2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Novel Adaptive MPC Design for Uncertain MIMO Discrete-time LTI Systems with Input Constraints

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***Abstract*— In this paper, a novel adaptive model predictive control (MPC) strategy is proposed for controlling a discrete-time multi-input multi-output (MIMO) linear system with constant uncertain parameters and subjected to input constraints. An adaptive law, designed to update the estimated parameters of the plant, is combined with a MPC algorithm for the estimated system. A normalizing factor is introduced in the adaptive update law, that removes its dependency on the bounds of the regressor vector as well as on the rate of adaptation gain. The proposed adaptive law guarantees stability of the parameter estimation error. The state estimation errors are proved to be bounded and asymptotically converging to zero. Stability analysis of the closed-loop system with the proposed adaptive MPC strategy is proved to show the boundedness and asymptotic convergence of the tracking error to zero.**

I. INTRODUCTION

Model predictive control (MPC) has gained much attention in the recent years due to its simple and effective approach of implementing control actions in constrained environment and its inherent robust properties towards external disturbances. Some excellent overviews of MPC can be found in [1]. The design methodology of this control strategy is well depicted in [2]. Overviews on MPC for nonlinear systems, also called nonlinear model predictive control, can be found in [3]. MPC has found its applications in various fields of engineering, such as in process control, flight control, control of electrical drives etc.

However, MPC being a model based control strategy, the systems with parametric uncertainties cannot be directly tackled by the conventional MPC strategy, owing to the difficulty in predicting the future states of the plant. For systems with unknown model, system identification has become a well accepted methodology to obtain a viable model for designing controllers. However, it is difficult to do offline system identification for an inherently unstable plant. In contrast, adaptive control does online adjustments of the controller/system parameters on the basis of measured input-output data, at the same time ensuring stability. Hence, a possible solution for the control of uncertain and constrained systems can be the amalgamation of adaptive control with MPC. Some contributions to this area of research include adaptive MPC based on comparison model [4], adaptive predictive control of nonlinear systems [5] and adaptive receding horizon predictive control for discrete-time linear systems [6].

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Although there have been significant contributions in the field of adaptive MPC, the control of a completely uncertain plant has been a challenging problem. A scheme for controlling a class of completely uncertain and unconstrained SISO systems is discussed in [5]. Adaptive MPC for controlling completely uncertain linear MIMO systems in the presence of constraints has remained an open research topic. Control strategies proposed in [4] and [6] guarantee closed-loop stability for partially uncertain and constrained linear systems. Some recent works [7], [8] have proposed effective adaptive MPC strategies to control completely uncertain systems. However, in [7], finite impulse response (FIR) approximation method is used to obtain a tractable approximated model corresponding to the uncertain plant for which the stability of the plant is a prerequisite. The method proposed in [8] is shown to control uncertain MIMO systems in the absence of any constraints. However, in order to guarantee boundedness of the dynamics of the parameter estimates, the adaptive law designed in [8] imposes a regressor vector dependent bound on the rate of adaptation, which is reformulated as a state dependent constraint on the input.

This paper presents a novel adaptive MPC method for solving set-point tracking problem for discrete-time MIMO linear time-invariant plant with complete (constant) parametric uncertainty, in the presence of input constraints. A gradient descent based adaptive law with a normalizing factor [*Section 3.2.1*, [9]] is used to update the estimated parameters, which in contrast with [8], does not impose any constraint on the system. The proposed adaptive law is proved to guarantee stability of the parameter estimation error as well as boundedness and asymptotic convergence of the state estimation error to zero. The adaptive scheme is combined with a constrained MPC formulation for solving the set-point tracking problem. It is proved that the errors occurring in the predicted states due to updating the system parameters are bounded and asymptotically converging to zero. Unlike [7], the stability of the uncertain nominal system is not a prerequisite in the developed adaptive MPC scheme. It is further theoretically proved that the tracking errors of the closed-loop system, with the proposed adaptive MPC, are bounded and asymptotically converging to zero.

II. PROBLEM STATEMENT

In this paper, the linear discrete-time MIMO system is considered to be of the form

$$x_r(k+1) = A_r x_r(k) + B_r u(k) \tag{1}$$
$$y_r(k) = C_r x_r(k) \tag{2}$$

where $x_r(k) \in \mathbb{R}^n$ is the system state vector, $u(k) \in \mathbb{R}^m$ is the input vector and $y_r(k) \in \mathbb{R}^l$ is the output vector. $A_r \in \mathbb{R}^{n\times n}$ and $B_r \in \mathbb{R}^{n\times m}$ are the uncertain and constant system matrix and input gain matrix, respectively. It is assumed that the pair (A_r, B_r) is controllable. Full state feedback is considered to be available and $C_r \in \mathbb{R}^{l\times n}$ is assumed to be known. The target is to make the system output $y_r(k)$ track a given set-point r, where $r = [r_1, r_2, ..., r_l]^T$ in the presence of the following input constraints

$$\|\Delta u(k)\| \leq \Delta U_{max} \tag{3}$$
$$\|u(k)\| \leq U_{max} \tag{4}$$

