2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Regional MPC with nonlinearly bounded regions of validity*

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Abstract— **The optimal solution to a linear model predictive control (MPC) problem is usually understood as the optimal input for the current state, i.e., a point in state space. However, the solution at a particular point contains more information and provides an optimal affine control law for a polytopic region in state-space. This control law and polytope can be used as long as the closed-loop system stays in the polytope. In the present paper we extend the polytopes to larger, nonlinearly constrained regions that result if optimality is no longer required, but only stability and feasibility of the closed-loop system are enforced. We show fewer quadratic programs need to be solved than in the optimal case, and the effort for evaluating the control laws is reduced compared to an earlier approach.**

I. INTRODUCTION

Model predictive control is an appropriate method for the control of multivariable constrained systems. It is based on solving an optimal control problem (OCP) in every time step, which is computational expensive. Many methods can be found in the literature that aim for a reduction of the computational effort. One group of methods proposes to accelerate the numerical solution of the OCP (see, e.g. [1], [2]). Another group aims to avoid the online optimization whenever possible (see, e.g. [3], [4], [5]). The second group includes event-triggered MPC, where feedback is not applied periodically but only when necessary. Event-triggered MPC has been proposed by several authors before. In [6], [7], [8], [9], [10] the difference between the predicted and actual trajectory is used as a triggering event. Other methods focus on the rate of change of the cost function as proposed in [11], [12].

Here we extend the regional MPC approach proposed in [13], [14], [15], [16], [17]. It exploits the piecewise affine structure of the solution to the linear MPC problem. The solution to the MPC problem at a point in state-space is interpreted as an optimal affine control law that is valid on an entire state-space polytope. The control law can be evaluated as long as the closed-loop system stays in the current polytope, i.e. a quadratic program (QP) is solved only if the current control law becomes invalid. The solution to this QP provides the next control law and polytope (see Fig. 1 for a sketch). The proposed method is appropriate for a networked control scheme since the control law and the corresponding polytope can be calculated on a powerful central node and transmitted to a local node. The evaluation of the control law and the polytope requires only simple arithmetic operations and thus can be evaluated on a lean local node [18].

The number of QPs solved can be reduced further by not only reusing a control law as long as it is optimal, but as long as it is feasible and stabilizing. This idea was already explored in [13], but the approach proposed there still required a computationally expensive evaluation of feasibility and stability criteria.

It is the purpose of the present paper to simplify the criterion for the reuse of affine control laws proposed in [13]. We show a nonlinearly constrained region can be described explicitly on which the current affine control law is feasible and stabilizing. This region of validity for the current control law is larger than the original polytope. Despite being nonlinearly constrained, it is computationally simple to check whether the system remains in this region in subsequent steps. Specifically, the region of validity is the intersection of a polytope and another region described by a simple quadratic inequality (a quadric). Checking whether subsequent states remain in the region of validity therefore is equivalent to evaluating a small number of linear inequalities and a single quadratic inequality. Furthermore, the evaluation of the new regions leads to a smaller online computational effort than required for evaluating the conditions used in [13]. We incorporate the new regions in an event-triggered MPC law and illustrate their use in a networked control scheme.

We state the system and problem class along with some preliminaries in Section II. The new approach is presented in Section III and applied to two examples in Section IV. An outlook is given in Section V.

A. Notation

For an arbitrary matrix $M \in \mathbb{R}^{a\times b}$, let $M^{\mathscr{L}}$ with $\mathscr{L} \subseteq \{1,\dots,a\}$ describe the submatrix with the rows indicated by $\mathscr{L}$. If $\mathscr{L}$ contains only one element, say $\mathscr{L} = \{i\}$, we write M^i. A *polytope* is the intersection of a finite number of halfspaces $\mathscr{P} = \{x \in \mathbb{R}^n | Tx \leq d\}$ with $T \in \mathbb{R}^{r\times n}$ and $d \in \mathbb{R}^r$.

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider a linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k), x(0) \text{ given,} \tag{1}$$

with state variables $x(k) \in \mathbb{R}^n$, input variables $u(k) \in \mathbb{R}^m$ and system matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. Assume state and input constraints

$$x(k) \in \mathscr{X} \subset \mathbb{R}^n,$$
$$u(k) \in \mathscr{U} \subset \mathbb{R}^m$$

* Support by the Deutsche Forschungsgemeinschaft (DFG) under grant MO 1086/15-1 is gratefully acknowledged.

