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Robust Synchronization Preserving Model Reduction of Lur'e Networks

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***Abstract*— This paper considers a model order reduction problem that reduces the complexity of interconnected Lur'e-type subsystems. The system consists of multiple identical Lur'e subsystems that evolve on a connected undirected graph. A sufficient condition of the robust synchronization of the Lur'e network is first related to the passivity of a linear time-invariant auxiliary system. Consequently, a passivity preserving model reduction scheme is applied to the auxiliary system, leading to reduced-order Lur'e subsystems, and the resulting reduced network system still robustly synchronizes. In addition, an *a prior* error bound between the full-order and reduced-order Lur'e subsystem is provided and the proposed method is illustrated by an example.**

I. Introduction

Network systems (or multi-agent systems) have received compelling attention from the system and control community. A network system is composed of multiple interconnected subsystems, and its behavior is a collection of responses from all individual subsystems and their interactions on a certain communication network. In a Lur'e network, each subsystems are Lur'e type dynamics. When the dynamic order of each subsystem becomes large, the overall network system will be of high complexity, which hinders fast prediction and transient analysis of the network states. Applications, such as controller design and fault diagnosis will also be inefficient. Hence, it is desired to apply some model reduction techniques to generate much smaller-sized models of the Lur'e subsystems that can approximate the input-output relation of the original subsystems. Meanwhile, we require the reduction process to preserve the synchronization property in the network, As network synchronization is an important property of networks.

There are a variety of methods developed for model order reduction. For example, moment matching and proper orthogonal decomposition, as efficient numerical techniques, can be applied to both linear and nonlinear systems [6]–[8]. However, they, in general, do not guarantee the stability of the reduced-order model and the bound on the approximation error. In contrast, balanced truncation and optimal Hankel norm approximation are well-known for the stability preservation and error boundedness, see [9]–[11] for an overview. These methods rely on the controllability and observability energy functionals of the system, i.e., the Gramian matrices. These concepts then have been extended to nonlinear balancing, see [12]–[14] and the references therein. In general, balancing nonlinear systems is expensive, as it requires the solutions of a large-scale nonlinear partial differential equation, namely the Hamilton-Jacobi equation. Moreover, as the other methods for model reduction of nonlinear systems, the truncated model from nonlinear balancing lacks an error bound on the approximation.

For linear network system, balanced truncation has only recently been studied. For example, [15] applies balanced truncation based on generalized Gramians to simplify the network topology. Some pioneering results in [16], [17] present methods to approximate network systems by reducing the complexity of linear subsystems. In this paper, we attempt to adopt balanced truncation to networked nonlinear systems, whose subsystems are identical and cast in Lur'e-type form. In [18], a reduction procedure is presented for absolutely stable Lur'e systems. The method essentially applies balanced truncation to the linear component of a Lur'e system and therefore is computationally cheap. Furthermore, both stability and the error bound for the reduced-order model are guaranteed.

However, in this paper, as the preservation of network synchronization is needed, we can not directly apply standard balanced truncation to each Lur'e subsystem. Instead, we establish a linear time-invariant auxiliary system associating the Lur'e network, and relate the robust synchronization condition to the passivity of this auxiliary system. Consequently, a passivity preserving model reduction on the auxiliary system yields reduced-order Lur'e subsystems and a synchronized reduced network system. Furthermore, the *a priori* bound on the approximation error is established to compare the behaviors of the full-order and reduced-order Lur'e subsystem.

This paper is organized as follows. Section II introduces the balanced truncation method and the model of a Lur'e network. Section III then presents the proposed method for synchronization preserving model reduction of Lur'e networks, and Section IV gives the analysis on the approximation error on each subsystem. The proposed method is illustrated in Section V by an example. Finally, Section VI concludes the paper.

Notation: The symbol $\mathbb{R}$ denotes the set of real numbers. $R(s)$ is the rational function field over R with variable s. I_n represents the identity matrix of size n. A symmetric matrix $A \succ 0$ ($A \prec 0$) means it is positive (negative) definite, while $A \succcurlyeq 0$ ($A \preccurlyeq 0$) means it is positive (negative) semidefinite. The Kronecker product of matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ is denoted by $A \otimes B \in \mathbb{R}^{mp \times nq}$.

