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A Unified Framework for Decentralized Control Synthesis

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Abstract— **This paper deals with the problem of decentralized control synthesis. We seek to find structured controllers that are stably realizable over the underlying network. We address the problem using an operator form of discrete-time linear systems. This allows for uniform treatment of various classes of linear systems, e.g., Linear Time Invariant (LTI), Linear Time Varying (LTV), or linear switched systems. We combine this operator representation for linear systems with the classical Youla parameterization to characterize the set of stably realizable controllers for a given network structure. We show that if the structure satisfies certain subspace like assumptions, then both the stability and performance problems can be formulated as convex optimization and more precisely as tractable model-matching problems. Furthermore, we show that the structured controllers found from our approach can be stably realized over the network.**

I. Introduction

Modern large-scale cyber-physical systems are composed of many interconnected subsystems that are usually spread over a large geographic area and communicate over a network. Many difficulties arise when designing a centralized controller for such systems due to communication delays, the structure of the underlying communication network, scalability, etc. Due to these issues, there has been a shift towards designing decentralized controllers, in which subcontrollers are designed and implemented for each subsystem and they can communicate over the network.

Decentralized, structured and distributed controller design has attracted the renewed attention of many researchers over the last 15 years or so. Several new developments occurred using state space methods (e.g., in the LMI framework [1], [2]) which suit quadratic criteria but could generally lead to suboptimal solutions. On the other hand, input-output approaches using the Youla-parametrization were found to be very powerful in providing truly optimal solutions for several classes of structured problems by reducing them to convex problems over the Youla parameter, encompassing a variety of criteria, including nonquadratic (e.g., [3], [4], [5], [6], [7]).

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In the input-output, or transfer function, domain, the stabilizing controllers are parametrized by the so-called Youla parameter, and the search for the optimal Youla parameter is carried out over the space of stable systems. Here, the order of the Youla parameter or that of the controller is not assumed a priori. However, unlike the state-space approaches, the realizability of the controller over the underlying communication network may become an issue, if not taken directly into account as pointed out in [8], [9]. That is, although the controller transfer function structure is compatible with the underlying network communication graph, it may lead to an internally unstable realization, i.e., a non-minimal realization with unstable pole zero cancellations (e.g., [10], [11] and [12]). Certain alternative input-output approaches have recently been proposed (e.g., [13] and references therein) that hold the potential to handle certain optimal and stably realizable structured design, by convex programming without resorting to Youla-parametrization. A potential drawback is the need to solve an exact model-matching problem, i.e., equations that, if possible to satisfy, may require infinite support of the LTI maps involved.

In this paper, we propose a unified way to synthesize stably realizable controllers with respect to any measure of performance, e.g., l_1, l_2, or l_∞ induced norms. Our approach is based on utilizing a state-space based operator form of the system and combining it with the ideas in the Youla-parameterization. This has been developed initially in the context of switching system analysis and design in [14], [15], and as it turns out, it fits well for optimally solving structured problems if and only if they are stably realizable. Although the reader can focus on the LTI case as a concrete example, these methods are general and hold for LTV systems as well.

II. Preliminaries

In this paper, $\mathbb{R}$ and $\mathbb{Z}$ denote the sets of real numbers and integers, respectively. The set of n-tuples $x = \{x(k)\}_{k=0}^{n-1}$ where $x(k)$s are real numbers is denoted by $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, its l_∞ and l_p norms are defined as $\|x\|_\infty = \max_{k\in\{0,1,\dots,n-1\}} |x(k)|$ and $\|x\|_p = \left(\sum_{k=0}^{n-1} |x(k)|^p\right)^{\frac{1}{p}}$, respectively. Let $g = \{g(k)\}_{k=0}^{\infty}$ be a sequence where $g(k) \in \mathbb{R}^n$. Then, the l_∞ and l_p norm of this sequence are defined as $\|g\|_\infty = \sup_{k\in\mathbb{Z}_+} \|g(k)\|_\infty$ and $\|g\|_p = \left(\sum_{k=0}^{\infty} \|g(k)\|_p^p\right)^{\frac{1}{p}}$ whenever they are finite. The set of $\mathbb{R}^n$-valued sequences whose l_p norm (l_∞ norm) is finite is denoted by l_p^n (l_∞^n). Given two normed spaces $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ and a linear operator $T : X \to Y$, its induced norm is defined as $\|T\|_{X-Y} := \sup_{f\neq 0} \frac{\|Tf\|_Y}{\|f\|_X}$. Whenever both vector spaces are X , we use the notation $\|T\|_{X\text{-ind}}$.

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Also, we may use $\|T\|$ without any subscript if the result holds for any induced norm.

