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Distributed Offset Correction for Time Synchronization in Networks with Random Delays

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***Abstract*— A new distributed asynchronous offset correction algorithm for time synchronization in networks with random communication delays, communication dropouts, and measurement noise is proposed. The algorithm is based on a specially constructed error function which includes delay compensation and unboundedly-increasing-time compensation parameters. Convergence of the algorithm in the mean square sense and w.p.1 is proved. A modification of the algorithm based on consensus on the delay compensation parameter is also proposed, offering better performance in practice. Illustrative simulation results are presented.**

I. INTRODUCTION

Cyber-Physical Systems (CPS), Internet of Things (IoT) and Sensor Networks (SN) have emerged as research areas of paramount importance, with many conceptual and practical challenges and numerous applications [1]–[3]. One of their basic requirements is *time synchronization*, providing a common notion of time to all the nodes. The problem of time synchronization has attracted a lot of attention, but still represents a challenge, *e.g.*, [4]. There are many approaches to time synchronization based on different assumptions and using different methodologies, *e.g.*, [4], [5]. Distributed schemes with the so-called gradient property [6], [7] have been proposed, including consensus-based algorithms, *e.g.*, [8]–[14]. A class of consensus based algorithms, called CBTS (Consensus-Based Time Synchronization) algorithms, have been treated in a unified way in [15]. Fundamental and yet unsolved problems in time synchronization are connected with communication delays and measurement noise; see [16] for basic issues, and [10], [17]–[19] for different aspects of delay influence.

In this paper we propose a new *asynchronous distributed* algorithm for *offset correction* in networks characterized by *random communication delays*, *communication dropouts* and *measurement noise*. We present an original recursion for offset correction parameter estimation starting from a specific average error function, obtained from the simple difference between local times by introducing two modifications aiming at: 1) eliminating the effect of linearly increasing absolute time, 2) coping with the influence of random delays by introducing additional *compensation parameters*. It is proved that the algorithm provides convergence in the mean square sense and w.p.1 to a set of finite random variables. To the authors' knowledge, the proposed algorithm represents the first method for offset correction able to handle both random delays and measurement noise. When used together with an appropriate drift correction algorithm (*e.g.*, algorithms proposed in [20]) it provides complete time synchronization in such lossy networks. The proof of convergence is derived using stochastic approximation arguments, taking into account asynchronous features of the recursions and properties of the average weighted Laplacian matrix of the underlying directed graph. An improvement of the algorithm based on the introduction of linear consensus iterations into the recursions for compensation parameters, aiming at decreasing the dispersion of the local offset convergence points, is introduced as an efficient tool in practice. Special cases related to delay and noise are discussed apart. Finally, some illustrative simulation results are presented.

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II. ALGORITHMS

A. Problem Definition

Assume a network consisting of n nodes, formally represented by a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is the set of nodes and $\mathcal{E}$ the set of arcs defining the structure of inter-node communications. Assume also that $\mathcal{N}_i^+$ is the out-neighborhood and $\mathcal{N}_i^-$ the in-neighborhood of node i.

Each node has a local clock defining the *local time*, given for any *absolute time* $t \in \mathcal{R}$ by

$$\tau_i(t) = \alpha_i t + \beta_i + \xi_i(t), \tag{1}$$

where α_i is the local *drift (gain)*, β_i is the local *offset*, while $\xi_i(t)$ is *measurement noise*, appearing due to equipment instabilities, round-off errors, thermal noise, *etc.* (see, *e.g.,* [12], [13], [21]–[23]). Each node i applies an affine transformation to $\tau_i(t)$, producing the *corrected local time*

$$\bar{\tau}_i(t) = a_i \tau_i(t) + b_i = g_i t + f_i + a_i \xi_i(t), \tag{2}$$

where a_i and b_i are local *correction parameters*, $g_i = a_i \alpha_i$ is the *corrected drift* and $f_i = a_i \beta_i + b_i$ the *corrected offset*, $i = 1, \ldots, n$.

In this paper we construct a new algorithm for *distributed real-time estimation* of the parameters b_i, assuming that parameters a_i, asymptotically providing equal corrected drifts g_i, are available (see, *e.g.*, [20]). Corrected drifts g_i and corrected offsets f_i should provide *global virtual time*, $i = 1, \ldots, n$.

The nodes communicate according to the *broadcast gossip scheme*, *e.g.*, [24]–[26], without requiring any fusion center. We assume that each node $j \in \mathcal{N}$ has its own *local broadcast clock* that ticks at the instants $t_l^j \in \mathcal{R}$, $l = 0, 1, 2, \ldots$, according to a Poisson process with the rate μ_j, independently of the other nodes. At each tick of its broadcast clock, node j broadcasts its current local time (together with its correction parameters, see the algorithms below) to its out-neighbors $i \in \mathcal{N}_j^+$. Each node $i \in \mathcal{N}_j^+$ hears the broadcast with probability $p_{ji} > 0$. The message of node j is received at node i at the time instant $t_l^{j,i} = t_l^j + \delta_l^{j,i}$,where $\delta_l^{j,i}$ represents the corresponding *communication delay* (see, *e.g.*, [10], [16]–[18], [27] for presentation of physical and technical sources of the delays). We assume in the sequel that the communication delay can be decomposed as

$$\delta_l^{j,i} = \bar{\delta}^{j,i} + \eta_i(t_l^{j,i}), \tag{3}$$

where $\bar{\delta}^{j,i}$ is assumed to be constant, while $\eta_i(t_l^{j,i})$ represents a stochastically time-varying component with zero mean. Then, all the nodes which have received the broadcast read their current local time $\tau_i(t)$, calculate their own current *corrected outputs* $\bar{\tau}_i(t_l^{j,i})$ and *update the values of their correction parameters* a_i and/or b_i. The process is repeated after each tick of the broadcast clock of any node in the network.

