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A Toolbox for Optimal Control Based on Symmetry Analysis with Applications to Aircraft Maximum Endurance and Maximum Range

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***Abstract*— Solutions of optimal control problems are the solutions of the Hamilton-Jacobi-Bellman equation. The main contribution of this paper is to propose a symmetry analysis approach to find solutions of the Hamilton-Jacobi-Bellman equation and to introduce a Toolbox to automate the procedure. Several examples will be presented illustrating the effectiveness of the procedure including the maximum endurance and maximum range of an aircraft.**

I. INTRODUCTION

In engineering, many problems can be formulated as optimal control problems (OCPs). There are several existing approaches to determine an optimal controller for a system. Unfortunately many of these cannot lead to an analytical solution when the dynamics of the system are nonlinear [1]. This paper will consider OCPs with nonlinear dynamics such that the necessary condition for optimality can be solved uniquely for the controller.

One common approach to obtaining an optimal controller is Pontryagin's Maximum Principle (PMP) which outlines several necessary conditions for optimality and results in ordinary differential equations (ODEs) with boundary conditions (a two point boundary value problem). Another approach is to invoke Bellman's Principle of optimality, which can be used to obtain a partial differential equation (PDE) known as the Hamilton-Jacobi-Bellman (HJB) equation. Both approaches have advantages and limitations. While the PMP approach can be used to obtain an optimal controller, this controller will be in general parameterized by time because the solution of the ODEs are functions of time. To bring out the feedback nature of the controller one needs to solve one of the equations of the state or costate response for the time variable and then replace the obtained expression for time in the other equations. This may not always be possible. The HJB approach has the ability to produce an optimal state feedback controller but requires solving a PDE which can be quite difficult. This paper proposes the method of symmetry analysis which is used to reduce the order of PDEs by relating state variables to each other along the optimal trajectory. Symmetry analysis has been used to reduce stochastic versions of the HJB equation that apply to finance (see [2], [3]). In [4] it was shown that a stochastic version of the HJB equation called the Heath equation can be reduced to the heat equation by means of symmetry analysis. However, the transformation that leads to this reduction depends explicitly on the coefficient of the second order term of the HJB equation. Therefore, such a reduction is not possible for deterministic versions of the HJB equation, which are first order. Symmetry analysis has been widely applied to the PMP approach to solve optimal control problems. For example, in reference [5], symmetries are exploited to obtain reduced control systems and reduced Hamiltonian systems. Reference [6] describes the reduction of a class of optimal control problems with symmetry using the geometry of Dirac structures. The paper notes that the PMP can be discussed from the viewpoint of Dirac structures. In [7], Young's inequality is used to derive conditions by which the solution of the associated HJB equation is simplified. To the best of the authors' knowledge the application of symmetry analysis to the HJB equation for general deterministic systems was only recently investigated for quadratic cost terms on the input in reference [8]. Symmetry analysis, while powerful, can be quite a complex tool as the symmetries themselves may be difficult to find. Furthermore, the choice of which symmetry to use in the reduction process may limit how far a PDE can be reduced. Therefore, this paper proposes an automated procedure developed in Maple which reduces HJB equations using symmetry analysis and, where possible, provides an explicit optimal controller.

The structure of the paper is as follows. Section II will cover some mathematical preliminaries followed by the presentation of the solution algorithm in Section III. Section IV presents the Maple Toolbox followed by several examples. The paper closes by presenting the conclusions.

II. MATHEMATICAL PRELIMINARIES

A. Optimal Control Problem Formulation and Assumptions

This paper will use the notation F_x to denote the partial derivative of the function F with respect to x. Consider the following OCP

$$\begin{aligned} V(t_0, x_0) = \inf_u \int_0^T L(x(\tau), u(\tau))d\tau \\ s.t. \quad \dot{x}(t) = f(x, u, t) \\ x(0) = x_0, x(T) = x_f \end{aligned} \tag{1}$$

where the input $u(t)$ is considered to be unconstrained and piecewise continuous.

Assumptions

1) There exists a unique solution of the optimal control problem.
2) All functions (including f, L and V) are C^1 functions and f has bounded first partial derivatives within its domain.

