2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Scalable Controller Synthesis for Heterogeneous Interconnected Systems Applicable to an Overlapping Control Framework

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Abstract— **A scalable $\mathscr{H}_\infty$ controller synthesis is proposed that is tractable for heterogeneous large-scale systems. It is applied to a control approach where a distributed controller is implemented in an overlapping augmented state space. The synthesis is based on an existing modeling approach where the coupling of the subsystems is cast as an interconnection channel. This modeling approach is extended to be applicable for the most general case of heterogeneous systems with different kinds of interconnections. An augmentation of the performance channel is introduced that leaves the system norm unchanged. For the interconnected model, the centralized synthesis conditions of the distributed controllers decompose into small problems of the size of the subsystems. The resulting conditions can be solved in a centralized way with considerably reduced complexity. Furthermore, the decomposition facilitates a distributed solution. For the special case of homogeneous subsystems a decentralized design is possible. Stability and performance results are given for the augmented overlapping controller applied to the physical system. The methods and results are also applicable for non-overlapping distributed systems as a special case of overlap. A numerical example is given which illustrates the modeling and the control performance.**

I. Introduction

To make the control of complex, possibly large-scale systems computationally tractable, in many cases decentralized or distributed control architectures are required. In applications where strong coupling between subsystems exists, as e.g., physical coupling between robots or coupled control goals as in formation control tasks, completely decentralized controllers may achieve a poor control performance. As introduced in [1], and extended in [2]–[4] and related literature, it can be beneficial to design the controller in an augmented overlapping state space, so that a decentralized controller also incorporates the coupling information. Following this idea, in [5] a framework has been introduced for decentralized state feedback based on a distributed estimation scheme in an augmented overlapping state space. In contrast to the existing similar approaches in the literature, [1]–[3], and references therein, the local controllers and estimators in [5] that are synthesized in the augmented state space are not transformed back to the original state space but implemented as such. They thus have an estimate of larger parts of the state space containing more than their local states. Through the estimates of the overlapping states, information of the overall system is incorporated in the local controllers. A centralized synthesis of overlapping estimators and decentralized controllers was proposed in [5], which is however only tractable for a limited system size.

Distributed control design for heterogeneous systems was considered in the literature, e. g. in [6]–[8] and references therein. The approaches are based on a model of subsystems that are interconnected over a graph. A linear fractional representation (LFR) is used to cast the interconnections into an interconnection channel. In [9], the case of homogeneous systems was considered and in [10], an extension to different groups of homogeneous subsystems was made.

Motivated by the overlapping framework in [5], the contribution of this paper is a scalable $\mathscr{H}_\infty$ synthesis approach that allows for a decomposition of the synthesis conditions. While presented for the overlapping framework, the results apply for non-overlapping distributed systems as a special case. First, an augmentation of the performance input and output is introduced that does not change the system norm but allows for assigning a performance input and output to each of the subsystems. Then, we present a generalization of the modeling approach in [10] that is applicable to systems coupled by different interconnections, e.g., by one interconnection structure for the states and by another one for the performance channel. The resulting linear fractional transformation (LFT) consists of a decentralized part and the interconnection. Based on this and motivated by results in [9], [11], the controller synthesis can be decomposed into problems of the size of the subsystems. This results in a tractable controller design for large-scale systems. Furthermore, it enables a decentralized synthesis for the special case of homogeneous subsystems. For heterogeneous subsystems, this decomposition is similar to the interconnection model in [12] and facilitates distributed solution methods.

For the overlapping framework we show that the controller designed in an augmented space, i.e., augmented state space and augmented performance channel, guarantees stability and performance for the physical system.

The paper is structured as follows. Section II gives the system descriptions in the original and the augmented state space. Section III presents the interconnected augmented system model, followed by the decomposed controller synthesis in Section IV and a numerical example in Section V.

Notation We denote a block-diagonal matrix D of submatrices $D_1,\dots,D_N$ by $D=\mathrm{diag}_{i=1}^N(D_i)$. The $n\times n$-identity matrix is denoted by I_n and the $n\times m$ matrix of all zeros as $0_{n\times m}$. If clear from the context, the indices are dropped. $\hat{I}_{\{i\}}$ is defined as a square matrix of appropriate dimensions of all zeros except that the (i,i)-entry is one and e_j is the j-th unit

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This research was supported by the NCCR Digital Fabrication, funded by the Swiss National Science Foundation (NCCR Digital Fabrication Agreement # 51NF40-141853).

vector. Minimum and maximum singular values are denoted as $\sigma_{\min}(\cdot)$ and $\sigma_{\max}(\cdot)$. The Kronecker product is denoted by $\otimes$ and the Moore-Penrose pseudoinverse by $\dagger$. Given a complex valued matrix $M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$ and P of appropriate dimensions, then the lower and upper LFT are defined as $\mathscr{F}_{\mathrm{l}}(M,P) = M_1 + M_2P(I-M_4)^{-1}M_3$ and $\mathscr{F}_{\mathrm{u}}(M,P) = M_4 + M_3P(I-M_1)^{-1}M_2$, respectively. We use the symbol $\star$ to simplify expressions as $M_1^\top M_2 M_1$, i.e., $\star^\top M_2 M_1 = M_1^\top M_2 M_1$.

II. Problem Formulation

As presented in [5], for the controller and estimator design, the given system is transformed to an augmented state space. This concept is briefly recalled in this section and the problem formulation is introduced.

A. Original System Description

A distributed continuous-time linear time-invariant (LTI) plant with the following dynamics is considered

$$G: \begin{cases} \dot{x} &= Ax + \sum_{i=1}^N B_{ui}u_i + B_w w, \\ y_i &= C_{yi}x + D_{ywi}w, \quad i = 1,...,N, \\ z &= C_z x + D_{zu}u, \end{cases} \tag{1}$$

with the state vector $x \in \mathbb{R}^{n_x}$. The N local input and output channels, $u_i \in \mathbb{R}^{n_{ui}}$ and $y_i \in \mathbb{R}^{n_{yi}}$, define N subsystems, denoted by the index $i \in \{1,...,N\}$. The disturbance and performance output, which in general are not local, are given by $w \in \mathbb{R}^{n_w}$ and $z \in \mathbb{R}^{n_z}$, respectively. The combined input and measurement matrices are defined as $B_u = \begin{bmatrix} B_{u1} & \dots & B_{uN} \end{bmatrix}$ and $C_y = \begin{bmatrix} C_{y1}^\top & \dots & C_{yN}^\top \end{bmatrix}^\top$, respectively, and D_{yw} equivalently. The pairs (A,B_u) and (A,C_y) are assumed to be controllable and observable, however, (A,B_{ui}) and (A,C_{yi}) need not be. System G will be referred to as the original system in the following. Note that in contrast to [5], continuous-time LTI systems are considered here because of their physically given sparsity structure.

B. Augmented System Description

For distributed systems with strong coupling in the dynamics or in the performance, it is beneficial for the subsystems to have shared information. Therefore, in [5], an estimator and controller are designed in an overlapping augmented state space. In this way, a subsystem can have an estimate of a part of the state that overlaps the parts that are estimated by other subsystems. To this end, the original state vector x is transformed into an augmented one $\xi \in \mathbb{R}^{n_\xi}$ containing copies of the overlapping states. The vector ξ is given by

$$\xi = \begin{bmatrix} \xi_1^\top & \cdots & \xi_N^\top \end{bmatrix}^\top = \begin{bmatrix} X_1^\top & \cdots & X_N^\top \end{bmatrix}^\top x = Xx,$$

where $\xi_i \in \mathbb{R}^{n_{\xi i}}$ belongs to subsystem i and $X \in \mathbb{R}^{n_\xi \times n_x}$ is a mapping of full column rank with $X_i \in \mathbb{R}^{n_{\xi i} \times n_x}$. The dynamics of the overall system in the augmented state space are given by

$$G_\xi: \begin{cases} \dot{\xi} &= A_\xi \xi + B_{\xi w}w + B_{\xi u}u, \\ y &= C_{y\xi}\xi + D_{yw}w, \\ z &= C_{z\xi}\xi + D_{zu}u, \end{cases} \tag{2}$$

with the augmented system matrices

$$\begin{aligned} A_\xi &= XAX^\dagger + M_A, \quad & B_{\xi u} &= XB_u, \quad & B_{\xi w} &= XB_w, \\ C_{y\xi} &= C_y X^\dagger + M_{C_y}, \quad & C_{z\xi} &= C_z X^\dagger + M_{C_z}. \end{aligned} \tag{3}$$

