2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Reachability and Controllability of Delayed Switched Boolean Control Networks

Amol Yerudkar, Carmen Del Vecchio, Navdeep Singh and Luigi Glielmo

Abstract— **In this paper we investigate the reachability and controllability of delayed switched Boolean control networks (DSBCNs). By resorting to the algebraic state space representation method built using semi-tensor product (STP) of matrices, we provide several necessary and sufficient conditions for these properties to hold which are based on input-state incidence matrix carrying entire network dynamics information. Also, to realize the reachability of DSBCNs in shortest time, an algorithm is presented which finds switching and control sequences forcing initial state trajectory to destination state. At last, an example is given to illustrate the main results.**

I. INTRODUCTION

Boolean network (BN) is a discrete-time finite dynamical system comprised of dynamical equations described by logical functions and whose variables are of Boolean type, usually labelled as "1" and "0". BN as a computational model for gene regulatory networks (GRNs) was firstly proposed by Kauffman [1]. Following the Kauffman's remarkable work, many scholars from various fields have proposed attractive results [2]–[4]. By adding suitable inputs and outputs we can manipulate BNs with the ease and develop control strategies. BNs with inputs and outputs are called Boolean control networks (BCNs).

Recently, a novel matrix product called semi-tensor product (STP) of matrices has been introduced for analysis and control of BNs and BCNs [5]. Using this approach, dynamics of BCN can be conveniently converted into a linear discrete-time system. Thus, one can study BCNs using classical control theory. Based on this algebraic representation many significant results have been presented for BCNs, covering several control theoretic problems which include controllability and observability [6]–[8], stability and stabilization [9]–[12] and many more. Detailed information can be found in some recent surveys [13], [14].

In control theory, switched systems being a special family of hybrid systems play an important role. In reality, many biological systems exhibit switching behaviour. For example, the cell's growth and division in eukaryotic cell commonly characterised by a progression of four processes, where each process is provoked by set of conditions and events [15], [16]. External and/or internal perturbations can also trigger switching behaviour in cellular systems. Such biological systems modelled using BNs are known as switched Boolean networks (SBNs). The research on SBNs has gained much attention from control community resulting in many impressive articles based on reachability, controllability and stabilization of SBNs [17], [18].

The modelling and analysis of GRNs often encounters a time delay due to slow biochemical reactions such as gene transcription and translation, and protein diffusion between the cytosol and nucleus [19]. In the last few decades BNs with time delay as an important model of GRNs, have attracted many scholars' interest from systems biology [19], and systems science [7]. BNs with time delay are called delayed Boolean networks (DBNs). Hence, it is clear that switching and time delay phenomenon are inherent in biological systems which makes it meaningful to study and analyse the behaviour of delayed switched systems. Therefore, it is essential to take both time delay and switching into consideration while modelling the GRNs [20]. A large number of results have been obtained on controllability and stability analysis of ordinary delayed switched systems [21], [22], since their study has attracted much attention. However, to the best of our knowledge there are no results available for the controllability or stability analysis of delayed switched Boolean control networks (DSBCNs).

Motivated by above discussion, we investigate the reachability and controllability of DSBCNs. Controllability being one of the fundamental concepts in control theory also plays an important role in systematic analysis of biological systems and GRNs. Although there exist extensive results on controllability of SBNs and DBNs [7], [23], [24] and references cited therein, the literature related to DSBCNs is very limited and there are many open problems. By resorting to the input-state incidence matrix approach based on STP for investigating the reachability and controllability of BNs [5], [23], we present analogous matrix conditions for analysing the reachability and controllability of DSBCNs.

The rest of this paper is organised as follows. In Section II, important preliminaries and notations related to STP are listed. Algebraic representation of DSBCNs and structure of input-state incidence matrix are discussed in Section III. Section IV investigates reachability and controllability of the DSBCNs. Section V gives an example for illustration and some conclusions are finally drawn in Section VI.

II. PRELIMINARIES

In this section, we list some necessary notations related to STP of matrices used in the subsequent sections.

- $\mathbb{R}$ and $\mathbb{Z}_+$ denote the set of real numbers and positive integers, respectively.

Yerudkar, Del Vecchio and Glielmo are with the Department of Engineering, University of Sannio, Benevento, Italy. {ayerudkar,c.delvecchio,glielmo}@unisannio.it

Singh is with the Department of Electrical Engineering, Veermata Jijabai Technological Institute, Mumbai, India. nmsingh@ee.vjti.ac.in

- $\mathscr{D} := \{0,1\}$, and $\mathscr{D}^n := \underbrace{\mathscr{D} \times \ldots \times \mathscr{D}}_{n}$. $1 \sim \delta_2^1$ and $0 \sim \delta_2^2$ is equivalent vector form for logical variables 1 and 0.
- δ_n^i : the i-th column of identity matrix I_n.
- $\Delta_n := \{\delta_n^i \mid i = 1,\ldots,n\}$, $\Delta := \Delta_2$.
- A matrix $L \in \mathbb{R}_{m\times n}$ is called a logical matrix if $\mathrm{Col}(L) \subset \Delta_m$ and $\mathscr{L}_{m\times n}$ is the set of all $m \times n$ logical matrices.
- Let $L \in \mathscr{L}_{n\times r}$. According to the definition, it is clear that $L = [\delta_n^{i_1}, \delta_n^{i_2}, \ldots, \delta_n^{i_r}]$. For the sake of compactness, it is shortly denoted as $L = \delta_n[i_1, i_2, \ldots, i_r]$.
- A $n \times m$ matrix $A = (a_{ij})$ is called a Boolean matrix, if $a_{ij} \in \mathscr{D}$, $\forall i = 1,\ldots,n,\ j = 1,\ldots,m$. Denote the set of $n \times m$ Boolean matrices by $\mathscr{B}_{n\times m}$.
- $\mathrm{Blk}_i(A)$ denotes the i-th $n \times n$ block of $n \times nm$ matrix A and $\mathrm{Col}_i(A)$ denotes the i-th column of the matrix A.
- $W_{[m,n]}$ represents the swap matrix with index $[m,n]$ defined by $W_{[m,n]} := [I_n \otimes \delta_m^1, I_n \otimes \delta_m^2, \ldots, I_n \otimes \delta_m^m]$, where $\otimes$ is the Kronecker product of matrices. When $m = n$, $W_{[m,n]}$ is denoted by $W_{[n]}$ or $W_{[m]}$.

Definition 2.1 ([5]): Let $A \in \mathbb{R}_{m\times n}$, $B \in \mathbb{R}_{p\times q}$. Denote the least common multiple of n and p by t =lcm$\{n, p\}$. Then the STP of A and B is defined as

$$A \ltimes B := (A \otimes I_{t/n})(B \otimes I_{t/p}).$$

Remark 1: If $n = p$, it is clear that above definition is degenerated to the conventional matrix product. Hence, the STP is a generalization of conventional matrix product. Throughout this paper we simply call it "STP" and omit the symbol "$\ltimes$" if no confusion is raised.

