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On the Distance between a Point and a Clothoid Curve

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Abstract— **We consider the problem of computing the minimum distance between a given point and an assigned clothoid curve. This models, for instance, the tracking error of a robotic vehicle such as a car-like one, from a prescribed trajectory at a certain instant. The proposed algorithm finds this minimum distance (not unique, in general) up to a specified tolerance with a few iterations of a robust hybrid Dekker/Brent numerical scheme with cubic order of convergence. We guarantee the convergence to the global optimum proving a theorem that gives the intervals where there are all the local solutions so that it is easy to select the global one. We show the performance of the method with a test case and compare the results with state of the art solutions.**

I. Introduction

The clothoid curve was discovered by Euler as the curve defined by the property that its curvature is linear with the arc length. Since Euler, it has been called with other names in various contexts, among them there is Euler Spiral and Cornu Spiral, see [1] for a beautiful historical introduction. The connection of the clothoid with the Fresnel Integrals, the Exponential Integral and the Gamma function, [2] renders its practical application a difficult problem because there are not any closed form solutions. However, the property of having linear curvature makes the clothoid superior to other classes of curves, which exhibit irrational curvature functions [3]. This importance has been recognised by a wide area of the scientific community, in many fields.

For instance, the clothoid is found in [4], [5], [6] as the solution of an optimal control problem for a car-like robot that has to find the shortest path connecting two points in the plane $\mathbb{R}^2$ with given initial and final angles and curvatures. In many contexts, it is often useful to measure the distance between a point and a (portion of) clothoid curve, which leads to the present study. A typical application is to measure the offset of a robot or a vehicle from the prescribed nominal trajectory, [7], [8], [9]. Other possible applications are in the field of computer graphics, e.g. in font design. Under the umbrella of path planning, many methods have been sought to solve the problem of connecting an initial with a final posture. A posture can be just the position in the x-y plane, but it can also include orientation and curvature. Well known tools to solve this problem are splines of lines and arcs (Dubin's curves), polynomial curves such as Bezier, B-splines and, recently, clothoid curves, [10], [11]. Once a trajectory is established, the vehicle has to follow it [12], [13], and the tracking requires knowing the distance between the vehicle's position and the nominal trajectory.

The problem and the State of the Art. The main issue of this distance problem is the efficient evaluation of the clothoid, as detailed in [14], [15]. For this we rely on our previous work and software [11], which permits us to easily and accurately compute the curves. In the present work we propose a robust numerical algorithm to find the (in principle infinitely countable many) points at minimum (or maximum) distance from a given clothoid. There is not much published literature that deals with this particular distance problem, and the problem is mostly solved the naïve way, by sampling points along a curve and computing the distance between them and the reference point. Of course this approach works, but is not accurate if this procedure is not refined many times, which makes it non-efficient. At best of our knowledge, variants of this method for solving the distance problem are discussed in Chapters 23-5 and 23-6 of [14] only. We take this publication for reference and it will be discussed in comparison with the presented solution. Novel techniques approximate the clothoids with curves that are easier to manipulate, often polynomials, but also with other spirals, [16].

In Section II we introduce the clothoid curve, in Section III we analyse the distance problem. First we exclude the cases of trivial clothoids, then we show how to recast a general clothoid into standard position and we prove the main result. In Section IV we discuss the numerical method used to solve the problem and in Section V we show a numerical example and a comparison with state of the art methods. In the last section we draw the conclusions and future research directions.

II. Background: Mathematical Formulation

In this section we review the analytic expressions for the clothoid curve and some results ([17]) necessary to tackle the problem of finding the points on a clothoid at critical distance from a given point (in general not on the clothoid). The expression of the curvature for a parametric plane curve $(x(s), y(s))$ parametrised by arc length s is given by $k = x'y'' - y'x''$, where we dropped the dependence of s of the functions. The arc length parametrisation condition is equivalent to $x'^2 + y'^2 = 1$. Imposing that the curvature is a linear (affine) function of the arclength yields the following system of ODEs:

$$x'y'' - y'x'' = \kappa' s + \kappa \tag{1}$$
$$x'^2 + y'^2 = 1, \tag{2}$$

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where κ' and κ are two real constants. The solution of this system is contained in the next theorem.

Theorem 1: The explicit expression of a clothoid curve is:

$$x(s) = x_0 + \int_0^s \cos\left(\frac{\kappa'}{2}\tau^2 + \kappa\tau + \theta_0\right) \mathrm{d}\tau, \quad (3)$$

$$y(s) = y_0 + \int_0^s \sin\left(\frac{\kappa'}{2}\tau^2 + \kappa\tau + \theta_0\right) \mathrm{d}\tau, \quad (4)$$

where (x_0, y_0) are integration constants that correspond to the base point of the curve, $\kappa', \kappa \in \mathbb{R}$ are the coefficients of the curvature $k(s) = \kappa' s + \kappa$, θ_0 is the initial angle with respect to the horizontal axis, and s is the arclength.

