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Polynomial stabilization of some control systems by smooth feedbacks

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Abstract— **In this article we are interested in polynomial stability for dynamical systems. Sufficient Lyapunov condition is given and the example of double integrator on $\mathbf{R}^n$ is solved by using two strategies: In the first one we use the backstepping techniques, while in the second we use homogeneous feedback laws with positive degree. As application, the problem of robot manipulators is considered.**

I. INTRODUCTION

Polynomial stability has received a considerable attention since last decades [2], [3], [12], [8]. In particular, with the recent progress in control theory, characterization of polynomial stability by Lyapunov approaches is reconsidered by several aspects; namely " converse Lyapunov theorem " and the homogeneity techniques for homogeneous systems.

In addition, with the progress of Lyapunov redesign, backstepping techniques and polynomial stability with applications to some cascaded systems are developed in [12]. Especially, both polynomial partial stabilization of satellite under two inputs and the ship model are treated. In [21], we have considered the polynomial stabilization by optimal control. Some results of calculus of variation as Hamilton-Bellman -Jacobi condition are used to characterize the optimization of the polynomial stabilizing feedback laws. As application, bilinear system with scalar control is treated where optimal feedbacks stabilizing polynomially all bilinear control system are established [15]. Moreover, Quin's result for bilinear control system [13] is impoved

The advantage of this polynomial stabilization is noted in many works and show how trajectories of the control system reaches the equilibrium point. In addition, it is remarked in many papers [12], [21] that stabilizing feedback laws can be $\mathscr{C}^2$ if we choose some parameters, in these feedbacks, large enough. However, this polynomial stabilizability is one of useful tools to overcome the Brockett's necessary condition [6] for many nonlinear controllable systems; especially systems of angular moment of satellite [10], Brockett's integrator [11], chained systems, chain of integrator, unicycle system, satellites [12] etc., by showing that we can stabilizes polynomially the important component while it is sufficient to know that "uncontrollable" part is converging.

In this paper, sufficient Lyapunov conditions characterizing this polynomial stability are given. In addition, the problem of double integrators in $\mathbf{R}^n$ is reconsidered by two aspects: backstepping techniques and homogeneity feedbacks. As application, the problem of robot manipulators is treated where homogeneous $\mathscr{C}^2$ feedback laws stabilizing the robot are reconstructed. The advantages of these feedbacks compared with other tools that are smooth, $\mathscr{C}^2$ on $\mathbf{R}^n$ and show how our system can reache the equilibrium point.

The paper is organized as follows. The second section is devoted to some preliminary results describing this polynomial stability. In Section 3, the stabilization problem of double integrator on $\mathbf{R}^n$ by two aspects is considered. As example, the model of robot manipulators is analyzed. Finally, the Conclusion is the subject of Section 5.

II. PRELIMINARIES

In this paper we use the following notations: The Euclidean vector norm is noted by $|.|$, $\langle .,. \rangle$ the usual scalar product in $\mathbf{R}^n$, $'$ is the symbol of transposition and $\mathscr{L}^1(\mathbf{R}_+)$ denotes the integrable space in Lebesgue sense on $\mathbf{R}_+$. Finally, $sgn(.)$ is the usually "sign" function, which is defined as follows $sgn(x) = 1$ if $x > 0$, $sgn(x) = -1$ if $x < 0$ and $sgn(0) = 0$, and $\mathbb{Q}^+_{odd} = \{r \in \mathbb{Q}_+ : r = \frac{p}{q} : p \text{ and } q \text{ are odd non negative integers}\}$. In this section we present several preliminary results and definitions which are related to the problem of polynomial stability of nonlinear system.

Consider a time-invariant system in the form of

$$\dot{x} = f(x), f(0) = 0 \tag{1}$$

$x \in \mathbf{R}^n$ where $f : \mathbf{R}^n \to \mathbf{R}^n$ is a smooth vector field.

Definition 2.1: ([16]) The system (1) is said polynomially stable if there exist $r > 0$ and $\alpha > 0$ such that if $|x(0)| < r$, then

$$\lim_{t \to +\infty} t^{\alpha} |x(t)| = 0. \tag{2}$$

Definition 2.2: Let be the control system $\dot{y} = h(y,u)$ where $y \in \mathbb{R}^n$ the state and $u \in \mathbb{R}^m$, and $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a smooth function satisfying $h(0,0) = 0$. This control system is polynomially stabilizable, if there exists a continuous feedback $y \mapsto u(y)$ such that, $u(0) = 0$, and $0 \in \mathbb{R}^n$ is polynomially stable for the closed loop system $\dot{y} = h(y,u(y))$.