For the feasibility of the MPC problem, the desired set-point must be reachable with the given input constraints, i.e., there must exist some steady control input u_{ss} satisfying (4) that can keep the output $y_r(k)$ equal to r as $k \to \infty$. Let the tracking error associated with the system be denoted by $e_r(k)$, given as

$$e_r(k) = y_r(k) - r \tag{5}$$

In case of set-point tracking, the knowledge of the steady value u_{ss} is required for solving the optimization problem in MPC. Since the system (1) is uncertain, obtaining exact knowledge of u_{ss} is not possible. Instead, the system given in equations (1) and (2) can be modified and represented in an incremental form [2], given as

$$x(k+1) = Ax(k) + B\Delta u(k) \tag{6}$$
$$y(k) = Cx(k) \tag{7}$$

where, $x(k) \triangleq [\Delta x_r(k)^T, e_r(k)^T]^T$, $\Delta x_r(k) \triangleq x_r(k) - x_r(k-1)$, $\Delta u(k) \triangleq u(k) - u(k-1)$ and $y(k) = e_r(k)$; and

$$A = \begin{bmatrix} A_r & 0_{n\times l} \\ C_r A_r & I_{l\times l} \end{bmatrix}, B = \begin{bmatrix} B_r \\ C_r B_r \end{bmatrix}, \quad C = \begin{bmatrix} 0_{l\times n} & I_{l\times l} \end{bmatrix}$$

where $A \in \mathbb{R}^{(n+l)\times(n+l)}$, $B \in \mathbb{R}^{(n+l)\times m}$ and $C \in \mathbb{R}^{l\times(n+l)}$. Such a formulation aids the concerned MPC problem as for successful set-point tracking, $\Delta u(k)$ will always go to zero. In the incremental model (6), we consider A and B as the uncertain constant system parameters. Define a lumped system parameter matrix Θ as $\Theta^T \triangleq [A, B]$. The system (6) can be represented using the lumped parameter matrix as

$$x(k+1) = \Theta^T \Phi(k) \tag{8}$$

where $\Phi(k) = [x(k)^T \quad \Delta u(k)^T]^T \in \mathbb{R}^{n+l+m}$ is the regressor vector.

Objective: To design a control input $\Delta u(k)$ (or $u(k)$), which makes the output $y(k)$ of the uncertain system (6)-(7) track zero (which corresponds to $y_r(k)$ tracking the set-point r), in the presence of the input constraints (3) and (4).

Definition 1: For any two vector $a = [a_1, a_2, \cdots, a_n]$ and $b = [b_1, b_2, \cdots, b_n]$ with $a, b \in \mathbb{R}^n$ for some finite $n \in \mathbb{I}$, the relation $a \leq b$ is equivalent to $a_i \leq b_i$ where $i = 1, 2, \cdots, n$.

III. Adaptive Model Predictive Control Design

A. Adaptive law for uncertain parameter estimation

We assume that the current knowledge of the state and the incremental input is available. An estimated system corresponding to the nominal system (6) can be chosen as

$$\widehat{x}_0(k+1) = \widehat{A}(k)x(k) + \widehat{B}(k)\Delta u(k) \tag{9}$$

where $\widehat{A}(k) \in \mathbb{R}^{(n+l)\times(n+l)}$ and $\widehat{B}(k) \in \mathbb{R}^{(n+l)\times m}$ are the estimates of the uncertain system parameters at the k^{th} instant. The lumped estimated parameter matrix is denoted as $\widehat{\Theta}(k)^T = [\widehat{A}(k) \quad \widehat{B}(k)]$ and the system (9) can be represented as

$$\widehat{x}_0(k+1) = \widehat{\Theta}(k)^T \Phi(k) \tag{10}$$

The corresponding state estimation error is given as $\tilde{x}_0(k) \triangleq x(k) - \widehat{x}_0(k)$ and its dynamic equation is found as

$$\tilde{x}_0(k+1) = x(k+1) - \widehat{x}_0(k+1) = -\tilde{\Theta}(k)^T \Phi(k) \tag{11}$$

where $\tilde{\Theta}(k) = \widehat{\Theta}(k) - \Theta$ is the parameter estimation error. The objective is to design an adaptive update law for $\widehat{A}(k)$ and $\widehat{B}(k)$ such that the parameter estimation error $\tilde{\Theta}(k)$ is at least bounded and the state estimation error $\tilde{x}_0(k+1)$ goes to zero. To achieve this, a gradient descent based adaptive law, given as

$$\widehat{\Theta}(k+1) = \widehat{\Theta}(k) - \lambda \triangledown J_x^T \tag{12}$$

