2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

A new look at the uncontrollable linearized quaternion dynamics with implications to LQR design in underactuated systems

Leandro R. Lustosa, Flávio Cardoso-Ribeiro, François Defaÿ and Jean-Marc Moschetta*

***Abstract*— Quaternion algebra is frequently employed for spacecraft attitude description due to its convenient numerical properties when compared to minimal formulations. In parallel, Linear Quadratic Control (LQR)-based attitude controllers are often applied to underactuated vehicles due to its intuitive tuning process and satisfactory stability robustness properties. However, nonlinear quaternion differential equations of motion linearization yields non-stabilizable systems. Thus, LQR techniques cannot be directly employed since the associated algebraic Riccati equation is ill-posed. The commonplace solution resorts to a reduced quaternion model where only three out of four quaternion coordinates are exploited. The present work shows that such choice exhibits numerically unstable regions that impedes solving the LQR problem for all possible operating points. Additionally, we propose two methods to obtain well-posed LQR problems over all operating points. The first is based on the reduced quaternion model with an appropriate change of coordinates. The second is to append a virtual stabilizing input (VSI) to the nonlinear system to attain controllable linearized systems. The VSI direction should be appropriately chosen to not disturb the controllable modes of the system. Finally, we show that a class of constant angular velocity tracking problems is time-invariant under an appropriate change of variables such that time-invariant LQR techniques are applicable.**

I. INTRODUCTION

The problem of spacecraft attitude determination and control (ADCS) calls for simple and robust solutions in view of strict fault tolerance and certification requirements. Guidance, navigation and control (GNC) routines often run on computers that are typically lag with what is commercially available due to their elevated reliability, low power consumption and high tolerance to vibration and radiation commonly encountered during launch and cruise flight, respectively.

Among the diverse attitude parametrization philosophies existent in the literature, quaternion algebra stands out due to its simplicity and uniform numerical stable appliance in $SO(3)$ in sharp contrast to Euler angles that possess singularities that preclude their global employment and calls for local charts switching that increases system complexity. Furthermore, an increasing use of quaternion algebra in the field of unmanned aerial vehicles (UAV) is apparent as atmospheric vehicles become more acrobatic [1] and/or allow for multiple flight modes [2].

On the control systems counterpart, undemanding control laws are available for quaternion attitude control in both linear and nonlinear worlds. For instance, previous work [3] obtained globally exponentially stable proportional-derivative control laws by means of energy-based Lyapunov functions for the rigid-body tracking problem. Central drawbacks that arrive, namely, quaternion unwinding (due to double cover of $SO(3)$) and chattering (due to measurement noise), can be efficiently handled by hysteresis mechanisms [4]. However, the aforementioned techniques require fully-actuated systems. Furthermore, controller tuning and stability margins in view of plant uncertainties are not accounted for.

Linear techniques, on the other hand, are well-established and capable of dealing with plant uncertainties and nonlinearities by means of linearization over a trajectory and gain scheduling. Linear Quadratic Control (LQR)-based attitude controllers are often applied to underactuated vehicles due to its intuitive tuning process and reasonable stability robustness properties. Furthermore, LQR scales and integrates well in underactuated systems planning algorithms [5]. Although unwinding can be handled by path-lifting techniques [8], nonlinear quaternion differential equations of motion linearization yields non-stabilizable systems that preclude LQR techniques employment. Previous work on spacecraft [6] and UAV [7] attitude control resort to a reduced quaternion model where only three out of four quaternion coordinates are exploited. Moreover, the latter proves global stability and local optimality of the proposed approach.

The present work draws a numerical stability figure of merit of the optimal solution of the LQR problem for the aforementioned reduced quaternion model and presents its numerically unstable regions. Additionally, we propose two methods to obtain well-posed LQR problems over all operating points. The first is based on the reduced quaternion model with an appropriate change of coordinates. The second is to append a virtual stabilizing input (VSI) to the nonlinear system to attain controllable linearized systems. The VSI direction should be appropriately chosen to not disturb the controllable modes of the system. Finally, we show that a class of constant angular velocity tracking problems are time-invariant under an appropriate change of variables such that time-invariant LQR techniques are applicable.

The paper layout is as follows: section II presents the quaternion notation employed herein and reviews the pertinent linearized quaternion dynamics properties while presenting the key coordinate transformation of the present work. Section III reviews the relevant LQR concepts and presents a numerical stability figure of merit for the associated algebraic Riccati equation. Sections IV and V derive the two proposed methods for linearized quaternion attitude LQR control design followed by concluding remarks in section VI.

