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Partial Phase Cohesiveness in Networks of Communitinized Kuramoto Oscillators

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Abstract— **Partial synchronization of neuronal ensembles are often observed in the human brain, which is believed to facilitate communication among anatomical regions demanded by cognitive tasks. Since such neurons are commonly modeled by oscillators, to better understand their partial synchronization behavior, in this paper we study community-driven partial phase cohesiveness in networks of communitinized Kuramoto oscillators, where each community itself consists of a population of all-to-all coupled oscillators. Sufficient conditions on the algebraic connectivity of the *selected* communities are obtained to guarantee the appearance of their phase cohesiveness, while leaving the remaining communities incoherent. These conditions are further reduced to the form of the lower bounds on the coupling strengths for the connections linking the selected communities. We also show that the ultimate level of the phase cohesiveness that the oscillators asymptotically converge to is predictable. Finally, numeral studies are performed to validate the obtained results.**

I. INTRODUCTION

Synchronization phenomena have been observed pervasively in complex networks in various scientific disciplines, including physics, chemistry, biology, social science and neural science [1]–[3]. The human brain is a typical example of such complex networks. It has been found that neuronal ensembles in the human brain have the intrinsic property to behave as oscillators, which are also connected by chemical and electrical couplings [4]. The pattern of synchronization (or phase cohesiveness), which has been seen across regions of human brain, is believed to facilitate communication among neuronal ensembles [5]. Only cohesively oscillating neuronal ensembles can exchange information effectively because their input and output windows are open at the same time [6]. However, abnormal patterns of synchronization can trigger neurological disorder, which is regarded as the cause of some serious diseases. In particular, epileptic patients often experience global phase cohesiveness in their brain [7], while healthy people do not. It suggests that a non-pathological brain has powerful regulation mechanisms not only to render synchronization, but also to prevent unnecessary synchronization among the neuronal ensembles that do not need communication [6].

The *Kuramoto model* and its variances serve as notable tools to describe the dynamics of coupled neuronal ensembles, enabling researchers to study the synchronization phenomena in the human brain analytically. It was first introduced by Kuramoto [8], and has been broadly applied to various scientific fields. Extensive attention has been attracted on the global or complete synchronization, where all the oscillators in the network reach phase cohesiveness under an identical frequency. Conditions on the critical coupling strengths to achieve global synchronization are obtained [9]–[11]. As a more complicated synchronization pattern, the partial synchronization or partial cohesiveness, whereby only a part of oscillators achieve phase cohesiveness and others remain incoherent, ideally characterizes the behaviors evidenced in the human brain. The number of studies addressing the partial synchronization, e.g., [12], [13], is increasing. However, despite some recent progresses, e.g., [14], [15], analytical results on partial synchronization are much fewer. Moreover, to the best of our knowledge, all the results for modeling dynamics in the human brain rely on the simplifying treatment that the dynamics of the neuronal ensemble in an anatomical region of the brain is modeled as a single oscillator. In fact, anatomical regions of the brain often exhibit a “network-of-networks” topology. Each anatomical region can consist of heterogeneous neuronal ensembles, which can interact not only within a region but also with neurons from other regions [16], [17]. This motivates us to study the more challenging and perhaps more precise model introduced in [18], where each neuronal ensemble is regarded as an oscillator, resulting in a community of all-to-all coupled oscillators in an anatomical region in the brain. At a higher level, these communities are also interconnected, but do not form a complete graph in general.

In this paper, we study the partial phase cohesiveness in such a network of coupled Kuramoto oscillator communities. Motivated by the observation that synchronization in the brain may emerge from one anatomical region [19], we investigate how a single community is able to drive some other communities and itself to phase cohesiveness by increasing the corresponding coupling strengths selectively, while the remaining communities behave incoherently. The contribution of this paper is threefold. First, different from the existing theoretical results, we consider the Kuramoto oscillators that coupled by a “network of networks” topology. Second, sufficient conditions on coupling strengths and algebraic connectivities within the selected communities are obtained to facilitate partial phase cohesiveness. Since we do not restrict ourselves to exact phase synchronization in the selected communities, the conditions provided in [15], such as the same external couplings and identity of the natural frequencies in these communities, are not required

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in our work. Third, we provide the analysis of region of attraction and present the ultimate level of the partial phase cohesiveness.

The remainder of this paper is organized as follows. Section II introduces the model we employ and formulates the problem formally. Sufficient conditions that enable partial phase cohesiveness are provided in Section III. Section IV contains our numeral validations. Finally, we draw the conclusion in Section V.

