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Output-Feedback Model Matching with Strong Stability for Hybrid Linear Systems with Periodic State Jumps

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Abstract— **This paper investigates the problem of model matching with strong stability for hybrid linear systems subject to periodic state jumps. The system and the model share the same hybrid time domain: i.e., they exhibit a continuous-time behavior except at certain isolated time instants, where their state shows abrupt discontinuities. The set of the admissible jump time sequences consists of the sequences of equally-spaced time instants whose period is no smaller than a positive real constant. The compensation scheme includes a hybrid dynamic regulator, defined over the same time domain, and a static output-feedback gain. The problem is to determine the hybrid regulator, the feedback gain and the positive time constant in such a way that: i) the forced response of the closed-loop hybrid system matches that of the hybrid model for all the admissible inputs and for all the admissible jump time sequences; ii) both the closed-loop hybrid system and the hybrid regulator are globally asymptotically stable for all the admissible jump time sequences. The existence of a solution to the stated problem is characterized by a necessary and sufficient condition.**

I. Introduction

Hybrid systems have been the object of extensive research during the last twenty years [1]–[6]. A hybrid linear system with state jumps (or linear impulsive system) is characterized by a continuous-time linear dynamics (flow dynamics) and a discrete-time linear dynamics (jump dynamics), which rule the state evolution on a time domain defined as a sequence of continuous-time intervals separated by single jump times. The interplay between the continuous and the discrete dynamics raises challenging issues when stabilization, control or observation problems are faced. Several topics have been investigated in the recent literature: e.g., stabilization [7]–[9], state estimation [10], linear quadratic control [11], output regulation [12]–[15], disturbance decoupling [16], unknown input observation [17] and model matching [18].

As to the discussion on model matching for hybrid linear systems presented in [18], it must be noticed that it only deals with the structural aspect of the problem: i.e., only the perfect match between the forced responses of the system and the model is required, while stability is not considered. On the contrary, model matching with stability has been extensively studied for other classes of hybrid systems, like switched systems [19]–[23], and has inspired interesting applications to fault detection [24], adaptive control [25], and control of asynchronous sequential machines [26]. Hence, in consideration of the wide use of hybrid linear systems with state jumps for handling real systems that encompass both analog and digital components, a more comprehensive study, also including stability, is worthwhile. The novel contribution of this work with respect to [18] is therefore that of presenting a thorough investigation, where the stability issues are taken into account, of the problem of model matching for hybrid linear systems.

In this work, the admissible sequences of jump times are assumed to be periodic, with the constraint that the length of the continuous-time interval is assumed to be greater than or equal to a certain positive time constant. Moreover, as far as stability is concerned, the focus is on strong stability: i.e., model matching is sought not only with global asymptotic stability of the closed-loop connection (by a nonzero gain matrix) of the system and the compensator, but also with global asymptotic stability of the compensator itself. It is worth mentioning that strong stability was introduced and discussed in [27]–[29], mainly for the benefits of achieving various control targets by means of a stable compensator.

Notation: $\mathbb{Z}_0^+$, $\mathbb{Z}^+$, $\mathbb{R}$, and $\mathbb{R}_0^+$ stand for the sets of nonnegative integer numbers, positive integer numbers, real numbers, and nonnegative real numbers, respectively. Matrices and linear maps are denoted by upper-case slanted letters, like A. The image and the kernel of A are denoted by $\operatorname{Im} A$ and $\operatorname{Ker} A$, respectively. The transpose of A is denoted by $A^\top$. The inverse of a nonsingular square matrix T is denoted by T^{-1}. The notation $\operatorname{diag}\{M_1,\dots,M_k\}$, where $M_1,\dots,M_k$ are square matrices, stands for a block diagonal matrix whose blocks on the main diagonal are $M_1,\dots,M_k$. Vector spaces and subspaces are denoted by calligraphic letters, like $\mathcal{V}$. The notation $B^{-1}\mathcal{V}$, where B is not required to be invertible, denotes the inverse image of $\mathcal{V}$ through B. The notation $[t_0,\, t_1)$ stands for the right-open real interval delimited by t_0 and t_1.

II. Output-Feedback Model Matching with Strong Stability — Problem Statement

Let τ be a positive real constant. The jump time sequence is a sequence of equally-spaced time instants defined as $\mathcal{T} = \{k\,\tau;\ k \in \mathbb{Z}^+\}$. Given a positive real constant τ_d, the set of all sequences $\mathcal{T}$ such that $\tau \geq \tau_d$ is defined as the set of the admissible jump time sequences and is denoted by $\mathscr{T}_{\tau_d}$.

