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Input-to-State Stability Mapping for Nonlinear Control Systems Using Quantifier Elimination

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Abstract— **The generalization of the classical nonlinear stability theory on systems with control inputs or disturbances naturally leads to the concept of input-to-state stability (ISS). Determining the ISS property for a given system is generally non-trivial, especially if it involves unknown or design parameters. To this end, in the present paper we present an algebraic method based on automated quantifier elimination (QE) procedures. Thereby we invoke parametrized ISS-Lyapunov functions. As a result, we obtain boundaries in the space of the unknown parameters, guaranteeing the ISS property. The proposed procedure is illustrated by three numerical examples.**

I. INTRODUCTION

The stability analysis of nonlinear systems with inputs plays a central role while studying control systems. The question which arises is, how do control inputs or disturbances influence the state trajectories. The stability under such conditions is described by the *input-to-state stability* (ISS), introduced in [1]. This property has been intensively studied during the last three decades [1]–[4].

Using the definition and conditions for ISS, in the present paper we start by providing corresponding quantifier based existence formulations. This requires the construction of so-called comparison functions and a Lyapunov candidate function to prove the ISS property for a given system. Usually an insight into the system dynamics as well as empirics is hereby helpful. This applies in particular with respect to uncertain or unknown parameters in the system's description. To overcome these problems we propose a procedure based on *quantifier elimination* (QE) techniques.

Quantifier elimination describes a method to reformulate formulas containing quantifiers in a quantifier-free equivalent. This idea has already been applied in systems analysis [5]–[9] as well as controller design [10], [11]. Here we use QE algorithms to tackle Lyapunov conditions for input-to-state stability. This approach leads us to a quantifier free formulation and results in boundaries dividing the parameter space into subspaces. These subspaces are only containing either parameter combinations with a provable ISS property or not.

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The basics of QE and the used algorithm are briefly presented in Section II, followed by an overview of ISS in Section III. After these theoretical preliminaries, an applicable ISS formulation for the QE algorithms is given. In Section V, three systems are analyzed and the performance and possibilities of the approach are illustrated. The paper ends up with a concluding summary and a comparison of the results.

II. QUANTIFIER ELIMINATION

In order to present the main ideas of QE, let us review some basic definitions from [12].

Definition 1 (Atomic formula): An *atomic formula* is defined as a polynomial expression of the form

$$\phi(x_1, \cdots, x_k)\,\tau\,0, \tag{1}$$

with $\tau \in \{>, =\}$ and $\phi \in \mathbb{Q}[x_1, \cdots, x_k]$, where the latter stands for the ring of polynomials of rational coefficients with real variables $x_1, \ldots, x_k$.

Definition 2 (Quantifier-free formula): A formula is said to be *quantifier-free*, if it is a propositional combination of atomic formulas with the boolean operators $\vee, \wedge, \neg, \implies$ and $\iff$.

Definition 3 (Prenex formula): A *prenex formula* in the variables $X = (x_1, \cdots, x_k)$ and $Y = (y_1, \cdots, y_l)$ has the form

$$PF(X, Y) := (Q_1 y_1) \cdots (Q_l y_l)\, F(X, Y), \tag{2}$$

where $Q_i \in \{\exists, \forall\}$ and $F(X, Y)$ is a quantifier-free formula. When a quantifier is corresponding to a variable, then the variable is called *quantified*, or *free* otherwise.

In the 1940s A. Tarski proved that over the real field always exists a quantifier-free formula which is equivalent to a prenex formula [13]. This is stated in the following theorem.

Theorem 1 (Quantifier Elimination over the Real Field): For any prenex formula $PF(X, Y)$ over the real field, there always exists a quantifier-free formula $QF(X)$ such that, for any $y \in R^l$, $QF(X)$ is true if and only if $PF(X, Y)$ is true.