Proof: From condition (2), it is possible to rewrite the ODE for the curvature in terms of one variable only, that is $y' = \sqrt{1 - x'^2}$ and $2y'y'' = -2x'x''$. The curvature (2) is thus

$$\frac{-x'x'x''}{\sqrt{1 - x'^2}} - x''\sqrt{1 - x'^2} = \kappa' s + \kappa.$$

A straightforward manipulation of the the previous equation leads to $x'' + (\kappa' s + \kappa)\sqrt{1 - x'^2} = 0$, which can be recast to a first order system with the change of variable $w = x'$,

$$w' = -(\kappa' s + \kappa)\sqrt{1 - w^2}.$$

This equation can be solved by the method of separation of variables and yields

$$-\frac{w'}{\sqrt{1 - w^2}} = \kappa' s + \kappa \Rightarrow -\int \frac{1}{\sqrt{1 - w^2}} \mathrm{d}w = \int \kappa' s + \kappa \mathrm{d}s.$$

It is easy to recognise the first integral as the integral of the inverse cosine function, whereas the integral at the r.h.s. is trivial. The solution is hence

$$\arccos(w) = \frac{\kappa'}{2}s^2 + \kappa s + \theta_0 \Rightarrow w = \cos\left(\frac{\kappa'}{2}s^2 + \kappa s + \theta_0\right),$$

where θ_0 is the integration constant. Then, the expression for $x(s)$ is the integral of w and is exactly (3). It is not possible to express this integral by means of elementary functions, because it is related to the standard Fresnel Integrals. The component $y(s)$ can be recovered observing that $y'(s) = \sqrt{1 - w(s)^2}$ and can be integrated to obtain (4). ■

III. Analysis of the Problem

Particular cases of clothoids are for $\kappa = 0$ or $\kappa' = 0$ or both. This gives origin to two particular cases that can be considered separately and that allows us to simplify the problem by recasting a general clothoid into standard form. When both curvature parameters are zero, the equation of the clothoid boils down to a straight line ℓ of equation

$$x(s) = x_0 + s\cos(\theta_0) \quad (5)$$

$$y(s) = y_0 + s\sin(\theta_0). \quad (6)$$

The distance between a point and a segment is given next.

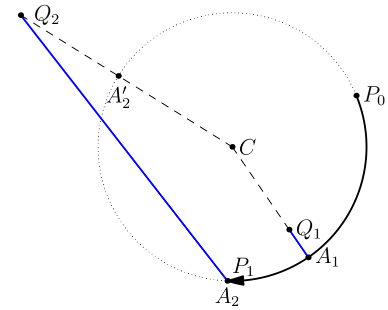

Fig. 1. Two examples of points at minimum distance from a circle of center C of radius $1/|\kappa|$. The point $Q = Q_1$ is projected to A_1 and is at distance $|1/|\kappa| - d(C, Q)|$. For another point $Q = Q_2$, the point at minimum distance is P_1, but its projection on the complete circle is A_2'.

Proposition 1 (Point-Segment): The distance of a point Q in the plane from the line segment ℓ given in (5)-(6) for $s \in [0, L]$ is

$$d(Q, \ell) = \begin{cases} d(Q, P_0) & \text{if} \quad \boldsymbol{w}_0 \cdot \boldsymbol{r} \leq 0 \\ d(Q, P_1) & \text{if} \quad \boldsymbol{w}_1 \cdot \boldsymbol{r} \geq 0 \\ ||\boldsymbol{w}_0 - (\boldsymbol{w}_0 \cdot \boldsymbol{r})\boldsymbol{r}|| & \text{otherwise,} \end{cases} \quad (7)$$

where $\boldsymbol{r} = (\cos(\theta_0), \sin(\theta_0))$ is the unit direction vector of the line, $\boldsymbol{w}_0 = Q - P_0$ and $\boldsymbol{w}_1 = Q - P_1$.

Proof: Omitted. ■

When only $\kappa' = 0$, then the clothoid reduces to an arc of circle of equation

$$x(s) = x_0 - [\sin(\theta_0) + \sin(\kappa s + \theta_0)]/\kappa \quad (8)$$

$$y(s) = y_0 + [\cos(\theta_0) - \cos(\kappa s + \theta_0)]/\kappa \quad (9)$$

If the point Q coincides with the center C of the circle $\mathcal{C}$, then the whole clothoid is at minimal distance $1/|\kappa|$ from the point Q. The center C has coordinates $C = P_0 + 1/\kappa\,[-\sin(\theta_0), \cos(\theta_0)]^T$.

