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# Output Feedback Stabilization and Asymptotic Performance Recovery for Input-affine Sampled-data Strict-feedback Systems

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*Abstract*— **In this paper output feedback stabilization and asymptotic performance recovery are considered for input-affine sampled-data strict-feedback systems. It is shown that output feedback stabilization and asymptotic performance recovery can be achieved by combining an emulation of continuous-time globally asymptotically stabilizing state feedback controllers with reduced-order observers that are semiglobal and practical in a sampling period. A numerical example is also given to illustrate the efficiency of the proposed output feedback controllers.**

## I. INTRODUCTION

Nonlinear control theory has been discussed mainly for continuous-time systems and several design methods of controllers and observers has been proposed (for details see [6], [9], [11], [19] and the references therein). From a practical point of view, sampled-data control theory is important, since most controllers are implemented by digital computers with zero-order holds (D/A converters) and samplers (A/D converters) [2]. For nonlinear sampled-data systems, the design frameworks of controllers and observers on the basis of approximate models have been given first ([1], [17], [18]). Then the design of controllers and observers are classified into two categories. In the first category, approximate discrete-time models of the sampled-data system such as the Euler model are used to design discrete-time controllers and observers ([7], [8], [12], [13], [16], [20]). In the second category, continuous-time controllers and observers are designed first by continuous-time control theory and then they are implemented digitally ([3], [4], [5], [10], [14], [15]). Moreover, an approximate reproduction of continuous-time response is discussed in the sampled-data control system. For input-affine nonlinear sampled-data systems, redesign of discrete-time state feedback controllers that give a good reproduction of the continuous-time response has been considered ([4], [5], [14]). For a class of nonlinear sampled-data systems, the design of high-gain observer-based output feedback controllers that reproduce the continuous-time response asymptotically has been also given ([3], [10]).

In this paper, we consider the input-affine strict-feedback system

$$\begin{aligned} \dot{x}_c &= f_1(x_c) + g_1(x_c)z_c, \\ \dot{z}_c &= f_2(x_c, z_c) + g_2(x_c, z_c)u_c \end{aligned} \tag{1}$$

where $x_c \in \mathbf{R}^{n_x}$, $z_c \in \mathbf{R}^{n_z}$ are the states and $u_c \in \mathbf{R}^m$ is the control input. Since many mechanical systems are of the form (1), the design of controllers for the system (1) is important from a practical point of view. Let

$$u_c(t) = u(k) \tag{2}$$

for any $t \in [kT, (k+1)T)$ and introduce the sampled observation

$$y(k) = x_c(kT) \tag{3}$$

for any $k \in \mathbf{N}_0 := \{0, 1, 2, ..\}$ where $T > 0$ is a sampling period. Then we want to design a discrete-time output feedback controller that semiglobally practically uniformly asymptotic (SPUA) stabilizes the sampled-data system (1)-(3) and achieves asymptotic performance recovery of the given continuous-time globally asymptotically stabilizing (GAS) state feedback controller, i.e., as a sampling period $T > 0$ goes to zero, the solutions of the sampled-data closed-loop system converge to those of the continuous-time closed-loop system.

We first summarize the design of a reduced-order observer that estimates $z_c(kT)$ of the sampled-data system (1)-(3) ([7], [8]). Then we combine the designed reduced-order observer with an emulation of the given continuous-time GAS state feedback controller to construct an output feedback controller. Then we show that the designed output feedback controller SPUA stabilizes the sampled-data system (1)-(3) and achieves asymptotic performance recovery. We also give a numerical example to illustrate the efficiency of the proposed output feedback controller.

*Notation:* Let $\|\cdot\|$ denote the norm of a vector. Let $\sigma(M)$ be a set of eigenvalues of a matrix $M$ and $\mathbf{D} = \{\lambda = a + ib : \sqrt{a^2+b^2} < 1\}$. A function $\alpha : \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$ is of class $\mathcal{K}$ (denoted by $\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A function $\beta$: $\mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for any fixed $t \geq 0$, $\beta(\cdot, t) \in \mathcal{K}$ and for each fixed $s \geq 0$, $\beta(s, \cdot)$ is decreasing to zero as its argument tends to infinity [9]. For a function $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$, we write $f(\cdot, \eta) = O(T^p)$ if for each compact set $\Omega \subset \mathbf{R}^n$, there exists a constant $C > 0$ such that $f(\cdot, \eta) \leq CT^p$ for any $\eta \in \Omega$ [4]. For simplicity of expression, we write $f(\eta_1(\cdot), \eta_2(\cdot)) = f(\eta_1, \eta_2)(\cdot)$.