Quantifier elimination over the real closed field has been proved in several different ways, in particular by Cohen [14], Hörmander [15] and especially by Seidenberg [16]. Thus Theorem 1 is often referred to the *Tarski-Seidenberg theorem* or *Tarski-Seidenberg principle* [17]. Tarski also gives the first algorithm for this problem, but this is practically not suitable. Actually, the most common strategies to handle a QE-problem are based on *cylindrical algebraic decomposition*

(CAD). This decomposition was developed by Collins 1973 [12] as the first practical relevant algorithm for quantifier elimination. A CAD decomposes a set of polynomials into connected semi-algebraic sets, called *cells*. The sign of each polynomial is constant in every cell. Afterwards these cells are projected from $\mathbb{R}^n$ onto $\mathbb{R}^{n-k}$, with $1 \leq k < n$. The projectors Π_k are *cylindrical*, which means that the projection of two cells a and b are equivalent ($\Pi_k(a) = \Pi_k(b)$) or disjoint ($\Pi_k(a) \cap \Pi_k(b) = \varnothing$). Furthermore, the projection is *algebraic* because every of its components is a semi-algebraic set. These cell descriptions can be organized as a tree data-structure, that enables to give a semi-algebraic set equivalently to the beginning-set of polynomials.

Since the development of CAD some software tools have been introduced to effectively handle QE-problems, e.g. the software packages QEPCAD [18], [19] and REDLOG [20], as well as the library RegularChains [21], [22] in Maple. Independently of the efficiency of implementation, these algorithms have an inherent computational complexity, which might be doubly exponential in the worst case. Such that a minimal formulation of the problem is inevitable for a successful computation. Prior to applying these algorithms, we recall the basics of ISS theory in the following section.

III. Input-to-State Stability

ISS extends the asymptotic stability property to systems of the form

$$\dot{x} = f(x, w), \tag{3}$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, the input or disturbance $w(t) \in \mathbb{R}^m$ and the equilibrium $(0, 0)$, which means that $\dot{x} = f(0, 0) = 0$. Before we state the concept and definition of input-to-state stability we need to define comparison functions [4].

Definition 4 (Class K functions): A continuous function $\alpha : [0, a) \to [0, \infty)$ belongs to class K if it is strictly increasing and $\alpha(0) = 0$. If $a = \infty$ and $\alpha(r) \to \infty$ for $r \to \infty$ then α belongs to class K_∞.

Definition 5 (Class KL functions): A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ belongs to class KL, if for each fixed t, the function $\beta(\cdot, t)$ belongs to class K and for each fixed r, the mapping $\beta(r, \cdot)$ is decreasing with $\beta(r, t) \to 0$ for $t \to 0$.

Using these comparison functions we can formulate a definition for input-to-state-stability [1].

Definition 6 (Input-to-state stability): A system (3) is called *input-to-state stable* (ISS), if there exist two functions $\beta \in KL$ and $\gamma \in K_\infty$, such that for every initial value $x_0 = x(0)$ and each measurable essentially bounded input function w the corresponding solution $X(t, x_0, w)$ exists on the entire real axis and the inequality

$$|x(t, x_0, w)| \leq \beta(|x_0|, t) + \gamma(||w||_\infty) \tag{4}$$

holds for $t \geq 0$, where $|\cdot|$ denotes the Euclidean norm and $||\cdot||_\infty$ the norm of the Lebesgue space L_∞, respectively.

Definition 6 means that all solutions remain in a ball with the radius $\beta(|x_0|, 0) + \gamma(||w||_\infty)$ and the solutions remain on a smaller ball with the radius $\gamma(||w||_\infty)$, as $t \to \infty$. If the input function is identically zero ISS leads to global asymptotic stability. For linear systems these two properties are equivalent, but not in the nonlinear case. The behavior described above is illustrated in Fig. 1.

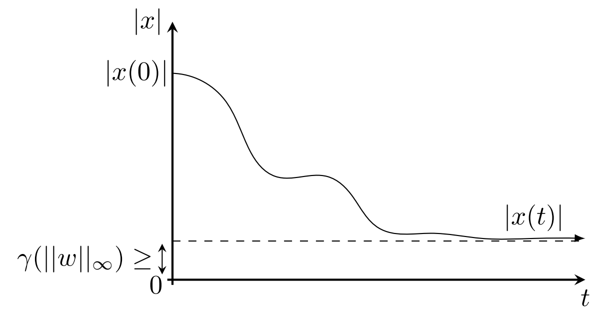

Fig. 1. Interpretation of Definition 6.

An equivalent characterization of the ISS property can be achieved using so called ISS-Lyapunov functions [2], [23].