The system Σ_P is defined as the hybrid linear system

$$\Sigma_P \equiv \begin{cases} \dot{x}_P(t) \;=\; A_P\, x_P(t) + B_P\, u(t), \\ \qquad\qquad t \in [k\,\tau, (k+1)\,\tau),\ \ k \in \mathbb{Z}_0^+, \\ x_P(t) \;=\; J_P\, x_P(t^-),\ \ t = k\,\tau,\ k \in \mathbb{Z}^+, \\ y_P(t) \;=\; C_P\, x_P(t),\ \ t \in \mathbb{R}_0^+, \end{cases}$$

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where $x_P \in \mathcal{X}_P = \mathbb{R}^{n_P}$ is the state, $u \in \mathbb{R}^p$ is the control input, and $y_P \in \mathbb{R}^q$ is the output, with $p, q \leq n_P$. A_P, B_P, J_P, and C_P are constant real matrices of suitable dimensions. B_P and C_P are assumed to be full-rank. The set of the admissible control input functions $u(t)$, with $t \in \mathbb{R}_0^+$, is defined as the set of all piecewise-continuous functions with values in $\mathbb{R}^p$. The differential state equation rules the flow dynamics, while the jump dynamics is governed by the algebraic state equation. Hence, the state motion $x_P(t)$ in $[0, \tau)$ is the solution of the differential equation, given the initial state $x_P(0) = x_{P,0}$ and the input function $u(t)$, with $t \in [0, \tau)$. When $t = k\,\tau$, with $k \in \mathbb{Z}^+$, the state $x_P(t)$ is the image of $x_P(t^-) = \lim_{\varepsilon \to 0^+} x_P(t-\varepsilon)$ through J_P. The state motion $x_P(t)$ in the time interval $[k\,\tau,\,(k+1)\,\tau)$, with $k \in \mathbb{Z}^+$, is the solution of the differential equation, given the initial state $x_P(k\,\tau)$ and the input function $u(t)$, with $t \in [k\,\tau,\,(k+1)\,\tau)$.

It is worth recalling that, given the period $\tau > 0$, the hybrid linear system Σ_P is globally asymptotically stable if and only if the state transition matrix over one period — i.e., $\Phi_P(\tau) = J_P\, e^{A_P\,\tau}$ — is Schur stable. It is also worth mentioning that, on the assumption that the flow dynamic matrix A_P is Hurwitz stable, if Σ_P is globally asymptotically stable for a jump time sequence with a given period τ_d, then it is globally asymptotically stable for all the jump time sequences with period $\tau \geq \tau_d$. Global asymptotic stability of Σ_P for all the jump time sequences with period $\tau \geq \tau_d$ is also referred to as global asymptotic stability over $\mathscr{T}_{\tau_d}$. Moreover, it is worth mentioning that global asymptotic stability of Σ_P over $\mathscr{T}_{\tau_d}$ implies asymptotic stability of the flow dynamics (as the limit case when $\tau \to \infty$ — i.e., when no jumps occur).

The model Σ_M is defined as the hybrid linear system

$$\Sigma_M \equiv \begin{cases} \dot{x}_M(t) \;=\; A_M\, x_M(t) + B_M\, d(t), \\ \qquad\qquad t \in [k\,\tau, (k+1)\,\tau),\; k \in \mathbb{Z}_0^+, \\ x_M(t) \;=\; J_M\, x_M(t^-),\; t = k\,\tau,\, k \in \mathbb{Z}^+, \\ y_M(t) \;=\; C_M\, x_M(t),\; t \in \mathbb{R}_0^+, \end{cases}$$

where $x_M \in \mathbb{R}^{n_M}$ is the state, $d \in \mathbb{R}^s$ is the input, and $y_M \in \mathbb{R}^q$ is the output. The set of the admissible input functions $d(t)$, with $t \in \mathbb{R}_0^+$, is defined as the set of all piecewise-continuous functions with values in $\mathbb{R}^s$.

Both the system Σ_P and the model Σ_M are assumed to be globally asymptotically stable over $\mathscr{T}_{\tau_d}$, for a given τ_d.

The output-feedback compensation scheme consists of a hybrid linear compensator and a nonzero output-feedback gain matrix. The compensator Σ_R is described by

$$\Sigma_R \equiv \begin{cases} \dot{x}_R(t) \;=\; A_R\, x_R(t) + B_R\, h(t), \\ \qquad\qquad t \in [k\,\tau, (k+1)\,\tau),\; k \in \mathbb{Z}_0^+, \\ x_R(t) \;=\; J_R\, x_R(t^-),\; t = k\,\tau,\, k \in \mathbb{Z}^+, \\ u(t) \;\;=\; C_R\, x_R(t) + D_R\, h(t),\; t \in \mathbb{R}_0^+, \end{cases}$$

where $x_R \in \mathbb{R}^{n_R}$ is the state and $h \in \mathbb{R}^s$ is the input. The gain matrix $K \in \mathbb{R}^{s \times q}$ realizes the feedback connection of the system output y_P on the compensator input h, so that $h(t) = v(t) - K\, y_P(t)$, where the new set of the admissible input functions $v(t)$, with $t \in \mathbb{R}_0^+$, is still defined as the set of all piecewise-continuous functions with values in $\mathbb{R}^s$. The assumption that the gain matrix K is different from zero excludes the trivial case of the open-loop cascade connection between Σ_R and Σ_P.