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apply for all $k \geq 0$. The system can be regulated to the origin by perpetually solving the optimal control problem

$$
\begin{aligned}
\min_{X,U} \quad & \tilde{x}(N)'P\tilde{x}(N) + \sum_{i=0}^{N-1} \left(\tilde{x}(i)'Q\tilde{x}(i) + \tilde{u}(i)'R\tilde{u}(i)\right) \\
\text{s.t.} \quad & \tilde{x}(0) = x, \\
& \tilde{x}(i+1) = A\tilde{x}(i) + B\tilde{u}(i), \quad i = 0,\dots,N-1, \\
& \tilde{x}(i) \in \mathscr{X}, \quad i = 0,\dots,N-1, \\
& \tilde{u}(i) \in \mathscr{U}, \quad i = 0,\dots,N-1, \\
& \tilde{x}(N) \in \mathscr{T},
\end{aligned}
\tag{2}
$$

with the horizon N, current state $x = x(k)$, state sequence $X = (\tilde{x}(1)',\dots,\tilde{x}(N)')'$, input sequence $U = (\tilde{u}(0)',\dots,\tilde{u}(N-1)')'$, the weighting matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$, $P \in \mathbb{R}^{n\times n}$ and the terminal set $\mathscr{T}$. In every time step Problem (2) is solved for the current state x and the first m elements of the resulting input sequence $\tilde{u}(0) = U^{\mathscr{M}}$ with $\mathscr{M} = \{1,\dots,m\}$ are applied to the system. We make the following assumptions throughout the paper.

Assumption 1 *Assume $Q \succ 0$, $R \succ 0$, $P \succ 0$, the pair (A,B) is stabilizable and the pair $(Q^{\frac{1}{2}},A)$ is detectable. Moreover, assume $\mathscr{X}$, $\mathscr{U}$ and $\mathscr{T} \subset \mathscr{X}$ are compact polytopes that contain the origin as an interior point.*

Stability can be guaranteed by choosing

$$
\begin{aligned}
& P = (A+BK_\infty)'P(A+BK_\infty) + K_\infty' R K_\infty + Q, \\
& K_\infty = -(R+B'PB)^{-1}B'PA, \\
& (A+BK_\infty)x \in \mathscr{T}
\end{aligned}
\tag{3}
$$

with $K_\infty x \in \mathscr{U}$ for every $x \in \mathscr{T}$. The matrix P is the positive definite solution of the discrete time algebraic Riccati equation. If the conditions (3) are satisfied, the optimal cost function is a Lyapunov function of the closed-loop system and decreases along all closed-loop trajectories [19]. The terminal set $\mathscr{T}$ is calculated according to [20].

After eliminating the state variables with (1) the optimal control problem (2) appears in the form

$$
\min_U \; V(x,U) \quad \text{s.t.} \quad GU \leq w + Ex, \tag{4}
$$

with the cost function $V(x,U) = \frac{1}{2}U'HU + x'FU + \frac{1}{2}x'Yx$, where $Y \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times mN}$, $H \in \mathbb{R}^{mN\times mN}$, $G \in \mathbb{R}^{q\times mN}$, $w \in \mathbb{R}^q$, $E \in \mathbb{R}^{q\times n}$ and q is the number of constraints. Note that $H \succ 0$, if the assumptions stated above are satisfied. A state $x \in \mathbb{R}^n$ is called *feasible* if a control input $U(x)$ exists that satisfies the constraints in (4). The set of feasible states is denoted $\mathscr{X}_f \subseteq \mathscr{X}$. The optimal solution $U^\star(x) : \mathscr{X}_f \to \mathbb{R}^{mN}$ is continuous and piecewise affine. More specifically, there exist a finite number $\bar{p}$ of polytopes $\mathscr{P}_j^\star$ with pairwise disjoint interiors such that $\mathscr{X}_f = \bigcup_{j=1}^{\bar{p}} \mathscr{P}_j^\star$ and

$$
U^\star(x) = \begin{cases} K_1^\star x + b_1^\star & \text{if} \quad x \in \mathscr{P}_1^\star \\ \vdots & \\ K_{\bar{p}}^\star x + b_{\bar{p}}^\star & \text{if} \quad x \in \mathscr{P}_{\bar{p}}^\star \end{cases} \tag{5}
$$

