

Stochastic Model Predictive Control with Discounted Probabilistic Constraints

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Abstract—This paper considers linear discrete-time systems with additive disturbances, and designs a Model Predictive Control (MPC) law to minimise a quadratic cost function subject to a chance constraint. The chance constraint is defined as a discounted sum of violation probabilities on an infinite horizon. By penalising violation probabilities close to the initial time and ignoring violation probabilities in the far future, this form of constraint enables the feasibility of the online optimisation to be guaranteed without an assumption of boundedness of the disturbance. A computationally convenient MPC optimisation problem is formulated using Chebyshev's inequality and we introduce an online constraint-tightening technique to ensure recursive feasibility based on knowledge of a suboptimal solution. The closed loop system is guaranteed to satisfy the chance constraint and a quadratic stability condition.

I. INTRODUCTION

Robust control design methods for systems with unknown disturbances must take into account the worst case disturbance bounds in order to guarantee satisfaction of hard constraints on system states and control inputs [1], [2], [3]. However, for problems with stochastic disturbances and constraints that are allowed to be violated up to a specified probability, worst-case control strategies can be unnecessarily conservative. This motivated the development of stochastic Model Predictive Control (MPC), which addresses optimal control problems for systems with chance constraints by making use of information on the distribution of model uncertainty [4], [5]. Although capable of handling chance constraints, existing stochastic MPC algorithms that ensure constraint satisfaction in closed loop operation typically rely on knowledge of worst case disturbance bounds to obtain such guarantees [6]. For the algorithms proposed in [7], [8], [9] for example, which simultaneously ensure closed loop constraint satisfaction and recursive feasibility of the online MPC optimisation, the degree of conservativeness increases as the disturbance bounds become more conservative.

This paper ensures both closed loop satisfaction of chance constraints and recursive feasibility but does not rely on disturbance bounds, instead requiring knowledge of only the first and second moments of the disturbance input. This is achieved by formulating the chance constraint as the sum over an infinite horizon of discounted violation probabilities, and implementing the resulting constraints using Chebyshev's inequality. Control problems involving discounted costs and constraints are common in financial engineering

applications (e.g. [10], [11], [12]), and allow system performance in the near future to be prioritised over long-term behaviour. This shift of emphasis is vital for ensuring recursive feasibility of chance-constrained control problems involving possibly unbounded disturbances. We describe an online constraint-tightening approach that guarantees the feasibility of the MPC optimisation, and, by considering the closed loop dynamics of the tightening parameters, we show that the closed loop system satisfies the discounted chance constraint as initially specified.

The paper is organised as follows. The control problem is described and reformulated with a finite prediction horizon in Section II. Section III proposes an online constraint-tightening method for guaranteeing recursive feasibility. Section IV summarises the proposed MPC algorithm and derives bounds on closed loop performance. In Section V, the closed loop behaviour of the tightening parameters is analysed and constraint satisfaction is proved. Section VI gives a numerical example illustrating the results obtained and the paper is concluded in Section VII.

Notation: The Euclidean norm is denoted $\|x\|$ and we define $\|x\|_Q^2 := x^T Q x$. The notation $Q \succcurlyeq 0$ and $R \succ 0$ indicates that Q and R are respectively positive semidefinite and positive definite matrices, and $\text{tr}(Q)$ denotes the trace of Q . The probability of an event A is denoted $\mathbb{P}(A)$. The expectation of x given information available at time k is denoted $\mathbb{E}_k[x]$ and $\mathbb{E}[x]$ is equivalent to $\mathbb{E}_0[x]$. The sequence $\{x_0, \dots, x_{N-1}\}$ is denoted $\{x_i\}_{i=0}^{N-1}$. We denote the value of a variable x at time k as x_k , and the i -step-ahead predicted value of x at time k is denoted $x_{i|k}$.

II. PROBLEM DESCRIPTION

Consider an uncertain linear system with model

$$x_{k+1} = Ax_k + Bu_k + \omega_k, \quad (1)$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$ are the system state and the control input respectively. The unknown disturbance input $\omega_k \in \mathbb{R}^{n_x}$ is independently and identically distributed with known first and second moments:

$$\mathbb{E}[\omega_k] = 0, \quad \mathbb{E}[\omega_k \omega_k^T] = W.$$

Unlike the approaches of [5], [13], which assume the additive disturbance lies in a compact set, the disturbance ω_k is not assumed to be bounded and its distribution may have infinite support. It is assumed that the system state is measured directly and available to the controller at each sample instant.

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The system (1) is subject to the constraint

$$\sum_{k=0}^{\infty} \gamma^k \mathbb{P}(\|Cx_k\| \geq t) \leq e, \quad (2)$$

for a given matrix $C \in \mathbb{R}^{n_c \times n_x}$, positive scalars t, e and discounting factor $\gamma \in (0, 1)$. This constraint gives a special feature to the control problem that the probability of future states violating the condition $\|Cx_k\| < t$ at time instants nearer to the initial time are weighted more heavily than those in the far future. For simplicity we refer to $\mathbb{P}(\|Cx_k\| \geq t)$ as a *violation probability*.

