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Stability of nonlinear impulsive differential equations with non-fixed moments of jumps

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Abstract— **This paper studies stability properties of the trivial solution to a system of nonlinear differential equations that undergo impulsive perturbations at non-fixed moments of time. We are motivated by modeling of networked control systems in which the communication between subsystems can be state-dependent. This leads to the impulsive system with multiple impulsive time sequences and a distinct jump map for each sequence. Lyapunov-like theorems equipped with novel dwell-time conditions for global asymptotic stability of the origin have been proven. We treat the cases of a stable continuous dynamics that is being destabilized by impulsive perturbations, and vice versa, the case of unstable continuous dynamics that is being stabilized by impulses. Our results are less conservative comparing to the existing ones since we propose the concept of a candidate Lyapunov function with multiple nonlinear rate functions to characterize its behaviour during flows and jumps and account the influence of impulses for each impulsive time sequence separately. Also, we demonstrate the application of the results to stability analysis of impulsive differential equations with fixed moments of jumps and compare them with the existing ones.**

I. INTRODUCTION

A motivation for this paper comes from engineering applications which comprise a network of control systems (NCS) that exchange information about their states. From the mathematical point of view, each system (plant) is a nonlinear ordinary differential equation equipped with input. The communication channels are being active and transmit information only when specific mixed algebraic and logic constraints are satisfied. Usually, these constraints are fulfilled only at some specific points of the time axis (on contrary to the plants whose states are changing continuously). These lead to a system for which a part of signals (e.g. states) evolves continuously, and another part of signals (e.g. control inputs) evolves discontinuously or in a piecewise continuous manner.

In order to study such important properties of the network as stability, or robustness w.r.t. perturbations of communication topology, or to design a control for some purpose one needs to represent the NCS in the form of an appropriate mathematical object. This object should capture a combination of continuous and discontinuous behaviour of the network.

Traditionally, control theory focuses on the study of interconnected dynamical systems linked through "ideal channels". On the contrary, communication theory studies the transmission of information over "imperfect channels" featuring varying transmission intervals, delays, and other communication constraints which can degrade the overall performance of the system and lead to qualitative changes in behaviour [1]. A combination of the two frameworks is needed to model networked control systems. One of the most popular design approaches to study properties of NCSs is emulation-based approach [2], [3]. The idea is to first design the controller for the plant while ignoring communication constraints, and then study constraints that should be imposed onto the system in order to preserve the desired properties or behaviour under communication constraints.

One of the conventional approaches to model NCSs within emulation-based methodology (which dominates in a recent literature [1], [4]–[11]) is to use hybrid dynamical systems framework introduced in [12]. The other approaches use the formalisms of mixed-logics-dynamics (MLD) [13], extended-linear-complementarity (ELC), and max-min-plus-scaling (MMPS) [14].

In this paper, we propose to use the framework of *impulsive differential equations with non-fixed moments of jumps* [15]. It has some important advantages, namely, it allows a combination of state-triggered and predefined time-triggered moments of jumps. Also, this framework is well-adapted to deal with time-varying ordinary differential equations, which makes it possible to expand the results to a further extent.

The questions of the existence of solutions to impulsive differential equations and their local stability properties have been summarized in [15]. These results also include a classification of impulsive differential equations depending on the character of impulsive actions [15], [16], stability characterization of solutions [17], [18], existence of periodic and almost-periodic solutions [19], [20] and invariant sets [21], Lyapunov-like theorems for local stability and associated dwell-time conditions [22], [23], justification of averaging method [24], [25]. In this paper, we contribute by a result for impulsive differential equations with non-fixed moments of jumps with a focus on less conservative sufficient Lyapunov-like conditions for global asymptotic stability of the trivial solution compared to [26]–[29].

The remainder of the paper is organized as follows. In Section 2, we consider a system of impulsive differential equations with multiple impulsive time sequences and introduce the concept of a candidate Lyapunov function with multiple rate functions which characterize its behaviour

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during flows and jumps. Next we provide Lyapunov-like sufficient conditions equipped with a novel dwell-time condition ensuring global asymptotic stability. In Section 3, we discuss impulsive sequence decomposition technique for impulsive systems with fixed moments of jumps and compare our result with previously known ones. A short conclusion and discussion complete the paper.

II. MAIN RESULTS

In this section, we consider impulsive system with non-fixed moments of jumps which is an appropriate model for networked system with communication constraints. Since the moments of data transmission in NCSs may occur completely independently for each subsystem, we consider impulsive system with multiple impulsive time sequences and distinct jump map for every impulsive time sequence [26], [29], [30].

