2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Patrolling and collision avoidance beyond classical Navigation Functions

C. Possieri and M. Sassano

Abstract— **In this paper, we consider the problem of navigating a single *unicycle-like* robot, while avoiding obstacles in a known environment, and, at the same time, of steering the agent itself to monitor and patrol an assigned path. To this end, we propose a novel framework that combines tools and algorithms borrowed from algebraic geometry with techniques inspired by those associated with classical navigation functions. The former aspect permits the systematic construction of Lyapunov functions that *certify* the convergence - with an *assignable* decaying rate - to the desired patrolling path in the absence of obstacles. This control action is then combined with an additional term and a *supervisory logic* obtained by relying on the collision avoiding abilities of the underlying navigation function. Such a mixed strategy may potentially lead beyond the current understanding and implementation of classical navigation functions. The paper is then concluded by several numerical simulations that corroborate the theoretical results.**

I. Introduction

The current trend in robotics and automation consists in envisioning distributed teams of simple and more agile robots that must cooperate to achieve common or individual behaviors and motions, possibly despite limited and structured communication. Such an approach, however, is only viable provided one is capable of designing efficient and reliable coordination strategies that permit the mobile robots to *safely*, *i.e.* avoiding any collision or dangerous maneuver, interact with each others or with the environment.

It appears evident that the task hinted at above can be approached from several different perspectives, which are in general encompassed by the theory concerning the so-called *multi-agent* models [1], [2], [3], which include for instance the problems of *collaborative*, *cooperative* and *formation control*, [4], [5], [6], [7], [8], [9], [10], [11]. Within the framework defined for the latter task, the majority of the existing solutions hinge upon the use of *navigation functions*, originally defined by [12] in the single agent case. Such functions are essentially based on the available knowledge on the topology of the environment, in terms of targets and (static or moving) obstacles, and are subsequently instrumental for the construction of (gradient-descent) control policies that are potentially capable of steering the mobile robot to the target while avoiding undesirable collisions. Interestingly, several results related to multi-agent systems are inspired by, and allow in turn to understand and characterize, naturally emerging behaviors, such as schools of fish, migrating birds and swarms of bees [13], [14], [15], [16], [17], [18], [19].

C. Possieri is with Dipartimento di Elettronica e Telecomunicazioni, Politecnico di Torino, Torino, Italy. M. Sassano is with the Dipartimento di Ingegneria Civile e Ingegneria Informatica, Università di Roma Tor Vergata, Roma, Italy. Emails:{corrado.possieri, mario.sassano}@uniroma2.it

Similar ideas have been recently successfully extended to the purely multi-agent setting, both in a centralized [20], [21] and decentralized [22], [23] implementation. In [6], [7] the problem of continuously monitoring a region using a team of unmanned aerial vehicles has been formulated as a differential game for which approximate solutions have been found using the methodology developed in [24]. The papers [25] and [26] deal with the challenge of employing the cooperation between mobile robots in a sensor network for exploration tasks, whereas the related problem of having a group of independent agents monitor a desired region while avoiding collisions is considered in [4] and [5].

In this paper, we focus on the problem of navigating a single *unicycle-like* robot while avoiding obstacles emerging in a known environment, and at same time of steering the agent itself to monitor and patrol an assigned path, as it is typical *e.g.* in surveillance scenarios. A similar problem has been considered in [27] and [28], by using dynamical window and harmonic potential fields, respectively. The main contribution of the paper consists in the definition of a navigation strategy that combines algorithms borrowed from algebraic geometry with techniques similar to those associated with navigation functions. The former aspect permits the systematic construction of Lyapunov functions that *certify* the convergence - with an *assignable* decaying rate - to the desired patrolling path in the absence of obstacles. This control action is then combined with an additional term obtained by relying on the collision avoiding abilities of the underlying navigation function. Such a mixed strategy may potentially lead beyond the current understanding of classical navigation functions, laying the foundations for novel approaches in the context of multiple robots.

The rest of the paper is organized as follows. The aim of Sections II and III consists in briefly recalling few basic facts and results concerning *algebraic geometry* and in defining the patrolling problem under investigation, respectively. The main results are then discussed in Section IV, in which the strategy that combines tools from algebraic geometry with classical navigation functions is presented. Several numerical simulations are then employed in Section V to validate the theoretical results, before drawing conclusions and hinting at future research directions in Section VI.

II. Review on attractive affine varieties

In this section, we briefly recall an algorithm (borrowed from [29], [30]) that permits the systematic computation of a set of vector fields $h(\cdot)$, whose associated dynamic system, namely $\dot{x} = h(x)$, $x(t) \in \mathbb{R}^n$, possesses a given affine variety $\mathcal{V}$ as an *attractive and h-invariant submanifold* of $\mathbb{R}^n$.

Algorithm 1

Input: $p_1, \ldots, p_s \in \mathbb{R}[x]$ such that $\mathcal{V} = \mathbf{V}(p_1, \ldots, p_s)$

Output: a class of $h \in \mathbb{R}^n$ and a set $\mathcal{D}_\mathcal{V} \subset \mathbb{R}^n$ such that $\mathcal{V}$ is h-invariant and attractive in $\mathcal{D}_\mathcal{V}$.