## II. PRELIMINARY RESULTS

Consider the system

$$\dot{\chi}_c = f(\chi_c, u_c),\ \ \chi_c(0) = \chi_0 \tag{4}$$

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where $\chi_c \in \mathbf{R}^n$ is the state, $u_c \in \mathbf{R}^m$ is the control input, $f$ is locally Lipschitz, and $f(0,0) = 0$. Let $u_c(t) = u(k)$ for any $t \in [kT, (k+1)T)$ and $k \in \mathbf{N}_0$. We assume the existence of a unique solution of the system (4) on some bounded interval of the form $[0, \eta)$ for each initial condition and each constant control. Then the difference equations corresponding to the exact model and the Euler model of the system (4) are given by

$$\begin{aligned} \chi(k+1) &= F_T^e(\chi, u)(k) \\ &= \chi(k) + \int_{kT}^{(k+1)T} f(\chi_c(s), u(k))ds \end{aligned} \quad (5)$$

and $\chi(k+1) = F_T^a(\chi, u)(k) = \chi(k) + Tf(\chi, u)(k)$, respectively. Note that $\chi(k) = \chi_c(kT)$ for the exact model $F_T^e$ and we must use the Euler model $F_T^a$ for the design purpose, since the exact model $F_T^e$ is not analytically computable. It is well-known that the Euler model $F_T^a$ is one-step consistent with the exact model $F_T^e$, i.e., for each compact set $\Omega \subset \mathbf{R}^n \times \mathbf{R}^m$, there exist $\gamma \in \mathcal{K}$ and $T^* > 0$ satisfying $\|F_T^e(\chi, u) - F_T^a(\chi, u)\| \leq T\gamma(T)$ for all $(\chi, u) \in \Omega$ and $T \in (0, T^*]$ [17].

Consider the parameterized state feedback controller

$$u(k) = u_T(\chi(k)) \quad (6)$$

where $u_T(0) = 0$ and the parameterized discrete-time system

$$\chi(k+1) = F_T(\chi, u_T(\chi))(k) \quad (7)$$

to define SPUA stability and an SPUA stabilizing pair [17].

*Definition 2.1:* The system (7) is called SPUA stable if there exists $\beta \in \mathcal{KL}$ such that for any pair of strictly positive real numbers $(D, d)$, there exists $T^* > 0$ such that the solutions of (7) satisfy $\|\chi(k)\| \leq \beta(\|\chi(0)\|, kT) + d$ for any $T \in (0, T^*]$, $k \in \mathbf{N}_0$, and $\chi(0)$ with $\|\chi(0)\| \leq D$.

*Definition 2.2:* Let $\bar{T} > 0$ be given and functions $U_T : \mathbf{R}^n \to \mathbf{R}_{\geq 0}$ and $u_T : \mathbf{R}^n \to \mathbf{R}^m$ defined for each $T \in (0, \bar{T}]$. Then $(u_T, U_T)$ is called an SPUA stabilizing pair for $F_T$ if there exist $\alpha_U^1$, $\alpha_U^2$, $\alpha_U^3 \in \mathcal{K}_\infty$ such that for any pair of strictly positive real numbers $(\Delta, \delta)$, there exist strictly positive real numbers $(T^*, L_U, M)$ with $T^* \leq \bar{T}$ satisfying

$$\alpha_U^1(\|\chi\|) \leq U_T(\chi) \leq \alpha_U^2(\|\chi\|), \quad (8)$$
$$U_T(F_T(\chi, u_T(\chi))) - U_T(\chi) \leq -T\alpha_U^3(\|\chi\|) + T\delta, \quad (9)$$
$$|U_T(\chi) - U_T(\bar{\chi})| \leq L_U\|\chi - \bar{\chi}\|, \quad (10)$$
$$\|u_T(\chi)\| \leq M \quad (11)$$

for any $\chi$, $\bar{\chi} \in \mathbf{R}^n$ with $\max\{\|\chi\|, \|\bar{\chi}\|\} \leq \Delta$ and $T \in (0, T^*]$.

*Theorem 2.1:* ([17], [18]) If $(u_T, U_T)$ is an SPUA stabilizing pair for $F_T^a$, then $(u_T, U_T)$ is an SPUA stabilizing pair for $F_T^e$ and $u_T$ SPUA stabilizes the exact model $F_T^e$. Consequently, the closed-loop sampled-data system

$$\dot{\chi}_c = f(\chi_c, u_T(\chi_c(kT))), \ ^\forall t \in [kT, (k+1)T) \quad (12)$$

is SPUA stable in the continuous-time sense, i.e., there exists $\beta \in \mathcal{KL}$ such that for any pair of strictly positive real numbers $(D, d)$ there exists $T^* > 0$ such that the solutions of the closed-loop system (12) satisfy $\|\chi_c(t)\| \leq \beta(\|\chi(0)\|, t) + d$ for any $\chi_c(0)$ with $\|\chi_c(0)\| \leq D$ and $T \in (0, T^*]$.

For the system (4), we consider the state feedback controller $u_c(t) = u_{sf}(\chi_c(t))$ where $u_{sf}(0) = 0$ and assume that there exist a smooth function $W : \mathbf{R}^n \to \mathbf{R}_{\geq 0}$ and $\alpha_W^1$, $\alpha_W^2$, $\alpha_W^3 \in \mathcal{K}_\infty$ satisfying

$$\alpha_W^1(\|\chi_c\|) \leq W(\chi_c) \leq \alpha_W^2(\|\chi_c\|), \quad (13)$$
$$\frac{\partial W}{\partial \chi_c}(\chi_c)f(\chi_c, u_{sf}(\chi_c)) \leq -\alpha_W^3(\|\chi_c\|). \quad (14)$$

In this case, the origin of the closed-loop system $\dot{\chi}_c = f(\chi_c, u_{sf}(\chi_c))$ is GAS. Let $\Omega_\chi \subset \mathbf{R}^n$ be a compact set. Then suppose also that $u_{sf}(\eta)$ is bounded for any $\eta \in \Omega_\chi$ and there exists $L_W > 0$ satisfying

$$|W(\eta) - W(\bar{\eta})| \leq L_W\|\eta - \bar{\eta}\| \quad (15)$$

for any $\eta$, $\bar{\eta} \in \Omega_\chi$.

