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Computation of Feasible Parametric Regions for Common Quadratic Lyapunov Functions

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***Abstract*— A parameter space method for computation of the feasible region of common quadratic Lyapunov function (CQLF) for a class of parameter-dependent switched systems is presented. The existence condition of CQLF is formulated as the solvability condition of a related parametric semi-algebraic system. To this end, a computer algebra method known as Rreal Root Classification (RRC) technique is utilized. We apply RRC to obtain symbolic conditions to the feasibility of CQLF in the parameter space. Numerical examples demonstrate that our method can compute the feasible region of CQLF without any conservativeness.**

Index Terms — Common quadratic Lyapunov function, switched linear systems, quantifier elimination, real root classification

I. Introduction

Finding all stabilizing parameters is often followed for analysis and design in control systems. Many studies of this kind in various contexts of linear control theory in the frequency and recently also in the time domain have been already conducted; e.g., see [1] and the references therein. To the present paper, a Lyapunov function approach for mapping the stability bounds of the continuous-time [9] systems is of relevance. This method can provide the feasibility bounds of the Lyapunov function, which divide the parameter space into a finite number of regions. Then, on the basis of continuity theorems of eigenvalues upon the underyling system parameters, a sample point from each such set is sufficient to check for the feasibility of the Lyapunov function in the entire corresponding set. An added value of the present paper, refers to the non-conservative computation of feasible regions for a common quadratic Lyapunov function for the class of switched linear systems involving arbitrary switching.

Finding the stabilizing parameter regions can be tackled by means of the quantifier elimination (QE) approach. This approach has been explored in previous works, e.g., for LTI systems [5] and switched linear systems [8]. The papers report yet unavoidable computational barriers (with the so-called QEPCAD software package, [2]). Our work follows similar lines, devising a constructive implementation of real root classification techniques instead. Constructive guidelines and not the numerical efficiency are of our main concern thereby. Numerical complexity scales with the dimension of parameter and state spaces, as well as the ingredients of the switched system at hand. Details of the numerical efficiency of the RRC algorithms in comparison with various QE packages the reader is directed to [4].

The real root classification (RRC) can be used to solve a special type of QE problems [14]. A RRC based method was proposed in [11] to compute the Lyapunov function for non-linear systems, in which the differential equations are polynomial. In [10] this method was extended to discover multiple Lyapunov functions for switched non-linear systems. However, no application of RRC for computation of the feasible region of Lyapunov function in the parameter space has been conducted so far, in particular not to our case study of the class of switched linear systems.

The remaining of the paper is organized as follows. The definition of our problem is given in Section II and the corresponding algebraic condition to the existence of CQLF will be derived in Section III. Then, Section IV will contribute to the presentation RRC method. In Section V, the functionality of our method will be verified by the numerical examples. Finally, we conclude the paper in Section VI.

II. Problem Definition

Consider the switched linear system with following form

$$\dot{x} = A_i(k)x, \quad i \in \mathcal{M} = \{1, \cdots, M\} \tag{1}$$

where $x \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n\times n}$ for all $i \in \mathcal{M}$, and $k \in \mathbb{R}^p$, referring to the unknown system parameters. The switching between the modi A_i is arbitrary up to finitely many jumps in a finite amount of time.

The exponential stability of switched linear systems can be guaranteed by the existence of a common quadratic Lyapunov function (CQLF) for all its subsystems (e.g., see the survey [12]). A symmetric positive definite (or semi positive definite) matrix P is denoted as $P > 0$ (or $P \geq 0$), and the negative definite (or semi negative definite) matrix P as $P < 0$ (or $P \leq 0$). Given a switched linear system (1), we say that there exists a CQLF for the system, if there exist a matrix $P > 0$ such

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that

$$A_i^T(k)P + PA_i(k) = -Q_i < 0, \quad \forall i \in \mathcal{M}. \tag{2}$$

Our problem is to find out all k which can make CQLF feasible, i.e, to find all the parameters k that can guarantee the existence of CQLF. Notice that in the equation, P is also k-dependent. Necessary and sufficient conditions that guarantee the existence of a common Lyapunov function $P = P_i$ for all i are generally unclear. Yet it is necessary that all the matrices $A_i(k)$ are to be Hurwitz. The problem of computing all the feasible parameters k such that $A_i(k)$ are Hurwitz has been studied in our previous work [13].

