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Event-Triggered Control for a Class of Cascade Systems

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***Abstract*— This paper addresses the event-triggered control of cascade connected systems. In particular, we present event-triggered mechanisms that guarantee the stabilization of cascade systems with partial state feedback without infinitely fast sampling. Our approach is based on growth conditions on the interconnection terms and does not follow the framework of input-to-state stability with respect to the subsystems and/or measurement errors.**

I. Introduction

During the last decade, the study of event triggered control has attracted considerable attention within the control systems community, see [1]-[5], [9]-[**?**], [16]-[18], [21], [25], [29], [30] and references therein. The main feature that characterizes the event-triggered feedback strategies is that information is exchanged only when a certain condition is violated resulting in aperiodic controller updates. Although event-based schemes are not a new concept (see for instance [2]), recent advances in systems and control provided powerful tools for the analysis and design of such methods.

The introduction of the input-to-state stability (ISS) property by Sontag [28], provided a new approach to the study of nonlinear systems with inputs. For event-triggered control, the ISS framework was firstly implemented in [29] for the design of a mechanism that is based on the state of the system and yields asymptotic stability of the closed-loop system. Specifically, under the assumption that the system is ISS with respect to measurement errors, a simple event trigger was proposed such that the sampling error is bounded by a specific threshold depending on the system's state and also the intervals between sampling instants are lower bounded by a positive constant. This technique proved to be very useful and inspired several authors to use analogous assumptions to study a variety of problems for linear and nonlinear systems. For instance, similar designs have been used for output-feedback and decentralized control in [5], for distributed network control [4], [30], for stabilization of systems subject to quantization and time-varying network induced delays in [10]. In the recent papers, [4], [16], and [17], the authors study the event-triggered control for large scale interconnected systems, by employing ISS small gain theorems. For the event-triggered control of nonlinear systems various other approaches and formalisms have also been considered; see for instance [11], [18], [21], [25].

For linear systems, the property of ISS follows immediately from the stabilizability of the system. However, for general nonlinear systems, this property holds for some special classes, see for instance [6], [8], [15] for such designs. In particular, in [6] and [8], there are counterexamples that show that the design of a feedback law that renders the system ISS wrt measurement errors is not always feasible.

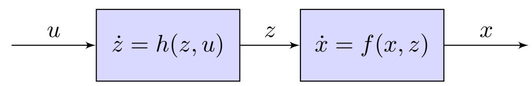

Fig. 1: Cascade connection of systems

In this work, we study the event-triggered control for a class of cascade connected systems, Fig.1. A feature that characterizes these systems is that the control input enters only the z-subsystem, and the state of the z-subsystem is considered as the input for the x-subsystem.

The problem of stabilization of cascade systems was addressed by various authors in the past years under different assumptions and techniques. For instance, in [23], it was proved that the uncontrolled cascade is globally asymptotically stable (GAS) if all trajectories are bounded and both subsystems are GAS. In [22], the authors studied the stabilization of partially linear cascade systems, i.e. the composition of a nonlinear system with a linear one. Several generalizations were also obtained in [7], [13] and [19] for general classes of cascade systems under growth restrictions on the interconnection terms. Various other approaches have also appeared in the literature, see for instance [20], [24] and references therein. Finally, the ISS framework provided new techniques for the study of cascades as in [27] and for general interconnected systems in [14].

This paper presents event-triggered control strategies for a class of cascade connected systems with partial state feedback designs, i.e. the feedback is based on the state of the second subsystem. A key feature of those strategies is that the stabilizing feedback is not necessarily ISS with respect to measurement errors. It should be noted that similar time-dependent triggering schemes have also appeared in [12] and [26]. However, here, we mainly deal with nonlinear systems. The paper is organized as follows. In Section II we recall some basic concepts and definitions. In Section III, a partially linear cascade system is considered to introduce the main ideas for the event-triggered control of cascade systems. Finally, in Section IV, we generalize the results of Section III for more general classes of systems. Examples and simulations are also included throughout the paper, to demonstrate the proposed techniques.

II. Preliminaries

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$, if it is continuous and strictly increasing with $\alpha(0) = 0$. If in

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addition $\lim_{s\to\infty}\alpha(s)=\infty$, then α is said to be of class $\mathcal{K}_\infty$. A function $\beta:\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each fixed t the mapping $\beta(\cdot,t)$ is of class $\mathcal{K}$ and for each fixed s it is decreasing to zero as $t\to\infty$. By $|x|$ we denote the Euclidean norm of a vector $x\in\mathbb{R}^n$. A function $f:\mathbb{R}^n\to\mathbb{R}^m$ is said to be Lipschitz continuous on compact sets if for every compact $S\subset\mathbb{R}^n$ there exists a constant L such that $|f(x)-f(y)|\leq L|x-y|$, for every $x,y\in S$. By $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ we denote the minimum and maximum real part of the eigenvalues of $A\in\mathbb{R}^{n\times n}$, respectively. Next we recall some known definitions.

Definition 2.1: The system $\dot{x}=f(x)$ with $f:\mathbb{R}^n\to\mathbb{R}^n$ being locally Lipschitz and $f(0)=0$, is *globally asymptotically stable* (GAS) if there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that for any initial state $x(0)$ the solution exists for all $t\geq 0$ and satisfies $|x(t)|\leq\beta(|x(0)|,t)$.

