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Stability Analysis of Systems with Delay-Dependent Coefficients Subject to Some Particular Delay Structure

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***Abstract*— Stability of systems with commensurate delays and delay-dependent coefficients is studied along the line of the τ-decomposition approach. This particular delay structure allows the use of some sophisticated result from the matrix theory to generalize the stability analysis method developed for systems with a single delay. Criterion for determining cross directions of imaginary roots are presented, leading to a systematic stability analysis with the aid of the graphs of some functions.**

I. Introduction

Time-delay systems with delay-dependent coefficients may arise from models of various scientific disciplines. These models include, for instance, population models with age structure [10], the stellar dynamos in [11] , or models of hematopoiesis dynamics [12], just to name a few. The delay parameter in the coefficients may also result from the linearization of a nonlinear model about some equilibria that depend on the delay.

There is a rich Literature on time-delay systems and various methods have been proposed for stability analysis of linear time-delay systems. See the books [3], [6], [7] for a comprehensive review and summaries of the progress made in this field. Most related to our approach are the D-decomposition and τ-decomposition methods [13], [16]. For the D-decomposition method, the delay value is fixed and consider the coefficients as variables, while for the τ-decomposition method the coefficients are constant and the delay is treated as a variable. Unfortunately, most of the approaches in the literature deal with systems with delay-free coefficients and do not lend themselves directly to the type of systems considered here. In [10], Berreta and Kuang proposed an effective method that combines analytical results with graphical information to determine the local stability of systems with a single delay and delay-dependent coefficients. In [8], we generalized the work of Berreta and Kuang for a wider class of systems.

In this paper, we extend the results in our previous work [8] for systems with a single delay to a class of systems with a particular delay structure, namely systems with commensurate delays. This class of systems can be represented by the following characteristic equation:

$$D(\lambda,\tau)=\sum_{k=0}^{M} P_k(\lambda,\tau)e^{-k\lambda\tau}=0, \tag{1}$$

where λ is the Laplace variable and each function $P_k(\lambda,\tau)$ is a polynomial in λ. Besides the commensurate-delay structure, often in practice there is a second kind of structure, which consists in the parameterization of each $P_k(\lambda,\tau)$. For fixed λ, these functions may also be polynomials of $g(\lambda)$ for some function $g(\cdot)$. The function g may takes a form of an exponential function, an reciprocal function, or a mixture of them. The reason will be briefly discussed in Section II. In this paper we will be focused on the commensurate-delay structure while leaving the second type of structure to be exploited in our future work.

Our method can be viewed as a generalized $\tau-$decomposition approach, which roughly proceeds as follows: Start with a delay value τ_L for which one knows the the number of roots of the characteristic equation on the right half pane. Let the delay parameter τ sweep through an interval of interest $\mathscr{I}=[\tau^l,\tau^u]$, one can identify all delay values for which the characteristic equation of the linearized system admits some roots on the imaginary axis. We arrange these delay values in ascending order as:

$$\tau^l\le\tau_1<\tau_2<\ldots<\tau_L\le\tau^u. \tag{2}$$

Provided sufficient continuity of the roots with respect to τ, the number of unstable roots can not change for τ in each interval (τ_k,τ_{k+1}), $k=1,...,L-1$ since no root crosses the imaginary axis. Therefore by identifying the direction of these roots crossing imaginary roots as τ sweeps through τ_k, one can determine the change of the number of roots on the right half plane. Consequently, the number of roots on the right half plane for each delay interval (τ_k,τ_{k+1}) can be determined accordingly. We will provide criterion to determine the crossing directions of the imaginary roots and show how system stability can be conveniently determined based on the graphs of some functions.

Some notation used in this paper is given here. Let $\mathrm{ord}(\cdot)$ be the order of a polynomial. For any complex number c, $\Re(c)$, $\Im(c)$ and $\bar{c}$ denotes its real part, imaginary part and conjugate, respectively. The unite disk and unit circle in the complex plane are denoted by $\mathbb{D}$ and $\partial\mathbb{D}$, respectively. $\mathbb{R}$ stands for the set of real numbers and $\mathbb{R}_+$ stands for the positive reals. We will use ∂ with a subscript to denote partial derivative. For instance, $\partial_\lambda D(\lambda,\tau):=\frac{\partial D(\lambda,\tau)}{\partial\lambda}$. Let $N^u(\tau)$ be the number of unstable characteristic roots for a given delay τ.

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II. MOTIVATING EXAMPLES

For linear systems represented by a state equation of the form

$$\dot{x}(t) = A_1(\tau)x(t) + A_2(\tau)x(t-\tau),$$

where A_1 and A_2 are matrices of appropriate dimensions, commensurate delays may appear in the corresponding characteristic equation even though there is only a single delay in the original state-space equation. In some cases, delay-dependent coefficients may be derived during the analysis of time-delay systems with delay-free coefficients. For instance, consider the α-stability of the following characteristic equation [9], [17]:

$$\lambda + e^{-\tau\lambda} + e^{-2\tau\lambda} = 0. \tag{3}$$

The characteristic equation is said to be $\alpha-$stable if the real part of all its roots is smaller than $-\alpha$. Replacing λ with $\lambda-\alpha$ in (3), we obtain

$$\lambda + \alpha + e^{\alpha\tau}e^{-\tau\lambda} + e^{2\alpha\tau}e^{-2\tau\lambda} = 0, \tag{4}$$

It is easy to see that the $\alpha-$stability of (3) is equivalent to the asymptotic stability of characteristic equation (4), which has delay-dependent coefficients. It is true for systems in general forms that this shift of variable will lead to a structured parameterization of system coefficients, i.e., each $P_k(\lambda,\tau)$ of the characteristic equation (1) being polynomial in $e^{\alpha\tau}$.