B. Offset Correction

We shall construct an algorithm for updating parameters b_i using the following error function:

$$\bar{\varphi}_i^b(t_l^j) = \bar{\tau}_j(t_l^j) - a_j\Delta\tau_j(t_l^j) - [\bar{\tau}_i(t_l^{j,i}) - a_i\Delta\tau_i(t_l^{j,i})] + c_i, \tag{4}$$

$i \in \mathcal{N}_j^+$, $j = 1, \ldots, n$, where $\Delta\tau_j(t_l^j)$ and $\Delta\tau_i(t_l^{j,i})$ are easily computable *increments of the local times*, given by $\Delta\tau_j(t_l^j) = \tau_j(t_l^j) - \tau_j(t_0^j) = \alpha_j \Delta t_l^j + \Delta\xi_j(t_l^j)$, $\Delta t_l^j = t_l^j - t_0^j$, $\Delta\xi_j(t_l^j) = \xi_j(t_l^j) - \xi_j(t_0^j)$, $\Delta\tau_i(t_l^{j,i}) = \alpha_i \Delta t_l^{j,i} + \Delta\xi_i(t_l^{j,i})$, $\Delta t_l^{j,i} = \Delta t_l^j + \Delta\delta_l^{j,i}$, where $\Delta\delta_l^{j,i} = \delta_l^{j,i} - \delta_0^{j,i}$, and $\Delta\xi_i(t_l^{j,i}) = \xi_i(t_l^{j,i}) - \xi_i(t_0^{j,i})$; c_i is an additional *compensation parameter*.

Starting from (4), we come up with the following updates for b_i and c_i, based on the assumption that an estimate of a_i is given:

$$\hat{b}_i(t_l^{j,i+}) = \hat{b}_i(t_l^j) + \varepsilon_i^b(t_l^j)\gamma_{ij}\varphi_i^b(t_l^j) \tag{5}$$

$$\hat{c}_i(t_l^{j,i+}) = \hat{c}_i(t_l^j) - \varepsilon_i^b(t_l^j)\gamma_{ij}\varphi_i^b(t_l^j) \tag{6}$$

where γ_{ij} are *a priori* chosen *nonnegative weights*, $\varphi_i^b(t_l^j) = \hat{\tau}_j(t_l^j) - \hat{a}_j(t_l^j)\Delta\tau_j(t_l^j) - [\hat{\tau}_i(t_l^{j,i}) - \hat{a}_i(t_l^j)\Delta\tau_i(t_l^{j,i})] + \hat{c}_i(t_l^j)$ is the current error resulting from (4), with $\hat{\tau}_j(t_l^j) = \hat{a}_j(t_l^j)\tau_j(t_l^j) + \hat{b}_j(t_l^j)$, $\hat{\tau}_i(t_l^{j,i}) = \hat{a}_i(t_l^j)\tau_i(t_l^{j,i}) + \hat{b}_i(t_l^j)$, $\hat{b}_i(t_l^j)$ and $\hat{c}_i(t_l^j)$ are the old estimates, $\hat{b}_i(t_l^{j,i+})$ and $\hat{c}_i(t_l^{j,i+})$ the new estimates, while $\varepsilon_i^b(t_l^j)$ is a positive step size. The estimates of the drift correction parameters $\hat{a}_j(t_l^j)$ are assumed to be generated by any adequate algorithm; when it is generated by one of the algorithms presented in [20], we obtain a consistent new time synchronization algorithm. The initial estimates are $\hat{b}_i(t_{0,i}) = \hat{c}_i(t_{0,i}) = 0$.

A *consensus-based modification* of (6) is considered apart. This modification is formally obtained by replacing $\hat{c}_i(t_l^j)$ at the right hand side of (6) by the following *convex combination*

$$\hat{c}_i^{con}(t_l^j) = \sigma_i\hat{c}_i(t_l^j) + (1 - \sigma_i)\hat{c}_j(t_l^j), \tag{7}$$

$0 < \sigma_i \leq 1$. The effects of the consensus-based modification will be analyzed below.

In terms of $\hat{g}_i(\cdot) = \hat{a}_i(\cdot)\alpha_i$ and $\hat{f}_i(\cdot) = \hat{a}_i(\cdot)\beta_i + \hat{b}_i(\cdot)$, (5) and (6) become:

$$\hat{f}_i(t_l^{j,i+}) + \Delta\hat{g}_i(t_l^{j,i+}) = \hat{f}_i(t_l^j) + \varepsilon_i^b(t_l^j)\gamma_{ij}\psi_i^b(t_l^j), \tag{8}$$

$$\hat{c}_i(t_l^{j,i+}) = \hat{c}_i(t_l^j) - \varepsilon_i^b(t_l^j)\gamma_{ij}\psi_i^b(t_l^j), \tag{9}$$

where $\Delta\hat{g}_i(t_l^{j,i+}) = \frac{\beta_i}{\alpha_i}[\hat{g}_i(t_l^{j,i}) - \hat{g}_i(t_l^{j,i+})]$, and $\psi_i^b(t_l^j) = [\hat{g}_j(t_l^j) - \hat{g}_i(t_l^j)]t_0^j + \hat{f}_j(t_l^j) - \hat{f}_i(t_l^j) - \hat{g}_i(t_l^j)[\bar{\delta}^{i,j} + \eta_i(t_0^{j,i})] + \hat{c}_i(t_l^j) + \frac{1}{\alpha_j}\hat{g}_j(t_l^j)\xi_j(t_0^j) - \frac{1}{\alpha_i}\hat{g}_i(t_l^j)\xi_i(t_0^{j,i})$.