For every subsystem, let

$$Q_i(p,k) = -A_i^T(k)P - PA_i(k), \tag{3}$$

where the matrix $P = (p_{ij})_{n\times n}$ is symmetric. Here the vector $p = (p_{11}, \cdots, p_{nn})$ is used to denote all the entries of P and contains $n(n+1)/2$ elements, because the matrix P is symmetric.

Also, note that for the sake of simplicity, we shall confine our deliberations in this study to the case of the switched linear systems with two subsystems only, i.e., $i \in \{1,2\}$.

III. Convex Cone and Lyapunov Function

Let us consider again the single LTI subsystem

$$\dot{x} = A(k)x. \tag{4}$$

The existence condition of Lyapunov function can be stated from the viewpoint of convex cone theory [7]. The Lyapunov function (3) is feasible if and only if the following convex cone

$$\mathcal{P}_A := \left\{ P = P^T > 0 : A^T P + PA < 0 \right\} \tag{5}$$

in the space of p-parameters $p_{11}, \cdots, p_{nn}$ is non-empty. For any element $P \in \mathcal{P}_A$, the function $V(x) = x^T P x$ is a quadratic Lyapunov function for the system (4).

The closure of the open convex cone $\mathcal{P}_A$ is given by

$$\bar{\mathcal{P}}_A = \left\{ P = P^T \geq 0 : A^T P + PA \leq 0 \right\} \tag{6}$$

and the boundary of $\bar{\mathcal{P}}_A$ can be described by the set difference

$$\begin{aligned} \bar{\mathcal{P}}_A / \mathcal{P}_A &= \left\{ P \in \bar{\mathcal{P}}_A : P \notin \mathcal{P}_A \right\} \\ &= \{ P \geq 0 : \det(A^T P + PA) = 0, \\ &\quad A^T P + PA \leq 0 \, \}. \end{aligned} \tag{7}$$

For the switched linear system $\dot{x} = A_i(k)x$, $i \in \mathcal{M}$, there exists a CQLF if and only if the intersection set

$$\mathcal{P}_\Pi = \bigcap_{i=1}^{m} \mathcal{P}_{A_i} \tag{8}$$

is not empty [7]. Here, $\mathcal{P}_{A_i}$ denotes the set of convex cone for the matrix A_i as defined by (5).

Theorem 1 (from [6]): Given two Hurwitz matrices A and B in $\mathbb{R}^{n\times n}$, then the cone $\mathcal{P}_A = \mathcal{P}_B$ if and only if $B = \mu A$ or $B = \mu A^{-1}$ for the real number $\mu > 0$.

Thus, the cones $\mathcal{P}_A$, $\mu \mathcal{P}_A$ and $\mu \mathcal{P}_{A^{-1}}$, $\forall \mu > 0$ are all identical, this yielding the following statement.

Corollary 1: If there exists a CQLF for the switched linear system (1), then there must be a positive definite matrix $P > 0$ with the form

$$P = \begin{bmatrix} 1 & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \tag{9}$$

i.e., the first entry of matrix is 1, to construct a CQLF for the switched linear system.

Proof: Because the matrix P is positive definite, its first entry must be positive, i.e., $p_{11} > 0$. If the convex cone $\mathcal{P}_\Pi$ exists, then according to Theorem 1 we have $\mathcal{P}_\Pi = p_{11}\mathcal{P}'_\Pi = \mathcal{P}'_\Pi$ and the matrix P with the form (9) must be a element of $\mathcal{P}'_\Pi$. ■

In other words, we can claim without loss of generality the existence of CQLF P in the form (9).

As already indicated, from now on we focus on a switching system with two ingredients only. More substituents call for extension of technical conditions which we do not discuss in this paper.

