2018 26th Mediterranean Conference on Control and Automation (MED)
Zadar, Croatia, June 19-22, 2018

A Set of Chance-Constrained Robust Stabilizing PID Controllers*

Andrey Tremba[1]

***Abstract*— Stabilization of linear SISO systems with random uncertain parameters is considered. A closed-loop system inherits random parameters, and it is stable with some probability. The probabilistic design problem is to tune a controller (e.g., a PID-controller) to stabilize the closed-loop system with high probability. Finding suitable parameters is a probabilistic robust stabilization problem (specifically, a chance-constrained problem). The ultimate goal is to describe the whole set of such controllers instead of individual representative. An algorithm is proposed for finding an inner approximation of the set of chance-constrained stabilizing PID-controllers. The method is based on the robust D-decomposition technique and deterministic error set representation of random uncertainty. A few examples with demonstration of the approach are provided and discussed.**

I. Introduction

Description of a control system often includes uncertainties, parametric or non-parametric. It is typical for the uncertainty to be unknown but time-invariant, i.e. static. Analysis and design of such system is called robust analysis/design problems. The uncertainty may be random or non-random. The *robustness* term is often related to non-random uncertainties. It is based on the "worst-case" approach. Within this approach, all possible values of the uncertainty are to be handled by the same controller. There are multiple tools and methods as for robust analysis, as for design, cf. [20] and the references therein. In certain cases, however, robust problems are barely solvable. For example, checking robust stability of a linear system with interval matrix uncertainty is an NP-hard problem [4].

To overcome this difficulty, a randomized approach was introduced. It is based on a reasonable assumption that the the uncertain parameters inherit randomness. For example, a random variable uniformly distributed in an n-dimensional box may be considered instead of an uncertain parameter in the box. Analysis of the system becomes probabilistic, and we can describe the system as being "stable with probability p". Design problems are also called *probabilistic* ones. Probabilistic stabilization problem is usually stated as: "find a controller so that the closed-loop system is stable with probability at least $1-\varepsilon$".

Notably, this approach admits of small probability of violation ε. Adjustment of this violation is summarized in the modulated robustness idea. The uncertainty set can be adjusted (reduced) by introducing violation probability in the originally robust design problem [8].

*This work was supported by Russian Science Foundation, project 16-11-10015

[1]Andrey Tremba is with Laboratory 7 of Adaptive and Robust Systems, V.A. Trapeznikov Institute of Control Sciences RAS, Moscow, Russia. atremba@ipu.ru

Apart from the deterministic→random conversion, there are also exist models, which are stated in terms of random uncertainties directly. To distinguish problems with non-random and random uncertainties, the term "robust" in the paper is mainly used below with respect to non-random uncertainty, while the term "probabilistic" is used for problems with random uncertainty.

Analytic solution of probabilistic analysis/design is a rare exception; e.g. [3]. Numerically, it can be solved via randomized methods via use of the Monte-Carlo *scenario* technique. Randomized algorithms collect multiple samples of random uncertainty. Then the algorithms process the sample set. It can be performed without giving a description of the uncertainty, but rather by just providing an uncertainty sample generator. This approach is applied to controller design as well [5], [6], [17], [24]. Various randomized algorithms are summarized in [25], and they were implemented in the Randomized Algorithms Control (RACT) toolbox for Matlab [27], with most functionality now moved into the R-ROMULOC toolbox [7].

Typically, both robust and probabilistic design algorithms yield a single controller, which satisfy given robustness requirements. For example, a probabilistic design problem may be formulated as "optimizing a closed-loop system performance index (damping ratio, peak, sensitivity etc.) in a way, so the system is stable with probability at least $1-\varepsilon$." This is the so-called *chance-constrained optimization problem*. In optimization, such problems are broadly studied under specific conditions, e.g. convexity [15]. Applications to robust control problems are less common, cf. [19].

There exist different probabilistic→robust problem transformations. Constructed design problem is simpler than the probabilistic one in some cases. Its solution is deterministic and tractable, though underlying data may be random. This approach has been recently applied to chance-constrained optimization problems and involves scenario technique [2], [18], [19]. The problem's setup, however, is often restrictive, e.g., by the convexity of the target function, randomness of output etc.

There are two main aims of the paper. The first one is to attract attention to set-wise probabilistic design control algorithms, which result in a set description of controllers. The second is to reuse the other approach instead of scenario one. The approach is of the oldest ones for tackling probabilistic problems numerically. Namely, a proper non-random set covering most frequent values of random uncertainty is chosen, then the robust problem is solved instead of the probabilistic (or chance-constrained) one. This transform is conservative, but proper choice of the non-random set decreases error.

