2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Sufficient stability condition for delayed port-Hamiltonian systems subject to input saturation*

Said Aoues[1], Damien Eberard[2], Warody Lombardi[3], Alexandre Seuret[4]

Abstract— **This work deals with a class of delayed port-Hamiltonian systems with control law given as a bounded output feedback. A sufficient stability condition of the resulting closed-loop dynamics is obtained combining Wirtinger-based integral inequality and Lyapunov-Krasovkii theorem. This condition is formulated in terms of (parametrized) linear matrix inequality. A numerical example shows reduction of the conservatism regarding literature result.**

I. Introduction

Time delays and actuators' saturation are inherent phenomenon in many engineering applications. These phenomenon require particular attention as both might generate erratic behavior and/or deteriorate performances (see *e.g.* [10], [11], [16], [7] or [3]). Generally speaking, stabilizability results have been obtained for linear systems (see [15] for bounded controls, and [19] with additional delays), and the nonlinear case remains an unreachable target. In this paper, a *sufficient* stability condition is derived for a particular class of nonlinear systems described by *delayed port-Hamiltonian* equations.

From a modeling viewpoint, port-Hamiltonian framework gives a characterization of power exchanges between interconnected subsystems in terms of energy functions, network interconnection topology, and dissipation ([8], [17]). Analyzing stability, the Hamiltonian (the total energy of the system) appears as a natural and successful Lyapunov candidate. However, this approach fails when the aforementioned phenomenon occur.

In the literature, sufficient stability conditions for delayed port-Hamiltonian systems have been derived in terms of (parametrized) *linear matrix inequality* (LMI) using Jensen inequality (*e.g.* [20], [5], [9], [12]) and Wirtinger-based integral inequality [2]. The latter inequality being known to reduce conservatism. Separately, the H_∞ control of port-Hamiltonian systems subject to input saturation have been studied in [18]. Regarding delay *and* saturation, a sufficient stability condition has been derived in terms of (parametrized) *bilinear matrix inequality* (BMI) [14], which lacks implementation tractability.

*This work was supported by the ANR project ANR-11-JS03-0004 financed by the French National Research Agency.

[1]S. Aoues is with Altran Company, 4 avenue Didier Daurat, 31700 Blagnac, France

[2]D. Eberard is with Ampere-lab, UMR CNRS 5005, Université de Lyon, INSA Lyon, 25 av. Jean Capelle, 69621 Villeurbanne Cedex, France

[3]W. Lombardi is with CEALETI, Minatec Campus, 17 rue des Martyrs, 38054 Grenoble Cedex, France

[4]A. Seuret is with LAAS-CNRS, Université de Toulouse, 7 avenue du Colonel Roche, F-31400 Toulouse, France

The current work is concerned with the stability analysis of delayed port-Hamiltonian systems with control law given as a saturated static output feedback. Using Wirtinger-based integral inequality and Lyapunov-Krasovskii theorem, a sufficient stability condition is given in terms of (parametrized) LMI. This result extends previous ones in two ways. First, it provides a more tractable criteria than the one given in terms of BMI. Second, as Wirtinger inequality is involved, the condition is less conservative than the ones derived using Jensen inequality. Last, the class of delayed port-Hamiltonian considered here encompasses those considered in the literature (compare [20], [5], [9], [2] with [12], see Remark 2). .

Organization of the paper is as follows. Section II recaps the notations used throughout the paper. Section III introduces the closed-loop dynamics of delayed port-Hamiltonian systems with saturated output feedback and its related stability issue. In Section IV, a sufficient stability condition is derived in terms of *parametrized* LMI. An affine (hence implementable) condition is deduced by weakening the class considered. In Section V, a numerical example illustrates the improvement of the result regarding the literature. Finally, Section VI concludes the paper.