*Authors are with Institut Supérieur de l'Aéronautique et de l'Espace SUPAERO, Toulouse 31400, France {leandro-ribeiro.lustosa, flavio-luiz.cardoso-ribeiro, francois.defay, jean-marc.moschetta}@isae.fr

II. QUATERNION ATTITUDE DYNAMICS

Quaternion algebra formulation varies to a small extent in literature, but herein a quaternion $\boldsymbol{q} \in (\mathbb{R}^R, \times)$ is defined as

$$\boldsymbol{q} = \begin{pmatrix} q_0 \\ \boldsymbol{q}_1 \end{pmatrix} \tag{1}$$

where $q_0 \in \mathbb{R}$ and $\boldsymbol{q} \in \mathbb{R}^3$ equipped with quaternion product $\times$ operation defined as

$$\boldsymbol{p} \times \boldsymbol{q} = \begin{pmatrix} p_0 q_0 - \boldsymbol{p}_1 \cdot \boldsymbol{q}_1 \\ p_0 \boldsymbol{q}_1 + q_0 \boldsymbol{p}_1 + \boldsymbol{p}_1 \times \boldsymbol{q}_1 \end{pmatrix} \tag{2}$$

The ill-posed property that the present paper highlights when performing LQR control design in the quaternion formulation is present in both under and fully actuated systems. Therefore, for presentation clarity, we shall focus on the fully actuated version. Our proposed control techniques are nonetheless seamlessly applicable to both cases. The fully actuated spacecraft attitude dynamics equations of motion in the quaternion formulation can be written as

$$\dot{\boldsymbol{q}} = \frac{1}{2} \begin{bmatrix} 0 & -\boldsymbol{\omega}^T \\ \boldsymbol{\omega} & -[\boldsymbol{\omega}\times] \end{bmatrix} \begin{pmatrix} q_0 \\ \boldsymbol{q}_1 \end{pmatrix} \tag{3}$$

where $\boldsymbol{\omega} \in \mathbb{R}^3$ denotes spacecraft angular velocity with respect to inertial frame described in body-frame coordinates and $[\boldsymbol{v}\times]$ denotes matrix representation of vector product

$$[\boldsymbol{v}\times] = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \tag{4}$$

A noteworthy property of (3) is that it preserves vector norm. This can be concluded by analyzing the quaternion Euclidean norm derivative as

$$\frac{d}{dt}|\boldsymbol{q}|^2 = \frac{d}{dt}\left(\boldsymbol{q}^T \boldsymbol{q}\right) = 2\boldsymbol{q}^T \frac{d\boldsymbol{q}}{dt} = \boldsymbol{q}^T \begin{bmatrix} 0 & -\boldsymbol{\omega}^T \\ \boldsymbol{\omega} & -[\boldsymbol{\omega}\times] \end{bmatrix} \boldsymbol{q} \tag{5}$$

and by noticing that the quadratic form of any skew-symmetric matrix is identically zero and therefore

$$\frac{d}{dt}|\boldsymbol{q}| = 0_{4\times 1} \quad \forall \boldsymbol{q} \in \mathbb{R}^4 \tag{6}$$

even though we are interested in only the unitary-norm manifold. It will be shown that the technique figured in this work is generalizable to norm-preserving systems.

Furthermore, $\boldsymbol{\omega}$ evolves in time due to torques $\boldsymbol{u} \in \mathbb{R}^3$ according to

$$\dot{\boldsymbol{\omega}} = J^{-1}\left(\boldsymbol{u} - [\boldsymbol{\omega}\times] J \boldsymbol{\omega}\right) \tag{7}$$

where $J \in \mathbb{R}^{3\times 3}$ denotes spacecraft inertia matrix. An appropriate choice of system state is therefore

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{\omega} \end{pmatrix} \tag{8}$$

with associated nonlinear affine control differential equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + B\boldsymbol{u} \tag{9}$$

composed of (3) and (7) with B given by

$$B = \begin{bmatrix} 0_{4\times 3} \\ J^{-1} \end{bmatrix} \tag{10}$$

During LQR control design, the nonlinear dynamics are linearized around a nominal trajectory $(\boldsymbol{x}(t), \boldsymbol{u}(t))$ yielding the time-variant linear system

$$\Delta\dot{\boldsymbol{x}}(t) = A(t)\Delta\boldsymbol{x}(t) + B\Delta\boldsymbol{u}(t) \tag{11}$$

For the quaternion attitude dynamics, straightforward differentiation yields

$$\begin{aligned} A(t) &= \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial \dot{q}}{\partial \boldsymbol{q}} & \frac{\partial \dot{q}}{\partial \boldsymbol{\omega}} \\ \frac{\partial \dot{\omega}}{\partial \boldsymbol{q}} & \frac{\partial \dot{\omega}}{\partial \boldsymbol{\omega}} \end{bmatrix} = \\ &= \begin{bmatrix} \frac{1}{2}\begin{bmatrix} 0 & -\boldsymbol{\omega}^T(t) \\ \boldsymbol{\omega}(t) & -[\boldsymbol{\omega}(t)\times] \end{bmatrix} & \frac{1}{2}\begin{bmatrix} -\boldsymbol{q}_1^T(t) \\ q_0(t)I + [\boldsymbol{q}_1(t)\times] \end{bmatrix} \\ 0_{3\times 4} & [\boldsymbol{\omega}(t)\times] - J^{-1}[\boldsymbol{\omega}(t)\times]J \end{bmatrix} \end{aligned} \tag{12}$$

