2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Gaussian mean-field models of linear systems

Marko Seslija and Rodolphe Sepulchre*

***Abstract*— This paper addresses the issue of modeling mean-field behavior in heterogeneous populations of linear time-invariant SISO systems. Our analysis is conducted in the frequency domain, where the heterogeneity of input-output mappings (transfer functions) is modeled as a complex-valued Gaussian process. The mean-field model of diffusively coupled agents is obtained as a Gaussian approximation of averaged input-output behavior. It is shown that the strong coupling and the large number of agents reduce the population variance.**

I. Introduction

Most complex systems in biophysics, neuroscience, and systems biology exhibit robust structures in the sense that failures of a certain fraction of individual units do not destroy the underlying functionality of the system. Modeling such systems *in toto* as large-scale networks is a formidable task, both from modeling and computational points of view. Because of this, the macroscopic viewpoint is considered as a more appropriate methodology in understanding the functionality of such systems.

When the size of network (interacting population) becomes large enough averaging effects appear, and modeling the collective behavior calls for an effective mean-field model, summarizing the effect of all the other agents on any given agent. Modeling heterogeneity and deriving the effective mean-field model from bottom-up is far from being straightforward. In simple physical models this can be achieved via the law of large numbers and the central limit theorem, under certain technical assumptions. The mean-field approach has been generalized to such fields as quantum field theory, non-equilibrium statistical mechanics, neural networks [1], [2], as well as to mean-field game theory in the control community [3], [4].

While mean-field modeling is intuitively clear, its rigorous derivation and the analysis of its properties is quite a challenge. Heterogeneity was previously studied in the multi-agent setting but chiefly in the context of consensus and synchronization [5]–[8], when the network behavior reduces to a zero- or one-dimensional behavior. In the systems and control community, the effect of network size on the average behavior has not received much attention. An exception is the recent paper [9] that explicitly introduces the notion of average dynamics as the formal sum of all states and studies the effect of number of agents on the robustness of synchronization for scalar systems. Such an approach is very much in the spirit of mean-field theory as it allows for the macroscopic (averaged) description of the underlying network.

*Department of Engineering, University of Cambridge, {m.seslija, r.sepulchre}@eng.cam.ac.uk

Frequently in practice, all we know about a system is its input-output behavior. In such circumstances, rather than building up a state-space model, it might be more natural to look at input-output mappings. For instance, the key identification technique for neural dynamics is the voltage clamp experiment. The voltage clamp experiment assigns a step input voltage to a neural circuit and measures the corresponding current trajectory. Locally, such a step response can be represented by a transfer function [10], and it appears natural to model such behavior in the frequency domain. In this paper, we adopt the formalism of Gaussian processes [11]–[13] and model the network heterogeneity in the frequency domain. Solely concentrating on the linear SISO case, the individual transfer functions of each node in the network are thought of as realizations of a complex-valued Gaussian process in the frequency domain. We define the mean-field model as the expected average input-output behavior.

For didactic purposes, firstly, we address the problem of constructing a mean-field approximation of a static SISO network, and then we move onto the problem of quasi-static mean-field model for LTI systems. For balanced networks with diffusive coupling, the mean-field model captures well the average macroscopic behavior of the network. Its mean does not depend on the coupling strength (as in [9]), while its variance (and pseudo-variance) scale inversely with the coupling strength.

II. Static model

We consider a population of N input-output agents, where the agents are at first disconnected from each other and then connected by diffusive coupling.

A. Population of heterogeneous static agents

Let an agent $i \in \mathcal{P} = \{1, \dots, N\}$ be given by the following input-output relation

$$y_i = a_i\, u_i\,, \tag{1}$$

where $y_i \in \mathbb{R}$ is the output of i-th node and u_i is the input taking real values. The scalars a_i's are realizations of the Gaussian random variable $a \sim \mathcal{N}\left(\bar{a}, \sigma_a^2\right)$, where $\bar{a}$ is the mean and σ_a^2 is the standard deviation.

We define the *average behavior* of the ensemble (1) as

$$\bar{y} = \frac{1}{N}\sum_{i=1}^{N} a_i\, u_i\big|_{u_i=u} = \frac{1}{N}\sum_{i=1}^{N} a_i\, u = a_{\text{avg}}\, u\,, \tag{2}$$

where $u \in \mathbb{R}$ is a generic input (e.g., the mean input).

The *expected average behavior* is

$$\bar{y}_E = \mathbb{E}\left[\bar{y}\right] = \bar{a}\, u\,, \tag{3}$$

with the variance being

$$\begin{aligned}\mathbb{V}\left[\bar{y}\right] &= \mathbb{E}\left[(\bar{y}-\bar{a}u)^2\right] \\ &= \frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\left[(a_i-\bar{a})^2\right]u^2 = \frac{1}{N}\sigma_a^2 u^2 \,.\end{aligned} \tag{4}$$

An alternative and equivalent way to look at the average behavior of the ensemble (1) is as realizations of a Gaussian process (see Figure 1). The mean and covariance function of such a process are given by (3) and (4), respectively. That is, the average behavior $\bar{y}$ is a Gaussian process (GP) given as

$$\bar{y}(u) \sim \mathcal{GP}\left(\bar{a}\,u, \frac{\sigma_a^2}{N}uu'\right) \,. \tag{5}$$

Such a behavior we shall call the ***mean-field model*** of the ensemble (1).