with $K_j^\star \in \mathbb{R}^{mN\times n}$ and $b_j^\star \in \mathbb{R}^{mN}$. We stress we would like to exploit the structure of (5) without ever calculating (5). It follows from (5) that the optimal solution to problem (4) *for a point $x \in \mathscr{X}_f$* provides a control law $K_j^\star x + b_j^\star$ and its region of validity $\mathscr{P}_j^\star$. This statement is summarized in Lemma 1, which follows from Theorem 2 in [4]. In preparation of Lemma 1 we introduce the sets of active and inactive constraints

$$
\begin{aligned}
\mathscr{A}(x) &= \{i \in \mathscr{Q} \mid G^i U^\star(x) = w^i + E^i x\}, \\
\mathscr{I}(x) &= \{i \in \mathscr{Q} \mid G^i U^\star(x) < w^i + E^i x\} = \mathscr{Q} \backslash \mathscr{A}(x),
\end{aligned}
\tag{6}
$$

where $\mathscr{Q} := \{1,\dots,q\}$ is the set of all constraint indices. We often write $\mathscr{A}$ and $\mathscr{I}$ instead of $\mathscr{A}(x)$ and $\mathscr{I}(x)$ for brevity.

Lemma 1 *Let $x \in \mathscr{X}_f$ be arbitrary and $\mathscr{A}(x) = \mathscr{A}$ the corresponding active set. Assume the matrix $G^{\mathscr{A}}$ has full row rank. Let*

$$
\begin{aligned}
K^\star &= H^{-1}(G^{\mathscr{A}})'(G^{\mathscr{A}}H^{-1}(G^{\mathscr{A}})')^{-1}S^{\mathscr{A}} - H^{-1}F', \\
b^\star &= H^{-1}(G^{\mathscr{A}})'(G^{\mathscr{A}}H^{-1}(G^{\mathscr{A}})')^{-1}w^{\mathscr{A}}, \\
T^\star &= \begin{pmatrix} G^{\mathscr{I}}H^{-1}(G^{\mathscr{A}})'(G^{\mathscr{A}}H^{-1}(G^{\mathscr{A}})')^{-1}S^{\mathscr{A}} - S^{\mathscr{I}} \\ (G^{\mathscr{A}}H^{-1}(G^{\mathscr{A}})')^{-1}S^{\mathscr{A}} \end{pmatrix}, \\
d^\star &= -\begin{pmatrix} G^{\mathscr{I}}H^{-1}(G^{\mathscr{A}})'(G^{\mathscr{A}}H^{-1}(G^{\mathscr{A}})')^{-1}w^{\mathscr{A}} - w^{\mathscr{I}} \\ (G^{\mathscr{A}}H^{-1}(G^{\mathscr{A}})')^{-1}w^{\mathscr{A}} \end{pmatrix}
\end{aligned}
\tag{7}
$$

where $S = E + GH^{-1}F'$, $S \in \mathbb{R}^{q\times n}$. Then $U^\star(x) = K^\star x + b^\star$ is the optimal control law on the polytope $\mathscr{P}^\star = \{x \in \mathbb{R}^n \mid T^\star x \leq d^\star\}$.

We claim without giving details that the computational effort for calculating the matrices in Lemma 1 is smaller than for solving a QP (see [18] for details). As proposed in [13], Lemma 1 can be used in an event-triggered MPC approach (see Fig. 1): After solving problem (4) for the current state, the active set can be determined according to (6). With the active set the optimal control law and its polytope that contains the current state can be calculated according to Lemma 1. Instead of solving a QP in the next time step, the control law can be evaluated for all states within the polytope. Leaving the current polytope is the event that triggers the solution of a new QP to determine the next control law and polytope. If the rank condition for $G^{\mathscr{A}}$ is not satisfied, the QP (4) has to be solved in the next time step.

