2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

# Broadcasting protocols for coordinating nonlinear network systems*

Sebastian Trip[1], Claudio De Persis[1] and Pietro Tesi[1,2]

*Abstract*— **We propose a new methodology to design broadcasting protocols for coordinating nonlinear network systems. Our design of the scheduling of information transmission is based on the introduction of clock variables, whose dynamics are regulated through a suitable storage function. Required clock dynamics, ensuring stability, follow then elegantly from Lyapunov like arguments. For illustrative purposes, we first consider an example of a consensus algorithm, whereafter we discuss a distributed integral controller in feedback interconnection to a network composed of output strictly incrementally passive subsystems. Finally, we show how the proposed method can be used to redesign a popular distributed controller in power grids, enabling a sampled-data implementation.**

## I. Introduction

The importance to consider the underlying communication layer when designing and analyzing distributed control protocols has been widely recognized. Recently, a considerable amount of effort has been put to find protocols enabling sampled-data communication, also with the aim of saving communication resources [1]. We do not aim here to give a complete overview, but we do note that particularly consensus algorithms, as well as the coordination of network systems with linear dynamics have been studied thoroughly [2], [3], [4], [5]. On the other hand, extensions towards the coordination of nonlinear network systems have been so far limited [6], [7], and we believe that the presented work will provide some new insights in this direction.

Among the possible implementations to realize a sampled-data distributed control system, the so-called broadcasting protocols, where a node transmits information at discrete times to all adjacent nodes, received a considerable amount of attention due to their simplicity. In this work, our focus is also on the design of a broadcasting protocol, for which we provide a scheduling of information transmission.

Particularly, our interest is in the redesign of distributed (averaging) integral controllers, that have been recently proposed to coordinate complex nonlinear network systems, such as power grids [8], [9], [10]. These controllers rely on a consensus protocol to synchronize the various control inputs. Despite initial results to explicitly incorporate the communication layer in these controllers [11], [12], one can notice that the discrete time nature of the communication and the scheduling of information exchange is not sufficiently understood yet.

### A. Main contributions

Based on the previous observations, we propose a new methodology to design broadcasting protocols that can, in comparison with existing results, more easily be incorporated within the stability study of nonlinear network systems. We summarize the main contributions of this work as follows:

1) We advocate the use of local clock variables, with a-priori unspecified dynamics, where broadcasting of information occurs once a clock variable reaches its threshold and gets reset. We formalize the obtained hybrid network system following [13], which is shown to be appropriate within similar settings [14], [7], [15].
2) We propose for the stability analysis a new storage function that depends, similarly as in [7] and [16], on the clock variables. To clarify the main idea, we initially focus on a simple coordination task, namely distributed average consensus for networks of integrators. Notably, the stability analysis explicitly suggests the design of the clock dynamics that were left unspecified. Furthermore, the analysis of the obtained clock dynamics leads, in the same spirit as [16], to required broadcasting frequencies.
3) For the distributed integral controller, in feedback interconnection to a network of nonlinear systems that are output strictly incrementally passive, we apply the same methodology as outlined before leading to different clock dynamics that depend on the excess of dissipation of the controlled network system. To illustrate the result, we show how the proposed method can be used to redesign continuous-time distributed controllers in power grids, enabling a sampled-data implementation.

### B. Outline

In Section II we briefly introduce the considered broadcasting protocols. Then, in Section III, we design a broadcasting protocol for some well studied consensus dynamics and study the stability of the obtained hybrid system. In Section IV, we study the redesign of a continuous-time consensus algorithm to coordinate nonlinear network systems that are output strictly incrementally passive. In Section V we apply the obtained results to the optimal power sharing problem in power grids. Finally, in Section VI, some conclusions and future directions are provided.

### C. Preliminaries

The combination of continuous time and discrete time dynamics results in an overall hybrid system. In this work,

*This work is part of the research programme ENBARK+ with project number 408.urs+.16.005, which is (partly) financed by the Netherlands Organisation for Scientific Research (NWO).

[1]S. Trip, C. De Persis and P. Tesi are with ENTEG, Faculty of Science and Engineering, University of Groningen, Nijenborgh 4, 9747 AG Groningen, the Netherlands. {s.trip, c.de.persis, p.tesi}@rug.nl.