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II. PRELIMINARIES

A. Balanced Truncation

Following [19], we recap some basic facts on model reduction by balanced truncation based on generalized Gramians. Consider a stable, linear time-invariant system in a state space representation

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, and $y \in \mathbb{R}^q$. Suppose A is Hurwitz. A pair of positive definite matrices, $P \succ 0$ and $Q \succ 0$, are the *generalized controllability and observability Gramians* of the system in (1), respectively, if they satisfy

$$\begin{aligned} AP + PA^T + BB^T &\preccurlyeq 0, \\ A^TQ + QA + C^TC &\preccurlyeq 0. \end{aligned} \tag{2}$$

Balancing the system in (1) amounts to find a nonsingular state space transformation T such that

$$TPT^{-1} = T^{-T}QT^T = \Theta, \tag{3}$$

with $\Theta := \mathrm{diag}\,(\theta_1, \theta_2, \cdots, \theta_n)$. The diagonal entries $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n > 0$ are called the *generalized Hankel singular values* (GHSVs) of the system. In the balanced realization, the state components corresponding to the smaller GHSVs have less influences on the input-output behavior, and therefore are truncated to yield an approximation. Furthermore, the upper bound of the model reduction error can be measured by the neglected GHSVs. Let the model dimension is reduced from n to r ($r < n$), then

$$\|G(s) - \hat{G}(s)\|_{\mathcal{H}_\infty} \leq 2 \sum_{i=r+1}^{n} \theta_i. \tag{4}$$

where $G(s)$ and $\hat{G}(s)$ denote the transfer matrices of the full-order and reduced-order systems, respectively.

B. Lur'e Networks

In this subsection, we present the mathematical model of a Lur'e network. Consider a weighted graph is defined by a triplet $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$. The sets $\mathcal{V}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ present the sets of nodes and edges, respectively. $\mathcal{W}$ is called *weighted adjacency matrix*. The (i, j)-th entry of $\mathcal{W}$, denoted by w_{ij}, is positive if edge $(i, j) \in \mathcal{E}$, and $w_{ij} = 0$ otherwise. The *Laplacian matrix* of graph $\mathcal{G}$, denoted by L, is then introduced with the (i, j)-th entry as

$$L_{ij} = \begin{cases} \sum_{j=1, j\neq i}^{n} w_{ij}, & i = j \\ -w_{ij}, & \text{otherwise.} \end{cases} \tag{5}$$

Assumption 1: We assume, in this paper, the network is defined on an **undirected connected graph** consisting of N ($N \geq 2$) nodes. In this case, we have $L = L^T \succcurlyeq 0$, and the eigenvalues of L are $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$.

The dynamics of nodes on the network are described by identical nonlinear dynamical systems as follows.

$$\boldsymbol{\Sigma_i} : \begin{cases} \dot{x}_i = Ax_i + Bu_i + Ez_i \\ y_i = Cx_i \\ z_i = -\phi(y_i, t) \end{cases}, i = 1, 2, \cdots, N, \tag{6}$$

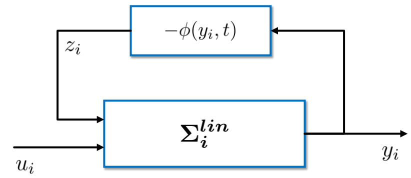

Fig. 1. The illustration of a Lur'e subsystem

where $x_i \in \mathbb{R}^n$, $u_i, y_i, z_i \in \mathbb{R}$. The subsystem $\boldsymbol{\Sigma_i}$ is in the Lur'e-type form, whose structure is illustrated in Fig. 1. Each Lur'e subsystem consists of a linear part $\boldsymbol{\Sigma_i^{lin}}$ formulated by

$$\boldsymbol{\Sigma_i^{lin}} : \begin{cases} \dot{x}_i = Ax_i + \begin{bmatrix} B & E \end{bmatrix} \begin{bmatrix} u_i \\ z_i \end{bmatrix} \\ y_i = Cx_i \end{cases}, i = 1, 2, \cdots, N, \tag{7}$$

and a continuous static output-dependent nonlinearity $\phi(y_i) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$.

Assumption 2: We assume that the uncertain feedback nonlinearity $\phi(\cdot)$ in (6) is **slope-restricted** as

$$0 \leq \frac{\phi(y_a) - \phi(y_b)}{y_a - y_b} \leq \mu, \tag{8}$$

for all $y_a, y_b \in \mathbb{R}$ and $y_a \neq y_b$, where $\mu > 0$ and $\phi(0) = 0$. Furthermore, the Lur'e dynamics $\boldsymbol{\Sigma_i}$ is assumed to be **absolutely stable**, i.e., A is Hurwitz, and the linear transfer function from z_i to y_i fulfills

$$\|C(sI_n - A)^{-1}E\|_{\mathcal{H}_\infty} < \mu^{-1}. \tag{9}$$

We refer to [3], [5] and the references therein for the definitions of absolute stability and slope-restrictedness, respectively.

