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Synchronization of discrete-time Lur'e systems under saturating control*

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***Abstract*— This article addresses the problem of control design to ensure the synchronization of discrete-time Lur'e-type systems subject to actuator saturation. Based on a quadratic Lyapunov function and sector conditions, linear matrix inequalities are derived to ensure that the difference between master and slave states converges asymptoticly to zero in the presence of control saturation. A convex optimization problem is formulated to design an error feedback control law that maximizes the set of admissible initial states. Less conservative conditions are also proposed to address the particular case where the Lur'e-type nonlinearity is described by a piecewise-linear function. Numerical results from a discrete-time chaotic system are considered to illustrate the method.**

I. INTRODUCTION

In the past few decades, the synchronization problem of chaotic circuits has been directly related to applications such as neural networks [1], secure data transmission [2] and phase-locked loop in communications [3]. This problem has attracted the control community attention since the seminal work [4], where it was proved that the synchronization of a master-slave system can be seen as a control design problem. In the following years, several control techniques have been applied to guarantee the master-slave synchronization, such as impulsive stabilization approach [5], robust $\mathcal{H}_\infty$ control [6] and adaptive control [7].

The nonlinear dynamics of some chaotic circuits, as for instance the Chua's circuit, n-scroll and hyperchaotic attractors [8], can be described by a Lur'e-type model, i.e. the interconnection of a linear system and a sector-bounded nonlinearity [9]. The synchronization of this class of systems may be analyzed in the context of absolute stability [10] and expressed in terms of linear matrix inequalities (LMI) [6]. The LMI formulation allows to address the stability analysis and control synthesis through the solution of a convex optimization problem, which can be efficiently solved by standard computational packages [11].

Most of the synchronization studies in the literature are carried out for continuous-time systems. Few publications focus on the synchronization of discrete-time Lur'e systems. See for instance [6], that considers $\mathcal{H}_\infty$ control, and [7] that proposes an adaptive control strategy considering fuzzy techniques. In both articles, stability is proved without considering constraints in the control signal.

On the other hand, control saturation is a nonlinearity that must be taken into account when dealing with most practical applications since actuators cannot deliver to the plant signals with arbitrarily high amplitude. In this case, the actuator saturation may lead to several effects in the closed-loop system such as degradation in the transient performance or even instability [12]. In particular, under control saturation, the global stability of the origin may not be achievable even if the plant dynamics is linear. Actually, if the open-loop dynamics is exponentially unstable [13], only regional (local) stability can be guaranteed. In this case, the convergence of the trajectories to the equilibrium point (origin) are ensured only if the initial condition belongs to the so-called region of attraction. When dealing with master-slave synchronization, guarantees that synchronization will occur may be provided by the stability analysis of the error dynamics between master and slave states. Hence, in the presence of control saturation, synchronization may not be achieved if the initial error between master and slave states is outside the region of attraction of the zero error equilibrium [14]. This problem has been addressed in the literature considering continuous-time systems in [14], [15] and [16]: in [14], a nonlinear output feedback controller is proposed from LMI conditions that ensure the asymptotic stability of the synchronization error; in [15], a fault tolerant sliding-mode adaptive controller is designed based on LMI constraints and a method to estimate the stability region independent of the fault information is presented; in [16], authors provided LMI conditions to ensure robust asymptotic stability and $\mathcal{H}_\infty$ disturbance attenuation for the local synchronization of time-delay systems. In parallel, in [1] the local synchronization of chaotic neural networks is achieved by a sampled-data controller and stability conditions are derived from a Lyapunov functional. It is important to point out that all these references considered a standard sector approach [9] to deal with the effects of the Lur'e-type nonlinearity.

This work addresses the master-slave synchronization of discrete-time systems, where stability analysis and control design are investigated taking into account the effects of control saturation. In particular, an error feedback controller is considered to guarantee that the synchronization error between master and slave states asymptotically converges to zero. Local stability conditions are derived from a quadratic Lyapunov candidate function using a traditional sector condition to deal with the plant nonlinearity and a modified sector condition to take into account the control saturation effect. Under these conditions, controller design is carried out by the solution of a convex optimization problem aiming to maximize the region of admissible initial master-slave

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error. Furthermore, inspired by the formulation presented in [17], less conservative results are derived for the particular case where the Lur'e system nonlinearity is described by a piecewise-linear function. The synchronization of discrete-time chaotic oscillators is considered in a numerical example to illustrate the proposed methodology.

Notations: $A^\intercal$ denotes the transpose of A, $A_{(i)}$ and $A_{(i,j)}$ are ith row and the (i,j) element of the matrix A, respectively. $\star$ means that the element is symmetric; $\mathrm{diag}\{A, B\}$ denotes a block diagonal matrix composed of matrices A and B. I and 0 represent the identity and zero matrices of appropriate dimensions. The notation x^+ refers to the successor of x, in a discrete-time framework. Matrices dimensions are often omitted when they can be inferred by the context.