II. Notations

Throughout the paper $\mathbb{R}^{n\times m}$ ($\mathbb{S}^n$) denotes the set of $n \times m$ real matrices ($n \times n$ real symmetric matrices). The notation $P \in \mathbb{S}^n_+$ means that $P \in \mathbb{S}^n$ and $P \succ 0$, that is P is a symmetric positive definite real matrix. For any matrices A, B of appropriate dimension, the notation $\mathrm{diag}(A, B)$ stands for the block-diagonal matrix $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. Symbol $*$ used in a matrix means *transpose complement*, that is $\left[\begin{smallmatrix} A & B \\ * & C \end{smallmatrix}\right] = \left[\begin{smallmatrix} A & B \\ B^T & C \end{smallmatrix}\right]$. Matrices I_n and $0_{n,m}$ represent the identity and null matrices of appropriate dimension, and when no confusion is possible the subscript will be omitted. For any function $f(x(\cdot)) : \mathbb{R} \to \mathbb{R}^n$, the notation f_{t-t_0} stands for $f(x(t - t_0))$, and ∇f is the gradient of f.

III. Problem statement

We consider a class of nonlinear systems, named as *delayed port-Hamiltonian systems*, defined by the following equations

$$\begin{aligned} \dot{x}(t) = [J(x(t)) - R(x(t))]\nabla H(x(t)) + g(x(t))\mathrm{sat}(u) \\ + M(x(t))\nabla H(x(t - \tau(t))) \qquad \text{(1a)} \end{aligned}$$

$$y = g^T(x(t))\nabla H(x(t)) \qquad \text{(1b)}$$

$$x(t) = \phi(t) \quad t \in [-\tau_{max}, 0] \qquad \text{(1c)}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, the state dependent matrices $J, R, M : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ satisfy $J^T = -J$, $R^T = R \succeq 0$, the Hamiltonian function $H : \mathbb{R}^n \to \mathbb{R}$ is assumed to be differentiable and bounded from below, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is the state dependent input vector fields, and $u, y \in \mathbb{R}^m$ are the port-input, port-output respectively.

The delay τ is a bounded differentiable map on $\mathbb{R}_+$, τ_{max} denotes the maximum time-delay and δ_{max} the maximum time-delay rate, that is for all $t > 0$

$$0 \leq \tau(t) \leq \tau_{max} \tag{2a}$$
$$0 \leq \dot\tau(t) \leq \delta_{max} < 1\,. \tag{2b}$$

The initial condition $\phi : [-\tau_{max}, 0] \to \mathbb{R}^n$ in (1c) is assumed to be differentiable.

The control law, given as a static output feedback

$$u = -K(t)y\,, \tag{3}$$

where $K : \mathbb{R}_+ \to \mathbb{R}^{m\times m}$ is a matrix-valued function, is subject to actuators' saturation: each control component u_i is restricted to the centered interval $[-\bar u_i\,, \bar u_i]$ (with $\bar u_i > 0$) by applying

$$\mathrm{sat}(u_i) = \begin{cases} \bar u_i & \text{if} \quad u_i > \bar u_i \\ u_i & \text{if} \quad |u_i| \leq \bar u_i \\ -\bar u_i & \text{if} \quad u_i < -\bar u_i \end{cases}. \tag{4}$$

Classically, nonlinearities of the actuators are transformed into sector nonlinearities by introducing a variable $v \in \mathbb{R}^m$ as follows

$$v = \mathrm{sat}(u) - u\,. \tag{5}$$

Remark 1: Note the first row of (1a) is the standard port-Hamiltonian dynamics ([8]), and a time-delay term $\nabla H(x(t-\tau(t)))$ appears in the second row. The assumption 'H bounded from below' is made to recover standard *passivity* analysis in case $M \equiv 0$.

Then plugging (3), (4), (5) into (1), the closed-loop dynamics with delay and saturation compactly rewrites as

$$\dot x(t) = [J - R - gKg^T]_t \nabla H_t + M_t \nabla H_{t-\tau(t)} + g_t v_t\,, \tag{6}$$

where the notation $f_{t-t_0} = f(x(t-t_0))$ has been employed.