II. PROBLEM FORMULATION

In this paper, we consider a network of interacting communities, each of which contains a population of interacting phase oscillators. Inspired by [17], [18], we assume that the behavior of each neuronal ensemble in an anatomical region is modeled by a Kuramoto oscillator and analyze synchronization phenomena in the human brain through the M communities, each of which itself consists of a number of all-to-all coupled Kuramoto oscillators. To simplify the analysis we assume, without loss of generality, that each community has the same number of oscillators, N. The formal model is given by

$$\dot{\theta}_i^p = \omega_i^p + \frac{K_p}{N}\sum_{n=1}^{N}\sin(\theta_n^p - \theta_i^p) + \sum_{l=1}^{M}\frac{a_{pl}}{N}\sum_{n=1}^{N}\sin(\theta_n^l - \theta_i^p), \quad i = 1, 2, \ldots, N, \tag{1}$$

where $\theta_i^p \in \mathbb{S}^1$ and $\omega_i^p \geq 0$ denote the phase and natural frequency of the ith oscillator in the pth community (anatomical region), and $\mathbb{S}^1$ represents the unit circle. The n-torus, the product of the n unit circles, is denoted by $\mathbb{T}^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. The second term on the right-hand side of (1) represents the interconnection within community p. Every community has the same number of oscillators, N, and thus the number of oscillators in the whole network is MN. We assume that each community is well mixed, i.e., the corresponding graph of N oscillators (within each community) is complete. Then, the corresponding adjacency matrix within community p is $C^p := K_p C$, where K_p is the intrinsic coupling strength, and $C := [c_{ij}] \in \mathbb{R}^{N\times N}$ satisfies $c_{ij} = 1$ for $i \neq j$, and $c_{ij} = 0$ otherwise. The third term represents the connections linking communities, which is modeled by a *connected*, undirected, and weighted graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ with the nodes $\mathcal{V} = \{1, \ldots, M\}$ and edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The positive weights $\mathcal{A}$ determined by the coupling strength between communities i and j satisfy $a_{ij} = a_{ji} > 0$ if $(i, j) \in \mathcal{E}$, otherwise, $a_{ij} = 0$. It is clear that $A := (a_{ij}) \in \mathbb{R}^{M\times M}$ is the weighted adjacency matrix of the graph $\mathcal{G}$.

Normally, global synchronization across all the regions hardly occurs, although it has been observed in patients' brain, e.g., people with epilepsy [18]. Instead, researchers have observed that some (not all) cortical regions in the brain achieve phase cohesive when a person is executing a cognitive task which needs these parts to coordinate, while the other regions of the brain remain incoherent [20]. We call this phenomenon *partial phase cohesiveness* in this paper. Denote a subset of communities by $\mathcal{R} = \{m_1, \ldots, m_r\} \subseteq \mathcal{V}$ with $1 \leq r \leq M$. The partial phase cohesiveness across communities $\mathcal{R}$ is defined formally as follows.

Definition 1: A solution $\theta : \mathbb{R}_{\geq 0} \to \mathbb{T}^{MN}$ to the M all-to-all coupled Kuramoto oscillators (1) is said to be *partially phase cohesive across* $\mathcal{R}$ *w.r.t* $\varphi \in [0, \pi/2)$ if $\theta(t) \in \mathcal{S}_\varphi$ for all $t > 0$ and all $\theta(0) \in \mathcal{S}_\varphi$, where

$$\mathcal{S}_\varphi = \{\theta \in \mathbb{T}^{MN} : |\theta_i^k - \theta_j^l| \leq \varphi, \; \forall i, j = 1, \ldots, N, \forall k, l \in \mathcal{R}\}.$$

In Definition 1, φ is a measure of cohesiveness, i.e., the smaller φ is, the more cohesive the phases of oscillators in $\mathcal{R}$ are. Synchronization across some anatomical regions of brain is sometimes triggered by a particular region when it needs other parts to cooperate. This motivates us to study *community-driven* partial cohesiveness. We assume that all the inter-community couplings a_{ij} and intrinsic couplings K_p are initially weak such that no phase cohesive can occur in the network. Then, we investigate how a community, which plays the role as a "leader", is able to achieve synchronization of the oscillators within itself by increasing the intrinsic coupling as well as to selectively drive some of other communities to phase cohesiveness by strengthening the connections to them. We call the "leader" *driving community* and the communities intended to synchronize *target communities*. Without loss of generality, we label the driving community by 1, and the target communities by $2, 3, \ldots, r, r \leq M$. We also regard 1 as a target community when we refer to "target communities". Although the target communities can in general have no direct inter-connection with the driving one, as a first step of analysis, we assume in this paper that the target communities are neighbors of the driving one, i.e., $a_{1m} > 0$ for all $2 \leq m \leq r$. Then, the problem reduces to seeking for conditions on the couplings K_1 and a_{1m} under which phase cohesiveness takes place in a subset of communities $\mathcal{R} := \{1, \ldots, r\}$.