Proposition 1: Given two Hurwitz matrices A_1 and A_2 in $\mathbb{R}^{n\times n}$, then there exists a CQLF for A_1 and A_2 if and only if at least one of the following conditions

$$\mathcal{P}_{A_1} \cap (\bar{\mathcal{P}}_{A_2} / \mathcal{P}_{A_2}) \neq \emptyset, \quad \mathcal{P}_{A_2} \cap (\bar{\mathcal{P}}_{A_1} / \mathcal{P}_{A_1}) \neq \emptyset \tag{10}$$

is satisfied.

It implies that a part of cone boundary of one subsystem is located in the interior of the cone corresponding to the other subsystem, because the cones are connected convex sets. Thus, if a CQLF exits, then one of the following two cases needs to apply:

1) The boundaries of two cones are intersecting, but non-tangential;
2) One convex cone is strictly a subset of the other one, i.e. the boundaries of two cones are non-intersecting.

A. Case 1: Non-tangential intersecting boundaries

Because both $\bar{\mathcal{P}}_{A_1}$ and $\bar{\mathcal{P}}_{A_2}$ are closed cones, this case implies a nonempty intersection of their interiors $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_2}$. Consequently, we have the following theorem for the parametric switched system with two subsystems.

Theorem 2: Given two parameter dependent matrices $A_1(k)$ and $A_2(k)$ in $\mathbb{R}^{n\times n}$, their Lyapunov cone boundaries are intersected and non-tangential if and only if there exists matrix P such that all the following algebraic conditions are satisfied:

$$\begin{cases} f_1(k,P) = \det(A_1(k)^T P + PA_1(k)) = 0, \\ f_2(k,P) = \det(A_2(k)^T P + PA_2(k)) = 0, \\ A_1^T P + PA_1 = -Q_1 \leq 0, \\ A_2^T P + PA_2 = -Q_2 \leq 0, \\ P \geq 0, \\ \nabla f_1(k,P) \neq \lambda \nabla f_2(k,P), \\ \lambda \neq 0, \end{cases} \tag{11}$$

where the $\nabla f_1(k,P)$ and $\nabla f_2(k,P)$ denote the gradients w.r.t P of scalar function of $f_1(k,P)$ and $f_2(k,P)$.

Proof: The intersection of boundaries $(\bar{\mathcal{P}}_{A_1}/\mathcal{P}_{A_1})$ and $(\bar{\mathcal{P}}_{A_1}/\mathcal{P}_{A_1})$ can be obtained directly from the definition of (7) by

$$\begin{aligned} &(\bar{\mathcal{P}}_{A_1}/\mathcal{P}_{A_1}) \cap (\bar{\mathcal{P}}_{A_2}/\mathcal{P}_{A_2}) \\ &= \Big\{ P \geq 0 : \det(A_1^T P + PA_1) = 0,\ A_1^T P + PA_1 \leq 0, \\ &\qquad \det(A_2^T P + PA_2) = 0,\ A_2^T P + PA_2 \leq 0 \Big\}. \end{aligned}$$

Then the boundaries are not tangential if and only if their gradients at the intersection point are not collinear. ■

Furthermore, with A_1 and A_2 being second-order systems, the cones $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_1}$ are three-dimensional convex cones. According to Corollary 1, we can set

$$P = \begin{bmatrix} 1 & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \tag{12}$$

and the algebraic conditions (11) can be formulated as a semi-algebraic system:

$$\begin{cases} f_1(k,P) = \det(A_1(k)^T P + PA_1(k)) = 0, \\ f_2(k,P) = \det(A_2(k)^T P + PA_2(k)) = 0, \\ \mu_{Q11}(k,P) \geq 0, \mu_{Q12}(k,P) \geq 0, \\ \mu_{Q21}(k,P) \geq 0, \mu_{Q22}(k,P) \geq 0, \\ \mu_{P1}(P) \geq 0, \mu_{P2}(P) \geq 0, \\ \frac{\partial f_1(k,P)}{\partial p_{21}} \frac{\partial f_2(k,P)}{\partial p_{22}} - \frac{\partial f_1(k,P)}{\partial p_{22}} \frac{\partial f_2(k,P)}{\partial p_{21}} \neq 0, \end{cases} \tag{13}$$

having at least one real solution for P. Here, the μ_{Qij} stands for the $j-$th leading principal minor of matrix $Q_i(k,P) = A_i(k)^T P + PA_i(k)$, $i \in \{1,2\}$, and μ_{Pj} for the $j-$th leading principal minor of matrix P. The last inequation is derived from the condition such that the gradients of $f_1(k,P)$ and $f_2(k,P)$ at the point P are not collinear.