Definition 2.2: The system $\dot{x}=f(x)$ with $f:\mathbb{R}^n\to\mathbb{R}^n$ being locally Lipschitz and $f(0)=0$, is *globally exponentially stable* (GES) if there exist positive constants κ, μ such that $|x(t)|\leq\kappa|x(0)|e^{-\mu t}$, $t\geq 0$, $x(0)\in\mathbb{R}^n$.

Lemma 2.1: [Gronwall-Bellman Lemma] Let $\chi,\psi:\mathbb{R}\to\mathbb{R}_{>0}$ be continuous functions and a, $b>0$. If $\chi^2(t)\leq a+b\int_0^t\psi(s)\chi^2(s)ds$, then, $\chi(t)\leq\Big(a\exp\Big[\,b\int_s^t\psi(\tau)d\tau\,\Big]\Big)^{1/2}$.

III. Partially Linear Cascade Systems

In this section we present then main ideas of event-triggered stabilization for partially linear cascade systems of the form:

$$\dot{x}=F(x)+G(x,z)z,\quad x\in\mathbb{R}^n,\, z\in\mathbb{R}^p, \tag{1a}$$
$$\dot{z}=Az+Bu,\quad u\in\mathbb{R}^m, \tag{1b}$$

where F, $G\in C^\infty$ and A, B are constant matrices of appropriate dimensions. In particular, we will study the behavior and performance of the cascade system when the (partial) state feedback $u(t)$, that stabilizes the z-subsystem, is implemented in a sample-and-hold fashion, i.e.: $u(t)=u(t_k),\ \forall t\in[t_k,t_{k+1})$, where t_k, $k\in\mathbb{N}$ are the time instants the controller is recomputed and updated.

Systems of the form (1) have been extensively studied and several results appear in the literature (see for instance [13], [22]). There are several examples where, even if both systems are GAS, the state of the x-subsystem may escape to infinity in finite time. In order to deal with such cases, the idea is to restrict the interconnection term $G(x,z)$ by a linear growth condition. Then, the stability of the cascade can be guaranteed as shown in [7], [13], [20], and [22]. Specifically, in [22], it was proved that the cascade (1) is GES if the following assumptions hold

(A1) The pair (A,B) is stabilizable.

(A2) There exist a C^1 function $V:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ and positive constants a_1, a_2, a_3, a_4 with

$$a_1|x|^2\leq V(x)\leq a_2|x|^2, \tag{2a}$$
$$\dot{V}=\tfrac{\partial V(x)}{\partial x}F(x)\leq -a_3|x|^2, \tag{2b}$$
$$\Big|\ \tfrac{\partial V(x)}{\partial x}\ \Big|\leq a_4|x|, \tag{2c}$$

that establish global exponential stability of the equilibrium $x=0$ of the nominal system $\dot{x}=F(x)$.

(A3) There exists a nondecreasing function $k(|z|)\geq 0$, such that $|G(x,z)|\leq k(|z|)|x|$ for all x, z.

Remark 3.1: By using standard converse Lyapunov theorems, there exists a Lyapunov function $V(x)$ satisfying (2), provided that $F(x)$ is globally Lipschitz.

Assume first that assumptions (A1), (A2), and (A3) hold. Then, by virtue of assumption (A1), the linear feedback $u=Kz$ renders the closed-loop system $\dot{z}=(A+BK)z$ globally asymptotically stable. Since between updates the value of u is held constant, the sampled closed-loop system is written $\dot{z}(t)=Az(t)+BKz(t_k)$. Let us define the state measurement error $e_r(t)=z(t_k)-z(t)$, $k\in\mathbb{N}$. Then, the cascade closed-loop system is written

$$\dot{x}=F(x)+G(x,z)z, \tag{3a}$$
$$\dot{z}=(A+BK)z+BKe_r. \tag{3b}$$

Our objective is to define a suitable triggering mechanism $\mathcal{T}(t,e_r(t))\geq 0$, with which the sample-and-hold implementation of the partial state feedback law $u=Kz$ will guarantee the stability of the cascade, and also that the inter-sampling times are lower bounded avoiding Zeno behavior. In particular, we have the following.

Proposition 3.1: Consider the partially linear cascade system (1) that satisfies assumptions (A1), (A2), and (A3). Suppose that the triggering condition is given by

$$\mathcal{T}(t,e_r(t))=|e_r(t)|-ce^{-at}, \tag{4}$$

with $c>0$, $0<a<\lambda=|\lambda_{\max}(A+BK)|$. Then, the closed-loop system (3) is exponentially stable and the inter-event times are bounded by a constant $\tau>0$. ◁

Proof: For brevity and conciseness, let us first define $\mathcal{A}:=A+BK$ and $\mathcal{B}:=BK$. Consider now the analytic solution of the closed loop system (3b) for $z_0\in\mathbb{R}^p$, $t\geq 0$: $z(t)=e^{\mathcal{A}t}z_0+\int_0^t e^{\mathcal{A}(t-s)}\mathcal{B}e_r(s)ds$, which implies, $|z(t)|\leq|e^{\mathcal{A}t}||z_0|+\int_0^t|e^{\mathcal{A}(t-s)}||\mathcal{B}||e_r(s)|ds$. Since the matrix $\mathcal{A}$ is Hurwitz, it follows that there exist constants κ, $\lambda>0$ such that $|e^{\mathcal{A}t}|\leq\kappa e^{-\lambda t}$, $t\geq 0$. Thus, we get

$$|z(t)|\leq\kappa|z_0|e^{-\lambda t}+\kappa\int_0^t e^{-\lambda(t-s)}|\mathcal{B}||e_r(s)|ds. \tag{5}$$