Let the equations $\Phi_j(t,x)=0$, $j=1,\dots,n$ define a set of smooth surfaces $\mathcal{M}_j$ in the extended phase space $\mathbb{R}^{N+1}$ that do not intersect:

$$\mathcal{M}_j=\{(t,x)\in\mathbb{R}\times\mathbb{R}^N:\Phi_j(t,x)=0\},\quad j=1,\dots,n.$$

We assume that the equations $\Phi_j(t,x)=0$ can be solved w.r.t. t and may have infinitely many solutions (e.g. impulsive system may undergo infinitely many impulsive jumps over an infinite interval of time). Let us denote these solutions as $t=\tau_j^i(x)$ and number them with a set of integers (or its subset) in such a way that $\tau_j^i(x)\to\infty$ when $i\to\infty$ and $\tau_j^i(x)\to-\infty$ when $i\to-\infty$. Consider the following system

$$\begin{array}{rcll}\dot{x}&=&f(x),&t\neq\tau_j^i(x),\\ x^+&=&g_j(x),&t=\tau_j^i(x),\quad j=1,\dots,n,\end{array}\tag{1}$$

where $t\in\mathbb{R}$, $x(t)\in X\subset\mathbb{R}^N$ is a state at time t, functions $f,g_j:X\to X$ are Lipschitz continuous and continuous w.r.t. x respectively; $f(0)=g_1(0)=\ldots=g_n(0)=0$. Constant $n\in\mathbb{N}$ stands for the number of different impulsive sequences. Impulsive perturbations occur when the integral curve of differential equation from (1) meets one of the surfaces $\mathcal{M}_j$. A behaviour of a solution to system (1) can be described in the following way: starting at (t_0,x_0), the point moves along the integral curve of differential equation from (1) until the moment t_1, where it meets a surface defined by function Φ_j, e.g. $\Phi_j(t_1,x_1)=0$ for some particular $j\in\{1,\dots,n\}$. At this moment, the point (t_1,x_1) is being instantly transferred to a new position $x_1^+:=g_j(t_1,x_1)$ and proceeds its evolution along the integral curve of $\dot{x}=f(x)$ with initial data (t_1,x_1^+) until the next intersection with the surfaces defined by Φ_k, for some particular $k\in\{1,\dots,n\}$, and so on. The state x is assumed to be left-continuous, and to have right limits at all times: $x(t+0)=\lim\limits_{s\to t+0}x(s)$. Throughout the paper, we assume that the integral curve of (1) intersects any surface $t=\tau_j^i(x)$ not more than once.

Remark 1: The class of impulsive systems with fixed moments of jumps falls into the class of equations (1). For this purpose impulsive surfaces $\mathcal{M}_j$, $j=1,\dots,n$ have to be defined as

$$\mathcal{M}_j=\{(t,x)\in\mathbb{R}\times\mathbb{R}^N:t=\tau_j^i\in\mathbb{R},i\in\mathbb{Z}\},j=1,\dots,n,$$

where $\{\tau_j^i\}_{i\in\mathbb{Z}}$, $j=1,\dots,n$ are some predefined strictly increasing sequences of points from $\mathbb{R}$.

The questions of the existence of solutions to (1) and their local stability properties for the case of a single surface

$$\mathcal{M}=\{(t,x)\in\mathbb{R}\times\mathbb{R}^N:\Phi(t,x)=0\}$$

have been studied in [15]. In this section, we aim to develop sufficient Lyapunov-like conditions for global asymptotic stability of the trivial solution to (1) in the case of several surfaces $\mathcal{M}_j$, $j=1,\dots,n$.

For a given point $x_0\in X$ let $x=\phi(t;t_0,x_0)$ be a solution to Cauchy problem (1) with initial condition $x(t_0)=x_0$. The assumptions on functions f,g_j, together with

$$\inf_i\left(\inf_{x\in X}\tau_j^{i+1}(x)-\sup_{x\in X}\tau_j^i(x)\right)\geq\theta_j>0\quad\forall j=1,\dots,n$$

guarantee the existence of solution for any $x_0\in X$ [15].