Any linear causal map $T : x \in l_p \to y \in l_q$, for $1 \leq p, q \leq \infty$, can be thought of as an infinite dimensional lower triangular matrix,

$$T = \left[\begin{array}{cccc} T_{0,0} & 0 & 0 & \cdots \\ T_{1,1} & T_{1,0} & 0 & \cdots \\ T_{2,2} & T_{2,1} & T_{2,0} & \\ \vdots & \vdots & & \ddots \end{array} \right]. \quad (1)$$

Given a sequence $g = \{g(k)\}_{k=0}^{\infty}$, the delay or shift operator Λ is defined by

$$\Lambda^k g = \left\{ \underbrace{0, ..., 0}_{k \text{ zeros}}, g(0), g(1), ... \right\},$$

and, with a slight abuse of notation, $\Lambda^{-k} g = \{g(k), g(k+1), ...\}$. A linear causal map T is called time-invariant if it commuted with the delay operator, i.e. $\Lambda T = T \Lambda$. If T is a Linear Time-Invariant (LTI), it is fully characterize by its impulse response denoted by $\{T(k)\}_{k=0}^{\infty}$. In this case, its infinite dimensional matrix representation is given by

$$T = \left[\begin{array}{cccc} T(0) & 0 & 0 & \cdots \\ T(1) & T(0) & 0 & \cdots \\ T(2) & T(1) & T(0) & \\ \vdots & \vdots & & \ddots \end{array} \right].$$

The standard delay operator is denoted by Λ. More precisely, for any $k \in \mathbb{Z}_+$ and any sequence $g = \{g(0), g(1), ...\}$,

$$\Lambda^k g = \left\{ \underbrace{0, ..., 0}_{k \text{ zeros}}, g(0), g(1), ... \right\},$$

A LTI system has the state-space representation of

$$G : \left\{ \begin{array}{c} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{array} \right., \text{ with } x(t_0) = x_0, \quad (2)$$

where $u(t) \in \mathbb{R}^m, x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, and $x_0 \in \mathbb{R}^n$ are input, state, output, and the initial condition of the system and A, B, C, and D are matrices with appropriate dimensions for all $t \in \mathbb{Z}_+$. Given a matrix S, we define $\hat{S}$ to be the diagonal operator

$$\hat{S} = \left[\begin{array}{ccc} S & 0 & \cdots \\ 0 & S & \\ \vdots & & \ddots \end{array} \right]. \quad (3)$$

Using this notation, we can define diagonal operators $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ and rewrite (2) as

$$G : \left\{ \begin{array}{c} x = \Lambda \hat{A} x + \Lambda \hat{B} u + \bar{x}_0 \\ y = \hat{C} x + \hat{D} u \end{array} \right., \quad (4)$$

where $\bar{x}_0 = \left\{ \underbrace{0, ..., 0}_{t_0 \text{ zeros}}, x_0, 0, 0, ... \right\}$, $x = \{x(t)\}_{t=0}^{\infty}$, $y = \{y(t)\}_{t=0}^{\infty}$, $u = \{u(t)\}_{t=0}^{\infty}$, and Λ is the delay operator. The above representation of G is referred to as the operator form. One can also write time delay systems in the operator form. Consider the system given by

$$H : \left\{ \begin{array}{c} x(t+1) = \sum_{i=0}^{N} A_i x(t-i) + \sum_{i=0}^{N} B_i u(t-i) \\ y(t) = \sum_{i=0}^{N} C_i x(t-i) + \sum_{i=0}^{N} D_i u(t-i) \end{array} \right.,$$

with initial condition $x_0 = \{x(k)\}_{k=-N}^{0}$. Define $\bar{A} = \sum_{i=0}^{N} \Lambda^i \hat{A}_i$. Similarly, we define $\bar{B}, \bar{C}$, and $\bar{D}$. Then, the time-delay system can be written in the operator form as

$$H : \left\{ \begin{array}{c} x = \Lambda \bar{A} x + \Lambda \bar{B} u + \bar{x}_0 \\ y = \bar{C} x + \bar{D} u \end{array} \right.. \quad (5)$$

Throughout this paper, we prefer to write the systems in the operator form (5) as it allows for treating various classes of systems (e.g. time-delay, switching, and LTV systems [14]) in a unified way. Henceforth, we consider the systems that have operator forms as in (5). Such systems can be seen as a mapping from $\left(\begin{array}{c} \bar{x}_0 \\ u \end{array} \right)$ to $\left(\begin{array}{c} x \\ y \end{array} \right)$. For this system, we adopt the following definitions of stability and gain.

