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Output Robust Control of Input-Saturated Plants with Anti-Windup Compensation

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Abstract— **In this paper the problem of output robust control for SISO systems with input saturation is addressed. The proposed approach is based on the high-gain control principle augmented with the integral loop and anti-windup compensation scheme. Stability analysis and simulation results are presented in the paper.**

I. INTRODUCTION

This paper is focused on the development of simple control algorithms under condition of bounded plant inputs. Regulators with a complicated structure might not be suitable for a range of applications due to the need of significant computing resources to process all signals. State controllers require additional sensors to measure output derivatives of the output variable, which might be unacceptable due to different reasons. Output robust control algorithms based on the high-gain principle are extremely useful in such applications. This research represents theoretical extension of the consecutive compensator approach [1] to systems with saturated inputs.

The performance decrease caused by the input saturation is a challenging issue arising from implementing control algorithms especially containing the integral term to real applications. Plant inputs are always bounded due to hardware constraints. If the control signal saturates, accumulating error results in the integral windup effect, which in turn leads to increasing overshoot value and settling time. This issue can be resolved using the so-called anti-windup technique. There is a series of its approaches and modifications [2]. The internal model control as the anti-windup scheme is developed in [3] and [4]. The design technique output regulator with nonlinear internal model is considered in [5], [6]. Stability questions of systems with anti-windup control are considered in [7]. Determination of stability regions for discrete-time linear system with saturating control through anti-windup schemes is considered in [8]. Problem of saturated control for a robotic boat is addressed in [9] and [10], for quadcopters in [11], [12], [13].

In this paper the robust output controller augmented with the integral loop and the anti-windup compensation scheme is designed. Addition of the integral loop together with anti-windup scheme has been done following the idea of the internal model principle [14]. This study provides stability analysis of the closed loop system represented as the interconnection of a linear part and saturation nonlinearity satisfying the sector conditions. Such approach of using Lurie form to represent the nonlinear system is used frequently to prove absolute stability of the closed-loop system (e.g. see [15], [16]).

This paper is organized as follows. The addressed problem is stated in Section II. The control design and stability analysis are provided in Section IV. A numerical example to illustrate efficiency of the proposed approach is given in Section V. Finally, the paper is summarized in Conclusions.

II. PROBLEM STATEMENT

Consider the SISO system

$$\dot{x} = Ax + bu + Rw, \tag{1}$$
$$y = c^T x, \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^{n_w}$ is the disturbance vector, $y \in \mathbb{R}^1$ is the measurable output, u is the control input, A, R, b, c are matrices and vectors of corresponding dimensions.

The disturbance vector w is considered as the state of the linear system

$$\dot{w} = Sw, \tag{3}$$

where in general S is a matrix of the corresponding dimension. The control signal u satisfies the saturation condition

$$u = \mathrm{sat}(\upsilon) = \begin{cases} u_{max} & \text{if} \quad \upsilon \geq u_{max}, \\ u & \text{if} \quad u_{min} < \upsilon < u_{max}, \\ u_{min} & \text{if} \quad \upsilon \leq u_{min}, \end{cases} \tag{4}$$

where u_{min} and u_{max} are the input saturation limits, υ is the control signal generated by the nominal linear controller.

Assumption 1: The plant (1), (2) is minimum phase (its zero dynamics is stable).

Assumption 2: The relative degree $\rho \geq 1$ of the transfer function $\frac{b(s)}{a(s)} = c^T(sI - A)^{-1}b$ is known.

Assumption 3: The pair (A, b) is controllable and the pair (A, c) is observable.

Assumption 4: The output of the disturbance generator (3) is a static signal, i.e. $S = 0$.

Assumption 5: The input saturation limits u_{min} and u_{max} satisfy

$$u_{min} = -C < 0, \quad u_{max} = C > 0,$$

where C is a positive number.

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Assumption 6: The disturbance is bounded $w \in \mathcal{L}_\infty$ and the nominal control signal u_0 needed for its compensation at steady state satisfies

$$u_{min} \le |u_0| \le u_{max}.$$

The purpose of this study is to design a control law u based on the output variable y such that

$$\lim_{t\to\infty} y(t) = 0$$

under conditions of external static disturbance and integral windup.

III. PRELIMINARY TRANSFORMATIONS

Perform the change of coordinates to extract zero dynamics of the plant. The model (1), (2) in the input-output representation is given as

$$y(s) = \frac{b(s)}{a(s)}\bar{u}(s) + \frac{r(s)}{a(s)}w(s), \tag{5}$$

where $b(s)$ is the Hurwitz polynomial due to Assumption 1.

