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A Real-Time Iteration Scheme with Quasi-Newton Jacobian Updates for Nonlinear Model Predictive Control

Pedro Hespanhol[1], Rien Quirynen[1]

***Abstract*— Nonlinear model predictive control (NMPC) requires the solution of a dynamic optimization problem at each sampling instant under strict timing constraints, involving nonlinear dynamics that can often be stiff or implicitly defined. The real-time iteration (RTI) scheme has been shown to allow real-world embedded applications of NMPC. The present paper proposes an extension of the standard RTI algorithm with a block-structured quasi-Newton method to obtain low-rank Jacobian updates that preserve the block structure of the optimal control problem. In addition, a particular structure-exploiting implementation is presented for implicit integration schemes such that no Jacobian evaluation is needed neither any matrix factorization. Based on a proof of concept implementation in C code, the computational performance of the algorithm is illustrated for multiple NMPC case studies.**

I. INTRODUCTION

Optimization based control and estimation techniques, such as nonlinear model predictive control (NMPC) and moving horizon estimation (MHE), allow a model-based design framework in which the system dynamics and constraints can directly be taken into account [1]. One main practical challenge lies in the ability to solve the following nonlinear optimal control problem (OCP) at each sampling instant, under strict timing constraints:

$$\min_{x(t),u(t)} \quad \int_0^T l(x(t),u(t))\,\mathrm{d}t \tag{1a}$$

$$\text{s.t.} \quad x_0 - \hat{x}_0 \;=\; 0, \tag{1b}$$

$$0 \;=\; f(\dot{x}(t),x(t),u(t)), \quad \forall t \in [0,T], \tag{1c}$$

$$p(x(t),u(t)) \leq 0, \qquad \forall t \in [0,T], \tag{1d}$$

where T denotes the control horizon length, $x(t) \in \mathbb{R}^{n_\mathrm{x}}$ denotes the states and $u(t) \in \mathbb{R}^{n_\mathrm{u}}$ denotes the controls. The function $l(\cdot)$ defines the stage cost. The nonlinear system dynamics are formulated as an implicit system of ordinary differential equations (ODE) in (1c), which could be extended with algebraic equations. The problem is parametric in the current state estimate $\hat{x}_0$, through the initial value condition in (1b). The path constraints are defined in Eq. (1d) and they are assumed to be affine, for simplicity.

Direct optimal control methods then continue by forming a discrete-time approximation of the problem in (1), based on an appropriate parameterization of the state and control trajectories, resulting in a tractable nonlinear program (NLP). Popular examples of this include the direct multiple shooting method [2] and direct collocation [3], [4]. The resulting constrained optimization problem can be handled by standard Newton-type algorithms such as interior point methods and sequential quadratic programming (SQP) techniques [5]. For the purpose of real-time predictive control and estimation, continuation-based online algorithms have been proposed. An overview on such methods can be found in [6].

This work extends the real-time iteration (RTI) scheme, which is an online variant of an SQP algorithm for NMPC [7]. One typically avoids the evaluation of the Lagrangian Hessian by using a Gauss-Newton type approximation [5]. Moreover, inexact Jacobian matrices can be used to further reduce the computational effort, e.g., by reusing the linearized dynamics over multiple sampling instants [8], [9]. Recently, inexact Newton implementations were proposed in order to reduce the computational cost per iteration for lifted collocation schemes [10]. Instead, our focus is on quasi-Newton type methods which rely on low-rank updates of the Jacobian matrix to obtain good convergence properties at a strongly reduced computational cost. Unlike standard Broyden type methods [11], a two-sided rank-one (TR1) update scheme has been proposed [12], [13] for constrained optimization. A main contribution of this paper is an extension of the lifted collocation algorithm based on a sparsity-preserving and structure-exploiting variant of the quasi-Newton type TR1 update technique.

This results in a novel extension of the RTI scheme that avoids the computation of Jacobian matrices and, instead, only relies on relatively cheap adjoint derivative evaluations. The resulting lifted collocation type algorithm avoids any type of matrix factorization or the solution of linear systems, unlike standard algorithms that are based on an implicit integration scheme. The computational performance of both these algorithms is illustrated for multiple case studies, using a preliminary C code implementation. These numerical results include a real-world control application of NMPC based steering in autonomous vehicles.

The paper is organized as follows. Section II briefly summarizes direct optimal control and inexact SQP-type optimization. The block-structured quasi-Newton Jacobian update procedure is detailed in Section III. Then, in Section IV, we discuss our tailored structure exploitation within the lifted collocation algorithm. Section V illustrates the numerical performance of the proposed algorithms based on multiple case studies of nonlinear MPC.