The closed-loop hybrid linear system Σ_L is defined as the closed-loop connection of Σ_R and Σ_P through the output-feedback gain matrix K. Thus, with $x_L = [x_P^\top\, x_R^\top]^\top$, one gets

$$\Sigma_L \equiv \begin{cases} \dot{x}_L(t) \;=\; A_L\, x_L(t) + B_L\, v(t), \\ \qquad\qquad t \in [k\,\tau, (k+1)\,\tau),\; k \in \mathbb{Z}_0^+, \\ x_L(t) \;=\; J_L\, x_L(t^-),\; t = k\,\tau,\, k \in \mathbb{Z}^+, \\ y_P(t) \;=\; C_L\, x_L(t),\; t \in \mathbb{R}_0^+, \end{cases}$$

where

$$\begin{aligned} A_L &= \begin{bmatrix} A_P - B_P\, D_R\, K\, C_P & B_P\, C_R \\ -B_R\, K\, C_P & A_R \end{bmatrix}, \\ B_L &= \begin{bmatrix} B_P\, D_R \\ B_R \end{bmatrix}, \quad J_L = \begin{bmatrix} J_P & 0 \\ 0 & J_R \end{bmatrix}, \\ C_L &= \begin{bmatrix} C_P & 0 \end{bmatrix}. \end{aligned}$$

With reference to the dynamic systems previously introduced, the problem of output-feedback model matching with strong stability is stated as follows.

Problem 1: (Output-feedback model matching with strong stability) Let the hybrid linear system Σ_P and the hybrid linear model Σ_M be given. Find a hybrid linear compensator Σ_R and a nonzero gain matrix K, such that:

$\mathcal{R}\,1$. the output $y_P(t)$ of the closed-loop hybrid linear system Σ_L and the output $y_M(t)$ of the hybrid linear model Σ_M are equal for all $t \in \mathbb{R}_0^+$, when Σ_L and Σ_M are subject to the same input (i.e., $d(t) = v(t)$, with $t \in \mathbb{R}_0^+$) and their initial states are zero, for all the admissible inputs $v(t)$, with $t \in \mathbb{R}_0^+$, and for all the admissible jump time sequences $\mathcal{T} \in \mathscr{T}_{\tau_d}$;

$\mathcal{R}\,2$. there exists a positive real constant $\bar{\tau}_d$ such that both the hybrid linear dynamics of the closed-loop system Σ_L and of the compensator Σ_R are globally asymptotically stable over $\mathscr{T}_{\bar{\tau}_d}$.

It is worth mentioning that Problem 1 could have been given a more general formulation by referring to hybrid linear systems that feature a control input also in the jump dynamics and two different output equations (one for the continuos-time intervals and the other for the jump time instants). Nevertheless, dealing with such more general statement would have entailed more intricate technicalities but substantially equivalent concepts.

III. Structural Equivalence to Measurable Disturbance Decoupling — Problem Statement

This section deals with the structural version of the output-feedback model matching problem and states its equivalence to the structural measurable disturbance decoupling problem.

The structural output-feedback model matching problem (Problem 2) is a simpler version of Problem 1, where the global asymptotic stability of the dynamics involved is not considered and the gain K is assumed to be fixed and known.

Problem 2: (Structural output-feedback model matching) Let the hybrid linear system Σ_P, the hybrid linear model Σ_M and the nonzero gain matrix K be given. Find a hybrid linear compensator Σ_R such that Requirement $\mathcal{R}\,1$ of Problem 1 is satisfied.

A necessary and sufficient condition for the solution of Problem 2 is derived by exploiting the equivalence with the structural measurable disturbance decoupling problem stated for the so-called extended hybrid linear system. To this aim, a modified hybrid linear model Σ'_M is first obtained by closing a feedback of the output, with the gain matrix K, on the flow dynamics. Then, the extended hybrid linear system Σ is defined as the output-difference connection of the hybrid linear system Σ_P with the modified hybrid linear model Σ'_M.

The modified hybrid linear model Σ'_M is defined by

$$\Sigma'_M \equiv \begin{cases} \dot{x}_M(t) \,=\, (A_M + B_M\, K\, C_M)\, x_M(t) + B_M\, w(t), \\ \qquad\qquad t \in [k\,\tau, (k+1)\,\tau),\ k \in \mathbb{Z}_0^+, \\ x_M(t) \,=\, J_M\, x_M(t^-),\ t = k\,\tau,\ k \in \mathbb{Z}^+, \\ y_M(t) \,=\, C_M\, x_M(t), \quad t \in \mathbb{R}_0^+, \end{cases}$$

where the set of the admissible input functions is the set of all piecewise-continuous functions $w(t)$, with $t \in \mathbb{R}_0^+$, whose values are in $\mathbb{R}^s$. Moreover, the extended hybrid linear system Σ is defined by taking the input u of Σ_P and the input w of Σ'_M as its respective control input and measurable disturbance input and by taking the difference between the outputs y_P and y_M of Σ_P and Σ'_M as its output. Thus, with $x = [x_P^\top\, x_M^\top]^\top$, one gets

$$\Sigma \equiv \begin{cases} \dot{x}(t) \,=\, A\, x(t) + B\, u(t) + H\, w(t), \\ \qquad\qquad t \in [k\,\tau, (k+1)\,\tau),\ k \in \mathbb{Z}_0^+, \\ x(t) \,=\, J\, x(t^-),\ t = k\,\tau,\ k \in \mathbb{Z}^+, \\ y(t) \,=\, C\, x(t),\ t \in \mathbb{R}_0^+, \end{cases}$$

where

$$A \,=\, \begin{bmatrix} A_P & 0 \\ 0 & A_M + B_M\, K\, C_M \end{bmatrix},\ B = \begin{bmatrix} B_P \\ 0 \end{bmatrix}, \quad (1)$$

$$H \,=\, \begin{bmatrix} 0 \\ B_M \end{bmatrix},\ J = \begin{bmatrix} J_P & 0 \\ 0 & J_M \end{bmatrix}, \quad (2)$$

$$C \,=\, \begin{bmatrix} C_P & -C_M \end{bmatrix}. \quad (3)$$

Let the state space of Σ be denoted by $\mathcal{X}$ and let n denote its dimension, so that $\mathcal{X} = \mathbb{R}^n$, with $n = n_P + n_M$.