Definition 1: Given two normed spaces $(\mathcal{U}, \|.\|_{\mathcal{U}})$ and $(\mathcal{X}, \|.\|_{\mathcal{X}})$, we say that the system H in (5) is $\mathcal{U}$ to $\mathcal{X}$ stable if it is a bounded operator from $\left(\begin{array}{c} \bar{x}_0 \\ u \end{array} \right) \in \mathcal{X} \times \mathcal{U}$ to $\left(\begin{array}{c} x \\ y \end{array} \right) \in \mathcal{X} \times \mathcal{X}$. More precisely, H is $\mathcal{U}$ to $\mathcal{X}$ stable if, for some $\gamma_1, \gamma_2 \geq 0$, $\|x\|_{\mathcal{X}} \leq \gamma_1 \|\bar{x}_0\|_{\mathcal{X}} + \gamma_2 \|u\|_{\mathcal{U}}$ and $\|y\|_{\mathcal{X}} \leq \gamma_1 \|\bar{x}_0\|_{\mathcal{X}} + \gamma_2 \|u\|_{\mathcal{U}}$ whenever $\|\bar{x}_0\|_{\mathcal{X}}$ and $\|u\|_{\mathcal{U}}$ are finite.

Definition 2: Given two normed spaces $(\mathcal{U}, \|.\|_{\mathcal{U}})$ and $(\mathcal{X}, \|.\|_{\mathcal{X}})$, and a $\mathcal{U}$ to $\mathcal{X}$ stable system H, its gain is defined as $\|H\|_{\mathcal{U}-\mathcal{X}} = \sup_{\substack{u \neq 0 \\ x_0 = 0}} \frac{\|y\|_{\mathcal{X}}}{\|u\|_{\mathcal{U}}}$.

For simplicity, we let $\mathcal{X}$ and $\mathcal{U}$ to be the same (but possibly with different dimension) l_p spaces.

III. Basic Setup

A standard practice for designing a distributed controller for subsystems communicating over a given network is to aggregate all subsystems into one system P and design a controller for this system. The controller must be designed in a way so that it can be implemented as subcontrollers communicating over the given network. The aggregate system, in the operator form, can be written as

$$P : \left\{ \begin{array}{c} x = \Lambda \bar{A} x + \Lambda \bar{B}_1 w + \Lambda \bar{B}_2 u + \bar{x}_0 \\ z = \bar{C}_1 x + \bar{D}_{11} w + \bar{D}_{12} u \\ y = \bar{C}_2 x + \bar{D}_{12} w, \end{array} \right., \quad (6)$$

where x, y, and z are the states, measurements, and the regulated output; w and u are the exogenous and control inputs; and, $\bar{A}, \bar{B}_i, \bar{C}_j, \bar{D}_{ij}$, for $i.j \in \{1, 2\}$ are bounded operators.

Example 3: Consider a network with N subsystems. Each subsystem is given by

$$\begin{aligned} x_i(t+1) &= A^i x_i(t) + B_1^i w_i(t) + B_2^i u_i(t) \qquad (7)\\ &\quad + \sum_{j=1}^{N} B_3^{ij} \eta_{ij}(t), \\ z_i(t) &= C_1^i x_i(t) + D_{11}^i w_i(t) + D_{12}^i u_i(t), \\ y_i(t) &= C_2^i x_i(t) + D_{21}^i w_i(t), \\ \nu_{ji}(t) &= C_3^{ji} x_i(t) + D_{31}^{ji} w_i(t), \; j = 1,2,...,N, \end{aligned}$$

where x_i, y_i, and z_i are the states, measured output, and regulated output of the i^{th} subsystem; ν_{ji} is the signal that the i^{th} subsystem communicates to the j^{th} subsystem and η_{ij} is the signal that i^{th} subsystem receives through its communication link with the j^{th} subsystem. We let $B_2^{ij} = C_3^{ji} = D_{31}^{ji} = 0$ if there is no communication link between i^{th} and j^{th} subsystems. Furthermore, due to the delay in the communication links, we set

$$\eta_{ij}(t) = \nu_{ij}(t - \tau_{ij}), \qquad (8)$$

where $\tau_{ij} \in \mathbb{Z}_{\geq 0}$ is delay in communication from j^{th} to i^{th} subsystem. Substituting (8) in (7), the i^{th} subsystem, in the operator form, can be written as

$$\begin{aligned} x_i &= \Lambda \left[\hat{A}^i x_i + \sum_{j=1}^{3} \Lambda^{\tau_{ij}} \hat{A}^{ij} x_j \right] \\ &\quad + \Lambda \left[\hat{B}_1^i w_i + \sum_{j=1}^{3} \Lambda^{ij} \hat{B}_1^{ij} w_j \right] + \Lambda \hat{B}_2^i u_i, \end{aligned}$$

where $\hat{A}^{ij} = \hat{B}_3^{ij} \hat{C}_3^{ij}$ and $\hat{B}_1^{ij} = \hat{B}_3^{ij} \hat{D}_{31}^{ij}$. Based on the above expression, it can be easily seen that for properly defined operators $\bar{A}, \bar{B}_i, \bar{C}_i, \bar{D}_{ij}$, for $i \in \{1,2\}$, the aggregate system can be written as in (6).