II. SYNCHRONIZATION PROBLEM

Consider the discrete-time master and slave systems given by

$$x_M^+ = Ax_M + B\sigma(Cx_M) \tag{1}$$

$$x_S^+ = Ax_S + B\sigma(Cx_S) + Eu \tag{2}$$

where x_M, $x_S \in \mathbb{R}^n$ are respectively the master and slave states, $u \in \mathbb{R}^n$ is the control input and $\sigma(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is a vector-valued decentralized nonlinearity, i.e. $\sigma_{(i)}(f) = \sigma_{(i)}(f_{(i)})$, $i = 1, \ldots, m$. A, B, C and E are real matrices with appropriate dimensions.

Assumption 1: Each component of $\sigma(\cdot)$, i.e. $\sigma_{(i)}(\cdot)$, is supposed to satisfy the following statements:

(i) it is a function with odd symmetry, i.e. $\sigma_{(i)}(-f) = -\sigma_{(i)}(f)$, and $\sigma_{(i)}(0) = 0$;

(ii) it is slope-restricted, that is

$$0 \leqslant \frac{\sigma_{(i)}(f) - \sigma_{(i)}(\hat{f})}{f - \hat{f}} \leqslant \xi_{(i)}, \ \forall f, \hat{f}; \tag{3}$$

(iii) it globally belongs to the sector $[0, \xi_{(i)}]$, $\xi_{(i)} > 0$, i.e. $\sigma_{(i)}(f)(\sigma_{(i)}(f) - \xi_{(i)}f) \leqslant 0, \forall f$.

Assumption 2: The control signal is bounded as follows:

$$|u_{(j)}| \leqslant \bar{u}_{(j)}, \ j = 1, \ldots, n. \tag{4}$$

The amplitude constraint (4) is translated in a saturating control signal

$$u = \mathrm{sat}_{\bar{u}}(v) \tag{5}$$

where v denotes the signal computed by the controller and $\mathrm{sat}_{\bar{u}}(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a classical symmetric vector-valued saturation function described by:

$$\mathrm{sat}_{\bar{u}}(v)_{(j)} \triangleq \mathrm{sign}(v_{(j)}) \min(|v_{(j)}|, \bar{u}_{(j)}), \ j = 1, \ldots, n. \tag{6}$$

Defining the synchronization error e as the difference between the master and slave states, i.e. $e = x_M - x_S$, then it follows from (1) and (2) that its dynamics is given by

$$e^+ = Ae + B\rho(Ce, Cx_S) - E\mathrm{sat}_{\bar{u}}(v) \tag{7}$$

where $\rho(\cdot, \cdot) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a decentralized vector-valued nonlinearity defined as follows:

$$\begin{aligned} \rho(Ce, Cx_S) &= \sigma(Cx_M) - \sigma(Cx_S) \\ &= \sigma(Ce + Cx_S) - \sigma(Cx_S). \end{aligned} \tag{8}$$

Considering a static error feedback controller

$$v = Ke \tag{9}$$

and defining a decentralized dead-zone function

$$\psi_{\bar{u}}(v) = v - \mathrm{sat}_{\bar{u}}(v), \tag{10}$$

then the interconnection between (7) and (9) results in the following closed-loop system:

$$e^+ = (A - EK)e + B\rho(Ce, Cx_S) + E\psi_{\bar{u}}(Ke). \tag{11}$$

It is clear from (11) that the master-slave synchronization becomes a stabilization problem, i.e. if $\lim_{t\to\infty} e = 0$ then $x_S \to x_M$. However, due to the nonlinear nature of this system, the global stability of the origin, in general, cannot be guaranteed. In this case, synchronization will be achieved only if the initial error belongs to an admissible set $\mathcal{Z}$ included in the region of attraction of the origin. Hence, the following problem can be stated:

Problem 1: Find a gain matrix K such that all trajectories of (11) starting within a given admissible set $\mathcal{Z}$ converge asymptotically to the origin.

In particular, an interesting point is to compute a gain K that leads to a maximization of the region of admissible initial conditions $\mathcal{Z}$, for which the synchronization is ensured.

III. STABILITY OF GENERAL SECTOR-BOUNDED NONLINEARITIES

A. Sector-based conditions

In this section, sector-based relations will be presented regarding nonlinearities $\rho(Ce, Cx_S)$ and $\psi_{\bar{u}}(v)$. These relations will be used to derive the stability conditions in the next section.

