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Joint continuous and impulsive control of Markov chains

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Abstract— **This work considers the continuous and impulse control of the finite state Markov chains in continuous time. In general continuous control governs the transition rates between the states of Markov chain (MC), so the instants and directions of the state changes are random. Meanwhile sometimes there is an urgent necessity to realise the transition which leads to immediate change of the state. Since such transitions need different efforts and produce different effects on the function of the MC itself, one can consider this situation as an impulse control, and if at the same time there is a possibility of graduate control we are coming to the problem of joint continuous and impulse control. In the article we develop the martingale representation of the MC governed by joint continuous and impulse control and give an approach to the solution of the optimal control problem on the basis of the dynamic programming equation. This equation has a form of quasivariational inequality which in the case of finite state MC can be reduced to the solution of the system of ordinary differential equation with one switching line, which may be easily determined with the aid of numerical procedure.**

I. INTRODUCTION

Impulse control as an action which produce the immediate (really very fast) change of the dynamic system state was in the focus of researches from 70-ths. In the series of pioneering works A. Bensoussan and J.L. Lions created well developed theory of impulse control for stochastic systems, which generalised the dynamic programming approach in form of so-called quasivariational inequality [1]. These ideas lead to the new class of so-called discrete-continuous systems which behave continuously (deterministically or randomly) between the jumps which could occur at instants of the impulse control application. In stochastic analysis these ideas were followed by many researches which extend them to the class of piecewise deterministic Markov models (PDMM), where behaviour between random and perhaps controllable jumps obeys the continuous dynamics [2], [3], [4]. Further research in the area of impulse control show its nontrivial character and the necessity of development the new class of controllable differential equations, that are the differential equations with measures which join the impulsive and continous actions in the universal manner and gave rise to well developed theory of the optimal control existence in deterministic and stochastic cases [5], [6] and to the optimality conditions in the maximum principle form [7].

Recently one can observe the growing interest to the control of MC and particularly to PDMM since these models are more appropriate to control practics and relatively much simplier for modelling and simulation [8], [9], [10]. Moreover MDPs appear useful in various applied areas such as large dams management [11] where impulse-similar action corresponding to controlled water release can be applied, in natural resources management [14], [13], in the internet congestion control [15], in the graduate control of water resources [12] with the aid of the water prices manipulation [16], and even in the natural gas consumption management [19]. At the same time the usage of the impulse controls as far as continuous ones becomes urgent in various areas such as the Internet, where the buffer release to avoid the congestion is the typical impulse control [17], and in water management where the controlled water release plays the same role and also is the impulse control action [18].

Another reason of growing interest to MDP is the relative simplicity of modelling for continuous dynamic systems by MCs with the finite number of states [23], and since the accuracy of approximation depends on the number of states, nowadays the increasing capacity of accessible computers and distributed calculation permits to achieve the necessary accuracy of the optimal control problem solution without problems. Moreover the control theory for continuous and discrete time MC optimization is well developed and the appropriate numerical procedures may be easily realised [24].

The structure of the article is as follows: in the next Section II we describe a model, Section III gives the derivation of the optimality condition in the form of quasivariational inequality, in Section IV we describe the numerical approach to the determining of the optimal control and in the Section V the numerical example demonstrate the solution of some simple problem related to the dams management.

II. THE MODEL DESCRIPTION AND THE PROBLEM STATEMENT

Here we use the standard problem statement [24], Chapter 12, and previous results obtained in [15]. Suppose that the state of a MC is described by a vector $X_t \in \mathcal{S} = \{e_1, .., e_N\}$ where N is the number of states and e_k is the vector with all null entries, except of k-th which is 1. Vector-valued function $X_t, t \in [0,T]$ satisfies the stochastic differential equation