can be used, where $\lambda > 0$. J_x is given as $J_x = \frac{1}{2}\tilde{x}_0(k+1)^T \tilde{x}_0(k+1)$ and $\triangledown J_x = \frac{\partial J_x}{\partial \widehat{\Theta}(k)}$. However, with the estimated system (9) and the adaptive law (12), an additional condition, given as

$$\lambda \leq \frac{\psi}{\Phi(k)^T \Phi(k)} \quad , 0 < \psi < 2$$

needs to be satisfied in order to guarantee boundedness of the parameter estimation error [*Theorem 1*, [8]], [9]. This condition imposes a state dependent constraint on the incremental input. In this paper, an improved gradient descent based adaptive law [9] is proposed to update the estimated parameters, without imposing any additional state dependent constraint on the incremental input. For this purpose, the estimated system is redesigned as

$$\widehat{x}(k+1) = \widehat{A}(k+1)x(k) + \widehat{B}(k+1)\Delta u(k) \tag{13}$$
$$= \widehat{\Theta}(k+1)^T \Phi(k) \tag{14}$$

where $\widehat{A}(k+1)$ and $\widehat{B}(k+1)$ are the estimates of the uncertain system parameters at the $(k+1)^{th}$ instant. The corresponding state estimation error dynamics is given as

$$\tilde{x}(k+1) = x(k+1) - \widehat{x}(k+1) = -\tilde{\Theta}(k+1)^T \Phi(k) \tag{15}$$

where $\tilde{\Theta}(k+1) = \widehat{\Theta}(k+1) - \Theta$. The cost function associated with the state estimation error is chosen as

$$\bar{J}_x = \frac{1}{2}\tilde{x}(k+1)^T \tilde{x}(k+1)$$

The gradient of $\bar{J}_x$ is found as

$$\nabla \bar{J}_x(\widehat{\Theta}(k+1)) = \frac{\partial \bar{J}_x(\widehat{\Theta}(k+1))}{\partial \widehat{\Theta}(k+1)} = -\tilde{x}(k+1)\Phi(k)^T$$

Substituting the above result in (12), the following is obtained

$$\widehat{\Theta}(k+1) = \widehat{\Theta}(k) + \lambda\Phi(k)\tilde{x}(k+1)^T \tag{16}$$

In order to get the parameter update law (16) in terms of the state estimation error $\tilde{x}_0(k+1)$ (since $\tilde{x}(k+1)$ is dependent on $\widehat{\Theta}(k+1)$, thus making the update law (16) unimplementable), the following is carried out

$$\begin{aligned}\tilde{x}(k+1) =& x(k+1) - \widehat{x}_0(k+1) + \widehat{x}_0(k+1) - \widehat{x}(k+1)\\ =& \tilde{x}_0(k+1) - \Big(\widehat{\Theta}(k+1) - \widehat{\Theta}(k)\Big)^T \Phi(k)\end{aligned} \tag{17}$$

Utilizing (16) and (17) we obtain

$$\tilde{x}(k+1) = \frac{\tilde{x}_0(k+1)}{(1+\lambda\Phi(k)^T\Phi(k))} \tag{18}$$

Replacing the above result in (16), the final gradient descent based adaptive law is obtained as

$$\widehat{\Theta}(k+1) = \widehat{\Theta}(k) + \frac{\lambda\Phi(k)\tilde{x}_0(k+1)^T}{(1+\lambda\Phi(k)^T\Phi(k))} \tag{19}$$

Remark 1: The state estimation error dynamics $\tilde{x}_0(k+1)$ is dependent on $\widehat{\Theta}(k)$ as seen in (11) (since Θ is constant and $\tilde{\Theta}(k) = \widehat{\Theta}(k) - \Theta$). Hence the adaptive law (19) is implementable using $\tilde{x}_0(k+1)$.

Theorem 1: For the system (6), with the uncertain parameters A and B, the adaptive law defined in (19) guarantees the following

1) The dynamics of parameter estimation error $\tilde{\Theta}(k)$ is stable.
2) If there exists some bounded and stabilizing $\Delta u(k)$ (or $u(k)$), then the state estimation error $\tilde{x}_0(k)$ is bounded and asymptotically converging to zero.