Remind the common way to analyze stability within port-Hamiltonian framework is to take the Hamiltonian H as Lyapunov candidate. Let us compute $\frac{\mathrm{d}}{\mathrm{d}t}H$ to illustrate the inability to conclude when time-delay and saturation occur. Indeed, from (6) straightforward computations lead to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H_t =& -[\nabla H^T(R + gKg^T)\nabla H]_t + \nabla H_t^T M_t \nabla H_{t-\tau(t)} \\ &+ [(\nabla H^T g)v]_t \\ \leq& -[\nabla H^T(R + gKg^T)\nabla H]_t + \nabla H_t^T M_t \nabla H_{t-\tau(t)} \\ &+ \frac{1}{2}\left[(\nabla H^T g)(\nabla H^T g)^T + v^T v\right]_t \\ \leq& -[\nabla H^T(R + gKg^T)\nabla H]_t + \nabla H_t^T M_t \nabla H_{t-\tau(t)} \\ &+ \frac{1}{2}\left[\nabla H^T(gg^T)\nabla H + (Kg^T\nabla H)^T(Kg^T\nabla H)\right]_t\,,\end{aligned}$$

where the first inequality comes from the fact that $2a^T b \leq a^T a + b^T b$ for any column vector a, b, and the second inequality comes from the fact that $v^T v \leq u^T u$ by definition of v given in (5). Rearranging the terms, one finally gets

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H_t \leq& -\left[\nabla H^T\left(R + g(K - \tfrac{1}{2}K^T K - \tfrac{1}{2}\mathrm{I})g^T\right)\nabla H\right]_t \\ &+ \nabla H_t^T M_t \nabla H_{t-\tau(t)}\,.\end{aligned} \tag{7}$$

From the first row of (7), one recognizes the classical condition to be satisfied by gain K to ensure stabilizing output feedback in case no delay occurs. Nevertheless, the indefinite sign of the additional term $\nabla H_t^T M_t \nabla H_{t-\tau(t)}$ prevent from concluding.

This highlights that solely the Hamiltonian function no longer contains enough information to deal with closed-loop stability when time-delay and saturation occur. Instead a Lyapunov-Krasovskii candidate is required.

Remark 2: Several classes of delayed port-Hamiltonian systems have been introduced in the literature, and it is hard to include one into another. For instance, the class considered in [12], formatted with our notation, reads

$$\dot x(t) = [J_t - R_t - T]\nabla H_t + T\nabla H_{t-\tau(t)} \tag{8}$$

where T is an arbitrary matrix. Let us briefly compare (8) with (6) (setting $v_t \equiv 0$). On the one hand, (8) avoids the restriction $T = [gKg^T]_t$ (omitting the x-dependecy). On the other hand, (8) requires $[gKg^T]_t = M_t$ to fit in (6). A more general formulation would consider both arbitrary matrices T and M in the dynamics.

IV. Main result

Before stating our main result, let us first recall a Wirtinger-based integral result presented in [13]. It will be instrumental in deriving a lower bound when differentiating the Lyapunov-Krasovkii candidate.

Lemma 1: Given $Q \in \mathbb{S}_+^n$ and a continuously differentiable function $f :]a, b[\to \mathbb{R}^n$, then the following inequality holds

$$(b-a)\int_a^b \frac{\mathrm{d}f}{\mathrm{d}s}^T(s)\, Q\, \frac{\mathrm{d}f}{\mathrm{d}s}(s)\mathrm{d}s \geq \begin{bmatrix} F_0 \\ F_1 \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & 3Q \end{bmatrix} \begin{bmatrix} F_0 \\ F_1 \end{bmatrix} \tag{9}$$

where $F_0 = f(b) - f(a)$ and $F_1 = f(b) + f(a) - \frac{2}{b-a}\int_a^b f(s)\mathrm{d}s$.

A sufficient stability condition for (6) is stated as follows.

Proposition 1: Assume the Hamiltonian H is a regular positive definite function in a neighborhood of the origin. Then, the trajectory of a delayed port-Hamiltonian system (1) with bounded inputs given by (3), (4) and (5) is (locally uniformly) asymptotically stable for all time-varying delays τ satisfying (2) if there exist matrices of appropriate dimensions S and Q both in $\mathbb{S}_+$ such that the following parametrized LMI hold for all x in a neighborhood of the origin

$$-\Gamma(x,\dot{\tau}) = -\Gamma_1(x) - \Gamma_2(\dot{\tau}) - \Gamma_3(x) \in \mathbb{S}_+ \tag{10}$$