1: Fix the $>_{\mathrm{Lex,POT}}$ monomial order and compute the reduced Groebner basis $\mathcal{C} = \{c_1, \ldots, c_\ell\}$ of $\mathcal{J} := \langle \frac{\partial p}{\partial x} \rangle$.
2: Let $\nu_i \in \mathbb{Z}$, $\nu_i \leqslant \ell$, be such that the first $n - i$ entries of all $c_1, c_2, \ldots, c_{\nu_i}$ are equal to zero, $i = 1, \ldots, n$.
3: Let $\nu_0 = 0$, $\lambda_i = \sum_{j=\nu_{i-1}+1}^{\nu_i} [c_j]_{n-i+1} c_j$, $i = 1, \ldots, n$.
4: Define $\Lambda = [\ \lambda_n \quad \cdots \quad \lambda_1\ ]$ and let $c > 0$ be such that $\Lambda \succeq 0$ in $\mathcal{D}_\mathcal{V} := \{x \in \mathbb{R}^n : \sum_{i=1}^s p_i^2(x) < c\}$.
5: Solve the polynomial equation $L_h p + \Lambda p = 0$ in h and verify that the greatest h–invariant subset of $\{x \in \mathcal{D}_\mathcal{V} : p(x) \in \mathrm{Ker}(\Lambda(x))\}$ is contained in $\mathcal{V}$.
6: **return** h and $\mathcal{D}_\mathcal{V}$.

Among the results of [29], it has also been shown that the set of $h \in \mathbb{R}^n[x]$ provided as outputs of Algorithm 1 is

$$h(x) = g(x) u_a + f(x), \tag{1}$$

with

$$\left[\begin{array}{c|c} g & f \\ \hline 0 & 1 \end{array}\right] = \mathrm{Syz}\left(\frac{\partial p}{\partial x}, \Lambda p\right),$$

where the polynomials $p_1, \ldots, p_s$ specify the target attractive variety, $p = [\ p_1 \quad \cdots \quad p_s\ ]^\top$, and u_a is an arbitrary vector. In the following sections, we envision a navigation strategy that exploits, on one hand, the possibility of rendering a specific patrolling path attractive and invariant for the dynamics of the mobile robot and, on the other hand, that suitably selects the additional parameterization in terms of u_a - whose arbitrary choice does not affect the above feature - to avoid collisions with obstacles.

III. Problem definition

The main objective of this section consists in the definition of the collision avoidance problem for a *unicycle-like* mobile robots, together with its formulation in terms of an (inverse) optimal control problem. Thus, consider *unicycle-like* mobile robots described by equations of the form [31]

$$\dot{X} = \cos(\vartheta) v, \quad \dot{Y} = \sin(\vartheta) v, \quad \dot{\vartheta} = \omega \tag{2}$$

where $(X(t), Y(t)) \in \mathbb{R}^2$ denotes the Cartesian position of the center of mass, $\vartheta(t) \in \mathbb{R}$ denotes the orientation with respect to the horizontal axis, $v(t) \in \mathbb{R}$ and $\omega(t) \in \mathbb{R}$ represent the linear and the angular velocity inputs, respectively. Before discussing the formal definition of the motion planning and patrolling with collision avoidance problem, consider the following definitions and standing assumptions.

Definition 1. (Obstacle). *The center of mass and the region in the configuration space occupied by the i-th obstacle are denoted by $m_{c,i} \in \mathbb{R}^2$ and $\mathcal{W}_i \subset \mathbb{R}^2$, respectively. Moreover, the obstacle $\mathcal{W}_i$ is characterized by*

$$\partial \mathcal{W}_i = \left\{x \in \mathbb{R}^2 : \|x - m_{c,i}\|_{E_i}^2 - \varrho_i^2 = 0\right\}, \tag{3}$$

where $\varrho_i > 0$, $E_i = E_i^\top > 0$ and the standard notation $\partial \mathcal{W}_i$ describes the boundary of the region $\mathcal{W}_i$. ⋄

As equation (3) entails, the structure of the obstacles is assumed to be elliptical. As it is customary in this context, if the obstacles is in fact not elliptical, it is possible to enclose the obstacle within an associated ellipse of minimal size.

Definition 2. (Collision). *Suppose there are m obstacles described by $\mathcal{M}_i$, $i = 1, ..., m$. A collision occurs if there exists $t_c \in \mathbb{R}$ such that $(X(t_c), Y(t_c)) \in \bigcup_{i=1}^m \mathcal{W}_i$.* ⋄

Let now $\mathcal{V} \subset \mathbb{R}^2$ denote the desired *patrolling path* of the mobile robot - characterized as the affine variety associated to a set of certain polynomials $p_1, \ldots, p_s \in \mathbb{R}[X, Y]$, namely $\mathcal{V} := \mathbf{V}(p_1(x), \ldots, p_s(x))$ - and consider the following *feasibility* assumption.