Further, to state the structural problem of measurable disturbance decoupling, the compensated hybrid linear system $\hat{\Sigma}$ is defined as

$$\hat{\Sigma} \equiv \begin{cases} \dot{\hat{x}}(t) \,=\, \hat{A}\, \hat{x}(t) + \hat{H}\, w(t),\ t \in [k\tau, (k+1)\tau),\ k \in \mathbb{Z}_0^+, \\ \hat{x}(t) \,=\, \hat{J}\, \hat{x}(t^-),\ t = k\,\tau,\ k \in \mathbb{Z}^+, \\ y(t) \,=\, \hat{C}\, \hat{x}(t),\ t \in \mathbb{R}_0^+, \end{cases}$$

where

$$\hat{A} \,=\, \begin{bmatrix} A & B\, C_R \\ 0 & A_R \end{bmatrix},\ \hat{H} = \begin{bmatrix} B\, D_R + H \\ B_R \end{bmatrix}, \quad (4)$$

$$\hat{J} \,=\, \begin{bmatrix} J & 0 \\ 0 & J_R \end{bmatrix},\ \hat{C} = \begin{bmatrix} C & 0 \end{bmatrix}. \quad (5)$$

Problem 3: (Structural measurable disturbance decoupling) Let the extended hybrid linear system Σ be given. Find a hybrid linear compensator Σ_R such that the compensated hybrid linear system $\hat{\Sigma}$ satisfies the requirement that the output $y(t)$ is zero, for all $t \in \mathbb{R}_0^+$, when the initial state is zero, for all the admissible inputs $w(t)$, with $t \in \mathbb{R}_0^+$, and for all the admissible jump time sequences $\mathcal{T} \in \mathscr{T}_{\tau_d}$.

The equivalence between Problem 2 and Problem 3 is formalized in the following proposition, whose proof, which is a matter of simple algebraic manipulation, is omitted.

Proposition 1: A hybrid linear compensator Σ_R solves Problem 3 if and only if it solves Problem 2.

IV. Structural Equivalence to Measurable Disturbance Decoupling — Problem Solution

The aim of this section is threefold: i) the notions needed to characterize the solvability of the structural measurable disturbance decoupling problem are presented; ii) a necessary and sufficient condition for solvability of the structural measurable disturbance decoupling problem is established; iii) the choice of the nonzero gain matrix is shown to not influence the existence of structural solutions.

The structural notions needed to state the necessary and sufficient condition for solvability of Problem 3 are first reviewed [7], [10], [12]–[15]. Such concepts are expressed with reference to the extended hybrid linear system Σ (however the structure of the matrices of Σ in (1)–(3) plays no role). The symbols $\mathcal{B}$, $\mathcal{H}$, and $\mathcal{C}$ respectively stand for $\operatorname{Im} B$, $\operatorname{Im} H$, and $\operatorname{Ker} C$. The notions of hybrid invariance and hybrid controlled invariance (denoted by $\mathscr{H}$) are introduced as follows. A subspace $\mathcal{V} \subseteq \mathcal{X}$ is said to be an $\mathscr{H}$-invariant subspace if $A\,\mathcal{V} \subseteq \mathcal{V}$ and $J\,\mathcal{V} \subseteq \mathcal{V}$. A subspace $\mathcal{V}$ is said to be an $\mathscr{H}$-controlled invariant subspace if $A\,\mathcal{V} \subseteq \mathcal{V} + \mathcal{B}$ and $J\,\mathcal{V} \subseteq \mathcal{V}$. Moreover, a subspace $\mathcal{V}$ is an $\mathscr{H}$-controlled invariant subspace if and only if there exists a linear map F such that $(A + B\,F)\,\mathcal{V} \subseteq \mathcal{V}$ holds along with $J\,\mathcal{V} \subseteq \mathcal{V}$. Any such F is said to be a friend of $\mathcal{V}$. Moreover, a subspace $\mathcal{V}$, with dimension ν and a basis matrix V, is an $\mathscr{H}$-controlled invariant subspace if and only there exist $L_A, L_J \in \mathbb{R}^{\nu \times \nu}$ and $M \in \mathbb{R}^{p \times \nu}$ such that

$$A\, V \,=\, V\, L_A + B\, M, \quad (6)$$

$$J\, V \,=\, V\, L_J. \quad (7)$$

In general, if such matrices exist, they are not unique.

