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Repetitive Process based Design of PD-Type Iterative Learning Control Laws

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***Abstract*— New results on the design of iterative learning control laws are developed. The analysis is in the repetitive process setting. The iterative learning law combines a PD-type learning function and state feedback, where only a relatively small number of parameters need to be tuned. The analysis is extended to allow different finite frequency range performance specifications, where this facility is relevant to many applications. The design computations required are linear matrix inequality based. The new design is illustrated by a simulation based study on robotic manipulator behavior, where the model used has been constructed from experimental data.**

I. INTRODUCTION

Iterative learning control (ILC) is used to improve the tracking response in the situations where the same finite duration operation is repeated, which is common in engineering applications. An industrial robotics example is the robot arm action to realize a repeatable pick and place task, which is used in this paper as the case study. Each operation execution is known in ILC framework as a trial and the trial length is used to denote the duration of this finite duration operation. In this paper discrete dynamics are considered and $y_k(p)$, $0 \leq p \leq \alpha - 1$, denotes the scalar or vector valued system output, where $k \geq 0$, represents the trial number, p the sample number along the trial and $\alpha < \infty$ is the constant for each trial. Once a trial is complete all information generated on the previous trial is available for use in computing the control input for the next trial. This allows, for example, that at sample p on trial $k+1$ information from $p + \lambda \leq \alpha$, $\lambda > 0$, on the previous trial can be used.

Research on ILC theory and applications has remained an active area of research and applications since the first results reported in [1]. Often new theoretical results are supported by at least experimental laboratory validation. The first work was motivated by robotics and other application areas include wafer stage motion systems, chemical batch processes and electrical motor control. A starting point for the literature is [2], [3], where an overview of developments can be found. More recently, ILC has been applied to robotic-assisted upper limb stroke rehabilitation with supporting clinical trials [4], [5], where ILC is used to regulate the assistive stimulation applied to assist patients making repeated attempts at a finite duration task such as lifting and reaching out with the affected arm.

Let u_k denote the control input on trial k, then the ILC design problem is to construct a control sequence $\{u_k\}_k$ such that the corresponding output sequence $\{y_k\}_k$, converges in k to a supplied reference vector or signal. Let $||\cdot||$ denote the norm on the underlying function space. Then the design problem can be stated as

$$\lim_{k\to\infty} ||e_k|| = 0, \ \lim_{k\to\infty} ||u_k - u_\infty|| = 0, \tag{1}$$

where u_∞ is termed the learned control.

The finite trial length means that trial-to-trial (in k) error convergence can be achieved even for unstable systems and hence the presence of transient response terms that increase over $0 \leq p \leq \alpha - 1$. One way of removing this problem is to first design a stabilizing control law, from the resulting controlled system and then apply the ILC design, see, e.g. [2], [3]. For discrete dynamics, a common setting for ILC analysis and design is known as lifting, which is based on the use of so-called supervectors. Consider, for example, the single-input single-output case. Then the supervector for the output is formed by the samples of this variable along the trial and likewise for the state and input variables. The result is a matrix linear difference equation linking the error on any trial to that on the next, to which standard linear systems results can be applied to design the ILC law.

Another approach to ILC is to use the two-dimensional (2D) systems/repetitive processes setting [6], [7], i.e., systems that propagate information in two independent directions, where in the case of ILC they are trial-to-trial and along each trial directions respectively. Repetitive process dynamics evolve over a subset of the upper right quadrant of the 2D plane and make repeated sweeps through dynamics defined over a finite duration. Hence, they are better suited for ILC analysis than other 2D system models. The main advantage of repetitive process based ILC design is that trial-to-trial error convergence and the dynamics along the trial can be considered in one setting and also the analysis extends directly to differential dynamic. Background on repetitive processes can be found in [8], [9].

One simple structure ILC law that has found wide application [3] is the PD-type, consisting of proportional and derivative gains acting on the tracking error. This paper extends the design of PD-type ILC laws using the repetitive process approach. The analysis shows that the problem involved can be completed by formulation as a stability problem for discrete linear repetitive processes, which can be solved by the use of Linear Matrix Inequality (LMI)

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computations. The ILC design is extended to finite frequency regions, allowing the achievement of different performance specifications over them. This analysis extensively uses the generalized Kalman-Yakubovich-Popov (KYP) lemma.