Transform the model (5) as

$$\frac{a(s)}{b(s)}y(s) = \bar{u}(s) + \frac{r(s)}{b(s)}w(s),$$

$$\left(c(s) + \frac{d(s)}{b(s)}\right)y(s) = \bar{u}(s) + \left(r_2(s) + \frac{r_1(s)}{b(s)}\right)w(s),$$

$$c(s)y(s) = \left(\bar{u}(s) - \frac{d(s)}{b(s)}y(s) + \frac{r_1(s)}{b(s)}w(s)\right) + \\ + r_2(s)w(s). \tag{6}$$

Degrees of the polynomials $d(s)$, $r_2(s)$ $c(s)$, $b(s)$, $r_1(s)$ equal $n-\rho-1$, $n-\rho-1$, ρ, $n-\rho$, $\rho-1$, respectively.

Rewrite the plant model (6) as

$$\begin{aligned} z(s) &= \frac{d(s)}{b(s)}y(s) - \frac{r_1(s)}{b(s)} \\ y(s) &= \frac{1}{c(s)}\left(\bar{u}(s) - z(s)\right) + \frac{r_2(s)}{c(s)}w(s), \end{aligned}$$

Write the state-space plant model

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \bar{u} + \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} w, && (7) \\ y &= \begin{bmatrix} 0 & c_2^T \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, && (8) \end{aligned}$$

where A_{11} is the Hurwitz matrix due to Assumption 1, if the matrix A_{22} is chosen in companion form, the vectors b_2 and c_2 can be considered as

$$b_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

IV. CONTROL DESIGN

A. Control Strategy

In order to increase preciseness of the control system and ensure convergence of the steady-state error to zero set $S = 0$ of the disturbance generator (3) and choose the control law

$$\begin{aligned} \upsilon &= -\kappa(c_q^T\xi + y) - \gamma\eta, && (9) \\ \dot{\xi} &= A_q\xi + b_q y, && (10) \\ \dot{\eta} &= \kappa(c_q^T\xi + y) + \nu\psi(\upsilon), && (11) \\ \psi(\upsilon) &= \upsilon - \mathrm{sat}(\upsilon), && (12) \end{aligned}$$

where $\psi(\upsilon)$ is the memoryless nonlinearity, $\kappa > 0$, $\gamma > 0$, $\nu > 0$, A_q, b_q, c_q are given as

$$A_q = \begin{bmatrix} -q'_\rho\sigma & 1 & 0 & \cdots & 0 \\ -q'_{\rho-1}\sigma^2 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ -q'_2\sigma^{\rho-1} & 0 & 0 & \cdots & 1 \\ -q'_1\sigma^\rho & 0 & 0 & \cdots & 0 \end{bmatrix}, b_q = \begin{bmatrix} q'_\rho\sigma \\ q'_{\rho-1}\sigma^2 \\ \vdots \\ q'_2\sigma^{\rho-1} \\ q'_1\sigma^\rho \end{bmatrix}, c_q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{\rho-1} \\ q_\rho \end{bmatrix},$$

where $\sigma > 0$ and q'_i $(i = \overline{1,\rho})$ are chosen for the system (10) to be Hurwitz, q_i $(i = \overline{1,\rho})$ are coefficients of an arbitrary polynomial of the degree $\rho - 1$

$$q(s) = q_\rho s^{\rho-1} + \cdots + q_2 s + q_1.$$

Combining the plant model (7), (8) and the control law (9)–(12) get the model of the closed-loop system (13).

Perform the change of coordinates $\chi = z_2 - \xi$ and compute the derivative

$$\begin{aligned} \dot{\chi} &= \dot{z}_2 - \dot{\xi} = \\ &= A_{21}z_1 + A_{22}z_2 - \kappa b_2 c_2^T z_2 - \kappa b_2 c_q^T(z_2 - \chi) - \\ &\quad -b_2\gamma\eta - b_2\psi + R_2 w - b_q c_2^T z_2 - A_q(z_2 - \chi) = \\ &= A_{21}z_1 + (A_{22} - I_0 - \kappa b_2(c_q^T + c_2^T))z_2 + \\ &\quad +(A_q + \kappa b_2 c_q^T)\chi - b_2\gamma\eta - b_2\psi + R_2 w, \end{aligned}$$

where

$$I_0 = A_q + b_q c_2^T = \begin{bmatrix} 0\,1\,0 \cdots 0 \\ 0\,0\,1 \cdots 0 \\ \vdots \\ 0\,0\,0 \cdots 0 \end{bmatrix}.$$

This change of coordinates puts the system (13) into the form (14).