III. A SUBOPTIMAL EVENT-TRIGGERED MPC APPROACH

An optimal control law $\hat{U}(x) := K_j^\star x + b_j^\star$ (see Fig. 1) can be used even if its polytope of validity $\mathscr{P}_j^\star$ is left, as long as $\hat{U}(x)$ satisfies the constraints and results in a stable closed-loop system [13]. We use the symbol $\hat{U}(x)$ for an affine law that may no longer be optimal. A control law $\hat{U}(x)$ satisfies

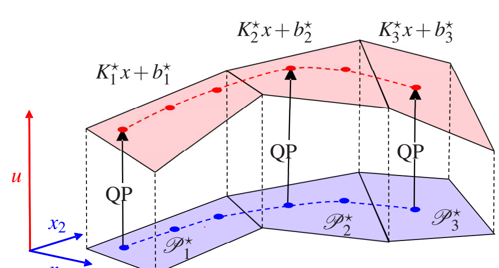


Fig. 1: Regional control laws $K_i^\star x + b_i^\star$ on polytopes $\mathscr{P}_i^\star$. A control law can be used as long as the system stays in the corresponding polytope. A QP only needs to be solved when the system leaves the current polytope.

the constraints, i.e. is feasible, at time step k if

$$G\hat{U}(x(k)) - Ex(k) \leq w \tag{8}$$

holds. In order to ensure stability, the cost function has to decrease along the closed-loop trajectories that result with $\hat{U}(x)$. This implies we need to enforce

$$V(x(k), \hat{U}(x(k))) < V(x(k-1), \hat{U}(x(k-1))). \tag{9}$$

In this section, the region implicitly defined by the conditions (8) and (9) is characterized explicitly (see Fig. 2). The resulting explicit representation improves upon the implicit use of (8) and (9) proposed in [13]. Firstly, the extension of the region of validity for a control law compared to the optimal polytope can be visualized for simple examples. Secondly, it is much easier to evaluate the new criterion, since it depends only on the current state. In contrast to the approach proposed in [13], the cost function value at the previous time step is not necessary here. As a consequence, the computational effort on the local node in a networked setting can be reduced. We determine the new representation in Section III-A and show how to use it in regional MPC in Section III-B. Subsequently, we discuss the reduction of the computational effort on the local node in Section III-C.

A. Calculation of suboptimal state-space regions of validity

Upon substitution of the current optimal control law $K^\star x + b^\star$, condition (8) is transformed into a state-space polytope, on which feasibility of the current control law is guaranteed. This is stated more precisely in the following proposition, which is based on the results in [3].

Proposition 1 *Let $x \in \mathscr{X}_f$ be arbitrary and $\mathscr{A}(x) = \mathscr{A}$ the corresponding active set. Assume the matrix $G^{\mathscr{A}}$ has full row rank and let $K^\star$ and $b^\star$ be defined as in Lemma 1. Let*

$$\begin{aligned} T_1 &= G^{\mathscr{I}} H^{-1}(G^{\mathscr{A}})'(G^{\mathscr{A}} H^{-1}(G^{\mathscr{A}})')^{-1} S^{\mathscr{A}} - S^{\mathscr{I}}, \\ d_1 &= -G^{\mathscr{I}} H^{-1}(G^{\mathscr{A}})'(G^{\mathscr{A}} H^{-1}(G^{\mathscr{A}})')^{-1} w^{\mathscr{A}} + w^{\mathscr{I}}, \end{aligned} \tag{10}$$

where $S = E + GH^{-1}F'$, $S \in \mathbb{R}^{q\times n}$. Then $\hat{U}(x) = K^\star x + b^\star$ is a feasible control law on the polytope $\mathscr{F} = \{x \in \mathbb{R}^n \mid T_1 x \leq d_1\}$. Moreover, $\mathscr{F} \supseteq \mathscr{P}^\star$ holds for the polytope $\mathscr{P}^\star$ from Lemma 1.

We only sketch the proof of Proposition 1, since it requires standard steps only. Since H is invertible, the stationarity conditions of the optimality conditions to (4) can be solved for U. After substituting the resulting equation for U in the active constraints and by using the rank condition for $G^{\mathscr{A}}$, the affine Lagrange multipliers $\lambda^{\mathscr{A}}$ can be determined. Substituting the expressions for U and $\lambda^{\mathscr{A}}$ into the inactive constraints leads to $T_1 x \leq d_1$. A comparison of T_1 and d_1 with $T^\star$ and $d^\star$ from (7) shows, that $\mathscr{F} \supseteq \mathscr{P}^\star$ holds, since $\mathscr{F}$ is defined by a subset of the halfspaces that constitute $\mathscr{P}^\star$ (see Fig. 3a for a sketch).

