2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Consensus Speed of Static Pinning Consensus Control of Multi-Agent Systems

Akinori Sakaguchi and Toshimitsu Ushio

Abstract— **This paper is concerned with the pinning consensus problem of multi-agent systems with single integrator agents in general directed networks. In this problem, the consensus speed is critical for the analysis of the convergence rate of the system. We show that the upper bound of the consensus speed of static pinning control is the eigenvalue with the smallest real part of the matrix reduced by deleting rows and columns corresponding to indexes of pinned agents from the Laplacian matrix. We also investigate the limit of eigenvalues of the controlled multi-agent system as the control gains go to infinity. In examples, we discuss the two cases where the network topologies are a directed small-scale and a directed scale-free graph.**

I. INTRODUCTION

Multi-agent systems consist of multiple intelligent agents which interact with each other through a network topology. Recently cooperative control of multi-agent systems has attracted a lot of attention and been studied actively in many different fields. Cooperative control of multi-agent systems is a control method which realizes cooperative motions [1] including rendezvous [2], formation control [3], [4], flocking [5], [6], and attitude alignment [7].

In cooperative control, a problem of matching the states of all agents to a common value is called a consensus problem [8], [9]. In the consensus control with exchanging information among adjacent agents, a consensus value depends on initial states of agents [10]. Our control goal is to achieve a consensus at a desirable consensus value which is independent of initial states. If we control all agents directly, the desirable consensus is achieved. It is, however, impossible to control all agents directly when the number of agents becomes enormous. Therefore, pinning control which injects control inputs to a few agents is useful for the desirable consensus control [11], [12]. A problem to achieve the consensus to the desirable value using pinning control is called a pinning consensus problem. In this problem, the consensus speed is defined as a convergence rate to the desired consensus value. In order to clarify the convergence of the pinning controlled multi-agent system, it is an important issue to identify an upper bound of the consensus speed [13].

On the other hand, a problem that leaders guide followers to reach consensus at a given value is called a leader-follower problem [8], [14], [15]. The pinning consensus problem can be regarded as the leader-follower problem. In the leader-follower problem, the convergence rate of the system is characterized by the grounded Laplacian matrix obtained by deleting rows and columns corresponding to indexes of leaders in the Laplacian matrix. Its eigenvalue with the smallest real part corresponds to the consensus speed. For undirected graphs, many studies on spectral properties of the grounded Laplacian matrix have been done [16], [17]. However, there are few studies for directed graphs [18].

In our previous work [19], for directed graphs, it has been shown that the upper bound of the consensus speed in a case where there exists a single pinned agent is the absolute value of the largest real part of all zeros of the reducible transfer function with respect to the pinned agent. In this paper, we consider the case where there exist multiple pinned agents and show that the upper bound of the consensus speed of static pinning control is the eigenvalue with the smallest real part of the matrix reduced by deleting rows and columns corresponding to indexes of pinned agents from the Laplacian matrix. We also investigate the limit of eigenvalues of the controlled multi-agent system as the control gains go to infinity. In examples, we discuss the two cases where the network topologies are a directed small-scale and a directed scale-free graph.

The remainder of this paper is organized as follows. In Section II, we give some notations and preliminaries. In Section III, we state the problem formulation of static pinning controlled multi-agent systems. In Section IV, we show the main result regarding the consensus speed of the static pinning control. To illustrate the main result, examples of a small-scale and a scale-free multi-agent system are presented in Section V. Finally, Section VI concludes the paper.

II. PRELIMINARIES

Notation: Let $\mathbb{R}$ and $\mathbb{C}$ be the sets of real numbers and complex numbers, respectively. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. For a set α, α^c and $|\alpha|$ denote the complement and the cardinality of α, respectively. $\mathbf{1}_n$ denotes the n-dimensional vector with all elements being 1. I_n denotes the $n \times n$ identity matrix and $\mathbf{0}_{m\times n}$ denotes the $m \times n$ zero matrix. Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a real matrix. $A^\top$ denotes the transpose of A and $A[\alpha, \beta]$ denotes the matrix obtained by selecting the rows of A indexed by a set α and the columns of A indexed by a set β. We write $A[\alpha]$ if $\alpha = \beta$. We write $A \geq 0$ if all $a_{ij} \geq 0$, and such a matrix A is called a nonnegative matrix, and write $A \geq B$ if

*This work was supported by JST ERATO Grant Number JPMJER1603, Japan.

A. Sakaguchi and T. Ushio are with the Department of Systems Innovation, Graduate School of Engineering Science, Osaka University, Machikaneyama 1-3, Toyonaka-shi, Osaka, 560–8531, Japan sakaguchi@hopf.sys.es.osaka-u.ac.jp

$A - B \geq 0$. $\rho(A)$ denotes the spectral radius of A. $\lambda_{\min}(A)$ denotes the smallest real part of eigenvalues of A.