[2]P. Tesi is also with the Department of Information Engineering, University of Florence, Firenze 50139, Italy. pietro.tesi@unifi.it.

we study hybrid systems using the formalism in [13], which considers systems of the following form:

$$\begin{aligned} \dot{x} &\in F(x) \quad \text{for } x \in C \\ x^{+} &\in G(x) \quad \text{for } x \in D, \end{aligned} \tag{1}$$

where $C$ is the flow set, $F$ is the flow map, $D$ is the jump set and $G$ is the jump map. The hybrid system, with state $x \in \mathbb{R}^n$, is denoted as $\mathcal{H} = (C, F, D, G)$, or briefly $\mathcal{H}$. A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a *hybrid time domain* if for all $(T, K) \in E$, $E \cap ([0, T] \times \{0, \ldots, K\}) = \bigcup_{k \in \{0,\ldots,K-1\}} ([t_k, t_{k+1}], k)$ for some finite sequence of times $0 = t_0 \leq t_1, \ldots, \leq t_K$. A function $\phi : E \to \mathbb{R}^n$ is a *hybrid arc* if $E$ is a hybrid time domain and if for each $k \in \mathbb{Z}_{\geq 0}$, $t \to \phi(t, k)$ is locally absolutely continuous on $I^k = \{t : (t, k) \in E\}$. The hybrid arc $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ is a *solution* to (1) if: (i) $\phi(0, 0) \in C \cup D$; (ii) for any $k \in \mathbb{Z}_{\geq 0}, \phi(t, k) \in C$ and $(d/dt)\phi(t, k) \in F(\phi(t, k))$ for almost all $t \in I^k$ (recall that $I^k = \{t : (t, k) \in \operatorname{dom} \phi\}$; (iii) for every $(t, k) \in \operatorname{dom} \phi$ such that $(t, k + 1) \in \operatorname{dom} \phi$, $\phi(t, k) \in D$ and $\phi(t, k + 1) \in G(\phi(t, k)$. A solution to (1) is: *nontrivial* if $\operatorname{dom} \phi$ contains at least two points; *maximal* if it cannot be extended; *complete* if $\operatorname{dom} \phi$ is unbounded; *precompact* if it is complete and the closure of its range is compact, where the range of $\phi$ is $\operatorname{rge} \phi := \{y \in \mathbb{R}^n : \exists(t, k) \in \operatorname{dom} \phi \text{ such that } y = \phi(t, k)\}$.

## II. From continuous consensus to broadcasting at discrete times

In this section we introduce the broadcasting implementation of the thoroughly studied consensus algorithm

$$\dot{\theta} = -\mathcal{L}\theta, \tag{2}$$

where $\theta \in \mathbb{R}^n$ is the state and $\mathcal{L} \in \mathbb{R}^{n \times n}$ is the Laplacian matrix associated to a connected and undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \{1, \ldots, n\}$ denoting the set of nodes and $\mathcal{E}$ denoting the set of edges. With broadcasting we, informally, mean that a node $i \in \mathcal{V}$ transmits (broadcasts) its current value of $\theta_i$ to its adjacent nodes, at discrete time instances. To exchange information, an underlying communication network is exploited, on which we impose a few assumptions:

**Assumption 1 (Communication network)**

- *The communication network is connected and undirected, i.e. if node $i$ can communicate with node $j$, then also the converse is possible.*
- *Each node knows (an upper bound on) its degree, i.e. on the number of connected adjacent nodes.*
- *The communication is without delays and errors.*

It is worth mentioning that in comparison with [7], where clocks or events are defined on the edges of the network, we focus on the design of 'rules', determining the broadcasting instances, that are implemented at the nodes, which is more in line with implementation practises. The main objective is to determine a scheduling of information transmission on each node, using only information that is locally available at a node, such that convergence to a consensus state is guaranteed. In the next section, we first study the autonomous system (2) that provides preparatory results for the remainder, where we will study system (2) to coordinate a network of output strictly incrementally passive subsystems.

## III. Average preserving consensus

In this section we propose a modification of the continuous consensus algorithm (2). Specifically, we consider the case where every node broadcasts at certain time instances, determined by a local clock at each node, its actual value of $\theta_i$ to its adjacent nodes, resulting in the system

$$\dot{\theta} = -\mathcal{L}\hat{\theta}, \tag{3}$$

where, $\hat{\theta}_i$, is the latest broadcasted value of $\theta_i$. First, note that (3) preserves the average of initial values, since $\mathbb{1}_n^T\dot{\theta} = -\mathbb{1}_n^T\mathcal{L}\hat{\theta} = 0$, where $\mathbb{1}_n \in \mathbb{R}^n$ is the vector consisting of all ones. Therefore, we aim at characterizing a scheduling of information transmission under which the solutions to system (3) converge to consensus, i.e. $\lim_{t\to\infty} \theta(t) = \overline{\theta}$, where

$$\overline{\theta} = \mathbb{1}_n \frac{\sum_i^n \theta_i(0)}{n} \in \operatorname{Im}(\mathbb{1}_n). \tag{4}$$

### A. Hybrid system

To formalize the idea outlined above we introduce at every node $i \in \mathcal{V}$ a local 'clock variable' $\phi_i$ whose dynamics are given by

$$\dot{\phi}_i = -2\deg(i)(\frac{1}{4} + \phi_i + \phi_i^2) - \epsilon_i, \tag{5}$$

with $\epsilon_i \in \mathbb{R}_{>0}$ an arbitrarily small constant. With $\deg(i)$ we denote the degree of node $i \in \mathcal{V}$, i.e. $\deg(i) = \mathcal{L}_{ii}$. Broadcasting of $\theta_i$ occurs when $\phi_i = a_i$, whereafter the clock is reset to $\phi_i^+ = b_i$, where $0 < a_i < b_i$, for all $i \in \mathcal{V}$. It is immediate to see from (5) that, with these choices of $a_i$, $b_i$ and $\epsilon_i$, we have that $\dot{\phi}_i < 0$.