The complementary matrices M_A, M_{C_y}, M_{C_z}, as in [3], are a design choice, which will be discussed in more detail later. They are chosen to fulfill

$$M_A X = 0, \quad M_{C_y}X = 0, \quad M_{C_z}X = 0. \tag{4}$$

Proposition 1: Given the augmented system in (2) with (3) and (4), and given the original system G in (1), then for the choice $\xi(0) = Xx(0)$, the following holds for the trajectory of the state $x(t)$ and the augmented state $\xi(t)$,

$$\xi(t,\xi(0),u(t),w(t)) = Xx(t,x(0),u(t),w(t)). \tag{5}$$

Furthermore, it holds that

$$z(t,\xi(t)) = z(t,x(t)), \quad \text{and} \quad y(t,\xi(t)) = y(t,x(t)). \tag{6}$$

Proof: We first note that the augmented system in (2) with the choice of complementary matrices in (4) is an extension of the original system, as defined in [3], where a nominal system without performance channel is considered. For the nominal case, it is proven in [3] that $\xi(t,\xi(0),u) = Xx(t,x(0),u)$. Redefining the input as $[u^\top, w^\top]^\top$ and equivalently the system matrix as $[B_u \ B_w]$, the results in [3] can directly be applied to show that (5) holds. With this, it holds that $C_{z\xi}\xi(t,\xi(0),u(t),w(t)) = \underbrace{(C_zX^\dagger + M_{C_z})X}_{C_z} x(t,x(0),u(t),w(t))$. The same can be shown for $y(t)$, and thus the result in (6) follows. ■

Due to Proposition 1, also the input-output behavior from w to z of the original system G and of the augmented system G_ξ under the same input u are equivalent.

Note that in [3] the general case of extension allows for augmenting also the output y and the control input u. This is not desirable here, as the synthesized controllers and estimators are implemented in the augmented state space rather than being transformed back to the original one. Therefore, the local inputs and outputs are not augmented so that they can be locally implemented at the physical system.

C. Controller and Estimator Structure

The general augmented state estimate dynamics and the related control inputs of subsystem i, as proposed in [5], are

$$\dot{\hat{\xi}}_i = \Phi_i\hat{\xi}_i + \Gamma_i y_i + \sum_{j\neq i}\left(\Psi_{ij}\hat{\xi}_j + \Theta_{ij}y_j\right), \quad u_i = K_{\xi i}\hat{\xi}_i,$$

where $\Phi_i \in \mathbb{R}^{n_{\xi i}\times n_{\xi i}}$ is a propagation gain, $\Gamma_i \in \mathbb{R}^{n_{\xi i}\times n_{yi}}$ is a measurement matrix, and $\Psi_{ij} \in \mathbb{R}^{n_{\xi i}\times n_{\xi j}}$ and $\Theta_{ij} \in \mathbb{R}^{n_{\xi i}\times n_{yj}}$ are communication gains of state estimates and measurements, respectively, from subsystems j to i. Communication can be necessary in the presence of fixed modes in the augmented system, which can limit either the achievable performance or even the stability of the closed-loop system. We refer to [13] for a check and elimination algorithm of fixed modes. In the following, the augmented system is assumed

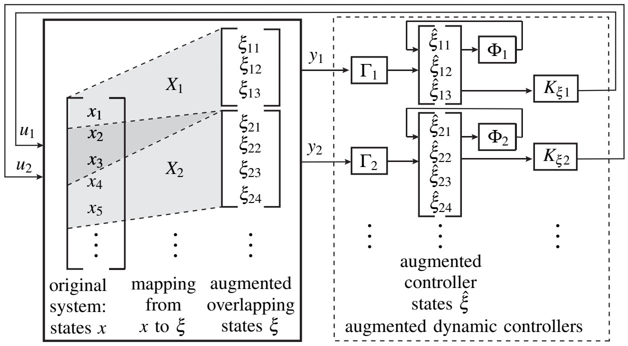

Fig. 1: Block-diagram of the original system and the dynamic output feedback controllers in the augmented state space.

to be stabilizable under a decentralized augmented controller and estimator structure. Then, no explicit communication of $\hat{\xi}_j$ or y_j from agents j to i is considered, i.e., Ψ_{ij} and Θ_{ij} are zero. Using the modeling approach in [14], the synthesis approach presented here can, w. l. o. g., be extended to a distributed one, where the interconnection structure of the controller is a design choice.

The design problem considered in this paper is to find the local estimator gains $\Phi = \mathrm{diag}_{i=1}^{N}(\Phi_i)$ and $\Gamma = \mathrm{diag}_{i=1}^{N}(\Gamma_i)$ and the controller gains $K_\xi = \mathrm{diag}_{i=1}^{N}\left(K_{\xi i}\right)$. This can be understood as the following dynamic output feedback controller,

$$H_\xi : \left\{ \begin{bmatrix} \dot{\hat{\xi}} \\ u \end{bmatrix} = \begin{bmatrix} \Phi & \Gamma \\ K_\xi & 0 \end{bmatrix} \begin{bmatrix} \hat{\xi} \\ y \end{bmatrix} \right. . \tag{7}$$

Figure 1 illustrates the transformation from the original into the augmented state space and the implementation of the augmented dynamic output feedback controllers.

III. Interconnected Model of the Augmented System

This section presents an LFR of the augmented system in a decentralized part and the interconnection part. To do so, we first introduce an augmentation of the performance channel and an extended modeling approach that is applicable for heterogeneous subsystems with different interconnections.

A. Augmentation of the Performance Channel

In order to decompose the system into local interconnected subsystems, the global performance input and output w and z, defined for the original system, are augmented such that to each of the subsystems a performance input and output can be assigned. This augmentation is given by

$$\bar{z} = Sz \quad \text{and} \quad \bar{w} = Tw,$$

with $\bar{z} = \begin{bmatrix} \bar{z}_1^\top & \dots & \bar{z}_N^\top \end{bmatrix}^\top$ and $\bar{w} = \begin{bmatrix} \bar{w}_1^\top & \dots & \bar{w}_N^\top \end{bmatrix}^\top$ and S and T having full rank. The augmentation of the related system matrices is defined as

$$\begin{aligned} C_{\bar{z}\xi} &= \bar{Q}^{\frac{1}{2}} S C_{z\xi} + M_{C_{z\xi}} \qquad & D_{\bar{z}u} &= \bar{Q}^{\frac{1}{2}} S D_{zu}, \\ B_{\xi\bar{w}} &= B_{\xi w} T^\dagger \bar{R}^{\frac{1}{2}} & D_{y\bar{w}} &= D_{yw} T^\dagger \bar{R}^{\frac{1}{2}}, \end{aligned} \tag{8}$$

where $M_{C_{z\xi}} X = 0$. The matrices $\bar{Q}$ and $\bar{R}$ are weightings which will be defined in Section IV for the controller

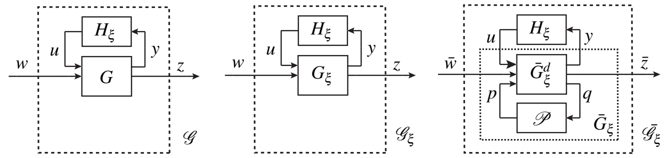

Fig. 2: Closed-loops of original system G and augmented systems G_ξ and $\bar{G}_\xi$ with the controller H_ξ. The block-diag. parts of $\bar{G}_\xi$ are in $\bar{G}_\xi^d$ and the off-block-diag. parts in the interconnection $\mathscr{P}$.

synthesis, so that the system norm is not changed. The complementary matrix $M_{C_{z\xi}}$ is again a design choice specified in the following.

