2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

# Consensus of overflowing clocks via repulsive Laplacian laws

G. de Carolis, S. Galeani and M. Sassano

*Abstract*— **The main objective of this paper consists in imposing consensus in a network of overflowing clocks with identical speeds but potentially different initial offsets. Each (overflowing) clock is modeled as a single integrator with the state confined in a bounded set such that, whenever the state reaches its maximum allowed value, it is immediately reset to zero (*overflowing* phenomenon), thus exhibiting both continuous-time and discrete-time behaviours. In this framework, control techniques inspired by the classical Laplacian philosophy lead to a somewhat unexpected result. In fact, it is shown that both an *attractive* and a *repulsive* Laplacian law induce two periodic orbits of the closed-loop system, characterized by the feature that along only one of these trajectories consensus is reached. It is then proved that the error-zeroing periodic orbit is unstable with the *attractive* Laplacian, hence agreement of the clocks is not achieved, while an asymptotic convergence on it is guaranteed with the *repulsive* Laplacian, hence consensus is reached with a repulsive control law among the clocks.**

## I. INTRODUCTION

In the last decade, a wide variety of basic devices have been endowed with significant sensing and actuating capacities, hence acquiring the ability to autonomously collect and share data. Such an unprecedented trend has allowed the accomplishment of increasingly complex tasks even in the presence of devices that are spatially distributed over large and remote regions. However, the key enabling technology to exploit the above architecture in a reliable way consists in ensuring that a *consistent* exchange of information is maintained over the entire network and, clearly, time synchronization of the internal clocks of the individual devices is a crucial aspect of such a problem. As a consequence, it is not surprising that a significant research effort has been devoted to the design of consensus algorithms for interconnected devices, see *e.g.* [1], [2], [3], [4].

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In a centralized implementation the solution is somewhat trivial; a single system is elected as a master and the other systems (slaves) are coordinated by the master system, like in [5], [6]. Although a centralized strategy may appear easier to implement, if the master device fails the network needs to be substantially redesigned; hence such a choice is less appealing, in terms of robustness, than a distributed approach, as in [7], [8]. In this context, a typical problem related to a distributed approach for network consensus is the time delay. Different strategies have been proposed to tackle the time discrepancy between the reading and transmission as in [9] and [10].

Each device is usually equipped with an internal clock, which, due to tolerances and hardware imprecisions, usually has a slightly different speed and offset with respect to the others. Therefore a synchronization policy essentially consists in the design of a local correction term required to have an acceptable timing. Few algorithms based on consensus protocol have been proposed to compensate the clocks offset, as in [11], or to compensate the clocks speed, as in [8], or to correct both as the Average TimeSynch algorithm in [12].

All the existing implementations model each clock of the devices in the network as a dynamic continuous-time system (typically a single integrator) defined over $\mathbb{R}$ the state of which is ideally allowed to evolve indefinitely without constraints. However, since each device implements a local clock via software and, in order to reduce memory consumption and in general to contain costs, only few bits are usually reserved to realize such a component, the *overflowing phenomenon* is indeed a crucial aspect and cannot be ignored in the modeling. In this paper, we show that neglecting such a feature may lead to dramatic consequences in terms of synchronization. As a matter of fact, every internal (overflowing) clock is supposed to take values in a predefined and bounded range, and if during its evolution it reaches the boundary of such a range, the state of the clock is reset to the lower bound, namely zero. Since the evolution of the clocks exhibits both continuous and discrete behaviours, the network is essentially modeled as a hybrid system [13]. To the best of our knowledge,

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this paper represents one of the very first attempts to tackle consensus for hybrid systems with state-driven jumps. A similar issue has been addressed under the name of *Flashing Fireflies* problem in [13], where the proposed solution involves an impulsive control law based on global, compared to *relative*, information. However, in a distributed network, global information may be a difficult requirement to fulfill.

The main contribution of this paper consists in analysing the dynamics of the proposed network in order to reach consensus using a continuous control law based only on relative information. Focusing on a network of two devices and modelling it as a two overflowing clock scenario, we show that, in the presence of the impulsive behaviour (overflow and reset) the classical *attractive* Laplacian law fails to induce agreement between the two clocks, and it actually drives the clocks error on an undesirable limit cycle. Then, we prove, on the other hand, that somewhat unexpectedly, a *repulsive* Laplacian law steers the error to zero for a system of overflowing clocks.