Let $u(k) = u_{sf}(\chi_c(kT)) = u_{sf}(\chi(k))$ that is an emulation of a continuous-time state feedback controller and is a special form of (6). Suppose that the solutions of the sampled-data system

$$\dot{\chi}_c = f(\chi_c, u_{sf}(\chi(k))), \ ^\forall t \in [kT, (k+1)T) \quad (16)$$

are well-defined for any $\chi_c(0) \in \mathbf{R}^n$. Then by [13], [14], and Theorem 2.1, we have the following result.

*Theorem 2.2:* Suppose that there exist a smooth function $W : \mathbf{R}^n \to \mathbf{R}_{\geq 0}$ and $\alpha_W^1$, $\alpha_W^2$, $\alpha_W^3 \in \mathcal{K}_\infty$ satisfying (13)-(15). Then $(u_{sf}, W)$ is an SPUA stabilizing pair for both the Euler model and the exact model of the system (16) and the sampled-data system (16) is SPUA stable in the continuous-time sense.

## III. Output Feedback Stabilization and Performance Recovery

Consider the input-affine sampled-data strict-feedback system (1)-(3) where $f_1$, $f_2$, $g_1$ and $g_2$ are sufficiently many times continuously differentiable, $f_1(0) = 0$, and $f_2(0,0) = 0$. Recall that the difference equations corresponding to the exact model and the Euler model of the sampled-data system (1)-(3) are given by

$$\begin{aligned} x(k+1) &= F_{1T}^e(x, z, u)(k), \\ z(k+1) &= F_{2T}^e(x, z, u)(k), \\ y(k) &= x(k) \end{aligned} \quad (17)$$

and

$$\begin{aligned} x(k+1) &= F_{1T}^a(x, z)(k) \\ &= x(k) + T[f_1(x) + g_1(x)z](k), \\ z(k+1) &= F_{2T}^a(x, z, u)(k) \\ &= z(k) + T[f_2(x, z) + g_2(x, z)u](k), \\ y(k) &= x(k), \end{aligned} \quad (18)$$

respectively where $F_{1T}^e$ and $F_{2T}^e$ are defined as in (5).

In the following, we first summarize the design of a reduced-order observer that estimates $z(k)$ of the exact model of the sampled-data system (1)-(3) ([7], [8]). Then we combine the designed reduced-order observer with an emulation of the given continuous-time GAS state feedback controller to construct an output feedback controller. Then we show that the designed output feedback controller SPUA stabilizes the sampled-data system (1)-(3) and achieves asymptotic performance recovery.

### A. Design of Reduced-order Observers

For the design of reduced-order observers, we assume
**A1**: The $n_z \times n_z$ matrix $\Phi = g_1^T g_1$ is nonsingular and its inverse is bounded over the compact domain of interest.
Note that $z(k)$ of the Euler model (18) can be obtained by

$$\begin{aligned} z(k) &= \Phi^{-1}(y) g_1^T(y) \left\{ \frac{\rho y - y}{T} - f_1(y) \right\}(k) \\ &= \Psi_T(y, \rho y)(k) \end{aligned}$$

at time $k+1$ where $\rho$ is the shift operator, i.e., $(\rho y)(k) = y(k+1)$. Then we consider the system

$$\begin{aligned} \hat{z}(k+1) &= O_T(\hat{z}, y, \rho y, u)(k) \\ &= (I - TH)\hat{z}(k) + T\Pi_T(y, \rho y, u)(k) \end{aligned} \quad (19)$$

as a candidate of a reduced-order observer for the Euler model (18) where $H = \mathrm{diag}\{h(1), \cdots, h(n_z)\}$ and $\Pi_T(y, \rho y, u) = H\Psi_T(y, \rho y) + f_2(y, \Psi_T(y, \rho y)) + g_2(y, \Psi_T(y, \rho y))u$. Let $e = z - \hat{z}$. Then we have $e(k+1) = (I - TH)e(k)$ and we obtain the following results ([7], [8]).

*Lemma 3.1:* Assume **A1** and
**A2:** $|1 - Th(i)| < 1$ for given $\hat{T} > 0$, any $T \in (0, \hat{T}]$, and $i = 1 \ldots, n_z$.
Then the system (19) is a reduced-order observer of the Euler model (18), i.e., $z(k) - \hat{z}(k) \to 0$ as $k \to \infty$ for any $T \in (0, \hat{T}]$.