Following lemma provides the fundamental result for matrix expression of logical functions, where the logical variable is identified as a vector in Δ.

Lemma 2.1 ([5]): Let $f : \mathscr{D}^n \longrightarrow \mathscr{D}$ be a logical function and $y = f(x_1,\ldots,x_n)$. Then there exists a unique logical matrix $M_f \in \mathscr{L}_{2\times 2^n}$, called the structure matrix of f, such that when the logical variables x_i, $i = 1,2,...,n$, y are expressed into their vector forms we have

$$y = M_f \ltimes_{i=1}^n x_i. \tag{1}$$

III. DELAYED SWITCHED BOOLEAN CONTROL NETWORK

The dynamics of the delayed switched Boolean control network with n nodes, m control inputs and p sub-networks can be described as follows:

$$\begin{cases} x_1(t+1) = f_1^{\sigma(t)}(U(t), X(t-\mu+1), \ldots, X(t)), \\ x_2(t+1) = f_2^{\sigma(t)}(U(t), X(t-\mu+1), \ldots, X(t)), \\ \vdots \\ x_n(t+1) = f_n^{\sigma(t)}(U(t), X(t-\mu+1), \ldots, X(t)), \end{cases} \tag{2}$$

where $\mu \in \mathbb{Z}_+$ denotes the time delay, $\sigma : \mathbb{Z}_+ \to P = \{1,2,\ldots,p\}$ is the switching signal, $X(i) := (x_1(i), x_2(i), \ldots, x_n(i)) \in \mathscr{D}^n$, $i = t-\mu+1,\ldots,t$ the state at time i, $U(t) := (u_1(t), u_2(t), \ldots, u_m(t)) \in \mathscr{D}^m$ the control input at time t and $f_i^j : \mathscr{D}^{\mu n+m} \to \mathscr{D}$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,p$ are logical functions. Here, along with switching we consider the higher order delay phenomenon where the present state of the logical network is updated based on its past μ values. For the case of simplicity we assume equal time delay in the states as in [7] and references therein.

Given a finite time switching signal $\sigma : \{0,1,\ldots,l\} \to P$ with l a given positive integer, set $\sigma(k) = i_k$, $k = 0,1,\ldots,l$. Then we obtain the switching sequence as $\pi := \{(0,i_0),(1,i_1),\ldots,(l,i_l)\}$ [23].

In the following we convert the DSBCN (2) into an algebraic form by using the STP, and then define the input-state incidence matrix for DSBCN.

Using the canonical vector form of logical variables and letting $x(t) = \ltimes_{i=1}^n x_i(t) \in \Delta_{2^n}$, $z(t) = \ltimes_{i=t-\mu+1}^t x(i) \in \Delta_{2^{\mu n}}$ and $u(t) = \ltimes_{i=1}^m u_i(t) \in \Delta_{2^m}$, by Lemma 2.1, we get the following component-wise algebraic form of DSBCN (2),

$$\begin{cases} x_1(t+1) = Q_1^{\sigma(t)} u(t) z(t), \\ x_2(t+1) = Q_2^{\sigma(t)} u(t) z(t), \\ \vdots \\ x_n(t+1) = Q_n^{\sigma(t)} u(t) z(t), \end{cases} \tag{3}$$

where $Q_i^{\sigma(t)} \in \mathscr{L}_{2\times 2^{\mu n+m}}$ is the structure matrix of $f_i^{\sigma(t)}$. Using the Khatri-Rao product, one can obtain the following algebraic form of (2):

$$x(t+1) = Q_{\sigma(t)} u(t) z(t), \tag{4}$$

where $Q_{\sigma(t)} = Q_1 * Q_2 * \ldots * Q_n \in \mathscr{L}_{2^n\times 2^{\mu n+m}}$ and "$*$" is Khatri-Rao product.

Hence, using the basic properties of STP we obtain that

$$z(t+1) := L_{\sigma(t)} u(t) z(t), \tag{5}$$

where $L_{\sigma(t)} = \left(I_{2^{(\mu-1)n}} \otimes Q_{\sigma(t)}\right) \ltimes W_{[2^{n+m}, 2^{(\mu-1)n}]} \ltimes [I_{2^{n+m}} \otimes \phi_{(\mu-1)n}]$, $L_{\sigma(t)} \in \mathscr{L}_{2^{\mu n}\times 2^{\mu n+m}}$ and ϕ_n is called the power reducing matrix for 2^n-valued logical vectors and $\phi_n = \delta_{2^{2n}}[1, 2^n+2, 2.2^n+3, \ldots, (2^n-2).2^n+2^n-1, 2^{2n}]$ [7]. According to [5] it is evident that, the system (5) with switching signal $\sigma(t)$ is equivalent to the DSBCN in (2).

First, we define the input-state product space for (2):

Definition 3.1: Consider the DSBCN (2) with integer time delay μ and switching signal $\sigma(t)$. The input-state product space for the system is defined as $\mathfrak{S} = \{\sigma,\ U,\ Z \mid \sigma \in P,\ U = (u_1, u_2, \ldots, u_m) \in \mathscr{D}^m,\ Z = (X(t-\mu+1), \ldots, X(t)) \in \mathscr{D}^{\mu n}\}$.

1. Let $P_i = (\sigma^i,\ U^i,\ Z^i) \in \mathfrak{S}$ and $P_j = (\sigma^j,\ U^j,\ Z^j) \in \mathfrak{S}$, where $U^i = (u_1^i, u_2^i, \ldots, u_m^i)$, $Z^i = (X^i(t-\mu+1), \ldots, X^i(t))$ and $X = (x_1, x_2, \ldots, x_n)$. (P_i, P_j) is said to be a directed edge, if Z^i, U^i and Z^j satisfy (2), that is, $x_k^j = f_k^{\sigma^i}(u_1^i, \ldots, u_m^i, X^i(t-\mu+1), \ldots, X^i(t))$, $k = 1,\ldots,n$. The set of edges is denoted by $\Psi \subset \mathfrak{S} \times \mathfrak{S}$.
2. The pair $(\mathfrak{S}, \Psi)$ forms a directed graph, which is called the input-state transfer graph.