The rest of the paper is organized as follows. In Section II the overflowing clock consensus problem is introduced; in Section III focusing on the two-overflowing-clock case a solution is proposed in Section IV, numerical results are presented, showing a comparison between the classical consensus approach and the alternative one proposed herein.

## II. Problem Formulation

Consider a network composed of $N$ devices equipped with an internal overflowing clock and suppose that the underlying network topology is captured by an undirected graph $\mathcal{G} = \{V, \mathcal{E}\}$, where $V$ is the set of nodes and $\mathcal{E}$ is the set of edges. Let $v_i \in V$ describe the *i-th* node, the presence of an edge $\varepsilon_{ij}$ in $\mathcal{E}$ between the vertices $v_i$ and $v_j$ means that the device $i$ is a neighbour of the device $j$ and they are allowed to exchange relative information. Since $\mathcal{G}$ is an undirected graph, then $\varepsilon_{ij} \in \mathcal{E}$ if and only if $\varepsilon_{ji} \in \mathcal{E}$. Dealing with a device in term of its internal clock, consider the above network as a (overflowing) *multi-clock model* consisting of a collection of $N$ hybrid systems described by the equations

$$\begin{array}{rclll} \dot{x}_i & = & \alpha + u_i & & x_i \in [0,1]\,, \\ x_i^+ & = & 0 & & x_i \geq 1\,, \end{array} \tag{1}$$

for $i = 1, \ldots, N$, where the initial value $x_i(t_0, 0) \in \mathbb{R}_{[0,1]}$ is called the *initial offset*, $\alpha \in \mathbb{R}_{>0}$ represents the *clock speed* and $u_i(t,k) \in \mathbb{R}$ is the *i-th* component of the control law based on the relative distances between the clock states of the node $i$ and its neighbours. Therefore, the agreement error can be compactly described by the vector

$$\eta(t,k) := \begin{bmatrix} x_1(t,k) - x_2(t,k) \\ x_2(t,k) - x_3(t,k) \\ \vdots \\ x_{N-1}(t,k) - x_N(t,k) \end{bmatrix}. \tag{2}$$

**Problem 1** (overflowing clock consensus problem)**.** *Consider a multi-clock model as in (1). The overflowing clock consensus problem consists in designing a feedback control law* $u(t,k) = \begin{bmatrix} u_1(t,k) & \ldots & u_N(t,k) \end{bmatrix}'$, *using only relative information of the clocks, such that*

$$\lim_{t+k \to \infty} \eta(t,k) = 0\,.$$

∘

**Remark 1.** In a *multi-clock model* as in (1), let $\mathcal{J}_k$ be the event that chronologically defines the end of the *k-th* flow and the start of the *(k+1)-th* flow, assuming that $\mathcal{J}_k$ occurs at the time instant $t_k$, then $x_{i,k} := x_i(t_k, k)$ and $x_{i,k}^+ := x_i(t_k, k+1)$ represent the value of the *i-th* component of the state at the end of the *k-th* flow and at the beginning of the *(k+1)-th* flow, respectively. ▲

The following standing assumption characterizes the scenario under investigation, which is rather standard in this context.

**Assumption 1.** *The network of interconnected overflowing-clock models possesses the following properties:*

*(RI) (Relative Information) Every clock is allowed to exchange only relative information (distances) with the set of its neighbours;*

*(AC) (All-To-All Connectivity) There exists a direct communication from each clock to any other clock.*

Consensus problems, under Assumption 1, are typically tackled by envisioning control laws described by

$$u(t,k) = -\frac{\gamma}{N-1} L x(t,k), \tag{3}$$

where $L \in \mathbb{R}^{N \times N}$ is the *Laplacian matrix* associated to the undirected graph $\mathcal{G}$ in presence of an all-to-all communication topology, namely

$$L := \begin{bmatrix} N-1 & -1 & -1 & \ldots & -1 \\ -1 & N-1 & -1 & \ldots & -1 \\ -1 & -1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ -1 & -1 & \ldots & -1 & N-1 \end{bmatrix}, \tag{4}$$

with $\gamma$ selected equal to one, hence yielding an *attractive Laplacian* control law. One of the main objectives of this paper consists in showing that an *attractive Laplacian* control law does not lead to consensus in a overflowing *multi-clock model*, but conversely the consensus can be obtained, somewhat unexpectedly, with a *repulsive Laplacian* control law, namely posing $\gamma = -1$ in (3).

