2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

The l_p Induced Norm and the Small-Gain Theorem for Discrete-time Stochastic Systems

Yuji Nagira[1], Yohei Hosoe[1] and Tomomichi Hagiwara[1]

***Abstract*— This paper discusses l_p induced norms and a small-gain theorem for discrete-time stochastic systems. The induced norm of systems generally depends on the associated input space of the systems. At the beginning, we consider a general stochastic l_p space and introduce the primitive l_p induced norm based on the space. However, the primitive l_p induced norm can actually be confirmed not to reflect the nature of the randomness behind the stochastic systems, which leads us to overestimating its value. Hence, we take a proper subspace of the stochastic l_p space and introduce another l_p induced norm based on the subspace. We demonstrate with examples that the norm based on the subspace successfully reflects the nature of the system randomness. Then, we give the small-gain theorem based on the norm.**

I. INTRODUCTION

In this paper, we give a small-gain theorem for discrete-time input-output (nonlinear) stochastic systems. The small-gain theorem was first derived and discussed extensively by Zames (1966) in [1],[2] for deterministic systems. Roughly speaking, the theorem states that for closed-loop systems consisting of two subsystems, if the subsystems are both l_p stable and the product of their l_p induced norms is less than 1, then the closed-loop system is also l_p stable. The theorem played a crucial role in the development of robust control theory for deterministic systems in recent decades. Compared to the deterministic systems case, however, robust control for stochastic systems has not been studied sufficiently. The aim of this paper is to facilitate such studies by properly deriving a small-gain theorem for stochastic systems.

In giving the small-gain theorem for stochastic systems, we need to deal with a stochastic version of the l_p induced norm, which generally depends on the input space of systems. At first, we introduce the general stochastic l_p space and the primitive l_p induced norm based on the space. However, this l_p induced norm in fact does not reflect the nature of the randomness behind stochastic systems and this leads us to overestimating its value. Hence, we take a proper subspace of the stochastic l_p space and introduce another l_p induced norm based on the subspace. Note that the idea of taking such a subspace itself was already considered in conventional studies such as [3],[4]. The significance of our arguments lies in properly giving the associated definitions of stability and causality rather than using the idea itself. In addition, since the class of our input-output (nonlinear) stochastic systems is larger than those in the conventional studies, our results can also be seen as a generalization of those conventional results.

For reference, we summarize other conventional studies related to this paper in the following. We first note that the small-gain theorem was derived for stochastic linear systems in [5]–[7], and for stochastic nonlinear systems from the viewpoint of input-to-state stability in [8],[9]. On the other hand, since we discuss the small-gain theorem for stochastic (nonlinear) systems from the viewpoint of input-output stability as already stated, the problem and the associated arguments essentially differ from those studies. Although a similar problem was already dealt with in [10], the paper failed to introduce the extended stochastic l_p space and thus the proof of the small-gain theorem was not properly completed; in the deterministic systems case [1],[2],[11],[12], the extended space is introduced for the l_p space as a key of the proof. If we do not introduce such an extended l_p space, we have to begin with the assumption that the inner signals in the closed-loop system belong to the l_p space, which implies the closed-loop system is l_p stable, and thus the proof does not make sense. This paper also resolves this issue by introducing the extended stochastic l_p space (and its subspace).

This paper is organized as follows. In Section II, we introduce the stochastic l_p space and its extended space. In addition, we state the definitions of the l_p stability and the primitive l_p induced norm. In Section III, we show that the primitive l_p induced norm in Section II is not in fact reflecting the nature of the randomness behind stochastic systems. Hence, we introduce in Section IV a proper subspace of the stochastic l_p space and consider the l_p induced norm based on the subspace. In Section V, we give the small-gain theorem for stochastic systems.

We use the following notation in this paper. $\mathbf{R}$ and $\mathbf{N}_0$ denote the set of real numbers and that of non-negative integers, respectively. The p-norm of the vector $(\cdot)$ is denoted by $|(\cdot)|_p$. The expectation of the random variable $(\cdot)$ is denoted by $E[(\cdot)]$.