While admittedly, these assumptions will constrain the problems that can be solved to the class of problems for which smooth dynamics, smooth functionals, and smooth value functions exist, several important problems in applications are in this class (as shown in the examples). However, we note that there are several other problems with input constraints that lead to bang-bang solutions and are also of interest in applications. Such applications are not included in the class of problems addressed in this paper.

The Hamilton-Jacobi-Bellman equation associated with problem (1) is [1]

$$\inf_u \Big(L + V_x \dot{x}\Big) + V_t = 0 \tag{2}$$

If we define the Hamiltonian as

$$H = L + V_x \dot{x} \tag{3}$$

then (2) becomes

$$\inf_u (H) + V_t = 0 \tag{4}$$

and the necessary condition for optimality according to the PMP is

$$H_u = 0 \tag{5}$$

because it was assumed that H is differentiable.

Although this section considered the general case of V depending on both t and x, the examples in the rest of the paper do not consider time-varying dynamics nor time-varying value functions. We will next discuss symmetry analysis, which is a powerful tool for solving HJB (2).

B. Symmetry Analysis

As equation (2) suggests, the HJB equation is a PDE that may be very hard to solve analytically. In fact there might not even exist a solution to that equation. In what follows we assume that a solution exists and is unique. To obtain a solution there is a notable technique that can reduce the number of variables in the PDE, eventually transforming it into an ODE if enough symmetries exist [9], [10]. Let

$$x = [x_1, ..., x_n] \in \mathbb{R}^n$$

and suppose that there exists a single parameter s, and a vector of functions $F^i(x, s)$ that are analytic in s such that

$$[\tilde{x}_1 = F^1(x, s), ..., \tilde{x}_n = F^n(x, s)]$$

are a group of diffeomorphisms with identity element $x_i = F^i(x, 0),\ i = 1, \ldots, n$. The paths $\tilde{x}^i$ for $i = 1, ..., n$ with parameter s form what is called a one parameter Lie group in n variables [9], [10]. Because all of the functions F^i are assumed to be analytic, it is possible for small enough s to write the Taylor series of $\tilde{x}$ around x as

$$\begin{aligned} \tilde{x} &= F(x, s) \\ &\approx F(x, 0) + sF_s(x, s)|_{s=0} \\ &= x + s\Big[F_s\Big]_{s=0} \end{aligned} \tag{6}$$

The derivatives of the various F^i with respect to the parameter s evaluated at $s = 0$ are called infinitesimals of the Lie group and are denoted by ξ_i. Let us define the vector $\xi = [\xi_1, \xi_2, ..., \xi_n]$. From (6) we can write

$$\tilde{x} \approx x + s\xi(x)$$

This approximation becomes exact in the limit when s goes to zero. For a constant c, consider a level curve of an analytic function Φ of x defined by

$$c = \Phi(x)$$

Thus for Φ defined on the tilde variable,

$$\tilde{c} = \Phi(\tilde{x})$$

When the two curves read the same $\Phi(\tilde{x}) = \Phi(x)$, we say that Φ is left invariant under the group transformation F [9], [10]. Taking the Taylor series expansion of $\Phi(\tilde{x})$ about x

$$\Phi(\tilde{x}) = \Phi(x) + s\xi\frac{\partial \Phi}{\partial x} + \frac{s^2}{2}\xi\frac{\partial}{\partial x}\Big(\xi\frac{\partial \Phi}{\partial x}\Big) + ...$$

The condition $\Phi(\tilde{x}) = \Phi(x)$ is satisfied if and only if

$$\xi\frac{\partial \Phi}{\partial x} = 0$$

The operator

$$\xi\frac{\partial}{\partial x}$$

is called the infinitesimal generator and is denoted by X. Thus Φ is invariant under a group transformation F, if and only if $X(\Phi) = 0$, and the transformation F is called a *symmetry* of Φ. For $n = 2$, and considering only the first order Taylor development of F, we have from (6),

$$\begin{aligned} \frac{d\tilde{x}_1}{ds} &= \Big[F_s^1\Big]_{s=0} = \zeta(x_1, x_2), \\ \frac{d\tilde{x}_2}{ds} &= \Big[F_s^2\Big]_{s=0} = \nu(x_1, x_2) \end{aligned} \tag{7}$$

