

A Generalized Discrete-Time Altafini Model

L. Wang¹, J. Liu², A. S. Morse¹, B. D. O. Anderson³, and D. Fullmer¹

Abstract—A discrete-time modulus consensus model is considered in which the interactions between the members of a networked family of n agents is described by a time-dependent gain graph whose vertices correspond to agents and whose arcs are assigned complex numbers from a prescribed cyclic group. Limiting behavior of the model's state is studied using a graphical approach. It is shown that a certain type of clustering of agents' "opinions" or states will be reached exponentially fast for almost all initial conditions if and only if the sequence of gain graphs is "repeatedly jointly structurally balanced" corresponding to the type of clustering being reached, where the number of clusters is at most the order of the prescribed cyclic group. It is also shown that the agents' states will all converge to zero asymptotically if the sequence of gain graphs is repeatedly jointly strongly connected and structurally unbalanced. In the special case when the cyclic group is of order two, the model simplifies to the so-called Altafini model whose gain graph is simply a signed graph.

I. INTRODUCTION

With the rapid expansion of online social services, there has been an increasing interest in understanding how individuals' opinions and behaviors evolve over time in a social network [1]. Opinion dynamics has a long history in the social sciences [2]. Probably the simplest and most well-known model of opinion dynamics is the classical DeGroot model which originated in statistics [3]. The DeGroot model deals with a time-invariant connected network of individuals, each of which updates his or her *opinion* {i.e., scalar-valued state} by taking a convex combination of the opinions of his or her neighbors at each discrete time step. The model is also called a consensus model and has attracted considerable attention in the systems and control community [4]–[6] with a focus on time-varying networks. It is well known that under appropriate joint connectivity assumptions, the DeGroot model with time-varying neighbor relationships causes all individuals' opinions to reach common value.

In a recent paper [7] the so-called Altafini model incorporates in the DeGroot model a binary social relationship among individuals. Specifically, the Altafini model uses a signed, directed graph to depict the neighbor relationships among the individuals; the graph's vertices correspond to

individuals while its arcs describe directions of information flow. Each (directed) arc is labeled with either a positive or negative sign where positive signs indicate friendly or cooperative relationships and negative signs indicate antagonistic or competitive relationships. The continuous-time Altafini model has been considered in [7]–[9] and its discrete-time counterpart has been studied in [8]–[11]. For the discrete-time Altafini model over time-varying signed directed graphs, it was shown in [10] that for any "repeated jointly strongly connected" sequence of graphs, the absolute values of all individuals' opinions will asymptotically reach a consensus, which has consensus and two-clustering as special cases. Necessary and sufficient conditions for exponential convergence with respect to each possible type of limit state were established in [9] in terms of structural balance/unbalance, a concept from social sciences [12].

The Altafini model is restricted to two clusters. In a realistic social network, multiple clusters of opinions occur from time to time. Consider for example, the current dialogue about what kinds of regulations should be introduced in the United States to limit public the access to guns. There is ample motivation to generalize or modify the Altafini model to a model capable of achieving multiple clusters. In [13], a generalization of the continuous-time Altafini model was proposed by allowing the gains of a time-invariant neighbor graph to be the elements of a finite group, with the order of the group determining the largest possible number of clusters. Another generalization was introduced in [14] which allowed the weights to be arbitrary complex numbers. It was shown in [14] that when the complex-weighted neighbor graph is fixed and strongly connected, either all individuals' opinions converge to zero, or the magnitudes of the opinions reach a common value, which is called a modulus consensus; the paper also considers a special discrete-time model. A discrete-time counterpart of the model in [14] was studied in [15] which utilizes time-varying graphs and establishes sufficient conditions for exponential convergence. It is worth emphasizing that both the models in [14] and [15] require nontrivial matrix analysis to determine the maximum possible number of clusters.

This paper considers a generalized discrete-time Altafini model defined over time-varying directed graphs, in which the gains are complex numbers from a cyclic group whose order determines the maximum possible number of clusters. Although the model is a special case of the model in [15], such a setting enables one analyze the model using a graphical approach and establish a necessary and sufficient condition for exponentially fast nonzero modulus consensus, whereas only a sufficient condition was provided in [15].

This work was supported by NSF grant 1607101.00, AFOSR grant FA9550-16-1-0290, and ARO grant W911NF-17-1-0499

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A sufficient condition for asymptotic consensus at zero is also provided. It turns out that the cyclic group composed of complex numbers is a special case of the group considered in [13]. This paper focus on the cyclic group for simplicity. It appears likely that the results derived here can be generalized to any point group; this will be addressed in future work.