Now, let us turn our attention to the subsystem (3a). Consider the derivative of V given in assumption (A2) for the subsystem (3a): $\dot{V}=\frac{\partial V}{\partial x}F(x)+\frac{\partial V}{\partial x}G(x,z(t))z(t)$, where $z(t)$ is the solution of the system (3b) and satisfies (5). Thus, by taking into account assumptions (A2), (A3), and (5) we obtain:

$$\begin{aligned}\dot{V}\leq&-a_3|x|^2+\Big|\ \tfrac{\partial V(x)}{\partial x}\ \Big||G(x,z(t))||z(t)|\\ \leq&-\tfrac{a_3}{a_2}V+k(|z(t)|)\tfrac{a_4}{a_1}\kappa|z_0|e^{-\lambda t}V\\ &+k(|z(t)|)\tfrac{a_4}{a_1}\kappa V\int_0^t e^{-\lambda(t-s)}|\mathcal{B}||e_r(s)|ds.\end{aligned} \tag{6}$$

The triggering condition (4) with $c>0$ and $0<a<\lambda=|\lambda_{\max}(A+BK)|$, will guarantee boundedness of (5) which will result in exponential stability of the cascade system (3). Indeed, with this triggering condition we get from (6),

$$\begin{aligned}\dot V \le & -\tfrac{a_3}{a_2}V + k(|z(t)|)\tfrac{a_4}{a_1}\kappa|z_0|e^{-\lambda t}V \\ & + k(|z(t)|)\tfrac{a_4\kappa|\mathcal{B}|c}{a_1(\lambda-a)}(e^{-at}-e^{-\lambda t})V, \qquad (7)\end{aligned}$$

where $\lambda - a > 0$. Notice first, that since $k(|z|)$ is nondecreasing and bounded for bounded z, it follows that

$$\begin{aligned} k(|z(t)|) & \le k\Big(\kappa|z_0|e^{-\lambda t} + \kappa\int_0^t e^{-\lambda(t-s)}|\mathcal{B}||e_r(s)|ds\Big) \\ & \le k\Big(\kappa|z_0|e^{-\lambda t} + \tfrac{\kappa|\mathcal{B}|c}{\lambda-a}(e^{-at}-e^{-\lambda t})\Big) \le k\Big(\kappa|z_0| + \tfrac{\kappa|\mathcal{B}|c}{\lambda-a}\Big).\end{aligned}$$

From the previous inequality and (7) we have that

$$\begin{aligned}\dot V \le & -\tfrac{a_3}{a_2}V + k\Big(\kappa|z_0| + \tfrac{\kappa|\mathcal{B}|c}{\lambda-a}\Big)\tfrac{a_4}{a_1}\kappa|z_0|e^{-\lambda t}V \\ & + k\Big(\kappa|z_0| + \tfrac{\kappa|\mathcal{B}|c}{\lambda-a}\Big)\tfrac{a_4\kappa|\mathcal{B}|c}{a_1(\lambda-a)}(e^{-at}-e^{-\lambda t})V. \qquad (8)\end{aligned}$$

It follows from (8) and the Gronwall-Bellman inequality that $V(x(t)) \le \gamma e^{-\frac{a_3}{a_2}t}V(x(0))$, where $\gamma := \exp\{k(\kappa|z_0| + \frac{\kappa|\mathcal{B}|c}{\lambda-a})\frac{a_4\kappa}{a_1\lambda}(|z_0| + \frac{|\mathcal{B}|}{a})\}$, which implies exponential stability for the cascade system (3).

Next, we prove that Zeno behavior is excluded, namely, that there exist a lower bound on the inter-event times. Without loss of generality, let us assume first that $|A| \neq 0$. Then, between two consecutive events $[t_k, t_{k+1})$ it holds that $\dot e_r(t) = -\dot z(t) = -Az(t) - BKz(t_k)$, or equivalently, $|\dot e_r(t)| = |A(z(t_k) - e_r) + BKz(t_k)| \le |A + BK||z(t_k)| + |A||e_r|$. At the event time t_k it also holds that $|e(t_k)| = 0$. Thus, by solving the previous differential inequality for $t \in [t_k, t_{k+1})$, we have that

$$|e_r(t)| \le \tfrac{|A+BK||z(t_k)|}{|A|}\Big(e^{|A|(t-t_k)} - 1\Big). \qquad (9)$$

Recall now that, according to the triggering condition (4), the next event instant occurs when, $|e_r(t_{k+1})| = ce^{-at_{k+1}}$. Thus, it follows from (9) that $ce^{-at_{k+1}} \le \Gamma|z(t_k)|(e^{|A|(t_{k+1}-t_k)} - 1)$, where $\Gamma = \frac{|A+BK|}{|A|}$. Then, from the previous inequality, a lower bound on the inter-event intervals is given by,

$$t_{k+1} - t_k \ge \tfrac{1}{|A|}\ln\Big(1 + \tfrac{ce^{-at_{k+1}}}{\Gamma|z(t_k)|}\Big), \ |A| \neq 0. \qquad (10)$$

