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Controlling the collective behaviour of networks of heterogenous Kuramoto oscillators with phase lags

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Abstract— **We investigate the problem of controlling a network of heterogeneous Kuramoto oscillators affected by phase lags in the communication with their neighbours. We find analytical conditions that allow to determine the control effort required to guarantee convergence of all the oscillators towards a common collective evolution despite the presence of heterogeneities and phase lags. After presenting some numerical simulations that confirm the theoretical results, we discuss the application of the theory to the problem of inducing motor coordination in a mixed group of human players and artificial agents performing a joint task.**

I. Introduction

Synchronisation in multi-agent systems has been extensively investigated over the past few decades [1], [2], [3], [4], [5], particularly in networks of agents exhibiting oscillatory dynamics [6], [7], [8], [9]. Oscillator models are used to model the emergence of coordination in several areas of Science and Technology [10], [11]. They were first proposed in the pioneering work of Winfree [12] and later by Kuramoto [13]. It has been proven that synchronisation can be achieved for a certain combination of the coupling strength among the agents, the level of heterogeneity among their dynamics [14], and the structure of their interconnections. When synchronized, the oscillators can be considered as acting as an individual unit [15].

Despite synchronisation being desirable in many complex systems, spanning from biology and engineering to economics [16], increasing the coupling among the agents, changing their interconnections, or decreasing their heterogeneity is not always feasible. Moreover, maintaining synchronization in the presence of uncertainties and/or perturbations can be cumbersome at times. Hence, often an external control action is required to steer all the nodes towards a synchronised behaviour. This action is usually referred to as *pinning control* [17] or, more specifically in the case of oscillator networks, as a *pacemaker* input [18]. Nodes receiving information from the pinner are referred to as *driver nodes* [19]. Choosing the number of driver nodes in order for all the agents to converge towards a desired collective trajectory, as well as estimating the required control effort as a function of the underlying network topology, is a problem generally referred to as *pinning controllability* [20], [21], [22].

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Studying the influence of the pacemaker in a network of oscillators is of crucial importance, both theoretically and practically [23]. Indeed, it is essential for power grids to operate at a specified common frequency [24]; in medicine, an artificial pacemaker may be implanted to regulate the heartbeat rate of patients [25]; in wireless networks, individual nodes are synchronised through an external coordination signal such as that received from a GPS [26]; in human ensembles, an external signal might be used to entrain all the group members and enhance their coordination [27], [28].

Previous literature has mostly focused on the case of networks of Kuramoto oscillators in the absence of pinners, where analytical conditions relating coupling strength, inner dynamics and topological structure of the network have been found in order for all the nodes to exhibit a synchronised behaviour [29], [30], [31], [32]. Surprisingly, the problem of studying the effects of a pacemaker on the synchronisation dynamics from a control viewpoint has been mainly overlooked.

In the few existing results addressing this problem, synchronisation has been investigated mostly locally for undirected (i.e., node i is connected to node j if and only if node j is connected to node i) and unweighted (i.e., the interconnection strength is the same across different pairs of oscillators) networks of heterogeneous (i.e., nodes are characterised by different inner dynamics) Kuramoto oscillators [33]. More recently, some results were presented on global synchronisation of both undirected [18] and directed [16], [34] weighted networks of homogeneous as well as heterogeneous Kuramoto oscillators.

However, to the best of our knowledge, neither analytical nor experimental results have been presented taking into account the presence of both a pacemaker and possible phase delays (or phase lags) affecting the communication between pairs of oscillators [35]. Such phase lags are particularly relevant in power grids, where they model energy losses due to the transfer conductance [31], in brain networks, where they model the lags between neurons [36], in biological and other systems (as in the case of humans), where they model delay couplings due to reaction times [37].

In this work we consider the most general case of directed and weighted networks of heterogeneous Kuramoto oscillators driven by a pacemaker and affected by phase delay in the communication with their neighbours, also referred to as *nonuniform* Kuramoto oscillators [31], [38]. After presenting the problem statement in Section II, we then derive analytical conditions relating the underlying topology, the coupling strength among the nodes, the phase lag, and the pacemaker's control effort on the driver nodes that guarantee synchronisation of all the agents (Section III). Then, in Section IV we confirm our theoretical findings via numerical simulations. Finally, a summary of the main findings and future work is presented in Section **??**.