Example 2.3: In $\mathbf{R}$, the control system $\dot{x} = u$ is polynomially stabilizable by feedback law $u(x) = -x^{\beta}$ where $\beta \in \mathbb{Q}^+_{odd} \bigcap]1, +\infty[$. This result is easily obtained by integration of the differential equation $\dot{x} = -x^{\beta}$.

In the analysis of the polynomial stability, we need the following technical lemma.

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Lemma 2.4: [14] In $\mathbf{R}$, $\alpha > 1$ is a real number

$$(|x|+|y|)^\alpha \leq 2^{\alpha-1}(|x|^\alpha+|y|^\alpha). \quad (3)$$

In the next, a comparison lemma is given by the following.

Lemma 2.5: Let $r>1$, and we consider u a continuous positive function defined on $[0,+\infty)$. Let be c a positive real number, and we assume that we have the inequality

$$u(t) \leqslant c - \int_0^t u^r(s)ds, t \geqslant 0. \quad (4)$$

Then

$$u(t) \leqslant \left(c^{1-r}+(r-1)t\right)^{\frac{1}{1-r}}, \quad (5)$$

therefore the solution of the integral inequality (4) is polynomially stable.

Proof. The proof is omitted.

One of useful criteria for this polynomial stability can be given by the next Proposition.

Proposition 2.6: The dynamical system (1) is globally polynomially stable if there exist a smooth Lyapunov function $V:\mathbf{R}^n \to \mathbf{R}$, some positive constants c_1 and r_1 such that

- $V(0)=0$,
- $$c_1|x|^{r_1} \leqslant V(x)$$
- there exists $c>0$ and $\alpha>0$ such that

$$\dot{V}+cV^{1+\alpha} \leqslant 0.$$

Proof. By using Lemma 2.5, the differential inequality $\dot{V}+cV^{1+\alpha} \leqslant 0$ lead to $V(x(t)) \leqslant V(x(0)) - \int_0^t V^{1+\alpha}(x(s))ds$. Here $r:=\alpha+1>1$, and the Lemma permit us to conclude.

As application of the above polynomial stability, we begin with the double integrator on $\mathbf{R}^n$.

III. Polynomial stabilization of double integrator on $\mathbf{R}^n$

Since the double integrator is a key system that appears in several mechanical systems, for this reason, many authors gave a lot attention for the construction of stabilizing feedback laws for this system by diverse approaches [5], [18], [4]. In this section we deal with the polynomial stabilization of double integrator on $\mathbf{R}^n$ by two aspects: The first one is the stabilization by Hölderian feedback laws, and the construction of the stabilizing feedbacks are based on direct Lyapunov approach by the backstepping approach, while the second way is the uses of homogeneity theory combined with LaSalle's Theorem.

A. Backstepping approach

In this part, we give an explicit construction of Hölderian feedback laws that make the double integrator in $\mathbf{R}^n$ polynomially stable.

Let be the system

$$\dot{x}=y, \dot{y}=u, \quad (6)$$

where the state is $(x,y) \in \mathbf{R}^n \times \mathbf{R}^n$ and $u \in \mathbf{R}^n$ the input.

We define $x^r := (x_1^r, x_2^r, ..., x_n^r)' \in \mathbf{R}^n$ where r is an odd integer or odd positive rational number (i.e. $r=\frac{p_1}{p_2}$ where p_1, p_2 are odd integers).

Step 1 With this notation, the reduced system $\dot{x}=u$ is polynomially stable under the family of feedback laws defined by $u=-\mu x^r = -kx^{2p+1}$, $\mu>0$, where p is a positive integer or nonnegative odd rational.

