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Nonlinear Reference Tracking with Model Predictive Control: An Intuitive Approach

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Abstract— **In this paper, we study the system theoretic properties of a reference tracking Model Predictive Control (MPC) scheme for general reference signals and nonlinear discrete-time systems subject to input and state constraints. Contrary to other existing theoretical results for reference tracking MPC, we do not modify the optimization problem with artificial references or terminal ingredients. Instead, we consider a simple, intuitive, implementation and derive theoretical guarantees under a stabilizability property of the system and a reachability condition on the reference trajectory. We provide sufficient conditions for exponential reference tracking and analyze the region of attraction.**

I. Introduction

Model Predictive Control (MPC) [1] is an optimization based control method, that can handle general nonlinear dynamics and constraints. Most of the early theoretical results for MPC are based on setpoint stabilization with terminal constraints [2], [3]. These results have been extended to account for setpoint stabilization without terminal constraints [4], [5], [6], which avoids difficult and conservative offline computations for terminal ingredients. Many applications require a more general tracking of dynamic references. Correspondingly, there is a research effort to extend results for setpoint stabilization to reference tracking.

Previous results in this direction are mostly focused on modifying the optimization problem with terminal constraints and/or artificial references in order to guarantee stability [7], [8], [9].

In [7], [8] terminal sets and terminal costs are designed around a reference trajectory, which is an extension of [3] to the reference tracking case. In [9] the more general case of regulating an output or tracking unreachable reference trajectories is considered. These results require the usage of artificial reference trajectories and terminal constraints to ensure recursive feasibility. Unpredictably changing reference trajectories are treated in [10], [11]. The corresponding procedures either rely on robust control invariant sets or terminal constraints with artificial reference trajectories.

Contrary to the above mentioned results, in this work we consider a simple reference tracking formulation without terminal constraints and derive sufficient conditions involving the system dynamics and the reference trajectory to guarantee desirable closed-loop properties. In particular, we consider the notion of local incremental stabilizability and show that it is a sufficient system property.

If it is possible to compute terminal ingredients for a given reference trajectory as in [7], [8], then the scheme proposed in this paper is able to ensure exponential convergence to the reference trajectory without complex design procedures. This enables us to derive closed-loop guarantees for large classes of reference trajectories.

In this paper, we focus on the basic case of reachable reference trajectories and leave the more general case of unreachable reference trajectories and output regulation to future work.

The paper is structured as follows: Section II presents the reference tracking MPC scheme. Section III provides sufficient conditions for exponential stability of the error dynamics and analyzes the region of attraction. Section IV provides a numerical example to illustrate the theoretical results. The paper concludes with section V.

II. Reference Tracking Model Predictive Control

A. Notation

The quadratic norm with respect to a positive definite matrix $Q = Q^\top$ is denoted by $\|x\|_Q^2 := x^\top Q x$. The minimal and maximal eigenvalue of a symmetric matrix $Q = Q^\top$ is denoted by $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$, respectively. The positive real numbers are denoted by $\mathbb{R}_{>0} = \{r \in \mathbb{R} | r > 0\}$, $\mathbb{R}_{\geq 0} = \mathbb{R}_{>0} \cup \{0\}$.

B. Setup

We consider the following nonlinear discrete-time system

$$x(t+1) = f(x(t), u(t)). \tag{1}$$

with the state $x \in \mathbb{R}^n$, the control input $u \in \mathbb{R}^m$ and time step $t \in \mathbb{N}$. We impose point-wise in time[1] constraints on the state and input

$$(x(t), u(t)) \in \mathcal{Z} = \mathcal{X} \times \mathcal{U}. \tag{2}$$

Given a reference trajectory $x_{\text{ref}}(\cdot)$, we define the tracking error $e(t) := x(t) - x_{\text{ref}}(t)$. The control goal is to achieve constraint satisfaction $(x(t), u(t)) \in \mathcal{Z}$, $\forall t \geq 0$ and exponential error convergence $\lim_{t\to\infty} \|e(t)\| = 0$ for a set of initial conditions, called the region of attraction.

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[1]The following derivations can be extended to time-varying constraint sets $\mathcal{Z}(t)$ and time-varying dynamics $f(x, u, t)$.