Another example is related to the feedback control based on the measured output signal. When static output is not sufficient to achieve stability or a satisfactory performance, one may construct a control law based on the derivatives of the output signal $y(t)$, which can be approximated using a delay-difference scheme: $\dot{y}(t) \approx \frac{y(t)-y(t-\tau)}{\tau}$, or $\lambda y(\lambda) \approx \frac{1-e^{-\lambda\tau}}{\tau}y(\lambda)$ after the Laplace transform. In general one may use $(\frac{1-e^{-\lambda\tau}}{\tau})^n y(\lambda)$ to approximate the Laplace transform of the nth order derivative of the output signal. Such an approximation is considered in [18] for the stabilization of a chain of integrators, leading to a closed loop system with delay-dependent coefficients:

$$\lambda^n + \sum_{l=0}^{n-1} k_i(\frac{1-e^{-\lambda\tau}}{\tau})^l = 0. \tag{5}$$

A rescaling trick is taken in [18] to transform (5) in such a way that delay-dependent coefficients can be avoided. However such a trick does not apply to systems of more general forms. After re-arranging terms in (5) and rewritting it in the form of (1) , it is easy to see that each function $P_k(\lambda,\tau)$ is polynomial in both λ and $\frac{1}{\tau}$.

We note that some extension is needed for the stability analysis method in this paper to be applicable to control systems subject to the delay-difference feedback. This is because the analysis here requires the system coefficients to be continous in τ, however the approximation scheme discussed above is not defined for $\tau = 0$ as τ appears in the denominator. The required extension has been presented in [19] for control schemes using only the delay-difference approximation of the first order derivative of $y(t)$. For extensions to control schemes using also the approximation of higher order derivatives, interested readers can refer to [20].

III. PREREQUISITES

We consider time-delay systems with characteristic equations (1), where each $P_k(\lambda,\tau)$ is continuous in τ and is a polynomial of λ with real coefficients for any given $\tau \in \mathscr{I}$. Characteristic equation of the form (1) represents a system with commensurate delays. Suppose $\mathscr{I} = [\tau^l, \tau^u]$ is the delay interval of interest and $N^u(\tau^l)$ is known. Our objective to find all the sub-intervals contained in $\mathscr{I}$ for which (1) is asymptotically stable. We may write $D(\lambda,\tau)$ as $D_\tau(\lambda)$ and $P_i(\lambda,\tau)$ as $P_{i\tau}(\lambda)$ to emphasize that λ is viewed as the argument and τ is regarded as a parameter of these functions.

Define

$$\hat{D}(\lambda,\tau,x) = \sum_{k=0}^{M} P_k(\lambda,\tau)x^k, \tag{6}$$

where x can be a scalar or a matrix of any dimension. Denote $\hat{D}_{\omega\tau}(x) = \hat{D}(j\omega,\tau,x)$, then it is easy to see

$$\hat{D}_{\omega\tau}(e^{-j\omega\tau}) = \hat{D}(j\omega,\tau,e^{-j\omega\tau}).$$

Introduce the following Hermitian matrix:

$$\begin{aligned} \mathrm{H}(\lambda,\tau) = \quad & Q_a(\lambda,\tau,S)^H Q_a(\lambda,\tau,S) \\ & -\hat{D}(\lambda,\tau,S)^H \hat{D}(\lambda,\tau,S), \end{aligned} \tag{7}$$

where $Q_a(\lambda,\tau,S) = \sum_{k=0}^{M} \overline{P_k}(\lambda,\tau)S^{M-k}$ and

$$S = \begin{pmatrix} 0\,1\,0\cdots 0 \\ 0\,0\,1\cdots 0 \\ \cdot\;\cdot\;\cdot\;\cdots 1 \\ 0\,0\,0\cdots 0 \end{pmatrix}$$

is an $M \times M$ shift matrix. Define a function F as

$$F(\omega,\tau) = -\det \mathrm{H}(j\omega,\tau), \tag{8}$$

We may write $F_\tau(\omega)$ instead of $F(\omega,\tau)$ when it is regarded as a polynomial in ω. We claim that

$$F(\omega,\tau) = 0 \tag{9}$$

is a necessary condition for $(\omega,\tau) \in \mathbb{R} \times \mathscr{I}$ to satisfy

$$D(j\omega,\tau) = 0. \tag{10}$$

This readily follows from the following expression for F, which is proved in [14]:

$$F(\omega,\tau) = -|P_M(j\omega,\tau)|^{2M} \prod_{i,k=1}^{M} (1 - z_i\overline{z_k}), \tag{11}$$

where z_i, $i = 1,2,\ldots,M$ are the roots of the polynomial $\hat{D}_{\omega\tau}(x)$. Equation (9) can be viewed as a generalization of the magnitude condition proposed in [8] for systems with a single delay.

We introduce a set of standing assumptions for commensurate delay systems considered in this paper.