From (7) we can write

$$\frac{d\tilde{x}_2}{d\tilde{x}_1} = \frac{\nu}{\zeta}$$

or, after rearranging terms,

$$\frac{d\tilde{x}_2}{\nu} = \frac{d\tilde{x}_1}{\zeta} \tag{8}$$

This is called the characteristic equation. Its solution will lead to the transformation of variables that represents the Lie point symmetry. Consider now a function Φ given by the left hand side of equation (4) for a second order OCP (1) with state variables $x = [x_1, x_2]$.

$$\Phi(x_1, x_2, t, V_{x_1}, V_{x_2}, V_t) = \inf_u (H) + V_t$$

Then the solution of (4) is the level curve $\Phi(Y) = 0$ where $Y = (x_1, x_2, t, V_{x_1}, V_{x_2}, V_t)$. For the transformation

$$\begin{aligned} \tilde{x}_1 &= F^1(Y, s) \\ \tilde{x}_2 &= F^2(Y, s) \\ \tilde{t} &= F^3(Y, s) \\ \tilde{V} &= F^4(Y, s) \end{aligned} \tag{9}$$

the tilde co-states $\tilde{V}_{\tilde{x}_1}, \tilde{V}_{\tilde{x}_2}, \tilde{V}_{\tilde{t}}$ can be obtained by differentiation. If Φ is left invariant when $\Phi(Y)$ is replaced with $\Phi(\tilde{Y})$ for $\tilde{Y} = [\tilde{x}_1, \tilde{x}_2, \tilde{t}, \tilde{V}_{\tilde{x}_1}, \tilde{V}_{\tilde{x}_2}, \tilde{V}_{\tilde{t}}]$ then F is a symmetry of Φ, with characteristic equations that can be found using the same method leading to (8). If the characteristic equations can be integrated it will provide a change of variables that leaves Φ invariant, and that reduces the number of variables in (4) by one. This will be illustrated in several examples, one of which in the next section.

It is important to note that, as mentioned in the introduction, the biggest advantage of using symmetry analysis to solve the HJB equation relative to solving the canonical ODEs of PMP is that the solution of the HJB equation will be naturally a feedback controller. However, when it exists, the solution of the canonical ODEs would obviously be the same as the one obtained from solving the HJB equation using symmetry analysis.

C. Example: Linear Quadratic Regulator

Consider an OCP defined by

$$\begin{aligned} V(x_0) = \inf_u &\int_0^\infty q_1 x_1^2 + q_2 x_2^2 + r u^2 d\tau \\ s.t. \;\; &\dot{x}_1(t) = x_2 \\ &\dot{x}_2(t) = bu \\ &x(0) = 0, \;\; u \in C^1 \end{aligned} \tag{10}$$

where it is assumed that $V \in C^1$, $r > 0$, $q_1 \geq 0, q_2 \geq 0$, $b \neq 0$, $x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$. The boundary condition is $V(0) = 0$ [1]. The application of symmetry analysis to (10) was considered in [8]. According to (2), the HJB equation is

$$\inf_u \Big(q_1 x_1^2 + q_2 x_2^2 + r u^2 + V_{x_1} x_2 + V_{x_2} b u \Big) + V_t = 0 \tag{11}$$

The necessary condition for optimality (5) for (10) is

$$\begin{aligned} 2ru + bV_{x_2} &= 0 \\ u &= -\frac{bV_{x_2}}{2r} \end{aligned} \tag{12}$$

Because the cost $q_1 x_1^2 + q_2 x_2^2 + r u^2$ is quadratic in u with $r > 0$ and the dynamics of x_2 are linear in u, the necessary condition for optimality is guaranteed to produce a unique solution for u. Replacing (12) in (11) yields the following PDE which holds along the optimal trajectory

$$q_1 x_1^2 + q_2 x_2^2 + V_t + V_{x_1} x_2 - \frac{V_{x_2}^2 b^2}{4r} = 0 \tag{13}$$