Indeed, we consider the Lyapunov function defined by $V: \frac{1}{2}|x|^2$. The time derivative of V can be estimated as follows

$$\begin{aligned} -\dot{V} &= \mu\langle x^r, x\rangle \\ &= \mu\left((x_1^2)^{(r+1)/2}+(x_2^2)^{(r+1)/2}+...+(x_n^2)^{(r+1)/2}\right) \\ &\geqslant \mu\frac{1}{2^{\frac{r+1}{2}-1}}\left(x_1^2+x_2^2+...+x_n^2\right)^{(r+1)/2} \\ &\geqslant cV^{(r+1)/2}. \end{aligned} \quad (7)$$

Hence, $\dot{V} \leqslant -cV^{1+\alpha}$ where $\alpha=\frac{r-1}{2}=p$ and $c=\mu\frac{1}{2^{\frac{r+1}{2}-1}}$, and by Proposition 2.6, the reduced system $\dot{x}=u$ is polynomially stabilizable under the family of feedbacks $u(x)=-\mu x^{2p+1}$.

Step 2 For the augmented system (6), we denote by $x:=(x_1,x_2,...,x_n)'$ and $y:=(y_1,y_2,...,y_n)'$, and we consider the candidate Lyapunov function V defined as in [12] by

$$V=\sum_{i=1}^n \frac{1}{2k}x_i^{2k}+\sum_{i=1}^n \frac{1}{2}(y_i+x_i^{2p+1})^2. \quad (8)$$

The time derivative of V along the system (6) is given by

$$\dot{V}=-\sum_{i=1}^n x_i^{2(k+p)}+\sum_{i=1}^n (y_i+x_i^{2p+1})(u_i+x_i^{2k-1}+(2p+1)x_i^{2p}y_i),$$

hence, under the choice of

$$u_i(x,y)=-(2p+1)x_i^{2p}y_i-x_i^{2k-1}-(y_i+x_i^{2p+1})^{1+2\frac{p}{k}}, \quad (9)$$

where $k, p \in \mathbf{N}^*$ are odd integers; we get

$$\dot{V}=-\sum_{i=1}^n x_i^{2(p+k)}-\sum_{i=1}^n (y_i+x_i^{2p+1})^{2+2\frac{p}{k}}. \quad (10)$$

Lemma 3.1: There exists a positive constant c such that $\dot{V}$ satisfies the differential inequality

$$\dot{V} \leqslant -cV^{1+\frac{p}{k}}, \quad (11)$$

and therefore, system (6) is polynomially stable under the family of feedbacks $u(x,y)=(u_i(x,y))_{1\leqslant i\leqslant n}$

Proof.

From (10), we have $-\dot{V} \geqslant \sum_{i=1}^n x_i^{2(p+k)} \geqslant x_i^{2(p+k)}$ for all $i=1,...,n$.

Then $(-\dot{V})^{\frac{k}{k+p}} \geqslant x_i^{2k}$ and $n(-\dot{V})^{\frac{k}{k+p}} \geqslant \sum_{i=1}^n x_i^{2k}$. Again by the same way, we get $n(-\dot{V})^{\frac{k}{k+p}} \geqslant \sum_{i=1}^n (y_i+x_i^{2p+1})^{2+2\frac{p}{k}}$. Then

$$\dot{V} \leqslant -(\frac{1}{2n})^{1+p/k}V^{1+\frac{p}{k}}.$$

Thus, by Proposition 2.6, system (6) is globally polynomially stabilizable. This ends the proof.

Asymptotic estimation By a simple integration of the above inequality we get

$$V \leqslant \frac{1}{(at+b)^{\frac{k}{p}}}, \tag{12}$$

where a and b are two constants depending on initial conditions.

Then we get the following estimations

$$|x_i(t)| \leqslant \frac{\sqrt[2k]{2k}}{(at+b)^{\frac{1}{2p}}}. \tag{13}$$

$$|y_i(t)| \leqslant |y_i + x_i^{2p}| + |x_i^{2p+1}| \leqslant \frac{\sqrt{2}}{(at+b)^{\frac{k}{2p}}} + \frac{\sqrt[2k]{(2k)^{2p+1}}}{(at+b)^{\frac{2p+1}{2p}}}. \tag{14}$$

Therefore by integral comparison theorem, we get from (13) and (14) that if the odd rational p is in the open interval $(0, \frac{1}{2})$ and $k > \max(1, 2p)$, then x and y are Lebesgue-integrable.

