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Output regulation for redundant plants via orthogonal moments

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Abstract— **In this paper we address the output regulation problem for redundant plants, namely systems possessing more control inputs than regulated outputs, with focus on the specific *opportunity* and *difficulty* deriving from the additional inputs. The former consists in the fact that, having a wealth of input configurations achieving the same steady-state behavior, it is possible to optimize additional performance criteria while preserving the primary task of output regulation. The latter stems from the fact that the *naive* approach of replicating the required internal model of the exosystem on each input channel leads to loss of observability/detectability of the cascaded interconnection of the internal model and the plant, thus preventing the achievement of overall closed-loop stability. The main result of this paper consists in the design of an inner auxiliary control loop that allows to break the above conflict between advantages and drawbacks of redundant plants. The result is then revisited by exploiting the *orthogonal moments* of the plant at the exosystem's frequencies. Differently from classic *moments*, which, for an asymptotically stable plant, describe the relation between inputs and steady-state output response, *orthogonal moments* characterize the input directions that yield zero steady-state output response at given frequencies.**

I. Introduction

Output regulation is the problem of ensuring that certain outputs of a plant converge to a desired reference, despite the influence of unmeasured disturbances and unknown initial conditions, see *e.g.* [1], [2], [3] for detailed discussions on the topic in the case of linear systems. Compared to the plain stabilization task, the fascinating feature of output regulation consists in the fact that complex steady-state evolutions are considered, as induced by the exogenous signals (the reference and disturbance, which are supposed to be generated by a known exosystem, with unknown initial state). Classic strategies to tackle the output regulation problem, specifically for SISO plants, typically hinge upon the use of the so-called *Internal Model*, which is a copy of the dynamics contained in the exosystem, that must be necessarily included in the closed-loop system. Several alternative approaches have been envisioned in recent years for linear and nonlinear systems, see *e.g.* [4], [5] and references therein. In [5], [6], [7], *external models*, compared to *internal models*, are studied. In particular, while the latter are included in the overall control scheme and a controller is subsequently designed to asymptotically stabilize the cascaded interconnection of the internal model and the plant, the former are interconnected (open-loop) in cascade to a pre-stabilized plant and their internal state is *discretely* updated (slowly enough) to determine the state that generates the correct steady-state input.

In the case of *square* MIMO systems, namely multi-input/multi-output plants possessing the same number of regulated outputs and controlled inputs, the above strategies may be straightforwardly extended provided the mentioned copy of the exosystem is replicated for each of the input channels. However, it has been immediately recognized [8], [9], [10], [11], [12], [13], [14], [15], [16] that considering *redundant* MIMO plants, namely systems possessing more control inputs than regulated outputs, simultaneously gives rise to specific *opportunity* and *difficulty* deriving from the presence of additional inputs, as mentioned in the abstract. As an interesting insight on a similar scenario, it must be pointed out that it has been recently showed [17], [18] that the presence of additional inputs, with respect to the regulated output, is in fact necessary in the setting of regulation for hybrid systems in the presence of time-driven jumps in order to define a well-posed problem.

The main contribution of this paper consists in envisioning few control architectures that allow to break the conflicting situation without renouncing to any of the interesting properties deriving from the additional inputs. All the proposed designs start with introducing a *complete* full-order copy of the internal models, namely one copy of the exosystem for each of the input channels, thus necessarily leading to loss of observability/detectability of the cascaded interconnection. This construction is particularly appealing for instance towards distributed and *decentralized* regulation or synchronization, in which it is typically imposed that each agent possesses its own local copy of the exosystem. Then, the difficulties naturally arising from the above desirable choice are circumvented by considering an additional control loop involving only the internal model unit; such control loop can take either the form of a mixed feedback/feedforward scheme, without any additional requirements, or the form of a purely feedback interconnection, provided certain minimum-phase assumptions are satisfied.

The rest of the paper is organized as follows: in Section II, the problem of interest is defined and a few relevant preliminary facts are highlighted; Section III presents the main results and a discussion of several features of the proposed solution, including a comparison with related results on input allocation; Section IV highlights the role of orthogonal moments and provides a related result, tailored at specific frequencies; finally, Section V presents numerical results showing the desirable performance achievable by the proposed method.

II. Problem definition and preliminaries

Consider the Linear Time invariant (LTI) plant $\mathcal{P}$ described by the equations

$$\dot{x} = Ax + Bu + Pw\,, \tag{1a}$$

$$e = Cx + Du + Qw\,, \tag{1b}$$

with $x \in \mathbb{R}^n$ denoting the state of the system, $u \in \mathbb{R}^m$ the control input, $e \in \mathbb{R}^p$ the regulated output and $w \in \mathbb{R}^s$ an exogenous input containing both references to be tracked and unmeasured disturbances to be rejected. The signal w is

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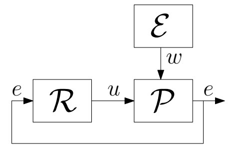

Fig. 1: The closed-loop system formed by the exosystem $\mathcal{E}$, the plant $\mathcal{P}$ and the regulator $\mathcal{R}$.

