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Constrained optimal end-effector positioning using pseudo-inversion and B-splines: numerical simulation on the one link flexible manipulator linearized model

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Abstract— **This paper investigates the potential possibility offered by the recently proposed pseudo inversion approach as a novel solution to the optimal output transition problem in the end-effector positioning task under saturation constraints on the control effort. Exact positioning and high precision tracking of the end-effector along a given path are key requirements. Due to flexibility and non minimum phase nature of the system, achieving high level performances is a challenging problem. To this purpose a two degrees of freedom control scheme is proposed: a feedforward action, denoted by $r(t)$, is applied to the stable closed loop system $\Sigma_{f,c}$, given by the feedback connection of the flexible arm with a stabilizing controller. The recently proposed pseudo inversion approach is adopted to compute $r(t)$ yielding an actual end-effector transient trajectory which is the best approximation of the desired transient one. The optimal $r(t)$ is assumed to be given by a B-spline function, and, to take into account saturating actuators, the actual control input $u(t)$ forcing the flexible arm is optimally approximated by a B-spline $\hat{u}(t)$, whose control points $\hat{u}(t)$ are chosen in such a way to satisfy the saturation constraints on $u(t)$. If $\hat{u}(t)$ is a sufficiently accurate approximation of $u(t)$, the exact fulfillment of saturation constraints by $\hat{u}(t)$ are transferred to $u(t)$. The main advantage of the approach is its generality: the method can be directly applied to the linear dynamic model of any multilink flexible arm.**

I. INTRODUCTION

To overcome the drawbacks of today 's industrial robots, such as high environmental impact, huge weight, high energy consumption, poor dexterity and low speed operations, several researchers have addressed the problem of the control of lightweight flexible manipulators in these last years.

The most difficult control problems posed by flexible robots are the end effector regulation around a final position and the end-effector trajectory tracking.
The aim is to drive the end point along a feasible path without any oscillations at the tip and to accomplish a fast and accurate vibration-free positioning.
It is always convenient to specify the intermediate trajectory so to guarantee smooth transition between initial and final configurations, and to impose an explicit time constraint on the overall motion.
The smoothness is necessary to avoid big deflections in robot components and, consequently, high peak torques which in turn could cause actuators saturation.

Feasible solutions for the above performance requirements have been investigated by most researchers in the ambit of the feedforward control. The input shaping based techniques proposed in [1]-[6] aim at reducing undesired vibrations in the links of a flexible robot to obtain an accurate end-point position. Input shaping is implemented by convolving a sequence of impulses, an input shaper, with a desired system command to produce a shaped input that is then used to drive the system. The goal of input shaping is to determine the amplitudes and timing of the impulses to eliminate or reduce residual vibration.

These parameters are obtained from the natural frequencies and damping ratios of the system. Using this approach, a response without vibration can be achieved, however, with a time delay approximately equal to the length of the impulse sequence and a decrease of the transient response speed.

Also stable inversion based approaches have shown to be very effective for obtaining accurate output tracking for flexible manipulators. In [7] Fourier transform is used to obtain a stable but non causal control input. To reduce the extensive computation, more efficient time domain stable inversion methods [8]-[13] have been proposed by several authors where non casual solutions are found by solving nontrivial ordinary differential equations (ODEs) through finite differences discretization techniques or by means of iterative and/or optimization methods.

Output redefinition techniques [14]-[16] have been proposed for single link flexible manipulators where the output is planned in such a way as to cancel the effect of the unstable systems zeros and to obtain causal control laws.

This paper situates in the recently proposed pseudo-inversion approach with preview [17]-[21]: given a desired trajectory $y_d(t)$ for the output of a linear stable closed loop system find a corresponding input $r(t)$ yielding an actual output $y(t)$ which is the optimal approximation of $y_d(t)$. Optimality is here intended as the achievement of an actual output which is the best approximation (in the least square sense) of a pre-specified output transient trajectory between two given set-points.

The advantage of using B-splines derives from being universal approximators belonging to the convex hull defined by the relative control points, and from their natural continuity at control points [22].

The actual external reference $r(t)$ driving the closed-loop

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system $\Sigma_{f,c}$ is assumed to be given by a B-spline, a second B-spline is used to model an optimal approximation $\hat{u}(t)$ of the actual control effort $u(t)$ produced by $r(t)$. The control points of $\hat{u}(t)$ are chosen in such a way to respect the saturation constraints on $u(t)$. It follows that $u(t)$ satisfies such constraints in a least square sense. Nevertheless, by the intrinsic continuity property of B-splines, the number of control points can be arbitrarily increased in the definition of $\hat{u}(t)$, thus indefinitely increasing the accuracy of the approximation of $u(t)$. In this way, the exact fulfillment of saturation constraints by $\hat{u}(t)$ can be transferred to $u(t)$.