The structure of the network is reflected in the coefficient operators involved in (6), e.g., as sparsity patterns [12]. Given a fixed network consisting of N nodes (subsystems) and a set of N inputs $\xi = [\xi_i]$ to, and N outputs $\zeta = [\zeta_i]$ from, these N nodes, let $\mathcal{S}$ denote the set of all input-output maps (or, transfer functions in the LTI case) T from ζ to ξ, i.e., $\xi = T\zeta$ that can be obtained from this network. That is, the input-output aggregation of all subsystem (or, subcontroller) dynamics, interconnected via the network, form an element T in $\mathcal{S}$ and, conversely, any element in $\mathcal{S}$ can be implemented, stably or unstably, as subsystems communicating over the given network. Consider the following example:

Example 4: **Nested network:** An example of a nested network is given in Figure 1. We adopt the notations introduced in Example 3. It can be easily verified that the aggregate system is given by (6) where $\bar{A} = \{\bar{A}(0), \bar{A}(1), 0, 0, ...\}$, $\bar{B}_1 = \{\bar{B}_1(0), \bar{B}_1(1), 0, ...\}$, $\bar{B}_2 = \{\bar{B}_2(0), 0, ...\}$, $\bar{C}_j = \{\bar{C}_j(0), 0, ...\}$, $\bar{D}_{ij} = \{\bar{D}_{ij}(0), 0, ...\}$ with

$$\bar{A}(0) = diag\{A^1, A^2, A^3\},$$

$$\bar{B}_j(0) = diag\{B_j^1, B_j^2, B_j^3\},$$

$$\bar{C}_j(0) = diag\{C_j^1, C_j^2, C_j^3\},$$

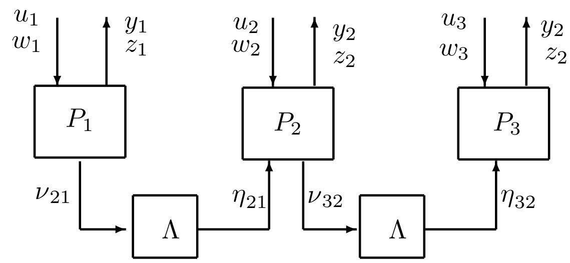

Fig. 1. A simple nested network.

$$\bar{D}_{ij}(0) = diag\{D_{ij}^1, D_{ij}^2, D_{ij}^3\},$$

and for $i, j \in \{1, 2\}$

$$\begin{aligned} \bar{A}(1) &= \begin{bmatrix} 0 & 0 & 0 \\ B_3^{21} C_3^{21} & 0 & 0 \\ 0 & B_3^{32} C_3^{32} & 0 \end{bmatrix}, \\ \bar{B}_1(1) &= \begin{bmatrix} 0 & 0 & 0 \\ B_3^{21} D_{31}^{21} & 0 & 0 \\ 0 & B_3^{32} D_{31}^{32} & 0 \end{bmatrix}. \end{aligned}$$

In this example, the structure of the network is reflected on the impulse response of the coefficient operators, e.g. $\bar{A}$. The terms in the impulse response of $\bar{A}$ are lower triangular, which conforms with the flow of communication from subsystem 1 to subsystem 2 and then to subsystem 3. And the sparsity structure in $\bar{A}(0)$ and $\bar{A}(1)$ is because each subsystem has immediate access to its own measurement signal but communicates with its neighbors with a delay. For this network, the set $\mathcal{S}$ is the space of all systems P whose impulse response $\{P(k)\}_{k=0}^{\infty}$ satisfies the following conditions: $P(k)$ is lower triangular for $k = 2, 3, ...$, $P(0)$ is diagonal, and

$$P(1) = \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ 0 & * & * \end{bmatrix},$$

where $*$ stands for a possibly non zero entry. Or, in transfer function terms,

$$P[\lambda] = \begin{bmatrix} * & 0 & 0 \\ \lambda * & * & 0 \\ \lambda^2 * & \lambda * & * \end{bmatrix},$$

where $P[\lambda] = \sum_{k=0}^{\infty} \lambda^k P(k)$ is the λ-transform. Accordingly, if K is a controller for P within the same communication network, $K[\lambda]$ should also be of the same form, i.e., $K \in \mathcal{S}$.

The set $\mathcal{S}$ is fully characterized by the underlying network. In this paper, given a (stabilizable and detectable in the usual sense) generalized plant $P = \begin{bmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{bmatrix}$ as in (6), we are interested in finding the controllers $K \in \mathcal{S}$ that are also stably realizable over the network. We should point out that even if K belongs to $\mathcal{S}$ and stabilizes P in the usual centralized sense, i.e., if

$$\begin{bmatrix} I & P_{yu} \\ K & I \end{bmatrix} \qquad (9)$$

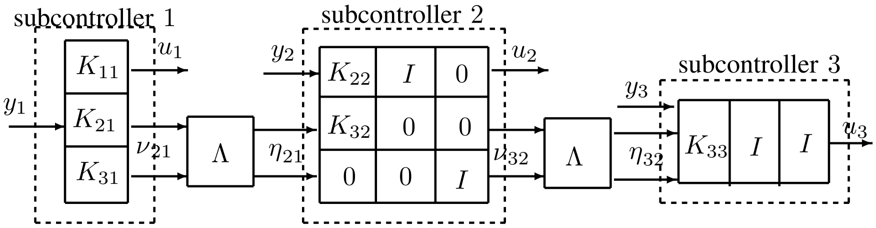

Fig. 2. Controller in Example 5 implemented over the network.