Standard notation is used from this point onwards. The null and identity matrices of compatible dimensions are denoted by 0 and I, respectively, and the notation $X \prec 0$ (respectively $X \succ 0$) means that the matrix X is negative definite (respectively, positive definite). Also, $\mathrm{sym}(X)$ denotes the matrix $X + X^T$ and $X^\perp$ is the orthogonal complement of X. Finally, the symbol $(\star)$ denotes block entries in symmetric matrices and $\rho(\cdot)$ and $\overline{\sigma}(\cdot)$ denote the spectral radius and maximum singular value, respectively, of their matrix arguments.

The new results in this paper are based on two important lemmas, where the first is the generalized KYP and the second is the Elimination (or Projection) lemma applied in many matrix transformations.

Lemma 1: [10] Given matrices $\mathbb{A}$, $\mathbb{B}_0$, Θ and

$$\Phi = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \Psi = \begin{bmatrix} 0 & \mathrm{e}^{\mathrm{j}\omega_c} \\ \mathrm{e}^{-\mathrm{j}\omega_c} & -2\cos(\omega_d) \end{bmatrix}, \quad (2)$$

with

$$\omega_c = \frac{(\omega_l + \omega_u)}{2},\ \omega_d = \frac{(\omega_u - \omega_l)}{2}$$

and ω_l, ω_u satisfying $-\pi \leq \omega_l \leq \omega_u \leq \pi$ and also $\det(\mathrm{e}^{\mathrm{j}\omega} I - \mathbb{A}) \neq 0$ for all $\omega \in [\omega_l, \omega_u]$, holds. Then the following statements are equivalent:

i) $\forall \omega \in [\omega_l, \omega_u]$

$$\begin{bmatrix} (\mathrm{e}^{\mathrm{j}\omega} I - \mathbb{A})^{-1}\mathbb{B}_0 \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (\mathrm{e}^{\mathrm{j}\omega} I - \mathbb{A})^{-1}\mathbb{B}_0 \\ I \end{bmatrix} \prec 0. \quad (3)$$

ii) There exist $\mathcal{Q} \succ 0$ and a symmetric $\mathcal{P}$ such that

$$\begin{bmatrix} \mathbb{A} & \mathbb{B}_0 \\ I & 0 \end{bmatrix}^\top (\Phi \otimes \mathcal{P} + \Psi \otimes \mathcal{Q}) \begin{bmatrix} \mathbb{A} & \mathbb{B}_0 \\ I & 0 \end{bmatrix} + \Theta \prec 0. \quad (4)$$

Lemma 2: [11] Given a symmetric matrix $\Gamma \in \mathbb{R}^{p \times p}$ and two matrices Λ, Σ of the same column dimension equal to p, there exists an unstructured matrix $\mathcal{W}$ such that the following inequality holds

$$\Gamma + \mathrm{sym}\{\Lambda^\top \mathcal{W} \Sigma\} \prec 0, \quad (5)$$

if and only if

$$\Lambda^{\perp\top} \Gamma \Lambda^\perp \prec 0,\ \Sigma^{\perp\top} \Gamma \Sigma^\perp \prec 0, \quad (6)$$

where $\Lambda^\perp$ and $\Sigma^\perp$ are arbitrary matrices whose columns form a basis of the null spaces of Λ and Σ, respectively.

The next section gives the required background and problem formulation. In the rest of the paper, the design is developed of the design and a simulation based case study is given based on an experimentally determined model of one axis of a gantry robot experimental faculty. This system emulates the commonly encountered pick and place task of collecting objects in sequence from a location, transferring them over a finite duration and placing them under synchronization on a moving conveyor.

II. Discrete Linear Repetitive Processes

The state-space model [9] of a discrete linear repetitive process is defined over $0 \leq p \leq \alpha - 1,\ k \geq 0$

$$\begin{aligned} x_{k+1}(p+1) &= \mathcal{A}x_{k+1}(p) + \mathcal{B}u_{k+1}(p) + \mathcal{B}_0 y_k(p), \\ y_{k+1}(p) &= \mathcal{C}x_{k+1}(p) + \mathcal{D}u_{k+1}(p) + \mathcal{D}_0 y_k(p), \end{aligned} \quad (7)$$

where $\alpha < \infty$ denotes the finite pass length, and on pass k, $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the pass profile (output) vector and $u_k(p) \in \mathbb{R}^l$ is the control input vector. The terms $\mathcal{B}_0 y_k(p)$ and $\mathcal{D}_0 y_k(p)$ represent the contribution of the previous pass profile to the current pass state and pass profile vectors respectively.

