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Plug-and-play Commissionable Models for Water Networks with Multiple Inlets*

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***Abstract*— Water scarcity is an increasing problem worldwide, yet at the same time huge amounts of water are lost through leakages in the distribution network. Fortunately, improved pressure control can lower the leakage problems. In this work, water networks with multiple pressure actuators and several consumers are considered. We show that a simple model mapping from the actuator pressures and flows to measured pressures inside the network can be derived, when the actuation is restricted to a well-defined subspace. This is shown under mild assumptions on the consumption pattern and hydraulic resistances of pipes. Numerical results of a real-life case-study comparing the output of an EPANET[1] model with the output of the proposed reduced network model support the theoretical results.**

I. INTRODUCTION

Water scarcity is an increasing problem worldwide, yet at the same time a huge amount of water is lost due to leaking water inside the distribution network. It is proven in practice that better pressure management is a cost-efficient way to lower the amount of leaking water and, at the same time, to reduce the number of bursts in the network [2], [3], [4], [5]. On top of this, a considerable amount of energy can be saved. This is the motivation for this paper, where we investigate an extension of our previous results [6] on single inlet distribution networks consisting solely of pipes towards distribution networks with multiple inlets. In particular, given a specific structure of the pressure reference controller, we show that a simple relation between measured pressures and flows at the inlets on one hand and measured pressures in the remaining network on the other hand can be established. This simple relation can be utilised in future extensions of the results presented here towards plug-and-play commissionable management of network pressures for instance. Research in control of water distribution systems has been focused on optimal pump scheduling in water networks with elevated reservoirs [7], [8], leakage detection [9], and contamination detection [10]. In the water industry, pressure management using various concepts has been considered and an overview can be found in [2]. In this paper, we will show that a simple relation between the supply flow and the pipe flows in the network exists. This relation can be used in developing a simplified network model that can be used for e.g. adaptive reference control in a pressure management scheme as in [6] or for generation of residuals for leakage isolation as in [11]. The paper starts by stating some general relations in a hydraulic network, using references to graph theory. Furthermore, we will present two graph theoretical Lemmata and a corollary, which will prove useful in the exposition. This is presented in Section II. In Section III, we present a partitioning of the network equations which will be instrumental in the derivation of the reduced network model. Section IV investigates the effect of a particular control-subspace on the network equations. In Section V, we establish a reduced order model between the total inflow (denoted σ), actuator pressures (denoted $\hat{p}$) and measured outputs (denoted y). In Section VI, the results from a numerical experiment on a real-life case-study are presented. The paper ends with some concluding remarks.

a) Nomenclature: Let $\langle x, y\rangle$ denote the scalar product between two vectors $x, y \in \mathbb{R}^n$. We will say a map $f : \mathbb{R}^n \to \mathbb{R}^n$ is *(strictly) monotonic increasing* if $\langle x-y, f(x)-f(y)\rangle \geq (>) 0$ for every $x, y \in \mathbb{R}^n$ such that $x \neq y$. With the notation $M(n, m; \mathbb{R})$ we denote the set of n-by-m matrices whose entries belong to the set $\mathbb{R}$. By $\mathbb{1}_n$ we denote the n-dimensional vector consisting of ones. By I_n we denote the n-dimensional identity matrix.

II. NETWORK MODEL

Water distribution networks can be described by a directed and connected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. The elements of the set $\mathcal{V} = \{\nu_1, \ldots, \nu_n\}$ are denoted vertices and represent pipe connections with possible end-user water consumption. The elements of the set $\mathcal{E} = \{e_1, \ldots, e_m\}$ are denoted edges and represent the pipes. The demand[2] at the vertex i is denoted by d_i, and the geodesic level[3] by h_i. The pressure drop due to hydraulic resistance of the edge j is denoted by f_j, and this pressure drop is a function of the flow q_j through the pipe (edge) j. To the graph $\mathcal{G}$, we associate the incidence matrix H, which we recall below

$$H_{i,j} = \begin{cases} -1 & , \text{ if the } j^{th} \text{ edge is entering } i^{th} \text{ vertex.} \\ 0 & , \text{ if the } j^{th} \text{ edge is not connected to} \\ & \quad \text{the } i^{th} \text{ vertex.} \\ 1 & , \text{ if the } j^{th} \text{ edge is leaving } i^{th} \text{ vertex.} \end{cases}$$

The incidence matrix has dimension $n \times m$, where m is the number of edges and n is the number of vertices in the graph. In the current exposition, we will exploit the following lemma on the incidence matrix and a corollary of it.

Lemma 1 *Let H be the incidence matrix of a connected and directed graph. Then* $\ker(H^T) = \operatorname{span}\{\mathbb{1}_n\}$.

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[1]EPANET is a standard modelling tool used at many water utilities [1]

[2]This includes unknown demands such as leaks.

[3]The geodesic level is the height above some predefined reference.

