2018 26th Mediterranean Conference on Control and Automation (MED)
Zadar, Croatia, June 19-22, 2018

Assignment of Invariant and Transmission Zeros in Linear Systems

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***Abstract*— The paper generalizes the result of Rosenbrock on the assignment of invariant and transmission zeros from systems (*A*, *B*, *C*, 0) with equal number of inputs and outputs to general (*A*, *B*, *C*, *D*) quadruples. The generalization, while straightforward, improves the solvability conditions and leads to a new construction of *C* and *D* matrices having least number of rows.**

***Keywords*— linear systems, invariant zeros, transmission zeros, system matrix, transfer matrix, zero assignment**

I. Introduction

Consider a linear, time-invariant system (A, B, C, D) of the form

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{1}$$

with A, B, C, D respectively $n \times n$, $n \times m$, $p \times n$, $p \times m$ constant matrices with entries in R, the field of real numbers. The system gives rise to the $p \times m$ proper rational transfer matrix

$$T(s) = C(sI_n - A)^{-1}B + D. \tag{2}$$

Define the $(n + p) \times (n + m)$ polynomial matrix

$$\Sigma(s) = \begin{bmatrix} -sI_n + A & B \\ C & D \end{bmatrix}.$$

Let

$$\Sigma_S(s) = \begin{bmatrix} \varepsilon_1(s) & & & & \\ & \varepsilon_2(s) & & & \\ & & \ddots & & \\ & & & \varepsilon_h(s) & \\ & & & & 0 \end{bmatrix} \tag{3}$$

be its Smith form [1, Section 6.3.3], where the invariant polynomials $\varepsilon_1(s)$, $\varepsilon_2(s)$, …, $\varepsilon_h(s)$ are monic polynomials arranged so that $\varepsilon_i(s)$ divides $\varepsilon_{i+1}(s)$, $i = 1, 2, \ldots, h-1$ and $h = \operatorname{rank} \Sigma(s)$. Then the roots of the polynomial $\varepsilon_1(s)\varepsilon_2(s) \cdots \varepsilon_h(s)$ are the *invariant zeros* of (A, B, C, D).

Let

$$T_{SM}(s) = \begin{bmatrix} \frac{\tau_1(s)}{\psi_1(s)} & & & & \\ & \frac{\tau_2(s)}{\psi_2(s)} & & & \\ & & \ddots & & \\ & & & \frac{\tau_k(s)}{\psi_k(s)} & \\ & & & & 0 \end{bmatrix} \tag{4}$$

be the Smith-McMillan form [1, Section 6.5.2] of $T(s)$, where the monic polynomials $\tau_i(s)$ and $\psi_i(s)$, $i = 1, 2, \ldots, k$ are coprime, $\tau_i(s)$ divides $\tau_{i+1}(s)$ and $\psi_{i+1}(s)$ divides $\psi_i(s)$, $i = 1, 2, \ldots, k-1$ and $k = \operatorname{rank} T(s)$. Then the roots of the polynomial $\psi_1(s)\psi_2(s) \cdots \psi_k(s)$ are the *poles* of $T(s)$ and the roots of the polynomial $\tau_1(s)\tau_2(s) \cdots \tau_k(s)$ are the (finite) *zeros* of $T(s)$, also known [2, p. 564] as the *transmission zeros* of (A, B, C, D).

If the pair (A, B) in (1) is controllable (that is, $sI_n - A$ and B are left coprime), then the system (A, B, I_n, 0) having the state as the output, gives rise to no invariant or transmission zeros. The zeros originate when the output is a linear combination of the state and input coordinates as specified by a choice of the matrices C and D in (1).

In his seminal book [3, Chapter 5, Section 4], Rosenbrock posed and solved the following two problems.

A. Assignment of Invariant Zeros

Let the $n \times n$ matrix A and the $n \times m$ matrix B be given in (1), with (A, B) controllable. Let the controllability indices of (A, B) in order of magnitude be $\lambda_1, \lambda_2, \ldots, \lambda_m$ with $\lambda_1 = \lambda_2 = \ldots = \lambda_{m-q} = 0$ where $q = \operatorname{rank} B$. Let $\varepsilon_1(s)$, $\varepsilon_2(s)$, …, $\varepsilon_m(s)$ be any prescribed monic polynomials.