All the subsystems on the network are interconnected according to the following static output-feedback protocol.

$$u_i = \sum_{j=1}^{N} w_{ij}(y_i - y_j), \; i = 1, 2, \cdots, N, \tag{10}$$

where $w_{ij} \in \mathbb{R} \geq 0$ is the (i, j)-th entry of weighted adjacency matrix of the underlying graph and stands for the intensity of the coupling between subsystem i and j. Then, combining (10) and (6) leads to a compact form of the Lur'e dynamical network as

$$\boldsymbol{\Sigma} : \begin{cases} \dot{x} = (I_N \otimes A - L \otimes BC)\, x - (I_N \otimes E)\, \Phi(y), \\ y = (I_N \otimes C)\, x, \end{cases} \tag{11}$$

where $\Phi(y) := [\phi(y_1), \phi(y_2), \cdots, \phi(y_N)]^T$, and $x := [x_1^T, x_2^T, \cdots, x_N^T]^T$, $y := [y_1, y_2, \cdots, y_N]^T$ are the collections of the states and outputs of the N subsystems. The vector $d := [d_1, d_2, \cdots, d_N]^T$ are external disturbances, and the matrix L is the graph Laplacian of underlying network.

In the context of networks, *synchronization* is one of the important properties, which substantially means that the states of the subsystems can achieve a common value. For a Lur'e network system in (11), the definition of *robust synchronization* is given as follows.

Definition 1: [1], [2] A Lur'e network system in form of (11) are called robustly synchronized if

$$\lim_{t\to\infty} (x_i(t) - x_j(t)) = 0, \ \forall i, j = 1, 2, \cdots, N,$$

for all initial conditions and all uncertain nonlinearities $\phi(\cdot, t)$ satisfying (8).

Moreover, if the nonlinear function $\phi(\cdot)$ in (6) is slope-restricted as in (8), then it is also *incremental passive* [20]. Thus, a sufficient condition for robust synchronization of the Lur'e network as in (11) can be obtained from [2] as follows.

Lemma 1: Consider the Lur'e network $\mathbf{\Sigma}$ as in (11) with a slope-restricted nonlinear function $\phi(\cdot)$. If there exists a matrix $Q \succ 0$ such that

$$QE = C^T \tag{12}$$

and

$$(A + \lambda_i BC)^T Q + Q(A + \lambda_i BC) \prec 0, \tag{13}$$

for all $i = 2, \cdots, N$, then $\mathbf{\Sigma}$ robustly synchronize. In (13), λ_i are the eigenvalues of the Laplacian matrix L.

III. Synchronization Persevering Model Reduction

In this section, the original Lur'e network $\mathbf{\Sigma}$ is assumed to be synchronized, and then we reduce the dimension of each subsystem in (6) such that synchronization is preserved in the reduced-order network.

First, the robust synchronization condition of the Lur'e network $\mathbf{\Sigma}$ is reinterpreted by the passivity of a set of auxiliary linear systems as follows.

$$\mathbf{\Gamma}(\lambda_i) : \begin{cases} \dot{\xi} = (A - \lambda_i BC)\xi + E\nu, \\ \eta = C\xi, \end{cases} \tag{14}$$

where $\xi \in \mathbb{R}^n$, $\nu, \eta \in \mathbb{R}$, and λ_i is the i-th smallest eigenvalue of the Laplacian matrix L in (11). Note that $\mathbf{\Gamma}(\lambda_i)$ is a single-input and single-output system. If the synchronization condition of the multi-agent system $\mathbf{\Sigma}$ in Lemma 1 is satisfied, then $A - \lambda_i BC$ is Hurwitz for all $i = 2, 3, \cdots, N$, due to (13). That is the auxiliary system $\mathbf{\Gamma}(\lambda_i)$ being asymptomatically stable.

Furthermore, following [21], the conditions in (12) and (13) coincide with the (strict) *passivity* of $\mathbf{\Gamma}(\lambda_i)$. Therefore, it provides us a way to preserve the robust synchronization in the reduced-order Lur'e network. That is firstly reducing the auxiliary system $\mathbf{\Gamma}(\lambda_i)$ with passivity preservation and then substituting the reduced system matrices to the network framework in (11) to generate the reduced-order network system.

Before implementing the above procedure, we introduce the following lemma, where the synchronization condition in Lemma 1 is further relaxed by only considering the largest eigenvalue λ_N.