Controllability and stabilizability of the system play a fundamental role in the design of LQR controllers and, therefore, are studied in the following. Firstly, notice that (6) implies that the reachable set of the nonlinear system is included in

$$M = \{\boldsymbol{q} \in \mathbb{R}^4 : |\boldsymbol{q}| = 1\} \subsetneq \mathbb{R}^4 \tag{13}$$

and therefore is not controllable. Additionally, the local linear model is not controllable either. This can be seen by a controllability normal form of the system which can be obtained by the following change of coordinates

$$\Delta\boldsymbol{x}'(t) = U(t)\Delta\boldsymbol{x}(t) \tag{14}$$

where

$$U(t) = \begin{bmatrix} U_q(t) & 0_{4\times 3} \\ 0_{3\times 4} & I_3 \end{bmatrix} \tag{15}$$

and

$$U_q(t) = \begin{bmatrix} q_0(t) & \boldsymbol{q}_1^T(t) \\ -\boldsymbol{q}_1(t) & q_0(t)I_3 - [\boldsymbol{q}_1(t)\times] \end{bmatrix} \tag{16}$$

and therefore

$$\Delta\dot{\boldsymbol{x}}'(t) = A'(t)\Delta\boldsymbol{x}'(t) + B'(t)\Delta\boldsymbol{u}(t) \tag{17}$$

where[1]

$$\begin{aligned} A'(t) &= \dot{U}(t)U^{-1}(t) + U(t)A(t)U^{-1}(t) = \\ &= \begin{bmatrix} \frac{1}{2}\begin{bmatrix} 0 & \boldsymbol{0}^T \\ \boldsymbol{0} & -[\boldsymbol{\omega}(t)\times] \end{bmatrix} & \frac{1}{2}\begin{bmatrix} \boldsymbol{0}^T \\ I_3 \end{bmatrix} \\ O_{3\times 3} & [\boldsymbol{\omega}(t)\times] - J^{-1}[\boldsymbol{\omega}(t)\times]J \end{bmatrix} \end{aligned} \tag{18}$$

and

$$B'(t) = U(t)B = \begin{bmatrix} 0_{4\times 3} \\ J^{-1} \end{bmatrix} \tag{19}$$

Notice that (15) is a valid coordinate transformation since the matrix U_q represents a conjugate left product in quaternion space and, therefore, it is invertible for all non-zero quaternions. From the linear system form in (18) and (19), we can conclude that the linear system is non-stabilizable

[1] The algebra is tedious and therefore omitted. If an adventurous reader wishes to check the computations, the following identity is valuable: $[\boldsymbol{y}\times]^2 + ||\boldsymbol{y}||^2 I = \boldsymbol{y}\boldsymbol{y}^T$.

(therefore uncontrollable) for all trajectories by looking at the first transformed state variable $\Delta x'_1$ dynamics, which is

$$\Delta \dot{x}'_1 = 0 \quad \forall \boldsymbol{x}(t), \boldsymbol{u}(t) \tag{20}$$

Notice that the transformed system (A', B') is time-variant in general. However, if we consider only reference trajectories that are constant in $\boldsymbol{\omega}(t) = \boldsymbol{\omega}_0$ (hereafter denoted ω-trajectories), the system becomes time-invariant[2]. Furthermore, the linear system has 6 controllable modes for all ω-trajectories. This can be seen by means of the transformed system (A', B') Kalman controllability matrix K:

$$K = \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix} = \\ = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \cdots \\ 0_{3\times3} & \frac{1}{2}J^{-1} & -\frac{1}{2}J^{-1}[\boldsymbol{\omega}_0\times] & \cdots \\ J^{-1} & \Delta J^{-1} & \Delta^2 J^{-1} & \cdots \end{bmatrix} \tag{21}$$

where

$$\Delta \triangleq [\boldsymbol{\omega}_0\times] - J^{-1}[\boldsymbol{\omega}_0\times]J \tag{22}$$

The lower triangular structure guarantees row rank 6 for all ω-trajectories. The controllability properties exploited in this section are fundamental notions in the LQR design to follow.