In this interpretation of heterogeneity, the Gaussian process gives a distribution of input-output mappings, as shown in Figure 1. An example of such a representation would be a family of I–V curves corresponding to a heterogeneous population of linear resistors.

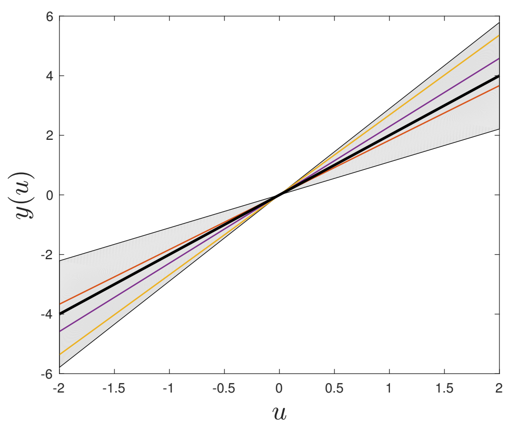


Fig. 1. The figure shows three realizations of the Gaussian process $y(u) \sim \mathcal{GP}\left(\bar{a}\,u, \sigma_a^2 uu'\right)$. The shaded area shows the mean ($\bar{a} = 2$) plus and minus two times the standard deviation ($\sigma_a^2 = 0.2$) for each input value (corresponding to the 95% confidence region). The black bold line represents the mean input-output relation.

B. Diffusively coupled static agents

Let the agents (1) be diffusively coupled in an all-to-all network, that is

$$u_i = v + c\sum_{j=1}^{N}(y_j - y_i),$$

where $v \in \mathbb{R}$ is a (free) input and $c \in \mathbb{R}$ is a strength of coupling. The network behavior is

$$y_i = a_i\left(v + c\sum_{j=1}^{N}(y_j - y_i)\right), \quad i \in \mathcal{P}. \tag{6}$$

We will introduce a ***Gaussian mean-field relaxation*** of (6) as

$$y_i = a_i v + c\,\bar{a}\sum_{j=1}^{N}(y_j - y_i), \quad i \in \mathcal{P}, \tag{7}$$

where v is a mean input. Since $\sum_{i=1}^{N}\sum_{j=1}^{N}(y_j - y_i) = 0$, we have $\sum_{i=1}^{N} y_i = \sum_{i=1}^{N} a_i v$, and consequently

$$y_i = \frac{1}{Nc\bar{a}+1}\left(c\,\bar{a}\sum_{j=1}^{N}a_j + a_i\right)v \,. \tag{8}$$

Remark 1: In this paper, we do not address the problem of how close two models (6) and (7) are to each other. All the numerical simulations[1] show that (7) is a good approximation of (6) as N becomes large. Furthermore, note that while the average $\sum_{i=1}^{N} y_i$ in (7) is independent of the coupling strength c, the weighted average $\sum_{i=1}^{N} a_i^{-1} y_i$ in (6) is independent of c.

The average behavior of (7) is

$$\bar{y}(v) = \frac{1}{N}\sum_{i=1}^{N} y_i(v) = \frac{1}{N}\sum_{i=1}^{N} a_i v = a_{\text{avg}} v \,, \tag{9}$$

while its expected value is

$$\bar{y}_E(v) = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} a_i v\right] = \bar{a}\,v \,, \tag{10}$$

and covariance

$$\text{cov}\left(\bar{y}(v), \bar{y}(v')\right) = \mathbb{E}\left[\frac{1}{N^2}\sum_{i=1}^{N}\left(a_i - \bar{a}\right)^2 vv'\right] = \frac{\sigma_a^2}{N} vv' \,.$$

Interestingly enough, the mean-field (9) is a Gaussian process given by (5). This means that the coupling strength c does not configure the parameters of the mean-field model of (7). Here it is important to emphasize that while the y_i's in (7) conform to a Gaussian process, the y_i's in (6) *do not* obey a Gaussian distribution. This means that the interconnection structure destroys the population's Gaussian property on the network level.