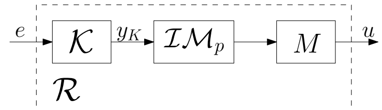

Fig. 2: The standard structure of the regulator $\mathcal{R}$ for the case $m > p$, with squaring down matrix M, the p-copy internal model $\mathcal{IM}_p$ and the stabilizer $\mathcal{K}$.

generated by an exosystem $\mathcal{E}$ described by the equations

$$\dot{w} = Sw\,. \tag{2}$$

For brevity, only the case in which the exosystem generates *constant* references or disturbances is considered in this paper, so that the following assumption holds.

Assumption 1: The exosystem (2) is characterized by the matrix $S = 0_{s\times s}$. $\circ$

The control task hinges upon the design of a regulator $\mathcal{R}$

$$\dot{x}_r = A_r x_r + B_c u_r, \tag{3a}$$

$$y_r = C_r x_c + D_r u_r, \tag{3b}$$

where $x_r \in \mathbb{R}^{n_r}$, $u_r \in \mathbb{R}^p$ and $y_r \in \mathbb{R}^m$ denote the state, the input and the output of the controller, respectively.

The following assumption is stated mainly in order to ensure *well-posedness* of the control task under investigation.

Assumption 2: $\mathcal{P}$ is *stabilizable* and *detectable*, *i.e.*

$$\operatorname{rank}\begin{bmatrix} \lambda I - A & B \end{bmatrix} = n, \quad \forall \lambda \in \sigma(A), \operatorname{re}(\lambda) \geq 0\,, \tag{4a}$$

$$\operatorname{rank}\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n, \quad \forall \lambda \in \sigma(A), \operatorname{re}(\lambda) \geq 0\,. \tag{4b}$$

Moreover, the following *non-resonance condition*

$$\operatorname{rank}\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = n + p, \quad \forall \lambda \in \sigma(S)\,, \tag{5}$$

(also known as *Davison condition*) holds. $\circ$

The non-resonance condition (5) implies, on one hand, that $m \geq p$, while ensuring, on the other hand, *solvability* of the output regulation problem. The formal statement characterizing the problem of interest is contained in the following definition (see also Fig. 1).

Problem 1: Consider the plant $\mathcal{P}$ as in (1) together with the exosystem $\mathcal{E}$ as in (2), and suppose that Assumptions 1 and 2 hold. Determine, if possible, a *regulator* as in (3) interconnected to $\mathcal{P}$ according to

$$u_r = e, \quad u = y_r, \tag{6}$$

such that

(R0) the closed-loop system (1), (3), (6) with $w(t) = 0$ for all $t \geq 0$, has a globally asymptotically stable (GAS) equilibrium point at the origin;

(R1) for any $w(0) \in \mathbb{R}^s$, the overall closed-loop system (1), (2), (3) admits a unique constant globally asymptotically stable motion with steady-state output equal to zero;

(R2) the cost function

$$J(x_{ss}, u_{ss}) = x_{ss}' Q_o x_{ss} + u_{ss}' R u_{ss}, \tag{7}$$

is minimized for the closed-loop system. $\diamond$

Note that the possibility of considering the objective hinted at in item *(R2)*, in addition to the tasks in items *(R0)* and *(R1)*, stems precisely from the fact that we are considering *redundant plants*, for which output regulation problems typically admit non-unique solutions. Solvability of the classical formulation of the output regulation problem, namely conditions *(R0)* and *(R1)*, requires that the plant $\mathcal{P}$ possesses at least the same number of controlled inputs, m, as that of the regulated outputs, p, namely $m \geq p$. Under very mild assumptions, a constructive solution to the regulation task is then provided by introducing p copies, *i.e.* as many as the number of outputs, of the internal model replicating the modes of the exosystem $\mathcal{E}$ in $\mathcal{R}$. However, if $m > p$ additional care must be taken towards this design strategy. In the following subsections, two somewhat alternative approaches are suggested and their advantages as well as intrinsic drawbacks are discussed.

A. Squaring-down approach

The *squaring-down approach* is inspired by the design in the case of square plants: it is typically capable of satisfying items *(R0)* and *(R1)*, but the optimality requirement in *(R2)* is not ensured in general. More precisely, whenever $m = p$, and provided Assumption 2 is verified, the correct steady-state input achieving output regulation is *uniquely* determined, as entailed also by the existence of a unique solution Π, Γ to the well-known *Francis Equations* $\Pi S = A\Pi + B\Gamma + P$, $0 = C\Pi + Q$ that yield the full-information control law $u_{ss}(t) = \Gamma w(t)$. In an error-feedback framework, an identical control action can be obtained by introducing a copy of the internal model on each input (or, equivalently, on each output) and then subsequently adding a stabilizer to the resulting cascaded interconnection.