Proof: Since each row in H^T corresponds to an edge in the graph, each row has exactly one entry with the value -1 and one entry with the value 1, while all other entries are zero. From this we can establish that $\mathbb{1}_n \in \ker(H^T)$. Furthermore, from graph theory (see e.g. [12]) we know that $\mathrm{rank}(H^T) = n-1$ for a connected graph. From the fundamental theorem of linear algebra [13], we then have that $\dim(\ker(H^T)) = 1$, from which we can conclude that $\ker(H^T) = \mathrm{span}\{\mathbb{1}_n\}$. ∎

Corollary 1 *Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a connected and directed graph with incidence matrix H. Furthermore, let $\mathcal{V} = \{\bar{\mathcal{V}}, \hat{\mathcal{V}}\}$ be a partitioning such that $\hat{\mathcal{V}} = \{\hat{\nu}_1, \ldots, \hat{\nu}_c\}$ is non-empty and $\mathcal{E} = \{\mathcal{E}_{\mathcal{T}}, \mathcal{E}_{\mathcal{C}}\}$ be a partitioning such that the corresponding sub-matrix $\bar{H}_{\mathcal{T}}$ of H is square and invertible. Then*

$$-\bar{H}_{\mathcal{T}}^{-T}\hat{H}_{\mathcal{T}}^{T}\mathbb{1}_c = \mathbb{1}_{n-c}. \tag{1}$$

Proof: By the chosen partitioning, we have

$$H = \begin{bmatrix} \bar{H}_{\mathcal{T}} & \bar{H}_{\mathcal{C}} \\ \hat{H}_{\mathcal{T}} & \hat{H}_{\mathcal{C}} \end{bmatrix} \tag{2}$$

where $\bar{H}_{\mathcal{T}}$ is square and invertible. By Lemma 1 it follows in particular that

$$\bar{H}_{\mathcal{T}}^{T}\mathbb{1}_{n-c} + \hat{H}_{\mathcal{T}}^{T}\mathbb{1}_c = 0. \tag{3}$$

By invertibility of $\bar{H}_{\mathcal{T}}$ it follows that

$$\mathbb{1}_{n-c} = -\bar{H}_{\mathcal{T}}^{-T}\hat{H}_{\mathcal{T}}^{T}\mathbb{1}_c \tag{4}$$

which proves the claim. ∎

We will also use the following lemma on partitioning of the incidence matrix

Lemma 2 *Let $n > c > 1$ and $\bar{H} \in M(n-c, m; \{-1,0,1\})$ be a sub-matrix of an incidence matrix H of a connected graph. Furthermore, let $\bar{H} = [\bar{H}_{\mathcal{T}}\ \bar{H}_{\mathcal{C}}]$ where $\bar{H}_{\mathcal{T}} \in M(n-c, n-c; \{-1,0,1\})$ is full rank. Lastly, let $B \in M(m-n+c, m; \mathbb{R})$ be the matrix*

$$B = [-\bar{H}_{\mathcal{C}}^{T}\bar{H}_{\mathcal{T}}^{-T}\ I_{m-n+c}]. \tag{5}$$

Then it follows that $\mathrm{im}(\bar{H}^T) = \ker(B)$.

Proof: First, notice that $B\bar{H}^T = 0$, which means that $\mathrm{im}(\bar{H}^T) \subset \ker(B)$. Since $\bar{H}$ is full rank it follows that $\dim(\mathrm{im}(\bar{H}^T)) = n-c$. Furthermore, since $\mathrm{rank}(B) = m-n+c$ it follows that $\dim(\ker(B)) = m-(m-n+c) = n-c$. Then the claim follows since $\mathrm{im}(\bar{H}^T) \subset \ker(B)$ and $\dim(\mathrm{im}(\bar{H}^T)) = \dim(\ker(B))$. ∎

The network must fulfill Kirchhoff's vertex law, which corresponds to conservation of mass in each vertex:

$$Hq = d, \tag{6}$$

where $q \in \mathbb{R}^m$ is the vector of flows in edges and $d \in \mathbb{R}^n$ is the vector of nodal demands, with $d_i > 0$ when demand flow is into vertex i. Because of mass conservation in the network, there can only be $n-1$ independent nodal demands which means that $\sum_{i=1}^{n} d_i = 0$ (this is also a consequence of Lemma 1). Let p be the vector of absolute pressures at the vertices and Δp be the vector of differential pressures across the edges, then the "Ohm law" for water networks gives

$$\Delta p = H^T p = f(q) - H^T h, \tag{7}$$

where $p \in \mathbb{R}^n$, $f : \mathbb{R}^m \to \mathbb{R}^m$, $f(x) = (f_1(x_1), \ldots, f_m(x_m))$ with f_i strictly increasing. f_i describes the flow dependent pressure drop due to the hydraulic resistance. The term $H^T h$ is the pressure drop across the components due to difference in geodesic level between the ends of the components with $h \in \mathbb{R}^n$ the vector of geodesic levels at each vertex expressed in units of potential (pressure). In the following exposition, we will assume that each $f_i : \mathbb{R} \to \mathbb{R}$ has the following structure motivated by turbulent flow in the pipes

$$f_i(x) = \rho_i |x| x, \tag{8}$$

with $\rho_i > 0$ a constant parameter of the pipe. The contribution in this paper comprises an extension of our work in [6]. In particular, we here consider networks with multiple inlets; whereas, the focus of [6] was on single-inlet networks.