Proposition 2 (Point-Circle): Let $\boldsymbol{r}$ be the unit vector of the segment that joins the center C of the circle with a point Q in the plane, $\boldsymbol{r} = (r_x, r_y) = (Q - C)/||Q - C||$, then

$$\bar{s} = \frac{1}{\kappa}\big(\arctan(r_x, -r_y) - \theta_0\big) \bmod \frac{2\pi}{|\kappa|} \quad (10)$$

is the (positive) curvilinear coordinate of the projection of Q on the complete circle $\mathcal{C}$, $d_0 = d(Q, P_0)$ and $d_1 = d(Q, P_1)$ are the distances of Q from the extremal points of the clothoid, respectively P_0 and P_1. The distance of the point Q from an arc of circle $\mathcal{C}$ given by a clothoid of form (8)-(9) for $s \in [0, L]$ is, see Figure 1:

$$d(Q, \mathcal{C}) = \begin{cases} d_0 & \text{if } \bar{s} > L \text{ and } d_0 < d_1 \\ d_1 & \text{if } \bar{s} > L \text{ and } d_1 < d_0 \\ \left|\dfrac{1}{|\kappa|} - d(C, Q)\right| & \text{if } 0 \leq \bar{s} \leq L. \end{cases}$$

Proof: Omitted. ■

The value $\bar{s}$ in (10) represents the projection of Q on the complete circle given by the clothoid (8)-(9). If the clothoid arc is shorter than the complete circle, this projection point A may not be part of the clothoid (see point A_2' in Figure 1),

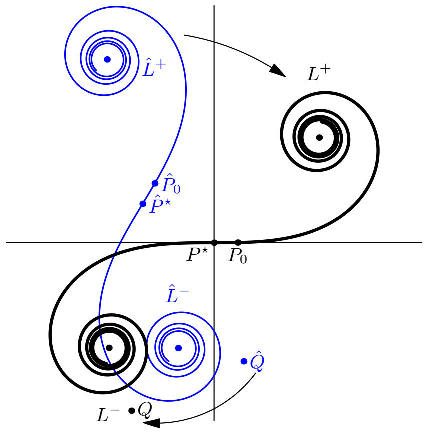

Fig. 2. A clothoid in general position (blue) with base point $\hat{P}_0$ and inflection point $\hat{P}^\star$. Its transform to standard position is shown in black. The hat superscript identifies the parameters of the clothoid in general position. $L^\pm$ are the limit points at $\pm$ infinity.

thus the minimum distance must be chosen between the two extremal points of the clothoid, respectively P_0 and P_1. The last case is verified when A lies inside the clothoid, $\bar{s}$ is then the curvilinear abscissa that realizes the minimum distance, which is therefore the difference between the radius of the circle and the distance of A from Q. Because the clothoid can loop, the minimum distance remains the same, but it can be periodically realized at different values of s by adding multiples of the circumference $2\pi/|\kappa|$.

A. Reduction to Standard Position

To efficiently compute the distance between a point and a clothoid with $\kappa' \neq 0$, it is convenient to apply a rigid transform, an isometry, to a spiral in general position as in (3)-(4) to bring it in standard form, with the inflection point (i.e. the point of zero curvature) at the origin, oriented with the horizontal axis and with positive κ'.

Suppose to have the parameters $\kappa' \neq 0$, $\hat{\kappa}$ and θ_0 of a clothoid with respect to a base point $P_0 = (x_0, y_0)$. Here we discuss the isometry that brings the clothoid to the standard setting, e.g. with respect to the inflection point where the curvature is zero as well as the angle, see Figure 2.

We have to find the inflection point $P^\star$ of zero curvature, which is located at the abscissa $s^\star = -\hat{\kappa}/\kappa'$, then we rotate the clothoid by the angle $\theta^\star = \frac{1}{2}\kappa' s^{\star 2} + \hat{\kappa} s^\star + \theta_0$. The new clothoid starts now from the origin with the same parameter κ' and with $\kappa = \hat{\kappa} + \kappa' s^\star = 0$. The general isometry that maps a point $(\hat{x}, \hat{y})$ from the previous reference to the new (x, y) is given by:

$$(x, y) = \begin{bmatrix} \cos(\theta^\star) & \sin(\theta^\star) \\ -\sin(\theta^\star) & \cos(\theta^\star) \end{bmatrix} \begin{bmatrix} \hat{x} - P_x^\star \\ \hat{y} - P_y^\star \end{bmatrix}. \qquad (11)$$

Remark 1: Depending on the sign of $\kappa' \neq 0$ it may be necessary to apply a reflection about the x axis to the clothoid curve and to the point Q to obtain a final $\kappa' > 0$.

In the sequel we assume, without loss of generality, that the given clothoid is in standard position.