II. STOCHASTIC l_p SPACE AND THE PRIMITIVE l_p INDUCED NORM FOR STOCHASTIC SYSTEMS

In this paper, we consider the closed-loop system Σ in Fig. 1, which consists of the discrete-time stochastic systems G_1 and G_2. In this figure, $u := [u^{1T}, u^{2T}]^T$ is regarded as the input while $w = [w^{1T}, w^{2T}]^T$ and $z = [z^{1T}, z^{2T}]^T$ are regarded as the output of the closed-loop system Σ. We express the randomness behind the stochastic

*This work was not supported by any organization

[1]Department of Electrical Engneering, Graduate School of Engineering, Kyoto University, Nishikyo-ku, Kyoto 615-8510, Japan nagira@jaguar.kuee.kyoto-u.ac.jp, {hosoe,hagiwara}@kuee.kyoto-u.ac.jp

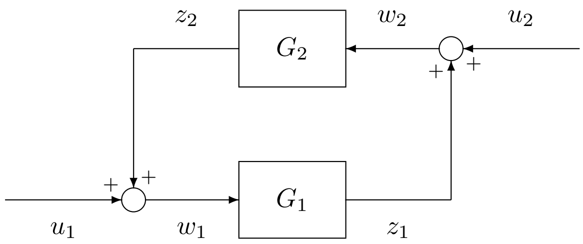

Fig. 1. Closed-loop system Σ.

subsystems G_1 and G_2 by the stochastic process ξ on the complete probability space $(\Omega, \mathcal{F}, P)$ (i.e., $G_1 = G_{1\xi}$ and $G_2 = G_{2\xi}$), where Ω, $\mathcal{F}$ and P denote the abstract space, the σ-algebra and the probability measure, respectively. For $G_\xi = G_{i\xi}$ $(i = 1, 2)$, if we take a sample $\omega \in \Omega$, then the input-output relation of $G_{\xi(\omega)}$ is determined and fixed through the process $\xi(\omega)$. For notational simplicity, G_ξ and $G_{\xi(\omega)}$ are respectively abbreviated to G and $G(\omega)$ when ξ is obvious from the context. In addition, we suppose the input u of the closed-loop system Σ belongs to the class of stochastic processes on the same probability space $(\Omega, \mathcal{F}, P)$ throughout this paper.

In this section, we begin by introducing the stochastic l_p space in a fashion similar to the deterministic systems case [1],[2],[11],[12]. Then, we further introduce the extended space of the stochastic l_p space and make the definitions of the l_p stability and the primitive l_p induced norm for stochastic systems. In what follows, we suppose $p \in [1, \infty]$ is fixed.

A. Stochastic l_p space

For $p \in [1, \infty]$, the space $l_p(\mathbf{N}_0 \times \Omega, \mathbf{R}^m)$ is defined as the set of the stochastic processes $x = \{x_k\}_{k=0}^{\infty} : \mathbf{N}_0 \times \Omega \to \mathbf{R}^m$ such that

$$\|x_k\|_{L_p} := \begin{cases} (E[|x_k|_p^p])^{1/p} & (p \in [1, \infty)) \\ \operatorname*{ess\,sup}_{\omega \in \Omega} |x_k(\omega)|_\infty & (p = \infty) \end{cases} \tag{1}$$

is well-defined (i.e., bounded) for each $k \in \mathbf{N}_0$ and

$$\|x\|_p := \begin{cases} \left(\sum_{k=0}^{\infty} \|x_k\|_{L_p}^p \right)^{1/p} & (p \in [1, \infty)) \\ \sup_{k \in \mathbf{N}_0} \|x_k\|_{L_\infty} & (p = \infty) \end{cases} \tag{2}$$

is well-defined. We call this space the stochastic l_p space. We give two remarks on this space, where the first one is for mathematical rigor in its definition while the second plays a key role in Section III.

Remark 1: To make (1) (and thus (2) too) a norm rather than a semi-norm, we actually consider the equivalence class for the random vector x_k defined as

$$[x_k] := \{x'_k : x'_k(\omega) = x_k(\omega) \text{ (a.s.)}\} \tag{3}$$

(where $\{x'_k\}_{k=0}^{\infty}$ is a stochastic process with the same underlying probability space as x) and identify x_k with $[x_k]$, and similarly for the stochastic process x.

Remark 2: Regardless of the underlying Ω, the stochastic l_p space virtually includes the usual deterministic l_p space, which is denoted by $l_{p\mathrm{d}}$ in this paper. To see this, consider the following subspace of the stochastic l_p space:

$$l_{p\mathrm{D}}(\mathbf{N}_0 \times \Omega, \mathbf{R}^m) := \{x \in l_p(\mathbf{N}_0 \times \Omega, \mathbf{R}^m)) : x(\omega) = x(\omega') \quad \forall \omega, \omega' \in \Omega\}. \tag{4}$$

The above space is the set of stochastic processes that are in fact 'deterministic' in the sense that for each element of $l_{p\mathrm{D}}(\mathbf{N}_0 \times \Omega, \mathbf{R}^m)$, its every sample path leads to the same signal (determined by the element). Since $\|x_k\|_{L_p}$ in (1) reduces to the norm of the common sample path evaluated at k (whatever element of $l_{p\mathrm{D}}(\mathbf{N}_0 \times \Omega, \mathbf{R}^m)$ one may take), it is easy to see from (2) that the subspace consisting of such 'deterministic' processes coincides with $l_{p\mathrm{d}}$. To summarize, we can say that the stochastic l_p space has a subset that may be identified with $l_{p\mathrm{d}}$, where the subset is given by (4).