III. PHASE COHESIVENESS ANALYSIS

A. Within One Community

We first consider a special case when the driving community provokes phase cohesiveness among oscillators within. In this case, a natural expectation is that if the coupling strength K_1 within the community is sufficiently larger than the others, then phase cohesiveness occurs within the community. We present a lower bound on K_1.

To study phase cohesiveness within community 1, we use the incremental dynamics between oscillators θ_i^1 and θ_j^1, $i, j = 1, \ldots, N$ in the community. From (1), their dynamics can be described by

$$\begin{aligned}\dot{\theta}_i^1 - \dot{\theta}_j^1 =& \omega_i^1 - \omega_j^1 \\ &+ \frac{K_1}{N}\sum_{n=1}^{N}\left(\sin(\theta_n^1 - \theta_i^1) - \sin(\theta_n^1 - \theta_j^1)\right) \\ &+ \sum_{m=2}^{M}\frac{a_{1m}}{N}\sum_{n=1}^{N}\left(\sin(\theta_n^m - \theta_i^1) - \sin(\theta_n^m - \theta_j^1)\right).\end{aligned} \tag{2}$$

Define $\theta^p := [\theta_1^p, \dots, \theta_N^p]^\top \in \mathbb{T}^N$ and $\omega^p := [\omega_1^p, \dots, \omega_N^p]^\top \in \mathbb{R}_{\geq 0}^N$ for $p \in \mathcal{V}$, where $\mathbb{R}_{\geq 0}^N$ is the set of non-negative real numbers. Also, define a vector of phase differences of oscillators in community 1,

$$\delta_1 := B_c^\top \theta^1 = \left[\begin{array}{ccc} \theta_1^1 - \theta_2^2, & \dots, & \theta_{N-1}^1 - \theta_N^1 \end{array}\right]^\top,$$

where B_c is the incidence matrix of the complete graph. By using this δ_1, we have the following compact form of (2),

$$\dot{\delta_1} = B_c^\top \omega^1 - \frac{K_1}{N} B_c^\top B_c \mathbf{sin}(\delta_1) + \bar{a}_1 \mathbf{u}_1, \tag{3}$$

where $\mathbf{sin}(x) := [\sin(x_1), \dots, \sin(x_n)]$, $\bar{a}_1 := \max_{m \in \mathcal{V}} a_{1m}$, and $\mathbf{u}_1 := (u_{ij})_{i<j} \in \mathbb{R}^{N(N-1)/2}$ with

$$u_{ij} := \sum_{m=2}^{M} \frac{a_{1m}}{\bar{a}_1 N} \sum_{n=1}^{N} \left(\sin(\theta_n^m - \theta_i^1) - \sin(\theta_n^m - \theta_j^1) \right). \tag{4}$$

This $\mathbf{u}_1$ can be viewed as the disturbance to the incremental dynamics (3).

Let us provide a sufficient condition on K_1 such that phase cohesiveness can occur in community 1. A similar theorem can be found in [21, Theorem 4.4], which, however, cannot be directly applied to our case. We treat the external disturbance $\mathbf{u}_1$ in a more relaxed way, and consequently the required analysis is technically more involved. We explain the detailed technical differences after the sketch of proof.

Theorem 1: Assume that the coupling strength K_1 in the driving community 1 satisfies the following condition

$$K_1 > K_{\text{critical}} := \|B_c^\top \omega_1\| + \sqrt{N(N-1)} d_1, \tag{5}$$

where

$$d_1 := \sum_{2=1}^{M} a_{1m} \tag{6}$$

is the degree of community 1. Next, let $\gamma_1 \in (\pi/2, \pi)$ and $\varphi_1 \in [0, \pi/2)$ be the unique solutions to the equations

$$(\pi/2) K_1 \operatorname{sinc}(\gamma_1) - \sqrt{2N(N-1)} d_1 \sin(\gamma_1/2) = \|B_c^\top \omega_1\|, \tag{7}$$

$$K_1 \sin(\varphi_1) - \sqrt{N(N-1)} d_1 = \|B_c^\top \omega_1\|, \tag{8}$$

respectively, where $\operatorname{sinc}(\gamma_1) := \sin(\gamma_1)/\gamma_1$. Then, for any $\gamma \in [\varphi_1, \gamma_1)$, the set