where

$$\Gamma_1(x) = \begin{bmatrix} -\frac{1}{2}(\alpha+\alpha^T)(x) & \frac{1}{2}M(x) & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11a}$$

$$\Gamma_2(\dot{\tau}) = \begin{bmatrix} S & 0 & 0 \\ 0 & -(1-\dot{\tau})S & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11b}$$

$$\Gamma_3(x) = \Gamma_{31}(x) + \Gamma_{32}(x) + \Gamma_{33}$$

with

$$\begin{aligned} \Gamma_{31}(x) &= (G_0 - G_1)^T(x) 2\beta(x)(G_0-G_1)(x) \\ \Gamma_{32}(x) &= G_1^T(x) 2\beta(x) G_1(x) \\ \Gamma_{33} &= G_2^T \begin{bmatrix} Q & 0 \\ 0 & 3Q \end{bmatrix} G_2 \,. \end{aligned} \tag{11c}$$

and

$$\begin{aligned} \alpha(x) &= R(x) + g(x)\left(K(t) - \tfrac{1}{2}K^T(t)K(t) - \tfrac{1}{2}\mathrm{I}\right) g^T(x) \\ \beta(x) &= \tau_{max}^2 \nabla^2 H^T(x) Q \nabla^2 H(x) \\ G_0(x) &= \begin{bmatrix} (J-R)(x) & M(x) & 0 \end{bmatrix} \\ G_1(x) &= \begin{bmatrix} g(x)K(t)g^T(x) & 0 & 0 \end{bmatrix} \\ G_2 &= \begin{bmatrix} \mathrm{I} & -\mathrm{I} & 0 \\ \mathrm{I} & \mathrm{I} & -2\mathrm{I} \end{bmatrix} \end{aligned}$$

Proof: Consider the positive definite functional

$$V(t) = V_1(t) + V_2(t) + V_3(t)\,, \tag{12}$$

where

$$V_1(t) = H_t \tag{13a}$$

$$V_2(t) = \int_{t-\tau(t)}^{t} \nabla H_s^T \, S \, \nabla H_S \mathrm{d}s \tag{13b}$$

$$V_3(t) = \tau_{max} \int_{-\tau_{max}}^{0} \int_{t+s}^{t} \frac{\mathrm{d}}{\mathrm{d}u}(\nabla H_u^T) Q \frac{\mathrm{d}}{\mathrm{d}u}(\nabla H_u) \mathrm{d}u\mathrm{d}s \tag{13c}$$

Define the extended vector

$$\varepsilon(t) = \mathrm{col}\left(\nabla H_t \,, \nabla H_{t-\tau(t)} \,, \frac{1}{\tau(t)} \int_{t-\tau(t)}^{t} \nabla H_s \mathrm{d}s \right)\,, \tag{14}$$

and let us compute the time derivative of V.

From (7), it follows

$$\dot{V}_1(t) \le \varepsilon^T(t) \Big[\Gamma_1\Big]_t \varepsilon(t)\,. \tag{15}$$

Derivative of V_2 along trajectory of (6) reads

$$\begin{aligned} \dot{V}_2(t) &= \nabla H_t^T S \nabla H_t - (1-\dot{\tau}(t)) \nabla H_{t-\tau(t)}^T S \nabla H_{t-\tau(t)} \\ &= \varepsilon^T(t) \Big[\Gamma_2\Big]_t \varepsilon(t)\,. \end{aligned} \tag{16}$$

Finally, derivative of V_3 along trajectory of (6) is

$$\begin{aligned} \dot{V}_3(t) &= \tau_{max}^2 [\nabla^2 H_t \dot{x}(t)]^T Q [\nabla^2 H_t \dot{x}(t)] \\ &- \tau_{max} \int_{t-\tau_{max}}^{t} \frac{\mathrm{d}}{\mathrm{d}u}(\nabla H_u^T) Q \frac{\mathrm{d}}{\mathrm{d}u}(\nabla H_u) \mathrm{d}u\,. \end{aligned} \tag{17}$$