It should be mentioned that complex-weighted graphs and the associated complex-valued adjacency matrices find applications in formation control [16], [17] and localization problems [18]. The work in this paper is also related to “group consensus” [19] and “cluster synchronization” [20]–[22].

II. PROBLEM FORMULATION

We are interested in a network of n agents labeled $1, 2, \dots, n$ which are able to receive information from their neighbors where by the neighbor of agent i is meant any other agent in agent i 's reception range. We write $\mathcal{N}_i(t)$ for the set of labels of agent i 's neighbors at discrete time $t = 0, 1, 2, \dots$ and we take agent i to be a neighbor of itself. A directed graph \mathbb{G} with n vertices labeled $1, 2, \dots, n$ is a *gain graph* if each arc (k, i) is assigned a *gain* g_{ik} where g_{ik} is a complex number from the cyclic group $\mathcal{G} = \{e^{\frac{2\pi(l-1)}{m}j} : l \in \mathbf{m}\}$; here m is a prescribed positive integer greater than 1 and $\mathbf{m} = \{1, 2, \dots, m\}$. We say that \mathbb{G} is a gain graph associated with the gain set \mathcal{G} . The simplest example of a gain graph is when $m = 2$ in which case the set of possible gains is $\{1, -1\}$ and \mathbb{G} is typically called a *signed graph* [7]. One interpretation on a signed graph is that agent i is a friend of agent k if arc (k, i) is assigned a 1, or a foe of agent k if arc (k, i) is assigned a -1 . It is more difficult to assign meaning to a gain graph if $m > 2$. Nonetheless such graphs have found applications in network flow theory, geometry, and physics [23]. Neighbor relations at time t are characterized by a gain graph $\mathbb{N}(t)$ associated with the gain set \mathcal{G} with n vertices, and a set of arcs defined so that there is an arc from vertex k to vertex i whenever agent k is a neighbor of agent i . It is natural to assume that each self-arc in $\mathbb{N}(t)$ is assigned with a gain “1”.

Each agent i in the network has a complex-valued state or opinion $x_i(t)$ which it updates using a discrete-time rule

$$x_i(t+1) = \frac{1}{m_i(t)} \sum_{j \in \mathcal{N}_i(t)} g_{ij}(t) x_j(t), \quad t \geq 0 \quad (1)$$

where $m_i(t)$ is the number of neighbors of agent i at time t , and $g_{ij}(t) \in \mathcal{G}$ is the gain assigned to the arc (j, i) .

The n scalar update equations of the form in (1) can be written as the single linear recursion equation

$$x(t+1) = G(t)x(t), \quad t \geq 0 \quad (2)$$

where each $x(t)$ for $t \geq 0$ is a vector in \mathbb{C}^n whose i th entry is $x_i(t)$, and $G(t)$ is an $n \times n$ matrix whose ij th entry is $\frac{1}{m_i(t)} g_{ij}(t)$ if $j \in \mathcal{N}_i(t)$, or 0 if $j \notin \mathcal{N}_i(t)$. We call $G(t)$ the *gain matrix* of the gain graph $\mathbb{N}(t)$. By the *flocking matrix* of $\mathbb{N}(t)$, written $F(t)$, is meant that $n \times n$ real-valued matrix whose ij -th entry $f_{ij}(t)$ is $\frac{1}{m_i(t)}$ if $j \in \mathcal{N}_i(t)$, or 0 if

$j \notin \mathcal{N}_i(t)$. It is easy to see that $|G(t)| = F(t)$ where $|G(t)|$ is the $n \times n$ matrix which results when each entry of $G(t)$ is replaced by its modulus.

We are interested in the convergence of the state $x(t)$ of system (2) as $t \rightarrow \infty$. System (2) achieves *modulus consensus* if

$$\lim_{t \rightarrow \infty} |x_i(t)| = \lim_{t \rightarrow \infty} |x_j(t)|, \quad \forall i, j \in \mathbf{n}$$

where $\mathbf{n} = \{1, 2, \dots, n\}$. Moreover, system (2) achieves *m-modulus consensus* if it achieves modulus consensus and \mathbf{n} can be partitioned into at most m subsets such that for any $i, j \in \mathbf{n}$, $\lim_{t \rightarrow \infty} x_i(t) = \lim_{t \rightarrow \infty} x_j(t)$ if i and j are in the same subset, and $\lim_{t \rightarrow \infty} x_i(t) \neq \lim_{t \rightarrow \infty} x_j(t)$ if i and j are in different subsets.

The problem of interest is to derive necessary and sufficient graphical conditions on a sequence of $\mathbb{N}(t)$ under which system (2) will achieve *m-modulus consensus* exponentially fast.