Perform the second change of variables $\zeta = (b_2^T b_2)^{-1} b_2^T z_2 + \eta$ and compute the derivative

$$\begin{aligned} \dot{\zeta} &= (b_2^T b_2)^{-1} b_2^T \dot{z}_2 + \dot{\eta} = \\ &= (b_2^T b_2)^{-1} b_2^T A_{21} z_1 + (b_2^T b_2)^{-1} b_2^T (A_{22} + \gamma I) z_2 - \\ &\quad -\gamma\zeta + (\nu - 1)\psi + (b_2^T b_2)^{-1} b_2^T R_2 w, \end{aligned} \tag{17}$$

where I is the identity matrix of the corresponding dimension.

For the sake of simplicity denote $\bar{b}_2^T = (b_2^T b_2)^{-1} b_2^T$. This change of coordinates puts the system (14) into the form (15), which can be then rewritten as (16).

$$\begin{bmatrix}\dot z_1\\ \dot z_2\\ \dot\xi\\ \dot\eta\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12} & 0 & 0\\ A_{21} & A_{22}-\kappa b_2 c_2^T & -\kappa b_2 c_q^T & -b_2\gamma\\ 0 & b_q c_2^T & A_q & 0\\ 0 & \kappa c_2^T & \kappa c_q^T & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ \xi\\ \eta\end{bmatrix} + \begin{bmatrix}0\\ -b_2\\ 0\\ \nu\end{bmatrix}\psi + \begin{bmatrix}R_1\\ R_2\\ 0\\ 0\end{bmatrix}w. \tag{13}$$

$$\begin{bmatrix}\dot z_1\\ \dot z_2\\ \dot\chi\\ \dot\eta\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12} & 0 & 0\\ A_{21} & A_{22}-\kappa b_2(c_q^T+c_2^T) & \kappa b_2 c_q^T & -b_2\gamma\\ A_{21} & A_{22}-I_0-\kappa b_2(c_q^T+c_2^T) & A_q+\kappa b_2 c_q^T & -b_2\gamma\\ 0 & \kappa(c_q^T+c_2^T) & -\kappa c_q^T & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ \chi\\ \eta\end{bmatrix} + \begin{bmatrix}0\\ -b_2\\ -b_2\\ \nu\end{bmatrix}\psi + \begin{bmatrix}R_1\\ R_2\\ R_2\\ 0\end{bmatrix}w. \tag{14}$$

$$\begin{bmatrix}\dot z_1\\ \dot z_2\\ \dot\chi\\ \dot\zeta\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12} & 0 & 0\\ A_{21} & A_{22}-\kappa b_2(c_q^T+c_2^T)+\gamma b_2\bar b_2^T & \kappa b_2 c_q^T & -b_2\gamma\\ A_{21} & A_{22}-I_0-\kappa b_2(c_q^T+c_2^T)+\gamma b_2\bar b_2^T & A_q+\kappa b_2 c_q^T & -b_2\gamma\\ \bar b_2^T A_{21} & \bar b_2^T(A_{22}+\gamma I) & 0 & -\gamma\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ \chi\\ \zeta\end{bmatrix} + \begin{bmatrix}0\\ -b_2\\ -b_2\\ (\nu-1)\end{bmatrix}\psi + \begin{bmatrix}R_1\\ R_2\\ R_2\\ \bar b_2^T R_2\end{bmatrix}w. \tag{15}$$

$$\begin{bmatrix}\dot z_1\\ \dot\zeta\\ \dot z_2\\ \dot\chi\end{bmatrix} = \begin{bmatrix}A_{11} & 0 & A_{12} & 0\\ \bar b_2^T A_{21} & -\gamma & \bar b_2^T(A_{22}+\gamma I) & 0\\ A_{21} & -b_2\gamma & A_{22}-\kappa b_2(c_q^T+c_2^T)+\gamma b_2\bar b_2^T & \kappa b_2 c_q^T\\ A_{21} & -b_2\gamma & A_{22}-I_0-\kappa b_2(c_q^T+c_2^T)+\gamma b_2\bar b_2^T & A_q+\kappa b_2 c_q^T\end{bmatrix}\begin{bmatrix}z_1\\ \zeta\\ z_2\\ \chi\end{bmatrix} + \begin{bmatrix}0\\ (\nu-1)\\ -b_2\\ -b_2\end{bmatrix}\psi + \begin{bmatrix}R_1\\ \bar b_2^T R_2\\ R_2\\ R_2\end{bmatrix}w. \tag{16}$$

The matrix A_{11} is Hurwitz due to Assumption 1 and the chosen basis of the model (7), (8). Further, the block

$$\mathbf{A}_1 = \begin{bmatrix}A_{11} & 0\\ \bar b_2^T A_{21} & -\gamma\end{bmatrix}$$

is Hurwitz due to the choice of the parameter $\gamma > 0$. Indeed, a system with the same state matrix can be seen as a serial connection of two stable subsystems.