Synthesis of robust stabilizing proportional-integral-derivative (PID) controllers for Linear Time Invariant (LTI) Single Input Single Output (SISO) systems is one of the interesting cases. The stabilization criteria is to have all roots of the closed-loop characteristic polynomial in the left half of the complex plane. It can be stated as minimization problem, with the degree of stability $\max_i \mathbf{Re}\lambda_i(k_i, k_p, k_d)$ in the role of the objective function. The function depends on the parameters k_i, k_p, k_d of the PID-controller and is not convex.

The main problem is to describe the *whole set* of chance-constrained stabilizing PID-controllers. PID-controllers stabilizing the system with given probability are referred to as chance-constrained robust stabilizing PID-controllers (or chance-constrained PID-controllers for short). Motivations for finding this set is twofold. First, the system property (stability) may be highly sensitive to the parameters of the controller [16]. To avoid this "fragility," the controller parameters are to be chosen "deep inside" the stability domain; e.g., as far from its boundary as possible. Secondly, the stability may be just one of few properties of interest, on top of, say, bounded peak response, settling time, etc. Intersection of corresponding sets of suitable controllers in the parameter space describes the controllers satisfying the constraints altogether [22].

For a substitute robust problem, we seek the whole set of robust stabilizing parameters, rather that just one representative (in contrast to [19]). The contribution of this note is an algorithm that generates an inner approximation of the set of all chance-constrained PID-controllers. It is done through use of the underlying robust D-partition technique.

The paper is organized as follows. A generic chance-constrained robust stabilization problem is stated, and notions related to random uncertainty are introduced in the next section. Then, use of robust methods for approximate solution of probabilistic design problems is outlined in Section III. It includes a brief description of the robust D-decomposition method. Illustrative examples are presented in Section IV. The discussion of extensions finalize the paper.

II. Uncertainties and Definitions

An explanation of the basic probabilistic-to-robustness idea is introduced in this section. It is followed by the definitions of the uncertainty and the related issues.

A. Chance-constrained Stabilization

Consider a system which depends on the vectors $\theta \in \mathbb{R}^m$ and $\Delta \in \mathbb{R}^n$ of the design parameters and random parameters, respectively. Let the system be equipped with a $0/1$ binary function $J(\theta, \Delta)$ indicating the stability property of the system.

Probabilistic design problem is to find a parameter $\theta^* \in \Theta$ such that the system is stable with high probability

$$\mathrm{Prob}_\Delta(J(\theta^*, \Delta) = 1) = \mathbb{E}_\Delta J(\theta^*, \Delta) \geq 1 - \varepsilon.$$

where $1 > \varepsilon \geq 0$ is a given violation probability. The goal is to describe the whole set of such parameters Θ^*_ε:

$$\Theta^*_\varepsilon = \{\theta : \mathbb{E}_\Delta J(\theta^*, \Delta) \geq 1 - \varepsilon\}.$$

Unfortunately, the problem of finding the exact description of this set may be intractable, even if the probability distribution of Δ is known. Finding only one suitable point of the set $\theta^* \in \Theta^*$ is a non-trivial chance-constrained (risk-constrained) problem itself.

If the dimension of the vector of the control parameters is low, the problem may be solved approximately by gridding. This approach requires multiple membership checks of tested parameter with respect to $\Theta^*(\varepsilon)$. It is a probabilistic analysis problem. It can be solved via the scenario approach by testing a large number of samples [25]. The output of the randomized algorithms is random as well, leaving possibility of giving wrong answer. The possibility is controlled by *confidence* or *assurance* parameter; higher assurance value results in larger number of samples to be tested.

The main idea of this article is to emphasis idea: a *robust* design method may be efficiently used for solving *hard probabilistic* design problems. As an example, we can efficiently describe a solution set of a probabilistic design problem with assurance 1.

We address to the approach which allow to get an inner approximation $\widehat{\Theta}^*_\varepsilon \subseteq \Theta^*_\varepsilon$ with the assurance equal to 1, i.e. with probability 1. The idea is simple: first we choose a set Q in uncertainty space $\mathbb{R}^n$, so that the uncertainty Δ resides in Q with given probability. Then a robust design algorithm ("find θ so that the given criteria $J(\theta, \Delta)$ holds for all $\Delta \in Q$") is applied, resulting in some θ^*. The value θ^* is a probabilistic design solution by construction. The approach of choosing a non-random set – a representative of random uncertainty is closely related to the ideas of [8], [18]. There is freedom of choosing size and shape of the set Q. It can be done in order to simplify robust design problem, or with respect to uncertainty distribution, or both. The choice is discussed at the end of the section.

To formalize the idea, uncertainty properties and notion of error sets are introduced. The error sets then used in the robust D-decomposition method explained in Section III.