To prove that there is no Zeno behavior, it suffices to show that $t_{k+1} - t_k > 0$. Recall that by the triggering condition (4) we have $|e_r(t_{k+1})| = |z(t_k) - z(t_{k+1})| \ge |z(t_k)| - |z(t_{k+1})|$, or equivalently

$$|z(t_k)| \le ce^{-at_{k+1}} + |z(t_{k+1})|. \qquad (11)$$

Notice now that from (4), (5), and the fact that $(e^{-at} - e^{-\lambda t}) \ge 0$ for all $t \ge 0$, the state $z(t)$ is bounded by

$$|z(t)| \le \kappa|z_0|e^{-\lambda t} + \tfrac{\kappa|\mathcal{B}|c}{\lambda-a}(e^{-at} - e^{-\lambda t}) \le \kappa_1|z_0|e^{-at}, \ (12)$$

where $\kappa_1 := \max\{\kappa|z_0|, \frac{\kappa|\mathcal{B}|c}{\lambda-a}\}$. Then, it follows from (11) and (12) that the following holds $\frac{ce^{-at_{k+1}}}{|z(t_k)|} \ge \frac{ce^{-at_{k+1}}}{ce^{-at_{k+1}} + |z(t_{k+1})|} \ge \frac{ce^{-at_{k+1}}}{ce^{-at_{k+1}} + \kappa_1|z_0|e^{-at_{k+1}}} = \frac{c}{c+\kappa_1|z_0|} > 0$. Thus, (10) and the previous inequality imply that for $|A| \neq 0$,

$$t_{k+1} - t_k \ge \tfrac{1}{|A|}\ln\Big(1 + \tfrac{c}{\Gamma(c+\kappa_1|z_0|)}\Big) > 0, \forall k \in \mathbb{N}. \qquad (13)$$

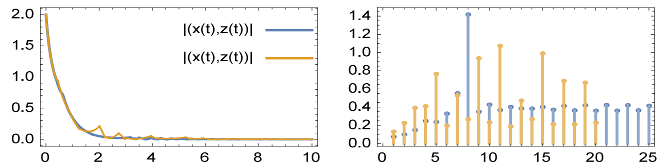

Fig. 2: Evolution of state of system (14) and inter-sampling times.

Finally, we consider the case where $|A| = 0$. Then, necessarily we have that the matrix B has full rank due to assumption (A1). With similar manipulations as above and by taking into account (4) and the fact that $\dot e_r(t) = -BKz(t_k)$ and $|e_r(t_k)| = 0$, we get $t_{k+1} - t_k \ge ce^{-at_{k+1}}/(|BK||z(t_k)|)$, with $0 < a < \lambda = \lambda_{\max}(BK)$. Then, with similar arguments as above we can obtain the lower bound $t_{k+1} - t_k \ge \frac{c}{|BK|(c+\kappa_1|z_0|)} > 0$, $\forall k \in \mathbb{N}$. ∎

Remark 3.2: We note that the triggering condition (4) is not new and variants of it have appeared in the literature, see for instance [12], [26].

Example 3.1: Consider the system

$$\begin{cases}\dot x_1 = -(3 + \cos(x_2))x_1 + x_2 + x_1z_1^2 \\ \dot x_2 = -x_1\sin(x_1) - 2x_2\end{cases} \qquad (14a)$$

$$\begin{cases}\dot z_1 = -z_1 + z_2 \\ \dot z_2 = 2z_1 + z_2 + 2u\end{cases} \qquad (14b)$$

Notice first that the nominal system $\dot x = F(x)$, $x = (x_1, x_2)^T$, $F(x) := (-(2 + \cos(x_2))x_1 + x_2, -x_1\sin(x_1) - x_2)^T$ is GES with Lyapunov function $V := 1/2x_1^2 + 1/2x_2^2$ and the interconnection term satisfies $|G(x,z)| \le k(|z|)|x|$ with $k(|z|) = |z_1|$. Thus, both assumptions A2 and A3 hold. Finally, the feedback law $u = -2z_1 - 5/2z_2$ stabilizes the linear subsystem (14b). Therefore, according to the previous analysis, the triggering condition is chosen as $|e_r(t)| \le ce^{-at}$, with $c > 0$ and a satisfying the following condition $0 < a < \lambda = |\lambda_{max}(A + BK)|$, where $\lambda_{max}(A + BK)$ is the maximum real part of the eigenvalues of $(A + BK)$. In Fig. 2, are shown the state of the cascade (14) with $a = 0.55$, $c = 0.38$ (blue) and with $a = 0.55$ and $c = 0.6$ (yellow), where we have experience with a number of 26 and 21 events respectively. It can be seen that larger values of c reduce the number of controller updates at the cost of performance.