Proof: Subtracting Θ from both sides of (19)

$$\tilde{\Theta}(k+1) = \tilde{\Theta}(k) + \frac{\lambda\Phi(k)\tilde{x}_0(k+1)^T}{(1+\lambda\Phi(k)^T\Phi(k))} \tag{20}$$

Replacing $\tilde{x}_0(k+1)$ from (11) in the above equation

$$\tilde{\Theta}(k+1) = \Big[I - \frac{\lambda\Phi(k)\Phi(k)^T}{(1+\lambda\Phi(k)^T\Phi(k))}\Big]\tilde{\Theta}(k) \tag{21}$$

A Lyapunov function candidate associated with $\tilde{\Theta}(k+1)$ is chosen as

$$V_0(k+1) = tr\Big(\tilde{\Theta}(k+1)^T\tilde{\Theta}(k+1)\Big) \tag{22}$$

where $tr(\cdot)$ denotes the trace of the argument matrix. Replacing $\tilde{\Theta}(k+1)$ from (21) in the above equation

$$\begin{aligned}V_0(k+1) &= tr(\tilde{\Theta}(k)^T\tilde{\Theta}(k)) - \frac{\lambda}{(1+\lambda\left\|\Phi(k)\right\|^2)}\\ &tr\Big(\tilde{\Theta}(k)^T\Phi(k)\Big[2 - \frac{\lambda\left\|\Phi(k)\right\|^2}{1+\lambda\left\|\Phi(k)\right\|^2)}\Big]\Phi(k)^T\tilde{\Theta}(k)\Big)\end{aligned} \tag{23}$$

Let $\alpha(k) = 2 - \frac{\lambda\|\Phi(k)\|^2}{1+\lambda\|\Phi(k)\|^2}$ and with $\lambda > 0$, it can be seen that $1 < \alpha(k) < 2, \forall\, k \in \mathbb{N}$. Therefore (23) can be modified as

$$\begin{aligned}&V_0(k+1) = V_0(k)\\ &- \frac{\lambda\alpha(k)}{(1+\lambda\left\|\Phi(k)\right\|^2)}tr\Big((\tilde{\Theta}(k)^T\Phi(k))(\Phi(k)^T\tilde{\Theta}(k))\Big)\\ &= V_0(k) - \frac{\lambda\alpha(k)}{(1+\lambda\left\|\Phi(k)\right\|^2)}tr\Big(\tilde{x}_0(k+1)\tilde{x}_0(k+1)^T\Big)\end{aligned} \tag{24}$$

$$\Delta V_0(k+1) = -\frac{\lambda\alpha(k)}{(1+\lambda\left\|\Phi(k)\right\|^2)}\Big(\tilde{x}_0(k+1)^T\tilde{x}_0(k+1)\Big) \tag{25}$$

where $\Delta V_0(k+1) = V_0(k+1) - V_0(k)$ and from (25), it is found that $\Delta V_0(k+1)$ is negative semi-definite, which proves that $V_0(k)$ is stable and hence bounded. This proves 1 of *Theorem* 1. From (24), it follows that

$$V_0(k+1) = V_0(0) - \sum_{i=0}^{k}\beta(i)\Big(\tilde{x}_0(i+1)^T\tilde{x}_0(i+1)\Big)$$

where $\beta(\cdot) \triangleq \frac{\lambda\alpha(\cdot)}{(1+\lambda\|\Phi(\cdot)\|^2)}$. Consequently

$$\sum_{i=0}^{+\infty}\beta(i)\Big(\tilde{x}_0(i+1)^T\tilde{x}_0(i+1)\Big) = V_0(0) - \lim_{k\to\infty}V_0(k) \tag{26}$$

The right hand side of the above equation is finite. Since $\Delta u(k)$ is assumed to be bounded and stabilizing, $\|\Phi(k)\|$, hence $\beta(k)$ is bounded $\forall\ k \in \mathbb{N}$. Hence the infinite sum will be finite indicating that state estimation error $\tilde{x}_0(k)$ is bounded and asymptotically converging to zero. ■

Remark 2: A bounded and stabilizing $\Delta u(k)$ will be computed through an MPC design which, independent of the convergence of the state estimation error $\tilde{x}_0(k+1)$, guarantees the boundedness of the regressor vector $\|\Phi(k)\|$.

B. MPC Design for the Estimated System

In this section, an MPC is developed for the estimated system (9). Suppose N_p is the length of the prediction horizon and N_c is the length of the control horizon. Throughout the rest of the paper it is assumed that $N_p = N_c = N$. Since the state estimation error $\tilde{x}_0(k+1)$ is used in the adaptive update law (19), the estimated system (9) will be used for designing the MPC. Let $\widehat{x}_0(k+i|k)$ denote the predicted state of the estimated system (9) at the $(k+i)^{th}$ instant, using the knowledge of $\widehat{x}_0(k)$ (here $\widehat{x}_0(k) = x(k)$). Let $\sigma = \min(m-1, N-1)$, where $m = 1, 2, ..., N$. Then following the classical MPC design presented in [2], the predictive equations can be formulated as

$$\widehat{x}_0(k+m|k) = \widehat{A}(k)^m x(k) + \sum_{j=0}^{\sigma}\widehat{A}(k)^{\sigma-j}\widehat{B}(k)\Delta u(k+j|k) \tag{27}$$

where $m = 1, \cdots, N$. Let $\Delta U(k) = [\Delta u(k|k) \quad \Delta u(k+1|k) \quad \cdots \Delta u(k+N-1|k)$ be the sequence of the future control inputs, required to obtain the desired predicted states. Then the MPC is given as