Assumption I. For all $\tau \in \mathscr{I}$, $P_{0\tau}$ satisfies $\mathrm{ord}(P_{0\tau}) = n$ and

$$\mathrm{ord}(P_{i\tau}) \le n, \; i = 1, 2, \cdots, M.$$

Furthermore, all the roots of following equation

$$s^M + c_{1\tau}s^{M-1} + c_{2\tau}s^{M-2} + \cdots + c_{M-1,\tau}s + c_M = 0$$

satisfy $|s| < 1$, where

$$c_{i\tau} = \lim_{\lambda \to \infty} \frac{P_{i\tau}(\lambda)}{P_{0\tau}(\lambda)}.$$

Assumption II. No $(\omega, \tau) \in \mathbb{R} \times \mathscr{I}$ satisfies simultaneously (9) and

$$P_M(j\omega, \tau) = 0 \tag{12}$$

Assumption III. There are only a finite number of (ω, τ) in $\mathbb{R}_+ \times \mathscr{I}$ that simultaneously satisfy (9) and

$$\partial_\omega F(\omega, \tau) = 0. \tag{13}$$

Moreover, $\mathrm{ord}(F_\tau(\omega))$ is constant for $\tau \in \mathscr{I}$.

Assumption IV. Let $(\omega^*, \tau^*) \in \mathbb{R}_+ \times \mathscr{I}$ be any pair that satisfies (10), then (13) does not hold for $\omega = \omega^*$ and $\tau = \tau^*$. Furthermore, each $P_k(\lambda, \tau)$, $k = 0, \ldots, M$ is differentiable in a neighborhood of $(j\omega^*, \tau^*)$.

Assumption V Only a finite triples of the form $(\omega, \tau, x) \in \mathbb{R} \times \mathscr{I} \times \mathbb{C}$ satisfy simultaneously

$$\begin{cases} \hat{D}(j\omega, \tau, x) & = 0 \\ \partial_x \hat{D}(j\omega, \tau, x) & = 0 \end{cases}$$

as well as (9). Let Φ_τ be the set that collects exactly all the τ values appearing in these triples, then further assume that (1) does not admit imaginary roots for any $\tau \in \Phi_\tau$.

Assumption I indicates that all the characteristic roots of (1) with real parts larger than $-c$, where c is some positive number, vary continuously as τ changes [4]. Assumption II together with Assumption IV ensure that if $\lambda = j\omega^*$ is an imaginary root for some critical delay τ^*, then λ must be locally a continuous function of τ. Moreover, this assumption allows us to define the phase functions for commensurate-delay systems. It will be shown that Assumption V is needed to ensure that the uniqueness of the unit solution of $\hat{D} = 0$ in x defined except for some specific τ values. Notice that this assumption is automatically satisfied if $M = 1$. It can be expected that these assumptions should hold for general systems in practice. For instance, Assumption V essentially requires five real equations to admit finite number of solutions in four real numbers, which can be satisfied except for some degenerated cases.

IV. Stability Analysis

A. Identifying imaginary roots

We address the first critical aspect of our analysis, that is to identify all τ values such that the characteristic equation admits imaginary roots. Another key aspect is to determine how these imaginary roots move as τ increases and sweeps through these critical values, which will be discussed subsequently.

Recall that if $\lambda = j\omega$ is an imaginary root of $D_\tau(\lambda)$, ω must be a real root of $F_\tau(\omega)$. Therefore it is natural to expect that the real roots of $F_\tau(\omega)$ will play an important role in the analysis. We first show that $\mathscr{I}$ can be decomposed into sub-intervals in which the number of real roots of $F_\tau(\omega)$ is invariant. Let

$$\tau^{(1)} < \tau^{(2)} < \ldots < \tau^{(K-1)}$$

be exactly all the τ value contained in all the pairs $(\omega, \tau) \in \mathbb{R}_+ \times \mathscr{I}$ that simultaneously satisfy (8) and (13). We also write $\tau^{(0)} = \tau^l$ and $\tau^{(K)} = \tau^u$ and then decompose $\mathscr{I}$ into K subintervals

$$\mathscr{I}^{(i)} = [\tau^{(i-1)}, \tau^{(i)}], \; i = 1, \ldots, K. \tag{14}$$

It has been shown in [8] that for any given index i, $F_\tau(\omega)$ admits a fixed number of unrepeated real roots for τ in the interior of $\mathscr{I}^{(i)}$. These roots can be regarded as a continuous function of τ, denoted as $\omega_k^{(i)}(\tau)$, $k = 1, 2, \ldots, m(i)$. The definition of these functions is extended to $\mathscr{I}^{(i)}$ by requiring them to be continuous at $\tau^{(i-1)}$ and $\tau^{(i)}$.

Proposition 1: Suppose $(\omega^*, \tau^*) \in \mathbb{R} \times \mathscr{I}$ satisfies (9) but does not satisfy (13). Furthermore, suppose $\tau^* \notin \Phi_\tau$, then $\hat{D}_{\omega^*\tau^*}(x)$ admits a unique root on $\partial\mathbb{D}$, which is simple.