*Remark 3.1:* Consider the Euler model (18) and the reduced-order observer (19). To simplify the expression, let $F_{jT}^i = F_{jT}^i(y, z, u)$, and $O_T^i = O_T(\hat{z}, y, F_{1T}^i, u)$ for $i = e, a$ and $j = 1, 2$. Note that $\rho y = F_{1T}^a$ and $e(k+1) = F_{2T}^a(k) - O_T^a(k) = (I - TH)e(k)$ for the Euler model (18) and by **A2**, $\sigma(I - TH) \subset \mathbf{D}$ for any $T \in (0, \hat{T}]$.

*Lemma 3.2:* Assume **A1** and **A2**. Then the reduced-order observer (19) is semiglobal and practical in $T$ for the exact model (17), i.e., there exists $\beta \in \mathcal{KL}$ such that for any pair of strictly positive real numbers $(D, d)$ and a compact set $\Omega \subset \mathbf{R}^{n_x} \times \mathbf{R}^{n_z} \times \mathbf{R}^m$, there exists $T^* > 0$ such that $\|z(0) - \hat{z}(0)\| \le D$ and $(x, z, u)(k) \in \Omega$ for any $k \in \mathbf{N}_0$ imply $\|z(k) - \hat{z}(k)\| \le \beta(\|z(0) - \hat{z}(0)\|, kT) + d$ for any $T \in (0, T^*]$ and $k \in \mathbf{N}_0$ where $(x, z) = (x_c, z_c)(kT)$.

### B. Design of Output Feedback Controllers

For the continuous-time system (1), consider the state feedback controller

$$u_c(t) = u_{sf}(x_c, z_c)(t) \quad (20)$$

and the closed-loop system

$$\dot{\chi}_c = f(x_c, z_c, u_{sf}(x_c, z_c)) \quad (21)$$

where $\chi_c = [\, x_c^T \quad z_c^T \,]^T$ and

$$f(\eta_1, \eta_2, \tau) = \begin{bmatrix} f_1(\eta_1) + g_1(\eta_1)\eta_2 \\ f_2(\eta_1, \eta_2) + g_2(\eta_1, \eta_2)\tau \end{bmatrix}.$$

For simplicity of notation, we also write $f(\chi_c, u) = f(x_c, z_c, u)$ and $u_{sf}(\chi_c) = u_{sf}(x_c, z_c)$. Let $\phi(t; \chi_0)$ be the solution of the system (21) with $\chi_c(0) = \chi_0$. Then we assume:
**A3:** There exist a smooth function $W : \mathbf{R}^n \to \mathbf{R}_{\ge 0}$ and $\alpha_W^1$, $\alpha_W^2$, $\alpha_W^3 \in \mathcal{K}_\infty$ satisfying (13)-(15) where $n = n_x + n_z$.
**A4:** $u_{sf}$ is continuous and $u_{sf}(0) = 0$.

*Remark 3.2:* By a backstepping design, we can find $u_{sf}$, $W : \mathbf{R}^n \to \mathbf{R}_{\ge 0}$ and $\alpha_W^1$, $\alpha_W^2$, $\alpha_W^3 \in \mathcal{K}_\infty$ satisfying **A3** and **A4** for some class of the system (1) ([9], [11]).

Consider the discrete-time output feedback controller

$$\begin{aligned} u(k) &= u_{sf}(y, \hat{z})(k), \\ \hat{z}(k+1) &= O_T(\hat{z}, y, F_{1T}^e, u)(k) \\ &= (I - TH)\hat{z} + T\Pi_T(y, F_{1T}^e, u)(k). \end{aligned} \quad (22)$$

Then the closed-loop system of the sampled-data system (1)-(3) and the controller (22) is given by

$$\begin{aligned} \dot{x}_c &= f_1(x_c) + g_1(x_c) z_c, \\ \dot{z}_c &= f_2(x_c, z_c) + g_2(x_c, z_c) u(k), \\ u(k) &= u_{sf}(y, z - e)(k), \\ e(k+1) &= F_{2T}^e(k) - O_T(z - e, y, F_{1T}^e, u)(k) \end{aligned} \quad (23)$$

for any $t \in [kT, (k+1)T)$ and $k \in \mathbf{N}_0$ where $(y, z) = (x_c, z_c)(kT)$. Let $[\, x_c^T \quad z_c^T \,]^T (t) = \phi_T(t; \chi_0)$ be the solution of the system (23). Moreover, the difference equations corresponding to the exact model and the Euler model of the system (23) are given by

$$\begin{aligned} \mu(k+1) &= \tilde{F}_T^e(\mu, u)(k), \\ u(k) &= u_{sf}(y, z - e)(k) \end{aligned} \quad (24)$$

and

$$\begin{aligned} \mu(k+1) &= \tilde{F}_T^a(\mu, u)(k), \\ u(k) &= u_{sf}(y, z - e)(k), \end{aligned} \quad (25)$$

respectively, where $\chi = [\, x^T \quad z^T \,]^T$, $\mu = [\, \chi^T \quad e^T \,]^T$, $F_T^i = [\, (F_{1T}^i)^T \quad (F_{2T}^i)^T \,]^T$, and

$$\tilde{F}_T^i(\mu, u) = \begin{bmatrix} F_T^i \\ F_{2T}^i - O_T(z - e, y, F_{1T}^i, u) \end{bmatrix}$$

for $i = e, a$, $j = 1, 2$. Note that $\chi(k) = \chi_c(kT) = \phi_T(kT; \chi_0)$ for the exact model (24). Then we have the following result.