B. Extended stochastic l_p space

In this subsection, we introduce the extended space of the stochastic l_p space. The small-gain theorem we aim at deriving in this paper is such that the closed-loop system consisting of two stochastic subsystems is ensured to be l_p stable (for a prescribed $p \in [1, \infty]$) under some conditions on the subsystems. Here, l_p stability of the closed-loop system is, roughly speaking, defined as the following requirement: the stochastic processes corresponding to the output signals in the closed loop belong to the stochastic l_p space. In studying whether or not this requirement is satisfied, we need to consider a space larger than the stochastic l_p space, to which the output in the loop always belong (even if the closed-loop system is not l_p stable); otherwise, the arguments can only begin by assuming that the output belong to l_p, which can only lead to circular arguments because it is exactly what the small-gain theorem intends to show. Since the attempt in [10] for deriving a small-gain theorem was made without introducing the extended space, the asserted proof in fact did not make sense. This is why we need to introduce the extended l_p space.

For the stochastic process $x : \mathbf{N}_0 \times \Omega \to \mathbf{R}^m$, we denote by $x_{[K]}$ the truncated stochastic process of x with $K \in \mathbf{N}_0$, i.e.,

$$x_{[K]k} = \begin{cases} x_k & (0 \le k \le K) \\ 0 & (K < k) \end{cases}. \tag{5}$$

Then, the extended space $l_{pe}(\mathbf{N}_0 \times \Omega, \mathbf{R}^m)$ of the stochastic l_p space is defined as the set of x such that $x_{[K]} \in l_p(\mathbf{N}_0 \times \Omega, \mathbf{R}^m)$ for all $K \in \mathbf{N}_0$. For notational simplicity, $l_p(\mathbf{N}_0 \times \Omega, \mathbf{R}^m)$ and $l_{pe}(\mathbf{N}_0 \times \Omega, \mathbf{R}^m)$ are respectively abbreviated to $l_p^m(\Omega)$ and $l_{pe}^m(\Omega)$ in the following. In addition, when the space Ω and/or the dimension m are obvious from or insignificant in the context, they are occasionally dropped and such shorthand notations as l_{pe}^m and $l_p(\Omega)$ will be used.

In Fig. 2, we illustrate the spaces to which the signals in the closed-loop system Σ belong. In particular, it should be noted that the internal signals in Σ belong not to the

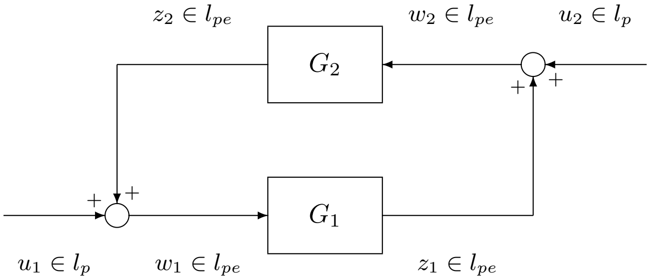

Fig. 2. Closed-loop system Σ with the signal space explicitly referred to.

stochastic l_p space but to its extended space, as stated at the beginning of this subsection.

C. l_p stability and the primitive l_p induced norm for stochastic systems

We now describe the class of the stochastic systems G_1 and G_2 that we can deal with in this paper. The class actually consists of the stochastic systems denoted by $G : (l_{pe}(\Omega), l_{pe}(\Omega))$, by which we mean that G is a stochastic system whose input and output both belong to $l_{pe}(\Omega)$. To describe additional assumptions required on G_1 and G_2, we introduce the notation

$$(w, z) \sim G \tag{6}$$

when $w \in l_{pe}(\Omega)$ and $z \in l_{pe}(\Omega)$ are consistent with the input-output relation of G (i.e., for each $\omega \in \Omega$, the input-output relation of the 'deterministic sample system $G(\omega)$' admits such a situation that its output equals the sample path (corresponding to ω) of the stochastic process z when its input is given by the corresponding sample path of the stochastic w). Then, we define

$$\mathcal{W}_G := \{w \in l_{pe}(\Omega) : \exists z \in l_{pe}(\Omega) \text{ s.t. } (w, z) \sim G\} \tag{7}$$

and call it the admissible input space of G. If we consider the case where the closed-loop system consists of nonlinear stochastic systems G_1 and G_2, the admissible input spaces for $G = G_i$ $(i = 1, 2)$ could often be smaller than $l_{pe}(\Omega)$, and $z \in l_{pe}(\Omega)$ such that $(w, z) \sim G$ may not be unique for some $w \in \mathcal{W}_G$. Dealing with such situations is important but actually rather hard when we aim at deriving the small-gain theorem for stochastic systems. Indeed, even the small-gain theorem for deterministic systems does not successfully deal with such kind of situations. Thus, we actually assume that $G = G_i$ $(i = 1, 2)$ satisfy the following assumption.