III. MAIN RESULTS

In this section, we first introduce some definitions and notation. Then main results of the paper are given.

Given a gain graph \mathbb{G} associated with gain set \mathcal{G} , a *walk* in \mathbb{G} is a sequence of vertices connected by arcs corresponding to the order of the vertices in the sequence. The *gain along a walk* of a gain graph is the product of gains assigned to arcs in the walk. A *semi-walk* in \mathbb{G} is a sequence of vertices connected by arcs in which the arc directions are ignored. The *gain along a semi-walk* of a gain graph is the value of the product of gains assigned to arcs whose directions are consistent with the order of the vertices in the semi-walk multiplying the product of the inverse of gains assigned to arcs whose directions are consistent with the reverse order of the vertices in the semi-walk. A walk is called a *path* if there is no repetition of vertices in the walk. A walk is *closed* if it has the same starting vertex and ending vertex. A walk is a *cycle* if it is closed and there is no repetition of vertices in the walk except the starting and ending vertices. A semi-walk is called a *semi-path* if there is no repetition of vertices in the semi-walk. A semi-walk is *closed* if it has the same starting vertex and ending vertex. A semi-walk is a *semi-cycle* if it is closed and there is no repetition of vertices in the semi-walk except the starting and ending vertices. The gain graph \mathbb{G} associated with the gain set \mathcal{G} , is said to be *structurally m-balanced* if all semi-walks joining the same ordered pair of vertices in \mathbb{G} have the same gain. Otherwise, the gain graph \mathbb{G} is *structurally unbalanced*. The following lemma from [13] can be used to check whether a gain graph associated with the gain set \mathcal{G} is structurally *m-balanced* or unbalanced.

Lemma 1: A gain graph \mathbb{G} associated with the gain set \mathcal{G} is *structurally m-balanced* if and only if all the semi-cycles of \mathbb{G} have gain 1. If, in addition, \mathbb{G} is strongly connected, \mathbb{G} is *structurally m-balanced* if and only if all the cycles of \mathbb{G} have gain 1.

More can be said. By a *clustering vector* is meant any vector $b \in \mathbb{C}^n$ whose first element b_1 satisfies $b_1 = 1$ and

i -th element $b_i \in \mathcal{G}$ for $i \in \{2, 3, \dots, n\}$; \mathcal{I} is the set of all clustering vectors in \mathbb{C}^n . Let \mathbb{G} be an n -vertex gain graph with associated with the gain set \mathcal{G} . Corresponding to each given clustering vector b , there is a decomposition of vertex set \mathbf{n} of \mathbb{G} into m disjoint subsets V_1, V_2, \dots, V_m such that $\cup_{i=1}^m V_i = \mathbf{n}$. In particular, $1 \in V_1$ and for vertex $i > 1$, $i \in V_p$ where p is that integer for which $b_i = e^{\frac{2\pi(p-1)}{m}j}$. If all semi-walks from vertex $i \in V_p$ to vertex $k \in V_q$ have the same gain $e^{\frac{2\pi(p-q)}{m}j}$ for $p \geq q \in \mathbf{m}$, the gain graph \mathbb{G} associated with \mathcal{G} is said to be *structurally m -balanced with respect to the clustering vector b* . Otherwise we say \mathbb{G} is *not structurally m -balanced with respect to the clustering vector b* . Later in Section IV, we will show that any gain graph \mathbb{G} is structurally m -balanced if and only if there exist a clustering vector b such that \mathbb{G} is structurally m -balanced with respect to the vector b .

A finite sequence of directed graphs $\mathbb{G}(1), \mathbb{G}(2), \dots, \mathbb{G}(p)$ with the same vertex set is *jointly strongly connected* if the union¹ of the directed graphs in this sequence is strongly connected. Meanwhile, an infinite sequence of directed graphs $\mathbb{G}(1), \mathbb{G}(2), \dots$ with the same vertex set is *repeatedly jointly strongly connected* if there exist positive integers p and q such that each the finite sequence $\mathbb{G}(q + kp), \mathbb{G}(q + kp + 1), \dots, \mathbb{G}(q + kp + p - 1)$ for all $k \geq 0$ is jointly strongly connected. A finite sequence of directed gain graphs $\mathbb{G}(1), \mathbb{G}(2), \dots, \mathbb{G}(p)$ with the same vertex set and same gain set \mathcal{G} is *jointly structurally m -balanced* with respect to a given clustering vector $b \in \mathcal{I}$ if the union² of the gain graphs in this sequence is structurally m -balanced with respect to the vector b . If there is no $b \in \mathcal{I}$ such that the union of the digraphs in this sequence is structurally m -balanced with respect to, this sequence of graphs is *jointly structurally unbalanced*. Meanwhile, an infinite sequence of directed gain graphs $\mathbb{G}(1), \mathbb{G}(2), \dots$ with the same vertex set and same gain set \mathcal{G} is *repeatedly jointly structurally m -balanced* with respect to a clustering vector $b \in \mathcal{I}$ (or *repeatedly jointly structurally unbalanced*) if there exist positive integers p and q such that each finite sequence $\mathbb{G}(q + kp), \mathbb{G}(q + kp + 1), \dots, \mathbb{G}(q + kp + p - 1)$ is structurally m -balanced with respect to b (or jointly structurally unbalanced) for all $k \geq 0$. It is worth emphasizing that the converse of repeatedly jointly structurally m -balanced is not repeatedly jointly structurally unbalanced.