III. Network Partitioning

Similar to [6], we seek to derive a reduced order network model which captures the dependence of the measured output pressures on the flows and pressures at the inlets. As a stepping stone towards this goal, we present in this section a partitioning of the network and derive an instrumental proposition, that gives a control structure under which an exact reduced order model can be formulated. To this end, consider the partitioning of the network graph presented in Corollary 1. Here, we will let the set $\hat{\mathcal{V}} = \{\hat{\nu}_1, \ldots, \hat{\nu}_c\}$ with $c > 1$ represent vertices in the graph corresponding to inlet vertices in the distribution network. The set $\bar{\mathcal{V}} = \{\bar{\nu}_1, \ldots, \bar{\nu}_{n-c}\}$ represents the remaining vertices in the graph. We will assume that the vectors $\hat{p}$, $\hat{d}$ of (inlet-) pressures and demands are measured. Furthermore, there exists a vector $y \in \mathbb{R}^o$ where $\{y_1, \ldots, y_o\} \subset \{\bar{p}_1, \ldots, \bar{p}_{n-c}\}$ of measured pressures at the remaining vertices. The vector $\bar{d}$ of demands at non-inlet vertices is unknown. As in Corollary 1, the partitioning of edges is chosen such that the sub-matrix $\bar{H}_{\mathcal{T}}$ of the incidence matrix is invertible. We will here consider the vector $\hat{p} \in \mathbb{R}^c$ associated to the set $\hat{\mathcal{V}}$ as a vector of control inputs (typically set by pumps or pressure reducing valves located at the vertices). The output vector y is a subset of the pressure vector $\bar{p} \in \mathbb{R}^{n-c}$, which is the vector of pressures associated with the set $\bar{\mathcal{V}}$. To explain the partitioning in a real network example, we consider the network illustrated in Fig. 1. The network consists of 326 vertices and 366 edges ($n = 326$ and $m = 366$). This particular network has two inlets indicated with the arrows marked 'Input #' (so $c = 2$) and $\hat{p} \in \mathbb{R}^2$ ($\hat{d} \in \mathbb{R}^2$) is the vector of pressures (demands) at these two vertices. The vector $\bar{p} \in \mathbb{R}^{324}$ ($\bar{d} \in \mathbb{R}^{324}$) consists of the pressures (demands) at the remaining 324 vertices in the network. Four of the pressures in the vector $\bar{p}$ are measured, and the measurement points are indicated by the

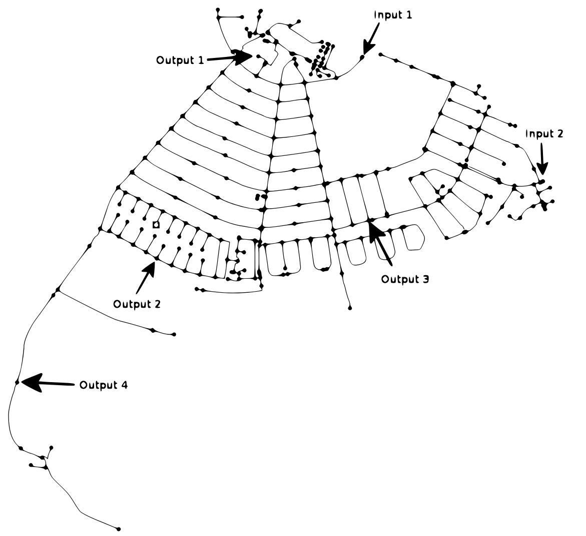

Fig. 1. Network illustration of the EPANET model of the distribution network used in the simulations in Section VI.

arrows marked 'Output #' (so $y \in \mathbb{R}^4$). With the chosen partitioning, we can rewrite (6) and (7) to the following

$$\begin{aligned} \bar{d} &= \bar{H}_{\mathcal{T}} q_{\mathcal{T}} + \bar{H}_{\mathcal{C}} q_{\mathcal{C}} \\ \hat{d} &= \hat{H}_{\mathcal{T}} q_{\mathcal{T}} + \hat{H}_{\mathcal{C}} q_{\mathcal{C}} \\ f_{\mathcal{T}}(q_{\mathcal{T}}) &= \bar{H}_{\mathcal{T}}^T(\bar{p} + \bar{h}) + \hat{H}_{\mathcal{T}}^T(\hat{p} + \hat{h}) \\ f_{\mathcal{C}}(q_{\mathcal{C}}) &= \bar{H}_{\mathcal{C}}^T(\bar{p} + \bar{h}) + \hat{H}_{\mathcal{C}}^T(\hat{p} + \hat{h}). \end{aligned} \tag{9}$$

From (9) we can derive the following expression for the vector $q_{\mathcal{T}}$ of flows in edges $\mathcal{E}_{\mathcal{T}}$

$$q_{\mathcal{T}} = -\bar{H}_{\mathcal{T}}^{-1} \bar{H}_{\mathcal{C}} q_{\mathcal{C}} + \bar{H}_{\mathcal{T}}^{-1} \bar{d}. \tag{10}$$