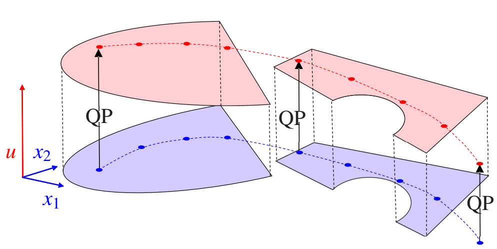


Fig. 2: Regional MPC with extended regions of validity resulting from suboptimality. The regions implicitly defined by conditions (8) and (9) can be transformed into an explicit representation, which is much easier to evaluate.

Similarly to the feasibility condition (8), the stability condition (9) can be transformed into inequalities that depend only on the current state x by substituting the optimal control law and the system dynamics (1). However, condition (9) is quadratic in contrast to (8). Thus, a nonlinear bounded state-space region $\mathscr{V}$ results. This is stated more precisely in Proposition 2. In preparation of Proposition 2 we introduce

$$\begin{aligned} T_2 &= M_2 - 2M_4' M_1 M_3 - M_2 M_3, \\ T_3 &= M_1 - M_3' M_1 M_3, \\ d_2 &= M_4' M_1 M_4 + M_2 M_4 \end{aligned} \tag{11}$$

with $M_1 = \frac{1}{2} K^{\star\prime} H K^\star + F K^\star + \frac{1}{2} Y$, $M_2 = b^{\star\prime} H K^\star + b^{\star\prime} F'$, $M_3 = (A + BK^{\star\mathscr{M}})^{-1}$ and $M_4 = -M_3 B b^{\star\mathscr{M}}$.

Proposition 2 *Let $\hat{U}$ be an arbitrary affine law from (5). Let $x^- \in \mathscr{X}_f$ be arbitrary and $x = Ax^- + B\hat{U}^{\mathscr{M}}(x^-)$ be the following state after applying $\hat{U}$ to system (1). Let $\mathscr{V}$ be defined by*

$$\mathscr{V} = \left\{\zeta \in \mathbb{R}^n \middle| \zeta' T_3 \zeta + T_2 \zeta < d_2\right\}. \tag{12}$$

If $(A + BK^{\star\mathscr{M}})^{-1}$ exists, then it holds

$$V(x, \hat{U}(x)) < V(x^-, \hat{U}(x^-)) \iff x \in \mathscr{V}. \tag{13}$$

Proof: The expression $A + BK^{\star\mathscr{M}}$ is invertible by assumption and $x = Ax^- + B(K^{\star\mathscr{M}} x^- + b^{\star\mathscr{M}})$ can be rearranged to

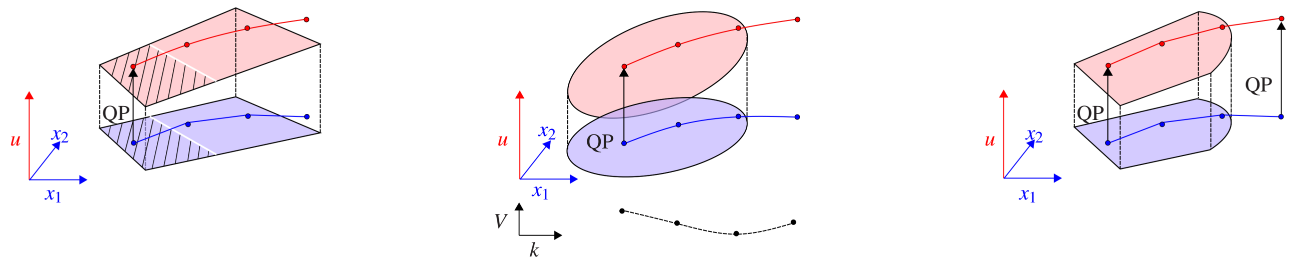
$$x^- = M_3 x + M_4 \tag{14}$$

with M_3 and M_4 from (11). By using the control law $\hat{U}(x) = K^\star x + b^\star$, the condition for a decreasing cost function $V(x, \hat{U}(x)) < V(x^-, \hat{U}(x^-))$ can be expressed in terms of x and x^-. It follows

$$\tilde{V}(x) < \tilde{V}(x^-) \tag{15}$$

with $\tilde{V}(x) = x' M_1 x + M_2 x + M_5$, the auxiliary expressions (11) and $M_5 = \frac{1}{2} b^{\star\prime} H b^\star$. The combination of (14) and (15) yields an expression depending on x only. The resulting expression can be transformed into $x' T_3 x + T_2 x < d_2$, i.e. the region $\mathscr{V}$. ■