The boundary conditions are the initial state vector on each pass $x_{k+1}(0),\ k \geq 0$, and the initial (i.e., on pass 0) pass profile $y_0(p)$. In this paper these are taken as $x_{k+1}(0) = d_{k+1},\ k \geq 0$, where d_{k+1} is the known $n \times 1$ vector and the entries in initial pass profile vector $y_0(p)$ are also known over the pass length.

The sequence of pass profiles generated by (7) can contain oscillations that increase in amplitude in the pass-to-pass direction (k). Consequently the stability theory for linear repetitive processes [9] demands that a bounded initial pass profile produces a bounded sequence of pass profiles. In the strongest form, termed stability along the pass, this property must hold for all possible values of the finite pass length, which can be analyzed mathematically by letting $\alpha \to \infty$. The following lemma gives necessary and sufficient conditions for this property.

Lemma 3: [9] Suppose that the pair $\{\mathcal{A}, \mathcal{B}\}$ is controllable and the pair $\{\mathcal{C}, \mathcal{A}\}$ observable. Then a discrete linear repetitive process described by (7) is stable along the pass if and only if

i) $\rho(\mathcal{D}_0) < 1$,
ii) $\rho(\mathcal{A}) < 1$,
iii) all eigenvalues of $\mathcal{G}(z) = \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B}_0 + \mathcal{D}_0,\ \forall |z| = 1$ have modulus strictly less than unity.

The first and the second conditions in this last result are just standard linear systems stability conditions and it is easy to test by, for example, solving a Lyapunov inequality. The third condition represents the coupling condition from the previous pass and requires computations for all points on the unit circle, which could cause computational problems. This condition also requires frequency attenuation of the previous pass profile over the complete frequency spectrum, which also could be very difficult to meet.

In some applications it may be required to impose stability and performance conditions only over finite frequency ranges of particular interest whose choice depends on the particular case under consideration. This can be accomplished by use of the generalized KYP lemma and the conditions of Lemma 3 can be written in LMI form. To proceed, the frequency range is assumed to be divided into H intervals such that

$$[0, \pi] = \bigcup_{h=1}^{H} [\omega_{h-1}, \omega_h], \quad (8)$$

where $\omega_0 = 0$ and $\omega_H = \pi$. Then Lemma 1 can be applied at any frequency in any of these intervals and gives the following result.

Theorem 1: A discrete linear repetitive process of (7), where the entire frequency range is divided into H different frequency intervals as shown in (8) is stable along the pass if there exist matrices $S \succ 0$, W, $Q_h \succ 0$ and symmetric P_h such that the following matrix inequalities are feasible

$$\begin{bmatrix} S-W-W^T & W^T\mathcal{A} \\ \mathcal{A}^T W & -S \end{bmatrix} \prec 0, \tag{9}$$

$$\begin{bmatrix} -P_h & e^{j\omega_{ch}}Q_h - W & 0 & 0 \\ (\star) & \Upsilon_1 & W^T\mathcal{B}_0 & \mathcal{C}^T \\ (\star) & (\star) & -I & \mathcal{D}_0^T \\ (\star) & (\star) & (\star) & -I \end{bmatrix} \prec 0, \tag{10}$$

for all $h = 1, \ldots, H$, where

$$\begin{aligned} \omega_{ch} &= \frac{\omega_{h-1}+\omega_h}{2},\ \omega_{dh} = \frac{\omega_h - \omega_{h-1}}{2}, \\ \Upsilon_1 &= P_h - 2\cos(\omega_{dh})Q_h + \mathcal{A}^T W + W^T\mathcal{A}. \end{aligned} \tag{11}$$