The main advantage of the approach is its generality: the method can be directly applied to the linear dynamic model of any multilink flexible arm and also works in presence of saturation constraints on the control effort.

The paper is organized in the following way. Some mathematical preliminaries are briefly recalled in Section 2, the control problem is formulated in Section 3 and its solution is reported in Section 4. The numerical simulation, reported in Section 5, has been performed on the linearized siso model given in [15]. The concluding remarks of Section 6 end the paper.

II. PRELIMINARIES

A. B-spline functions [22]

B-splines are piecewise polynomial functions derived from a slight adjustment of Bezier's curves in order to obtain polynomial curves that automatically tie together smoothly.

Let $(c_i)_{i=1}^{\ell}$ be a set of ℓ control points for a spline curve $s(v)$ of degree d, with non decreasing knots $(\hat{v}_i)_{i=1}^{\ell+d+1}$

$$s(v) = \sum_{i=1}^{\ell} c_i B_{i,d}(v) \qquad v \in \mathcal{V} \triangleq [\hat{v}_1, \hat{v}_{\ell+d+1}] \qquad (1)$$

where $B_{i,d}(v)$, $d > 1$, is given by the Cox-de Boor recursion formula

$$B_{i,d}(v) = \frac{v - \hat{v}_i}{\hat{v}_{i+d} - \hat{v}_i} B_{i,d-1}(v) + \frac{\hat{v}_{i+1+d} - v}{\hat{v}_{i+1+d} - \hat{v}_{i+1}} B_{i+1,d-1}(v), \qquad (2)$$

and

$$B_{i,0}(v) = \begin{cases} 1, & \hat{v}_i \le v < \hat{v}_{i+1} \\ 0, & otherwise. \end{cases} \qquad (3)$$

Property 1 (Convex Hull Property). Any value assumed by $s(v)$, $\forall v \in \mathcal{V}$, lies in the convex hull of its $\ell + 1$ control points $(c_i)_{i=1}^{\ell}$.

Property 2. Suppose that the number $\hat{v}_{i+1}$ occurs m times among the knots $(\hat{v}_j)_{j=i-d}^{m+d}$ with m some integer bounded $1 \le m \le d + 1$, e.g $\hat{v}_i < \hat{v}_{i+1} = \cdots = \hat{v}_{i+m} < \hat{v}_{i+m+1}$, then the spline function $s(v)$ has continuous derivative up to order $d - m$ at knot $\hat{v}_{i+1}$.

This property implies that the spline smoothness can be changed using multiple knot points. It is common choice to set $m = d+1$ multiple knot points for the initial and the last knot points. In this way (1) assumes the first and the final control points as initial and final values.

B. Constrained Least Squares

The continuous-time least square problem has the general form

$$\min_f J(e) = \min_f \|Q^{1/2}e\|_2^2 = \min_f \int_0^T e(t)^T Q(t) e(t) dt, \quad (4)$$

$$e(t) \triangleq b(t) - D(t) f \qquad (5)$$

where $e(t)$ is the residual, $b(t)$ is the observation vector, $D(t)$ is the design matrix, $Q(t)$ is a positive definite weight matrix for each fixed $t \in [0, T]$ and f is the vector of model parameters. The norm functional $J(e)$ defines the squared weighted L_2 norm of $e(t)$, $t \in [0, T]$. The unique unconstrained solution is known to be given by

$$\hat{f} = \left(\int_0^T D(t)^T Q D(t) dt \right)^{-1} \int_0^T D(t)^T Q b(t) dt, \qquad (6)$$

provided the inverse exists.

Box constraints on the parameters vector to be estimated are represented by the following inequalities

$$\underline{f} \le f \le \bar{f} \ , \qquad (7)$$

where $\underline{f}$ ($\bar{f}$) is the lower (upper) bound of the parameter vector f.

The constrained least square problem given by (4),(5) and (7) does not admit a closed form solution, nevertheless a numerical solution can be obtained reformulating such problem in discrete form. Discretizing the residual vector $e(t)$ at $t_k = k \cdot t'_s$, $k = 1, 2, \cdots, N$, (t'_s is the numerical discretization interval), and defining the following vectors

$$\bar{e} \triangleq \begin{bmatrix} \bar{e}_1^T & \bar{e}_2^T & \cdots & \bar{e}_N^T \end{bmatrix}^T \qquad (8)$$

$$\bar{b} \triangleq \begin{bmatrix} \bar{b}_1^T & \bar{b}_2^T & \cdots & \bar{b}_N^T \end{bmatrix}^T \qquad (9)$$

$$\bar{D} \triangleq \begin{bmatrix} \bar{D}_1^T & \bar{D}_2^T & \cdots & \bar{D}_N^T \end{bmatrix}^T \qquad (10)$$

where $\bar{e}_k \triangleq e(t_k) = b(t_k) - D(t_k) f$, $\bar{b}_k \triangleq b(t_k)$ and $\bar{D}_k \triangleq D(t_k)$ for $k = 1, \cdots, N$, the following discrete version of the original problem (4),(5) and (7) is obtained

$$\min_f \bar{J}(\bar{e}) = \min_f \|\bar{Q}^{1/2} \bar{e}\|_2^2, \qquad (11)$$

$$\bar{e} \triangleq \bar{b} - \bar{D} f \qquad (12)$$

$$\underline{f} \le f \le \bar{f} \qquad (13)$$

where $\bar{Q}$ is the diagonal matrix obtained with the samples $Q(t_k)$ for $k = 1, \cdots, N$.