A. Graph Theory

Information flow among nodes is described by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ are sets of nodes and edges, respectively. Each node $i \in \mathcal{V}$ represents agent i and each edge (j, i) means that a state of agent j is transmitted to agent i. Therefore, self-loops are excluded, that is, we have $(i, i) \notin \mathcal{E}$ for any $i \in \mathcal{V}$. The matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ is the adjacency matrix of $\mathcal{G}$, and then the element $a_{ij} = 1$ if and only if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The degree matrix of $\mathcal{G}$ is defined as $\mathcal{D} = \mathrm{diag}\{d_1, \ldots, d_n\} \in \mathbb{R}^{n \times n}$, where d_i denotes the in-degree of agent i. The Laplacian matrix of $\mathcal{G}$ is $\mathcal{L} = \mathcal{D} - \mathcal{A} \in \mathbb{R}^{n \times n}$.

B. Basic Facts on Matrices

We review the following lemmas [20] which will be used in this paper.

Lemma 1: Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be a matrix and let

$$R_i = \left\{ s \in \mathbb{C} \ : \ |s - a_{ii}| \leq \sum_{j=1,\ j \neq i}^{n} |a_{ij}| \right\}, \ i = 1, \ldots, n, \tag{1}$$

denote the Geršgorin disc. Then, all eigenvalues of A are located in the union $\bigcup_{i=1}^{n} R_i$ of n discs. Furthermore, if a union of ℓ discs ($\ell < n$) forms a connected region that is disjoint from all the remaining $n - \ell$ discs, then there are precisely ℓ eigenvalues of A in this region.

Lemma 2: For any nonsingular matrix $A \in \mathbb{R}^{n \times n}$ and sets $\alpha, \ \beta \subseteq \{1, \ldots, n\}$ with $|\alpha| = |\beta|$,

$$\det A^{-1}[\alpha, \beta] = (-1)^{(\sum_{i \in \alpha} i + \sum_{j \in \beta} j)} \frac{\det A[\beta^c, \alpha^c]}{\det A}. \tag{2}$$

III. Problem Formulation

We consider a multi-agent system whose topology is described by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $n \geq 2$ single integrator agents as follows:

$$\dot{x}_i(t) = -\sum_{j=1}^{n} a_{ij} \left(x_i(t) - x_j(t) \right), \ i = 1, \ldots, n, \tag{3}$$

where $x_i(t) \in \mathbb{R}$ is the state of agent i at time $t \in \mathbb{R}_+$, and a_{ij} is the element of $\mathcal{A}$. Then, the consensus value of the consensus protocol (3) depends on initial states [10].

Our control goal is to achieve a consensus at a desirable consensus value $x_d \in \mathbb{R}$ which is independent of initial states. The pinning control which injects control inputs to a few agents is useful for the desirable consensus control [11]. The agent to which control input is injected is called a pinned agent. In this paper, we consider the pinning controlled multi-agent system with ℓ pinned agents and reorder the label of all agents from pinned agents. Let $\mathcal{P} = \{1, \ldots, \ell\}$ be a set of pinned agents. The dynamics of pinned agents $i \in \mathcal{P}$ is described by

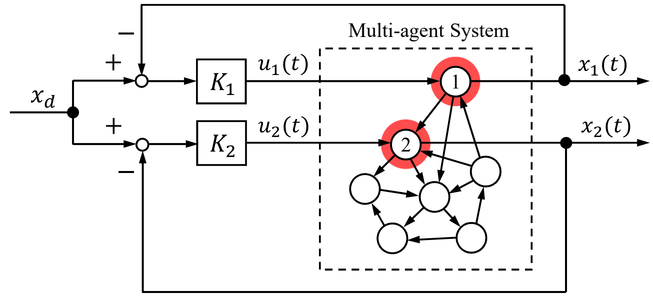


Fig. 1. An illustration of the static pinning controlled multi-agent system, where nodes 1 and 2 are pinned agents.

$$\dot{x}_i(t) = -\sum_{j=1}^{n} a_{ij} \left(x_i(t) - x_j(t) \right) + u_i(t), \ i = 1, \ldots, \ell, \tag{4}$$

where $u_i(t) \in \mathbb{R}$ is a pinning control input for pinned agent i. In this paper, we consider the case where $u_i(t)$ is given by the following static feedback control:

$$u_i(t) = K_i(x_d - x_i(t)), \tag{5}$$

where $K_i > 0$ is the gain for pinned agent i. Fig. 1 shows an illustration of the static pinning control with two pinned agents. Considering the state of the pinned agents $y(t) = [x_1(t), \ldots, x_\ell(t)]^\top \in \mathbb{R}^\ell$ as the output of the system, the state equation of the pinning controlled system is given by

$$\begin{aligned} \dot{x}(t) &= -\mathcal{L}x(t) + Pu(t), \\ y(t) &= P^\top x(t), \end{aligned} \tag{6}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^\top \in \mathbb{R}^n$ is the state of the system, $P = [I_\ell, \mathbf{0}_{\ell \times (n-\ell)}]^\top \in \mathbb{R}^{n \times \ell}$ is the matrix which denotes pinned agents $\mathcal{P}$ and $u(t) = [u_1(t), \ldots, u_\ell(t)]^\top \in \mathbb{R}^\ell$ is the static pinning control input for the pinned agents $\mathcal{P}$ as follows:

$$u(t) = K(x_d \mathbf{1}_\ell - y(t)), \tag{7}$$

where $K = \mathrm{diag}\{K_1, \ldots, K_\ell\} \in \mathbb{R}^{\ell \times \ell}$ is a gain matrix.