The population variance: The population variance σ_p^2 is given as

$$\begin{aligned}\sigma_p^2 &= \frac{1}{N}\sum_{i=1}^{N}\left(y_i - \bar{y}\right)^2 \\ &= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{Nc\bar{a}+1}\left(c\bar{a}\sum_{j=1}^{N}a_j + a_i\right)v - a_{\text{avg}}v\right)^2 \\ &= \frac{v^2}{(Nc\bar{a}+1)^2}\frac{1}{N}\sum_{i=1}^{N}\left(c\,\bar{a}\sum_{j=1}^{N}a_j + a_i - (Nc\bar{a}+1)a_{\text{avg}}\right)^2 \\ &= \frac{v^2}{(Nc\bar{a}+1)^2}\frac{1}{N}\sum_{i=1}^{N}\left(a_i - a_{\text{avg}}\right)^2 \,.\end{aligned}$$

[1]Rather than showing a numerical example here, we direct the reader to Section IV (in particular Figure 3).

Observe that σ_p^2 is a random variable whose expected value can be expressed in terms of the generating probability distribution for the variable a_i as

$$\begin{aligned}\mathbb{E}[\sigma_p^2] &= \frac{v^2}{(Nc\bar{a}+1)^2}\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[(a_i-a_{\mathrm{avg}})^2] \\ &= \frac{v^2}{(Nc\bar{a}+1)^2}\sigma_a^2 \,.\end{aligned} \tag{11}$$

Note that the population variance depends on the size of population N and coupling strength c.

III. Population of LTI systems

Next we turn our attention to a network of SISO LTI agents, where the input-output mapping of each agent i is given by the transfer function G_i.

The transfer functions G_i's will be treated as realizations of the ***complex-valued*** **Gaussian process**

$$G(s) \sim \mathcal{CGP}\left(G_0(s), \Gamma(s,s'), C(s,s')\right), \tag{12}$$

where $G_0(s)$ is the mean function, $\Gamma(s,s')$ and $C(s,s')$ are the covariance and pseudo-covariance functions[2] given by

$$\begin{aligned} G_0(s) &= \mathbb{E}\left[G(s)\right] \\ \Gamma(s,s') &= \mathbb{E}\left[\left(G(s)-G_0(s)\right)\left(G(s')-G_0(s')\right)^*\right] \\ C(s,s') &= \mathbb{E}\left[\left(G(s)-G_0(s)\right)\left(G(s')-G_0(s')\right)\right], \end{aligned} \tag{13}$$

with s being purely imaginary, $s = \mathrm{j}\omega \in \mathbb{I}$, $\omega \in \mathbb{R}$. This means that our discussion will be conducted in the frequency domain.

Analogous to the definition of the real Gaussian process, the mapping $G : \mathrm{j}\mathcal{F} \to \mathbb{C}$, $\mathcal{F} \subset \mathbb{R}$, is a complex Gaussian process if every finite collection of random variables $\{G(\mathrm{j}\omega_i)\}_{\omega_i\in\mathcal{F}}$, is jointly distributed according to a complex Gaussian distribution (see, e.g., [14]). The complex-valued Gaussian processses that as input take purely real (or imaginary) arguments are isomorphic to two-output GPs. The complex-valued $G(\mathrm{j}\omega) = G_r(\mathrm{j}\omega) + \mathrm{j}\,G_i(\mathrm{j}\omega)$, where G_r and G_i take their values in $\mathbb{R}$, can be represented as bivariate process $(G_r(\mathrm{j}\omega), G_i(\mathrm{j}\omega))^T$ with the mean $(\overline{G}_r(\mathrm{j}\omega), \overline{G}_i(\mathrm{j}\omega))^T$ and the covariances $\mathrm{cov}(G_r(\mathrm{j}\omega), G_r(\mathrm{j}\omega')) = C_{rr}(\omega,\omega')$, $\mathrm{cov}(G_i(\mathrm{j}\omega), G_i(\mathrm{j}\omega')) = C_{ii}(\omega,\omega')$, $\mathrm{cov}(G_r(\mathrm{j}\omega), G_i(\mathrm{j}\omega')) = C_{ri}(\omega,\omega')$, and $\mathrm{cov}(G_i(\mathrm{j}\omega), G_r(\mathrm{j}\omega')) = C_{ir}(\omega,\omega')$, where due to the symmetry of the real-valued covariance functions we have $C_{rr}(\omega,\omega') = C_{rr}(\omega',\omega)$ and $C_{ii}(\omega,\omega') = C_{ii}(\omega',\omega)$ are nonnegative kernels, and $C_{ri}(\omega,\omega') = C_{ir}(\omega',\omega)$. The covariance and pseudo-covariances of the complex Gaussian process G can then be expressed as $\Gamma(\mathrm{j}\omega,\mathrm{j}\omega') = C_{rr}(\omega,\omega') + C_{ii}(\omega,\omega') + \mathrm{j}(C_{ir}(\omega,\omega') - C_{ri}(\omega,\omega'))$ and $C(\mathrm{j}\omega,\mathrm{j}\omega') = C_{rr}(\omega,\omega') - C_{ii}(\omega,\omega') + \mathrm{j}(C_{ri}(\omega,\omega') + C_{ir}(\omega,\omega'))$. Furthermore, the covariance functions are chosen such that the block matrix

$$\begin{bmatrix} [C_{rr}(\omega_i,\omega_j)]_{i,j=1}^m & [C_{ri}(\omega_i,\omega_j)]_{i,j=1}^m \\ [C_{ir}(\omega_i,\omega_j)]_{i,j=1}^m & [C_{ii}(\omega_i,\omega_j)]_{i,j=1}^m \end{bmatrix}$$

is nonnegative definite for any $m \in \mathbb{N}$ and any $\omega_1, \ldots, \omega_m \in \mathcal{F}$. Note that $[C_{ir}(\omega_i,\omega_j)]_{i,j=1}^m = \left([C_{ri}(\omega_i,\omega_j)]_{i,j=1}^m\right)^T$.