B. Interconnected System Model

The system in the augmented state space with augmented performance input and output is given as

$$\bar{G}_\xi : \begin{cases} \dot{\xi} &= A_\xi \xi + B_{\xi\bar{w}} \bar{w} + B_{\xi u} u, \\ y &= C_{y\xi} \xi + D_{y\bar{w}} \bar{w}, \\ \bar{z} &= C_{\bar{z}\xi} \xi + D_{\bar{z}\bar{w}} \bar{w} + D_{\bar{z}u} u. \end{cases} \tag{9}$$

Motivated by the approach in [10], the system is modeled as an LFT of a decentralized part containing the block-diagonal system parts, and an interconnection part containing the off-block-diagonal system parts. To do so, an interconnection channel from p to q, modeled by the interconnection matrix $\mathscr{P}$, is introduced, which captures the system coupling. This leads to the LFT given by $\bar{G}_\xi = \mathscr{F}_l(\bar{G}_\xi^d, \mathscr{P})$ with

$$\bar{G}_\xi^d : \left\{ \begin{aligned} &\begin{bmatrix} \dot{\xi} \\ y \\ \bar{z} \\ q \end{bmatrix} = \left[\begin{array}{c|c|c|c} A_\xi^d & B_{\xi u}^d & B_{\xi\bar{w}}^d & B_p^d \\ \hline C_{y\xi}^d & 0 & D_{y\bar{w}}^d & D_{yp}^d \\ \hline C_{\bar{z}\xi}^d & D_{\bar{z}u}^d & D_{\bar{z}\bar{w}}^d & D_{\bar{z}p}^d \\ \hline C_q^d & D_{qu}^d & D_{q\bar{w}}^d & 0 \end{array} \right] \begin{bmatrix} \xi \\ u \\ \bar{w} \\ p \end{bmatrix}, \\ &p = \mathscr{P} q, \end{aligned} \right. \tag{10}$$

where all system matrices in $\bar{G}_\xi^d$ are block-diagonal. The most right block-diagram of Figure 2 shows this structure enabled by the augmentation of $\bar{w}$ and $\bar{z}$.

The sparsity of the augmented system matrices in (9) depends on the mapping X, the original system matrices and the complementary matrices M_A, M_{C_y}, M_{C_z} in (3) and $M_{C_{z\xi}}$ in (8). In the controller synthesis considered in the following, the block-diagonal entries in the system matrices are dealt with in a decomposed way. The off-block-diagonal entries modeled through the interconnection channel need to be accounted for in multiplier conditions that are coupled over the subsystems. Therefore, we choose the complementary matrices such that the augmented system matrices have as few off-diagonal blocks as possible, i.e., we seek the augmented system description where the corresponding interconnection topology is as sparse as possible. This augmentation is illustrated in the example in Section V.

In order to reach the structure in (10), note that every system matrix in (9) can be written in the form

$$M_\xi = \underbrace{\mathrm{diag}_{i=1}^{N}\left(M_{\xi ii}\right)}_{M_\xi^d} + \sum\nolimits_{i=1}^{N} \sum\nolimits_{j=1}^{L} \left(\hat{I}_{\{i\}} \mathscr{P}_j \otimes M_{\xi ij} \right). \tag{11}$$

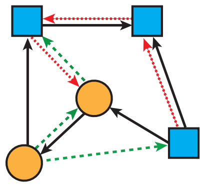

Fig. 3: Two groups of homogeneous subsystems interconnected by three different interconnection patterns ($L=3$) symbolized by the different arrow types.

The interconnection matrices $\mathscr{P}_j \in \mathbb{R}^{N\times N}$ have zeros everywhere except for ones at all the entries (i,k) corresponding to an interconnection from subsystems k to i treated by $M_{\xi ij}$. The matrices $M_{\xi ii}$ correspond to the diagonal blocks of M_ξ. In general, they can all be different, which we call a heterogeneous system, in contrast to the special case of a homogeneous system, where they are all equal. Figure 3 shows an example with two groups of homogeneous subsystems. The matrices $M_{\xi ij}$ correspond to those off-diagonal blocks of M_ξ which represent the influence from all subsystems specified by the structure of $\mathscr{P}_j$ to the subsystems i. There can be L different interconnection matrices $\mathscr{P}_j$ for the interconnection signals, e.g., the states can be interconnected through a different structure than the input signals. This is also illustrated in Figure 3. The overall interconnection matrix in (10) is defined as

$$\mathscr{P} = \operatorname{diag}_{j=1}^{L}\left(\mathscr{P}_j \otimes I_{n_{p_j}}\right).$$

The matrices A^d_ξ, $B^d_{\xi u}$, $B^d_{\xi \bar{w}}$, $C^d_{y\xi}$, $C^d_{\bar{z}\xi}$, $D^d_{\bar{z}u}$, $D^d_{y\bar{w}}$ and $D^d_{\bar{z}\bar{w}}$ in (10) are the diagonal parts of the system matrices in (9). They are defined by the first term in (11), e.g., $A^d_\xi = \operatorname{diag}_{i=1}^{N}\left(A^d_{\xi i}\right)$, with $A^d_{\xi i} = A_{\xi ii}$. The remaining system matrices in (10) are given as $B^d_p = \operatorname{diag}_{i=1}^{N}\left(B^d_{pi}\right)$ and $D^d_{\bar{z}p}$, D^d_{yp}, C^d_q, $D^d_{q\bar{w}}$, and D^d_{qu} have the same structure, where the submatrices, e.g., B^d_{pi}, are composed of the off-diagonal parts of the system matrices in (9), appearing as $M_{\xi ij}$ in (11). For the ease of presentation, their exact definitions are given in the Appendix in (20).

Remark 1: This modeling approach extends the special case in [10], where different groups of homogeneous subsystems are connected by only one interconnection pattern.

IV. Controller and Estimator Design

This section gives the main results of the paper. Based on the interconnected model of the augmented system, a scalable synthesis for the augmented overlapping controller and estimator is presented.

A. Closed-Loop Systems

The closed-loop dynamics of the system $\bar{G}_\xi$ under the controller H_ξ in (7) is given by $\bar{\mathscr{G}}_\xi = \mathscr{F}_{\mathrm{u}}(\bar{G}_\xi, H_\xi)$ as

$$\bar{\mathscr{G}}_\xi : \left\{ \begin{aligned} &\begin{bmatrix} \dot{\xi} \\ \dot{\hat{\xi}} \\ \bar{z} \\ q \end{bmatrix} = \begin{bmatrix} A^d_\xi & B^d_{\xi u}K_\xi & B^d_{\xi\bar{w}} & B^d_p \\ \Gamma C^d_{y\xi} & \Phi & 0 & 0 \\ C^d_{\bar{z}\xi} & D^d_{\bar{z}u}K_\xi & D^d_{\bar{z}\bar{w}} & D^d_{\bar{z}p} \\ C^d_q & D^d_{qu}K_\xi & D^d_{q\bar{w}} & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \hat{\xi} \\ \bar{w} \\ p \end{bmatrix}, \\ &p = \mathscr{P} q, \end{aligned} \right. \qquad (12)$$

with the system matrices as defined in (11) and in the Appendix in (20).

Figure 2 shows all three closed-loop structures of the original system G, and the augmented systems G_ξ and $\bar{G}_\xi$, under the controller H_ξ, given by $\mathscr{G} = \mathscr{F}_{\mathrm{u}}(G, H_\xi)$, $\mathscr{G}_\xi = \mathscr{F}_{\mathrm{u}}(G_\xi, H_\xi)$, and $\bar{\mathscr{G}}_\xi$ as in (12), respectively. The controller will be designed using the structure of $\bar{\mathscr{G}}_\xi$, and physically be applied as in $\mathscr{G}$. $\mathscr{G}_\xi$ is an intermediate system for proving stability and performance of the closed-loop $\mathscr{G}$.