III. The time-invariant linear quadratic regulator

Consider the time-invariant linear system in state-space representation given by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \tag{23}$$

where $\boldsymbol{x}(t) : \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{u}(t) : \mathbb{R} \to \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are, respectively, state vector, input (or control) vector, system matrix and input matrix. We are interested in computing the optimal control policy $\boldsymbol{u}^*(t)$ that minimizes the cost

$$J(\boldsymbol{x}_0, \boldsymbol{u}) = \int_0^\infty \left(\boldsymbol{x}^TQ\boldsymbol{x} + \boldsymbol{u}^TR\boldsymbol{u}\right) dt \tag{24}$$

where $\boldsymbol{x}_0 = \boldsymbol{x}(0)$, $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ are, respectively, initial state, positive semi-definite state penalty and positive definite actuator penalty matrices. This strategy is called the linear quadratic regulator (LQR) [11]. By means of the Hamilton-Jacobi-Bellman equation, one can show [11] that the control policy

$$\boldsymbol{u}^*(t) = \arg\min_{\boldsymbol{u}} J(\boldsymbol{x}_0, \boldsymbol{u}) \tag{25}$$

for the system described by (23) is independent of $\boldsymbol{x}_0$ and is given by

$$\boldsymbol{u}^*(t) = -R^{-1}B^TP\boldsymbol{x}(t) \tag{26}$$

where P is the unique solution of the algebraic Riccati equation (ARE)

$$PA + A^TP - PBR^{-1}B^TP + Q = 0 \tag{27}$$

if (A, B) is stabilizable and $(A, Q^{1/2})$ is detectable. The solution can be numerically computed [12], [13] and a figure of merit for its numerical stability, represented here by means of its relative condition number, is [14]

$$c_{rel}(A, B, Q, R) = \frac{1}{||P||_F}\Big(||Q||_F||Z_1||_2 + \\ + ||A||_F||Z_2||_2 + ||BR^{-1}B^T||_F||Z_3||_2\Big) \tag{28}$$

where

$$Z_1 = T^{-1} \tag{29}$$

$$Z_2 = T^{-1}\Big(I \otimes P + (P \otimes I)\Pi\Big) \tag{30}$$

$$Z_3 = T^{-1}(P \otimes P) \tag{31}$$

and

$$T = I_n \otimes (A - BR^{-1}B^TP)^T + (A - BR^{-1}B^TP)^T \otimes I_n \tag{32}$$

with $||\cdot||_F$, $||\cdot||_2$ and $\otimes$ denoting, respectively, Frobenius norm, 2-norm and the Kronecker product.

The LQR can be extended [11] to time-varying linear systems with an increase of algorithm complexity and required look-up table memory for implementation in spacecraft embedded systems. Fortunately, section II proved that the spacecraft attitude regulation and ω-trajectory tracking problems are time-invariant under appropriate change of coordinates. Be that as it may, quaternion linearization was shown to suffer from lack of controllability in all operating points precluding a theoretical LQR solution to exist or a numerically stable computational solution to be found. This short-coming is addressed in the next two sections.

IV. The reduced quaternion model revisited

Intuitively, the lack of controllability in quaternion systems arrives from its non-minimal representation that lives in a proper subset of $\mathbb{R}^4$. To address this issue, previous work [6] rewrites (12) with, for instance, the first coordinate q_0 of a quaternion in terms of the other components such that

$$q_0 = \pm\sqrt{1 - q_1^2 - q_2^2 - q_3^2} \tag{33}$$

and drops the redundant associate q_0 lines and columns from the system. In a neighborhood of an operating point in $\{(\boldsymbol{q}, \boldsymbol{\omega}) : \boldsymbol{q}^T\boldsymbol{q} = 1, \boldsymbol{\omega} \in \mathbb{R}^3\}$ the signal ambiguity can be resolved.[3] Furthermore, previous work assumes a regulation problem, i.e., reference trajectory with $\boldsymbol{\omega}(t) = \mathbf{0}$ for all t. These yield the so-called reduced quaternion model (A_r, B_r) given by

$$A_r = \begin{bmatrix} 0_{3\times3} & \frac{1}{2}\left[\pm\sqrt{1 - q_1^2 - q_2^2 - q_3^2}I + [\boldsymbol{q}_1\times]\right] \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix} \tag{34}$$

and

$$B_r = \begin{bmatrix} 0_{3\times3} \\ J^{-1} \end{bmatrix} \tag{35}$$

[2]Notice that our approach yields an alternative to Floquet's theorem approach to computing $U(t)$ for the present problem.

[3]The careful reader will notice that a neighborhood of $q_0 = 0$ might yield an ill-defined ambiguity. Nevertheless, one can argue that we should never choose such points as trimming points and proceed with controller design.

such that

$$\begin{pmatrix}\Delta\dot{\boldsymbol{q}}_1\\ \Delta\dot{\boldsymbol{\omega}}\end{pmatrix} = A_r \begin{pmatrix}\Delta\boldsymbol{q}_1\\ \Delta\boldsymbol{\omega}\end{pmatrix} + B_r\Delta\boldsymbol{u} \tag{36}$$

Notice that the regulation problem assumption reduces the model to a linear time-invariant system whereas the deletion of redundant columns and lines potentially renders the system controllable (and therefore eligible for LQR framework employment). Indeed, the controllability matrix K_r of the reduced system yields

$$\begin{aligned} K_r = \begin{bmatrix} B_r & A_rB_r & A_r^2B_r & \cdots \end{bmatrix} = \\ = \begin{bmatrix} 0_{3\times3} & \frac{1}{2}\Theta J^{-1} & 0_{3\times3} & \cdots \\ J^{-1} & 0_{3\times3} & 0_{3\times3} & \cdots \end{bmatrix} \end{aligned} \tag{37}$$

where

$$\Theta \triangleq \pm\sqrt{1-q_1^2-q_2^2-q_3^2}I + [\boldsymbol{q}_1\times] \tag{38}$$