To begin the process of symmetry analysis consider the mapping

$$\tilde{t} = t + s$$

for any $s \in \mathbb{R}$. Under this change of variables, note that $V_{\tilde{t}} = V_t$. As this is the only term in (11) in which t appears, and the boundary condition $V(0) = 0$ is independent of time, (13) remains invariant under this change of variables. Therefore, there must exist a solution to (13) which does not depend on time. Replacing $V_t = 0$ in (13) then yields

$$q_1 x_1^2 + q_2 x_2^2 + V_{x_1} x_2 - \frac{V_{x_2}^2 b^2}{4r} = 0 \tag{14}$$

We see that the number of variables in (13) has been reduced by one using the time translation invariance. To reduce the PDE in (14) further to an ODE, consider the dilation

$$\begin{aligned} \tilde{x}_1 &= e^s x_1 \\ \tilde{x}_2 &= e^s x_2 \\ \tilde{V} &= e^{2s} V \end{aligned} \tag{15}$$

Noting that $\tilde{V}_{\tilde{x}_1} = e^s V_{x_1}$ and $\tilde{V}_{\tilde{x}_2} = e^s V_{x_2}$, equation (14) in the tilde variables becomes

$$e^{2s} \Big(q_1 x_1^2 + q_2 x_2^2 + V_{x_1} x_2 - \frac{V_{x_2}^2 b^2}{4r} \Big) = 0 \tag{16}$$

which holds if and only if (14) also holds. Thus under the dilation mapping (15), equation (14) is left invariant, implying that the dilation (15) is a *symmetry* of the equation.

From (8) and (15) the characteristic equations for this problem are

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2} = \frac{dV}{2V}$$

Integrating and rearranging terms we see that (14) is invariant under the change of variables: $\alpha = \frac{x_2}{x_1}$, $V = x_1^2 G(\alpha)$. Substituting these in (14) and factoring out an x_1^2 term produces

$$2\alpha G - \alpha^2 G' + q_1 + q_2 \alpha^2 - \frac{1}{4r}(G')^2 b^2 = 0 \tag{17}$$

which is an ODE, the solution to which is

$$G(\alpha) = k_0 + k_1 \alpha + k_2 \alpha^2$$

where

$$k_0 = \frac{\sqrt{b^2 q_1 q_2 + 2bq_1\sqrt{q_1 r}}}{b}$$

$$k_1 = \frac{2\sqrt{q_1 r}}{b}$$

$$k_2 = \frac{\sqrt{r(b^2 q_2 + 2b\sqrt{q_1 r})}}{b}$$

Substituting for the original variables, and rearranging terms yields

$$V(x_1, x_2) = k_0 x_1^2 + k_1 x_1 x_2 + k_2 x_2^2 \tag{18}$$

and thus

$$u = -\frac{V_{x_2} b}{2r} = -\frac{b}{2r}(k_1 x_1 + 2k_2 x_2) \tag{19}$$

Thus the method of symmetry analysis successfully produced an optimal (linear) state feedback controller.

III. Symmetry Analysis Algorithm

One of the advantages of symmetry analysis is that it is quite general and is not constrained to linear optimal control problems. Therefore, the objective of the algorithm to be described in this section is also to be general and applicable to systems with n state variables, where n is any integer. Although the procedure is general, it will be applied in this paper to examples with a maximum of two state variables to avoid cluttering the paper with long solutions. Note however that for systems with two states the HJB depends on three variables, the third variable being time. Therefore, two symmetries are enough to transform the PDE into an ODE. In the presence of only two symmetries we note that there are more efficient ways to compute them using properties of solvable Lie algebras. In fact, one can retrieve the solvable Lie subalgebras from the initial PDE. However, this procedure does not allow to deal with hidden symmetries. For more details on this process please see [10].