Problem (11)- (13) can by solved by the numerical algorithm given in [19] and implemented by the MATLAB® function *lsqlin*.

III. PROBLEM STATEMENT

Let Σ denote the 2DOF control system given by the series connection of a feedforward filter Σ_{ff} with a square closed loop system $(\Sigma_{f,c}, x)$ consisting of the feedback connection of a flexible onelink manipulator (Σ_p, x_p) with a LTI stabilizing controller (Σ_c, x_c) also providing the internal model of constant signals for robustness issues. The state space representation of $\Sigma_{f,c}$ is

$$\dot{x}(t) = Ax(t) + Br(t), \tag{14}$$

$$y(t) = Cx(t) \tag{15}$$

where: $x(t) = [x_p(t)^T, x_c(t)^T]^T \in \mathbb{R}^n$, $r(t) \in \mathbb{R}^q$, $y(t) \in \mathbb{R}^q$. The plant Σ_p is forced by the control input $u(t) \in \mathbb{R}^p$ ($p \geq q$) given by $u(t) = F_c x_c(t)$.

It is assumed that: A1) $\Sigma_{f,c}$ is asymptotically internally stable, A2) $\Sigma_{f,c}$ has no transmission zero at $s = 0$. The transfer matrices of $\Sigma_{f,c}$ and of the system $\Sigma_{f,u}$ between $r(t)$ and $u(t)$ are denoted by $W_{r,y}(s)$ and $W_{r,u}(s)$ respectively.

A preliminary informal definition of the considered problem states that the output $y(t)$ is required to optimally approximate a pre-specified desired trajectory $y_d(t)$, the external reference $r(t)$ is required to be uniformly bounded and the control effort $u(t)$ is required to satisfy the following saturation constraints

$$\underline{u}_i \leq u_i(t) \leq \bar{u}_i, i = 1, \cdots p, \, t \geq 0 \tag{16}$$

where $u_i(t)$ is the i-th component of $u(t)$.

The purpose of Σ_{ff} is to pre-filter the given $y_d(t)$ to yield the input signal $r(t)$ allowing the fulfillment of the above requirements.

To formally state the above problem, the following notations and preliminary considerations are given.

Assumption A1) allows the partition of $y_d(t)$ and $r(t)$ in transient and steady state components.

$$y_d(t) = \begin{cases} y_{d,t}(t), \; t \in [\,0, t_s\,] \stackrel{\triangle}{=} T_s \\ \tilde{y}_d, \; t \in (\,t_s, \infty\,) \\ y_{d,t}(t_s) = \tilde{y}_d \end{cases} \tag{17}$$

$$r(t) = \begin{cases} r_t(t), \; t \in [\,0, t_t\,] \stackrel{\triangle}{=} T_t, \;\; t_t > t_s \\ \tilde{r}, \; t \in (\,t_t, \infty\,) \end{cases} \tag{18}$$

where T_s is the time interval over which the transient desired output $y_{d,t}(t)$ is required to converge towards the steady state value $\tilde{y}_d$.

Choosing t_t sufficiently larger than t_s, the transient external reference $r_t(t)$ at time t_t will be practically coinciding with the steady state value $\tilde{r}$. By A2) and recalling that $\Sigma_{f,c}$ contains an internal model of constant signals one has $\tilde{r} = W_{r,y}(0)^{-1}\tilde{y}_d = \tilde{y}_d$.

Also the actual control effort $u(t)$ forcing Σ_p and the actual output $y(t)$ of $\Sigma_{f,c}$ are partitioned in a similar way

$$u(t) = \begin{cases} u_t(t), \; t \in T_t \\ \tilde{u}(t), \; t \in (\,t_t, \infty\,) \end{cases} \tag{19}$$

$$y(t) = \begin{cases} y_t(t), \; t \in T_t \\ \tilde{y}(t), \; t \in (t_t, \infty\,) \end{cases} \tag{20}$$

Unlike (17) and (18), definitions (19) and (20) represent approximate partitions in transient and steady-state components because $u(t)$ and $y(t)$ reach the respective steady state values $\tilde{u}$ and $\tilde{y}$ for $t \to \infty$. Nevertheless, choosing t_t sufficiently larger than t_s guarantees that the actual output $y(t)$ has converged to a pseudo steady-state trajectory $\tilde{y}(t)$ practically coinciding with $\tilde{y}$, $\forall t \geq t_t$ and asymptotically converging to $\tilde{y}$. Moreover, such a t_t also guarantees a practically negligible post action effect on the control effort (Devasia (2003)). Namely, analogously to $y(t)$, also the actual control effort $u(t)$, converges to a pseudo steady-state trajectory $\tilde{u}(t)$ almost coinciding with $\tilde{u}$ $\forall t \geq t_t$, and asymptotically converging to $\tilde{u}$.