Now, we define the input-state incidence matrix for DSBCN similar to [23]. Identifying the delay μ and switching signal $\sigma = l \sim \delta_p^l \in \Delta_p$, the input-state product space for DSBCN becomes $\Delta_p \times \Delta_{2^m} \times \Delta_{2^{\mu n}}$ and a state in this space can be expressed as $P_i = \delta_p^{i_1} \times \delta_{2^m}^{i_2} \times \delta_{2^{\mu n}}^{i_3}$. These states can be arranged according to the definition of ordered multi-index $Id(i_1, i_2, i_3; p, 2^m, 2^{\mu n})$ given in [5] as follows. Let i_1, i_2, and i_3 run from 1 to p, 2^m and $2^{\mu n}$, respectively, where i_3 runs first, i_2 second and i_1 last.

$$P_1 = \delta_p^1 \times \delta_{2^m}^1 \times \delta_{2^{\mu n}}^1, \dots, P_{2^{\mu n}} = \delta_p^1 \times \delta_{2^m}^1 \times \delta_{2^{\mu n}}^{2^{\mu n}}, \\ \dots, P_{p2^{\mu n+m}} = \delta_p^p \times \delta_{2^m}^{2^m} \times \delta_{2^{\mu n}}^{2^{\mu n}}. \quad (6)$$

Definition 3.2: The input-state incidence matrix $\mathscr{I}$ of the DSBCN (2) is a $p2^{\mu n+m} \times p2^{\mu n+m}$ matrix, which is defined as

$$\mathscr{I}_{ij} = \begin{cases} 1, & \text{if } (P_j, P_i) \in \mathfrak{S}, \\ 0, & \text{otherwise}, \end{cases} \quad (7)$$

where P_i and P_j are any two states defined as per the ordered multi-index (6), $\mathfrak{S}$ is the same as in Definition 3.1 and $\mathscr{I}_{ij}$ denotes the (i,j)-th element of the $\mathscr{I}$.

We have following property about $\mathscr{I}$.

Proposition 3.1: Consider the DSBCN (2) with its algebraic form (5). The input-state incidence matrix of the system (2) can be given as

$$\mathscr{I} := \left.\begin{bmatrix} \tilde{L} \\ \tilde{L} \\ \vdots \\ \tilde{L} \end{bmatrix}\right\} p2^m \in \mathscr{B}_{p2^{\mu n+m} \times p2^{\mu n+m}}, \quad (8)$$

where $\tilde{L} = [L_1 L_1 \dots L_p] \in \mathscr{L}_{2^{\mu n} \times p2^{\mu n+m}}$.

Proof: For any two states $P_i, P_j \in \mathfrak{S}$, suppose that $P_i = \delta_p^{i_1} \times \delta_{2^m}^{i_2} \times \delta_{2^{\mu n}}^{i_3}$ and $P_j = \delta_p^{j_1} \times \delta_{2^m}^{j_2} \times \delta_{2^{\mu n}}^{j_3}$. By Definition 3.1, (P_i, P_j) is an edge if and only if $x(t+1) = \delta_{2^{\mu n}}^{i_3}$ is reachable from $x(t) = \delta_{2^{\mu n}}^{j_3}$ under $\sigma(t) = \delta_p^{j_1}$ and control $u(t) = \delta_{2^m}^{j_2}$, that is $\delta_{2^{\mu n}}^{i_3} = L_{j_1} \ltimes \delta_{2^m}^{j_2} \ltimes \delta_{2^{\mu n}}^{j_3} = \mathrm{Col}_{j_3}(\mathrm{Blk}_{j_2}(L_{j_1}))$.

Since i_1 and i_2 are independent of P_j, without loss of generality, we first let $i_1 = i_2 = 1$. Then, it can be seen clearly from (6) that, $i = i_3$. Setting $\mathrm{Col}_{j_3}(\mathrm{Blk}_{j_2}(L_{j_1})) = \delta_{2^{\mu n}}^{\zeta}$ we get,

$$\begin{bmatrix} \mathscr{I}_{1j} \\ \mathscr{I}_{2j} \\ \vdots \\ \mathscr{I}_{2^{\mu n} j} \end{bmatrix} = \delta_{2^{\mu n}}^{\zeta} = \mathrm{Col}_{j_3}(\mathrm{Blk}_{j_2}(L_{j_1})) = \delta_{2^{\mu n}}^{\zeta}.$$

For P_j, $1 \le j \le p2^{\mu m+n}$, one can obtain from the order given in (6) that $j = (j_1 - 1)2^{\mu n+m} + (j_2 - 1)2^{\mu n} + j_3$ and $\mathrm{Col}_{j_3}(\mathrm{Blk}_{j_2}(L_{j_1})) = \delta_{2^{\mu n}}^{\zeta} = \mathrm{Col}_{(j_2-1)2^{\mu n}+j_3}(L_{j_1})$. Then, for a fixed integer j_1, we have

$$\begin{bmatrix} \mathscr{I}_{1,(j_1-1)2^{\mu n+m}+1} & \cdots & \mathscr{I}_{1,(j_1-1)2^{\mu n+m}+2^{\mu n+m}} \\ \mathscr{I}_{2,(j_1-1)2^{\mu n+m}+1} & \cdots & \mathscr{I}_{2,(j_1-1)2^{\mu n+m}+2^{\mu n+m}} \\ \vdots & \vdots & \vdots \\ \mathscr{I}_{2^{\mu n},(j_1-1)2^{\mu n+m}+1} & \cdots & \mathscr{I}_{2^{\mu n},(j_1-1)2^{\mu n+m}+2^{\mu n+m}} \end{bmatrix} \\ = [\mathrm{Col}_1(L_{j_1}) \ \dots \ \mathrm{Col}_{2^{\mu n+m}}(L_{j_1})] = L_{j_1}.$$

Letting $j_1 = 1, 2, \dots, p$, respectively, we obtain

$$\begin{bmatrix} \mathrm{Row}_1(\mathscr{I}) \\ \mathrm{Row}_2(\mathscr{I}) \\ \vdots \\ \mathrm{Row}_{2^{\mu n}}(\mathscr{I}) \end{bmatrix} = [L_1 \ \dots \ L_p] = \tilde{L}.$$

Rest of the (block) rows of $\mathscr{I}$ can be obtained in the same way by letting $i_1 = 1, 2, \dots, p$ and $i_2 = 2, \dots, 2^m$. Each block is equal to $\tilde{L}$. This concludes that (8) is true. ■

A matrix $A \in \mathbb{R}_{m \times n}$ is called a row-periodic matrix with period τ if, τ is a proper factor of m such that, $\mathrm{Row}_{i+\tau}(A) = \mathrm{Row}_i(A), 1 \le i \le m - \tau$. Equivalently, $A \in \mathbb{R}_{m \times n}$ is a row-periodic matrix with period τ (where $m = \tau k$) if and only if $A = \mathbf{1}_k A_0$, where $A_0 \in \mathbb{R}_{\tau \times n}$ called the basic block of A and $\mathbf{1}_k = \left[\underbrace{1\ 1\ \dots\ 1}_{k}\right]^{\mathrm{T}}$. Moreover, if $A \in \mathbb{R}_{m \times n}$ is a row-periodic matrix with period τ, then so is A^s, $s \in \mathbb{Z}_+$. Denote the basic block of A^s by A_0^s, then $A_0^{s+1} = A_0 A^s = A_0 \mathbf{1}_k A_0^s = \sum_{i=1}^k \mathrm{Blk}_i(A_0) A_0^s$. Detailed description of row-periodic matrix and its properties are listed in [5].