II. Problem Statement

Consider a network of N nonuniform Kuramoto oscillators of the form:

$$\dot{\theta}_i(t) = \omega_i + \frac{c}{N}\sum_{j=1}^{N} a_{ij}\sin\left(\theta_j(t) - \theta_i(t) - \beta_{ij}\right) \tag{1}$$

where θ_i and $\dot{\theta}_i$ represent phase and angular velocity of the ith oscillator with $i = 1, 2, \ldots, N$, respectively, $\omega_i > 0$ refers to the natural oscillation frequency of the ith oscillator, $c > 0$ is the coupling strength assumed to be the same among all the agents, a_{ij} represents the local influence that node j has on node i, and β_{ij} is the phase lag affecting their communication. For directed and weighted topologies, in general, $a_{ij} \neq a_{ji}$ with $a_{ij} \geq 0$ $\forall i, j$. Specifically, $a_{ij} > 0$ is representative of the fact that node i is influenced by node j.

Remark 1. *For a network of heterogeneous Kuramoto oscillators (i.e., $\omega_i \neq \omega_j$),* phase synchronisation *where $\lim_{t\to\infty}|\theta_i(t) - \theta_j(t)| = 0$ $\forall i, j$ can never be achieved [39]. However, for sufficiently high values of the coupling strength c compensating for the heterogeneity of the nodes in their natural oscillation frequencies, it is possible to achieve* frequency synchronisation *or* phase-locking *[31], where $\lim_{t\to\infty}\dot{\theta}_i(t) = \omega^*$ $\forall i$, and consequently $\lim_{t\to\infty}|\theta_i(t) - \theta_j(t)| = const.$ $\forall i, j$. In other words, despite heterogeneous oscillators not being able to reach a consensus phase value, they can synchronise in their angular velocity, so that they oscillate at the same frequency and the phase differences among all the agents remain constant.*

Given a network of N nonuniform Kuramoto oscillators as that described in Eq. (1), if the coupling among its agents is not strong enough, the heterogeneity of the nodes and phase lags is too high or the underlying topology is not sufficiently connected, synchronisation cannot be achieved. Therefore, the general problem of interest is that of designing an appropriate external signal $u = [u_1\ u_2\ \ldots\ u_N]^T$ so that the controlled system:

$$\dot{\theta}_i(t) = \omega_i + \frac{c}{N}\sum_{j=1}^{N} a_{ij}\sin\left(\theta_j(t) - \theta_i(t) - \beta_{ij}\right) + u_i(t) \tag{2}$$

can achieve phase synchronisation onto a desired collective trajectory $\theta_d(t)$, when $\omega_i = \omega_j$, or frequency synchronisation onto a desired collective angular velocity ω^*, when $\omega_i \neq \omega_j$ (which is the case we consider in this paper).

Let us refer to $\theta_i^*(t)$ as the phase trajectory that the ith node would have reached if frequency synchronisation had been achieved in a network of heterogeneous oscillators in the absence of control, with $\lim_{t\to\infty}\dot{\theta}_i^*(t) = \omega^*$ (the value of θ_i^* will be computed in the next section). Note that, in general, $\theta_i^*(t) \neq \theta_j^*(t)$. We are interested in unveiling whether each ith node can be steered onto $\theta_i^*(t)$ with an appropriate choice of $u_i(t)$, so as to stabilise the phase-locking regime.

In this case, by controlling the network with a pacemaker, the equation becomes:

$$\begin{aligned}\dot{\theta}_i(t) = \omega_i &+ \frac{c}{N}\sum_{j=1}^{N} a_{ij}\sin\left(\theta_j(t) - \theta_i(t) - \beta_{ij}\right)\\ &+ k_i\sin\left(\theta_i^*(t) - \theta_i(t)\right)\end{aligned} \tag{3}$$

where $k_i \geq 0$ represents the control gain for the ith oscillator. Therefore, in this scenario the problem is that of identifying a subset of the nodes to be pinned:

$$k_i\begin{cases} = 0, & \text{node } i \text{ is not a driver}\\ > 0 & \text{node } i \text{ is a driver}\end{cases} \tag{4}$$

and estimating the control strength required for them in order for the network to achieve frequency synchronisation, that is:

$$\lim_{t\to\infty}\theta(t) = \theta^*(t) \tag{5}$$

where $\theta(t) := [\theta_1(t)\quad \theta_2(t)\quad \ldots\quad \theta_N(t)]^T$, $\dot{\theta}(t) := [\dot{\theta}_1(t)\ \dot{\theta}_2(t)\ \ldots\ \dot{\theta}_N(t)]^T$, and with $\dot{\theta}^*(t) = \omega^* 1_N$.

