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# Robust $H_2$ control of ladder circuits modeled as a subclass of $2$D systems*

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***Abstract*— Spatially interconnected systems are the special case of $2$D systems, since there exist 2 independent variables in the model. In the paper RLC ladder circuit uncertain model is taken as a particular case of such systems. Due to the model structure and uncertainties solving problems of stability testing and stabilization for such models requires developing its own, $2$D based, approach. Hence on, a short introduction into $2$D system theory is provided, then a methodology towards stability analysis and stabilizing controller design with the application of Linear Matrix Inequalities (LMI) techniques are presented. The main results of this paper is providing the solution for problems of the robust control and $H_2$ robust control of ladder circuits in terms of LMIs.**

## I. INTRODUCTION

Spatially interconnected systems are formed as a result of the structured connection of subsystems, which possess its own time dynamics and are mutually interconnected. according to some prescribed rule/graph. For example, they can form a series or a circle, they can be displaced on the plane or in the 3D space. Hence, they can be assumed to have two-fold dynamics, i.e. the spatial one and time [1]. In what follows, they be considered as the particular case of the multidimensional (nD) systems.

In this paper results are developed for particular case of spatially interconnected systems, i.e. RLC ladder circuits. Such systems (ladder circuits/networks) are used in many different applications, as filters, delay lines and equivalent circuits for transmission lines. They also can represent a model for the analysis and simulation of transmission lines, chains of transmission gate or long wire interconnections [2] from different points of view. The ladder models can be applied towards the analysis of electric ladder and ring networks, in approximation of some distributed parameter systems e.g. [3], in the simulation of physical systems such as mechanical, chemical, thermal e.g. [4], [5] or in analysis of chaotic oscillations in the ladder network, energy transfer in an RC ladder networks [6].

In general, facing problems of analysis and synthesis for such systems, two approaches are possible to follow. The first is based on the direct 2D characteristics of the considered model and in general lead to new results from system and control theory (for first results see e.g. [7]). It is to be noted that not in every case it provides the applicable solution. On the other hand, the second approach is based on the so-called lifting procedure, when the dynamics along the finite indeterminate is ”hidden” into the model matrices. Then, taking into account the internal structure of considered models, known from 1D control theory results can be adopted in order to solve problems from 2D ($n$D) systems area. There are also some drawbacks of that approach - it can lead to the very large overall model dimensions, which in turns can be the source of numerical problems. The similar situation occurs for the 2D systems subclass, so-called repetitive processes - see e.g. [8] and references therein.

As aforementioned, in this paper a particular case of RLC ladder circuits is considered. It is assumed that both: the structure of each node in the ladder and nominal values of elements in the node are homogeneous. Since values of real electronic elements that constitute the ladder circuit are always given with some tolerance, there appears a natural need to use uncertain model and solve the considered control problems in terms of robustness. For the purposes of this paper the norm bounded uncertainty model has been chosen. The stability tests and controller designs are developed using the Lyapunov theory and are given as a Linear Matrix Inequality conditions. Due to the fact that the disturbance influence onto the model is assumed also the solution for $H_2$ robust control task in order to suppress disturbances is presented.

Presented results have been also numerically evaluated and simulated. All numerical computations were done in MATLAB with Sedumi SDP solver installed, however there are no apparent problems in implementing the code in other computational environments, e.g. SCILAB (for similar case refer to [9]).

Throughout this paper $M \succ 0$ (respectively $\prec 0$) denotes a real symmetric positive (negative) definite matrix, $I$ and $0$ denotes the identity and zero block matrices of appropriate dimensions and also

$$\mathrm{tri}\{\beta,\gamma,\eta\} \hat{=} \begin{bmatrix} \gamma & \beta & & & 0 \\ \eta & \gamma & \beta & & \\ & \ddots & \ddots & \ddots & \\ & & \eta & \gamma & \beta \\ 0 & & & \eta & \gamma \end{bmatrix}.$$

## II. SPATIALLY INTERCONNECTED SYSTEMS AS A SUB-CLASS OF 2D HYBRID SYSTEMS

Consider the dynamical system presented at Figure 1. It is constructed with a series of homogenous cells (blocks or nodes). For any selected node $p$ ($0 < p < \alpha - 1$) besides

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the time dynamics, there is an influence of the neighboring nodes $p-1$ and $p+1$. Hence, it is natural to treat such a system as a subclass of 2D systems. It is straightforward to

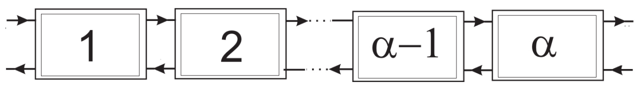


Fig. 1. The idea of considered spatially interconnected system

see that considered system can be described by the following state-space model over $p = 0, 1, \ldots, \alpha-1$, and $t \geq 0$ where $p$ denotes the block number (or system node) and $t$ - time, respectively

$$
\begin{aligned}
\dot{x}(p,t) =& \widetilde{\mathcal{A}}_1 x(p-1,t) + \widetilde{\mathcal{A}}_2 x(p,t) + \widetilde{\mathcal{A}}_3 x(p+1,t) + \widetilde{\mathcal{B}} u(p,t) \\
&+ \widetilde{\mathcal{E}}_1 w(p-1,t) + \widetilde{\mathcal{E}}_2 w(p,t) + \widetilde{\mathcal{E}}_3 w(p+1,t), \\
y(p,t) =& \mathcal{C} x(p,t).
\end{aligned} \tag{1}
$$