We can divide $\tilde{L} = [L_1 L_1 \ \dots \ L_p] \in \mathscr{L}_{2^{\mu n} \times p2^{\mu n+m}}$ into $p2^m$ blocks as $\tilde{L} = \left[\mathrm{Blk}_1(\tilde{L}) \ \dots \ \mathrm{Blk}_{p2^m}(\tilde{L})\right]$, where $\mathrm{Blk}_i(\tilde{L}) \in \mathscr{L}_{2^{\mu n} \times 2^{\mu n}}$, $i = 1, 2, \dots, p2^m$. Based on the definition of row-periodic matrix, following property is straightforward for the input-state incidence matrix.

Proposition 3.2: Input-state incidence matrix $\mathscr{I}$ presented in (8) is a row-periodic matrix with period $2^{\mu n}$ and $\mathscr{I} = \mathbf{1}_{p2^m} \tilde{L}$. Furthermore, $\mathscr{I}^l$, $l \in \mathbb{Z}_+$ is also a row-periodic matrix with period $2^{\mu n}$, and its basic block is

$$\mathscr{I}_0^l = \tilde{M}^{l-1} \tilde{L} \in \mathbb{R}^{2^{\mu n} \times p2^{\mu n+m}}, \quad (9)$$

where $\tilde{M} = \sum_{i=1}^{p2^m} \mathrm{Blk}_i(\tilde{L})$, and $\tilde{L} = [L_1 L_1 \ \dots \ L_p]$.

According to the definition of $\mathscr{I}$, it is clear that $\mathscr{I}_{ij}$ implies the connectivity between P_i and P_j; i.e. when $\mathscr{I}_{ij} = 1$, P_i is reachable from P_j in one step. On similar lines, following theorem provides the condition for P_i, P_j connectivity at the l-th step.

Theorem 3.3: Consider the DSBCN (2) and assume that the (i,j)-th element of the l-th power of its input-state incidence matrix, $(\mathscr{I}^l)_{ij}$, is equal to c, $l \in \mathbb{Z}_+$. Then, with P_i and P_j defined as per the ordered multi-index set, there are c paths such that P_i is reachable from P_j at the l-th step.

Proof: We prove this by mathematical induction. When $l = 1$, the conclusion follows from the Definition 3.2. Now, assume that $(\mathscr{I}^l)_{ij}$ is the number of paths from the state P_j to P_i at the l-th step. Next, a path from P_j to P_i at the $(l+1)$-th step can be decomposed into a path from P_j to P_k at the l-th step and then a path from P_k to P_i at one step. We know that P_k has $p2^{\mu n+m}$ choices, then the number of paths from P_j to P_i at the $(l+1)$-th step is given as

$$\sum_{k=1}^{p2^{\mu n+m}} \mathscr{I}_{ik} \left(\mathscr{I}^l\right)_{kj} = \left(\mathscr{I} \mathscr{I}^l\right)_{ij} = \left(\mathscr{I}^{l+1}\right)_{ij} = c, \quad (10)$$

which means that the conclusion is true for $l + 1$. By mathematical induction, the conclusion holds. ■

Corollary 1: Consider the system (2). P_i is reachable from P_j at the l-th step if and only if $(\mathscr{I}^l)_{ij} > 0$.

Remark 2: Theorem 3.3 and Corollary 1 illustrate that the entire reachability information for the DSBCN (2) is contained in $\{\mathscr{I}^l \mid l \in \mathbb{Z}_+\}$. By the Cayley-Hamilton theorem [25], it is easy to see that if $(\mathscr{I}^l)_{ij} = 0$, $\forall l \le$

$p2^{\mu n+m}$, then $(\mathscr{I}^l)_{ij}=0$, $\forall l\in\mathbb{Z}_+$. Thus, we consider only $\{\mathscr{I}^l \mid l\le p2^{\mu n+m}\}$.

IV. REACHABILITY AND CONTROLLABILITY

Definition 4.1: Consider the DSBCN (2). For any initial trajectory $r=\ltimes_{i=1-\mu}^{0}x(i)\in\Delta_{2^{\mu n}}$ and any given destination state $x_d\in\Delta_{2^n}$,

1) x_d is reachable from $r=\ltimes_{i=1-\mu}^{0}x(i)\in\Delta_{2^{\mu n}}$ at the l-th step ($l>0$) if we can find a switching signal $\sigma:\{0,\dots,l-1\}\to P$ and a sequence of controls $U(0):=\{u_1(0),\dots,u_m(0)\},\dots,U(l-1):=\{u_1(l-1),\dots,u_m(l-1)\}$, such that under the switching signal σ and controls $\{U(t),\ t=0,1,\dots,l-1\}$ $r=\ltimes_{i=1-\mu}^{0}x(i)\in\Delta_{2^{\mu n}}$ can be driven to the destination state x_d at time l;
2) the reachable set of $r=\ltimes_{i=1-\mu}^{0}x(i)\in\Delta_{2^{\mu n}}$ at time l is denoted by $R_l\left(\ltimes_{i=1-\mu}^{0}x(i)\right)$; The reachable set of $r=\ltimes_{i=1-\mu}^{0}x(i)\in\Delta_{2^{\mu n}}$ is denoted by $R\left(\ltimes_{i=1-\mu}^{0}x(i)\right)$ and it is obvious that $R\left(\ltimes_{i=1-\mu}^{0}x(i)\right)=\bigcup_{l=1}^{\infty}R_l\left(\ltimes_{i=1-\mu}^{0}x(i)\right)$;
3) the system is said to be controllable at $\ltimes_{i=1-\mu}^{0}x(i)$ if $R\left(\ltimes_{i=1-\mu}^{0}x(i)\right)=\Delta_{2^n}$. The system is said to be controllable, if it is controllable at all initial trajectories.

At this point, before continuing the analysis of reachability and controllability we present the bijective mapping between the state and the trajectory of the system (2).