Given that $\sigma(\cdot)$ is a sector-bounded nonlinear function, satisfying Assumption 1, the following lemma can be stated:

Lemma 1: If $\sigma(f)$ satisfies Assumption 1, then the nonlinear function $\rho(f_1, f_0) = \sigma(f_1 + f_0) - \sigma(f_0)$ is such that relation

$$\rho(f_1, f_0)^\intercal S_1(\rho(f_1, f_0) - \Xi f_1) \leqslant 0, \ \forall f_1, f_0 \tag{12}$$

with $\Xi = \mathrm{diag}\{\xi_{(1)}, \xi_{(2)}, \ldots, \xi_{(m)}\}$ is globally verified for any diagonal positive definite matrix S_1.

Proof: The proof follows the same reasoning considered in [14] for the proof of Lemma 2 and therefore is omitted. ■

The saturation of the control input rewritten as a dead-zone nonlinearity will be analyzed by means of a modified sector condition through the following lemma [12]:

Lemma 2: If v and w belong to the set $\mathcal{W}(v, w)$ with $\mathcal{W}(v, w) = \{v, w \in \mathbb{R}^n | \, |v_{(j)} - w_{(j)}| \leqslant \bar{u}_{(j)}, j = 1, \ldots, n\}$, then relation

$$\psi_{\bar{u}}(v)^\intercal S_2(\psi_{\bar{u}}(v) - w) \leqslant 0 \tag{13}$$

is verified for any diagonal positive definite matrix S_2.

B. Stability conditions

Based on a quadratic Lyapunov function and the sector-based relations presented in the previous section, conditions to ensure the asymptotic stability of the closed-loop system (11), i.e. to solve the synchronization problem, are formally stated in the next theorem:

Theorem 1: If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n\times n}$, diagonal positive definite matrices $T_1 \in \mathbb{R}^{m\times m}$, $T_2 \in \mathbb{R}^{n\times n}$, matrices Y and $Z \in \mathbb{R}^{n\times n}$ satisfying the relations

$$\begin{bmatrix} W & \star & \star & \star \\ -\Xi C W & 2T_1 & \star & \star \\ -Z & 0 & 2T_2 & \star \\ AW - EY & BT_1 & ET_2 & W \end{bmatrix} > 0 \tag{14}$$

$$\begin{bmatrix} W & \star \\ Y_{(j)} - Z_{(j)} & \bar{u}^2_{(j)} \end{bmatrix} > 0, j = 1, \ldots, n \tag{15}$$

with $K = YW^{-1}$, then all trajectories of the system (11) starting at the set

$$\mathcal{Z} = \{e \in \mathbb{R}^n \mid e^\intercal P e \leqslant 1\} \tag{16}$$

with $P = W^{-1}$ converge asymptotically to the origin.

Proof: Choosing a quadratic Lyapunov function $V(e) = e^\intercal P e$ and computing $\Delta V = V(e^+) - V(e)$ yields:

$$\Delta V = e^{+\intercal} P e^+ - e^\intercal P e. \tag{17}$$

Suppose that $\sigma(\cdot)$ satisfies Assumption 1. Then, from Lemma 1 it is guaranteed that

$$\Upsilon_1 = \rho(Ce, Cx_S)^\intercal S_1 \left[\rho(Ce, Cx_S) - \Xi C e\right] \leqslant 0 \tag{18}$$

for any diagonal matrix $S_1 > 0$.

In addition, considering (9) and taking $w = Ge$, with G being a free matrix to be determined, such that $\mathcal{W}(v, w) = \mathcal{W}(e) = \{e \in \mathbb{R}^n | \, |(K_{(j)} - G_{(j)})e| \leqslant \bar{u}_{(j)}, \, j = 1, \ldots, n\}$, then, from Lemma 2 it follows that

$$\Upsilon_2 = \psi_{\bar{u}}{}^\intercal S_2 (\psi_{\bar{u}} - Ge) \leqslant 0 \tag{19}$$

is satisfied with any diagonal matrix $S_2 > 0$, provided that $e \in \mathcal{W}(e)$. Hence, from (11), (18) and (19) it results that

$$\begin{aligned} \Delta V \leqslant{} & [(A - EK)e + B\rho(Ce, Cx_S) \\ & + E\psi_{\bar{u}}(v)]^\intercal P[(A - EK)e + B\rho(Ce, Cx_S) + E\psi_{\bar{u}}(v))] \\ & - e^\intercal P e - 2\Upsilon_1 - 2\Upsilon_2 \end{aligned} \tag{20}$$

provided that $e \in \mathcal{W}(e)$.

Defining now $\zeta^\intercal = \begin{bmatrix} e^\intercal & \rho^\intercal & \psi_{\bar{u}}{}^\intercal \end{bmatrix}$, then (20) may be rewritten as $\Delta V \leqslant -\zeta^\intercal M \zeta$, with

$$\begin{aligned} M = {} & \begin{bmatrix} P & \star & \star \\ -S_1 \Xi C & 2S_1 & \star \\ -S_2 G & 0 & 2S_2 \end{bmatrix} \\ & - \begin{bmatrix} A - EK & B & E \end{bmatrix}^\intercal P \begin{bmatrix} A - EK & B & E \end{bmatrix}. \end{aligned} \tag{21}$$

Thus $M > 0$ implies that $\Delta V < 0$ if $e \in \mathcal{W}(e)$.