Here $x(p,t) \in \mathbb{R}^n$, $u(p,t) \in \mathbb{R}^m$, $y(p,t) \in \mathbb{R}^r$, $w(p,t) \in \mathbb{R}$ are the state, the input, the output and disturbance vectors, respectively of the $p$-th subsystem. $\dot{x}(.,t)$ denotes a derivative on the variable $t$. Assume also that matrices in (1) are uncertain using the norm bounded uncertainty definition, i.e.

$$
\begin{aligned}
\widetilde{\mathcal{A}}_i &= \mathcal{A}_i + \Delta \bar{A}_i = \mathcal{A}_i + H_i F_i E_i, \; i = 1,2,3 \\
\widetilde{\mathcal{B}} &= \mathcal{B} + \delta \bar{B} = \mathcal{B} + H_B F_B E_B, \\
\widetilde{\mathcal{E}}_i &= \mathcal{E}_i + \Delta \bar{E}_i = \mathcal{E}_i + H_{Ei} F_{Ei} E_{Ei}, \; i = 1,2,3,
\end{aligned} \tag{2}
$$

where $\mathcal{A}_i$, $\mathcal{E}_i$, $i = 1,2,3$, $\mathcal{B}$ are the nominal model matrices i.e. computed for the nominal values of elements and $\Delta \bar{A}_i$, $\Delta \bar{E}_i$, $i = 1,2,3$ $\Delta \bar{B}$ denote uncertainty factors of underlying matrices. What's more, using the norm bounded uncertainty, $H_i$, $E_i$, $H_{Ei}$, $E_{Ei}$, $i = 1,2,3$ and $H_B$, $E_B$, are known, appropriately dimensioned matrices and $F_i$, $F_{Ei}$, $i = 1,2,3$ and $F_B$ are matrices of appropriate dimensions with unknown entries which denote the uncertainty. It is also assumed that $\|F_i\| < 1$, $\|F_{Ei}\| < 1, i = 1,2,3$, $\|F_B\| < 1$. Note that since $\mathcal{C}$ is a selector matrix (it selects which state/states is/are taken as output/s), it is assumed to be fixed.

To complete the model, it is necessary to provide the appropriate boundary conditions which can be given now in the following form:

$$
\begin{aligned}
x(-1,t) &= f_{-1}(t), \; x(\alpha,t) = f_\alpha(t), \quad t \geq 0, \\
x(p,0) &= f_0(p) = 0, \; 0 \leq p \leq \alpha - 1,
\end{aligned} \tag{3}
$$

where $f_{-1}(t)$, $f_\alpha(t)$ and $f_0(p)$ are bounded functions.

### A. Lifting to 1D equivalent model

Define state, input, output and disturbance supervectors $\mathbf{x}(t)$, $\mathbf{u}(t)$, $\mathbf{y}(t)$, $\mathbf{w}(t)$, i.e.:

$$
\begin{aligned}
\mathbf{x}(t) &= \left[x(0,t)^T, \; x(1,t)^T, \; \ldots, \; x(\alpha-1,t)^T\right]^T, \\
\mathbf{u}(t) &= \left[u(0,t)^T, \; u(1,t)^T, \; \ldots, \; u(\alpha-1,t)^T\right]^T, \\
\mathbf{y}(t) &= \left[y(0,t)^T, \; y(1,t)^T, \; \ldots, \; y(\alpha-1,t)^T\right]^T, \\
\mathbf{w}(t) &= \left[w(0,t), \; w(1,t), \; \ldots, \; w(\alpha-1,t)\right]^T.
\end{aligned}
$$

Application of the "lifting" procedure to (1) provides the following 1D equivalent model

$$
\begin{aligned}
\frac{d}{dt}\mathbf{x}(t) &= \widetilde{\Phi}\mathbf{x}(t) + \widetilde{\Psi}\mathbf{u}(t) + \widetilde{\Xi}\mathbf{w}(t), \\
\mathbf{y}(t) &= \Lambda \mathbf{x}(t),
\end{aligned} \tag{4}
$$

where

$$
\begin{aligned}
\widetilde{\Phi} &= \mathrm{tri}\{\widetilde{\mathcal{A}}_3, \widetilde{\mathcal{A}}_2, \widetilde{\mathcal{A}}_1\}, \; \widetilde{\Psi} = \mathrm{diag}\{\widetilde{\mathcal{B}}, \cdots, \widetilde{\mathcal{B}}\}, \\
\widetilde{\Xi} &= \mathrm{tri}\{\widetilde{\mathcal{E}}_3, \widetilde{\mathcal{E}}_2, \widetilde{\mathcal{E}}_1\}, \; \Lambda = \mathrm{diag}\{\mathcal{C}, \cdots, \mathcal{C}\}.
\end{aligned} \tag{5}
$$

Note that above defined matrices are uncertain, however only for $\widetilde{\Psi}$ there exists straight decomposition $\widetilde{\Psi} = \Psi + \Delta\Psi = \Psi + H_\Psi F_\Psi E_\Psi$. To get the proper norm bounded uncertainty for matrices $\widetilde{\Phi}$ and $\widetilde{\Xi}$ some manipulations have to be done. Since considered interconnected systems are homogenous, block matrices that constitute 1D equivalent model are fixed and made of the same block matrices $\widetilde{\mathcal{A}}_i$, $\widetilde{\mathcal{E}}_i$, $i = 1,2,3$, $\widetilde{\mathcal{B}}$, $\mathcal{C}$. The case where nominal values of elements in nodes differ refer e.g. to [10].