B. Random Uncertainties and Error Sets

The uncertain parameters Δ are defined with a probability space $(\mathbb{D}, \mathcal{F}, \mathcal{P})$ with the sample set $\mathbb{D}$, the events set $\mathcal{F}$ and the probability measure $\mathcal{P} : \mathcal{F} \to [0, 1]$. For simplicity we associate each random vector parameter with a point of the sample set, taking the sample set in vector space $\mathbb{D} \subseteq \mathbb{R}^m$. Then $\mathcal{F}$ simply contains all subsets[1] of $\mathbb{D}$. Avoiding minor technical intricacy we assume that $\mathbb{D}$ has non-empty interior, i.e. $\mathbb{D}$ does not reside in a low-dimensional manifold in $\mathbb{R}^m$. The random variable Δ with the given distribution is denoted as "$\Delta \sim \mathcal{F}$".

Let's fix violation level $0 \leq \epsilon < 1$. A set Q_ε is called "ε-error set" (or "error set with parameter ε") if $\mathcal{P}(Q_\varepsilon \cap \mathbb{D}) \geq 1 - \varepsilon$. Note that the definition slightly differs from common definition of error set, as we require probability of violation not to be *exactly* ε, but to be *less or equal* ε.

[1]Strictly speaking, the random variable Δ is a function $\Delta : \mathbb{D} \to \mathbb{R}^m$ with identity transformation, and by *all* subsets we assume elements of Borel σ-algebra, generated by open balls in $\mathbb{R}^m$.

Informally speaking for any ε-error set Q_ε holds $\mathrm{Prob}(\Delta \in Q_\varepsilon) \geq 1-\varepsilon$; i.e. probability of falling out of the set is as low as $\mathrm{Prob}(\Delta \notin \Theta_\varepsilon) < \varepsilon$.

As example, consider uniform 2D distribution on a support set $\mathbb{D}$ with unit area, for example $\mathbb{D} = \{(\Delta_1, \Delta_2) : 0 \leq \Delta_1 \leq 1, 0 \leq \Delta_2 \leq 1\}$. Then any set with area of intersection with $\mathbb{D}$ being greater than $1-\varepsilon$ is an ε-error set; e.g. a square with edge length at least $\sqrt{1-\varepsilon}$ within $\mathbb{D}$ is suitable.

Another important case is the multivariate normal distribution $\Delta \sim \mathcal{N}(\mu, \Sigma),\ \Delta \in \mathbb{R}^n$ with pdf

$$f(x) = \frac{1}{(2\pi)^{-n/2}\sqrt{\det \Sigma}} e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}.$$

It has natural ellipsoidal error sets with center at μ and properly scaled "shape" Σ. Namely, probability of the uncertainty Δ being inside the ellipsoid

$$Q_\varepsilon = \{x : (x-\mu)^T\Sigma^{-1}(x-\mu) \leq \xi_n^2(1-\varepsilon)\} \qquad (1)$$

is exactly $1-\varepsilon$. Here $\xi_n^2(\cdot)$ is inverse distribution function (quantile function) of the chi-square distribution with n degrees of freedom, [1]. The ellipsoid has the smallest volume among other error sets with same parameter ε, that's crucial for application of robust design problem. It is also convenient to use ellipsoidal error sets for other elliptical distributions (also known as elliptically contoured distributions) as well.

There are few evident properties of the ε-error sets.

1) an error set Q with parameter ε for a random variable Δ may be also considered as error set for the same variable with larger parameter $\overline{\varepsilon} : \overline{\varepsilon} \geq \varepsilon$.
2) if Q is an error set with parameter ε for a random variable Δ, then any outer set $\overline{Q} \supseteq Q$ is also error set for Δ with same parameter ε.
3) if a set Q contains whole sample set $\mathbb{D}$ or the support set $\mathrm{supp}\,(\Delta)$ of a random variable $\Delta \in \mathbb{D}$, then it is an error set with parameter $\varepsilon = 0$.

Note that error sets may have arbitrary form, i.e. balls in various norms, ellipsoids, parallelotopes, boxes or other sets which have simple description and/or convenient for the robust analysis or design. For a given distribution of random variable Δ, its typical sets or high-probability sets can be used as ε-error set. Next will be shown that the small-volume ε-error sets are preferred.

III. Chance-constrained PID-controllers

Let's consider a probabilistic stabilization design problem for LTI SISO systems with feedback. We introduce the notion of the probabilistic D-decomposition and propose its approximate solution in this section.

Without loss of generality, a continuous LTI system is considered. The plant has a rational transfer function

$$G(s, \Delta) = \frac{N(s, \Delta)}{D(s, \Delta)},$$

with $N(s, \Delta), D(s, \Delta)$ being polynomials of variable s, and $\Delta \sim \mathcal{F}$ is a vector random parameter. Feedback PID controller has transfer function

$$C(s, k_i, k_p, k_d) = \frac{k_i + k_p s + k_d s^2}{D_c(s)}.$$

This formulae includes both filtered PID-controller ($D_c(s) = s(1 + \alpha T s + T^2 s^2)$) and the ideal one ($D_c(s) = s$).