Therefore, full row rank of K_r is conditioned to

$$\begin{aligned} &\det\left(\frac{1}{2}\left(\pm\sqrt{1-q_1^2-q_2^2-q_3^2}I + [\boldsymbol{q}_1\times]\right)J^{-1}\right) = \\ &= \frac{1}{8\det J}\begin{vmatrix} q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{vmatrix} = \frac{\pm\sqrt{1-q_1^2-q_2^2-q_3^2}}{8\det J} \end{aligned} \tag{39}$$

which allows one to conclude that the quaternion reduced model is globally controllable except for $q_0 = 0$ operating points. It is expected therefore a variable numerical stability in the space of configurations. Figure 1 illustrates the condition number $c_{rel}(A_r, B_r, I, I)$ of the associated Riccati problem in function of different *ZYX*-order Euler angles $\{\psi, \theta, \phi\}$, respectively, yaw, pitch and roll. We conclude that near-uncontrollable operating points are numerical unstable delivering unreliable local optimal controllers.

Previous work [6] regulates for $\boldsymbol{q} = (1, 0, 0, 0)$, and, therefore, does not encounter problems. However, a wide attitude envelope spacecraft quaternion operating point controller can run into numerical problems calling for a more stable description. Furthermore, numerical stability over the whole configuration space is one of the most attractive properties of quaternions that is undermined by the present LQR workflow.

We propose a solution to this problem by means of the transformation given in (14). The associated controllability matrix is given by

$$\begin{aligned} K_r' = \begin{bmatrix} B_r' & A'B_r' & (A_r')^2B_r' & \cdots \end{bmatrix} = \\ = \begin{bmatrix} 0_{3\times3} & \frac{1}{2}J^{-1} & 0_{3\times3} & \cdots \\ J^{-1} & 0_{3\times3} & 0_{3\times3} & \cdots \end{bmatrix} \end{aligned} \tag{40}$$

which is clearly full row rank thus controllable. Additionally, we obtain constant numerical stability $\log(\log(\log(c))) = -0.49688$ over the entire configuration space. Based on the foregoing discussion, algorithm 1 illustrates how the proposed transformation enters in a typical LQR design workflow for arbitrary spacecraft attitude control problems containing quaternion formulation. Notice that this design workflow requires Jacobian matrices A' and B' that are

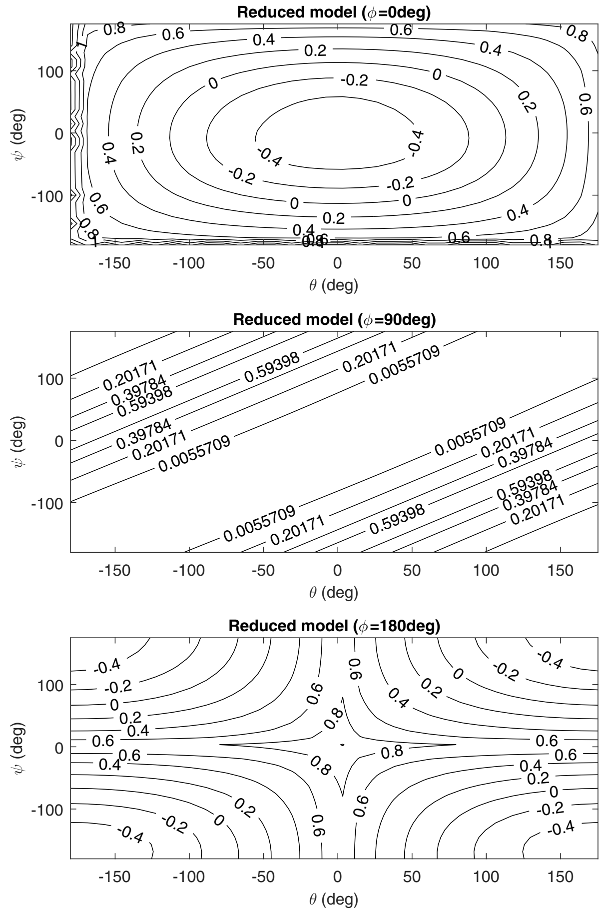

Fig. 1. Numerical stability of the reduced formulation of the linearized quaternion LQR by means of the contour curves of $log(log(log(c)))$. Three log levels are employed due to ill-conditioning and exponentially growing conditioning number of the reduced model Riccati equation. The two other methods yield constant conditioning over all configuration space and are equal to $c_{rel} = -0.49688$ and $c_{rel} = -0.42785$ for the transformed reduced and VSI models, respectively.

not the original system derivatives and, therefore, precludes employment of automated LQR controller generators (i.e., automatic tools that deliver LQR controllers for arbitrary nonlinear systems by means of numerical Jacobian computation and direct insertion of those in LQR gain computation routines). The next section proposes an alternative solution to the non-stabilizable quaternion LQR problem that allows for automatic tools application and yields a cleaner workflow.