B. Distance from a Clothoid

We consider a clothoid in standard position with $\kappa' > 0$ as discussed before and an arbitrary point $Q = (Q_x, Q_y)$ in the plane. We are interested in the critical points of the distance between the curve and the point. The Euclidean distance of a point Q from a clothoid $\mathcal{C}$ in standard position is given by

$$d(Q, \mathcal{C}(s)) = \sqrt{(x(s) - Q_x)^2 + (y(s) - Q_y)^2}, \qquad (12)$$

where $x(s)$ and $y(s)$ are the parametric expressions obtained in (3)-(4), with $\kappa = \theta_0 = 0$. To study the critical points of the distance (12) we analyse the zeros of its derivative. Instead of dealing with $d'(Q, \mathcal{C}(s))$, it is convenient to introduce the function $f(s) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s} d^2(Q, \mathcal{C}(s))$ and find its roots:

$$f(s) = (x(s) - Q_x)\cos\frac{\kappa' s^2}{2} + (y(s) - Q_y)\sin\frac{\kappa' s^2}{2}. \qquad (13)$$

The first derivative of $f(s)$ is

$$\begin{aligned} f'(s) &= 1 + \kappa' s g(s), \qquad (14) \\ g(s) &= (y(s) - Q_y)\cos\frac{\kappa' s^2}{2} - (x(s) - Q_x)\sin\frac{\kappa' s^2}{2} \end{aligned}$$

where $g(s)$ is an auxiliary function. It is important to notice that the computation of $g(s)$ does not require a particular computational effort, because all the quantities are obtained from the evaluation of $f(s)$ and just two multiplications and a subtraction are needed. There is also a functional relation between f and g, namely, $g'(s) = -\kappa' s f(s)$, which allows us to write the second derivative of f at the cheap computational cost of

$$f''(s) = \kappa' g(s) - (\kappa' s)^2 f(s). \qquad (15)$$

This information will be useful in the sequel, as it allows us to write a numerical scheme with a higher convergence order than the usual Newton method. The critical points of the distance function are the curvilinear abscissas s for which $f(s) = 0$. Their nature is established by means of the sign of the second derivative of the distance, which is the same sign of $f'(s)$. There are countable many isolated critical points, because the clothoid loops around the limit points $L^\pm$ for $s \to \pm\infty$. In general, we limit ourselves to a portion of the clothoid, say for $s \in [L_0, L_1]$, but our approach remains valid also for the case $s \in \mathbb{R} \cup \{\pm\infty\}$.

Remark 2: We consider apart the special cases when $Q = L^+$ or $Q = L^-$. For instance, when $Q = L^+$, it has been shown in [5] that the distance is a monotone decreasing positive function for $s > 0$ with $d(Q, L^+) = 0$. Moreover, the angle between Q and the tangent to the clothoid is also monotone decreasing ([5]) from $\frac{3}{4}\pi$ to $\frac{\pi}{2}$, hence the only critical point for $s > 0$ is the (minimum) value $s = +\infty$. Indeed, points for $s < 0$ are not special cases and are discussed next. With symmetry arguments, we obtain an analogous result if $Q = L^-$.

Lemma 2: The roots of $f(s)$ are countable and separable.

Proof: We have to study two cases: (a) far from $s = 0$; (b) in proximity of $s = 0$. In the first case we have that being $Q \neq L^\pm$, the differences $(x(s) - Q_x)$ or $(y(s) - Q_y)$ cannot both vanish and at least one becomes an almost constant value for $s \to \pm\infty$, in fact $x(s), y(s) \to L^{\pm}$. Hence, $f = 0$ is possible only when the combination of $(x(s) - Q_x)$ and $(y(s) - Q_y)$ with the trigonometric functions is zero, thus this case reduces to find the roots of an expression of kind

$$0 = K_x \cos\left(\frac{1}{2}\kappa' s^2\right) + K_y \sin\left(\frac{1}{2}\kappa' s^2\right), \qquad (16)$$

for two values K_x and K_y that we consider constant. It follows that the roots are countable and separable between the roots of the trigonometric functions. Those values for the sine and the cosine are given respectively (for $n = 0, 1, 2, \ldots$) by

$$s = \pm\sqrt{\frac{2\pi n}{\kappa'}}, \qquad s = \pm\sqrt{\frac{2\pi n + \pi}{\kappa'}}. \qquad (17)$$