Assumption 1: The stochastic system $G : (l_{pe}(\Omega), l_{pe}(\Omega))$ satisfies the following conditions.

1. $\mathcal{W}_G = l_{pe}(\Omega)$.
2. For each $w \in l_{pe}(\Omega)$, there exists a unique $z \in l_{pe}(\Omega)$ such that $(w, z) \sim G$.
3. $(0, 0) \sim G$.

The above unique output z will be denoted by Gw in the following. Under the above assumption, the input-output stability of the stochastic system G is defined as follows.

Definition 1: The stochastic system $G : (l_{pe}(\Omega), l_{pe}(\Omega))$ satisfying Assumption 1 is said to be $l_p(\Omega)$ stable (or l_p stable, for short) if the following two conditions are satisfied.

(i) The unique $z \in l_{pe}(\Omega)$ such that $(w, z) \sim G$ in fact satisfies $z \in l_p(\Omega)$ whenever $w \in l_p(\Omega)$.

(ii) There exists a positive γ such that

$$\|z\|_p \leq \gamma \|w\|_p \quad (\forall w \in l_p(\Omega)), \tag{8}$$

where z denotes the unique $z \in l_p(\Omega)$ such that $(w, z) \sim G$.

Stability of the closed-loop system Σ with $w = [w^{1T}, w^{2T}]^T$ and $z = [z^{1T}, z^{2T}]^T$ viewed as its output is defined in accordance with the above.

If G is $l_p(\Omega)$ stable, then its primitive l_p induced norm is defined as

$$\|G\|_p := \sup_{w \in l_p \setminus \{0\}} \frac{\|z\|_p}{\|w\|_p}. \tag{9}$$

III. LIMITATION ON THE ANALYSIS BASED ON THE PRIMITIVE l_p INDUCED NORM FOR STOCHASTIC SYSTEMS

In the preceding section, we introduced the primitive l_p induced norm for stochastic systems. However, as stated in Section I, this norm is not reflecting the nature of the randomness behind the stochastic systems (i.e., the distribution of ξ) effectively, which finally leads only to severely conservative stability analysis. In this section, we theoretically confirm this through comparing the primitive l_p induced norm with a sort of 'worst' gain for stochastic systems. Then, in the following section (i.e., Section IV), we will take an appropriate subspace of the stochastic l_p space as the input space for stochastic systems and further introduce another l_p induced norm based on the subspace for less conservative stability analysis.

For each fixed $\omega \in \Omega$, we can view $G(\omega) = G_i(\omega)$ $(i = 1, 2)$ as a usual deterministic time-varying system. For the arguments in this section, we consider the case when the following assumption is satisfied.

Assumption 2: $G(\omega)$ is l_{pd} stable (i.e., l_p stable in the usual sense of deterministic systems [11],[12]) for each $\omega \in \Omega$.

By the way, $l_{p\mathrm{D}}(\Omega)$, a shorthand notation for $l_{p\mathrm{D}}(\mathbf{N}_0 \times \Omega, \mathbf{R}^m)$, is a subset of the space $l_p(\Omega)$ regardless of Ω, as stated in Remark 2. Thus, the extended space of $l_{p\mathrm{D}}(\Omega)$, defined in an obvious fashion and denoted by $l_{pe\mathrm{D}}(\Omega)$, is also a subset of $l_{pe}(\Omega)$. Since every stochastic process in $l_{pe}(\Omega)$ is an admissible input of G by Assumption 1.1, it follows that

(i) every stochastic process in $l_{pe\mathrm{D}}(\Omega)$ is an admissible input of G.

Furthermore, it readily follows once again from Remark 2 that for each stochastic process $w \in l_{pe\mathrm{D}}(\Omega)$, its sample path is independent of $\omega \in \Omega$ and thus w can be identified with an element of l_{ped} and that

(ii) the collection of all stochastic process $w \in l_{pe\mathrm{D}}(\Omega)$ can be identified with l_{ped}.

It is now obvious from (i) and (ii) that the admissible input space of $G(\omega)$ is the whole $l_{pe\mathrm{d}}$ for each $\omega \in \Omega$. Furthermore, it follows from Assumption 1.2 that $G(\omega)$ admits a unique output for each input $w \in l_{pe\mathrm{d}}$, regardless of $\omega \in \Omega$. The output will be denoted by $G(\omega)w$ in the following.