As an illustration, consider a complex-valued Gaussian process generated by the mean (transfer function)

$$G_0(\mathrm{j}\omega) = \frac{1}{(\mathrm{j}\omega)^2 + 2\zeta\omega_n\mathrm{j}\omega + \omega_n^2},$$

where $\zeta = 0.2$ and $\omega_n = 10$. Let all the covariances be the square exponential functions given as

$$C_{rr}(\omega,\omega') = C_{ii}(\omega,\omega') = 2\,C_{ir}(\omega,\omega') = \sigma^2 \mathrm{e}^{-\frac{(\omega-\omega')^2}{\ell^2}},$$

where we choose $\sigma = 0.002$ and $\ell = 200$. Figure 2 shows the mean G_0 and three functions drawn from a Gaussian process G, while the shaded areas represent 95% confidence region.

The reader will notice that while $\Re G$ and $\Im G$ are Gaussian processes, the magnitude $|G(\mathrm{j}\omega)|$ and $\angle G(\mathrm{j}\omega)$ are *not* Gaussian.

Remark 2: A simple example of heterogeneity would be

$$G_i(s) = G_0(s) + \sum_{j=1}^{d}\delta_{ij}\Delta_j(s)\,, \tag{14}$$

where G_j's are "generating" transfer functions and Δ_{ij}'s are random variables having the expectation $\mathbb{E}[\Delta_{ij}] = 0$ and the variance $\mathbb{E}[\Delta_{ij}^2] = \sigma_j^2$, $\sigma_j > 0$, and $j \in \mathcal{D} = 1, \ldots, d$. We assume that G_i, Δ_{ij} are proper and stable rational transfer functions, that is $G_i, \Delta_{ij} \in R\mathcal{H}_\infty$. This model of heterogeneity has a direct interpretation in the framework of Gaussian processes.

A. Network of heterogeneous LTI systems

In the case of disconnected population of agents, the input-output relationship of an agent $i \in \mathcal{P}$ is $Y_i(s) = G_i(s)V(s)$, where V is the input frequency response. Considering the fact that the disconnected population of agents can be treated as a population of diffusely coupled agents when the coupling strength c is zero, throughout this section we shall be concerned with the diffusively interconnected population.

Analogously to (6), the network model in the frequency domain is

$$Y_i(\mathrm{j}\omega) = G_i(\mathrm{j}\omega)\left(V(\mathrm{j}\omega) + c\sum_{j=1}^{N}\left(Y_j(\mathrm{j}\omega) - Y_i(\mathrm{j}\omega)\right)\right) \tag{15}$$

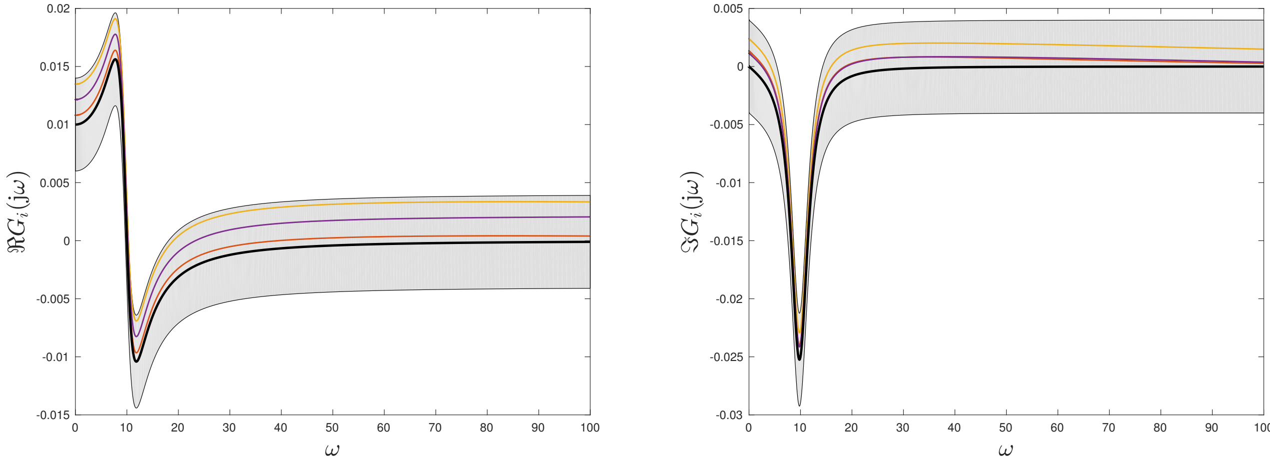
for all $i \in \mathcal{P}$.