Between two of such consecutive values, f can vanish at most once. Thus, by looking at the variation of the sign of the distance function evaluated at those points we can detect the presence of at most one root of f.
Regarding case (b), $(x(s)-Q_x) \to 0$ or $(y(s)-Q_y) \to 0$ can happen only in proximity of the origin, otherwise we have $\lim_{s\to\pm\infty}(x(s), y(s)) = L^{\pm}$, but $Q \neq L^{\pm}$ by hypothesis. Thus we look for the possible root of $(x(s) - Q_x) = 0$ or $(y(s) - Q_y) = 0$ for s contained between zero and the value of the first peak of the Fresnel Cosine and Sine functions, respectively. The peak value of the Fresnel Cosine is obtained when $\frac{\kappa'}{2}s^2 = \pm\frac{\pi}{2}$, that is for $s = \pm\sqrt{\pi/\kappa'}$; the peak of the Fresnel Sine is obtained for $s = \pm\sqrt{2\pi/\kappa'}$; also $s = 0$ makes the Fresnel integrals vanish. In particular, $s = 0$ is a root of $f(s)$ if and only if Q lies on the y axis. Inside such intervals, there can be at most one root because the Fresnel Integrals are monotone from zero to the first maximum. Because Q is assigned, there can be at most two roots, one for the Fresnel Cosine and one for the Sine. Let us name those two possible roots $s_c^\star$ and $s_s^\star$, respectively for the Cosine and the Sine. We sort the two roots, so for simplicity assume that $0 \le s_c^\star \le s_s^\star$. Then, f can have a single root in:

$$[0, s_c^\star], \quad [s_c^\star, s_s^\star], \quad [s_s^\star, \sqrt{2\pi/\kappa'}] \qquad (18)$$

and analogously for the symmetric negative intervals. ■
Once a root has been found, for example via a numeric algorithm such as one described in the next section, the nature of the critical point can be established by looking at the sign of the second derivative of the distance function. This gives, in principle, a robust method to compute all points on a clothoid at critical distance from an arbitrary point Q. Moreover, because of the periodic behaviour of f, maxima and minima are alternating.

Theorem 3 (Distance Point-Clothoid): Let $\mathcal{C}$ be a clothoid in standard position with $\kappa' > 0$ and Q a point in the plane, let $n \in \mathbb{N}$ and $s_1 = \min\{|s_c^\star|, |s_s^\star|\}$, $s_2 = \max\{|s_c^\star|, |s_s^\star|\}$ be given in Lemma 2. Then for $n = 0, 1, 2, \ldots$

$$\begin{aligned} \mathcal{S}^+ &= \left\{0, s_1, s_2, \sqrt{\frac{2\pi}{\kappa'}}, \sqrt{\frac{2\pi n}{\kappa'}}, \sqrt{\frac{2\pi n + \pi}{\kappa'}}, \ldots\right\}, \\ \mathcal{S}^- &= -\mathcal{S}^+, \end{aligned} \qquad (19)$$

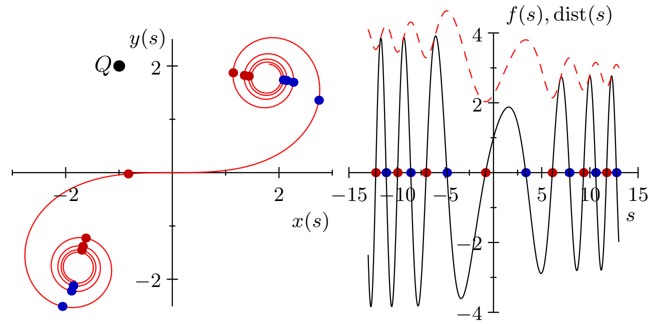

Fig. 3. The case of generic $Q \neq L^{\pm}$. *Left*: the clothoid and the point Q, the red dots are points at minimum distance and the blue are points at maximum distance. *Right*: the distance function is red dashed, in black the plot of $f(s)$. The dots are the values on the curvilinear abscissa s of the points at minimum (red) and maximum (blue) distance corresponding to the plot on the left.

are the sequences of numbers constructed in (17)-(18) that define the intervals of Lemma 2 which contain at most one critical point of (12) each. Additional roots can be the values $s = \{0, \pm\infty\}$, possibly. Moreover, a critical point z is a minimum if $f'(z) > 0$.

Proof: The sequences $\mathcal{S}^+$ and $\mathcal{S}^-$ are constructed by means of Lemma 2, the value $s = 0$ is an extra root if and only if Q is on the y axis. It must be considered apart $s = \pm\infty$ if $Q = L^{\pm}$. The roots $s_c^\star, s_s^\star$ are at most two, as discussed in Lemma 2. Finally, the sign of the second derivative of the distance function (12) is the same sign of f'. This is an useful property because $f'(z)$ does not need to be recomputed *a posteriori* as it is a side product of the numerical scheme used to find z, e.g. Newton-Raphson. ■

Corollary 4 (Distance Point-Portion of clothoid): The distance between a point Q and a portion of a clothoid in standard position for $s \in [L_0, L_1]$ can be recovered from Theorem 3 by intersecting the sequences $\mathcal{S}^+$ and $\mathcal{S}^-$ with $[L_0, L_1]$ and by checking the extrema $s = L_0$ and $s = L_1$, with care of the special cases $s = \{0, \pm\infty\}$, possibly.