Lemma 2: If there exists a positive definite matrix $\mathcal{Q}$ such that

$$\begin{bmatrix} A^T\mathcal{Q} + \mathcal{Q}A + C^TC & \lambda_N \mathcal{Q}B & \mathcal{Q}E - C^T \\ \lambda_N B^T\mathcal{Q} & -I & 0 \\ E^T\mathcal{Q} - C & 0 & 0 \end{bmatrix} \prec 0, \tag{15}$$

with λ_N is the largest eigenvalue of L in (11), then the auxiliary linear system $\mathbf{\Gamma}(\lambda_i)$ in (14) is positive real for all $\lambda_i \leq \lambda_N$, and hence the Lur'e network $\mathbf{\Sigma}$ robustly synchronizes.

Proof: The above LMI is equivalent to

$$A^T\mathcal{Q} + \mathcal{Q}A + \lambda_N^2 \mathcal{Q}BB^T\mathcal{Q} + C^TC \prec 0, \ \mathcal{Q}E = C^T. \tag{16}$$

For a matrix $\mathcal{Q} \succ 0$, we have

$$\begin{aligned} &(A - \lambda_i BC)^T\mathcal{Q} + \mathcal{Q}(A - \lambda_i BC) \\ &= A^T\mathcal{Q} + \mathcal{Q}A - \lambda_i(B^TC^T\mathcal{Q} + \mathcal{Q}BC). \end{aligned} \tag{17}$$

Note that

$$\begin{aligned} &\lambda_i(B^TC^T\mathcal{Q} + \mathcal{Q}BC) \\ &= (\lambda_i B^T\mathcal{Q} + C)^T(\lambda_i B^T\mathcal{Q} + C) - \lambda_i^2\mathcal{Q}BB^T\mathcal{Q} - C^TC. \end{aligned} \tag{18}$$

For any $\lambda_i \leq \lambda_N$, it then leads to

$$\begin{aligned} &(A - \lambda_i BC)^T\mathcal{Q} + \mathcal{Q}(A - \lambda_i BC) \\ &\preccurlyeq A^T\mathcal{Q} + \mathcal{Q}A + \lambda_i^2\mathcal{Q}BB^T\mathcal{Q} + C^TC \\ &\preccurlyeq A^T\mathcal{Q} + \mathcal{Q}A + \lambda_N^2\mathcal{Q}BB^T\mathcal{Q} + C^TC \prec 0 \end{aligned} \tag{19}$$

where the facts that $(B^T\mathcal{Q} + \lambda_N C)^T(B^T\mathcal{Q} + \lambda_N C) \succcurlyeq 0$ and $\mathcal{Q}BB^T\mathcal{Q} \succcurlyeq 0$ are used. Together with the relation $\mathcal{Q}E = C^T$ in (16), we conclude that the system $\mathbf{\Gamma}(\lambda_i)$ with the dynamics in (14) is passive for all $\lambda_i \leq \lambda_N$. ■

Remark 1: Lemma 2 is convenient to be applied as a sufficient condition to check the robust synchronization of the Lur'e network $\mathbf{\Sigma}$, since one just need to show the existence of $\mathcal{Q}$ in (15) rather than the solutions of (12) and (13) for all λ_i with $i = 2, \cdots, N$. Furthermore, only using the solutions of the LMI in (15), we can reduce the dimension of the auxiliary system $\mathbf{\Gamma}(\lambda_i)$, for any $\lambda_i \leq \lambda_N$.

The passivity-preserving model reduction is now introduced in the framework of balanced truncation. Let K_M and K_m be the maximum and minimal solutions of (15). Let T be a nonsingular coordinate transformation such that

$$TK_M^{-1}T^T = T^{-T}K_mT^{-1} = \Theta = \text{diag}(\theta_1, \theta_2, \cdots, \theta_n), \tag{20}$$

with $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n > 0$. We define the balanced system of $\mathbf{\Gamma}(\lambda_i)$ by $\bar{\mathbf{\Gamma}}(\lambda_i) := (\bar{A} - \lambda_i \bar{B}\bar{C}, \bar{E}, \bar{C})$, where

$$\bar{A} := TAT^{-1}, \ \bar{B} := TB, \ \bar{E} := TE, \text{ and } \bar{C} := CT^{-1}. \tag{21}$$

Now, we truncate the $n - k$ states corresponding to the smallest θ_i of the balanced system $\bar{\mathbf{\Gamma}}(\lambda_i)$. To do so, we consider the following matrix partitions.