Definition 7 (ISS-Lyapunov function): A smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an *ISS-Lyapunov function* of (3) if the following conditions hold for all x and w

$$\underline{\alpha}(|x|) \leq V(x) \leq \bar{\alpha}(|x|), \tag{5}$$

$$|x| \geq \gamma(|w|) \implies \dot{V}(x, w) \leq -\alpha(|x|), \tag{6}$$

with $\underline{\alpha}, \bar{\alpha} \in K_\infty$ and $\alpha, \gamma \in K$.

Alternatively to condition (6) the ISS-Lyapunov function can be defined in a so called "dissipation" type of characterization [2, Remark 2.4].

Remark 1: A smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ is an ISS-Lyapunov function of (3) if and only if the following conditions are fulfilled for all x and w

$$\underline{\alpha}(|x|) \leq V(x) \leq \bar{\alpha}(|x|) \tag{7}$$

$$\dot{V}(x, w) \leq -\alpha(|x|) + \gamma(|w|), \tag{8}$$

with $\underline{\alpha}, \bar{\alpha}, \alpha, \gamma \in K_\infty$.

The Eqs. (5) or (7) guarantee the positive definiteness and the radially unboundedness of V. Such a function is called *globally positive definite*. In addition, Eqs. (6) and (8) ensure that $\dot{V}$ is negative definite for all input magnitudes if $|x|$ is large enough. Furthermore, an ISS-Lyapunov function is a classical Lyapunov function of the autonomous System $\dot{x} = f(x, 0)$.

As stated in Section II, the QE-algorithm is very sensitive to the number of free variables. So it is advantageous to decrease the number of variables. One approach is to decompose the problem in minor ones. This might not be possible in every cases, but the following lemma presents one possibility [1].

Lemma 1: Consider a cascade interconnected system shown in Fig 2. If every subsystem of such an interconnected system is ISS, then is the whole system is ISS as well.

That means, if the system (3) can be interpreted in a cascaded interconnection (Fig. 2) we can analyze the subsystems, with an appropriate smaller dimension, separately. These results introduced in this section are used in the

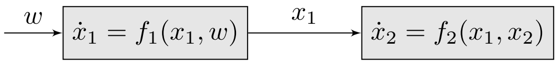

Fig. 2. Cascade interconnection.

following to propose a quantifier elimination based procedure to determine the input-to-state stability of a given system.

IV. Quantifier Elimination for ISS Analysis

The conditions (7) and (8) are the initial point for the following considerations. Certainly, we need to assure that the comparison functions are of class K_∞. Ichihara [3] suggested to handle that problem with Sum-of-Squares (SOS) decomposition and stated the following lemma.

Lemma 2: A univariate real even polynomial without constant term

$$\alpha(s) = \sum_{i=1}^{N} c_{2i} s^{2i}, \tag{9}$$

with at least one coefficient $c_{2i} \neq 0$, belongs to class K_∞ if and only if

$$s \cdot \frac{d\alpha(s)}{ds} \geq 0 \tag{10}$$

holds for all $s \in \mathbb{R}$.

Remark 2: Quantifier elimination can be used to check the conditions of this lemma. Furthermore, QE can also be used to derive conditions for more general polynomials (i.e., with odd monoms) to belong to the class $\mathcal{K}_\infty$. Consider a real polynomial

$$\alpha(s) = \sum_{i=1}^{L} c_i s^i. \tag{11}$$

As above, we have $\alpha(0) = 0$ since the constant part of the polynomial is omitted. If the implication

$$s > 0 \quad \Longrightarrow \quad \frac{d\alpha(s)}{ds} > 0 \tag{12}$$

holds for all but finitely many $s \in \mathbb{R}$, the polynomial is strictly increasing. As a polynomial, the function α is also unbounded, i.e., we have $\alpha(s) \to \infty$ as $s \to \infty$. Hence, it belongs to the class $\mathcal{K}_\infty$. Using QE software, conditions (10) and (12) can be checked with the "for all" quantifier. In addition, QEPCAD provides a quantifier "for all but finitely many" [19].

Remark 3: A quadratic Lyapunov candidate function

$$V(x) = x^T P x \quad \text{with} \quad P = P^T \in \mathbb{R}^{n \times n}$$

fulfills conditions (5) or (7), respectively, if and only if the matrix P is positive definite due to Courant-Fischer-Theorem (Min-Max-Principle):

$$\underbrace{\lambda_{\min}(P) \cdot \|x\|^2}_{\underline{\alpha}(\|x\|)} \leq x^T P x \leq \underbrace{\lambda_{\max}(P) \cdot \|x\|^2}_{\bar{\alpha}(\|x\|)}$$

Using the Equs. (7), (8) and (10) we can formulate the subsequent theorem.