II. NEWTON-TYPE OPTIMIZATION FOR DIRECT OPTIMAL CONTROL

In direct optimal control, the OCP in Eq. (1) is first approximated by a tractable NLP. We formulate an equidistant

[1]Mechatronics, Mitsubishi Electric Research Laboratories, Cambridge, MA, 02139, USA. quirynen@merl.com

grid over the control horizon T consisting of the collection of time points $\{t_i\}_{i=0}^N$. We also consider a piecewise constant control parameterization $u(\tau) = u_i$ for $\tau \in [t_i, t_{i+1})$.

A. Direct Multiple Shooting and Direct Collocation

A popular approach for direct optimal control is based on direct multiple shooting [2]. We consider an explicit fixed-step integration scheme [14] that defines the discrete-time system dynamics for each shooting interval. The resulting block-structured OCP reads as:

$$
\begin{aligned}
\min_{X,\,U} \quad & \sum_{i=0}^{N-1} l_i(x_i,u_i) + l_N(x_N) && \text{(2a)}\\
\text{s.t.} \quad & x_0 - \hat{x}_0 \;=\; 0, && \text{(2b)}\\
& F_i(x_i,u_i) \;=\; x_{i+1}, \qquad i = 0,\ldots,N-1, && \text{(2c)}\\
& p(x_i,u_i) \leq 0, \qquad i = 0,\ldots,N. && \text{(2d)}
\end{aligned}
$$

The optimization variables are the state $X = [x_0^\top, \ldots, x_N^\top]^\top$ and control trajectory $U = [u_0^\top, \ldots, u_{N-1}^\top]^\top$. In direct transcription or direct collocation [3], [4], an implicit integration scheme and its intermediate variables are directly made part of the nonlinear optimization problem. The discrete-time optimal control problem reads as:

$$
\begin{aligned}
\min_{X,\,U,\,K} \quad & \sum_{i=0}^{N-1} l_i(x_i,u_i) + l_N(x_N) && \text{(3a)}\\
\text{s.t.} \quad & x_0 - \hat{x}_0 \;=\; 0, && \text{(3b)}\\
& G_i(x_i,u_i,K_i) \;=\; 0, \qquad i = 0,\ldots,N-1, && \text{(3c)}\\
& x_i + B_i\,K_i \;=\; x_{i+1}, \qquad i = 0,\ldots,N-1, && \text{(3d)}\\
& p(x_i,u_i) \leq 0, \qquad i = 0,\ldots,N, && \text{(3e)}
\end{aligned}
$$

where the additional trajectory $K = [K_0^\top, \ldots, K_{N-1}^\top]^\top$ denotes the intermediate variables. These variables are defined implicitly by the equations in (3c), such that the continuity condition reads as in Eq. (3d). The Jacobian $\frac{\partial G_i}{\partial K_i}(\cdot)$ will generally be invertible for an integration scheme applied to a well-defined set of differential equations in (1c).

B. Adjoint-based Inexact SQP and Lifted Collocation

We focus on SQP methods to solve the forementioned NLPs, where at each iteration a quadratic program (QP) approximation is obtained by linearizing the constraint functions and forming a quadratic objective. Hence, the linearization point corresponds to the state $\bar{X} = [\bar{x}_0^\top, \ldots, \bar{x}_N^\top]^\top$ and control values $\bar{U} = [\bar{u}_0^\top, \ldots, \bar{u}_{N-1}^\top]^\top$ from the previous iteration. We consider the adjoint-based SQP algorithm for fast NMPC [8], [9]. In case of the multiple shooting NLP (2), each SQP iteration solves a convex QP subproblem:

$$
\begin{aligned}
\min_{\Delta W} \quad & \sum_{i=0}^{N} \frac{1}{2}\Delta w_i^\top H_i\,\Delta w_i \;+\; h_i^\top \Delta w_i && \text{(4a)}\\
\text{s.t.} \quad & \Delta x_0 \;=\; \hat{x}_0 - \bar{x}_0, && \text{(4b)}\\
& a_i + A_i \Delta w_i \;=\; \Delta x_{i+1}, \qquad i = 0,\ldots,N-1, && \text{(4c)}\\
& p(\Delta w_i) \leq -p_i, \qquad i = 0,\ldots,N, && \text{(4d)}
\end{aligned}
$$