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[24]

$$X_t = X_0 + \int_0^t A(s,u(s))X_s ds + W_t, \quad \text{where} \quad W_t \text{ is square integrable martingal} \tag{1}$$

with control dependent generator $A(t,u)$ and some $\mathcal{F}_t^X$ - predictable controls $u(t) \in U(X_t)$, where $U(X)$ is the set of admissible control values admissible in the state X. Impulse control is the set of pairs $\mathcal{I} = \{(V_i, \tau_i), i = 1, 2, ..., N\}$ where $\tau_i < \tau_{i+1} \le T$, $V_i \in \mathcal{V}(X_{\tau_i})$, where $\mathcal{V}(X)$ is the set of impulse controls admissible in the state X, and $N \le \infty$. The application of the impulse control at time τ_i produces an immediate random change of the state for X such that

$$\Delta X_{\tau_i} = \psi(\tau_i, V_i, X_{\tau_i -}) = \bar{A}(\tau_i, V_i)X_{\tau_i -} + \Delta W_{\tau_i}, \tag{2}$$

where

$$E[\Delta W_{\tau_i} | \mathcal{F}_{\tau_i -}^X] = E[\Delta W_{\tau_i} | X_{\tau_i -}] = 0, \quad \text{and}$$

$$\langle \Delta W_{\tau_i} \rangle = diag(\bar{A}(\tau_i, V_i)X_{\tau_i -}) - \bar{A}(\tau_i, V_i)X_{\tau_i -}\bar{A}^T(\tau_i, V_i).$$

Assumption 1: Here, $\psi(\tau_i, V_i, X_{\tau_i -})$ is the random operator such that

$$X_{\tau_i} = \Delta X_{\tau_{i-}} + X_{\tau_{i-}} \in \mathcal{S}, \quad \text{for any} \quad X_{\tau_{i-}} \in \mathcal{S}, V_i \in \mathcal{V}(X_{\tau_{i-}}), \tau_i \in [0,T]. \tag{3}$$

The function $\bar{A}(t,V)$ assumed to be continuous in t, V. Therefore, the equation for controled MC has a form

$$X_t = X_0 + \int_0^t A(s,u(s))X_s ds + \sum_{\tau_i \le t} \bar{A}(\tau_i, V_i)X_{\tau_i -} + W_t^{'}, \tag{4}$$

where $W_t^{'}$ is square integrable martingale with quadatic variation [24]

$$\langle W_t \rangle = \sum_{\tau_i \le t} \langle \Delta W_{\tau_i} \rangle - \int_0^t (diag\, X_{s-})A^T(s,u(s))ds - \int_0^t A(s,u(s))(diag\, X_{s-})ds + diag \int_0^t A(s,u(s))X_{s-}ds. \tag{5}$$

The aim of the control is to minimize the cost function (6) over all $\mathcal{F}_t^X$ - predictable controls $u(t) \in U$, and impulse controls $\mathcal{I}$. In (6) $I\{\cdot\}$ is the indicator function that is $I\{A\} = 1$ if the event A occurs and equals zero otherwise.

Assumption 2: Vector-valued functions g_0, ψ_0 are assumed to be continuous in t, u respectively, while vector-valued function $\psi_0(t,V)$ is continuous in t and V and satisfies the inequality $\psi_0^l(t,V) \ge C > 0$ for all $l = 1,..,N$. The sets U and $\mathcal{V}$ are compact.

III. Dynamic Programming Equation in the Form of Quazivariational Inequality

The value function of the problem (1)-(6) has a form (6), where function $\phi(t) \in R^N$ is the solution of quazivariational inequality (7)

with terminal condition $\phi(T) = \phi_0$.

Function $\phi(t)$ must be determined from equation (7) that is much easier than to solve the general quazivariational inequalities since the continuous part of (7) is the system of ordinary differential equations which must be solved from the end point $\phi(T) = \phi_0$ up to the switching line

$$0 = \langle \phi(t), X \rangle - \min_{V \in \mathcal{V}(X)} \left\{ \langle \phi(t), [I + \bar{A}(t,V)]X \rangle + \langle \psi_0(t,V), X \rangle \right\}$$

or

$$G(t,X) = \min_{V \in \mathcal{V}(X)} \langle \bar{A}^*(t,V)\phi(t) + \psi_0(t,V), X \rangle = 0. \tag{8}$$

For any particular X one can determine the switching point $t^*(X)$, where expression $G(t,X)$ changes the sign from negative to positive. The corresponding value of the impulse control equals

$$V^* = \underset{V \in \mathcal{V}(X)}{argmin} \langle \bar{A}^*(t^*(X), V)\phi(t^*(X)) + \psi_0(t^*(X), V), X \rangle. \tag{9}$$

Then it remains to find the solution of Quasivariational Inequality (7).