*Theorem 3.1:* Assume **A1**-**A4**. Then
1) The output feedback controller (22) SPUA stabilizes the exact model (17) and satisfies $\|\phi(kT; \chi_0) - \phi_T(kT; \chi_0)\| = O(T)$ for any $k \in \mathbf{N}_0$.
2) The output feedback controller (22) SPUA stabilizes the sampled-data system (1)-(3) and satisfies $\|\phi(t; \chi_0) - \phi_T(t; \chi_0)\| = O(T)$ for any $t \ge 0$.

*Remark 3.3:* In [3] and [10], high-gain observers and high-gain observer-based output feedback controllers for a class of nonlinear sampled-data systems have been considered. When high-gain observers and output feedback controllers are designed in continuous-time, their sampled-data implementation and asymptotic performance recovery of continuous-time controller have been discussed. The design method of reduced-order observers in this paper is based on the Euler model of sampled-data strict-feedback system and the structure of the designed observers and controllers is different from that of sampled-data implementation of continuous-time high-gain observers and controllers given in [3] and [10].

### C. Proof of Theorem 3.1

Let $\bar{\phi}_T(t;\chi_0)$ be the solution of the sampled-data closed-loop system $\dot{\chi}_c = f(\chi_c, u_{sf}(\chi(k)))$ for any $t \in [kT, (k+1)T)$ with $\chi_c(0) = \chi_0$. To prove Theorem 3.1, we introduce the following result.

*Lemma 3.3:* ([4], [5], [14]) Assume **A3**. Then

$$\begin{aligned} \|\phi(kT;\chi_0) - \bar{\phi}_T(kT;\chi_0)\| &= O(T), \\ \|\phi(t;\chi_0) - \bar{\phi}_T(t;\chi_0)\| &= O(T) \end{aligned}$$

for any $k \in \mathbf{N}_0$ and $t \geq 0$.

Let $\Omega \subset \mathbf{R}^n \times \mathbf{R}^m$ be a given compact set. Then by **A4** and the smoothness of the functions in (1), there exist positive real numbers $L_f$ and $L_u$ satisfying $\|f(\eta,\tau) - f(\bar{\eta},\bar{\tau})\| \leq L_f(\|\eta-\bar{\eta}\| + \|\tau-\bar{\tau}\|)$ and $\|u_{sf}(\eta) - u_{sf}(\bar{\eta})\| \leq L_u\|\eta-\bar{\eta}\|$ for any $(\eta,\tau)$, $(\bar{\eta},\bar{\tau}) \in \Omega$.

**Proof of Theorem 3.1:** Let strictly positive real numbers $\Delta_\chi$ and $\Delta$ with $\Delta_\chi < \Delta$ be given and $\Omega_u \in \mathbf{R}^m$ a compact set. By the one-step consistency between the exact model $F_T^e$ and the Euler model $F_T^a$, there exist $T_1^* > 0$ and $\gamma \in \mathcal{K}$ satisfying

$$\max\{\|(F_T^e - F_T^a)(\chi,u)\|, \|(\tilde{F}_T^e - \tilde{F}_T^a)(\mu,u)\|\} \leq T\gamma(T)$$

for any $\|\chi\| \leq \Delta_\chi$, $\|\mu\| \leq \Delta$, $u \in \Omega_u$, and $T \in (0, T_1^*]$. Note that $u_{sf}(\chi)$ and $u_{sf}(\check{\chi})$ are bounded for any $\|\mu\| \leq \Delta$ where $u_{sf}(\check{\chi}) = u_{sf}(\chi - [\,0 \quad e^T\,]^T) = u_{sf}(y, z-e)$.

1) Similar to the proof of Theorem 3.1 in [7], we can show that the closed-loop exact model (24) is SPUA stable and hence it is enough to show $\|\phi(kT;\chi_0) - \phi_T(kT;\chi_0)\| = O(T)$ for any $k \in \mathbf{N}_0$. By Lemma 3.3, we have

$$\begin{aligned} \|(\phi - \phi_T)(kT;\chi_0)\| &\leq \|\phi(kT;\chi_0) - \bar{\phi}_T(kT;\chi_0)\| \\ &\quad + \|\bar{\phi}_T(kT;\chi_0) - \phi_T(kT;\chi_0)\| \\ &\leq O(T) + \|(\bar{\psi}_T - \psi_T)(k;\chi_0)\| \end{aligned}$$

where $\psi_T(k;\chi_0) = \phi_T(kT;\chi_0)$ and $\bar{\psi}_T(k;\chi_0) = \bar{\phi}_T(kT;\chi_0)$. Hence it is enough to show

$$\|\bar{\psi}_T(k;\chi_0) - \psi_T(k;\chi_0)\| = O(T) \tag{26}$$

for any $k \in \mathbf{N}_0$. Note that $\bar{\psi}_T(0;\chi_0) - \psi_T(0;\chi_0) = 0$ and assume (26) is satisfied for $k = l$. For simplicity of notation, let $\bar{\psi}_T = \bar{\psi}_T(l;\chi_0)$, $\psi_T = \psi_T(l;\chi_0)$, and $\check{\psi}_T = \psi_T - [\,0 \quad e^T\,]^T$. Then by the one-step consistency between $F_T^e$ and $F_T^a$, we have