where $w_i := (x_i,u_i)$ and $w_N := x_N$. The notation $\Delta W = [\Delta w_0^\top, \ldots, \Delta w_N^\top]^\top$ is used to denote the deviation variables $\Delta w_i := w_i - \bar{w}_i$. The stage cost is defined by a (nonlinear) least squares term $l_i(x_i,u_i) = \frac{1}{2}\|R(x_i,u_i)\|_2^2$ such that the generalized Gauss-Newton (GGN) method from [15] uses the block-structured Hessian approximation $H_i := \nabla R(\bar{w}_i)\nabla R(\bar{w}_i)^\top \approx \nabla_{w_i}^2 \mathcal{L}$, where $\mathcal{L}(\cdot)$ refers to the Lagrangian of (2). The matrix $A_i \approx \frac{\partial F_i}{\partial w_i}(\bar{w}_i)$ in Eq. (4c) denotes the Jacobian approximation and $a_i := F_i(\bar{w}_i) - \bar{x}_{i+1}$ for the continuity condition on each interval. The gradient term h_i in the objective (4a) reads as in [8], [9], which can be evaluated efficiently using backward differentiation [16].

Similarly, the same adjoint-based SQP method can be applied directly to the direct collocation problem in (3):

$$
\begin{aligned}
\min_{\Delta W,\,\Delta K} \quad & \sum_{i=0}^{N} \frac{1}{2}\Delta w_i^\top H_i\,\Delta w_i \;+\; h_i^{c^\top} \begin{bmatrix} \Delta w_i \\ \Delta K_i \end{bmatrix} && \text{(5a)}\\
\text{s.t.} \quad & \Delta x_0 \;=\; \hat{x}_0 - \bar{x}_0, && \text{(5b)}\\
& c_i + D_i \Delta w_i + C_i \Delta K_i \;=\; 0, && \text{(5c)}\\
& e_i + \Delta x_i + B_i\,\Delta K_i \;=\; \Delta x_{i+1}, && \text{(5d)}\\
& p(\Delta w_i) \leq -p_i, \qquad i = 0,\ldots,N, && \text{(5e)}
\end{aligned}
$$

based on the evaluation $c_i := G_i(\bar{w}_i, \bar{K}_i)$ and the Jacobian approximations $D_i \approx \frac{\partial G_i}{\partial w_i}(\bar{w}_i, \bar{K}_i)$ and $C_i \approx \frac{\partial G_i}{\partial K_i}(\bar{w}_i, \bar{K}_i)$. The corresponding gradient correction h_i^c can be computed in an analogous fashion as the gradient term h_i in (4a). Recently, it has been shown in [10] how tailored embedded optimization algorithms [17], [18] can also be used to efficiently solve the direct collocation structured QP in (5) based on a numerical elimination and expansion of the collocation variables. For this purpose, the step direction for the collocation variables can be obtained as $\Delta K_i = -C_i^{-1}(c_i + D_i \Delta w_i)$ directly from Eq. (5c). The variables ΔK_i in the QP (5) can then be numerically eliminated in order to obtain the condensed QP subproblem of the block-structured multiple shooting form in Eq. (4). We will present a quasi-Newton type variant of this condensing and expansion procedure further in Section IV.

C. Real-Time Iteration Scheme for Nonlinear MPC

In embedded NMPC applications, one needs to solve the nonlinear OCP of Eq. (2) or (3) at each sampling instant under strict timing constraints. For this purpose, we instead use the real-time iteration (RTI) scheme [6], [7], which is a continuation based variant of a fixed-step SQP method. More specifically, by warm-starting the algorithm with the shifted solution to the previous problem, only one SQP iteration is performed at each time step. The general idea is that one prefers to obtain new measurement information from the system, rather than iterating until convergence for an optimization problem that is becoming outdated. It has been shown in [7], under some reasonable assumptions in a simplified setting, that the stability of the closed-loop system based on the RTI scheme can be guaranteed also in presence of inaccuracies and external disturbances.

III. BLOCK-TR1 JACOBIAN UPDATES FOR DIRECT OPTIMAL CONTROL

This paper aims at proposing a quasi-Newton type algorithm [19] to keep the Jacobian approximations sufficiently accurate while preserving the block sparsity structure of the QP subproblem in Eq. (4) or (5). Existing literature on rank-one approximation-based SQP methods such as [12] analyze several rank-one updates for the constraint Jacobian, such as the classical good and bad Broyden schemes in [20]. In particular, the work in [12] proposes an alternative two-sided rank-one (TR1) update as a generalization of the symmetric rank-one (SR1) update scheme in [21]. The TR1 approach enjoys several benefits over the classical methods, such as heredity and invariance to scaling. Moreover, the work in [13] presents convergence results for an SQP method based on SR1 to approximate the Hessian of the Lagrangian and TR1 to approximate the constraint Jacobian.