Proof: Assume that (9) and (10) are feasible. Then $W + W^T \succ S \succ 0$ implies that W is non-singular and hence invertible. An application of Lemma 2 yields $\mathcal{A}^T S\mathcal{A} - S \prec 0$. Also it is the straightforward consequence that the LMI

$$\begin{bmatrix} -I & \mathcal{D}_0 \\ \mathcal{D}_0^T & -I \end{bmatrix} \prec 0,$$

is feasible and by Schur's complement formula $\mathcal{D}_0^T\mathcal{D}_0 - I \prec 0$ must hold. Hence, from Lyapunov stability theory we have that conditions $i)$ and $ii)$ of Lemma 3 must be satisfied. Next, after standard transformations (see [12] for the details of these transformations), (10) can be written in the form of (4)

$$\begin{bmatrix} A & B_0 \\ I & 0 \end{bmatrix}^\top \Xi \begin{bmatrix} A & B_0 \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \prec 0,$$

where

$$\Xi = \begin{bmatrix} -P_h & e^{j\omega_{ch}}Q_h \\ Q_h e^{j\omega_{ch}} & P_h - 2cos(\omega_{dh})Q_h \end{bmatrix}, \Pi = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}. \tag{12}$$

Using Lemma 1 it follows that

$$\overline{\sigma}(\mathcal{G}(e^{j\omega})) < 1,\ \forall\omega \in [\omega_{h-1}, \omega_h], h = 1, \ldots, H,$$

where $\mathcal{G}(e^{j\omega}) = \mathcal{C}(e^{j\omega}I - \mathcal{A})^{-1}\mathcal{B}_0 + \mathcal{D}_0$. Since $\overline{\sigma}(\mathcal{G}(e^{j\omega})) < 1$ holds in each frequency interval then from (8) $\overline{\sigma}(\mathcal{G}(e^{j\omega})) < 1$ holds over entire frequency range. This means that condition $iii)$ of Lemma 3 is satisfied and the proof is complete. ■

The convergence rate of the sequence of pass profiles generated by a discrete linear repetitive process depends on $\overline{\sigma}(\mathcal{G}(e^{j\theta})) < 1$. This rate can be increased by minimizing parameter $0 < \gamma < 1$ subject to $\overline{\sigma}(\mathcal{G}(e^{j\theta})) < \gamma$ over all frequency ranges.

III. Embedding ILC into a repetitive process setting

Assume that the process dynamics can be modeled by the discrete linear time-invariant state-space model

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p), \\ y_{k+1}(p) &= Cx_{k+1}(p), \end{aligned} \tag{13}$$

where in the ILC setting k denotes the trial number and $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the output vector and $u_k(p) \in \mathbb{R}^l$ is the control input vector on trial number k. Also we introduce the forward shift operator z along the p - axis as

$$zx_{k+1}(p) = x_{k+1}(p+1).$$

For the details of how the z-transform can be applied over the finite trial length without errors arising from that the basic definition is over an infinite interval, see [3].

Next, with the assumption that the pair $\{A, B\}$ is controllable and the pair $\{C, A\}$ observable, (13) can be equivalently represented by the transfer-function matrix

$$G(z) = C(zI - A)^{-1}B.$$

Introduce the reference trajectory as $y_d(p)$ and then the error on trial k is equal to

$$e_k(p) = y_d(p) - y_k(p).$$

For the remainder of this paper and to emphasize the repetitive process setting for design, each ILC trail will be termed a pass.

In ILC, the current pass input is commonly constructed as the sum of that used on the previous pass plus a correction term, i.e., a control law of the form

$$u_{k+1}(p) = u_k(p) + \Delta u_{k+1}(p), \tag{14}$$

where $\Delta u_{k+1}(p)$ is to be designed. Suppose also that

$$\begin{aligned} \Delta u_{k+1}(p) = &Kx_{k+1}(p) + K_1 e_k(p) \\ &+ K_2(e_k(p+1) - e_k(p)), \end{aligned} \tag{15}$$

where the control law matrices K, K_1 and K_2 are to be designed. Hence the correction term in this ILC design is the sum of state feedback control on the current pass aimed to correct the transient response along the pass plus a learning term of the PD-type depending on the previous pass error (e_k), whose aim is to enforce pass-to-pass error convergence. For clarity and ease of explanation how the proposed control law of (15) is implemented, the block diagram is shown in Fig. 1.