The main results of this paper are as follows.

Theorem 1: Suppose that the sequence of neighbor graphs $\mathbb{N}(0), \mathbb{N}(1), \mathbb{N}(2), \dots$ with the same gain set \mathcal{G} is repeatedly jointly strongly connected. The system described by (2) reaches an m -modulus consensus corresponding to $b \in \mathcal{I}$ exponentially fast for almost all initial conditions if and

¹The union of a finite sequence of directed graphs with the same vertex set is a directed graph with the same vertex set and the arc set which is the union of the arc sets of all directed graphs in the sequence.

²The union of a finite sequence of directed gain graphs with the same vertex set is a directed multi-graph with the same vertex set and an arc set consisting of the union of all the arc sets of the gain graphs comprising the union; it is understood that in such a multi-graph, termed a *gain multi-graph*, the multiple arcs from a given vertex i to another given vertex j have distinct gains.

only if the graph sequence $\mathbb{N}(0), \mathbb{N}(1), \mathbb{N}(2), \dots$ is repeatedly jointly structurally m -balanced with respect to the clustering vector b .

Theorem 2: Suppose that the sequence of neighbor graphs $\mathbb{N}(0), \mathbb{N}(1), \mathbb{N}(2), \dots$ with the same gain set \mathcal{G} is repeatedly jointly strongly connected. For each initial state, the state x of the system (2) converges to zero asymptotically if the graph sequence $\mathbb{N}(0), \mathbb{N}(1), \mathbb{N}(2), \dots$ is repeatedly jointly structurally unbalanced.

Both theorems are proved in the next section.

IV. ANALYSIS

In this section, we first give a result on the graph structurally m -balanceness. Analysis on Theorem 1 and Theorem 2 will be provided as well.

Proposition 1: Let \mathbb{G} be a gain graph with n vertices labeled $1, 2, \dots, n$ and associated with gain set \mathcal{G} . \mathbb{G} is structurally m -balanced if and only if there exist a clustering vector b such that \mathbb{G} is structurally m -balanced with respect to the vector b .

The proof of Proposition 1 is omitted due to space limitations and will be given in the full length version of this paper.

When $m = 2$, the model becomes the Altafini model which has been well studied in [7], [9]–[11]. As defined in [7], a graph \mathbb{G} with the gain set $\{1, -1\}$ is structurally 2-balanced if the vertices of \mathbb{G} can be partitioned into two sets such that each arc connecting two agents in the same set has a positive gain and each arc connecting two agents in different sets has a negative gain. This definition also satisfies Proposition 1 for which we can say the graph is structurally 2-balanced with respect to a clustering vector composed of 1 and -1 .

In the following, we are going to show each element b in \mathcal{I} uniquely defines a clustering pattern of all the agents in the connected network by the gains of the entries of b . That is if two entries say $b(i)$ and $b(j)$ of vector b have the same gain, agents i and j are in the same clustering. For a structurally m -balanced graph \mathbb{G} with gain set \mathcal{G} , the agents in the same vertex set V_i for $i \in \mathbf{m}$ will converge to the same value.

Define a time-dependent mn -dimensional vector $z(t)$ such that for each time t ,

$$z(t) = \begin{bmatrix} \alpha_0 x(t) \\ \alpha_1 x(t) \\ \vdots \\ \alpha_{m-1} x(t) \end{bmatrix}$$

where $\alpha_i = e^{\frac{2\pi i}{m}t}$ for $i \in \mathbf{m} = \{0, 1, 2, \dots, m-1\}$.