By defining $x(t)=\ltimes_{i=1}^{n}x_i(t)$ and $z(t)=\ltimes_{i=t-\mu+1}^{t}x(i)$, we get a bijective mapping $\ltimes_{i=t-\mu+1}^{t}:\Delta_{2^n}\to\Delta_{2^{\mu n}}$. We call $x(t)$ the state of the DSBCN (2) and $(x(t-\mu+1),x(t-\mu+2),\dots,x(t))$ the trajectory of length μ. Thus, bijection between the state of the BCN and the trajectory can be derived [5]. Here we investigate the relationship between the state $x(t+k-\mu)=\delta_{2^n}^{\alpha}$ and the trajectory $z(t)$. Since $z(t)=x(t+1-\mu)x(t+2-\mu)\dots x(t)$, let $\alpha\in\{1,2,\dots 2^n\}$, then with $k=1$ and $x(t+1-\mu)=\delta_{2^n}^{\alpha}$, we get

$$z(t)=\delta_{2^n}^{\alpha}\delta_{2^{(\mu-1)n}}^{j}=\delta_{2^{\mu n}}^{(\alpha-1)2^{(\mu-1)n}+j},\ j=1,2,\dots,2^{(\mu-1)n}. \tag{11}$$

Thus, we have $x(t+1-\mu)=\delta_{2^n}^{\alpha}$ if and only if,

$$z(t)\in\Xi_1^{\alpha}=\left\{\delta_{2^{\mu n}}^{(\alpha-1)2^{(\mu-1)n}+j},\ j=1,2,\dots,2^{(\mu-1)n}\right\}. \tag{12}$$

More details on bijective mapping can be found in [5], [7].

In accordance with the above definition, now we explore the reachability and controllability of the DSBCN (2) based on the physical meaning of the input-state incidence matrix. We have the following results for the reachability and controllability of the system (2).

Theorem 4.1: Consider the system (2) with its input-state incidence matrix $\mathscr{I}$. Then,

1) $x(l+1-\mu)=\delta_{2^n}^{\alpha}$ is reachable from the initial state sequence $\ltimes_{i=1-\mu}^{0}x(i)=\delta_{2^{\mu n}}^{\beta}$ at the l-th step, if and only if there exists $\gamma=(\alpha-1)2^{(\mu-1)n}+j$ such that, $\delta_{2^n}^{\alpha}\delta_{2^{(\mu-1)n}}^{j}=\delta_{2^{\mu n}}^{\gamma}$, where $j=1,2,\dots,2^{(\mu-1)n}$, and
$$\sum_{i=1}^{p2^m}\left(\mathrm{Blk}_i(\mathscr{I}_0^l)\right)_{\gamma\beta}=\left(\tilde{M}^l\right)_{\gamma\beta}>0, \tag{13}$$
where $\mathscr{I}_0^l=\tilde{M}^{l-1}L$, $\tilde{M}=\sum_{i=1}^{p2^m}\mathrm{Blk}_i(\tilde{L})$ and $\tilde{L}=[L_1L_1\ \dots\ L_p]$;
2) $x=\delta_{2^n}^{\alpha}$ is reachable from the initial state sequence $\ltimes_{i=1-\mu}^{0}x(i)=\delta_{2^{\mu n}}^{\beta}$, if and only if there exists $\gamma=(\alpha-1)2^{(\mu-1)n}+j$ such that, $\delta_{2^n}^{\alpha}\delta_{2^{(\mu-1)n}}^{j}=\delta_{2^{\mu n}}^{\gamma}$ where $j=1,2,\dots,2^{(\mu-1)n}$, and
$$\sum_{l=1}^{p2^{\mu n+m}}\left(\tilde{M}^l\right)_{\gamma\beta}>0; \tag{14}$$
3) The system is controllable at $\ltimes_{i=1-\mu}^{0}x(i)=\delta_{2^{\mu n}}^{\beta}$ if and only if
$$\sum_{l=1}^{p2^{\mu n+m}}\mathrm{Col}_{\beta}\left(\tilde{M}^l\right)>0; \tag{15}$$
4) The system is controllable if and only if the controllability matrix, $\mathscr{C}$, satisfies
$$\mathscr{C}=\sum_{l=1}^{p2^{\mu n+m}}\left(\tilde{M}^l\right)>0. \tag{16}$$

Proof: By Definition 4.1, the destination state $x_d=x(l-\mu+1)=\delta_{2^n}^{\alpha}$ is reachable from the initial trajectory $\ltimes_{i=1-\mu}^{0}x(i)=\delta_{2^{\mu n}}^{\beta}$, if and only if one can find a switching sequence $\{\sigma(0)=\delta_p^{\beta_1},\dots,\sigma(l-1)\}$ and a control sequence $\{u(0)=\delta_{2^m}^{\beta_2},\dots,u(l-1)\}$, such that the trajectory of the system (2) starting from $\ltimes_{i=1-\mu}^{0}x(i)$ reaches $x(l-\mu+1)$ at time l. Since $\sigma(l)$ and $u(l)$ are arbitrary, without loss of generality, we set $\sigma(l)=\delta_p^1$ and $u(l)=\delta_{2^m}^1$. Then, we get two states in the input-state space as $P_{\beta_0}=\delta_p^{\beta_1}\times\delta_{2^m}^{\beta_2}\times\delta_{2^{\mu n}}^{\beta}$ and $P_\gamma=\delta_p^1\times\delta_{2^m}^1\times\delta_{2^{\mu n}}^{\gamma}$, where $\beta_0=(\beta_1-1)2^{\mu n+m}+(\beta_2-1)2^{\mu n}+\beta$. Here, the destination state is $x_d\in\Delta_{2^n}$. Referring to the above discussion on bijection between the state and the trajectory of length μ we get $\gamma=(\alpha-1)2^{(\mu-1)n}+j$, $j=1,2,\dots,2^{(\mu-1)n}$. Using this relation one can calculate γ to determine the reachability of the system (2). In this manner, the reachability of the DSBCN (2) is converted to the reachability of the input-state transfer graph.

It is clear from Corollary 1 that P_γ is reachable from P_{β_0} at the l-th step, if and only if $(\mathscr{I}^l)_{\gamma\beta_0}>0$. Since $(\mathscr{I}^l)$ is a row-periodic matrix, setting $\beta_3=(\beta_1-1)2^m+\beta_2$ and considering the basic block of $(\mathscr{I}^l)$ as defined in (9) we get $(\mathscr{I}_0^l)_{\gamma\beta_0}=[\mathrm{Blk}_{\beta_3}(\mathscr{I}_0^l)]_{\gamma\beta}$. Now, with $1\le\beta_3\le p2^m$ we know that $x(l-\mu+1)$ is reachable from $\ltimes_{i=1-\mu}^{0}x(i)$ if and only if $\sum_{\beta_3=1}^{p2^m}[\mathrm{Blk}_{\beta_3}(\mathscr{I}_0^l)]_{\gamma\beta}>0$. By (9) and $\tilde{M}=\sum_{i=1}^{p2^m}\mathrm{Blk}_i(\tilde{L})$, one can obtain

$$\sum_{\beta_3=1}^{p2^m}\mathrm{Blk}_{\beta_3}(\mathscr{I}_0^l)=\tilde{M}^{l-1}\sum_{\beta_3=1}^{p2^m}\mathrm{Blk}_{\beta_3}(\tilde{L})=\tilde{M}^l, \tag{17}$$

where $\tilde{L}=[L_1L_1\ \dots\ L_p]$. Therefore $x(l-\mu+1)=\delta_{2^n}^{\alpha}$ is reachable from $\ltimes_{i=1-\mu}^{0}x(i)=\delta_{2^{\mu n}}^{\beta}$, if and only if we can find a γ such that, $\left(\tilde{M}^l\right)_{\gamma\beta} > 0$. From the above analysis, it is clear that 1) is true. Next, we prove $2)-4)$ is true.