Assumption 1. *The intersection of $\mathcal{V}$ and $\bigcup_{i=1}^m \mathcal{M}_i$ is empty, and there exists a continuous path $\mathcal{P}$ between $(X(0), Y(0)) \in \mathbb{R}^2$ and $\mathcal{V}$ such that $\mathcal{P} \cap (\bigcup_{i=1}^m \mathcal{W}_i) = \emptyset$.* ∘

In the following statement, we provide a formulation of the main problem investigated in this paper.

Problem 1. *(Motion planning and patrolling).* Consider a mobile robot with dynamics as in (2) and a desired patrolling path $\mathcal{V} \subset \mathbb{R}^2$. The *motion planning and patrolling with collision avoidance problem* consists in determining a feedback control input that steers the robot from its initial position to $\mathcal{V}$, that makes it move along $\mathcal{V}$ with a prefixed patrolling speed $u_p \in \mathbb{R}$, and that avoids collisions. ∘

IV. Motion planning and patrolling with collision avoidance

To begin with, by employing well-known techniques for instance illustrated in [32], [33], the relative dynamics of any point on the robot that does not belong to the segment connecting the two wheels, such as those of the *center of mass*, can be feedback linearized and consequently arbitrarily assigned. Therefore, we initially suppose, for path planning and navigation purposes, that the dynamics of the mobile robot are described by a *virtual* single integrator of the form

$$\dot{x} = u, \tag{4}$$

where $x(t) \in \mathbb{R}^2$ denotes the Cartesian position of the virtual mobile robot and $u(t) \in \mathbb{R}^2$ its speed. The proposed strategy combines two different logics and control policies that are described in the following in details. Roughly speaking, we first design, by relying on the results of Algorithm 1, a feedback control based on the knowledge of the patrolling path $\mathcal{V}$ that steers the mobile robots towards such a path, regardless of the presence of static obstacles. This is achieved by means of the construction of a Lyapunov *certificate* of such convergence, with an assignable decaying rate. Then, an inverse optimal control formulation for collision avoidance is provided for such a (partially) closed-loop system, namely exploiting the degrees of freedom left by the parameterization of the vector fields yielded by Algorithm 1. The resulting control policy does not affect the decaying rate property of the Lyapunov function, while allowing the robot to deviate from the original trajectory to avoid the obstacles.

To begin with, the following theorem provides a procedure to select u such that the affine variety $\mathcal{V} := \mathbf{V}(p_1(x), \ldots, p_s(x))$ is attractive for system (4).

Theorem 1. *Consider system* (4) *and let $u_p \in \mathbb{R}$ be given. Let $h \in \mathbb{R}[x]$ as in* (1) *be the output of Algorithm 1 with input $p_1, \ldots, p_s$ such that $\mathcal{V} := \mathbf{V}(p_1(x), \ldots, p_s(x))$. Then, $\mathcal{V}$ is invariant and attractive for $x(0) \in \mathcal{D}_{\mathcal{V}}$ provided that*

$$u = \delta(x) f(x) + \gamma(x) g(x) u_s + g(x) u_p, \tag{5}$$

for any $\gamma : \mathbb{R}^2 \to \mathbb{R}^m$, $\delta : \mathbb{R}^2 \to \mathbb{R}_{>0}$, and $u_s : \mathbb{R}^2 \to \mathbb{R}$. ■

In order to guarantee that the mobile robot is steered from its initial position to $\mathcal{V}$, in the following, we will assume implicitly that the initial condition $x(0)$ of system (4) belongs to the domain of attraction of $\mathcal{V}$. However, attractiveness and invariance of $\mathcal{V}$ do not *a priori* ensure that the path of the mobile robot is collision free. Therefore, in the following subsection we provide an alternative formulation of Problem 1 in terms of an (inverse) optimal control problem.

A. Collision avoidance as inverse optimal control problem

By taking advantage of the statement of Theorem 1, assume that the (partial) closed-loop system is given by

$$\dot{x} = \delta(x) f(x) + \gamma(x) g(x) u_s + g(x) u_p, \tag{6}$$

where $x(t) \in \mathbb{R}^n$, $u_s(t) \in \mathbb{R}$, $f : \mathbb{R}^2 \to \mathbb{R}^2$, $g : \mathbb{R}^2 \to \mathbb{R}^m$, $\delta : \mathbb{R}^2 \to \mathbb{R}_{>0}$ and $\gamma : \mathbb{R}^2 \to \mathbb{R}$. By Theorem 1, the affine variety $\mathcal{V} := \mathbf{V}(p_1, \ldots, p_s)$ is invariant and attractive for system (6). Thus, informally speaking, the mobile robot tends to the affine variety $\mathcal{V}$ regardless of any selection of the control input u_s. Before a formal statement of the problem that we aim to solve, the collision free motion planning can be summarized as follows. The dynamics given in (6) are such that the mobile robot is steered from its initial position to the set $\mathcal{V}$. The goal here is to tune the control input u_s so that collisions between agents and with obstacles are avoided. Hence, the motion planning with collision avoidance problem can be formulated as an infinite–horizon optimal control problem, as detailed in the following statement.