C. MPC formulation

The open-loop cost of an input sequence $u(\cdot|t) \in \mathbb{R}^{m\times N}$ at time step t is defined as

$$J_N(x(t), t, u(\cdot|t)) = \sum_{k=0}^{N-1} \|e(k|t)\|_Q^2, \quad (3)$$
$$x(0|t) = x(t), \quad x(k+1|t) = f(x(k|t), u(k|t)),$$
$$e(k|t) = x(k|t) - x_{\mathrm{ref}}(t+k), \quad 0 \le k \le N-1,$$

with the current measured state $x(t)$, the predicted tracking error $e(\cdot|t)$ and a positive definite weighting matrix $Q = Q^\top \in \mathbb{R}^{n\times n}$. The proposed reference tracking MPC scheme is characterized by the solution of the following optimization problem

$$V_N(x(t), t) = \min_{u(\cdot|t)} J_N(x(t), t, u(\cdot|t)) \quad (4)$$
$$\text{s.t. } x(k|t) \in \mathcal{X},\ u(k|t) \in \mathcal{U}.$$

The solution is the value function V_N, and optimal state and input trajectories $(x^*(\cdot|t),\ u^*(\cdot|t))$ that satisfy the constraints (2) and minimize the deviation from the reference trajectory over the prediction horizon $N \in \mathbb{N}$. We define $V_N(x(t), t) = \infty$, in case there exists no feasible solution. We denote the resulting MPC feedback by $\mu_N(x(t), t) = u^*(0|t)$. The resulting closed-loop system is given by

$$x(t+1) = f(x(t), \mu_N(x(t), t)) = x^*(1|t), \quad t \ge 0. \quad (5)$$

The error dynamics can be expressed as a time-varying system

$$e(t+1) = g(e(t), t), \quad t \ge 0. \quad (6)$$

III. Exponential Reference Tracking

This section establishes suitable conditions, under which the proposed reference tracking scheme stabilizes the desired reference trajectory. The result established in Theorem 2 is a straightforward extension of established results for setpoint stabilization [5]. The main contributions in this context are in Section III-B and III-C, framing the required assumptions in a context suitable for reference tracking MPC, and the resulting Theorem 10, which is an extension and variation of the results in [12].

A. Local Stability

We use an assumption, similar to the assumption used in MPC without terminal constraints for setpoint stabilization [12].

Assumption 1. *There exist constants $\gamma,\ c \in \mathbb{R}_{>0}$, such that for all $\|e(t)\|_Q^2 \le c$ and all $N \ge 2$*

$$V_N(x(t), t) \le \gamma \|e(t)\|_Q^2. \quad (7)$$

This assumption closely relates to the exponential stabilizability of the reference trajectory. The restriction to a local property via the constant c enables us to verify this assumption in Section III-B based on a local stabilizability property and suitable conditions on the reference trajectory.

The following theorem establishes local exponential stability of the error dynamics of the closed-loop system.

Theorem 2. *Let Assumption 1 hold. Assume that the initial state $x(0)$ satisfies $V_N(x(0), 0) \le c\gamma$. There exists a N_0, such that for all $N > N_0$, there exists a $\alpha_N \in \mathbb{R}_{>0}$, such that the closed-loop system (5) satisfies*

$$\|e(t)\|_Q^2 \le V_N(x(t), t) \le \gamma \|e(t)\|_Q^2,$$
$$V_N(x(t+1), t+1) - V_N(x(t), t) \le -\alpha_N \|e(t)\|_Q^2,$$

for all $t \ge 0$. The origin $e = 0$ is uniformly exponentially stable under the closed-loop error dynamics (6).

Proof. The proof is closely related to the stability proof of MPC without terminal constraints [5, Variant 1] and split into three parts. Part I and II show the desired properties at time $t \ge 0$, assuming $V_N(x(t), t) \le c\gamma$. Part III establishes that $V_N(x(t), t) \le c\gamma$ holds recursively.

Part I: A lower bound on $V_N(x(t), t)$ is given by

$$V_N(x(t), t) = \sum_{k=0}^{N-1} \|e^*(k|t)\|_Q^2 \ge \|e(t)\|_Q^2.$$

Assumption 1 in combination with $V_N(x(t), t) \le c\gamma$ implies $V_N(x(t), t) \le \gamma \|e(t)\|_Q^2$. To see this we make a case distinction, whether $\|e(t)\|_Q^2 \le c$ or not. If $\|e(t)\|_Q^2 \le c$ the assertion directly follows from Assumption 1. Else if $\|e(t)\|_Q^2 > c$, we have $V_N(x(t), t) \le c\gamma \le \gamma\|e(t)\|_Q^2$. Thus, an upper bound to $V_N(x(t), t)$ is given by

$$V_N(x(t), t) \le \gamma \|e(t)\|_Q^2.$$

Part II: Show that V_N decreases for large enough N.