Remark 2. *Note that no phase delay has been considered for the pacemaker, as it represents an external input on which it is assumed to have control.*

III. Analytical Results

In this section we first evaluate the unique solution $\theta^*(t)$ that an uncontrolled network of nonuniform Kuramoto oscillators would achieve in the case of frequency synchronisation. We then shed light on how to control such network with a pacemaker in order to steer each ith node towards $\theta_i^*(t)$ and stabilise the phase-locking regime.

A. Open-loop network

In the following theorem, we analyse the behaviour of the uncontrolled network supposing that phase-locking regime is achieved.

Theorem 1. *Consider a network of nonuniform Kuramoto oscillators as that described in Eq.* (1)*. The solution $\theta^*(t)$, with $\dot{\theta}^*(t) = \omega^* 1_N$, achieved by the network in the hypothesis of frequency synchronisation depends on the number of nodes N, the coupling strength c, the adjacency matrix A, the oscillation frequencies ω_i and the phase lags β_{ij} as expressed later in Eq.* (14)*.*

Proof. It is possible to rewrite Eq. (1) as:

$$\begin{aligned}\dot{\theta}_i(t) = \omega_i &+ \frac{c}{N}\sum_{j=1}^{N} a_{ij}\sin\left(\theta_j(t) - \theta_i(t)\right)\cos\beta_{ij}\\ &- \frac{c}{N}\sum_{j=1}^{N} a_{ij}\cos\left(\theta_j(t) - \theta_i(t)\right)\sin\beta_{ij}\end{aligned} \tag{6}$$

In the phase-locking regime, it holds that $r(t) \simeq 1$, where:

$$r(t)e^{j\psi(t)} = \frac{1}{N}\sum_{k=1}^{N} e^{j\theta_k(t)} \quad (7)$$

is the standard Kuramoto order parameter (note that $r(t)=1$ can only be achieved in the ideal case of phase synchronisation for a network of homogeneous Kuramoto oscillators). For sufficiently high values of the coupling strength c, the oscillators form a tight cluster such that $|\theta_i(t)-\theta_j(t)| \ll 1$ [31], [33], thus leading to:

$$\dot{\theta}_i(t) \simeq \omega_i + \frac{c}{N}\sum_{j=1}^{N} a_{ij}\left[(\theta_j(t)-\theta_i(t))\cos\beta_{ij} - \sin\beta_{ij}\right] \quad (8)$$

which yields:

$$\dot{\theta}_i(t) \simeq \omega_i - \frac{c}{N}\sum_{j=1}^{N} a_{ij}\sin\beta_{ij} + \frac{c}{N}\sum_{j=1}^{N} a_{ij}\cos\beta_{ij}\left(\theta_j(t)-\theta_i(t)\right) \quad (9)$$

Let us define the following quantities:

- delayed adjacency matrix $\hat{A} := \{\hat{a}_{ij} = a_{ij}\cos\beta_{ij}\}$;
- delayed degree matrix $\hat{D} := \{\hat{\delta}_i = \sum_{j=1}^{N}\hat{a}_{ij}\}$;
- delayed Laplacian matrix $\hat{L} := \hat{D} - \hat{A}$;
- delayed oscillation frequency $\hat{\omega}_i := \omega_i - \frac{c}{N}\sum_{j=1}^{N} a_{ij}\sin\beta_{ij}$.

Thanks to the previous definitions, the oscillator system described in Eq. (9) can be rewritten as:

$$\dot{\theta}_i(t) \simeq \hat{\omega}_i + \frac{c}{N}\sum_{j=1}^{N}\hat{a}_{ij}\ \left(\theta_j(t)-\theta_i(t)\right) \quad (10)$$

or equivalently:

$$\dot{\theta}(t) \simeq \hat{\omega} - \frac{c}{N}\hat{L}\theta(t) \quad (11)$$

where $\hat{\omega} := [\hat{\omega}_1\ \hat{\omega}_2\ \ldots\ \hat{\omega}_N]^T$.