The aim of this work is to design a controller that minimises the cost function

$$\mathbb{E} \left[\sum_{k=0}^{\infty} \|x_k - x^r\|_Q^2 + \|u_k - u^r\|_R^2 \right] \quad (3)$$

and ensures a quadratic stability condition on the closed loop system while the constraint (2) is satisfied. The weighting matrices in (3) are assumed to satisfy $Q \succcurlyeq 0$ and $R \succ 0$, and we assume knowledge of reference targets x^r and u^r for the state and the control input that satisfy the steady state conditions

$$(I - A)x^r = Bu^r, \quad \|Cx^r\| < t. \quad (4)$$

Assumption 1: (A, B) is controllable and $(A, Q^{\frac{1}{2}})$ is observable.

A. Finite horizon formulation

The problem stated above employs an infinite horizon and is subject to a constraint defined on infinite horizon. If the infinite sequence of control inputs $\{u_k\}_{k=0}^{\infty}$ were considered to be decision variables, then clearly the optimisation problem would be infinite dimensional and thus in principle computationally intractable [6]. However, the use of an infinite horizon can impart desirable properties, notably stability [14], [15]. It is therefore beneficial to design an MPC law using a cost function and constraints that are defined on a finite horizon in such a way that they are equivalent to the infinite horizon cost and constraints of the original problem. The finite horizon cost function for a prediction horizon of N steps is given by

$$\mathbb{E} \left[\sum_{i=0}^{N-1} \|x_{i|k} - x^r\|_Q^2 + \|u_{i|k} - u^r\|_R^2 + F(x_{N|k}) \right] \quad (5)$$

where $\mathbb{E}[F(x_{N|k})]$ is the terminal cost and $F(x) \geq 0$ for all x . The constraint (2) is likewise truncated to a finite horizon:

$$\sum_{i=0}^{N-1} \gamma^i \mathbb{P}(\|Cx_{i|k}\| \geq t) + f(x_{N|k}) \leq \varepsilon_k. \quad (6)$$

Here $f(x_{N|k})$ is a terminal term chosen (as will be specified in (14) and Lemma 4) to approximate the infinite sum in (2) so that $\sum_{i=N}^{\infty} \gamma^i \mathbb{P}(\|Cx_{i|k}\| \geq t) \leq f(x_{N|k})$, and ε_k is a bound on the lhs of (6) that is achievable at time k . Although ε_k may increase or decrease over time since it is conditioned on the system state at time

k , we show in Section V that (2) is satisfied if $\varepsilon_0 \leq e$ and ε_k is defined as described in Section III.

Even with the cost and constraints defined as in (5)-(6) on a finite horizon, the probability distribution of states may be unknown at each time step and the finite horizon version of the problem is therefore still intractable in general. Even if the probability distribution of ω_k is known explicitly, computing (5) and (6) requires the solution of a set of multivariate convolution integrals, which in principle is still difficult to manage [5].

B. Constraint handling and open loop optimisation

This section considers how to approximate the finite horizon constraint (6) using the two-sided Chebyshev inequality [16, Section V.7] and gives the explicit form of the MPC cost function. The cost and constraints are then combined to construct the MPC optimisation problem that is repeatedly solved online. We define the sequence of control inputs predicted at time k as

$$u_{i|k} = K(x_{i|k} - \bar{x}_{i|k}) + m_{i|k}, \quad i = 0, \dots, N-1 \quad (7)$$

$$u_{N+i|k} = K(x_{N+i|k} - x^r) + u^r, \quad i = 0, 1, \dots \quad (8)$$

where $m_{i|k}$ is the i -step-ahead prediction of the nominal control input given information at time k , that is, $\mathbb{E}_k[u_{i|k}] = m_{i|k}$, and $\bar{x}_{i|k}$ is the i -step-ahead prediction of the nominal state given information at time k , that is, $\mathbb{E}_k[x_{i|k}] = \bar{x}_{i|k}$.

Assumption 2: $\Phi := A + BK$ is strictly stable.

Given the predicted control law (7)-(8), the first two moments of the predicted state and control input sequences can be computed. Thus, the predicted nominal state trajectory is given by $\bar{x}_{0|k} = x_k$ and

$$\bar{x}_{i|k} = A^i \bar{x}_{0|k} + \sum_{j=0}^{i-1} A^{i-1-j} B m_{j|k}, \quad i = 1, \dots, N \quad (9)$$

$$\bar{x}_{N+i|k} = \Phi^i (\bar{x}_{N|k} - x^r) + x^r, \quad i = 1, 2, \dots \quad (10)$$

whereas the covariance matrix, $X_{i|k}$, of the i -step-ahead predicted state is given by $X_{0|k} = 0$ and

$$X_{i|k} = \sum_{j=0}^{i-1} \Phi^j W (\Phi^j)^T, \quad i = 1, 2, \dots \quad (11)$$

Clearly $X_{i|k}$ is independent of k , and in the following development we simplify notation by letting $\hat{X}_i := X_{i|k}$.