Chance-constrained stabilizing controller design problem is to find a set of all stabilizing controllers

$$\Theta_\varepsilon^* = \{\theta : \mathbb{E}_\Delta J(k_i, k_p, k_d, \Delta) \geq 1-\varepsilon\}, \qquad (2)$$

for a given violation probability ε. A function $J(\cdot)$ is the indicator function of stability property of the closed-loop system with fixed controller parameters and uncertainty value.

Stability of the closed-loop system is determined by its characteristic polynomial

$$p(s, k_i, k_p, k_d, \Delta) = D(s, \Delta) D_c(s) + (k_i + k_p s + k_d s^2) N(s, \Delta). \qquad (3)$$

Thus the set of chance-constrained PID-controllers is

$$\Theta_\varepsilon^* = \{\theta : \mathrm{Prob}_\Delta(p(s, k_i, k_p, k_d, \Delta) \text{ is Hurwitz}) \geq 1-\varepsilon\}. \quad (4)$$

As mentioned above, finding a point inside Θ_ε^* is a chance-constrained optimization problem. Checking if a point within Θ_ε^* is a probabilistic analysis problem. Finding a point on the boundary of the set is the subject of an even harder VaR (value-at-risk) problem. All these problems are time-consuming or practically intractable except for trivial cases (see Example 1).

The problem is to describe the set of chance-constrained stabilizing PID controllers Θ_ε^* constructively. In the paper, solution of substitute robust design problem for an inner approximation of the set is proposed. The solution is built by robust D-decomposition: a method characterizing the set of robust stabilizing controllers.

A. Robust D-decomposition (Robust D-partition)

The classic D-decomposition, also known as *D-partition*, is a technique which constructively describes whole set of stabilizing parameters $\{\theta : p(s, \theta) \text{ is Hurwitz}\}$, for a polynomial without uncertainties. It is also known as the *parameter space decomposition*. For review of the technique, including various extensions cf. [9], [10], [11], [23]. Main restriction of the method is dimensionality of the design parameters. It is most effective in the case of 2 parameters, and the area of interest can be visualized and plotted. In practice for 3-parametric controllers (as PID) gridding over one of the parameters is used.

Now consider an example problem of checking if multi-parametric polynomial of degree r

$$p(s, \theta, q) = a_0(\theta, q)s^r + a_1(\theta, q)s^{r-1} + \ldots + a_r(\theta, q),$$
$$\theta \in \mathbb{R}^m, q \in Q \subseteq \mathbb{R}^n,$$

is stable for all uncertainty values $q \in Q$.

The robust D-decomposition outlines "root-invariant" regions $D_k,\ k = 0, \ldots, r$ in the space of parameter θ. In each of the regions the polynomial has the same number of stable roots *regardless* uncertainty value:

$$D_k(Q) = \{\theta : p(s, \theta, q) \text{ has } k \text{ stable roots for all } q \in Q\}.$$

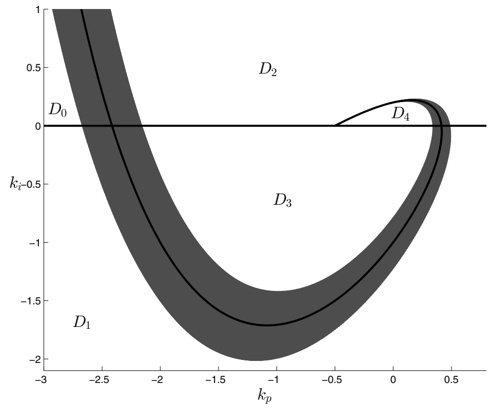

Fig. 1. Example of robust D-decomposition for a 4th order polynomial.

The most important region D_r is exactly the set of interest, i.e. stability set:

$$D_r(Q) = \{\theta : p(s, \theta, q) \text{ is Hurwitz for all } q \in Q\}.$$

If the uncertainty set Q contains only one point, the robust D-decomposition coincides with the standard D-decomposition, with "curved line" boundaries between regions (thick line on Figure 1). Expanding Q results in "blurring" the boundaries as shown on Figure 1. Of course, the stability region D_r may disappear if the uncertainty set Q becomes large enough.

The robust D-decomposition was introduced in [21]. It is related with the zero exclusion principle and the Mapping Theorem, [4]. The robust D-decomposition in the case of affine uncertainty bounded in ℓ_p-norms, is studied in [26]. Recently an attempt to reconstruct boundaries of D_k analytically was taken by [14].