Given $V_N(x(t), t) \le \gamma\|e(t)\|_Q^2$, there exists a $k_x \in \{0, \dots, N-1\}$, such that

$$\|e^*(k_x|t)\|_Q^2 \le \frac{V_N(x(t), t)}{N} \le \frac{\gamma}{N}\|e(t)\|_Q^2.$$

For $N > \gamma$, we further know that $k_x \ge 1$. Correspondingly, the value function at the next time step satisfies

$$\begin{aligned}
&V_N(x(t+1), t+1) \\
\le &\sum_{k=1}^{k_x - 1} \|e^*(k|t)\|_Q^2 + V_{N-k_x+1}(x^*(k_x|t), t+k_x) \\
= &\sum_{k=1}^{k_x} \|e^*(k|t)\|_Q^2 - \|e^*(k_x|t)\|_Q^2 + V_{N-k_x+1}(x^*(k_x|t), t+k_x).
\end{aligned}$$

We have $\|e^*(k_x|t)\|_Q^2 \le \frac{V_N(x(t),t)}{N} \le \frac{c\gamma}{N} \le c$ and thus we can use Assumption 1 to bound V_{N-k_x+1}, which yields

$$\begin{aligned}
&V_N(x(t+1), t+1) \\
\le &\underbrace{\sum_{k=1}^{k_x} \|e^*(k|t)\|_Q^2}_{\le V_N(x(t),t) - \|e(t)\|_Q^2} + (\gamma - 1)\underbrace{\|e^*(k_x|t)\|_Q^2}_{\le \gamma/N \|e(t)\|_Q^2} \\
\le &V_N(x(t), t) - \underbrace{\frac{N - \gamma(\gamma-1)}{N}}_{:=\alpha_N} \|e(t)\|_Q^2. \quad (8)
\end{aligned}$$

For $N > N_0 := \gamma \max\{1, \gamma - 1\}$, we have

$$V_N(x(t+1), t+1) - V_N(x(t), t) \leq -\alpha_N \|e(t)\|_Q^2,$$

with $\alpha_N \in \mathbb{R}_{>0}$.

Part III: Recursive Feasibility and Exponential Stability

Based on the decrease condition in Part II, the condition $V_N(x(t), t) \leq c\gamma$ is recursively satisfied and the derivations in Part I and II hold for all $t \geq 0$.

Thus, for any prediction horizon $N > N_0$ and any initial condition $V_N(x(0), 0) \leq c\gamma$, the tracking error $e = 0$ is uniformly exponentially stable using standard Lyapunov arguments [13] □

Remark 3. *Less conservative estimates on the sufficient prediction horizon N_0 can be obtained using the methods proposed in [5, Variant 2], but are avoided for simplicity.*

B. Conditions based on System and Reference Trajectory

Assumption 1 might be difficult to verify directly, especially in the case of generally dynamic reference trajectories and hard state and input constraints. To obtain verifiable sufficient conditions, we propose to formulate this assumption in a Lyapunov context of incremental stability [14], [15], [16] and to decouple the assumption on the dynamics, the reference trajectory and the constraints.

Assumption 4. *There exists a control law $\kappa : \mathcal{X} \times \mathcal{X} \times \mathcal{U} \to \mathbb{R}^m$, a δ-Lyapunov function $V_\delta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$, that is continuous in the first argument and satisfies $V_\delta(x, x) = 0\ \forall x \in \mathcal{X}$, and parameters $c_{\delta,l},\ c_{\delta,u},\ \delta_{loc} \in \mathbb{R}_{>0}$, $\rho \in (0, 1)$, such that the following properties hold for all $x, z \in \mathcal{X}$, with $V_\delta(x, z) \leq \delta_{loc}$:*

$$c_{\delta,l}\|x - z\|^2 \leq V_\delta(x, z) \leq c_{\delta,u}\|x - z\|^2, \tag{9}$$
$$V_\delta(x^+, z^+) \leq \rho V_\delta(x, z), \tag{10}$$

with

$$x^+ = f(x, \kappa(x, z, v)),$$
$$z^+ = f(z, v) \in \mathcal{X}, \quad v \in \mathcal{U}.$$