**Remark 1 (Clock dynamics)** *The choice of (5) and its rationale is a result of the stability analysis, which is provided in [17]. We provide the clock dynamics here for the sake of exposition. It is important to note that every node possesses its own clock, allowing for asynchronous communication and that (5) does not depend on any global information of the network. Furthermore, (5) allows us to determine the minimum broadcasting frequency under which convergence to consensus is guaranteed and we do so in Subsection III-D.*

**Remark 2 (Broadcasting with time-varying frequencies)** *For a given $b_i$ and $a_i$, equation (5) determines the broadcasting frequency. The analysis in this section permits to include a time-varying broadcasting frequency [18] by considering instead of (5), the differential inclusion*

$$\dot{\phi}_i \in \big[ - M_i, -2\deg(i)(\frac{1}{4} + \phi_i + \phi_i^2) - \epsilon_i \big], \tag{6}$$

*where $\epsilon_i \in \mathbb{R}_{>0}$, and $M_i \in \mathbb{R}_{>0}$ are constants. The corresponding analysis is pursued in a future publication. However, periodic broadcasting has its advantages due to its simplicity and can be implemented straightforwardly and permits e.g. a 'round robin' scheduling of transmission.*

To analyze the convergence properties under the broadcasting protocol, we formulate the system within the hybrid system framework discussed in [13]. For the present study, the corresponding flow set is $C := \{(\theta, \hat{\theta}, \phi) \in \mathbb{R}^n \times \mathbb{R}^n \times [a, b]\}$, with $[a, b] := [a_1, b_1] \times \ldots \times [a_n, b_n]$. The flow map $F(\theta, \hat{\theta}, \phi)$ is

$$\left.\begin{array}{rl} \dot{\theta} = & -\mathcal{L}\hat{\theta} \\ \dot{\hat{\theta}} = & 0 \\ \dot{\phi} = & \alpha \end{array}\right\} := F(\theta, \hat{\theta}, \phi), \tag{7}$$

where $\alpha_i = -2\deg(i)(\frac{1}{4} + \phi_i + \phi_i^2) - \epsilon_i$, given by (5) above. The jump set is $D := \{(\theta, \hat{\theta}, \phi) \in \mathbb{R}^n \times \mathbb{R}^n \times [a, b] : \exists i \in \{1, \ldots, n\} \text{ s.t. } \phi_i = a_i\}$. The jump map $G(\theta, \hat{\theta}, \phi)$ is defined by

$$G(\theta, \hat{\theta}, \phi) := \{G_i(\theta, \hat{\theta}, \phi) : i \in \{1, \ldots, n\} \text{ and } \phi_i = a_i\}, \tag{8}$$

with

$$\left.\begin{array}{rll} \theta^+ = & \theta & \\ \hat{\theta}_i^+ = & \theta_i & \\ \hat{\theta}_j^+ = & \hat{\theta}_j & j \neq i \\ \phi_i^+ = & b_i & \\ \phi_j^+ = & \phi_j & j \neq i \end{array}\right\} := G_i(\theta, \hat{\theta}, \phi). \tag{9}$$

The definition of the jump map (8) ensures that at each jump, only one clock variable is reset. In case multiple clocks have reached their lower bound, multiple, but a finite number, successive jumps occur without flows in between. The hybrid system with the data above will be represented by the notation $\mathcal{H}_1 = (C, F, D, G)$ or, briefly, by $\mathcal{H}_1$. The associated stability analysis is based on an invariance principle for hybrid systems that requires the system to be nominally well posed [13, Definition 6.2]. This property is established for system at hand in the following lemma:

**Lemma 1 (Nominally well posed)** *The system $\mathcal{H}_1$ is nominally well posed.*

The proofs throughout this paper are omitted due to space restrictions, but are provided in [17].

*B. Precompactness of solutions*

In this subsection, we show that the maximal solutions to $\mathcal{H}_1$ are precompact, i.e. they are complete and the closure of their range is compact, which are the solutions to which the invoked invariance principle applies. Before showing the precompactness of the solutions, we establish an important lemma that is essential to the remainder of this section. Consider the storage function[1]

$$V_1(\theta, \hat{\theta}, \phi) = \underbrace{\frac{1}{2}(\theta - \overline{\theta})^T(\theta - \overline{\theta})}_{\text{Continuous dynamics}} + \underbrace{(\theta - \hat{\theta})^T[\phi](\theta - \hat{\theta})}_{\text{Discrete modification}}, \tag{10}$$

with $\overline{\theta}$ given by (4). Note that (10) consists of a 'usual' quadratic term that is typically used to prove stability of the continuous consensus algorithm (2) and an additional term to take into account the broadcasting induced modifications.