It follows that Assumption 2 validates our referring to the deterministic l_p induced norm

$$\|G(\omega)\|_{p\mathrm{d}} = \sup_{w \in l_{p\mathrm{d}} \setminus \{0\}} \frac{\|G(\omega)w\|_{p\mathrm{d}}}{\|w\|_{p\mathrm{d}}} < \infty \tag{10}$$

for each $\omega \in \Omega$, where $\|(\cdot)\|_{p\mathrm{d}}$ on the right-hand side denotes the usual deterministic l_p norm for discrete-time signals. Let us further assume that

$$\gamma_p^\star(G) := \operatorname*{ess\,sup}_{\omega \in \Omega} \|G(\omega)\|_{p\mathrm{d}} \tag{11}$$

is well-defined. Then, it corresponds to the 'worst' gain of $G(\omega)$ for $\omega \in \Omega$.

We can show the following theorem with regard to the above 'worst' gain.

Theorem 1: Suppose G satisfies Assumptions 1 and 2. Then, $\|G\|_p \geq \gamma_p^\star(G)$ for $p \in [1, \infty]$.

IV. SUBSPACE OF STOCHASTIC l_p SPACE AND THE ξ-RESTRICTED l_p INDUCED NORM

In the preceding section, we showed in Theorem 1 that the primitive l_p induced norm $\|G\|_p$ cannot be smaller than the 'worst' gain of G with respect to $\omega \in \Omega$. In this section, we introduce an appropriate subspace of the stochastic l_p space by taking the stochastic process ξ underlying G into consideration, and call it the ξ-restricted subspace (of the stochastic l_p space). In a form relevant to this subspace, we explicitly define the system causality, which will be used in the small-gain theorem later. In addition, we also consider the l_p induced norm based on the subspace and call it the ξ-restricted l_p induced norm. Then, we provide examples on the analytical calculation of the primitive and ξ-restricted l_p induced norms, in which the ξ-restricted norm is shown to be indeed reflecting the nature of the stochastic process ξ underlying the stochastic systems. To stress the fact that the arguments here are precisely devoted to the treatment of ξ underlying the stochastic system G, it is denoted by G_ξ throughout this section.

A. ξ-restricted subspace of stochastic l_p space

In this subsection, we introduce what we call the ξ-restricted subspace of the stochastic l_p space by taking the stochastic process ξ underlying $G_\xi = G_{\xi i}$ $(i = 1, 2)$ into consideration.

Suppose ξ_k $(k \in \mathbf{N}_0)$ are independent with respect to k. Define $\mathcal{F}_k$ as the σ-algebra generated by $\xi_0, \ldots, \xi_k$ for $k \in \mathbf{N}_0$, and $\mathcal{F}_{-1} := \{\emptyset, \Omega\}$. Then, we denote by $l_{p\xi}(\mathbf{N}_0 \times \Omega, \mathbf{R}^m)$ the set of $x \in l_p(\mathbf{N}_0 \times \Omega, \mathbf{R}^m)$ such that x_k is $\mathcal{F}_{k-1}$-measurable at each $k \in \mathbf{N}_0$ and the norm (2) is well-defined. In the following, $l_{p\xi}(\mathbf{N}_0 \times \Omega, \mathbf{R}^m)$ is abbreviated to $l_{p\xi}(\Omega)$ or $l_{p\xi}$. If $x \in l_{p\xi}(\Omega)$, x_0 is a usual deterministic vector, and thus is irrelevant to ξ. On the other hand, x_k $(k \geq 1)$, roughly speaking, depends on some/all of the past sequence $\xi_0, \ldots, \xi_{k-1}$ (or can be simply a deterministic vector). However, this x_k is irrelevant to the present and the future ξ_i $(i \geq k)$, which is due to the i.i.d.-nature of ξ.

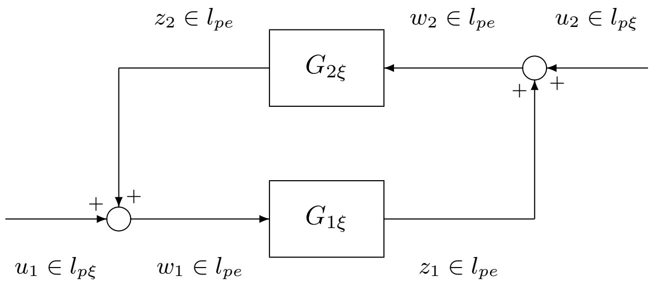

Fig. 3. Closed-loop system Σ in case $u \in l_{p\xi}$.

B. System causality

Let us denote by $l_{pe\xi}(\Omega)$ the extended space of $l_{p\xi}(\Omega)$. Then, we define the causality for our stochastic systems as follows, in which ξ is taken into consideration.