Proof: Let z_l, $l = 1, 2, \ldots, M$ be the roots of $\hat{D}_{\omega\tau}(x)$ for (ω, τ) in some neighborhood of (ω^*, τ^*). In this neighborhood, Assumption V together with $\tau^* \notin \Phi_\tau$ ensures that $\hat{D}'_{\omega\tau}(z_l) \neq 0$ for each l. Therefore by the implicit function theorem, each z_l is locally a continuous function of (ω, τ) denoted as $z_l(\omega, \tau)$ and is differentiable in ω. We first prove the existence of a solution on the unit disk. Suppose such a solution on $\partial\mathbb{D}$ does not exist, then it follows from $F(\omega^*, \tau^*) = 0$ and (11) that there exist two roots z_i, z_k of $\hat{D}_{\omega^*\tau^*}(x)$, with $i, k \le M$ and $i \neq k$ such that $\overline{z_i}(\omega^*, \tau^*) z_k(\omega^*, \tau^*) = 1$. It is implied by (11) that $F(\omega, \tau)$ can be decomposed as

$$F(\omega, \tau) = g_1(\omega, \tau)(1 - z_l\overline{z_k})(1 - z_k\overline{z_l}),$$

where g_1 is a differentiable function at (ω^*, τ^*) and the arguments of z_l, z_k are omitted for brevity. The last equality further implies that $\partial_\omega F(\omega^*, \tau^*) = 0$, which contradicts Assumption III. Therefore there must exist at least one z_l on $\partial\mathbb{D}$. To see that such a root is simple and unique, suppose there exist two solutions z_l, z_k, $l \neq k$ both on $\partial\mathbb{D}$. Then using (11), we can locally express F as

$$F(\omega, \tau) = g_2(\omega, \tau)(1 - z_l\overline{z_l})(1 - z_k\overline{z_k}),$$

where g_2 is differentiable at (ω^*, τ^*) and the arguments of z_l, z_k are again suppressed. This further implies that $\partial_\omega F(\omega^*, \tau^*) = 0$, which is again a contradiction. ∎

In the light of the last proposition, for given i and k, we introduce function $z_k^{(i)}(\tau)$ as the unique solution of

$$\hat{D}(\omega_k^{(i)}(\tau), \tau, x) = 0, \tag{15}$$

in x on $\partial\mathbb{D}$, for any $\tau \in (\tau^{(i-1)}, \tau^{(i)}) - \Phi_\tau$. Since the roots of $\hat{D}_{\omega\tau}(x)$ is continuous with respect to ω and τ, we can extend the definition of $z_k^{(i)}(\tau)$ to entire $\mathscr{I}^{(i)}$ by requiring the function $z_k^{(i)}(\tau)$ to be continuous on $\mathscr{I}^{(i)}$. Further define:

$$\theta_k^{(i)}(\tau) = \angle z_k^{(i)}(\tau) + \omega_k^{(i)}(\tau)\tau, \tag{16}$$

where $\angle z_k^{(i)}(\tau)$ is a continuous function in $\mathscr{I}^{(i)}$, which measures the phase angel of the complex number $z_k^{(i)}(\tau)$. Notice, the value of $\angle z_k^{(i)}(\tau)$ is not necessarily restricted to any 2π interval.

Proposition 2: For any given i, a pair $(\omega^*, \tau^*) \in \mathbb{R}_+ \times \mathscr{I}^{(i)}$ satisfies (10) if and only if there exist some integers k such that $\omega^* = \omega_k^{(i)}(\tau^*)$ and

$$\theta_k^{(i)}(\tau^*) = 2r\pi, \ r \text{ integer.} \tag{17}$$

Going through each interval $\mathscr{I}^{(i)}$ and each curve $\omega_k^{(i)}(\tau)$, we may identify all $\tau = \tau_l$, $l = 1, 2, \cdots, L$ such that (17) hold for some integer k if $\tau_l \in \mathscr{I}^{(i)}$. For each given τ_l, it is possible that more than one k satisfies (17), and we denote the corresponding $\omega_k^{(i)}(\tau_l) \geq 0$ as ω_{lh}, $h = 1, 2, \cdots, H_l$. In this way we can identify all the pairs (ω_{lh}, τ_l), $h = 1, 2, \cdots, H; l = 1, 2, \cdots, L$, that satisfy (10).

Suppose $\tau^* \in \mathscr{I}^{(i)}$ for some index i is a critical delay, then it is easy to see that the derivative of $\theta_k^{(i)}(\tau)$, $k = 1, 2, ..., m(i)$ is exits. Indeed we have

$$\frac{d\omega_k^{(i)}(\tau^*)}{d\tau} = \left.\frac{\partial_\tau F}{\partial_\omega F}\right|_{\substack{\omega=\omega^* \\ \tau=\tau^*}}, \frac{d\theta_k^{(i)}(\tau^*)}{d\tau} = \left.\frac{\partial_\tau \hat{D}}{\partial_x \hat{D}}\right|_{\substack{\omega=\omega^* \\ \tau=\tau^*}}.$$

Then by differentiating both sides of the relation:

$$\exp\left(j\theta_k^{(i)}(\tau) - j\omega_k^{(i)}(\tau)\tau\right) = z_k^{(i)}(\tau),$$

at $\tau = \tau^*$, we have

$$\frac{d\theta_k^{(i)}(\tau^*)}{d\tau} = \frac{1}{j}\frac{dz_k^{(i)}(\tau^*)}{d\tau}\left(z_k^{(i)}(\tau^*)\right)^{-1} + \frac{d\left(\omega_k^{(i)}(\tau^*)\tau^*\right)}{d\tau}. \tag{18}$$

B. Counting unstable roots

We will first show that each imaginarycharacteristic root of $D_\tau(\lambda)$ corresponding to some critical delay is locally a differentiable function of τ. For this purpose, we introduce the following formula, which will also be useful for determining the cross-direction of the imaginary roots.