Recalling that $\tilde{r} = \tilde{y}_d$, the steady-state value of $u(t)$ is given by

$$\tilde{u} = W_{r,u}(0)\tilde{y}_d, \tag{21}$$

The value $\tilde{y}_d$ must be chosen in such a way that the corresponding steady state value $\tilde{u}$ exactly satisfy the saturation constraints (16). By definitions of vector and matrix infinity norm, it readily follows that it is enough to choose $\tilde{y}_d$ according to : $\|\tilde{y}_d\|_\infty \leq \|W_{r,u}(0)\|_\infty^{-1} u_m$, where $u_m \stackrel{\triangle}{=} \min_{i=1,\cdots,p}\{|\underline{u}_i|, |\bar{u}_i|.\}$ If $W_{r,u}(0) = 0_{p\times q}$ (namely $s = 0$ is a blocking zero of $W_{r,u}(s)$) then $\tilde{y}_d$ can be arbitrarily chosen.

The above considerations show that the main question about the investigated optimization problem concerns the transient state. To this purpose the transient reference $r_t(t)$ is assumed to be given by a B-spline function and the actual $u_t(t)$ produced by $r_t(t)$ is optimally approximated by a B-spline $\hat{u}_t(t)$, which is required to exactly satisfy the same box-constraints of $u(t)$, i.e. :

$$\underline{u}_i \leq \hat{u}_{t,i}(t) \leq \bar{u}_i, i = 1, \cdots p, \; t \in T_t, \tag{22}$$

where $\hat{u}_{t,i}(t)$ is the i-th component of $\hat{u}_t(t)$.

The idea is to achieve a so accurate least square approximation $\hat{u}_t(t)$ such that also $u_t(t)$ practically satisfies (16). In this way the exact constraint on $u_t(t)$ is transformed on a least square constraint on the approximation error $\hat{u}_t(t) - u_t(t)$, whose quadratic norm should be as small as possible.

Defining $e(t)$ over T_t as $e(t) \stackrel{\triangle}{=} [e_1^T(t), e_2^T(t)]^T \stackrel{\triangle}{=} [y_{d,t}^T(t) - y_t^T(t), \hat{u}_t^T(t) - u_t^T(t)]^T$, the problem considered in this paper can be formally stated as follows:

Problem statement It is required to find the parameter vector (denoted by f) defining the two B-spline functions $r_t(t)$ and $\hat{u}_t(t)$, $t \in T_t$ with $r_t(t)$ and $\hat{u}_t(t)$ smoothly converging to $\tilde{r} = \tilde{y}_d$, and $\tilde{u} = W_{r,u}(0)\tilde{y}_d$ such that the following functional is minimized:

$$J(e(t)) = \int_0^{t_t} e^T(t)Q(t)e(t)dt = \\ = \int_0^{t_t} (e_1^T(t)Q_1(t)e_1(t) + e_2^T(t)Q_2(t)e_2(t))dt \tag{23}$$

subject to (22) where $Q(t) = \text{diag}\{Q_1(t), Q_2(t)\}$ is a suitably chosen positive definite weight matrix.

Remark Some comments on the feasibility and stability of the method are in order. For any imposed saturation constraint on $\hat{u}_t(t)$, the above approach defines a constrained optimization problem with guaranteed feasibility. In fact, modeling $\hat{u}_t(t), t \in T_t$, as a B-spline function implies its membership to the convex hull of its control points, that can be arbitrarily chosen on the basis of the saturations constraints. The feasibility of the corresponding optimization problem $\forall t \in T_t$ directly follows. Clearly, tightening the saturation constraints increases the minimum of the cost functional. Also the stability of Σ is a straightforward consequence of the proposed method: internal stability derives by A1) and by the uniform boundedness of $r(t)$, directly implied by the way it is computed.