Problem 2. Consider the dynamics given in (6) where f and g are as in (1) and h is the output of Algorithm 1 with input $p_1, \ldots, p_s$. A solution to Problem 1 is obtained by minimizing the cost functional

$$J(x(0), u_s) := \frac{1}{2} \int_0^\infty (q(x(t)) + u_s(t)^2) dt, \tag{7}$$

with $q : \mathbb{R}^2 \to \mathbb{R}$, $q(x) = 0$ for all $x \in \mathcal{V}$,

$$q(x) := \alpha(x) \sum_{i=1}^{s} (p_i(x))^2 + \beta(x) \sum_{i=1}^{s} (p_i(x))^4, \tag{8}$$

where $\alpha : \mathbb{R}^2 \to \mathbb{R}$, and $\beta : \mathbb{R}^2 \to \mathbb{R}_{\geqslant 0}$ is such that $\lim_{x \to \bar{x}} \beta(x) = \infty$, for any $\bar{x} \in \bigcup_{i=1}^{m} \mathcal{M}_i$. ◦

Remark 1. Differently from classical optimal control problems, where the function $q(x)$ appearing in (7) is generally positive definite with respect to x to imply asymptotic stability of the optimal closed-loop system, in Problem 2 we admit cost functionals J that are not sign definite. As a matter of fact, the function $\alpha(x)$ appearing in (8) may be negative for some $x \in \mathbb{R}^2$. This relaxed assumption on the cost functional is due to the fact that, by Theorem 1, $\mathcal{V}$ is asymptotically stable in $\mathbb{R}^2 \setminus \{0\}$ for *any* u_s. Therefore, we can admit cost functionals that are not sign definite since asymptotic stability of $\mathcal{V}$ is already guaranteed. Hence, the value function has not to be used as a candidate Lyapunov function, as it is typical pursued in optimal control. △

Note that the function α can be arbitrarily assigned, hence equation (7) describes in fact a *family of cost functionals*, parameterized with respect to α, and therefore Problem 2 can be interpreted as an *inverse optimal control problem*, namely a task in which also the cost function itself can be partially selected. On the other hand, the function β appearing in (8) is a *barrier function* that penalizes the mobile robot from approaching the obstacles (see, *e.g.*, [34], [7] for some examples of barrier functions). As a consequence, determine a control law that provides a *finite* optimal cost for Problem 2 ensures that the mobile robot avoids collisions.

Computing a solution to Problem 2 corresponds to determine a feedback policy $k(x)$ such that if $u_s^\star = k(x)$, then

$$J(x(0), u_s^\star) \leqslant J(x(0), u_s)$$

for any admissible u_s and $x(0) \in \mathcal{D}_{\mathcal{V}}$. By [35], determining a solution to the optimal control problem (6),(7) corresponds to find a solution to the Hamilton–Jacobi–Bellman (HJB) PDEs

$$-\frac{1}{2}\gamma(x)^2 \frac{\partial V}{\partial x}(x) g(x) g(x)^\top \left(\frac{\partial V}{\partial x}(x)\right)^\top + \frac{\partial V}{\partial x}(x)(\delta(x) f(x) + g(x) u_p) + \frac{1}{2} q(x) = 0. \tag{9}$$

with $V(x) = 0$, for any $x \in \mathcal{V}$. Hence, if a solution to the HJB PDE (9) is determined, then the motion planning with collision avoidance problem is solved by

$$u_s = k(x) = -\left(\gamma(x) \frac{\partial V}{\partial x}(x) g(x)\right)^\top. \tag{10}$$

Solving the HJB PDE (9) for generic vector fields f and g, which are determined and related to the specific selection of the patrolling path $\mathcal{V}$, is a challenging task [36]. Therefore, in the rest of this section the attention is focused on the solution to Problem 2 for a specific selection of the path $\mathcal{V}$ and for a generic q. More precisely, we suppose that $\mathcal{V}$ is generated by the polynomial $p(X, Y) = X^2 + Y^2 - 1$ that essentially describes a *circle* of radius one in the state-space. It is, however, important to stress that this choice that significantly simplifies the following constructions is not restrictive, as entailed by the following statement.

Proposition 1. *Let $\mathcal{C}$ be a closed, simple, regular curve in $\mathbb{R}^2$. Then $\mathcal{V} := \mathbf{V}(X^2 + Y^2 - 1)$ and $\mathcal{C}$ are diffeomorphic.* ■

The meaning of Proposition 1 is that, even if in this section the attention is focused on solving Problem 2 with respect to the unit circle, the proposed technique can be used to patrol any closed, simple, regular curve in $\mathbb{R}^2$, modulo a suitably defined change of coordinates.

By using Algorithm 1 to determine a set of vector fields whose associated dynamic system has $\mathcal{V} = \mathbf{V}(X^2+Y^2-1)$, we obtain that the affine variety $\mathcal{V}$ is attractive and invariant for the dynamical system

$$\dot{x} = \gamma(x)g(x)u_s + \delta(x)f(x) + g(x)u_p, \tag{11}$$

where $x = [\ X \quad Y\ ]^\top$, $u_s(t) \in \mathbb{R}$ and

$$f(x) := \begin{bmatrix} -X(X^2+Y^2-1) \\ -Y(X^2+Y^2-1) \end{bmatrix}, \quad g(x) = \begin{bmatrix} -Y \\ X \end{bmatrix}, \tag{12}$$

whereas $\gamma : \mathbb{R}^2 \to \mathbb{R}$ and $\delta : \mathbb{R}^2 \to \mathbb{R}_{>0}$ are arbitrary. It can be verified that the domain of attraction of $\mathcal{V}$ is $\mathbb{R}^2 \setminus \{0\}$. Thus, initial conditions $x(0) \in \mathbb{R}^2 \setminus \{0\}$ are considered.