From the definition of leading principal minors, we know that here the leading principal minors $\mu_{Q12}(k,P)$ and $\mu_{Q22}(k,P)$ are equal to the determinant of matrix Q_1 and Q_2 respectively, i.e.,

$$\mu_{Q12}(k,P) = f_1(k,P) = 0,$$
$$\mu_{Q22}(k,P) = f_2(k,P) = 0.$$

Therefore, in the semi-algebraic system (13) the inequalities $\mu_{Q12}(k,P) \geq 0$ and $\mu_{Q22}(k,P) \geq 0$ can be omitted.

Theorem 2 has a geometrical interpretation, when we consider $\bar{\mathcal{P}}_{A1}$ and $\bar{\mathcal{P}}_{A2}$ as three-dimensional closed convex cones. They both must have cross sections with the plane $\mathcal{H} = \{P : h(P) = p_{11} - 1 = 0\}$. The intersected but non-tangential cross sections guarantee the intersection of convex cones $\mathcal{P}_{A1}$ and $\mathcal{P}_{A2}$, i.e., the existence of CQLF.

We denote the set of feasible parameters as $\Omega_1(k)$, in which the semi-algebraic system (13) has at least one real solution for P:

$$\begin{aligned} \Omega_1(k) = \{k \mid &\text{The semi-algebraic system (13) has} \\ &\text{at least one real solution for } P\}. \end{aligned} \tag{14}$$

B. Case 2: One of the convex cones is a strict subset of the other cone

For this case, given any matrix $P > 0$ in the smaller convex cone, it must be also in the other convex cone. Accordingly, any $P > 0$ from the smaller convex cone represents a CQLF for the two subsystems.

Theorem 3: If the convex cone $\mathcal{P}_{A1}$ is a subset of the convex cone $\mathcal{P}_{A2}$, i.e., $\mathcal{P}_{A1} \subseteq \mathcal{P}_{A2}$, then there exists a unique positive definite matrix $P > 0$ such that both following Lyapunov functions are feasible

$$\begin{cases} A_1^T P + PA_1 = -I < 0 \\ A_2^T P + PA_2 < 0, \end{cases} \tag{15}$$

where I represents the identity matrix in $\mathbb{R}^{n \times n}$. The matrix $P > 0$ represents a CQLF for A_1 and A_2.

For the parametric switched linear systems, we define the following matrix functions

$$W_1(k,P) := A_1^T(k)P + PA_1(k) + I, \tag{16}$$

$$Q_2(k,P) := A_2^T(k)P + PA_2(k) \tag{17}$$

then the feasible Lyapunov functions (15) can be equivalent to following semi-algebraic system in terms of a symmetric real solution P

$$\begin{cases} w_{111}(k,P) = 0, \cdots, w_{1ij}(k,P) = 0, \cdots, w_{1nn}(k,P) = 0 \\ \mu_{Q21}(k,P) > 0, \cdots, \mu_{Q2j}(k,P) > 0, \cdots, \mu_{Q2n}(k,P) > 0, \\ \mu_{P1}(P) > 0,\ \cdots,\ \mu_{Pj}(P) > 0,\ \cdots\ \mu_{Pn}(P) > 0. \end{cases} \tag{18}$$

Here the polynomial w_{1ij} represents the ij-th entry of the matrix $W_1(k,P)$, μ_{Qij} and $\mu_i(P)$ stands for the $j-$th leading principal minor of matrix $Q_i(k,P)$ and P respectively. We denote the set of feasible parameters as $\Omega_2(k)$, by which the semi-algebraic system (18) has real solutions for P

$$\begin{aligned} \Omega_2(k) = \{k \mid &\text{The semi-algebraic system (18) has} \\ &\text{real solutions for } P\}. \end{aligned} \tag{19}$$