$$\mathcal{D}_\gamma := \{\theta \in \mathbb{T}^{MN} : \|\delta_1\| \leq \gamma\}$$

is a positively invariant set of the N all-to-all coupled Kuramoto oscillators (1). In addition, there always exists $\mu(\gamma) < \pi/2$, defined by

$$\mu(\gamma) := \frac{\|B_c^\top \omega_1\| + \sqrt{2N(N-1)} d_1 \sin(\gamma/2)}{K_1 \operatorname{sinc}(\gamma)}, \tag{9}$$

such that any solution to (1) starting from $\theta(0) \in \{\theta \in \mathbb{T}^{MN} : \|\delta_1\| \leq \gamma\}$, $\gamma \in [\varphi_1, \gamma_1)$, asymptotically converges to the set $\{\theta \in \mathbb{T}^{MN} : \|\delta_1\| \leq \mu(\gamma)\}$.

Proof: Due to the page limit, we only provide a sketch of proof. First, it is not hard to show the each of the solutions to (7) and (8) exists and is unique, respectively.

Next, we prove that for any $\gamma \in [\varphi_1, \gamma_1]$, the set $\mathcal{D}_\gamma$ is positively invariant. Towards this end, we construct a positive definite function $V_1(\delta_1) = \frac{1}{2}\|\delta_1\|^2$ and show that the time derivative of $V_1(\delta_1)$ along the trajectory of (3) is negative for any $\mu(\gamma) < \|\delta_1\| \leq \gamma$. An important step to show this is to estimate an upper bound of the external disturbance $\|\mathbf{u}_1\|$. We estimate it according to

$$\|\mathbf{u}_1\| \leq \sqrt{2N(N-1)} d_1 \sin(\gamma/2)/\bar{a}_1, \tag{10}$$

which is more relaxed compared to [21, Theorem 4.4].

Finally, since $\dot{V}_1(\delta_1) < 0$ for $\|\delta_1\| > \mu(\gamma)$, any solution to (1) starting from $\theta(0) \in \{\theta \in \mathbb{T}^{MN} : \|\delta_1\| \leq \gamma\}$, $\gamma \in [\varphi_1, \gamma_1)$, asymptotically converges to the set $\{\theta \in \mathbb{T}^{MN} : \|\delta_1\| \leq \mu(\gamma)\}$, then the asymptotic convergence is proven. ■

Remark 1: As mentioned before Theorem 1, Theorem 4.4 in [21] cannot directly be applied even though we use a similar Lyapunov function. The disturbance in [21, Theorem 4.4] is regarded to be bounded by a constant, but the upper bound of the disturbance in our case is a function of γ (see (10)). This makes the analysis more challenging compared to [21].

In addition, we identify a set to which any solution to (1) converges exponentially.

Proposition 1: There always exists a $\bar{\gamma} \in [\varphi_1, \gamma)$, such that starting from $\theta(0) \in \{\theta \in \mathbb{T}^{MN} : \|\delta_1\| < \gamma_1\}$ any solution to (1) exponentially converges to the set $\{\theta \in \mathbb{T}^{MN} : \|\delta_1\| < \bar{\gamma}\}$.

The proof of Proposition 1 is omitted here due to the page limit.

B. Community-Driven Partial Cohesiveness

In the previous subsection, we have studied a special case when phase cohesiveness occurs only in the driving community. In this subsection, we address a more general problem in which the number of target communities is r. Sufficient conditions on the driving couping strengths a_{1m}, $m \in \mathcal{R} \backslash \{1\}$, and K_1 are obtained to ensure that phase cohesiveness takes place among oscillators in communities $\mathcal{R}$.

Similar to the single community case, we introduce several notations. We put the communities in $\mathcal{R}$ into a cluster and rewrite the model (1) into

$$\begin{aligned} \dot{\theta}_i^p =& \omega_i^p \\ &+ \frac{K_p}{N} \sum_{n=1}^{N} \sin(\theta_n^p - \theta_i^p) + \sum_{m=1}^{r} \frac{a_{pm}}{N} \sum_{n=1}^{N} \sin(\theta_n^m - \theta_i^p) \\ &+ \sum_{m=r+1}^{M} \frac{a_{pl}}{N} \sum_{n=1}^{N} \sin(\theta_n^m - \theta_i^p), p \in \mathcal{R}, i = 1, \dots, N. \end{aligned} \tag{11}$$