Similar ideas have been extended also to the case of redundant (non-square) plants, by arbitrarily selecting a constant matrix $M \in \mathbb{R}^{m\times p}$ with the property that the conditions of Assumption 2 still hold also for the modified system with B and D replaced by BM and DM, respectively (see also Fig. 2). The rationale behind the squaring-down philosophy consists in the fact that the above choice of the matrix M essentially generates an *auxiliary* output regulation problem, sharing the same properties of the original one, but stated for a square plants, in which only some of the original inputs (or a fixed linear combination of them) are employed to enforce regulation. However, as it appears evident squaring-down, hence arbitrarily limiting the abundance of control inputs with the only aim of obtaining a unique solution, implies that further optimization (*e.g.* item *(R2)*) of the steady-state inputs becomes impossible, since the squaring down gives a fixed, and unique, relation between the values of each scalar input at steady-state; for further discussion see [15], [16].

B. Naive approach

An alternative approach to the squaring-down one consists instead of feeding each input channel with an *independent* copy of the exosystem, regardless of the number of outputs;

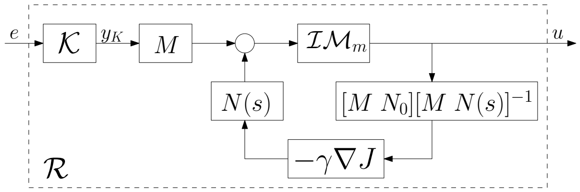

Fig. 3: The structure of the regulator $\mathcal{R}$ with feedback internal model for the case $m > p$, equivalent to the one in Fig. 4 if the inverse of $[M \;\; N(s)]$ is stable.

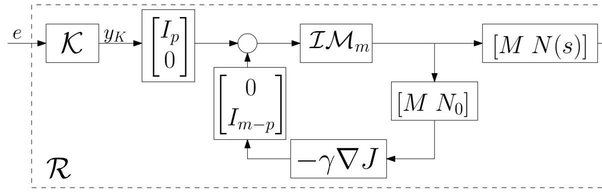

Fig. 4: The structure of the regulator $\mathcal{R}$ with feedforward-feedback internal model for the case $m > p$, with the m-copy internal model $\mathcal{IM}_m$, the annihilator $N(s)$ and its steady-state gain N_0.

this corresponds essentially to the classic construction of a *servocompensator* [2]. This strategy requires, under Assumption 1, the use of m independent single integrators connected to the plant $\mathcal{P}$, and in the case $m > p$ it inevitably leads to loss of structural properties of observability/detectability of the cascaded interconnection as shown by the application of the PBH test to the system with the stabilizer removed. This aspect renders item *(R0)* impossible to achieve. On the other hand, items *(R1)* as well as *(R2)* can be easily obtained, and in particular the fact that the achievement of *(R2)* is not impaired can be clearly seen by the fact that in the considered scheme each input can be fed with an arbitrary (hence, possibly the optimal) signal generated by a single integrator. Despite the latter desirable feature, such a *naive* approach may lead to serious consequences deriving from the absence of item *(R0)* (see, for instance, Fig. 6 in Section V for a visualization of the detrimental effects of lack of internal asymptotic stability). In fact, even when the exosystem's dynamics only generates bounded signals, it easy to show that the existence of undetectable dynamics in the cascade between the internal model and the plant may allow for responses that are invisible in the output but unbounded in the state, which *e.g.* in the context of cyber-physical systems might be exploited by a malicious attacker.

The main objective of this paper consists in proposing a systematic design that allows to suitably modify the naive approach in order to retain all the desirable features of both the naive and the squaring-down approaches, without renouncing to the spirit behind each of them.

III. Regulator design

As already mentioned, in this paper for brevity only the problem under Assumption 1 (namely, the case of constant exogenous signals w) will be dealt with, although similar results hold under more general choices for S.

A. Preliminaries

Two alternative architectures are envisioned in this section, which extend the standard squared-down regulator in Fig. 2, in order to simultaneously take items *(R0)*, *(R1)* and *(R2)* into account. Preliminarily, few main features of the control scheme in Fig. 2 must be discussed. First, $M \in \mathbb{R}^{m\times p}$, *i.e.* the *squaring-down* matrix, can be any matrix such that the conditions of Assumption 2 still hold also for the modified system with B and D replaced by BM and DM, respectively. Second, $\mathcal{IM}_p$ is a p-copy internal model of the exosystem $\mathcal{E}$, in the sense that the transfer matrix of $\mathcal{IM}_p$ consists of an identity matrix of dimension p multiplied by a scalar, strictly proper, transfer function with denominator equal to s, and clearly its numerator does not possess roots at $s = 0$. Finally, the following assumption essentially constructively characterizes and suggests the role played by the controller $\mathcal{K}$ in the scheme of Fig. 2, which will be borrowed also by the architectures proposed herein.

Assumption 3: The controller $\mathcal{K}$ in Fig. 2 is such that the cascade interconnection of $\mathcal{IM}_p$, M and plant $\mathcal{P}$ in closed-loop with $\mathcal{K}$ possesses a globally asymptotically stable equilibrium point at the origin. $\circ$

Note that, by the structure of $\mathcal{IM}_p$ and by definition of M, Assumption 3 contains rather mild requirements. In fact, the choice of the matrix M guarantees that the cascade interconnection of $\mathcal{IM}_p$, M and plant $\mathcal{P}$ is detectable and stabilizable, hence the controller $\mathcal{K}$ may be, for instance, designed, by relying on the *separation principle*, as an observer combined with a static control law assigning the eigenvalues of the closed-loop system. In addition, any *generic* (random) selection of M of rank p satisfies the required conditions.

