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A chain observer for a class of nonlinear systems with long multiple delays in output measurements

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Abstract— **The main contribution of this paper is to present a chain observer for a class of nonlinear systems when the output measurements are affected by a long known and multiple bounded time-varying delay. The proposed observer is composed of a chain of observers with similar algorithm. Each observer estimates the state over a time horizon, while the first observer of the chain estimates the current state. The structure of each observer of the chain is based on the presence of a dynamical term which permits the compensation of the delay. The observer gains are computed by a set of parameterized Linear Matrix Inequalities (LMIs) which depends on the delay. A Lyapunov-Krasovskii functional is used to demonstrate the asymptotical convergence to zero of the observation error. The performance of the proposed observer is evaluated through a numerical example.**

Index Terms— **Chain observer, multiples delay, Lyapunov-Krasovskii, Linear Matrix Inequalities, long delay**

I. INTRODUCTION

The delay phenomenon is often encountered in various engineering systems such as mechanical and electrical systems, communication networks, among others. The source of a time-delay can be due to the nature of the system or induced into the system due to the transmission delays, which are associated to other components interacting with the system. For example, when the system is controlled or monitored by a remote controller through a communication system, or when the measurement process intrinsically causes a non-negligible time-delay. A time-delay can generate instability or oscillations in a system. For this reason, many researchers are devoted to investigate the different fields of automatic control for time-delayed systems, such as stability, observability, controllability and system identification, among others, e.g. [11], [15], [16]. Other researches are described in [13], [18], where different control approaches for delayed systems are presented.

For several years a great effort has been devoted to the study of the observer design problem for state estimation of linear and nonlinear systems with a single and same delay in output measurements [4], [7], [22]. One of the first works of observer for nonlinear systems with delayed output is presented in [5], where a chain-observer is proposed for a class of nonlinear drift observable systems, where the output measurements are affected by a single known constant delay. Here each observer in the chain is used to estimate the system state for a suitable fraction of the total delay. A similar approach has been used in [9], where some restrictions of the chain-observer in [5] have been overcome. Also, in [1] another predictor for nonlinear systems with constant delayed output is proposed, which is based on a cascade observer. Concrete conditions for the convergence of this predictor have been derived by the use of Linear Matrix Inequalities (LMIs). In [8] a prediction cascade observer has been extended to triangular nonlinear systems with a single and same delay in the output measurements by a chain of high-gain observer. In a recent work by [20], a chain observer is proposed for nonlinear systems with a same single constant delays, which are present in the state, the input and the output, here the LPV approach is used to provide a resolution of the LMIs.

There are several recent publications documenting the observer design for systems with single same time-varying delay measurements. The particular case of piecewise constant delay is presented by [19] where the state observer is only given to the case of linear systems with delayed output. In [2], a state observer for drift observable nonlinear systems with a single time-varying delay in the output measurements is presented with a stability condition that depends of the maximum value of the time-varying delay. Lyapunov-Razumikhin approach was used to prove the asymptotic convergence to zero of the observation error. Similarly, a high-gain observer with time-varying delayed measurements is proposed by [21]. This observer considers a piecewise continuous bounded delay, without any information about its derivative. Recently an interesting approach to this issue has been proposed by [3], this work introduces an innovative chain observer, which is extended to delayed systems for drift observable nonlinear systems with multiple time-varying delays in outputs.

The contribution of this work is the synthesis of a chain observer for Lipschitz nonlinear systems in the presence of long multiple delayed measurements. The known delays present in the outputs are different and variables. The main advantage of our approach is to consider a more general class

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of Lipschitz nonlinear systems, involving less restrictive LMI stability conditions. The structure of the presented observer is based on a chain of m observers with similar algorithm. Each observer estimates the state in a time horizon equal to $\frac{\tau_M}{m}$, while the first observer estimates the current state. The delay is partitioned on a fraction of the time delay h, which depends on the upper bound of the delay τ_M and on the observer number m. The gains of the chain observer are delay-dependent. These gains are related to the delay of each partition h. Indeed, the Lyapunov-Krasovskii functional is used to prove the asymptotic convergence to zero of the observation error.

This paper is organized as follows: in Section II some preliminaries and the problem statement are described. In Section III the proposed chain observer is presented. In Section IV numerical simulations are presented in order to evaluate the performance of the proposed observer. Finally, conclusions are discussed in Section V.

II. Preliminaries and problem statement

Consider the following nonlinear system with multiple time-varying delayed outputs:

$$\left\{\begin{array}{l} \dot{x}(t) = Ax(t) + \varphi(u(t),x(t)) \\ y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{pmatrix} = Cx(t), \\ \bar{y}(t) = \begin{pmatrix} \bar{y}_1(t) \\ \bar{y}_2(t) \\ \vdots \\ \bar{y}_p(t) \end{pmatrix} = \begin{pmatrix} y_1(t-\tau_1(t)) \\ y_2(t-\tau_2(t)) \\ \vdots \\ y_p(t-\tau_p(t)) \end{pmatrix} \end{array}\right. \tag{1}$$

where $x \in \mathbb{R}^n$ represents the system state, $u(t) \in \mathbb{R}$ is the input, $y(t) \in \mathbb{R}^p$ denotes the unmeasured system output and $\bar{y}(t) \in \mathbb{R}^p$ expresses the delayed output measurement. A and C are constant matrices with appropriate dimensions. The pair (A,C) is assumed to be detectable. $\tau_i(t), i=1:p$, represent the known time-varying delay of the $i-th$ output and $\tau_i(t), i=1:p$ is subject to the following hypothesis:

Assumption 1: The variable delay $\tau_i(t), i=1:p$ is a known continuous bounded function, e.g. $0 \leq \tau_i(t) \leq \tau_{M,i}, \quad i=1:p$, where $\tau_{M,i}, i=1:p$ denotes the upper bound of each time-varying delay $\tau_i(t), i=1:p$ and set the maximum value $\tau_M = max\{\tau_{M,i}, i=1:p\}$.

In additon the function $\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz nonlinearity with a Lipschitz constant γ, i.e., φ satisfies the following assumption:

Assumption 2: The function $\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz nonlinearity with a Lipschitz constant γ, i.e.:

$$\|\varphi(u,x) - \varphi(u,z)\| \leq \gamma\|x-z\|, \qquad \forall x,z \in \mathbb{R}^n \tag{2}$$

There are several publications documenting the continuous observer for Lipschitz nonlinear systems. For instance, [6], [24], proposed a reformulation of the Lipschitz property, which provides a best less conservative Lipschitz condition. In order to achieve this condition, for system (1), definitions and lemmas are required.

Definition 1: [24] Consider two vectors $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ and $z = (z_1, \cdots, z_n)^T \in \mathbb{R}^n$. The auxiliary vector $x^{z_i} \in \mathbb{R}^n$ is defined as $x^{z_0} = x$ and:

$$x^{z_i} = (z_1, \cdots, z_i, x_{i+1}, \cdots, x_n)^T \qquad \text{for} \qquad i = 1:n \tag{3}$$

Lemma 1: [24] Consider a Lipschitz function φ: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. Then for all, $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ and $z = (z_1, \cdots, z_n)^T \in \mathbb{R}^n$. There exist functions $\Psi_j : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\varphi(u,x) - \varphi(u,z) = \left(\sum_{j=1}^{j=n} \Psi_j(u, x^{z_{j-1}}, x^{z_j}) e_n^T(j)\right)(x-z) \tag{4}$$

where $e_n(i) = (0, \cdots, 0, 1, 0, \cdots, 0)^T \in \mathbb{R}^n$.

Lemma 2: [24] For the function $\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, the following conditions are equivalent:

- Lipschtiz property: φ is a Lipschitz function w.r.t. its argument, i.e.:

$$\|\varphi(u,x) - \varphi(u,z)\| \leq \gamma\|x-z\| \qquad \forall x,z \in \mathbb{R}^n \tag{5}$$

- For all $i,j = 1:n$, there exist functions $\Psi_{ij} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and constants a_{ij} and b_{ij}, such that

$$\varphi(u,x) - \varphi(u,z) = \left(\sum_{i=1}^{i=n}\sum_{j=1}^{j=n} \Psi_{ij} H_{ij}\right)(x-z) \tag{6}$$

  with $a_{ij} \leq \Psi_{ij} \leq b_{ij}$, where $|a_{ij}|, |b_{ij}| \leq \gamma$, $\Psi_{ij} \triangleq \Psi_{ij}(u, x^{z_{j-1}}, x^{z_j})$ and $H_{ij} = e_n(i) e_n^T(j)$.

Proofs of Lemmas 1 and 2 are in [24].

III. Chain observer for nonlinear systems with outputs affected by multiple delays

In the following section, a chain observer for Lipschitz nonlinear systems of the form (1) with multiple delayed outputs is introduced. This chain observer is composed by m observation algorithms. Each of them reconstructing the states of the previous m observation algorithms in the time instant $h = \dfrac{\tau_M}{m}$, where τ_M denotes the maximum value which is defined in Assumption 1. Indeed, the first observer $\hat{x}(t)$ estimates the current state of the system $x(t)$.

In order to design a chain observer for system (1), we adopt the following notations and definitions, as in [3] and [5]:

The observation horizon is $h = \dfrac{\tau_M}{m}$.

The new state and input $x_j(t)$ and $u_j(t)$ are respectively defined by:

$$\left\{\begin{array}{lll} x_j(t) & = x(t-(j-1)h), & j=1:m \\ u_j(t) & = u(t-(j-1)h), & j=1:m \end{array}\right. \tag{7}$$

note that for $j=1$, $x_1(t) = x(t)$ is the current state, whereas $\hat{x}_1(t)$ denotes the current state estimate which is the output of the chain observer.