Applying Schur's complement, pre- and post-multiplying M by diag$\{W, T_1, T_2, I\}$ and its transpose and performing the change of variables $T_1 = S_1^{-1}$, $T_2 = S_2^{-1}$ and $Z = GW$, then it is possible to conclude that the satisfaction of (14) implies $M > 0$.

LMI (15) guarantees that the ellipsoidal set (16) is included in $\mathcal{W}(e)$ [12]. As a consequence, the satisfaction of (14) and (15) implies effectively $\Delta V < 0$ for all trajectories of the closed-loop system (11) starting in $\mathcal{Z}$.

■

IV. STABILITY OF PIECEWISE-LINEAR SECTOR-BOUNDED NONLINEARITIES

In this section, the stability results from Theorem 1 will be extended to the case where the nonlinear function $\sigma(\cdot)$ is described by a piecewise-linear function.

A. System description

Consider now that Assumption 1 holds and that each component of $\sigma(\cdot)$ may be described by a piecewise-linear function

$$\sigma_{(i)}(f) = \begin{cases} \lambda_{1(i)} f_{(i)}, & f_{(i)} \in [0, b_{1(i)}] \\ \lambda_{2(i)} f_{(i)} + c_{2(i)}, & f_{(i)} \in (b_{1(i)}, b_{2(i)}] \\ \quad \vdots & \\ \lambda_{N(i)} f_{(i)} + c_{N(i)}, & f_{(i)} \in (b_{N-1(i)}, \infty) \end{cases} \tag{22}$$

where $c_{l+1(i)} \triangleq b_{l(i)}(\lambda_{l(i)} - \lambda_{l+1(i)}) + c_{l(i)}$, $l = 1, \ldots, N-1$, and $c_{1(i)} \triangleq 0$. The values of $\sigma_{(i)}(f)$ for $f_{(i)} < 0$ may be determined by odd symmetry.

Employing the formulation in [17] for N slopes (or $N-1$ bends), $\sigma(f)$ may be described by a sum of a linear element and saturation functions as follows

$$\sigma(f) = \Lambda_N f + \sum_{l=1}^{N-1} (\Lambda_l - \Lambda_{l+1}) \text{sat}_{b_l}(f), \tag{23}$$

where $\Lambda_l = \text{diag}\{\lambda_{l(1)}, \ldots, \lambda_{l(m)}\}$.

Thus, based on a vector-valued decentralized deadzone function (10), $\sigma(\cdot)$ may be rewritten as

$$\begin{aligned} \sigma(f) &= \Lambda_N f + \sum_{l=1}^{N-1} \left[(\Lambda_l - \Lambda_{l+1})(f - \psi_{b_l}(f))\right] \\ &= \Lambda_1 f - \sum_{l=1}^{N-1} \bar{\Lambda}_l \psi_{b_l}(f) \end{aligned} \tag{24}$$

with $\bar{\Lambda}_l = \Lambda_l - \Lambda_{l+1}$. In this case, it follows that

$$\begin{aligned} & \rho(Ce, Cx_S) = \\ & \quad \Lambda_1 Ce - \sum_{l=1}^{N-1} \bar{\Lambda}_l \left[\psi_{b_l}(Ce + Cx_S) - \psi_{b_l}(Cx_S)\right]. \end{aligned} \tag{25}$$

Thus, the synchronization error dynamics (7) is given by

$$e^+ = (A + B\Lambda_1 C)e - B\sum_{l=1}^{N-1} \bar{\Lambda}_l \Gamma_l(Ce, Cx_S) - E\text{sat}_{\bar{u}}(v) \tag{26}$$

where $\Gamma_l(\cdot,\cdot) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a vector-valued decentralized nonlinearity defined by:

$$\Gamma_l(Ce, Cx_S) \triangleq \psi_{b_l}(Ce + Cx_S) - \psi_{b_l}(Cx_S). \tag{27}$$

Considering the static error feedback controller (9) yields

$$e^{+} = (A + B\Lambda_1 C - EK)e - B\sum_{l=1}^{N-1}\bar{\Lambda}_l\Gamma_l(Ce, Cx_S) + E\psi_{\bar{u}}(Ke). \quad (28)$$

B. Sector-based conditions

Similarly to Lemma 1, the following lemma regarding $\Gamma_l(Ce, Cx_S)$ may be stated:

Lemma 3: The nonlinear function $\Gamma_l(f_1, f_0) = \psi_{b_l}(f_1 + f_0) - \psi_{b_l}(f_0)$ is such that

$$\Gamma_l(f_1, f_0)^{\intercal} Q_l(\Gamma_l(f_1, f_0) - f_1) \leqslant 0,\ \forall f_1, f_0 \quad (29)$$

is globally verified for any diagonal positive definite matrix Q_l.