Remark 5. *This system property is comparable to local (exponential) stabilizability around arbitrary reference trajectories. This assumption is satisfied, if f is locally Lipschitz and the linearized dynamics around any point $(z, v) \in \mathcal{X} \times \mathcal{U}$ are stabilizable in a quasi-linear parameter varying sense. A more general and detailed description of this system property can be found in [17].*

For the following derivations it suffices if this assumption is satisfied only locally around the considered reference trajectory. Instead, we consider this assumption on the constraint set $\mathcal{X}$ to provide guarantees for generic classes of reference trajectories. The controller κ and the $\delta-$Lyapunov function V_δ are only used for the analysis and are not required in the implementation. If this assumption is satisfied with $\kappa = v$, the system has a well-defined steady state response [18].

Assumption 6. *The reference trajectory $(x_{ref}(\cdot), u_{ref}(\cdot))$ is such that*

$$x_{ref}(t + 1) = f(x_{ref}(t), u_{ref}(t))$$

and $V_\delta(x(t), x_{ref}(t)) \leq \delta_{ref}$ implies

$$x(t) \in \mathcal{X}, \quad \kappa(x(t), x_{ref}(t), u_{ref}(t)) \in \mathcal{U}$$

with V_δ, κ from Assumption 4 and some $\delta_{ref} \in \mathbb{R}_{>0}$.

The first part of this assumption ensures that the system can track the reference trajectory. The second part of this assumption is similar to using tighter constraints for the reference trajectory (compare tube-based robust MPC [19], [20]). This assumption is satisfied for to strictly reachable reference trajectories, in the sense that the reference is reachable and lies in the interior of the constraint set. The following proposition shows that we can indeed decouple the assumption on the system dynamics and the reference trajectory.

Proposition 7. *Suppose that Assumptions 4 and 6 are satisfied. Then Assumption 1 is satisfied.*

Proof. The idea of this proposition is that the controller in Assumption 4 can locally stabilize any trajectory. Using Assumption 6, this candidate solution satisfies the state and input constraints.

Part I: Consider problem (4) and the following corresponding candidate solution

$$\begin{aligned} u(k|t) =& \kappa(x(k|t), x_{\text{ref}}(t + k), u_{\text{ref}}(t + k)), \\ x(k + 1|t) =& f(x(k|t), u(k|t)), \quad x(0|t) = x(t), \\ e(k|t) =& x(k|t) - x_{\text{ref}}(t + k). \end{aligned}$$

Abbreviate

$$V_\delta(k|t) = V_\delta(x(k|t), x_{\text{ref}}(k + t)).$$

Due to the assumed continuity of V_δ and (9), $\|e(t)\|_Q^2 \leq c$, with a small enough c implies

$$V_\delta(x(t), x_{\text{ref}}(t)) \leq c_{\delta,u}\|e(t)\|^2 \leq \frac{c_{\delta,u}}{\lambda_{\min}(Q)}\|e(t)\|_Q^2.$$

To ensure feasibility of the candidate solution we set

$$\delta := \min\{\delta_{\text{loc}}, \delta_{\text{ref}}\}, \quad c := \lambda_{\min}(Q)\delta/c_{\delta,u}, \tag{11}$$

which implies that for all $\|e(t)\|_Q^2 \leq c$, we have

$$V_\delta(0|t) \leq c_{\delta,u}\|e(t)\|^2 \leq c_{\delta,u}\frac{c}{\lambda_{\min}(Q)} = \delta.$$

Using Assumption 4 and $V_\delta(0|t) \leq \delta \leq \delta_{\text{loc}}$, (10) ensures recursive contractiveness of V_δ with

$$V_\delta(k|t) \leq \rho^k\delta \leq \delta \leq \delta_{\text{ref}}.$$

Thus, Assumption 6 ensures $(x(k|t), u(k|t)) \in \mathcal{X} \times \mathcal{U}$, for all $k \in \{0, \ldots, N - 1\}$.