Let the graph $\mathcal{G}_\mathcal{R} := (\mathcal{V}_\mathcal{R}, \mathcal{E}_\mathcal{R}, \mathcal{A}_\mathcal{R})$ describe the couplings among all oscillators in $\mathcal{R}$, where $\mathcal{V}_\mathcal{R} := \{\theta_i^p, i = 1, \dots, N, p \in \mathcal{R}\}$. Define $\boldsymbol{\theta} := [{\theta^1}^\top, {\theta^2}^\top, \dots, {\theta^1}^\top]^\top \in \mathbb{T}^{Nr}$ and $\boldsymbol{\omega} := [{\omega^1}^\top, {\omega^2}^\top, \dots, {\omega^r}^\top]^\top \in \mathbb{R}^{Nr}$. Let $Z := [z_{ij}] \in$

$\mathbb{R}^{Nr\times Nr}$ be the adjacency matrix of the graph $\mathcal{G}_\mathcal{R}$, which is

$$Z=\begin{bmatrix} \frac{K_1}{N}C & \frac{a_{12}}{N}\mathbf{1}_{N\times N} & \cdots & \frac{a_{1r}}{N}\mathbf{1}_{N\times N}\\ \frac{a_{12}}{N}\mathbf{1}_{N\times N} & \frac{K_2}{N}C & \cdots & \frac{a_{2r}}{N}\mathbf{1}_{N\times N}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{a_{1r}}{N}\mathbf{1}_{N\times N} & \frac{a_{2r}}{N}\mathbf{1}_{N\times N} & \cdots & \frac{K_r}{N}C \end{bmatrix}, \quad (12)$$

where C is the adjacency matrix of a complete graph within each community as defined in Section II. Let L be the Laplacian matrix given by $L=\operatorname{diag}(Z\mathbf{1}_{Nr})-Z$. Recall that the communities $2,\dots,r$ are neighbors of community 1, and thus $\mathcal{G}_\mathcal{R}$ is connected. Then, the smallest eigenvalue of L is 0, and let $\lambda_2(L)$ denote the second smallest one. This eigenvalue $\lambda_2(L)$ is also called *algebraic connectivity* of graph $\mathcal{G}_\mathcal{R}$ [22]–[24].

The dynamics of coupled oscillators in the cluster (11) can be rewritten into a compact form as follows

$$\begin{aligned}\dot{\boldsymbol{\theta}}_i &= \boldsymbol{\omega}_i+\sum_{n=1}^{Nr} z_{in}\sin(\boldsymbol{\theta}_n-\boldsymbol{\theta}_i)\\ &\quad +\sum_{m=r+1}^{M}\frac{a_{\mu(i)m}}{N}\sum_{n=1}^{N}\sin(\theta_n^m-\boldsymbol{\theta}_i), i=1,...Nr,\end{aligned}$$

where $i=1,2,\dots,Nr$ and $\mu(i):=1+\lfloor i/N\rfloor$. Note that for $x\in\mathbb{R}$, $\lfloor x\rfloor$ represents the maximum integer less than or equal to x. In a similar manner as the one community case, we consider the incremental dynamics between any oscillators i,j in the cluster,

$$\begin{aligned}\dot{\boldsymbol{\theta}}_i-\dot{\boldsymbol{\theta}}_j &= \boldsymbol{\omega}_i-\boldsymbol{\omega}_j+\sum_{n=1}^{Nr} z_{in}\sin(\boldsymbol{\theta}_n-\boldsymbol{\theta}_i)\\ &\quad -\sum_{n=1}^{Nr} z_{jn}\sin(\boldsymbol{\theta}_n-\boldsymbol{\theta}_j)+\bar{a}_o u_{ij}, i,j=1,...Nr, \quad (13)\end{aligned}$$

where

$$\begin{aligned}\bar{a}_o &:= \max_{\substack{1\le i\le r\\ r+1\le j\le M}} a_{ij},\\ u_{ij} &:= \sum_{m=r+1}^{M}\frac{a_{\mu(i)m}}{\bar{a}_oN}\sum_{n=1}^{N}\sin(\theta_n^m-\boldsymbol{\theta}_i)\\ &\quad -\sum_{m=r+1}^{M}\frac{a_{\mu(j)m}}{\bar{a}_oN}\sum_{n=1}^{N}\sin(\theta_n^m-\boldsymbol{\theta}_j). \quad (14)\end{aligned}$$

Similar to Subsection III-A, u_{ij} is taken as a disturbance to the incremental dynamics (13). Define the phase differences vector $\delta := B_c^\top\boldsymbol{\theta}\in\mathbb{R}^{Nr(Nr-1)/2}$. Then, the incremental dynamics (13) can be described in a compact form,

$$\dot{\delta}=B_c^\top\boldsymbol{\omega}-B_c^\top B\operatorname{diag}(\{c_{ij}\}_{(i,j)\in\mathcal{E}})\mathbf{sin}(B^\top\theta)+\bar{a}_o\mathbf{u}_R, \quad (15)$$

where B and B_c are the incidence matrices of the graph $\mathcal{G}_\mathcal{R}$ and the complete graph with the same nodes $\mathcal{V}_\mathcal{R}$, respectively, and $\mathbf{u}_R:=(u_{ij})_{i<j}\in\mathbb{R}^{Nr(Nr-1)/2}$.