Application of the control law (14) to (13) results in controlled system dynamics of the form

$$\begin{aligned} x_{k+1}(p+1) = &(A + BK)x_{k+1}(p) + BLe_k(p) \\ &+ BK_1 e_k(p+1) + Bu_k(p), \\ y_{k+1}(p) = &Cx_{k+1}(p), \end{aligned} \tag{16}$$

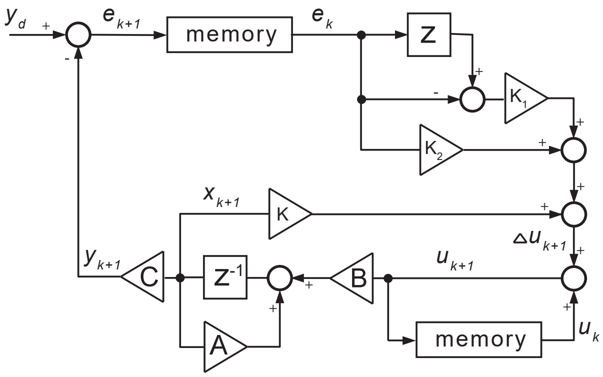

Fig. 1. Block diagram representation of PD-type ILC.

where $L = K_1 - K_2$. Also the transfer-function matrix coupling $e_k(p)$ to y_{k+1} in this last state-space model is

$$\begin{aligned} H(z) =& C(zI-(A+BK))^{-1}B(L+zK_1) \\ =& C(zI-(A+BK))^{-1}BL \\ &+Cz(zI-(A+BK))^{-1}BK_1. \end{aligned} \tag{17}$$

Assuming that $zI - (A + BK)$ is nonsingular it is easily shown that

$$(zI-(A+BK))^{-1}(zI-(A+BK))=I,$$

and hence

$$z(zI-(A+BK))^{-1}=I+(zI-(A+BK))^{-1}(A+BK).$$

Therefore

$$\begin{aligned} H(z) =& C(zI-(A+BK))^{-1}BL \\ &+Cz(zI-(A+BK))^{-1}BK_1 \\ =& C(zI-(A+BK))^{-1}BL \\ &+C(I+(zI-(A+BK))^{-1}(A+BK))BK_1 \\ =& C(zI-(A+BK))^{-1}(BL+(A+BK)BK_1) \\ &+CBK_1. \end{aligned}$$

Also the tracking error on pass k is given in the z domain by

$$E_k(z) = Y_d(z) - Y_k(z) = Y_d(z) - G(z)U_k(z),$$

where $Y_d(z)$ is the z-transform of the reference trajectory. Hence $E_{k+1}(z)$ can be written as

$$E_{k+1}(z)-E_k(z)=-G(z)\left(U_{k+1}(z)-U_k(z)\right). \tag{18}$$

To focus on pass-to-pass error convergence, introduce $M(z)$ as

$$M(z) = -H(z) + I,$$

where $H(z)$ is defined in (17) and also the pass-to-pass error propagation can be written in the form

$$E_{k+1}(z) = M(z)E_k(z).$$

Hence the tracking error converges as $k \to \infty$, if and only if all eigenvalues of $M(z)$ are less than one in magnitude, i.e.

$$\rho\left(M(e^{j\omega})\right) < 1,\ \forall\omega \in [-\pi, \pi]. \tag{19}$$

Practical experience however shows that some ILC laws lead to poor transients during the convergence process even if the above condition is satisfied (e.g., the tracking error grows over some number of initial passes before eventually converging). To avoid these obstacles, a stronger convergence requirement is required. Note that the Euclidean norm of the tracking error decreases monotonically from pass-to-pass if $M(\mathrm{e}^{j\omega})$ satisfies the sufficient stability condition

$$\overline{\sigma}(M(\mathrm{e}^{j\omega})) < 1,\ \forall\omega \in [-\pi, \pi]. \tag{20}$$

Although (19) is less conservative, (20) prevents pass-to-pass error oscillations and hence is more frequently used for design. Also

$$\|M(z)\|_\infty \triangleq \max_{\omega\in[-\pi,\pi]} \overline{\sigma}(M(\mathrm{e}^{j\omega})),$$

and hence the pass-to-pass error convergence problem for ILC schemes can be reformulated as an $\mathcal{H}_\infty$ control problem. Moreover, let $||\cdot||_2$ denote the ℓ_2 norm and it is obvious that

$$\|e_k(p)\|_2 \leq \|M(z)\|_\infty^k \|e_0(p)\|_2\,,$$

and therefore if the condition (20) is satisfied then monotonic trial-to-trial error convergence occurs in ℓ_2 for $k \to \infty$.