In the proof of Proposition 2 the previous state x^- is used and expressed in terms of the closed-loop model. For that reason the calculation of the inverse $(A + BK^{\star\mathscr{M}})^{-1}$ is required. This is necessary, since condition (9) detects a decrease of the cost function for the current state x compared to the previous state x^-. Note that this is different from a formulation with x and x^+, which would detect a decrease in the next time step resulting in a different region of validity. Further note that the state x^- is not required to be in $\mathscr{P}^\star$ in Proposition 2. Consequently, reusability of $\hat{U}$ can be checked not just in the first time step after leaving the current polytope, but for later time steps, too. Note that $\mathscr{V}$ does in general not result in a convex region, since $\mathscr{V}$ may be an n-dimensional hyperbola, for example. Nevertheless checking $x \in \mathscr{V}$ for some $x \in X_f$ is simple, since it merely requires to check the condition $x' T_3 x + T_2 x < d_2$. If the region $\mathscr{V}$ is left, we can no longer guarantee stability and therefore need to compute a new optimal input sequence and control law. If the inverse and therefore the region $\mathscr{V}$ cannot be



(a) The control law guarantees feasibility on a polytopic region. The new region results from removing some constraints (white line) from the optimal polytope (hatched).

(b) The control law is stabilizing on a region that can be described by a quadratic inequality.

(c) The control law can be reused for all states inside the nonlinearly constrained region that results from the intersection of the feasibility (see 3a) and stability region (see 3b).

Fig. 3: The intersection of a feasibility and stability region defines a nonlinearly constrained region of validity.

calculated, the optimal region $\mathscr{P}^\star$ according to Lemma 1 must be determined instead by solving a QP. Now, let

$$\mathscr{E} = \mathscr{F} \cap \mathscr{V} \tag{16}$$

be the intersection of the regions $\mathscr{F}$ and $\mathscr{V}$ defined in Proposition 1 and Proposition 2, respectively. Then it is obvious that for all $x \in \mathscr{E}$ the control law $\hat{U}(x) = K^\star x + b^\star$ is feasible and results in a decreasing value of the cost function.

B. Event-triggered control law update

The control law $\hat{U}$ and the regions $\mathscr{E}$ can be used in the following regional MPC algorithm: Solving problem (2) or QP (4) for an arbitrary state $x \in \mathscr{X}_f$ defines an active set $\mathscr{A}(x)$, which can be used to determine the control law $K^\star(x) + b^\star$ with Lemma 1 and a corresponding region of validity $\mathscr{R}$. If $(A + BK^{\star\mathscr{M}})$ is invertible, the region $\mathscr{R} = \mathscr{E}$ is determined according to (16). Otherwise, the optimal region $\mathscr{R} = \mathscr{P}^\star$ is calculated according to Lemma 1. The control law can be used as long as the system stays in $\mathscr{R}$. Leaving the region $\mathscr{R}$ is the event that triggers the solution of a new QP to determine the next active set. The described procedure results in the time varying control law

$$\begin{aligned} \tilde{U}(x) &= \tilde{K}(k)x(k) + \tilde{b}(k), \\ \tilde{u}(k) &= \tilde{K}^{\mathscr{M}}(k)x(k) + \tilde{b}^{\mathscr{M}}(k) \end{aligned} \tag{17}$$

and the region $\mathscr{R}(k)$ with $\mathscr{R}(0) = \mathscr{R}_{new}(0)$, $\tilde{K}(0) = K^\star$, $\tilde{b}(0) = b^\star$ and

$$\begin{pmatrix} \tilde{K}(k) \\ \tilde{b}(k) \\ \mathscr{R}(k) \end{pmatrix} = \begin{cases} \begin{pmatrix} \tilde{K}(k-1) \\ \tilde{b}(k-1) \\ \mathscr{R}(k-1) \end{pmatrix} & \text{if } x(k) \in \mathscr{R}(k-1) \\ \begin{pmatrix} K^\star \\ b^\star \\ \mathscr{R}_{new} \end{pmatrix} & \text{otherwise} \end{cases} \tag{18}$$

for all $k > 0$, where

$$\mathscr{R}_{new}(k) = \begin{cases} \mathscr{E} & \text{if } (A + B\tilde{K}^{\mathscr{M}}(k))^{-1} \textit{exists} \\ \mathscr{P}^\star & \text{otherwise.} \end{cases} \tag{19}$$

We state the feasibility and stability properties of the new update rule in Proposition 3. As a preparation, we summarize results on the feasibility of successive states and the cost function decrease of a MPC-controlled system ([19], Lemma 3 in [13]) in the following lemma.