In the following results, $\mathcal{K}$ and M are assumed to be designed as specified above, that is, as in the classic squaring-down approach: any choice of $\mathcal{K}$ and M in Fig. 2 can be extended to, at least, one of the two architectures proposed in this paper (cfr Fig. 3 and Fig. 4). The key point herein is that $\mathcal{IM}_p$ instead must be replaced by $\mathcal{IM}_m$ (having the same structure but dimension m) in order to achieve item *(R2)* of Problem 1 (see the end of the discussion in Section II-B).

B. Proposed architectures

The aim of this section consists in introducing two alternative architectures to re-design the naive approach in order to solve Problem 1. The most suggestive control scheme is the *purely feedback* architecture presented in Fig. 3. Ignoring at first the feedback loop containing $N(s)$, the scheme is a direct extension of the servocompensator in the square plant case, since it contains an internal model $\mathcal{IM}_m$ that essentially corresponds to placing m identical internal models in parallel, one for each input. As already mentioned, the cascade of $\mathcal{IM}_m$ and $\mathcal{P}$ would not be detectable, hence preventing the achievement of item *(R0)*. The additional feedback loop in Fig. 3 allows to recover *asymptotic stability* of the overall control system (once the regulator is connected to the plant), thus achieving *(R0)*, meanwhile optimizing the steady-state value of the cost criterion, thus achieving *(R2')*. The result is obtained by relying on two novel ingredients: *i)* $N(s) \in \mathbb{R}^{m\times(m-p)}$ which is an *annihilator* [13] for plant $\mathcal{P}$, namely a stable system of size $m \times (m-p)$ such that the cascade of $N(s)$ and $\mathcal{P}$ has a zero transfer matrix; *ii)* a block $-\gamma\nabla J$ where ∇J denotes the computation of suitable gradient information for the considered optimality criterion, and $\gamma \in \mathbb{R}_{>0}$ is an adjustable parameter used to tune the convergence along the gradient. The matrix N_0, also appearing in Fig. 3, is the static gain of $N(s)$. Although particularly appealing, the architecture in Fig. 3 is feasible only if the *inverse* of the transfer matrix $[M \quad N(s)]$ is stable. For this reason, a generally applicable scheme is shown in Fig. 4. In this architecture, the blocks $[I_p \quad 0]'$ and $[0 \quad I_{m-p}]'$ clearly entail that *i)* the stabilizer $\mathcal{K}$ only affects directly the first p copies of the internal model in $\mathcal{IM}_m$, which in turn influence the plant via the squaring-down gain M alone; *ii)* the optimizer block $-\gamma\nabla J$ only affects directly the last $(m-p)$ copies in $\mathcal{IM}_m$, which in turn influence the plant via the annihilator $N(s)$ alone.

Note that for the architecture in Fig. 4 the inverse of the transfer matrix $[M \quad N(s)]$ is not required, and then no stability requirement has to be imposed on it. The following result relating the two architectures in Fig. 3 and Fig. 4 holds.

Proposition 1: The regulators in Fig. 3 and Fig. 4 have the same transfer matrix (from e to u). ◇

C. Design procedure

The proposed design procedure can be outlined as follows, where $P(s)$ is the transfer matrix of the plant $\mathcal{P}$, and $\mathcal{IM}_h$ denotes any h-copy internal model unit, having transfer matrix $I_h\varphi(s)^{-1}$, where I_h is an identity matrix of size h and $\varphi(s)$ is the minimal polynomial of S. Note that at Step 4 it is assumed that $\det(A) \neq 0$, thus ruling out the somewhat trivial case in which the plant already contains an internal model of the modes of the exosystem.

Step 1. Select $M \in \mathbb{R}^{m\times p}$ such that the squared down system $\bar{\mathcal{P}}$ described by $(\bar{A}, \bar{B}, \bar{C}, \bar{D}) = (A, BM, C, DM)$ satisfies Assumption 2.

Step 2. Design the stabilizer $\mathcal{K}$ as any feedback stabilizer for the cascade of $\mathcal{IM}_p$ and $\bar{\mathcal{P}}$.

Step 3. Design the annihilator $\mathcal{N}$ with transfer matrix $N(s) = Q(s)\psi(s)^{-1}$, where $Q(s)$ is any full rank polynomial matrix of size $m \times (m-p)$ such that $P(s)Q(s) = 0$, with $P(s)$ being the transfer matrix of the plant $\mathcal{P}$, and $\psi(s)$ is any Hurwitz polynomial of degree not less then the highest degree among the scalar entries of $Q(s)$.