Then the $m \times n = p \times n$ matrix C in (1) can be chosen so that the invariant zeros of (A, B, C, 0) are the roots of the polynomial $\varepsilon_1(s)\varepsilon_2(s) \cdots \varepsilon_m(s)$ if and only if the following conditions are all satisfied.

(a) $\varepsilon_{r+1}(s) = \varepsilon_{r+2}(s) = \cdots = \varepsilon_m(s) = 0$, for some $r \le q$,

Supported by the Czech Science Foundation, project GA16-17398S.

(b) $\varepsilon_i(s)$ divides $\varepsilon_{i+1}(s)$, $i = 1, 2, \ldots, r-1$,

(c) the degrees of the nonzero $\varepsilon_i(s)$ satisfy

$$\sum_{i=1}^{j} \deg \varepsilon_i(s) \le \sum_{i=1}^{j} (\lambda_{m-r+i} - 1), \quad j = 1, 2, \ldots, r.$$

Note that $n + r = h$ in (3).

B. Assignment of Transmission Zeros

Let A, B and $\lambda_1, \lambda_2, \ldots, \lambda_m$ and q be as in *Problem A* above. Let $\varepsilon_1(s), \varepsilon_2(s), \ldots, \varepsilon_m(s)$ be any prescribed monic polynomials. Let the conditions (a) to (c) above hold true.

If $r = q$ and $\varepsilon_q(s)$ is coprime with $\psi_1(s)$, then the $m \times n = p \times n$ matrix C in (1) can be chosen so that C and $sI_n - A$ are right coprime and the roots of the polynomial $\varepsilon_1(s)\varepsilon_2(s) \cdots \varepsilon_q(s)$ are the transmission zeros of $(A, B, C, 0)$.

Note that $r = q = k$ in (4).

The assignment of invariant zeros is a problem related to that of matrix pencil completion [4]. Given the pencil $[-sI_n + A \quad B]$, whose invariants under two-sided nonsingular constant transformations are as follows: no finite elementary divisors, unity infinite elementary divisors, column minimal indices $\lambda_1, \lambda_2, \ldots, \lambda_m$, and no row minimal indices. One seeks for constant matrices C and D in order to complete the pencil to the system matrix $\Sigma(s)$ with finite elementary divisors given by $\varepsilon_1(s), \varepsilon_2(s), \ldots, \varepsilon_r(s)$ for some $r \le m$ and with the remaining invariants not specified.

II. Preliminaries

Let the $n \times n$ matrix A and the $n \times m$ matrix B be given in (1), with (A, B) controllable ($sI_n - A$ and B left coprime). Let $N(s)$ and $D(s)$ be right coprime polynomial matrices such that

$$(sI_n - A)^{-1} B = N(s) D^{-1}(s).$$

Denote V the set of m-row polynomial vectors $v(s)$ such that $v(s)\, D^{-1}(s)$ is strictly proper. The following result is due to Hautus and Heymann.

Lemma 1 [5, Corollary 4.11]. The set V is a linear space over R of dimension $\deg \det D(s)$ and the rows of $N(s)$ form a basis for V. □

Further, let $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_l)$ be lists of nonnegative integers, arranged in nondecreasing order, of length l. Denote $\operatorname{sum} \mu = \mu_1 + \mu_2 + \ldots + \mu_l$ and $\operatorname{sum} \nu = \nu_1 + \nu_2 + \ldots + \nu_l$. We say that μ *dominates* ν, and write $\mu \succ \nu$, if

$$\sum_{i=1}^{j} \mu_i \ge \sum_{i=1}^{j} \nu_i, \quad j = 1, 2, \ldots, l.$$

The following result is an important property of polynomial matrices.

Lemma 2 [1, Lemma 7.2-2]. Let $P(s)$ be a $p \times r$ polynomial matrix of rank r, with column degrees $\mu_1 \le \mu_2 \le \ldots \le \mu_r$ and invariant polynomials $p_1(s), p_2(s), \ldots, p_r(s)$. Define the lists $\mu := (\mu_1, \mu_2, \ldots, \mu_r)$ and $\delta := (\deg p_1(s), \deg p_2(s), \ldots, \deg p_r(s))$. Then $\mu \succ \delta$. Furthermore, if $P(s)$ is column reduced [1, p. 384], then $\operatorname{sum} \mu = \operatorname{sum} \delta$. □

Rosenbrock discovered that a converse result is also true.