IV. A General Case

In this section we address the problem of event triggered control for the following general case of cascade systems

$$\dot x = F(x) + G(x,z)z, \ \ x \in \mathbb{R}^n, z \in \mathbb{R}^m, \qquad (15a)$$

$$\dot z = f(z) + g(z)u, \ \ u \in \mathbb{R}^p, \qquad (15b)$$

with f, $g \in C^1$. In particular, we will present two triggering schemes, with which, Zeno behavior is avoided and the cascade system (15) is GAS. For the latter we will exploit assumption (A3) as well as, the following two hypotheses

(A4) There exist a C^1 function $V_z : \mathbb{R}^m \to \mathbb{R}$, a locally Lipschitz map $h : \mathbb{R}^m \to \mathbb{R}$ with $h(0) = 0$, and constants $b_i > 0$, $i = 1, \ldots, 4$ such that

$$b_1|z|^2 \leq V_z(z) \leq b_2|z|^2 \quad (16a)$$

$$\dot{V}_z = \tfrac{\partial V_z(z)}{\partial z}(f(z) + g(z)h(z)) \leq -b_3|z|^2 \quad (16b)$$

$$\left| \tfrac{\partial V_z(z)}{\partial z} \right| \leq b_4|z|. \quad (16c)$$

(A5) The equilibrium $x = 0$ of $\dot{x} = F(x)$ is GAS and there exists a C^1 function $V_x : \mathbb{R}^n \to \mathbb{R}$ with $V_x(0) = 0$ such that $a_1|x|^2 \leq V_x(x) \leq a_2|x|^2$, $|\frac{\partial V_x}{\partial x}| \leq a_3|x|$, a_1, a_2, $a_3 > 0$ and $\frac{\partial V_x(x)}{\partial x}F(x) \leq 0$, $x \in \mathbb{R}^n$.

Note that, assumption (A4) implies that the closed-loop system $\dot{z} = f(z) + g(z)h(z)$ is GES and assumption (A5) requires a Lyapunov function for the nominal system $\dot{x} = F(x)$ to be known. Next, we recall a theorem from [23]:

Theorem 4.1: Consider the cascade system

$$\dot{x} = f_1(x, z),\ x \in \mathbb{R}^n,\ z \in \mathbb{R}^m, \quad (17a)$$

$$\dot{z} = f_2(z), \quad (17b)$$

where f_1, f_2 are locally Lipschitz and $f_2(0) = f_1(0, 0) = 0$. Then, if $\dot{x} = f_1(x, 0)$ and $\dot{z} = f_2(z)$ are both GAS, and every trajectory of (17) is bounded for $t > 0$, then (17) is GAS. ◁

We will present next, two triggering mechanisms that guarantee the stability of the cascade closed-loop system (15) and prevent the occurrence of Zeno behavior. In particular, the following result holds

Proposition 4.1: Under assumptions (A3), (A4), and (A5) the system (15) with each of the triggering conditions

(i) $|z(t_k) - z(t)| \leq \sigma|z(t)|$, $\sigma = \sigma(z_0) > 0$;
(ii) $|z(t_k) - z(t)| \leq ce^{-at}$, c, $a > 0$,

is asymptotically stable and Zeno behavior is avoided.

Proof: Suppose that we sample and hold the measurement at time $t = t_k$ and use the constant feedback $u = h(z(t_k))$, $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. Then the subsystem (15b) is written $\dot{z} = f(z) + g(z)h(z(t_k))$, $t \in [t_k, t_{k+1})$, or equivalently

$$\begin{aligned}\dot{z}(t) =& f(z(t)) + g(z(t))h(z(t)) \\ &+ g(z(t))(h(z(t_k)) - h(z(t))). \end{aligned} \quad (18)$$

(i) Let $z_0 \in \mathbb{R}^m$, and define the set $\Omega_s := \{z \in \mathbb{R}^m : |z| \leq \sqrt{\frac{b_2}{b_1}}|z_0|\}$, where b_1, $b_2 > 0$ are given in assumption (A4). Then, by taking into account (16b), (16c), and the facts that $g \in C^1$ is bounded on Ω_s by a constant $k_g > 0$, and h is locally Lipschitz, with Lipschitz constant L_h on the compact set Ω_s, we calculate the derivative $\dot{V}_z$ of V_z along (18) to obtain the following estimate

$$\dot{V}_z(z(t)) \leq -b_3|z(t)|^2 + b_4k_gL_h|z(t)||z(t_k) - z(t)|. \quad (19)$$

If we enforce the triggering condition

$$|z(t_k) - z(t)| \leq \sigma|z(t)|, \quad (20)$$

with $1 > \sigma > 0$ sufficiently small in such a way that $\sigma b_4k_gL_h < b_3$, then, we get from (16a) and (19)

$$\dot{V}_z(z(t)) \leq -\tfrac{b_3}{b_2}(1 - \sigma b_4k_gL_h)V_z(z(t)). \quad (21)$$

Notice that Ω_s is a forward invariant set for the closed-loop system, since the triggering rule (20) guarantees that $\dot{V} \leq 0$. Indeed, suppose on the contrary that there exists $T \in (t_0, t_1)$, where t_1 is the first event, such that $z(T) \notin \Omega_s$. Due to continuity of $z(\cdot)$, there exists $\bar{t} \in (t_0, T)$ such that $V_z(z(\bar{t})) = \frac{b_2}{b_1}|z_0|^2$ and $z(t) \notin \Omega_s$, $t \in (\bar{t}, T]$. Then we get from (16a) that $\frac{b_2}{b_1}|z_0|^2 < V_z(z(T)) = V_z(z(\bar{t})) + \int_{\bar{t}}^T \dot{V}_z(z(s))ds \leq V_z(z(\bar{t})) = \frac{b_2}{b_1}|z_0|^2$, which is a contradiction. Finally, from the Gronwall-Bellman inequality, (16a), and (21) we get that

$$|z(t)| \leq \sqrt{\frac{b_2}{b_1}}|z_0|e^{-\frac{b_3}{2b2}(1-\sigma b_4k_gL_h)t}. \quad (22)$$