In order to achieve the design of the proposed chain observer for system (1), let us define the following new output:

$$\bar{\bar{y}}_j(t) = \begin{cases} \bar{\bar{y}}_m(t), & \text{If } j = m \\ C\hat{x}_{j+1}(t), & \text{If } j = 1 : m-1 \end{cases} \tag{8}$$

where,

$$\bar{\bar{y}}_m(t) = \begin{pmatrix} \bar{y}_1(t-(\tau_M - \tau_1(t))) \\ \bar{y}_2(t-(\tau_M - \tau_2(t))) \\ \vdots \\ \bar{y}_p(t-(\tau_M - \tau_p(t))) \end{pmatrix} = \begin{pmatrix} y_1(t-\tau_M) \\ y_2(t-\tau_M) \\ \vdots \\ y_p(t-\tau_M) \end{pmatrix} = Cx(t-\tau_M) \tag{9}$$

Indeed, for $j = m$ each measured output $\bar{y}_i(t), \quad i = 1 : p$ is further delayed of an amount $\tau_M - \tau_i(t), \quad i = 1 : p$ and consequently we obtain the new output $\bar{\bar{y}}_m(t)$. For $j = 1 : m-1$ the measured output is replaced by $C\hat{x}_{j+1}$, that is the estimate of $y(t - jh)$ coming from the observer number $(j+1)$ of the chain.

Equation (8) can be rewritten as follows:

$$\bar{\bar{y}}_j(t) = \begin{cases} Cx(t-\tau_M), & \text{for } j = m \\ C\hat{x}_{j+1}(t), & \text{for } j = 1 : m-1 \end{cases} \tag{10}$$

using notations (7) for $j = m$, we obtain:

$$\bar{\bar{y}}_j(t) = \begin{cases} Cx_m(t-h), & \text{for } j = m \\ C\hat{x}_{j+1}(t), & \text{for } j = 1 : m-1 \end{cases} \tag{11}$$

Now, by using the equations (1), (7) and (11) the dynamics of $x_j(t), \quad j = 1 : m$, can be deduced and consequently:

$$\begin{cases} \dot{x}_j(t) &= Ax_j(t) + \varphi(u_j(t), x_j(t)), \qquad j = 1 : m \\ \bar{\bar{y}}_j(t) &= Cx_m(t-h), \quad \text{for } j = m \\ \bar{\bar{y}}_j(t) &= C\hat{x}_{j+1}(t), \quad \text{for } j = 1 : m-1 \end{cases} \tag{12}$$

System (12) is used to propose a chain observer for the nonlinear system (1), therefore, the proposed chain observer for system (1) is given by:

$$\begin{cases} \dot{\hat{x}}_m(t) &= A\hat{x}_m(t) + \varphi(u_m(t), \hat{x}_m(t)) + w_m(t) \\ \dot{\hat{x}}_{m-1}(t) &= A\hat{x}_{m-1}(t) + \varphi(u_{m-1}(t), \hat{x}_{m-1}(t)) + w_{m-1}(t) \\ \vdots & \\ \dot{\hat{x}}_{j+1}(t) &= A\hat{x}_{j+1}(t) + \varphi(u_{j+1}(t), \hat{x}_{j+1}(t)) + w_{j+1}(t) \\ \dot{\hat{x}}_j(t) &= A\hat{x}_j(t) + \varphi(u_j(t), \hat{x}_j(t)) + w_j(t) \\ \vdots & \\ \dot{\hat{x}}_2(t) &= A\hat{x}_2(t) + \varphi(u_2(t), \hat{x}_2(t)) + w_2(t) \\ \dot{\hat{x}}_1(t) &= A\hat{x}_1(t) + \varphi(u_1(t), \hat{x}_1(t)) + w_1(t) \\ \hat{x}_j(\theta) &= \phi(\theta - (j-1)h) \quad \theta \in [-h, 0], j = 1 : m \\ u_j(\theta) &= \omega(\theta - (j-1)h) \quad \theta \in [-h, 0], j = 1 : m \end{cases} \tag{13}$$

where

$$\begin{cases} \bullet \ \text{for} \quad j = m \\ \quad w_m(t) = -K_1(C\hat{x}_m(t) - \bar{\bar{y}}_m(t)) + \mu_m(t), \\ \quad \mu_m(t) = K_1 C \int_{t-h}^{t} (A\hat{x}_m(s) + \varphi(u_m(s), \hat{x}_m(s))) ds, \\ \quad \bar{\bar{y}}_m(t) = Cx(t-\tau_M) = Cx_m(t-h) \\ \bullet \ \text{for} \quad j = 1 : m-1 \\ \quad w_j(t) = -K_2(C\hat{x}_j(t) - C\hat{x}_{j+1}(t)) + \mu_j(t), \\ \quad \mu_j(t) = K_2 C \int_{t-h}^{t} (A\hat{x}_j(s) + \varphi(u_j(s), \hat{x}_j(s))) ds \end{cases} \tag{14}$$

K_1 and K_2 are the observer gains which will be specified later and ϕ, ω are two known functions used to initialize the system (13) for $\theta \in [-h, 0]$.