Finally, in section II, we concluded that the quaternion system tracking problem (with constant velocity trajectory $\boldsymbol{\omega}(t) = \boldsymbol{\omega}_0$) is time-invariant in transformed coordinates allowing for time-invariant LQR design. Therefore, a small modification of algorithm 1 allows for tracking control as illustrated in algorithm 2. Figure 2 illustrates a simple ω-trajectory tracking in the y-axis with magnitude $|\boldsymbol{\omega}_0| = 20$ deg/s with initial errors of 30deg and $0.5|\boldsymbol{\omega}_0|$ in each Euler angle and each angular velocity component, respectively.

Data: given $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$, $\boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{\omega}, \dots) \in \mathbb{R}^n$
Result: find LQR control $\Delta\boldsymbol{u} = -K\Delta\boldsymbol{x}$
operating points to be regulated:
$\boldsymbol{x} \in X_0 = \{(\boldsymbol{q}, \boldsymbol{0}, \dots) : \boldsymbol{q} \in Q\}$ and $\boldsymbol{u} \in U_0$;
for $\boldsymbol{x}_i \in X_0$, $\boldsymbol{u}_i \in U_0$ **do**
compute Jacobians $A = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}$, $B = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}$;
compute $U = \begin{bmatrix} U_q(\boldsymbol{q}_i) & 0 \\ 0 & I_{n-4} \end{bmatrix}$;
compute $A' = UAU^T$ and $B' = UB$;
compute $K' = \mathrm{lqr}(A', B', Q, R)$;
remove first line of U;
compute $K = -K'U$;
end

Algorithm 1: LQR reduced regulation controllers for a set Q of desirable attitude points.

As expected, the LQR controller converges to the desired trajectory notwithstanding initial large tracking errors.

Data: given $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$, $\boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{\omega}, \dots) \in \mathbb{R}^n$
Result: find LQR control $\Delta\boldsymbol{u} = -K\Delta\boldsymbol{x}$
operating points to be regulated:
$\boldsymbol{x} \in X_0 = \{(\boldsymbol{q}, \boldsymbol{0}, \dots) : \boldsymbol{q} \in Q\}$ and $\boldsymbol{u} \in U_0$;
for $\boldsymbol{x}_i \in X_0$, $\boldsymbol{u}_i \in U_0$ **do**
compute Jacobians $A = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}$, $B = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}$;
compute $U = \begin{bmatrix} U_q(\boldsymbol{q}_i) & 0 \\ 0 & I_{n-4} \end{bmatrix}$;
compute $A' = UAU^T$ and $B' = UB$;
compute $K' = \mathrm{lqr}(A', B', Q, R)$;
remove first line of U;
compute $K = -K'U$;
end

Algorithm 2: LQR reduced regulation controllers for a set Q of desirable attitude points.

V. The virtual stabilizing input solution

An alternative intuitive solution to the problem is to append an extra control input $u_a(t)$ to turn the non-stabilizable mode of the system into a controllable one. This control input is not part of the physical system and must be orthogonal to the system modes to not disburb real physical dynamics while affecting the zero dynamics found in (18). A natural direction is given, therefore, by the first line of the $U(t)$ matrix given by (15) yielding the augmented system

$$\begin{pmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{\omega}} \end{pmatrix} = A \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{\omega} \end{pmatrix} + \left[B \;\middle|\; \begin{matrix} \boldsymbol{q} \\ \boldsymbol{0} \end{matrix} \right] \begin{pmatrix} \boldsymbol{u} \\ u_a \end{pmatrix} \tag{41}$$

Notice that it matches our intuition in the sense that the virtual input should point in the direction in which we have no control, which is the direction of the quaternion itself. The reasoning behind this is illustrated by (5) and (6). Quaternions live in a 4-dimensional sphere and its derivatives are restricted to the tangent of the sphere in a given point of operation $\boldsymbol{q}$. Therefore, the direction $\boldsymbol{q}$ is, indeed, the most suitable direction for the artificial input.