The destination state $x_d = \delta_{2^n}^{\alpha}$ is reachable from initial state sequence $\ltimes_{i=1-\mu}^{0} x(i) = \delta_{2^{\mu n}}^{\beta}$, if and only if there exists $\gamma = (\alpha-1)2^{(\mu-1)n} + j$ and an integer $l \in \mathbb{Z}_+$, such that (13) holds. From Remark 2, we just consider $l \le p2^{\mu n+m}$. Thus, (14) is true if and only if there exists an integer $l \le p2^{\mu n+m}$, such that (13) holds, we conclude that 2) is true.

The system is controllable at the initial state sequence $\ltimes_{i=1-\mu}^{0} x(i) = \delta_{2^{\mu n}}^{\beta}$, if and only if any state $x = \delta_{2^n}^{\alpha}$, $1 \le \alpha \le 2^n$ i.e. equivalently $\delta_{2^{\mu n}}^{\gamma}$, $1 \le \gamma \le 2^{\mu n}$ is reachable from $\ltimes_{i=1-\mu}^{0} x(i) = \delta_{2^{\mu n}}^{\beta}$. Thus, from 2), $\sum_{l=1}^{p2^{\mu n+m}} (\tilde{M}^l)_{\gamma\beta} > 0$, $\forall 1 \le \gamma \le 2^{\mu n}$, that is $\sum_{l=1}^{p2^{\mu n+m}} \mathrm{Col}_{\beta}(\tilde{M}^l) > 0$, which implies that 3) holds.

The system is said to be controllable if it is controllable at any initial state trajectory $\ltimes_{i=1-\mu}^{0} x(i) = \delta_{2^{\mu n}}^{\beta}$; we have $\sum_{l=1}^{p2^{\mu n+m}} \mathrm{Col}_{\beta}(\tilde{M}^l) > 0$, $\forall 1 \le \gamma \le 2^{\mu n}$, which implies that 4) is true. ■

When controllability of DSBCN is considered using the input-state incidence matrix approach, the only concern is presence of the connection from one state to another. Hence, it is enough to show whether or not $\sum_{l=1}^{p2^{\mu n+m}} (\tilde{M}^l) > 0$. Since, the real value of each entry of $\mathscr{I}_0^l$ is of less interest, we can simply use the Boolean algebra in the above calculations.

Finally, we investigate how to design a sequence of controls and proper switching sequence to realise the reachability of the system (2), and establish an algorithm to deal with this problem.

Proposition 4.2 ([23]): For any integer $1 \le i \le p2^m$ there exist unique positive integers i_1 and i_2 such that

$$\delta_{p2^m}^{i} = \delta_p^{i_1} \ltimes \delta_{2^m}^{i_2}, \tag{18}$$

where

$$i_1 = \begin{cases} k, & \text{if } i = k2^m,\ k = 1,2,\ldots,p; \\ \left[\dfrac{i}{2^m}\right] + 1, & \text{otherwise}, \end{cases} \tag{19}$$

$\left[\frac{i}{2^m}\right]$ denotes the largest integer less than or equal to $\frac{i}{2^m}$ and $i_2 = i - (i_1 - 1)2^m$.

Algorithm 1: Consider the DSBCN (2). Assume that, due to bijective mapping $x_d = \delta_{2^{\mu n}}^{\gamma} \in R\left(\ltimes_{i=1-\mu}^{0} x(i)\right)$ where $\ltimes_{i=1-\mu}^{0} x(i) = \delta_{2^{\mu n}}^{\beta}$. Then the switching sequence and controls which force $\ltimes_{i=1-\mu}^{0} x(i)$ to x_d in the shortest time l can be designed by the following steps:

- Step 1: Find the smallest integer l such that for $\mathscr{I}_0^l = [\mathrm{Blk}_1(\mathscr{I}_0^l)\mathrm{Blk}_2(\mathscr{I}_0^l)\ldots\mathrm{Blk}_{p2^m}(\mathscr{I}_0^l)]$, there exists a block, say, $\mathrm{Blk}_{\mathfrak{a}}(\mathscr{I}_0^l)$ satisfying $[\mathrm{Blk}_{\mathfrak{a}}(\mathscr{I}_0^l)]_{\gamma\beta} > 0$.
- Step 2: Calculate $\mathfrak{a}_1$ and $\mathfrak{a}_2$ by Proposition 4.2 such that $\delta_{p2^m}^{\mathfrak{a}} = \delta_p^{\mathfrak{a}_1} \ltimes \delta_{2^m}^{\mathfrak{a}_2}$. Set $\sigma(0) = \delta_p^{\mathfrak{a}_1}$, $u(0) = \delta_{2^m}^{\mathfrak{a}_2}$ and $x(l) = \delta_{2^{\mu n}}^{\gamma}$. Stop if $l = 1$, otherwise go to step 3.
- Step 3: Find k and $\mathfrak{b}$ such that $[\mathrm{Blk}_{\mathfrak{b}}(\mathscr{I}_0^l)]_{\gamma k} > 0$ and $[\mathrm{Blk}_{\mathfrak{a}}(\mathscr{I}_0^{l-1})]_{k\beta} > 0$. Calculate $\mathfrak{b}_1$ and $\mathfrak{b}_2$ by Proposition 4.2 such that, $\delta_{p2^m}^{\mathfrak{b}} = \delta_p^{\mathfrak{b}_1} \ltimes \delta_{2^m}^{\mathfrak{b}_2}$. Set $\sigma(l-1) = \delta_p^{\mathfrak{b}_1}$, $u(l-1) = \delta_{2^m}^{\mathfrak{b}_2}$ and $x(l-1) = \delta_{2^{\mu n}}^{k}$.
- Step 4: If $l-1 = 1$, stop. Otherwise replace l and γ by $l-1$ and k, respectively, and go back to step 3.