Simple algebraic arguments show that the set of all $\mathscr{H}$-controlled invariant subspaces contained in a given subspace $\mathcal{C} \subseteq \mathcal{X}$ is an upper semilattice with respect to the sum and the inclusion. The maximum of the semilattice is denoted by $\mathcal{V}^*_{\mathscr{H}}$. The subspace $\mathcal{V}^*_{\mathscr{H}}$ is the last term of the sequence $\mathcal{V}_0 = \mathcal{C}$, $\mathcal{V}_i = \mathcal{V}_{i-1} \cap A^{-1}\,(\mathcal{V}_{i-1} + \mathcal{B}) \cap J^{-1}\,\mathcal{V}_{i-1}$, with $i = 1, \ldots, k$, where k is the least integer such that $\mathcal{V}_{k+1} = \mathcal{V}_k$. The subspace $\mathcal{V}^*_{\mathscr{H}}$ plays a crucial role in the necessary and sufficient condition for the existence of a solution to Problem 3.

Theorem 1: Let the extended hybrid system Σ be given. Problem 3 has a solution if and only if

$$\mathcal{H} \subseteq \mathcal{V}^*_{\mathscr{H}} + \mathcal{B}. \quad (8)$$

The proof of Theorem 1 is omitted since it can be derived along the same lines of that of Theorem 1 in [18]. Actually, in [18], the jump times are not uniformly spaced and the output-feedback gain matrix is the identity. However, none of these features affect the existence of structural solutions. In fact, as to the periodic jump time sequence, this only influences stability, as will be shown later. As to the (nonzero) output-feedback gain matrix, the irrelevance of its specific value to the existence of structural solutions is explicited below.

Proposition 2: Let the hybrid linear system Σ_P, the hybrid linear model Σ_M and the nonzero gain matrix K be given. Consider the extended hybrid linear system Σ. Let (8) hold. Then, $\mathcal{V}^*_{\mathscr{H}}$ is independent of K.

Proof: Let

$$V = \begin{bmatrix} V_P^\top & V_M^\top \end{bmatrix}^\top, \tag{9}$$

where the partition is consistent with that of (1)–(3), be a basis matrix of $\mathcal{V}^*_{\mathscr{H}}$. Since $\mathcal{V}^*_{\mathscr{H}}$ is an $\mathscr{H}$-controlled invariant subspace of Σ, there exist L_A, L_J, and M such that (6)–(7) hold. In light of (1) and (9), (6) can be written as

$$\begin{bmatrix} A_P & 0 \\ 0 & A_M + B_M\, K\, C_M \end{bmatrix} \begin{bmatrix} V_P \\ V_M \end{bmatrix} = \begin{bmatrix} V_P \\ V_M \end{bmatrix} L_A + \begin{bmatrix} B_P \\ 0 \end{bmatrix} M. \tag{10}$$

Moreover, since (8) holds, there exist L'_A and M' such that

$$\begin{bmatrix} 0 \\ B_M \end{bmatrix} = \begin{bmatrix} V_P \\ V_M \end{bmatrix} L'_A + \begin{bmatrix} B_P \\ 0 \end{bmatrix} M'. \tag{11}$$

Let $K + K'$ be a different nonzero gain matrix. Note that

$$\begin{bmatrix} A_P & 0 \\ 0 & A_M + B_M\, K\, C_M + B_M\, K'\, C_M \end{bmatrix} \begin{bmatrix} V_P \\ V_M \end{bmatrix} = \begin{bmatrix} A_P & 0 \\ 0 & A_M + B_M\, K\, C_M \end{bmatrix} \begin{bmatrix} V_P \\ V_M \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & B_M\, K'\, C_M \end{bmatrix} \begin{bmatrix} V_P \\ V_M \end{bmatrix}.$$

Hence, by (10) and (11), $L''_A = L_A + L'_A\, K'\, C_M\, V_M$ and $M'' = M + M'\, K'\, C_M\, V_M$ are such that

$$\begin{bmatrix} A_P & 0 \\ 0 & A_M + B_M\, K\, C_M + B_M\, K'\, C_M \end{bmatrix} \begin{bmatrix} V_P \\ V_M \end{bmatrix} = \begin{bmatrix} V_P \\ V_M \end{bmatrix} L''_A + \begin{bmatrix} B_P \\ 0 \end{bmatrix} M'',$$

which proves that $\mathcal{V}^*_{\mathscr{H}}$ is an $\mathscr{H}$-controlled invariant subspace for the extended hybrid linear system Σ', whose flow dynamic matrix is $A' = \operatorname{diag}\{A_P,\, A_M + B_M\,(K + K')\,C_M\}$. Then, it will be shown that $\mathcal{V}^*_{\mathscr{H}}$ is the maximal $\mathscr{H}$-controlled invariant subspace for Σ' contained in $\mathcal{C}$. Let ${\mathcal{V}^*_{\mathscr{H}}}' \supseteq \mathcal{V}^*_{\mathscr{H}}$ be the maximal $\mathscr{H}$-controlled invariant subspace for Σ' contained in $\mathcal{C}$. Then, the same arguments above show that ${\mathcal{V}^*_{\mathscr{H}}}'$ is an $\mathscr{H}$-controlled invariant subspace for Σ contained in $\mathcal{C}$, which implies, in particular, that ${\mathcal{V}^*_{\mathscr{H}}}' \subseteq \mathcal{V}^*_{\mathscr{H}}$, since the latter is the maximal $\mathscr{H}$-controlled invariant subspace for Σ contained in $\mathcal{C}$. Therefore, ${\mathcal{V}^*_{\mathscr{H}}}' = \mathcal{V}^*_{\mathscr{H}}$. ■