Lemma 3 [3, Chapter 5, Lemma 4.1]. Let $P(s)$ be an $r \times r$ column-reduced polynomial matrix with invariant polynomials $p_1(s), p_2(s), \ldots, p_r(s)$ and let $\delta := (\deg p_1(s), \deg p_2(s), \ldots, \deg p_r(s))$. Let $\nu := (\nu_1, \nu_2, \ldots, \nu_r)$ be an arbitrary prescribed list of nonnegative integers, in nondecreasing order, such that $\nu \succ \delta$ and $\operatorname{sum} \nu = \operatorname{sum} \delta$. Then there exist unimodular polynomial matrices $U_1(s)$ and $U_2(s)$ such that the matrix $Q(s) := U_1(s)P(s)U_2(s)$ is column reduced with column degrees $\nu_1, \nu_2, \ldots, \nu_r$ and with an identity highest-column-degree coefficient matrix. □

III. Assignment of Invariant Zeros

We shall improve Rosenbrock's result in that

(i) the number of outputs, p, of (1) is not constrained to equal m, the number of inputs;

(ii) the matrix D in (1) is not bound to be zero, thus providing a less restrictive solvability condition;

(iii) the matrices C and D having a least number of rows, p, are determined;

(iv) a new proof of sufficiency is proposed that yields a simple construction of C and D.

Theorem 1. Let the $n \times n$ matrix A and the $n \times m$ matrix B be given in (1). Let (A, B) be controllable, with controllability indices $\lambda_1, \lambda_2, \ldots, \lambda_m$ of which $q = \operatorname{rank} B$ is nonzero and arranged in order of magnitude, $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_q$, and $\lambda_{q+1} = \lambda_{q+1} = \ldots = \lambda_m = 0$.

Let $\varepsilon_1(s), \varepsilon_2(s), \ldots, \varepsilon_q(s)$ be any prescribed monic polynomials. Then the $p \times n$ matrix C and the $p \times m$ matrix D in (1) can be chosen so that the invariant zeros of (A, B, C, D) are the roots of the polynomial $\varepsilon_1(s)\varepsilon_2(s) \cdots \varepsilon_q(s)$ if and only if

(a) $\varepsilon_{r+1}(s) = \varepsilon_{r+2}(s) = \cdots = \varepsilon_q(s) = 0$,
for some $r \le \min(p, q)$,

(b) $\varepsilon_i(s)$ divides $\varepsilon_{i+1}(s)$, $i = 1, 2, \ldots, r-1$,

(c) the list $\lambda = (\lambda_{q-r+1}, \lambda_{q-r+2}, \ldots, \lambda_q)$ of the r largest controllability indices and the list $\delta = (\delta_1, \delta_2, \ldots, \delta_r)$ of the degrees of the nonzero $\varepsilon_i(s)$ satisfy $\lambda \succ \delta$.

Proof. The proof of necessity is based on existence results and draws on [3, Theorem 4.1].

The matrices A and B are first transformed to a standard form. If $q < m$, there is a constant nonsingular matrix G such that

$$BG = \begin{bmatrix} B_1 & 0 \end{bmatrix}$$

and the $n \times q$ matrix B_1 has rank q. The controllability indices of (A, B_1) are $\lambda_1, \lambda_2, \ldots, \lambda_q$.

Let Q_1 and Q_2 be constant nonsingular matrices that transform the matrix $[-sI_n + A \quad B_1]$ to the Brunovský standard form [6]

$$Q_1\left[-sI_n+A \;\; B_1\right]Q_2=$$

$$\begin{bmatrix} -sI_{\lambda_1}+N_1 & & & & E_1 \\ & -sI_{\lambda_2}+N_2 & & & E_2 \\ & & \ddots & & \vdots \\ & & & -sI_{\lambda_q}+N_q & E_q \end{bmatrix} := \left[-sI_n+A_2 \;\; B_2\right]$$

where

$$N_i=\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix},\ \text{size } \lambda_i \times \lambda_i,\ i=1,2,\ldots,q$$

and E_i is a $\lambda_i \times q$ matrix whose entries are all zero but the entry in column i and row λ_i, which is 1.