Step 4. Choose $\gamma > 0$ and design the matrix ∇J according to the formulae:

$$\nabla J = N_0' Q_e \,[M \quad N_0]\,, \tag{8a}$$

$$Q_e = \begin{bmatrix} -A^{-1}B \\ I_m \end{bmatrix}' \begin{bmatrix} Q_o & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} -A^{-1}B \\ I_m \end{bmatrix}, \tag{8b}$$

where $N_0 = N(0)$.

Step 5. Realize the regulator as in Fig. 4.

If $[M \quad N(s)]$ is minimum phase (and then the control scheme in Fig. 3 can be used, see Corollary 1), then the following step can be used in place of Step 5 above.

Step 5'. Realize the regulator as in Fig. 3.

Two remarks are in order about the above procedure.

Remark 1: It is worth noting that a *separation principle* holds in the above design. In fact, all components (the stabilizer $\mathcal{K}$, the internal model $\mathcal{IM}_m$, the annihilator $\mathcal{N}$ and the gain ∇J) can be designed independently from each other and put together according to the proposed scheme. ▲

Remark 2: While the requirement of minimum phaseness of $[M \quad N(s)]$ might appear demanding, in the simulations it was found that, perhaps at the price of redesigning the squaring down gain M, it was always possible to find a suitable choice of M. Although a formal statement ensuring that such a suitable M always exists is not yet available, the numerical evidence suggest that it is actually the case. ▲

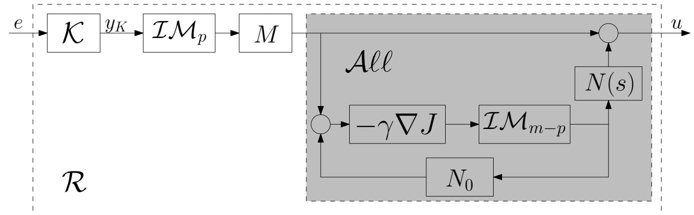

Fig. 5: The standard regulator in Fig. 2 with a feedforward allocator $\mathcal{A}\ell\ell$.

D. Main results

The main results in this paper are formalized in the following statements, in which we rule out the somewhat trivial case in which the plant already contains an internal model of the modes of the exosystem.

Theorem 1: Suppose that $\sigma(A) \cap \sigma(S) = \emptyset$ and that Assumptions 2, 1 and 3 hold. Then, Problem 1 is solved by the regulator in Fig. 4 designed as in Section III-C. ◇

Remark 3: It is interesting to point out that, despite the fact that the schemes in Fig. 3 and in Fig. 5 are equivalent from an input-output point of view, the spirit behind their design is fundamentally different. The design of $\mathcal{IM}_p$ and $\mathcal{IM}_{m-p}$ in Fig. 5 arises from a classic squaring-down approach (see Section II-A) to regulator design, followed by the application of an input allocation scheme. On the other hand, the design of $\mathcal{IM}_m$ in Fig. 3 stems from a suitable modification of the naive approach of Section II-B, generalizing the classic servocompensator design. ▲

Corollary 1: Consider the same assumptions of Theorem 1. If $[M \quad N(s)]$ is *minimum-phase*, then the conclusions of Theorem 1 hold for the regulator in Fig. 3. ◇

Remark 4: Interestingly, due to the presence of the annihilator $N(s)$ combined with the squaring-down matrix M, both the schemes in Fig. 3 and Fig. 4 are such that the *transient responses* induced by the controller $\mathcal{K}$ alone (*i.e.* without further control actions) are completely preserved also in the presence of the additional feedback/feedforward loops containing $N(s)$; such a property is termed *output-invisibility* of such loops [12], [13]. Nice consequences of output invisibility are that the design can be based on the *separation principle* mentioned in Remark 1, and that by choosing sufficiently high values for the gain γ, the convergence of the input correction (provided by the feedback loop containing $N(s)$) to its optimized steady-state can be made arbitrarily fast. ▲

IV. Use of Orthogonal Moments

As entailed by the discussions of the previous section, knowledge of the annihilator transfer matrix $N(s)$ is instrumental for the achievement of the additional *output-invisibility property*, mentioned in Remark 4. However, despite complete information on the annihilator *at any frequency* seems to be crucial, it is possible - by renouncing to the *output-invisibility property*, but still preserving *steady-state invisibility* [14] - to design an alternative scheme that requires only knowledge of the annihilator at specific frequencies (essentially those associated to the exosystem), as summarized in the following definition.

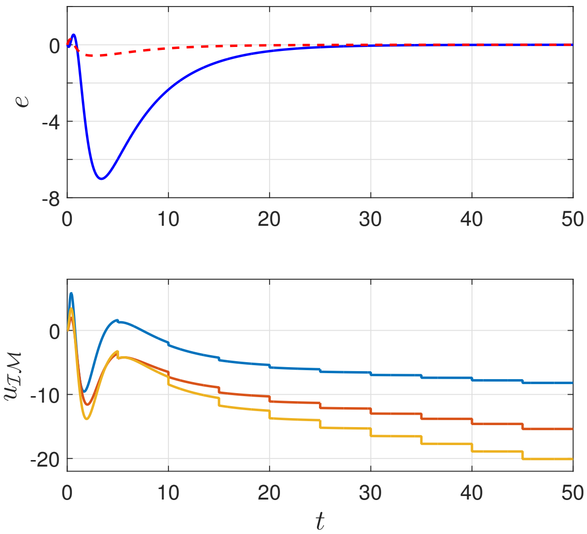

Fig. 6: Top graph: time histories of the output error $e(t)$ of system (9) in closed-loop with the *naive* architecture consisting only of the internal model unit defined by $(A_{\mathcal{IM}}, B_{\mathcal{IM}}, C_{\mathcal{IM}})$ and the controller (12). Bottom graph: time histories of the outputs of the Internal Model unit.