$$\begin{aligned} &\bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \bar{E} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \\ &\bar{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \Theta = \begin{bmatrix} \Theta_1 & \\ & \Theta_2 \end{bmatrix}, \end{aligned} \tag{22}$$

where $A_{11} \in \mathbb{R}^{k\times k}$, $B_1 \in \mathbb{R}^k$, $E_1 \in \mathbb{R}^k$, $C_1 \in \mathbb{R}^{1\times k}$, and $\Sigma_1 := \text{diag}(\theta_1, \theta_2, \cdots, \theta_k)$. Hereafter, denote

$$\hat{A} = A_{11}, \ \hat{B} = B_1, \ \hat{E} = E_1, \text{ and } \hat{C} = C_1. \tag{23}$$

It can be shown that the matrix $\hat{A} - \lambda_i \hat{B}\hat{C}$ is the k-th order principal submatrix of the system matrix, $\bar{A} - \lambda_i \bar{B}\bar{C}$, in the balanced system $\bar{\mathbf{\Gamma}}(\lambda_i)$. Thus, the truncated model of $\mathbf{\Gamma}(\lambda_i)$ is presented as

$$\hat{\mathbf{\Gamma}}(\lambda_i) : \begin{cases} \dot{\hat{\xi}} = (\hat{A} - \lambda_i \hat{B}\hat{C})\hat{\xi} + \hat{E}\nu, \\ \hat{\eta} = \hat{C}\hat{\xi}, \end{cases} \tag{24}$$

where $\hat{\xi} \in \mathbb{R}^k$ and $\hat{\eta} \in \mathbb{R}$.

Remark 2: Lemma 2 requires a stricter synchronization condition than Lemma 1. However, this is computationally cheaper, as it need to check the feasibility of (15) with only λ_N. Moreover, the solution K_M and K_m can be used for balanced truncation of $\mathbf{\Gamma}(\lambda_i)$ for all $\lambda_i \leq \lambda_N$.

Then, the reduced-order dynamics of each agent is constructed by substituting the truncated matrices $\hat{A}$, $\hat{B}$, $\hat{E}$, and $\hat{C}$ to the Lur'e form as in (6).

$$\hat{\mathbf{\Sigma}}_i : \begin{cases} \dot{\hat{x}}_i = \hat{A}\hat{x}_i + \hat{B}u_i + \hat{E}\hat{z}_i \\ \hat{y}_i = \hat{C}\hat{x}_i \\ \hat{z}_i = -\phi(\hat{y}_i) \end{cases}, i = 1, 2, \cdots, N, \tag{25}$$

with $\hat{x}_i \in \mathbb{R}^k$, and $\hat{z}_i, \hat{y}_i \in \mathbb{R}$. Consequently, it leads to reduced-order Lur'e network dynamics as

$$\hat{\mathbf{\Sigma}} : \begin{cases} \dot{\hat{x}} = (I_N \otimes \hat{A} - L \otimes \hat{B}\hat{C})\hat{x} - (I_N \otimes \hat{E})\Phi(\hat{y}), \\ \hat{y} = (I_N \otimes \hat{C})\hat{x}, \end{cases} \tag{26}$$

comparing to (11).

The following theorem shows that the synchronization property is preserved after the above reduction process.

Theorem 1: Consider the full-order Lur'e network $\mathbf{\Sigma}$ in (11) and its reduced-order model $\hat{\mathbf{\Sigma}}$ in (26). If $\mathbf{\Sigma}$ robustly synchronizes due to Lemma 2, then $\hat{\mathbf{\Sigma}}$ also robustly synchronizes.

Proof: By Lemma 1, the system $\hat{\mathbf{\Sigma}}$ robustly synchronizes if the auxiliary system of $\hat{\mathbf{\Sigma}}$ is passive. Thus, it is sufficient to prove that $\hat{\mathbf{\Gamma}}(\lambda_i)$ in (24) is passive for all $\lambda_i \leq \lambda_N$.

Lemma 2 implies that the auxiliary system $\mathbf{\Gamma}(\lambda_i)$ is passive, so does the balanced system $\bar{\mathbf{\Gamma}}(\lambda_i)$. It means that

$$(\bar{A} - \lambda_i \bar{B}\bar{C})^T \Theta + \Theta(\bar{A} - \lambda_i \bar{B}\bar{C}) \prec 0, \text{ and } \Theta\bar{E} = \bar{C}^T, \tag{27}$$

for $i = 2, 3, \cdots, N$, where Θ is given in (20), and $\bar{A}$, $\bar{B}$, $\bar{E}$, and $\bar{C}$ are in (21). Expanding (27) using the partitions as in (22) then yields

$$(\hat{A} - \lambda_i \hat{B}\hat{C})^T \Theta_1 + \Theta_1(\hat{A} - \lambda_i \hat{B}\hat{C}) \prec 0, \text{ and } \Theta_1\hat{E} = \hat{C}^T.$$

for $i = 2, 3, \cdots, N$. Hence, the reduced auxiliary system is also passive, which gives the synchronization of the reduced-order model $\hat{\mathbf{\Sigma}}$ by Lemma 1. ■