**Lemma 2 (Evolution of $V_1$)** *The storage function $V_1$ given in (10) satisfies*

$$\begin{array}{ll} \dot{V}_1(\theta, \hat{\theta}, \phi) = \begin{bmatrix} \theta \\ \hat{\theta} \end{bmatrix}^T Z_1 \begin{bmatrix} \theta \\ \hat{\theta} \end{bmatrix} \leq 0 & (\theta, \hat{\theta}, \phi) \in C \\ V_1(\theta^+, \hat{\theta}^+, \phi^+) \leq V_1(\theta, \hat{\theta}, \phi) & (\theta, \hat{\theta}, \phi) \in D, \end{array} \tag{11}$$

*along the solutions to $\mathcal{H}_1$, where $Z_1$ is given by*

$$Z_1 = \begin{bmatrix} [\alpha] & -[\alpha] - \frac{1}{2}(I + 2[\phi])\mathcal{L} \\ -[\alpha] - \frac{1}{2}((I + 2[\phi])\mathcal{L})^T & [\alpha] + [\phi]\mathcal{L} + \mathcal{L}[\phi] \end{bmatrix}. \tag{12}$$

Exploiting the previous lemma we can now establish the following result:

**Lemma 3 (Precompactness of solutions)** *Every maximal solution to system $\mathcal{H}_1$ is precompact.*

*C. Stability analysis*

According to the previous subsections, the system $\mathcal{H}_1$ is nominally well posed and its maximal solutions are precompact. To infer that the solutions approach the desired (consensus) state, we rely on an invariance principle for hybrid systems [13, Theorem 8.2]. Before we do so, we establish the following lemma that we will use to characterize the set that the solutions to $\mathcal{H}_1$ approach.

**Lemma 4 (Nullspace of $Z_1$)** *The nullspace of $Z_1$ given in (12) satisfies* $\mathrm{Ker}(Z_1) = \mathrm{Im}(\mathbb{1}_{2n})$.

Using the lemma above, we are now ready to state the main result of this section.

**Theorem 1 (Approaching consensus)** *The maximal solutions of system $\mathcal{H}_1$ approach the point where $\theta = \hat{\theta} = \overline{\theta}$, with*

$$\overline{\theta} = \mathbb{1}_n \frac{\sum_i \theta_i(0)}{n} \in \mathrm{Im}(\mathbb{1}_n). \tag{13}$$

*D. Minimum broadcasting frequency*

The choice of $a_i$, $b_i$ and the clock dynamics

$$\dot{\phi}_i = -2\deg(i)(\frac{1}{4} + \phi_i + \phi_i^2) - \epsilon_i, \tag{14}$$

determine the broadcasting frequency of node $i \in \mathcal{V}$. To determine the required broadcasting frequency, we study the clock dynamics given by

$$\dot{\phi}_i = -2\deg(i)(\frac{1}{4} + \phi_i + \phi_i^2), \tag{15}$$

that provides the upper bound on the maximum inter sampling time for a given $a_i$ and $b_i$. We turn our attention to the question how much time $T_i$ elapses between a clock reset $\phi_i(t) = b_i$ until the following reset at $\phi_i(t + T_i) = a$. The following result can be obtained by solving (15) analytically, obtaining an explicit expression for $\phi(t)$.

[1] We denote $\mathrm{diag}(\phi_1, \ldots, \phi_n)$ by $[\phi]$.

**Theorem 2 (Inter broadcasting time)** *The inter broadcasting time $T_i$, induced by dynamics (15), is given by*

$$T_i = \frac{b_i - a_i}{\deg(i)(2a_ib_i + a_i + b_i + \frac{1}{2})}. \quad (16)$$

We define the upper bound on the allowed time between to broadcasting instances $T_i^{max}$, as $T_i^{max} = \lim_{b_i\to\infty, a_i\to 0} T_i(a_i, b_i)$. From Theorem 2 the following result is immediate:

**Corollary 1 (Minimum broadcasting frequency)** *The maximum allowed time between two sampling instances satisfies $T_i^{max} < \frac{1}{\deg(i)}$, such that the broadcasting frequency of node $i \in \mathcal{V}$ needs to be faster than the degree of node $i$.*

The implication of this section is that despite the nonlinear clock dynamics $\dot{\phi}_i = -2\deg(i)(\frac{1}{4}+\phi_i+\phi_i^2)-\epsilon_i$, we can rely on a simple counter to determine the broadcasting instances.