$$\Delta U^*(k) = \underset{\Delta U(k)}{\operatorname{argmin}}\, \widehat{J}(k) \tag{28}$$

subject to (3) and (4), where

$$\begin{aligned}\widehat{J}(k) = &\sum_{m=1}^{N} \widehat{x}_0(k+m|k)^T Q \widehat{x}_0(k+m|k) + \\ &\sum_{m=0}^{N-1} \Delta u(k+m|k)^T r_u \Delta u(k+m|k)\end{aligned} \tag{29}$$

where $Q = C^T C$ and $r_u \in \mathbb{R}^{m \times m}$ is positive definite. The constraint (4) can be converted into constraints on $\Delta u(k)$, following method presented in [2], and can be combined with (3) to give a single LMI in $\Delta U(k)$. The proposed control strategy is implemented using receding horizon control

$$\Delta u(k) = [I_{m \times m}, 0, ..., 0] \Delta U^*(k) \tag{30}$$

C. Stability of the Closed-Loop System

In order to guarantee stability of the closed-loop estimated system, the following terminal constraint is introduced [*Section 3.7.1.* [2]]

$$\widehat{x}_0(k+N|k) = 0 \tag{31}$$

Definition 2: The MPC optimization problem (28), subjected to (3), (4) and (31), is said to be feasible at some instant k, if there exists some control input sequence $\Delta U(k)$ (dependent on $x(k)$), which satisfies the constraints (3),(4) and (31) and returns a finite value of the cost function.

Assumption 1: If there exists some $x(k)$, for which the MPC optimization problem is feasible for the initial choice $\widehat{\Theta}(k)$, then the optimization problem will be feasible for all $\widehat{\Theta}(k+i)$ and $x(k)$, where $i > k$.

It is assumed that for some initial choices of $\widehat{x}(k)$ and $\widehat{\Theta}(k)$, the optimization problem (28) is feasible at the k^{th} instant and there exists an optimal $\Delta U^*(k)$ satisfying (3),(4) and (31), where

$$\Delta U^*(k) = [\Delta u^*(k|k)^T, ..., \Delta u^*(k+N-1|k)^T]^T \tag{32}$$

At the $(k+1)^{th}$ instant, the initial value of the state would be $\widehat{x}(k+1) = \widehat{x}^*(k+1|k)$, for which a feasible sequence exists with the system parameters being $\widehat{\Theta}(k)$. Hence, using *Assumption* 1, it can be claimed that, there exists some feasible control input sequence[1] for the estimated system at $(k+1)^{th}$ instant with the initial states and system parameters respectively being $\widehat{x}(k+1)$ and $\widehat{\Theta}(k+1)$. Therefore, a feasible solution for the optimization problem (28) at $(k+1)^{th}$ instant can be given as

$$\begin{aligned}&\Delta U(k+1) = [\Delta u(k+1|k+1)^T, ..., \Delta u(k+N|k+1)^T]^T \\ &= [\Delta u^*(k+1|k)^T, ..., \Delta u^*(k+N-1|k)^T, \\ &\Delta u(k+N|k+1)^T]^T\end{aligned} \tag{33}$$

[1]A control input sequence $\Delta U(k)$ is feasible if it satisfies the imposed input constraints but may or may not optimize the cost function $\widehat{J}_y(k)$

where $\Delta u(k+m|k+1) = \Delta u^*(k+m|k)$, $m = 1(1)N-1$. If *Assumption* 1 holds, then some $\Delta u(k+N|k+1)$, satisfying (3), (4) and the terminal constraint $\widehat{x}(k+N+1|k+1) = 0$, will exist. Thus recursive feasibility of the adaptive MPC problem is established.

Theorem 2: If the optimization (28), subject to (3),(4) and (31), is feasible at the initial time, then with the receding horizon control (30), the tracking error of the closed-loop system defined by (1) and (2) (or (6) and (7)), with the adaptive update law (19), is bounded and asymptotically converging to zero.

Proof: It is assumed that the optimization (28) is possible at the k^{th} instant and there exists a $\Delta U^*(k)$, satisfying (3),(4) and (31), so that the cost function (29) reaches its optimal value $\widehat{J}^*(k)$. Considering $\widehat{x}^*(k|k) = x(k)$ a Lyapunov function candidate is now chosen as

$$\begin{aligned}V(k) = x(k)^T Q x(k) + \widehat{J}^*(k) = \sum_{m=0}^{N} \widehat{x}^*(k+m|k)^T Q \\ \widehat{x}^*(k+m|k) + \sum_{m=0}^{N-1} \Delta u^*(k+m|k)^T r_u \Delta u^*(k+m|k)\end{aligned}$$