Given a nonlinear OCP of the class defined in Section II, the general symmetry analysis approach for solving the HJB equation is as follows:

1) Construct the HJB equation for the OCP.
2) Solve (5) for the controller u^* (assumed to have a unique solution).
3) Replace u^* in the HJB equation to obtain a new equation. Define this new equation as $HJB[0]$
4) Determine the symmetries S of $HJB[0]$ indexed by the parameter i
5) Repeat for each i until step 8): Choose the i^{th} available symmetry $S[i]$ from 4) and reduce $HJB[0]$ to obtain equation $HJB[0,i]$ of 1 less variable than $HJB[0]$.
6) Determine the symmetries $S[i]$ of $HJB[0,i]$ indexed by the parameter j
7) Choose the j^th available symmetry $S[i][j]$ from 6) and reduce $HJB[0,i]$ to obtain equation $HJB[0,i,j]$.
8) Do 6) and 7) until $HJB[0,i,j,...,k]$ for some $k > 0$ is obtained which is either an ODE or possesses no more symmetries.
9) Determine which choice of symmetries $[i,j,...,k]$ used to obtain $HJB[0,i,j,...,k]$ results in an equation of the lowest possible number of variables, and return that equation.
10) If the equation with the lowest possible number of variables is a solvable ODE, solve it and express the answer in terms of the original state variables.

We may think of the possible symmetries and reductions as a tree where the nodes represent equations and the vertices represent symmetries (see Fig. 1 for the case related to the LQR problem). Although it was not shown, equation (13) actually possesses three symmetries $S[1], S[2], S[3]$, the second and third of which lead to reductions called $HJB[0,2], HJB[0,3]$, respectively. The first symmetry $S[1]$ does not lead to a reduction. The first reduction $HJB[0,2]$ possesses two symmetries $S[2][1], S[2][2]$, the second of which leads to an ODE called $HJB[0,2,2]$. The symmetry $S[2][1]$ does not lead to a reduction. The second reduction $HJB[0,3]$ has one symmetry which leads to an ODE $HJB[0,3,1]$. Solving this ODE will result in the same solution as the solution of ODE $HJB[0,2,2]$.

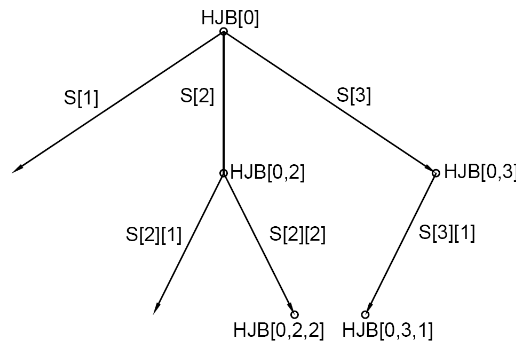

Fig. 1. Tree of Symmetries

Define a *path* along the symmetry tree as a set of nodes connected by forward vertices (in the direction of the arrows in Fig. 1) to each other, starting at $HJB[0]$. Determining the greatest possible reduction of the HJB equation consists of finding the longest path(s) in the tree. As the order of the OCP increases, so too does the symmetry tree. It is therefore desirable to generalize and automate the process of symmetry analysis. Maple's symbolic computation makes it an ideal tool to automate this process.

IV. Toolbox for Symmetry Analysis

The generalized, automated process follows the algorithm from Section III. The examples below demonstrate the use of symmetry analysis in determining an optimal controller for an OCP of the class defined in Section II, and illustrate the functionality of the toolbox.

A. Maximum Endurance OCP

In [11] an aircraft's maximum endurance is formulated as

$$\begin{aligned} V = \max_{v} \int_0^T dt \\ s.t. \quad \dot{W} = -S_{FC} D \\ W(0) = W_c, W(T) = W_d \\ v > 0 \end{aligned} \tag{20}$$

where W is the weight of the aircraft, S_{FC} is the specific fuel consumption (assumed to be constant), the final time T is free, and D is the drag given by

$$D = \frac{1}{2} C_{D,0} \rho S v^2 + \frac{2 C_{D,2} W^2}{\rho S v^2} \tag{21}$$

where v is the velocity, ρ is the density, S is the wing area, and $C_{D,0}$, $C_{D,2}$ are constant drag coefficients.