Remark 1: The proposed offset correction estimation scheme represented by (5), (6), (8) and (9) is obtained by *two major modifications* of the basic error function $\psi_i^b(t_l^j)^0 = \bar{\tau}_j(t_l^j) - \bar{\tau}_i(t_l^{j,i})$, which has been utilized in all the existing CBTS algorithms (see [15] and the references therein). The first modification introduces two additional terms $a_j\Delta\tau_j(t_l^j)$ and $a_i\Delta\tau_j(t_l^j)$, the role of which is to replace the unboundedly increasing term t_l^j in the expression for $\psi_i^b(t_l^j)^0$ by the bounded term t_0^j in $\psi_i^b(t_l^j)$. The second modification consists of introducing a *new variable* $\hat{c}_i(t_l^j)$, the main role of which is to cope directly with the effects of communication delays and enable convergence of offset correction parameter estimates.

Notice that the algorithm for the offset synchronization proposed in [12] cannot handle delays and measurement noise, while the algorithm [14] does not ensure convergence of corrected offsets.

C. Global Model

In order to obtain a global model for the whole network, we introduce at this point a *global virtual broadcast clock* with the rate equal to $\mu = \sum_{i=1}^n \mu_i$, that ticks whenever any of the local broadcast clocks tick (see, *e.g.*, [24], [25]). Let the k-th tick of the virtual clock, $k = 1, 2, \ldots$, correspond to the k-th update of the parameter estimates, or k-th iteration of the whole time synchronization algorithm. Let j be the index of the node that broadcasts at the k-th tick. Following [15], [24], we replace the variable t_l^j by the corresponding index k in all the above defined functions of time, $\tau_j(t_l^j) = \tau_j(k)$, $\bar{\tau}_j(t_l^j) = \bar{\tau}_j(k)$, $\xi_j(t_l^j) = \xi_j(k)$, etc. We also replace $t_l^{j,i}$ by k, so that: $\tau_i(t_l^{j,i}) = \tau_i(k)$, $\bar{\tau}_i(t_l^{j,i}) = \bar{\tau}_i(k)$, $\xi_i(t_l^{j,i}) = \xi_i(k)$, etc. In the case of the delays, we write $\bar{\delta}^{j,i} = \bar{\delta}_i(k)$; analogously, we also write $\eta_i(t_l^{j,i})$ as $\eta_i(k)$.

For notational simplicity we now introduce the following assumption on the probabilities of receiving a message sent by an in-neighbor:

(A1) $p_{ji} = 1$; $j, i = 1, \ldots, n$.

Let $\hat{g}(k) = [\hat{g}_1(k) \cdots \hat{g}_n(k)]^T$, $\hat{f}(k) = [\hat{f}_1(k) \cdots \hat{f}_n(k)]^T$ and $\hat{c}(k) = [\hat{c}_1(k) \cdots \hat{c}_n(k)]^T$, where $\hat{g}_i(k) = \hat{a}_i(k)\alpha_i$, $\hat{a}_i(k) = \hat{a}_i(t_l^j)$, $\hat{f}_i(k) = \hat{a}_i(k)\beta_i + \hat{b}_i(k)$, $\hat{b}_i(k) = \hat{b}_i(t_l^j)$, $\hat{c}_i(k) = \hat{c}_i(t_l^j)$, $i = 1, \dots, n$, $\hat{f}(k+1) = [\hat{f}_1(t_l^{j,1+}) \dots \hat{f}_n(t_l^{j,n+})]^T$, $\hat{g}(k+1) = [\hat{g}_1(t_l^{j,1+}) \dots \hat{g}_n(t_l^{j,n+})]^T$, $\varepsilon^b(k) = \operatorname{diag}\{\varepsilon_1^b(k), \dots, \varepsilon_n^b(k)\}$, $\varepsilon_i^b(k) = \varepsilon_i^b(t_l^j)$, $A = \operatorname{diag}\{\alpha_1, \dots, \alpha_n\}$, $\Gamma(k) = [\Gamma(k)_{\mu\nu}]$, with $\Gamma(k)_{ii} = -\gamma_{ij}$ and $\Gamma(k)_{ij} = \gamma_{ij}$, with $\Gamma(k)_{\mu\nu} = 0$ otherwise, $\Gamma_d(k) = \operatorname{diag}\{\operatorname{diag}\{\gamma_{1j}, \dots, \gamma_{nj}\}\omega(k)\}$, $\omega(k) = [\omega_1(k) \dots \omega_n(k)]^T$, $\omega_i(k) = 1$, $\omega_\mu(k) = 0$ for $\mu \neq i$.