Noting (6) writes $\dot{x}(t) = (G_0 - G_1)_t \varepsilon(t) + g_t v_t$ and using the inequality $2a^T b \le a^T a + b^T b$, one has

$$\begin{aligned} &\tau_{max}^2 [\nabla^2 H_t \dot{x}(t)]^T Q [\nabla^2 H_t \dot{x}(t)] \\ &\quad= \tau_{max}^2 \dot{x}^T(t) \beta_t \dot{x}(t) \\ &\quad\le \varepsilon^T(t) \Big((G_0 - G_1)_t^T (2\beta)_t (G_0 - G_1)_t \Big) \varepsilon(t) \\ &\qquad + v_t^T \Big(g_t^T (2\beta)_t g_t \Big) v_t \\ &\quad\le \varepsilon^T(t) \Big[\Gamma_{31} + \Gamma_{32} \Big]_t \varepsilon(t) \end{aligned} \tag{18}$$

where the last inequality is due to $v^T v \le u^T u$ by (5). The integral term in the right-hand side of (17) can be handled first noting it is upper bounded by the integral running from $t - \tau(t)$ to t (as the function to be integrated is nonnegative), and second by applying lemma 1 (noting F_0, F_1 in lemma 1 write $\left[\begin{smallmatrix} F_0 \\ F_1 \end{smallmatrix}\right] = G_2 \varepsilon(t)$). It follows

$$\begin{aligned} &-\tau_{max} \int_{t-\tau_{max}}^{t} \frac{\mathrm{d}}{\mathrm{d}u}(\nabla H_u^T) Q \frac{\mathrm{d}}{\mathrm{d}u}(\nabla H_u) \mathrm{d}u \\ &\quad\le -\tau(t) \int_{t-\tau(t)}^{t} \frac{\mathrm{d}}{\mathrm{d}u}(\nabla H_u^T) Q \frac{\mathrm{d}}{\mathrm{d}u}(\nabla H_u) \mathrm{d}u \\ &\quad\le \varepsilon^T(t) \Big[\Gamma_{33}\Big]_t \varepsilon(t)\,. \end{aligned} \tag{19}$$

Summarizing equations (15), (16), (18) and (19), we have so far obtained

$$\dot{V}(t) \le \varepsilon^T(t) \Big[\Gamma\Big]_t \varepsilon(t)\,. \tag{20}$$

If condition (10) holds, then there exists a constant $c > 0$ such that $\dot{V}(t) \le -c\|\varepsilon(t)\|^2 \le -c\|\nabla H_t\|^2$. Since H is assumed regular around the origin, $\|\nabla H(x)\|$ is a continuous positive definite function in a neighborhood of the origin. Thus there exists a class-$\mathcal{K}_\infty$ function κ such that $\kappa(\|x\|) \le \|\nabla H(x)\|$ in a neighborhood of the origin (see *e.g.* comparison lemma IV.1 in [1] or lemma 18 in [6]). One can therefore apply Lyapunov-Krasovskii theorem to conclude on (local uniform) stability of the closed-loop (6). ■

Remark 3: The assumption 'H regular' guarantees that the equation $\nabla H(x) = 0$ has no accumulation point in a neighborhood of the origin. Hence, positive definiteness of H reads $\nabla H(x) = 0 \Longrightarrow x = 0$ around the origin. Analog property is obtained assuming H to be a $\mathcal{C}^2$ *strongly convex* positive definite function in a neighborhood of the origin.

It should be noticed that the *parametrized* LMIs (10) is not numerically tractable: the feasibility problem cannot be solved with standard LMI routines. We shall restrict the class considered to derive an implementable criteria. Let us start with the following observation: by a Schur[1] complement argument, the term Γ_{32} can be *linearized* relative to $\nabla^2 H$. Indeed (omitting x-dependency for clarity)

$$\begin{aligned}\Gamma_{32} \prec 0 &\iff G_1^T \nabla^2 H^T Q \nabla^2 H G_1 \prec 0 \\ &\iff \begin{bmatrix} 0 & (\nabla^2 H G_1)^T \\ * & -Q^{-1} \end{bmatrix} \prec 0\,, \end{aligned} \tag{21}$$

since $Q \succ 0$. Applying the argument twice shows Γ in (10) is actually *affine* in the Hessian. It is also clear that Γ is affine in the delay rate $\dot{\tau}$. Consequently, there is a natural way to weaken Prop. 1 in order to obtain a *nonparametrized* set of LMI.