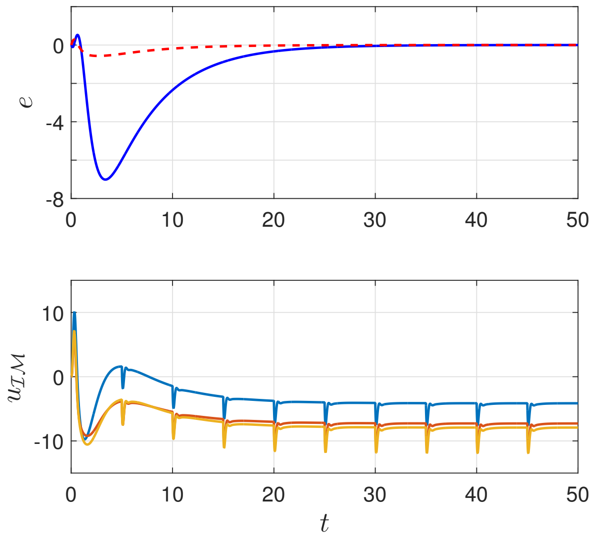

Fig. 7: Top graph: time histories of the regulated output error $e(t)$ of system (9) in closed-loop with the architecture in Fig. 3. Bottom graph: time histories of the outputs of the Internal Model unit.

Definition 1: (Orthogonal Moment). Consider the plant $\mathcal{P}$ as in (1) with transfer matrix $P(s) \in \mathbb{R}^{p\times m}$ such that $\mathrm{rank}(P(s)) = p$. An *orthogonal moment* of $\mathcal{P}$ at $s^\star \in \mathbb{C}$ is any complex matrix $N^\star \in \mathbb{C}^{m\times(m-p)}$ that is a basis for the null space of the *moment* of $\mathcal{P}$ at $s^\star$, namely, such that

(i) $P(s^\star)N^\star = 0$;

(ii) $\mathrm{rank}(N^\star) = m - p$.

$\circ$

Remark 5: The meaning of *orthogonal moments* is essentially that of characterizing the directions in the input space that allow for a reconfiguration of the input actions without affecting the *steady-state* value of the output of the plant $\mathcal{P}$. On the other hand, the *annihilator* $N(s)$ has the property of leaving unaffected the whole forced response (not just its steady-state component). ▲

Focusing, for brevity, only on the orthogonal moments at $s = 0$, the following proposition discusses the use of the *orthogonal moments* in the architectures of Fig. 3 and Fig. 4, with $N(s)$ replaced by the orthogonal moment N_0.

Proposition 2: Consider the plant $\mathcal{P}$ as in (1) and suppose that Assumptions 1, 2 and 3 hold. Then, for any given $\mathcal{K}$ designed as in Section II, there exists $\gamma^\star \in \mathbb{R}_{>0}$ such that the regulators in Fig. 3 and Fig. 4 solve Problem 1 for any $\gamma \in (0, \gamma^\star)$. $\diamond$

Remark 6: The bound $\gamma^\star$ is actually a function of the chosen stabilizer $\mathcal{K}$, and then also the *separation principle* in Remark 1 holds in a weaker fashion here. ▲

Remark 7: Orthogonal moments are particularly prone to a data-driven implementation in the presence of an uncertain plant of the above schemes. ▲

V. Numerical Simulations

Consider equations as in (1), with

$$A = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \tag{9}$$

$\alpha_i \in \mathbb{R}$, $i = 1, 2, 3$, and $B = I_3$, whereas the exosystem is described by the equations (2), with

$$S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{10}$$

The above scenario may be, for instance, interpreted as the collection of three independent agents with state $x_i(t) \in \mathbb{R}$ - each of which implementing its own local *internal model* - which share the objective of achieving a common *formation* characterized by assigned (constant) distances, dictated by the initial condition $w_0 \in \mathbb{R}^2$ of the exosystem. The *naive* design of Section II-B in this context consists in the construction of an Internal Model unit with state $x_{\mathcal{IM}}(t) \in \mathbb{R}^3$, replicating a single integrator on each individual input channel, *i.e.* described by the matrices $A_{\mathcal{IM}} = 0_{3\times 3}$, $B_{\mathcal{IM}} = I_3$ and $C_{\mathcal{IM}} = I_3$. Such an architecture leads to the loss of observability for the cascaded interconnection between the Internal Model unit and the plant, compactly described by