Then for all $i \in \{1, 2, \dots, mn\}$,

$$z_i(t+1) = \sum_{j=1}^{mn} \bar{a}_{ij}(t) z_j(t)$$

in which if $g_{ij}(t) = 1$ for $p \in \mathbf{m}$ and $i, j \in \mathbf{n}$

$$\bar{a}_{i+pn, j+pn} = \max\{f_{ij}(t), 0\},$$

if $g_{ij}(t) = \alpha_q$ for a fixed $q \in \underline{m}$, for each $p \in \underline{m}$, and $i, j \in \mathbf{n}$,

$$\bar{a}_{i+pn, j+((p+q) \bmod m)n} = \max\{f_{ij}(t), 0\}$$

where $(p+q) \bmod m$ is the remainder of $p+q$ divided by m . It is obvious that the expanded system is equivalent to system (2). The system can be written in the form of a state equation

$$z(t+1) = \bar{G}(t)z(t) \quad (3)$$

where $\bar{G}(t) = [\bar{a}_{ij}(t)]$ is an $mn \times mn$ stochastic matrix. With this fact, the graph of $\bar{G}(t)$ is a directed graph with mn vertices. It is not difficult to see that $\bar{G}(t)$ can be seen as an $n \times n$ block circulant matrix with blocks of size $m \times m$ where each row block vector is rotated one block to the right relative to the preceding row block vector. Let $\bar{N}(t)$ be the graph of $\bar{G}(t)$. $\bar{N}(t)$ has the following properties.

Lemma 2: For any $i, j \in \mathbf{n}$, if $g_{ij}(t) = \alpha_q$ where $q \in \underline{m}$, $\bar{N}(t)$ has an arc from vertex $j + ((p+q) \bmod m)n$ to vertex $i + pn$ for $p \in \underline{m}$. In particular, $\bar{N}(t)$ has self-arcs at all mn vertices.

Lemma 3: Suppose that $N(t)$ has a directed path from vertex i to vertex j with $i, j \in \mathbf{n}$. Then $\bar{N}(t)$ has a directed path from vertex i to vertex $j + ((m-q) \bmod m)n$ if the directed path from i to j in $N(t)$ has a gain α_q for $q \in \underline{m}$.

Lemma 4: For a fixed $q \in \underline{m}$, $\bar{N}(t)$ has a directed path from vertex $i + pn$ to vertex $j + ((m-q+p) \bmod m)n$ with $i, j \in \mathbf{n}$ $p \in \underline{m}$, if and only if it has a directed path from vertex i to vertex $j + ((m-q) \bmod m)n$.

Consider the following example in which self-arcs are omitted for simplicity. Fig. 1 is the graph of G , which is a three vertex graph associated with the gain set $\{1, \alpha_1, \alpha_2\}$ where $\alpha_1 = e^{\frac{2\pi}{3}j}$ and $\alpha_2 = e^{\frac{4\pi}{3}j}$. Correspondingly, the matrix G for system (2) associated with Fig. 1 is

$$G = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2}\alpha_1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2}\alpha_2 & \frac{1}{2} \end{bmatrix}$$

The matrix \bar{G} for system (3) is

$$\bar{G} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

It is easy to see that \bar{G} is a 3×3 block circulate matrix with blocks of size 3×3 . The graph \bar{N} of \bar{G} is shown in Fig. 2.

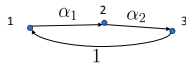


Fig. 1. Graph N associated with the sign set $\{1, \alpha_1, \alpha_2\}$

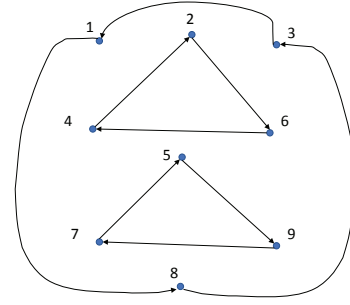


Fig. 2. Graph \bar{N}

Proposition 2: Suppose that an n -vertex gain graph of $G(t)$ associated with the gain set \mathcal{G} is strongly connected and structurally m -balanced with respect to a clustering vector $b \in \mathcal{I}$. Then, the graph of $\bar{G}(t)$ consists of m disjoint strongly connected components of the same size, n .