Corollary 1: Assume system (1) satisfies

(A1) H is regular around the origin,
(A2) Hessian $\nabla^2 H$ is embedded in a polytop $\mathbb{P}$,
(A3) J, R, M, g, K are constant matrices.

Then, the sufficient stability condition (10) only need to be checked on the set of vertices of the polytop $[0, \delta_{max}] \times \mathbb{P}$.

Proof: Remind that under polytopic embedding, there exist constant matrices $\{H^{(j)}\}_j$ such that for all x in $\mathbb{P}$, there exist $(\lambda_1, \dots)$ in $[0,1]$ with $\sum_j \lambda_j = 1$ such that

$$\nabla^2 H(x) = \sum_{j=1}^{N} \lambda_j H^{(j)}\,. \tag{22}$$

Hence, since Γ is affine in the parameters $\dot{\tau}$ and $H^{(j)}$, and therefore no longer depends on x, positive definiteness inside a polytop is equivalent to positive definiteness on its vertices. ■

Remark 4: If, given a scalar μ, we consider initial conditions ϕ such that $V(x(t,\phi)) \leq \mu$ for all $t > 0$, then there exists an open set containing the origin such that (22) holds.

Remark 5: Under assumptions (A1),(A2),(A3), condition Γ as stated in (10) can be implemented with standard LMI routine.

As quoted by a reviewer, the results given here are local and considering saturation issue might be questionable. However, a global stability result can be obtained adding a (strong) regularity assumption on H (the key point being that ∇H has no accumulation point).

Corollary 2: Assume system (1) satisfies

(A1') H is a regular (or strongly convex) function over $\mathbb{R}^n$

together with (A2) and (A3) given above. Then, the sufficient stability condition (10), satisfied on the set of vertices of the polytop $[0, \delta_{max}] \times \mathbb{P}$, ensures global (uniform) stability.

[1]Schur complement is as follows

$$\begin{bmatrix} Q & S \\ * & R \end{bmatrix} \prec 0 \iff \begin{cases} Q - SR^{-1}S^T \prec 0 \\ \qquad\qquad R \prec 0 \end{cases}.$$

V. Numerical example

As a proof of concept, a numerical example is treated: stability condition (10) is compared with [14].

The system's characteristics are given by

$$H(q,p) = \sin^2(q) + \frac{1}{2}p^2 \tag{23a}$$

$$[J - R] = \begin{bmatrix} -2 & 1 \\ -1 & -1 \end{bmatrix} \tag{23b}$$

$$M = \begin{bmatrix} -0.5 & 0.7 \\ -1.1 & -1 \end{bmatrix} \tag{23c}$$

$$g = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{23d}$$

Dynamics (1) then reads

$$\begin{aligned} \dot{q}(t) &= -2\sin(2q(t)) + p(t) + \mathrm{sat}(u) \\ &\qquad - 0.5\sin(2q(t-\tau(t))) + 0.7p(t-\tau(t)) \\ \dot{p}(t) &= -\sin(2q(t)) - p(t) \\ &\qquad - 1.1\sin(2q(t-\tau(t))) - p(t-\tau(t)) \\ y &= \sin(2q(t))\,. \end{aligned} \tag{24}$$

Observe the Hessian is

$$\nabla^2 H(q,p) = \begin{bmatrix} 2\cos(2q) & 0 \\ 0 & 1 \end{bmatrix}, \tag{25}$$

assumptions (A1),(A2),(A3) of Cor. 1 are thus satisfied.

In a first step, stability of the open-loop (*i.e.* set $u = 0$ in (1a) or, equivalently, $K = 0$ and $v = 0$ in (6)) is studied as follows: given a maximum time-delay rate δ_{max}, the maximum time-delay τ_{max} is computed. The results are reported in Tab.I. This comparison grades the conservatism of the stability condition that is correlated with the maximum time-delay amplitude.

As expected, it appears that condition (10) is less conservative than the condition stated in the literature, thanks to Wirtinger-based integral lower bound.