Theorem 2: Considering the polynomials $f(x, w, k)$ and $V(x, q)$. Let the polynomials $\underline{\alpha}(|x|, p)$, $\bar{\alpha}(|x|, r)$, $\gamma(|w|, s)$ and $\alpha(|x|, t)$ be in form (9) The variables k, q, p, r, s, t are unknown or design parameters. All parameters $\tilde{k}$ for which

$$\exists(q, p, r, c, d), \forall(x, w) \begin{cases} V(x, q) - \underline{\alpha}(|x|, p) \geq 0. \\ \bar{\alpha}(|x|, r) - V(x, q) \geq 0. \\ \gamma(|w|, c) - \alpha(|x|, d) \geq \frac{\partial V}{\partial x} f(x, w, \tilde{k}) \\ s \cdot \frac{d\xi(s)}{ds} \geq 0, \xi \in \{\underline{\alpha}, \bar{\alpha}, \alpha, \gamma\} \end{cases} \tag{13}$$

is solvable, gives the ISS systems $f(x, w, \tilde{k})$.

Proof: The last row of (13) guaranties that the functions $\underline{\alpha}, \bar{\alpha}, \alpha, \gamma$ belong to the class K_∞. The other three inequalities imply the conditions (7) and (8). So if (13) is solvable for a value $\tilde{k}$ then is V an ISS-Lyapunov function by Remark 1 and thus $f(x, w, \tilde{k})$ is ISS. ■

Combining the inequality expressions (13) with the boolean operator $\wedge$ we directly obtain a prenex formula. If we apply the QE procedure to this prenex formula we get boundaries in the space of k for which we can guarantee the ISS property.

Remark 4: The assumption that the comparison functions are of form (9) or (11), is reasonable, because CAD is just able to analyze polynomial functions and so we need to quit the roots resulting of the norm.

Remark 5: The boundaries which are calculated by the proposed procedures are just a subset of the real ISS area in the parameter space of k. The conservatism depends on the chosen structure of V, as well as the comparison functions $\underline{\alpha}, \bar{\alpha}, \alpha$ and γ. Nevertheless, a chosen V generally enables the determination of a suitable $\underline{\alpha}$ and $\bar{\alpha}$.

In contrast to SOS it is also possible to apply the proposed procedure to the conditions (5) and (6), but the implication in (6) often results in a much more computational intensive calculation. Therefore, we prefer the usage of the conditions (7) and (8).

V. Examples

In this section we illustrate our approach on three example systems. The quantifier elimination was carried out using the open source software package QEPCAD B, which extends the original QEPCAD [18], [19].

A. Example 1

The first system we want to analyze is

$$\dot{x} = -x - \frac{x^2}{10} - x^3 + \frac{w}{10}. \tag{14}$$

To determine the ISS property of system (14), we choose $V(x) = qx^2$ as Lyapunov candidate. Calculating the first derivative we get

$$
\begin{aligned}
&\frac{\partial V}{\partial x} \cdot (-x - x^2/10 - x^3 + w/10) \\
&= -2qx^4 - \frac{1}{5}qx^3 + \frac{1}{5}qwx - 2qx^2 \\
&\le -2qx^4 - \frac{1}{5}qx^3 + \frac{1}{5}qw^2 + \frac{1}{5}qx^2 - 2qx^2 \\
&= -2qx^4 - \frac{1}{5}qx^3 - \frac{9}{5}qx^2 + \frac{1}{5}qw^2 \\
&\le \underbrace{-\frac{11}{5}qx^4 - \frac{10}{5}qx^2}_{-\alpha(|x|)} + \underbrace{\frac{1}{5}qw^2}_{+\gamma(|w|)} .
\end{aligned}
$$

Condition (5) is always satisfied for Lyapunov candidates $V(x) = qx^2$ with $q > 0$, see Remark 3. Since condition (6) is also fulfilled, is system (14) input-to-state stable.