IV. Numerical Algorithm

In this section, we discuss the complete numerical procedure to compute the minimum (or maximum) distance between a point Q and a clothoid $\mathcal{C}$ in general position. The first step is to identify whether $\mathcal{C}$ is a line segment or a circle: in these cases we use respectively Proposition 1 and Proposition 2. In the case of a full clothoid, i.e. $\kappa' \neq 0$, we recast $\mathcal{C}$ (and accordingly Q) into standard position. Then, because we cannot list all the possible minima, we suppose to study only a portion of the clothoid, for $s \in [L_0, L_1]$. We split this interval into the intervals suggested by Theorem 3. For each interval $[s_j, s_{j+1}]$, we compute the value of $f(s)$ given in (13), if $f(s_j)f(s_{j+1}) > 0$, then in that interval there is no solution. Otherwise, we use the Halley-Brent method described below to find the single root. The last stage of the root finding algorithm requires to compute the derivative of f at the root and its sign characterises the critical point as minimum or maximum. We repeat this computation for all the intervals plus the special cases $s = 0$, if $0 \in [L_0, L_1]$,

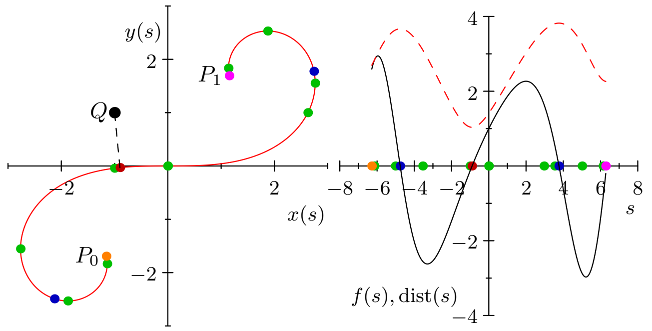

Fig. 4. *Left*: the portion of clothoid and the point Q, the red dot is the critical point at minimum distance and the blue are the critical points at maximum distance. *Right*: the distance function is red dashed, in black the plot of $f(s)$. The dots are the values on the curvilinear abscissa s of the points at minimum (red) and maximum (blue) distance corresponding to the plot on the left, green dots are the values that separates the roots of Corollary 4.

Interval	Halley	Bisect.	Newton	Bisect.
$[-\sqrt{8\pi}, \sqrt{4\pi}]$	4	0	3	1
$[s_c^\star, 0]$	2	0	4	1
$[\sqrt{4\pi}, \sqrt{8\pi}]$	4	0	5	2

Fig. 5. The number of iterations required to solve the discussed example. In the first column the interval where there is a root, in the second and third column the iterations required by present method with the Halley solver, in the fourth and fifth the case of the Newton method. The method based on Halley exhibits better steps as it does not require bisection steps. The specified error tolerance was 10^{-10}.

Sampled pts	max d err.	mean d err.	max s err.	mean s err.
100	6×10^{-2}	7×10^{-4}	7×10^{0}	3×10^{-2}
1000	6×10^{-3}	8×10^{-6}	6×10^{-3}	3×10^{-3}
10000	6×10^{-4}	8×10^{-8}	6×10^{-4}	3×10^{-4}

Fig. 6. Maximum and mean error of the distance and the corresponding curvilinear abscissa s computed with the proposed algorithm with precision 10^{-10} and the result obtained by sampling the clothoid with 100, 1000 and 10000 points. The minimum error is 0 for points Q on the clothoid.

$s = L_0$ and $s = L_1$. We have thus the complete (finite) list of all minima and maxima points of the clothoid and we can select the global minimum or maximum. Next we describe the root finding algorithm for the root inside $[s_j, s_{j+1}]$.

A. An Hybrid Numerical Method for the Distance Problem

The knowledge of an interval $[s_j, s_{j+1}]$ containing a root of the function $f(s)$ allows us to construct a performing numeric scheme that improves the classic bisection/secant method of Dekker/Brent (see [18]) by using a derivative approach. The additional computational simplicity of the derivatives of $f(s)$ given in (14) and (15) enters in the construction of the third order Halley method together with a check that the new step remains in the interval $[s_j, s_{j+1}]$. If this is not the case, the Halley step is replaced by a bisection step; however, the experiments suggest that this is only seldom the case, hence the convergence is of order 3 most of the times. The Halley iteration ([19]), for an initial guess $s = z_0$ is given, for $n = 0, 1, 2, 3, \ldots$, by

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}\left(1 - \frac{f(z_n)}{f'(z_n)}\frac{f''(z_n)}{2f'(z_n)}\right)^{-1}. \qquad (20)$$

The derivatives of $f(s)$ are relatively easy to compute, indeed we can write them only as a combination of $f(s)$ and $g(s)$. This leads, in principle, to the construction of even higher order derivative methods. The check that the new step of the Halley method remains in the interval $[s_j, s_{j+1}]$ with an eventual bisection step guarantees the convergence.