A. Solution of Quasivariational Inequality

In the optimal control problems with impulse controls the existence and uniqueness of the Quasivariational inequality, which takes the place of the Dynamic Programming Equation (DPE) is the principal moment [1]. In the case of the finite state MC equation (7) breaks up (into) the finite number of difference-diffrential equations, corresponding to the number of states. For each $l = 1,..,N$ define a switching function $G^l(\phi,t) : (\phi,t) \in R^N \times [0,T] \to R^1$ such that

$$G^l(\phi,t) = \min_{V \in \mathcal{V}(e_l)} \left\{ \sum_{k=1}^N \phi_k \bar{A}_{l,k}(t,V) + \psi_0^l(t,V) \right\}. \tag{10}$$

Due to continuity of $\bar{A}$ and ψ_0 in t the switching function is continuous in (ϕ,t) and determines for each l the switching line

$$Sw^l = \{(\phi,t,l) : G^l(\phi,t) = 0\}. \tag{11}$$

So if

$$\phi(t) = \{\phi_1(t), ..., \phi_N(t)\},$$

$$J_0[u(\cdot),\mathcal{I}] = E\left\{\langle\phi_0, X_T\rangle + \int\limits_0^T \langle g_0(s,u(s)), X_s\rangle ds + \sum_{\tau_i\le T}\langle\psi_0((\tau_i,V_i), X_{\tau_i-}\rangle\right\}$$
$$= E\left\{\sum_{k=1}^N \phi_0^k I\{X_T = e_k\} + \int\limits_0^T g_0^k(s,u(s))I\{X_s = e_k\}ds + \sum_{\tau_i\le T}\langle\psi_0^k((\tau_i,V_i)I\{X_{\tau_i-} = e_k\}\right\} \tag{6}$$

$$\phi(t,X) = \inf_{u(\cdot),\mathcal{I}} E\left\{\langle\phi_0, X_T\rangle + \int\limits_t^T \langle g_0(s,u(s)), X_s\rangle ds + \sum_{t<\tau_i\le T}\langle\psi_0(\tau_i,V_i), X_{\tau_i-}\rangle\,|X_{t-} = X\right\} = \langle\phi(t), X\rangle, \tag{6}$$

$$\min\left\{\left\langle\frac{d\phi(t)}{dt}, X\right\rangle + \min_{u\in U}\langle A^*(t,u)\phi(t) + g_0(t,u), X\rangle, \langle\phi(t), X\rangle - \min_{V\in\mathcal{V}}\langle[I + \bar{A}^*(t,V)]\phi(t) + \psi_0(t,V), X\rangle\right\} = 0 \tag{7}$$

then the entries $\phi_l(t), \quad l = 1,..,N$ satisfy the system of the following equations

$$\frac{d\phi_l(t)}{dt} + \min_{u\in U}\left\{\sum_{k=1}^N \phi_k(t)A_{l,k}(t,u) + g_0^l(t,u)\right\} = 0,$$
if
$$G^l(\phi(t),t) > 0. \tag{12}$$

By solving the system (12) in inverse time with terminal conditions $\phi(T) = \phi_0$ for each of $l = 1,..,N$ one may achive the switching line at time $\{t^l : G^l(\phi(t^l),t^l) = 0\}$. Due to the condition (8) function ϕ remains to be continuous. Then the following jump condition for ϕ apply

$$0 = \min_{V\in\mathcal{V}(\mathcal{X})}\left\{\langle\bar{A}^*(t,V)\phi(t), X\rangle + \langle\psi_0(t,V), X)\rangle\right\} \tag{13}$$

where the minimizing impulse action

$$V = \underset{V\in\mathcal{V}(\mathcal{X})}{argmin}\left\{\langle\bar{A}^*(t,V)\phi(t), X\rangle + \langle\psi_0(t,V), X)\rangle\right\} \tag{14}$$

exists and further solution of (12) continiues backward until next hitting of the switching line. Therefore, at jump point T_i with impulse control V_i defined by relations (13),(14), and function $\phi(t)$ remains to be continuous. The procedure described above defines the solution of quasivariational inequality (7).