In the phase-locking regime, all the oscillators rotate with the same value of angular velocity ω^*. Therefore, it is possible to study the dynamics of the system in a frame rotating with such value of angular velocity, where we define $\tilde{\theta}_i := \theta_i^*(t) - \omega^* t$ and $\tilde{\theta} := [\tilde{\theta}_1\ \tilde{\theta}_2\ \ldots\ \tilde{\theta}_N]^T$. In such a frame all the oscillators hold still, that is $\frac{d\tilde{\theta}}{dt} = 0_N$.

Consequently, from Eq. (11) we obtain:

$$\tilde{\theta} = \frac{N}{c}\hat{L}^{\dagger}\hat{\omega} \quad (12)$$

where $\hat{L}^{\dagger} = V\Sigma^{\dagger}Z^T = \sum_{j=2}^{N}\frac{v^j z^{jT}}{\sigma_j}$ [14], which in the original static frame yields:

$$\theta^*(t) = \frac{N}{c}\hat{L}^{\dagger}\hat{\omega} + \omega^* t\ 1_N \quad (13)$$

with $Z = [z^1\ z^2\ \ldots\ z^N]$ and $V = [v^1\ v^2\ \ldots\ v^N]$ being matrices whose columns are defined by the left and right eigenvectors of L, and with $\Sigma = diag(\sigma_i) : 0 = \sigma_1 < \sigma_2 \leq \ldots \leq \sigma_N$ being a diagonal matrix whose entries are the singular values of L.

Supposing that $-\frac{\pi}{2} < \beta_{ij} < \frac{\pi}{2}\ \forall i,j$, the topological structure described by $\hat{A}$ is the same as that described by $A := \{a_{ij}\}$. Therefore, assuming that A defines a strongly connected topology, so does $\hat{A}$. In such scenario, denoting with $\hat{\Omega}$ the average value of the delayed oscillation frequencies, the value ω^* towards which all the angular velocities $\dot{\theta}_i$ converge is given by:

$$\omega^* = \hat{\Omega} + \frac{z^{1T}\cdot[\hat{\omega} - \hat{\Omega}1_N]}{z^{1T}\cdot 1_N} \quad (14)$$

where $\hat{L} = Z\Sigma V^T = \sum_{j=2}^{N}\sigma_j z^j v^{jT}$ [41].

□

Note that frequency synchronisation of the network described in Eq. (1), or equivalently phase-locking, might not be stable. It is possible to shed light on this issue by analysing the sign of the real part of the eigenvalues of $J(\theta^*(t))$, where $J(\theta) := \{J_{ij} = \frac{\partial\dot{\theta}_i}{\partial\theta_j}\}$ is the *Jacobian matrix* of the system:

$$J_{ij}(\theta^*(t)) = \begin{cases} -\frac{c}{N}\sum_{k\neq i} a_{ik}\cos(\tilde{\theta}_k - \tilde{\theta}_i - \beta_{ik}), & i=j \\ \frac{c}{N}\ a_{ij}\cos(\tilde{\theta}_j - \tilde{\theta}_i - \beta_{ij}), & i\neq j \end{cases} \quad (15)$$

According to the Gershgorin circle theorem [42], if one or more off-diagonal entries are negative, the Jacobian matrix described in Eq. (15) might admit the presence of positive eigenvalues [33], thus leading to instability of the phase-locking regime.

B. Closed-loop network

In the following theorem we investigate the effects of adding an external contribution u_i on a subset of the nodes in the network.

Theorem 2. *Consider a network of nonuniform Kuramoto oscillators controlled by a pacemaker, as that described in Eq. (3). Frequency synchronisation is stabilised if $k_i > K_i$, where*

$$K_i := \frac{c}{N}\left[\min\left\{\sum_{j\neq i}|R_{ij}|, \sum_{i\neq j}|R_{ij}|\right\} - \sum_{j\neq i}R_{ij}\right] \quad (16)$$

with $R_{ij} := a_{ij}\cos\left(\tilde{\theta}_j - \tilde{\theta}_i - \beta_{ij}\right)$.