Proposition 4.3: As long as $x_d \in R\left(\ltimes_{i=1-\mu}^{0} x(i)\right)$, the control sequence $\{u(0),\ldots,u(l-1)\}$ and switching sequence $\{(0,\sigma(0)),\ldots,(l-1,\sigma(l-1))\}$ generated by Algorithm 1 can drive the initial state trajectory to desired state in shortest time.

Proof: Consider an algebraic form (5) of DSBCN (2). Assume $x_d = \delta_{2^n}^{\alpha}$ and using bijective relation we have $x_d = \delta_{2^{\mu n}}^{\gamma}$, initial state sequence $\ltimes_{i=1-\mu}^{0} x(i) = \delta_{2^{\mu n}}^{\beta}$ and $x_d \in R\left(\ltimes_{i=1-\mu}^{0} x(i)\right)$. Since the trajectory driven by proper sequence of controls and switching sequence from initial state sequence to x_d is generally not unique, we only try to find the shortest path. By the construction of controllability matrix $\mathscr{C}$ given in the Theorem 4.1, there exists a smallest l such that $\left[\mathrm{Blk}_{\mathfrak{a}}(\mathscr{I}_0^l)\right]_{\gamma\beta} > 0$. Which implies that, for arbitrary $\sigma(l) \in \Delta_p$ and $u(l) \in \Delta_{2^m}$ the state $\sigma(l) \times u(l) \times \delta_{2^{\mu n}}^{\gamma}$ is reachable from $\sigma(0) \times u(0) \times \delta_{2^{\mu n}}^{\beta}$ at the l-th step, where $\sigma(0) \times u(0) = \delta_{p2^m}^{a}$. One can find unique integers $\mathfrak{a}_1$ and $\mathfrak{a}_2$ using Preposition 4.2, such that $\delta_{p2^m}^{\mathfrak{a}} = \delta_p^{\mathfrak{a}_1} \ltimes \delta_{2^m}^{\mathfrak{a}_2}$. We then know that, $\ltimes_{i=1-\mu}^{0} x(i) = \delta_{2^{\mu n}}^{\beta}$ can reach $x_d = \delta_{2^{\mu n}}^{\gamma}$ at the l-th step if $\sigma(0) = \delta_p^{\mathfrak{a}_1}$ and $u(0) = \delta_{2^m}^{\mathfrak{a}_2}$.

Now, according to the Theorem 3.3, a path from P_j to P_i at the l-th step can be decomposed into a path from P_j to P_v at the $(l-1)$-th step and a path from P_v to P_i at one step. A state $P_v = \delta_p^{\mathfrak{b}_1} \times \delta_{2^m}^{\mathfrak{b}_2} \times \delta_{2^{\mu n}}^{k}$ can be found with $\delta_{p2^m}^{\mathfrak{b}} = \delta_p^{\mathfrak{b}_1} \ltimes \delta_{2^m}^{\mathfrak{b}_2}$, such that $x(l-1) = \delta_{2^{\mu n}}^{k}$ is reachable from initial state sequence $\ltimes_{i=1-\mu}^{0} x(i)$ at the $(l-1)$-th step under the switching $\sigma(0) = \delta_p^{\mathfrak{a}_1}$ and input $u(0) = \delta_{2^m}^{\mathfrak{a}_2}$. Further, x_d is reachable from $x(l-1)$ at one step under $\sigma(l-1) = \delta_p^{\mathfrak{b}_1}$ and $u(l-1) = \delta_{2^m}^{\mathfrak{b}_2}$. Equivalently, one can find k and $\mathfrak{b}$ such that $[\mathrm{Blk}_{\mathfrak{b}}(\mathscr{I}_0)]_{\gamma k} > 0$ and $[\mathrm{Blk}_{\mathfrak{a}}(\mathscr{I}_0^{l-1})]_{k\beta} > 0$. Repeating this procedure we can obtain switching and control sequence which force initial state sequence to destination state in the shortest time. ■

Remark 3: In this paper, obtaining algebraic state representation for the delayed switched BCN is similar to the augmentation in the state space. Presence of delay augments the network transition matrix but does not affect the topological structures of the system after converting the system to (5). Moreover, presence of controlled switches makes it more convenient to design the switching path.

V. NUMERICAL EXAMPLE

In this section, we give an illustrative example to study reachability and controllability of the DSBCNs.

Consider the following DSBCN,

$$\begin{cases} x_1(t+1) = f_1^{\sigma(t)}(U(t),\ X(t-\mu+1),\ X(t)), \\ x_2(t+1) = f_2^{\sigma(t)}(U(t),\ X(t-\mu+1),\ X(t)), \end{cases} \tag{20}$$

where $\sigma : \mathbb{Z}_+ \to P = \{1,2\}$ is the switching signal and $\mu = 2$. Moreover,

$$\sigma = 1 \begin{cases} x_1(t+1) = u(t) \wedge x_1(t-1) \wedge x_2(t) \\ x_2(t+1) = u(t) \to x_1(t-1) \wedge x_2(t), \end{cases} \tag{21}$$

$$\sigma = 2 \begin{cases} x_1(t+1) = u(t) \wedge x_2(t-1) \vee x_1(t) \\ x_2(t+1) = u(t) \to x_2(t-1) \vee x_1(t). \end{cases} \quad (22)$$

Converting it to algebraic form[1] we get,

$$\begin{aligned} Q_1 = &\delta_4[1\ 4\ 1\ 4\ 1\ 4\ 1\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4 \\ &\quad 3\ 4\ 3\ 4\ 3\ 4\ 3\ 4\ 3\ 4\ 3\ 4\ 3\ 4\ 3\ 4], \\ Q_2 = &\delta_4[1\ 1\ 1\ 1\ 1\ 1\ 4\ 4\ 1\ 1\ 1\ 1\ 1\ 1\ 4\ 4 \\ &\quad 1\ 1\ 3\ 3\ 1\ 1\ 3\ 3\ 1\ 1\ 3\ 3\ 1\ 1\ 3\ 3]. \end{aligned}$$

Next, using (5) we obtain $L_{\sigma(t)}$ as follows:

$$\begin{aligned} L_1 = &\delta_{16}[1\ 8\ 9\ 16\ 1\ 8\ 9\ 16\ 4\ 8\ 12\ 16\ 4\ 8\ 12\ 16 \\ &\ 3\ 8\ 11\ 16\ 3\ 8\ 11\ 16\ 3\ 8\ 11\ 16\ 3\ 8\ 11\ 16], \\ L_2 = &\delta_{16}[1\ 5\ 9\ 13\ 1\ 5\ 12\ 16\ 1\ 5\ 9\ 13\ 1\ 5\ 12\ 16 \\ &\ 1\ 5\ 11\ 15\ 1\ 5\ 11\ 15\ 1\ 5\ 11\ 15\ 1\ 5\ 11\ 15]. \end{aligned}$$

By Proposition 3.1, input-state incidence matrix of the DS-BCN is given as

$$\mathscr{I} = \begin{bmatrix} \tilde{L} \\ \tilde{L} \end{bmatrix},$$

where $\tilde{L} = [L_1 L_2]$.