Next, there exists a number κ_0 such that for $\forall\kappa \geq \kappa_0$ the block

$$\mathbf{A}_2 = \begin{bmatrix}A_{11} & 0 & A_{12}\\ \bar b_2^T A_{21} & -\gamma & \bar b_2^T(A_{22}+\gamma I)\\ A_{21} & -b_2\gamma & A_{22}-\kappa b_2(c_q^T+c_2^T)+\gamma b_2\bar b_2^T\end{bmatrix}$$

is Hurwitz. Indeed, only the element $(3,3)$ depends on the parameter κ. The remaining blocks will not be affected if this parameter increases. As a consequence, a sufficiently large value of κ ensures Hurwitzness of this block matrix.

Finally, the overall state matrix of the closed-loop system (16) can be forced to be Hurwitz due to the structure of the matrix A_q. Indeed, the parameter σ is included only in the block $(4,4)$, while the remaining block elements do not depend on it. As a result, its increase ensures shift of eigenvalues to the left from the imaginary axis of the complex plane. Summarize all these steps in Statement.

Statement 1: Consider the plant model given at state-space (7), (8) and the control law (9)–(12). Hurwitzness of the state matrix of the resultant closed-loop system (16) follows from Hurwitzness of all its diagonal blocks, which is forced by the parameters γ, κ, σ.

B. Stability Analysis of System with Input Saturation

Consider the case without external disturbances ($w = 0$). Represent the closed-loop system (16) in a compact form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\psi(\upsilon), \tag{18}$$
$$\upsilon = \mathbf{c}^T\mathbf{x}, \tag{19}$$

where

$$\mathbf{x}=\begin{bmatrix}x\\ \xi\\ \eta\end{bmatrix}, \mathbf{A}=\begin{bmatrix}A-\kappa bc^T & -\kappa bc_q^T & -\gamma b\\ b_q c^T & A_q & 0\\ \kappa c^T & \kappa c_q^T & 0\end{bmatrix}, \mathbf{b}=\begin{bmatrix}-b\\ 0\\ \nu\end{bmatrix}, \mathbf{c}=\begin{bmatrix}-\kappa c\\ -\kappa c_q\\ -\gamma\end{bmatrix}.$$

Statement 2: The closed-loop system consisting of the linear part (18)–(19) with Hurwitz state matrix $\mathbf{A}$ and the memoryless nonlinear function $\psi(\upsilon)$ is absolutely stable for all initial conditions.

Proof: In accordance to the Popov criterion [17], the closed-loop system consisting of the linear part (18)–(19) with Hurwitz state matrix $\mathbf{A}$ and the memoryless nonlinear function $\psi(\upsilon)$ is absolutely stable for all initial conditions if there exists a constant $\varpi \geq 0$ such that

$$(1 + \lambda_i\varpi) \neq 0$$

for each eigenvalue λ_i of the matrix $\mathbf{A}$ and the transfer function

$$W(s) = 1 + (1 + s\varpi)W_\ell(s)$$

strictly positive real, where $W_\ell(s)$ is the transfer function of the system (18)–(19).

Choose $\varpi = 0$, then to prove the absolute stability of the system it is sufficient to show strictly positive realness of the transfer function

$$W(s) = 1 + W_\ell(s).$$

Calculate the transfer function $W_\ell(s)$

$$\begin{aligned} W_\ell(s) &= \mathbf{c}^T(sI-\mathbf{A})^{-1}\mathbf{b}\\ &= \begin{bmatrix}-\kappa c^T & -\kappa c_q^T & -\gamma\end{bmatrix}\mathcal{A}\begin{bmatrix}-b\\ 0\\ \nu\end{bmatrix}. \end{aligned}$$

where

$$\mathcal{A} = (sI-\mathbf{A})^{-1} = \begin{bmatrix}sI-A+\kappa bc^T & \kappa bc_q^T & \gamma b\\ -b_q c^T & sI-A_q & 0\\ -\kappa c^T & -\kappa c_q^T & s\end{bmatrix}^{-1},$$