Now adjoin to the matrix $[-sI_n + A_2 \quad B_2]$ a $p \times (n+q)$ matrix $[C \quad D]$ to give

$$\begin{bmatrix} -sI_n+A_2 & B_2 \\ C & D \end{bmatrix}. \tag{5}$$

Write C and D in terms of columns,

$$C=\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix},\quad D=\begin{bmatrix} d_1 & d_2 & \cdots & d_q \end{bmatrix}$$

and by column operations eliminate s from rows 1 to n of (5). Explicitly, add s times column $n+1$ to column λ_1, then s times column λ_1 to column $\lambda_1 - 1$, …, and then add s times column 2 to column 1. Deal similarly with the other columns. Then by row operations reduce the last p rows to zero except for entries in columns $1, \lambda_1 + 1, \ldots, n - \lambda_q + 1$. Explicitly, do this by adding to rows $n+1, n+2, \ldots, n+p$ suitable multiples of rows $1, 2, \ldots, \lambda_1 - 1, \lambda_1 + 1, \ldots, \lambda_1 + \lambda_2 - 1, \lambda_1 + \lambda_2 + 1, \ldots, n-1$. The final matrix has the form

$$\begin{bmatrix} N_1 & & & & E_1 \\ & N_2 & & & E_2 \\ & & \ddots & & \vdots \\ & & & N_q & E_q \\ F_1(s) & F_2(s) & \cdots & F_q(s) & 0 \end{bmatrix} \tag{6}$$

where the $p \times \lambda_i$ matrix $F_i(s)$ has all its entries equal to zero but the first column, which is

$$F_{i,1}(s)=c_{\mu_i+1}+c_{\mu_i+2}s+\ldots+c_{\mu_i+\lambda_i}s^{\lambda_i-1}+d_i s^{\lambda_i},$$
$$\mu_i=\lambda_1+\lambda_2+\ldots+\lambda_{i-1},$$
$$i=1,2,\ldots,q.$$

Then by interchanges of columns, the matrix (6) can be brought to the form

$$\begin{bmatrix} I_n & \\ & F(s) \end{bmatrix},\quad F(s)=\begin{bmatrix} F_{1,1}(s) & F_{2,1}(s) & \cdots & F_{q,1}(s) \end{bmatrix}.$$

The $p \times q$ matrix $F(s)$ has its column i of degree less than or equal to λ_i. Assume that the invariant polynomials of $F(s)$ are $\varepsilon_1(s), \varepsilon_2(s), \ldots, \varepsilon_r(s)$ for some r. Clearly, $r \le \min(p, q)$, which proves the claims (a) and (b). Furthermore, define a $p \times r$ matrix $F_1(s)$ by selecting r nonzero columns of highest degree from $F(s)$. Applying Lemma 2 to $F_1(s)$, we verify condition (c).

The proof of sufficiency is based on constructive arguments and is new. Suppose that conditions (a) to (c) hold true for some nonzero polynomials $\varepsilon_1(s), \varepsilon_2(s), \ldots, \varepsilon_r(s)$ and $\varepsilon_{r+1}(s) = \varepsilon_{r+2}(s) = \cdots = \varepsilon_q(s) = 0$. Form the matrix

$$H(s)=\begin{bmatrix} \varepsilon_1(s) & & & \\ & \varepsilon_2(s) & & \\ & & \ddots & \\ & & & \varepsilon_r(s) \end{bmatrix}.$$

Apply Lemma 3 to $H(s)$ so as to make the degree of its i-th column less than or equal to λ_{q-r+i}, $i = 1, 2, \ldots, r$ without changing its invariant polynomials. Call the resulting $r \times r$ matrix $H_1(s)$. Select $p = r$, the least value of p achievable, and form a $p \times q$ matrix $H_2(s)$ by adjoining $q - r$ zero columns to $H_1(s)$ as follows,

$$H_2(s)=\begin{bmatrix} 0 & H_1(s) \end{bmatrix}.$$

This does not change the invariant polynomials either.