Then, (8) and (9) give

$$\hat{f}(k+1) + \Delta\hat{g}(k+1) = \hat{f}(k) + \varepsilon^b(k)Y(k) \tag{10}$$

$$\hat{c}(k+1) = \hat{c}(k) - \varepsilon^b(k)Y(k), \tag{11}$$

where $\Delta\hat{g}(k+1) = \operatorname{diag}\omega(k)(\hat{g}(k+1) - \hat{g}(k))$, $Y(k) = \Gamma(k)\hat{f}(k) + [t^0(k)\Gamma(k) - \Gamma_d(k)\bar{\delta}_d(k) - \Gamma_d(k)\eta_d^0(k) + \Gamma(k)\xi_d^0(k)A^{-1}]\hat{g}(k) + \Gamma_d(k)\hat{c}(k)$, $t^0(k) = t_0^j$ (t_0^j depends on k, in the sense that it depends on the chosen j), $\bar{\delta}_d(k) = \operatorname{diag}\bar{\delta}(k)$, $\bar{\delta}(k) = [\bar{\delta}^{1,j} \cdots \bar{\delta}^{n,j}]^T$, ($\bar{\delta}^{\mu,j} = 0$ for $\mu \notin \mathcal{N}_j^+$), $\eta_d^0(k) = \operatorname{diag}\eta^0(k)$, $\eta^0(k) = [\eta_1^0(k) \cdots \eta_n^0(k)]^T$, ($\eta_i^0(k) = \eta_i(t_0^{j,i})$ for $i \in \mathcal{N}_j^+$, otherwise, $\eta_i^0(k) = 0$), $\xi_d^0(k) = \operatorname{diag}\xi^0(k), \xi^0(k) = [\xi_1^0(k) \cdots \xi_n^0(k)]^T$ ($\xi_j^0(k) = \xi_j(t_0^j)$, $\xi_i^0(k) = \xi_i(t_0^{j,i})$ for $i \in \mathcal{N}_j^+$, otherwise, $\xi_i^0(k) = 0$), and $\hat{g}_d(k) = \operatorname{diag}\{\hat{g}_1(k), \dots, \hat{g}_n(k)\}$. Notice that $\{t^0(k)\}$, $\{\eta^0(k)\}$ and $\{\xi^0(k)\}$ are random sequences with a finite set of possible realizations of t_0^j, $\eta_i(t_0^{j,i})$ and $\xi_j(t_0^j)$ (or $\xi_i(t_0^{j,i})$), $j = 1, \dots, n$, selected at random for each k by the choice of j.

For the algorithm with consensus, $\hat{c}(k)$ is replaced by $\hat{c}^{con}(k) = C(k)\hat{c}(k)$, where $C(k) = [C(k)_{\mu\nu}]$, with $C(k)_{\mu\mu} = \sigma_\mu$ and $C(k)_{\mu j} = 1 - \sigma_\mu$ for all $\mu \in \mathcal{N}_j^+$, with $C(k)_{\mu\nu} = 0$ otherwise.

III. Convergence Analysis

A. Preliminaries

Within the exposed general setting, we additionally assume:

(A2) Graph $\mathcal{G}$ has a spanning tree.

(A3) $\{\xi_i(k)\}$ and $\{\eta_i(k)\}$, $i = 1, \dots n$, are mutually independent zero mean i.i.d. random sequences, bounded w.p.1.

(A4) The step sizes $\varepsilon_i^b(k)$ are given by $\varepsilon_i^b(k) = \nu_i(k)^{-\zeta}$ and $\nu_i(k) = \sum_{m=1}^k I\{$ node i received a message $\}$, which represents the number of updates of node i up to the instant k ($I\{\cdot\}$ denotes the indicator function), while $\frac{1}{2} < \zeta \leq 1$.

Remark 2: The assumptions are typical for similar problems. Notice that (A4) eliminates the need for a centralized clock which would provide k to all the nodes.

Asymptotics of the step size $\varepsilon_i^b(k)$ are discussed in detail in [20].

Properties of matrix $\Gamma(k)$ are essential for convergence of (10) and (11); its expectation $\bar{\Gamma} = E\{\Gamma(k)\}$ contains all the information about the network structure and the weights of particular links. It has the structure of a weighted Laplacian for $\mathcal{G}$:

$$\bar{\Gamma} = \begin{bmatrix} -\sum\limits_{j,j\neq 1}\gamma_{1j}\pi_{1j} & \gamma_{12}\pi_{12} & \cdots & \gamma_{1n}\pi_{1n} \\ \gamma_{21}\pi_{21} & -\sum\limits_{j,j\neq 2}\gamma_{2j}\pi_{2j} & \cdots & \gamma_{2n}\pi_{2n} \\ & & \ddots & \\ \gamma_{n1}\pi_{n1} & \gamma_{n2}\pi_{n2} & \cdots & -\sum\limits_{j,j\neq n}\gamma_{nj}\pi_{nj} \end{bmatrix} \tag{12}$$

($\gamma_{ij} = 0$ when $j \notin \mathcal{N}_i^-$), where π_{ij} is the probability that the node i updates its parameters as a consequence of a tick of node j (π_{ij} follow, in general, from the Poisson rates μ_j and the transmission probabilities p_{ij}; in the case of no communication dropouts, $\pi_{ij} = \pi_j$, $i \in \mathcal{N}_j^+$, where π_j is the probability of node j to broadcast).

According to [20], $\bar{\Gamma}$ has one eigenvalue at the origin and the remaining ones in the left half plane by virtue of (A1) [28], [29]. If $T = \left[\mathbf{1} \vdots T_{n\times(n-1)}\right]$, where $T_{n\times(n-1)}$ is such that $\operatorname{span}\{T_{n\times(n-1)}\} = \operatorname{span}\{B\}$ ($\mathbf{1} = [1 \cdots 1]^T$), then, $T^{-1}\bar{\Gamma}T = \begin{bmatrix} 0 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & \bar{\Gamma}^* \end{bmatrix}$,where $\bar{\Gamma}^*$ is Hurwitz [20]. This fact is used within the proof of the theorems below.