Let's explain the idea behind robust D-partition. Assume that polynomial $p(s, \overline{\theta}, q)$ is robustly stable for some $\overline{\theta}$, i.e. the point belongs to the stability set: $\overline{\theta} \in D_r$. Let the parameter be continuously changing, moving at $\mathbb{R}^m$ space. Roots of the polynomial $p(s, \theta, q)$ will continuously change as well, and property of robust stability may be lost. It means that for some uncertainty $q \in Q$ a root of the polynomial "touches" imaginary line. This way our moving parameter $\overline{\theta}$ enters the "blurred boundary" set Θ_{sep}, leaving stability set D_r. There exists special case of degree drop, which is omitted for brevity, cf. [11] for details. The separation set

$$\Theta_{sep} = \{\theta \in \mathbb{R}^m : p(j\omega, \theta, q) = 0,\, q \in Q, \omega \in (-\infty, +\infty)\}$$

is parametrized by $\omega \in \mathbb{R}$ and $q \in Q$ (shadowed area on Figure 1). The description is used for the constructive characterization and visualization of the robust D-decomposition, including the stability set D_r.

As it mentioned, practical application of the D-decomposition is limited to 2- or 3- design parameter cases, mainly because of visualization and tractability issues. Note that each of the sets D_k may consist of few disjoint regions, with total number in 2D case as much as $O(r^2)$ ones. For 3-parametric case gridding on one of the parameters is is usually used. PID-, first-order and lead-lag controllers design are also well-suited for the decomposition. Due to the dimension restriction the D-decomposition is also referred to as a graphical method. The (robust) D-partition method also works for outlining of Schur stable polynomial for the discrete systems; or other cases when all roots should reside in given subset of complex plane, etc. Applications to time-delay systems, with characteristic quasi-polynomials instead of polynomials, are also possible, [13].

B. Algorithm to Find a Set of Chance-constrained PID-controllers

The problem is to describe a set of chance-constrained PID-controllers (2) for LTI system with random uncertainty $\Delta \sim \mathcal{F}$. It is described as set (4) via characteristic polynomial (3) of the closed-loop system. The set is parametrized by the probability of violation ε.

Algorithm (Chance-constrained robust PID-controllers)

Input: characteristic polynomial $p(s, k_i, k_p, k_d, \Delta)$ of degree r, uncertainty $\Delta \sim \mathcal{F}$, violation level $\varepsilon > 0$.

- Choose an ε-error set Q_ε according to the distribution of Δ;
- Build the robust D-decomposition for the uncertain polynomial $p(s, k_i, k_p, k_d, q)$, $q \in Q_\varepsilon$. It results in the sets $D_k(Q_\varepsilon) \subseteq \mathbb{R}^3, k = 0, \dots, r$.

Output: stability set $D_r(Q_\varepsilon)$.

Theorem 1: The Algorithm's output set $D_r(Q_\varepsilon)$ is an inner approximation of target set (4) for any ε-error set:

$$D_r(Q_\varepsilon) \subseteq \Theta^*_\varepsilon.$$

Proof: With probability at least $1 - \varepsilon$ uncertain parameters are within Q_ε by definition. A point (k_i, k_p, k_d) in $D_r(Q_\varepsilon)$ corresponds to robustly stabilizing controller for all uncertainties in Q_ε. Thus the closed-loop system is stable with probability at least $1 - \varepsilon$. So the point (k_i, k_p, k_d) is in Θ^*_ε as well, because it represents a chance-constrained PID-controller with violation level ε. ■

While set of chance-constrained stabilizing PID-controllers $D_r(Q_\varepsilon)$ depends the error set Q_ε, the following lemma give hint of choice of the set.

Lemma 1: If $Q^1 \subseteq Q^2$, then $D_r(Q^1) \supseteq D_r(Q^2)$.

Its proof follows from construction of Θ_{sep} and omitted. The lemma means that for a fixed violation level ϵ, any *inner* error set always gets *larger* approximation of the set of chance-constrained PID-controllers. It does not mean that the *smallest* error set results in the largest approximation yet.

The algorithm may be implemented by gridding over one of PID-controller parameters, e.g. k_d, to get planar robust D-decomposition images. The sets $D_k(Q_\varepsilon)$ are in $\mathbb{R}^2$ then. Combination of the planar sets produces three-dimensional set of the inner approximation of the whole chance-constrained PID-controllers set Θ^*_ε.