If $q < m$, there exists a nonsingular constant matrix G such that

$$BG=\begin{bmatrix} B_1 & 0 \end{bmatrix} \tag{7}$$

and the $n \times q$ matrix B_1 has rank q. Let $N_1(s)$ and $D(s)$ be right coprime polynomial matrices such that

$$(sI_n - A)^{-1}B_1 = N_1(s)D^{-1}(s). \tag{8}$$

There exists [1, p. 386] a unimodular polynomial matrix $U(s)$ such that

$$N_1(s)U(s)=N_2(s),\quad D(s)U(s)=D_2(s) \tag{9}$$

and the $q \times q$ matrix $D_2(s)$ is column reduced with column degrees $\lambda_1, \lambda_2, \ldots, \lambda_q$ and with an identity highest-column-degree coefficient matrix.

Then solve the polynomial matrix equation

$$XD_2(s)+YN_2(s)=H_2(s) \tag{10}$$

for *constant* matrices X and Y. Explicitly, set

$$X=\begin{bmatrix} H_{2,1} & H_{2,2} & \cdots & H_{2,q} \end{bmatrix}$$

where $H_{2,1} = H_{2,2} = \ldots = H_{2,q-r} = 0$ and $H_{2,i}$, $i = q - r + 1, \ldots, q$ is the column coefficient of s^{λ_i} in column i of $H_2(s)$. Then $H_2(s) - XD_2(s)$ has its i-th column either zero or of degree less than λ_i. So has the matrix $N_2(s)$. Invoking Lemma 1, a constant matrix Y exists that satisfies (10).

The matrices C and D can now be chosen as

$$C=Y,\quad D=\begin{bmatrix} X & 0 \end{bmatrix}G^{-1}. \quad \square \tag{11}$$

Note that the dominance condition (c) of Theorem 1 is less restrictive than the corresponding condition (c) of Rosenbrock. Indeed, when D is not bound to be zero, the invariant polynomial degrees need to be dominated by the list λ rather than $\lambda - 1$.

Further note that the $q - r$ zero columns adjoined to $H_1(s)$ when forming $H_2(s)$ can actually be inserted in $H_1(s)$ at any arbitrary positions.

IV. Assignment of Transmission Zeros

We shall improve Rosenbrock's result so that all the claims (i) to (iv) of Section III apply and, moreover,

(v) a tighter solvability condition is established that is not only sufficient but also necessary.

Theorem 2. Let the roots of the polynomial $\varepsilon_1(s)\varepsilon_2(s)\ldots\varepsilon_r(s)$ be the invariant zeros of (A, B, C, D) assigned according to Theorem 1, with the $r \times n$ matrix C and the $r \times m$ matrix D determined by (11).

Let $\Delta(s)$ be any $q \times q$ greatest common right divisor of the polynomial matrices $D_2(s)$ and $H_2(s)$ in (10) and let

$$D_2(s) := D_3(s)\Delta(s), \quad H_2(s) := H_3(s)\Delta(s). \tag{12}$$

Then the roots of the polynomial $\varepsilon_1(s)\varepsilon_2(s)\ldots\varepsilon_r(s)$ are the transmission zeros of (A, B, C, D) if and only if $H_2(s)$ and $H_3(s)$ have the same invariant polynomials.

Proof. The zeros of $T(s)$ are those of $T(s)G$ for any nonsingular constant matrix G. Then (2) implies

$$\begin{aligned} T(s)G &= C(sI_n - A)^{-1}\begin{bmatrix} B_1 & 0 \end{bmatrix} + DG \\ &= \begin{bmatrix} CN_1(s)D^{-1}(s) & 0 \end{bmatrix} + DG \\ &= \begin{bmatrix} CN_2(s)D_2^{-1}(s) & 0 \end{bmatrix} + DG \\ &= \begin{bmatrix} \left(YN_2(s) + XD_2(s)\right)D_2^{-1}(s) & 0 \end{bmatrix} \\ &= \begin{bmatrix} H_2(s)D_2^{-1}(s) & 0 \end{bmatrix} \\ &= \begin{bmatrix} H_3(s)D_3^{-1}(s) & 0 \end{bmatrix} \end{aligned}$$

on successively using (7) through (12).

Since (A, B) is controllable, the system (A, B, C, D) has no input-decoupling zeros [3, p. 64]. Then the invariant zeros will coincide with the transmission zeros if and only if no invariant zero is simultaneously an output-decoupling zero [3, p. 65].