B. Control Objective

As control objective, the $\mathscr{H}_\infty$-norm of the closed-loop transfer function from w to z of the physical system G under the controller H_ξ, i.e. $\|\mathscr{G}\|_{\mathscr{H}_\infty}$, is minimized. For a scalable synthesis of H_ξ, we exploit the structure of $\bar{\mathscr{G}}_\xi$ and thus minimize $\|\bar{\mathscr{G}}_\xi\|_{\mathscr{H}_\infty}$. We first show that under the same controller H_ξ this norm is equal to the norm of $\mathscr{G}_\xi$, i.e., $\|\bar{\mathscr{G}}_\xi\|_{\mathscr{H}_\infty} = \|\mathscr{G}_\xi\|_{\mathscr{H}_\infty}$, before showing that $\|\mathscr{G}_\xi\|_{\mathscr{H}_\infty} = \|\mathscr{G}\|_{\mathscr{H}_\infty}$. This does not hold for any arbitrary augmentation of the system matrices in (8), but the weightings $\bar{Q}$ and $\bar{R}$, which are assumed to have full rank, need to be chosen such that the system norm is invariant to the input-output transformation of the system. Therefore, the weightings are chosen as

$$\bar{Q} = S^{\dagger\top} S^\dagger + M_Q, \qquad \bar{R} = TT^\top + M_R, \qquad (13)$$

with $M_Q = M_Q^\top$, $M_R = M_R^\top$, $S^\top M_Q S = 0$, and $T^\dagger M_R T^{\dagger\top} = 0$. The complementary matrices M_Q and M_R are again chosen such that $C_{z\xi}$ and $B_{\xi\bar{w}}$ are as decentralized as possible, i.e., have as few off-diagonal blocks as possible. Equality of $\|\bar{\mathscr{G}}_\xi\|_{\mathscr{H}_\infty}$ and $\|\mathscr{G}_\xi\|_{\mathscr{H}_\infty}$ is stated in the following

Theorem 1: The closed-loop transfer functions $\mathscr{G}_\xi = \mathscr{F}_{\mathrm{u}}(G_\xi, H_\xi)$ and $\bar{\mathscr{G}}_\xi = \mathscr{F}_{\mathrm{u}}(\bar{G}_\xi, H_\xi)$, from w to z with G_ξ in (2), and from $\bar{w}$ to $\bar{z}$ with $\bar{G}_\xi$ in (9), respectively, are given. With the transformation in (8) and the weightings in (13), it holds

$$\|\bar{\mathscr{G}}_\xi\|_{\mathscr{H}_\infty} = \|\mathscr{G}_\xi\|_{\mathscr{H}_\infty}.$$

Proof: Similarly as in Lemma 9 in [9], for a transformation of a system G to $\bar{G} = T_l G T_r^\dagger$, the following performance bounds can be proven

$$\frac{\sigma_{\min}(T_r)}{\sigma_{\max}(T_l)}\|\bar{G}\|_{\mathscr{H}_\infty} \le \|G\|_{\mathscr{H}_\infty} \le \frac{\sigma_{\max}(T_r)}{\sigma_{\min}(T_l)}\|\bar{G}\|_{\mathscr{H}_\infty}. \qquad (14)$$

The transfer functions $\mathscr{G}_\xi$ and $\bar{\mathscr{G}}_\xi$ are given by

$$\mathscr{G}_\xi = \begin{bmatrix} C_{z\xi} & D_{zu}K_\xi \end{bmatrix}\left(sI - \begin{bmatrix} A_\xi & B_{\xi u}K_\xi \\ \Gamma C_{y\xi} & \Phi \end{bmatrix}\right)^{-1}\begin{bmatrix} B_{\xi w} \\ 0 \end{bmatrix} + D_{zw},$$

$$\bar{\mathscr{G}}_\xi = \left(S^{\dagger T} S^\dagger + M_Q\right)^{\frac{1}{2}} S \left[\mathscr{G}_\xi\right] T^\dagger \left(TT^T + M_R\right)^{\frac{1}{2}}.$$

Then (14) is considered with $G := \mathscr{G}_\xi$, $\bar{G} := \bar{\mathscr{G}}_\xi$, $T_l := \left(S^{\dagger T} S^\dagger + M_Q\right)^{\frac{1}{2}} S$ and $T_r := \left(TT^T + M_R\right)^{-\frac{1}{2}} T$. It remains to be shown that the transformation matrices T_l and T_r are semi-orthogonal, which is a generalization of orthogonal for rectangular matrices, i.e., $T_r^\top T_r = I$ and $T_l T_l^\top = I$. Then, the bounds in (14) are tight such that the $\mathscr{H}_\infty$-norm is not changed under the system transformation. To show that T_r is semi-orthogonal, we see that

$$\begin{aligned} T_r^\top T_r &= \left((TT^\top + M_R)^{-\frac{1}{2}} T\right)^\top \left((TT^\top + M_R)^{-\frac{1}{2}} T\right) \\ &= T^\top \left((TT^\top + M_R)^{-\frac{1}{2}}\right)^\top (TT^\top + M_R)^{-\frac{1}{2}} T \\ &= T^\top (TT^\top + M_R)^{-1} T = I, \end{aligned}$$

which holds because of $M_R = M_R^\top$. Showing that $T_l^\top T_l = I$ follows along the same lines. ∎

Proposition 2: Given $\mathscr{G} = \mathscr{F}_{\mathrm{u}}(G, H_\xi)$ of the original system G in (1) and the augmented closed-loop system $\bar{\mathscr{G}}_\xi$ in (12), the following holds

$$\|\bar{\mathscr{G}}_\xi\|_{\mathscr{H}_\infty} = \|\mathscr{G}\|_{\mathscr{H}_\infty}.$$

Proof: We show that the following equalities hold $\|\bar{\mathscr{G}}_\xi\|_{\mathscr{H}_\infty} = \|\mathscr{G}_\xi\|_{\mathscr{H}_\infty} = \|\mathscr{G}\|_{\mathscr{H}_\infty}$, where the first one has been proven in Theorem 1. For the second equality, recall that G and G_ξ under the same input u have the same input-output behavior from w to z, and the same output $y(t)$, as shown in Proposition 1. This means that the closed-loops of G and G_ξ under the same output feedback controller H_ξ, i.e., $\mathscr{G} = \mathscr{F}_{\mathrm{u}}(G, H_\xi)$ and $\mathscr{G}_\xi = \mathscr{F}_{\mathrm{u}}(G_\xi, H_\xi)$, have the same input-output behavior from w to z and thus the norms of the transfer functions $\mathscr{G}$ and $\mathscr{G}_\xi$ are equal. ∎

The following proposition suggests that the augmentation of the performance channel is also applicable for an $\mathscr{H}_2$-based controller synthesis.

Proposition 3: Given $\mathscr{G} = \mathscr{F}_{\mathrm{u}}(G, H_\xi)$ of the original system G in (1) and the augmented closed-loop system $\bar{\mathscr{G}}_\xi$ in (12), it holds

$$\|\bar{\mathscr{G}}_\xi\|_{\mathscr{H}_2} = \|\mathscr{G}\|_{\mathscr{H}_2}.$$

Proof: As shown in [9], the $\mathscr{H}_2$-norm is also unitary-invariant, and therefore the same proof as in Theorem 1 can be applied to show that $\|\bar{\mathscr{G}}_\xi\|_{\mathscr{H}_2} = \|\mathscr{G}_\xi\|_{\mathscr{H}_2}$. Furthermore, equality between $\|\mathscr{G}_\xi\|_{\mathscr{H}_2}$ and $\|\mathscr{G}\|_{\mathscr{H}_2}$ follows from equality of the input-output behavior of $\mathscr{G}$ and $\mathscr{G}_\xi$ which has been proven in Proposition 2. ∎

C. Full Block S-Procedure for the Interconnected System

For synthesizing the gains $\Phi_i, \Gamma_i, K_{\xi i}$ in (7), the full block S-procedure in [15] is employed, which is briefly recalled in the following theorem.