has a stable inverse, it does not mean that the controller K can automatically be realized stably, although we can implement it as the interconnection of subcontrollers consistent with the network. Unless we guarantee that the implementation of K does not have internal hidden unstable modes, the closed-loop may not be stable. This is because a stabilizing controller in a centralized sense, by design, guarantees the boundedness (stability) of the measured output and control input, y and u. Also, under the detectability assumption of the plant, the boundedness of y and u translates to that of x and z in (6). However, one cannot infer that the signals travelling between the subcontrollers are bounded. Therefore, in designing a controller that is implementable over the network in a stable way, one needs to investigate and guarantee the boundedness of the signals travelling between the subcontrollers.

Example 5: Consider the nested network in Example 4 and let $K \in \mathcal{S}$ be a stabilizing controller in the centralized sense as in (9). Since $K = \{K(k)\}_{k=0}^{\infty} \in \mathcal{S}$, it can be partitioned as

$$K = \begin{bmatrix} K_{11} & 0 & 0 \\ \Lambda K_{21} & K_{22} & 0 \\ \Lambda^2 K_{31} & \Lambda K_{32} & K_{33} \end{bmatrix}.$$

One way to implement this controller over the network is illustrated in Figure 2. By definition, the signals y, z, and u are bounded when w and $\bar{x}_0$ are bounded. However, as mentioned above, it is not guaranteed that the signals travelling between the subcontrollers, i.e. ν_{21} and ν_{32}, are bounded. Therefore, special attention should be paid to the stable implementability of structured controllers.

Accordingly, we define the set $\mathcal{S}_K \subseteq \mathcal{S}$ to be the set of stabilizing controllers K that can be stably implemented over the network without loosing stability, i.e., the subcontrollers communicate bounded signals provided that the measured outputs, y, and control inputs, u, are bounded.

More precisely, the closed-loop system, in the operator form, is given by

$$G_{cl}: \begin{cases} x = \left[I - \Lambda\left(\bar{A} + \bar{B}_2 K \bar{C}_2\right)\right]^{-1} \times \\ \qquad \left[\Lambda\left(\bar{B}_1 + \bar{B}_2 K \bar{D}_{12}\right) w + \bar{x}_0\right] \\ z = \left(\bar{C}_1 + \bar{D}_{12} K \bar{C}_2\right) x + \left(\bar{D}_{11} + \bar{D}_{12} K \bar{D}_{12}\right) w \\ y = \bar{C}_2 x + \bar{D}_{12} w, \\ u = K \bar{C}_2 x + K \bar{D}_{12} w. \end{cases} \quad (10)$$

The closed-loop system in (10), can be thought of as a linear operator from $\begin{pmatrix} \bar{x}_0 \\ w \end{pmatrix}$ to signals x, y, z, and u. In conjunction with Definition 1, we adopt the following definition for stabilizing controllers which are also stably implementable.

Definition 6: A controller $K : y \to u \in \mathcal{S}_K$ is said to be stabilizing the plant P (6) if the closed loop system (10) is a bounded operator from $\bar{x}_0$ and w to x, y, z, and u.

We note that from now on, when we say K stabilizes P we refer to K that are stably implementable, i.e., $K \in \mathcal{S}_K$.

In this paper, given a structure $\mathcal{S}$, we develop necessary and sufficient conditions in terms of convex problems under which it is possible to find optimal structured controllers $K \in \mathcal{S}_K$ that are stably implementable over the network.

A typical S of interest consists of controllers with certain sparsity or delay patterns. More generally, we assume that $\mathcal{S}$ is a subspace that satisfies the following:

Assumption 7: The set $\mathcal{S}$ is delay-invariant, contains identity, and is closed under addition and multiplication, i.e., $I \in \mathcal{S}$, $\Lambda\mathcal{S} \in \mathcal{S}$ and for any $X, Y \in \mathcal{S}$, $X + Y \in \mathcal{S}$ and $XY \in \mathcal{S}$.

Assumption 8: The set $\mathcal{S}$ contains the coefficient operator $\bar{A}$ and $\bar{C}_2\left(I - \Lambda\bar{A}\right)^{-1}\Lambda\bar{B}_2$. Furthermore, for any $X \in \mathcal{S}$, it holds that $\bar{A} + \bar{B}_2 X \in \mathcal{S}$ and $\bar{A} + X\bar{C}_2 \in \mathcal{S}$.