To introduce appropriate notions of stability and a candidate Lyapunov function, we recall the following standard definitions of comparison functions: a function $\alpha:[0,\infty)\to[0,\infty)$ is of class $\mathcal{K}$, and we write $\alpha\in\mathcal{K}$, when α is continuous, strictly increasing, and $\alpha(0)=0$. If α is also unbounded, then we say it is of class $\mathcal{K}_\infty$, and we write $\alpha\in\mathcal{K}_\infty$. A continuous function $\beta:[0,\infty)\times[0,\infty)\to[0,\infty)$ is of class $\mathcal{KL}$, and we write $\beta\in\mathcal{KL}$, when $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each fixed $t\geq0$, and $\beta(r,t)$ decreases to 0 as $t\to\infty$ for each fixed $r\geq0$. A function $\rho:[0,\infty)\to[0,\infty)$ is positive definite, and we write $\rho\in\mathcal{P}$, if $\rho(s)>0$ for all $s>0$ and $\rho(0)=0$.

Definition 1: System (1) is called globally asymptotically stable (GAS) if there exists function $\beta\in\mathcal{KL}$ such that for any initial point $x_0\in X$ the corresponding solution $x=\phi(t;t_0,x_0)$ exists for all $t\geq t_0$ and satisfies

$$|\phi(t;t_0,x_0)|\leq\beta(|x_0|,t-t_0)\quad\forall t\geq t_0.\tag{2}$$

Definition 2: A continuous function $V:X\to[0,\infty)$, $0\in\operatorname{int}(X)$ is called a candidate Lyapunov function for system (1) if $\exists\alpha_1,\alpha_2\in\mathcal{K}_\infty$, such that

$$\alpha_1(|x|)\leq V(x)\leq\alpha_2(|x|)\quad\forall x\in X\tag{3}$$

holds and $\exists\varphi\in\mathcal{P}$ and $\exists\psi_j\in\mathcal{K}_\infty$, $j=1,\dots,n$ such that $\forall x\in X$ it holds that

$$\dot{V}(x)\leq-\varphi(V(x)),\tag{4}$$

$$V(g_j(x))\leq\psi_j(V(x)).\tag{5}$$

The Dini derivative in (4) is defined as

$$\dot{V}(x)=\lim_{t\to+0}\frac{1}{t}\left(V(\phi_c(t;0,x))-V(x)\right),$$

where ϕ_c is a transition map that corresponds to the continuous part of the system (1), i.e. $\phi_c(t;0,x)$ is the state of the system (1) at time t if the state at time $t_0:=0$ was x and no impulses occur.

We will characterize global stability property of system (1) in terms of a candidate Lyapunov function equipped with an appropriate dwell-time condition. Every pair of functions g_j and ψ_j corresponds to the appropriate impulsive time sequence $\{\tau_j^i(x)\}_{i=-\infty}^{\infty}$. Such separate treatment of impulsive jumps enables to establish less conservative sufficient conditions for stability comparing to conventional approaches [15], [28], which use a single function ψ to characterize the behaviour of a candidate Lyapunov function on jumps. Moreover, in a closely related class of hybrid dynamical systems, the dwell-time conditions ensuring global asymptotic stability have been developed only for the cases of linear functions φ and ψ (see e.g. [31, Proposition 1]). The following is the main result of the paper.

Theorem 1: Let for system (1) there exist a candidate Lyapunov function V with the corresponding rate functions $\varphi \in \mathcal{P}$ and $\psi_j \in \mathcal{K}_\infty$, $j = 1, \ldots, n$ and

$$\inf_i \left(\inf_{x \in X} \tau_j^{i+1}(x) - \sup_{x \in X} \tau_j^i(x) \right) \geq \theta_j > 0 \quad \forall j = 1, \ldots, n \tag{6}$$

hold. If functions $\varphi \in \mathcal{P}$, $\psi_j \in \mathcal{K}_\infty$ and constants θ_j, $j = 1, \ldots, n$ are such that for some $\gamma > 0$

$$\int_a^{\psi_j(a)} \frac{ds}{\varphi(s)} \leq \frac{1}{n} \min_{j=1,\ldots,n} \theta_j - \gamma \quad \forall j = 1, \ldots, n \tag{7}$$

holds for any $a > 0$, then system (1) is GAS.

Proof: Fix any $\varepsilon > 0$ and let $l = \inf_{|x| \geq \varepsilon} V(x)$. Then pick $\delta = \delta(\varepsilon)$ small enough that

$$m = \sup_{|x| < \delta} V(x) < l$$

and let $x(\cdot) := \phi(\cdot; t_0, x_0)$, $x_0 \in J_\delta$ be any solution to (1) that starts in the δ-neighbourhood of the origin J_δ.