## III. CONSENSUS OF TWO OVERFLOWING CLOCKS

In this section we focus on the basic scenario of the consensus between two overflowing clocks. Therefore, assume without loss of generality $\alpha = 1$ and consider a *multi-overflowing-clock model* as in (1) with $N = 2$, namely

$$\begin{aligned} \dot{x}_1 &= 1 + u_1, \quad x_1 \in [0,1]\,, \quad x_1^+ = 0, \quad x_1 \geq 1\,, \\ \dot{x}_2 &= 1 + u_2, \quad x_2 \in [0,1]\,, \quad x_2^+ = 0, \quad x_2 \geq 1\,. \end{aligned} \tag{5}$$

We refer to the model in (5) as the *two-overflowing-clock model*. The *error* in (2) belongs to $\mathbb{R}$, since

$$\eta(t,k) := x_1(t,k) - x_2(t,k)\,, \tag{6}$$

and it is characterized by the following *flow-dynamics*

$$\begin{aligned} \dot{\eta}(t,k) &= \dot{x}_1(t,k) - \dot{x}_2(t,k) \\ &= u_1(t,k) - u_2(t,k)\,, \end{aligned} \tag{7}$$

which, under the control law in (3), becomes

$$\begin{aligned} \dot{\eta}(t,k) &= -\gamma \begin{bmatrix} 1 & -1 \end{bmatrix} L \begin{bmatrix} x_1(t,k) & x_2(t,k) \end{bmatrix}' \\ &= -\gamma \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t,k) \\ x_2(t,k) \end{bmatrix} \\ &= -2\gamma\eta(t,k)\,. \end{aligned} \tag{8}$$

**Remark 2.** The equation (8) entails that the modulus of the *error* has a monotonically increasing behaviour, during flows, if $\gamma = -1$ (*repulsive Laplacian*), otherwise a decreasing behaviour if $\gamma = 1$ (*attractive Laplacian*). ▲

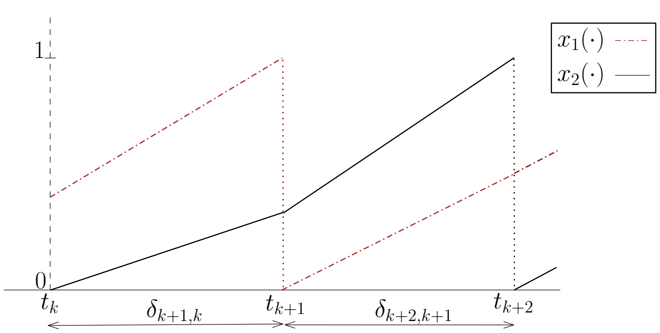


Fig. 1: Time histories of two overflowing clocks over three reset times in closed loop with a repulsive Laplacian law.

On the other hand, denoting $t_k$ as the time instant in which a reset occurs (*i.e.* when either of the clocks reaches its upper-bound), the error *jump-dynamics* is such that

$$\eta_k^+ := x_{1,k}^+ - x_{2,k}^+ = \eta_k - sign(\eta_k)\,,$$

from which, since $sign(\eta_k^+) = -sign(\eta_k)$, it appears evident that the absolute value of the error satisfies $|\eta_k^+| = 1 - |\eta_k|$.

In the remaining of this section, we first derive an equation relating the time intervals between three consecutive jumps, which is then instrumental for characterizing all the periodic orbits of (5) under the action of (3). The section is concluded by the stability analysis of such trajectories.

**Proposition 1.** *Consider a two-overflowing-clock model, with $u(t,k)$ as in (3) for $\gamma \in \{-1, 1\}$, let $t_k$ and $t_h$ be two reset times and assume $t_h > t_k$. Define $\delta_{h,k}$ as the generic time interval between $t_k$ and $t_h$, then the following relation holds*

$$\delta_{h,k} = \frac{1}{2}(x_{1,h} + x_{2,h} - x_{1,k}^+ - x_{2,k}^+) + \frac{h-k-1}{2}\,. \tag{9}$$

◇

*Proof.* To begin with, note that the *flow-dynamic* in (5) under the action of the control law $u(t,k)$ defined in (3), namely

$$\begin{bmatrix} \dot{x}_1(t,k) \\ \dot{x}_2(t,k) \end{bmatrix} = -\gamma \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t,k) \\ x_2(t,k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

with $\gamma \in \{-1, 1\}$, can be equivalently rewritten as

$$\begin{cases} \dot{x}_1(t,k) = 1 - \gamma\eta(t,k)\,, \\ \dot{x}_2(t,k) = 1 + \gamma\eta(t,k)\,. \end{cases} \tag{10}$$