Proposition 3: Suppose $(\omega^*, \tau^*) \in \mathbb{R} \times \mathscr{I}$ satisfies (10). Let z_k^*, $k = 1, \ldots, M$ be all the solutions of $D_{\omega^*\tau^*}(x)$. Without loss of generality, let z_1^* be the unique solution on $\partial\mathbb{D}$. Then the following holds:

$$\partial_\omega F(\omega^*, \tau^*) = -2c|P_M(j\omega^*, \tau^*)|^{2M}\Re\left(\overline{j\partial_\lambda D(j\omega^*, \tau^*)} \times \sum_{i=1}^{M} iP_i(j\omega^*, \tau^*)e^{-ij\omega^*\tau^*}\right), \tag{19}$$

where

$$c = \Big|\sum_{i=1}^{M} iP_i(j\omega^*, \tau^*)e^{-ij\omega^*\tau^*}\Big|^{-2} \times \prod_{\substack{i,k=1 \\ (i,k)\neq(1,1)}}^{M} (1 - z_i^*\overline{z_k^*}). \tag{20}$$

The proof can be found in [20]. This proposition combined with Assumption III indicates that given a pair(ω_{lh}, τ_l) that satisfies (10), then $\partial_\lambda D(j\omega_{lh}, \tau_l) \neq 0$. Therefore by the implicit function theorem, (10) determines λ as a differentiable function of τ, for (λ, τ) in a neighborhood of (ω_{lh}, τ_l). We shall write this function as $\lambda(\tau)$. To count the number of unstable roots for a given delay value, we introduce some quantities as follows. If $\tau_l \neq \tau^l$, define

$$\text{Inc}(\omega_{lh}, \tau_l) = \lim_{\varepsilon \to 0^+} \frac{\Re(\lambda(\tau_l + \varepsilon)) - \Re(\lambda(\tau_l - \varepsilon))}{2}. \tag{21}$$

If $\tau_l = \tau^l$, which implies $\tau_l = \tau_1$, define instead

$$\text{Inc}(\omega_{1h}, \tau_1) = \max\{0, \lim_{\varepsilon \to 0^+} \Re(\lambda(\tau_1 + \varepsilon))\}. \tag{22}$$

The limits in the definitions above exist because Assumption IV guarantees that in some neighborhood of τ_{lh}, $\Re(\lambda(\tau))$ is continuous and equals 0 if and only $\tau = \tau_l$. Now let the number of right half plane roots of $D_\tau(\lambda)$ be $N^u(\tau)$. For any $\tau \in \mathscr{I}$, $\tau \neq \tau_l$, $l = 1, 2, \ldots, L$, it is easy to see the following relation holds:

$$N^u(\tau) = N^u(\tau^l) + \sum_{l=1}^{L_\tau} \text{Inc}(\tau_l), \tag{23}$$

where $L_\tau = \max\{l \mid \tau_l < \tau\}$ and

$$\text{Inc}(\tau_l) = 2\sum_{h=1}^{H_l} \text{Inc}(\omega_{lh}, \tau_l) \tag{24}$$

It is clear that the quantity $\text{Inc}(\omega_{lh}, \tau_l)$ indicates the cross direction of the imaginary root, namely towards right or left half plane these imaginary roots moves as τ sweeps through some critical delay values. The reader can refer to [8] for more detailed discussion of this respect. In the case $\lambda'(\tau_l) \neq 0$, we can compute $\text{Inc}(\omega_{lh}, \tau_l)$ using

$$Inc(\omega_{lh}, \tau_l) = \begin{cases} \text{sign}\left(\Re(\lambda'(\tau_l))\right) & \text{if } \tau_l \neq \tau^l \\ \max\{\text{sign}\left(\Re(\lambda'(\tau_l))\right), 0\} & \text{if } \tau_l = \tau^l \end{cases} \tag{25}$$

We next provide a formula to compute the right hand side of (25).

C. Cross-direction analysis

Theorem 1: Suppose $(\omega^*, \tau^*) \in \mathbb{R}_+ \times \mathscr{I}$ satisfy (10) and let i, k be such numbers that $\tau^* \in \mathscr{I}^{(i)}$ and $\omega_k^{(i)}(\tau^*) = \omega^*$. Assume each $P_i(\lambda, \tau)$ is analytic at $\tau = \tau^*$, then (1) defines λ as an analytic function of τ in a sufficiently small neighborhood of $(j\omega^*, \tau^*)$ Assume further that n_d is a number that may depend on i such that $(\frac{d}{d\tau})^l \theta_k^{(i)}(\tau^*) = 0$, for $l = 1, 2, \cdots, n_d - 1$, then the following holds for $l = 1, 2, \cdots, n_d$:

$$\operatorname{sgn}\left(\Re\left((\tfrac{d}{d\tau})^l \lambda(\tau^*)\right)\right) = (-1)^{N_x(j\omega^*,\tau^*)} \operatorname{sgn}(\partial_\omega F(\omega^*, \tau^*)) \times \operatorname{sgn}\left((\tfrac{d}{d\tau})^l \theta_k^{(i)}(\tau^*)\right), \quad (26)$$

where $N_x(\omega^*, \tau^*)$ is the number of roots of $\hat{D}_{\omega^*\tau^*}(x)$ that are outside the unit disk.

Due to the space limitation, we can only provide a sketch of the proof. For the complete version, readers are refered to [20].