V. Output-Feedback Model Matching with Strong Stability — Problem Solution

The aim of this section is to provide a necessary and sufficient condition for the solvability of the output-feedback model matching problem with strong stability. To this aim, it is worth noting that the necessary and sufficient condition of Theorem 1 can be formulated in a modified form which considers, in place of $\mathcal{V}^*_{\mathscr{H}}$, any $\mathscr{H}$-controlled invariant subspace, $\mathcal{V}_{\mathscr{H}}$ that satisfies

$$\mathcal{V}_{\mathscr{H}} \subseteq \mathcal{C}, \quad \mathcal{H} \subseteq \mathcal{V}_{\mathscr{H}} + \mathcal{B}. \tag{12}$$

Hence, in the computation of a hybrid linear compensator that solves Problem 3 (or, equivalently, Problem 2, by Proposition 1), the subspace $\mathcal{V}^*_{\mathscr{H}}$ can be replaced by $\mathcal{V}_{\mathscr{H}}$ — the procedure for synthesizing the compensator based on $\mathcal{V}_{\mathscr{H}}$ can be simply derived by repeating the constructive proof of sufficiency of [18, Theorem 1] with $\mathcal{V}_{\mathscr{H}}$ in place of $\mathcal{V}^*_{\mathscr{H}}$. The disadvantage of considering a generic $\mathcal{V}_{\mathscr{H}}$ in place of $\mathcal{V}^*_{\mathscr{H}}$ is that no algorithm is available for computing such $\mathcal{V}_{\mathscr{H}}$ satisfying (12). However, as will be shown in the following, a special choice of $\mathcal{V}_{\mathscr{H}}$, characterized by maximality with respect to suitably-defined stabilizability properties, will allow us to state a necessary and sufficient constructive condition also for the output-feedback model matching problem with strong stability. To this aim, some definitions and properties concerning a generic $\mathscr{H}$-controlled invariant subspace $\mathcal{V}_{\mathscr{H}}$ need to be introduced.

Given an $\mathscr{H}$-controlled invariant subspace $\mathcal{V}_{\mathscr{H}}$ and a friend $F : \mathcal{X} \to \mathcal{U}$, let Σ_F be the compensated hybrid linear system defined by

$$\Sigma_F \equiv \begin{cases} \dot{x}(t) \;=\; (A + B\,F)\,x(t) + H\,w(t), \\ \qquad\qquad t \in [k\,\tau, (k+1)\,\tau),\ k \in \mathbb{Z}_0^+, \\ x(t) \;=\; J\,x(t^-),\ t = k\,\tau,\ k \in \mathbb{Z}^+, \\ y(t) \;=\; C\,x(t),\ t \in \mathbb{R}_0^+. \end{cases}$$

Since $\mathcal{V}_{\mathscr{H}}$ is an $\mathscr{H}$-invariant subspace for Σ_F, the hybrid linear dynamics of Σ_F induces, by restriction, a hybrid linear dynamics on $\mathcal{V}_{\mathscr{H}}$, henceforth denoted by $\Sigma_F|_{\mathcal{V}_{\mathscr{H}}}$. Hence, the following definition is well-posed.

Definition 1: An $\mathscr{H}$-controlled invariant subspace $\mathcal{V}_{\mathscr{H}}$ is said to be inner stabilizable if there exists a friend $F : \mathcal{X} \to \mathcal{U}$ such that $\Sigma_F|_{\mathcal{V}_{\mathscr{H}}}$ is globally asymptotically stable over $\mathscr{T}_{\tau_d}$, for some $\tau_d > 0$.

If V is a basis matrix $\mathcal{V}_{\mathscr{H}}$, the matrices L_A and L_J which solve (6) for $M = -\,F\,V$ and (7) respectively define the flow and the jump dynamics of $\Sigma_F|_{\mathcal{V}_{\mathscr{H}}}$. $\mathscr{H}$-inner stabilizability is therefore equivalent to the existence of $\tau_d > 0$ such that $L_J\, e^{L_A\, \tau}$ is Schur stable for all $\tau \geq \tau_d$. If L_A is Hurwitz stable, $\mathcal{V}_{\mathscr{H}}$ is inner stabilizable. In particular, if the $\mathscr{H}$-controlled invariant subspace $\mathcal{V}_{\mathscr{H}}$ is inner stabilizable with respect to the linear system which define the flow dynamics, then it is inner stabilizable.

An important notion that is newly introduced here is that of maximal inner stabilizable $\mathscr{H}$-controlled invariant subspace for Σ contained in a given subspace $\mathcal{K}$. In order to show the existence of such subspace, that will be called *good* and denoted by $\mathcal{V}^*_{\mathscr{H}\,g}(\mathcal{K})$, and to provide a procedure to derive it, let us recall that the set of all controlled invariant subspaces for the flow dynamics that are contained in a given subspace $\mathcal{K} \subseteq \mathcal{X}$ and are inner stabilizable has a maximal element, which is called the maximal good controlled invariant subspace and which is denoted by $\mathcal{V}_g(\mathcal{K})$. Such subspace was introduced, together with a procedure to derive it, in [30, Section 5.6]. For the sake of brevity, the following proposition (like the next one) is presented without proof.