Remark 3.2: If we choose k and p in (12) sufficiently large such that $p > k$ we get a family of stabilizing feedback laws of classes $\mathscr{C}^\infty$ that make trajectories of (6) to decrease like $\frac{1}{t^\beta}$. But, the disadvantage of these feedbacks that are not be bounded and the saturation function does not work in this case. The next approach solves this problem and give a bounded polynomial homogeneous stabilizing feedback laws.

B. Homogeneous feedback laws

In this subsection, we give a new homogeneous feedbacks stabilize polynomially the double integrator in $\mathbf{R}^n$. Consider again system (6) and let be $\alpha > 2$. We take the Lyapunov candidate function of the form

$$V(x, y) = \sum_{i=1}^{n} \frac{1}{\alpha_i}|x_i|^\alpha + \frac{1}{2}|y_i|^2, \tag{15}$$

then, his time derivative along (6) is given by

$$\dot{V}(x, y) = \sum_{i=1}^{n} y_i[u_i + |x_i|^{\alpha-1} sgn(x_i)],$$

hence, with the control laws $u(x, y) = (u_i(x, y))_{1 \leqslant i \leqslant n}$ where

$$u_i(x, y) = -sgn(x_i)|x_i|^{\alpha-1} - sgn(y_i)|y_i|^{\frac{2(\alpha-1)}{\alpha}}, \tag{16}$$

we get

$$\dot{V}(x, y) = -\sum_{i=1}^{n} |y_i|^{\frac{3\alpha_i - 2}{\alpha}} \leqslant 0.$$

Because $\dot{V}(x, y) = 0$ together with (16), we get $x = 0$, then by LaSalle's principal invariance, the equilibrium $(x, y) = (0, 0)$ is asymptotically stable for the closed loop system (6).

In addition, if we consider the dilation defined by

$$\delta_\varepsilon(x, y) = (\varepsilon x_i, \varepsilon^{\frac{\alpha}{2}} y_i) = (\varepsilon x_1, ..., \varepsilon x_n, \varepsilon^{\frac{\alpha}{2}} y_1, ..., \varepsilon^{\frac{\alpha}{2}} y_n)',$$

and in closed loop the function f defined on $\mathbf{R}^{2n}$ by $f(x, y) = (f_1(x, y), f_2(x, y)) = (y, u(x, y))$, then a simple calculation leads to

$$f_1(\delta_\varepsilon(x, y)) = \varepsilon^{\frac{\alpha}{2}} y = \varepsilon^{1+(\frac{\alpha}{2}-1)} y = \varepsilon^{1+d} f_1(x, y), \tag{17}$$

$$f_2(\delta_\varepsilon(x, y)) = u(\varepsilon x_i, \varepsilon^{\frac{\alpha}{2}} y_i) = \varepsilon^{\alpha-1} f_2(x, y) = \varepsilon^{\frac{\alpha}{2}+d} f_2(x, y). \tag{18}$$

Then, the closed-loop system (6) is homogeneous of positive degree $d = \frac{\alpha}{2} - 1 > 0$. Hence, by [3, Corollary 5.4 (i), pp. 185] we conclude the trivial equilibrium of (6) is globally polynomially stable. Thus we have proved the following Proposition.

Proposition 3.3: under the family of feedback laws $u(x, y) = (u_i(x, y))_{1 \leqslant i \leqslant n}$ where for all $i = 1, ..., n$, u_i are given by (16), system (6) is globally polynomially stable. Moreover, for α is large enough, these feedbacks are smooth.

IV. Application to robot manipulators

In this section, we provide at leats a $\mathscr{C}^2$ stabilizing feedback laws making the rigid-body model of robot manipulators polynomially stable. In order to put the reader in the context, the motion of robot manipulators are given by the robot control [9]

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = u, \tag{19}$$

where $q, \dot{q}, \ddot{q} \in \mathbf{R}^n$ are respectively the vectors of joint angular position, velocity and acceleration. The matrix $M(q) \in \mathbf{R}^{n \times n}$ is the inertia matrix supposed to be positive, $C(q, \dot{q})\dot{q}$ regroups the Coriolis and centrifugal forces, and the vector $G(q) \in \mathbf{R}^n$ represents the gravitational force and $u \in \mathbf{R}^n$ is vector of joint control input torques.