B. Convergence

The main assumption is availability of the estimates of *corrected drifts*, satisfying:

(A5) Corrected drifts $\hat{g}(k)$ utilized in the offset correction algorithm are available and satisfy w.p.1

$$\hat{g}(k) = \chi(k)\mathbf{1} + \hat{g}(k)^{[2]}, \tag{13}$$

where $\chi(k) = \chi^* + o(1)$, χ^* is a scalar random variable and $\|\hat{g}(k)^{[2]}\| = o(\frac{1}{k^d})$, $d > 0$.

We start the analysis by introducing the following expressions in (10) and (11):

$$\begin{gathered} \Gamma(k) = \bar{\Gamma} + \tilde{\Gamma}(k), \;\; \Gamma_d(k) = \bar{\Gamma}_d + \tilde{\Gamma}_d(k), \\ \xi^0(k) = \bar{\xi}^0 + \tilde{\xi}^0(k), \;\; \eta^0(k) = \bar{\eta}^0 + \tilde{\eta}^0(k), \\ \bar{\delta}(k) = \bar{\delta} + \tilde{\delta}(k), \;\; t^0(k) = \bar{t}^0 + \tilde{t}^0(k), \end{gathered} \tag{14}$$

where $\bar{\Gamma} = E\{\Gamma(k)\}$, $\bar{\Gamma}_d = E\{\Gamma_d(k)\}$, $\bar{\xi}^0 = E\{\xi^0(k)\} = \sum_{j=1}^n \xi(t_0^j)\pi_j$, $\bar{\eta}^0 = E\{\eta^0(k)\} = \sum_{j=1}^n \eta(t_0^j)\pi_j$, $\bar{\delta}^0 = E\{\bar{\delta}(k)\} = \sum_{j=1}^n [\bar{\delta}_0^{1,j} \cdots \bar{\delta}_0^{n,j}]^T\pi_j$ and $\bar{t}^0 = E\{t^0(k)\} = \sum_{j=1}^n t_0^j\pi_j$. Therefore, $\{\tilde{\Gamma}(k)\}$, $\{\tilde{\Gamma}_d(k)\}$, $\{\tilde{\xi}^0(k)\}$, $\{\tilde{\eta}^0(k)\}$, $\{\tilde{\delta}^0(k)\}$ and $\{\tilde{t}^0(k)\}$ are zero mean i.i.d. random sequences (due to randomness in determining the transmitting node for a given k).

Theorem 1: Let assumptions (A1)–(A5) be satisfied. Then, for all $\zeta \in (\frac{1}{2}, 1]$ satisfying $\zeta + d > 1$, $\hat{f}(k)$ from (10) converges to $\hat{f}^*$ and $\hat{c}(k)$ from (11) to $\hat{c}^*$ in the mean square sense and w.p.1, where $\hat{f}^*$ and $\hat{c}^*$ satisfy the equation

$$[\bar{\Gamma} \vdots \bar{\Gamma}_d]\hat{h}^* = 0, \tag{15}$$

where $\hat{h}^* = [(\hat{f}^* + \chi^* \bar{\xi}_d^0 A^{-1}\mathbf{1})^T \vdots (\hat{c}^* - \chi^* A(\bar{\eta}^0 + \bar{\delta}))^T]^T$.

The algorithm with consensus on $\hat{c}$ is of special practical interest.

Theorem 2: Let the assumptions of Theorem 1 hold. Then, for all $\zeta \in (\frac{1}{2}, 1]$ satisfying $\zeta + d > 1$, $\hat{f}(k)$ and $\hat{c}(k)$, generated by the algorithm (10), (11) with consensus on $\hat{c}(k)$ (using (7)) converge in the mean square sense and w.p.1 to $\hat{f}^*$ and $\hat{c}^* = \hat{c}^{con}\mathbf{1}$, respectively ($\hat{c}^{con}$ is a scalar), where $\hat{f}^*$ and $\hat{c}^{con}$ satisfy the equation $M_1^{con}\hat{h}^{con} = 0$, where

$$M_1^{con} = \left[\begin{array}{c:c} \bar{\Gamma} & \mathrm{vec}\{\bar{\Gamma}_d\} \\ \hdashline -\sum_{i=1}^{n} \bar{\phi}_i \bar{\Gamma}^{(i)} & -\sum_{i=1}^{n} \bar{\phi}_i \mathrm{vec}\{\bar{\Gamma}_d\}_i \end{array}\right], \tag{16}$$

$\hat{h}^{con} = [(\hat{f}^* + \chi^* \bar{\xi}_d^0 A^{-1}\mathbf{1})^T \vdots \hat{c}^{con} - \sum_{i=1}^n \bar{\phi}_i \chi^* (A\bar{\eta}^0 + A\bar{\delta})_i]^T$, with $\bar{\phi} = [\bar{\phi}_1 \cdots \bar{\phi}_n]$, $\bar{\phi}\bar{C} = \bar{\phi}$ and $\bar{C} = E\{C(k)\}$; $\bar{\Gamma}^{(i)}$ denotes i-th row of the matrix $\bar{\Gamma}$, and $\mathrm{vec}\{\bar{\Gamma}_d\}_i$ i-th element of $\mathrm{vec}\{\bar{\Gamma}_d\}$.