Definition 1: The pinning controlled system (6) is said to achieve an exponential pinning consensus if there exist positive constants $\gamma > 0$ and $v > 0$ such that, for any initial state $x(0)$ and $t \in \mathbb{R}_+$,

$$\|x(t) - x_d \mathbf{1}_n\| \leq \gamma \|x(0) - x_d \mathbf{1}_n\| e^{-vt}, \tag{8}$$

where the maximum value of v is called a consensus speed.

Note that the consensus speed of the static pinning control corresponds to the eigenvalue with the smallest real part of the matrix $\mathcal{L} + PKP^\top$. We consider a solution $x_0(t) = x_d$ of the agent, labeled by node 0, which satisfies

$$\dot{x}_0(t) = 0, \ x_0(0) = x_d. \tag{9}$$

Note that node 0 gives the desired consensus value x_d to pinned agents $\mathcal{P}$ and can be viewed as a grounded node whose state is kept constant value [18]. Let $\bar{\mathcal{G}} = \big\{\mathcal{G} \cup \{0\}, \mathcal{E} \cup \{(0,i) : i \in \mathcal{P}\}\big\}$ be an extended graph of $\mathcal{G}$ with the grounded node 0. The static pinning controlled system (6) achieves the exponential pinning consensus if and only if there exists a directed spanning tree whose root is the grounded node 0 in the graph $\bar{\mathcal{G}}$ as explained in [21].

Assumption 1: There exists a directed spanning tree whose root is the grounded node 0 in the graph $\bar{\mathcal{G}}$.

IV. MAIN RESULT

In the following theorem, the upper bound of the consensus speed of the static pinning control (7) will be presented.

Theorem 1: If *Assumption 1* holds, the consensus speed of the static pinning control is bounded by

$$v^{\mathcal{P}}_{\mathrm{ub\_s}} = \lambda_{\min}(\mathcal{L}[\mathcal{P}^c]). \tag{10}$$

Proof: Let $K = k\tilde{K}$ with $k \in \mathbb{R}$ and $\tilde{K} = \mathrm{diag}\{\tilde{K}_1, \dots, \tilde{K}_\ell\} \in \mathbb{R}^{\ell\times\ell}$. Using (7) for (6), the static pinning controlled system is given by

$$\dot{x}(t) = -(\mathcal{L} + kP\tilde{K}P^\top)x(t) + x_d kP\tilde{K}\mathbf{1}_\ell. \tag{11}$$

Note that the consensus speed corresponds to the eigenvalue with the smallest real part of the matrix $\mathcal{L}+kP\tilde{K}P^\top$. First, we investigate the eigenvalue of the matrix $\mathcal{L}+kP\tilde{K}P^\top$ as k equals to 0 and goes to infinity. Let the characteristic equation $\phi_k(s) = \det\{sI - (\mathcal{L}+kP\tilde{K}P^\top)\}$ with respect to k. Note that $\phi_0(s) = \det(sI - \mathcal{L})$.

$$\begin{aligned}
\phi_k(s) &= \det\{sI - (\mathcal{L}+kP\tilde{K}P^\top)\} && (12)\\
&= \det\Big\{(sI-\mathcal{L})(I - k(sI-\mathcal{L})^{-1}P\tilde{K}P^\top)\Big\} && (13)\\
&= \det(sI-\mathcal{L})\det\Big\{I - k(sI-\mathcal{L})^{-1}P\tilde{K}P^\top\Big\}. && (14)
\end{aligned}$$

Let $M = (sI-\mathcal{L})^{-1}P$ and $N = \tilde{K}P^\top$. It follows from $\det(I - MN) = \det(I - NM)$ that

$$\begin{aligned}
\phi_k(s) &= \det(sI-\mathcal{L})\det\Big\{I - k\tilde{K}P^\top(sI-\mathcal{L})^{-1}P\Big\} && (15)\\
&= (-k)^\ell \det(sI-\mathcal{L})\det\bigg\{\tilde{K}\frac{\Delta(s)}{\det(sI-\mathcal{L})} - \frac{1}{k}I\bigg\} && (16)
\end{aligned}$$

where

$$\Delta(s) = \begin{bmatrix} \Delta_{11}(s) & \cdots & \Delta_{\ell 1}(s) \\ \vdots & \ddots & \vdots \\ \Delta_{1\ell}(s) & \cdots & \Delta_{\ell\ell}(s) \end{bmatrix}, \tag{17}$$

and $\Delta_{ij}(s)$ is the (i,j) cofactor of the matrix $sI - \mathcal{L}$. If k goes to infinity,

$$\begin{aligned}
\lim_{k\to\infty}\phi_k(s) &= (-k)^\ell \det(sI-\mathcal{L})\det\bigg\{\tilde{K}\frac{\Delta(s)}{\det(sI-\mathcal{L})}\bigg\} && (18)\\
&= (-k)^\ell \det\tilde{K}\det(sI-\mathcal{L})\det\bigg\{\frac{\Delta(s)}{\det(sI-\mathcal{L})}\bigg\} && (19)\\
&= (-k)^\ell \det\tilde{K}\det(sI-\mathcal{L})^{1-\ell}\det\Delta(s). && (20)
\end{aligned}$$