Proof: (Sketch) Let $\omega(\tau)$ be the differentiable function implicitly defined by (9) in some neighborhood of τ^*, which satisfies $\omega(\tau^*) = \omega^*$. For notational convenience, denote

$$a(\tau) = \theta_k^{(i)}(\tau). \quad (27)$$

With Assumption III, Proposition 3 implies that $\partial_\lambda D(j\omega^*, \tau^*) \neq 0$. Then by the implicit function theorem, $D(\lambda, \tau) = 0$ determines λ as a differentiable function of τ in a neighborhood of τ^*. Let $\lambda(\tau) = x(\tau) + jy(\tau)$, then x, y are real differentiable functions in a neighborhood of $\tau = \tau^*$, $\lambda = \lambda(\tau^*)$. We define a function $h(x, y, \tau)$ as

$$h(x, y, \tau) = D(x + jy, \tau), \quad (28)$$

then clearly in a neighborhood of $\tau = \tau^*$ we have

$$h(x(\tau), y(\tau), \tau) = 0. \quad (29)$$

Define $g(a, \omega, \tau) = \hat{D}(j\omega, \tau, \exp(ja - j\omega\tau))$ then it is easy to see

$$g(a(\tau), \omega(\tau), \tau) = 0, \quad (30)$$

for τ in some neighborhood of τ^*. Noticing the definition (27) and $x(\tau^*) = 0$, the following is obvious:

$$g(a(\tau^*), \cdot, \cdot) = h(x(\tau^*), \cdot, \cdot) = h(0, \cdot, \cdot). \quad (31)$$

Furthermore, it is possible to prove that for $k \leq n_d$

$$(\tfrac{d}{d\tau})^k x(\tau^*) = \frac{\Re(j\partial_y h(p_1)\overline{\partial_a g(p_2)})}{\Re(j\partial_y h(p_1)\overline{\partial_x h(p_1)})} (\tfrac{d}{d\tau})^k a(\tau^*). \quad (32)$$

where $P_1 = (0, \omega^*, \tau^*)$, $P_2 = (a(\tau^*), \omega^*, \tau^*)$.

Since at $\tau = \tau^*$, $\lambda = j\omega^*$ it holds that $\partial_a g(p_2) = j\sum_{i=1}^M iP_i e^{-ij\omega^*\tau^*}$, $\partial_y h(p_1) = j\partial_\lambda D(j\omega^*, \tau^*)$, $\partial_x h = \partial_\lambda D(j\omega^*, \tau^*)$, we obtain from (32)

$$\begin{aligned} \frac{d\Re(\lambda(\tau^*))}{d\tau} &= \frac{\Re(j\sum_{i=1}^M iP_i e^{-ij\omega^*\tau^*} \cdot \overline{\partial_\lambda D})}{|\partial_\lambda D|^2} a' \Big|_{\substack{\tau=\tau^* \\ \lambda=j\omega^*}} \\ &= \frac{-\Re(\sum_{i=1}^M iP_i e^{-ij\omega^*\tau^*} \cdot \overline{j\partial_\lambda D})}{|\partial_\lambda D|^2} a' \Big|_{\substack{\tau=\tau^* \\ \lambda=j\omega^*}}. \quad (33) \end{aligned}$$

Proposition 3 gives

$$\partial_\omega F(\omega^*, \tau^*) = -2c|P_M|^{2M} \Re(\sum_{i=1}^M iP_i e^{-ij\omega^*\tau^*} \cdot \overline{j\partial_\lambda D})_{\substack{\tau=\tau^* \\ \lambda=j\omega^*}},$$

where functions are evaluated at $\tau = \tau^*$, $\lambda = j\omega^*$, and from (20) it is easy to see $\operatorname{sgn}(c) = (-1)^{N_x(\omega^*,\tau^*)}$. By definition we also have $(\frac{d}{d\tau})^k a(\tau^*) = (\frac{d}{d\tau})^k \theta_k^{(i)}(\tau^*)$. Substitute these expressions into (33) and notice that $x(\tau)$ is differentiable at τ^*, (26) is proved. ■

The last theorem establishes an interesting link between system stability and the phase curves. In any given interval $\mathscr{I}^{(i)}$, following the graph of each $\theta_k^{(i)}(\tau)$, one can identify τ^* as a critical delay if the graph of $\theta_k^{(i)}(\tau)$ intersects any horizontal line located at $2r\pi$ for some integer r and deduce that $\pm j\omega_k^{(i)}(\tau^*)$ is a pair of imaginary characteristic roots corresponding to τ^*. Whether this pair of roots become stable or unstable as τ increases depends partially on how the graph of $\theta_k^{(i)}(\tau)$ crosses $2r\pi$, namely from below to above or vise versa. It also depends on the sign of $\partial_\omega F$ as well as N_x at $(\omega, \tau) = (\omega_k^{(i)}(\tau^*), \tau^*)$.