V. Tests and Comparison with State of the Art

We consider here the test case of a clothoid already in standard position of parameter $\kappa' = \frac{1}{4}$, the point Q is located at $Q = (-1, 1)$. The interval of the abscissa s is limited to $s \in [-2\pi, 2\pi]$, see Figure 4. We will find the critical points with the herein discussed algorithm based on the Dekker/Brent method combined with the Halley method, we compare its performance with the Newton-Raphson scheme used instead of Halley.

To find the critical points we split the interval $[-2\pi, 2\pi]$ as described in Corollary 4. We are not in the special case $Q = L^{\pm}$ but the case of $s = 0 \in [-2\pi, 2\pi]$, has to be considered apart. Indeed $f(0) = -Q_x \neq 0$, hence also this special case does not occur. Next we determine the eventual roots $s_c^\star$, $s_s^\star$. There is a root of $x(s) - Q_x$ for $s_c^\star \approx -1.002$ and a root for $y(s) - Q_y$ for $s_s^\star \approx 2.9696$. The other values that split the interval are given by 0, $\pm\sqrt{4\pi}$, $\pm\sqrt{8\pi}$,$\pm\sqrt{12\pi}$. Those values are the dots on the horizontal axis of Figure 4 (right). Hence we have a total of 10 intervals to find the critical points. Looking at the sign of f evaluated on each of the 11 points that subdivide $[L_0, L_1]$ it is easy to conclude that there is a zero only between $[-\sqrt{8\pi}, \sqrt{4\pi}]$, $[s_c^\star, 0]$, $[\sqrt{4\pi}, \sqrt{8\pi}]$. Figure 5 contains the total number of iterations required by the Dekker/Brent solver based on Newton-Raphson and on Halley, with the number of sub steps of bisection performed. As expected, the method based on Halley gives superior performance in terms of number of iterations and number of bisection sub steps.

A. Comparison and Comments

We compare the algorithm herein presented with the techniques available in literature. In the example discussed above, the presented method requires 11 function evaluations. A classic method used to compute the distance of a point to a general curve involves sampling the curve and for each sampled point computing the distance point-curve. This method has the advantage of working for all curves and is effective as it converges to the global minimum distance. The price to pay is the computational inefficiency, as many samples are required to have an approximate solution, without control on the precision. This is evident from Figure 6. However this approach is derivative free and easy to implement. The presented numerical experiments show that the number of samples required to achieve the distance within a reasonable error tolerance is not satisfactory for less than 100 evaluations.

Another method discussed in [14], Chapters 23-5, 23-6, approximates the clothoid with a polyline and computes the distance between each segment of the polyline and the point Q. This approach works better than the previous, but it still

requires many samples to produce an accurate solution.
In [14] a local search for the minimum distance is discussed as well. The problem is setup as a NonLinear Programming (NLP) via the Lagrangian Multipliers. When the clothoid reduces to a line or a circle, the NLP is simple and is solved analytically; the results are in agreement with our Propositions 1 and 2, although [14] finds only one local solution and there can be more. For the full clothoid problem, [14] accounts for the mandatory approximation of the clothoid with series expansions up to order 6, which yields a problem solvable only with elaborated software computations (NLP solvers). Their approach is not robust because an NLP solver may not converge in all cases and the solution is only local. If the method converges to the global solution, it is very accurate, however, it does not consider multiple solutions. Another important issue is that an interval where to look for a solution is not provided and the Newton iteration can converge to a local minimum far away, whereas the present method is safe remaining in the prescribed interval. It is important to point out that the Newton method often tries to leave the prescribed interval and the bisection method must be called, see Figure 6. This is only seldom the case for the Halley iteration.
As a final comment, the present method is competitive compared to other intuitive methods or other based on NLP. Sampling is effective only if a rough estimate of the distance is required, but has no control on the accuracy. The other methods that are not based on sampling, find in general only a local minimum, not the global. The present method collects the benefits of the accurate local methods with the global properties of the methods based on sampling. It has been included in a real time robotic application in [20], as it requires a computational time of the order of fractions of millisecond on an embedded system.

VI. Conclusions

We presented a numerical method to find the distance between a point and a clothoid with high accuracy, up to the specified error tolerance. The key point is Lemma 2, with the separation of the roots: together with Theorem 3 and its corollary, it gives a complete characterization of all the critical points and the intervals that contain at most one root, hence the sampling is very targeted and efficient. The solver required to find the root inside each interval can be very basic as the bisection method, but the structure of the trigonometric functions involved allows us to exploit the cheap computation of the derivatives needed for a better numerical scheme, like the proposed Dekker/Brent with Halley method. Therefore, the proposed method is accurate at desired precision and produces all the maxima and minima points, which for a clothoid are countable many, and also the global minimum (or maximum). The computation effort is reasonable as Theorem 3 explains where to sample the points. The numeric root finding requires few iterations to produce a solution and its characterisation as minimum or maximum point. Alternatives to present method are available, as discussed in the previous section, but with the same number of iterations or function evaluations the produced solution has considerably less accuracy than our method.
Future research directions are the study of numerically robust formulas that allow the smooth transition from the clothoid case to the circle case, and analogously, from the circle to straight line. For instance, the distance from the circle has a division by κ which produces numerical instabilities in the computation as $\kappa \to 0$, that is when the circle is close to a straight line.