IV. Error Analysis

We start with the analysis of the approximation of the linear dynamics $\mathbf{\Sigma}_i^{lin}$ in (7). The linear part of the reduced-order Lur'e subsystem truncation in (25) is given by

$$\hat{\mathbf{\Sigma}}_i^{lin} : \begin{cases} \dot{\hat{x}}_i = \hat{A}\hat{x}_i + \begin{bmatrix} \hat{B} & \hat{E} \end{bmatrix} \begin{bmatrix} u_i \\ \hat{z}_i \end{bmatrix} \\ \hat{y}_i = \hat{C}\hat{x}_i \end{cases}, i = 1, 2, \cdots, N. \tag{28}$$

For simplicity, we hereafter denote

$$\begin{aligned} g_B(s) &:= C(sI_n - A)^{-1}B, \ g_E(s) := C(sI_n - A)^{-1}E, \\ \hat{g}_B(s) &:= \hat{C}(sI_k - \hat{A})^{-1}\hat{B}, \ \hat{g}_E(s) := \hat{C}(sI_k - \hat{A})^{-1}\hat{E}. \end{aligned} \tag{29}$$

Then, we provide the *a priori* error bounds on the linear part in the following lemma.

Lemma 3: Consider the following transfer functions in (29) corresponding to the linear parts of the full-order and reduced-order Lur'e subsystems, $\mathbf{\Sigma}_i^{lin}$ and $\hat{\mathbf{\Sigma}}_i^{lin}$, respectively. The following error bounds hold.

$$\|g_B(s) - \hat{g}_B(s)\|_{\mathcal{H}_\infty} \leq \frac{2}{\lambda_N} \sum_{k=r+1}^{n} \theta_k, \tag{30a}$$

$$\|g_E(s) - \hat{g}_E(s)\|_{\mathcal{H}_\infty} \leq 2 \sum_{k=r+1}^{n} \theta_k, \tag{30b}$$

where θ_k are defined in (20).

Proof: The minimal solution of (15), K_m, satisfies

$$K_m \succ 0, \ A^T K_m + K_m A + C^T C \prec 0. \tag{31}$$

Analogously, the maximum solution of (15), $K_M \succ 0$, satisfies (16), which is reformulated as

$$\begin{aligned} &AK_M^{-1} + K_M^{-1}A^T + K_M^{-1}C^TCK_M^{-1} + \lambda_N^2 BB^T \prec 0, \\ &E = K_M^{-1}C^T. \end{aligned} \tag{32}$$

Substituting the latter equation into the inequality yields

$$AK_M^{-1} + K_M^{-1}A^T + EE^T + \lambda_N^2 BB^T \prec 0. \tag{33}$$

Thus, the following two Lyapunov inequalities hold.

$$AK_M^{-1} + K_M^{-1}A^T + \lambda_N^2 BB^T \prec 0, \tag{34a}$$

$$AK_M^{-1} + K_M^{-1}A^T + EE^T \prec 0. \tag{34b}$$

Recall the definitions of generalized Gramians in Section II-A. From (31), (34a) and (34b), the pair K_M^{-1}, K_m can be regarded as generalized Gramians of the systems $\lambda_N \cdot g_B(s)$ and $g_E(s)$. Since the reduced matrices $\hat{A}$, $\hat{B}$, $\hat{E}$ and $\hat{C}$ in the reduced-order models in (29) are obtained by the balanced truncation based on the pair K_M^{-1} and K_m. Application of (4) in Section II-A then leads to the error bounds in (30a) and (30b). ■

Moreover, the approximation error for $\mathbf{\Sigma}_i^{lin}$ in (7) is also bounded.

Proposition 1: Let $\mathbf{\Sigma}_i^{lin}$ and $\hat{\mathbf{\Sigma}}_i^{lin}$ be the linear parts of the full-order and reduced-order Lur'e subsystems, respectively. The approximation error between the two system is bounded by

$$\|\mathbf{\Sigma}_i^{lin} - \hat{\mathbf{\Sigma}}_i^{lin}\|_{\mathcal{H}_\infty} \leq \frac{2}{\min\{\lambda_N, 1\}} \sum_{k=r+1}^{n} \theta_k, \tag{35}$$

with θ_k given in (20).