δ_{max}	0.2	0.4	0.6	0.8	0.9
τ_{max} via [14]	0.226	0.222	0.214	0.193	0.181
τ_{max} via Prop. 1	0.541	0.446	0.342	0.250	0.211

TABLE I
MAXIMUM TIME-DELAY τ_{max} COMPUTED FOR GIVEN VALUES OF THE MAXIMAL TIME-DELAY RATE δ_{max}

In a second step, the closed-loop stability is our concern. Take $\tau(t) = \frac{1}{2}\arctan(t)$, then $\tau_{max} = \pi/4$, $\delta_{max} = 1/2$ and (2) is satisfied. The feedback gain $K = 0.75$ has been iteratively computed to maximize the convergence rate. The saturation constraints are imposed by $|\mathrm{sat}(u)| \leq 0.3$.

Numerical simulations have been processed using a discrete gradient scheme (see *e.g.* [4]) with time-step $\Delta t = 10^{-2}$. The initial condition is taken as $\phi = 0$ except at $t = 0$ where $(q(0), p(0)) = (-2; 2)$. Note the open-loop trajectory converges to the origin.

In order to illustrate the result given in Corollary 1, the *desired* equilibrium is imposed by adding a constant input $u_e = 0.28$ leading to the control law

$$u(t) = \mathrm{sat}\left(-Ky(t) + u_e\right) . \tag{26}$$

One then verifies sufficient condition (10) holds on the vertices of the polytop $[0, \delta_{max}] \times \mathbb{P}$.

Open-loop and closed-loop dynamics are depicted in Fig. 1. Observe the modified equilibrium according to (26). Mind the closed-loop curve illustrates robustness of the control law relative to the *a priori* unknown time-varying delay $\tau(t)$. The corresponding saturated control is compared with the unconstrained one in Fig. 2.

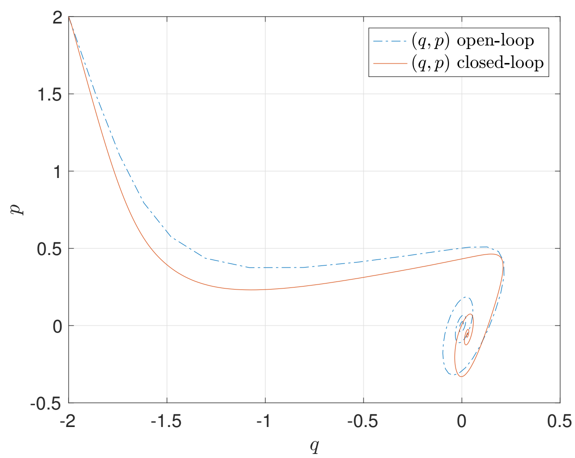

Fig. 1. Phase portraits: open-loop with delay (—·), and closed-loop with delay and input saturation (solid line)

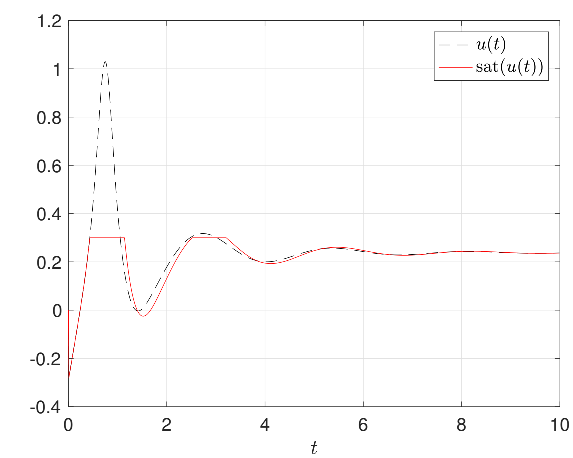

Fig. 2. Control laws: without saturation (——), and with saturation constraints (solid line)

VI. Conclusion

In this paper, a sufficient stability condition has been derived for a class of (nonlinear) delayed port-hamiltonian systems with control law given as output feedback. The condition is formulated in terms of parametrized LMIs. Compared with the literature, an expanded Lyapunov-Krasovkii candidate is considered (in the sense it contains more information about the past history of the system), and a less conservative bounding inequality is used in the proof (Wirtinger-based integral inequality). As a result, the stability criteria we obtained is more accurate than the ones given in the literature. Moreover, weakening the class yields a nonparametrized condition that can be tested with standard LMI routine.