Represent the matrix $\mathcal{A}$ in the block form

$$\mathcal{A} = \bar{\mathcal{A}}^{-1} = \begin{bmatrix} \bar{\mathcal{A}}_{11} \, \bar{\mathcal{A}}_{12} \\ \bar{\mathcal{A}}_{21} \, \bar{\mathcal{A}}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \mathcal{A}_{11} \, \mathcal{A}_{12} \\ \mathcal{A}_{21} \, \mathcal{A}_{22} \end{bmatrix}, \tag{20}$$

where

$$\bar{\mathcal{A}}_{11} = sI - A + \kappa b c^T, \qquad \bar{\mathcal{A}}_{12} = \begin{bmatrix} \kappa b c_q^T \, \gamma b \end{bmatrix},$$
$$\bar{\mathcal{A}}_{21} = \begin{bmatrix} -b_q c^T \\ -\kappa c^T \end{bmatrix}, \qquad \bar{\mathcal{A}}_{22} = \begin{bmatrix} sI - A_q \; 0 \\ -\kappa c_q^T \;\; s \end{bmatrix},$$

and use the Frobenius formula for block matrices inversion.

Calculate the inverse matrix of the block $\bar{\mathcal{A}}_{22}$

$$\bar{\mathcal{A}}_{22}^{-1} = \begin{bmatrix} sI - A_q \; 0 \\ -\kappa c_q^T \;\; s \end{bmatrix}^{-1} = \begin{bmatrix} \mathcal{A}_q & 0 \\ \frac{\kappa}{s} c_q^T \mathcal{A}_q & \frac{1}{s} \end{bmatrix},$$

where $\mathcal{A}_q = (sI - A_q)^{-1}$.

Calculate the block $\mathcal{A}_{11}$

$$\begin{aligned} \mathcal{A}_{11} &= \left(\bar{\mathcal{A}}_{11} - \bar{\mathcal{A}}_{12} \bar{\mathcal{A}}_{22}^{-1} \bar{\mathcal{A}}_{21} \right)^{-1} \\ &= \left(sI - A + b \frac{\kappa b_q(s)(s+\gamma)}{s a_q(s)} c^T \right)^{-1}, \end{aligned}$$

where

$$\frac{b_q(s)}{a_q(s)} = c_q^T \mathcal{A}_q(s) b_q + 1. \tag{21}$$

Remark 1: Note that, Hurwitzness of the numerator and denominator of the transfer function (21) can be achieved by choosing matrix A_q and vectors b_q and c_q.

Calculate the block $\mathcal{A}_{12}$

$$\begin{aligned} \mathcal{A}_{12} &= -\mathcal{A}_{11} \bar{\mathcal{A}}_{12} \bar{\mathcal{A}}_{22}^{-1} \\ &= \begin{bmatrix} -\frac{\kappa(s+\gamma)}{s} \mathcal{A}_{11} b c_q^T \mathcal{A}_q(s) & -\frac{\gamma}{s} \mathcal{A}_{11} b \end{bmatrix}. \end{aligned}$$

Calculate the block $\mathcal{A}_{21}$

$$\begin{aligned} \mathcal{A}_{21} &= -\bar{\mathcal{A}}_{22}^{-1} \bar{\mathcal{A}}_{21} \mathcal{A}_{11} \\ &= \begin{bmatrix} \mathcal{A}_q(s) b_q c^T \\ \frac{\kappa}{s} (c_q^T \mathcal{A}_q(s) b_q + 1) c^T \end{bmatrix} \mathcal{A}_{11} \\ &= \begin{bmatrix} \mathcal{A}_q(s) b_q c^T \mathcal{A}_{11} \\ \frac{\kappa b_q(s)}{s a_q(s)} c^T \mathcal{A}_{11} \end{bmatrix}. \end{aligned}$$

Calculate the block $\mathcal{A}_{22}$

$$\begin{aligned} \mathcal{A}_{22} &= \bar{\mathcal{A}}_{22}^{-1} - \mathcal{A}_{21} \bar{\mathcal{A}}_{12} \bar{\mathcal{A}}_{22}^{-1} \\ &= \begin{bmatrix} \left(1 - \frac{\kappa(s+\gamma)}{s} \mathcal{A}_q(s) b_q \frac{\beta(s)}{\alpha(s)} c_q^T \right) \mathcal{A}_q(s) & -\frac{\gamma}{s} \mathcal{A}_q(s) b_q \frac{\beta(s)}{\alpha(s)} \\ \left(1 - \frac{\kappa(s+\gamma) b_q(s)}{s a_q(s)} \frac{\beta(s)}{\alpha(s)} \right) \frac{\kappa}{s} c_q^T \mathcal{A}_q(s) & \left(1 - \frac{\gamma\kappa b_q(s)}{s a_q(s)} \frac{\beta(s)}{\alpha(s)} \right) \frac{1}{s} \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \frac{\beta(s)}{\alpha(s)} &= c^T \mathcal{A}_{11} b \\ &= \frac{b(s) s a_q(s)}{s a_q(s) a(s) + \kappa b_q(s)(s+\gamma) b(s)}. \end{aligned} \tag{22}$$