Regarding the last two factors, we have the following observation. First, the signature of the quantity $\partial_\omega F(\omega_k^{(i)}(\tau), \tau)$ is invariant in $(\tau^{(i-1)}, \tau^{(i)})$. This can be shown easily using a continuity argument as in [8]. Second, for any interval $U \in \mathscr{I}^{(i)}$ such that $U \cap \Phi_\tau = \{\phi\}$, the quantity $N_x(\omega_k^{(i)}(\tau), \tau)$ is also invariant over U. To see this is indeed true, first notice that for the roots of $\hat{D}_{\omega\tau}(x)$ to enter or leave the unite disk, it must first lie on $\partial\mathbb{D}$ since the roots of $\hat{D}_{\omega\tau}(x)$ is continuous with respect to the parameters. However, Proposition 1 indicates that the root of $\hat{D}_{\omega\tau}(x)$ on $\partial\mathbb{D}$ exists uniquely and is unrepeated given $\omega = \omega_k^{(i)}(\tau)$ and $\tau \in U$. Therefore we conclude that $N_x(\omega_k^{(i)}(\tau), \tau)$ must be constant over U. In the case when $\mathscr{I}^{(i)}$ is disjoint with Φ_τ, the quantity $N_x(\omega_k^{(i)}(\tau), \tau)$ is constant on $\mathscr{I}^{(i)}$.

In practice the term $\frac{d\theta_k^{(i)}(\tau^*)}{d\tau}$ can be determined simply based on the graphs of the phase functions. Here we provide a formula to compute it:

$$\frac{d\theta_k^{(i)}(\tau^*)}{d\tau} = -\frac{\sum_{l=0}^M \frac{d_F P_l}{d\tau} e^{lj\omega^*\tau^*}}{\sum_{l=1}^M lP_l e^{lj\omega^*\tau^*}}\Bigg|_{\substack{\lambda=j\omega^* \\ \tau=\tau^*}}, \quad (34)$$

where

$$\frac{d_F P_l}{d\tau}\Bigg|_{\substack{\lambda=j\omega^* \\ \tau=\tau^*}} = j\partial_\lambda P_l \frac{d\omega_k^{(i)}}{d\tau} + \partial_\tau P_l \Bigg|_{\substack{\lambda=j\omega^* \\ \tau=\tau^*}}. \quad (35)$$

This formula can be readily derived by differentiating the left hand side of (15) and using (18). However, since the crossing direction criterion (26) essentially depends on the monotonicity of the phase function at the critical delays, in practice one can determine the crossing direction simply based on the graph of the phase functions in the same spirit of the analysis in [10].

V. Numerical Example

We consider the characteristic equation (4) with the delay interval $\mathscr{I} = [0, 0.8]$ and $\alpha = 1.5$. By definition, we have

$$P_0 = \lambda - \alpha, \; P_1 = e^{\alpha\tau}, \; P_2 = e^{2\alpha\tau}.$$

The expression of function F can be obtained using (7) and (9) as

$$F = -\omega^4 + a_1\omega^2 + a_2,$$

where

$$a = e^{\tau\lambda},\ a_1 = 2a^4 + a^2 - \frac{9}{2},$$
$$a_2 = -a^8 + a^6 + \frac{15}{2}a^4 + \frac{9}{4}a^2 - \frac{81}{16}.$$

By solving (9) and (13) together for $(\omega, \tau) \in \mathbb{R}_+ \in \mathscr{I}$, we find that $\mathscr{I}$ can be decomposed into $\mathscr{I}^{(1)} = [\tau^{(0)}, \tau^{(1)}]$, $\mathscr{I}^{(2)} = [\tau^{(1)}, \tau^{(2)}]$ and $\tau^{(0)} = 0$, $\tau^{(1)} \approx 0.4045$, $\tau^{(2)} = 0.8$. Polynomial $F_\tau(\omega)$ has one real solutions in $\mathscr{I}^{(1)}$ and two real solutions in $\mathscr{I}^{(2)}$. Consequently functions $\omega_1^{(1)}$, $\omega_2^{(1)}$, $\omega_2^{(2)}$ are well defined in the corresponding intervals and the associated phase curves are plotted in Fig. 1. The graph of $\theta_1^{(1)}$ crosses the horizontal line 0 at $\tau_1 = 0.2368$ and the graphs of $\theta_1^{(2)}$, $\theta_2^{(2)}$ cross the horizontal line 2π at $\tau_2 = 0.6878$, $\tau_3 = 0.6976$ respectively. By definition, we have $\omega_{11} = \omega_1^{(1)}(\tau_1) \approx 2.9010$, $\omega_{12} = \omega_2^{(2)}(\tau_2) \approx 5.4195$, $\omega_{22} = \omega_1^{(2)}(\tau_3) \approx 5.6381$.

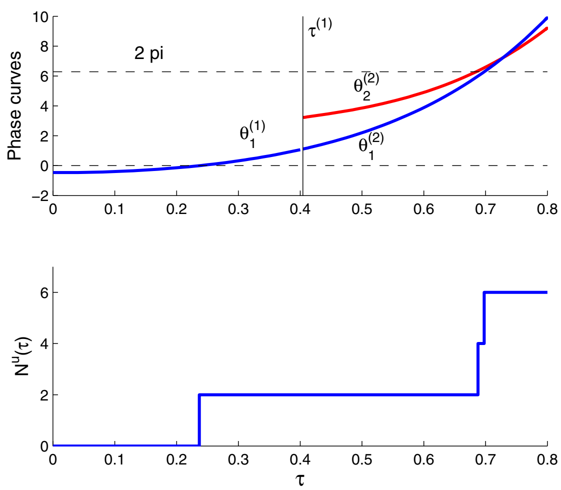

Fig. 1. The phase curves and the number of unstable roots for (4)

.