Using (9) and (10), we can derive the following

$$\begin{aligned} & f_{\mathcal{C}}(q_{\mathcal{C}}) - \bar{H}_{\mathcal{C}}^T \bar{H}_{\mathcal{T}}^{-T} f_{\mathcal{T}}(-\bar{H}_{\mathcal{T}}^{-1} \bar{H}_{\mathcal{C}} q_{\mathcal{C}} + \bar{H}_{\mathcal{T}}^{-1} \bar{d}) = \\ & (\hat{H}_{\mathcal{C}}^T - \bar{H}_{\mathcal{C}}^T \bar{H}_{\mathcal{T}}^{-T} \hat{H}_{\mathcal{T}}^T)(\hat{p} + \hat{h}). \end{aligned} \tag{11}$$

The vector $\bar{d} \in \mathbb{R}^{n-c}$ represents the vertex demands at non-inlet vertices. If we let $\sigma \in \mathbb{R}_+$ denote the total demand in the network, then we can write $\bar{d} = -v\sigma$, where $v \in \mathbb{R}^{n-c}$ with $v_i \in (0, 1)$ and $\sum_i v_i = 1$. Now, let $q_{\mathcal{C}} = a_{\mathcal{C}} \sigma$, then we can re-write the left-hand side of (11) to

$$f_{\mathcal{C}}(a_{\mathcal{C}})\sigma^2 - \bar{H}_{\mathcal{C}}^T \bar{H}_{\mathcal{T}}^{-T} f_{\mathcal{T}}(-\bar{H}_{\mathcal{T}}^{-1} \bar{H}_{\mathcal{C}} a_{\mathcal{C}} - \bar{H}_{\mathcal{T}}^{-1} v)\sigma^2, \tag{12}$$

where we have exploited the homogeneity properties of $f_i(\cdot)$ (see e.g. [14]). Let the function $F_v : \mathbb{R}^{m-n+c} \to \mathbb{R}^{m-n+c}$ (parameterised by v) be given by

$$F_v(x) = f_{\mathcal{C}}(x) - \bar{H}_{\mathcal{C}}^T \bar{H}_{\mathcal{T}}^{-T} f_{\mathcal{T}}(-\bar{H}_{\mathcal{T}}^{-1} \bar{H}_{\mathcal{C}} x - \bar{H}_{\mathcal{T}}^{-1} v). \tag{13}$$

Using arguments similar to the ones used in [6], it can be shown that $F_v(\cdot)$ is a homeomorphism. This means that there exists an inverse map $F_v^{-1} : \mathbb{R}^{m-n+c} \to \mathbb{R}^{m-n+c}$ such that from (11) we obtain

$$a_{\mathcal{C}} = F_v^{-1}\left(\frac{1}{\sigma^2}(\hat{H}_{\mathcal{C}}^T - \bar{H}_{\mathcal{C}}^T \bar{H}_{\mathcal{T}}^{-T} \hat{H}_{\mathcal{T}}^T)(\hat{p} + \hat{h})\right). \tag{14}$$

That is, $a_{\mathcal{C}}$ can be expressed in terms of v, σ, $\hat{p}$ and $\hat{h}$. However, we only know the existence of such an expression, not its structure.

Remark 1 *Notice that, if the matrix*

$$A = \hat{H}_{\mathcal{C}}^T - \bar{H}_{\mathcal{C}}^T \bar{H}_{\mathcal{T}}^{-T} \hat{H}_{\mathcal{T}}^T \in M(m - n + c, c; \mathbb{R}) \tag{15}$$

has a non-trivial kernel and $\hat{p} \neq 0$ *is chosen such that* $\hat{p} \in \ker(A)$, *then the expression* (14) *for* $a_{\mathcal{C}}$ *becomes independent of* $\hat{p}$. *In addition, if we let* $0 \neq (\hat{p} + \hat{h}) \in \ker(A)$ *then* $a_{\mathcal{C}}$ *becomes independent of* σ, $\hat{p}$ *and* $\hat{h}$. *The expression for* $a_{\mathcal{C}}$ *is thereby only dependent on the parameter* v.

This is the motivation for the following proposition

Proposition 1 *Let* $A \in M(m - n + c, c; \mathbb{R})$ *be defined by* (15). *Then,* A *has a non-trivial kernel, and* $\ker(A) = \mathrm{span}\{\mathbb{1}_c\}$.