Lemma 2 *Let the weighting matrix P and the terminal set $\mathscr{T}$ be defined as in* (3)*. Let $k > 0$ and $x(k-1) \in \mathscr{X}_f$ be arbitrary and let $\hat{U} \in \mathbb{R}^{mN}$ be an arbitrary control law that satisfies the constraints* (4) *for $x(k-1)$. Then the successor state $x(k) = Ax(k-1) + B\hat{U}^{\mathscr{M}}$ is feasible and*

$$V^\star(x(k)) \leq V(x(k-1), \hat{U}) - \lambda_{\min}(Q)||x(k-1)||_2^2$$

holds, where $\lambda_{\min}(Q)$ denotes the smallest eigenvalue of Q, which is positive since $Q \succ 0$.

Proposition 3 *Let $\mathscr{P}$ and $\mathscr{T}$ be defined as in* (3) *and $\mathscr{X}_f$ the set of states, such that* (4) *has a feasible solution. Let the control law* (17)-(19) *be applied to the system* (1)*. Then the origin is an asymptotically stable steady state of the controlled system with domain of attraction $\mathscr{X}_f$.*

Proof: We first prove the statement

$$G(\tilde{K}(k)x(k) + \tilde{b}(k)) - Ex(k) \leq w \quad \forall x(0) \in \mathscr{X}_f, k \in \mathbb{N} \tag{20}$$

by induction. Let $x(0) \in \mathscr{X}_f$ be arbitrary and $k = 0$. For $k = 0$, relations (17)-(19) lead to $\tilde{K}(0)x(0) + \tilde{b}(0) = K^\star x(0) + b^\star = U^\star(x(0))$ and with (20) it follows $GU^\star(x(0)) - Ex(0) \leq w$. Obviously, the inequality is satisfied, since the optimal solution fulfills the constraints. Now assume (20) holds for $k-1$ and consider k. Then feasibility of $x(k)$ results from feasibility of $x(k-1)$ according to Lemma 2 and we either have

(i) $x(k) \in \mathscr{R}(k-1) = \mathscr{P}^\star(k-1)$ or
(ii) $x(k) \in \mathscr{R}(k-1) = \mathscr{E}(k-1)$ or
(iii) $x(k) \notin \mathscr{R}(k-1)$.

In cases (i) and (iii) the relations (17)-(19) yield the optimal solution $U^\star(x(k))$, which satisfies the constraints (20). In case (ii), $x(k) \in \mathscr{E}(k-1)$ and according to Proposition 1 the control law is feasible and satisfies (20). It remains to prove asymptotic stability. For this purpose, we show that $\tilde{V}(x) := V(x, \tilde{U}(x))$ with $\tilde{U}(x)$ according to (17) is a Lyapunov function, i.e. it is positive definite and strictly decreasing along closed-loop trajectories. The optimal cost function $V^\star(x)$ is a Lyapunov function on $\mathscr{X}_f$ and therefore positive definite on $\mathscr{X}_f$. Due to $\tilde{V}(x) = V(x, \tilde{U}(x)) \geq V(x, U^\star(x)) = V^\star(x)$ the cost function $\tilde{V}(x)\colon \mathscr{X}_f \to \mathbb{R}$ is also positive definite, where feasibility of $\tilde{U}(x)$ is ensured as shown above. A decrease of the cost function can be shown with cases (i)-(iii). In case (i) the optimal control law is used in time steps $k-1$ and k. Due to the decrease of the optimal cost function $V^\star(x)$, it follows $\tilde{V}(x(k-1)) = V^\star(x(k-1)) > V^\star(x(k)) = \tilde{V}(x(k))$. In case (ii) the required decrease follows with Proposition 2, where

x^- and x are identified with $x(k-1)$ and $x(k)$, respectively. The required inverse in Proposition 2 exists according to the definition of $\mathscr{R}_{new}$ in (19). In case (iii), the optimal control law $U^\star(x)$ is used in time step k and the optimal $U^\star(x(k-1))$ or a suboptimal but feasible $\tilde{U}(x(k-1))$ in time step $k-1$. According to Lemma 2 it follows $V^\star(x(k)) \leq \tilde{V}(x(k-1)) - \lambda_{\min}(Q)||x(k-1)||_2^2$. Since $\lambda_{\min}(Q) > 0$, we have $\tilde{V}(x(k)) = V^\star(x(k)) < \tilde{V}(x(k-1))$. ■