Applying the proposed procedure and set

$$
\begin{aligned}
\underline{\alpha}(|x|, p) &= px^2 \\
\bar{\alpha}(|x|, r) &= rx^2 \\
\alpha(|x|, d) &= dx^2 \\
\gamma(|w|, c) &= cw^2,
\end{aligned}
$$

we can check the ISS property by Theorem 2. Note, that for positive parameters p, r, c, d the functions $\underline{\alpha}, \bar{\alpha}, \alpha, \gamma$ belong to the class K_∞. The ISS property can be verified based on (13) using the following QEPCAD script:

```
[ Example 1 without parameter ]
(q,p,r,c,d,x,w)
0
(E q)(E p)(E r)(E c)(E d)(A x)(A w)
[q>0 /\ p>0 /\ r>0 /\ c>0 /\ d>0 /\
 p x^2 <= q x^2 /\
 q x^2 <= r x^2 /\
 2 q x (-x-x^2-x^3)<=-d x^2+c w^2].
finish
```

The first line contains a comment. The variables are given in the next line. We have no free variables, i.e., all variables are quantified. The variables q, p, r, c, d are associated with the "exists" quantifier, the variables x, w with the "for all" quantifier. The calculation yields the result `true`. This means, that the system is ISS.

Since we have a quadratic Lyapunov candidate function, the comparison functions $\underline{\alpha}, \bar{\alpha}$ and the inequalities (7) be omitted due to Remark 3. This results in the following simplified QEPCAD script:

```
[ Example 1 simplified ]
(q,c,d,x,w)
0
(E q)(E c)(E d)(A x)(A w)
[q>0 /\ c>0 /\ d>0 /\
 2 q x (-x-x^2-x^3)<=-d x^2+c w^2].
finish
```

If we multiply a parameter k to the quadratic term of system (14) it results

$$\dot{x} = -x - kx^2 - x^3 + \frac{w}{10}. \tag{15}$$

Keeping the comparison functions α, γ and the Lyapunov candidate V as in the non-parametric case, the region for the parameter k, where system (15) is ISS, can be computed based on (13) using the following QEPCAD script:

```
[ Example 1 with free parameter ]
(k,q,c,d,x,w)
1
(E q)(E c)(E d)(A x)(A w)
[q>0 /\ c>0 /\ d>0 /\
 2 q x (-x-k x^2-x^3)<=-d x^2+c w^2].
finish
```

Now, we have 1 free variable (namely k), which is the first element in the list of variables. The calculation yields the following equivalent quantifier-free formula: `k - 2 < 0 /\ k + 2 > 0`. This result corresponds to the stability boundaries $-2 < k < 2$.

Keep in mind that such boundaries generally depend on the chosen structure of the comparison functions as well as the Lyapunov candidate, as we will see in the following example. In our case, system (15) with $w = 0$ becomes stable (but not anymore asymptotically stable) in the sense of Lyapunov for $|k| = 2$ and unstable for $|k| > 2$. Therefore, the inequality $-2 < k < 2$ mentioned above gives the strict ISS boundaries regarding to the parameter k.

B. Example 2

The second example is

$$
\begin{aligned}
\dot{x}_1 &= -x_1^3 + x_1 x_2 \\
\dot{x}_2 &= kx_1^2 - x_2 + w.
\end{aligned} \tag{16}
$$

Let us suppose $V_1 = \frac{1}{2}(x_1^2 + x_2^2)$ as Lyapunov candidate and set $\alpha = d_1 x_1^4 + d_2 x_2^2$ and $\gamma = cw^2$. The proposed function α is not a comparison function of the norm $|x|$, so that we can not directly reason the ISS property, but it holds

$$dx^4 \ge \frac{d}{4}|x|^{1+\frac{1}{|x|}}. \tag{17}$$

A similar estimation exists for dx^2. Therefore, we can estimate the function $\alpha = d_1 x_1^4 + d_2 x_2^2$ with the norm based estimation $\tilde{\alpha} = \min(\frac{d_1}{4}, \frac{d_2}{4})|x|^{1+\frac{1}{|x|}}$. This relation is illustrated in Fig. 3. So if $\alpha = d_1 x_1^4 + d_2 x_2^2$ fulfills (13), we can always find a proper comparison function. By choosing this simpler α we reduce the complexity of the resulting polynomial and therefore we significantly reduce the computational effort, as well. By applying the proposed procedure we get

$$k < 1 \wedge k + 3 > 0 \iff -3 < k < 1. \tag{18}$$

The influence of the parameter k to the autonomous system is illustrated in Fig. 4. For $k = 0$ it results in the global asymptotic behavior shown in Fig. 4(a) for the autonomous system. By increasing k and crossing the boundary $k = 1$ the system becomes unstable. This is exemplarily demonstrated for $k = 1.1$ in Fig. 4(b). We might expect the same system behavior if we decrease k under -3, but Fig. 4(c) depicts that the system is at least asymptotically stable. This fact