By *Lemma 2*, we have $\det\Delta(s) = \det(sI-\mathcal{L})^{\ell-1}\det(sI - \mathcal{L}[\mathcal{P}^c])$, which implies that

$$\lim_{k\to\infty}\phi_k(s) = (-k)^\ell \det\tilde{K}\det(sI - \mathcal{L}[\mathcal{P}^c]). \tag{21}$$

Therefore, $n-\ell$ eigenvalues of the matrix $\mathcal{L}+kP\tilde{K}P^\top$ converge to those of the matrix $\mathcal{L}[\mathcal{P}^c] \in \mathbb{R}^{(n-\ell)\times(n-\ell)}$.

The Geršgorin disc with respect to the i-th row of the matrix $\mathcal{L}+kP\tilde{K}P^\top$ is given by

$$R_i = \Big\{s \in \mathbb{C} \;:\; \Big|s - (k\tilde{K}_i + d_i)\Big| \le d_i\Big\},\; i = 1,\dots,\ell, \tag{22}$$

$$R_i = \Big\{s \in \mathbb{C} \;:\; |s - d_i| \le d_i\Big\},\; i = \ell+1,\dots,n. \tag{23}$$

The centers of Geršgorin discs R_i $(i = 1,\dots,\ell)$ depend on k, whereas those of Geršgorin discs R_i $(i = \ell+1,\dots,n)$ are independent of k. Therefore, the union $\bigcup_{i=1}^{\ell} R_i$ of ℓ discs and the union $\bigcup_{i=\ell+1}^{n} R_i$ of $n-\ell$ discs are disjoint as k goes to infinity. By *Lemma 1*, ℓ eigenvalues of the matrix $\mathcal{L}+kP\tilde{K}P^\top$ go to infinity as k goes to infinity.

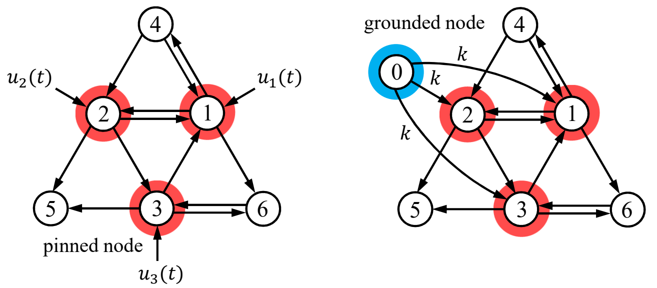
Next, we show properties of the eigenvalue with the smallest real part. For any $k > 0$, we can take a sufficiently large positive constant $\alpha > 0$ such that $S_k = \alpha I - (\mathcal{L} + kP\tilde{K}P^\top) \ge 0$ is the nonnegative matrix, which implies that the spectral radius $\rho(S_k)$ of S_k is an eigenvalue of S_k. Obviously, $\rho(S_k) = \alpha - \lambda_{\min}(\mathcal{L}+kP\tilde{K}P^\top)$ since the eigenvalue with the smallest part of $\mathcal{L}+kP\tilde{K}P^\top$ is real.

Finally, we show the monotonicity of $\lambda_{\min}(\mathcal{L}+kP\tilde{K}P^\top)$ with respect to k. For any k_1 and k_2 with $k_2 > k_1 > 0$, we have $S_{k_1} \ge S_{k_2} \ge 0$. Hence [20],

$$\begin{aligned}
\rho(S_{k_1}) &\ge \rho(S_{k_2}), && (24)\\
\lambda_{\min}(\mathcal{L}+k_2P\tilde{K}P^\top) &\ge \lambda_{\min}(\mathcal{L}+k_1P\tilde{K}P^\top). && (25)
\end{aligned}$$

Therefore, $\lambda_{\min}(\mathcal{L}+kP\tilde{K}P^\top)$ is monotonically increasing as k increases. That is, $\lambda_{\min}(\mathcal{L}+kP\tilde{K}P^\top)$ starts from the simple eigenvalue 0 of the Laplacian matrix $\mathcal{L}$, monotonically increases along with the real axis, and converges to the eigenvalue $\lambda_{\min}(\mathcal{L}[\mathcal{P}^c])$ with the smallest real part of $\mathcal{L}[\mathcal{P}^c]$. ∎

Remark 1: The matrix $\mathcal{L}[\mathcal{P}^c]$ is the grounded Laplacian matrix of the following Laplacian matrix $\mathcal{L}^\dagger$ of the weighted



(a) A directed graph $\mathcal{G}_1$.

(b) An interpretation of (a) $\mathcal{G}_1$ with the grounded node 0.