$$A_{\mathrm{cas}} = \begin{bmatrix} A & BC_{\mathcal{IM}} \\ 0 & A_{\mathcal{IM}} \end{bmatrix}, \quad B_{\mathrm{cas}} = \begin{bmatrix} 0 \\ B_{\mathcal{IM}} \end{bmatrix} \tag{11}$$

and $C_{\mathrm{cas}} = [\, C \quad 0 \,]$. Moreover, let $\mathcal{X}_i \in \mathbb{R}^6$ denote a basis for the subspace of unobservable states, *i.e.* $\mathrm{im}(\mathcal{X}_i) = \ker([C'_{\mathrm{cas}}, (C_{\mathrm{cas}}A_{\mathrm{cas}})', ..., (C_{\mathrm{cas}}A^5_{\mathrm{cas}})']')$. Finally, the *naive* construction is completed by the design of a controller $\mathcal{K}$ that is in charge of asymptotically stabilizing the observable part of the cascaded interconnection, by employing measurements of the output error $e = Cx + Qw$ and by relying on perfect knowledge of the data of the plant as well as of the exosystem. This is achieved, as suggested by Assumption 3, by letting $\mathcal{K}$ be described by

$$\begin{aligned} \dot{x}_{\mathcal{K}} &= (A_r - B_r K_r - G_r C_r)\, x_{\mathcal{K}} + G_r e \\ u_{\mathcal{K}} &= -M K_r x_{\mathcal{K}} \,, \end{aligned} \tag{12}$$

with the triplet A_r, B_r and C_r derived by slightly modifying A_{cas}, B_{cas} and C_{cas}, respectively, according to

$$A_r = \begin{bmatrix} A & BMH'C_{\mathcal{IM}}H \\ 0 & H'A_{\mathcal{IM}}H \end{bmatrix}, \qquad B_r = \begin{bmatrix} 0 \\ H'B_{\mathcal{IM}}H \end{bmatrix}$$

and $C_r = H'C_{\mathrm{cas}}H$, with $H = [\begin{array}{cc} I_2 & 0_{2\times 1} \end{array}]'$, where $M \in \mathbb{R}^{3\times 2}$ is any matrix such that the pairs (A_r, B_r) and (A_r, C_r) are reachable and observable, respectively. In addition, $K_r \in \mathbb{R}^{2\times 5}$ and $G_r \in \mathbb{R}^{5\times 2}$ are selected such that $\Lambda(A_r - B_r K_r) \subset \mathbb{C}^-$ and $\Lambda(A_r - G_r C_r) \subset \mathbb{C}^-$, respectively. In the following numerical simulations we let $\alpha_1 = 0.5$, $\alpha_2 = 1$, $\alpha_3 = 1.5$,

$$M = \begin{bmatrix} -4 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix},$$

and

$$K_r = \begin{bmatrix} 1.8 & 64.7 & -76.4 & 14.6 & -0.8 \\ -21 & -96.2 & 158.2 & -0.8 & 14.9 \end{bmatrix}$$
$$G_r = \begin{bmatrix} 1.3 & -16.4 & -13.4 & -7.8 & -6.1 \\ -48 & -45 & -62.3 & 6.1 & -7.8 \end{bmatrix}.$$

It is assumed that the overall regulator consists only of the somewhat *naive* control scheme discussed above, namely, the regulator in Fig. 3 without the feedback loop including ∇J. Moreover - in order to point out the intrinsic drawbacks of such strategy that result in the loss of observability for the closed-loop plant - the agents, namely the state of the plant (9) and the individual internal models, are affected by a potentially malicious attack that consists in a periodic reset of the state $[x', x'_{\mathcal{IM}}]' \in \mathbb{R}^6$ in the direction of $\mathcal{X}_i$, namely $[x(t_k^+)', x_{\mathcal{IM}}(t_k^+)']' = [x(t_k^-)', x_{\mathcal{IM}}(t_k^-)']' + \mu\mathcal{X}_i$, for some constant μ and for any $t_k = kT$, $k \in \mathbb{N}$. Let the initial condition of the plant and the Internal Model unit be defined as $x_0 = [4, 3, 1]'$ and $x_{\mathcal{IM},0} = [0, 0, 0]'$, respectively, while $x_{\mathcal{K},0} = [0, 0, 0, 0, 0]'$. As it can be immediately visualized in the top graph of Fig. 6, depicting the time histories of the output error in closed-loop with the *naive* control scheme, despite the fact that the above attack is in fact *invisible* via the regulated output, it is indeed affecting the internal state of the regulator, resulting in a constant *drift* of such states at each reset along the direction of the unobservable subspace.