Part II: Since u is a feasible solution to (4), we have

$$V_N(x(t), t) \leq J_N(x(t), t, u(\cdot|t)).$$

Given the contractivity and the bounds on V_δ, we have

$$c_{\delta,l}\|e(k|t)\|^2 \le V_\delta(k|t) \le \rho^k c_{\delta,u}\|e(t)\|^2,$$
$$\|e(k|t)\|_Q^2 \le \frac{\lambda_{\max}(Q)c_{\delta,u}}{\lambda_{\min}(Q)c_{\delta,l}}\rho^k\|e(t)\|_Q^2.$$

Correspondingly we get

$$J_N(x(t),t,u(\cdot|t)) = \sum_{k=0}^{N-1}\|e(k|t)\|_Q^2 \le \gamma\|e(t)\|_Q^2,$$

with

$$\gamma := \frac{\lambda_{\max}(Q)c_{\delta,u}}{\lambda_{\min}(Q)c_{\delta,l}}\sum_{k=0}^{\infty}\rho^k = \frac{\lambda_{\max}(Q)c_{\delta,u}}{\lambda_{\min}(Q)c_{\delta,l}}\frac{1}{1-\rho}. \quad (12)$$

We have shown that there exists a tube around the reference trajectory, in which the value function V_N can be bounded based on the stabilizing controller κ, thus showing satisfaction of Assumption 1 with γ, c according to (11),(12). □

The above proposition shows that we can bound V_N based on a local stabilizability property. Similar results for setpoint stabilization can be found in [5], [21, Sec. 6.2].

Remark 8. *For linear systems $f(x,u) = Ax + Bu$, Assumption 4 is equivalent to stabilizability of (A,B) and can be satisfied with the discrete-time linear quadratic regulator (DLQR) solution (K,P)*

$$\kappa(x,z,v) = v + K(x-z), \quad V_\delta(x,z) = \|x-z\|_P^2,$$
$$\rho = 1-\lambda_{\min}(QP^{-1}), \quad \gamma = \lambda_{\max}(PQ^{-1}).$$

Furthermore, Assumption 6 reduces to

$$x_{ref}(t) \in \overline{\mathcal{X}} := \mathcal{X} \ominus \mathcal{X}_\delta, \quad u_{ref}(t) \in \overline{\mathcal{U}} := \mathcal{U} \ominus K\mathcal{X}_\delta,$$

where $\ominus$ denotes the Pontryagin set difference, $\mathcal{X}_\delta = \{x | x^\top Px \le \delta_{ref}\}$ and $c = \delta_{ref}\lambda_{\min}(QP^{-1})$.

Remark 9. *Proposition 7 can be extended to include input costs $\|u(k|t) - u_{ref}(k+t)\|_R^2$ in the MPC formulation* (4)*. This requires a local Lipschitz bound on the controller κ. In addition, the input reference $u_{ref}(\cdot)$ needs to be known for the implementation, which is not the case for the considered formulation.*

C. Region of Attraction

Theorem 2 is only valid locally ($V_N \le c\gamma$) and thus only guarantees a small region of attraction. In the following, we demonstrate that this result can be extended to enlarge the region of attraction by increasing the prediction horizon N.

Theorem 10. *Let Assumptions 4 and 6 hold. For any $V_{\max} \in \mathbb{R}_{>0}$, there exists a N_1 and $\gamma_{V_{\max}} \in \mathbb{R}_{>0}$, such that for all $N > N_1$, there exists a $\alpha_{2,N} \in \mathbb{R}_{>0}$, such that for all initial conditions $V_N(x(0),0) \le V_{max}$, the closed-loop system* (5) *satisfies*

$$\|e(t)\|_Q^2 \le V_N(x(t),t) \le \gamma_{V_{\max}}\|e(t)\|_Q^2,$$
$$V_N(x(t+1),t+1) - V_N(x(t),t) \le -\alpha_{2,N}\|e(t)\|_Q^2,$$

for all $t \ge 0$. The origin $e = 0$ is uniformly exponentially stable under the closed-loop error dynamics (6)*.*

Proof. Part I and II show the desired properties at time $t \ge 0$, assuming $V_N(x(t),t) \le V_{\max}$. Part III establishes that $V_N(x(t),t) \le V_{\max}$ holds recursively.