Proof: It then follows from (33) that

$$AK_M^{-1} + K_M^{-1}A + \min\{\lambda_N^2, 1\} \begin{bmatrix} B & E \end{bmatrix} \begin{bmatrix} B^T \\ E^T \end{bmatrix} \prec 0. \tag{36}$$

Therefore, K_M^{-1} and K_m in (31) are generalized Gramians for the linear system $\min\{\lambda_N, 1\} \cdot \mathbf{\Sigma}_i^{lin}$. Following the same reasoning in the proof of Lemma 3, we obtain (35). ■

Now, we investigate the approximation error of each Lur'e subsystem with uncertain nonlinearity $\phi(\cdot)$. Following the procedure in [18], the condition of absolute stability of the reduced-order Lur'e system $\hat{\mathbf{\Sigma}}_i$ is firstly discussed.

Proposition 2: By performing the balanced truncation in Section III, the reduced Lur'e-type agent $\hat{\mathbf{\Sigma}}_i$ in (25) is absolutely stable if

$$\|g_E(s)\|_{\mathcal{H}_\infty} < \mu^{-1} - 2 \sum_{k=r+1}^{n} \theta_k, \tag{37}$$

where μ is defined in (8).

Proof: The result follows from [18]. If the equation (37) holds, then it can be shown that $\|\hat{g}_E(s)\|_{\mathcal{H}_\infty} < \mu^{-1}$. With a Hurwitz $\hat{A}$ matrix in $\hat{\mathbf{\Sigma}}_i$, we conclude that $\hat{\mathbf{\Sigma}}_i$ is absolutely stable. ■

Then, by assuming that $\hat{\mathbf{\Sigma}}_i$ in (25) is absolutely stable, namely satisfying (37), we explore an error bound for the reduction of Lur'e subsystems $\mathbf{\Sigma}_i$. The proof of the following theorem outline is inspired by [18]. Differently, an *a priori* error bound is provided.

Theorem 2: Consider the full-order Lur'e system $\mathbf{\Sigma}_i$ in (6) satisfying the slope-restrictedness condition in Assumption 2 and the synchronization condition in Lemma 2. Denote the reduced-order model of $\mathbf{\Sigma}_i$ by $\hat{\mathbf{\Sigma}}_i$ in (25), which satisfies the absolute stability condition in (37). Then, the error of the outputs of $\mathbf{\Sigma}_i$ and $\hat{\mathbf{\Sigma}}_i$ is bounded by

$$\|y_i(t) - \hat{y}_i(t)\|_2 \leq \frac{\delta(1+\delta+\epsilon)}{\lambda_N \mu \epsilon(\delta+\epsilon)} \|u_i(t)\|_2, \tag{38}$$

where $\delta = 2\sum_{k=r+1}^{n} \theta_k$ and $\epsilon := \mu^{-1} - \delta - \|g_E(s)\|_{\mathcal{H}_\infty} > 0$. The full proof will be presented in our later version.

V. Illustrative Example

The feasibility of the proposed method is illustrated through a simulation, which considers a network consisting of 4 Lur'e subsystems. The interconnection topology is given in Fig 2.

The dynamics of Lur'e subsystems $\mathbf{\Sigma}_i$ are in (6) are given by matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & -3 & 0.25 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0.5 & -2 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0.5 & -2 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & -2 & 2 \end{bmatrix}^T,$$
$$C = E^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and a nonlinearity

$$\phi(y) = 0.5\left(|y+1| - |y-1|\right). \tag{39}$$

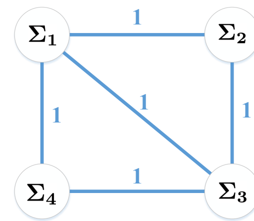

Fig. 2. Interconnection topology of the Lur'e network

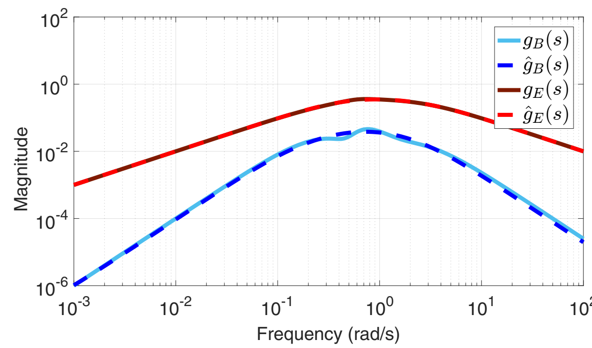

Fig. 3. The comparison of the frequency responses of linear parts in the full-order and reduced-order Lur'e subsystems, which show that the linear part of each Lur'e subsystem is well-approximated.