## III. LADDER CIRCUITS AS AN EXAMPLE OF INTERCONNECTED SYSTEMS

Consider the particular example of ladder circuits shown at Figure 2. It is assumed that it consists of cells of identical structure but values of parameters can vary. This assumption is a basis for the forthcoming uncertainty analysis. It is straightforward to see that such a circuit can be described by (1). In Figure 2, $i(p,t)$ and $E(p,t)$ are current and voltage controlled sources respectively, added to each $p$-th node. The first type sources are assumed being intrinsic parts of the considered circuit and the second realize the control actions. Applying Kirchoff's laws for a $p$th node yields the 2D state-space model, written over nodes $p = 0, 1, \ldots, \alpha-1$ and time $t \geq 0$ in the form of (1). In this case,

$$
x(p,t) = \begin{bmatrix} U_c(p,t) & i_L(p,t) \end{bmatrix}^T
$$

denotes the local state of the $p$-th cell and $E(p,t) = u(p,t)$ denotes voltage controlled sources representing the control actions. Assume also that a controlled source of each $p$-th circuit node is of the form

$$
i(p,t) = \gamma U_c(p-1,t) + w(p,t),
$$

where $\gamma$ is the gain of the current source controlled by voltage and hence is a conductance and $w(p,t)$ denotes disturbances influencing.

In what follows, matrices of the model (1) can be presented as

$$
\begin{aligned}
&\widetilde{\mathcal{A}}_1 = \begin{bmatrix} \frac{\gamma}{C} & 0 \\ \frac{1}{L} & 0 \end{bmatrix}, \widetilde{\mathcal{A}}_2 = \begin{bmatrix} -\frac{1}{R_2 C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_1}{L} \end{bmatrix}, \widetilde{\mathcal{A}}_3 = \begin{bmatrix} 0 & -\frac{1}{C} \\ 0 & 0 \end{bmatrix}, \\
&\widetilde{\mathcal{B}} = \begin{bmatrix} 0 \\ -\frac{1}{L} \end{bmatrix}, \widetilde{\mathcal{E}}_1 = \widetilde{\mathcal{E}}_3 = 0, \; \widetilde{\mathcal{E}}_2 = \begin{bmatrix} 0 \\ -\frac{1}{L} \end{bmatrix}.
\end{aligned} \tag{6}
$$

Again, to complete the model description, it is necessary to provide the boundary conditions, which for this case become

$$
\begin{aligned}
&x(-1,t) = \begin{bmatrix} U(t) & 0 \end{bmatrix}^T, \; x(\alpha,t) = \begin{bmatrix} 0 & i(t) \end{bmatrix}^T, \\
&x(p,0) = 0, \; 0 \leq p \leq \alpha - 1
\end{aligned} \tag{7}
$$

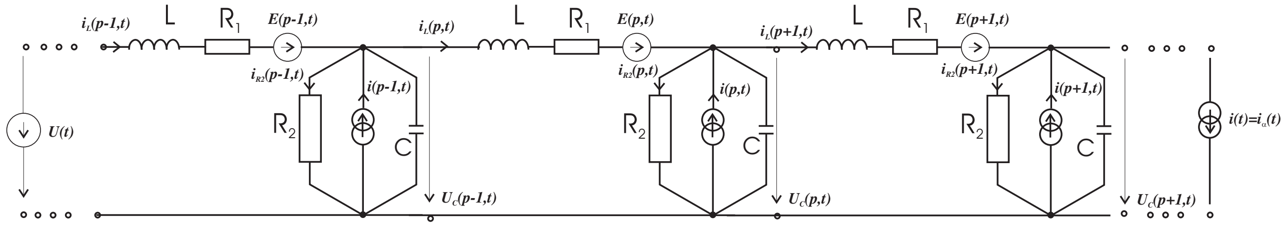


Fig. 2. The considered ladder chain

where $i(t)$ and $U(t)$ are known functions representing boundary sources.