$$\begin{aligned} & \|\bar{\psi}_T(l+1;\chi_0) - \psi_T(l+1;\chi_0)\| \\ = \; & \|F_T^e(\bar{\psi}_T, u_{sf}(\bar{\psi}_T)) - F_T^e(\psi_T, u_{sf}(\check{\psi}_T))\| \\ \leq \; & \|(F_T^e - F_T^a)(\bar{\psi}_T, u_{sf}(\bar{\psi}_T))\| \\ & + \|(F_T^e - F_T^a)(\psi_T, u_{sf}(\check{\psi}_T))\| \\ & + \|F_T^a(\bar{\psi}_T, u_{sf}(\bar{\psi}_T)) - F_T^a(\psi_T, u_{sf}(\check{\psi}_T))\| \\ \leq \; & 2T\gamma(T) + \|F_T^a(\bar{\psi}_T, u_{sf}(\bar{\psi}_T)) - F_T^a(\psi_T, u_{sf}(\check{\psi}_T))\| \\ = \; & O(T^2) + \|F_T^a(\bar{\psi}_T, u_{sf}(\bar{\psi}_T)) - F_T^a(\psi_T, u_{sf}(\check{\psi}_T))\| \end{aligned}$$

and by direct calculation, we also have

$$\begin{aligned} & \|F_T^a(\bar{\psi}_T, u_{sf}(\bar{\psi}_T)) - F_T^a(\psi_T, u_{sf}(\check{\psi}_T))\| \\ = \; & \|\bar{\psi}_T - \psi_T\| \\ & + T\|f(\bar{\psi}_T, u_{sf}(\bar{\psi}_T)) - f(\psi_T, u_{sf}(\psi_T))\| \\ & + T\|f(\psi_T, u_{sf}(\psi_T)) - f(\psi_T, u_{sf}(\check{\psi}_T))\| \\ \leq \; & [1 + TL_f(1+L_u)]\|\bar{\psi}_T - \psi_T\| + TL_fL_u\|e\| \\ = \; & O(T) + TL_fL_u\|e\|. \end{aligned}$$

By the SPUA stability of the closed-loop exact model (24), $\|e\|$ is uniformly bounded and we have $\|F_T^a(\bar{\psi}_T, u_{sf}(\bar{\psi}_T)) - F_T^a(\psi_T, u_{sf}(\check{\psi}_T))\| = O(T)$. Hence by induction, we obtain (26) for any $k \in \mathbf{N}_0$.

2) By Theorem 2.1, it is enough to show $\|\phi(t;\chi_0) - \phi_T(t;\chi_0)\| = O(T)$ for any $t \geq 0$. From 1), we have

$$\begin{aligned} & \|\phi(t;\chi_0) - \phi_T(t;\chi_0)\| \\ \leq \; & \|\phi(t;\chi_0) - \phi(lT;\chi_0)\| + \|\phi_T(t;\chi_0) - \phi_T(lT;\chi_0)\| \\ & + \|\phi(lT;\chi_0) - \phi_T(lT;\chi_0)\| \\ = \; & O(T) + \|\phi(t;\chi_0) - \phi(lT;\chi_0)\| \\ & + \|\phi_T(t;\chi_0) - \phi_T(lT;\chi_0)\| \end{aligned}$$

for any $t \in [lT, (l+1)T)$. Here we can use the Bellman-Gronwall's inequality [9] to obtain

$$\begin{aligned} & \|\phi(t;\chi_0) - \phi(lT;\chi_0)\| \\ \leq \; & \frac{1}{L_f(1+L_u)}[e^{L_f(1+L_u)(t-lT)} - 1]\|\hat{f}(\phi)(lT;\chi_0)\| \\ \leq \; & \frac{1}{L_f(1+L_u)}[e^{L_f(1+L_u)T} - 1]\|\hat{f}(\phi)(lT;\chi_0)\|, \\ & \|\phi_T(t;\chi_0) - \phi_T(lT;\chi_0)\| \\ \leq \; & \frac{1}{L_f}[e^{L_f(t-lT)} - 1]\|f(\psi_T, u_{sf}(\check{\phi}_T))\| \\ \leq \; & \frac{1}{L_f}[e^{L_fT} - 1]\|f(\psi_T, u_{sf}(\check{\phi}_T))\| \end{aligned}$$

for any $t \in [lT, (l+1)T)$ and $l \in \mathbf{N}_0$ where $\hat{f}(\phi)(\cdot;\cdot) = f(\phi(\cdot;\cdot), u_{sf}(\phi(\cdot;\cdot)))$. Since $e^{\eta T} - 1 = Th(T,\eta)$ and a function $h(T,\eta)$ is bounded on any compact set, we have $\|\phi(t;\chi_0) - \phi(lT;\chi_0)\| = O(T)$ and $\|\phi_T(t;\chi_0) - \phi_T(lT;\chi_0)\| = O(T)$. Hence we have the assertion.