$V(k)$ can be upper and lower bounded as ([3], [8])

$$\left.\begin{aligned}V(k) \geq \alpha_1\Big(\|e_r(k)\|^2 + \|\Delta u^*(k)\|^2\Big) \\ V(k) \leq \alpha_2\Big(\|e_r(k)\|^2 + \|\Delta u^*(k)\|^2\Big)\end{aligned}\right\} \tag{34}$$

where $\alpha_1(\cdot), \alpha_2(\cdot) \in \kappa_\infty$ are invertible functions. Accordingly, the Lyapunov function candidate, at $(k+1)^{th}$ instant, is then formulated as

$$\begin{aligned}&V(k+1) = x(k+1)^T Q x(k+1) + \widehat{J}^*(k+1) \\ &= \sum_{m=1}^{N+1} \widehat{x}_0^*(k+m|k+1)^T Q \widehat{x}_0^*(k+m|k+1) + \\ &\sum_{m=1}^{N} \Delta u^*(k+m|k+1)^T r_u \Delta u^*(k+m|k+1)\end{aligned} \tag{35}$$

A feasible control sequence at $(k+1)^{th}$ instant can be chosen according to (33) with the assumption that some feasible $\Delta u(k+N|k+1)$ exists, satisfying

$$\widehat{x}(k+1+N|k+1) = 0 \tag{36}$$

Thus $\Delta U(k+1)$ satisfies the constraint given by (31). Hence, feasibility of the optimization problem (28) is guaranteed at $(k+1)^{th}$ instant. Hence at the $(k+1)^{th}$ instant, some positive definite function $\overline{V}(k+1)$, is obtained as

$$\begin{aligned}\overline{V}(k+1) = &\sum_{m=1}^{N} \widehat{x}(k+m|k+1)^T Q \widehat{x}(k+m|k+1) \\ &+ \sum_{m=1}^{N-1} \Delta u^*(k+m|k)^T r_u \Delta u^*(k+m|k) \\ &+ \Delta u(k+N|k+1)^T r_u \Delta u(k+N|k+1)\end{aligned} \tag{37}$$

It is to be noted that, (37) is obtained by replacing the optimal input sequence $\Delta U^*(k+1)$ by the feasible sequnece $\Delta U(k+$

1) in (35). Therefore, $\overline{V}(k+1)$ is an upper bound on $V(k+1)$ and hence the following hold true

$$V(k+1) - V(k) \leq \overline{V}(k+1) - V(k) \tag{38}$$

Due to the update of the system parameters from $[\widehat{A}(k), \widehat{B}(k)]$ to $[\widehat{A}(k+1), \widehat{B}(k+1)]$ from k^{th} instant to $(k+1)^{th}$ instant, the following holds true

$$\widehat{x}(k+i|k+1) = \widehat{x}^*(k+i|k) + \delta_{i-1}(\Delta\Theta(k+1)) \tag{39}$$

where $i = 1, \cdots, N$, $\Delta\Theta(k+1) \triangleq \widehat{\Theta}(k+1) - \widehat{\Theta}(k)$ and

$$\delta_i(k+1) = \Delta\Theta_{\widehat{A}}(k+1, i)\widehat{x}^*(k+1|k) + \Delta\Theta_{\widehat{A}\widehat{B}}(k+1, i) \tag{40}$$

where $\Delta\Theta_{\widehat{A}}(k+1, c) \triangleq \widehat{A}(k+1)^c - \widehat{A}(k)^c$ and $\Delta\Theta_{\widehat{A}\widehat{B}}(k+1, c) \triangleq \sum_{j=1}^{c} \big(\widehat{A}(k+1)^{j-1}\widehat{B}(k+1) - \widehat{A}(k)^{j-1}\widehat{B}(k)\big)\Delta u^*(k+j|k)$. $\delta_i(k+1)$ is the explicit expression for $\delta_i(\Delta\Theta(k+1))$. It is seen that $\Delta\Theta(k+1) \to 0$ asymptotically, because of (19) and *Theorem* 1, which results in $\big(\Delta\Theta_{\widehat{A}}(k+1, c), \Delta\Theta_{\widehat{A}\widehat{B}}(k+1, c)\big) \to 0$. Consequently, it is claimed that

$$\lim_{k\to\infty} \delta_i(k+1) = 0 \tag{41}$$

Using the result from (27), (37) and (39)

$$\begin{aligned}
&\overline{V}(k+1) - V(k) = -x(k)^T Q x(k) - \Delta u^*(k)^T r_u \Delta u^*(k) \\
&+ 2\sum_{m=1}^{N} \Big(\widehat{A}(k)^m x(k) + \sum_{j=0}^{\sigma} \widehat{A}(k)^{\sigma-j}\widehat{B}(k)\Delta u^*(k+j|k)\Big)^T \\
&Q\delta_{m-1}(k+1) + \sum_{m=0}^{N_{p-1}} \delta_m(k+1)^T Q \delta_m(k+1) \\
&+ \Delta u(k+N|k+1)^T r_u \Delta u(k+N|k+1)
\end{aligned} \tag{42}$$