When directly applying the above mentioned approaches to the NLPs in Eq. (2) or (3), then the optimal control block-structured sparsity is destroyed immediately. Therefore, the solution of each of the resulting QP subproblems would then become considerably more expensive. Structure exploiting or *partitioned* updates have been proposed in the context of unconstrained optimization in [2], [22], [23]. To the best of our knowledge, a tailored SQP algorithm based on block-wise updates for the constraint Jacobian matrices has not yet been proposed, in order to exploit the particular multiple shooting structure in optimal control.

A. Block-wise TR1 Update Formula

Let us present the block-wise TR1 update formula initially for the multiple shooting structured NLP in Eq. (2). When applying the SQP method, the resulting QP subproblem in Eq. (4) requires an approximation matrix $A_i^{\circ} \approx \frac{\partial F_i}{\partial w_i}(w_i^{\circ})$ for each of the continuation constraints $i = 0, \ldots, N-1$. After solving this QP, we then need to update each of these matrices independently. Following the work in [12], we would like that each updated approximation matrix A_i^{+} satisfies the following two secant conditions:

$$\begin{aligned} &\text{Adjoint Condition (AC): } \sigma_i^\top A_i^{+} = \mu_i^\top, \\ &\text{Forward Condition (FC): } \; A_i^{+} s_i = y_i, \end{aligned} \tag{6}$$

where we define the adjoint vector $\mu_i^\top = \sigma_i^\top \frac{\partial F_i}{\partial w_i}(w_i^{+})$, given $\sigma_i = \lambda_i^{+} - \lambda_i^{\circ}$, and the difference in function evaluations $y_i = F(w_i^{+}) - F(w_i^{\circ})$. Note that λ_i^{+} and λ_i°, respectively, denote the new and old Lagrange multipliers for the linearized continuation constraints in Eq. (4c). Similarly, $w_i^{\circ} := (x_i^{\circ}, u_i^{\circ})$ and $w_i^{+} := w_i^{\circ} + \Delta w_i^{\star}$ denote respectively the old and new primal variables such that $s_i := w_i^{+} - w_i^{\circ}$. As discussed also in [12], the gradient $\sigma_i^\top \frac{\partial F_i}{\partial w_i}(w_i^{+})$ can be computed efficiently using the backward mode of algorithmic differentiation (AD), e.g., see [16].

The proposed block-wise TR1 update formula then reads as follows:

$$A_i^{+} = A_i^{\circ} + \alpha \left(y_i - A_i^{\circ} s_i\right)\left(\mu_i^\top - \sigma_i^\top A_i^{\circ}\right), \tag{7}$$

for $i = 0, \ldots, N-1$ and where α is a scalar that will be defined further. Aside from the case where the function $F(\cdot)$ is affine, the two conditions, AC and FC in Eq. (6), are not consistent with each other and they can therefore not both be satisfied by the updated matrix A_i^{+}. Thus, similar to the standard TR1 update in [12], the block-wise update will only be able to satisfy one or the other. In the adjoint variant of the update, the scaling value is defined as:

$$\alpha_{\mathrm{A}} = 1/(\sigma_i^\top y_i - \sigma_i^\top A_i^{\circ} s_i), \tag{8}$$

such that the adjoint condition in (6) is satisfied exactly and the forward condition holds up to some accuracy. This value reads similarly for the forward variant α_F, replacing $\sigma_i^\top y_i$ by $\mu_i^\top s_i$, such that the forward condition is satisfied exactly. Note that the block-TR1 update results in a rank-N update for the complete Jacobian matrix of the QP in (4).

B. RTI Implementation for Nonlinear MPC

We propose to use the block-TR1 update formula in an inexact but computationally cheaper variant of the RTI scheme for nonlinear MPC. The resulting implementation is illustrated in Algorithm 1. The use of costly differentiation techniques is restricted to merely two adjoint derivative directions [16] for each shooting interval instead of a complete Jacobian matrix evaluation. One adjoint is needed to compute μ_i for the block-TR1 update formula and one adjoint is needed in the gradient correction. Algorithm 1 is called at each sampling instant of the NMPC scheme and, since the TR1 update is part of the *preparation* step, the computational delay between obtaining the new state estimate and applying the next control input to the system is restricted to the solution of the QP in the *feedback* phase.

Note that similar to the standard RTI scheme [6], the proposed algorithm requires an initialization step prior to the real-time feedback control loop, in order to initialize the state and control variables, the Lagrange multipliers and the approximate Jacobian matrices. Depending on the particular application, such an initialization can be computed offline, e.g., based on a steady state solution.