Remark 1: By differentiating $w_j(t), j = 1 : m$ in (14). Thus, we can represent system (13)-(14) by the following implementable recursive form of the candidate chain observer, as follows:

$$\dot{\hat{x}}_j(t) = A\hat{x}_j(t) + \varphi(u_j(t), \hat{x}_j(t)) + w_j(t), \qquad j = 1 : m \tag{15}$$

where,

$$\begin{cases} \bullet \ \text{for} \quad j = m \\ w_m(t) &= -K_1(C\hat{x}_m(t) - \bar{\bar{y}}_m(t)) + \mu_m(t), \\ \dot{\mu}_m(t) &= K_1 C(A\hat{x}_m(t) + \varphi(u_m(t), \hat{x}_m(t)) \\ & \quad -(A\hat{x}_m(t-h) + \varphi(u_m(t-h), \hat{x}_m(t-h)))) \\ \mu_m(0) &= K_1 C \int_{-h}^{0} (A\hat{x}_m(s) + \varphi(u_m(s), \hat{x}_m(s))) ds, \\ \bar{\bar{y}}_m(t) &= Cx(t-\tau_M) = Cx_m(t-h) \\ \bullet \ \text{for} \quad j = 1 : m-1 \\ w_j(t) &= -K_2(C\hat{x}_j(t) - C\hat{x}_{j+1}(t)) + \mu_j(t), \\ \dot{\mu}_j(t) &= K_2 C(A\hat{x}_j(t) + \varphi(u_j(t), \hat{x}_j(t)) - (A\hat{x}_j(t-h) \\ & \quad + \varphi(u_j(t-h), \hat{x}_j(t-h)))) \\ \mu_j(0) &= K_2 C \int_{-h}^{0} (A\hat{x}_j(s) + \varphi(u_j(s), \hat{x}_j(s))) ds \end{cases} \tag{16}$$

The integral terms $\mu_j(0), \quad j = 1 : m$ presented in (16), have constant bounds. They can be calculated using numerical methods (Trapezoidal, Simpson, etc.).

Define now the following observation errors:

$$\begin{cases} \tilde{x}_m(t) &= \hat{x}_m(t) - x_m(t), \quad \text{for} \quad j = m \\ \tilde{x}_j(t) &= \hat{x}_j(t) - x_j(t), \quad \text{for} \quad j = 1 : m-1 \end{cases} \tag{17}$$

By combining (12) and (15), from (7), we can see that $x_{j+1}(t) = x_j(t-h)$, therefore, we can establish the dynamics of the observation error:

$$\begin{cases} \dot{\tilde{x}}_m(t) &= A\tilde{x}_m(t) - K_1 C(\hat{x}_m(t) - x_m(t-h)) \\ & \quad + \varphi(u_m(t), \hat{x}_m(t)) - \varphi(u_m(t), x_m(t)) \\ & \quad + K_1 C \int_{t-h}^{t} (A\hat{x}_m(s) + \varphi(u_m(s), \hat{x}_m(s))) ds, \\ & \quad \text{for } j = m \\ \dot{\tilde{x}}_j(t) &= A\tilde{x}_j(t) - K_2 C(\hat{x}_j(t) - x_j(t-h)) + K_2 C\tilde{x}_{j+1}(t) \\ & \quad + \varphi(u_j(t), \hat{x}_j(t)) - \varphi(u_j(t), x_j(t)) \\ & \quad + K_2 C \int_{t-h}^{t} (A\hat{x}_j(s) + \varphi(u_j(s), \hat{x}_j(s))) ds, \\ & \quad \text{for } j = 1 : m-1 \end{cases} \tag{18}$$

Using the Newton-Leibniz formula for the terms $x_m(t-h)$ and $x_j(t-h)$, and using (12), we have:

$$\begin{cases} \dot{\tilde{x}}_m(t) &= (A - K_1 C)\tilde{x}_m(t) + \varphi(u_m(t), \hat{x}_m(t)) \\ & \quad - \varphi(u_m(t), x_m(t)) \\ & \quad + K_1 C \int_{t-h}^{t} (A\tilde{x}_m(s) + \varphi(u_m(s), \hat{x}_m(s)) \\ & \quad - \varphi(u_m(s), x_m(s))) ds, \qquad \text{for } j = m \\ \dot{\tilde{x}}_j(t) &= (A - K_2 C)\tilde{x}_j(t) + K_2 C\tilde{x}_{j+1}(t) \\ & \quad + \varphi(u_j(t), \hat{x}_j(t)) - \varphi(u_j(t), x_j(t)) \\ & \quad + K_2 C \int_{t-h}^{t} (A\tilde{x}_j(s) + \varphi(u_j(s), \hat{x}_j(s)) \\ & \quad - \varphi(u_j(s), x_j(s))) ds, \qquad \text{for } j = 1 : m-1 \end{cases} \tag{19}$$

Now, since φ is a Lipschitz function, therefore, according to Lemmas 1-2 there are functions Ψ_{ij} such that:

$$\begin{cases} \varphi(u_m(t),\hat{x}_m(t)) - \varphi(u_m(t),x_m(t)) & = \left(\sum_{i=1}^{i=n}\sum_{j=1}^{j=n}\Psi_{ij}H_{ij}\right)\tilde{x}_m(t), \\ & \text{for } j=m \\ \varphi(u_j(t),\hat{x}_j(t)) - \varphi(u_j(t),x_j(t)) & = \left(\sum_{i=1}^{i=n}\sum_{j=1}^{j=n}\Psi_{ij}H_{ij}\right)\tilde{x}_j(t), \\ & \text{for } j=1:m-1 \end{cases} \tag{20}$$

