\sigma = \{0\}$ and scalars $\beta_1 = \{0.01\}$, $\beta_2 = \{1\}$ and $\xi = 10$, it is obtained $\gamma = 0.1603$ ($\gamma_{analysis} = 0.1122$). Subsequently using an iterative procedure based on Theorem 2, the following reduced system is obtained

$$\left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right] = \left[\begin{array}{cc|c} -0.4217 & -1.8979 & 0.0591 \\ 0.1139 & -1.3286 & -0.6349 \\ \hline 0.3910 & 0.1921 & 0.1749 \end{array}\right]$$

with $\gamma = 0.0117$ ($\gamma_{analysis} = 0.0115$) after 6 iterations which took 9.4 seconds. This result is better than the ones provided by the methods: [15] $\gamma = 0.0998$ and [16] $\gamma = 0.0389$, as corroborated by the Bode-magnitude plot presented in Figure 1, where the fourth-order model, the second-order reduced models obtained by Theorems 1–2, [15] and [16] are represented by the lines blue, red, black and green, respectively, and the dashed lines are out of the chosen range.

Example 2: The second example considers an identified 10th order model of a flexible plate that is actively controlled by means of three piezo actuators [33]. To create an uncertain model, the stiffness and damping are allowed to increase up to 50% (jointly). In this case is easy to obtain a polytopic model with two vertices by combining the extreme values. In this practical example, the pre-specified frequency range chosen was $|\omega| \leq 330$ rad/s. The investigation is divided into two cases: a SISO version, where it is considered only the first input and the first output, and a MIMO version, where all three inputs-outputs are considered.

As an additional design requirement, an integrator was

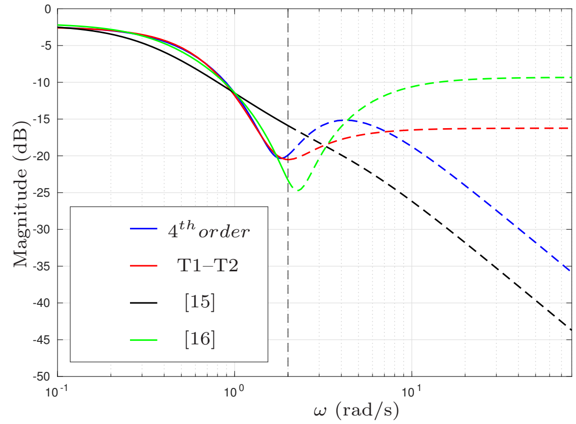

Fig. 1. Bode-magnitude diagrams of the original fourth-order system, the second-order system obtained by Theorems 1–2, [15] and [16].

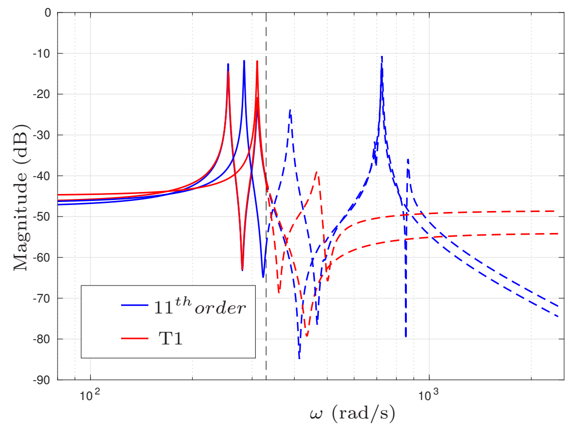

Fig. 2. Frequency responses of the two vertices of the polytope for the original eleventh-order system and the reduced fifth-order system obtained by Theorem 1.

introduced to eliminate the steady-state error, increasing the order of the system by 1 (SISO case - V-.1) and by 3 (MIMO case - V-.2).

1) SISO Case: When applying Theorem 1 with $\sigma = \{2\}$ and scalars $\beta_1 = \{0.1\}$, $\beta_2 = \{0.1\}$ and $\xi = 0.1$, it is obtained $\gamma = 0.0827$. In this case, the employment of Theorem 2 did not produce significant improvements. Figure 2 shows Bode-magnitude plots associated to the two vertices of the polytope for the eleventh-order model (in blue) and the fifth-order reduced model (in red) obtained by Theorem 1. Considering the chosen frequency range (dashed lines are out of the chosen range) and the Bode plot behavior, a good, graphically wise, result was obtained for a fifth-order reduced model.

2) MIMO Case: When applying Theorem 1 with $\sigma = \{3\}$ and scalars $\beta_1 = \{1\}$, $\beta_2 = \{1\}$ and $\xi = 1$, it is obtained $\gamma = 0.1327$. It was not possible to improve the results with Theorem 2. For the chosen frequency, a good approximation was obtained for an eighth-order reduced model (dashed lines are out of the chosen range). Figure 3 shows singular value-magnitude plots associated

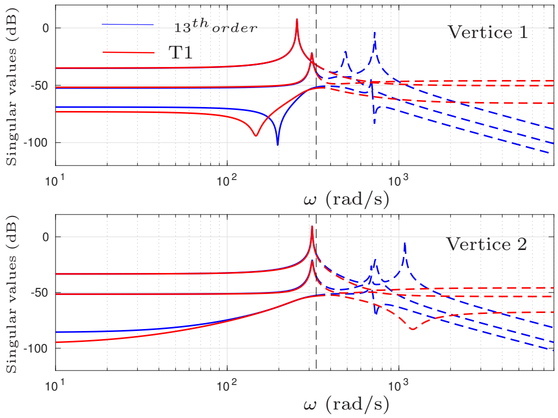

Fig. 3. Singular value diagram of the original thirteenth-order system and reduced eighth-order system obtained by Theorem 1.

to the two vertices of the polytope for the thirteenth-order model (in blue) and the eighth-order reduced model (in red) obtained by Theorem 1.

VI. CONCLUSIONS

This paper presented an LMI-based approach to cope with $\mathcal{H}_\infty$ model order reduction of uncertain linear systems over frequency ranges. The approach is more general than the methods available in the literature in the sense of also being able to handle uncertain systems and to provide a full control of the parametric dependency of the reduced matrices. Numerical examples illustrated the advantages of the approach with respect to previous results available in the literature.

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