In this paper, we use Chebyshev's inequality to handle probabilistic constraints. The advantages of this approach are that it can cope with arbitrary or unknown disturbance probability distributions (the only information required being the first two moments of the predicted state trajectory), and furthermore it results in quadratic inequalities that are straightforward to implement. Approximating (6) by direct application of the two-sided Chebyshev inequality [17], we obtain

$$\frac{\text{tr}(C^T C \hat{X}_i) + \|C \bar{x}_{i|k}\|^2}{t^2} \leq \beta_{i|k}, \quad i = 0, \dots, N-1 \quad (12)$$

$$\sum_{i=0}^{N-1} \gamma^i \beta_{i|k} + f(\bar{x}_{N|k}) \leq \varepsilon_k, \quad (13)$$

where $\{\beta_{i|k}\}_{i=0}^{N-1}$ is a sequence of non-negative scalars. The terminal term $f(\bar{x}_{N|k})$ in (13) is chosen so that

$$f(\bar{x}_{N|k}) = \frac{\text{tr}(C^T C \tilde{S})}{t^2} + \frac{\gamma^N}{t^2} \left[\|\bar{x}_{N|k} - x^r\|_{\tilde{P}}^2 + \frac{\|x^r\|_{C^T C}^2}{(1-\gamma)} \right] + \frac{2\gamma^N (x^r)^T C^T C (I - \gamma\Phi)^{-1} (\bar{x}_{N|k} - x^r)}{t^2} \quad (14)$$

where $\tilde{S} \succ 0$, $\tilde{P} \succ 0$, and $I - \gamma\Phi$ is invertible since $\gamma\Phi$ is strictly stable. The design of \tilde{S}, \tilde{P} is discussed in Section V.

In terms of the predicted nominal state trajectory in (9)-(10), the predicted cost is defined

$$J(\bar{x}_{0|k}, \{m_{i|k}\}_{i=0}^{N-1}, \varepsilon_k) := \|\bar{x}_{N|k} - x^r\|_P^2 + \sum_{i=0}^{N-1} \left(\|\bar{x}_{i|k} - x^r\|_Q^2 + \|m_{i|k} - u^r\|_R^2 \right) \quad (15)$$

whenever a sequence $\{\beta_{i|k}\}_{i=0}^{N-1}$ exists satisfying (12)-(13) for the given $\bar{x}_{0|k}$, $\{m_{i|k}\}_{i=0}^{N-1}$ and ε_k . On the other hand, if $\bar{x}_{0|k}$, $\{m_{i|k}\}_{i=0}^{N-1}$ and ε_k are such that constraints (12)-(13) are infeasible, we set $J(\bar{x}_{0|k}, \{m_{i|k}\}_{i=0}^{N-1}, \varepsilon_k) := \infty$. Note that $\|\bar{x}_{N|k} - x^r\|_P^2$ in (15) represents the terminal cost, and that $P \in \mathbb{S}_{++}^n$. The choice of P is discussed in Section IV.

To summarise, the MPC optimisation solved at time k is

$$J^*(x_k, \varepsilon_k) := \min_{\{m_{i|k}\}_{i=0}^{N-1}} J(x_k, \{m_{i|k}\}_{i=0}^{N-1}, \varepsilon_k), \quad (16)$$

and its solution for any feasible x_k and ε_k is denoted

$$\{m_{i|k}^*(x_k, \varepsilon_k)\}_{i=0}^{N-1} := \arg \min_{\{m_{i|k}\}_{i=0}^{N-1}} J(x_k, \{m_{i|k}\}_{i=0}^{N-1}, \varepsilon_k). \quad (17)$$

For simplicity we write this solution as $\{m_{i|k}^*\}_{i=0}^{N-1}$, with the understanding that this sequence depends on x_k and ε_k . The corresponding nominal predicted state trajectory is given by

$$\bar{x}_{i|k}^* = A^i x_k + \sum_{j=0}^{i-1} A^{i-1-j} B m_{j|k}^*, \quad i = 1, \dots, N \quad (18)$$

$$\bar{x}_{N+i|k}^* = \Phi^i (\bar{x}_{N|k}^* - x^r) + x^r, \quad i = 1, 2, \dots \quad (19)$$

The MPC law at time k is defined by

$$u_k := m_{0|k}^*, \quad (20)$$

and the closed loop system dynamics are given by

$$x_{k+1} = A x_k + B m_{0|k}^*(x_k, \varepsilon_k) + \omega_k, \quad (21)$$

where ω_k is the disturbance realisation at time k .