### A. Modeling of uncertainty factors for ladder circuits

Note that since electronic elements values in (6) are always given with some tolerance, it is a justification for requirements of uncertainty analysis. In what follows, the goal here is to obtain uncertainty factors $H_i$, $H_{Ei}$, $E_i$, $E_{Ei}$, $i = 1, 2, 3$ and $H_B$, $E_B$ based upon tolerances of every element in the ladder node. Hence, for any electronic element (say $Z$)

$$Z^{real} \in Z^{nominal} + \tau^Z \bar{Z},$$

where $\tau^Z \in R : -1 < \tau^Z < 1$ and $\bar{Z} = \frac{tol_Z}{100} Z^{nominal}$. In fact, in the uncertainty analysis, maximal deviation from the nominal values is considered. In order to compute the uncertainty factors the following approach is used. First, compute system matrices $\mathcal{A}_i$, $\mathcal{E}_i$, $i = 1, 2, 3$, $\mathcal{B}$ using the nominal values of electronic elements in the node. Then, by checking all combinations $Z^{min}$, $Z^{max}$ for every uncertain element in the node find such elements combination that provides maximum norm matrix $\Delta\eta = \check{\eta} - \eta$, where $\eta$ is the nominal matrix taken into account (one of $\mathcal{A}_i$, $\mathcal{E}_i$, $i = 1, 2, 3$, $\mathcal{B}$) and $\check{\eta}$ is the current combination of elements matrix. Finally, apply any decomposition method (e.g. $QR$ decomposition) to compute $H_\eta$, $E_\eta$ ($\Delta\eta = H_\eta E_\eta$). Note also that in the above procedure the maximum possible deviation from the nominal values is considered, hence it in fact can be assumed that $F_\eta = I$. In a real system the maximum deviation is not attained hence $\|F_\eta\| < 1$.

*Remark 1:* Note that for models of ladder circuits considered in this papers there are 4 elements in a node, hence 16 combinations of $Z_{min}$, $Z_{max}$ are to be checked for each system matrix examined. However, in general the number of considered combinations can be stated as $2^\nu$, where $\nu$ denotes the number of uncertain elements in a model.

### B. Stability and stabilization

Application of Lyapunov stability theory to (4) gives the following results.

*Lemma 1:* [11] The system described by (4) with no control inputs and no disturbances influencing the model is stable, if and only if, $\exists\, \mathcal{P} \succ 0$ such that

$$\mathcal{P}\widetilde{\Phi} + \widetilde{\Phi}^T \mathcal{P} \prec 0. \qquad (8)$$

Note that the fully populated matrix $\mathcal{P}$ denotes that each node directly influences all others and this is not necessary in case of considered system. Hence, the following sufficient stability condition is used which also eases implementation.

*Lemma 2:* [11] The system described by (4) with no control inputs and no disturbances influencing the model is stable, if $\exists\, Q \succ 0$ such that

$$\text{tri}\{Q\widetilde{\mathcal{A}}_1^T + \widetilde{\mathcal{A}}_3 Q, Q\widetilde{\mathcal{A}}_2^T + \widetilde{\mathcal{A}}_2 Q, Q\widetilde{\mathcal{A}}_3^T + \widetilde{\mathcal{A}}_1 Q\} \prec 0. \qquad (9)$$

Note that since the above results can be applied only for fixed (no uncertainties existing) models, they cannot be applied for the case considered in this paper.

## IV. UNCERTAIN MODEL STABILITY AND STABILIZATION

First, recall the following result given e.g. in [12]

*Lemma 3:* Let $\mathcal{H}$, $\mathcal{E}$ be given real matrices of appropriate dimensions and $\mathcal{F}$ satisfy $\|\mathcal{F}\| \leq 1$. Then, for any $\epsilon > 0 \in \mathbb{R}$ the following holds

$$\mathcal{HFE} + \mathcal{E}^T\mathcal{F}^T\mathcal{H}^T \leq \epsilon\mathcal{H}\mathcal{H}^T + \epsilon^{-1}\mathcal{E}^T\mathcal{E}.$$

### A. Robust stability

*Theorem 1:* The uncertain system described by (1) with uncertainties defined as (2), with no inputs and no disturbances influencing the model is stable, if $\exists Q \succ 0$ and $\epsilon > 0$, such that

$$\begin{bmatrix} \Omega & \epsilon H & E^T \\ \epsilon H^T & -\epsilon I & 0 \\ E & 0 & -\epsilon I \end{bmatrix} \prec 0,\ \epsilon > 0, \qquad (10)$$

where

$$\Omega = \text{tri}\{Q\mathcal{A}_1^T + \mathcal{A}_3 Q, Q\mathcal{A}_2^T + \mathcal{A}_2 Q, Q\mathcal{A}_3^T + \mathcal{A}_1 Q\},$$
$$H = \begin{bmatrix}\text{tri}(H_3, 0, 0) & \text{diag}(H_2, \dots, H_2) & \text{tri}(0, 0, H_1),\end{bmatrix}$$
$$E = \begin{bmatrix}\text{diag}(E_3Q, \dots, E_3Q) \\ \text{diag}(E_2Q, \dots, E_2Q) \\ \text{diag}(E_1Q, \dots, E_1Q)\end{bmatrix}.$$

**Proof.** Rewrite (9) for a model with uncertainties defined as (2), i.e.

$$\Omega + \Upsilon + \Upsilon^T \prec 0,$$
$$\text{where } \Upsilon = \text{tri}\{H_3F_3E_3Q,\ H_2F_2E_2Q,\ H_1F_1E_1Q\}.$$