Proof: Since the graph of $G(t)$, $N(t)$, is structurally m -balanced with respect to a clustering vector $b \in \mathcal{I}$, according to Proposition 1, there exist m disjoint vertex sets V_1, V_2, \dots, V_m such that $\bigcup_{p=1}^m V_p = \mathbf{n}$ and for any vertex $i \in \mathbf{n}$, $i \in V_p$ if $b(i) = \alpha_{p-1}$ for any $p \in \underline{m}$. For the m disjoint vertex sets V_1, V_2, \dots, V_m , $1 \in V_1$, and all semi-walks from vertex $i \in V_q$ to $j \in V_p$ have the same gain α_{p-q} for any $p \geq q \in \underline{m}$. Let V_{pq} for any $p, q \in \underline{m}$ be a vertex set such that

$$V_{pq} = \{(p-1)n + i | \forall i \in V_q\}. \quad (4)$$

We get that V_{pq} are disjoint for different p, q , and $\bigcup_{p=1}^m \bigcup_{q=1}^m V_{pq}$ is the vertex set of \bar{N} . Note $V_{1q} = V_q$ for any $q \in \underline{m}$. Next we are going to show that the following m components for $p \in \underline{m}$

$$\mathcal{C}_p = \{V_{p,1}, V_{p-1,2}, \dots, V_{1,p}, V_{p+1,m}, V_{p+2,m-1}, \dots, V_{m,p+1}\} \quad (5)$$

are disjoint. Moreover, each component is strongly connected and has size n .

Since the size of V_{pq} is same as the size of V_q . The size of \mathcal{C}_p is n . To begin with, we are going to show that any two vertices in \mathcal{C}_p are mutually reachable for $p \in \underline{m}$. Since \bar{G} is an $n \times n$ block circulate matrix, if any two vertices in \mathcal{C}_m are mutually reachable, any two vertices in \mathcal{C}_p for any $p \in \underline{m}$ are mutually reachable. Now look at $\mathcal{C}_m = \{V_{m1}, V_{m-1,2}, \dots, V_{1m}\}$. Arbitrarily choose two nonempty elements of \mathcal{C}_m . Say $V_{m-p+1,p}$ and $V_{m-q+1,q}$ where $1 \leq p < q \leq m$. Due to the definition, in graph N , there is a path from a vertex $i \in V_p$ to a vertex $j \in V_q$ with a gain α_{q-p} . According to Lemma 3, \bar{N} has a directed path from vertex i to vertex $j + (m-q+p)n$. Moreover, according to Lemma 4, \bar{N} has a directed path from a vertex $i + (m-p)n$ which is in $V_{m-p+1,p}$ to a vertex $j + (m-q)n$ which is in $V_{m-q+1,q}$. Since the graph N is strongly connected and structurally m -balanced, if there is a path from a vertex $i \in V_p$ to a vertex $j \in V_q$ with a gain α_{q-p} , there must be a path from the vertex $j \in V_q$ to a vertex $i \in V_p$ with a gain α_{m-q+p} . Similarly, we get the result that \bar{N} has a directed path from a vertex $j + (m-q)n$ which is in $V_{m-q+1,q}$ to a vertex $i + (m-p)n$ which is in $V_{m-p+1,p}$.

That is, any two vertices from $V_{m-p+1,p}$ and $V_{m-q+1,q}$ are mutually reachable. Thus any two vertices in C_p for $p \in \mathbf{m}$ are mutually reachable.

Next, we prove that all the m components are disconnected by contradiction. Suppose there is a path from a vertex in C_p to a vertex in C_q for $p < q$. Arbitrarily choose two vertices i and j from V_1 . Since we have shown that any two vertices in C_p and C_q are mutually reachable respectively, there is a path from a vertex $i + (p-1)n$ in V_{p1} to a vertex $j + (q-1)n$ in V_{q1} . According to Lemma 4, there is a path from a vertex i to vertex $j + (q-p)n$. This means that there is a path from vertex i to vertex j with a gain α_{m-q+p} . But both i and j belong to V_1 which means that the path between these two vertices should have gain 1. A contradiction. Thus all the m components are disconnected. ■

Look at the graph \mathbb{N} in Fig. 1. \mathbb{N} is a strongly connected and structurally 3-balanced graph. Here we can get $V_1 = \{1, 3\}$, $V_2 = \{2\}$, and $V_3 = \emptyset$ where the semi-walks from a vertex in V_1 to a vertex in V_2 have gain α_1 . Correspondingly, we get the expanded graph $\bar{\mathbb{N}}$ as shown in Fig. 2. For graph $\bar{\mathbb{N}}$, we have $V_{11} = \{1, 3\}$, $V_{12} = \{2\}$, $V_{21} = \{4, 6\}$, $V_{22} = \{5\}$, $V_{31} = \{7, 9\}$, $V_{32} = \{8\}$, and $V_{13} = V_{23} = V_{33} = \emptyset$. Three disjoint strongly connected components of size 3 are achieved as shown in Fig 3. The first component consists of vertex 1, 3, and 8. The second component consists of vertex 2, 4, and 6. And the last component consists of vertex 5, 7, and 9.