In the remainder of this paper we discuss how to choose ε_k , K , P , \tilde{P} and \tilde{S} so as to guarantee quadratic stability and satisfaction of the constraint (2) under the MPC law (20).

III. RECURSIVE FEASIBILITY

Recursively feasible MPC strategies have the property that the MPC optimisation problem is guaranteed to be feasible at every time-step if it is initially feasible. This property can be ensured by imposing a terminal constraint that requires the predicted system state to lie in a particular set at the end of the prediction horizon [6]. For a deterministic MPC problem, if an optimal solution can be found at current time, then the

tail sequence, namely the optimal control sequence shifted by one time-step, will be a feasible suboptimal solution at the next time instant if the terminal constraint is defined in terms of a suitable invariant set for the predicted system state [18], [19]. For a robust MPC problem with bounded additive disturbances, recursive feasibility can likewise be guaranteed under either open or closed loop optimisation strategies by imposing a terminal constraint set that is robustly invariant. However, this approach is not generally applicable to systems with unbounded additive disturbances, and in general it is not possible to ensure recursive feasibility in this context while guaranteeing constraint satisfaction at every time instant.

In this section we propose a method for guaranteeing recursive feasibility of the MPC optimisation that does not rely on terminal constraints. Instead recursive feasibility is ensured, despite the presence of unbounded disturbances, by allowing the constraint on the discounted sum of probabilities to be time-varying. For all time-steps $k > 0$, the approach uses the optimal sequence computed at time $k-1$ to determine a value of ε_k that is necessarily feasible at time k . Using this approach it is possible to choose ε_0 so that the original constraint (2) is satisfied, as we discuss in Section V.

We use the notation $\mathcal{S}(\{m_{i|k}^*\}_{i=0}^{N-1})$ to denote a nominal control sequence derived from a time-shifted version of $\{m_{i|k}^*\}_{i=0}^{N-1}$, defined by

$$\mathcal{S}(\{m_{i|k}^*\}_{i=0}^{N-1}) := \{m_{i+1|k}^* + K \Phi^i \omega_k\}_{i=0}^{N-1}, \quad (22)$$

with $m_{N|k}^* := K(\bar{x}_{N|k}^* - x^r) + u^r$. Note that the disturbance realisation ω_k can be computed given the measured state x_{k+1} and hence the sequence $\mathcal{S}(\{m_{i|k}^*\}_{i=0}^{N-1})$ is available to the controller at time $k+1$.

Lemma 1: The MPC optimisation (16) is recursively feasible if ε_k is defined at each time $k = 1, 2, \dots$ as

$$\varepsilon_k := \min \left\{ \varepsilon \mid J(x_k, \mathcal{S}(\{m_{i|k-1}^*\}_{i=0}^{N-1}), \varepsilon) < \infty \right\}. \quad (23)$$

Proof: The definition of the MPC predicted cost implies that, for any given sequence $\{m_{i|k}\}_{i=0}^{N-1}$, there necessarily exists a value of ε such that $J(x_k, \{m_{i|k}\}_{i=0}^{N-1}, \varepsilon)$ is finite. Moreover $\mathcal{S}(\{m_{i|k-1}^*\}_{i=0}^{N-1})$ is (with probability 1) well-defined at time k if the MPC optimisation is feasible at time $k-1$. It follows that the minimum value of ε defining ε_k in (23) exists if the MPC optimisation is feasible at time $k-1$, and this establishes recursive feasibility ■

The sequence $\mathcal{S}(\{m_{i|k}^*\}_{i=0}^{N-1})$ can be regarded as the tail of the minimiser (17) with adjustments. With equations (9) and (10), the minimisation (23) can be solved to give an explicit expression for ε_k for all $k > 0$ as

$$\varepsilon_k = \sum_{i=0}^{N-1} \gamma^i \frac{\text{tr}(C^T C \hat{X}_i) + \|C(\bar{x}_{i+1|k-1}^* + \Phi^i \omega_{k-1})\|^2}{t^2} + f(\bar{x}_{N+1|k-1}^* + \Phi^N \omega_{k-1}). \quad (24)$$

Essentially, the optimisation problem to be solved at each time step is feasible because the parameter ε_k is updated via (24) using knowledge of the disturbance w_{k-1} obtained from the measurement of the current state x_k . In this respect

the approach is similar to constraint-tightening methods that have previously been applied in the context of stochastic MPC (e.g. [7], [8], [9]) in order to ensure recursive feasibility and constraint satisfaction in closed loop operation. However, each of these methods requires that the disturbances affecting the controlled system are bounded, and they become more conservative as the degree of conservativeness of the assumed disturbance bounds increases. The approach proposed here avoids this requirement and instead ensures closed loop constraint satisfaction using the analysis of Section V.