Next, consider the Lyapunov function V_x given in assumption (A5). Then we have $\dot{V}_x \leq \frac{\partial V_x}{\partial x}G(x, z(t))z(t)$ from which it follows that $V_x(x(t)) \leq V_x(x_0) + \int_0^t |\frac{\partial V_x}{\partial x}||G(x(s), z(s))||z(s)|ds$. From the latter and by taking now into account assumptions (A3) and (A5), we obtain

$$|x(t)|^2 \leq \tfrac{V_x(x_0)}{a_1} + \tfrac{a_3}{a_1}\int_0^t |x(s)|^2k(|z(s)|)|z(s)|ds. \quad (23)$$

Consider now the estimate (22) for the state $z(t)$. Since $k(|z(t)|) \geq 0$ is nondecreasing, we get from (23) that $|x(t)|^2 \leq \beta_1 + \beta_2\int_0^s |x(s)|^2k(\kappa|z_0|)e^{-\mu s}ds$, where $\beta_1 = V_x(x_0)/a_1$, $\beta_2 = (\kappa|z_0|a_3)/a_1$, $\kappa = \sqrt{b_2/b_1}$, and $\mu = b_3(1 - \sigma b_4k_gL_h)/(2b_2)$. By applying Lemma 2.1, with $a = \beta_1$, $b = \beta_2k(\kappa|z_0|)$ and $\psi(t) = e^{-\mu t}$ we get

$$|x(t)| \leq \left(\beta_1 \exp\left\{\int_0^t k(\kappa|z_0|)\beta_2e^{-\mu s}ds\right\}\right)^{\frac{1}{2}},\ t \geq 0, \quad (24)$$

which implies that $x(t)$ is bounded. Since $z(t)$ is also bounded by (22), we conclude from Theorem 4.1, that the cascade is asymptotically stable.

Finally, we prove that the sampling intervals are bounded from below. The proof follows similar arguments to [29]. First, define $e = \hat{z} - z$, $\hat{z} := z(t_k)$ and notice that since f, g are C^1 and h is Lipschitz on compact sets, it follows that $|\dot{z}| \leq (L_f + k_gL_h)|z| + (k_gL_h)|e| \leq (L_f + k_gL_h)(|z| + |e|) := L(|z| + |e|)$, where $L_f > 0$ and $L_h > 0$ are the Lipschitz constants of f and h respectively, $k_g > 0$ a bound of g on the compact set Ω_s. Similar to [29], it can be shown that the derivative of $|e|/|z|$ satisfies the estimate $\frac{d}{dt}\frac{|e|}{|z|} \leq L(1 + \frac{|e|}{|z|})^2$, with $L := L_f + k_gL_h$. By denoting $y = |e|/|z|$, we have $\dot{y} \leq L(1 + y)^2$ and by the Comparison Principle, it follows that y satisfies the bound $y(t) \leq \phi(t, \phi_0)$, where $\phi(t, \phi_0)$ is the solution of the differential equation $\dot{\phi} = L(1 + \phi)^2$, $\phi(0, \phi_0) = \phi_0$. Thus, the inter-event times are bounded by the time τ that satisfies $\phi(\tau, 0) = \sigma$, with σ satisfying (20). From the solution $\phi(\tau, 0) = \tau L/(1 - \tau L)$ we get the lower bound $\tau = \sigma/(L(1 + \sigma))$ which completes the proof that Zeno behavior is excluded.

(ii) Assume now that we enforce $|z(t_k) - z(t)|$ to satisfy

$$|z(t_k) - z(t)| \leq ce^{-at}, \quad (25)$$

for some positive constants c and a. Notice first that the triggering condition (25) implies that

$$|z(t)| \leq ce^{-at} + |z(t_k)|,\ \forall t \in [t_k, t_{k+1}). \quad (26)$$

Let $0 < C \leq 1$ and define the sets $B(|z_0|) := \{z \in \mathbb{R}^m : |z| \leq |z_0|\}$, $Q_1 := B(|z_0|) + C$, and $Q_2 := V_z^{-1}(V_z(B(|z_0|))) + C$, where $V_z^{-1}(V_z(B(|z_0|))) := \{z \in \mathbb{R}^m : V_z(z) \in V_z(B(|z_0|))\}$. Then it holds that

$$Q_1 \subset Q_2. \tag{27}$$

Let $t_0 = 0$. Then, until the first event at t_1 occurs, (26) implies that for $0 < c < C$, $z(t) \in Q_1$ and consequently from (27) that $z(t) \in Q_2$. By taking into account, assumption (A4), (19), and (25) we obtain

$$\dot{V}_z(z(t)) \leq -b_3|z(t)|^2 + cb_4k_gL_h|z(t)|e^{-at}, \tag{28}$$

where $L_h > 0$ is a Lipschitz constant of $h(\cdot)$ on the compact set Q_2, and $k_g > 0$ is a bound of g on the same set. Let $0 < c < \min\{C, \frac{b_3}{b_4k_gL_h}\}$, where $0 < C \leq 1$. Then, if it holds that $|z(t)| \geq e^{-at}$, it follows from (16a) and (28) that