Substitute derived expressions of the blocks $\mathcal{A}_{11}$, $\mathcal{A}_{12}$, $\mathcal{A}_{21}$ and $\mathcal{A}_{22}$ into the matrix $\mathcal{A}$ (for the sake of brevity the middle column is omitted)

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & * & -\frac{\gamma}{s} \mathcal{A}_{11} b \\ \mathcal{A}_q(s) b_q c^T \mathcal{A}_{11} & * & -\frac{\gamma}{s} \mathcal{A}_q(s) b_q \frac{\beta(s)}{\alpha(s)} \\ \frac{\kappa b_q(s)}{s a_q(s)} c^T \mathcal{A}_{11} & * & \left(1 - \frac{\gamma\kappa b_q(s)}{s a_q(s)} \frac{\beta(s)}{\alpha(s)} \right) \frac{1}{s} \end{bmatrix}.$$

Calculate the transfer function $W_\ell(s)$

$$\begin{aligned} W_\ell(s) &= \begin{bmatrix} -\kappa c^T \; -\kappa c_q^T \; -\gamma \end{bmatrix} \mathcal{A} \begin{bmatrix} -b \\ 0 \\ \nu \end{bmatrix} \\ &= \kappa c^T \mathcal{A}_{11} b + \kappa c_q^T \mathcal{A}_q(s) b_q c^T \mathcal{A}_{11} b \\ &\quad + \frac{\gamma\kappa b_q(s)}{s a_q(s)} c^T \mathcal{A}_{11} b + \frac{\gamma\nu\kappa}{s} c^T \mathcal{A}_{11} b \\ &\quad + \frac{\gamma\nu\kappa}{s} c_q^T \mathcal{A}_q(s) b_q \frac{\beta(s)}{\alpha(s)} - \frac{\gamma\nu}{s} \\ &\quad + \frac{\gamma^2 \nu\kappa b_q(s)}{s^2 a_q(s)} \frac{\beta(s)}{\alpha(s)} \\ &= \Bigg(1 + c_q^T \mathcal{A}_q(s) b_q + \frac{\gamma b_q(s)}{s a_q(s)} \\ &\quad + (1 + c_q^T \mathcal{A}_q(s) b_q) \frac{\gamma\nu}{s} + \frac{\gamma^2 \nu b_q(s)}{s^2 a_q(s)} \Bigg) \kappa \frac{\beta(s)}{\alpha(s)} \\ &\quad - \frac{\gamma\nu}{s} \\ &= \frac{\kappa(s+\gamma\nu)(s+\gamma) b_q(s)}{s^2 a_q(s)} \frac{\beta(s)}{\alpha(s)} - \frac{\gamma\nu}{s} \\ &= \frac{\kappa(s+\gamma\nu)(s+\gamma) b_q(s) b(s)}{s(s a_q(s) a(s) + \kappa b_q(s)(s+\gamma) b(s))} - \frac{\gamma\nu}{s} \\ &= \frac{\beta_\kappa(s) - \gamma\nu a_q(s) a(s)}{s a_q(s) a(s) + \beta_\kappa(s)}, \end{aligned}$$

where

$$\beta_\kappa(s) = \kappa b_q(s)(s+\gamma) b(s). \tag{23}$$

Next, calculate the transfer function $W(s)$ and show its strictly positive realness

$$\begin{aligned} W(s) &= W_\ell(s) + 1 \\ &= \frac{2\beta_\kappa(s) + (s - \gamma\nu) a_q(s) a(s)}{s a_q(s) a(s) + \beta_\kappa(s)} \\ &= \frac{2\kappa b_q(s)(s+\gamma) b(s) + (s - \gamma\nu) a_q(s) a(s)}{s a_q(s) a(s) + \kappa b_q(s)(s+\gamma) b(s)}. \end{aligned}$$

Indeed, together with Remark 1 and Assumption 1 it easy to show, that there exists a number κ_0 such that for $\kappa \geq \kappa_0$ both the numerator and denominator of the transfer function $W(s)$ are Hurwitz. The relative degree of $W(s)$ is zero. As a consequence, strictly positive realness follows, i.e.