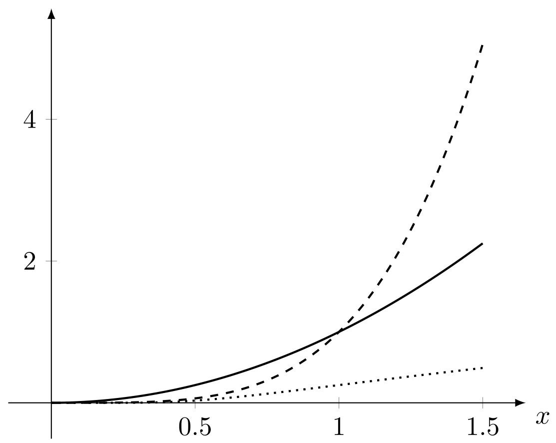

Fig. 3. Comparison of the functions x^2 (solid), x^4 (dashed) and $\frac{1}{4}|x|^{1+\frac{1}{|x|}}$ (dotted).

is not a proof for ISS, but it indicates that the system might be ISS for $k < -3$. Remembering Remark 5 the calculated boundaries (18) can be conservative resulting by the initial choice of the Lyapunov candidate and the comparison functions. The Lyapunov candidate V_1 can be generalized with

$$V_2 = q(x_1^2 + x_2^2), \tag{19}$$

unfortunately the resulting stability boundaries in k are the same (cf. (18)). A better result can be generated using

$$V_3 = q_1 x_1^2 + q_2 x_2^2 \tag{20}$$

which gives the exact boundary $k < 1$. This relation can be explained using the following inequality

$$k^2 q_2^2 + 2k q_1 q_2 + q_1^2 - 4 q_1 q_2 < 0, \text{ with } q_1, q_2 > 0, \tag{21}$$

which gives the stability condition in k, q_1 and q_2. Considering V_2 we have $q_1 = q_2 = q$ and (21) simplifies to

$$(k^2 + 2k + 1)q^2 - 4q^2 = (k^2 + 2k - 3)q^2 < 0. \tag{22}$$

This leads directly to the two boundaries 1 and -3. However, if $q_1 > 0$ and $q_2 > 0$ are arbitrary coefficients only the boundary $k = 1$ remains.

C. Example 3

Considering the system

$$\dot{x}_1 = -\beta x_1 - x_2 x_3 \tag{23}$$
$$\dot{x}_2 = \sigma(-x_2 + x_3) \tag{24}$$
$$\dot{x}_3 = -x_3 + w, \tag{25}$$

with the feedback $w = \rho x_2 - x_1 x_2$ it results the well-known Lorenz system. Obvious, system (23)-(25) has a cascade structure, see Fig. 5. Thus Lemma 1 can be applied. Eq. (25) is ISS with respect to w and (24) is ISS for $\sigma > 0$ with respect to x_3. This can be shown using quadratic functions