By extending Theorem 1, let us provide a sufficient condition on the algebraic connectivity $\lambda_2(L)$ such that phase cohesiveness occurs among oscillators in $\mathcal{R}$. Since the proof is similar to Theorem 1, it is omitted here.

Theorem 2: Assume that the algebraic connectivity $\lambda_2(L)$ of $\mathcal{G}_\mathcal{R}$ is larger than a critical value, i.e., $\lambda_2(L)$ satisfies the following condition

$$\lambda_2(L)>\lambda^c_{\text{critical}}:=\|B_c^\top\boldsymbol{\omega}\|+\sqrt{2Nr(Nr-1)}d_R, \quad (16)$$

where $d_R:=\max_{i\in\mathcal{R}}\sum_{m=r+1}^{M}a_{im}$. Next, let $\gamma_R\in(\pi/2,\pi]$ and $\varphi_R\in[0,\pi/2)$ be the unique solutions to the following equations

$$\begin{aligned}&(\pi/2)\lambda_2(L)\operatorname{sinc}(\gamma_R)-\sqrt{2Nr(Nr-1)}d_R=\|B_c^\top\boldsymbol{\omega}\|,\\ &\lambda_2(L)\sin(\varphi_R)-\sqrt{2Nr(Nr-1)}d_R=\|B_c^\top\boldsymbol{\omega}\|,\end{aligned}$$

respectively. Then for any $\gamma\in[\varphi_R,\gamma_R]$, the set

$$\mathcal{D}_{\gamma_R}:=\{\theta\in\mathbb{T}^{MN}:\|\delta\|\le\gamma\}$$

is a positively invariant set of the soution to the model (1). In addition, there always exists $\mu_R(\gamma)<\pi/2$, defined by

$$\mu_R(\gamma):=\frac{\|B_c^\top\boldsymbol{\omega}\|+\sqrt{2Nr(Nr-1)}d_R}{\lambda_2(L)\operatorname{sinc}(\gamma)}, \quad (17)$$

such that any solution to (1) starting from $\theta(0)\in\{\theta\in\mathbb{T}^{MN}:\|\delta\|\le\gamma\}$, $\gamma\in[\varphi_R,\gamma_R)$, asymptotically converges to the set $\{\theta\in\mathbb{T}^{MN}:\|\delta\|\le\mu_R(\gamma)\}$. In other words, the coupled Kuramoto oscillators (1) are partially phase cohesive across $\mathcal{R}=\{1,\dots,r\}$ w.r.t $\mu_R(\gamma)\in[0,\pi/2)$. Moreover, there exists $\bar{\gamma}_R\in[\varphi_R,\gamma)$ such that the solution to (1) exponentially converges to the set $\{\theta\in\mathbb{T}^{MN}:\|\delta\|<\bar{\gamma}_R\}$.

One can interpret $B_c^\top\boldsymbol{\omega}$ and $\mathbf{u}_R$ in (15) as the internal and external disturbances to the cluster. To achieve partially phase cohesive irrespective of these disturbances, it is natural requirement that the algebraic connectivity $\lambda_2(L)$ is strong enough. Note that $B_c^\top\boldsymbol{\omega}$ is a constant in our context, while $\mathbf{u}_c$ is time-varying, depending on the phase differences between oscillators within and outside of the cluster. Theorem 2 suggests that no matter how different these phase differences are, if the algebraic connectivity within this cluster is sufficiently large compared to the external couplings, phase cohesiveness can still occur in the cluster.

Remark 2: Conditions (5) and (16) in Theorems, respectively, 1 and 2 are similar but different even if $r=1$. Acutually, for $r=1$, the condition (16) reduces to

$$\lambda_2(L)>\lambda^c_{\text{critical}}:=\|B_c^\top\omega_1\|+\sqrt{2N(N-1)}d_1, \quad (18)$$

where d_1 is defined in Theorem 1. Therefore, (18) implies (5), but not vise versa. That is, Theorem 1 provides a less conservative condition than Theorem 2 when $r=1$.

In what follows, we show the important role that individual driving couplings, $a_{12},\dots,a_{1r}$, play in making up the algebraic connectivity $\lambda_2(L)$.