Fig. 2. A directed small-scale graph $\mathcal{G}_1$ with 6 agents and 3 pinned agents $\mathcal{P} = \{1, 2, 3\}$.

graph $\mathcal{G}^\dagger$ whose grounded node is the merged node of all pinned agents $\mathcal{P}$.

$$\mathcal{L}^\dagger = \left[\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline -\epsilon_{\ell+1} & & & \\ \vdots & & \mathcal{L}[\mathcal{P}^c] & \\ -\epsilon_n & & & \end{array}\right] \tag{26}$$

where $\epsilon_i = \sum_{j \in \mathcal{P}} a_{ij}$ is the weight that is the total number of the directed edges from pinned agents to agent i. The first row and column of $\mathcal{L}^\dagger$ correspond to the merged node 0 of all pinned agents $i \in \mathcal{P}$, that is, the grounded node indexed by 0. The other rows and columns correspond to the other agents $\mathcal{V} \setminus \mathcal{P}$. According to *Assumption 1*, all eigenvalues of the matrix $\mathcal{L}[\mathcal{P}^c]$ have positive real parts.

Remark 2: In [18], the eigenvalue with the smallest real part of the grounded Laplacian matrix is discussed. For the grounded Laplacian matrix of a directed graph, it is shown that an upper bound of $\lambda_{\min}(\mathcal{L} + kP\tilde{K}P^\top)$ is $k \max_{i \in \mathcal{P}} \tilde{K}_i$ [18]. This upper bound depends on k. However, the upper bound $\lambda_{\min}(\mathcal{L}[\mathcal{P}^c])$ derived from *Theorem 1* is the convergence value as k goes to infinity, and is independent of k. Therefore, $\lambda_{\min}(\mathcal{L}[\mathcal{P}^c])$ is the exact upper bound of $\lambda_{\min}(\mathcal{L} + kP\tilde{K}P^\top)$.

The following corollary is derived immediately from the proof of *Theorem 1* and states properties of the eigenvalues of $\mathcal{L} + kP\tilde{K}P^\top$ that determines the convergence rate.

Corollary 1: If *Assumption 1* holds, then, $\ell = |\mathcal{P}|$ eigenvalues of $\mathcal{L} + kP\tilde{K}P^\top$ go to infinity and its remaining $n - \ell = |\mathcal{V} \setminus \mathcal{P}|$ eigenvalues converge to those of $\mathcal{L}[\mathcal{P}^c]$ with positive real parts as k goes to infinity.

V. Examples

A. Directed Small-Scale Graph

We consider the case where a network topology is a directed small-scale graph $\mathcal{G}_1$ with $n = 6$ agents and $\ell = 3$ pinned agents $\mathcal{P} = \{1, 2, 3\}$ as shown in Fig. 2. Since *Assumption 1* holds from Fig. 2 (b), the exponential pinning consensus is achieved. The Laplacian matrix $\mathcal{L}$ of $\mathcal{G}_1$ is given by

$$\mathcal{L} = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}, \tag{27}$$

and the following matrix $\mathcal{L}[\mathcal{P}^c]$ is obtained by deleting rows and columns corresponding to indexes of pinned agents $\mathcal{P}$ from (27),

$$\mathcal{L}[\mathcal{P}^c] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{28}$$

Then, using the static pinning control, the consensus speed is bounded by

$$v^{\mathcal{P}}_{\text{ub\_s}} = 1. \tag{29}$$

For simplicity, let all gains of the pinned agents be k, that is, $K_1 = K_2 = K_3 = k$. Table I shows the eigenvalues of the three matrices $\mathcal{L}$, $\mathcal{L} + kPP^\top$ with the high gain $k = 10^3$, and $\mathcal{L}[\mathcal{P}^c]$. By Table I, $n - \ell = 3$ eigenvalues of $\mathcal{L} + kPP^\top$ with $k = 10^3$ are close to those of $\mathcal{L}[\mathcal{P}^c]$ and its remaining $\ell = 3$ eigenvalues are large. Also, Fig. 3 shows the root loci of $\mathcal{L} + kPP^\top$ as k goes from 0 to 10^3. In Fig. 3, $\times$ and $\bigcirc$ denote the eigenvalues of $\mathcal{L}$ and $\mathcal{L}[\mathcal{P}^c]$, respectively. From Fig. 3, we can see that the eigenvalue $\lambda_{\min}(\mathcal{L} + kPP^\top)$ with the smallest real part is real, increases monotonically, and converges to that of $\mathcal{L}[\mathcal{P}^c]$ and $\ell = 3$ eigenvalues as many as the number of pinned agents go to infinity as k increases. Therefore, we can confirm that the upper bound of the consensus speed is 1.

For $\mathcal{G}_1$, we confirm the interpretation of *Theorem 1* described by *Remark 1*. Fig. 4 shows the transient response of all agents where the desirable consensus value $x_d = 1$, the

TABLE I

Eigenvalues of the matrix $\mathcal{L}$, $\mathcal{L} + kPP^\top$, and $\mathcal{L}[\mathcal{P}^c]$.