Similarly, when $\mathcal{P}_{A2} \subseteq \mathcal{P}_{A1}$ then the following Lyapunov functions must be feasible

$$A_2^T P + PA_2 = -I < 0, \\ A_1^T P + PA_1 < 0. \tag{20}$$

By defining the matrix function $W_2(k,P)$ as

$$W_2(k,P) := A_2^T(k)P + PA_2(k) + I, \tag{21}$$

$$Q_1(k,P) := A_1^T(k)P + PA_1(k) \tag{22}$$

the following semi-algebraic system is required to solve for a symmetric real solution P

$$\begin{cases} w_{211}(k,P) = 0, \cdots, w_{2ij}(k,P) = 0, \cdots, w_{2nn}(k,P) = 0, \\ \mu_{Q11}(k,P) > 0, \cdots, \mu_{Q1j}(k,P) > 0, \cdots, \mu_{Q1n}(k,P) > 0, \\ \mu_{P1}(P) > 0, \cdots, \mu_{Pj}(P) > 0, \cdots, \mu_{P2}(P) > 0. \end{cases} \tag{23}$$

$\Omega_3(k)$ is used to represent the set of feasible parameters of semi-algebraic system (23) as

$$\Omega_3(k) = \{k \mid \text{The semi-algebraic system (23) has real solutions for } P\}. \tag{24}$$

Finally, the set of all the feasible parameters guaranteeing the existence of CQLF can be given by

$$\Omega(k) = \Omega_1(k) \cup \Omega_2(k) \cup \Omega_3(k). \tag{25}$$

IV. Quantifier Elimination and Real Root Classification for Semi-Algebraic Systems

A. Quantifier Elimination

The computation of feasible parameters set $\Omega(k)$ can be formulated as a Quantifier Elimination (abbr. QE) problem with existential quantifiers [3], i.e., $\exists P$ such that at least one of the formulas (13), (18) and (23) is satisfied. Then we want to find a *quantifier-free* formula, which is equivalent to the quantified formulas. Furthermore, the quantifier-free formula contains only the free variables k and the propositional operators $\vee, \wedge$ and $\neg$.

Since A. Tarski had proven the existence of a solution to the QE problem over the reals and provided the first algorithmic technique for real quantifier elimination in the 1940s, a variety of computer-algebra tools have been developed for efficient implementation of the QE procedures. For instance, the software packages QEPCAD [2], REDLOG, as well as the library of RegularChains [4] in Maple are commonly used. However, these algorithms issue unavoidably computational barriers in solving practical QE problems. In the worst case, their computational complexity is doubly exponential in terms of the number of variables [4].

B. Real Root Classification

Due to the high computational complexity of QE, we apply an algorithm from [14], which can be implemented to solve a specific class of QE problems. Let us consider the following QE problem: find a sufficient and necessary condition w.r.t the parameter k such that a real solution x exists for the following system

$$\begin{cases} f_1(k,x) = 0, \cdots, f_i(k,x) = 0, \cdots, f_n(k,x) = 0, \\ g_1(k,x) \geq 0, \cdots, g_j(k,x) \geq 0, \cdots, g_s(k,x) \geq 0, \\ p_1(k,x) > 0, \cdots, p_p(k,x) > 0, \cdots, p_t(k,x) > 0, \\ h_1(k,x) \neq 0, \cdots, h_q(k,x) \neq 0, \cdots, h_m(k,x) \neq 0, \end{cases} \tag{26}$$

where $x \in \mathbb{R}^n$ and $k \in \mathbb{R}^d$ represent the variable and the parameter respectively, and f_i, g_j, p_p, h_q are polynomials in $\mathbb{Q}[k,x]$ with $n \geq 1$ and $s,t,m \geq 0$. Note that $\mathbb{Q}[k,x]$ denotes the set of polynomials with the elements of k and x as variables and the rational number as coefficients. A system in the above form is called a semi-algebraic system. It can be also denoted by $[F, G, P, H]$, where F, G, P and H stand for the polynomial equations $[f_1 = 0, \cdots, f_n = 0]$, the non-negative polynomial inequalities $[g_1 \geq 0, \cdots, g_s \geq 0]$, the strictly positive polynomial inequalities $[p_1 > 0, \cdots, g_t > 0]$, and the polynomial inequations $[h_1 \neq 0, \cdots, h_m \neq 0]$, respectively.