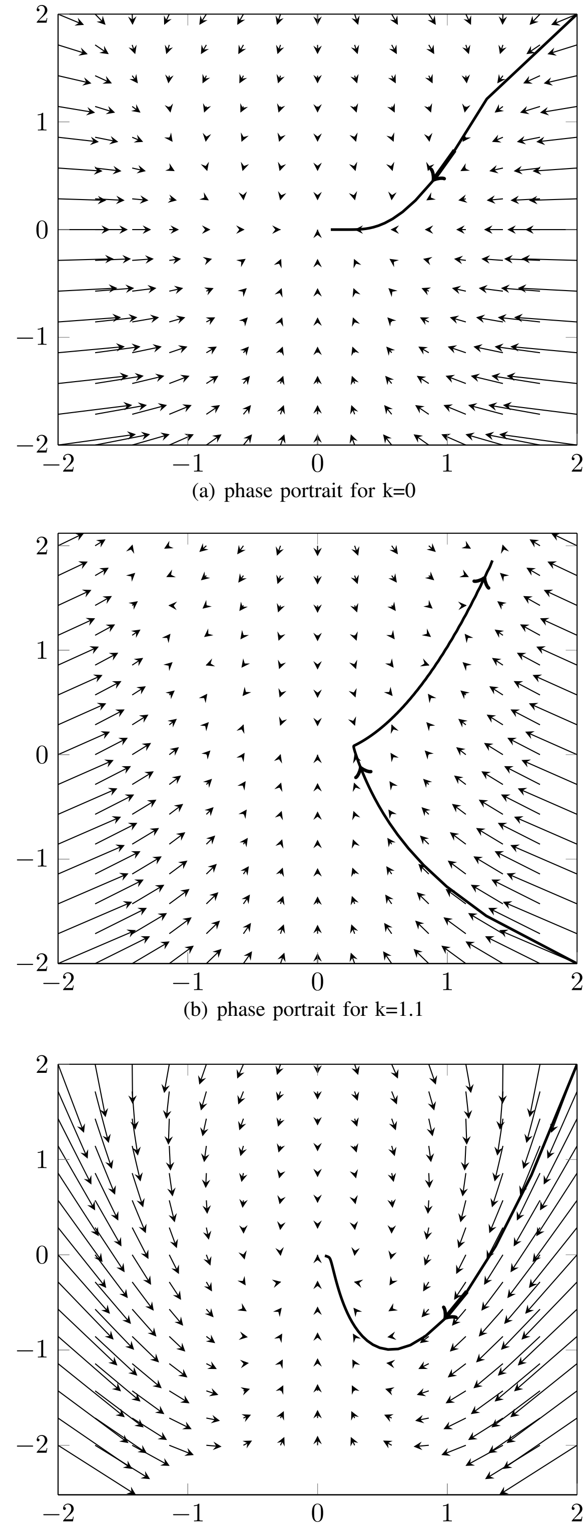

Fig. 4. Phase-portraits of system (17) for different values of k with $w = 0$.

for V and the comparison functions due to the linearity of that equations. For Eq. (23) we used

$$V = x_1^2 \tag{26}$$

$$\gamma = c(x_2^2 + x_3^2) \tag{27}$$

$$\alpha = dx_1^2. \tag{28}$$

The straight forward approach

$$\exists(c,d)\,\forall(x_1,x_2,x_3) : -\dot{V} - \alpha + \gamma \geq 0 \wedge c > 0 \wedge d > 0, \tag{29}$$

leads to `false`, because the multiplicative term x_2x_3 can not be upper bounded with the chosen γ. Nevertheless, using $\gamma = c(x_2^2 + x_3^2)^2$ results in $\beta > 0$. Angeli [24] showed that the saturated system

$$\dot{x}_1 = -\beta x_1 - \mathrm{sat}(x_2)\,\mathrm{sat}(x_3), \tag{30}$$

with $\mathrm{sat}(x)$ being a piecewise linear saturation function, is incremental ISS and thus ISS. This leads to the condition

$$\exists(c,d)\,\forall(x_1,x_2,x_3) : \tag{31}$$

$$|x_2| \leq \delta \wedge |x_3| \leq \delta \implies -\dot{V} - \alpha + \gamma \geq 0 \wedge c > 0 \wedge d > 0.$$

Applying the QE procedure gives $\beta > 0$, even for the quadratic γ.

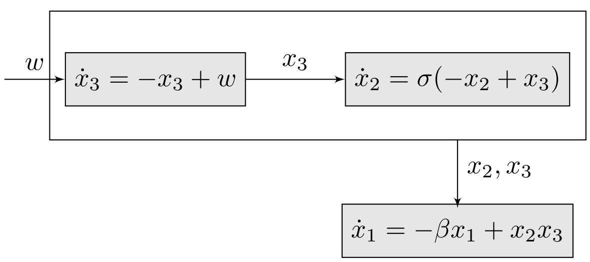

Fig. 5. Cascade interpretation of system (23)-(25).