## IV. Coordination of dynamical systems

In this section we investigate the broadcasting implementation of the consensus algorithm to coordinate nonlinear network systems. Consider therefore (see also Remark 3) a nonlinear system

$$\begin{aligned} \dot{x} =&\ f(x, u) \\ y =&\ h(x), \end{aligned} \quad (17)$$

where $x \in \mathbb{R}^p$ is the state, $u \in \mathbb{R}^n$ is the input and $y \in \mathbb{R}^n$ is the output, and that satisfies the following assumption:

**Assumption 2 (Incremental passivity)** *There exists a continuous differentiable, positive definite storage function $\mathcal{S}(x, \overline{x}) : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$ that satisfies the inequality*

$$\dot{\mathcal{S}} \leq -(y-\overline{y})[Y](y-\overline{y}) + (y-\overline{y})^T(u-\overline{u}), \quad (18)$$

*along to solutions to (17), for some positive definite diagonal matrix $[Y]$, and where the triplet $(\overline{x}, \overline{u}, \overline{y})$ satisfies the steady state equations*

$$\begin{aligned} \mathbf{0} =&\ f(\overline{x}, \overline{u}) \\ \overline{y} =&\ h(\overline{x}). \end{aligned} \quad (19)$$

The focus of this work is to propose a broadcasting redesign of a class of existing continuous distributed controllers. Therefore, we assume that the following (continuous) distributed controller is suitable to control (17):

$$\begin{aligned} \dot{\theta} =& -\mathcal{L}\theta - K^T(y-\overline{y}) \\ u =&\ K\theta, \end{aligned} \quad (20)$$

which admits the steady state solution

$$\begin{aligned} \mathbf{0} =& -\mathcal{L}\overline{\theta} \\ \overline{u} =&\ K\overline{\theta}, \end{aligned} \quad (21)$$

where $(\overline{u}, \overline{y})$ are as in (19) and $K \in \mathbb{R}^{n\times n}$ is a diagonal matrix. An explicit example of (17) where this assumption is valid, is provided in the case study in the next section, where a distributed controller is used to control a power network.

**Remark 3 (Network systems)** *We write (17) to allow for a concise notation. However, note that (17) models network systems in case the dynamics at node $i$ are given by*

$$\begin{aligned} \dot{x}_i =& f_i(x, u_i) \\ y_i =& h_i(x), \end{aligned} \quad (22)$$

*where the dynamics of component $x_i$ depend only on $x_j$, $j \neq i$, if node $j$ is adjacent (connected) to node $i$. Furthermore, we note that, since $[Y]$ is diagonal, the dissipation inequality (18) can be written as a sum of 'node-wise' dissipation terms:*

$$\dot{\mathcal{S}} \leq \sum_{i\in\mathcal{V}} -(y_i-\overline{y}_i)Y_i(y_i-\overline{y}_i) + (y_i-\overline{y}_i)(u_i-\overline{u}_i).$$

We can now establish the following lemma for (17) in feedback interconnection to (20):

**Lemma 5 (Regulation with continuous communication)** *To solutions to system (17) in closed loop with (20) approach (locally) the set where $y = \overline{y}$ and $\theta \in \mathrm{Im}(\mathbb{1}_n)$.*

Note that the result of Lemma 5 holds globally if we additionally assume that $\mathcal{S}$ is radially unbounded. We now aim at deriving a similar result as Lemma 5, in case a sampled-data implementation of the communication is considered. Based on the results of the previous section, we propose a broadcasting implementation of (20):

$$\dot{\theta} = -\mathcal{L}\hat{\theta} - K^T(h(x) - h(\overline{x})), \quad (23)$$

where $\hat{\theta}_i$ is again the latest broadcasted value of $\theta_i$. We define a clock variable $\phi$ having the following dynamics[2]

$$\dot{\phi} = \alpha + \beta, \quad (24)$$

where

$$\alpha_i = -2\deg(i)(\frac{1}{4} + \phi_i + \phi_i^2) - \epsilon_i \quad \forall i \in \mathcal{V}, \quad (25)$$

and

$$\beta_i = -\frac{4(K_i\phi_i)^2}{Y_i} - \zeta_i \quad \forall i \in \mathcal{V}. \quad (26)$$

Here, $\epsilon_i \in \mathbb{R}_{>0}$ and $\zeta_i \in \mathbb{R}_{>0}$ are two constants. Note that (25) is identical to (5) and that (26) is an additional term imposed by the interconnected system (17). As in the previous section, the clock dynamics follow from the stability analysis and are stated here for the sake of exposition.

**Remark 4 (Different distributed controllers)** *In this work we focus on a broadcasting implementation of distributed controllers with dynamics (20). Nevertheless, we believe that the chosen approach is suitable to study other variations of distributed controllers and this will be considered in a future research.*

[2]With a slight abuse of notation, we use, for the sake of notational simplicity, the same symbols $\phi$, $C$, $D$ $F$ and $G$ as in the previous section.