A semi-algebraic system is constant when it contains no parameters, i.e., $d = 0$, otherwise it is parametric. For a given parametric SAS S, the problem of *real root classification (RRC)* refers to compute all the parameters k such that the semi-algebraic system has exactly $0, 1, 2, \ldots$ distinct real solutions.

It was shown in [15] that the semi-algebraic system (26) can be transformed into an equivalent set of triangular system (abbr. TS) $\{TS_1, \cdots, TS_e\}$ given by

$$TS_i : \begin{cases} f_1(k,x_1) = 0, \\ f_2(k,x_1,x_2) = 0, \\ \vdots \\ f_n(k,x_1,x_2,\cdots,x_n) = 0, \\ g_1(k,x) \geq 0, \cdots, g_t(k,x) \geq 0, \end{cases} \tag{27}$$

where $i \in \{1,\ldots,e\}$. The equivalence means that an x is a solution of (26) if and only if it is a solution of one of the systems $\{TS_1, \cdots, TS_e\}$.

In [14] a complete algorithm for computation of the RRC for a given triangular system is given. For a given parametric TS, the so called border polynomials $BP(k)$ need to be first computed. The border polynomials BPs will decompose the parameter space into finite number of algebraic sets. In each set $BPs \neq 0$, the system TS has constant number of distinct real solutions. Thus, it is sufficient to determine the number of distinct real solutions of TS by checking any one of points from the corresponding set.

More details about the complete algorithm of RRC can be found in [14]. Starting with Maple 13, the RRC function has been integrated into the the Maple package *RegularChains*. Note that the examples in the next section are implemented in Maple 2016.

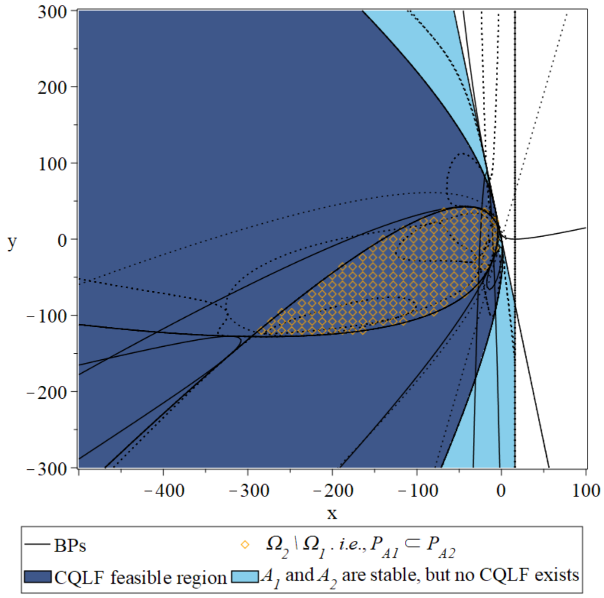

Fig. 1. Illustrating Example 1: For some pairs (x, y) the Lyapunov stability cones intersect (dark blue), for the others, one cone is a strict subset of the other one (brown marked).

V. Example

Example 1: Consider first a second-order switched linear system

$$A_1 = \begin{pmatrix} x & 3 \\ y & -16 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -3 & 2 \\ -8 & -1 \end{pmatrix}, \tag{28}$$

with free system parameter (x, y). The goal consists in computing the region in the (x, y) parameter space, where A_1 and A_2 share a CQLF (so-called CQLF feasibility region).

We first apply the parametric semi-algebraic system (13) and then run the RRC algorithm thereupon. The outcome is graphically depicted in Figure 1. The CQLF region is depicted in dark blue, while the common Hurwitz region for the two subsystems is depicted in light blue. We can see that the feasible region of CQLF is smaller than the intersection of stability region of the subsystems - as it should be. The symbolical formulation to the feasible region of CQLF - which is produced by the Maple RRC package is omitted here, because it involves large-order polynomials and is therefore space consuming.