Proof. The Jacobian matrix for the network described in Eq. (3), evaluated in the phase-locking solution $\theta^*(t)$, is given by:

$$J_{ij}(\theta^*(t)) = \begin{cases} -\frac{c}{N}\sum_{k\neq i} a_{ik}\cos(\tilde{\theta}_k - \tilde{\theta}_i - \beta_{ik}) - k_i, & i=j \\ \frac{c}{N}\ a_{ij}\cos(\tilde{\theta}_j - \tilde{\theta}_i - \beta_{ij}), & i\neq j \end{cases} \quad (17)$$

so that possible eigenvalues with positive real part can be shifted onto the complex left half plane by appropriately acting on the control gains k_i.

According to the Gershgorin circle theorem [42], $J_{ij}(\theta^*(t))$ in Eq. (17) has all eigenvalues with negative real part if and only if $k_i > K_i$, where K_i is defined in Eq. (16). This guarantees that $\theta^*(t)$ is a stable solution for Eq. (3),

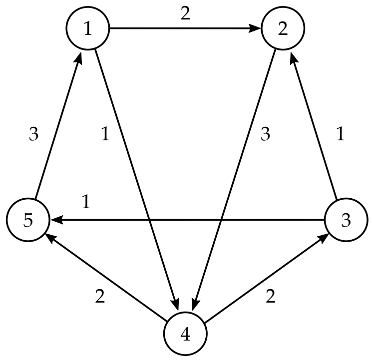

Fig. 1. Strongly connected weighted topology of $N = 5$ nonuniform Kuramoto oscillators. An arrow going from node j to node i is representative of the fact that $a_{ij} > 0$. The specific value of the interconnection strength is represented on the edge itself.

hence phase-locking is achieved and all the nodes oscillate with same angular velocity ω^* defined in Eq. (14).

□

Remark 3. *It is possible to identify the driver nodes by studying the sign of K_i. Indeed:*

- $K_i \leq 0 \Rightarrow k_i = 0$*, hence the ith node does not need to be pinned (i.e., the ith eigenvalue of the Jacobian of the system without pacemaker has negative real part);*
- $K_i > 0 \Rightarrow k_i > 0$*, hence the ith node needs to be pinned (i.e., the ith eigenvalue of the Jacobian of the system without pacemaker has strictly positive real part, hence it needs to be shifted onto the complex left half plane).*

Remark 4. *Our approach extends previous investigations on control of networks of Kuramoto oscillators. Indeed, we here consider directed and weighted networks of heterogeneous oscillators affected by inhomogeneous phase lags, as opposed to the study in [33] where undirected and unweighted networks are considered, and to that in [34] where the case of directed and weighted networks is studied in the absence of phase delays.*

IV. Validation

In this section we confirm numerically the analytical results obtained in Section III. We consider a strongly connected weighted network of $N = 5$ nonuniform Kuramoto oscillators, with natural oscillation frequencies $\omega = [0.2\ 0.4\ 0.6\ 0.8\ 1]^T$ rad s^{-1} (Fig. 1). We set the coupling strength to $c = 0.5$, and the phase delay between connected nodes to $\beta_{ij} = \frac{\beta}{a_{ij}}$ with $\beta = \frac{\pi}{4}$ rad, as done in [38].

In the absence of a pacemaker, the oscillators do not reach synchronisation, as shown by the trajectories of their phases θ_i [Fig. 2(a)], their angular velocity $\dot{\theta}_i$ [Fig. 2(b)], and by the Kuramoto order parameter r [Fig. 2(c)]. It is therefore necessary to build matrix $R := \{R_{ij}\}$ and pin some of the nodes in the network accordingly:

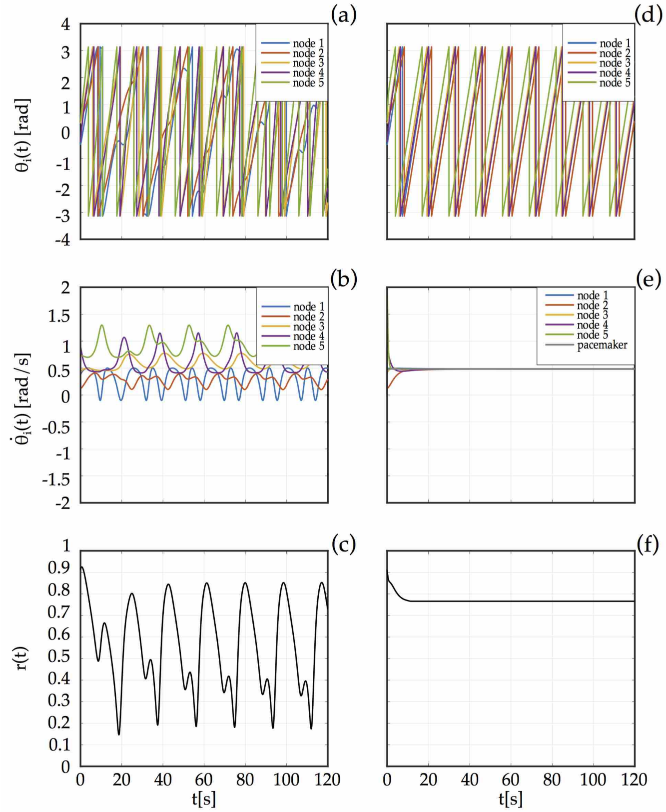

Fig. 2. Simulation results for a strongly connected weighted network of $N = 5$ nonuniform Kuramoto oscillators. The left column refers to simulation results obtained in the absence of a pacemaker, whereas the right column refers to those obtained when pinning the 5th node. Different colours refer to different nodes. Phase trajectories θ_i (a,d), angular velocities $\dot{\theta}_i$ (b,e) and Kuramoto order parameter r (c,f) are shown for both scenarios, respectively.

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & 1.8755 \\ 1.6155 & 0 & 0.8491 & 0 & 0 \\ 0 & 0 & 0 & 1.0142 & 0 \\ 0.8970 & 1.7266 & 0 & 0 & 0 \\ 0 & 0 & -0.0530 & -0.6037 & 0 \end{bmatrix}$$

As $K_5 \simeq 0.13$ while $K_1 = 0$, $K_2 \simeq -0.07$, $K_3 \simeq -0.01$, $K_4 \simeq -0.1$, only the 5th node is pinned ($k_5 = 20K_5 \simeq 2.6$), which leads to $\tilde{\theta} \simeq [-0.02\ \ -1.05\ \ 0.29\ \ -0.35\ \ 1.13]^T$ in the rotating frame, and hence phase-locking. Indeed, the phase difference between any pair of nodes keeps constant [Fig. 2(d)], the angular velocities of all the oscillators converge to that of the pacemaker [$\omega^* \simeq 0.48$ rad s^{-1}, Fig. 2(e)], and the Kuramoto order parameter converges to a constant value of $r \simeq 0.77$ [Fig. 2(f)].

V. A Possible Application to Enhancing Joint Task Coordination in Human Groups

Motor coordination and synchronisation is an essential feature of many human activities, where a group of individuals perform a joint task. Examples include hands clapping in an audience [43], walking in a crowd [44], [45], music playing [46], [47], sports [48], [49] or dance [50],

[51], [52]. Achieving synchronisation in the group involves perceptual interaction through sound, feel, or sight, and the establishment of mental connectedness and social attachment among group members [53], [54]. In [59], networks of heterogeneous Kuramoto oscillators were proposed as a good model to describe the emergence of coordination in the paradigmatic example where a group of human players are asked to generate an oscillatory hand motion and coordinate it with that of the others (a multiplayer version of the so-called Mirror game, a paradigmatic joint task often studied in the literature).

It was envisaged that the availability of a mathematical description of the players' dynamics can be instrumental for designing better control architectures driving virtual agents (e.g., robots, computer avatars) to coordinate their motion within groups of humans [55], [56], [57], [58], as well as for predicting the coupling strength needed to restore synchronisation based on initial knowledge of individual consistency, group variance and topology.

We wish to emphasize that the work presented in this paper can be instrumental for the design of control architectures able to drive virtual players in a group to monitor and "influence" the motion of some of the human players with the aim of enhancing the emergence of synchronous coordinated behaviour. This can allow, for example, new strategies for the diagnosis and rehabilitation of patients affected by social disorders as recently proposed in [60]. A platform for the deployment and testing of these virtual players in human groups performing a joint oscillatory task was recently presented in [61] and can be instrumental to test experimentally the strategies derived in this paper. This is the subject of ongoing research. (It is expected that some preliminary results will be included here in the final version of this paper.)

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