IV. LIFTED COLLOCATION ALGORITHM WITH BLOCK-TR1 JACOBIAN UPDATES

Algorithm 1 can de directly applied to the direct collocation problem in (3), where the block-TR1 update formula is used for the Jacobian approximation matrices $\begin{bmatrix} D_i & C_i \end{bmatrix} \approx \frac{\partial G_i}{\partial (w_i, K_i)}(\bar{w}_i, \bar{K}_i)$. However, the resulting QP subproblem is relatively large due to the amount of additional collocation variables and corresponding equations in (5c). Motivated by the work on lifted collocation integrators as discussed in Section II-B, we aim to exploit the sparsity structure of the QP in (5) directly in the block-wise rank-one update formula. Unlike the standard lifted collocation scheme, this results in a tailored block-TR1 based SQP method for direct collocation without the use of a Jacobian evaluation and without any matrix factorization technique.

Algorithm 1 Block-TR1 based Jacobian Updates within a Real-Time Iteration Scheme for Nonlinear MPC.

Input: $w_i^\circ = (x_i^\circ, u_i^\circ)$, λ_i° and A_i° for $i = 0, \ldots, N-1$.

```
Problem linearization
```
1: Formulate the QP in (4) with matrices A_i° and evaluate vectors a_i and h_i for $i = 0, \ldots, N-1$.

```
Computation of step direction
```
2: Obtain current state estimate $\hat{x}_0$. ▷ from system

3: Solve the QP subproblem in Eq. (4): $w_i^+ \leftarrow w_i^\circ + \Delta w_i^\star$ and $\lambda_i^+ \leftarrow \lambda_i^\star$.

4: Apply new control input u_0^+. ▷ to system

```
Block-wise TR1 update
```
5: **for** $i = 0, \ldots, N-1$ **do in parallel**

6: $\quad A_i^+ \leftarrow A_i^\circ + \alpha\,(y_i - A_i^\circ s_i)\left(\mu_i^\top - \sigma_i^\top A_i^\circ\right)$.

7: **end for**

Output: $w_i^+ = (x_i^+, u_i^+)$, λ_i^+ and A_i^+ for $i = 0, \ldots, N-1$.

A. Tailored Structure Exploitation for Direct Collocation

As mentioned earlier, the Jacobian matrix $\frac{\partial G_i}{\partial K_i}$ for the collocation equations needs to be invertible. Therefore, given an invertible approximation $C_i \approx \frac{\partial G_i}{\partial K_i}(w_i^\circ, K_i^\circ)$, we can rewrite the linearized expression in Eq (5c) as follows

$$\Delta K_i = -C_i^{-1}\left(c_i + D_i \Delta w_i\right). \tag{9}$$

By substituting the above expression for ΔK_i back into the direct collocation structured QP in (5), one obtains the equivalent but condensed formulation

$$\min_{\Delta W} \quad \sum_{i=0}^{N} \frac{1}{2} \Delta w_i^\top H_i\, \Delta w_i \,+\, \tilde{h}_i^{c\top} \Delta w_i \tag{10a}$$

$$\text{s.t.} \quad \Delta x_0 \,=\, \hat{x}_0 - x_0^\circ, \tag{10b}$$

$$d_i + \Delta x_i - B_i\, C_i^{-1} D_i \Delta w_i \,=\, \Delta x_{i+1}, \tag{10c}$$

$$p(\Delta w_i) \le -p_i, \qquad i = 0, \ldots, N, \tag{10d}$$

where $d_i = e_i - B_i\, C_i^{-1} c_i$ is defined and the condensed gradient reads as

$$\tilde{h}_i^c = \nabla_{w_i} l(w_i^\circ) + \left(\frac{\partial G_i}{\partial w_i} - \frac{\partial G_i}{\partial K_i}\, C_i^{-1} D_i\right)^\top \omega_i^\circ. \tag{11}$$

Note that the resulting QP formulation in Eq. (10) is of the same problem dimensions and exhibits the same sparsity as the multiple shooting structured QP subproblem in Eq. (4). Therefore, state of the art block-structured QP solvers can be used for which an overview can be found in [17]. After solving the condensed QP in (10), the collocation variables can be obtained from the expansion step in Eq. (9). Based on the optimality conditions of the original direct collocation structured QP in (5), the corresponding Lagrange multipliers can be updated as follows

$$\omega_i^+ = \omega_i^\circ - C_i^{-\top} \frac{\partial G_i}{\partial K_i}^\top \omega_i^\circ - C_i^{-\top} B_i^\top \lambda_i^+, \tag{12}$$

where λ_i^+ denote the new Lagrange multipliers for the continuity conditions in (10c) or in (5d).