$\mathcal{L}$	$\mathcal{L} + kPP^\top$ with $k = 10^3$	$\mathcal{L}[\mathcal{P}^c]$
0	0.999	1
1.16	1.99	2
2	2	2
2	$10^3 + 1$	
$3.42 \pm j0.606$	$10^3 + 3 \pm j0.0223$	

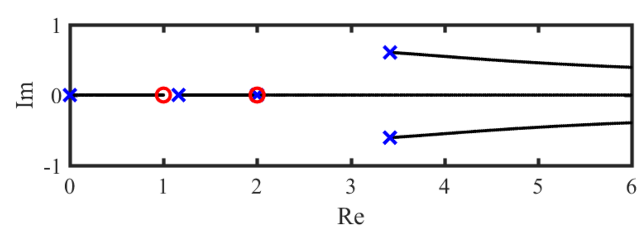


Fig. 3. Root loci of the directed graph $\mathcal{G}_1$ with pinned agents 1, 2, and 3, where $\times$ and $\bigcirc$ denote the eigenvalues of $\mathcal{L}$ and $\mathcal{L}[\mathcal{P}^c]$, respectively.

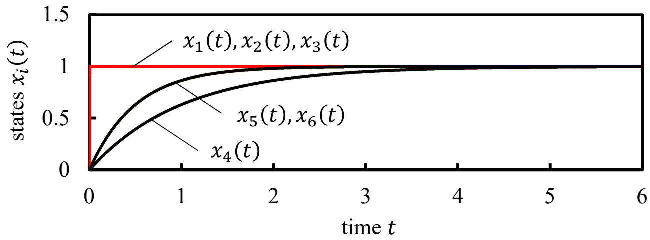
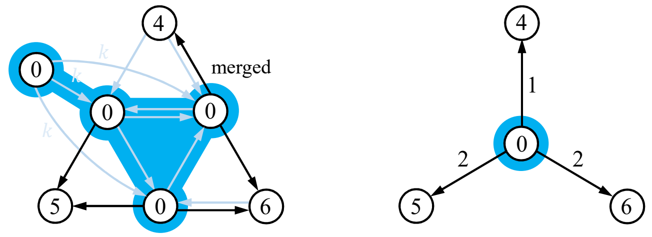


Fig. 4. A transient response of the multi-agent system with $\mathcal{G}_1$.



(a) An interpretation of Fig. 2 (b) as k goes to infinity.

(b) A weighted directed graph $\mathcal{G}_1^\dagger$.

Fig. 5. A weighted directed graph $\mathcal{G}_1^\dagger$ obtained by k going to infinity.

initial state $x(0) = 0$, and the high gain $k = 10^3$ in Fig. 2 (b). The states $x_1(t)$, $x_2(t)$, and $x_3(t)$ of pinned agents quickly converge to the desirable consensus value x_d. Therefore, the dynamics of the pinned agents can be approximated by (9) that describes the dynamics of the grounded node. Fig. 5 (a) shows the directed graph $\mathcal{G}_1$ in Fig. 2 (b) as k goes to infinity. From Fig. 4, pinned agents 1, 2, and 3 become the grounded node 0 since the dynamics of the pinned agents can be approximated by (9) as k goes to infinity. Also, the directed edges from the other agents 4, 5, and 6 to the multiple grounded node 0 and among multiple grounded nodes 0 can be deleted since the grounded node is not influenced by other agents. Furthermore, the weighted directed graph $\mathcal{G}_1^\dagger$ in Fig. 5 (b) is obtained since the multiple grounded nodes in Fig. 5 (a) can be merged as a single node. The Laplacian matrix $\mathcal{L}^\dagger$ of $\mathcal{G}_1^\dagger$ is given by

$$\mathcal{L}^\dagger = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline -1 & 1 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{array}\right] = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline -1 & & & \\ -2 & & \mathcal{L}[\mathcal{P}^c] & \\ -2 & & & \end{array}\right], \tag{30}$$

and the grounded Laplacian matrix reduced by deleting the first row and column corresponding to the grounded node from $\mathcal{L}^\dagger$, corresponds to $\mathcal{L}[\mathcal{P}^c]$ in (28). Therefore, the consensus speed is the eigenvalue $\lambda_{\min}(\mathcal{L}[\mathcal{P}^c])$ with the smallest real part of $\mathcal{L}[\mathcal{P}^c]$ as k goes to infinity.

B. Directed Scale-Free Graph

We consider a directed scale-free graph $\mathcal{G}_2$ with $n = 100$ and $m_0 = m = 5$ using the Barabási-Albert algorithm [22] as shown in Fig. 6, where m_0 is the size of the seed and m is the average of the in-degree and out-degree. Let $\ell = 3$. We consider the following two sets $\mathcal{P}_1$ and $\mathcal{P}_2$ of pinned agents. The agents in $\mathcal{P}_1$ have the first, second, and third largest out-degree. All agents in $\mathcal{P}_2$ have the out-degree 1. For simplicity, let all gains of the pinned agents be k. Table II shows the eigenvalues with the smallest real part of the two matrices $\mathcal{L} + kPP^\top$ with the high gain $k = 10^3$ and $\mathcal{L}[\mathcal{P}^c]$ in both cases $\mathcal{P}_1$ and $\mathcal{P}_2$. By Table II, the upper bound of the consensus speeds in both cases $\mathcal{P}_1$ and $\mathcal{P}_2$ are given by respectively

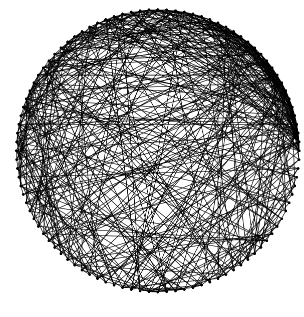


Fig. 6. A directed scale-free graph $\mathcal{G}_2$ with $n = 100$ and $m_0 = m = 5$.