Proper choice of ε-error set is very serious issue, see also Example 1. In some cases robustness property (e.g. robust stability) is violated on the uncertainty sets with "icicle geometry". These violation sets, having a low probabilistic measure, are not far away from nominal (central-like) values of uncertainty, but rather thin inserted needles, bursting robustness. If we are able to expel these violation set from a ε-error set, the ε-error set become non-convex. It may be a problem for chance-constrained optimization problems, but the robust D-decomposition may be modified for the non-convex sets as well. Lemma 1 hints that smaller error shall be preferred over larger ones. As the mapping of error set to the approximation set D_r is highly nonlinear, the smallest error (for a fixed violation level ε) is not necessarily is an optimal one. This is demonstrated in Example 1 with the smallest set Q^1_ε and the optimal one Q^2_ε.

IV. Examples

The first example is trivial. It demonstrates both conservatism of the robust approach for the probabilistic design problem, and importance of choosing the ε-error set properly. In the second example a probabilistic design problem of the stabilization of an LTI system with a PID-controller is considered.

A. Example 1. Stability of a 1st order polynomial

Consider a first-order characteristic polynomial

$$p(s, \theta, \Delta) = s + \theta - \Delta, \tag{5}$$

with the design parameter θ and the random uncertainty with the standard normal distribution $\Delta \sim \mathcal{N}(0, 1/2)$. The problem is to choose the parameter θ to ensure Hurwitz stability of the polynomial with probability $1-\varepsilon$. The only root of (5) is $s_1 = \Delta - \theta$, and it should be negative.

The support set $\mathbb{D}$ of Δ is whole line $(-\infty, +\infty)$, and there exists no θ making polynomial $p(s)$ Hurwitz for *all* uncertainty values (i.e. with zero violation level), so there is no robust solution for the uncertainty set.

However, for fixed θ we can exactly calculate probability of the polynomial being Hurwitz, using the Gauss error function [1].

$$\mathrm{Prob}(p(s,\theta,\Delta) \text{ is Hurwitz}) = \mathrm{Prob}(\Delta < \theta) = \tfrac{1}{2}(1 + \mathrm{erf}\,(\theta)).$$

Minimal parameter θ, satisfying the chance constraint $\mathrm{Prob}_\Delta(p(s, \theta, \Delta) \text{ is Hurwitz}) \geq 1 - \varepsilon$ is

$$\theta^* = -\,\mathrm{erf}^{\,-1}(2\varepsilon - 1).$$

It is calculated via inverse error function $\mathrm{erf}^{\,-1}(\cdot)$. The set $\Theta^*_\varepsilon = \{\theta : \theta > \theta^*\} = [\theta^*, \infty)$ is exactly the set (2) of all chance-constrained stabilizing "controllers" for the polynomial (5) in the design space $\theta \in \mathbb{R}^1$.

In order to apply the robust design approximation approach we choose few error sets Q^i_ε, $i = 1, 2, 3$, such that $\mathrm{Prob}(\Delta \in Q^i_\varepsilon) \geq 1 - \varepsilon$. Then we find inner approximations of Θ^*_ε, corresponding to the sets, as the stability sets

$$\begin{aligned} D_1(Q_i) &= \{\theta : \text{polynomial (5) is Hurwitz for all } q \in Q^i_\varepsilon\} = \\ &= \{\theta : \theta > \max_{q \in Q^i_\varepsilon} q\}, \quad i = 1, 2, 3. \end{aligned}$$

One of the sets is the ellipsoid set (1), which in 1D case is just a symmetric interval $Q^1_\varepsilon = [-q^*_1, q^*_1]$ with

$$q^*_1 = -\,\mathrm{erf}^{\,-1}(\varepsilon - 1).$$

We also consider two unbounded intervals $Q^2_\varepsilon = (-\infty, q^*_2]$ and $Q^3_\varepsilon = [-q^*_2, \infty)$ as error sets, with

$$q^*_2 = -\,\mathrm{erf}^{\,-1}(2\varepsilon - 1).$$

The three chosen sets are error sets indeed with $\mathrm{Prob}(\Delta \in Q^i_\varepsilon) = 1 - \varepsilon$, with the Q^1_ε is the smallest one. The error sets match stability sets $D_1(Q^1_\varepsilon) = (q^*_1, \infty) \subset D_1(Q^2_\varepsilon) = (q^*_2, \infty)$, and $D_1(Q^3_\varepsilon) = \emptyset$. That set Q^2_ε on the happy occasion is essentially the "optimal" error-set, as inner approximation $D_1(Q^2_\varepsilon) \subseteq \Theta^*_\varepsilon$ coincides with the *exact* stabilizing set Θ^*_ε. Note that for the poorly chosen error set Q^3_ε there is no robustly stabilizing parameters θ at all.

Following heuristics is proposed: a) in most cases bounded error sets are preferred to unbounded ones, and b) geometrically smaller error sets (with smaller volume, or high-probability sets) are preferred to larger ones. In fact, good error set Q_ε is to be chosen with respect to the structure and properties of the given problem.