First, we show that this solution remains inside the ball J_δ all the time it moves along the flow (along the trajectory of the differential equation from (1)). Let $v(t) := V(x(t))$. It is sufficient to show that $v(t) < l$ $\forall t \geq 0$. Suppose the opposite: Let $x(t)$ leave the ball J_δ at some time t^δ. Then $v(t^\delta) = V(x(t^\delta)) \geq l$. But from (4), function v is not increasing while $x(t) \in \bar{J}_\delta$. Hence, $v(t^\delta) \leq m < l$. So we got a contradiction.

Next, we study the behaviour of solutions under impulsive perturbations. Let $x(t)$ meet the surface $t = \tau_{j_1}^1(x)$ at the point $x_1 \in J_\delta \subset X$. From (4), for the times $t \in [t_0, \tau_{j_1}^1(x_1)]$ the following inequality $\dot{v}(t) \leq -\varphi(v(t))$ holds. From here by integrating we get

$$\int_{t_0}^{\tau_{j_1}^1(x_1)} \frac{dv(t)}{-\varphi(v(t))} \geq \tau_{j_1}^1(x_1) - t_0.$$

Substituting $v(t) = s$ we obtain

$$\int_{v(\tau_{j_1}^1(x_1))}^{v(t_0)} \frac{ds}{\varphi(s)} \geq \tau_{j_1}^1(x_1) - t_0. \tag{8}$$

At the point x_1 the solution is being instantly transferred to a new position $x_1^+ = x(\tau_{j_1}^1(x_1) + 0)$ and prolonged until the next hit of some surface $t = \tau_{j_2}^1(x)$ at the point x_2. In a similar way, we derive that

$$\int_{v(\tau_{j_2}^1(x_2))}^{v(\tau_{j_1}^1(x_1)+0)} \frac{ds}{\varphi(s)} \geq \tau_{j_2}^1(x_2) - \tau_{j_1}^1(x_1). \tag{9}$$

From (5) it follows that

$$\int_{v(\tau_{j_2}^1(x_2))}^{\psi_{j_1}(v(\tau_{j_1}^1(x_1)))} \frac{ds}{\varphi(s)} \geq \tau_{j_2}^1(x_2) - \tau_{j_1}^1(x_1). \tag{10}$$

Considering the time interval $[t_0, t^*]$ with

$$t^* = t_0 + \min_{j=1,\ldots,n} \theta_j$$

we are assured that the solution to system (1) can meet each surface $\mathcal{M}_j$, $j = 1, \ldots, n$ at most once in this interval. Hence, the total number of impulses over the interval $[t_0, t^*]$ is $k \leq n$:

$$\int_{v(\tau_{j_k}^1(x_k))}^{\psi_{j_{k-1}}(v(\tau_{j_{k-1}}^1(x_{k-1})))} \frac{ds}{\varphi(s)} \geq \tau_{j_k}^1(x_k) - \tau_{j_{k-1}}^1(x_{k-1}), \tag{11}$$

$$\int_{v(t^*)}^{\psi_{j_k}(v(\tau_{j_k}^1(x_k)))} \frac{ds}{\varphi(s)} \geq t^* - \tau_{j_k}^1(x_k). \tag{12}$$

By summation of (8) – (12) we get

$$\int_{v(\tau_{j_1}^1(x_1))}^{v(t_0)} \frac{ds}{\varphi(s)} + \int_{v(\tau_{j_2}^1(x_2))}^{\psi_{j_1}(v(\tau_{j_1}^1(x_1)))} \frac{ds}{\varphi(s)} + \ldots$$

$$+ \int_{v(\tau_{j_k}^1(x_k))}^{\psi_{j_{k-1}}(v(\tau_{j_{k-1}}^1(x_{k-1})))} \frac{ds}{\varphi(s)} + \int_{v(t^*)}^{\psi_{j_k}(v(\tau_{j_k}^1(x_k)))} \frac{ds}{\varphi(s)} \geq t^* - t_0.$$