Proposition 3: Given the hybrid linear system Σ and a subspace $\mathcal{K} \subseteq \mathcal{X}$, consider the sequence of subspaces $\mathcal{V}^j$ defined by $\mathcal{K}^0 = \mathcal{K}$, $\mathcal{W}^0 = \mathcal{V}_g(\mathcal{K}^0)$, $\mathcal{V}^0 = \mathcal{V}^*_{\mathscr{H}}(\mathcal{W}^0)$ and $\mathcal{K}^j = \mathcal{V}^{j-1}$, $\mathcal{W}^j = \mathcal{V}_g(\mathcal{K}^j)$, $\mathcal{V}^j = \mathcal{V}^*_{\mathscr{H}}(\mathcal{W}^j)$, for $j = 0, 1, \ldots, \ell$, where ℓ, with $0 \leq \ell \leq \dim \mathcal{K}$, is the least integer such that $\mathcal{V}^\ell$ is inner stabilizable. The last term of the above sequence, namely $\mathcal{V}^\ell$, is the maximal inner stabilizable $\mathscr{H}$-controlled invariant subspace for Σ contained in $\mathcal{K}$.

In order to state the necessary and sufficient condition for solvability of Problem 1, the maximal inner stabilizable $\mathscr{H}$-controlled invariant subspace contained in $\mathcal{C}$ is considered. To avoid heavier notation, such subspace is simply denoted by $\mathcal{V}^*_{\mathscr{H}\,g}$. The next proposition shows that if the inclusion

$$\mathcal{H} \subseteq \mathcal{V}^*_{\mathscr{H}\,g} + \mathcal{B} \tag{13}$$

holds for a given nonzero gain matrix K with the property of preserving global asymptotic stability over $\mathscr{T}_{\tau_d}$ of the modified model Σ'_M, then $\mathcal{V}^*_{\mathscr{H}\,g}$ is independent of K and, consequently, (13) holds for any K with the same property.

Proposition 4: Let the hybrid linear system Σ_P and the hybrid linear model Σ_M be given. Let K be a nonzero gain matrix such that the modified model Σ'_M is globally asymptotically stable over $\mathscr{T}_{\tau_d}$. Consider the hybrid linear system Σ. Let $\mathcal{V}^*_{\mathscr{H}\,g}$ be the maximal inner stabilizable $\mathscr{H}$-controlled invariant subspace of Σ contained in $\mathcal{C}$. Then, $\mathcal{V}^*_{\mathscr{H}\,g}$ is independent of the specific K as long as global asymptotic stability over $\mathscr{T}_{\tau_d}$ of Σ'_M is maintained.

Lemma 1: Let the hybrid linear system Σ_P and the hybrid linear model Σ_M be given. Let K be a nonzero gain matrix such that the modified model Σ'_M is globally asymptotically stable over $\mathscr{T}_{\tau'}$, for some $\tau' > 0$. Let the hybrid linear compensator Σ_R solve the corresponding Problem 3. Then, Σ_L is globally asymptotically stable over $\mathscr{T}_{\tau_L}$, for some $\tau_L > 0$, if and only if Σ_R is globally asymptotically stable over $\mathscr{T}_{\tau_R}$, for some $\tau_R > 0$.

Proof: Recall that, since the hybrid linear compensator Σ_R solves, by assumption, Problem 3, it also solves Problem 2, by virtue of Proposition 1.

If. The Kalman decomposition of the flow dynamics of the cascade connection between Σ_R and Σ_P shows that the static output feedback loop through K only modifies the modes that are output-observable and input-excitable. In other terms, the modes of the flow dynamics of Σ_L that are not output-observable or that are not input-excitable coincide with the modes of the same kind of the flow dynamics of the cascade connection. Since both Σ_R and Σ_P are globally asymptotically stable over $\mathscr{T}_{\tau_R}$ and $\mathscr{T}_{\tau_d}$, respectively, all the modes of the associated flow dynamics are convergent and so are the modes of the cascade composition. On the other hand, since Requirement $\mathcal{R}\,1$ is met, the modes of the flow dynamics of Σ_L that are output-observable and input-excitable coincide with the modes of the same kind of the flow dynamics of Σ_M, that are convergent since Σ_M is globally asymptotically stable over $\mathscr{T}_{\tau_d}$. Hence, all modes of the flow dynamics of Σ_L are convergent, which means that the flow dynamics of Σ_L is asymptotically stable. Thus, the hybrid linear system Σ_L is globally asymptotically stable over $\mathscr{T}_{\tau_L}$, for some $\tau_L > 0$.