Assumption 9: We assume $\bar{B}_2$ and $\bar{C}_2$ have respectively trivial right and left null spaces.

In the next section, we synthesize the state feedback controller that is stably implementable over the network. A useful result, which can be proved using the Youla parametrization as in [14], is as follows:

Lemma 10: A linear system, with operator form as in (5), is stable if and only if there exists a stable system Q such that one of the following conditions holds:

$$\begin{aligned} \left\|\bar{A}\left(I + \Lambda Q\right) - Q\right\| &< 1, \\ \left\|\left(I + Q\Lambda\right)\bar{A} - Q\right\| &< 1. \end{aligned}$$

IV. State Feedback Problems

In this section, we focus on the state-feedback problems when the set $\mathcal{S}$ satisfies Assumptions 7 and 8. The output-feedback problems is the subject of our future research where similar type of approach can be utilized to deal with such problems. Given a state-feedback controller $K \in S : x \to u$, the closed system is given by

$$G_{cl}: \begin{cases} x = \Lambda\left(\bar{A} + \bar{B}_2 K\right) x + \Lambda\bar{B}_1 w + \bar{x}_0 \\ z = \left(\bar{C}_1 + \bar{D}_{12} K\right) x + \bar{D}_{11} w \\ u = Kx \end{cases} \quad (11)$$

Our first result states the necessary and sufficient condition for the existence of a stabilizing controller that is also stably implementable over the network.

Theorem 11: There exists a stabilizing $K \in \mathcal{S}_K$ if and only if there exist stable systems $Q \in \mathcal{S}$ and $Z \in \mathcal{S}$ such that

$$\left\|\bar{A}\left(I+\Lambda Q\right)+\bar{B}_2 Z-Q\right\|<1. \tag{12}$$

In this case,

$$K=Z\left(I+\Lambda Q\right)^{-1}. \tag{13}$$

Proof: First, suppose $K \in \mathcal{S}_K$ is stabilizing. That is the closed loop system (11) is a stable map from $\bar{x}_0$ and w to x, z, and u. In particular, for $w=0$ in (11), we have

$$\begin{aligned} x &= \left[I-\Lambda\left(\bar{A}+\bar{B}_2K\right)\right]^{-1}\bar{x}_0, \\ u &= K\left[I-\Lambda\left(\bar{A}+\bar{B}_2K\right)\right]^{-1}\bar{x}_0. \end{aligned}$$

Therefore, $\left[I-\Lambda\left(\bar{A}+\bar{B}_2K\right)\right]^{-1}$ and $K\left[I-\Lambda\left(\bar{A}+\bar{B}_2K\right)\right]^{-1}$ are stable maps. Define operators $Q := \left(\bar{A}+\bar{B}_2K\right)\left[I-\Lambda\left(\bar{A}+\bar{B}_2K\right)\right]^{-1}$ and $Z := K\left(I+\Lambda Q\right)$. These operators are stable and belong to the set $\mathcal{S}$ since they can be rewritten as

$$\begin{aligned} \Lambda Q &= -I+\left[I-\Lambda\left(\bar{A}+\bar{B}_2K\right)\right]^{-1} \\ &= \sum_{t=1}^{\infty}\left[\Lambda\left(\bar{A}+\bar{B}_2K\right)\right]^{t}, \\ Z &= K\left[I-\Lambda\left(\bar{A}+\bar{B}_2K\right)\right]^{-1}, \end{aligned}$$

and $\Lambda\left(\bar{A}+\bar{B}_2K\right) \in \mathcal{S}$, by Assumptions 7 and 8, if $K \in \mathcal{S}$. It is easy to verify that for Q and Z defined above the left hand side of (12) vanishes, i.e., $\bar{A}\left(I+\Lambda Q\right)+\bar{B}_2Z-Q=0$, and hence (12) holds.

Conversely, suppose (12) is satisfied for some stable $Q, Z \in \mathcal{S}$ and K is given as in (13). We need to show that K is stabilizing and stably implementable over the network. From (12), there exists a stable system T with $\|T\|<1$ such that $\left(\bar{A}+\bar{B}_2K\right)\left(I+\Lambda Q\right)-Q=T$. Then, direct calculation shows that the closed-loop $G_{cl}:\left(\bar{x}_0^T, w^T\right)^T \to \left(x^T, z^T, u^T\right)^T$ is given by

$$G_{cl}:\left[\begin{array}{cc} H_1 & H_1\Lambda\bar{B}_1 \\ H_2 & H_2\Lambda\bar{B}_1+\bar{D}_{11} \\ H_3 & H_3\Lambda\bar{B}_1 \end{array}\right],$$

where

$$\begin{aligned} H_1 &= \left(I+\Lambda Q\right)\left(I-\Lambda T\right)^{-1}, \\ H_2 &= \left[\bar{C}_1\left(I+\Lambda Q\right)+\bar{D}_{12}Z\right]\left(I-\Lambda T\right)^{-1}, \\ H_3 &= Z\left(I-\Lambda T\right)^{-1}. \end{aligned}$$