Now, we check whether $x_d = \delta_4^3$ is reachable form initial state sequence δ_{16}^6. By Theorem 4.1, using the condition given in 1) and 2), we get $\gamma = 9, 11$ or 12 such that,

$$\sum_{l=1}^{64} \left(\tilde{M}^l\right)_{9,6} > 0,\ \sum_{l=1}^{64} \left(\tilde{M}^l\right)_{11,6} > 0,\ \sum_{l=1}^{64} \left(\tilde{M}^l\right)_{12,6} > 0.$$

Then, $x_d = \delta_4^3$ is reachable from initial state sequence δ_{16}^6.

Also, for a given system (21), (22), one can check through the calculations that conditions 3) and 4) in Theorem 4.1 are not valid i.e. there are "0" entries present in the controllability matrix. Therefore, the system is not controllable.

VI. CONCLUSIONS

Reachability and controllability of the delayed switched Boolean control networks have been investigated. By using the semi-tensor product method, algebraic form of the system has been obtained. Input-state incidence matrix based necessary and sufficient conditions for the reachability and controllability of the DSBCN have been presented. Additionally, an algorithm to find the proper switching sequence and control sequence to realize the reachability of DSBCN is also presented. The study of illustrative example has shown that the presented results are effective.

Computational complexity is one of the major issues while using the STP based method for BCNs and eventually DS-BCNs. Main future direction is to develop new approaches to reduce the computational cost involved in solving the control problems for BCNs. Future directions also include addressing the cases of multi-time delays and time-variant delays in the states and solving observability and stability problems for DSBCNs.

[1]The calculation of algebraic form is based on the MATLAB STP toolbox available at `http://lsc.amss.ac.cn/~dcheng/`.

REFERENCES

[1] S. A. Kauffman, "Metabolic stability and epigenesis in randomly constructed genetic nets," *Journal of theoretical biology*, vol. 22, no. 3, pp. 437–467, 1969.

[2] T. Akutsu, M. Hayashida, W.-K. Ching, and M. K. Ng, "Control of boolean networks: Hardness results and algorithms for tree structured networks," *Journal of theoretical biology*, vol. 244, no. 4, pp. 670–679, 2007.

[3] Y. Zhao, J. Kim, and M. Filippone, "Aggregation algorithm towards large-scale boolean network analysis," *IEEE Transactions on Automatic Control*, vol. 58, no. 8, pp. 1976–1985, 2013.

[4] F. Ay, F. Xu, and T. Kahveci, "Scalable steady state analysis of boolean biological regulatory networks," *PloS one*, vol. 4, no. 12, p. e7992, 2009.

[5] D. Cheng, H. Qi, and Z. Li, *Analysis and control of Boolean networks: a semi-tensor product approach*. Springer Science & Business Media, 2010.

[6] D. Laschov and M. Margaliot, "Controllability of boolean control networks via the perron–frobenius theory," *Automatica*, vol. 48, no. 6, pp. 1218–1223, 2012.

[7] J. Lu, J. Zhong, D. W. Ho, Y. Tang, and J. Cao, "On controllability of delayed boolean control networks," *SIAM Journal on Control and Optimization*, vol. 54, no. 2, pp. 475–494, 2016.

[8] D. Cheng, H. Qi, T. Liu, and Y. Wang, "A note on observability of boolean control networks," *Systems & Control Letters*, vol. 87, pp. 76–82, 2016.

[9] D. Cheng, H. Qi, Z. Li, and J. B. Liu, "Stability and stabilization of boolean networks," *International Journal of Robust and Nonlinear Control*, vol. 21, no. 2, pp. 134–156, 2011.

[10] N. Bof, E. Fornasini, and M. E. Valcher, "Output feedback stabilization of boolean control networks," *Automatica*, vol. 57, pp. 21–28, 2015.

[11] R. Li, M. Yang, and T. Chu, "State feedback stabilization for boolean control networks," *IEEE Transactions on Automatic Control*, vol. 58, no. 7, pp. 1853–1857, 2013.

[12] H. Li and Y. Wang, "Output feedback stabilization control design for boolean control networks," *Automatica*, vol. 49, no. 12, pp. 3641–3645, 2013.

[13] E. Fornasini and M. E. Valcher, "Recent developments in boolean networks control," *Journal of Control and Decision*, vol. 3, no. 1, pp. 1–18, 2016.

[14] D. Cheng and T. Liu, "A survey on logical control systems," *Unmanned Systems*, vol. 4, no. 01, pp. 97–116, 2016.

[15] V. Hatzimanikatis, K. Lee, J. Bailey *et al.*, "A mathematical description of regulation of the g1-s transition of the mammalian cell cycle," *Biotechnology and bioengineering*, vol. 65, no. 6, pp. 631–637, 1999.

[16] B. Lewin, *Genes VII*. Oxford University Press, Oxford, UK, 2000.

[17] H. Li and Y. Wang, "Controllability analysis and control design for switched boolean networks with state and input constraints," *SIAM Journal on Control and Optimization*, vol. 53, no. 5, pp. 2955–2979, 2015.

[18] F. Li and Y. Tang, "Set stabilization for switched boolean control networks," *Automatica*, vol. 78, pp. 223–230, 2017.

[19] T.-H. Chueh and H. H.-S. Lu, "Inference of biological pathway from gene expression profiles by time delay boolean networks," *PloS one*, vol. 7, no. 8, p. e42095, 2012.

[20] L. Wang, Z.-P. Luo, H.-L. Yang, and J. Cao, "Stability of genetic regulatory networks based on switched systems and mixed time-delays," *Mathematical biosciences*, vol. 278, pp. 94–99, 2016.

[21] P. Yan and H. Özbay, "Stability analysis of switched time delay systems," *SIAM Journal on Control and Optimization*, vol. 47, no. 2, pp. 936–949, 2008.

[22] M. S. Mahmoud, "Switched time-delay systems," in *Switched Time-Delay Systems*. Springer, 2010, pp. 109–130.

[23] H. Li and Y. Wang, "On reachability and controllability of switched boolean control networks," *Automatica*, vol. 48, no. 11, pp. 2917–2922, 2012.

[24] F. Li and J. Sun, "Controllability of higher order boolean control networks," *Applied Mathematics and Computation*, vol. 219, no. 1, pp. 158–169, 2012.

[25] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*. Academic press, 2014.