B. Example 2. Chance-constrained stabilizing PID-controllers

The example is the modified example 1 from [12]. The plant $G(s, \Delta) = \frac{s-15}{(1+\Delta_1)s^2+(1+\Delta_2)s-1}$ with the random uncertainty Δ is to be stabilized by the ideal PID-controller. The uncertainty has the multivariate normal distribution with $\sigma_1 = 0.1, \sigma_2 = 0.2$:

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}\right).$$

PID controller is chosen in form $C(s, k_p, k_i, k_d) = k_p + \frac{k_i}{s} + k_d s$. One parameter of the controller is fixed to $k_d = -0.3$. Here we assume gridding over k_d. Multiple slices for the different k_d are stacked then. It gives the 3D visualization of the inner approximation of the set of all chance-constrained stabilizing PID-controllers.

The closed-loop system has the characteristic polynomial

$$\begin{aligned} p(s, k_p, k_i, \Delta_1, \Delta_2) =& (0.7 + \Delta_1)s^3 + (5.5 + k_p + \Delta_2)s^2 + \\ &+ (-1 + k_i - 15k_p)s - 15k_i. \end{aligned}$$

In order to find an approximation of set of chance-constrained stabilizing PID-controllers (2), the Algorithm was applied to the PID-controller with fixed k_d.

The uncertainty Δ is the multivariate normal one, and the error sets are chosen as an ellipses (1) with appropriate sizes. The error sets may be expressed as $\{(q_1, q_2) : q_1^2/\sigma_1^2 + q_2^2/\sigma_2^2 \leq r_\varepsilon\}$, which is weighted ℓ_2-bounded uncertainty. The robust D-decomposition for such uncertainties is quite simple, [21], [26].

On Figures 2, 3 the stability sets $D_3(Q_\varepsilon)$ are plotted within stability region (dashed line). Violation levels $\varepsilon = 0.8$ and $\varepsilon = 0.99$ are taken for the images.

For the comparison, true set of probabilistic stabilizing controllers was estimated by gridding over (k_p, k_i) pairs

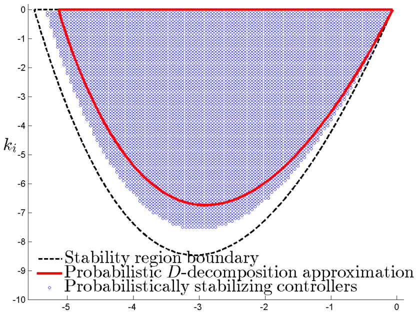

Fig. 2. Chance-constrained robust PID-constrollers for Example 2, $\varepsilon = 0.8$.

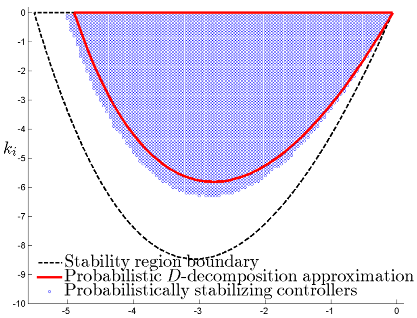

Fig. 3. Chance-constrained robust PID-constrollers for Example 2, ε=0.99.

inside stability region. Then a scenario-based probabilistic analysis algorithm with accuracy $\epsilon = 0.99$ and confidence level $\delta = 0.001$ (assurance 0.999) was applied for estimating probability of Hurwitz stability. Estimation of the probability with this assurance requires 16506 uncertainty-sampled polynomials to be checked per each (k_i, k_p), [25]. Then the estimated probability $\widehat{P}_{k_p,k_i} \approx \mathrm{Prob}_\Delta(p(s,k_p,k_i)$ is Hurwitz) was checked versus given violation level. If it was greater then $1-\varepsilon$, the point was considered as representing a chance-constrained controller and was plotted as small blue circle. The simulation was performed within the RACT toolbox, [27].

From the figures follows that the proposed Algorithm gets moderately conservative sets. Its advantage over gridding of parameters k_i, k_p is less computation cost.

V. Conclusion

In the paper the set of probabilistically robust stabilizing PID controllers (chance-constrained robust PID-controllers) for SISO LTI systems is considered. Finding this set explicitly is the hard problem, and a substitute deterministic robust design method is suggested. The method is based on the robust D-decomposition. Given a high-probability error set, approximating random uncertainty, the algorithm returns inner approximation of the objective set with moderate efforts. Its outcome is deterministic and can be used for for 2- and 3-parametric controllers. Advantages and drawbacks of the approach are discussed.

The examples show that the application of the method has small conservatism. It can be used as an approximate solution of similar low-parametric intractable probabilistic design problems as well.

References

[1] M. Abramowitz and I.A. Stegun, Eds. Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables. National Bureau of Standards, 1970.