Now, as is defined in [24] the matrix $\Psi = (\Psi_{ij})$, where the matrix parameter Ψ belongs to a bounded convex set H for which the set of vertices is defined by:

$$V_H = \{\Phi \in R^{n\times n} : \Phi_{ij} \in \{a_{ij}, b_{ij}\}\} \tag{21}$$

and define the matrix:

$$\bar{A}(\Psi) = A + \left(\sum_{i=1}^{i=n}\sum_{j=1}^{j=n}\Psi_{ij}H_{ij}\right), \quad \forall\Psi = (\Psi_{ij}) \tag{22}$$

Using (19), (20) and (22) the observation error dynamics is rewritten as:

$$\begin{cases} \dot{\tilde{x}}_m(t) & = (\bar{A}(\Psi) - K_1C)\tilde{x}_m(t) + K_1C\int_{t-h}^{t}\bar{A}(\Psi)\tilde{x}_m(s)ds, \\ & \text{for } j=m \\ \dot{\tilde{x}}_j(t) & = (\bar{A}(\Psi) - K_2C)\tilde{x}_j(t) + K_2C\tilde{x}_{j+1}(t) \\ & +K_2C\int_{t-h}^{t}\bar{A}(\Psi)\tilde{x}_j(s)ds, \\ & \text{for } j=1:m-1 \end{cases} \tag{23}$$

Now, we can state the following theorem

Theorem 1: Consider system (1) with Lipschitz nonlinearity of Assumption 2 and the proposed observer (13) – (14). For any given delay τ_M the observation error $\tilde{x}_1(t) = \hat{x}_1(t) - x_1(t)$ converges asymptotically to zero if there exists an integer m, matrices $S_k > 0, k = 1:2$, matrices $R_k, k = 1:2$, and constants $\varepsilon_q, q = 1:3$, such that for any matrix parameter Ψ in the polytope V_H defined in (21) the following matrix inequalities holds:

$$\begin{bmatrix} M_1 & hR_1^TC \\ hC^TR_1 & -h\varepsilon_1 I \end{bmatrix} < 0 \tag{24}$$

where $M_1 = S_1\bar{A}(\Psi) + \bar{A}^T(\Psi)S_1 - R_1^TC - C^TR_1 + h\varepsilon_1\bar{A}^T(\Psi)\bar{A}(\Psi)$, and:

$$\begin{bmatrix} M_2 & hR_2^TC \\ hC^TR_2 & -h\varepsilon_2 I \end{bmatrix} < 0 \tag{25}$$

where $M_2 = S_2\bar{A}(\Psi) + \bar{A}^T(\Psi)S_2 - R_2^TC - C^TR_2 + h\varepsilon_2\bar{A}^T(\Psi)\bar{A}(\Psi) + \varepsilon_3 I$, therefore $K_1 = S_1^{-1}R_1^T$ and $K_2 = S_2^{-1}R_2^T$.

Proof of Theorem 1

• **Step m** (for $j = m$). From (23) the observation error $\tilde{x}_m(t) = \hat{x}_m(t) - x_m(t)$ is given by:

$$\dot{\tilde{x}}_m(t) = (\bar{A}(\Psi) - K_1C)\tilde{x}_m(t) + K_1C\int_{t-h}^{t}\bar{A}(\Psi)\tilde{x}_m(s)ds \tag{26}$$

Consider now the following Lyapunov-Krasovskii functional:

$$V_m = \tilde{x}_m^T(t)S_1\tilde{x}_m(t) + \varepsilon_1\int_{-h}^{0}\int_{\nu}^{0}\tilde{x}_m^T(t+s)\bar{A}^T(\Psi)\bar{A}(\Psi)\tilde{x}_m(t+s)ds\, d\nu \tag{27}$$

where $\varepsilon_1 > 0$ is a positive constant.

Using (26), we can compute the time derivative of V_m, as follows:

$$\begin{aligned} \dot{V}_m = & \tilde{x}_m^T(t)(S_1(\bar{A}(\Psi) - K_1C) + (\bar{A}(\Psi) - K_1C)^TS_1)\tilde{x}_m(t) \\ & + \int_{t-h}^{t} 2\tilde{x}_m^T(t)S_1K_1C\bar{A}(\Psi)\tilde{x}_m(s)ds \\ & + h\varepsilon_1\tilde{x}_m^T(t)\bar{A}^T(\Psi)\bar{A}(\Psi)\tilde{x}_m(t) \\ & - \varepsilon_1\int_{t-h}^{t}\tilde{x}_m^T(s)\bar{A}^T(\Psi)\bar{A}(\Psi)\tilde{x}_m(s)ds \end{aligned} \tag{28}$$