Recalling that $t_{k+1}$ denotes the reset time following $t_k$, then the evolution of the system (10) over $[t_k, t_{k+1}]$ is

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} \delta_{k+1,k} + \frac{1}{2}(x_{1,k}^+ + x_{2,k}^+) + \frac{1}{2}e^{-2\gamma\delta_{k+1,k}}\eta_k^+ \\ \delta_{k+1,k} + \frac{1}{2}(x_{1,k}^+ + x_{2,k}^+) - \frac{1}{2}e^{-2\gamma\delta_{k+1,k}}\eta_k^+ \end{bmatrix}.$$

By adding the two latter relations, the time interval between $t_k$ and $t_{k+1}$ can be obtained as

$$\delta_{k+1,k} = \frac{1}{2}(x_{1,k+1} + x_{2,k+1} - x_{1,k}^+ - x_{2,k}^+)\,. \tag{11}$$

Iteratively exploiting (11) and assuming that the fol-

$$\delta_{h,k} = \delta_{h,h-1} + \cdots + \delta_{k+2,k+1} + \delta_{k+1,k} = \frac{1}{2}(x_{1,h} + x_{2,h}) - \frac{1}{2}(x^+_{1,h-1} + x^+_{2,h-1}) + \cdots$$
$$\cdots + \frac{1}{2}(x_{1,k+2} + x_{2,k+2}) - \frac{1}{2}(x^+_{1,k+1} + x^+_{2,k+1}) + \frac{1}{2}(x_{1,k+1} + x_{2,k+1}) - \frac{1}{2}(x^+_{1,k} + x^+_{2,k}) \,. \tag{12}$$

lowing reset times occur at $t_{k+2},\ t_{k+3},\ \ldots,\ t_h$, then the overall length of the time interval between $t_k$ and $t_h$ is given by (12).

Since at each jump every state can either change from one to zero or remain unchanged, as briefly shown in figure Fig. 1, then the following equation holds

$$(x_{1,j} + x_{2,j}) - (x^+_{1,j} + x^+_{2,j}) = 1 \,,$$

for $j = k+1, \ldots, h-1$, then (12) implies (9). □

**Proposition 2.** *Consider a two-overflowing-clock model under Assumption 1, with $u(t,k)$ as in (3) for $\gamma \in \{-1, 1\}$, let $t_k$, $t_{k+1}$ and $t_{k+2}$ be the time instants in which three consecutive resets occur, then it holds*

$$\delta_{k+2,k} = t_{k+2} - t_k = 1 \,, \tag{13}$$

*along any orbit periodic over $[t_k, t_{k+2}]$.* ◇

*Proof.* The equation in (9) implies that the time interval between $t_k$ and $t_{k+2}$ can be rewritten as

$$\delta_{k+2,k} = \frac{1}{2} + \frac{x_{1,k+2} - x^+_{1,k}}{2} + \frac{x_{2,k+2} - x^+_{2,k}}{2} \,. \tag{14}$$

Since by definition of periodic orbit $x^+_{1,k+2} = x^+_{1,k} - 1$ and $x^+_{2,k+2} = x_{2,k}$, then (14) yields (13). □

The relation in (13), which entails that the time interval between a complete cycle of jumps of $x_1$ and $x_2$ must be necessary equal to 1 along any periodic orbit, is now employed to determine the number of admissible periodic orbits for the error dynamics in a *two-overflowing-clock* model.

**Proposition 3.** *Consider a two-overflowing-clock model under Assumption 1, with $u(t,k)$ as in (3) for $\gamma \in \{-1, 1\}$, then only two periodic orbits are admissible and their intersections to the $x_2$-axis correspond to*

$$x_1 = 0 \,, \tag{15a}$$
$$x_1 = \frac{1}{1 + e^{-\gamma}} \,. \tag{15b}$$

◇

**Remark 3.** Since along the periodic trajectory defined starting from $x_0 = \begin{bmatrix} x_{1,0} & x_{2,0} \end{bmatrix}' = \begin{bmatrix} 0 & 0 \end{bmatrix}'$ the error remains equal to zero, then it is referred to as the *error-zeroing* orbit; instead, along the periodic trajectory