B. Block-TR1 Update for Direct Collocation

The block-TR1 update formula from Eq. (7) can be readily applied to direct collocation, resulting in

$$[D_i^+\; C_i^+] = [D_i^\circ\; C_i^\circ] + \alpha\,(y_i - [D_i^\circ\; C_i^\circ] s_i)\left(\mu_i^\top - \sigma_i^\top [D_i^\circ\; C_i^\circ]\right) \tag{13}$$

where the quantities $\mu_i^\top = \sigma_i^\top \frac{\partial G_i}{\partial (w_i, K_i)}(w_i^+, K_i^+)$ and $\sigma_i = \omega_i^+ - \omega_i^\circ$ are defined. In addition, $s_i := \begin{bmatrix} w_i^+ - w_i^\circ \\ K_i^+ - K_i^\circ \end{bmatrix}$ and $y_i = G_i(w_i^+, K_i^+) - G_i(w_i^\circ, K_i^\circ)$ is defined. In order to use this block-TR1 update formula in combination with the lifted collocation scheme, one needs to be able to efficiently form the condensed QP in Eq. (10). For this purpose, we need to avoid the costly computations of the inverse matrix C_i^{-1} as well as the matrix-matrix multiplication $C_i^{-1} D_i$. In what follows, we present a procedure to directly obtain a rank-one update formula for the inverse matrix $C_i^{+^{-1}}$ and for the corresponding product $E_i^+ := C_i^{+^{-1}} D_i^+$.

C. Avoiding Expensive Matrix-Matrix Operations

Based on the Sherman-Morrison formula, one can directly update the matrix inverse given the previous invertible approximation $C_i^{\circ^{-1}} \approx \frac{\partial G_i}{\partial K_i}^{-1}$. Let us first rewrite the block-TR1 update from (13) as follows

$$D_i^+ = D_i^\circ + \alpha\, U V_{\mathrm{D}}^\top \quad \text{and} \quad C_i^+ = C_i^\circ + \alpha\, U V_{\mathrm{C}}^\top, \tag{14}$$

where $U = y_i - [D_i^\circ\; C_i^\circ] s_i$ and $[V_{\mathrm{D}}^\top\; V_{\mathrm{C}}^\top] = \mu_i^\top - \sigma_i^\top [D_i^\circ\; C_i^\circ]$. The Sherman-Morrison formula then reads as

$$C_i^{+^{-1}} = C_i^{\circ^{-1}} - \alpha\beta\, C_i^{\circ^{-1}} U V_{\mathrm{C}}^\top C_i^{\circ^{-1}}, \tag{15}$$

where $\beta = \frac{1}{1 + \alpha V_{\mathrm{C}}^\top C_i^{\circ^{-1}} U}$.

Let us define $\tilde{U} = C_i^{\circ^{-1}} U$ such that we obtain the following update

$$\begin{aligned} E_i^+ = C_i^{+^{-1}} D_i^+ &= C_i^{\circ^{-1}}\left(D_i^\circ + \alpha U V_{\mathrm{D}}^\top\right) \\ &\quad - \alpha\beta C_i^{\circ^{-1}} U V_{\mathrm{C}}^\top C_i^{\circ^{-1}}\left(D_i^\circ + \alpha U V_{\mathrm{D}}^\top\right) \\ &= E_i^\circ + \alpha \tilde{U} V_{\mathrm{D}}^\top - \alpha\beta \tilde{U} V_{\mathrm{C}}^\top (E_i^\circ + \alpha \tilde{U} V_{\mathrm{D}}^\top) \\ &= E_i^\circ + \alpha \tilde{U} \tilde{V}^\top, \end{aligned} \tag{16}$$

where $\tilde{V}^\top = V_{\mathrm{D}}^\top - \beta V_{\mathrm{C}}^\top (E_i^\circ + \alpha \tilde{U} V_{\mathrm{D}}^\top)$. It is readily seen that the update for E_i is a rank-one update.

D. Lifted Collocation based RTI Scheme for NMPC

The novel block-TR1 update formula for the matrix $E_i^+ = C_i^{+^{-1}} D_i^+$ in Eq. (16) provides an efficient manner to directly compute the rank-one update to the matrices in the condensed QP formulation of Eq. (10), without the need for a matrix factorization, inversion and without any matrix-matrix multiplications. Instead, the proposed implementation merely requires matrix-vector multiplications and outer products, resulting in a quadratic instead of cubic computational complexity. However, this comes at the cost of a slightly increased memory footprint, since the matrices C_i^{-1} and E_i need to be stored from one iteration to the next as well. The implementation of the proposed block-TR1 method for direct collocation is presented in Algorithm 2, resulting in a novel lifted collocation type RTI scheme for nonlinear MPC.