The key to this method lies in the definition of the sequence $\mathcal{S}(\{m_{i|k}^*\}_{i=0}^{N-1})$. If this sequence were optimised simultaneously with ε_k , rather than defined by the suboptimal control sequence (22), then it would be possible to reduce the MPC cost (16). However this would require more computational effort than is needed to evaluate (24). For deterministic MPC problems it can be shown that the cost of using the tail sequence is no greater than the optimal cost at the current time with an appropriate terminal weighting matrix [15], but this property cannot generally be ensured in the presence of unbounded disturbances. In fact the optimal cost defined by (16) is not necessarily monotonically non-increasing if ε_k is defined by (24), but the proposed approach based on the adjusted tail sequence (22) ensures a quadratic closed loop stability bound, as we discuss next.

IV. SMPC ALGORITHM

This section analyses the stability of the MPC law and shows that the closed loop system satisfies a quadratic stability condition. We first state the MPC algorithm based on the optimisation defined in (16).

Algorithm 1: At each time-step $k = 0, 1, \dots$:

- (i). Measure x_k , and if $k > 0$, compute ε_k using (24).
- (ii). Solve the quadratically constrained quadratic programming (QCQP) problem:

$$\begin{aligned} & \underset{\{m_{i|k}, \beta_{i|k}\}_{i=0}^{N-1}}{\text{minimise}} \sum_{i=0}^{N-1} \left(\|\bar{x}_{i|k} - x^r\|_Q^2 + \|m_{i|k} - u^r\|_R^2 \right) \\ & \quad + \|\bar{x}_{N|k} - x^r\|_P^2 \end{aligned}$$

subject to (12), (13), and (9) with $\bar{x}_{0|k} = x_k$.

- (iii). Apply the control law $u_k = m_{0|k}^*$.

Although the MPC optimisation in step (ii) involves a quadratic constraint as well as linear constraints, it can be solved efficiently, for example using a second-order conic program (SOCP) solver, since the objective and the quadratic constraint are both convex.

Theorem 2: Given initial feasibility at $k = 0$, the minimisation in step (ii) of Algorithm 1 is feasible for $k = 1, 2, \dots$ and the closed loop system satisfies the quadratic stability condition

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E} \left[\|x_k - x^r\|_Q^2 + \|u_k - u^r\|_R^2 \right] \leq \text{tr}(WP) \quad (25)$$

provided K in (7)-(8) and P in (15) are chosen so that

$$P = \Phi^T P \Phi + K^T R K + Q. \quad (26)$$

Proof: From Lemma 1, the sequence $\mathcal{S}(\{m_{i|k}^*\}_{i=0}^{N-1})$ provides a feasible suboptimal solution at time $k+1$. Hence by optimality we necessarily have

$$J^*(x_{k+1}, \varepsilon_{k+1}) \leq J(x_{k+1}, \mathcal{S}(\{m_{i|k}^*\}_{i=0}^{N-1}), \varepsilon_{k+1}),$$

and since this inequality holds for every realisation of ω_k , by taking expectations conditioned on the state x_k we obtain

$$\mathbb{E}_k [J^*(x_{k+1}, \varepsilon_{k+1})] \leq \mathbb{E}_k [J(x_{k+1}, \mathcal{S}(\{m_{i|k}^*\}_{i=0}^{N-1}), \varepsilon_{k+1})]. \quad (27)$$

Evaluating $\bar{x}_{i|k+1}$ by setting $\bar{x}_{0|k+1} = x_{k+1}$ and $m_{i|k+1} = m_{i+1|k}^* + K\Phi^i \omega_k$ in (9)-(10) gives the feasible sequence

$$\bar{x}_{i|k+1} = \bar{x}_{i+1|k}^* + \Phi^i \omega_k, \quad i = 0, \dots, N,$$

and from (26) and (27) it follows that

$$\begin{aligned} \mathbb{E}_k [J^*(x_{k+1}, \varepsilon_{k+1})] & \leq J^*(x_k, \varepsilon_k) - \|x_k - x^r\|_Q^2 \\ & \quad - \|u_k - u^r\|_R^2 + \text{tr}(WP). \end{aligned} \quad (28)$$

Summing both sides of this inequality over $k \geq 0$ after taking expectations given information available at time $k = 0$, and making use of the property that $\mathbb{E}_0 [\mathbb{E}_k [J^*(x_{k+1}, \varepsilon_{k+1})]] = \mathbb{E}_0 [J^*(x_{k+1}, \varepsilon_{k+1})]$, gives (25). ■

Stability is the overriding requirement and in most recent MPC literature the cost function is chosen so as to provide a Lyapunov function suitable for analysing closed loop stability [15]. Theorem 2 is proved via cost comparison, and, given the quadratic form of the cost function, this analysis results in the quadratic stability condition (25). Similar asymptotic bounds on the time average of a quadratic expected stage cost are obtained in [5], [20]. However, in the current context, Theorem 2 demonstrates that an MPC algorithm can ensure closed loop stability without imposing terminal constraints derived from an invariant set.