Next, note that

$$\begin{aligned}\Upsilon =& \text{tri}(H_3, 0, 0)\text{diag}(F_3, \dots, F_3)\text{diag}(E_3Q, \dots, E_3Q) \\ &+ \text{diag}(H_2, \dots, H_2)\text{diag}(F_2, \dots, F_2)\text{diag}(E_2Q, \dots, E_2Q) \\ &+ \text{tri}(0, 0, H_1)\text{diag}(F_1, \dots, F_1)\text{diag}(E_1Q, \dots, E_1Q) \\ =& H\,F\,E,\end{aligned}$$

where $F = \text{diag}(F_3, \ldots, F_3, F_2, \ldots, F_2, F_1, \ldots, F_1)$ and since $\|F_i\| < 1,\ i = 1, 2, 3$, it follows that $\|F\| < 1$. Application of Lemma 3 provides

$$\Omega + \epsilon H H^T + \epsilon^{-1} E^T E \prec 0,\ \epsilon > 0$$

Apply the Schur complement to get

$$\begin{bmatrix} \Omega & H & E^T \\ H^T & -\epsilon^{-1} I & 0 \\ E & 0 & -\epsilon I \end{bmatrix} \prec 0,\ \epsilon > 0$$

Finally left- and right- multiply it by $\text{diag}(I, \epsilon I, I)$ to obtain the LMI of (10). That finishes the proof. □

### B. Controllability

First step here is the controllability assumption i.e., the circuit described by (4) is controllable. Using one of many equivalent conditions it can be given by:

$$rank(W) = \texttt{dim}(x)\alpha \tag{11}$$

$\forall z \in \mathbb{C}$, where

$$W = \begin{bmatrix} zI - \widetilde{\Phi} & \widetilde{\Psi} \end{bmatrix}$$

It follows immediately from the structure of $\widetilde{\Phi}$ and $\widetilde{\Psi}$ that the model (4) is controllable, if the pair $\{\widetilde{\mathcal{A}}_2,\ \widetilde{\mathcal{B}}\}$ is also controllable. Since $\widetilde{\Phi}$, $\widetilde{\Psi}$ are uncertain, hence the justified doubt about the rank of the above matrix can arise. In general, there can happen that the existence of uncertainty factors influences the rank of $W$ and (11) is not satisfied. However, such a critical situation can arise very rarely. For models of considered ladder circuit, since uncertainty factors depend directly on tolerances of electronic elements, the situation when (11) is not satisfied would arise when the tolerance of elements is equal 100% or more, which is obviously the academic only case and will be not considered here. Hence, it is assumed that (11) is satisfied, what allows to follow to the robust stabilization.

### C. Robust stabilization

Consider again the model of spatially interconnected system described by (4) and choose the following state feedback law

$$\mathbf{u}(t) = K\mathbf{x}(t) \text{ and } K = \text{tri}\{K_1, K_2, K_3\}. \tag{12}$$

It is straightforward to see that for any single node $p = 0, 1, \ldots, \alpha-1$ (12) can be rewritten in the 2D fully equivalent form as

$$\begin{aligned} u(p,t) =& u^1(p,t) + u^2(p,t) + u^3(p,t) \\ =& K_1 x(p-1,t) + K_2 x(p,t) + K_3 x(p+1,t). \end{aligned} \tag{13}$$

It leads to the following result.

*Theorem 2:* Suppose that a control law of the form (12) is applied to the uncertain system described by (4) and no disturbances influence the model. Then, the resulting circuit is stable, if there exist matrices $Q \succ 0$, $N_1$, $N_2$, $N_3$ and a scalar $\epsilon > 0$, such that the following LMI is feasible

$$\begin{bmatrix} \Omega_c & \epsilon H_c & E_c^T \\ \epsilon H_c^T & -\epsilon I & 0 \\ E_c & 0 & -\epsilon I \end{bmatrix} \prec 0,\ \epsilon > 0, \tag{14}$$

where

$$\Omega_c = \Omega + \text{tri}\{N_1^T \mathcal{B}^T + \mathcal{B} N_3,\ N_2^T \mathcal{B}^T + \mathcal{B} N_2,\ N_3^T \mathcal{B}^T + \mathcal{B} N_1\},$$
$$H_c = [\text{tri}(H_3, 0, 0)\ \text{tri}(H_B, 0, 0)\ \text{diag}(H_2, \ldots, H_2)\ \ldots$$
$$\ldots\ \text{diag}(H_B, \ldots, H_B)\ \text{tri}(0, 0, H_1)\ \text{tri}(0, 0, H_B)],$$
$$E_c = \begin{bmatrix} \text{diag}(E_3 Q, \ldots, E_3 Q) \\ \text{diag}(E_B N_3, \ldots, E_B N_3) \\ \text{diag}(E_2 Q, \ldots, E_2 Q) \\ \text{diag}(E_B N_2, \ldots, E_B N_2) \\ \text{diag}(E_1 Q, \ldots, E_1 Q) \\ \text{diag}(E_B N_1, \ldots, E_B N_1) \end{bmatrix}.$$