For the stability analysis, we assume that $M(q)$,$C(q, \dot{q})$, and $G(q)$ are all smooth. In addition, some well-known hypotheses [9] of (19) are considered:

- $M(q) \geqslant 0$ and bounded,
- the matrix $\dot{M}(q) - 2C(q, \dot{q})$ is skew-symmetric,
- $G(q)$ is bounded.

1) Regulation problem: :

The regulation problem consists to find a control law $u = u(q, \dot{q})$ such that the equilibrium of the closed-loop system at $(q, \dot{q}) = (q^d, 0)$ is globally polynomially stable, where q^d and q respectively denotes the desired and actual joint variables. We consider the change of variables $z = (q, \dot{q})'$, then

$$\dot{z} = \begin{pmatrix} \dot{q} \\ M^{-1}(q)(u - C(q, \dot{q})\dot{q} - G(q)) \end{pmatrix} := f(z, u), \tag{20}$$

and we denotes respectively by $x := q - q^d$ and for $u = (u_1, ..., u_n)' \in \mathbf{R}^n$, $sgn([u])[u]^s := (sgn(u_1)|u_1|^s, ..., sgn(u_n)|u_n|^s)'$. Then we have the following polynomial stability result.

Proposition 4.1: Let be $\alpha > 2$, then under the family of feedback laws

$$u(z) = M(q)\big(G(q)+C(q,\dot q)\dot q - sgn([x])[x]^{\alpha-1} - sgn([\dot q])[\dot q]^{\frac{2(\alpha-1)}{\alpha}}\big), \tag{21}$$

then (19) is globally polynomially stable.

Proof. For $x = q - q^d$ and $y = \dot x = \dot q$, let $X = (x, y)'$, then the closed-loop system under the control law (21) takes the form

$$\begin{aligned} \dot x_i &= y_i, \\ \dot y_i &= -sgn(x_i)|x_i|^{\alpha-1} - sgn(y_i)|y_i|^{\frac{2(\alpha-1)}{\alpha}}, \end{aligned} \tag{22}$$

where $x := (x_i)_{1\leqslant i\leqslant n} \in \mathbf{R^n}$ and $y := (y_i)_{1\leqslant i\leqslant n} \in \mathbf{R^n}$.

From Proposition 3.3, the system (22) is polynomially stable. Consequently, (20) is polynomially stable.

Next, we provide a smooth and bounded stabilizing polynomially feedback laws for the system (20).

Corollary 4.2: Given $\varepsilon > 0$ and $\alpha > 2$. Then, system (22) is globally polynomially stable under the family of saturated feedback laws

$$u_{i,\alpha}(x,y) = -sat_\varepsilon\big(sgn(x_i)|x_i|^{\alpha-1}\big) - sat_\varepsilon\big(sgn(y_i)|y_i|^{\frac{2(\alpha-1)}{\alpha}}\big), \tag{23}$$

where the saturation function sat_ε is defined by

$$\begin{aligned} sat_\varepsilon(x) :\ &= x,\ if\ |x| < \varepsilon, \\ &= \varepsilon\, sgn(x),\ if\ |x| \geqslant \varepsilon. \end{aligned} \tag{24}$$

Remark 4.3: The system (20) is again globally polynomially stable under the family of feedback laws

$$u_i(x,y) = -(2p+1)x_i^{2p}y_i - x_i^{2k-1} - (y_i + x_i^{2p+1})^{1+2\frac{p}{k}}, \tag{25}$$

where $k, p \in \mathbf{N}^*$ are odd integers.

But the disadvantage of these feedbacks that are not bounded.

Next, we give another family of stabilizing feedback laws the system (20).

Proposition 4.4: The feedback control

$$u(q,\dot q) = G(q) - sgn([q-q^d])[q-q^d]^{\alpha-1} - sgn([\dot q])[\dot q]^\beta,\ \alpha > 2. \tag{26}$$

with $\beta = \frac{2(\alpha-1)}{\alpha}$, stabilizes polynomially the system (20) .