Since $\|T\|<1$, we have $\left\|\left(I-\Lambda T\right)^{-1}\right\| \leq \frac{1}{1-\|T\|}$ and hence the mappings H_1, H_2, H_3, and G_{cl} are stable. It remains to show that K, given in (13), can be stably implemented. Notice that, $K: y \to u$ can be written as

$$K:\left\{\begin{array}{c} \xi=y-\Lambda Q\xi \\ u=Z\xi \end{array}\right. . \tag{14}$$

Since Q and Z are stable and belong to $\mathcal{S}$, they are implementable over the network. However, in order for K to be stably implementable, one needs to make sure that ξ is a bounded signal if y and u are. By the way of contradiction, suppose ξ is not bounded while y and u are. Then, from (12), we have

$$\bar{A}\left(I+\Lambda Q\right)\xi+\bar{B}_2Z\xi-Q\xi=T\xi,$$

for some T with $\|T\|<1$. Premultiplying this expression by Λ and using (14), we obtain

$$\Lambda\bar{A}y+\Lambda\bar{B}_2u+\xi-y=\Lambda T\xi,$$

where we used $\Lambda Q\xi=y-\xi$. This implies

$$\xi=\left(I-\Lambda T\right)^{-1}\left(I-\Lambda\bar{A}\right)y-\left(I-\Lambda T\right)^{-1}\Lambda\bar{B}_2u. \tag{15}$$

Notice that the right hand side of (15) is bounded while we assumed ξ was unbounded. This is a contradiction and completes the proof. ■

A. Control Synthesis

In this section, we address the problem of finding a stably implementable controller that minimizes the input-output gain of the closed-loop. In this context, it is conventional to only consider the gain from w to z in (10). For state-feedback, this problem amounts to solving the following minimization:

$$\inf_{K\in\mathcal{S}_K}\left\|G_{cl}\left(P,K\right)\right\|,$$

where

$$\begin{aligned} &G_{cl}\left(P,K\right)= \\ &\left(\bar{C}_1+\bar{D}_{12}K\right)\left[I-\Lambda\left(\bar{A}+\bar{B}_2K\right)\right]^{-1}\Lambda\bar{B}_1+\bar{D}_{11}. \end{aligned}$$

Theorem 12: Let γ be a positive real number. Then, the following conditions are equivalent:

i) There exists a stabilizing controller $K \in \mathcal{S}_K$ and $\left\|G_{cl}\left(P,K\right)\right\|<\gamma$.
ii) For any $\delta \in [0,1)$, there exist stable systems $Q \in \mathcal{S}$ and $Z \in \mathcal{S}$ such that (16)-(18) hold.
iii) There exist some $\delta \in [0,1)$ and stable systems $Q \in \mathcal{S}$ and $Z \in \mathcal{S}$ such that (16)-(18) hold.

$$\rho \leq \delta, \tag{16}$$

$$\left\|\bar{A}\left(I+\Lambda Q\right)+\bar{B}_2Z-Q\right\| \leq \rho, \tag{17}$$

$$\begin{aligned} &\left\|\left[\bar{C}_1\left(I+\Lambda Q\right)+\bar{D}_{12}Z\right]\Lambda\bar{B}_1+\bar{D}_{11}\right\| \\ &\quad+\frac{\rho}{1-\delta}\left\|\bar{C}_1\left(I+\Lambda Q\right)+\bar{D}_{12}Z\right\|\left\|\bar{B}_1\right\|<\gamma. \end{aligned} \tag{18}$$

In either case, a controller is given by

$$K=Z\left(I+\Lambda Q\right)^{-1}. \tag{19}$$

Proof: Note that iii) is a special case of ii). That is, ii) implies iii) since if for any value of $\delta \in [0,1)$, conditions (16)-(18) hold, they should also hold for particular value of δ, for example $\delta=\frac{1}{2}$. Thus, to prove this theorem, it remains to show that i) implies ii) and iii) implies i).

i)⇒ii): First, suppose there exists a controller $K \in \mathcal{S}_K$ such that $\left\|G_{cl}\left(P,K\right)\right\|<\gamma$. Define Q and Z as given in the proof of Theorem 11. Similarly to the proof of Theorem 11, one can argue about the stability of Q and Z. Furthermore,

Q and Z belong to the set $\mathcal{S}$ and direct calculation shows that (17) is satisfied with $\rho = \delta = 0$. It remains to show (18). To this end, notice that for the Q and Z defined in (**??**)-(**??**) and $\rho = \delta = 0$, the left hand side of (18) reduces to

$$\begin{aligned} & \left\| \left(\bar{C}_1 + \bar{D}_{12}K\right)\left[I - \Lambda\left(\bar{A} + \bar{B}_2 K\right)\right]^{-1} \Lambda \bar{B}_1 + \bar{D}_{11} \right\| \\ = \; & \left\| G_{cl}\left(P, K\right) \right\|, \end{aligned}$$

which is less than γ and hence (18) holds.