References

[1] R. Levien, “The euler spiral: a mathematical history,” EECS Department, University of California, Berkeley, Tech. Rep. UCB/EECS-2008-111, Sep 2008.

[2] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. National Bureau of Standards Applied Mathematics Series, 1964, no. 55.

[3] E. Bakolas and P. Tsiotras, “On the generation of nearly optimal, planar paths of bounded curvature and bounded curvature gradient,” in *2009 American Control Conference*, June 2009, pp. 385–390.

[4] A. Scheuer and T. Fraichard, “Continuous-curvature path planning for car-like vehicles,” in *Intelligent Robots and Systems, 1997. IROS ’97., Proceedings of the 1997 IEEE/RSJ International Conference on*, vol. 2. Intelligent Robots and Systems. IROS ’97.,, sep 1997, pp. 997–1003.

[5] V. Kostov and E. Degtiariova-Kostova, “Some Properties of Clothoids,” INRIA, Tech. Rep. RR-2752, Dec. 1995.

[6] E. Degtiariova-Kostova and V. Kostov, “Irregularity of Optimal Trajectories in a Control Problem for a Car-like Robot,” INRIA, Tech. Rep. RR-3411, Apr. 1998.

[7] Y. Shan, W. Yang, C. Chen, J. Zhou, L. Zheng, and B. Li, “Cf-pursuit: A pursuit method with a clothoid fitting and a fuzzy controller for autonomous vehicles,” *Int. J. of Advanced Robotic Systems*, vol. 12, no. 9, p. 134, 2015.

[8] M. Frego, E. Bertolazzi, F. Biral, D. Fontanelli, and L. Palopoli, “Semi-analytical minimum time solutions with velocity constraints for trajectory following of vehicles,” *Automatica*, vol. 86, pp. 18 – 28, 2017.

[9] ——, “Semi-analytical minimum time solutions for a vehicle following clothoid-based trajectory subject to velocity constraints,” in *2016 European Control Conference (ECC)*, June 2016, pp. 2221–2227.

[10] D. S. Meek and D. J. Walton, “A two-point $g1$ hermite interpolating family of spirals,” *J. Comp. Appl. Math.*, vol. 223, no. 1, pp. 97–113, 2009.

[11] E. Bertolazzi and M. Frego, “G1 fitting with clothoids,” *Mathematical Methods in the Applied Sciences*, vol. 38, no. 5, pp. 881–897, 2015.

[12] P. F. Lima, M. Trincavelli, J. Mrtensson, and B. Wahlberg, “Clothoid-based model predictive control for autonomous driving,” in *2015 European Control Conference (ECC)*, July 2015, pp. 2983–2990.

[13] M. Frego, P. Bevilacqua, E. Bertolazzi, F. Biral, D. Fontanelli, and L. Palopoli, “Trajectory planning for car-like vehicles: A modular approach,” in *2016 IEEE 55th Conference on Decision and Control (CDC)*, Dec 2016, pp. 203–209.

[14] E. W. Grafarend, R. J. You, and R. Syffus, *Map Projections – Cartographic Information Systems*, 2nd ed. Springer-Verlag Berlin Heidelberg, New York, 2014.

[15] A. M. Lekkas, A. R. Dahl, M. Breivik, and T. I. Fossen, “Continuous-Curvature Path Generation Using Fermat’s Spiral,” *Modeling, Identification and Control*, vol. 34, no. 4, pp. 183–198, 2013.

[16] M. Candeloro, A. M. Lekkas, and A. J. Srensen, “A voronoi-diagram-based dynamic path-planning system for underactuated marine vessels,” *Control Engineering Practice*, vol. 61, pp. 41 – 54, 2017.

[17] E. Bertolazzi and M. Frego, “Interpolating clothoid splines with curvature continuity,” *Mathematical Methods in the Applied Sciences*, vol. 41, no. 4, pp. 1723–1737, 2018, mma.4700.

[18] K. E. Atkinson, *An introduction to numerical analysis, 2nd Edition*. John Wiley, 1991, vol. 528.

[19] J. Traub, *Iterative Methods for the Solution of Equations*, ser. AMS Chelsea Publishing Series. Chelsea, 1982.

[20] P. Bevilacqua, M. Frego, D. Fontanelli, and L. Palopoli, “Reactive planning for assistive robots,” *IEEE Robotics and Automation Letters*, vol. 3, no. 2, pp. 1276–1283, April 2018.