Proof. Let $x = q - q^d$, $y = \dot x = \dot q$, and $X = (x,y)' \in \mathbf{R}^{2n}$, then the closed-loop system under the control law (26) have the following form

$$\begin{cases} \dot x = & y \\ \dot y = & -M(q^d)^{-1}\big(sgn[x][x]^{\alpha-1} + sgn([y])[y]^\beta\big] + g(x,y). \end{cases} \tag{27}$$

where

$$\begin{aligned} &g(x,y) = -M(x+q^d)^{-1}C(x+q^d,y)y - \\ &\tilde M(q^d,x)(sgn([x])[x]^{\alpha-1} + sgn([y])[y]^\beta). \end{aligned} \tag{28}$$

with $\tilde M(q^d,x) = M(x+q^d)^{-1} - M(q^d)^{-1}$

The function g can be decomposed on two functions g_1 and g_2 where $g_1(x,y) := -M(x+q^d)^{-1}C(x+q^d,y)y$ and $g_2(x,y) := -\tilde M(q^d,x)(sgn([x])[x]^{\alpha-1} + sgn([y])[y]^\beta)$.

The system

$$\begin{cases} \dot x = & y, \\ \dot y = & -M(q^d)^{-1}(sgn([x])[x]^{\alpha-1} + sgn([y])[y]^\beta). \end{cases} \tag{29}$$

is homogeneous of degree $d = \frac{\alpha}{2} - 1 > 0$ with respect to dilation $\delta_\varepsilon(x,y) = (\varepsilon x, \varepsilon^{\frac{\alpha}{2}} y)'$.

Proposition 4.5: The origin of the system (27) is globally polynomially stable.

Proof.

Consider a Lyapunov function

$$V(X) = \sum_{i=1}^n \frac{1}{\alpha_i}|x_i|^{\alpha_i} + \frac{1}{2}y'M(x+q^d)y.$$

Then

$$\dot V(X) = \sum_{i=1}^n |x_i|^{\alpha_i-1} sgn(x_i)y_i + \dot y' M(x+q^d)y + \frac{1}{2}y'M(\dot{x+q^d})y = -\sum_{i=1}^n |y_i|^{1+\beta} \leqslant 0.$$

It is now straightforward to prove by invoking the LaSalle's invariant set theorem, the equilibrium of the closed-loop system is globally asymptotically stable.

But we cannot conclude the homogeneity of system (27) (because the presence of the function g). We take the Lyapunov function of the closed-loop system (29)), defined by $V(X) = \sum_{i=1}^n \frac{1}{\alpha_i}|x_i|^{\alpha_i} + \frac{1}{2}y'M(q^d)y$. So, $\dot V(X) = -\sum_{i=1}^n |y_i|^{1+\beta} \leqslant 0$. The system (29) is globally asymptotically stable and homogeneous of positive degree ($d = \frac{\alpha}{2} - 1 > 0$). From LaSalle's theorem and Bacciotti-Rosier Theorem [3, Corollary 5.4 (i), pp. 185], the equilibrium of the system (29) is globally polynomially stable.

Moreover, for the function g we have the following assumptions:

- $M(x+q^d)^{-1}$ and $C(x+q^d,y)$ are bounded by 1,
- $\tilde M(q^d,x)$ is bounded by 1.

The function g is seen as a nonlinear perturbation satisfying the condition [7, Theorem 12.16]). Indeed, both the vector functions $(0, g_1(x,y))$ and $(0, g_2(x,y))$ are homogeneous with respect to δ_ε with upper degree respectively, $\alpha/2$ and $\alpha - 1$ (in fact, this is the notion of interval homogeneity introduced by Z. Su et al. [17]). Since the nominal system (29) is homogeneous $\alpha/2 - 1$ and we have the relationship:

$$\alpha/2 - 1 < \alpha/2 < \alpha - 1.$$

Then, we conclude that system (27) is globally polynomially stable under the control law (26).

Remark 4.6: We often have constraints on position, velocity, acceleration of the manipulator joints:

1) For example, we consider an robotic manipulator described by the equation [1]

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+G(q)+F(\dot{q})=u, \tag{30}$$

where $q,\dot{q},\ddot{q}\in\mathbf{R}^n$ are respectively the vectors of joint angular position, velocity and acceleration. The matrix $M=M^T>0$, C, and g are matrices with smooth inputs and proper dimensions, $u\in\mathbf{R}^n$ is the control input. Morover, the friction force

$$F(\dot{q})=F_v\dot{q}+F_c sgn(\dot{q}) \tag{31}$$

where $F_c=\mathrm{diag}(f_{c_i})_{i=1}^m$ is the viscous terms, and $F_v=\mathrm{diag}(f_{v_i})_{i=1}^m$ is the Coulomb friction terms. [1] and [19] show that Coulomb friction can extend the system stability bounds but may lead to an input-dependent stability, and it causes an actuator limit cycle.