iii)⇒i): Suppose (16)-(18) hold for some $Q, Z \in \mathcal{S}$, $\delta.\rho \in [0, 1)$. Then the stability and implementability of the controller follows from Theorem 11. It remains to show that $\|G_{cl}(P, K)\| < \gamma$. Notice that, for K defined in (19), the closed loop-map can be rewritten as

$$\begin{aligned} G_{cl}\left(P, K\right) = & \left[\bar{C}_1\left(I + \Lambda Q\right) + \bar{D}_{12} Z\right] \Lambda \bar{B}_1 + \bar{D}_{11} \\ + & \left[\bar{C}_1\left(I + \Lambda Q\right) + \bar{D}_{12} Z\right] \Lambda T \left(I - \Lambda T\right)^{-1} \Lambda \bar{B}_1, \end{aligned} \quad (20)$$

where $T = \bar{A}\left(I + \Lambda Q\right) + \bar{B}_2 Z - Q$. From (17), we have $\|T\| \leq \rho < 1$ and consequently from (20) one obtains

$$\begin{aligned} \|G_{cl}\| \leq \; & \left\| \left[\bar{C}_1\left(I + \Lambda Q\right) + \bar{D}_{12} Z\right] \Lambda \bar{B}_1 + \bar{D}_{11} \right\| \\ & + \frac{\rho}{1 - \delta} \left\| \bar{C}_1\left(I + \Lambda Q\right) + \bar{D}_{12} Z \right\| \left\| \bar{B}_1 \right\|, \end{aligned}$$

where we used $\left\| \Lambda T \left(I - \Lambda T\right)^{-1} \right\| \leq \frac{\rho}{1-\rho} \leq \frac{\rho}{1-\delta}$. Notice that the right hand side of the above expression is less than γ from (18). and hence the proof is complete. ■

Remark 13: From condition ii) of the theorem, one can fix $\delta \in [0, 1)$ without loss of generality. In this case, the feasibility of (16)-(18) can be cast as optimization in (ρ, Q, Z). This is, in general, not a convex problem. However, for a fixed ρ, (16)-(18) become convex in Q and Z. In fact, they form the so-called model matching problems and their solutions can be tractably computed with arbitrary accuracy through utilizing Finite Impulse Response (FIR) approximation of Q and Z. Therefore, the feasibility of (16)-(18) can be cast as a line search for $\rho \in [0, \delta]$, which can be carried out efficiently via a bisection algorithm, together with model matching optimization Q and Z.

Remark 14: Alternatively, one can solve the convex optimization

$$\begin{aligned} \min_{Q,Z} \frac{1}{\varepsilon} & \left\| \bar{A}\left(I + \Lambda Q\right) + \bar{B}_2 Z - Q \right\| \\ + & \left\| \left[\bar{C}_1\left(I + \Lambda Q\right) + \bar{D}_{12} Z\right] \Lambda \bar{B}_1 + \bar{D}_{11} \right\|, \end{aligned}$$

where $\varepsilon > 0$ is small. Suppose, (Q^*, Z^*) is a solution of this optimization. Then, $g_1 := \left\| \left[\bar{C}_1\left(I + \Lambda Q^*\right) + \bar{D}_{12} Z^*\right] \Lambda \bar{B}_1 + \bar{D}_{11} \right\|$ and $g_2 := g_1 + \frac{\rho^*}{1-\rho^*} \left\| \bar{C}_1\left(I + \Lambda Q^*\right) + \bar{D}_{12} Z^* \right\| \left\| \bar{B}_1 \right\|$ are lower and upper bounds on $\|G_{cl}(P, K^*)\|$ provided that $\rho^* := \left\| \bar{A}\left(I + \Lambda Q^*\right) + \bar{B}_2 Z^* - Q^* \right\| < 1$.

Remark 15: From condition ii) of the theorem, one can set $\delta = 0$ without loss of generality. In this case, (16) implies that $\rho = 0$. Hence, the abovementioned combination of the line search and model-matching optimizations reduces only to model matching ones. The resulting model matching problems, however, are exact model matching problems and their solutions may not be in general not in the span of FIR Q and Z systems.

V. Conclusion

In this paper, we proposed a framework to synthesize structured controllers that can be stably realized over the network. This framework is unifying in the sense that various linear system, e.g., LTI, LTV, and linear switched systems, can be treated analogously with respect to any measure of performance, e.g., l_1, l_2, or l_∞ induced norms. Our approach is based on utilizing an operator representation of the system and combining it with the classical Youla-parameterization. We formulated the stability and performance problems as tractable model-matching convex optimization. Furthermore, the controllers can be stably realized over the network. Although, we presented the results for state-feedback, our approach can be extended to output-feedback problems as well.

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