Proposition 2: For the adjacency matrix of the graph $\mathcal{G}_\mathcal{R}$, the algebraic connectivity $\lambda_2(L)$ satisfies

$$\underline{a}_d\le\lambda_2(L)\le\frac{Nr}{Nr-1}\min_{1\le i\le r}\left(\frac{N-1}{N}K_i+\sum_{m=1}^{r}a_{im}\right), \quad (19)$$

where $\underline{a}_d$ is the minimum value among the intrinsic coupling in the driving community and the driving couplings, which is defined by $\underline{a}_d:=\min\{a_{12},\dots,a_{1m},K_1\}$.

Due to the page limit, we omit the proof of Proposition 2. Proposition 2 provides an estimate of how the algebraic connectivity $\lambda_2(L)$ depends on the intrinsic coupling in the driving community and the driving couplings. The corollary below, which follows from Theorem 2 directly, presents a sufficient condition on these couplings such that the oscillators in $\mathcal{R}$ achieve phase cohesiveness, no matter how weak the intrinsic coupling within each community is.

Corollary 1: If the intrinsic coupling in the driving community and all the driving couplings are lower bounded by $\underline{a}_d$ which satisfies

$$\underline{a}_d > \lambda^c_{\text{critical}}, \tag{20}$$

where the critical value $\lambda^c_{\text{critical}}$ is defined in (16), then all the statements in Theorem 2 hold for $\gamma_R \in (\pi/2, \pi]$ and $\varphi_R \in [0, \pi/2)$ that are the unique solutions to the following equations

$$(\pi/2)\underline{a}_d N \,\text{sinc}(\gamma_R) - \sqrt{2Nr(Nr-1)}d_1 = \|B_c^\top \boldsymbol{\omega}\|,$$
$$\underline{a}_d N \sin(\varphi_R) - \sqrt{2Nr(Nr-1)}d_1 = \|B_c^\top \boldsymbol{\omega}\|,$$

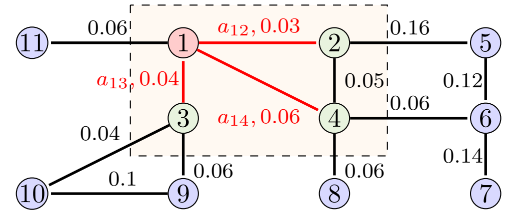
respectively.

Theorem 2 and Corollary 1 suggest that by just increasing the intrinsic coupling strength and the coupling strengths among communities correspondingly, the driving community is able to drive some of its neighbors to phase cohesiveness selectively. It is worth mentioning that the results in Theorem 2 is still applicable when the target communities are *not direct neighbors* of the driving community, as long as the corresponding algebraic connectivity is sufficiently large.

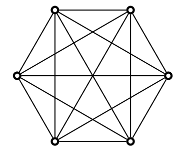
We observe that the algebraic connectivities, quantified by K_1 and $\lambda_2(L)$ in Theorems 1 and 2, respectively, characterize the convergence speeds for partially phase cohesiveness. From Theorem 1, the convergence speed within the driving community can be improved by only increasing K_1. However, from Theorem 2 and Proposition 1, this is not enough for improving the convergence speed within the target communities because it also depends on the coupling strength $a_{1m}, m = 2, \ldots, r$, between communities 1 and m. Therefore, it is expected if $K_1 \gg a_{1m}$, the convergence speed within the driving community is much faster than that of the target communities. The difference of the convergence speeds are further studied through numerical simulations.

IV. Numeral Examples

To validate the results we obtained in Section III, we perform some numeral studies in this section. We consider a connected network consisting of 11 communities (see Fig.1(a)), each of which contains 6 well mixed oscillators (see Fig.1(b)). The initial inter-community coupling strengths are given next to the edges respectively in Fig.1(a). We choose the natural frequencies of the oscillators from a normal distribution with the mean 3π and the standard deviation 1. The initial intrinsic coupling strengths $K_m, 1 \le m \le 11$, are randomly generated to be small enough such that no phase cohesiveness can occur in any of the communities. Let 1 be the driving community.

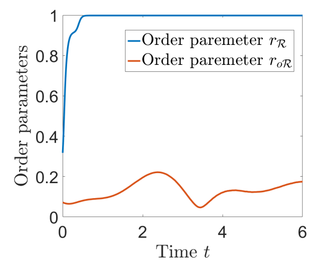


(a) The inter-community network: driving community 1 (red) and target communities 2, 3, 4 (green).