[2] Ch. Bandi and D. Bertsimas. Tractable stochastic analysis in high dimensions via robust optimization. *Math. Program., Ser. B*, 134:23–70, 2012.

[3] F. Dabbene. Exact solution of uncertain convex optimization problems. *Proc. 26th American Control Conference*, 2654–2659, 2007.

[4] S.P. Bhattacharyya, H. Chapellat, and L.H. Keel. Robust Control - The Parametric Approach. Prentice Hall, 2000.

[5] G. Calafiore and M.C. Campi. The scenario approach to robust control design. *IEEE Trans. Autom. Control*, 51(5):742–753, 2006.

[6] M.C. Campi, S. Garatti, and M. Prandini. The scenario approach for systems and control design. *Annual Reviews in Control*, 33(2):149–157, 2009.

[7] M. Chamanbaz, F. Dabbene, D. Peaucelle, and R. Tempo. R-RoMulOC: A unified tool for randomized and robust multiobjective control. *Proc. 8th IFAC Symposium on Robust Control Design, IFAC-PapersOnline*, 48(14):144–149, 2015.

[8] S. Garatti and M. Campi. Modulating robustness in control design: Principles and algorithms. *IEEE Control Syst. Mag.*, 33(2):36–51, 2013.

[9] Grey Violet. Continuity argument revisited: geometry of root clustering via symmetric products. arXiv preprint 1512.08645, 2016.

[10] E.N. Gryazina and B.T. Polyak. Stability regions in the parameter space: D-decomposition revisited. *Automatica*, 42(1):13–26, 2006.

[11] E.N. Gryazina, B.T. Polyak, and A.A. Tremba. D-decomposition technique state-of-the-art. *Automation and Remote Control*, 69(12): 1991–2026, 2008.

[12] M.T. Ho and Ch.Y. Lin. PID Controller Design for Robust Performance. *IEEE Trans. Autom. Control*, 48(8):1404–1409, 2003.

[13] N. Hohenbichler. All stabilizing PID controllers for time delay systems. *Automatica*, 45(11):2678–2684, 2009.

[14] Ch. Hwang, L.-F. Hwang and J.-H. Hwang. Robust D-partition. *Journal of the Chinese Institute of Engineers*, 33(6):811–821, 2010.

[15] P. Kall and S.W. Wallace. Stochastic Programming. John Wiley & Sons, Chichester, 1994.

[16] L.H. Keel and S.P. Bhattacharyya. Robust, fragile, or optimal? *IEEE Trans. on Autom. Control*, 42(8):1098–1105, 1997.

[17] P. Khargonekar and A. Tikku. Randomized algorithms for robust control analysis and synthesis have polynomial complexity. *Proc. 35th IEEE conference on Decision and Control*, 4370–3475, 1996.

[18] K. Margellos, P. Goulart, and J. Lygeros. On the Road Between Robust Optimization and the Scenario Approach for Chance Constrained Optimization Problems. *IEEE Trans. Autom. Control*, 59(8):2258–2263, 2014.

[19] P. Mercader, K. Soltesz, and A. Baños. Robust PID Design by Chance-Constrained-Optimization. *Journal of the Franklin Institute*. 354(18): 8217–8231, 2017.

[20] I.R. Petersen and R. Tempo. Robust control of uncertain systems: Classical results and recent developments. *Automatica*, 50(5):1315–1335, 2014.

[21] N.P. Petrov and B.T. Polyak. Robust D-partition. *Automation and Remote Control*, 52(11):1513-1523, 1990.

[22] Z. Shafiei, A.T. Shenton, Frequency-domain design of PID controllers for stable and unstable systems with time delay. *Automatica*, 33(12): 2223–2232, 1997.

[23] D.D. Siljak. Parameter space methods for robust control design: A guided tour. *IEEE Trans. on Autom. Control*, AC-34, 674–688, 1989.

[24] R. Tempo, E.-W. Bai, F. Dabbene. Probabilistic robustness analysis: explicit bounds for the minimum number of samples. *IEEE Trans. on Autom. Control*, 30:237–242, 1997.

[25] R. Tempo, G. Calafiore, and F. Dabbene. Randomized Algorithms for Analysis and Control of Uncertain Systems. Second Edition. Springer, 2013.

[26] A. Tremba. Robust D-decomposition under l_p-bounded parametric uncertainties. *Automation and Remote Control*, 67(12):1878–1892, 2006.

[27] A. Tremba, G.C. Calafiore, F. Dabbene, E. Gryazina, B. Polyak, P. Shcherbakov, and R. Tempo. RACT: randomized algorithms control toolbox for MATLAB. *Proc. 17th World Congress of IFAC, Seoul*, 390–395, 2008.