$$\dot{V}_z(z(t)) \leq -\tfrac{1}{b_2}(b_3 - cb_4k_gL_h)V_z(z(t)), \tag{29}$$

with $b_3 - cb_4k_gL_h > 0$. Indeed, assume first that $\min\{C, \frac{b_3}{b_4k_gL_h}\} = C \leq 1$. This would imply that $b_4k_gL_h < b_3$ and (29) holds with $0 < c < C \leq 1$. On the other hand, if $\min\{C, \frac{b_3}{b_4k_gL_h}\} = \frac{b_3}{b_4k_gL_h}$, then with $c = \frac{b_3}{\alpha b_4k_gL_h}$, $\alpha > 1$ we again obtain (29) with $(b_3 - cb_4k_gL_h) = b_3(1 - \frac{1}{\alpha}) > 0$. Additionally, it follows from (29) that $|z(t)| \leq \sqrt{\frac{b_2}{b_1}}|z_0|e^{-\mu t}$, where $\mu = (b_3 - cb_4k_gL_h)/(2b_2)$. Notice now that, at time $t_0 = 0$, due to continuity of $z(\cdot)$ the triggering will not happen instantaneously. Thus, until the next event occurs at time $t_1 > t_0$, $\dot{V}_z \leq 0$. The latter implies that $z(t) \in V_z^{-1}(V_z(B(|z_0|)))$, $t \in [t_0, t_1)$. Similar arguments hold for all intervals $[t_k, t_{k+1})$, $k \in \mathbb{N}$. In particular, let $k \in \mathbb{N}$. Then, since $z(t_k) \in V_z^{-1}(V_z(B(|z_0|)))$, we get from (26) that $z(t) \in Q_2$ for all $t \in [t_k, t_{k+1})$. The latter, in conjunction with the fact that $\dot{V}_z \leq 0$, implies that $z(t) \in V_z^{-1}(V_z(B(|z_0|)))$, $t \in [t_k, t_{k+1})$.

Finally, if $|z(t)| \leq e^{-at}$, $t \geq 0$, we get from (16a) that $|z(t)| \leq \sqrt{b_2/b_1}e^{-at}$, $t \geq 0$. On the other hand, if there exists time $T > 0$ such that $|z(T)| \geq e^{-aT}$, then from continuity of $z(\cdot)$, there exists $\hat{t} < T$ such that $|z(\hat{t})| = e^{-a\hat{t}}$ and $|z(t)| \geq e^{-at}$, $t \in [\hat{t}, T]$. Then, from (16a), (29), and by further restricting $0 < a \leq \mu = (b_3 - cb_4k_gL_h)/(2b_2)$, we obtain $b_1|z(T)|^2 \leq V(z(T)) \leq V(z(\hat{t})) \leq b_2|z(\hat{t})|^2 \leq b_2e^{-2\mu\hat{t}}$, which implies that $|z(T)| \leq \sqrt{b_2/b_1}\max\{|z_0|, 1\}e^{-a\hat{t}}$. Next, for every $\tau > 0$ such that $|z(t)| \geq e^{-at}$, $t \in [0, \tau]$, it follows from (29) that $|z(t)| \leq \sqrt{b_2/b_1}|z_0|e^{-at}$, $t \geq 0$ with $0 < a \leq \mu$. If there exists $T > 0$ such that $|z(T)| \leq e^{-aT}$ it follows with similar arguments as before that $|z(t)| \leq \sqrt{b_2/b_1}\max\{|z_0|, 1\}e^{-at}$, $t \geq T$. Hence, $|z(t)| \leq \sqrt{b_2/b_1}\max\{|z_0|, 1\}e^{-at}$, $t \geq 0$, provided that $0 < a \leq \mu$.

Next, we will show that, with $0 < c < \min\{C, b_3/(b_4k_gL_h)\}$, $0 < C \leq 1$ and $0 < a \leq \mu = (b_3 - cb_4k_gL_h)/(2b_2)$, we can avoid infinitely fast sampling. In order to show that the inter-event times are lower bounded, we follow similar arguments with those of Proposition 3.1. First, define $e_r(t) := z(t_k) - z(t)$. By taking into account that f, $g \in C^1$ and h is Lipschitz on compact sets with constants L_f and L_h respectively, we have $|\dot{e}_r(t)| \leq (L_f + L_hk_g)|z(t_k)| + L_f|e_r(t)|$. Solving this differential inequality with $|e(t_k)| = 0$ we get $|e_r(t)| \leq \frac{(L_f+L_hk_g)|z(t_k)|}{L_f}(e^{(L_f+L_hk_g)(t-t_k)} - 1)$. From the triggering condition (25), we know that the next event instant occurs when $|e_r(t_{k+1})| = ce^{-at_{k+1}}$. Hence, from the previous inequality we have that the inter-event times satisfy $t_{k+1} - t_k \geq \frac{1}{(L_f+L_hk_g)}\ln(1 + \frac{L_fce^{-at_{k+1}}}{(L_f+L_hk_g)|z(t_k)|})$. Thus, it suffices to prove that the argument of the logarithm is greater than one. With similar arguments to those in the proof of Proposition 3.1, it follows, by taking into account that $|z(t_{k+1})| \leq \sqrt{b_2/b_1}|z_0|e^{-at_{k+1}}$ and the fact that $|z(t_k)| \leq ce^{-at_{k+1}} + |z(t_{k+1})|$, that

$$t_{k+1} - t_k \geq \frac{1}{\mathrm{L}}\ln(1 + \frac{cL_f}{\mathrm{L}(c + \sqrt{b_2/b_1}|z_0|)}) > 0$$

where $\mathrm{L} := L_f + L_hk_g$.