VI. Conclusion

We discussed the computation of the input-to-state stability property and feasible parameter regions for which this property holds. This is done by quantifier elimination. Therefore we formulated a prenex formula based Lyapunov conditions for ISS. On the basis of this formula we applied a quantifier elimination method to calculate quantifier-free conditions. These conditions are polynomials in terms of the free parameters and can directly be mapped to the parameter space. For illustration purposes, we tested this approach in three numerical systems and outlined the potential and pitfalls of that approach. While the concept of quantifier elimination is a very universal tool and has a multiplicity of options in system analysis and control, it has inherent insuperable computational barriers. It turns out that preprocessing of the problem formulation may be helpful in improving the efficiency, this sometimes even yielding viability of the approach.

In further studies we will address other stability and robustness notions like (strong) integral input-to-state stability or integral input-to-state stability as well as polynomial recast methods to consider non-polynomial systems and Lyapunov functions.

Acknowledgment

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References

[1] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34(4):435–443, April 1989.

[2] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Syst. Control Lett.*, 24(5):351–359, April 1995.

[3] H. Ichihara. Sum of squares based input-to-state stability analysis of polynomial nonlinear systems. *SICE Journal of Control, Measurement, and System Integration*, 5(4):218–225, 2012.

[4] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, Upper Saddle River, third edition, 2002.

[5] R. Liska and S. Steinberg. Applying quantifier elimination to stability analysis of difference schemes. *The Computer Journal*, 36(5):497–503, 1993.

[6] M. Jirstrand. Nonlinear control system design by quantifier elimination. *Journal of Symbolic Computation*, 24(2):137–152, August 1997.

[7] T. V. Nguyen, Y. Mori, T. Mori, and Y. Kuroe. Qe approach to common lyapunov function problem. *Journal of Japan Society for Symbolic and Algebraic Computation*, 10(1):52–62, 2003.

[8] Z. She, B. Xia, R. Xiao, and Z. Zheng. A semi-algebraic approach for asymptotic stability analysis. *Nonlinear Analysis: Hybrid Systems*, 3(4):588–596, 2009.

[9] R. Voßwinkel, J. Tong, K. Röbenack, and N. Bajcinca. Stability bounds for systems and mechanisms in linear descriptor form. In *Ilmenau Scientific Colloquium (SSC)*, September 2017.

[10] P. Dorato. Non-fragile controller design: an overview. In *Proc. of the American Control Conference (ACC)*, volume 5, pages 2829–2831. IEEE, 1998.

[11] P. Dorato, D. Famularo, C. T. Abdallah, and W. Yang. Robust nonlinear feedback design via quantifier elimination theory. *International Journal of Robust and Nonlinear Control*, 9(11):817–822, 1999.

[12] G. E. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decomposition–preliminary report. *ACM SIGSAM Bulletin*, 8(3):80–90, 1974.

[13] A. Tarski. *A Decision Method for a Elementary Algebra and Geometry*. Project rand. Rand Corporation, 1948.

[14] P. J. Cohen. Decision procedures for real and p-adic fields. *Communications on Pure and Applied Mathematics*, 22(2):131–151, 1969.

[15] L. Hörmander. *The analysis of linear partial differential operators II*. Springer, 1983.

[16] A. Seidenberg. A new decision method for elementary algebra. *Annals of Mathematics*, 60(2):365–374, 1954.

[17] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in Real Algebraic Geometry*. Springer, Berlin, Heidelberg, 2nd edition, 2006.

[18] G. E. Collins and H. Hong. Partial cylindrical algebraic decomposition for quantifier elimination. *Journal of Symbolic Computation*, 12(3):299 – 328, 1991.

[19] C. W. Brown. Qepcad b: a program for computing with semi-algebraic sets using cads. *ACM SIGSAM Bulletin*, 37(4):97–108, 2003.

[20] A. Dolzmann and T. Sturm. Redlog: Computer algebra meets computer logic. *Acm Sigsam Bulletin*, 31(2):2–9, 1997.

[21] C. Chen and M. M. Maza. Quantifier elimination by cylindrical algebraic decomposition based on regular chains. *Journal of Symbolic Computation*, 75:74–93, 2016.

[22] C. Chen and M. M. Maza. Real quantifier elimination in the regularchains library. In *ICMS*, pages 283–290, 2014.

[23] E. D. Sontag and A. Teel. Changing Supply Functions in Input/State Stable Systems. *IEEE Transactions on Automatic Control*, 40(8):1476–1478, 1995.

[24] D. Angeli. A Lyapunov approach to incremental stability properties. *IEEE Transactions on Automatic Control*, 47(3):410–421, 2002.