$$v_{\text{ub\_s}}^{\mathcal{P}_1} = 1.433, \tag{31}$$

$$v_{\text{ub\_s}}^{\mathcal{P}_2} = 3.864 \times 10^{-5}, \tag{32}$$

and the eigenvalues with the smallest real part of $\mathcal{L} + kPP^\top$ with $k = 10^3$ are close to those of $\mathcal{L}[\mathcal{P}^c]$. Also, Figs. 7 and 8 show the root loci of $\mathcal{L} + kPP^\top$ in cases $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively as k goes from 0 to 10^3. Fig. 8 (b) shows the enlarged figure of Fig. 8 (a) around the origin. From Figs. 7 and 8 (b), the eigenvalue $\lambda_{\min}(\mathcal{L} + kPP^\top)$ is real, increases monotonically, and converges to that of $\mathcal{L}[\mathcal{P}^c]$ in both cases $\mathcal{P}_1$ and $\mathcal{P}_2$. However, the upper bound of the consensus speed of $\mathcal{P}_2$ is very small compared with that of $\mathcal{P}_1$. Fig. 9 shows the transient response of all agents where the desirable consensus value $x_d = 1$, the initial state $x(0) = 0$, and the high gain $k = 10^3$. In the both (a) and (b), the states $x_1(t)$, $x_2(t)$, and $x_3(t)$ of pinned agents quickly converge to the desirable consensus value x_d. Comparing (a) and (b), it is shown that pinned agents $\mathcal{P}_1$ with the first, second, and third largest out-degree have better convergence

TABLE II
EIGENVALUES WITH THE SMALLEST REAL PART OF THE MATRIX $\mathcal{L} + kPP^\top$ AND $\mathcal{L}[\mathcal{P}^c]$.

set of pinned agents	$\mathcal{L} + kPP^\top$ with $k = 10^3$	$\mathcal{L}[\mathcal{P}^c]$
$\mathcal{P}_1$	1.426	1.433
$\mathcal{P}_2$	3.844×10^{-5}	3.864×10^{-5}

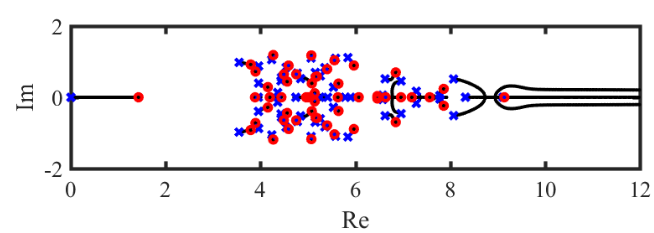
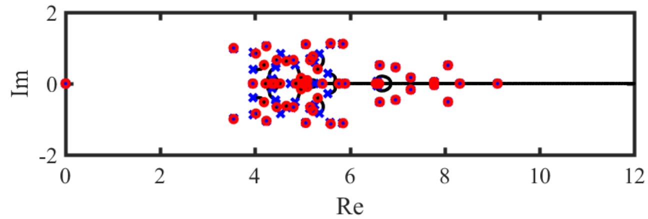
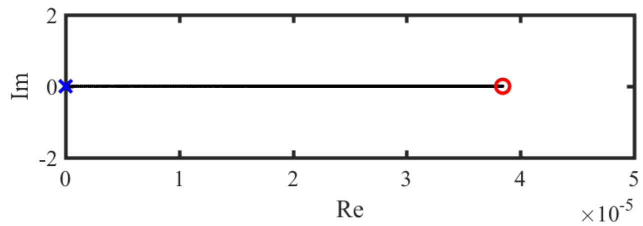


Fig. 7. Root loci of the directed scale-free graph $\mathcal{G}_2$ with pinned agents $\mathcal{P}_1$, where $\times$ and $\bigcirc$ denote the eigenvalues of $\mathcal{L}$ and $\mathcal{L}[\mathcal{P}^c]$, respectively.



(a) Root loci of the directed scale-free graph $\mathcal{G}_2$ with pinned agents $\mathcal{P}_2$.



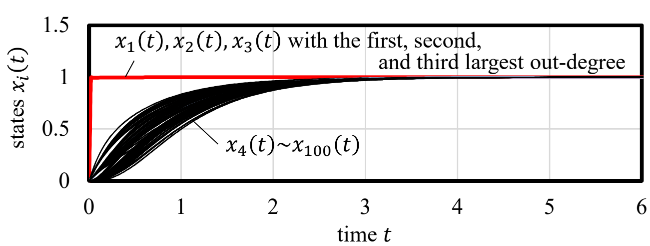
(b) The enlarged figure of root loci (a) around the origin.

Fig. 8. Root loci of the directed scale-free graph $\mathcal{G}_2$ with pinned agents $\mathcal{P}_2$, where $\times$ and $\bigcirc$ denote the eigenvalues of $\mathcal{L}$ and $\mathcal{L}[\mathcal{P}^c]$, respectively.