Remark 3: Theorems 1 and 2 guarantee convergence of all the corrected offsets, but, in general, not to the same value for all the nodes. However, comparison between the relations (15) and (16), indicates that it could be expected to achieve lower dispersion of the components of $\hat{f}^*$ within $\hat{h}^{con}$. Simulation results presented in Section IV confirm this statement.

Remark 4: Assumption (A1) can readily be removed at the expense of additional complexity of notation. Basically, the convergence proofs would remain unchanged, except for different convergence points and practically lower convergence rate.

Proofs of the theorems are given in the Appendix.

C. Special Cases

When communication delays and measurement noise can be neglected, the proposed algorithm (10), (11) *with* (7) is able *to achieve consensus* on corrected offsets $\hat{f}_i(k)$. Namely, in this case we have

$$\bar{\Gamma}\hat{f}^* + \bar{\Gamma}_d\mathbf{1}\ \hat{c}^{con} = 0$$
$$\sum_{i=1}^{n} \bar{\phi}_i \{\bar{\Gamma}^{(i)}\hat{f}^* + (\bar{\Gamma}_d\mathbf{1})_i \hat{c}^{con}\} = 0. \tag{17}$$

The equation $\bar{\Gamma}\hat{f}^* = -\bar{\Gamma}_d\mathbf{1}\ \hat{c}^{con}$ has a nontrivial solution for $\hat{f}^*$ only for $\hat{c}^{con} = 0$, having in mind that $\bar{\Gamma}_d\mathbf{1}$ does not belong to the column space of $\bar{\Gamma}$. Therefore, we have $\bar{\Gamma}\hat{f}^* = 0$, wherefrom the result follows. However, according to Theorem 1, the basic algorithm containing $\hat{c}(k)$, but without consensus cannot guarantee convergence of $\hat{f}(k)$ to consensus, due to additional degrees of freedom in the solution of $\bar{M}_1\hat{h}^* = 0$. In the case when delays are constant ($\delta_l^{j,i} = \bar{\delta}^{j,i}$) and the noise absent, the results of the theorems hold. Elimination of the recursion for $\hat{c}(k)$ leads to divergence.

When the stochastic terms $\xi(\cdot)$ and $\eta(\cdot)$ are equal to zero, it is possible to achieve exponential convergence rate by adopting constant step size. However, the offset synchronization algorithm again does not provide consensus, in general.

When the delay is equal to zero as well, the algorithm can be further simplified. We come up with a synchronization algorithm in which b_i is estimated using (5), where $\varphi_i^b(t_l^j) = \hat{\tau}_j(t_l^j) - \hat{\tau}_i(t_l^{j,i})$. The resulting algorithm is then able to achieve *exponential convergence to consensus* for corrected offsets. This result is obtained for the first time in [22] for pseudo periodic communication sequences (convergence properties of such an algorithm are equivalent to those from [12]). In the case when delays are constant ($\delta_l^{j,i} = \bar{\delta}^{j,i}$), the corrected offsets diverge. Introduction of a recursion for $\hat{c}(k)$ leads to convergence, in the sense of Theorem 1.

IV. Simulations

Simulation results are presented as illustrations of the main theoretical conclusions given above. A network of ten nodes forming a directed graph has been simulated, with drifts α_i and offsets β_i randomly chosen in the intervals $(0.95, 1.05)$ and $(-0.25, 0.25)$, respectively. Average communication delays $\bar{\delta}^{j,i}$ have been chosen to be 0.1, while $\{\eta(k)\}$ and $\{\xi(k)\}$ have been simulated as Gaussian white noise zero mean sequences with standard deviation equal to 0.01. It has been adopted that $\zeta = 0.95$.

Typical behavior of the proposed offset correction algorithms ((10), (11) with and without consensus iterations (7)) are presented in Figs. 1 and 2; drift correction parameters are generated by the algorithm b) from [20]. Convergence of all the components of the vector $\hat{f}(k)$ is evident in both cases. Convergence to consensus is not possible due to stochastic delays. However, both algorithms asymptotically decrease the distance between the corrected offsets $\hat{f}_i(k)$ compared to the initial values β_i. The algorithm with consensus provides lower dispersion of the asymptotic values, as expected.

To illustrate the importance of the introduced compensation parameters in the error function for offset correction (4), we set $\Delta\tau_i(k) = 0$, $i = 1, \ldots, n$ and obtain the typical behavior of the corrected offsets as in Fig. 3, which is a consequence of the linearly increasing weight of the estimation error of the drift correction algorithm. Furthermore, if we set $\hat{c}(k) = 0$, corrected offsets diverge to infinity for all realizations. The same conclusion holds for all existing distributed time synchronization algorithms for sensor networks.

V. Conclusion

In this paper a new distributed asynchronous algorithm has been proposed for offset correction for time synchronization in networks with random communication delays and measurement noise. The new algorithm is constructed starting from local time differences and adding: 1) special terms that take care of the influence of increasing time due to drift estimates which is an inevitable input to the algorithm, and 2) compensation parameters that take care of communication delays. It has been proved that the corrected offsets converge in the mean square sense and w.p.1 to finite random variables. An efficient algorithm for practical applications based on introducing consensus on the compensation parameters has