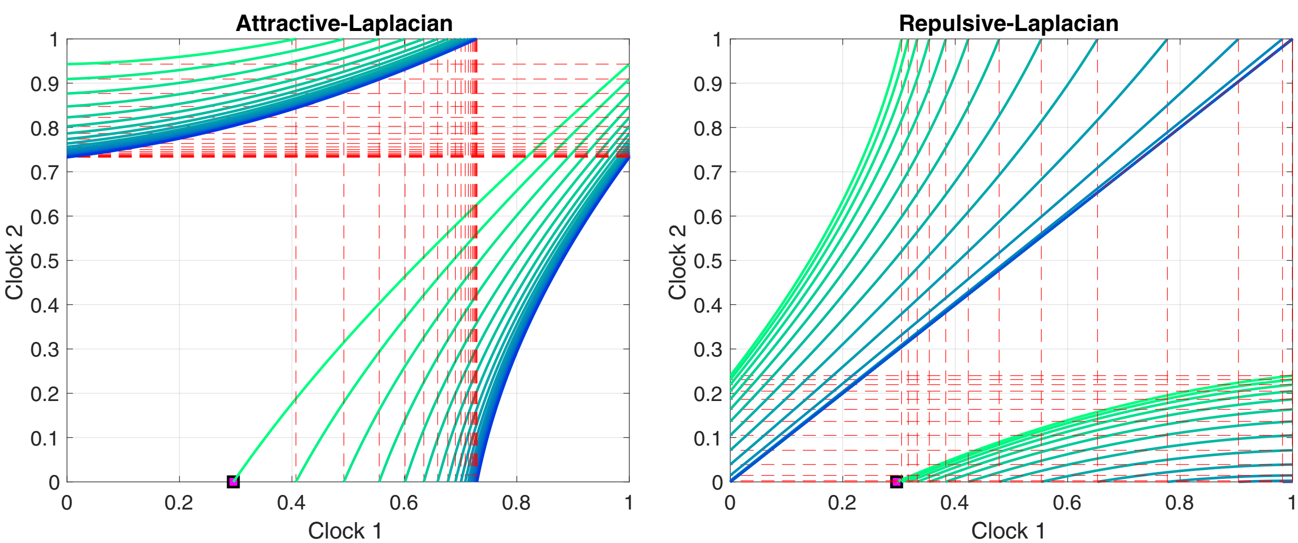


Fig. 2: Evolution of the two clocks in (5) under an *attractive Laplacian* control law (top graph) and under a *repulsive Laplacian* control law (bottom graph). The initial condition is marked by a magenta rectangle. The forward time evolution of the trajectories is replicated by colors, ranging from green shades to blue shades. The red dashed lines define the jumps.

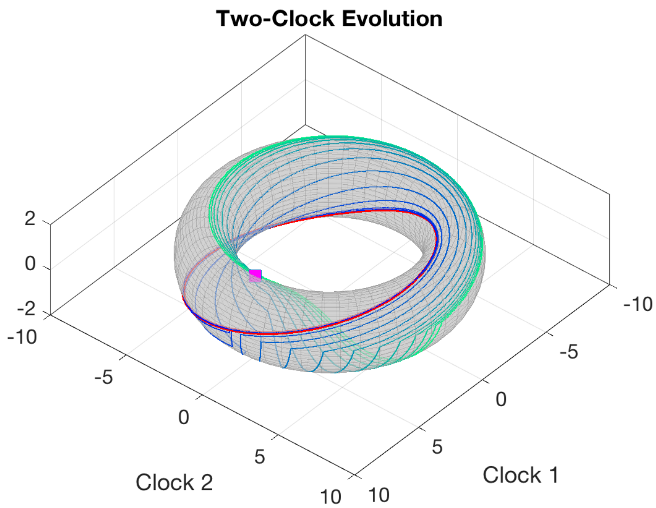


Fig. 3: Three dimensional representation of the two overflowing clocks evolution under a *repulsive Laplacian* control law. The initial condition is marked by a magenta cube. The forward time evolution of the trajectories is replicated by colors, ranging from green shades to blue shades.