If the above LMI holds, stabilizing control law matrices are computed by

$$K_i = N_i Q^{-1},\ i = 1, 2, 3. \tag{15}$$

**Proof.** Rewrite (9) for the closed loop uncertain model with uncertainties defined as (2).

$$\Omega_c + \Upsilon_c + \Upsilon_c^T \prec 0,$$

$$\begin{aligned} \text{where } \Omega_c = \text{tri}\{& Q(\mathcal{A}_1 + \mathcal{B} K_1)^T + (\mathcal{A}_3 + \mathcal{B} K_3) Q, \\ & Q(\mathcal{A}_2 + \mathcal{B} K_2)^T + (\mathcal{A}_2 + \mathcal{B} K_2) Q, \\ & Q(\mathcal{A}_3 + \mathcal{B} K_3)^T + (\mathcal{A}_1 + \mathcal{B} K_1) Q\}, \\ \Upsilon_c = \text{tri}\{& (H_3 F_3 E_3 + H_B F_B E_B K_3) Q, \\ & (H_2 F_2 E_2 + H_B F_B E_B K_2) Q, \\ & (H_1 F_1 E_1 + H_B F_B E_B K_1) Q\}. \end{aligned}$$

Substitute $K_i = N_i Q^{-1},\ i = 1, 2, 3$. Next, note that

$$\begin{aligned} \Omega_c =& \Omega + \text{tri}\{N_1^T \mathcal{B}^T + \mathcal{B} N_3,\ N_2^T \mathcal{B}^T + \mathcal{B} N_2,\ N_3^T \mathcal{B}^T + \mathcal{B} N_1\}, \\ \Upsilon_c =& \text{tri}(H_3, 0, 0)\text{diag}(F_3, \ldots, F_3)\text{diag}(E_3 Q, \ldots, E_3 Q) \\ &+ \text{tri}(H_B, 0, 0)\text{diag}(F_B, \ldots, F_B)\text{diag}(E_B N_3, \ldots, E_B N_3) \\ &+ \text{diag}(H_2, \ldots, H_2)\text{diag}(F_2, \ldots, F_2)\text{diag}(E_2 Q, \ldots, E_2 Q) \\ &+ \text{diag}(H_B, \ldots, H_B)\text{diag}(F_B, \ldots, F_B) \\ &\text{diag}(E_B N_2, \ldots, E_B N_2) \\ &+ \text{tri}(0, 0, H_1)\text{diag}(F_1, \ldots, F_1)\text{diag}(E_1 Q, \ldots, E_1 Q) \\ &+ \text{tri}(0, 0, H_B)\text{diag}(F_B, \ldots, F_B)\text{diag}(E_B N_1, \ldots, E_B N_1) \\ =& H_c F_c E_c, \end{aligned}$$

where $F_c = \text{diag}(F_3, \ldots, F_3, F_B, \ldots, F_B, \ldots$
$\ldots F_2, \ldots, F_2, F_B, \ldots, F_B, F_1, \ldots, F_1, F_B, \ldots, F_B)$.

Note that $\|F_c\| < 1$. Application of Lemma 3 provides

$$\Omega_c + \epsilon H_c H_c^T + \epsilon^{-1} E_c^T E_c \prec 0,\ \epsilon > 0.$$

Next apply the Schur complement to get

$$\begin{bmatrix} \Omega_c & H_c & E_c^T \\ H_c^T & -\epsilon^{-1} I & 0 \\ E_c & 0 & -\epsilon I \end{bmatrix} \prec 0,\ \epsilon > 0.$$

Left- and right- multiply it by $\text{diag}(I, \epsilon I, I)$ finally to obtain the LMI of (14). □

## V. $H_2$ ROBUST CONTROL

Since in the considered model there are disturbances, natural seems to try to deal with them and eventually remove or at least minimize their influence. Consequently, together with assuring the robust stability of the model, the aim here is to limit the influence of disturbances onto the model. This formally can be defined as providing such control action that minimizes the $H_2$ norm of the transfer function from disturbances (which are considered as additional input) to model outputs (see e.g. [13]). This leads to providing a controller $K$ such that the following equation is satisfied

$$G(s) = \|\Lambda(sI - \widetilde{\Phi}_c)^{-1}\widetilde{\Xi}\|_2 < \varphi,$$

where $\widetilde{\Phi}_c = \widetilde{\Phi} + \widetilde{\Psi}K$, $\varphi$ denotes the attenuation level.

Then for $\varphi = \rho^{\frac{1}{2}}$ the above condition can be expressed in the LMI form as

$$\begin{aligned} &\min \rho, \\ &\text{s. t. } \widetilde{\Phi}_c\mathcal{Q} + \mathcal{Q}\widetilde{\Phi}_c^T + \widetilde{\Xi}\widetilde{\Xi}^T \prec 0, \\ &\quad \text{trace } (\Lambda\mathcal{Q}\Lambda^T) < \rho, \\ &\quad \mathcal{Q} = \text{diag}(Q, \dots, Q) \succ 0. \end{aligned} \tag{16}$$

Again, it is straightforward to see that the above LMI problem cannot be solved directly since uncertain matrices there appear (only $\Lambda$ matrix is fixed). Hence, it has to be reformulated in order to address model uncertainties.