Part I: The lower bound on $V_N(x(t),t)$ is obvious. The upper bound on $V_N(x(t),t)$ can be established based on Proposition 7 and the bound $V_{\max}$. For $\|e(t)\|_Q^2 \le c$, we have $V_N(x(t),t) \le \gamma\|e(t)\|_Q^2$, with $c,\ \gamma$ according to (11),(12). Thus, we have $V_N(x(t),t) \le \gamma_{V_{\max}}\|e(t)\|_Q^2$, with $\gamma_{V_{\max}} := \max\{\gamma, \frac{V_{\max}}{c}\}$, assuming $V_N(x(t),t) \le V_{\max}$, compare [12].

Part II: Show that V_N decreases for large enough N.

Given $V_N(x(t),t) \le V_{\max}$, there exists a $k_x \in \{0,\dots,N-1\}$, such that

$$\|e^*(k_x|t)\|_Q^2 \le \frac{1}{N}V_N(x(t),t) \le \frac{V_{\max}}{N}$$

For $N > \gamma_{V_{\max}}$, we have $\|e^*(k_x|t)\|_Q^2 \le c$ and thus $V_{N-k_x+1}(x^*(k_x|t), t+k_x) \le \gamma\|e^*(k_x|t)\|_Q^2$ with Proposition 7. From $\|e^*(k_x|t)\|_Q^2 < \|e(t)\|_Q^2$ we further know $k_x \ge 1$. This yields

$$\begin{aligned}
&V_N(x(t+1),t+1)\\
\le&\sum_{k=1}^{k_x}\|e^*(k|t)\|_Q^2 - \|e^*(k_x|t)\|_Q^2\\
&+ V_{N-k_x+1}(x^*(k_x|t), t+k_x)\\
\le&V_N(x(t),t) - \|e(t)\|_Q^2 + (\gamma-1)\|e^*(k_x|t)\|_Q^2\\
\le&V_N(x(t),t) - \|e(t)\|_Q^2 + \frac{\gamma-1}{N}\underbrace{V_N(x(t),t)}_{\le\gamma_{V_{\max}}\|e(t)\|_Q^2}\\
\le&V_N(x(t),t) - \underbrace{\frac{N-(\gamma-1)\gamma_{V_{\max}}}{N}}_{=:\alpha_{2,N}}\|e(t)\|_Q^2. \quad (13)
\end{aligned}$$

For

$$N > N_1 := \gamma_{V_{\max}}\max\{1, \gamma-1\}, \quad (14)$$

we have $\alpha_{2,N} \in \mathbb{R}_{>0}$.

Part III: Recursive Feasibility and Exponential Stability:

Based on the decrease condition in Part II, $V_N(x(t),t) \le V_{\max}$ is recursively satisfied and the derivations in Part I and II hold for all $t \ge 0$. Uniform exponential stability of the error $e = 0$ follows analogous to Theorem 2.

In conclusion, for any prediction horizon $N > N_1$ and any initial condition with $V_N(x(0),0) \le V_{\max}$ the closed-loop system stabilizes the reference trajectory. □

Remark 11. *Theorem 10 implies the following closed-loop performance bound:*

$$\lim_{K\to\infty}\sum_{t=0}^{K-1}\|e(t)\|_Q^2 \le \frac{V_{\max}}{\alpha_{2,N}} \quad (15)$$

The main idea in Theorem 10 in order to establish the larger region of attraction is as follows. Due to the assumed bound on the value function ($V_{\max}$) and for a large enough prediction horizon N, there exists a point k_x with $\|e^*(k_x|t)\|_Q^2 \le c$. Thus, at this point we can append the trajectory $x^*(\cdot|t)$ with the controller κ, (which is implicitly done by invoking Proposition 7), in order to obtain a candidate solution $x(\cdot|t+1)$ for the next time step $t+1$. The corresponding candidate solution is sketched in Figure 1. Given the stabilizing property of the controller κ, we can use Proposition 7 to obtain an upper bound on the cost of the appended trajectory piece, which can be made arbitrarily small by increasing the prediction horizon N. Thus, if N is chosen large enough, the appended cost is small and the value function decreases, which implies closed-loop stability of the reference trajectory.