By assumption, the invariant zeros of (A, B, C, D) are given by $\varepsilon_1(s)$, $\varepsilon_2(s)$, …, $\varepsilon_r(s)$, which are the invariant polynomials of $H_2(s)$. The transmission zeros of (A, B, C, D) are the zeros of $T(s)$. Since $D_3(s)$ and $H_3(s)$ in (12) are right coprime, the transmission zeros are given by the invariant polynomials of $H_3(s)$. Therefore, the two sets of zeros will coincide if and only if the invariant polynomials of $H_2(s)$ and $H_3(s)$ coincide. □

The flexibility in forming $H_2(s)$ from $H_1(s)$ is instrumental in achieving this property. While the $q - r$ zero columns may be inserted at any arbitrary positions as far as the invariant polynomials of $H_2(s)$ are concerned, the insertion may affect the greatest common right divisor of $D_2(s)$ and $H_2(s)$, hence affect the invariant polynomials of $H_3(s)$. The invariant zeros removed from $H_3(s)$ in this way become output-decoupling zeros. Such a situation must be avoided in order to have the transmission and invariant zeros coincide.

It is also to be noted that when $r < q$ then (A, B, C, D) may have some output-decoupling zeros, which are not invariant, however.

V. Example

Let the matrices A and B be given such that

$$\begin{bmatrix} -sI_n + A & B \end{bmatrix} = \left[\begin{array}{cccc:ccc} -s-1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -s & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -s & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s & 0 & 0 & 1 \end{array}\right]$$

with $m = 3$, $n = 4$, $q = 3$ and $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 2$. We wish to complete the pencil to a system matrix $\Sigma(s)$ having the invariant polynomials

$$\varepsilon_1(s) = s + 1, \quad \varepsilon_2(s) = s + 1.$$

Thus, $r = 2$. The conditions (a) to (c) of Theorem 1 are satisfied for λ_2 and λ_3, so we set

$$H(s) = H_1(s) = \begin{bmatrix} s+1 & 0 \\ 0 & s+1 \end{bmatrix}.$$

Select $p = r = 2$, the least value of p achievable, and form a $p \times q$ matrix $H_2(s)$ by inserting $q - r = 1$ zero column in $H_1(s)$ at an arbitrary position. One such a choice is

$$H_2(s) = \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+1 & 0 \end{bmatrix}. \tag{13}$$

Since $q = m = 3$, we calculate $B = B_1$ and

$$N_1(s) = N_2(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & s \end{bmatrix}, \quad D_2(s) = \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s^2 \end{bmatrix}.$$

Equation (10) admits a solution (11) of the form

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and the system (A, B, C, D) has the invariant zeros $\{-1, -1\}$, as prescribed.

Furthermore, we wish to have the transmission zeros of (A, B, C, D) coincide with the invariant zeros. Since

$$\Delta(s)=\begin{bmatrix} s+1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s^2 \end{bmatrix},$$

the matrix

$$H_3(s)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & s+1 & 0 \end{bmatrix}$$

possesses different invariant polynomials than (13). As a result, the system has a single transmission zero at –1; the other invariant zero at –1 has become an output-decoupling zero.

Therefore, we have to make a selection for $H_2(s)$ other than (13). Inserting the zero column in $H_1(s)$ so as to make it column 1 rather than column 3, we obtain

$$H_2(s)=\begin{bmatrix} 0 & s+1 & 0 \\ 0 & 0 & s+1 \end{bmatrix}. \tag{14}$$

Then

$$\Delta(s)=\begin{bmatrix} s+1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the invariant polynomials of the matrix

$$H_3(s)=\begin{bmatrix} 0 & s+1 & 0 \\ 0 & 0 & s+1 \end{bmatrix}$$

now equal those of (14). Therefore, the condition of Theorem 2 is satisfied.

Now (10) admits a solution (11) of the form

$$C=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad D=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the system (A, B, C, D) has two transmission zeros {–1, –1}, as desired.

Note that the resulting system has eigenvalues {–1, 0, (0, 0)}, the invariant zeros {–1, –1}, and an output-decoupling zero at –1, which is not invariant. The transfer matrix of the system has poles {0, (0, 0)} and zeros (the transmission zeros of the system) equal to {–1, –1}.

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