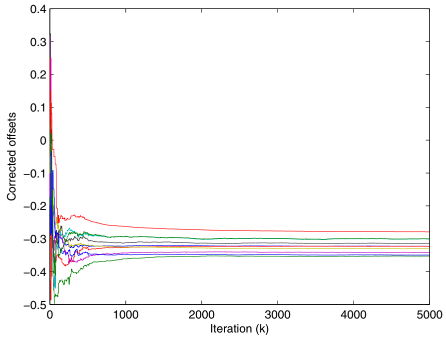

Fig. 1. Corrected offsets: no consensus on $\hat{c}(k)$

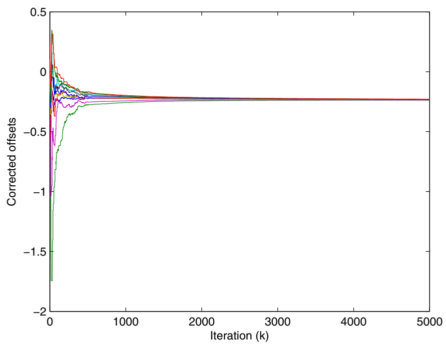

Fig. 2. Corrected offsets: consensus on $\hat{c}(k)$ included

also been proposed. Simulation results provide illustrations of the presented theoretical results.

Appendix

Proof of Theorem 1. Let $\hat{h}(k) = [(\hat{f}(k) + \chi(k)\bar{\xi}_d^0 A^{-1}\mathbf{1})^T \vdots (\hat{c}(k) - \chi(k)A(\bar{\eta}^0 + \bar{\delta}))^T]^T$. From (10), (11) and (14), after applying the results from [20], we obtain

$$\hat{h}(k+1) = \hat{h}(k) + \frac{1}{k^{\zeta''}} P_d^{-\zeta''}[M_1(k)(\hat{h}(k) + u_1(k) + u_2(k)) + M_2(k)\hat{G}(k)], \quad (18)$$

where

$$u_1(k) = o(\frac{1}{k^{\zeta^*}})[(A^{-1}\bar{\xi}^0)^T \vdots (A(\bar{\eta}^0 + \bar{\delta}))^T]^T,$$

$$u_2(k) = [(\hat{g}_d(k)A^{-1}\tilde{\xi}^0(k))^T \vdots (\hat{g}_d(k)A(\tilde{\eta}^0(k) + \tilde{\delta}(k)))^T]^T,$$

$M_1(k) = \bar{M}_1 + \tilde{M}_1(k)$, with

$$\bar{M}_1 = \begin{bmatrix} \bar{\Gamma} & \bar{\Gamma}_d \\ -\bar{\Gamma} & -\bar{\Gamma}_d \end{bmatrix}, \tilde{M}_1(k) = \begin{bmatrix} \tilde{\Gamma}(k) & \tilde{\Gamma}_d(k) \\ -\tilde{\Gamma}(k) & -\tilde{\Gamma}_d(k) \end{bmatrix},$$

$M_2(k) = \bar{M}_2 + \tilde{M}_2(k)$, $\bar{M}_2 = \mathrm{diag}\{\bar{t}^0\bar{\Gamma}, \bar{t}^0\bar{\Gamma}\}$, $\tilde{M}_2(k) = \mathrm{diag}\{\tilde{t}^0(k)\Gamma(k) + \bar{t}^0\tilde{\Gamma}, \tilde{t}^0(k)\Gamma(k) + \bar{t}^0\tilde{\Gamma}\}$, $\hat{G}(k) = [\hat{g}(k)^T \vdots \hat{g}(k)^T]^T$ and $P_d^{-\zeta} = \mathrm{diag}\{P^{-\zeta}, P^{-\zeta}\}$.

From (18) we realize that $P_d^{-\zeta}\bar{M}_1$ has n eigenvalues at the origin and n eigenvalues in the left half plane (according

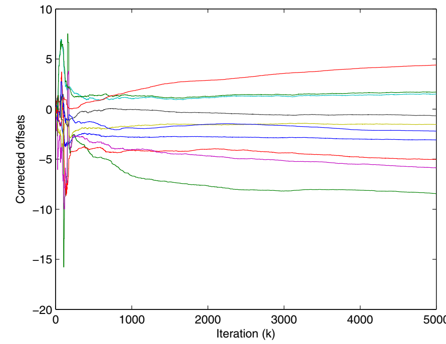

Fig. 3. Corrected offsets without compensation ($\Delta\tau_i(k) = 0$, $i = 1, \dots, n$)

to the properties of $\bar{\Gamma}$). Therefore, there exists a nonsingular transformation S such that

$$S^{-1}P_d^{-\zeta}\bar{M}_1 S = \begin{bmatrix} 0 & 0 \\ 0 & \bar{M}^* \end{bmatrix}, \quad (19)$$

where $\bar{M}^*$ is Hurwitz ([29]). Introduce $\tilde{h}(k) = S^{-1}\hat{h}(k)$, with $\tilde{h}(k) = [\tilde{h}(k)^{[1]T} \vdots \tilde{h}(k)^{[2]T}]^T$, where $\dim \tilde{h}(k)^{[1]} = \dim \tilde{h}(k)^{[2]} = n$. We obtain from (18) the following two recursions:

$$\tilde{h}(k+1)^{[1]} = \tilde{h}(k)^{[1]} + \frac{1}{k^{\zeta}}\{\Psi(k)^{[1]}\tilde{h}(k) + p(k)^{[1]} + q(k)^{[1]}\} \quad (20)$$

$$\tilde{h}(k+1)^{[2]} = \tilde{h}(k)^{[2]} + \frac{1}{k^{\zeta}}\{\bar{M}^*\tilde{h}(k)^{[2]} + \Psi(k)^{[2]}\tilde{h}(k) + p(k)^{[2]} + q(k)^{[2]}\}, \quad (21)$$

where

$$S^{-1}P_d^{-\zeta}\tilde{M}_1(k)S = \begin{bmatrix} \Psi(k)^{[1]} \\ \Psi(k)^{[2]} \end{bmatrix},$$

$$S^{-1}P_d^{-\zeta}[\tilde{M}_1(k)u_1(k) + M_1(k)u_2(k) + \tilde{M}_2(k)\hat{g}(k)] = \begin{bmatrix} p(k)^{[1]} \\ p(k)^{[2]} \end{bmatrix}, \quad (22)$$

$$S^{-1}P_d^{-\zeta}[\bar{M}_1(k)u_1(k) + \bar{M}_2\hat{g}(k)] = \begin{bmatrix} q(k)^{[1]} \\ q(k)^{[2]} \end{bmatrix}.$$