$$\operatorname{Re} W(j\omega) > 0, \quad \forall \omega \in [-\infty, \infty]$$

or equivalently

$$\operatorname{Re} W_\ell(j\omega) > -1, \quad \forall \omega \in [-\infty, \infty],$$

and the absolute stability of the system (18), (19) follows in accordance to the Popov criterion [17]. ■

C. Analysis of Steady-State Error Convergence

Return to the case of external disturbance presence ($w \neq 0$) and analyze the steady error. Obviously, the following relation holds at steady state

$$\upsilon = \mathrm{sat}(\upsilon),$$

consequently

$$\psi(\upsilon) = \upsilon - \mathrm{sat}(\upsilon) = 0.$$

In order to determine steady state behavior use the Sylvester equation applied to the model (13)

$$\begin{bmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_\xi \\ \Sigma \end{bmatrix} S = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} - \kappa b_2 c_2^T & -\kappa b_2 c_q^T & -b_2\gamma \\ 0 & b_q c_2^T & A_q & 0 \\ 0 & \kappa c_2^T & \kappa c_q^T & 0 \end{bmatrix} \begin{bmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_\xi \\ \Sigma \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \\ 0 \\ 0 \end{bmatrix},$$

where focus on the fourth row

$$\Sigma S = \kappa(c_2^T \Pi_2 + c_q^T \Pi_\xi),$$

which shows that at steady state

$$c_2^T \Pi_2 + c_q^T \Pi_\xi = 0. \tag{24}$$

Find the relation between $c_2^T \Pi_2$ and $c_q^T \Pi_\xi$. Consider the auxiliary variable z_0

$$z_0 = y + c_q^T \xi, \tag{25}$$

taking into account (24), the steady state value of which is zero.

From (10) follows

$$\xi(s) = (sI - A_q)^{-1}(b_q y(s) + \xi(0)),$$

where $\xi(0)$ is a vector of initial conditions.

Then, rewrite (25)

$$\begin{aligned} z_0(s) &= y(s) + c_q^T (sI - A_q)^{-1}(b_q y(s) + \xi_1(0) + \xi_2(0)) = \\ &= (c_q^T (sI - A_q)^{-1} b_q + 1) y(s) + \varepsilon(s), \end{aligned}$$

where $\varepsilon(s) = c_q^T (sI - A_q)^{-1}(\xi_1(0) + \xi_2(0))$ corresponds to the exponentially decaying function $\varepsilon(t)$.

If c_q^T is chosen so that the numerator of the transfer function $(c_q^T (sI - A_q)^{-1} b_q + 1)$ is Hurwitz and the relative degree is zero, then from

$$y(s) = (c_q^T (sI - A_q)^{-1} b_q + 1)^{-1}(z_0(s) - \varepsilon(s))$$

find that the steady-state error of y as well as $c_q^T \xi$ converges to zero.

V. NUMERICAL EXAMPLE

Consider the system (1), (2) with parameters

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$R = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

saturation limits $u_{max} = 5$ and $u_{min} = -5$ and disturbance generator (3) with parameters

$$S = 0, \quad w(0) = 2.$$

Find the transfer function of the plant

$$\frac{b(s)}{a(s)} = c^T (sI - A)^{-1} b = \frac{1}{s^2 + s},$$

from where define the relative degree $\rho = 2$, taking into account what choose the control law (9)–(12) with parameters $\kappa = 2$, $\gamma = 1$, $\nu = 1$,

$$A_q = \begin{bmatrix} -10 & 1 \\ -100 & 0 \end{bmatrix}, \quad b_q = \begin{bmatrix} 10 \\ 100 \end{bmatrix}, \quad c_q = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $\sigma = 10$.

Transfer function $W(s)$ form:

$$W(s) = \frac{s^5 + 10s^4 + 103s^3 + 474s^2 + 1180s + 800}{s^5 + 11s^4 + 112s^3 + 342s^2 + 640s + 400}. \tag{26}$$

The simulation results are shown in Fig. 1. The Nyquist plot of the transfer function (26) is located strictly to the right of the imaginary axis (see Fig 1a). The Popov plot [17] is located to the right of the vertical line passing through the point $-1 + j0$ (see Fig. 1b). The plot shown in Fig. 1c illustrates the control signal bounded with the saturation limits $u_{max} = 5$ and $u_{min} = -5$. Output signals of three simulation runs are given in Fig. 1d. As one can see deactivation of the integral loop ($\gamma = 0$ and $\nu = 0$) causes the steady state error. The controller augmented with the integral component but excluding the anti-windup scheme ($\gamma \neq 0$ and $\nu = 0$) allows to compensate this error with the significant output value overshoot. The proposed control algorithm with all the terms activated ($\gamma \neq 0$ and $\nu \neq 0$) provides smooth transient behavior with the steady state error eliminated and the overshoot reduced.