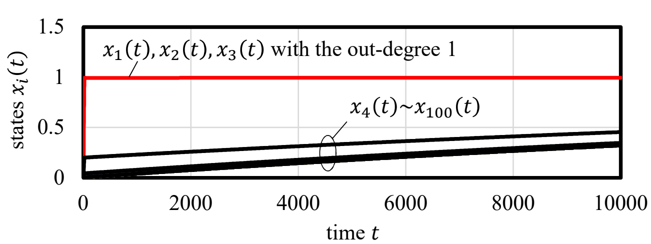
to the desired consensus value than $\mathcal{P}_2$ with the out-degree 1. Also, it takes huge time to achieve the exponential pinning consensus in Fig. 9 (b). Hence, we can see that it is better to inject pinning control inputs to the agents with the larger out-degree in the directed scale-free graph.

VI. Conclusion

We considered the static pinning consensus control for multi-agent systems. Using the static pinning control, the consensus speed is bounded by the eigenvalue with the smallest real part of the matrix reduced by deleting rows and columns corresponding to indexes of pinned agents from the Laplacian matrix. We also investigate the limit of eigenvalues of the controlled multi-agent system as the control gains go to infinity. For future work, we will investigate the upper bound of the consensus speed for several classes of directed graphs and design a dynamic pinning controller to increase the consensus speed compared with the static one. Also, we will consider a centrality measure of the network as another performance measure of the convergence to the desired state.



(a) Pinned agents $\mathcal{P}_1$ with the first, second, and third largest out-degree.



(b) Pinned agents $\mathcal{P}_2$ with the out-degree 1.

Fig. 9. A transient response of the multi-agent system with $\mathcal{G}_2$.

References

[1] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[2] J. Cortés, S. Martínez, and F. Bullo, "Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions," *IEEE Trans. Autom. Control*, vol. 51, no. 8, pp. 1289–1298, 2006.

[3] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1465–1476, 2004.

[4] G. Lafferriere, A. Williams, J. Caughman, and J. J. P. Veerman, "Decentralized control of vehicle formations," *Syst. Control Lett.*, vol. 54, no. 9, pp. 899–910, 2005.

[5] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: algorithms and theory," *IEEE Trans. Autom. Control*, vol. 51, no. 3, pp. 401–420, 2006.

[6] D. Lee and M. W. Spong, "Stable flocking of multiple inertial agents on balanced graphs," *IEEE Trans. Autom. Control*, vol. 52, no. 8, pp. 1469–1475, 2007.

[7] W. Ren, "Distributed attitude alignment in spacecraft formation flying," *Int. J. Adapt. Control Signal Process.*, vol. 21, pp. 95–113, 2007.

[8] M. Cao, A. S. Morse, and B. D. O. Anderson, "Reaching a consensus in a dynamically changing environment: a graphical approach," *SIAM J. Control Optim.*, vol. 47, no. 2, pp. 575–600, 2008.

[9] W. Ren, R. W. Beard, and E. M. Atkins, "A survey of consensus problems in multi-agent coordination," in *Proc. Amer. Contr. Conf.*, pp. 1859–1864, 2005.

[10] W. Ren, R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Syst. Mag.*, vol. 27, no. 2, pp. 71–82, 2007.

[11] X. F. Wang and G. Chen, "Pinning control of scale-free dynamical networks," *Phys. A*, vol. 310, pp. 521–531, 2002.

[12] H. Su and X. Wang, *Pinning control of complex networked systems*, Springer, 2013.

[13] A. Olshevsky and J. N. Tsitsiklis, "Convergence speed in distributed consensus and averaging," *SIAM J. Control Optim.*, vol. 48, no. 1, pp. 33–55, 2009.

[14] M. Cao, A. Olshevsky, and W Xia, "Focused first-followers accelerate aligning followers with the leader in reaching network consensus," *IFAC Proceedings Volumes*, vol. 47, no. 3, pp. 10042–10047, 2014.

[15] W. Ni and D. Cheng, "Leader-following consensus of multi-agent systems under fixed and switching topologies," *IEEE Syst. Control Lett.*, vol. 59, pp. 209–217, 2010.

[16] M. Pirani and S. Sundaram, "Spectral properties of the grounded laplacian matrix with applications to consensus in the presence of stubborn agents," in *Proc. Amer. Contr. Conf.*, pp. 2160–2165, 2014.

[17] M. Pirani and S. Sundaram, "On the smallest eigenvalue of grounded laplacian matrices," *IEEE Trans. Autom. Control*, vol. 61, no. 2, pp. 509–514, 2016.

[18] W. Xia and M. Cao, "Analysis and applications of spectral properties of grounded laplacian matrices for directed networks," *Automatica*, vol. 80, pp. 10–16, 2017.

[19] A. Sakaguchi and T. Ushio, "Dynamic pinning consensus control of multi-agent systems," *IEEE Control Syst. Lett.*, vol. 1, no. 2, pp. 340–345, 2017.

[20] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[21] W. Ren, "Multi-vehicle consensus with a time-varying reference state," *Syst. Control Lett.*, vol. 56, no. 7–8, pp. 474–483, 2007.

[22] A. L. Barabási and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, pp. 509–512, 1999.