Future works will address stabilizing control law synthesis (with *a priori* performances rather than checking *a posteriori*) and estimation of domain of attraction.

References

[1] D. Angeli, E.D. Sontag, and Y. Wang. A characterization of integral input-to-state stability. *IEEE Trans. Autom. Control*, 45(6):1082–1097, 2000.

[2] S. Aoues, W. Lombardi, D. Eberard, and A. Seuret. Robust stability for delayed port-Hamiltonian systems using improved Wirtinger-based inequality. *IEEE CDC*, pages 3119–3124, 2014.

[3] E. Fridman. *Introduction to Time-Delay Systems: Analysis and Control*. Birkhäuser, 2014.

[4] T. Itoh and K. Abe. Hamiltonian-conserving discrete canonical equations based on variational difference quotients,. *Journal Comput. Phys*, 77:85–102, 1998.

[5] C.-Y. Kao and R. Pasumarthy. Stability analysis of interconnected Hamiltonian systems under time delays. *IET Control Theory and Appl.*, 6(4):570–577, 2012.

[6] C.M. Kellett. A compenduim of comparison function results. *Math. Control Signals Syst.*, 26:339–374, 2014.

[7] V.L. Kharitonov. *Time-Delay Systems*. Birkhüser, 2013.

[8] B.M. Maschke and A.J. van der Schaft. Port controlled Hamiltonian systems: modeling origins and system theoretic properties. In *proc of the IFAC symposium on NOLCOS*, pages 282–288, 1992.

[9] P. Mukhija, I.N. Kar, and R. K P Bhatt. Delay-distribution based stability analysis of time-delayed port-hamiltonian systems. In *Signal Processing, Computing and Control (ISPCC), IEEE International Conference on*, pages 1–5, 2012.

[10] S. I. Niculescu, E. Verriest, L. Dugard, and J. M Dion. Stability and robust stability of time-delay systems: A guided tour. *Stability and control of time-delay systems*, pages 1–71, 1998.

[11] J.-P. Richard. Time-delay systems:an overview of some recent advances and open problems. *Automatica*, 39(10):1667–1694, 2003.

[12] J. Schiffer, E. Fridman, R. Ortega, and J. Raisch. Stability of a class of delayed port-Hamiltonian systems with application to microgrids with distributed rotational and electronic generation. *Automatica*, 74:71–79, 2016.

[13] A. Seuret and F. Gouaisbaut. Wirtinger-based integral inequality: Application to time-delay systems. *Automatica*, 49:2860–2866, 2013.

[14] W.W. Sun. Stabilization analysis of time-delay Hamiltonian systems in the presence of saturation. *Applied Mathematics and Computation*, 217(23):9625 – 9634, 2011.

[15] H.J. Sussmann, E.D. Sontag, and Y. Yang. A general result on the stabilization of linear systems using bounded controls. *IEEE Trans. Autom. Control*, 39(12):2411–2425, 1994.

[16] S. Tarbouriech, G. Garcia, M. Gomes da Silva Jr, and I. Queinnec. *Stability and Stabilization of Linear Systems with Saturating Actuators*. Springer, 2011.

[17] A.J. van der Schaft. *L_2-gain and passivity techniques in nonlinear control*. Springer Series in Comp. Math. 31, 1999.

[18] A. Wei and Y. Wang. Stabilization and H_∞ control of nonlinear port-controlled Hamiltonian systems subject to actuator saturation. *Automatica*, 46(6):2008–2013, 2010.

[19] K. Yakoubi and Y. Chitour. Linear systems subject to input saturation and time delay: global asymptotic stabilization. *IEEE Trans. Autom. Control*, 52(5), 2007.

[20] Y.J. Yoo, J.H. Koo, and S.C. Won. Delay dependent stability condition for the port-Hamiltonian systems with time varying delay. In *Asian Control Conference (ASCC)*, pages 834–838, 2011.