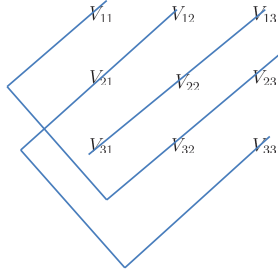


Fig. 3. Vertex sets of strongly connected components of graph $\bar{\mathbb{N}}$

Proposition 3: Suppose that an n -vertex gain graph of $G(t)$ is strongly connected and structurally unbalanced. Then, the graph of $\bar{G}(t)$ consists of at most $\lfloor \frac{m}{2} \rfloor$ disjoint strongly connected components, of at least size $2n$.

Proof: Since the graph of $G(t)$, i.e., $\mathbb{N}(t)$ is strongly connected and structurally unbalanced, for a fixed vertex i and any other vertex j in $\mathbb{N}(t)$, suppose there is a path from i to j with a gain α_p and there is a path from j to i with a gain α_q such that $1 \leq p + q < m$. According to Lemma 3, there is a path from i to $j + (m-p)n$ in graph $\bar{\mathbb{N}}$, and a path from j to $i + (m-q)n$ in $\bar{\mathbb{N}}$. Based on Lemma 4, there is a path from $j + (m-p)n$ to $i + ((2m-p-q) \bmod m)n$ in $\bar{\mathbb{N}}$. Since $p + q < m$, $(2m-p-q) \bmod m \neq i$ that is $i + ((2m-p-q) \bmod m)n$ can not be vertex i . Based on Lemma 4, there is a path from $i + ((2m-p-q) \bmod m)n$

to $j + ((3m-2p-q) \bmod m)n$. Repeat this procedure, eventually there is a path from vertex $j + (((2k-1)m-kp-(k-1)q) \bmod m)n$ to $i + ((2km-kp-kq) \bmod m)n$ in $\bar{\mathbb{N}}$ where k is an integer which is great than 1. There exist k such that $(2km-kp-kq) \bmod m = 0$. The easiest choice is to let $k = m$. It means that there is a cycle starting from vertex i , passing vertex $j + (m-p)n$, vertex $i + ((2m-p-q) \bmod m)n \dots$ and eventually ending with vertex i again. Since j can be any other vertex in $\bar{\mathbb{N}}$, for a component in graph $\bar{\mathbb{N}}$ consisting of agent i , it is strongly connected and the size must be greater than $2n$. ■

Next we are going to prove the main results of this paper.

Proof of Theorem 1: (Sufficiency) Since the sequence of neighbor graphs $\mathbb{N}(1), \mathbb{N}(2), \dots$ is repeatedly jointly strongly connected, and repeatedly jointly structurally m -balanced with respect to the clustering vector b , without loss of generality, suppose there exists positive integers p and q such that each finite sequence of graphs $\mathbb{N}(q+kp), \mathbb{N}(q+kp+1), \dots, \mathbb{N}(q+kp+p-1)$ is jointly strongly connected and jointly structurally m -balanced with respect to the clustering vector b for $k \geq 0$. Let

$$\mathbb{H}(k) = \mathbb{N}(q+kp) \cup \mathbb{N}(q+kp+1) \cup \dots \cup \mathbb{N}(q+kp+p-1)$$

Then $\mathbb{H}(k)$ is strongly connected and structurally m -balanced with respect to the clustering vector b . According to Proposition 1, there exist m disjoint vertex sets V_1, V_2, \dots, V_m such that $\cup_{p=1}^m V_p = V$. Without loss of generality, let $1 \in V_1$, and all walks from vertex $i \in V_q$ to $j \in V_p$ have the same gain α_{p-q} for any $p \geq q \in \mathbf{m}$. Now consider the expanded graph $\bar{\mathbb{H}}(k)$ which is $\bar{\mathbb{H}}(k) = \bar{\mathbb{N}}(q+kp) \cup \bar{\mathbb{N}}(q+kp+1) \cup \dots \cup \bar{\mathbb{N}}(q+kp+p-1)$. According to Proposition 2, graph $\bar{\mathbb{H}}(k)$ consists of m disjoint strongly connected components of the same size n . V_{pq} and C_p are defined the same as Eq. (4) and (5). Each C_p is a strongly connected component of size n . According to the result of discrete-time linear consensus process [24], all the vertices in C_p achieve consensus exponentially fast for almost all initial conditions. From the structural of C_p and system (3), for any fixed $p, q \in \mathbf{m}$, $i \in V_p$, $j \in V_q$,

$$\lim_{t \rightarrow \infty} \alpha_{p-1} x_i(t) = \lim_{t \rightarrow \infty} \alpha_{q-1} x_j(t) \neq 0.$$

That is the same as we say system (1) or (2) achieves m -modulus consensus corresponding to the clustering vector b exponentially fast for almost all initial states.