IV. Problem solution

In the minimization of functional (23) the presence of box constraints on $u_t(t)$ is dealt with requiring the B-spline $r_t(t)$ to be estimated to yield the best tracking performance and, at the same time, to give a B-spline $\hat{u}_t(t)$ optimally approximating the actual $u_t(t)$ and exactly respecting the box-constraints (16). Thanks to the B-spline convex hull property, it is possible to represent constraints (16) on $\hat{u}_t(t)$ through a suitable definition of control points. Solving the constrained least square algorithm, the box constraints on the approximating $\hat{u}_t(t)$ are transformed into softened least squares constraints on the actual $u_t(t)$. Increasing the number of control points, the approximation error tends to zero and an exact fulfillment of constraints by $u_t(t)$ can be also obtained. In general it is not possible to state "a priori" the exact number of control points, this can only be verified "a posteriori" by simulation. The numerical example illustrates this situation.

A. B-spline representation of signals

Identifying the parameter v of (1) with the time variable $t \in [0, t_t]$, the corresponding B-spline curve $s(t)$ can be used to represent a scalar time function defined over the transient interval T_t.

Let ℓ and d be the number of control points and degree respectively of a generic scalar B-spline $s(t)$, $t \in [0, t_t]$, $t_t < \infty$. Any control point c_i, $i = 1, \cdots, \ell$, of $s(t)$ belongs to $\mathbb{R}^2$ and is completely defined by its (c_x, c_y) coordinates

$$c_i \triangleq [c_{x_i}, c_{y_i}]^T \tag{24}$$

The coordinate c_{x_i} indicates the time instant $t_i \in [0, t_t]$ at which the c_y-coordinate of the control point c_i assumes the corresponding value c_{y_i}. To preserve the causality of the signal, each t_i is a priori assigned partitioning the interval $[0, t_t]$ in $\ell - 1$ equispaced intervals.

Choosing a knot vector $(\hat{t}_i)_{i=1}^{\ell+d+1}$ with $\hat{t}_1 = \cdots = \hat{t}_{d+1} = 0$ and $\hat{t}_{\ell+1} = \cdots = \hat{t}_{\ell+d+1} = t_t$, Property 2 of B-splines implies that $s(t)$ starts from the c_y-coordinate of the initial control point c_1 at time $t = 0$ and ends at the c_y-coordinate of final control point c_ℓ at time instant $t = t_t$. The remaining knot points $\hat{t}_j$, $j = d + 2, \cdots, \ell$ are chosen uniformly equispaced within $[0, t_t]$. Defining the vectors

$$\mathbf{c}_s \triangleq \begin{bmatrix} c_{y_1} & c_{y_2} & \cdots & c_{y_\ell} \end{bmatrix}^T \tag{25}$$

and

$$\mathbf{B}_d(t) \triangleq \begin{bmatrix} B_{1,d}(t) & B_{2,d}(t) & \cdots & B_{\ell,d}(t) \end{bmatrix} \tag{26}$$

it is possible to rewrite (1) as

$$s(t) = \sum_{i=1}^{\ell} c_{y_i} B_{i,d}(t) = \mathbf{B}_d(t)\mathbf{c}_s. \tag{27}$$

Equation (27) is now used to define a compact B-spline representation of the $q \times 1$ external reference vector $r_t(t)$ and of the $p \times 1$ control effort vector $\hat{u}_t(t)$, $t \in T_t$. To this purpose the following vectors and matrix are defined:

$$\mathbf{c}_{r_t} \triangleq \left[\mathbf{c}^{1T}_{r_t}, \cdots, \mathbf{c}^{qT}_{r_t} \right]^T, \qquad \bar{\mathbf{B}}_{rd}(t) \triangleq \operatorname{diag}\left[\mathbf{B}_d(t)\right]. \tag{28}$$

Each $\mathbf{c}^i_{r_t} \triangleq \left[c^{i,1}_{r_t}, \cdots, c^{i,\ell}_{r_t} \right]^T$, $i = 1, \cdots, q$, is defined as in (25). The dimensions of $\mathbf{c}_{r_t}$ are $(q\ell \times 1)$. The dimensions of the block diagonal matrix $\bar{\mathbf{B}}_{rd}(t)$ are $(q \times q\ell)$.

The above notation and (27) allow $r_t(t)$ to be rewritten as

$$r_t(t) = \bar{\mathbf{B}}_{rd}(t)\mathbf{c}_{r_t}, \quad t \in T_t \tag{29}$$

The same procedure can be applied to the B-spline $\hat{u}_t(t)$ approximating the actual control effort $u_t(t)$, one has

$$\hat{u}_t(t) = \bar{\mathbf{B}}_{ud}(t)\mathbf{c}_{u_t}, \quad t \in T_t. \tag{30}$$

where $\mathbf{c}_{u_t}$ is defined as $\mathbf{c}_{r_t}$, but has dimensions $(p\ell \times 1)$, and $\bar{\mathbf{B}}_{ud}(t)$ is defined analogously to $\bar{\mathbf{B}}_{rd}(t)$, but has dimensions $(p \times p\ell)$.