The Gaussian mean-field relaxation introduced in the previous section leads to

$$Y_i(\mathrm{j}\omega) = G_i(\mathrm{j}\omega)V(\mathrm{j}\omega) + c\,G_0(\mathrm{j}\omega)\sum_{j=1}^{N}\left(Y_j(\mathrm{j}\omega) - Y_i(\mathrm{j}\omega)\right), \tag{16}$$

for all $i \in \mathcal{P}$.

[2]An important subclass of complex Gaussian processes, extensively employed in signal processing, is called the *circularly-symmetric complex GP's* and corresponds to the case of zero mean and zero pseudo-covariance function: $G_0 = 0$ and $C = 0$.



(a) (b)

Fig. 2. The panel shows three realizations of the complex-valued Gaussian process. The real and imaginary means are shown in bold lines.

Given the input $V(\mathrm{j}\omega)$, the average output is

$$\begin{aligned}\overline{Y}(\mathrm{j}\omega) &= \frac{1}{N}\sum_{i=1}^{N} Y_i(\mathrm{j}\omega) \\ &= \frac{1}{N}\sum_{i=1}^{N} G_i(\mathrm{j}\omega)V(\mathrm{j}\omega)\,,\end{aligned} \tag{17}$$

while the expected average input-output behavior is

$$\overline{Y}_E(\mathrm{j}\omega) = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} G_i(\mathrm{j}\omega)V(\mathrm{j}\omega)\right] = G_0(\mathrm{j}\omega)\,V(\mathrm{j}\omega)\,. \tag{18}$$

The covariance of $\overline{Y}(\mathrm{j}\omega)$ is

$$\begin{aligned}\Gamma_{\overline{Y}}(\mathrm{j}\omega,\mathrm{j}\omega') &= \operatorname{cov}\left(\overline{Y}(\mathrm{j}\omega), \overline{Y}^*(\mathrm{j}\omega')\right) \\ &= \mathbb{E}\left[\left(\overline{Y}(\mathrm{j}\omega) - \overline{Y}_E(\mathrm{j}\omega)\right)\left(\overline{Y}(\mathrm{j}\omega') - \overline{Y}_E(\mathrm{j}\omega')\right)^*\right] \\ &= \frac{\Gamma(\mathrm{j}\omega,\mathrm{j}\omega')}{N}\,V(\mathrm{j}\omega)V^*(\mathrm{j}\omega')\,.\end{aligned}$$

The pseudo-covariance of $\overline{Y}(\mathrm{j}\omega)$ is

$$\begin{aligned}C_{\overline{Y}}(\mathrm{j}\omega,\mathrm{j}\omega') &= \operatorname{cov}\left(\overline{Y}(\mathrm{j}\omega), \overline{Y}(\mathrm{j}\omega')\right) \\ &= \frac{C(\mathrm{j}\omega,\mathrm{j}\omega')}{N}\,V(\mathrm{j}\omega)V(\mathrm{j}\omega')\,.\end{aligned}$$

This means that the mean-field model is the complex-valued GP given by

$$\overline{Y}(\mathrm{j}\omega) \sim \mathcal{CGP}\left(G_0(\mathrm{j}\omega)V(\mathrm{j}\omega), \Gamma_{\overline{Y}}(\mathrm{j}\omega,\mathrm{j}\omega'), C_{\overline{Y}}(\mathrm{j}\omega,\mathrm{j}\omega')\right)\,. \tag{19}$$

Remark 3: In the case of heterogeneity (14), the frequency (complex) average response of the ensemble $\mathcal{P}$ is given by

$$\begin{aligned}\overline{Y}(\mathrm{j}\omega) &= \frac{1}{N}\sum_{i=1}^{N} G_i(\mathrm{j}\omega)V(\mathrm{j}\omega) \\ &= G_0(\mathrm{j}\omega)V(\mathrm{j}\omega) + \sum_{j=1}^{d} \bar{\delta}_j \Delta_j(\mathrm{j}\omega)V(\mathrm{j}\omega)\,,\end{aligned} \tag{20}$$

while the expected average response is

$$\overline{Y}_E(\mathrm{j}\omega) = G_0(\mathrm{j}\omega)V(\mathrm{j}\omega)\,. \tag{21}$$

The disagreement between the averaged behavior and the expected averaged behavior, that is, $\overline{Y} - \overline{Y}_E$, goes to zero as N tends to infinity.