Finally, by taking into account assumption (A5) and with similar arguments as in part (i), we again obtain (24) with $\beta_1 = V(x_0)/a_1$, $\beta_2 = (\kappa a_3)/a_1$, $\kappa = \sqrt{b_2/b_1}\max\{1, |z_0|\}$. The latter, together with Theorem (4.1), implies asymptotic stability of the cascade system (15). ■

Remark 4.1: In (28), we used the fact that the set Q_2 is compact in order to obtain the positive constants k_g and L_h. In practice it is not always easy to calculate explicitly this set. However, for any $z_0 \in \mathbb{R}^m$, it is easier to calculate the constants k_g and L_h on the set $Q_3 := B(\sqrt{b_2/b_1}|z_0|) + C$ which includes Q_2.

Remark 4.2: For the triggering mechanism (25), analogous results were obtained in [9], where system (18) was considered as a perturbed system with perturbation terms resulting from sampling.

Finally, we provide an extension of Proposition 3.1 for the general system (15), that relaxes the need for (A5). In particular, the following holds

Proposition 4.2: Under assumptions (A2), (A3), and (A4), the cascade system (15) with each of the triggering conditions (20) and (25) is exponentially stable and Zeno behavior is excluded.

Proof: [Sketch] Similarly to the proof of Proposition 4.1, we can obtain an estimate fo the form $|z(t)| \leq \kappa|z_0|e^{-\mu t}$, with κ and μ depending on each of the triggering conditions (20) and (25). Specifically, for the case (20), $\kappa = \sqrt{b_2/b_1}$ and $\mu = (1 - \sigma b_4k_gL_h)b_3/(2b_2)$, $0 < \sigma < 1$; and for the case (25), $\kappa = \sqrt{b_2/b_1}$, $0 < c < \min\{1, 1/(k_gL_h)\}$ and $0 < \mu \leq (1 - c)b_3/(2b_2)$. Then, from assumption (A2), there exists a positive definite and proper Lyapunov function V_x and positive constants a_1, a_2, a_3, a_4 such that (2) holds. Then, by taking into account (2), the previous estimate of $z(t)$, and assumption (A3) we obtain

$$\dot{V}_x \leq -\tfrac{a_3}{a_2}V_x + \tfrac{a_4}{a_1}k(\kappa|z_0|)e^{-\mu t}V_x,$$

for $\mu, \kappa > 0$ as previously defined and therefore, $V_x(x(t)) \leq \gamma\exp\{-a_3t/a_2\}V_x(x(0))$, for some $\gamma > 0$, which implies exponential stability. Similar to the proof of Proposition 4.1, it can be shown that Zeno behavior is excluded. ■

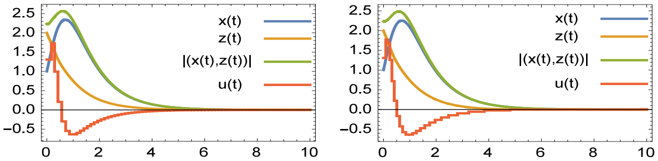

Fig. 3: Evolution of states with state-dependent threshold (left) and time-dependent threshold (right).

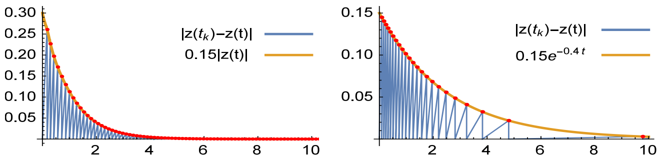

Fig. 4: Evolution of $|z(t_k) - z(t)|$ with state-dependent threshold (left) and time-dependent threshold (right).

Remark 4.3: It should be noted that the feedback $u = h(z)$ that exponentially stabilizes the z-subsystem in assumption (A4), does not necessarily render the closed-loop system ISS with respect to measurement errors $e \in \mathbb{R}^m$. This is illustrated in the following example.

Example 4.1: Consider the planar system

$$\dot{x} = -x + xz^2$$
$$\dot{z} = -z\sin^2(z^2) + u\cos(z^2)$$

This system satisfies assumptions (A2), (A3) and (A4). Indeed, the state feedback $u = -z\cos(z^2)$ renders the closed-loop system exponentially stable with Lyapunov function $V_z = 1/2z^2$. However, as has been proved in [6], this feedback does not render the z-subsystem ISS with respect to measurement errors. From the time-derivative of $V_z = 1/2z^2$ we get $\dot{V}_z = -z^2$ and thus (16) holds with $b_1 = b_2 = 1/2$, $b_3 = 1$, $b_4 = 1$. Similarly, we also have for the nominal system $\dot{x} = -x$, that $a_1 = a_2 = 1/2$, $a_3 = a_4 = 1$. With initial conditions $(x_0, z_0) = (1, 2)$, we can find $L_h = 5.5$, $k_g = 1$ and we can select $\sigma = 0.15$, $c = 0.15$ and $a = 0.4$. The simulation results are depicted in Fig. 3 and 4.

V. Conclusion

This paper presents results for the event-triggered control of a class of cascade systems with partial state feedback. The proposed mechanisms ensure stability of the system and avoid infinitely fast sampling. A main feature of those techniques is that no assumption of ISS with respect to measurement errors is required. Future work will include event-triggered control of cascade systems with full state feedback and also more general classes of cascades, i.e., the first subsystem being also augmented by the control input.

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