B. Constrained Least Square Spline Identification

Equations (29) and (30) indicate that the B-spline representation of $r_t(t)$ and $\hat{u}_t(t)$ requires the identification of vectors $\mathbf{c}_{r_t}$ and $\mathbf{c}_{u_t}$. These parameter vectors are computed using a constrained least squares procedure as indicated in Section 2.2.

Some passages, not reported for brevity, show that, in the present case, the residual equation (5) related to functional (23) is defined by the following vectors and matrices

$$e(t) = [e_1(t)^T, e_2(t)^T]^T \tag{31}$$

$$f \triangleq \begin{bmatrix} \mathbf{c}^T_{r_t} & \mathbf{c}^T_{u_t} \end{bmatrix}^T \tag{32}$$

$$D(t) \triangleq \begin{bmatrix} G_a(t) & 0 \\ G_b(t) & -\bar{\mathbf{B}}_{ud}(t) \end{bmatrix} \tag{33}$$

$$b(t) \triangleq [(y_{d,t}(t) - Ce^{At}x(0))^T \ (-F_c e^{At}x(0))^T]^T \tag{34}$$

In the present case, the vector component $\mathbf{c}_{u_t}$ must be such that, the corresponding B-spline $\hat{u}_t(t)$ satisfy (16) , $\forall t \in T_t$. By Property 1 of B-spline functions, this will be satisfied imposing the following constraints to the control points of $\hat{u}_t(t)$

$$\underline{u}_i \le c^{i,j}_{u_t} \le \bar{u}_i, \quad i = 1, ..., p \quad j = 1, ..., \ell. \tag{35}$$

The solution $\hat{f} \triangleq \begin{bmatrix} \hat{\mathbf{c}}^T_{r_t} & \hat{\mathbf{c}}^T_{u_t} \end{bmatrix}^T$ of the constrained least square problem (31)-(35) can be numerically computed as explained in the second part of Section 2.2: discretization of the continuous constrained least square problem according to (8)-(13) and successive application of the algorithm [19] implemented by the MATLAB® function *lsqlin*. It is easy to see that definition (33) implies that $D(t)$ is full column rank, hence the constrained least square problem admits a unique solution that can be computed in polynomial time. This solution is used to compute $r_t(t)$ and $u_t(t)$, $t \in T_t$ according to (29) and (30) respectively, putting $\hat{\mathbf{c}}_{r_t}^T = \mathbf{c}_{r_t}^T$ and $\hat{\mathbf{c}}_{u_t}^T = \mathbf{c}_{u_t}^T$.

Remark 1: The presented approach can be easily extended to the case of a switching desired output set-point. At some transition instants t_h, $h \in \mathbb{Z}^+$, the actual set point $\tilde{y}_{d,h}$ is reset to a new value $\tilde{y}_{d,h+1}$, according to the particular application.

To obtain a constrained optimal output transition over each interval $T_h \triangleq [t_h, t_{h+1})$ ($t_0 = 0$), the proposed method can be still applied provided that $t_{h+1} - t_h > t_{t,h}$, $\forall h \in \mathbb{Z}^+$, where $t_{t,h}$ has the same meaning of t_t. Namely each transient interval $T_{t,h} \triangleq [t_h, t_h + t_{t,h})$ should be long enough to guarantee that the actual output $y(t)$, starting from the previous set point $\tilde{y}_{d,h}$, has practically reached the new set point $\tilde{y}_{d,h+1}$. The next transition instant and the new set-point may be known in advance with a very short preview time. In practice, the preview interval is only determined by the computation time for estimating the spline coefficients, namely the control points, so that the method can be practically implemented on line.

Also for this more general problem, the uniform boundedness of the reference input for uniformly bounded sequences $\{\tilde{y}_{d,h}\}$, is a straightforward consequence of the way $r(t)$ is computed.

V. Numerical results

The method has been tested on the Single Flexible Link Manipulator (SFLM) considered in [15], [16]. Choosing the base torque T as the control input and the end-effector displacement y_{tip} as the output, the dynamic model is given by [15]:

$$\frac{y_{tip}}{T} = \frac{-0.295s^4 + 98.736s^2 + 3331956.636}{s^2\left(s^4 + 19015.243s^2 + 25179356.808\right)}. \tag{36}$$

This is a non hyperbolic transfer function with zeros $z_{1,2} = \pm 56.5i$, $z_{3,4} = \pm 59.43$, and poles $p_{1,2} = 0$, $p_{3,4} = \pm 132.26i$, $p_{5,6} = \pm 37.84i$. It is noted that the transfer function has two null poles. With standard procedures, the following minimal reachable and observable state space realization can be associated to (36):

$$\dot{x}_p(t) = A_p x_p(t) + B_p\, u(t), \quad y(t) = C_p x_p(t)$$

where A_p,B_p,C_p and x_p are not reported for brevity. The closed loop system Σ_f is given by the unitary feedback connection of the plant with a stabilizing controller which exploits the knowledge of state vector (see [23] for details), and also provides the internal model of constant signals, [24]. This also holds if, as in this case, the internal model is already provided by the plant. The closed loop eigenvalues are: $\{-573.2, -3.435 \pm 60i, -3.445 \pm 3.456i, -0.5858, -1\}$.