The following theorem proposes a solution to Problem 2.

Theorem 2. *Let $s = 1$ and $p_1(x) = X^2+Y^2-1$. Let $u_p \in \mathbb{R}$, a constant $\sigma \in \mathbb{R}\setminus\{0\}$, and a barrier function $\beta : \mathbb{R}^2 \to \mathbb{R}_{\geqslant 0}$ be given. Consider system* (11) *with $\gamma(x) = \frac{1}{|\sigma|}\sqrt{\beta(x)}$ and the cost functional* (7) *where $q(x)$ is given by* (8) *with*

$$\alpha(x) = \sigma(2u_p - 8\delta(x)\left(X^2+Y^2\right)\operatorname{atan2}(X,Y)).$$

Then, the feedback control input

$$u_s = k(x) = -\sigma\sqrt{\beta(x)}(X^2+Y^2-1)^2 \tag{13}$$

solves Problem 2. ■

In Theorem 2, a *family* of infinite (inverse) optimal control problems, parameterized with respect to σ, has been solved. Each of these solutions provide a finite cost for (7), hence implying avoidance of the obstacles. It is worth noting that both the functions γ and α depend on the design parameter σ, leading to different optimal control strategies. However, despite σ varies in $\mathbb{R} \setminus \{0\}$, the different closed-loops that may arise as functions of the parameter σ are limited to the two following vector fields

$$\dot{x} = \begin{cases} \delta(x)f(x) + g(x)u_p + \beta(x)p_1^2(x)g(x), & \text{if } \sigma \in \mathbb{R}_{>0}, \\ \delta(x)f(x) + g(x)u_p - \beta(x)p_1^2(x)g(x), & \text{if } \sigma \in \mathbb{R}_{<0}, \end{cases}$$

where $\delta : \mathbb{R}^2 \to \mathbb{R}_{>0}$. Therefore, in the following, a supervisory control that selects whether to use one or the other vector field is proposed with the objective of minimizing the value of the function $\beta(x)$, *i.e.* maximizing the distance of the mobile robot from the obstacles, along the trajectories of the closed-loop system. Namely, the collision avoidance supervisory logic consists in selecting either of the two admissible vector fields according to the following rule

$$\dot{x} = \begin{cases} \delta f + g u_p + \beta p_1^2 g, & \text{if } X\frac{\partial\beta}{\partial Y} - Y\frac{\partial\beta}{\partial X} \leqslant 0, \\ \delta f + g u_p - \beta p_1^2 g, & \text{if } X\frac{\partial\beta}{\partial Y} - Y\frac{\partial\beta}{\partial X} < 0, \end{cases} \tag{14}$$

where $\delta : \mathbb{R}^2 \to \mathbb{R}_{>0}$. With such a choice, one has that $\dot{x} = f(x) + \beta(x)p_1^2(x)g(x)$ if $\frac{\partial\beta}{\partial x}(f + gu_p + \beta p_1^2 g) \leqslant \frac{\partial\beta}{\partial x}(f + gu_p - \beta p_1^2 g)$, whereas $\dot{x} = f(x) - \beta(x)p_1^2(x)g(x)$, otherwise. Therefore, since $\dot{\beta} = \frac{\partial\beta}{\partial x}(f \pm \beta p_1^2 g)$, selecting the closed–loop behavior as in (14) corresponds to choice the vector field that minimizes the increase of the barrier function β and hence the trajectories of system (14) stay as far as possible from the obstacles while (locally) minimizing the cost functional (7). Note that, since the right–hand side of (14) need not be continuous, solutions to system (14) have to be understood in the Filippov sense [37].

Remark 2. The degree of freedom provided by the arbitrary selection of the function δ is not exploited here. The effect of this choice results in a different decaying rate of the Lyapunov function associated to the convergence of the mobile robot to the patrolling path $\mathcal{V}$, hence it may be used to accelerate or slow down such a convergence rate depending on the distance of the robot from one of the obstacles. △

V. Numerical Simulations

In this section, several examples of application of the techniques given in Section IV are reported to illustrate the effectiveness of the proposed motion planning procedure.

A. Effect of the obstacles on trajectories

In this section, we illustrate how the iterative addition of obstacles affects the resulting trajectories of closed-loop system (14). For simplicity and for comparison purposes, throughout this section we assume that $\delta(x) = 1$ for all $x \in \mathbb{R}^2$, $u_p = 1$, and $x(0) = x_0 := [\ 5 \quad 0\ ]^\top$.