Define

$$\begin{aligned}
\xi_1(k) &\triangleq 2\sum_{m=1}^{N} \Big(\widehat{A}(k)^m\Big)^T Q\delta_{m-1}(k+1) \\
\xi_2(k) &\triangleq 2\sum_{m=1}^{N} \Big(\sum_{j=0}^{\sigma} \widehat{A}(k)^{\sigma-j}\widehat{B}(k)\Delta u^*(k+j|k)\Big)^T \\
&\quad Q\delta_{m-1}(k+1)
\end{aligned}$$

According to *Theorem* 1, $\widehat{\Theta}(k)$ is ultimately bounded and due to the constraints (3), (4) and (31) imposed on the input, $\Delta U^*(k)$ will also be always bounded. Hence, $\xi_1(k)$ and $\xi_2(k)$ will also be bounded at all instants. Using these assumptions, $\overline{V}(k+1) - V(k)$ can be upper bounded as follows

$$\begin{aligned}
&\overline{V}(k+1) - V(k) \leq -\lambda_{min}(Q)\left\|\widehat{x}(k)\right\|^2 - r_u \left\|\Delta u^*(k)\right\|^2 \\
&+ \left\|\xi_1(k)\right\|\left\|x(k)\right\| + \left\|\xi_2(k)\right\| + \sum_{m=0}^{N_{p-1}} \left\|Q\right\|\left\|\delta_m\right\|^2 \\
&+ r_u \left\|\Delta u(k+N|k+1)\right\|^2
\end{aligned} \tag{43}$$

where $\lambda_{min}(\cdot)$ is the minimum eigen value of the argument matrix. Since Q is a positive definite matrix, $\Theta_{min}(Q) > 0$. Suppose $\lambda_{min}(Q) = \lambda_1 + \lambda_2$, where $\lambda_1, \lambda_2 > 0$. Using this result in (43), the following is obtained

$$\begin{aligned}
&\overline{V}(k+1) - V(k) \leq -\lambda_1 \left\|x(k)\right\|^2 - \lambda_2 \left\|x(k)\right\|^2 \\
&- r_u \left\|\Delta u(k)\right\|^2 + \left\|\xi_1(k)\right\|\left\|x(k)\right\| + \left\|\xi_2(k)\right\| \\
&+ \sum_{m=0}^{N_{p-1}} \left\|Q\right\|\left\|\delta_m\right\|^2 + r_u \left\|\Delta u(k+N|k+1)\right\|^2
\end{aligned}$$

Completing the squares using the second and the fourth terms on the right hand side of the last inequality, the following is obtained

$$\overline{V}(k+1) - V(k) \leq -\lambda_1 \left\|x(k)\right\|^2 - r_u \left\|\Delta u(k)\right\|^2 + \Omega \tag{44}$$

where

$$\begin{aligned}
\Omega = &\frac{\left\|\xi_1(k)\right\|^2}{4\lambda_2} + \left\|\xi_2(k)\right\| + \sum_{m=0}^{N_{p-1}} \left\|Q\right\|\left\|\delta_m\right\|^2 \\
&+ r_u \left\|\Delta u(k+N|k+1)\right\|^2
\end{aligned}$$

Since $\xi_1(k), \xi_2(k)$ and δ_m are bounded and $\Delta u(k+N|k+1)$ is finite, Ω is also bounded. Moreover, as $\delta_m \to 0$ asymptotically, it implies

$$\lim_{k\to\infty} \xi_1(k), \xi_2(k) = 0$$

Also, the control input $\Delta u(k+N|k+1)$ is present due to the presence of model mismatch in between two consecutive instants. Since $\Delta\Theta(k+1) \to 0$ as $k \to \infty$ (from *Theorem* 1), $\Delta u(k+N|k+1) \to 0$ as $k \to \infty$. Therefore, it can be claimed that

$$\lim_{k\to\infty} \Omega = 0 \tag{45}$$

Consequently, from (38) and (44) it follows

$$V(k+1) - V(k) \leq -\varrho\Big[\left\|x(k)\right\|^2 + \left\|\Delta u^*(k)\right\|^2\Big] + \Omega \tag{46}$$

where $\varrho = \min(\lambda_1, r_u)$. Since $x(k) \triangleq [\Delta x_r(k)^T, e_r(k)^T]^T$, it implies $\left\|e_r(k)\right\| < \left\|x(k)\right\|$. Therefore,

$$V(k+1) - V(k) \leq -\varrho\Big[\left\|e_r(k)\right\|^2 + \left\|\Delta u^*(k)\right\|^2\Big] + \Omega \tag{47}$$

Using the result from (34), equation (47) can be reformulated as

$$V(k+1) - V(k) \leq -\varrho\alpha_1^{-1}\Big(V(k)\Big) + \Omega \tag{48}$$

Using *Definition* 3.2 of [10], it is claimed that $V(k)$ is an ISS (input to state stable)-Lyapunov function. Since Ω is bounded and asymptotically converging to zero, it is further claimed that $V(k)$ is bounded and asymptotically converging to zero.Therefore, it can be claimed that the state $x(k)$ and consequently tracking error $e_r(k)$ of the actual system (1)-(2) is bounded and asymptotically converging to zero.