*Theorem 3:* Suppose that a control law of the form (12) is applied to the model (4) (with disturbances present). Then, the resulting circuit is stable and disturbances are dumped with attenuation level $\varphi = \rho_p^{\frac{1}{2}}\alpha$, if there exist matrices $Q \succ 0$, $N_1$, $N_2$ and $N_3$ and a scalar $\epsilon > 0$, such that the following LMI-based problem is feasible

$$\min \rho_p, \tag{17}$$

$$\text{s. t. } \begin{bmatrix} \Omega_c & \Xi & \epsilon H_c & \epsilon H_\Xi & E_c^T & 0 \\ \Xi^T & -I & 0 & 0 & 0 & E_\Xi^T \\ \epsilon H_c^T & 0 & -\epsilon I & 0 & 0 & 0 \\ \epsilon H_\Xi^T & 0 & 0 & -\epsilon I & 0 & 0 \\ E_c & 0 & 0 & 0 & -\epsilon I & 0 \\ 0 & E_\Xi & 0 & 0 & 0 & -\epsilon I \end{bmatrix} \prec 0, \tag{18}$$

$$\text{trace } (\mathcal{C}Q\mathcal{C}^T) < \rho_p, \tag{19}$$

where $\Omega_c$, $H_c$, $E_c$ have been defined in Theorem 2 and

$$H_\Xi = \big[\text{diag}(H_{E2}, \dots, H_{E2}) \quad \text{tri}(H_{E3}, 0, 0) \quad \text{tri}(H_{E1}, 0, 0)\big],$$

$$E_\Xi = \begin{bmatrix} \text{diag}(E_{E2}, \dots, E_{E2}) \\ \text{diag}(E_{E3}, \dots, E_{E3}) \\ \text{diag}(E_{E1}, \dots, E_{E1}) \end{bmatrix}.$$

**Proof.** Note that the inequality (18) can be viewed as a robust stabilization condition considered in Theorem 2 with the additionally uncertain factor $\widetilde{\Xi}\widetilde{\Xi}^T$. In what follows, it can be rewritten as

$$\Omega_c + H_cF_cE_c + E_c^TF_c^TH_c^T + \widetilde{\Xi}\widetilde{\Xi}^T \prec 0,$$

Using the Schur complement

$$\begin{bmatrix} \Omega_c + H_cF_cE_c + E_c^TF_c^TH_c^T & \Xi + H_\Xi F_\Xi E_\Xi \\ \Xi^T + E_\Xi^T F_\Xi^T E_\Xi^T & -I \end{bmatrix} \prec 0,$$

or

$$\begin{bmatrix} \Omega_c & \Xi \\ \Xi^T & -I \end{bmatrix} + \begin{bmatrix} H_c & H_\Xi \\ 0 & 0 \end{bmatrix}\begin{bmatrix} F_c & 0 \\ 0 & F_\Xi \end{bmatrix}\begin{bmatrix} E_c & 0 \\ 0 & E_\Xi \end{bmatrix} + \begin{bmatrix} E_c^T & 0 \\ 0 & E_\Xi^T \end{bmatrix}\begin{bmatrix} F_c^T & \\ 0 & F_\Xi^T \end{bmatrix}\begin{bmatrix} H_c^T & 0 \\ H_\Xi^T & 0 \end{bmatrix} \prec 0,$$

where $F_\Xi = \text{diag}(F_{EE2}, \dots, F_{EE2}, F_{EE3}, \dots, F_{EE3}, \dots$
$\dots F_{EE1}, \dots, F_{EE1})$.

Note that $\|F_\Xi\| < 1$. Application of Lemma 3 provides

$$\begin{bmatrix} \Omega_c & \Xi \\ \Xi^T & -I \end{bmatrix} + \epsilon \begin{bmatrix} H_c & H_\Xi \\ 0 & 0 \end{bmatrix}\begin{bmatrix} H_c^T & 0 \\ H_\Xi^T & 0 \end{bmatrix} + \epsilon^{-1}\begin{bmatrix} E_c & 0 \\ 0 & E_\Xi \end{bmatrix}\begin{bmatrix} E_c^T & 0 \\ 0 & E_\Xi^T \end{bmatrix} \prec 0, \epsilon > 0.$$

Again, apply the Schur complement to get

$$\begin{bmatrix} \Omega_c & \Xi & H_c & H_\Xi & E_c^T & 0 \\ \Xi^T & -I & 0 & 0 & 0 & E_\Xi^T \\ H_c^T & 0 & -\epsilon^{-1}I & 0 & 0 & 0 \\ H_\Xi^T & 0 & 0 & -\epsilon^{-1}I & 0 & 0 \\ E_c & 0 & 0 & 0 & -\epsilon I & 0 \\ 0 & E_\Xi & 0 & 0 & 0 & -\epsilon I \end{bmatrix} \prec 0,$$

Left- and right- multiply it by $\text{diag}(I, I, \epsilon I, \epsilon I, I, I)$ to get (18). Also note that since $\Lambda = \text{diag}(\mathcal{C}, \dots, \mathcal{C})$ is fixed, the expression $\text{trace}(\mathcal{C}Q\mathcal{C}^T)$ is devoted to a single node in the ladder and can be rewritten as (19). That finishes the proof. □

*Remark 2:* Note that the above procedure relies on minimization of $\rho_p$. Hence, taking into account (19), it is straightforward to see that minimization of $\rho_p$ leads to minimization of $\text{trace}(Q)$. In what follows, entries of controller matrices computed as (15) can have very large values. If computed controllers are too large, the minimization problem stated in Theorem 3 can be reformulated as a feasibility problem, i.e. for a'priori chosen $\rho$, solve (18)-(19).