We begin with the simple case of obstacle-free motion planning and patrolling. In this scenario $\beta = 0$ and hence system (14) reduces to

$$\dot{x} = f(x) + g(x)u_p. \tag{15}$$

The trajectory of system (15) starting at x_0 and the affine variety $\mathcal{V}$ are depicted in Figure 1.

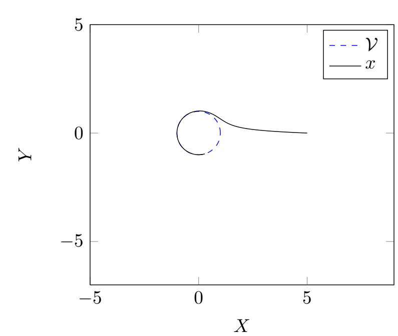

Fig. 1. State-space trajectory of the mobile robot described by (14), in the absence of obstacles and with patrolling path $\mathcal{V} = \mathbf{V}(X^2+Y^2-1)$.

As shown in Figure 1, the trajectory of system (15) firstly approaches the affine variety $\mathcal{V}$ and secondly patrols the unit circle with the desired speed u_p. Suppose now that there is an obstacle that occupies the region

$$\mathcal{W}_1 = \{x \in \mathbb{R}^2 : w(x) \leqslant 0\},$$

with

$$w_1 = \frac{1}{2}(X-3.5)^2 + \frac{1}{4}Y^2 - 1.$$

Thus, define the inverse barrier function $\beta = w_1^{-1}$ and

$$\Lambda := \left\{ x \in \mathbb{R}^2 : X\frac{\partial \beta}{\partial Y}(x) - Y\frac{\partial \beta}{\partial X}(x) \leqslant 0 \right\}. \quad (16)$$

Following the logic of (14), consider the system

$$\dot{x} = \begin{cases} f(x) + g(x) + \beta(x)p_1^2(x)g(x), & \text{if } x \in \Lambda, \\ f(x) + g(x) - \beta(x)p_1^2(x)g(x), & \text{otherwise.} \end{cases} \quad (17)$$

Figure 2 depicts the trajectory of system (17) starting at x_0, and the sets $\mathcal{V}$, $\mathcal{W}$, and Λ.

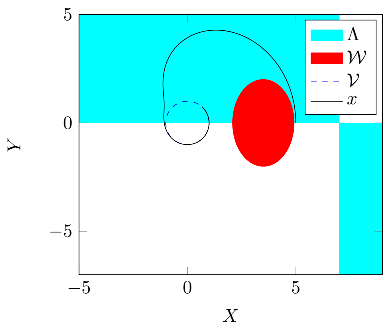

Fig. 2. State-space trajectory of the mobile robot described by (17), in the presence of the obstacle $\mathcal{W}_1$ and with patrolling path $\mathcal{V}$.

As shown by such a figure, the mobile robot is firstly steered within the space towards $\mathcal{V}$ avoiding the obstacle and secondly patrols the unit circle with the desired speed u_p.

Assume now that there are two additional obstacles that occupy the semi–algebraic sets $\mathcal{W}_i := \{x \in \mathbb{R}^2 : w_i(x) \leqslant 0\}$, $i = 2, 3$, where $w_2 = (X + 2)^2 + (Y - 2)^2 - 1$, $w_3 = (X + 3)^2 + (Y + 2)^2 - 1$. Let $\mathcal{W} = \bigcup_{i=1}^{3} \mathcal{W}_i$, define the barrier function $\beta = \sum_{i=1}^{3} w_i^{-1}$ and consider system (17) where Λ is defined as in (16). Figure 2 depicts the trajectory of system (17) starting at x_0, and the sets $\mathcal{V}$, $\mathcal{W}$, and Λ.

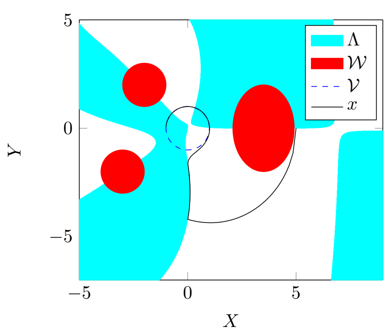

Fig. 3. Trajectory with three obstacles.

As shown by such a figure, the trajectory of system (17) reaches the unit circle avoiding collisions.

We conclude this section by adding four further obstacles $\mathcal{W}_i := \{x \in \mathbb{R}^2 : w_i(x) \leqslant 0\}$, $i = 4, \ldots, 7$, where $w_4 = (X-3)^2 + \frac{1}{2}(x+5)^2 - 1$, $w_5 = \frac{1}{4}(X-3)^2 + (X-4)^2 - 1$, $w_6 = (x+1)^2 + (x+4)^2 - 1$, and $w_7 = (x-6)^2 + (x+4)^2 - 1$. Hence, let $\mathcal{W} = \bigcup_{i=1}^{7} \mathcal{W}_i$, define the barrier function $\beta = \sum_{i=1}^{7} w_i^{-1}$ and consider system (17) where Λ is defined as in (16). Figure 4 depicts the trajectory of system (17) starting at x_0, and the sets $\mathcal{V}$, $\mathcal{W}$, and Λ.