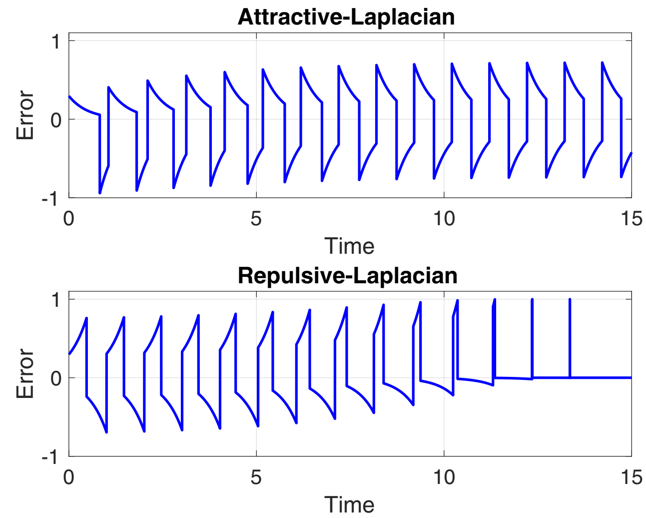


Fig. 4: Time evolution of the error in (6) under an *attractive Laplacian* control law (on the top) and under a *repulsive Laplacian* control law (on the bottom).

defined starting from $x_0 = \begin{bmatrix} \frac{1}{1+e^{-\gamma}} & 0 \end{bmatrix}'$, the error remains different from zero, then it can be called the *non-error-zeroing* orbit. ▲

It now remains to assess the stability properties of these orbits, stating the following, somewhat surprising, result.

**Proposition 4.** *Consider a two-overflowing-clock model under Assumption 1, with $u(t, k)$ as in (3), then the error-zeroing orbit is (almost) globally asymptotically stable if $\gamma = -1$, i.e. under a repulsive Laplacian control law.* ⋄

Finally, it is proved that consensus is not achieved with the *attractive Laplacian* control law. The following remark stresses the impossibility to use the previous approach to demonstrate the attractiveness property also in the *attractive Laplacian* case, and an alternative approach is pursued.

**Proposition 5.** *Consider a two-overflowing-clock model under Assumption 1, with $u(t, k)$ as in (3), then the* non-error-zeroing orbit *is (almost) globally asymptotically stable if $\gamma = 1$, i.e. under an attractive Laplacian control law.* ⋄

## IV. Numerical Simulation

In this section the theoretical results related to the use of a *repulsive Laplacian* and an *attractive Laplacian* control laws in a *two-overflowing-clock* model are supported by a simulation. Consider the system in (5) under the control law in (3), with $\alpha = 1$ and $\gamma \in \{-1, 1\}$. Defining the initial condition as $x_0 = \begin{bmatrix} \frac{1.1}{1+e} & 0 \end{bmatrix}'$, namely a condition that does not belong to a periodic orbit, then the two overflowing clocks evolve as illustrated in Fig. 2, in which it is clearly shown, using a color map from green to blue, that starting from the same initial condition (marked by a magenta rectangle) in the *attractive Laplacian* case the trajectory converges to the *non-error-zeroing* orbit, otherwise, under the *repulsive Laplacian* control law, the two clock evolution converges to the *error-zeroing* orbit. Since the nature of the jump events is such that the two clock values are limited between zero to one, then the state-space, namely a square box in 2D (as seen in Fig. 2) can be graphically represented in 3D by a toroid. Hence, the performance of the *repulsive Laplacian* control law can be better appreciated through a 3D representation of the same *two-overflowing-clock* model evolution as done in Fig. 3, where the inner radius has been deliberately made larger for visualization purposes. In addition, to stress even more the attractivity properties of the *error-zeroing* orbit under a classical *attractive Laplacian* law rather than the *repulsive Laplacian* law, a sketch of the time evolution of the error is proposed in Fig. 4.

The latter figure entails that if $\gamma = 1$ (*attractive Laplacian*) the error converges to a limit cycle, on the other hand if $\gamma = -1$ (*repulsive Laplacian*) the error grows during flows and jumps until it is reset to zero, hence achieving consensus of the two clocks. Note that, due to numerical approximation, in the bottom of Fig. 4 the error seems to go to zero in finite time, but in fact the convergence is asymptotic.

## V. Conclusion

A solution to the *two-overflowing-clocks* consensus problem, based on relative information, is proposed in this paper. The perfomance of two Laplacian control techniques (attractive and repulsive) have been considered for the proposed hybrid system and the stability properties of the periodic orbits are analysed under the control action. The almost global convergence to consensus of a two-overflowing-clock model under the *repulsive Laplacian* control law, assuming an instantaneous communication between the two clocks, is demonstrated, while it is stressed the inability of the *attractive Laplacian* control law in achieving the same issue.

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