Because the subsystem matrix A_2 contains no parameters, the parameter set $\Omega_3(k)$ can not be considered here. Then the region for all feasible parameters of CQLF is mapped by the union set $\Omega_1(k) \cup \Omega_2(k)$. Here the set $\Omega_1(k)$ collects the parameters by which convex cones $\bar{\mathcal{P}}_{A1}$ and $\bar{\mathcal{P}}_{A2}$ share intersected boundaries and interiors, while the parameters in the set $\Omega_2(k)$ are computed by setting the CQLF as in (15). The set $\Omega_2(k) \setminus \Omega_1(k)$ - depicted by brown markers - is the set of parameters such that the convex cone $\mathcal{P}_{A1}$ is a strict subset of $\mathcal{P}_{A2}$, i.e. $\mathcal{P}_{A1} \subset \mathcal{P}_{A2}$, and these parameters are unable to be mapped by $\Omega_1(k)$, because in this case the convex cones $\bar{\mathcal{P}}_{A1}$ and $\bar{\mathcal{P}}_{A2}$ have non-intersected boundaries.

Example 2: We illustrate how the developed method can be used for the design of a control system. Consider a PID controlled switched linear system, whose two subsystems are given by

$$A_1 = \begin{pmatrix} 0 & 1 \\ \frac{1+0.75K_I}{9+0.75K_D} & \frac{-3+0.75K_P}{9+0.75K_D} \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 0 & 1 \\ \frac{1+0.25K_I}{4+0.25K_D} & \frac{-2+0.25K_P}{4+0.25K_D} \end{pmatrix}.$$

Here, $k = (K_P, K_I, K_D)^T$ denotes the PID controller parameters. For illustration purposes, we fix the proportional coefficient at $K_P = 2$. By implementation of our method, the CQLF feasibility region in the selected parameter subspace (K_I, K_D) turns out to be specified by the inequality constraints:

$$K_D > -12$$
$$K_D^2 + \frac{15}{4}K_I K_D + \frac{9}{4}K_I^2 + 38K_D + \frac{129}{2}K_I + \frac{1417}{4} < 0$$

which is depicted in Figure 2 (dark blue region). Again, it is clearly observable that the CQLF feasibility region is a strict subset of the intersecting Hurwitz stability region for the two subsystems (light blue region).

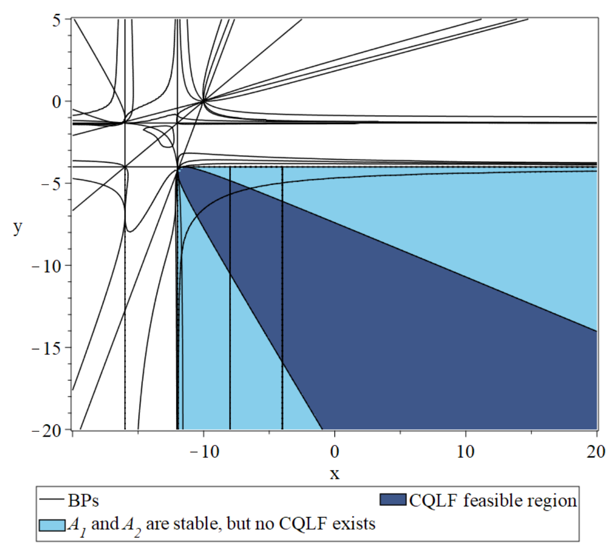

Fig. 2. Illustrating Example 2 with $K_P = 2$ ($x \widehat{=} K_D$, $y \widehat{=} K_I$).

VI. Conclusion

A computer algebra algorithm for the computation of the feasibility region of CQLF for a class of parameter dependent switched linear systems has been presented. The feasibility of a CQLF is formulated by means of the solvability condition of a related parametric semi-algebraic system. Utilizing the real root classification (RRC) technique, the set of all feasible parameters guaranteeing the existence of CQLF is then symbolically computed.The proposed method provides non-conservative CQLF feasible regions. A technical extension of the method will cover switched linear systems with more than two subsystems.

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