(Necessity) Prove by contradiction. Suppose the sequence of neighbor graphs $\mathbb{N}(1), \mathbb{N}(2), \dots$ is repeatedly jointly strongly connected, but not repeatedly jointly structurally m -balanced with respect to a clustering vector b . Two scenarios need to be considered.

First, if the sequence of neighbor graphs is repeatedly jointly structurally m -balanced with respect to another clustering vector b_1 , according to sufficiency we just proved system (1) achieves m -modulus consensus corresponding to b_1 not b .

Second, the sequence of the neighbor graphs is not repeatedly jointly structurally m -balanced with respect to any

clustering vector. It means that either structurally unbalanced graphs or more than one class of structurally m -balanced graphs, or both appear infinitely many times. Since the sequence of neighbor graphs $\mathbb{N}(1), \mathbb{N}(2), \dots$ is repeatedly jointly strongly connected, there exist two positive integers q and p such that each finite sequence of graphs $\mathbb{N}(q + kp), \mathbb{N}(q + kp + 1), \dots, \mathbb{N}(q + kp + p - 1)$ is jointly strongly connected for $k \geq 0$. Let

$$\mathbb{F}(k) = \mathbb{N}(q + kp) \cup \mathbb{N}(q + kp + 1) \cup \dots \cup \mathbb{N}(q + kp + p - 1)$$

Then $\mathbb{F}(k)$ is strongly connected. If a gain graph \mathbb{N} is structurally unbalanced, then any finite sequence of gain graphs which contains \mathbb{N} must be jointly structural unbalanced. If two gain graphs \mathbb{N}_1 and \mathbb{N}_2 are structurally m -balanced with respect to two different clustering vectors b_1 and b_2 correspondingly, then any finite sequence of gain graphs which contains \mathbb{N}_1 and \mathbb{N}_2 must be jointly structural unbalanced. Since either structurally unbalanced gain graphs or more than one class of structurally m -balanced graphs, or both appear infinitely many times, the graphs in the sequence $\mathbb{F}(k)$ will be structurally unbalanced for infinitely many times. There must exist two integers n_1, n_2 satisfying $1 \leq n_1 + n_2 < m$ such that the graphs in the sequence $\mathbb{F}(k)$, which has a path from i to j with a gain α_{n_1} and a path from j to i with a gain α_{n_2} , appear infinitely many times. From the proof of Proposition 3, if $\mathbb{F}(k)$ has a path from i to j with a gain α_{n_1} and a path from j to i with a gain α_{n_2} , the union of expanded graph $\bar{\mathbb{F}}(k) = \bar{\mathbb{N}}(q + kp) \cup \bar{\mathbb{N}}(q + kp + 1) \cup \dots \cup \bar{\mathbb{N}}(q + kp + p - 1)$ must have at most $\lfloor \frac{m}{2} \rfloor$ disjoint strongly connected components, of at least size $2n$. Moreover, vertex i , vertex $j + (m - n_1)n$, vertex $i + ((2m - n_1 - n_2) \bmod m)n$ must belong to one strongly component. According to [4], state $z(i)$ and $z(i + ((2m - n_1 - n_2) \bmod m)n)$ would achieve consensus asymptotically which means that $z(i)$ would converge to zero asymptotically fast. Since i is randomly chosen, z converges to zero asymptotically fast. Thus the sequence of the neighbor graphs is repeatedly jointly structurally m -balanced with respect to the clustering vector b . ■

Proof of Theorem 2: Since the sequence of neighbor graphs $\mathbb{N}(1), \mathbb{N}(2), \dots$ with the same gain set \mathcal{G} is repeatedly jointly strongly connected and repeatedly jointly structurally unbalanced, system (2) converges to zero asymptotically fast for almost all initial conditions based the analysis of necessity of theorem 1. ■

V. CONCLUSION

In this paper, a generalized discrete-time Altafini model over time-varying gain graphs, in which the arcs are assigned complex numbers from a cyclic group whose order determines the maximum possible number of clusters, has been studied through a graphical approach. Necessary and sufficient conditions for exponential convergence of the system with respect to nonzero limit states have been established under the assumption of repeatedly jointly strong connectivity. A sufficient condition for asymptotic consensus

at zero has also been provided. The results in this paper can be extended to the case where the gains of the neighbor graph are the elements of a finite abelian group. Necessary and sufficient conditions for exponential convergence at zero of the system will be studied in the future. The time-varying case without the strong connectivity assumption is another direction for future research.

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