VI. CONCLUSIONS

This paper is devoted to development of output robust control algorithm of input-saturated plants with anti-windup compensation. The absolute stability of closed-loop system is proved by the Popov criterion. The developed algorithm takes into account input saturation effects and can be applied for plants with uncertain parameters. This is significantly useful for real technical systems.

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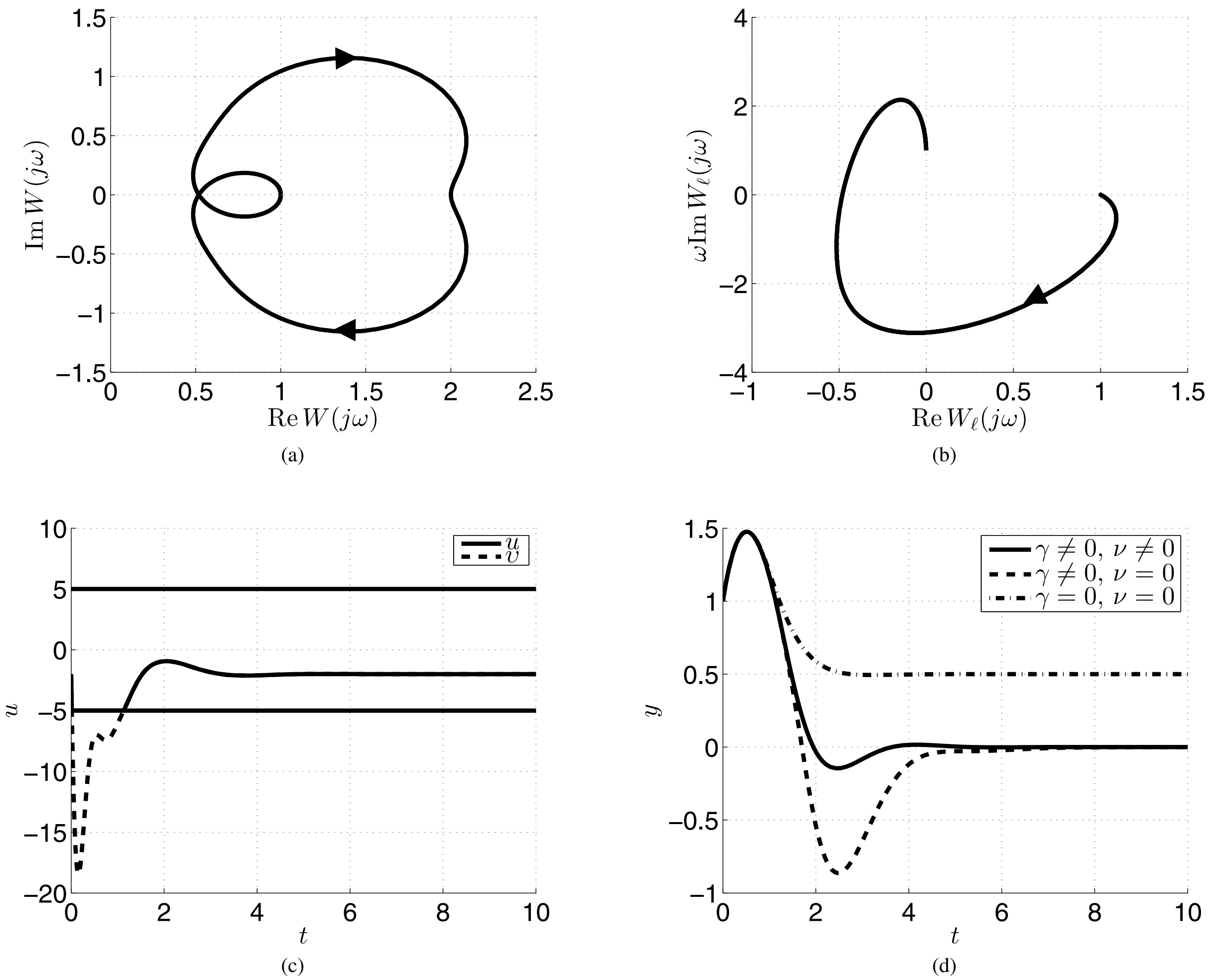

Fig. 1: Simulation results

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