Next, we study, similarly to the previous section, the following hybrid system with flow set $C := \{(x, \theta, \hat{\theta}, \phi) \in \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^n \times [a, b]\}$. The flow map $F(x, \theta, \hat{\theta}, \phi)$ is

$$\left.\begin{aligned} \dot{x} &= f(x, K\theta) \\ \dot{\theta} &= -\mathcal{L}\hat{\theta} - K^T(h(x) - h(\overline{x})) \\ \dot{\hat{\theta}} &= \mathbf{0} \\ \dot{\phi} &= \alpha + \beta \end{aligned}\right\} := F(x, \theta, \hat{\theta}, \phi) \quad (27)$$

where $\alpha_i$ and $\beta_i$ are given by (15) and (26), respectively. The jump set is $D := \{(x, \theta, \hat{\theta}, \phi) \in \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^n \times [a, b] : \exists i \in \{1, \ldots, n\}$ s.t. $\phi_i = a_i\}$. The jump map $G(x, \theta, \hat{\theta}, \phi)$ is defined by

$$G(x, \theta, \hat{\theta}, \phi) := \{G_i(x, \theta, \hat{\theta}, \phi) : i \in \{1, .., n\} \text{ and } \phi_i = a_i\},$$

with

$$\left.\begin{aligned} x^+ &= x \\ \theta^+ &= \theta \\ \hat{\theta}_i^+ &= \theta_i \\ \hat{\theta}_j^+ &= \hat{\theta}_j \quad j \neq i \\ \phi_i^+ &= b_i \\ \phi_j^+ &= \phi_j \quad j \neq i \end{aligned}\right\} := G_i(x, \theta, \hat{\theta}, \phi). \quad (28)$$

The hybrid system with the data above will be represented by the notation $\mathcal{H}_2 = (C, F, D, G)$ or, briefly, by $\mathcal{H}_2$. Before analyzing the asymptotical behaviour of $\mathcal{H}_2$, we establish the following useful lemma:

**Lemma 6 (Evolution of $V_1 + \mathcal{S}$)** *The storage function $V_2 = V_1 + \mathcal{S}$, with $V_1$ as (10) and $\mathcal{S}$ as in Assumption 2, satisfies for $(x, \theta, \hat{\theta}, \phi) \in C$*

$$\dot{V}_2 = \begin{bmatrix} \theta \\ \hat{\theta} \end{bmatrix}^T Z_1 \begin{bmatrix} \theta \\ \hat{\theta} \end{bmatrix} + \begin{bmatrix} \theta - \hat{\theta} \\ h(x) - h(\overline{x}) \end{bmatrix}^T Z_2 \begin{bmatrix} \theta - \hat{\theta} \\ h(x) - h(\overline{x}) \end{bmatrix} \leq 0 \quad (29)$$

*and for $(x, \xi, \theta, \hat{\theta}, \phi) \in D$*

$$V_2(x^+, \xi^+, \theta^+, \hat{\theta}^+, \phi^+) \leq V_2(x, \xi, \theta, \hat{\theta}, \phi), \quad (30)$$

*along the solutions to $\mathcal{H}_2$, where $Z_1$ and $Z_2$ are given by*

$$Z_1 = \begin{bmatrix} [\alpha] & -[\alpha] - \frac{1}{2}(I + 2[\phi])\mathcal{L} \\ -[\alpha] - \frac{1}{2}((I + 2[\phi])\mathcal{L})^T & [\alpha] + [\phi]\mathcal{L} + \mathcal{L}[\phi] \end{bmatrix}, \quad (31)$$

$$Z_2 = \begin{bmatrix} [\beta] & [\phi]K + K^T[\phi] \\ [\phi]K + K^T[\phi] & -Y \end{bmatrix}. \quad (32)$$

The following two lemmas can be established, similarly to Lemma 1 and Lemma 3 in the previous section, and we omit the details.

**Lemma 7 (Nominally well posed)** *The system $\mathcal{H}_2$ is nominally well posed.*

**Lemma 8 (Precompactness of solutions)** *Every maximal solution to system $\mathcal{H}_2$ is precompact.*

We are now ready to state the main result of this section.

**Theorem 3 (Regulation with broadcasting)** *The maximal solutions of system $\mathcal{H}_2$ approach (locally) the set where $h(x) = y = \overline{y}$ and $\theta = \hat{\theta} = \overline{\theta} \in \operatorname{Im}(\mathbb{1}_n)$.*

### A. Inter broadcasting time

Similarly as is done in Section III-D, solving

$$\dot{\phi}_i = -2\deg(i)(\frac{1}{4} + \phi_i + \phi_i^2) - \frac{4K_i^2\phi_i^2}{Y_i}, \quad (33)$$

analytically leads to the following result:

**Corollary 2 (Inter broadcasting time)** *The maximum allowed time between two sampling instances satisfies*

$$T_i^{max} < \frac{2}{\sqrt{c_{1i}c_{2i}}}\left(\frac{\pi}{2} - \arctan\left(\frac{\sqrt{c_{1i}}}{\sqrt{c_{2i}}}\right)\right), \quad (34)$$

*with $c_{1i} = 2\deg(i)$ and $c_{2i} = \frac{4K_i^2}{Y_i}$.*

A further analysis shows also that $T_i^{max}$ given above reduces to the expression in Corollary 1, if one takes $c_{2i} \to 0$.