Proof: Since $\psi_{b_l}(\cdot)$ is a decentralized nonlinearity, to prove this lemma we assume without loss of generality that $m = 1$, i.e. f_1 and f_0 are scalars. Then, consider the following cases:

(a) $f_0 > 0$ and $f_1 > 0$

- I $f_0 < b_l$ and $f_1 + f_0 < b_l$. In this case, $\Gamma_l(f_1, f_0) = 0$ since $\psi_{b_l}(f_1 + f_0) = 0$ and $\psi_{b_l}(f_0) = 0$, resulting in $\Gamma_l(f_1, f_0)^{\intercal} Q_l(\Gamma_l(f_1, f_0) - f_1) = 0$;
- II $f_0 < b_l$ and $f_1 + f_0 > b_l$. This is precisely the case illustrated in Fig. 1. Thus, $\psi_{b_l}(f_1 + f_0) = f_1 + f_0 - b_l$ and $\psi_{b_l}(f_0) = 0$, then $\Gamma_l(f_1, f_0) = f_1 + f_0 - b_l > 0$ and $\Gamma_l(f_1, f_0)^{\intercal} Q_l(\Gamma_l(f_1, f_0) - f_1) = \Gamma_l(f_1, f_0)^{\intercal} Q_l(f_1 + f_0 - b_l - f_1) = \Gamma_l(f_1, f_0)^{\intercal} Q_l(f_0 - b_l)$. Since $\Gamma_l(f_1, f_0) > 0$, $Q_l > 0$ and $f_0 - b_l < 0$ we conclude that $\Gamma_l(f_1, f_0)^{\intercal} Q_l(\Gamma_l(f_1, f_0) - f_1) < 0$;
- III $f_0 > b_l$ and $f_1 + f_0 > b_l$. Here we have $\psi_{b_l}(f_1 + f_0) = f_1 + f_0 - b_l$ and $\psi_{b_l}(f_0) = f_0 - b_l$, hence $\Gamma_l(f_1, f_0) = f_1 + f_0 - b_l - (f_0 - b_l) = f_1 > 0$ which implies $\Gamma_l(f_1, f_0)^{\intercal} Q_l(\Gamma_l(f_1, f_0) - f_1) = \Gamma_l(f_1, f_0)^{\intercal} Q_l(f_1 - f_1) = 0$;

(b) $f_0 > 0$ and $f_1 < 0$

- I $f_0 < b_l$ and $-b_l < f_1 + f_0 < b_l$. This case is analogous to (a)-I;
- II $f_0 < b_l$ and $f_1 + f_0 < -b_l$. Note that $\psi_{b_l}(f_1 + f_0) = f_1 + f_0 + b_l$ and $\psi_{b_l}(f_0) = 0$, resulting in $\Gamma_l(f_1, f_0) = f_1 + f_0 + b_l < 0$ and $\Gamma_l(f_1, f_0)^{\intercal} Q_l(\Gamma_l(f_1, f_0) - f_1) = \Gamma_l(f_1, f_0)^{\intercal} Q_l(f_1 + f_0 + b_l - f_1) = \Gamma_l(f_1, f_0)^{\intercal} Q_l(f_0 + b_l)$. Since $\Gamma_l(f_1, f_0) < 0$, $Q_l > 0$, $f_0 > 0$ and $b_l > 0$ then it follows $f_0 + b_l > 0$ and $\Gamma_l(f_1, f_0)^{\intercal} Q_l(\Gamma_l(f_1, f_0) - f_1) < 0$;
- III $f_0 > b_l$ and $f_1 + f_0 > b_l$. See case (a)-III;
- IV $f_0 > b_l$ and $-b_l < f_1 + f_0 < b_l$. In this case $\psi_{b_l}(f_1 + f_0) = 0$ and $\psi_{b_l}(f_0) = f_0 - b_l$, hence $\Gamma_l(f_1, f_0) = -f_0 + b_l < 0$ and, consequently $\Gamma_l(f_1, f_0)^{\intercal} Q_l(\Gamma_l(f_1, f_0) - f_1) = \Gamma_l(f_1, f_0)^{\intercal} Q_l(-f_0 + b_l - f_1) = \Gamma_l(f_1, f_0)^{\intercal} Q_l(b_l - (f_0 + f_1))$. Since $\Gamma_l(f_1, f_0) < 0$, $Q_l > 0$ and $f_1 + f_0 < b_l$, then $b_l - (f_0 + f_1) > 0$ and $\Gamma_l(f_1, f_0)^{\intercal} Q_l(\Gamma_l(f_1, f_0) - f_1) < 0$;
- V $f_0 > b_l$ e $f_1 + f_0 < -b_l$. This case results in $\psi_{b_l}(f_1 + f_0) = f_1 + f_0 + b_l$ and $\psi_{b_l}(f_0) = f_0 - b_l$ such that $\Gamma_l(f_1, f_0) = f_1 + f_0 + b_l - (f_0 - b_l) = f_1 + 2b_l < 0$. This is derived from the fact that $f_0 > b_l$, which leads to $f_1 + b_l < f_1 + f_0 < -b_l$. Hence, $\Gamma_l(f_1, f_0)^{\intercal} Q_l(\Gamma_l(f_1, f_0) - f_1) = \Gamma_l(f_1, f_0)^{\intercal} Q_l(f_1 + 2b_l - f_1) = 2\Gamma_l(f_1, f_0)^{\intercal} Q_l(b_l)$ and $\Gamma_l(f_1, f_0)^{\intercal} Q_l(\Gamma_l(f_1, f_0) - f_1) < 0$ since $\Gamma_l(f_1, f_0) < 0$, $Q_l > 0$ and $b_l > 0$.