Algorithm 2 Lifted Collocation with Block-TR1 Updates within a Real-Time Iteration Scheme for Nonlinear MPC.

Input: $w_i^\circ = (x_i^\circ, u_i^\circ)$, K_i°, λ_i°, ω_i°, C_i°, D_i°, ${C_i^\circ}^{-1}$ and E_i°.

`Problem linearization`

Formulate the QP in (10) with matrices E_i° and evaluate vectors d_i and $\tilde{h}_i^c$ in (11) for $i = 0, \dots, N-1$.

`Computation of step direction`

Obtain current state estimate $\hat{x}_0$. ▷ from system

Solve the QP subproblem in Eq. (10): $w_i^+ \leftarrow w_i^\circ + \Delta w_i^\star$ and $\lambda_i^+ \leftarrow \lambda_i^\star$.

Apply new control input u_0^+. ▷ to system

`Block-wise TR1 update`

for $i = 0, \dots, N-1$ **do in parallel**

$K_i^+ \leftarrow K_i^\circ - {C_i^\circ}^{-1} c_i - E_i^\circ \Delta w_i^\star$,

$\omega_i^+ \leftarrow \omega_i^\circ - C_i^{-\top} \frac{\partial G_i}{\partial K_i}^\top \omega_i^\circ - C_i^{-\top} B_i^\top \lambda_i^+$,

$D_i^+ \leftarrow D_i^\circ + \alpha U V_{\mathrm{D}}^\top$ and $C_i^+ \leftarrow C_i^\circ + \alpha U V_{\mathrm{C}}^\top$,

${C_i^+}^{-1} \leftarrow {C_i^\circ}^{-1} - \alpha \beta \tilde{U} V_{\mathrm{C}}^\top {C_i^\circ}^{-1}$,

$E_i^+ \leftarrow E_i^\circ + \alpha \tilde{U} \tilde{V}^\top$.

end for

Output: w_i^+, K_i^+, λ_i^+, ω_i^+, C_i^+, D_i^+, ${C_i^+}^{-1}$ and E_i^+.

V. NUMERICAL CASE STUDIES

In this section, we proceed to analyze how the proposed block-TR1 method can be used in the context of nonlinear MPC using the RTI scheme [7]. Each iteration of the MPC scheme consists of two parts:

1) *Preparation phase*: linearize the system by computing the necessary derivatives and build the optimal control structured QP subproblem.
2) *Feedback phase*: solve the QP and obtain the next control input to apply feedback to the system.

The block-wise TR1 based Jacobian update in both Algorithm 1 and 2 becomes part of the preparation step and therefore does not affect the computational delay between obtaining the new state estimate and applying the next control input to the system. The QP solution in the feedback phase will be obtained by the method proposed in [18].

We validate the closed-loop performance of these novel RTI variants based on two numerical case studies. Motivated by real embedded applications, we illustrate the computation times for the presented NMPC algorithms using the ARM Cortex-A53 processor in the Raspberry Pi 3. A detailed local convergence analysis is part of ongoing research.

A. NMPC for the Chain of Masses

Similar to the work in [10], the nonlinear chain of masses can be used to validate the computational performance of an optimal control algorithm for a range of numbers of masses n_{m} (we refer to [9] for the OCP formulation). Figure 1 illustrates the computation times of both the preparation and feedback steps of an NMPC implementation with $n_{\mathrm{m}} = 2, \dots, 8$ masses, using the lifted collocation based SQP scheme in Algorithm 2. It can be observed that the preparation time scales quadratically with the number of states for the block-TR1 implementation, instead of the cubic complexity when using the exact Jacobian. Specifically, the Jacobian evaluation, factorization and matrix-matrix multiplications are replaced by adjoint differentiation sweeps and matrix-vector operations in Algorithm 2. On the other hand, the feedback time remains essentially the same because, after the linearization, both approaches lead to the solution of a similarly structured QP in (4) or (10).