B. The population mean-field model

Since the variance of $\overline{Y}$ is proportional to $\frac{1}{N}$, $\overline{Y}$ may be written as

$$\overline{Y}(\mathrm{j}\omega) = \overline{Y}_E(\mathrm{j}\omega) + O(\tfrac{1}{\sqrt{N}}),$$

where $O(\frac{1}{\sqrt{N}})$ denotes a small fluctuation term of order $\frac{1}{\sqrt{N}}$. When the fluctuation term is disregarded, the mean-field quantity $\overline{Y}$ may be regarded as nonrandom quantity $\overline{Y}_E$, and the model (16) may be rewritten as

$$Y_i(\mathrm{j}\omega) = G_i(\mathrm{j}\omega)V(\mathrm{j}\omega) + c\,G_0(\mathrm{j}\omega)\left(N\overline{Y}_E(\mathrm{j}\omega) - NY_i(\mathrm{j}\omega)\right)\,. \tag{22}$$

More explicitly, the model (22) can be understood as

$$Y(\mathrm{j}\omega) = \frac{G(\mathrm{j}\omega) + c\,N\,G_0^2(\mathrm{j}\omega)}{1 + c\,N\,G_0(\mathrm{j}\omega)}V(\mathrm{j}\omega), \tag{23}$$

with G_i's and Y_i's being realizations of random processes G and Y, respectively.

The population variance in the frequency domain: Consider the population generating model (23). Then the **population covariance** Γ_Y is the complex-valued function given as

$$\begin{aligned}\Gamma_Y(\mathrm{j}\omega,\mathrm{j}\omega') &= \operatorname{cov}\left(Y(\mathrm{j}\omega), Y^*(\mathrm{j}\omega')\right) \\ &= \mathbb{E}\left[\left(Y(\mathrm{j}\omega) - \overline{Y}_E(\mathrm{j}\omega)\right)\left(Y(\mathrm{j}\omega') - \overline{Y}_E(\mathrm{j}\omega')\right)^*\right] \\ &= \mathbb{E}\Bigg[\frac{V(\mathrm{j}\omega)}{1 + c\,N\,G_0(\mathrm{j}\omega)}\left(G(\mathrm{j}\omega) - G_0(\mathrm{j}\omega)\right) \\ &\qquad \left(\frac{V(\mathrm{j}\omega')}{1 + c\,N\,G_0(\mathrm{j}\omega')}\right)^*\left(G(\mathrm{j}\omega') - G_0(\mathrm{j}\omega')\right)^*\Bigg] \\ &= \frac{V(\mathrm{j}\omega)}{1 + c\,N\,G_0(\mathrm{j}\omega)}\left(\frac{V(\mathrm{j}\omega')}{1 + c\,N\,G_0(\mathrm{j}\omega')}\right)^*\Gamma(\mathrm{j}\omega,\mathrm{j}\omega').\end{aligned}$$

In the similar fashion, the population pseudo-covariance C_Y is

$$\begin{aligned}
C_Y(\mathrm{j}\omega,\mathrm{j}\omega') &= \operatorname{cov}\left(Y(\mathrm{j}\omega),Y(\mathrm{j}\omega')\right)\\
&= \mathbb{E}\left[\left(Y(\mathrm{j}\omega)-\overline{Y}_E(\mathrm{j}\omega)\right)\left(Y(\mathrm{j}\omega')-\overline{Y}_E(\mathrm{j}\omega')\right)\right]\\
&= \mathbb{E}\Bigg[\frac{V(\mathrm{j}\omega)}{1+c\,N\,G_0(\mathrm{j}\omega)}\left(G(\mathrm{j}\omega)-G_0(\mathrm{j}\omega)\right)\\
&\qquad\left(\frac{V(\mathrm{j}\omega')}{1+c\,N\,G_0(\mathrm{j}\omega')}\right)\left(G(\mathrm{j}\omega')-G_0(\mathrm{j}\omega')\right)\Bigg]\\
&= \frac{V(\mathrm{j}\omega)}{1+c\,N\,G_0(\mathrm{j}\omega)}\frac{V(\mathrm{j}\omega')}{1+c\,N\,G_0(\mathrm{j}\omega')}C(\mathrm{j}\omega,\mathrm{j}\omega').
\end{aligned}$$

The **population mean-field model is the complex-valued GP** given by

$$Y(\mathrm{j}\omega)\sim\mathcal{CGP}\left(G_0(\mathrm{j}\omega)V(\mathrm{j}\omega),\Gamma_Y(\mathrm{j}\omega,\mathrm{j}\omega'),C_Y(\mathrm{j}\omega,\mathrm{j}\omega')\right).$$

We see that the strong coupling and a large number of agents reduce the heterogeneity in the network. Numerical experiment show that the same conclusion also holds for the original network (15) (see Section IV).

Remark 4: In the heterogeneous networks, the strong coupling leads to a practical synchronization [5], [6], [15]. The same effects can be associated with the network size, which implies that increasing the population size enhances the robustness of synchronization/consensus against heterogeneity.