The proof of the remaining cases where $f_0 < 0$ can be derived from symmetry arguments since $\psi_{b_l}(f)$ is an odd symmetric function. ■

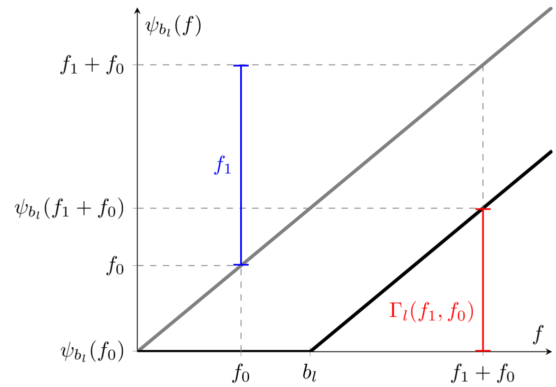


Fig. 1: Sector-bounded nonlinear function $\psi_{b_l}(f)$.

C. Stability conditions

The following theorem addresses the stability of the closed-loop (28):

Theorem 2: If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n\times n}$, diagonal positive definite matrices $T_2 \in \mathbb{R}^{n\times n}$, $R_l \in \mathbb{R}^{m\times m}$, $l = 1, \ldots, N-1$, matrices Y and $Z \in \mathbb{R}^{n\times n}$ satisfying relations (15) and

$$\begin{bmatrix} W & \star & \star & \star & \star & \star \\ -CW & 2R_1 & \star & \star & \star & \star \\ \vdots & \vdots & \ddots & \star & \star & \star \\ -CW & 0 & \cdots & 2R_{N-1} & \star & \star \\ -Z & 0 & \cdots & 0 & 2T_2 & \star \\ \Omega & \Theta_1 & \cdots & \Theta_{N-1} & ET_2 & W \end{bmatrix} > 0 \quad (30)$$

with $\Omega = AW + B\Lambda_1 CW - EY$, $\Theta_l = -B\bar{\Lambda}_l R_l$, $l = 1, \cdots, N-1$ and $K = YW^{-1}$, then all trajectories of the system (28) starting at set $\mathcal{Z}$ defined in (16), with $P = W^{-1}$, converge asymptotically to the origin.

Proof: The proof of this theorem follows the same procedure employed in the proof of Theorem 1, where Υ_1 is replaced by

$$\Upsilon_3 = \sum_{l=1}^{N-1}\Gamma_l(Ce, Cx_S)^{\intercal} Q_l\left[\Gamma_l(Ce, Cx_S) - Ce\right] \leqslant 0 \quad (31)$$

with $R_l = Q_l^{-1}, l = 1, \ldots, N-1$.

Then, from (19), (28) and (31) it follows that

$$\Delta V \leqslant [(A+B\Lambda_1 C - EK)e - B\sum_{l=1}^{N-1}\bar{\Lambda}_l \Gamma_l(Ce, Cx_S)$$
$$+ E\psi_{\bar{u}}(v)]^\intercal P[(A+B\Lambda_1 C-EK)e - B\sum_{l=1}^{N-1}\bar{\Lambda}_l\Gamma_l(Ce, Cx_S)$$
$$+ E\psi_{\bar{u}}(v))] - e^\intercal Pe - 2\Upsilon_2 - 2\Upsilon_3. \quad (32)$$