Lemma 3: If K in (7)-(8) is the unconstrained LQ-optimal feedback gain, K_{LQ} , for the system (1) with cost (3), then for the closed loop system under the control strategy of Algorithm 1, the control law $u_k = m_{0|k}^*$ converges as $k \rightarrow \infty$ to the unconstrained optimal feedback law $u_k = K_{LQ} x_k$.

Proof: Consider a system with the same model parameters A, B, W as (1), and a stabilizing linear feedback law with gain K . Denoting the states and control inputs of this system respectively as \hat{x}_k and $\hat{u}_k = K\hat{x}_k$, we have

$$\lim_{k \rightarrow \infty} \mathbb{E} \left[\|\hat{x}_k - x^r\|_Q^2 + \|\hat{u}_k - u^r\|_R^2 \right] = \text{tr}(WP) \quad (29)$$

where P is the solution of (26). However the certainty equivalence theorem [21] implies that $\text{tr}(WP)$ is minimized with $K = K_{LQ}$. Therefore (25) implies

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E} \left[\|x_k - x^r\|_Q^2 + \|u_k - u^r\|_R^2 \right] \\ = \lim_{T \rightarrow \infty} \mathbb{E} \left[\|\hat{x}_T - x^r\|_Q^2 + \|\hat{u}_T - u^r\|_R^2 \right], \end{aligned}$$

so that $u_k \rightarrow K_{LQ} x_k$ as $k \rightarrow \infty$ under Assumption 1. ■

The convergence result in Lemma 3 is to be expected because of the discounted constraint (2). Since $\gamma^k \rightarrow 0$ as $k \rightarrow \infty$, the probabilistic constraint places greater emphasis

on near-future predicted states and ignores asymptotic behaviour. Under this condition the unconstrained LQ-optimal feedback control law is asymptotically optimal for (3).

V. THE BEHAVIOUR OF THE SEQUENCE $\{\varepsilon_k\}_{k=0}^\infty$ AND CONSTRAINT SATISFACTION

This section considers the properties of the sequence $\{\varepsilon_k\}_{k=0}^\infty$ in closed loop operation under Algorithm 1. We first give expressions for the parameters \tilde{S} and \tilde{P} in the definition (14) of the terminal term $f(\bar{x}_{N|k})$. Then, using the explicit expression for ε_k in (24), we derive a recurrence equation relating the expected value of ε_{k+1} to x_k and ε_k . This allows an upper bound to be determined for the sum of discounted violation probabilities on the left hand side of (2). With this bound we can show that the closed loop system under the control law of Algorithm 1 satisfies the chance constraint (2) if ε_k is initialised with $\varepsilon_0 = e$.

Lemma 4: Let \tilde{S} and \tilde{P} be the solutions of

$$\tilde{P} = \gamma \Phi^T \tilde{P} \Phi + C^T C \quad (30)$$

$$\tilde{S} = \gamma \Phi \tilde{S} \Phi^T + \frac{\gamma^{N+1}}{1-\gamma} W + \gamma^N \hat{X}_N. \quad (31)$$

Then $f(\bar{x}_{N|k})$ defined in (14) satisfies

$$f(\bar{x}_{N|k}) = \sum_{i=N}^{\infty} \gamma^i \frac{\text{tr}(C^T C \hat{X}_i) + \|C \bar{x}_{i|k}\|^2}{t^2} \quad (32)$$

where $\bar{x}_{i|k}$ is given by (10) for all $i \geq N$.

Proof: Writing $\|C \bar{x}_{i|k}\|^2 = \|C(\bar{x}_{i|k} - x^r) + Cx^r\|^2$ and using (10), we obtain

$$\begin{aligned} \|C \bar{x}_{i|k}\|^2 &= \|C \Phi^{i-N} (\bar{x}_{N|k} - x^r)\|^2 \\ &\quad + 2(x^r)^T C^T C \Phi^{i-N} (\bar{x}_{N|k} - x^r) + \|Cx^r\|^2 \end{aligned}$$

for all $i \geq N$, and since \tilde{P} satisfies (30), we have

$$\begin{aligned} \sum_{i=N}^{\infty} \frac{\gamma^i}{t^2} \|C \bar{x}_{i|k}\|^2 &= \frac{\gamma^N}{t^2} \|\bar{x}_{N|k} - x^r\|_{\tilde{P}}^2 + \frac{\gamma^N}{(1-\gamma)} \frac{\|x^r\|_{C^T C}^2}{t^2} \\ &\quad + \frac{2\gamma^N (x^r)^T C^T C (I - \gamma \Phi)^{-1} (\bar{x}_{N|k} - x^r)}{t^2}. \end{aligned} \quad (33)$$