We introduce two main Lyapunov functions $V^h(k) = E\{\|\tilde{h}(k)^{[1]}\|^2\}$ and $W^h(k) = E\{\tilde{h}(k)^{[2]T}R^h \tilde{h}(k)^{[2]}\}$, where $R^h > 0$ satisfies the Lyapunov equation $R^h\bar{M}^* + \bar{M}^{*T}R^h = -Q^h$, for any given $Q^h > 0$ (according to (19)).

At the first step, we set $q(k)^{[1]} = 0$ and $q(k)^{[2]} = 0$ and analyze the corresponding solutions of (20) and (21) by $\tilde{h}_1(k)^{[1]}$ and $\tilde{h}_1(k)^{[2]}$, respectively, by introducing $V_1^h(k) = E\{\|\tilde{h}_1(k)^{[1]}\|^2\}$ and $W_1^h(k) = E\{\tilde{h}_1(k)^{[2]T}R^h\tilde{h}_1(k)^{[2]}\}$. The results from [30] can be directly applied to (20) and (21) (Theorem 11 and Lemma 12 therein), leading to the conclusion that $\sup_k V_1^h(k) < \infty$ and that $W_1^h(k)$ tends to zero when $k \to \infty$.

At the second step, we analyze the zero state responses $\tilde{h}_2(k)^{[1]}$ and $\tilde{h}_2(k)^{[2]}$ of (20) and (21) to the inputs $q(k)^{[1]}$ and $q(k)^{[2]}$, respectively. If $V_2^h(k) = E\{\|\tilde{h}_2(k)^{[1]}\|^2\}$ and $W_2^h(k) = E\{\tilde{h}_2(k)^{[2]T} R^h \tilde{h}_2(k)^{[2]}\}$, we obtain, after some technicalities that

$$\begin{aligned} V_2^h(k+1) \leq & (1 + c' \frac{1}{k^{2\zeta}}) V_2^h(k) + (\frac{1}{k^\zeta} q(k)^{[1]})^2 \\ & + E\{\tilde{h}_2(k)^{[1]}\} \frac{1}{k^\zeta} q(k)^{[1]}, \end{aligned} \tag{23}$$

$(c'\infty)$. Having in mind that $q(k)^{[1]} = o(\frac{1}{k^d})$ w.p.1 by assumption, we conclude that $\sup_k E\{\tilde{h}_2(k)^{[1]}\}^2 < \infty$ and, consequently, $\sup_k V_2^h(k) < \infty$ for all $\zeta \in (\frac{1}{2}, 1]$ satisfying $\zeta + d > 1$ (owing to the fact that $\sum_k \frac{1}{k^{\zeta+d}} < \infty$).

Analysis of $\tilde{h}_2(k)^{[2]}$ relies on classical results from stochastic approximation [31], wherefrom we obtain that $\lim_{k\to\infty} W_2^h(k) = 0$.

Using the arguments exposed in [30], we conclude that $\tilde{h}(k)^{[1]}$ tends to a random n-vector $\tilde{h}^{[1]*}$, and that $\tilde{h}(k)^{[2]}$ tends to zero in the mean square sense and w.p.1. Therefore, $\hat{h}^* = \lim_{k\to\infty} \hat{h}(k) = S\tilde{h}^*$, where $\tilde{h}^* = [\tilde{h}^{[1]*T} \vdots 0_{1\times n}^T]^T$. The result of the theorem follows after taking into account that $\chi(k) \to \chi^*$ w.p.1 and that $\bar{M}_1 \hat{h}^* = \bar{M}_1 S \tilde{h}^* = P_d^{-\zeta} S \begin{bmatrix} 0 & 0 \\ 0 & \bar{M}^* \end{bmatrix} \tilde{h}^* = 0$.

Proof of Theorem 2. We pay attention only to the possible convergence points: the rest can be derived by following the proof of Theorem 1. Namely, according to [32], we formulate the ODE characterizing the asymptotic behavior of the algorithm, and obtain that

$$\begin{aligned} & \bar{\Gamma} \hat{f}^* - \chi^* [\bar{\Gamma}_d (\bar{\delta} + \bar{\eta}^0) - \bar{\Gamma} \bar{\xi}^0] + \bar{\Gamma}_d \mathbf{1} \; \hat{c}^{con} = 0 \\ & \sum_{i=1}^n \bar{\phi}_i \{ \bar{\Gamma}^{(i)} \hat{f}^* - \chi^* [(\bar{\Gamma}_d [\bar{\delta} + \bar{\eta}^0])_i - (\bar{\Gamma} \bar{\xi}^0)_i] \\ & \quad + (\bar{\Gamma}_d \mathbf{1})_i \hat{c}^{con} \} = 0, \end{aligned} \tag{24}$$

wherefrom the result directly follows.

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