Only if. Arguments similar to those presented above show that all the modes of the flow dynamics of the cascade connection between Σ_R and Σ_P that are not output-observable or that are not input-excitable coincide with the modes of the same kind of the flow dynamics of the cascade connection Σ_L, where the latter are all convergent since Σ_L is globally asymptotically stable over $\mathscr{T}_{\tau_L}$, for some $\tau_L > 0$. On the other hand, since Problem 3 is solved, the modes of the flow dynamics of the cascade composition between Σ_R and Σ_P that are output-observable and input-excitable coincide with the modes of the same kind of the flow dynamics of Σ'_M, that are convergent since Σ'_M is globally asymptotically stable over $\mathscr{T}_{\tau'}$. It follows that all the modes of the flow dynamics of the cascade connection between Σ_R and Σ_P are convergent. Since the matrix which defines such dynamics has an upper block-triangular form with the flow dynamic matrices of Σ_P and Σ_R on the main diagonal, it follows that A_R is a Hurwitz matrix and, therefore, Σ_R is globally asymptotically stable over $\mathscr{T}_{\tau_R}$, for some $\tau_R > 0$. ∎

Theorem 2: Let the hybrid linear system Σ_P and the hybrid linear model Σ_M be given. Let K be a nonzero gain matrix such that the modified model Σ'_M is globally asymptotically stable over $\mathscr{T}_{\tau'}$, for some $\tau' > 0$. Then, Problem 1 has a solution (in particular, with the given K) if and only if (13) holds.

Proof: If. Let (13) hold. Then, a hybrid linear compensator Σ_R that solves Problem 3 (or, equivalently, that satisfies Requirement $\mathcal{R}\,1$ of Problem 1 with K given in the statement) can be obtained with the mentioned procedure, where $\mathcal{V}^*_{\mathscr{H}}$ is replaced by $\mathcal{V}^*_{\mathscr{H}\,g}$. In particular, by picking F as an inner stabilizing friend of $\mathcal{V}^*_{\mathscr{H}\,g}$, one gets that the hybrid linear dynamics of Σ_R is globally asymptotically stable over $\mathscr{T}_{\tau_R}$ for some $\tau_R > 0$. Consequently, Σ_L is globally asymptotically stable over $\mathscr{T}_{\tau_L}$, for some $\tau_L > 0$, by Lemma 1. Hence, the hybrid linear compensator Σ_R thus determined and the nonzero gain matrix K taken as specified in the statement solve Problem 1, since a positive constant $\bar{\tau}_d$, such that Requirement $\mathcal{R}\,2$ is also met, can be chosen as the greater between τ_R and τ_L.

Only if. Let Problem 1 have a solution, given by the hybrid linear compensator Σ_R and by the nonzero gain matrix K. Since, in particular, Requirement $\mathcal{R}\,1$ is satisfied, there exists an $\mathscr{H}$-invariant subspace $\hat{\mathcal{V}}$ for the overall compensated hybrid linear system $\hat{\Sigma}$ such that $\hat{\mathcal{H}} \subseteq \hat{\mathcal{V}} \subseteq \hat{\mathcal{C}}$, where, with the usual notation, $\hat{\mathcal{H}}$ and $\hat{\mathcal{C}}$ respectively stand for the subspace spanned by the columns of $\hat{H}$ and for the kernel of $\hat{C}$. Let $\hat{V}$ be a basis matrix of $\hat{\mathcal{V}}$. Hence, by the above-recalled properties of $\hat{\mathcal{V}}$, there exist matrices L_1, L_2, and N such that $\hat{A}\,\hat{V} = \hat{V}\,L_1$, $\hat{J}\,\hat{V} = \hat{V}\,L_2$, and $\hat{H} = \hat{V}\,N$. Moreover, $\hat{C}\,\hat{V} = 0$. Taking (4)–(5) into account and considering $\hat{V}$ partitioned accordingly, one gets

$$\begin{bmatrix} A & B\,C_R \\ 0 & A_R \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} L_1, \tag{14}$$

$$\begin{bmatrix} J & 0 \\ 0 & J_R \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} L_2, \tag{15}$$

$$\begin{bmatrix} B\,D_R + H \\ B_R \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} N, \tag{16}$$

$$\begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0. \tag{17}$$

Let $\mathcal{V}_1$ be the subspace of $\mathcal{X}$ spanned by the columns of V_1. From (14)–(17), it ensues that $\mathcal{V}_1$ is an $\mathscr{H}$-controlled invariant subspace for Σ containing $\mathcal{H}$ and contained in $\mathcal{C}$. Let $\mathcal{V}_2$ be the subspace of $\mathcal{X}_R$ spanned by the columns of V_2. From (14)–(15), it follows that $\mathcal{V}_2$ is an $\mathscr{H}$-invariant subspace for Σ_R. Since Requirement $\mathcal{R}\,2$ is also satisfied, Σ_R is globally asymptotically stable over $\mathscr{T}_{\tau_R}$ for some $\tau_R > 0$. Hence, in particular, its flow dynamics is asymptotically stable. Since L_1 represents the dynamics induced on $\mathcal{V}_2$ by the flow dynamics of Σ_R, L_1 is Hurwitz. Meanwhile, L_1 represents the dynamics induced on $\mathcal{V}_1$ by the flow dynamics obtained by compensating the hybrid linear system Σ by the control input $u(t) = C_R\, x_R(t)$, which implies that $\mathcal{V}_1$ is inner stabilizable. Then, (13) follows from maximality of $\mathcal{V}^*_{\mathscr{H}\,g}$. ■

VI. Conclusions

The problem of model matching by output feedback has been stated for hybrid linear systems with periodic jumps. In particular, the matching between the forced responses of the closed-loop compensated system and of the model is required, along with global asymptotic stability of both the closed-loop compensated system and the compensator itself (strong stability), provided that the time between the subsequent jumps be sufficiently large. A necessary and sufficient solvability condition has been proven.

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