Furthermore, if $\tilde{S} = \sum_{i=N}^{\infty} \gamma^i \hat{X}_i$, then \tilde{S} is the solution of the Lyapunov equation (31) since (11) implies

$$\begin{aligned} \gamma \Phi \tilde{S} \Phi^T &= \sum_{i=N}^{\infty} \gamma^{i+1} \Phi \hat{X}_i \Phi^T = \sum_{i=N}^{\infty} \gamma^{i+1} (\hat{X}_{i+1} - W) \\ &= \tilde{S} - \gamma^N \hat{X}_N - \frac{\gamma^{N+1}}{1-\gamma} W, \end{aligned}$$

and it follows that

$$\sum_{i=N}^{\infty} \frac{\gamma^i}{t^2} \text{tr}(C^T C \hat{X}_i) = \frac{\text{tr}(C^T C \tilde{S})}{t^2}. \quad (34)$$

Combining (33) and (34), it is clear that (32) is equivalent to (14) if \tilde{P} and \tilde{S} are defined by (30) and (31). ■

The following result gives the relationship between ε_k and the expected value of ε_{k+1} for the closed loop system.

Theorem 5: If ε_k is defined by (24) at all times $k \geq 1$, then in closed loop operation under Algorithm 1 we have

$$\gamma \mathbb{E}_k [\varepsilon_{k+1}] \leq \varepsilon_k - \frac{\|Cx_k\|^2}{t^2} \quad (35)$$

for all $k \geq 0$.

Proof: Evaluating ε_{k+1} using (24) and (32) gives

$$\varepsilon_{k+1} = \sum_{i=0}^{\infty} \gamma^i \frac{\text{tr}(C^T C \hat{X}_i) + \|C(\bar{x}_{i+1|k}^* + \Phi^i \omega_k)\|^2}{t^2},$$

where $\bar{x}_{i|k}^*$ is given by (18)-(19) and ω_k is the realisation of the disturbance at time k . Taking expectations conditioned on information available at time k , this implies

$$\begin{aligned} \gamma \mathbb{E}_k [\varepsilon_{k+1}] &= \sum_{i=0}^{\infty} \gamma^{i+1} \frac{\text{tr}(C^T C \hat{X}_i) + \|C \bar{x}_{i+1|k}^*\|^2}{t^2} \\ &\quad + \sum_{i=0}^{\infty} \gamma^{i+1} \frac{\text{tr}(C^T C \Phi^i W (\Phi^i)^T)}{t^2}, \end{aligned}$$

but feasibility of the sequence $\{m_{i|k}^*\}_{i=0}^{N-1}$ at time k implies $\sum_{i=0}^{\infty} \frac{\gamma^i}{t^2} [\text{tr}(C^T C \hat{X}_i) + \|C \bar{x}_{i|k}^*\|^2] \leq \varepsilon_k$ and therefore

$$\begin{aligned} \gamma \mathbb{E}_k [\varepsilon_{k+1}] &\leq \varepsilon_k - \frac{\|Cx_k\|^2}{t^2} \\ &\quad + \sum_{i=0}^{\infty} \frac{\gamma^i}{t^2} \text{tr}[C^T C (\gamma \Phi^i W \Phi^{iT} + (\gamma - 1) \hat{X}_i)]. \end{aligned}$$

To complete the proof we note that the sum on the RHS of this inequality is zero since

$$\gamma^{i+1} \Phi^i W \Phi^{iT} + (\gamma^{i+1} - \gamma^i) \hat{X}_i = \gamma^{i+1} \hat{X}_{i+1} - \gamma^i \hat{X}_i,$$

and because $\hat{X}_0 = 0$ and $\lim_{i \rightarrow \infty} \gamma^i \hat{X}_i = 0$. ■

The main result of this section is given next.

Theorem 6: The closed loop system under Algorithm 1 satisfies the chance constraint (2) if $\varepsilon_0 = e$.

Proof: Theorem 5 implies that the closed loop evolution of ε_k satisfies

$$\gamma^{i+1} \mathbb{E}_k [\varepsilon_{k+i+1}] \leq \gamma^i \mathbb{E}_k [\varepsilon_{k+i}] - \frac{\gamma^i}{t^2} \mathbb{E}_k [\|Cx_{k+i}\|^2]$$

for all non-negative integers k, i . Summing both sides of this equation over $i \in \{0, 1, \dots\}$ gives

$$\varepsilon_k \geq \sum_{i=0}^{\infty} \gamma^i \frac{\mathbb{E}_k [\|Cx_{k+i}\|^2]}{t^2} + \lim_{i \rightarrow \infty} \gamma^i \mathbb{E}_k [\varepsilon_{k+i}]. \quad (36)$$