■

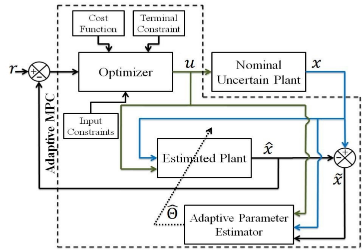

Fig. 1: Architecture of Adaptive MPC.

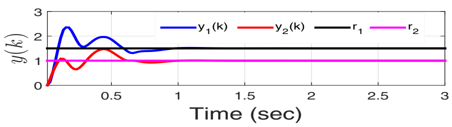

Fig. 2: Tracking Performance.

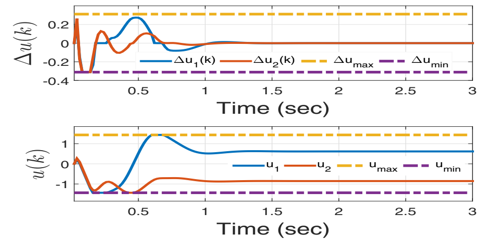

Fig. 3: The control inputs.

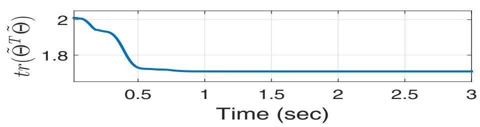

Fig. 4: The parameter estimation error.

IV. Simulation Results

The following simulation example is used from [8], to illustrate the theoretical results. The plant under consideration is MIMO with two inputs, given as $[u_1, u_2]^T$ and two outputs given as $[y_{r_1}, y_{r_2}]^T$. The uncertain system matrix, input matrix and the known output matrix are given as

$$A_r = \begin{bmatrix} 0.8 & 0.4 & 1.1 \\ 0.6 & 1.5 & -0.1 \\ 0.1 & -1.2 & 1.8 \end{bmatrix}, B_r = \begin{bmatrix} 0.7 & 0 \\ 0.2 & 0 \\ -0.6 & 1.4 \end{bmatrix},$$

$$C_r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The constraints imposed on the above system is given as

$$\|\Delta u(k)\| \leq 0.44\ ,\ \ \|u(k)\| \leq 2.028 \tag{49}$$

The above constraints can be represented as

$$-0.31 * \mathbf{1}_m \leq \Delta u(k) \leq 0.31 * \mathbf{1}_m \tag{50}$$

$$-1.43 * \mathbf{1}_m \leq u(k) \leq 1.43 * \mathbf{1}_m \tag{51}$$

The initial values of the estimated parameter matrices are considered as

$$\widehat{A}_r(0) = \begin{bmatrix} 1 & 0.5 & 0.75 \\ 0.2 & 0.8 & 0.5 \\ 0 & 0 & 2 \end{bmatrix}; \quad \widehat{B}_r(0) = \begin{bmatrix} 0.5 & 0.1 \\ 1 & 0.1 \\ -0.5 & 1.75 \end{bmatrix}$$

The initial conditions of the system states are chosen as $x(0) = [0.15, 0.1, -0.2, 0, 0]^T$ and that of the estimated states are chosen as $\widehat{x}_0(0) = [0, 0, 0, 0, 0]^T$. The prediction horizon and the control horizon are chosen as $N = 5$. The sampling interval is chosen as $h = 0.02$. The rate of adaptation is chosen to be $\lambda = 0.1$ and $r_u = 0.1$. The reference time varying signals are given as $r = [1.5\ 1]^T$. The block diagram representation of the Adaptive MPC architecture is shown in Fig. 1. The tracking performance of the closed-loop system is shown in Fig. 2. The plots of the control inputs and incremental control inputs are displayed in Fig. 3. The plot of the norm of squared parameter estimation error is shown in Fig. 4.

V. Conclusion

A novel adaptive MPC strategy is proposed for solving set-point tracking problem for a completely uncertain discrete-time linear MIMO system in the presence of input constraints. A normalizing factor is introduced in the gradient descent based adaptive update law, making the adaptive law independent of the magnitude of the regressor vector and the rate of adaptation. With the proposed adaptive update law, it is proved that the parameter estimation error dynamics is stable and the state estimation errors is bounded and asymptotically converging to zero. It is further proved that the tracking errors of closed-loop system with the proposed adaptive MPC are bounded and asymptotically converging to zero. The theoretical results are substantiated by the provided simulation results.

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