### A. Case study

Consider the circuit built from elements of the nominal values $C = 1 \times 10^{-4}$ $[F]$, $L = 5 \times 10^{-1}$ $[H]$, $R_1 = 1000$ $[\Omega]$, $R_2 = 20000$ $[\Omega]$, $\gamma = 0.005$. The tolerances of elements have been assumed as $R_1 : 5\%$, $R_2 : 10\%$, $C : 10\%$, $L : 5\%$. The boundary sources are taken as $U(t) = 10$ and $i(t) = 0$. The length of ladder is set to $\alpha = 20$. The disturbances along the ladder have been chosen as a random vector generated with the following Matlab formula: $dist = (rand(alpha, 1) * 2 - 1) * 50$. The considered model is unstable.

The application of Theorem 2 results in computing the following controller matrices:

$$K_1 = [-3 \;\; -1223], \; K_2 = [162 \; 23361], \; K_3 = [-5 \;\; -97].$$

The simulations have been performed over time $t \in (0, \; 2)$[s]. Figure 3 shows the voltages on capacitors generated by the controlled circuit. Note that the control action ensures the stability however the influence of disturbances

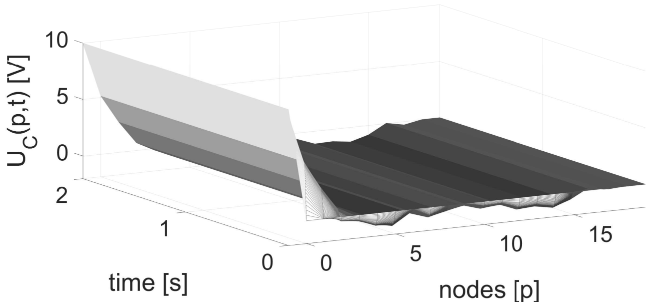


Fig. 3. Simulation results for the active controlled circuit — $U_C(p,t)$ over time and nodes

is exactly visible. Hence, Theorem 3 has been applied to provide the following controllers (with the attenuation level value $\rho_p = 0.296$)

$$K_1 = 10^9 \times [0.3\ 6.7],\ K_2 = 10^{11} \times [0.1\ 26.1], \qquad K_3 = 10^9 \times [0.3\ 67.8].$$

Entries in above controllers are enormous and its practical applicability is limited. Instead, solve again this problem for the prescribed $\rho_p = 0.005$ (as described in Remark 2). It leads to obtaining the following controllers:

$$K_1 = 10^3 \times [-0.4\ \ -59.4],\ K_2 = 10^4 \times [0.3\ 55.7], \qquad K_3 = 10^4 \times [0.2\ 26.6].$$

Note that relatively large entries of controllers do not imply large values of control input - see Figure 5. Note that both:

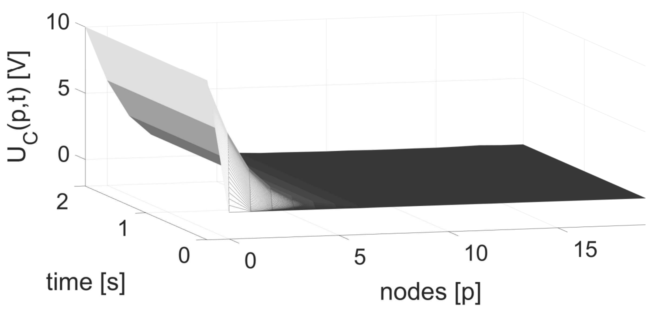


Fig. 4. Simulation results for the active circuit with $H_2$ controllers applied — $U_C(p,t)$ over time and nodes

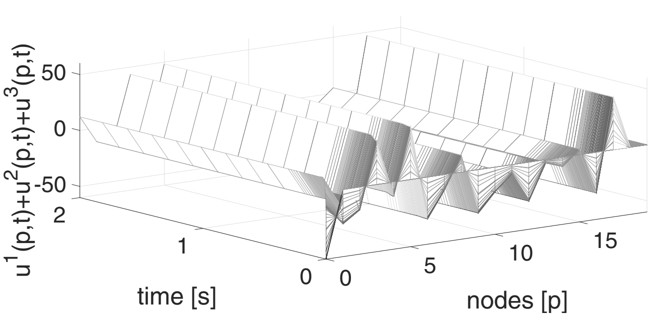


Fig. 5. Sum of inputs over the nodes

the stability and the disturbance attenuation are now reached.

## VI. Conclusions

In this paper ladder circuits models which are a particular case of spatially interconnected systems has been considered. The modeling process of active RLC ladder circuit has provided a 2D state space model that includes the norm bounded uncertainties. Then, the "lifting" procedure has been applied to provide the uncertain 1D equivalent model. This, in turn, has allowed to formulate the problems of stability testing and stabilization in terms of robustness. Also, due to disturbances influencing the model, the $H_2$ robust control task has been stated. Formulated problems have been solved providing the LMI-based conditions, numerically evaluated and simulated.

Future works will be towards comparison of the results presented in this paper with solving the same problems via 'direct' 2D approach.

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