## IV. Numerical Example

Consider a simplified Moore-Greitzer model of a jet engine with the assumption of no stall:

$$\dot{x}_c = -\frac{3}{2}x_c^2 - \frac{1}{2}x_c^3 - z_c, \quad \dot{z}_c = -u_c \tag{27}$$

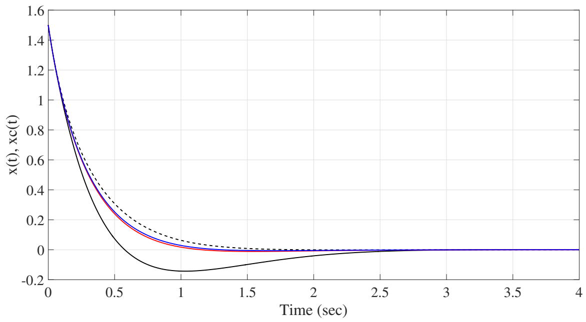


Fig. 1. Simulation result of the time responses of $x_c(t)$ of the continuous-time closed-loop system (black broken line) and $x(t)$ of the sampled-data closed-loop system ($T = 0.1$: black line, $T = 0.01$: red line, $T = 0.001$: blue line).

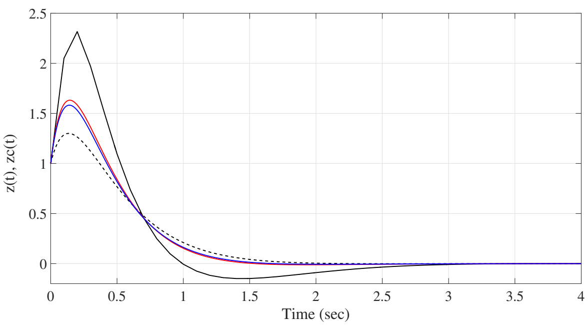


Fig. 2. Simulation result of the time responses of $z_c(t)$ of the continuous-time closed-loop system (black broken line) and $z(t)$ of the sampled-data closed-loop system ($T = 0.1$: black line, $T = 0.01$: red line, $T = 0.001$: blue line).

([4], [11], [14]). By a backstepping method, a continuous-time state feedback controller

$$u_c = u_{sf}(x_c, z_c) = -7x_c + 5z_c \tag{28}$$

GAS the continuous-time model (27) [11]. For the system (27) we assume $u_c(t) = u(k)$ for any $t \in [kT, (k+1)T)$ and introduce the sampled observation $y(k) = x_c(kT)$. Then the reduced-order observer-based output SPUA stabilizing controller is given by

$$\begin{aligned} u(k) &= u_{sf}(y, \hat{z})(k) = -7y(k) + 5\hat{z}(k), \quad (29)\\ \hat{z}(k+1) &= (1 - TH)\hat{z}(k) + T[H\Psi(y, \rho y) - u](k) \end{aligned}$$

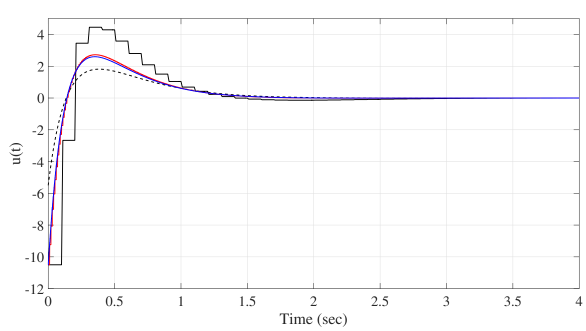


Fig. 3. Simulation result of the time responses of $u_c(t)$ given by (28) (black broken line) and $u(k)$ given by (29) ($T = 0.1$: black line, $T = 0.01$: red line, $T = 0.001$: blue line).

where $\Psi(y, \rho y) = -(\rho y - y)/T - (3/2)y^2 - (1/2)y^3$.

Let $(x_c(0), z_c(0)) = (1.5, 1)$ and $H = 8$. Then Figs 1 and 2 are the simulation results of the time responses of $(x_c(t), z_c(t))$ of the continuous-time closed-loop system given by (28) and $(x(t), z(t))$ of the sampled-data closed-loop system given by (29). Fig 3 is the simulation result of the time responses of $u_c(t)$ given by (28) and $u(k)$ given by (29). We can see that $(x(t), z(t))$ becomes close to $(x_c(t), z_c(t))$ as $T > 0$ becomes small, i.e., the output feedback controller (29) achieves asymptotic performance recovery of the continuous-time state feedback controller (28). Moreover, we can obtain better asymptotic performance recovery by letting $H > 0$ large such that **A2** is satisfied as $T > 0$ becomes small (Figs 4-7). This implies the output feedback controller (29) with high observer gains gives better performance recovery than that with low observer gains. Fig 8 is the simulation result of $x_c(t)$ of the continuous-time closed-loop system and $x(t)$ of the sampled-data closed-loop system for $(T, H) = \{(0.1, 8), (0.01, 80), (0.001, 800)\}$ when $(x_c(0)z_c(0)) = (22, 21)$. Since the initial state $(22, 21)$ is farther away from the origin, asymptotic performance recovery becomes worse and we need both smaller sampling periods less than 0.001 (s) and higher observer gains greater than 800 to obtain better performance recovery.