Defining $\nu^\intercal = \begin{bmatrix} e^\intercal & \Gamma_1^\intercal & \cdots & \Gamma_{N-1}^\intercal & \psi_{\bar{u}}{}^\intercal \end{bmatrix}$, then (32) may be rewritten as $\Delta V \leqslant -\nu^\intercal L \nu$, with L resulting from transformations similar to those applied in the proof of Theorem 1. Thus $L > 0$ implies that $\Delta V < 0$ if $e \in \mathcal{W}(e)$. ■

Remark 1: In the particular case of piecewise-linear functions with only one bend, conditions in Theorem 2 are equivalent to those in Theorem 1 with a loop transformation [9] to convert a sector $[\Lambda_1, \Lambda_2]$ in a sector $[0, \Xi]$. Nevertheless, for the general case with more than one bend, conditions in Theorem 2 take into account the full information of the nonlinearity and not only its maximal and minimal slopes.

Based on Theorem 1 and Theorem 2, the controller gain K can be determined by the solution of an optimization problem under LMI constraints (14) and (15) (or (15) and (30)). In particular, it is of main interest to find a matrix K that leads to the maximization of the volume of the ellipsoidal set $\mathcal{Z}$. As presented in [12], this is indirectly achieved by the solution of

$$\begin{gathered} \max \ \mathrm{trace}(W) \\ \text{subject to (14) and (15) (or (15) and (30)).} \end{gathered} \quad (33)$$

Since the stability conditions are LMIs, this problem is convex and can be solved by standard computational packages.

V. EXAMPLE

Consider a discrete-time system represented in state-space by

$$x^+ = Ax + B\sigma(Cx) + Eu \quad (34)$$

where:

$$A = \begin{bmatrix} 0.91 & 0.09 & 0 \\ 0.01 & 0.99 & 0.01 \\ 0 & -0.14 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0.09 \\ 0 \\ 0 \end{bmatrix}, \ E = 0.01I$$

and $\sigma(Cx) = -\phi(x_1)$, with $\phi(x_1)$ described by the following piecewise-linear function:

$$\phi(x_1) = \begin{cases} -g_2(x_1+1) + g_1, & \text{if } x_1 < -1 \\ -g_1 x_1, & \text{if } |x_1| \leqslant 1 \\ -g_2(x_1 - 1) - g_1, & \text{if } x_1 > 1 \end{cases}. \quad (35)$$

Parameters $g_1 = 1.15$ and $g_2 = 0.7$ have been chosen such that the state trajectory exhibits a chaotic behavior [18] (see Fig. 2). Note that this nonlinearity can be described as in (22) with $N = 2$, $b_1 = 1$, $\Lambda_1 = g_1$ and $\Lambda_2 = g_2$. The saturation levels of the control signal are given by $\bar{u}_{(j)} = 2$, $j = 1, 2, 3$.

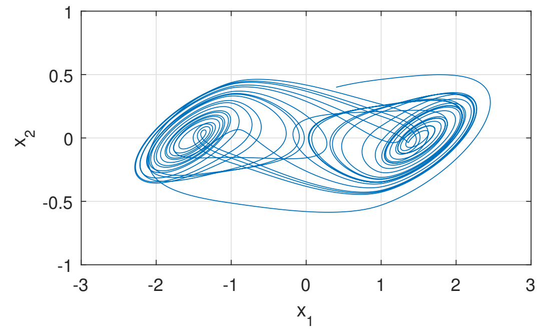


Fig. 2: Chaotic behavior of system (34) with nonlinearity (35) in the $x_1 \times x_2$ plane.

Solving[1] the optimization problem (33) with the conditions of Theorem 1 results in trace$(W) = 20.27$ and the following matrices:

$$P = \begin{bmatrix} 0.2993 & 0.3607 & 0.08664 \\ 0.3607 & 1.215 & 0.04416 \\ 0.08664 & 0.04416 & 0.1133 \end{bmatrix}$$
$$K = \begin{bmatrix} 1.369 & 1.888 & 0.4663 \\ 1.379 & 2.932 & 0.4185 \\ 0.9183 & 1.199 & 0.7122 \end{bmatrix}.$$

Then, solving the optimization problem with the conditions of Theorem 2 results in trace$(W) = 49.13$ and

$$P = \begin{bmatrix} 0.3037 & 0.2515 & 0.1456 \\ 0.2515 & 0.6824 & 0.05045 \\ 0.1456 & 0.05045 & 0.1130 \end{bmatrix}$$
$$K = \begin{bmatrix} 1.419 & 1.277 & 0.6818 \\ 1.259 & 2.121 & 0.4822 \\ 1.196 & 0.8831 & 0.7410 \end{bmatrix}.$$

Figs. 3 and 4 illustrate the closed-loop behavior of the state trajectories and the resulting control signal, respectively. Master and slave initial states were set to $x_M(0) = \begin{bmatrix} 0.4 & 0.4 & 0.6 \end{bmatrix}^\intercal$ and $x_S(0) = \begin{bmatrix} -0.2 & -0.2 & 0.1 \end{bmatrix}^\intercal$ such that $e(0)^\intercal Pe(0) \leqslant 1$, i.e. the initial error belongs to the set $\mathcal{Z}$ for both cases. It can be seen that synchronization is achieved as the slave states asymptotically converge to the master ones. It should also be noted the occurrence of saturation for both controllers.