On the other hand, the implementation of the scheme in Fig. 3, with $\gamma = 1$, prevents loss of observability, which in turns constitutes an obstruction for the achievement of internal asymptotic stability of the overall closed-loop system, and avoids the drifting phenomenon hinted at in the comment of Fig. 6. The top graph of Fig. 7 displays the time histories of the output error that are identical to those obtained only by the classical control architecture (compare top graphs of Fig. 6 and Fig. 7). Nonetheless, the *output-invisible* action of the feedback loop including ∇J in Fig. 3 can be appreciated in the bottom graph of Fig. 7, which shows the time histories of the outputs of the compensated Internal Model unit. The numerical simulation is obtained by implementing, as required by Corollary 1, the *annihilator* $N(s)$ with state $x_a(t) \in \mathbb{R}$ and the *inverse system* $[M \quad N(s)]^{-1}$ with state $x_i(t) \in \mathbb{R}$ described in the state space, respectively, by the matrices

$$A_{\mathrm{ann}} = -1.5, B_{\mathrm{ann}} = 1, C_{\mathrm{ann}} = \begin{bmatrix} -2 \\ -2.5 \\ -3 \end{bmatrix}, D_{\mathrm{ann}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and $A_{\mathrm{inv}} = -1.5$, $B_{\mathrm{inv}} = [\begin{array}{ccc} 1 & 4 & -4 \end{array}]$,

$$C_{\mathrm{inv}} = \begin{bmatrix} 0.5 \\ -3 \\ 0 \end{bmatrix}, \qquad D_{\mathrm{inv}} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & -4 & 5 \\ 1 & 4 & -4 \end{bmatrix}.$$

VI. Conclusions

The problem of output regulation for plants having more inputs than outputs has been revisited, and a new regulator design has been proposed, based on the independent construction (by a *separation principle*) of two units, namely a *stabilizer* and a *smart servocompensator* which is able to optimize its steady-state output meanwhile preserving output regulation. Ongoing work deals with the extension of the control scheme to the case of uncertain plants.

References

[1] B. A. Francis and W. M. Wonham, "The internal model principle of control theory," *Automatica*, vol. 12, no. 5, pp. 457–465, 1976.

[2] E. Davison, "The robust control of a servomechanism problem for linear time-invariant multivariable systems," *IEEE Trans. Aut. Cont.*, vol. 21, no. 1, pp. 25–34, 1976.

[3] W. Wonham, *Linear Multivariable Control. A Geometric Approach*, 3rd ed., ser. Applications of Mathematics. New York, NY: Springer Verlag, 1985, vol. 10.

[4] A. Serrani, A. Isidori, and L. Marconi, "Semi-global nonlinear output regulation with adaptive internal model," *IEEE Trans. Aut. Cont.*, vol. 46, no. 8, pp. 1178–1194, 2001.

[5] A. Serrani, "Rejection of harmonic disturbances at the controller input via hybrid adaptive external models," *Automatica*, vol. 42, no. 11, pp. 1977–1985, 2006.

[6] D. Carnevale, S. Galeani, M. Sassano, and A. Serrani, "External models for output regulation based on moment estimation from input-output data," in *IFAC World Congress*, 2017, pp. 7777–7782.

[7] G. de Carolis, S. Galeani, and M. Sassano, "Data driven, robust output regulation via external models," in *Med. Control Conf.*, 2017, pp. 1263–1268.

[8] L. Zaccarian, "Dynamic allocation for input-redundant control systems," *Automatica*, vol. 45, pp. 1431–1438, 2009.

[9] S. Galeani, A. Serrani, G. Varano, and L. Zaccarian, "On linear over-actuated regulation using input allocation," in *Conf. on Decision and Control*, 2011, pp. 4471–4476.

[10] S. Galeani, A. Serrani, G. Varano, and L. Zaccarian, "On input allocation-based regulation for linear over-actuated systems," *Automatica*, vol. 52, no. 2, pp. 346–354, 2015.

[11] D. Sigthorsson and A. Serrani, "Tracking with steady-state optimization: an application to air-breathing hypersonic vehicle control," in *Med. Control Conf.*, Ancona, Italy, 2006, pp. 1–6.

[12] A. Serrani, "Output regulation for over-actuated linear systems via inverse model allocation," in *Conf. on Decision and Control*, 2012, pp. 4871–4876.

[13] A. Cristofaro and S. Galeani, "Output invisible control allocation with steady-state input optimization for weakly redundant plants," in *Conf. on Decision and Control*, Dec. 2014, pp. 4246–4253.

[14] S. Galeani and S. Pettinari, "On dynamic input allocation for fat plants subject to multi-sinusoidal exogenous inputs," in *Conf. on Decision and Control*, Dec. 2014, pp. 2396–2403.

[15] S. Galeani and G. Valmorbida, "Nonlinear regulation for linear fat plants: the constant reference/disturbance case," in *Med. Control Conf.*, Jun. 2013, pp. 683–690.

[16] G. Valmórbida and S. Galeani, "Nonlinear output regulation for over-actuated linear systems," in *Conf. on Decision and Control*, Dec. 2013, pp. 4485–4490.

[17] D. Carnevale, S. Galeani, and L. Menini, "Output regulation for a class of linear hybrid systems. Part 1: trajectory generation," in *Conf. on Decision and Control*, 2012, pp. 6151–6156.

[18] D. Carnevale, S. Galeani, and M. Sassano, "Necessary and sufficient conditions for output regulation in a class of hybrid linear systems," in *Conf. on Decision and Control*, 2013, pp. 2659–2664.