In [11], the maximum endurance OCP was solved analytically resulting in a well known expression for the optimal velocity [12] as follows

$$v_{optimal} = \left(\frac{2W}{\rho S} \sqrt{\frac{C_{D,2}}{C_{D,0}}} \right)^{1/2} \tag{22}$$

This problem may now be solved using the Toolbox proposed in this paper. Let $v^2 = u$. The necessary condition for optimality (5) will no longer give a unique solution for u. It will instead be a quadratic equation of u. However, this example does not violate the assumption of a unique controller because u, as the square of the velocity v must be positive. Therefore, the negative solution of u must not be considered. The limitation $u > 0$ may be added to the problem when creating the HJB, by adding an optional input `mode=positive_controller`. However, in order for the procedure to be able to choose a positive controller, a declaration of positive constants must be made first, as follows

```
PositiveValues([Cd0, Cd2, rho, S, W], [])
EX2:=CreateHJB(1
                   [W],
                   [-SFC*Drag],
                    V,
                    u,
                   mode=positive_controller
```

The result is a table assigned to the variable `EX2` from which we may extract the HJB equation

```
EX2[Equation]
```

$$1 - V_W(W,t)SFC\left(\frac{1}{2}Cd0\rho Su + \frac{2Cd2W^2}{\rho Su}\right) + V_t(W,t) = 0$$

as well as the equation in which the dependency on u has been removed:

```
EX2[0][Equation]
```

$$1 - V_W(W,t)SFC\left(\sqrt{Cd0Cd2}W + \frac{Cd2WCd0}{\sqrt{Cd0Cd2}}\right) + V_t(W,t) = 0$$

Examining paths that lead to a reduction of the HJB yields

```
FindReductions(EX2)
```

$$[0,2],[0,3],[0,5]$$

We may look at the equations for each of the possible reductions:

```
EX2[0,2][Equation]
```

$$1 + \frac{d}{d\_t1}\_u1(\_t1) = 0$$

```
EX2[0,3][Equation]
```

$$-2Cd0Cd2\left(\frac{d}{d\_t1}\_u1(\_t1)\right)SFC\_t1 + \sqrt{Cd0Cd2} = 0$$

```
EX2[0,5][Equation]
```

$$-2Cd0Cd2\left(\frac{d}{d\_t1}\_u1(\_t1)\right)SFC\_t1 + \sqrt{Cd0Cd2} = 0$$

We notice that these are all ODEs. If we consider the equation `EX2[0,3][Equation]` (which is the same as `EX2[0,5][Equation]`) a solution can be found as

```
CompleteSolution(EX2, [0,3])
```

$$\left[V(W,t) = \frac{1}{2}\frac{2\_C1\sqrt{Cd0}\sqrt{Cd2}SFC + \ln W}{\sqrt{Cd0}\sqrt{Cd2}SFC}\right.$$
$$\left. u = \frac{2\sqrt{Cd2}W}{\sqrt{Cd0}\rho S}\right] \tag{23}$$

where $\_C1$ is a constant that depends on $W(T) = W_d$. A complete solution including the initial conditions, can be found using the `CompleteSolution` procedure with the optional input `mode=withboundary`. If this input is used, the procedure inputs must also include the independent variables of the cost function, in this case $[W,t]$, the terminal values of these variables, $[W_d, t_d]$, the terminal value of the cost function, 0, and the constant that must be solved for, $\_C1$. The procedure yields

```
CompleteSolution(EX2, [0,3],
                   mode=withboundary,
                   [W,t],
                   [Wd, td]
                   0, _C1)
```

$$\left[V(W,t) = \frac{1}{2}\frac{-\ln W_d + lnW}{\sqrt{Cd0}\sqrt{Cd2}SFC}\right.$$
$$\left. u = \frac{2\sqrt{Cd2}W}{\sqrt{Cd0}\rho S}\right]$$

Note that u is the square of the optimal velocity in (22) and the value function V corresponds to the optimal endurance.