Additionally, we observe that instead of appending the virtual input at the linearized mode, one can add it directly in the nonlinear model such that

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) + \begin{bmatrix} \boldsymbol{q}(t) \\ \boldsymbol{0} \end{bmatrix} u_a(t) \tag{42}$$

since its linearization yields (with respect to operating point $u_a = 0$)

$$\Delta\dot{\boldsymbol{x}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \Delta\boldsymbol{x} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} \Delta\boldsymbol{u} + \begin{bmatrix} \boldsymbol{q}(t) \\ \boldsymbol{0} \end{bmatrix} u_a(t) \tag{43}$$

which results in a linearized artificial input in the desirable direction. The resulting linearized dynamics are given by

$$A_a = A, \quad B_a = \left[B \;\middle|\; \begin{matrix} \boldsymbol{q} \\ \boldsymbol{0} \end{matrix} \right] \tag{44}$$

with controllability matrix (for the regulation problem) given by

$$\begin{aligned} K_a &= \begin{bmatrix} B_a & A_a B_a & A_a^2 B_a & \cdots \end{bmatrix} = \\ &= \begin{bmatrix} 0_{4\times 3} & \boldsymbol{q} & \frac{1}{2} \begin{bmatrix} -\boldsymbol{q}_1^T(t) \\ q_0(t) I + [\boldsymbol{q}_1(t)\times] \end{bmatrix} J^{-1} & \cdots \\ J^{-1} & \boldsymbol{0} & \boldsymbol{0} & \cdots \end{bmatrix} \end{aligned} \tag{45}$$

Therefore, full row rank of K_a is conditioned to

$$\det \left(\begin{bmatrix} \boldsymbol{q} & \frac{1}{2} \begin{bmatrix} -\boldsymbol{q}_1^T(t) \\ q_0(t) I + [\boldsymbol{q}_1(t)\times] \end{bmatrix} J^{-1} \end{bmatrix} \right) \neq 0 \tag{46}$$

since

$$\begin{vmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{vmatrix} = 1 \tag{47}$$

which allows one to conclude that the quaternion virtual input method is globally controllable. Additionally, its associated ARE condition number in the space of configurations is contant and equal to $\log(\log(\log(c))) = -0.42785$ thus numerically stable for all operating points with only a marginal difference when compared to the reduction model due to the increase of the matrices orders involved. The major gain in this strategy is when integrated in a LQR automated design workflow. If the nonlinear model is assumed with virtual input, than no modifications on the LQR flow are required, i.e., Jacobians can be numerically computed and readily inserted in the ARE solver as the algorithm 3 illustrates. The simplicity and clarity of the proposed approach is evident. We reinforce that $u_a(t)$ is not related to any physical input in any sense and its corresponding component in the LGR gain K should be disregarded.

Finally, notice that the present technique is restricted to the regulation problem. The tracking problem requires transformation of coordinates to be eligible to time-invariant LQR design.