B. Verifcation theorem for joint impulse and graduate optimal control

Theorem 1: a) Let the function $\phi(t) \in R^n$ be defined by procedure described in Subsection III-A by equation

$$\left\langle\frac{d\phi(t)}{dt}, X\right\rangle + \min_{u\in U(X)}\langle A^*(t,u)\phi(t) + g_0(t,u), X\rangle = 0, \tag{15}$$

in each of intervals $[T_i, T_{i+1})$ with terminal condition $\phi(T) = \phi_0$.

b) Let the controls $u^*(t)$ on $[T_i, T_{i+1})$ are choosen Markovian as $u(t) = u(t, X_t)$ and provide the minimum in the r.h.s. of (15).

c) Let the switching times $T_i^* = t^*(X)$ are choosen according relations

$$\min_{V\in\mathcal{V}(X)}\langle[\bar{A}^*(t^*(X),V)]\phi(t^*(X))$$
$$+\psi_0(t^*(X),V), X\rangle = 0, \tag{16}$$

and the impulse action V_i^* are choosen according relation (14). Then the joint graduate and impulse controls $\{u^*(\cdot), T_i^*, V_i^*\}$ are optimal.

Remark 1: We do not give a complete proof of the theorem, since it is rather cumbersome and repeats in general terms similar results, see for example in [10], [17].

IV. Example of the Impulse Controls

A. Deterministic Impulse Control

In this paper, we consider a model similar to [20] with an extension admitting impulse control. Under impulse control, we mean an emergency level reduction in order to prevent overflow of the dam above a critical level. Along with the impulse control, we also use the usual "continuous" control, in which the level change occurs with the intensity of the transition determined by the finite control. To illustrate the approach, we use a model with $N = 4$ states:

- 1 "below normal",
- 2 "normal",
- 3- "critical",
- 4 - "catastrophic overflow".

Acceptable transitions are: $1 \to 2$, the intensity of which $\lambda(t)$ is determined by the seasonal external precipitation; $2 \to 3$, or $2 \to 1$, the intensity of which is determined as above by precipitation $\lambda(t)$, evaporation loss $\mu(t)$ and the intensity of controlled "continuous" release $u(t) \in [0, \mathbf{u}]$; $3 \to 2$, which corresponds to the impulse control and also $3 \to 2$ due to "continous" transition caused by evaporation loss $\mu(t)$ and the intensity of controlled "continuous" release $u(t) \in [0, \mathbf{u}]$; and finally $3 \to 4$, the intensity of which is determined, as above, by external precipitation $\lambda(t)$. State 4 is "absorbing" and transitions from it does not happen. The purpose of control is to maintain state 2, while keeping the probability of state 4 at an acceptable level. The corresponding vector of state $X \in R^4$ and the matrix of the transitions intensities $A(t,u)$ has a form (for the sake of simplicity we omit dependence on t)

$$A(t,u) = \begin{pmatrix} -\lambda & \mu+u_2 & 0 & 0 \\ \lambda & -(\lambda+\mu+u_2) & \mu+u_3 & 0 \\ 0 & \lambda & -(\lambda+\mu+u_3) & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix}.$$

The impulse control here has just one value and the transition matrix corresponding to the impulse action applied at τ_i has a form

$$E + \bar{A}(\tau_i, V_i) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Assume that payment for the impulse action equals

$$\psi_0(t,V) = (0,0,1,0)^*, \tag{17}$$

then the condition of the impulse switch at point T_i as a function of $\phi(t)$, according to (8) is given by relation

$$\langle \phi(T_i-), X\rangle = \langle \phi(T_i), E + \bar{A}(T_i,V_i)X\rangle + \langle \psi_0(T_i,V_i), X\rangle,$$

and the condition for ϕ at T_i by relation (**??**) gives

$$\begin{aligned} \phi_1(T_i-) &= \phi_1(T_i), \\ \phi_2(T_i-) &= \phi_2(T_i), \\ \phi_3(T_i-) &= \phi_2(T_i) + 1, \\ \phi_4(T_i-) &= \phi_4(T_i). \end{aligned} \tag{18}$$

The terminal conditions are assumed as

$$\phi_0 = \phi(T) = (4.0, 1.0, 4.0, 15.0)^*. \tag{19}$$

The running cost function, taking into account the deviation from "normal" state 2, and the payment for controllable release, and the penalty for "catastrophic overflow" equals

$$g_0(t,u) = \begin{pmatrix} 1 \\ 0 + u_2^2 \\ 1 + u_3^2 \\ 2 \end{pmatrix}. \tag{20}$$

Here

$$u_2 \in [0, u_2^{max}], u_3 \in [0, u_3^{max}]$$

are controllable releases from state 2 and 3, respectively.