Denoting $\theta := \min_{j=1,\ldots,n} \theta_j$, the last inequality can be rewritten as

$$\int_{v(t^*)}^{v(t_0)} \frac{ds}{\varphi(s)} + \int_{v(\tau_{j_k}^1(x_k))}^{\psi_{j_k}(v(\tau_{j_k}^1(x_k)))} \frac{ds}{\varphi(s)} +$$

$$\int_{v(\tau_{j_{k-1}}^1(x_{k-1}))}^{\psi_{j_{k-1}}(v(\tau_{j_{k-1}}^1(x_{k-1})))} \frac{ds}{\varphi(s)} + \ldots + \int_{v(\tau_{j_1}^1(x_1))}^{\psi_{j_1}(v(\tau_{j_1}^1(x_1)))} \frac{ds}{\varphi(s)} \geq \theta.$$

From the dwell-time condition (7) it follows then that

$$\int_{v(t^*)}^{v(t_0)} \frac{ds}{\varphi(s)} + \frac{k}{n}\theta - k\gamma \geq \theta.$$

Finally,

$$\int\limits_{v(t^*)}^{v(t_0)} \frac{ds}{\varphi(s)} \geq \theta - \frac{k}{n}\theta + k\gamma := \xi,$$

where ξ is a positive number. It means that $v(t^*) < v(t_0)$. Continuing the above described process one may build an infinite sequence of time-points $t_0 := t_0^* < t^* =: t_1^* < t_2^* < \ldots$ such that the corresponding values of the candidate Lyapunov function satisfy $v(t_0) > v(t_1^*) > v(t_2^*) > \ldots$ and

$$\int\limits_{v(t_{i+1}^*)}^{v(t_i^*)} \frac{ds}{\varphi(s)} \geq \xi \quad \forall i \in \mathbb{N} \cup \{0\}.$$

Since $v(t)$ is bounded by zero from below, the sequence $\{v(t_i^*)\}_{i=0}^{\infty}$ converges. Let us prove that it converges to zero. Suppose the opposite: Let

$$\lim_{i\to\infty} v(t_i^*) = a > 0 \quad \text{and} \quad c = \min_{a \geq s \geq v(t_0)} \varphi(s).$$

Then

$$\xi \leq \int\limits_{v(t_{i+1}^*)}^{v(t_i^*)} \frac{ds}{\varphi(s)} \leq \frac{1}{c}\left(v(t_i^*) - v(t_{i+1}^*)\right) \Rightarrow$$

$$v(t_i^*) - v(t_{i+1}^*) \geq \xi c = \text{const} \quad \forall i \in \mathbb{N} \cup \{0\}.$$

Hence

$$v(t_{i+2}^*) \leq v(t_{i+1}^*) - \xi c \leq v(t_i^*) - 2\xi c.$$

This means that for a sufficiently large index j the corresponding value $v(t_j^*)$ becomes negative, which contradicts the positive definitiveness of the candidate Lyapunov function. Hence, $\lim\limits_{i\to\infty} v(t_i^*) = 0$.

The fact of convergence of the sequence $\{v(t_i^*)\}_{i=0}^{\infty}$ does not readily imply the convergence of the corresponding solution $x(\cdot)$ to zero. Because of possible impulses during time interval $(t_i^*, t_{i+1}^*]$, there may exist a sequence of points $\{\hat{t}_i^*\}_{i=0}^{\infty}$ with $\hat{t}_i^* \in (t_i^*, t_{i+1}^*]$, $i \in \mathbb{N} \cup \{0\}$ such that

$$\lim_{i\to\infty} \hat{t}_i^* = \infty, \quad \lim_{i\to\infty} v(\hat{t}_i^*) \neq 0$$

(see Figure 1 for clarification). However, the monotonicity property of functions $\psi_j \in \mathcal{K}_\infty$ can be used to prove the asymptotic stability of the origin (note that for the case of impulsive systems with fixed moments of jumps and a single jump map it is sufficient to require $\psi \in \mathcal{P}$ [28] to guarantee GAS). Function β from Definition 1 can be constructed as follows.

For any number $s > 0$ define the function $\Psi : [0,\infty) \to [0,\infty)$ as follows

$$\Psi(s) = \max_{\substack{j_1,\ldots,j_n \in 1,\ldots,n \\ j_i \neq j_k, j \neq k}} \tilde{\psi}_{j_1}(\tilde{\psi}_{j_2}(\ldots(\tilde{\psi}_{j_n}(s))\ldots),$$

where $\tilde{\psi}_j(s) = \max\{s, \psi_j(s)\}$ $\forall s \in [0,\infty)$. Since all the functions $\psi_j \in \mathcal{K}_\infty$, $j = 1,\ldots,n$, the corresponding value of the candidate Lyapunov function is bounded by