The controlled dynamics can be written as

$$\begin{aligned} x_{k+1}(p+1) =& \mathcal{A}x_{k+1}(p) + \mathcal{B}_0 e_k(p), \\ e_{k+1}(p) =& \mathcal{C}x_{k+1}(p) + \mathcal{D}_0 e_k(p), \end{aligned} \tag{21}$$

where

$$\begin{aligned} &\mathcal{A} = A+BK,\ \mathcal{B}_0 = (BL + (A+BK)BK_1), \\ &\mathcal{C} = -C, \mathcal{D}_0 = I - CBK_1. \end{aligned} \tag{22}$$

The state-space model (21) is in the form of a discrete linear repetitive process, see (7), where on pass $k+1$, $x_{k+1}(p)$ is the state vector, $e_k(p)$ is the pass profile vector and there is no external input term. Stability along the pass of this repetitive process guarantees that the error sequence $\{e_k\}$ converges to zero as $k \to \infty$ independently of the pass length.

IV. LMI BASED DESIGN OVER FINITE FREQUENCY DOMAIN

In this section the main result of the paper is developed, which guarantees monotonic pass-to-pass error convergence under finite frequency design specifications. The possibility of specifying different performance requirements is very important in practice since performance issues occur over different frequency ranges. For example, the trial-to-trial error convergence rate is realised in the 'low' frequency range whereas low sensitivity to disturbances and sensor noise are realised in the 'high' frequency range.

Theorem 2: Consider an ILC scheme described in the form of a discrete linear repetitive process (21) and (22). Suppose also that the entire frequency range is divided into H different frequency intervals as in (8). Then a discrete linear repetitive process representing an ILC scheme of (7) is stable along the pass, which guarantees monotonic pass-to-pass error convergence, if there exist matrices $\mathcal{S} \succ 0$, $\mathcal{W}$, N,

L, K_1, $\mathcal{Q}_h \succ 0$ and symmetric $\mathcal{P}_h$ such that the following LMIs hold

$$\begin{bmatrix} \mathcal{S}-\mathcal{W}-\mathcal{W}^T & A\mathcal{W}+BN \\ (A\mathcal{W}+BN)^T & -\mathcal{S} \end{bmatrix} \prec 0, \tag{23}$$

$$\begin{bmatrix} -\mathcal{P}_h & \mathrm{e}^{\mathrm{j}\omega_{ch}}\mathcal{Q}_h-\mathcal{W} & 0 & 0 & 0 \\ (\star) & \Upsilon_2 & \Upsilon_3 & -\mathcal{W}C^T & BN \\ (\star) & (\star) & -I & (I-CBK_1)^T & K_1^TB^T \\ (\star) & (\star) & (\star) & -I & 0 \\ (\star) & (\star) & (\star) & 0 & -\mathcal{W}-\mathcal{W}^T \end{bmatrix} \prec 0, \tag{24}$$

for all $h = 1, \dots, H$, where ω_{ch}, ω_{dh} are defined in (11) and

$$\Upsilon_2 = \mathcal{P}_h - 2\cos(\omega_{dh})\mathcal{Q}_h + \mathrm{sym}\{A\mathcal{W}+BN\}, \Upsilon_3 = BL+ABK_1.$$

If this set of LMIs is feasible, the required control law matrices K, K_2 and K_1 in (14) can be computed directly from the solution of this set as

$$K = N\mathcal{W}^{-1}, K_2 = K_1 - L.$$

Proof: Suppose that the LMIs (23) and (24) are feasible. Then $\mathcal{W}+\mathcal{W}^T \succ \mathcal{S} \succ 0$, which implies that $\mathcal{W}$ is invertible. Next post- and pre-multiply (23) by $\mathrm{diag}\{\mathcal{W}^{-1}, \mathcal{W}^{-1}\}$ to obtain (9) on also setting $W = \mathcal{W}^{-1}$ and $S = \mathcal{W}^{-T}\mathcal{S}\mathcal{W}^{-1}$.

Introduce the notation

$$\Omega = \begin{bmatrix} -\mathcal{P}_h & \mathrm{e}^{\mathrm{j}\omega_{ch}}\mathcal{Q}_h-\mathcal{W} & 0 & 0 \\ (\star) & \Upsilon_2 & BL+ABK_1 & -\mathcal{W}C^T \\ (\star) & (\star) & -I & (I-CBK_1)^T \\ (\star) & (\star) & (\star) & -I \end{bmatrix},$$

$$U^T = \begin{bmatrix} 0 & K^TB^T & 0 & 0 \end{bmatrix}, J = \begin{bmatrix} 0 & 0 & K_1^TB^T & 0 \end{bmatrix}, N = K\mathcal{W},$$

and rewrite (24) as

$$\begin{bmatrix} \Omega & J \\ J^T & 0 \end{bmatrix} + \mathrm{sym}\left\{ \begin{bmatrix} 0 \\ I \end{bmatrix} \mathcal{W}^T \begin{bmatrix} U^T & -I \end{bmatrix} \right\} \prec 0.$$