Therefore, the continuous part for system of equations (7) has a form

$$\begin{aligned} \frac{d\phi_1}{dt} &= [-\lambda\phi_1 + (\mu+u_2)\phi_2] + 1, \\ \frac{d\phi_2}{dt} &= \min_{u_2 \in U}[\lambda\phi_1 - (\lambda+\mu+u_2)\phi_2 + (\mu+u_3)\phi_3 + u_2^2], \\ \frac{d\phi_3}{dt} &= \min_{u_3 \in U}[\lambda\phi_2 - (\lambda+\mu+u_3)\phi_3 + u_3^2] + 1, \\ \frac{d\phi_4}{dt} &= \lambda\phi_3 + 2. \end{aligned} \tag{21}$$

So the system (21) must be solved from terminal point T, and even if the conditions (18) apply its solution remains to be continuous and determines the possible instants of the impulse action application T_i, defined by relation (18) for

$$\phi_3(T_i) \geq \phi_2(T_i) + 1. \tag{22}$$

B. Implementation of the impulse control

We should emphasize the essential difference between cases with a continuous and discrete set of states, as for the Markov chain. In the first case, impulse control is usually applied at the initial time, if the system is higher (relatively speaking) of the switching surface. Then if, as a result of further movement, the system again falls on the switching surface, the impulse control becomes singular and tries to keep the system below the switching surface by realizing so-called repulsive action [28]. In this case, the pulses are not separated, but so-called singular control apply. In the case of the Markov chain, if, as a result of impulse control, the condition

$$\mathcal{V}(X_{\tau_i}) = \emptyset$$

is fulfilled, the control pulses are separated and the next impulse action becomes permissible after a finite time, when the jumping trajectory commes to the set where

$$\mathcal{V}(X_{t-}) \neq \emptyset.$$

C. Numerical example

We consider the problem with deterministic impulse control, described in Subsection IV-A and take the following parameters,

$$T = 3, \quad u_2^{max} = 0.2, \quad u_3^{max} = 0.02,$$

the terminal conditions (19), the running cost (20), and the impulse cost (17). Solution of the system (21) in backward time is shown on Fig. 1.

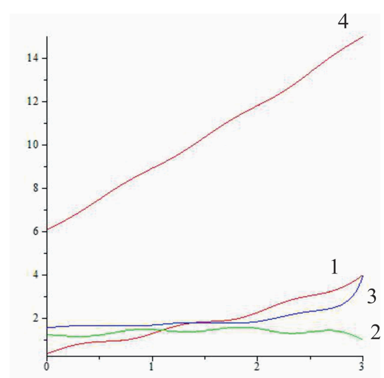

Fig. 1. Solution of the system (21) in backward time with terminal conditions (19). 1 - $\phi_1(t)$, 2 - $\phi_2(t)$, 3 - $\phi_3(t)$, 4 - $\phi_4(t)$.

Switch line (22) calculated as $SW(t) = \phi_3(t) - \phi_2(t) - 1$ is given on Fig 2. So if the system is in the state 3 and at the same time $SW(t) \geq 0$ the impulse action must be applied. Therefore, the application of impulse action is possible on the terminal interval $\approx [2.7, 3.0]$ (see Fig. 2). At the same time on the whole interval $[0, T]$ controls $u_2(t), u_3(t)$ determined from system (21) apply.

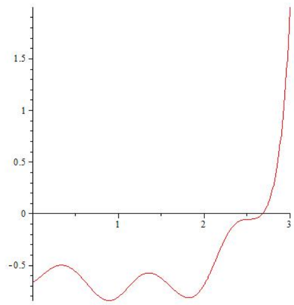

Fig. 2. Switch line (22) calculated as $\phi_3(t) - \phi_2(t) - 1$.

V. Conclusions

This work presents the attempt to develop an approach to the numerical solution of a stochastic problem with impulsive and "continuous" controls. Model of MC makes the realization of the numerical solution much easier than in the case of continuous states. Further research will be aimed to applied problems arising in the distribution of natural resources (water, gas) [19], [20], and the data transmission of signals through communication channels with fluctuating characteristics [21], [22].

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