Definition 2: The stochastic system $G_\xi : (l_{pe}(\Omega), l_{pe}(\Omega))$ is said to be ξ-restricted causal if the following conditions are satisfied for each $w \in l_{pe\xi}(\Omega)$ $(\subset \mathcal{W}_G)$ (and $z = G_\xi w \in l_{pe}(\Omega)$) and every $K \in \mathbf{N}_0$.

1. The unique $z \in l_{pe}(\Omega)$ in fact satisfies $z \in l_{pe\xi}(\Omega)$.
2. The output $z' = G_\xi w_{[K]} \in l_{pe\xi}(\Omega)$ for $w_{[K]}$ satisfies $z'_{[K]} = z_{[K]}$.

Although the definition of causality is not mentioned clearly in [3],[4],[10] because these studies deal with a class of stochastic linear systems represented by state equations, we explicitly showed the above definition since our systems are nonlinear input-output stochastic systems.

Here, let us consider the case with the signals belonging to $l_{p\xi}(\Omega)$ as the external input u of Σ as in Fig. 3; this restriction would not be strong in the practical sense. Then, when $G_{1\xi}$ and $G_{2\xi}$ are both ξ-restricted causal, which will be an essential assumption in the small-gain theorem, the output $[w^T, z^T]^T \in l_{pe}(\Omega)$ of Σ can be confirmed to satisfy $[w^T, z^T]^T \in l_{pe\xi}(\Omega)$ by Def. 2.1 as shown in Fig. 4. This fact plays an important role in deriving the small-gain theorem later.

C. ξ-restricted l_p induced norm

We next define the stability based on $l_{p\xi}(\Omega)$ as follows.

Definition 3: The stochastic system $G_\xi : (l_{pe}(\Omega), l_{pe}(\Omega))$ is said to be $l_{p\xi}(\Omega)$ stable if the following two conditions are satisfied.

(i) The unique $z \in l_{pe}(\Omega)$ in fact satisfies $z \in l_p(\Omega)$ whenever $w \in l_{p\xi}(\Omega)$.
(ii) There exists a positive γ such that

$$\|z\|_p \leq \gamma \|w\|_p \quad (\forall w \in l_{p\xi}(\Omega)). \tag{12}$$

By definition, if G_ξ is $l_p(\Omega)$ stable then it is also $l_{p\xi}(\Omega)$ stable, although the opposite assertion is not generally true; we will confirm this in Example 2 later.

Associated with the above stability, we define the ξ-restricted l_p induced norm

$$\|G_\xi\|_{p\xi} := \sup_{w \in l_{p\xi} \setminus \{0\}} \frac{\|z\|_p}{\|w\|_p}. \tag{13}$$

D. Examples of analytical calculations of l_p induced norms

Since $l_{p\xi}(\Omega)$ is a subspace of $l_p(\Omega)$, it is obvious that $\|G_\xi\|_{p\xi} \le \|G_\xi\|_p$. In this subsection, we take $p = 2$ and provide two examples of analytical calculations of $\|G_\xi\|_2$ and $\|G_\xi\|_{2\xi}$, aiming at showing the difference between the two norms. With these examples, we show that unlike the primitive l_p induced norm, the ξ-restricted l_p induced norm actually reflects the nature of the stochastic process ξ underlying the stochastic systems.

We first provide the following example about the stochastic system G_ξ such that $\|G_\xi\|_{2\xi} < \|G_\xi\|_2$.

Example 1: Consider the system G_ξ given by

$$G_\xi : z_k = \xi_k w_k, \tag{14}$$

where w and z is the input and the output of G_ξ respectively and ξ is the stochastic process underlying G_ξ. In addition, ξ_k $(k \in \mathbf{N}_0)$ are independent and identically distributed (i.i.d.) with respect to k and obeys the uniform distribution $U(0,1)$. We can confirm that this G_ξ satisfies Assumption 2, and hence, $\|G_\xi\|_2 \ge \gamma_2^\star(G_\xi)$ by Theorem 1. For $\epsilon > 0$, consider

$$S_\epsilon := \{\omega \in \Omega : 1 - \epsilon \le \xi_k(\omega) \ (\forall k \in \mathbf{N}_0)\} \tag{15}$$

(note that $P(S_\epsilon) > 0$ for any $\epsilon > 0$). Then, we have

$$\begin{aligned} \gamma_2^\star(G_\xi)^2 &\ge \operatorname*{ess\,sup}_{\omega \in S_\epsilon} \|G(\omega)\|_{2\mathrm{d}}^2 \\ &= \operatorname*{ess\,sup}_{\omega \in S_\epsilon} \sup_{w \in l_{2\mathrm{d}}} \frac{\sum_{k=0}^\infty \xi_k(\omega)^2 w_k^2}{\|w\|_{2\mathrm{d}}^2} \\ &\ge (1-\epsilon)^2, \end{aligned} \tag{16}$$

where the third line is obtained by

$$\sum_{k=0}^\infty \xi_k(\omega)^2 w_k^2 \ge (1-\epsilon)^2 \|w\|_{2\mathrm{d}}^2 \qquad (\forall \omega \in S_\epsilon). \tag{17}$$

Hence, we have $\|G_\xi\|_2 \ge 1 - \epsilon$, which further leads to $\|G_\xi\|_2 \ge 1$ since $\epsilon > 0$ is arbitrary.