Proof: First, notice that we can write A as

$$A = [-\bar{H}_{\mathcal{C}}^T \bar{H}_{\mathcal{T}}^{-T} \; I] \begin{bmatrix} \hat{H}_{\mathcal{T}}^T \\ \hat{H}_{\mathcal{C}}^T \end{bmatrix} = B\hat{H}^T. \tag{16}$$

Now, assume $x \in \mathbb{R}^c \setminus \{0\}$ and $x \in \ker(A)$. As a consequence of Lemma 2, there exists $y \in \mathbb{R}^{n-c} \setminus \{0\}$ such that

$$0 = B\bar{H}^T y + B\hat{H}^T x = BH^T \begin{bmatrix} y \\ x \end{bmatrix}. \tag{17}$$

From (17) it follows that

$$z = H^T \begin{bmatrix} y \\ x \end{bmatrix} \in \ker(B). \tag{18}$$

As a consequence of Lemma 2 it follows that there exists $v \in \mathbb{R}^{n-c} \setminus \{0\}$ such that

$$\bar{H}^T v = \bar{H}^T y + \hat{H}^T x, \tag{19}$$

which is rewritten as

$$0 = \bar{H}^T(y - v) + \hat{H}^T x = H^T \begin{bmatrix} y - v \\ x \end{bmatrix}, \tag{20}$$

and hence

$$\begin{bmatrix} y - v \\ x \end{bmatrix} \in \ker(H^T). \tag{21}$$

Using Lemma 1 it can thus be concluded that $x \in \mathrm{span}\{\mathbb{1}_c\}$, which proves the claim. ∎

IV. Reference Control Structure

In this section, we investigate further the degree of freedom one can allow when designing a pressure reference controller. This investigation leads to an additional proposition, which gives additional substantiation to the use of the control structure discussed in Remark 1.

We will assume that a requirement for the reference controller is to fulfil the constraint

$$\hat{d}_i \geq 0. \tag{22}$$

This constraint is natural since it restricts the demand flow at the inlet vertices to be oriented into the distribution network at all times. If $\hat{d}_i < 0$, then water is flowing out of the network at the inlet vertex. We will assume that we want our pressure controller to fulfil the constraint (22) at all times. This means, in particular, that the constraint should be fulfilled whenever the demand in the network is low. For low demands, we have $\bar{d} \approx 0$. This motivates the following proposition.

Proposition 2 *Let each $f_i(\cdot)$ be given by* (8) *and the partitioning of the network graph be given as in* (9)*. Assume $\bar{d} = 0$ and the constraint* (22) *is fulfilled, then $(\hat{p}+\hat{h}) \in \text{span}\{\mathbb{1}_c\}$.*

Proof: Using $\bar{d} = 0$ in (9) and (10), we obtain

$$\hat{d} = (\hat{H}_{\mathcal{C}} - \hat{H}_{\mathcal{T}}\bar{H}_{\mathcal{T}}^{-1}\bar{H}_{\mathcal{C}})q_{\mathcal{C}} = A^T q_{\mathcal{C}}. \tag{23}$$

Moreover, from mass conservation, we have that $\sum_j \bar{d}_j = -\sum_i \hat{d}_i$. Since $\bar{d} = 0$ and we have the constraint (22), then necessarily $\hat{d}_i = 0$ and consequently it follows from (23) that

$$q_{\mathcal{C}} \in \ker(A^T). \tag{24}$$

Next, we define the function $G : \mathbb{R}^{m-n+c} \to \mathbb{R}^{m-n+c}$ as

$$G(x) = f_{\mathcal{C}}(x) - \bar{H}_{\mathcal{C}}^T\bar{H}_{\mathcal{T}}^{-T} f_{\mathcal{T}}(-\bar{H}_{\mathcal{T}}^{-1}\bar{H}_{\mathcal{C}}x). \tag{25}$$

Due to the properties of $f_i(\cdot)$, it can be shown (using arguments similar to [6]) that $G(\cdot)$ is a strictly monotonically increasing homeomorphism with $G(0) = 0$. This, in particular, means that $G(\cdot)$ has a strictly monotonically increasing inverse, say $G^{-1}(\cdot)$ [15]. Since $q_{\mathcal{C}}$ needs to fulfil (11) also for $\bar{d} = 0$, we have (recall (15))

$$q_{\mathcal{C}} = G^{-1}(A(\hat{p} + \hat{h})). \tag{26}$$

Furthermore, due to (24) we have

$$A^T G^{-1}(A(\hat{p} + \hat{h})) = 0. \tag{27}$$

Now, we can prove the thesis by contradiction. To this end, assume that $(\hat{p}+\hat{h}) \neq 0$ and $(\hat{p}+\hat{h}) \notin \ker(A) = \text{span}\{\mathbb{1}_c\}$, then, due to strict monotonicity of $G^{-1}(\cdot)$, for every $y \in \mathbb{R}^c \setminus (\hat{p} + \hat{h})$ we have

$$\langle A(\hat{p} + \hat{h}) - Ay, G^{-1}(A(\hat{p} + \hat{h})) - G^{-1}(Ay)\rangle > 0. \tag{28}$$

In particular, since $G^{-1}(0) = 0$

$$\langle A(\hat{p} + \hat{h}), G^{-1}(A(\hat{p} + \hat{h}))\rangle > 0, \tag{29}$$

from which it follows that

$$A^T G^{-1}(A(\hat{p} + \hat{h})) \neq 0. \tag{30}$$

However, this contradicts (27). Since $G^{-1}(0) = 0$ and $G^{-1}(\cdot)$ is a homeomorphism, we conclude that $(\hat{p} + \hat{h}) \in \ker(A)$. Since $\ker(A) = \text{span}\{\mathbb{1}_c\}$ the proposition follows. ∎

Proposition 2 provides a guideline for which control strategies can be used. That is, if one wants to guarantee that the constraint (22) is fulfilled, then necessarily one needs to use a control vector which fulfils $(\hat{p} + \hat{h}) \in \text{span}\{\mathbb{1}_c\}$. We can also note that when the demand $\bar{d}$ is zero, then this choice of control vector has the consequence that $q_{\mathcal{C}} = 0$. The latter follows from (26), Proposition 1 and the fact that $G^{-1}(0) = 0$.