As $W_{r,u}(0) = 0$, by (21) one has that any chosen $\tilde{y}_d$ is compatible with the saturation constraints. The pre-specified desired trajectory for the end effector is the following S-shape function

$$y_d(t) = \begin{cases} y(0), & 0 \le t \le a \\ 2\left(\tilde{y}_d - y(0)\right)\left(\frac{t-a}{b-a}\right)^2 + y(0), & a \le t \le \frac{a+b}{2} \\ \left(\tilde{y}_d - y(0)\right)\left(1 - 2\left(\frac{t-b}{b-a}\right)^2\right) + y(0), & \frac{a+b}{2} \le t \le b \\ \tilde{y}_d = 1, & t \ge b \end{cases}$$

with $a = 0$, $b = 0.3$, $y(0) = 0$, and $\tilde{y}_d = 1$. The corresponding diagram is reported in Fig (1).

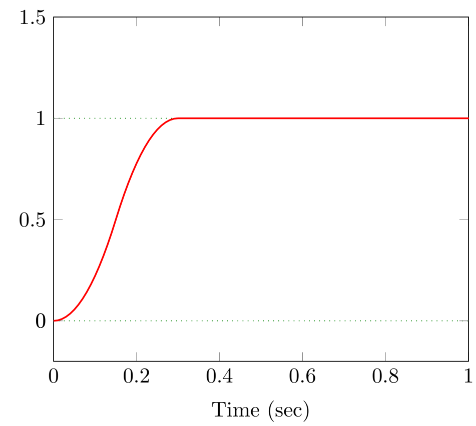

Fig. 1. The desired output trajectory $y_d(\cdot)$ for the end effector displacement. Its settling time is $t_s = 0.3$.

To evidence the effects of the proposed method, two different sets of simulations, corresponding to two different approaches have been performed. The first one does not explicitly consider any constraint on the control effort, namely (16) is not taken into account. To this purpose: $e(t) = e_1(t)$, $Q(t) = Q_1(t)$, and the parameter vector f to be estimated reduces to $f = c_{r_t}$ because $\hat{u}_t(t)$ and its respective constraints (22) are not defined. The solution of the unconstrained least square problem is of the kind (6).

The second set of simulations refers to the direct application of the present approach. The choice $Q_1(t) = Q_2(t) = 1, t \in T_t$, has been adopted to equally weigh both components of both errors $e_1(t)$ and $e_2(t)$.The two sets of simulations are referred to as Unconstrained and Constrained Approach (UA and CA) respectively.

The following choices hold in both cases: 1) x(0) is the null vector, 2) the transient interval has been chosen as $T_t = [0, t_t]$ where $t_t = 15 \ge 4\tau_m$, and $\tau_m = 1.7$ is the maximum time constant of $\Sigma_{f,c}$, 3) each set of simulations refers to $\ell = 120$ control points. As for the CA, the saturation constraints $\underline{u}_1 = \underline{u}_2 = -4$, $\bar{u}_1 = \bar{u}_2 = 4$ have been assumed. The constrained least squares problem corresponding to the CA has been solved using the Matlab function *lsqlin* considering a uniformly discretized residual vector $e(kt_s)$ with $k = 0, 1, \cdots$, and $t_s = 0.01$ sec.

Figure 2 evidences that , though a constrained control effort, a satisfactory transient $y_t(t)$ smoothly converging towards $\tilde{y}_d = 1$ is obtained. The zoom in Figure 3 shows that the UA yields slightly better tracking performance with

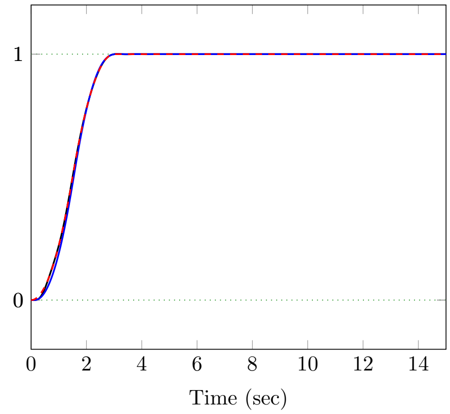

Fig. 2. The behavior of the actual $y_t(t)$ in the unconstrained case (black dashed line) and the constrained case (blue continuous line) over T_t. The red dashed line represents the desired output trajectory $\tilde{y}_d(t)$.