Theorem 2: [15] Given the stable continuous-time LTI system

$$\begin{bmatrix} \dot{x} \\ z \\ q \end{bmatrix} = \begin{bmatrix} \mathscr{A} & \mathscr{B}_1 & \mathscr{B}_2 \\ \mathscr{C}_1 & \mathscr{D}_{11} & \mathscr{D}_{12} \\ \mathscr{C}_2 & \mathscr{D}_{21} & \mathscr{D}_{22} \end{bmatrix} \begin{bmatrix} x \\ w \\ p \end{bmatrix}, \quad p = \mathscr{P} q,$$

then the system has an $\mathscr{L}_2$-gain from w to z smaller than γ if and only if there exist variables $\mathscr{X} = \mathscr{X}^\top > 0$, $R = R^\top$, $Q = Q^\top$ and S of appropriate dimensions such that

$$[\star]^T \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix}^\top \begin{bmatrix} \mathscr{P} \\ I \end{bmatrix} > 0, \quad (15)$$

$$[\star]^\top \left[\begin{array}{cc|cc|cc} 0 & \mathscr{X} & 0 & 0 & 0 & 0 \\ \mathscr{X} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma I & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma} I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q & S \\ 0 & 0 & 0 & 0 & S^\top & R \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ \mathscr{A} & \mathscr{B}_1 & \mathscr{B}_2 \\ \hline 0 & I & 0 \\ \mathscr{C}_1 & \mathscr{D}_{11} & \mathscr{D}_{12} \\ \hline 0 & 0 & I \\ \mathscr{C}_2 & \mathscr{D}_{21} & \mathscr{D}_{22} \end{array}\right] < 0. \quad (16)$$

We show in the following proposition how the conditions in Theorem 2 applied to the interconnected system in (12) decompose into conditions of the size of the subsystems by appropriate structural assumptions on the multipliers Q, R and S and on the Lyapunov matrix $\mathscr{X}$. We introduce the system matrices of subsystem i in (12) as

$$\begin{bmatrix} \mathscr{A}_i & \mathscr{B}_{1,i} & \mathscr{B}_{2,i} \\ \mathscr{C}_{1,i} & \mathscr{D}_{11,i} & \mathscr{D}_{12,i} \\ \mathscr{C}_{2,i} & \mathscr{D}_{21,i} & \mathscr{D}_{22,i} \end{bmatrix} = \left[\begin{array}{cc|c|c} A^d_{\xi i} & B^d_{\xi u i} K_{\xi i} & B^d_{\xi \bar{w} i} & B^d_{p i} \\ \Gamma_i C^d_{y \xi i} & \Phi_i & 0 & 0 \\ \hline C_{\bar{z} \xi i i} & D^d_{\bar{z} u i} K_{\xi i} & D^d_{\bar{z} \bar{w} i} & D^d_{\bar{z} p i} \\ \hline C^d_{q i} & D^d_{q u i} K_{\xi i} & D^d_{q \bar{w} i} & 0 \end{array}\right].$$

Proposition 4: The augmented system $\bar{G}_\xi = \mathscr{F}_{\mathrm{l}}(\bar{G}^d_\xi, \mathscr{P})$ is given in (12) with $\mathscr{P}$ being normal. Then there exists a controller H_ξ as in (7) such that $\mathscr{G} = \mathscr{F}_{\mathrm{u}}(G, H_\xi)$ is stable and has an $\mathscr{L}_2$-gain less than γ, if there exist $\mathscr{X}_i = \mathscr{X}_i^\top > 0$, and $\tilde{R} = \tilde{R}^\top$, $\tilde{Q} = \tilde{Q}^\top$ and $\tilde{S}$ with $\tilde{R} = \mathrm{diag}_{j=1}^L(\tilde{R}_j)$, $\tilde{Q} = \mathrm{diag}_{j=1}^L(\tilde{Q}_j)$ and $\tilde{S} = \mathrm{diag}_{j=1}^L(\tilde{S}_j)$ such that

$$[\star]^T \begin{bmatrix} \tilde{Q}_j & \tilde{S}_j \\ \tilde{S}_j^\top & \tilde{R}_j \end{bmatrix} \begin{bmatrix} \lambda I_{n_{p_j}} \\ I_{n_{p_j}} \end{bmatrix} > 0, \ \forall \lambda \in \mathrm{spec}(\mathscr{P}_j), \quad \forall j \in \{1, \dots, L\}, \quad (17)$$

$$[\star]^\top \left[\begin{array}{cc|cc|cc} 0 & \mathscr{X}_i & 0 & 0 & 0 & 0 \\ \mathscr{X}_i & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma I & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma} I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \tilde{Q} & \tilde{S} \\ 0 & 0 & 0 & 0 & \tilde{S}^\top & \tilde{R} \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ \mathscr{A}_i & \mathscr{B}_{1,i} & \mathscr{B}_{2,i} \\ \hline 0 & I & 0 \\ \mathscr{C}_{1,i} & \mathscr{D}_{11,i} & \mathscr{D}_{12,i} \\ \hline 0 & 0 & I \\ \mathscr{C}_{2,i} & \mathscr{D}_{21,i} & \mathscr{D}_{22,i} \end{array}\right] < 0, \quad \forall i \in \{1, \dots, N\}. \quad (18)$$

Proof: When applying Theorem 2 for the interconnected system as in (12) with the structural assumptions on the Lyapunov matrix $\mathscr{X} = \mathrm{diag}_{i=1}^N(\mathscr{X}_i)$ and the multipliers $Q = \mathrm{diag}_{j=1}^L(I_N \otimes \tilde{Q}_j)$, $R = \mathrm{diag}_{j=1}^L(I_N \otimes \tilde{R}_j)$ and $S = \mathrm{diag}_{j=1}^L(I_N \otimes \tilde{S}_j)$, then the matrices in the nominal condition (16) are composed of only block-diagonal matrices and therefore completely decompose into one condition per subsystem as in (18). The multiplier condition (15) also decomposes into one for each of the different interconnection patterns as

$$[\star]^T \begin{bmatrix} I_N \otimes \tilde{Q}_j & I_N \otimes \tilde{S}_j \\ I_N \otimes \tilde{S}_j^\top & I_N \otimes \tilde{R}_j \end{bmatrix} \begin{bmatrix} \mathscr{P}_j \otimes I_{n_{p_j}} \\ I_{N n_{p_j}} \end{bmatrix}, \ \forall j \in \{1, \dots, L\}. \quad (19)$$

Furthermore, by applying Lemma 1 from [11], as given in the Appendix, we can state the following. If the pattern matrices $\mathscr{P}_j$ are normal, they can always be transformed into diagonal matrices with their eigenvalues on the diagonal. This further decomposes (19) into the ones given in (17).

So far, it has been shown that H_ξ stabilizes $\bar{\mathscr{G}}_\xi = \mathscr{F}_{\mathrm{u}}(\bar{G}_\xi, H_\xi)$ and leads to an $\mathscr{H}_\infty$-norm less than γ. Since only the performance channel is transformed, stability of $\bar{\mathscr{G}}_\xi$ implies stability of $\mathscr{G}_\xi = \mathscr{F}_{\mathrm{u}}(G_\xi, H_\xi)$.

Furthermore, as shown in [3], as G_ξ is an extension of G (with redefined input), a stabilizing augmented controller for G_ξ is also stabilizing for G and therefore, stability of $\mathscr{G}_\xi = \mathscr{F}_{\mathrm{u}}(G_\xi, H_\xi)$ also implies stability of $\mathscr{G} = \mathscr{F}_{\mathrm{u}}(G, H_\xi)$.

The equalities of the $\mathscr{H}_\infty$-norms of $\bar{\mathscr{G}}_\xi$, $\mathscr{G}_\xi$ and $\mathscr{G}$ is given by Theorem 1 and Proposition 2. ∎

Remark 2: Proposition 4 can be applied w. l. o. g. if any

synthesis	nominal cond. (nc)	multiplier cond. (mc)
(a) centralized	1 nc of $NX_\mathrm{n} \times NX_\mathrm{n}$	1 mc of $NLX_\mathrm{m} \times NLX_\mathrm{m}$
(b) decomposed heterogeneous	N nc of $X_\mathrm{n} \times X_\mathrm{n}$	$2L$ mc of $X_\mathrm{m} \times X_\mathrm{m}$
(c) decomposed homogeneous	1 nc of $X_\mathrm{n} \times X_\mathrm{n}$	$2L$ mc of $X_\mathrm{m} \times X_\mathrm{m}$

TABLE I: Numbers and dimensions of matrix inequalities for the centralized (a) and decomposed synthesis for heterogeneous (b) and homogeneous subsystems (c), with the dimensions of the nominal (X_n) and the multiplier conditions (X_m) for the single subsystems.