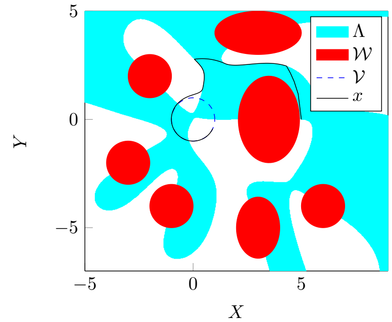

Fig. 4. Trajectory with seven obstacles.

As shown by such a figure, despite the increased amount of obstacles, the trajectory of system (17) reaches the unit circle avoiding obstacles and then patrols it with the desired speed. Considering the same setting, Figures 5 and 6 depict the resulting trajectory of the mobile robot together with the function $\beta(x)$ and the Lyapunov function $V(x)$, respectively. It is interesting to point out that the latter guarantees convergence to the patrolling path, with a certain decaying rate, while the former is similar in spirit to a classical navigation function. Note that, despite standard implementations, the combined action of the two control policies results in trajectories along which the value of the navigation function β may actually increase (see Figure 5).

VI. Conclusions

In this paper, a novel framework has been proposed to solve the problem of navigating a unicycle–like mobile robot within a known environment avoiding obstacles and patrolling an assigned path. To this end, we combined tools borrowed from algebraic geometry (that ensures the convergence of the mobile robot to the desired path) with techniques inspired by classical navigation functions (that allows to avoid collisions with the obstacles). The combination of these two techniques may potentially lead beyond the current understanding of navigation functions. As a matter of fact, in such a scenario, while the Lyapunov function is monotonically decreasing along the trajectories of the mobile robot (thus guaranteeing convergence to the assigned path), the navigation function defined by the obstacles may increase, thus possibly avoiding the drawbacks of classical implementations of navigation functions. By this reasoning, this paper is a stepping stone towards the definition of a new navigation paradigm that can be employed in multiple scenarios as, *e.g.*, for the coordination of multiple agents that have to be steered to patrol selected paths avoiding collisions with each other and with obstacles.

References

[1] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 1st ed., 2010.

[2] F. L. Lewis, H. Zhang, K. Hengster-Movric, and A. Das, *Cooperative Control of Multi-Agent Systems: Optimal and Adaptive Design Approaches*. Springer, 2014.

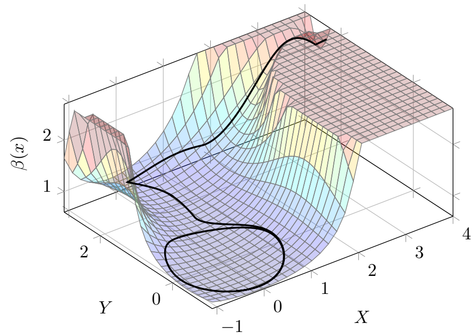

Fig. 5. Values of the barrier function β.

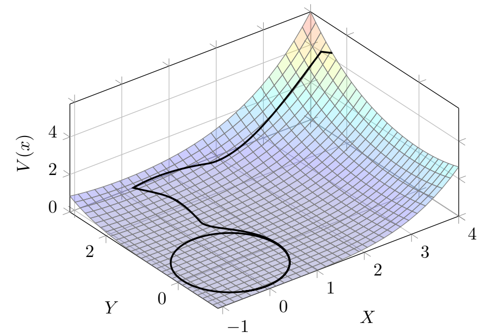

Fig. 6. Values of the Lyapunov function $V := \frac{1}{100}(X^2 + Y^2 - 1)^2$.

Fig. 7. Values of the functions β and V along the trajectory of system (17). Note that V is monotonically decreasing whereas β is not.

[3] N. E. Leonard, "Multi-agent system dynamics: Bifurcation and behavior of animal groups," in *9th IFAC Symposium on Nonlinear Control Systems*, pp. 1–11, Elsevier, 2013.

[4] I. I. Hussein, D. M. Stipanovic, and Y. Wang, "Reliable coverage control using heterogeneous vehicles," in *46th IEEE Conference on Decision and Control*, pp. 6142 –6147, 2007.

[5] I. I. Hussein and D. M. Stipanovic, "Effective coverage control for mobile sensor networks with guaranteed collision avoidance," *IEEE Transactions on Control Systems Technology*, vol. 15, no. 4, pp. 642 –657, 2007.

[6] T. Mylvaganam and A. Astolfi, "Approximate optimal monitoring: preliminary results," in *American Control Conference*, pp. 4745–4750, 2012.

[7] T. Mylvaganam and A. Astolfi, "Approximate optimal monitoring," in *14th European Control Conference*, pp. 1199–1204, 2014.

[8] M. Egerstedt and X. Hu, "Formation constriained multi-agent control," *IEEE Transations on Robotics and Automation*, vol. 17, no. 6, pp. 947–951, 2001.

[9] M. Jafarian and C. De Persis, "Exact formation control with very coarse information," in *American Control Conference (ACC)*, pp. 3026–3031, June 2013.

[10] H. Bai, M. Arcak, and J. Wen, *Cooperative Control Design: A Systematic, Passivity-Based Approach*. Communications and Control Engineering, Springer, 2011.