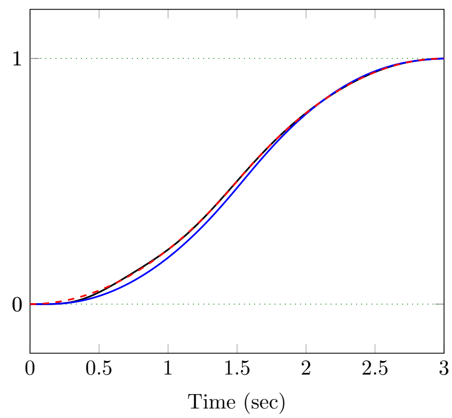

Fig. 3. Zoom of $\tilde{y}_d(t)$ (red dashed line) and of $y_t(t)$ in the unconstrained case (black dashed line) and the constrained case (blue continuous line) over $[0\,3] \subseteq T_t$.

respect to CA. Nevertheless, CA apparently satisfies the constraints on the control effort, while UA does not (see Figure 4). Figure 5 shows how the bounds on the estimated control effort affect the computed reference input $r_t(t)$. Figure 6 evidences an almost overlapped $u_t(t)$ and $\hat{u}_t(t)$ both satisfying the specified bounds.

In the UA and CA cases, the steady state has been practically attained by $r(t)$, $y(t)$ and $u(t)$ within the interval $[0,\, 6] \subseteq T_t$. The steady state value of $u(t)$ is $\tilde{u} = 0$.

VI. Conclusion

It has been shown how pseudo-inversion and B splines can be advantageously used to achieve a very accurate output tracking in the end-effector positioning problem also in presence of saturation constraints on the control effort. Hard constraints on a control variable are transferred on an optimal approximation of the variable itself. If the approximation is really satisfactory, also the actual control variable can satisfy the constraints almost exactly. To this purpose B-splines have been used because of their appealing features: 1) they belong to the convex hull generated by control points, 2) B-splines are defined in such a way to automatically satisfy the conti-

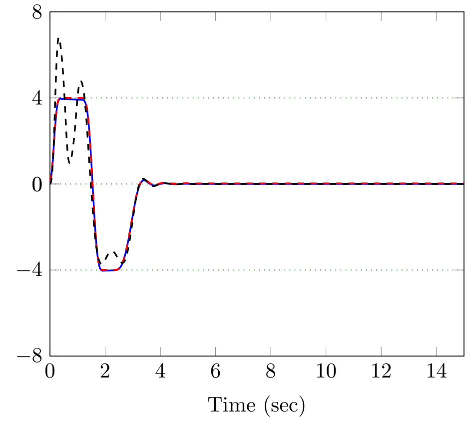

Fig. 4. The behavior of the control input $u_t(t)$ in the unconstrained case (black dashed line) and the constrained case (blue continuous line) over T_t. The red dashed line represents the computed B-spline $\hat{u}_t(t)$ that approximates $u_t(t)$.

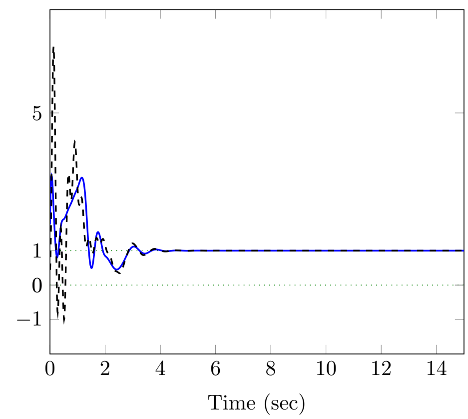

Fig. 5. The behavior of $r_t(t)$ in the unconstrained case (black dashed line) and the constrained case (blue continuous line) over T_t.

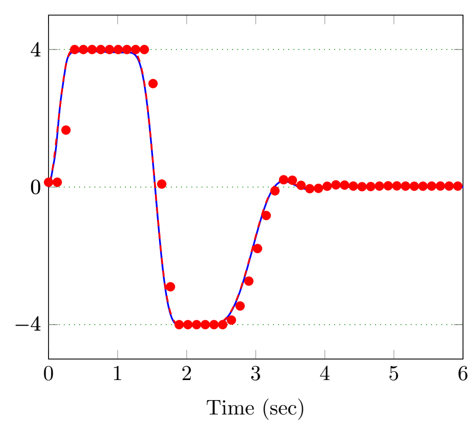

Fig. 6. Zoom of $u_t(t)$ (blue line) and of its optimal approximation $\hat{u}_t(t)$ (red line) over the interval [0, 6]. The red circles represents the computed control points.

nuity constraints at knot points. Property 1 can be exploited to impose the approximating spline to exactly satisfy hard constraints, property 2 can be exploited because it allows an arbitrary increase of knot points without increasing, at the same time, the number of continuity constraints to be satisfied. This allows the B-spline to practically overlie the function to be approximated.

The main merit of the pseudo inversion is in terms of its generality: the method can be directly applied to any flexible multilink arm. The preliminary simulation results show effectiveness of the approach. Future works concerns the application to more challenging robot models.

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