By (8), we have $\mu = 1$. Therefore, each subsystem is absolutely stable, due to the fact that

$$\|g_E(s)\|_{\mathcal{H}_\infty} = \|C(sI-A)^{-1}E\|_{\mathcal{H}_\infty} = 0.3606 < \mu^{-1} = 1$$

It can be checked that the LMI in (15) is feasible, i.e., the solutions of (15) exists. Thus, by Lemma 2, the original Lur'e network in form of (11) synchronizes under the interconnection topology as in Fig. 2.

Then, solving the LMI in (15) gives maximum and minimal solutions K_M and K_m, which can be simultaneously digitalized as

$$TK_M^{-1}T^T = T^{-T}K_mT^{-1} = \Theta =$$
$$\mathrm{diag}(1, 1, 0.1149, 0.1126, 0.0493, 0.0425, 0.0225, 0.0098).$$

Using the balanced truncation procedure of Section III to eliminate the last four states in the balanced system leads to the following reduced matrices.

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -3 & -1.0742 & 0.7406 \\ 0 & -0.1943 & -1.1737 & 0.3472 \\ 0 & 0.1412 & 1.2461 & -0.5473 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 & 0 & -0.1248 & 0.0841 \end{bmatrix}^T,$$
$$C = E^T = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix},$$

Therefore, the reduced Lur'e subsystems $\hat{\mathbf{\Sigma}}_i$ are obtained. For comparison, we depict the frequency responses of the transfer functions $g_B(s)$, $\hat{g}_B(s)$, $g_E(s)$, and $\hat{g}_E(s)$ in Fig 3, which indicates that the approximation error on the linear part of each Lur'e subsystem is small. Moreover,

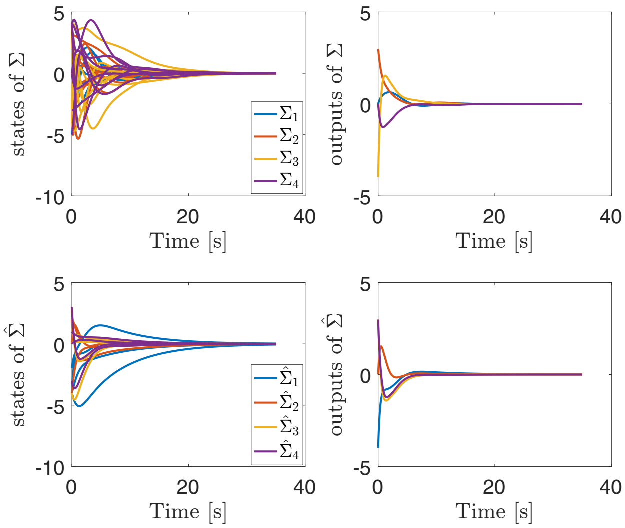

Fig. 4. The trajectories of the states and outputs of the full-order Lur'e network (in the top two figures) and reduced-order network (in the bottom two figures). The initial states are set to random values for both systems. It shows that the both full-order and reduced-order networks are synchronized.

we measure the linear reduction errors by $\mathcal{H}_\infty$ norm as $\|g_B(s) - \hat{g}_B(s)\|_{\mathcal{H}_\infty} \approx 0.0112$, and $\|g_E(s) - \hat{g}_E(s)\|_{\mathcal{H}_\infty} \approx 0.0183$, which using Lemma 3, the a priori bounds are obtained as 0.0620 and 0.2481, respectively.

Next, with the dynamics of the reduced-order Lur'e subsystems $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{i}}$, we construct the a lower-dimensional Lur'e network as in (26). Note that both original and reduced Lur'e network in (11) and ((26)) are autonomous. To investigate the synchronization phenomenon in both systems, we stimulate both systems by assigning random values as their initial states. The trajectories of the states and outputs of both networks are then plotted in Fig. 4. We can see that, by the proposed model reduction scheme, the reduced-order Lur'e network preserves the synchronization property.

VI. Conclusion

In this paper, we have investigated the problem of model order reduction for Lur'e networks. The proposed method aims at preserving the robust synchronization through the reduction process. To this end, a sufficient condition for robust synchronization of Lur'e networks was presented, which relates to the passivity of a linear auxiliary system. Using the maximum and minimum solutions of (15), we reduce the auxiliary system and then obtain the reduced system matrices for Lur'e subsystems. An *a priori* error bound for the Lur'e subsystems can be obtained, since the reduction process can also be regarded as Lyapunov balanced truncation based on generalized Gramians for the linear part of each Lur'e subsystem. In the time domain, an *a priori* bound on the input-output error between the full-order and reduced-order Lur'e subsystems are established using the error bound for the linear part and the slope-restrictedness of the uncertain nonlinear function.

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