[11] T. Mylvaganam and A. Astolfi, "A differential game approach to formation control for a team of agents with one leader," in *American Control Conference*, pp. 1469–1474, 2015.

[12] E. Rimon and D. Koditschek, "Exact robot navigation using artificial potential functions," *IEEE Transactions on Robotics and Automation*, vol. 8, no. 5, pp. 501–518, 1992.

[13] M. A. Haque, A. R. Rahmani, and M. B. Egerstedt, "Biologically inspired confinement of multi-robot systems," *ASME Journal of Dynamic Systems, Measurement and Control*, vol. 3, pp. 213–224, 2011.

[14] N. E. Leonard and E. Fiorelli, "Virtual leaders, artificial potentials and coordinated control of groups," in *40th IEEE Conference on Decision and Control*, pp. 2968–2973, 2001.

[15] P. Ögren, E. Fiorelli, and N. E. Leonard, "Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed environment," *IEEE Transactions on Automatic Control*, vol. 49, pp. 1292–1302, Aug 2004.

[16] P. Ögren, E. Fiorelli, and N. E. Leonard, "Formations with a mission: Stable coordination of vehicle group maneuvers," *Proc. Symp. Mathematical Theory of Networks and Systems*, pp. 1–22, 2002.

[17] H. Su, X. Wang, and Z. Lin, "Flocking of multi-agents with a virtual leader," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 293–307, 2009.

[18] B. Nabet, N. E. Leonard, I. Couzin, and S. Levin, "Dynamics and decision making in animal group motion," *Journal of Nonlinear Science*, vol. 19, pp. 399–495, 2009.

[19] D. A. Paley, N. E. Leonard, R. Sepulchre, D. Grünbaum, and J. K. Parrish, "Oscillator models and collective motion," *IEEE Control Systems Magazine*, pp. 89–105, 2007.

[20] H. Tanner and A. Boddu, "Multiagent navigation functions revisited," *IEEE Transactions on Robotics and Automation*, vol. 28, no. 6, pp. 1346–1359, 2012.

[21] H. Tanner and A. Kumar, "Towards decentralization of multi-robot navigation functions," in *IEEE International Conference on Robotics and Automation*, 2005.

[22] D. Dimarogonas, M. Zavlanos, S. Loizou, and K. Kyriakopoulos, "Decentralized motion control of multiple holonomic agents under input constraints," in *IEEE Conference on Decision and Control*, 2003.

[23] M. De Gennaro and A. Jadbabaie, "Formation control for a cooperative multi-agent system using decentralized navigation functions," in *American Control Conference*, 2006.

[24] M. Sassano and A. Astolfi, "Dynamic approximate solutions of the HJ inequality and of the HJB equation for input-affine nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 57, pp. 2490–2503, 2012.

[25] F. Zhang and N. E. Leonard, "Cooperative filters and control for cooperative exploration," *IEEE Transactions on Automatic Control*, vol. 55, no. 3, pp. 640–663, 2010.

[26] J. Cortes, S. Martinez, T. Karatas, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Transactions on Robotics and Automation*, vol. 20, no. 2, pp. 243–355, 2004.

[27] P. Ogren and N. E. Leonard, "A convergent dynamic window approach to obstacle avoidance," *IEEE Transactions on Robotics*, vol. 21, no. 2, pp. 188–195, 2005.

[28] J.-O. Kim and P. K. Khosla, "Real-time obstacle avoidance using harmonic potential functions," *IEEE Transactions on Robotics and Automation*, vol. 8, no. 3, pp. 338–349, 1992.

[29] C. Possieri and A. Tornambe, "On f-invariant and attractive affine varieties for continuous-time polynomial systems: the case of robot motion planning," in *53rd IEEE Conference on Decision Control*, pp. 3751–3756, 2014.

[30] C. Possieri and A. Tornambe, "On polynomial vector fields having a given affine variety as attractive and invariant set: application to robotics," *International Journal of Control*, vol. 88, no. 5, pp. 1001–1025, 2015.

[31] B. Siciliano and O. Khatib, *Handbook of Robotics*. Springer, 2008.

[32] F. Martinelli, C. Possieri, and A. Tornambe, "Motion planning for a unicycle-like mobile robot, using algebraic attractive curves," in *22nd Mediterranean Conference on Control and Automation*, pp. 1335–1340, IEEE, 2014.

[33] T. Mylvaganam, M. Sassano, and A. Astolfi, "A differential game approach to multi-agent collision avoidance," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 4229–4235, 2017.

[34] G. Leitmann and J. Skowronski, "Avoidance control," *Journal of Optimization Theory and Applications*, vol. 23, no. 4, pp. 581–591, 1977.

[35] D. P. Bertsekas, *Dynamic programming and optimal control*, vol. 2. Athena scientific Belmont, MA, 1995.

[36] T. Bian and Z. P. Jiang, "Value iteration, adaptive dynamic programming, and optimal control of nonlinear systems," in *55th IEEE Conference on Decision Control*, pp. 3375–3380, 2016.

[37] A. F. Filippov, *Differential equations with discontinuous right-hand side*. American Mathematical Society, 1988.