IV. Numerical example

As an example, consider the network model (15), where each G_i is drawn from the complex-valued Gaussian process with mean

$$G_0(\mathrm{j}\omega)=\frac{1}{\mathrm{j}\omega+10}, \tag{24}$$

and covariances given by

$$\begin{aligned}
C_{rr}(\omega,\omega') &= C_{ii}(\omega,\omega')\\
&= 0.015^2\,\mathrm{e}^{-\frac{(\omega-\omega')^2}{160^2}}\,\mathrm{e}^{-\frac{(\omega-10)^2}{25}}\,\mathrm{e}^{-\frac{(\omega'-10)^2}{25}},
\end{aligned} \tag{25}$$

and

$$C_{ri}(\omega,\omega')=C_{ir}(\omega,\omega')=0.03^2\,\mathrm{e}^{-\frac{(\omega-\omega')^2}{160^2}}. \tag{26}$$

Observe that the Gaussian process that models heterogeneity in the network is nonstationary. That is, the covariance and pseudocovariance are not only functions of $(\omega-\omega')$ but also depend on ω and ω'. The particular choice of covariances given in (25) and (26) insures that the heterogeneity in the network is centered around the frequency $\omega=10$.

Figure 3 shows three transfer functions drawn at random from a GP prior and their Bode plots. The well-known effects of strong coupling lead to a reduction of population variance. We also observe that increasing N (with c being positive and fixed) reduces population variance as illustrated by (e) and (f). While the outputs Y_i's of the Gaussian relaxation model (15) obey a Gaussian distribution, the outputs of the original model (16) *do not*. Nevertheless, the probability distribution behind the original model seems to be very close to the Gaussian approximation.

V. Conclusions

Unlike for networks with identical nodes, little is known about heterogeneous multi-agent systems. Their behavior frequently does not reduce to a simple phenomena such as consensus and synchronization. The mean-field model seems to be a first step in understanding how heterogeneous networks behave internally and how they interact with the environment. The paper addresses this problem within the popular framework of Gaussian processes [11]. While the results are very preliminary, we stress the importance of an input-output approach and the fundamental limitation that Gaussian properties are not preserved under diffusive interconnections. Extensions to the nonlinear case and application to neural circuits will be considered in the future work.

Acknowledgment

The research leading to these results has received funding from the European Research Council under the Advanced ERC Grant Agreement Switchlet n.670645.

References

[1] S. I. Amarimber, "Characteristics of Random Nets of Analog Neuron-Like Elements," IEEE Transactions on Systems, Man, and Cybernetics, vol. SMC-2, no. 5, pp. 643–657, Nov. 1972.
[2] HR Wilson and JD Cowan, "Excitatory and inhibitory interactions in localized populations of model neurons," Biophysical Journal, vol 12, Issue 1, 1972, pp. 1–24
[3] M. Huang, R.P. Malhamé, and P.E. Caines, "Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle," Communications in Information & Systems 6 (3), 2006, pp. 221–252
[4] M. Nourian and P. E. Caines, "ϵ–Nash mean field game theory for nonlinear stochastic dynamical systems with major and minor agents," SIAM Journal on Control and Optimization, 51 (4), 2013, pp. 3302–3331
[5] D.J. Hill and J. Zhao, "Global synchronization of complex dynamical networks with non-identical nodes," in Proc. 47th CDC, 2008, pp. 817–822.
[6] J. Zhao, D.J. Hill, and T. Liu, "Synchronization of Dynamical Networks With Nonidentical Nodes: Criteria and Control," IEEE Transactions on Circuits and Systems I: Regular Papers, Volume: 58, Issue: 3, March 2011
[7] A. Jadbabaie, N. Motee, and M. Barahona, "Robust Synchronization of Uncertain Linear Multi-Agent Systems," Proceedings of American Control Conference, 2004, pp. 1511–1523
[8] H.L. Trentelman, K. Takaba, and N. Monshizadeh, "Robust Synchronization of Uncertain Linear Multi-Agent Systems," IEEE Transactions on Automatic Control, Volume: 58, Issue: 6, 2013, pp. 1511–1523
[9] J. Kim, J. Yang, H. Shim, J.S. Kim, and J.H. Seo, "Robustness of Synchronization of Heterogeneous Agents by Strong Coupling and a Large Number of Agents," IEEE Transactions on Automatic Control, Volume: 61, Issue: 10, Oct. 2016, pp. 3096–3102
[10] R. Sepulchre, G. Drion, A. Franci, "Excitable Behaviors," In: Tempo R., Yurkovich S., Misra P. (eds) *Emerging Applications of Control and Systems Theory*, Lecture Notes in Control and Information Sciences – Proceedings, 2018, Springer, Cham
[11] C.E. Rasmussen and C.K.I. Williams, *Gaussian processes for machine learning,* The MIT Press, 2006.
[12] D.J.C. MacKay, *Information Theory, Inference, and Learning Algorithms,* Cambridge University Press, 2003, p. 535–548
[13] M. Seeger, "Gaussian Processes for Machine Learning," International Journal of Neural Systems. 14 (2), 2004, pp. 69–104
[14] F. Tobar and R. Turner, "Modelling of complex signals using Gaussian processes," In: Acoustics, Speech and Signal Processing (ICASSP), 2015 IEEE International Conference on, 2015, 2209–2213.
[15] J. M. Montenbruck, M. Bürger, and F. Allgöwer, "Practical synchronization with diffusive couplings," Automatica, vol. 53 pp. 235–243, 2015