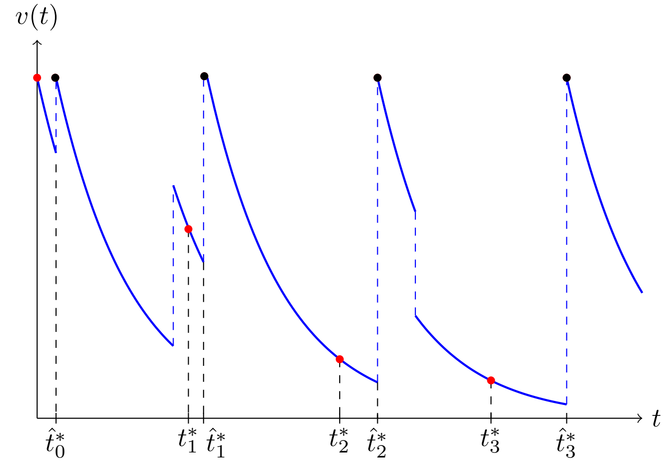

Fig. 1. The sequence $\{v(t_i^*)\}_{i=0}^{\infty}$ converges to zero (red dots), but the sequence $\{v(\hat{t}_i^*)\}_{i=0}^{\infty}$ may not converge to zero (black dots).

$v(t) \leq \Psi(v(t_i^*))$ for any $t \in [t_i^*, t_{i+1}^*]$, $i \in \mathbb{N} \cup \{0\}$. Here we have used the worst-case approach and assumed that all the impulses occur at the very beginning of the interval $[t_i^*, t_{i+1}^*]$, where the value of the candidate Lyapunov function is the largest. Since Ψ is monotonically increasing and the sequence $\{v(t_i^*)\}_{i=0}^{\infty}$ is monotonically decreasing and converges to zero, the sequence $\{\Psi(v(t_i^*))\}_{i=0}^{\infty}$ is also monotone and converges to zero. Utilizing the property of radially unboundedness of the candidate Lyapunov function (3), the piecewise linear function

$$\tilde{\beta}(r,t) = \begin{cases} -\frac{\eta}{t_1^* - t_0} t + \alpha_1^{-1}(\Psi(\alpha_2(r))) + \eta \\ \text{for} \quad t \in [0, t_1^* - t_0], \\ \frac{\alpha_1^{-1}(\Psi(v(t_1^*))) - \alpha_1^{-1}(\Psi(\alpha_2(r)))}{t_2^* - t_1^*}(t + t_0) + \\ \frac{\alpha_1^{-1}(\Psi(\alpha_2(r)))t_2^* - \alpha_1^{-1}(\Psi(v(t_1^*)))t_1^*}{t_2^* - t_1^*} \\ \text{for} \quad t \in (t_1^* - t_0, t_2^* - t_0], \\ \frac{\alpha_1^{-1}(\Psi(v(t_i^*))) - \alpha_1^{-1}(\Psi(v(t_{i-1}^*)))}{t_{i+1}^* - t_i^*}(t + t_0) + \\ \frac{\alpha_1^{-1}(\alpha_1^{-1}(\Psi(v(t_{i-1}^*)))t_{i+1}^* - \alpha_1^{-1}(\Psi(v(t_i^*)))t_i^*}{t_{i+1}^* - t_i^*} \\ \text{for} \quad t \in (t_i^*, t_{i+1}^*], \quad i = 2, \ldots \end{cases}$$

is a bound for the norm of solution

$$|\phi(t; t_0, x_0)| \leq \tilde{\beta}(|x_0|, t - t_0) \quad \forall t \geq t_0 \tag{13}$$

for arbitrary $\eta > 0$. Constant η is needed to estimate the norm of the solution on the 'first' time interval $[0, t_1^* - t_0]$ only. The estimate (13) is nonuniform since function $\tilde{\beta}$ depends on a particular solution to the system (1). Similarly to [28], one may check that due to (4),

$$v(t_i^*) \leq F^{-1}(F(v(t_0)) - i\xi) \quad \forall i = 1, \ldots, \hat{i}, \tag{14}$$

where F is a continuous strictly increasing function defined by $F(q) := \int\limits_r^q \frac{ds}{\varphi(s)}$, $q \in (0,\infty)$ for some fixed $r > 0$; the inverse F^{-1} in (14) exists for indices $i \in \{1,\ldots,\hat{i}\}$, where