On the other hand, $\|G_\xi\|_{2\xi}$ is calculated as follows:

$$\begin{aligned} \|G_\xi\|_{2\xi}^2 &= \sup_{w \in l_{2\xi} \setminus \{0\}} \frac{\sum_{k=0}^\infty E[\xi_k^2 w_k^2]}{\sum_{k=0}^\infty E[w_k^2]} \\ &\overset{\text{(d)}}{=} \sup_{w \in l_{2\xi} \setminus \{0\}} \frac{E[\xi_0^2] \sum_{k=0}^\infty E[w_k^2]}{\sum_{k=0}^\infty E[w_k^2]} \\ &= E[\xi_0^2] = \int_0^1 x^2 dx = \frac{1}{3}. \end{aligned} \tag{18}$$

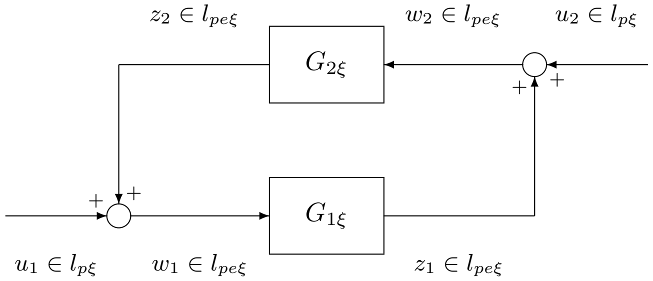

Fig. 4. Closed-loop system Σ in case when both $G_{1\xi}$ and $G_{2\xi}$ are ξ-restricted causal.

Since w_k is $\mathcal{F}_{k-1}$-measurable and ξ_k $(k \in \mathbf{N}_0)$ are i.i.d. with respect to k, w_k is independent of ξ_k at each k, and thus equality (d) holds. Hence, we see that there exists a gap between $\|G_\xi\|_{2\xi} = 1/\sqrt{3}$ and $\|G_\xi\|_2 \ge 1$ in this example, which in turn implies that $\|G_\xi\|_{2\xi}$ reflects the nature of ξ underlying G_ξ.

We next provide the following example, in which $\|G_\xi\|_{2\xi}$ is well-defined while $\|G_\xi\|_2$ is not; this implies that G_ξ is not $l_p(\Omega)$ stable but $l_{p\xi}(\Omega)$ stable.

Example 2: Consider the stochastic system G_ξ under the same problem setting as Example 1 except the distribution of ξ_k; here we consider the normal distribution $N(0,1)$ as the distribution of ξ_k, whose mean is 0 and standard deviation is 1. At first, let us show that $\|G_\xi\|_2$ is not well-defined. Since the system (14) is a static linear (stochastic) system, the following equality holds for the input $w \in l_2^\bullet(\Omega) := \{v \in l_2(\Omega) : v_i = 0 \ (\forall i \ge 1)\}$ without loss of generality:

$$\|G_\xi\|_2^2 = \sup_{w \in l_2^\bullet(\Omega),\ \|w\|_2 = 1} E[\xi_0^2 w_0^2]. \tag{19}$$

Now we take $a > 0$ and denote by $U(a)$ the set of $v \in l_2^\bullet(\Omega)$ such that $v(\omega) = 0$ for all $\omega \in \Omega$ satisfying $|\xi_0(\omega)| < a$. By (19), we have

$$\begin{aligned} \|G_\xi\|_2^2 &\ge \sup_{w \in U(a),\ \|w\|_2 = 1} E[\xi_0^2 w_0^2] \\ &\ge \sup_{w \in U(a),\ \|w\|_2 = 1} a^2 E[w_0^2] \\ &= a^2. \end{aligned} \tag{20}$$

Since $a > 0$ can be arbitrarily large, this indicates that $\|G_\xi\|_2$ is not well-defined.

On the other hand, $\|G_\xi\|_{2\xi}$ is calculated as $\|G_\xi\|_{2\xi}^2 = E[\xi_0^2]$, since w_0 and ξ_0 are independent, and thus $\|G_\xi\|_{2\xi}^2 = 1$ because ξ_0 obeys $N(0,1)$. Hence, $\|G_\xi\|_{2\xi}$ is well-defined.