But $\gamma^i \mathbb{E}_k [\varepsilon_{k+i}]$ is necessarily non-negative for all $k, i \geq 0$, so by Chebyshev's inequality this implies

$$\sum_{i=0}^{\infty} \gamma^i \mathbb{P}(\|Cx_{k+i}\| \geq t) \leq \varepsilon_k \quad (37)$$

for all $k \geq 0$. An obvious consequence of the bound (37) is that the closed loop system will satisfy the chance constraint (2) if ε_0 is chosen to be equal to e . ■

The presence of the factor $\gamma \in (0, 1)$ on the LHS of (35) implies that the expected value of ε_k can increase as well as decrease along closed loop system trajectories. In fact, for

values of γ close to zero, a rapid initial growth in ε_k is to be expected, which is in agreement with the interpretation that the constraint (2) penalises violation probabilities more heavily at times closer to the initial time in this case. On the other hand, for values of γ close to 1, ε_k can be expected to decrease initially, implying a greater emphasis on the expected number of violations over some initial horizon.

VI. NUMERICAL EXAMPLE

This section describes a numerical example illustrating the quadratic stability and constraint satisfaction of the closed loop system (21) under Algorithm 1. Consider a system with

$$A = \begin{bmatrix} 1 & 2 \\ 1.5 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1.2 \\ 1.5 \end{bmatrix},$$

and Gaussian disturbance $\omega_k \sim \mathcal{N}(0, W)$ with covariance matrix $W = 0.2I_{2 \times 2}$. The constraint (2) is defined by $\gamma = 0.9$, $t = 1$, $e = 3.5$, $C = \begin{bmatrix} 0.6 & 0.52 \end{bmatrix}$, and the weighting matrices in the cost (3) are given by

$$Q = C^T C = \begin{bmatrix} 0.3600 & 0.3120 \\ 0.3120 & 0.2704 \end{bmatrix}, \quad R = 1.$$

Input and state references are $u^r = -0.6$, $x^r = (0.72, 0.36)$, and the prediction horizon is chosen as $N = 7$. The feedback gain is chosen as $K = [-0.92 \ -0.85]$ for the cost (3), and matrices P , \tilde{P} and \tilde{S} are chosen to satisfy (26), (30) and (31). The initial value for ε_k is $\varepsilon_0 = e = 3.5$.

Two sets of simulations (A and B) demonstrate the closed loop stability result in Theorem 2 and the constraint satisfaction result in Theorem 6, respectively.

Simulation A: To estimate empirically the average cost,

$$\bar{J} := \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E} \left[\|x_k - x^r\|_Q^2 + \|u_k - u^r\|_R^2 \right],$$

we consider the mean value of the stage cost over 100 simulations. Each simulation has a randomly selected initial condition ($x_0 \sim \mathcal{N}(0, I)$, with infeasible values discarded), and a length of $T = 500$ time-steps. This gives the estimated average cost as $\bar{J} \approx 0.5036$, which is no greater than $\text{tr}(WP) = 0.5304$, and hence agrees with the bound (25). Moreover, the estimate of \bar{J} decreases considerably more slowly as the simulation length T increases.

Simulation B: To test numerically whether the chance constraint (2) is satisfied, we estimate the discounted sum of violation probabilities on the LHS of (2),

$$V := \sum_{k=0}^{\infty} \gamma^k \mathbb{P}(\|Cx_k\| \geq t),$$

by counting the number of violations at $k \in \{0, \dots, T-1\}$, for 10^3 simulations with $x_0 = (-1.1130, 1.1156)^T$ and $T = 100$. This gives $V \approx 0.8328$, which is less than $e = 3.5$ and hence satisfies the constraint (2). For this example we have $\gamma^{100} \approx 10^{-5}$, so increasing T beyond 100 time-steps has negligible effect on the estimate of V . Therefore the discrepancy between e and the estimated value of V can be attributed to the conservativeness of Chebyshev's inequality. In addition, if the unconstrained LQ-optimal feedback law $u_k = K_{LQ}(x_k - x^r) + u^r$ were employed, the value of the

bound $\sum_{k=0}^{\infty} \gamma^k \mathbb{E}_k [\|Cx_k\|^2] / t^2$ in (36) would be 4.6998, which exceeds e . Hence this control law may not satisfy (2) and is worse than the MPC law (20) in terms of this bound.

VII. CONCLUSIONS

A stochastic MPC algorithm that imposes constraints on the sum of discounted future constraint violation probabilities can ensure recursive feasibility of the online optimisation and closed loop constraint satisfaction. Key features are the design of a constraint-tightening procedure and closed loop analysis of the tightening parameters. The MPC algorithm requires knowledge of the first and second moments of the disturbance, and is implemented as a convex QCQP problem.

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