## V. Case study

To illustrate the results obtained in this paper, we perform an academic case study on the distributed control, aiming at optimal power sharing, of a power network with a circle topology, consisting of four interconnected control areas. The case study is taken from [8], to which we refer for additional background information and used numerical values. We assume, for the sake of exposition, that the voltages are constant. We incorporate the results from this paper to obtain a sampled-data implementation of the communication among the controllers proposed in [8], resulting in a hybrid system. Consider the flow set $C := \{(\eta, \omega, \theta, \hat{\theta}, \phi) \in \mathbb{R}^4\times, \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \times [a_i, b_i]^4\}$. Here, we have chosen $a_i = 1^{-10}, b_i = 2$ for all $i \in \{1, \ldots 4\}$. The flow map $F(\eta, \omega, \theta, \hat{\theta}, \phi)$ is given by (35), where the first two lines represent the dynamics of the power network, and the last three lines represent a distributed controller that exchanges information at discrete time instances. The clock dynamics are based on (24), where we take $\epsilon_i = \zeta_i = 0$ for which we compensate by choosing $b_i < \infty$.

$$\begin{aligned} \dot{\eta} &= \mathcal{B}_P^T\omega \\ M\dot{\omega} &= Q^{-1}\theta - \mathcal{B}_P\Gamma\sin(\eta) - D_g\omega - P_d \\ \dot{\theta} &= -\mathcal{L}\hat{\theta} - Q^{-1}\omega \\ \dot{\hat{\theta}} &= \mathbf{0} \\ \dot{\phi} &= -2\mathrm{Deg}\cdot(\frac{1}{4}\mathbb{1}_4 + \phi + [\phi]\phi) - 4D_g^{-1}Q^{-2}[\phi]\phi \end{aligned} \quad (35)$$

Here, $\eta, \omega, \theta, \hat{\theta}, \phi, P_d \in \mathbb{R}^4$ are the voltage angle differences, frequency deviations, weighted power generations, latest broadcasted values, clock variables and power demands, respectively. Furthermore, $\mathrm{Deg} = \mathrm{diag}(\deg(1), \ldots, \deg(4))$, $\mathcal{B}_P$ is the incidence matrix reflecting the topology of the power network and the remaining matrices are positive definte and diagonal of approriate dimenstions. The jump set is $D := \{(\eta, \omega, \theta, \hat{\theta}, \phi) \in \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \times [a_i, b_i]^n : \exists i \in$

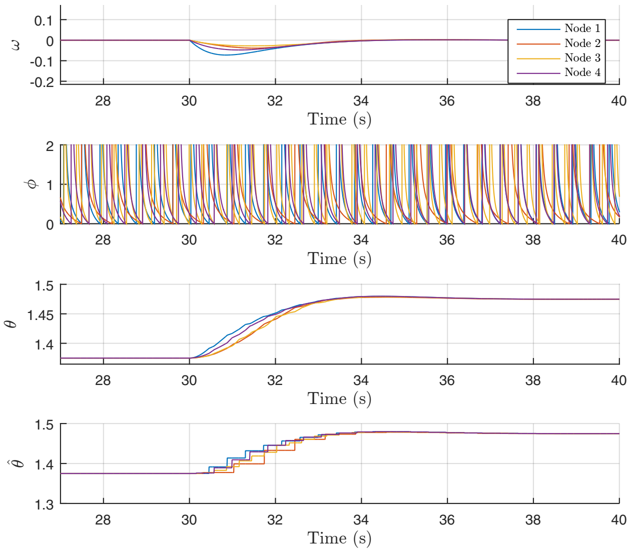


Fig. 1: Time evolution of the frequency deviation $\omega$, clock variable $\phi$, control input $\theta$, and broadcasted value $\hat{\theta}$.

$\{1,\ldots,n\}$ s.t. $\phi_i = 1^{-10}\}$. The jump map $G(\eta,\omega,\theta,\hat{\theta},\phi)$ is defined by $G(\eta,\omega,\theta,\hat{\theta},\phi) := \{G_i(\eta,\omega,\theta,\hat{\theta},\phi) : i \in \{1,\ldots,n\}$ and $\phi_i = a_i\}$, with

$$\left.\begin{array}{rcll} \eta^+ &=& \eta & \\ \omega^+ &=& \omega & \\ \theta^+ &=& \theta & \\ \hat{\theta}_i^+ &=& \theta_i & \\ \hat{\theta}_j^+ &=& \hat{\theta}_j & j \neq i \\ \phi_i^+ &=& b_i & \\ \phi_j^+ &=& \phi_j & j \neq i \end{array}\right\} := G_i(\eta,\omega,\theta,\hat{\theta},\phi). \quad (36)$$

The system is initially at steady state with a constant demand $P_d(t) = (2.00, 1.00, 1.50, 1.00)^T$ for $t \in [0, 30)$, whereafter the demand is increded to $P_d(t) = (2.20, 1.05, 1.55, 1.10)^T$ for $t \geq 30$. The frequency response to the control input is given in Figure 1. From Figure 1 we can see how the frequency drops due to the increased load. Furthermore we note that the controller regulates the power generation such that a new steady state condition is obtained where the frequency deviation is again zero and that the power generation $Q^{-1}\theta$ satisfies at steady state $Q^{-1}\overline{\theta} \in \mathrm{Im}(Q^{-1}\mathbb{1}_4)$.