V. SMALL-GAIN THEOREM FOR STOCHASTIC SYSTEMS

We consider the closed-loop system Σ depicted in Fig. 3 (more precisely Fig. 4, since we will assume ξ-restricted causality of G_1 and G_2) again, whose input is $u := [u^{1T}, u^{2T}]^T$ and output is $w = [w^{1T}, w^{2T}]^T$ and $z = [z^{1T}, z^{2T}]^T$. For this Σ, we assume the following, which is a sort of well-posedness assumption.

Assumption 3: There exists a unique output $[w^T, z^T]^T \in l_{pe}(\Omega)$ for each input $u \in l_{p\xi}(\Omega)$ in the closed loop system Σ.

Then, we can show the following theorem, which we call the small-gain theorem based on the ξ-restricted subspace of the stochastic l_p space.

Theorem 2: Suppose $p \in [1, \infty]$ is fixed, $G_i : (l_{pe}(\Omega), l_{pe}(\Omega))$ $(i = 1, 2)$ are ξ-restricted causal and $l_{p\xi}(\Omega)$ stable and the corresponding closed-loop system Σ (depicted in Fig. 3) satisfies Assumption 3. If there exist γ_1 and γ_2 such that $\|G_1\|_{p\xi} \le \gamma_1$, $\|G_2\|_{p\xi} \le \gamma_2$ and $\gamma_1\gamma_2 < 1$, then Σ is $l_{p\xi}$ stable.

VI. CONCLUSION

In this paper, we discussed l_p induced norms and the associated small-gain theorem for stochastic systems. At first, we introduced the general stochastic l_p space and its extended space in a fashion similar to the conventional studies [1],[2],[11],[12] about deterministic systems. Then, we gave definitions of $l_p(\Omega)$ stability and the primitive l_p induced norm for stochastic systems. However, this l_p induced norm was in fact not reflecting the nature of the stochastic process ξ underlying stochastic systems, which led us to overestimating its value. To circumvent this problem, we further introduced the ξ-restricted subspace of the stochastic l_p space and the ξ-restricted l_p induced norm. We confirmed through examples that this norm can take ξ into consideration. Then, through the use of the extended space of (the subspace of) the stochastic l_p space, which was not introduced in [10], we can give the small-gain theorem based on the norm by circumventing the arguments from becoming circular (as was indeed the case in [10]).

REFERENCES

[1] G. Zames, On the input-output stability of time-varying nonlinear feedback systems, Part I, *IEEE Trans. Auto. Control*, **11**-2, 228/238 (1966)

[2] G. Zames, On the input-output stability of time-varying nonlinear feedback systems, Part II, *IEEE Trans. Auto. Control*, **11**-2, 465/476 (1966)

[3] D. Hinrichsen and A. J. Pritchard, Stochastic H_∞, *SIAM J. Control and Optim.* **36**-5, 1504/1538 (1998)

[4] A. E. Bouhtouri, D. Hinrichsen and A. J. Pritchard, H_∞ type control for discrete-time stochastic systems, *Int. J. of Robust and Nonlinear Control*, **9**-13, 923/948 (1999)

[5] V. Dragan, A. Halanay and A. Stoica, A small gain theorem for linear stochastic systems, *Systems & control letters*, **30**-5, 243/251 (1997)

[6] V. Dragan, T. Morozan and A. Stoica, *Mathematical Methods in Robust Control of Discrete-Time Linear Stochastic Systems*, Springer (2010)

[7] J. Lu and R. E. Skelton, Mean-square small gain theorem for stochastic control:discrete-time case, *IEEE Trans. Auto. Control*, **47**-3, 490/494 (2002)

[8] Z. J. Wu, X. J. Xie and S. Y. Zhang, Adaptive backstepping controller design using stochastic small-gain theorem, *Automatica*, **43**-4, 608/620 (2007)

[9] J. Liu and M. French, A generalisation of the nonlinear small-gain theorem for systems with abstract initial Conditions, Proc 2013 European Control Conference, 1699/1704 (2013)

[10] S. Wan and L. Qiu, A mixed deterministic and stochastic small gain theorem and its application to networked stabilization, Proc. 52nd IEEE Conference on Decision and Control, 6385/6390 (2013)

[11] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed., Prentice Hall (1993)

[12] H. J. Marquez, *Nonlinear Control Systems: Analysis and Design*, John Wiley & Sons (2003)

[13] L. F. Richardson, *Measure and Integration: A Concise Introduction to Real Analysis*, John Wiley & Sons (2009)

[14] M. Capiński and E. Kopp, *Measure, Integral and Probability*, 2nd ed., Springer-Verlag London (2004)

[15] J. S. Rosenthal, *A First Look at Rigorous Probability Theory*, 2nd ed., World Scientific Publishing Co. Pte. Ltd. (2006)