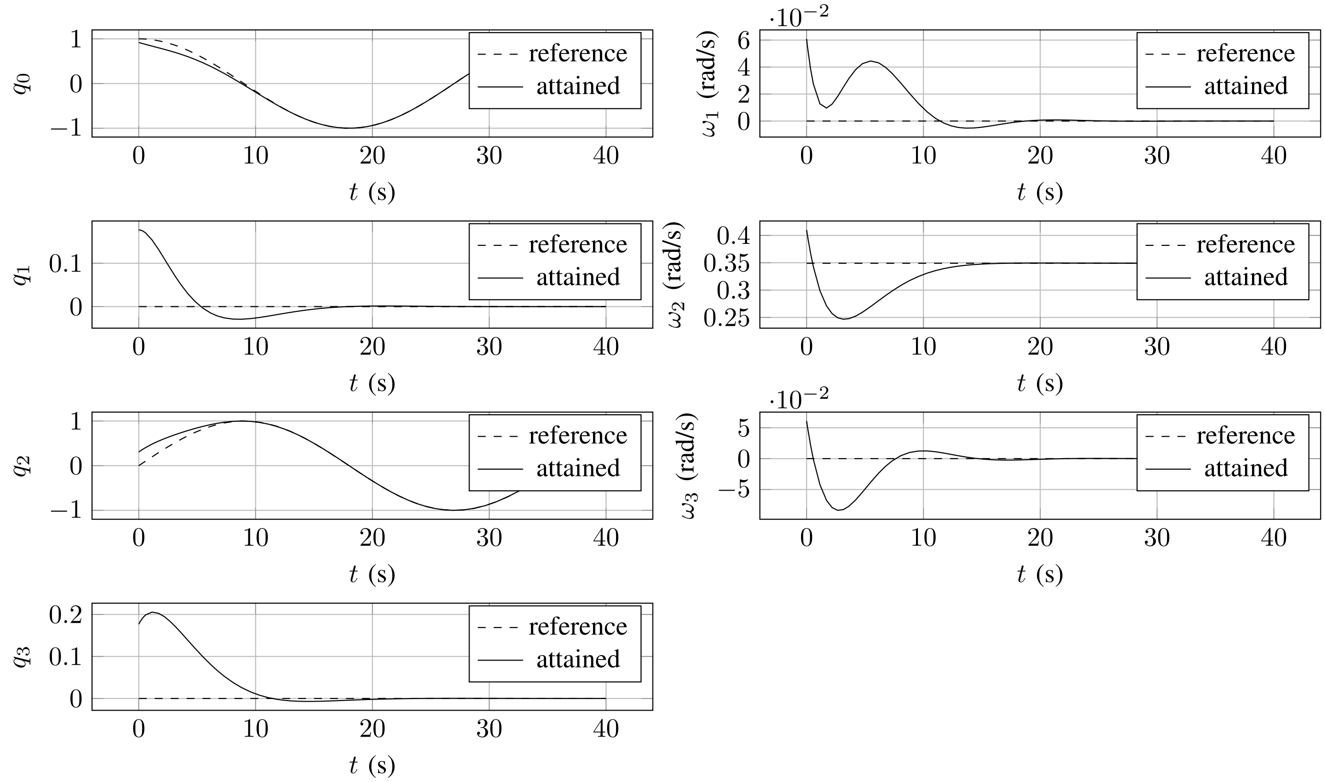

Fig. 2. Tracking of an ω-trajectory with transformed reduced quaternion model.

Data: given $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) + (\boldsymbol{q}, \boldsymbol{0}, \cdots)^T u_a$,
$\boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{\omega}, \dots) \in \mathbb{R}^n$
Result: find LQR control $\Delta(\boldsymbol{u}, u_a)^T = -K\Delta\boldsymbol{x}$
operating points to be regulated:
$\boldsymbol{x} \in X_0 = \{(\boldsymbol{q}, \boldsymbol{0}, \dots) : \boldsymbol{q} \in Q\}$ and $\boldsymbol{u} \in U_0$;
for $\boldsymbol{x}_i \in X_0$, $\boldsymbol{u}_i \in U_0$ **do**
  compute Jacobians $A = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}$, $B = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}$;
  compute $K = \text{lqr}(A, B, Q, R)$;
end

Algorithm 3: LQR virtual regulation controllers for a set Q of desirable attitude points.

VI. Conclusion

The present paper revisits the problem of employing LQR design in quaternion-based linearized systems for local control of complex underactuated systems. We warn of numerically unstable regions in a popular algorithm and propose two alternative solutions that yield constant low relative condition numbers for the entire configuration space. The first solution is built on top of the commonplace solution whereas the second one is an elegant and simple reformulation suited for applications that require minimal intrusion in the classical LQR design workflow. We also demonstrate how to adapt the first solution to the constant angular velocity tracking problem.

References

[1] Casau P., Cabecinhas D., Silvestre C., *Hybrid control strategy for the autonomous transition flight of a fixed-wing aircraft*, IEEE Transactions on Control Systems Technology, Vol. 21, No. 6, November 2013.

[2] Lustosa L. R., Defay F., Moschetta J. M., *Longitudinal study of a tilt-body vehicle: modeling, control and stability analysis*, International Conference on Unmanned Aircraft Systems, 2015.

[3] Wen J. T. Y., Delgado K. K., *The attitude control problem*, IEEE Transactions on automatic control, Vol. 36, No. 10, October 1991.

[4] Mayhew C.G., Sanfelice R.G., Teel A.R., *Quaternion-based hybrid control for robust global attitude tracking*, IEEE Transactions on Automatic Control, Vol. 56, No. 11, November 2011.

[5] Moore J., Tedrake R., *Control synthesis and verification for a perching UAV using LQR-trees*, In Proceedings of the IEEE Conference on Decision and Control, page 8, Maui, Hawaii, December 2012.

[6] Yang Y., *Controllability of spacecraft using only magnetic torques*, IEEE Transactions on Aerospace and Electronic Systems, Vol. 52, Issue 2, April 2016.

[7] Valeria E. R., Caldera R. E., Lara S. C., Guichard J., *LQR control for a quadrotor using unit quaternions: modeling and simulation*, Electronics, Communications and Computing (CONIELECOMP), 2013 International Conference on, p. 172-178, 2013.

[8] Mayhew C.G., Sanfelice R.G., Teel A.R. *On path-lifting mechanisms and unwinding in quternion-based attitude control*, IEEE Transactions on Automatic Control, Vol. 58, No. 5, May 2013.

[9] Kawamura Y., *An extension of the Algebraic Riccati Equation for the Stationary Control Problem Without Stabilizability Condition*, IEEE Transactions on Automatic Control, Vol. AC-30, No. 10, October 1985.

[10] Wu G., Sreenath K., *Variation-based linearization of nonlinear systems evolving on $SO(3)$ and $\mathbb{S}^2$*, IEEE Access, Vol. 3, pp 1592-1604, September 2015.

[11] Anderson B. D. O., Moore J. B., *Optimal control, Linear quadratic methods*, Dover, Mineola, 1990.

[12] Laub A. J., *A Schur method for solving algebraic Riccati equations*, Massachussets Institute of Technology, Laboratory for Information and Decision systems, LIDS, Report number 859, 1978.

[13] Petkov P., Christov N., Konstantinov M., *Computational methods for linear control systems*, New York: Prentice, 1991.

[14] Byers R., *Numerical condition of the algebraic Riccati equation, Linear Algebra and its role in system theory*, B. N. Datta, ed., 35-49, Providence, RI: AMS Contemporary Mathematics Series, vol. 47, 1985.