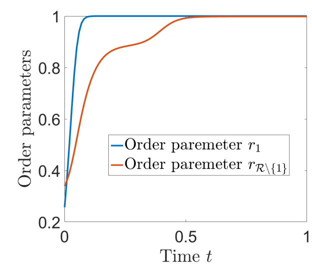


(b) The well mixed graph within each community

Fig. 1. The two-layer structure of the network of networks considered.



(a) Phase Cohesiveness in the cluster and incoherence outside



(b) Comparison of the convergence speed

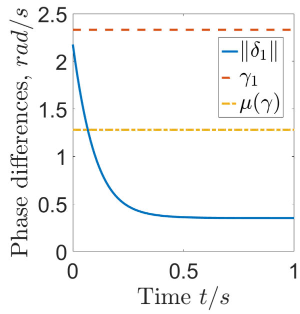
Fig. 2. Target phase cohesiveness driven by a community.

We first study the case when only the oscillators within community 1 are supposed to function cohesively. According to (5), we compute the critical coupling strength $K_{\text{critical}} = 4.5713$. We let $K_1 = 10$, and the trajectory of $\|\delta_1(t)\|$ is plotted in Fig. 3(a). It can be observed that, starting from $\|\delta_1(0)\| = \gamma < \gamma_1$, the trajectory of $\|\delta_1(t)\|$ converges to $\{\delta_1 : \|\delta_1(t)\| < \mu(\gamma)\}$, suggesting that phase cohesiveness w.r.t $\mu(\gamma)$ occurs in community 1. The results we obtained in Theorem 1 are verified.

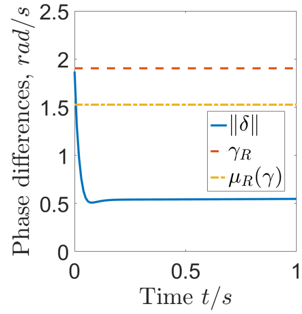
Next, we consider another case when the target communities is $1, 2, 3, 4$. From Theorem 2, we compute the critical value $\lambda_{\text{critical}} = 23.3827$, and let $K_1, a_{12}, a_{13}, a_{14} = 30$ such that $\lambda_1(L) > \lambda_{\text{critical}}$ according to Corollary 1. The trajectory of $\|\delta(t)\|$ is plotted in Fig. 3(b), from which one can observe that the trajectory of $\|\delta(t)\|$ converges to $\{\delta : \|\delta(t)\| < \mu_R(\gamma)\}$, suggesting that phase cohesiveness w.r.t $\mu(\gamma)$ occurs in the cluster $\mathcal{R} = \{1, 2, 3, 4\}$.

To further numerally investigate how different convergence speeds can emerge in the cluster $\mathcal{R}$, we relax the requirement for the initial phases by generating them randomly. We employ the order parameter introduced in [8] to measure the degree of phase cohesiveness, which is defined by

$$re^{i\psi} = \frac{1}{n}\sum_{i=1}^{n} e^{i\theta_j}, \tag{21}$$



(a) Target community: 1.



(b) Target communities: $1, 2, 3, 4$.

Fig. 3. The trajectories of $\|\delta_1\|$ (left) and $\|\delta\|$ (right).

where ψ is the average phase. The value of r satisfies $r \in [0, 1]$. The larger r is, the higher degree of phase cohesiveness becomes. By plotting the trajectory of r, it is sufficient to observe the phase evolution. Let $K_1, a_{12}, a_{13}, a_{14}$ be large enough to facilitate phase cohesiveness in the cluster. One can observe from Fig. 2(a) that phase cohesiveness occurs in the cluster, but oscillators outside remain incoherent. Interestingly, if K_1 is much greater than the driving couplings $r_{1i}, i = 2, 3, 4$, the convergence speed of the oscillators in driving community to phase cohesiveness is much faster than the rest oscillators in $\mathcal{R}$ do, which can be observed in Fig. 2(b). We are interested in studying this phenomenon further in the future.

V. Conclusion

In this paper, we have considered a network of networks of Kuramoto oscillators, which is motivated by the neuronal dynamics observed in the human brain. Partial rather than global or complete phase cohesiveness driven by a community has been investigated since it is the normal pattern in the brain. We have considered two cases. One is that the phase cohesiveness takes place only within the driving community, and the other is that the number of target communities is r. Sufficient conditions on the intrinsic coupling strength of the driving community and algebraic connectivity of the selected communities have been obtained, respectively. We have also presented lower bounds for the ultimate level of phase cohesiveness. To validate the obtained results, numerical simulations have been performed. We are currently working with cognitive neural scientists to apply our results to human subjects performing cognitive tasks requiring retrieval of memory.

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