Now, by considering the expression $2\tilde{x}_m^T(t)S_1K_1C\bar{A}(\Psi)\tilde{x}_m(s) = 2((S_1K_1C)^T\tilde{x}_m(t))^T\bar{A}(\Psi)\tilde{x}_m(s)$, it yields:

$$\begin{aligned} & \int_{t-h}^{t} 2\tilde{x}_m^T(t)S_1K_1C\bar{A}(\Psi)\tilde{x}_m(s)ds \leq \\ & \varepsilon_1^{-1}h\tilde{x}_m(t)^T(S_1K_1C)(S_1K_1C)^T\tilde{x}_m(t) \\ & +\varepsilon_1\int_{t-h}^{t}\tilde{x}_m^T(s)\bar{A}^T(\Psi)\bar{A}(\Psi)\tilde{x}_m(s)ds \end{aligned} \tag{29}$$

By substituting equations (29) into (28):

$$\begin{aligned} \dot{V}_m \leq & \tilde{x}_m^T(t)(S_1(\bar{A}(\Psi) - K_1C) + (\bar{A}(\Psi) - K_1C)^TS_1)\tilde{x}_m(t) \\ & + h\varepsilon_1^{-1}\tilde{x}_m^T(t)(S_1K_1C)(S_1K_1C)^T\tilde{x}_m(t) \\ & + h\varepsilon_1\tilde{x}_m^T(t)\bar{A}^T(\Psi)\bar{A}(\Psi)\tilde{x}_m(t) \\ = & \tilde{x}_m^T(t)\Lambda_1\tilde{x}_m(t) \end{aligned} \tag{30}$$

where $\Lambda_1 = S_1(\bar{A}(\Psi) - K_1C) + (\bar{A}(\Psi) - K_1C)^TS_1 + h\varepsilon_1^{-1}(S_1K_1C)(S_1K_1C)^T + h\varepsilon_1\bar{A}^T(\Psi)\bar{A}(\Psi)$.

Therefore, if $\Lambda_1 < 0$ then $\dot{V}_m < 0$, meaning that the observation error $\tilde{x}_m(t)$ is asymptotically stable, i.e., $\lim_{t\to+\infty}||\tilde{x}_m(t)|| \to 0$. Furthermore, by using the Schur complement, it can be shown that $\Lambda_1 < 0$ holds if:

$$\begin{bmatrix} M_3 & hS_1K_1C \\ hC^TK_1^TS_1^T & -h\varepsilon_1 I \end{bmatrix} < 0 \tag{31}$$

where $M_3 = S_1(\bar{A}(\Psi) - K_1C) + (\bar{A}(\Psi) - K_1C)^TS_1 + h\varepsilon_1\bar{A}^T(\Psi)\bar{A}(\Psi)$.

By considering $S_1K_1 = R_1^T$, the inequality (31) can be represented as inequality (24). The observer gain K_1 is given by $K_1 = S_1^{-1}R_1^T$.

• **Step j** (for $j = 1:m-1$). The asymptotic convergence of the observation error $\tilde{x}_j(t) = \hat{x}_j(t) - x_j(t),\ j = 1:m-1$ shall be proved by induction.

Let us now suppose that in the step $j+1$, the error $\tilde{x}_{j+1}(t)$ converges asymptotically to zero and we shall show that at step j the error $\tilde{x}_j(t)$ converges also asymptotically to zero. From (23) the observation error $\tilde{x}_j(t) = \hat{x}_j(t) - x_j(t), j = 1:m-1$ is given by:

$$\dot{\tilde{x}}_j(t) = (\bar{A}(\Psi) - K_2C)\tilde{x}_j(t) + K_2C\int_{t-h}^{t}\bar{A}(\Psi)\tilde{x}_j(s)ds + K_2C\tilde{x}_{j+1}(t) \tag{32}$$

The following Lyapunov-Krasovskii functional is taken into account:

$$V_j = \tilde{x}_j^T(t)S_2\tilde{x}_j(t) + \varepsilon_2\int_{-h}^{0}\int_{\nu}^{0}\tilde{x}_j^T(t+s)\bar{A}^T(\Psi)\bar{A}(\Psi)\tilde{x}_j(t+s)dsd\nu \tag{33}$$

where $\varepsilon_2 > 0$ is a positive constant.
By considering (32), the time derivative of V_j is:

$$\begin{aligned}\dot{V}_j &= \tilde{x}_j^T(t)(S_2(\bar{A}(\Psi) - K_2C) + (\bar{A}(\Psi) - K_2C)^T S_2)\tilde{x}_j(t) \\ &+ h\varepsilon_2 \tilde{x}_j^T(t)\bar{A}^T(\Psi)\bar{A}(\Psi)\tilde{x}_j(t) + 2\tilde{x}_j^T(t)S_2K_2C\tilde{x}_{j+1}(t) \\ &+ \int_{t-h}^{t} 2\tilde{x}_j^T(t)S_2K_2C\bar{A}(\Psi)\tilde{x}_j(s)ds \\ &- \varepsilon_2 \int_{t-h}^{t} \tilde{x}_j^T(s)\bar{A}^T(\Psi)\bar{A}(\Psi)\tilde{x}_j(s)ds \end{aligned} \quad (34)$$