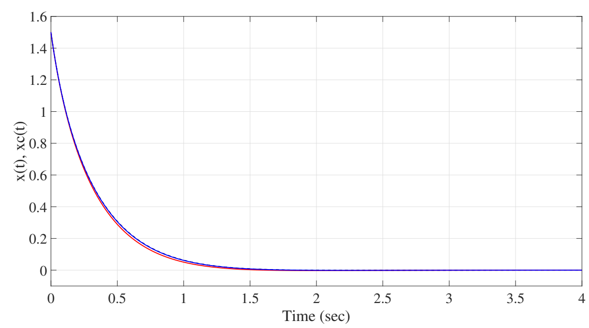


Fig. 4. Simulation result of the time responses of $x_c(t)$ of the continuous-time closed-loop system (black broken line) and $x(t)$ of the sampled-data closed-loop system ($(T, H) = (0.01, 80)$: red line, $(T, H) = (0.001, 800)$: blue line).

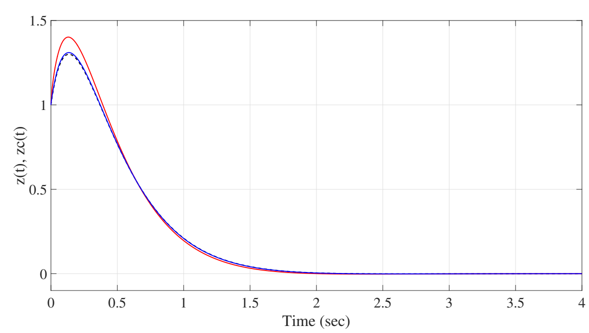


Fig. 5. Simulation result of the time responses of $z_c(t)$ of the continuous-time closed-loop system (black broken line) and $z(t)$ of the sampled-data closed-loop system ($(T, H) = (0.01, 80)$: red line, $(T, H) = (0.001, 800)$: blue line).

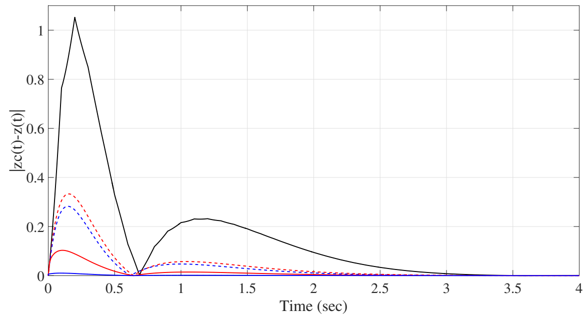


Fig. 6. Simulation result of the time responses of $|z_c(t)-z(t)|$ ($(T,H)=(0.1,8)$: black line, $T=(0.01,8)$: red broken line, $(T,H)=(0.01,80)$: red line, $(T,H)=(0.001,8)$: blue broken line, $(T,H)=(0.001,800)$: blue line).

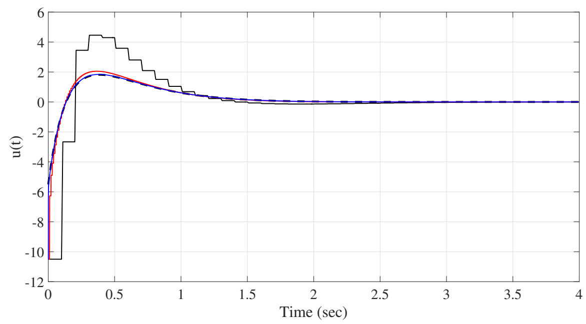


Fig. 7. Simulation result of the time responses of $u_c(t)$ given by (28) (black broken line) and $u(k)$ given by (29) ($(T,H)=(0.1,8)$:black line, $(T,H)=(0.01,80)$: red line, $(T,H)=(0.001,800)$: blue line).

## V. Conclusion

In this paper we have considered output feedback stabilization and asymptotic performance recovery for input-affine sampled-data strict-feedback systems. We have shown that output feedback stabilization and asymptotic performance recovery can be achieved by the combination of an emulation of continuous-time GAS state feedback controllers and reduced-order observers given in [7] and [8]. The extension to asymptotic performance recovery by high-gain reduced-order observer-based output feedback controllers is a future work.

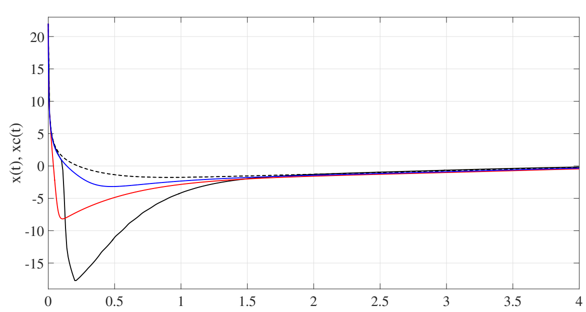


Fig. 8. Simulation result of the time responses of $x_c(t)$ of the continuous-time closed-loop system (black broken line) and $x(t)$ of the sampled-data closed-loop system ($(T,H)=(0.1,8)$: black line, $(T,H)=(0.01,80)$: red line, $(T,H)=(0.001,800)$: blue line).

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