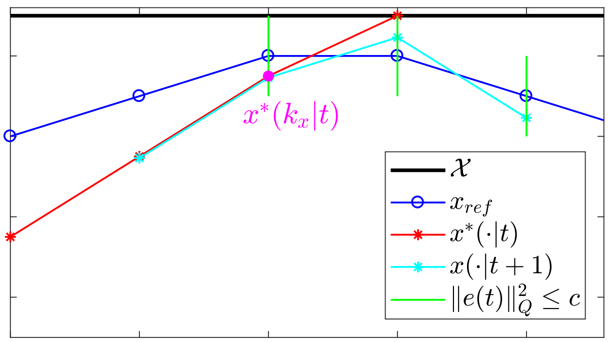

Fig. 1. Illustration - Exponential Reference Tracking

Remark 12. *The resulting theoretical guarantees are closely related to [12], where setpoint stabilization without terminal constraints based on a local stabilizability assumption around the setpoint is studied. While the assumptions and guarantees are of the same nature, the structure of the proof and the resulting bounds on the prediction horizon N are different, however.*

It is possible to derive a similar theorem based on Proposition 7 by using the method presented in [12]. This reframes the problem with a regional upper bound $\gamma_{V_{\max}} = \{\gamma, \frac{V_{\max}}{c}\}$ and then proceeds analogous to Theorem 2. In the limit for $V_{\max} \to \infty$, the bound in [12, Rk. 5] approaches $N \geq 2\frac{V_{\max}}{c}\log V_{\max}$, while the bound in Theorem 10 scales linearly with $V_{\max}$ and is thus tighter for large $V_{\max}$ (but potentially worse for large γ, see Rk. 3). In addition, the presented proof highlights some intrinsic properties in establishing regional results based on local properties, as discussed above.

Theorem 10 guarantees that we can proportionally increase the prediction horizon N and $V_{\max}$ to enlarge the region of attraction. Furthermore, $\alpha_{2,N}$ remains constant and the performance bound (15) *grows linearly with $V_{\max}$. Such simple linear connections are rather hard to conclude using the method in [12].*

To recapitulate the result of Theorem 10: Assume that the system is locally stabilizable (Ass. 4). Given any reference trajectory, that is strictly reachable in the sense of Assumption 6, there exists a large enough prediction horizon N, such that the closed-loop system stabilizes the reference trajectory (assuming the initial state can be stabilized with finite cost $V_{\max}$). Similarly, if the system is locally stabilizable on some subset $\mathcal{Z}_{\text{stab}}$, the same guarantees hold if the reference trajectory also lies in the interior of this region. This allows us to give guarantees on a whole set of reference trajectories, without adjusting the MPC algorithm to the specific reference trajectory.

Remark 13. *If it is possible to compute terminal sets around the reference based on a linearization as in [8], a time-varying version of Assumption 4 is satisfied with*

$$\kappa(x, x_{ref}(t), u_{ref}(t), t) = u_{ref}(t) + K(t)(x - x_{ref}(t)),$$
$$V_\delta(x, x_{ref}(t), t) = \|x - x_{ref}(t)\|^2_{P(t)}.$$

Thus, the proposed MPC scheme exponentially stabilizes all reference trajectories that can be stabilized with [8], provided a sufficiently large prediction horizon N is used.

The approaches in [7], [8] consider the linearization along a specific reference trajectory, leading to a linear time-varying system. The introduced notion of local incremental stabilizability on the other hand is independent of the reference trajectory and more similar to considering a linear parameter varying system.

It should be re-emphasized that the main contribution of this work is the fact that we can obtain sufficient conditions by using basic assumptions on the reference trajectory and the system dynamics to generate a local set around the reference with desired contractiveness properties. This is the tool that allows us to reframe the reference tracking problem in a way similar to the setpoint stabilization problem and thus to use well-established methods.

IV. Numerical Example

The following simple numerical example illustrates the theoretical results in this paper.

A. System

Consider the following nonlinear, unstable system

$$\begin{pmatrix} x_1^+ \\ x_2^+ \end{pmatrix} = \begin{pmatrix} \frac{3}{4}(x_1 + x_2) \\ x_1^2 + 2x_2 - (1 + x_1^2)u \end{pmatrix} = f(x, u),$$

with $x \in \mathbb{R}^2$, $u \in \mathbb{R}$ and $Q = I_2$. The controller

$$\kappa(x, z, v) = \frac{1}{1 + x_1^2}(x_1^2 - z_1^2 + K_1(x_1 - z_1) + K_2(x_2 - z_2)) + \frac{(1 + z_1^2)}{1 + x_1^2}v,$$

leads to a linear exponentially stable error dynamics. The gain K and positive definite matrix P are computed based on the DLQR and thus satisfy Assumption 4 (see Rk. 8) with $\rho = 0.62$ and $\gamma = 2.65$. We consider polytopic constraints

$$\mathcal{X} = \{x | \; |x_1| \leq 0.65, |x_2| \leq 0.35\}, \quad \mathcal{U} = \{u | \; |u| \leq 1\},$$

and a periodic reference x_{ref}, that satisfies Assumption 6 with $\delta_{\text{ref}} = 6.8 \cdot 10^{-3}$ and thus $c = 2.5 \cdot 10^{-3}$. The reference trajectory can be seen in Figure 2.