$\mathscr{P}_j$ is not normal, as any given interconnection matrix can be transformed and augmented into a normal one. For simplicity the normal case is considered here. Possible transformations to obtain normal interconnection matrices are given in [11].

With Proposition 4, both the nominal condition (18) and the multiplier condition (17) decompose into small matrix inequalities of dimensions on the scale of the single subsystems. This decomposition is possible due to the structural assumptions on the Lyapunov and the multiplier matrices, which introduces conservatism. Table I shows the number and dimensions of conditions to be solved for the centralized and the decomposed controller synthesis. For simplicity we assume that the dimensions of the single subsystems are equal although they can be heterogeneous. Whereas the centralized method scales quadratically with the number of subsystems N and the number of different interconnections L, the decomposed approach for heterogeneous subsystems scales linearly with N and L. The factor 2 applies to the normal case accounting for the smallest and largest eigenvalues of each $\mathscr{P}_j$. For this, we need to introduce the additional constraint $\tilde{Q}_j < 0$ in Proposition 4 to guarantee that the multiplier condition is a concave function in λ_j. If some P_j are simultaneously diagonalisable, L can be further reduced at the cost of increased conservatism.

For equal subsystems, the number of nominal conditions further reduces. For homogeneous subsystems, only one small nominal condition needs to be solved. The number of multiplier conditions again scales linearly with L.

Remark 3: The implementation of the synthesis problem for homogeneous subsystems can be done in a centralized way or each subsystem can solve it in a decentralized way, assuming knowledge of the eigenvalues of the interconnection matrices. If global information should not be shared amongst the subsystems, the results in Proposition 4 facilitate the implementation of distributed synthesis algorithms, where agreement on the multiplier parts is necessary.

D. Controller Synthesis

The synthesis problem for the dynamic output feedback controller in (16) is not convex. As proposed in [16], a variable transformation can be applied to transform (16) into bilinear matrix inequalities (BMI). Then, an iterative algorithm can be applied, iterating between fixing one set of variables and solving for the other one. Such an algorithm consists of two steps. First, an initial feasible solution is searched for by iteratively applying bisection steps starting from the nominal solution with $\mathscr{P} = 0$. Then, it is iteratively optimized for γ. Further details can be found in [16].

V. Numerical Example

A numerical example is given which rather than representing a large-scale system illustrates the modeling of a system by an interconnected augmented one and the proposed controller synthesis.

A. Robotic Example System

A simplified robotic system is considered, as presented in [5], which is composed of $N = 3$ robotic subsystems that cooperatively transport an object, to which they are physically coupled. The robots are controlled by lower-level decentralized impedance controllers [17], which allow us to model their closed-loop dynamics as second-order LTI systems [18]. In a reduced form, they are given by $m_i\ddot{x}_i + d_i\dot{x}_i + k_i x_i = f_i + b_i u_i$ with $x_i,\ \dot{x}_i,\ \ddot{x}_i,\ f_i$ being the position, velocity, acceleration, and force of the end-effector of the i-th robot. The parameters $m_i,\ d_i,\ k_i$ represent the virtual mass, damping, and stiffness of the i-th robot. The contact between the i-th robot and the object is also modeled as second-order dynamics with k_{io} and d_{io} the stiffness and damping parameters, respectively. The states of subsystem $i,\ i = 1,\dots,3$, are the position and velocity of robot i, denoted by $\underline{x}_i := [x_i,\ \dot{x}_i]^\top$. The states of the object are $\underline{x}_o := [x_o,\ \dot{x}_o]^\top$. The overall dynamics are

$$\underbrace{\begin{bmatrix}\dot{\underline{x}}_1\\ \dot{\underline{x}}_2\\ \dot{\underline{x}}_3\\ \dot{\underline{x}}_o\end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} a_{11} & 0 & 0 & a_{1o}\\ 0 & a_{22} & 0 & a_{2o}\\ 0 & 0 & a_{33} & a_{3o}\\ a_{o1} & a_{o2} & a_{o3} & a_{oo}\end{bmatrix}}_{A} \underbrace{\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\\ \underline{x}_3\\ \underline{x}_o\end{bmatrix}}_{x}$$

$$+ \underbrace{\begin{bmatrix} B_{u11} & 0 & 0\\ 0 & B_{u22} & 0\\ 0 & 0 & B_{u33}\\ 0 & 0 & 0\end{bmatrix}}_{[B_{u1}\quad B_{u2}\quad B_{u3}]} \underbrace{\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}}_{u} + \underbrace{\begin{bmatrix} B_{w11} & 0 & 0 & 0\\ 0 & B_{w22} & 0 & 0\\ 0 & 0 & B_{w33} & 0\\ 0 & 0 & 0 & B_{woo}\end{bmatrix}}_{B_w} \underbrace{\begin{bmatrix}w_1\\ w_2\\ w_3\\ w_o\end{bmatrix}}_{w},$$

$$\underbrace{\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}}_{y} = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \underbrace{\begin{bmatrix}1&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&1&0\\ 0&0&0&0&1&0&1&0\end{bmatrix}}_{C_y = [C_{y1}^\top,\ C_{y2}^\top,\ C_{y3}^\top]^\top} \underbrace{\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\\ \underline{x}_3\\ \underline{x}_o\end{bmatrix}}_{x},$$

$$\underbrace{\begin{bmatrix}z_x\\ z_u\end{bmatrix}}_{z} = \underbrace{\begin{bmatrix}C_{zx}\\ 0_{n_u \times n_x}\end{bmatrix}}_{C_z} x, + \underbrace{\begin{bmatrix}0_{n_x \times n_u}\\ D_{zuu}\end{bmatrix}}_{D_{zu}} u,$$

with

$$a_{ii} = \begin{bmatrix} 0 & 1\\ -\frac{1}{m_i}(k_i + k_{i,o}) & -\frac{1}{m_i}(d_i + d_{i,o})\end{bmatrix},\quad a_{io} = a_{oi} = \begin{bmatrix} 0 & 0\\ \frac{k_{i,o}}{m_o} & \frac{d_{i,o}}{m_o}\end{bmatrix},$$

$$a_{oo} = \begin{bmatrix} 0 & 1\\ -\frac{1}{m_o}(k_{1,o} + k_{2,o} + k_{3,o}) & -\frac{1}{m_o}(d_{1,o} + d_{2,o} + d_{3,o})\end{bmatrix},$$

$$B_{uii} = \begin{bmatrix}0 & \frac{10}{m_i}\end{bmatrix}^T,\quad B_{wii} = \begin{bmatrix}0 & \frac{10}{m_o}\end{bmatrix}^T,\quad C_{zx} = I_{n_x},\quad D_{zuu} = 100 I_{n_u}.$$

In the following, the parameters are given as $k_i = 0.4,\ d_i = 0.6,\ m_i = 5,\ k_{io} = 0.3,\ d_{io} = 0.8,\ m_o = 4,\ \forall i = 1,\dots,3$, and therefore define homogeneous subsystems.

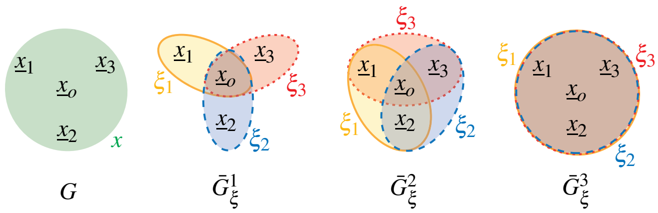

Fig. 4: Illustration of the original state space (with state vector x) and of the different overlaps within the augmented state spaces (with ξ_1, ξ_2, and ξ_3 of the subsystems 1, 2, 3).

B. Augmentation to Different Overlapping State Spaces

The model of the augmented system is considered for three different overlaps of the augmented states of the subsystems, which are illustrated in Figure 4. They are chosen to be symmetric for the subsystems. Therefore, for the homogeneous subsystems in the original state space, the augmented subsystems are also homogeneous.