B. Maximum Range for Cruise

Consider now the maximum range for a cruising aircraft. This problem was formulated as an OCP in [11] as

$$\begin{aligned} V = \max_u \int_0^T & S_{FC}Ddt \\ s.t.\ \dot{W} &= -S_{FC}D \\ \dot{x} &= v \\ x(0) &= x_0, x(T) = x_f, \\ v &> 0 \end{aligned} \tag{24}$$

where drag is defined by equation (21). For this problem, let us consider the controller $u = v$. The Hamiltonian of (24) is

$$H = S_{FC}D(1 - V_W) + V_x u$$

From the PMP, we have

$$H_x = 0 = \dot{V}_x$$

which implies that V_x as a function of time is a constant K. The `CreateHJB` procedure automatically checks the partial derivative of the Hamiltonian with respect to the state variables. If it is the case that one of the costates is a constant, the `CreateHJB` will take this into account. We begin by declaring positive and negative values, creating an HJB equation, and investigating the $HJB[0]$ equation

```
PositiveValues([Cd0, Cd2, rho, S, W], [])
EX3:=CreateHJB(
            SFC*Drag,
            [x,W],
            [u, -SFC*Drag], V, u)
EX3[0][Equation]
```

$$\begin{aligned} &SFC\left(\frac{1}{2}Cd0\rho Su^2 + \frac{2Cd2W^2}{\rho Su^2}\right) + Ku \\ &- \frac{Cd0SFCS^2\rho^2u^4 + KS\rho u^3 - 4Cd2SFCW^2}{Cd0S^2\rho^2u^4 - 4Cd2W^2} \\ &\times\left(\frac{1}{2}Cd0\rho Su^2 + \frac{2Cd2W^2}{\rho Su^2}\right) + V_t = 0 \end{aligned} \tag{25}$$

Here, the PMP replaced V_x with K. The necessary condition (5) was then solved for V_W in terms of K. The sensitivity V_W was then replaced in the HJB equation in terms of K. The symmetries that lead to reductions can be determined as
`FindReductions(EX3, [0])`

$$[0, 2], [0, 6], [0, 9], [0, 10], [0, 13], [0, 17] \tag{26}$$

The sixth symmetry, represented by path $[0, 6]$ is the time translation symmetry
`EX7[0][Infinitesimals][6]`

$$\begin{aligned}[\xi_x(x, W, t, J) = 0, \xi_W(x, W, t, J) = 0,\\ \xi_t(x, W, t, J) = 1, \eta_J(x, W, t, J) = 0]\end{aligned} \tag{27}$$

The time translation symmetry leads to reduction
`EX3[0,6][Equation]`

$$\frac{1}{2}\frac{Ku(Cd0S^2\rho^2u^4 - 12Cd2\_t2^2)}{Cd0S^2\rho^2u^4 - 4Cd2\_t2^2} = 0$$

If a reduction no longer depends on any co-states, as is the case for `EX3[0,6][Equation]`, the `CompleteSolution` procedure will return a solution of the equation in terms of the original variables
`CompleteSolution(EX3, [0,6])`

$$u = \frac{12^{1/4}(Cd2W^2Cd0^3S^2\rho^2)^{1/4}}{Cd0S\rho}$$

which is the maximum range speed found in [12].

V. Conclusions

This paper showed that symmetry analysis is an effective tool to reduce the number of variables in a HJB equation to solve optimal control problems. However, the algorithm for applying symmetry analysis to HJB equations, as described in Section III, can be too complex to be done without a computer. It is therefore desirable to automate the procedure and create a toolbox for future use as shown in Section IV. Examples showed how to use the toolbox to obtain solutions for the maximum endurance and the maximum range problems. The beta version of the toolbox can be obtained by contacting the authors.

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APPENDIX: Toolbox Functions

```
CreateHJB := proc (
             Cost :: algebraic,
             State_Vars :: list,
             dynamics :: list,
             dependent_variable :: name,
             controller :: name,
             mode :: identical(
             positive_controller,
             negative_controller))

RealPositiveValues := proc (
             pos_constants :: list,
             neg_constants :: list)

Infinitesimals := proc (
             hjb_table :: table,
             infinitesimal_tree:=[0])

FindReductions := proc (
             hjb_table :: table,
             path:=[0])

CompleteSolution := proc (
            hjb_table :: table,
            path :: list,
            mode :: identical(
            withboundary,
            withoutboundary)
            :=withoutboundary))
```

References

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