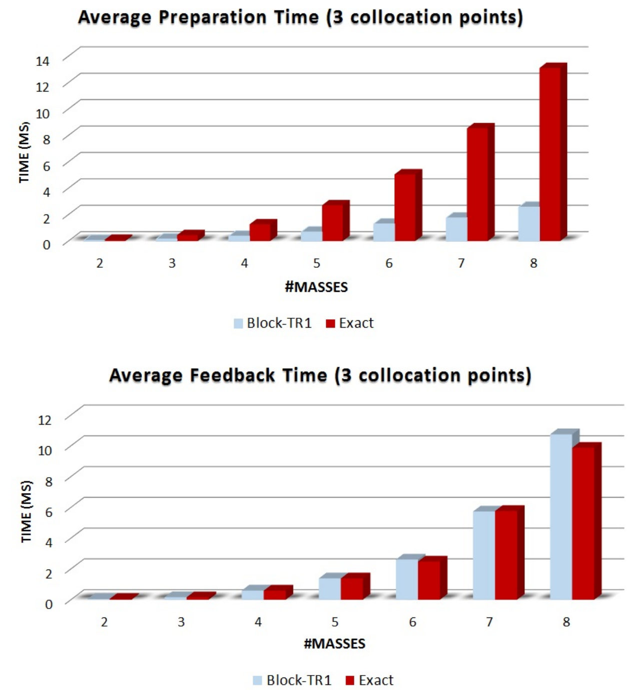

Fig. 1: Comparison of the average preparation and feedback times (in ms): block-TR1 method versus exact Jacobian. [1]

TABLE I: Average computation times (in ms) for nonlinear MPC on a chain of $n_{\mathrm{m}} = 6$ masses, i.e., 30 differential states (4 Gauss collocation nodes versus 10 steps of RK4).

	Explicit (RK4 in Alg. 1)			Implicit (GL4 in Alg. 2)		
	exact	block-TR1		exact	block-TR1	
Linearization	32.36	5.33	**16%**	291.37	35.99	**12%**
QP solution	23.22	37.82		26.33	27.86	
Total RTI step	56.39	43.99	78%	318.58	64.69	20%

Table I provides a more detailed comparison between the exact Jacobian and the proposed block-TR1 variant of the RTI scheme, using an ARM Cortex-A53 processor. The table shows these results for both the explicit Runge-Kutta method of order 4 (RK4) in combination with Algorithm 1 and using the implicit 4-stage Gauss-Legendre (GL4) method within Algorithm 2. The proposed block-TR1 scheme results in a considerable speedup of the problem linearization step of about factor $6-8$. In order to obtain a relatively fair comparison, the number of integration steps for RK4 has been chosen such that the numerical accuracy is close to

[1]The computation times in Figure 1 have been obtained using an Intel i7-7700k processor @ 4.20 GHz on Windows 10 with 64 GB of RAM.

TABLE II: Average computation times (in ms) for vehicle control based on a single-track vehicle model within NMPC (4 Gauss collocation nodes versus 30 steps of RK4).

	Explicit (RK4 in Alg. 1)			Implicit (GL4 in Alg. 2)		
	exact	block-TR1		exact	block-TR1	
Linearization	106.73	75.78	**71%**	52.22	18.27	**35%**
QP solution	4.46	4.51		4.59	4.72	
Total RTI step	111.79	80.94	72%	57.43	23.64	41%

that of the 4-stage GL method. However, since the system dynamics for the chain of masses are non-stiff, an explicit integration scheme typically performs better.

B. NMPC based Vehicle Control

The second case study is based on the NMPC based vehicle control scheme as presented recently in [24], based on single-track vehicle dynamics with a Pacejka-type tire model. The validated model parameters can be found in [25]. As often the case in practice, these vehicle dynamics are rather stiff such that an implicit integration scheme should preferably be used. Therefore, it forms an ideal case study for the proposed lifted collocation based RTI scheme of Algorithm 2. Let us perform the closed-loop NMPC simulations as presented in [24], but using the proposed block-TR1 based RTI algorithm. We carried out simulations for two successive double lane changes on snow, and the resulting closed-loop trajectories for both the exact Jacobian and the block-TR1 were indistinguishable from each other.

The corresponding computation times on the ARM Cortex-A53 processor are illustrated in Table II. Because of the relatively stiff wheel dynamics, the proposed lifted collocation method from Algorithm 2 becomes very attractive and additionally provides a speedup of about factor 3 over the standard exact Jacobian based implementation. Note that, even though the Raspberry Pi 3 is not an embedded processor by itself, it uses an ARM core of the same type as those that are used by multiple high-end automotive microprocessors.

VI. CONCLUSIONS

In this paper we proposed a block-wise sparsity preserving TR1 update for an adjoint-based inexact SQP method to efficiently solve the nonlinear optimal control problems arising in NMPC. We also showed how this approach can be implemented in the lifted collocation framework, in order to avoid matrix factorizations and matrix-matrix multiplications. This allows us to more efficiently handle both stiff and non-stiff system dynamics within an RTI based NMPC implementation, as illustrated by multiple case studies that show the closed-loop numerical performance.

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