Fig. 5 displays the cuts (on the planes $e_1 \times e_2$, $e_2 \times e_3$ and $e_1 \times e_3$) of the region $\mathcal{Z}$. It can be seen that Theorem 2 leads to a larger set $\mathcal{Z}$ when compared to Theorem 1, i.e. greater is the value for trace(W) (or equivalently) smaller is trace(P). This shows that by exploiting a particular structure of the $\sigma(f)$, in this case a piecewise-linear characteristic, less conservative results in terms of stability can be obtained.

VI. CONCLUSION

This work has addressed the design of static error feedback controllers to guarantee the master-slave synchronization of discrete-time Lur'e-type systems under saturating inputs. Based on a quadratic Lyapunov function and sector conditions, LMI conditions to ensure the regional stability of the

[1]CVX [11] package for Matlab was used with solver SDPT3.

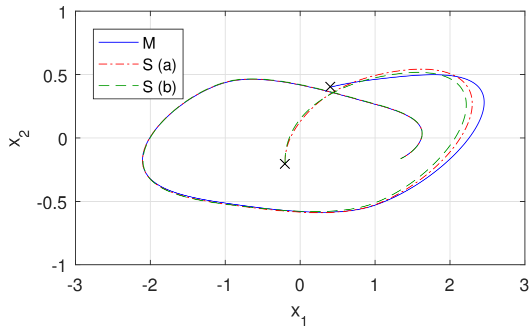
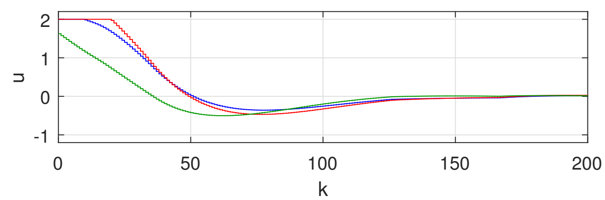
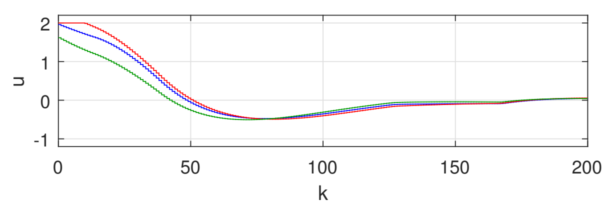


Fig. 3: Master and slave trajectories in the $x_1 \times x_2$ plane: (a) Theorem 1 and (b) Theorem 2.



(a)

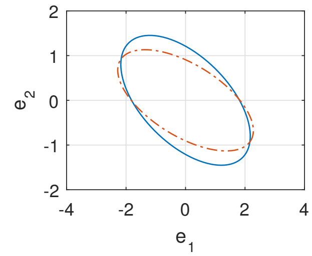


(b)

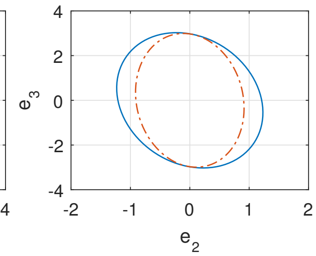
Fig. 4: Input control signal u: (a) Theorem 1 and (b) Theorem 2.

error between master and slave states have been derived. In addition, a formulation based on saturation functions has been considered to obtain stability conditions when the Lur'e-type nonlinearity is described by a piecewise-linear function. By means of a numerical example it was shown that conditions derived from the piecewise-linear formulation are less conservative, leading to a larger estimate for the set of admissible initial states.

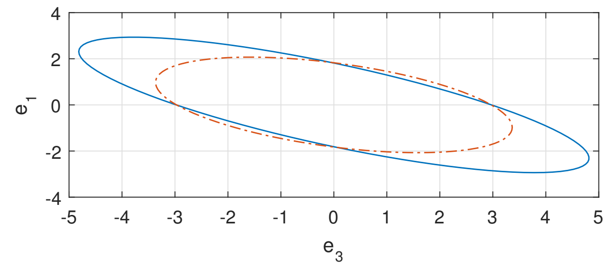
As possibilities of future work one can cite the use of a generalized sector condition to deal with the piecewise-linear nonlinearity and the design of dynamic output feedback controllers.



(a)



(b)



(c)

Fig. 5: Cuts of the region of guaranteed stability $\mathcal{Z}$: Theorem 1 (dash-dot line) and Theorem 2 (solid line).

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