$\hat{i}$ can be a natural number of ∞. Finally, function β from Definition 1 can be constructed as follows:

$$\beta(r,t) = \begin{cases} -\frac{\eta}{\theta}t + \alpha_1^{-1}(\Psi(\alpha_2(r))) + \eta \quad \text{for} \quad t \in [0,\theta], \\ \frac{t}{\theta}(\alpha_1^{-1}(\Psi(F^{-1}(F(\alpha_2(r)) - i\xi))) \\ -\alpha_1^{-1}(\Psi(F^{-1}(F(\alpha_2(r)) - (i-1)\xi)))) \\ +(i+1)\alpha_1^{-1}(\Psi(F^{-1}(F(\alpha_2(r)) - (i-1)\xi))) \\ -i\alpha_1^{-1}(\Psi(F^{-1}(F(\alpha_2(r)) - i\xi))) \\ \text{for} \quad t \in (i\theta, (i+1)\theta], \quad i = 1, \ldots, \hat{i}, \\ \beta(r, (\hat{i}+1)\theta)e^{-(t-(\hat{i}+1)\theta)} \text{ for } t \in ((\hat{i}+1)\theta, \infty). \end{cases}$$

The last case is needed if only $\hat{i} < \infty$. This completes the proof. ∎

Remark 2: The constant n in dwell-time condition (7) may be substituted with $m \leq n$, where m denotes the number of "bad" impulsive jump maps, i.e. impulses for which the corresponding rate function $\psi_j(s) > s$ for some $s \in (0,\infty)$. If $m = 0$, i.e. all jump maps are "good", then dwell-time condition (7) is always satisfied.

Remark 3: Even for the case of fixed predefined moments of impulsive perturbations, Theorem 1 provides a new result and generalizes Theorem 1 from [26], where the exponential candidate Lyapunov function has been used.

In the same fashion as the proof of Theorem 1, one may prove stability theorem for a system with possible unstable continuous dynamics and stabilizing impulses:

Theorem 2: Let there exist a continuous function $V : X \to [0,\infty)$, $0 \in \text{int}(X)$ such that

- $\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty$, such that
$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \forall x \in X;$$
- $\exists \varphi \in \mathcal{P}$ and $\exists \psi_j \in \mathcal{K}_\infty$, $j = 1, \ldots, n$ such that $\forall x \in X$ it holds that
$$\dot{V}(x) \leq \varphi(V(x)),$$
$$V(g_j(x)) \leq \psi_j(V(x)),$$

and constants $\theta_j, \tilde{\theta}_j > 0$, $j = 1, \ldots, n$ such that (6) and

$$\sup_i \left(\sup_{x \in X} \tau_j^i(x) - \inf_{x \in X} \tau_j^{i-1}(x) \right) \leq \tilde{\theta}_j \quad \forall j = 1, \ldots, n$$

hold. If for some $\gamma > 0$ the dwell-time condition

$$\int_{\psi_j(a)}^{a} \frac{ds}{\varphi(s)} \geq \frac{1}{n} \max_{j=1,\ldots,n} \tilde{\theta}_j + \gamma \quad \forall j = 1, \ldots, n$$

holds for any $a > 0$, then system (1) is GAS.

III. IMPULSIVE TIME SEQUENCE DECOMPOSITION

For the case of impulsive differential equation with fixed moments of jumps and single impulsive time sequence ($n = 1$), dwell-time condition (7) coincides with the one from [28] and reads as follows

$$\int_a^{\psi(a)} \frac{ds}{\varphi(s)} \leq \theta - \gamma, \tag{15}$$

where θ denotes the minimal distance between two consequent impulses. However, our approach provides an opportunity for impulsive time sequence decomposition into an appropriate number of impulsive subsequences such that the dwell-time condition (7) relaxes the condition (15). We will illustrate this on the following example:

Example 1: Consider the impulsive differential equation with the fixed moments of jumps at $\mathcal{T} = \{0.1, 0.15, 0.25, 0.3, 0.4, 0.45, 0.55, \ldots\}$

$$\begin{array}{rcll} \dot{x} & = & -x - x^3, & t \notin \mathcal{T}, \\ x^+ & = & x^{\frac{3}{2}}, & t \in \mathcal{T}. \end{array} \tag{16}$$

The minimal distance between two consecutive impulses $\theta = 0.05$. We pick radially unbounded function $V(x) = x^2$ as a candidate Lyapunov function. Now, let us check the conditions (4) and (5):

$$\dot{V}(x) = 2x(-x - x^3) = -2(V(x) + V^2(x)) \quad \Rightarrow$$
$$\varphi(s) = 2(s + s^2),$$
$$V(g(x)) = V(x^{\frac{3}{2}}) = x^3 = V^{\frac{3}{2}}(x) \quad \Rightarrow \quad \psi(s) = s^{\frac{3}{2}}.$$