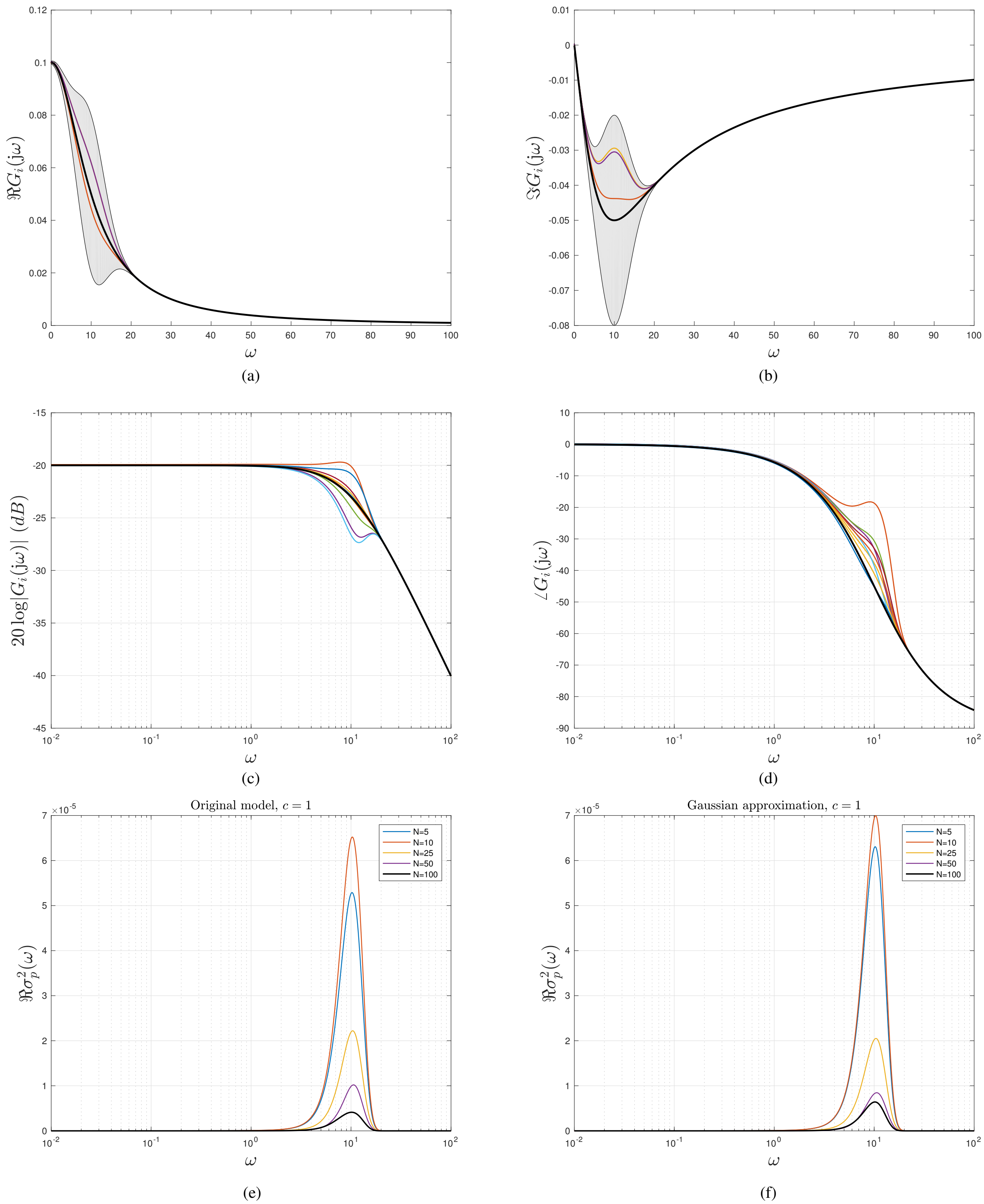


Fig. 3. The panels (a) and (b) show real and imaginary parts of three realizations (transfer functions) of the complex-valued Gaussian process. The real and imaginary mean are shown in black bold lines. Panels (c) and (d) show Bode plots for ten different realizations of the GP transfer function. Note that while real and imaginary parts are normally distributed, the magnitude and phase are *not*. Panels (e) and (f) show real parts of population variance for the original model and its Gaussian relaxation. Keeping the coupling strength c fixed and increasing the number of agents in the network reduces the population variance for both models.