1) Augmented System $\bar{G}^1_\xi$ with Overlap 1: The states of the object overlap for each augmented state vector of the subsystems. The overall augmented state vector is given by $\xi = \begin{bmatrix}\xi_1^\top & \xi_2^\top & \xi_3^\top\end{bmatrix}^\top$ with $\xi_i = \begin{bmatrix}\underline{x}_i^\top & \underline{x}_o^\top\end{bmatrix}^\top$. The disturbance input and performance output are augmented to $\bar{w} = \begin{bmatrix}\bar{w}_1^\top & \bar{w}_2^\top & \bar{w}_3^\top\end{bmatrix}^\top$, with $\bar{w}_i = \begin{bmatrix}w_i & w_o\end{bmatrix}^\top$, and $\bar{z} = \begin{bmatrix}\bar{z}_1^\top & \bar{z}_2^\top & \bar{z}_3^\top\end{bmatrix}^\top$, with $\bar{z}_i = \begin{bmatrix}\underline{x}_i & \underline{x}_o\end{bmatrix}^\top$. The augmented system matrices are

$$
\begin{aligned}
&A_\xi = XAX^\dagger + M_A \\
&= \begin{bmatrix}
a_{11} & \frac{a_{1o}}{3} & 0 & \frac{a_{1o}}{3} & 0 & \frac{a_{1o}}{3} \\
a_{1o} & \frac{a_{oo}}{3} & a_{2o} & \frac{a_{\xi oo}}{3} & a_{3o} & \frac{a_{oo}}{3} \\
0 & \frac{a_{2o}}{3} & a_{22} & \frac{a_{2o}}{3} & 0 & \frac{a_{2o}}{3} \\
a_{1o} & \frac{a_{oo}}{3} & a_{2o} & \frac{a_{oo}}{3} & a_{3o} & \frac{a_{oo}}{3} \\
0 & \frac{a_{3o}}{3} & 0 & \frac{a_{3o}}{3} & a_{33} & \frac{a_{3o}}{3} \\
a_{1o} & \frac{a_{oo}}{3} & a_{2o} & \frac{a_{oo}}{3} & a_{3o} & \frac{a_{oo}}{3}
\end{bmatrix}
+ \begin{bmatrix}
0 & \frac{2}{3}a_{1o} & 0 & -\frac{1}{3}a_{1o} & 0 & -\frac{1}{3}a_{1o} \\
0 & \frac{2}{3}a_{oo} & 0 & -\frac{1}{3}a_{oo} & 0 & -\frac{1}{3}a_{oo} \\
0 & -\frac{1}{3}a_{2o} & 0 & \frac{2}{3}a_{2o} & 0 & -\frac{1}{3}a_{2o} \\
0 & -\frac{1}{3}a_{oo} & 0 & \frac{2}{3}a_{oo} & 0 & -\frac{1}{3}a_{oo} \\
0 & -\frac{1}{3}a_{3o} & 0 & -\frac{1}{3}a_{3o} & 0 & \frac{2}{3}a_{3o} \\
0 & -\frac{1}{3}a_{oo} & 0 & -\frac{1}{3}a_{oo} & 0 & \frac{2}{3}a_{oo}
\end{bmatrix} \\
&= \left[\begin{array}{cc:cc:cc}
a_{11} & a_{1o} & 0 & 0 & 0 & 0 \\
a_{1o} & a_{oo} & a_{2o} & 0 & a_{3o} & 0 \\
\hdashline
0 & 0 & a_{22} & a_{20} & 0 & 0 \\
a_{1o} & 0 & a_{2o} & a_{oo} & a_{3o} & 0 \\
\hdashline
0 & 0 & 0 & 0 & a_{33} & a_{3o} \\
a_{1o} & 0 & a_{2o} & 0 & a_{3o} & a_{oo}
\end{array}\right]
= \begin{bmatrix}
A_{\xi 11} & A_{\xi 2o} & A_{\xi 3o} \\
A_{\xi 1o} & A_{\xi 22} & A_{\xi 3o} \\
A_{\xi 1o} & A_{\xi 2o} & A_{\xi 33}
\end{bmatrix}.
\end{aligned}
$$

For the homogeneous case, it holds that $A_{\xi 11} = A_{\xi 22} = A_{\xi 33} = A_{\xi ii} = A^d_{\xi i}$ and $A_{\xi 1o} = A_{\xi 2o} = A_{\xi 3o} = A_{\xi io} = A_{\xi i1}$. The augmentation of the other system matrices is similar. Thus, the augmented system is rewritten as in (11),

$$
\begin{aligned}
A_\xi &= I_N \otimes A^d_{\xi i} + \mathscr{P}_1 \otimes A_{\xi i1}, & C_{y\xi} &= I_N \otimes C^d_{y\xi i}, \\
B_{\xi u} &= I_N \otimes B^d_{\xi ui}, & C_{\bar{z}\xi} &= I_N \otimes C^d_{\bar{z}\xi i}, \\
B_{\xi \bar{w}} &= I_N \otimes B^d_{\xi \bar{w} i} + \mathscr{P}_1 \otimes B_{\xi \bar{w} i1},
\end{aligned}
$$

with the interconnection matrix $\mathscr{P}_1 = \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$. The augmented measurement output and control input matrices are local, and the augmented performance output, computed as in (8), is also local with $C^d_{\bar{z}\xi i} = \begin{bmatrix}I_{n_{x_i}} & 0\\ 0 & \frac{\sqrt{3}}{3}I_{n_{x_i}}\end{bmatrix}$. The augmented disturbance input matrix is computed as $B_{\xi\bar{w}} = XB_wT^\dagger\bar{R}^{-\frac{1}{2}}$, resulting in $B^d_{\xi\bar{w}i} = \begin{bmatrix}B_{wii} & 0\\ 0 & \frac{\sqrt{3}}{3}B_{woo}\end{bmatrix}$ and $B_{\bar{w}\xi i1} = \begin{bmatrix}0 & 0\\ 0 & \frac{\sqrt{3}}{3}B_{woo}\end{bmatrix}$.

2) Augmented System $\bar{G}^2_\xi$ with Overlap 2: For each robot, the augmented state contains its own state, the state of the object, and the state of one neighbor, i.e., $\xi = \begin{bmatrix}\xi_1^\top & \xi_2^\top & \xi_3^\top\end{bmatrix}^\top$ with $\xi_1 = \begin{bmatrix}\underline{x}_1^\top & \underline{x}_2^\top & \underline{x}_o^\top\end{bmatrix}^\top$, $\xi_2 = \begin{bmatrix}\underline{x}_2^\top & \underline{x}_3^\top & \underline{x}_o^\top\end{bmatrix}^\top$ and $\xi_3 = \begin{bmatrix}\underline{x}_3^\top & \underline{x}_1^\top & \underline{x}_o^\top\end{bmatrix}^\top$. The disturbance input and performance output are augmented to $\bar{w} = \begin{bmatrix}\bar{w}_1^\top & \bar{w}_2^\top & \bar{w}_3^\top\end{bmatrix}^\top$ and $\bar{z} = \begin{bmatrix}\bar{z}_1^\top & \bar{z}_2^\top & \bar{z}_3^\top\end{bmatrix}^\top$, with $\bar{w}_1 = \begin{bmatrix}w_1 & w_2 & w_o\end{bmatrix}^\top$, $\bar{z}_1 = \begin{bmatrix}\underline{x}_1 & \underline{x}_2 & \underline{x}_o\end{bmatrix}^\top$, and $\bar{w}_2$, $\bar{z}_2$, $\bar{w}_3$, and $\bar{z}_3$ similar, according to the structure of ξ_2 and ξ_3. The augmented system matrices are

$$
\begin{aligned}
A_\xi &= I_N \otimes A^d_{\xi i} + \mathscr{P}_1 \otimes A_{\xi i1}, & C_{y\xi} &= I_N \otimes C^d_{y\xi i}, \\
B_{\xi u} &= I_N \otimes B^d_{\xi ui} + \mathscr{P}_2 \otimes B_{u\xi i2}, & C_{\bar{z}\xi} &= I_N \otimes C^d_{\bar{z}\xi i}, \\
B_{\xi \bar{w}} &= I_N \otimes B^d_{\bar{w}\xi i} + \mathscr{P}_1 \otimes B_{\xi \bar{w} i1} + \mathscr{P}_2 \otimes B_{\xi\bar{w}i2},
\end{aligned}
$$

with $L = 2$, $\mathscr{P}_1 = \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$ and $\mathscr{P}_2 = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$. Again, $C_{\bar{z}\xi}$ and $B_{\xi\bar{w}}$ are computed as in (8) resulting, e.g., in $C^d_{\bar{z}\xi i} = \operatorname{diag}\left(\frac{\sqrt{2}}{2}I_{n_{x_i}}, \frac{\sqrt{2}}{2}I_{n_{x_i}}, \frac{\sqrt{3}}{3}I_{n_{x_o}}\right)$.