We verify that the following holds:

$$\partial_\omega F(\omega_1^{(1)}(\tau), \tau) < 0,\ N_x(\omega_1^{(1)}(\tau), \tau) = 1,\ \forall \tau \in \mathscr{I}_o^{(1)},$$
$$\partial_\omega F(\omega_1^{(2)}(\tau), \tau) < 0,\ N_x(\omega_1^{(2)}(\tau), \tau) = 1,\ \forall \tau \in \mathscr{I}_o^{(2)},$$
$$\partial_\omega F(\omega_2^{(2)}(\tau), \tau) > 0,\ N_x(\omega_2^{(2)}(\tau), \tau) = 0,\ \forall \tau \in \mathscr{I}_o^{(2)}.$$

Therefore, we conclude from (26) that $\mathrm{Inc}(\omega^*, \tau^*) = 1$ at all the three crossing points. In other words, the three pairs of imaginary roots $\pm j\omega_{11}$, $\pm j\omega_{12}$, $\pm j\omega_{22}$ all move toward the right half plane as the delay value increases through τ_1, τ_2, τ_3, respectively. It can be verified that $N^u(0) = 0$, then the number of unstable roots can be easily computed using (23) and is plotted against τ in the lower diagram of Fig. 1. It is clear that System (3) is $\alpha-$stable with $\alpha = 1.5$ for $\tau \in [0, \tau_1)$ and not $\alpha-$stable for $\tau \in [\tau_1, 0.8]$.

VI. Conclusion

A method of stability analysis for time-delay systems with coefficients depending on the delay is presented, which extends some of our previous results on systems with a single delay to systems with commensurate delays. The method partitions the delay interval of interest into disjoint subintervals so that a generalized magnitude condition yields a fixed number of solutions of frequencies ω as functions of the delay τ within each subinterval. We provided conditions for imaginary roots to appear at some critical delay values, followed by a criterion to identify cross-directions of these imaginary roots. Our analysis also shows an interesting connection between system stability and some functions' graphs, which allows us to conveniently carry out the stability analysis based on these graphs.

References

[1] K. L. Cooke and P. van den Driessche. "On zeroes of some transcendental equations," *Funkcialaj Ekvacioj*, vol. 29, pp. 77–90, 1986.

[2] F. G. Boese. "Stability with respect to the delay: On a paper by K.L. Cooke and P. van den Driessche," *J. Math. Anal. Appl*, vol. 228, pp. 293–321, 1998.

[3] K. Gu, V. L. Kharitonov, & J. Chen (2003). Stability of time-delay systems.

[4] Gu, K. (2012). A review of some subtleties of practical relevance for time-delay systems of neutral type. ISRN Applied Mathematics, Vol 2012, Article ID 725783, 46 pages, doi: 10.5402/2012/725783.

[5] K. Knopp. *Theory of Functions*, Parts I and II, Translated to English by F. Bagemihl, Dover, Mineola, NY, 1996.

[6] W. Michiels, & S. I. Niculescu, (2014). Stability, Control, and Computation for Time-Delay Systems: An Eigenvalue-Based Approach (Vol. 27). Siam.

[7] S. I. Niculescu, (2001). Delay effects on stability: a robust control approach, vol 269. Springer, Heidelberg.

[8] K. Gu, C. Jin, I. Boussaada and S. I. Niculescu, "Towards more general stability analysis of systems with delay-dependent coefficients," 2016 IEEE 55th Conference on Decision and Control (CDC), Las Vegas, NV, 2016, pp. 3161-3166.

[9] J. Chen, G. Gu, Carl N. Nett. A new method for computing delay margins for stability of linear delay systems, Systems & Control Letters, Volume 26, Issue 2, 22 September 1995, Pages 107-117.

[10] E. Beretta, Y. Kuang (2002). Geometric stability switch criteria in delay differential systems with delay dependent parameters. SIAM Journal on Mathematical Analysis, 33(5), 1144-1165.

[11] Wilmot-Smith, A. L., et al. "A time delay model for solar and stellar dynamos." The Astrophysical Journal 652.1 (2006): 696.

[12] F. Crauste, "Global Asymptotic Stability and Hopf Bifurcation for a Blood Cell Production Model." (2005), on-line document.

[13] K. Walton and J. E. Marshall. "Direct method for TDS stability analysis," *IEE Proc.* vol. 134, part D, pp. 101-107, 1987.

[14] Young NJ. An identity which implies Cohn's theorem on the zeros of a polynomial. Journal of Mathematical Analysis and Applications. 1979 Jul 31;70(1):240-8.

[15] L. E. El'Sgol'ts and S. B. Norkin. *Introduction to the Theory and Application of Differential Equations with Deviating Arguments,* Translated by J. L. Casti, Academic Press, New York, 1973.

[16] M. S. Lee and C. S. Hsu. "On the τ-decomposition method of stability analysis for retarded dynamical systems," *SIAM J. Control*, 7:249, 259, 1969.

[17] D. Hertz, E.J. Jury and E. Zeheb, Stability independent and dependent of delay for delay differential systems, J. Franklin Inst. 318(3) (1984) 143-150.

[18] S. -I. Niculescu and W. Michiels, "Stabilizing a chain of integrators using multiple delays," in IEEE Transactions on Automatic Control, vol. 49, no. 5, pp. 802-807, May 2004.

[19] C. Jin, S. I. Niculescu, I. Boussaada, K. Gu, "Stability Analysis of Control Systems subject to Delay-Difference Feedback," Proceeding of the 20th IFAC World Congress, 2017, Toulouse, France.

[20] C. Jin, " Stability Analysis of Systems with Delay-Dependent Coefficients, " PhD thesis, Paris-sud University, 2017.