V. Reduced Model Structure

We use (9) and (10) to derive the following expression for the vector $\bar{p}$ of pressures at non-inlet vertices

$$\bar{p} = \bar{H}_{\mathcal{T}}^{-T} f_{\mathcal{T}}(-\bar{H}_{\mathcal{T}}^{-1}\bar{H}_{\mathcal{C}}q_{\mathcal{C}} + \bar{H}_{\mathcal{T}}^{-1}\bar{d}) - \bar{H}_{\mathcal{T}}^{-T}\hat{H}_{\mathcal{T}}^T(\hat{p} + \hat{h}) - \bar{h}. \tag{31}$$

Next, using the expressions $q_{\mathcal{C}} = a_{\mathcal{C}}\sigma$ and $\bar{d} = -v\sigma$ and the homogeneity property of $f_i(\cdot)$ we can obtain the following

$$\bar{p} = \bar{H}_{\mathcal{T}}^{-T} f_{\mathcal{T}}(-\bar{H}_{\mathcal{T}}^{-1}\bar{H}_{\mathcal{C}}a_{\mathcal{C}} - \bar{H}_{\mathcal{T}}^{-1}v)\sigma^2 - \bar{H}_{\mathcal{T}}^{-T}\hat{H}_{\mathcal{T}}^T(\hat{p}+\hat{h}) - \bar{h}. \tag{32}$$

Again, we emphasise that in general (see (14)) $a_{\mathcal{C}}$ will be dependent on the parameters v and $\hat{h}$ and the signals $\sigma(t)$ and $\hat{p}(t)$. However, in the special case where $\hat{p}(t) + \hat{h} = \kappa\mathbb{1}$ for some $\kappa \in \mathbb{R}$, then $a_{\mathcal{C}}$ will only depend on the parameter vector v which represents the distribution of the total demand among the non-inlet vertices. In the case where v is a constant vector[4], we can write the following expression for the pressure $\bar{p}_i(t)$ at the ith non-inlet vertex

$$\bar{p}_i(t) = \alpha_i\sigma^2(t) + \sum_j \beta_{ij}\hat{p}_j(t) + \gamma_i, \tag{33}$$

where

$$\begin{aligned} \alpha_i &= (\bar{H}_{\mathcal{T}}^{-T})_i f_{\mathcal{T}}(-\bar{H}_{\mathcal{T}}^{-1}\bar{H}_{\mathcal{C}}a_{\mathcal{C}} - \bar{H}_{\mathcal{T}}^{-1}v) \\ \beta_{ij} &= (-\bar{H}_{\mathcal{T}}^{-T}\hat{H}_{\mathcal{T}}^T)_{ij}\ ,\ \gamma_i = -(\bar{H}_{\mathcal{T}}^{-T}\hat{H}_{\mathcal{T}}^T)_i\hat{h} - \bar{h}_i. \end{aligned} \tag{34}$$

If v is a time-varying parameter $v(t)$ and $\hat{p} + \hat{h} = \kappa\mathbb{1}$, then we can write

$$\bar{p}_i(t) = \alpha_i(t)\sigma^2(t) + \sum_j \beta_{ij}\hat{p}_j(t) + \gamma_i. \tag{35}$$

That is, in this case α_i will also be a time-varying parameter. Typically, in a water distribution network, the demand profiles exhibit a periodic behaviour which means that $v(t + T) = v(t)$ and $\sigma(t + T) = \sigma(t)$ where T denotes the length of the period. Likewise, the parameter $\alpha_i(t)$ will exhibit a periodic behaviour in this case. Since the model (35) of $\bar{p}_i(t)$ is linear in the parameters $\alpha_i(t)$, β_{ij} and γ_i, standard parameter identification methods [16] can be used to identify these parameters based on measurements of $\bar{p}_i(t)$, $\sigma(t)$ and $\hat{p}(t)$ over a number of time periods. This can even be done recursively when new data batches are received. Having identified the model parameters, the model (35) can be utilised to design the vector $\hat{p}(t)$ of pressure inputs similar to what was done in [6]. The model can also be utilised for residual generation with the purpose of leakage isolation similarly to [11]. Thus, the model can be utilised for these purposes as a plug-and-play commissionable alternative to more complex network models such as EPANET which has a plethora of parameters to adjust. As mentioned, a particularly interesting case is when the vector $\hat{p}(t)$ of control inputs is restricted to the subspace of $\mathbb{R}^c$ fulfilling

$$\hat{p}(t) + \hat{h} = \kappa(t)\mathbb{1} \Leftrightarrow \hat{p}(t) = \kappa(t)\mathbb{1} - \hat{h}, \tag{36}$$

[4]For instance if all consumers are residential, all non-reference vertices with non-zero demand have the same consumption profile and v will be constant.

where the scalar $\kappa(t)$ is the design parameter of the controller. By limiting to control signals fulfilling (36), we know that the model (35) is exact. Furthermore, we have the following result which simplifies the model further, when applying the control structure (36).