2) The equation of motion of UMS(underactuated mechanical system) is

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+G(q)=F(q)u, \tag{32}$$

where $F(q)\in\mathbf{R}^{n\times m}$ is a non-square matrix of external forces.
The reason of complexity to control this system [20] show that we can control this system with many methods: backstepping, direct Lyapunov method, IDA-PBC and controlled Lagrangian methods, optimal control, fuzzy control and sliding mode control.

Numerical simulation The proposed homogeneous feedback laws for the robot manipulator are tested numerically, and the simulation shows that trajectories of the system reach the equilibrium point.The simulation is presented in Fig. 1 and Fig.2.

Where

- q_1,q_2: the vectors of joints angular position 1 and 2.
- $\dot{q}_1,\dot{q}_2$: the vectors of velocity 1 and 2.

We take the robot with $n=2$. Then we consider

$$M(q)=\begin{pmatrix}(a_1+a_2) & 0\\ 0 & a_2\end{pmatrix}, \tag{33}$$

$C=0$, and

$$G(q)=g\begin{pmatrix}0\\ a_2q_2\end{pmatrix}, \tag{34}$$

$g=10N/Kg$ where $a_1=18.8$, $a_2=13.2$, $\alpha=2$ and $\beta=\frac{4}{3}$, with the feedback law

$$u=\begin{pmatrix}u_1\\ u_2\end{pmatrix}, \text{ with}$$

$$u_1=G(q)-sgn(q_1-q_1^d)|q_1-q_1^d|^{\alpha-1}-sgn(\dot{q}_1)|\dot{q}_1|^{\beta}, \tag{34}$$

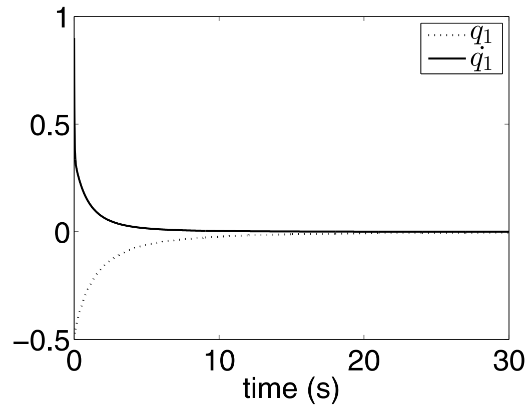

Fig. 1. Trajectories of the state $q_1,\dot{q}_1$

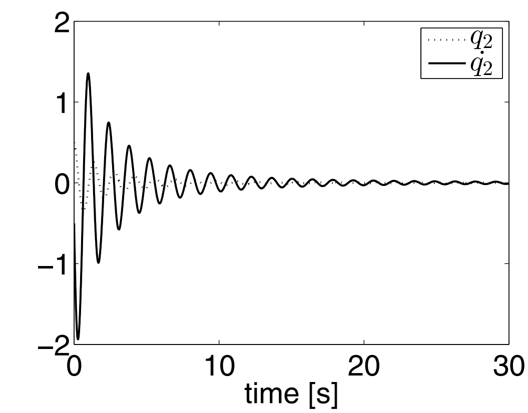

Fig. 2. Trajectories of the state $q_2,\dot{q}_2$

$$u_2=G(q)-sgn(q_2-q_2^d)|q_2-q_2^d|^{\alpha-1}-sgn(\dot{q}_2)|\dot{q}_2|^{\beta}. \tag{35}$$

The analysis shows that all trajectories of the system converge polynomially to the equilibrium point.

V. CONCLUSIONS

This paper has studied the problem of the polynomial stability of finite dimensional dynamical system on $\mathbf{R}^n$. Sufficient Lyapunov condition is given characterizing this polynomial stability. In addition, we have established $\mathscr{C}^2$ feedback laws stabilizing all integrator on $\mathbf{R}^n$, and thus permit us the construction of the stabilizing feedback laws for robot manipulators in polynomial sense.
The problems of observation under the two methods (i.e backstepping and homogeneity degree) are under investigation in the future work.

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