We try to check global asymptotic stability of system (16) using dwell-time condition (15)

$$\int_a^{\sqrt{a^3}} \frac{ds}{2(s+s^2)} = \frac{1}{2} \int_a^{\sqrt{a^3}} \frac{1}{s} - \frac{1}{s+1} ds = \frac{1}{2} \ln \frac{\sqrt{a^3} + \sqrt{a}}{\sqrt{a^3} + 1}.$$

Since the global maximum

$$\max_{a \in [0,\infty)} \frac{1}{2} \ln \frac{\sqrt{a^3} + \sqrt{a}}{\sqrt{a^3} + 1} \approx 0,056 \tag{17}$$

is larger than $\theta = 0.05$ we cannot conclude on global asymptotic stability of the origin. Also, we cannot employ averaged-type dwell-time conditions (e.g. [27, Corollary 1]) since the rate functions φ and ψ are nonlinear.

However, impulsive time sequence $\mathcal{T}$ has a special structure and can be decomposed into two time sequences

$$\mathcal{T}_1 = \{t \in [0,\infty) : t = 0.1 + 0.15i, i \in \mathbb{N} \cup \{0\}\}$$

and

$$\mathcal{T}_2 = \{t \in [0,\infty) : t = 0.15i, i \in \mathbb{N}\}$$

correspondingly. The minimal distances between two consecutive impulses within each sequence are $\theta_1 = \theta_2 = 0.15$. The jump maps for these time sequences are the same

$$g_1(x) = g_2(x) = x^{\frac{3}{2}}.$$

Let us check the newly established dwell-time condition (7) from Theorem 1:

$$\frac{1}{2} \ln \frac{\sqrt{a^3} + \sqrt{a}}{\sqrt{a^3} + 1} \leq \frac{1}{2} \min\{0.15, 0.15\} - \gamma. \tag{18}$$

From (17) it follows that the inequality (18) holds for $\gamma \in (0, 0.019)$. Hence, system (16) is GAS. A solution to system (16) is depicted on Figure 2.

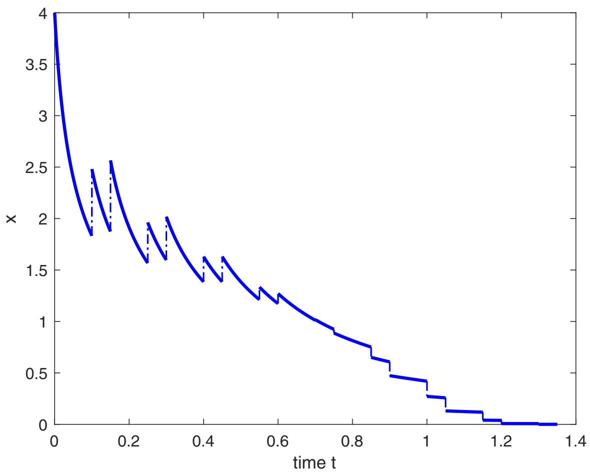

Fig. 2. A plot of a solution to system (16) with initial condition $x(0) = 4$. Dashed vertical lines denote instantaneous jumps.

IV. CONCLUSION AND OUTLOOK

The main contribution of the paper is new Lyapunov-like theorems equipped with novel dwell-time conditions that balance stable and unstable dynamics of the system to guarantee global asymptotic stability of the trivial solution. These theorems generalize the previously known results in two directions. First, our results handle the case of non-fixed state-dependent impulses on contrary to the papers [26]–[29] which study impulsive systems with fixed moments of jumps. Second, we extend the notion of a candidate Lyapunov function with multiple rate coefficients to describe discrete dynamics (proposed in [29] for the investigation of the input-to-state stability property) to the case of nonlinear rate functions. This allows to derive less conservative conditions for global asymptotic stability of the origin even for the case of fixed moments of jumps.

The further problems of our interest are to extend the proposed approach for studying the input-to-state stability property of impulsive systems with non-fixed moments of jumps and to prove small-gain type theorem for interconnection of this kind of systems. Another interesting challenge is to try to relax the requirement on rate functions ψ_j to be from the class $\mathcal{K}_\infty$ in Theorem 1.

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