Proposition 3 *Applying the control* (36) *to the vector* $\hat{p}$ *of pressures at inlet vertices results in the following expression for the pressure at the ith non-inlet vertex*

$$\bar{p}_i(t) = \alpha_i(t)\sigma^2(t) + \kappa(t) + \tilde{\gamma}_i. \tag{37}$$

Proof: Applying the constraint (36) to (35) gives the following

$$\bar{p}_i(t) = \alpha_i(t)\sigma^2(t) + \tilde{\beta}_i\kappa(t) + \tilde{\gamma}_i, \tag{38}$$

where

$$\tilde{\beta}_i = \sum_j \beta_{ij}\ ,\ \tilde{\gamma}_i = \gamma_i - \sum_j \beta_{ij}\hat{h}_j = -\bar{h}_i. \tag{39}$$

By applying Corollary 1 in (38) and the definition (39) of $\tilde{\beta}_i$ the thesis follows, since, by Corollary 1, we have $\tilde{\beta}_i = 1$. ■

Comparing the structure in (37) to the structure of the model of single inlet networks which was derived in [6], we see that they are identical. Again, we emphasise that this is only the case when the constraint (36) is fulfilled.

Remark 2 *In general, it cannot be assumed that the vector* $\hat{p}(t)$ *of pressures at inlet vertices fulfils the constraint* (36). *Therefore, in general, as* (14) *shows, the vector* a_C *will be dependent on both the inlet pressures as well as the total inlet flow* $\sigma(t)$. *Since we only know the existence of an expression for* a_C *in the general case and not its structure, we will here propose a heuristic model for the general case inspired by the structure in* (35). *In particular, we propose the following structure*

$$\bar{p}_i(t) = \alpha_i(t)\sigma^2(t) + \sum_j \beta_{ij}(t)\hat{p}_j(t). \tag{40}$$

As it can be seen in (40) *the difference to* (35) *is that the parameter* β_{ij} *is now also time dependent and* $\gamma_i = 0$.

In the next section, we will illustrate the performance of the models (37) and (40) using a numerical example of a real-life case-study.

VI. Results

This section contains numerical results showing the performance of the reduced order models compared to a high fidelity model implemented in EPANET. The high fidelity results are obtained using an EPANET model of the network illustrated in Fig. 1. The vector of demands in the EPANET reflects the demands of the users in the real-life case-study. This in particular means that there is a mix of residential and industrial consumption profiles in the model. This has the effect that the demand distribution vector $v(t)$ in this particular case is not constant. We have simulated two scenarios. In the first scenario, the vector $\hat{p}(t)$ fulfils the constraint (36) and in the second scenario a vector which does not fulfil the constraint has been used. In both scenarios, the vector $\hat{p}(t)$ is a Gaussian random input with mean and standard deviation matching real-life measurements. An initial period of eight days is simulated in EPANET and the data of $y(t)$, $\sigma(t)$ and $\hat{p}(t)$ from the last seven days of this initial period is used to estimate the parameters of the reduced model using standard linear regression. After the initial period used for parameter estimation, the standard deviation of the input $\hat{p}$ is doubled and the output $y(t)$ from EPANET is then compared with the output from the proposed reduced model for the following two days. In the first scenario, the reduced model (37) is used, and in the second scenario, the reduced model (40) is used. The results from the first scenario is illustrated in Fig. 2 and Fig. 3. Fig. 2 shows the inputs for the reduced model, where the first column is for day nine and the second is for day ten. The first row in Fig. 2 shows the total inflow $\sigma(t)$, the second row shows the pressure $\hat{p}_1(t)$ at the first inlet vertex, and the third row shows the pressure $\hat{p}_2(t)$ at the second inlet vertex. Fig. 3 compares the pressure outputs

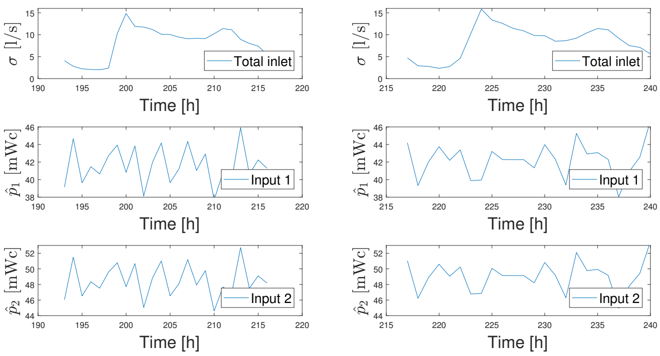

Fig. 2. Inputs for the estimation when using the proposed control structure.