Consider that $2\tilde{x}_j^T(t)S_2K_2C\bar{A}(\Psi)\tilde{x}_j(s) = 2((S_2K_2C)^T\tilde{x}_j(t))^T\bar{A}(\Psi)\tilde{x}_j(s)$. Therefore:

$$\begin{aligned}\int_{t-h}^{t} 2\tilde{x}_j^T(t)S_2K_2C\bar{A}(\Psi)\tilde{x}_j(s)ds &\leq h\varepsilon_2^{-1}\tilde{x}_j^T(t)(S_2K_2C)(S_2K_2C)^T\tilde{x}_j(t) \\ &+ \varepsilon_2 \int_{t-h}^{t} \tilde{x}_j^T(s)\bar{A}^T(\Psi)\bar{A}(\Psi)\tilde{x}_j(s)ds \end{aligned} \quad (35)$$

It yields:

$$2\tilde{x}_j^T(t)S_2K_2C\tilde{x}_{j+1}(t) \leq \varepsilon_3||\tilde{x}_j(t)||^2 + \varepsilon_3^{-1}||S_2K_2C||^2||\tilde{x}_{j+1}(t)||^2 \quad (36)$$

By substituting equations (35)-(36) into (34):

$$\dot{V}_j \leq \tilde{x}_j^T(t)\Lambda_2\tilde{x}_j(t) + \varepsilon_3^{-1}||S_2K_2C||^2||\tilde{x}_{j+1}(t)||^2 \quad (37)$$

where

$$\begin{aligned}\Lambda_2 &= S_2(\bar{A}(\Psi) - K_2C) + (\bar{A}(\Psi) - K_2C)^T S_2 + h\varepsilon_2\bar{A}^T(\Psi)\bar{A}(\Psi) \\ &+ \varepsilon_3 I + h\varepsilon_2^{-1}(S_2K_2C\bar{A}(\Psi))(S_2K_2C\bar{A}(\Psi))^T \end{aligned} \quad (38)$$

Using the fact that from the induction assumption at step $j+1$, the error $\tilde{x}_{j+1}$ converges asymptotically to zero, then from (37) if $\Lambda_2 < 0$ and using the comparison Lemma [10], the observation error $\tilde{x}_j(t), j = 1:m$, is also asymptotically stable with a result such that $\lim_{t\to+\infty} ||\tilde{x}_j(t)|| \to 0$, therefore, we conclude recursively, that all observation errors $\tilde{x}_j$ converge asymptotically to zero for $j = 1:m-1$. Furthermore, using the Schur complement, it can be shown that $\Lambda_2 < 0$ holds if:

$$\begin{bmatrix} M_4 & hS_2K_2C \\ hC^TK_2^TS_2 & -h\varepsilon_2 I \end{bmatrix} < 0 \quad (39)$$

where $M_4 = S_2(\bar{A}(\Psi) - K_2C) + (\bar{A}(\Psi) - K_2C)^T S_2 + h\varepsilon_2\bar{A}^T(\Psi)\bar{A}(\Psi) + \varepsilon_3 I$
Considering $S_2K_2 = R_2^T$ then inequality (39) can be represented by inequality (25), then we can obtain the observer gain $K_2 = S_2^{-1}R_2^T$. This completes the proof of Theorem 1.

IV. Example

As an example let us consider the dynamical model of the flexible joint robot manipulator presented in [17]:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0.1 & 0 & -0.1 & 0 \\ 1.95 & 0 & -1.95 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 21.6u \\ 0 \\ 0.33\sin x_4 \end{bmatrix} \quad (40)$$

$$y(t) = \begin{bmatrix} y_1(t) & y_2(t) & y_3(t) \end{bmatrix}^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = Cx(t)$$

where $x(t) = [x_1 \quad x_2 \quad x_3 \quad x_4]^T$, $y(t)$ denotes the unmeasured system output and $C = [I_{3\times3} \quad 0_{3\times1}]$, the input is denoted by $u(t) = sin(\pi t)$ and its Lipschitz constant is $\gamma = 0.33$.
In the following a cascade observer of the form (1) is used for the state estimation $\hat{x}(t)$. Simultions are presented in the two cases when the measurement $y(t)$ is available with multiple constant delays and multiple varible delays.

A. Multiple constant delays

In this case the available measure $\bar{y}(t)$ is given by:

$$\bar{y}(t) = \begin{bmatrix} \bar{y}_1(t) \\ \bar{y}_2(t) \\ \bar{y}_3(t) \end{bmatrix} = \begin{bmatrix} y_1(t-\tau_1) \\ y_2(t-\tau_2) \\ y_3(t-\tau_3) \end{bmatrix} = \begin{bmatrix} x_1(t-\tau_1) \\ x_2(t-\tau_2) \\ x_3(t-\tau_3) \end{bmatrix} \quad (41)$$

where τ_1, τ_2 and τ_3 are the different known constant delays. This simulation was carried out using the parameter $m = 2$ for values of the time-constant delays $\tau_1 = \tau_M$, $\tau_2 = (2\tau_M)/3$, $\tau_3 = \tau_M/3$ and $\tau_M = 0.2$ sec. and consequently $h = 0.1$sec. Now, we need to compute the observer gain for each step of the chain observer (K_1 and K_2), this was achieved through Theorem 1 for $h = 0.1sec$, the observer gains are found to be:

$$K_1 = \begin{bmatrix} 1.28 & 2.65 & 1.82 \\ 1.22 & 76.2 & 16.56 \\ 1.59 & 16.12 & 5 \\ 1.91 & -3.97 & 1.51 \end{bmatrix}, K_2 = \begin{bmatrix} 33.39 & 8.19 & 33.96 \\ 8.19 & 14.18 & 9.39 \\ 33.96 & 9.39 & 34.78 \\ 4.13 & 5.22 & 4.86 \end{bmatrix} \quad (42)$$

Unfortunate, for lack of space, the simulation results have not included in this case.