B. Exponential Tracking and Region of Attraction

According to Theorem 2, the prediction horizon $N = 5$ guarantees local exponential tracking. The local region of attraction $V_N(x(0)) \leq \gamma c$ is at least as large as $\|e(0)\|_Q^2 \leq c$. Because of the DLQR solution (Rk. 8), $V_\delta \leq \delta$ implies $V_N \leq c\gamma = \delta$ and thus we can guarantee local stability for $\|e(0)\|_P^2 \leq \delta$. The corresponding local region of attraction are the local ellipsoidal sets depicted in Figure 2.

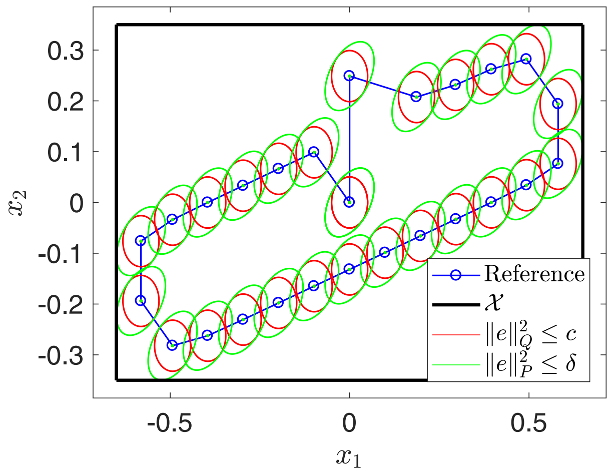

Fig. 2. Reference trajectory and local region of attraction

We choose a prediction horizon of $N = 10$, which according to Theorem 10 increases the region of attraction to $V_N(x(t), t) \leq V_{\max} := 1.39 \cdot 10^{-2}$. The guaranteed region of attraction is shown in Figure 3. As expected, we can see a clear increase in the region of attraction due to the increased prediction horizon. Note, that the constraint set $\mathcal{X}$ is not control invariant and thus the maximal region of attraction is always a strict subset of $\mathcal{X}$.

Remark 14. *Derivations similar to [12] (compare Rk. 12) guarantee $V_{\max} = 10^{-2}$ for $N = 10$. In order to guarantee exponential stability for $V_{\max} = 1.39 \cdot 10^{-2}$, these derivations require a prediction horizon of $N = 17$, which is roughly $\ln(V_{\max}/c)$ times larger then the bounds in Theorem 10. Thus, Theorem 10 yields less conservative bounds.*

V. Conclusion

We have analyzed the closed-loop properties of a simple reference tracking MPC formulation. Theorem 10 provides a sufficient prediction horizon N that guarantees exponential tracking for a class of systems and reference signals. The sufficient prediction horizon N depends on the stabilizability, the desired region of attraction and the distance of the reference to the constraints.

In conclusion, this paper bridges the gap from set-point stabilization to reference tracking by a suitable reformulation. As a by product we have introduced an alternative proof for setpoint stabilization for MPC without terminal constraints based on local stabilizability, resulting in less conservative bounds on the required prediction horizon compared to existing results.

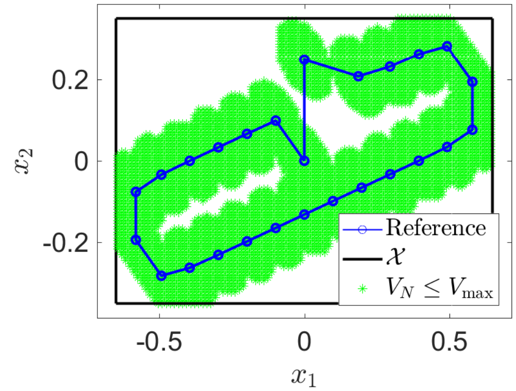

Fig. 3. Increased Region of Attraction

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