Hence, by the result of Lemma 2, (24) holds if and only if

$$\Omega + UJ^T + JU^T \prec 0,$$

and this last inequality can be rewritten as

$$\begin{bmatrix} -\mathcal{P}_h & \mathrm{e}^{\mathrm{j}\omega_{ch}}\mathcal{Q}_h-\mathcal{W} & 0 & 0 \\ (\star) & \Upsilon_2 & \mathcal{B}_0 & -\mathcal{W}C^T \\ (\star) & (\star) & -I & \mathcal{D}_0^T \\ (\star) & (\star) & (\star) & -I \end{bmatrix} \prec 0. \tag{25}$$

Next, post- and pre-multiply the above inequality by $\mathrm{diag}\{\mathcal{W}^{-1}, \mathcal{W}^{-1}, I, I\}$ and its transpose, respectively. Finally, set $P_h = \mathcal{W}^{-T}\mathcal{P}_h\mathcal{W}^{-1}$ and $Q_h = \mathcal{W}^{-T}\mathcal{Q}_h\mathcal{W}^{-1}$ to establish that (25) is equivalent to (10) and the proof is complete. ■

In practical situations the effects of high frequency noise and non-repeating disturbances degrade the performance of an ILC design. When the tracking error is attenuated, any high frequency noise present may be amplified and one solution to this problem is Q-filtering to limit the frequency range of the learning for stability and noise attenuation. In such a case, the result of Theorem 2 is used to design both the feedback and learning controllers for error convergence and performance in prescribed frequency range. This frequency range can be chosen by inspection of frequency spectrum of the signal to be tracked. The Q-filter must be a low-pass filter with (ideally) unity magnitude for low for the frequency range where reference tracking is required and zero at all other frequencies.

V. Application case study

To illustrate effectiveness of the new design, a gantry robot undertaking a pick and place operation is considered. This robot was previously used for testing and comparing the performance of other ILC schemes, see, e.g. [8]. Since the robot axes are orthogonal, it is assumed that they are decoupled and can be considered individually. Based on frequency response tests, the transfer-functions for each axis have been identified. In particular, the transfer-function for one axis is

$$G_x(s) = \frac{13077183.4436(s+113.4)}{s(s^2+61.57s+1.125\cdot 10^4)} \times \frac{(s^2+30.28s+2.13\cdot 10^4)}{(s^2+227.9s+5.647\cdot 10^4)(s^2+466.1s+6.142\cdot 10^5)},$$

which is an adequate model of the dynamics to use for ILC design. The transfer-function has been discretized with a sampling time of $T_s = 0.01$ seconds to obtain a discrete linear state-space model of the form (13) with

$$A = \begin{bmatrix} 1.5313 & -0.9716 & 0.3821 & 0.0056 & 0.1114 & -0.0571 & 0.0670 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1250 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1250 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.0313 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = 10^{-4} \cdot \begin{bmatrix} 166 & -4 & 92 & 34 & -52 & 42 & -26 \end{bmatrix}.$$

The state matrix A has all eigenvalues inside the unit circle except for one of value unity on the real axis of the complex plane and hence feedback action is necessary to obtain satisfactory response.

The reference trajectory of the gantry robot represents a "pick and place" process of duration 2 seconds, and this signal has been used in all ILC law tests to make all results comparable. The reference trajectory for the axis considered in this paper is given in Fig. 2. Examining the reference

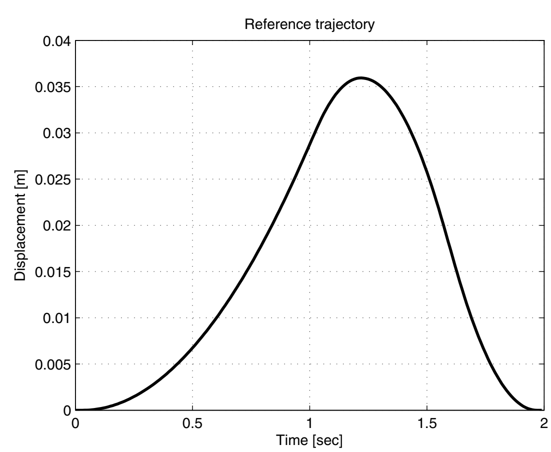

Fig. 2. The reference trajectory for the axis considered.