B. Example: Multiple variable delays

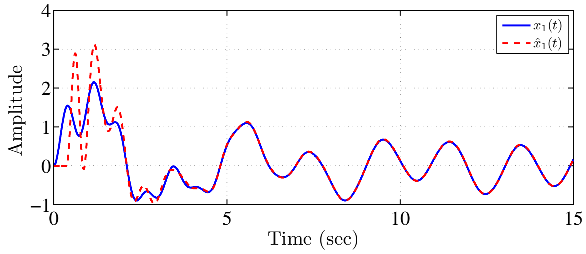
In this case the the available meusure $\bar{y}(t)$ is given by:

$$\bar{y}(t) = \begin{bmatrix} \bar{y}_1(t) \\ \bar{y}_2(t) \\ \bar{y}_3(t) \end{bmatrix} = \begin{bmatrix} y_1(t-\tau_1(t)) \\ y_2(t-\tau_2(t)) \\ y_3(t-\tau_3(t)) \end{bmatrix} = \begin{bmatrix} x_1(t-\tau_1(t)) \\ x_2(t-\tau_2(t)) \\ x_3(t-\tau_3(t)) \end{bmatrix} \quad (43)$$

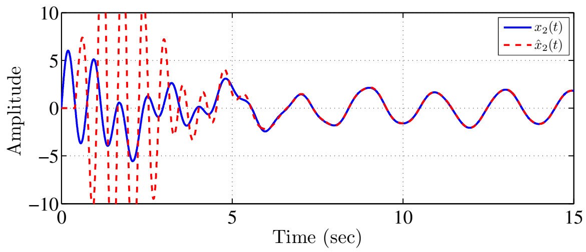
where $\tau_1(t)$, $\tau_2(t)$ and $\tau_3(t)$ are the different known variable delays. This simulation was carried out using the parameter $m = 4$ for values of the time-variable delays $\tau_1(t) = 0.3 + 0.1sin(0.1t)$, $\tau_2(t) = 0.2 + 0.1sin(0.2t)$, $\tau_3(t) = 0.15 + 0.1sin(0.3t)$ with $\tau_M = 0.4$ sec. and $h = 0.1$sec., consequently the observer gains used in this simulation are the same as in (42). Now, we validate the performance of the observer when the outputs are affected by long time-variable delays ($\tau_M = 0.4sec.$), to overcome these delays, the number of the estimators in the chain is adapted to $m = 4$, it is due to the fact that the observer gains were computed using $h = 0.1sec$, recalling that $m = \tau_M/h$. In Figure (1) the achieved responses of the proposed observer are presented. It is worth noting that the proposed observer was able to tackle long delays. These results clearly confirm the theoretical results.

V. Conclusions

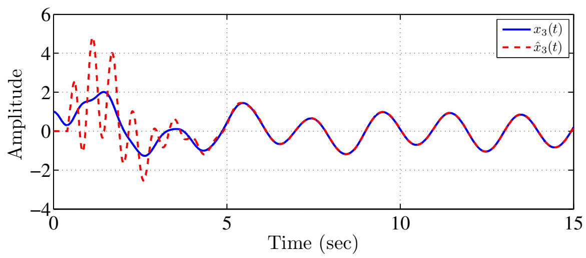
An observer consisting of a chain of observers is introduced for nonlinear systems in the presence of a long known and multiple time delay in output measurements. The delays present in the outputs are different and variables. Lyapunov-Krasovskii functional is used to prove the asymptotical



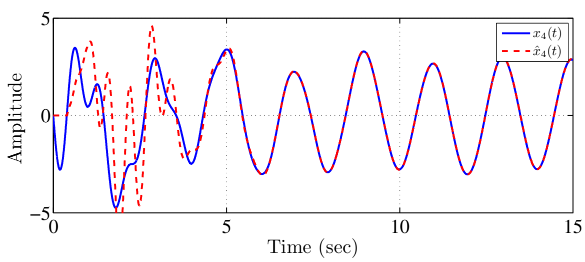
(a) Estimation of the x_1



(b) Estimation of the x_2



(c) Estimation of the x_3



(d) Estimation of the x_4

Fig. 1. Estimation of the state nonlinear system $\hat{x}_4(t)$, with $\tau_M = 0.4sec.$ and $m = 4$

convergence to zero of the observation error. Observer gains are achieved using an algorithm, solving a system of the Linear Matrix Inequalities. An advantage of the proposed observer is that the delay is dynamically compensated in the observer gain. The observer exhibits good estimation of the state of the system, even in the presence of significant delayed measurements.

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