3) Augmented System $\bar{G}^3_\xi$ with Overlap 3: In [19], the so-called parallel estimation case with a complete overlap of the augmented state estimates was considered. For this case of complete overlap, we define the augmented state vectors as $\xi = \begin{bmatrix}\xi_1^\top & \xi_2^\top & \xi_3^\top\end{bmatrix}^\top$ with $\xi_1 = \begin{bmatrix}\underline{x}_1^\top & \underline{x}_2^\top & \underline{x}_3^\top & \underline{x}_o^\top\end{bmatrix}^\top$, $\xi_2 = \begin{bmatrix}\underline{x}_2^\top & \underline{x}_3^\top & \underline{x}_1^\top & \underline{x}_o^\top\end{bmatrix}^\top$ and $\xi_3 = \begin{bmatrix}\underline{x}_3^\top & \underline{x}_1^\top & \underline{x}_2^\top & \underline{x}_o^\top\end{bmatrix}^\top$. The augmented system matrices are obtained similarly as before.

C. Augmented Controller and Estimator Design

In the following, we perform the decomposed synthesis of decentralized dynamic output feedback controllers as in Proposition 4 for the three different overlaps. Note that w. l. o. g. distributed controllers with communication can also be designed with these results. For the controller synthesis, the iterative procedure as briefly described in Section IV is applied. The synthesized controllers in the different augmented state spaces are denoted by H^i_ξ for the overlaps $i = 1, 2, 3$. The control performance bounds γ and $\mathscr{H}_\infty$-norms of the closed-loop transfer functions $\mathscr{G}^i = \mathscr{F}_{\mathrm{u}}(G, H^i_\xi)$ are given in Table II. Note that the system norms of the augmented systems G^i_ξ and $\bar{G}^i_\xi$, and of the original system G, under the controller H^i_ξ are equal, i.e., $\|\mathscr{G}^i\|_{\mathscr{H}_\infty} = \|\mathscr{G}^i_\xi\|_{\mathscr{H}_\infty} = \|\bar{\mathscr{G}}^i_\xi\|_{\mathscr{H}_\infty}$ with $\mathscr{G}^i_\xi = \mathscr{F}_{\mathrm{u}}(G^i_\xi, H^i_\xi)$ and $\bar{\mathscr{G}}^i_\xi = \mathscr{F}_{\mathrm{u}}(\bar{G}^i_\xi, H^i_\xi)$. For comparison, the case of a centralized controller in the original state space is also given, denoted by H^c with $\mathscr{G}^c = \mathscr{F}_{\mathrm{u}}(G, H^c)$. The results show an obvious trend. The larger the overlap of the augmented controllers, the better the control performance.

To illustrate the complexity and scalablity of the proposed synthesis method, let us consider the case of overlap 1 in more detail. The dimensions in Table II are then $X_n = 15$ and $X_m = 6$. Whereas we solve one nominal condition of 15×15 and two multiplier conditions of 6×6, the centralized method requires solving one nominal condition of 45×45 and a multiplier condition of 18×18 for $N = 3$. For a system with, e.g. $N = 20$, the dimension of the centralized nominal condition further grows to 300×300.

	$\mathscr{G}^1$	$\mathscr{G}^2$	$\mathscr{G}^3$	$\mathscr{G}^c$
γ	80.70	68.65	44.28	41.63
$\|\cdot\|_{\mathscr{H}_\infty}$	49.77	41.62	41.69	41.58

TABLE II: Dynamic output feedback closed-loop performance for $\mathscr{G}^i = \mathscr{F}_{\mathrm{u}}(G, H^i_\xi)$, with controllers H^i_ξ of overlaps $i = 1,2,3$, and $\mathscr{G}^c = \mathscr{F}_{\mathrm{u}}(G, H^c)$ with central controller H^c in original state space.

VI. Conclusion

We have introduced a scalable controller synthesis for distributed systems. Although presented for an augmented overlapping control framework, it is also applicable to non-overlapping distributed systems. An established interconnected system model was extended to the general case of heterogeneous subsystems with different interconnetions. To assign a single performance input and output to each subsystem, we augmented the performance channel without changing the system norm. For the interconnected system model, the controller synthesis conditions decompose into smaller conditions of the scale of the single subsystems. In the case of heterogeneous subsystems, the synthesis scales linearly with the number of different subsystems and different interconnections. For homogeneous subsystems the resulting synthesis conditions can be solved in a decentralized way and the complexity stays constant with the number of subsystems. For heterogeneous subsystems the decomposition of the synthesis conditions facilitates the implementation of distributed algorithms, which is subject of on-going work.

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Appendix

A. System Matrices of (10)

The submatrices, e.g., B^d_{pi}, of the matrices in (10), e.g., $B^d_p = \mathrm{diag}_{i=1}^N\left(B^d_{pi}\right)$, are given by

$$\begin{aligned}
B^d_{pi} &= \sum\nolimits_{j=1}^L \left(e_j^\top \otimes \left[A_{\xi ij} \;\; B_{\xi uij} \;\; B_{\xi\bar{w}ij}\right]\right),\\
D^d_{\bar{z}pi} &= \sum\nolimits_{j=1}^L \left(e_j^\top \otimes \left[C_{\bar{z}\xi ij} \;\; D_{\bar{z}uij} \;\; D_{\bar{z}\bar{w}ij}\right]\right),\\
D^d_{ypi} &= \sum\nolimits_{j=1}^L \left(e_j^\top \otimes \left[C_{y\xi ij} \;\; 0 \;\; D_{y\bar{w}ij}\right]\right),\\
C^d_{qi} &= \sum\nolimits_{j=1}^L \left(e_j \otimes \left[I_{n_{\xi_j}} \;\; 0 \;\; 0\right]^\top\right),\\
D^d_{q\bar{w}i} &= \sum\nolimits_{j=1}^L \left(e_j \otimes \left[0 \;\; 0 \;\; I_{n_{\bar{w}_j}}\right]^\top\right),\\
D^d_{qui} &= \sum\nolimits_{j=1}^L \left(e_j \otimes \left[0 \;\; I_{n_{u_j}} \;\; 0\right]^\top\right).
\end{aligned} \tag{20}$$

Lemma 1: [11] Given a normal, real-valued pattern matrix $\mathscr{P} = \mathscr{P}_j \otimes I_{n_{p_j}}$ with eigenvalues λ, then the following is equivalent

$$\text{(i)}\ [\star]^\top \begin{bmatrix} \tilde{Q}_j & \tilde{S}_j \\ \tilde{S}_j^\top & \tilde{R}_j \end{bmatrix} \begin{bmatrix} \lambda I_{n_{p_j}} \\ I_{n_{p_j}} \end{bmatrix} < 0,\ \forall \lambda \in \mathrm{spec}\left(\mathscr{P}_j\right),$$
$$\forall j \in \{1,...,L\},$$
$$\text{(ii)}\ \left[\star\right]^T \begin{bmatrix} I_N \otimes \tilde{Q}_j & I_N \otimes \tilde{S}_j \\ I_N \otimes \tilde{S}_j^\top & I_N \otimes \tilde{R}_j \end{bmatrix} \begin{bmatrix} \mathscr{P}_j \otimes I_{n_{p_j}} \\ I_{N n_{p_j}} \end{bmatrix},\ \forall j \in \{1,...,L\}.$$