trajectory in the frequency domain confirms that this signal consists of harmonics between 0 and 4 Hz only. Hence, it is reasonable to choose the cutoff frequency of Q-filter as approximately 4 Hz because frequencies from 0 to 4Hz have only to be emphasized in the design. Applying Theorem 2 for the frequency range 0 to 4 Hz gives

$$K = \begin{bmatrix} -41.92 & 18.3 & -12.83 & -3.26 & -2.4 & 0.29 & -0.92 \end{bmatrix},$$
$$K_1 = 580.4529,\ K_2 = 552.6674.$$

The resulting plot of $|M(\mathrm{e}^{\mathrm{j}\omega})|$ is given in Fig. 3 and confirms that the design specifications are met. The controlled

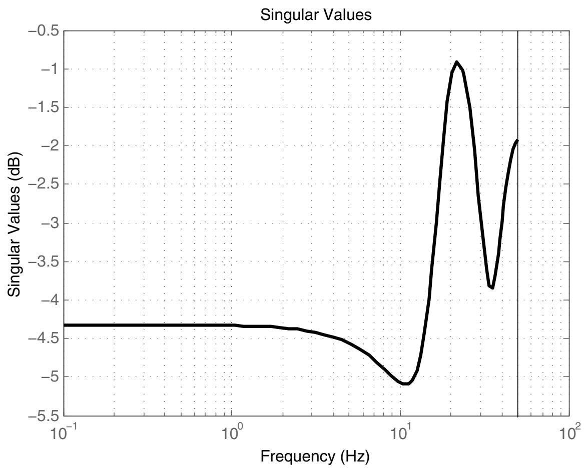

Fig. 3. Plot of $|M(\mathrm{e}^{\mathrm{j}\omega})|$.

dynamics were simulated over 30 passes and for each one the RMS value of the tracking error calculated using $\mathrm{RMS}(e) = \sqrt{\frac{1}{\alpha}\sum_{p=1}^{N} e(p)^2}$, where $N = 200$ for the 100 Hz sampling frequency and trial duration of 2 seconds. Fig. 4 shows the RMS values of the tracking error as a function of the pass number. Also noise with RMS=10^{-4} is included and the convergence curve stays at this level after 15 passes. Simulations of the response of the controlled system confirm

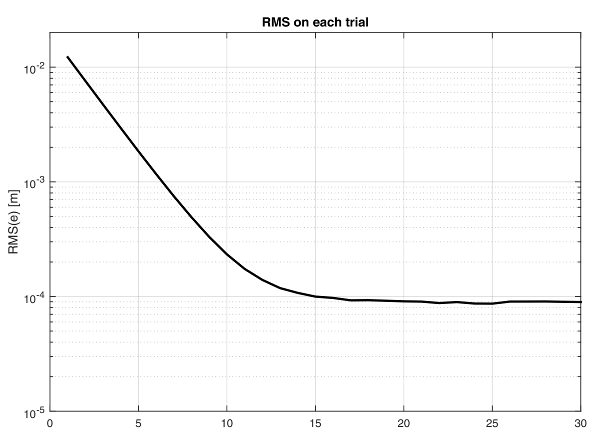

Fig. 4. RMS values of the error over 30 passes.

that trial-to-trial error convergence occurs.

VI. Conclusions

This paper has developed a new algorithm for the design of an ILC law for discrete linear systems that combines a PD-type learning function with state feedback. Also, the new design enforces a required frequency attenuation over a finite frequency range in comparison to many known results that demand this attenuation over the entire frequency range. Furthermore, previous results in the repetitive process setting required the computation of the difference between current and previous pass state vectors and hence this latter vector has to be stored. The design is this paper removes this requirement. Also the results have been established by using the generalized KYP lemma to transform frequency domain specifications to LMIs and therefore easily computed by numerical software. The theoretical findings have been illustrated by simulation study using a model for one axis of a gantry robot experimental facility used to benchmark ILC control schemes illustrates the effectiveness of the design. Ongoing research includes experimental validation on the same gantry robot as in [8].

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