## VI. Conclusions and Future Directions

We have proposed a methodology to design broadcasting protocols for coordinating nonlinear network systems. The scheduling of information transmission relies on introduced clock variables, whose dynamics are regulated through a suitable storage function. Since the required clock dynamics follow from Lyapunov like arguments, we argue that, in comparison with existing results, the prosed design can be more easily incorporated within the stability study of nonlinear network systems. Among possible nonlinear network systems, we have focused on the redesign of distributed (averaging) integral controllers, coordination a nonlinear network system composed of output strictly incrementally passive subsystems. An interesting future research direction is to apply the proposed design of clock dynamics to larger classes of distributed systems. Furthermore, allowing for communication delays and time-varying communication topologies are important extensions that will be considered.

## References


[1] D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, "Distributed event-triggered control for multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1291–1297, 2012.

[2] T. Liu, M. Cao, C. De Persis, and J. M. Hendrickx, "Distributed event-triggered control for asymptotic synchronization of dynamical networks," *Automatica*, vol. 86, pp. 199 – 204, 2017.

[3] C. Nowzari and J. Cortés, "Distributed event-triggered coordination for average consensus on weight-balanced digraphs," *Automatica*, vol. 68, pp. 237 – 244, 2016.

[4] E. Garcia, Y. Cao, and D. W. Casbeer, "Periodic event-triggered synchronization of linear multi-agent systems with communication delays," *IEEE Transactions on Automatic Control*, vol. 62, no. 1, pp. 366–371, 2017.

[5] T. H. Cheng, Z. Kan, J. R. Klotz, J. M. Shea, and W. E. Dixon, "Event-triggered control of multiagent systems for fixed and time-varying network topologies," *IEEE Transactions on Automatic Control*, vol. 62, no. 10, pp. 5365–5371, 2017.

[6] D. Liuzza, D. V. Dimarogonas, M. di Bernardo, and K. H. Johansson, "Distributed model based event-triggered control for synchronization of multi-agent systems," *Automatica*, vol. 73, pp. 1 – 7, 2016.

[7] C. De Persis and R. Postoyan, "A lyapunov redesign of coordination algorithms for cyber-physical systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 2, pp. 808–823, 2017.

[8] S. Trip, M. Bürger, and C. De Persis, "An internal model approach to (optimal) frequency regulation in power grids with time-varying voltages," *Automatica*, vol. 64, pp. 240 – 253, 2016.

[9] J. W. Simpson-Porco, F. Dörfler, and F. Bullo, "Synchronization and power sharing for droop-controlled inverters in islanded microgrids," *Automatica*, vol. 49, no. 9, pp. 2603–2611, 2013.

[10] M. Andreasson, D. V. Dimarogonas, K. H. Johansson, and H. Sandberg, "Distributed vs. centralized power systems frequency control," in *Proc. of the 12th European Control Conference (ECC)*, Zürich, Switzerland, 2013, pp. 3524–3529.

[11] J. Schiffer, F. Dörfler, and E. Fridman, "Robustness of distributed averaging control in power systems: Time delays & dynamic communication topology," *Automatica*, vol. 80, pp. 261 – 271, 2017.

[12] Y. Fan, G. Hu, and M. Egerstedt, "Distributed reactive power sharing control for microgrids with event-triggered communication," *IEEE Transactions on Control Systems Technology*, vol. 25, no. 1, pp. 118–128, 2017.

[13] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: modeling, stability, and robustness*. Princeton University Press, 2012.

[14] R. Postoyan, P. Tabuada, D. Nešić, and A. Anta, "A framework for the event-triggered stabilization of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 60, no. 4, pp. 982–996, 2015.

[15] S. Trip and C. De Persis, "Communication requirements in a master-slave control structure for optimal load frequency control," in *Proc. of the 2017 IFAC World Congress*, Toulouse, France, 2017, pp. 10 519–10 524.

[16] D. Carnevale, A. R. Teel, and D. Nešić, "A Lyapunov proof of an improved maximum allowable transfer interval for networked control systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 5, p. 892, 2007.

[17] S. Trip, "Distributed control of power systems; passivity, optimality and energy functions," *PhD Thesis, University of Groningen*, 2017.

[18] L. Hetel, C. Fiter, H. Omran, A. Seuret, E. Fridman, J.-P. Richard, and S. I. Niculescu, "Recent developments on the stability of systems with aperiodic sampling: An overview," *Automatica*, vol. 76, pp. 309 – 335, 2017.