from the model (37) for the four measurement vertices (rows in the figure) with the actual outputs from EPANET. Again,

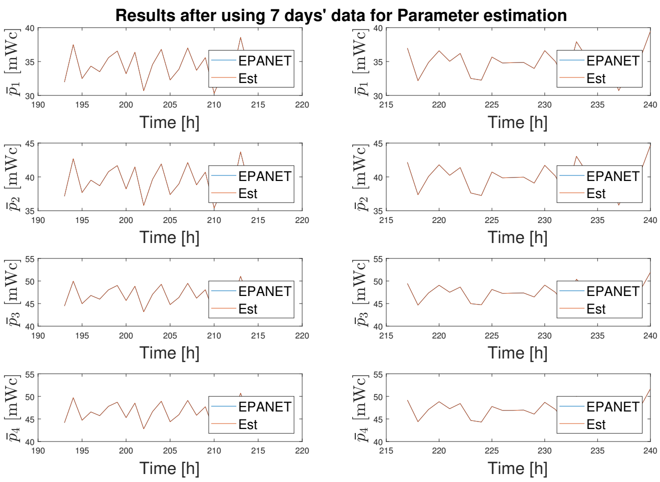

Fig. 3. Estimated outputs together with actual outputs under the proposed control structure.

the columns of the figure are the results from day nine/ten respectively. As it can be seen in Fig. 3, there is very low deviation (max 0.025 [mWc]) between the output from the reduced model and the one from the EPANET model. This is also expected since we know that the model structure (37) is exact when the pressure input (36) is used. The results from the second scenario are illustrated in Fig. 4 and Fig. 5. The structure of the rows and columns in Fig. 4 is the same as in Fig. 2 and the structure of Fig. 5 is the same as Fig. 3. As it can be seen in Fig.5, the outputs of the reduced

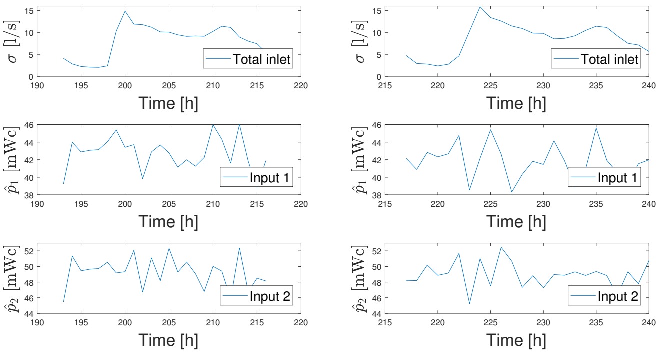

Fig. 4. Inputs for the estimation when not using the proposed control structure.

model (40) follows the output of the EPANET simulation well. However, it is also evident that the deviation between the two is higher (max 0.76 [mWc]) than in the first scenario. This is due to the fact that the model (37) is exact under the constraint (36), while the model (40) is a heuristic model, which appears not to capture exactly the dependence of $y(t)$ on $\hat{p}(t)$ and $\sigma(t)$.

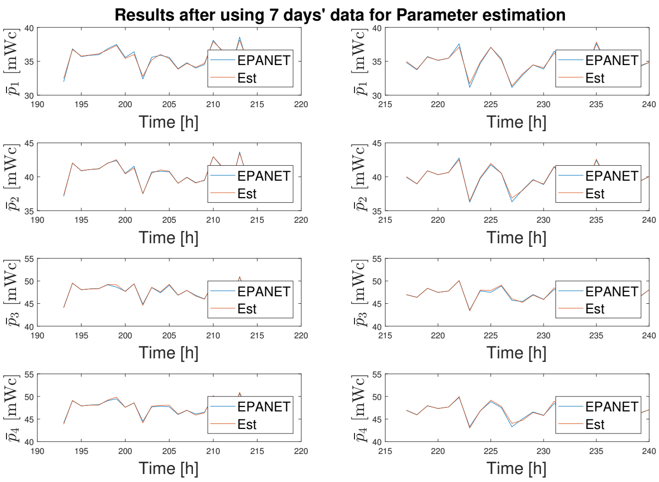

Fig. 5. Estimated outputs together with actual outputs when not using the proposed control structure.

VII. Conclusion

We have proposed a reduced model for estimation of pressure measurements in a water distribution network with multiple inlets. The proposed model has few parameters which can be estimated using historical data together with measurements available at the inlets. We have shown that if the vector of pressure inputs is constrained to a well defined subspace, then the reduced model is exact. For the case where the pressure input is not confined to this subspace, we have proposed a heuristic reduced model with a slightly different structure, which performs well in simulations. We anticipate future applications of the proposed models to be within pressure control of networks with multiple inlets as well as residual generation for leakage isolation in said networks. An extension of the current model framework towards networks with capacitive elements (elevated reservoirs) should also be considered. This we see as a non-trivial task.

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