C. Reduction of the computational effort on the local node

As in [13] we apply the new event-triggered MPC law (17)-(19) to a networked control setting (see Fig. 4). The matrices $(\tilde{K}^{\mathscr{M}}, \tilde{b}^{\mathscr{M}})$ defining the control law $\tilde{u}$ and the matrices $(T^\star, d^\star)$ or $(T_1, T_2, T_3, d_1, d_2)$ defining the corresponding region of validity $\mathscr{R}$ can be calculated on a central node and transmitted to a local one. On the local node the control law can be applied to the system as long as the current state stays in the corresponding region. Whenever the current region is left, a new control law and a new region are calculated on the central node by solving a QP. As in [13], by using this new event-triggered MPC law the number of QPs to be solved on the central node can be decreased compared to the optimal approach.

Obviously, it is of interest to reduce the computational effort on both the local and the central node. The suboptimal approach proposed in [13] reduces the number of QPs to be solved, but the computational effort on the local node is larger compared to the optimal approach. More precisely, checking whether a point in state space lies in the optimal region $\mathscr{P}^\star$ requires

$$2qn \tag{21}$$

floating point operations (flops) per time step, whereas the evaluation of conditions (8) and (9) requires

$$2(mN)^2 + 2mN + 2nmN + 2n^2 + n + 2qmN + 2qn + 2. \tag{22}$$

flops per time step. In contrast,

$$\begin{aligned} 2qn \quad &\text{if} \quad \mathscr{R} = \mathscr{P}^\star, \\ 2n^2 + 3n + 2\tilde{q}n \quad &\text{if} \quad \mathscr{R} = \mathscr{E} \end{aligned} \tag{23}$$

flops per time step are required to evaluate the region $\mathscr{R}$ in the approach proposed here, where $\tilde{q}$ denotes the number of inactive constraints.

Due to space limitations we do not explain the derivation of the expressions (21), (22) and (23) in detail. Note that the evaluation of an optimal polytope includes q inner products of length n and q additions resulting in expression (21). The leading term in expression (22) results from the evaluation of $U'HU$ in (4) that includes mN inner products of length mN. In contrast, the new regions do not depend on the input sequence and the leading term in (23) arises from the evaluation of $\zeta' T_3 \zeta$ in (12) that includes n inner products of length n. Consequently, especially for systems with a large number of optimization variables mN, the event-triggered MPC law (17)-(19) is more appropriate than the one in [13].

IV. EXAMPLES

We apply the new approach to two examples that differ with respect to their size and the complexity of the resulting MPC problem. We calculate closed-loop system trajectories for random initial states until $||x_k||_2 \leq 10^{-3}$ for all examples.

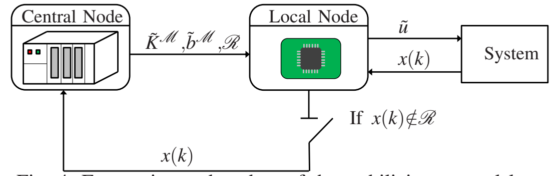


Fig. 4: Event-triggered update of the stabilizing control law.

SISO system. We consider system (1) with the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}. \tag{24}$$

The state and input constraints read $-10 \leq x_1 \leq 10$, $-5 \leq x_2 \leq 5$ and $-1 \leq u_1 \leq 1$. The weighting matrices are $Q = I^{2\times 2}$ and $R = 0.1$. The terminal state weighting matrix P and the terminal set $\mathscr{T}$ are determined as in (3). The horizon is set to $N = 5$. This example results in a QP with 34 inequalities and 5 optimization variables.

MIMO System. We consider system (1) with the matrices

$$A = \begin{pmatrix} 0.834 & -0.204 & 0 & 0 & 0 & 0 \\ 0.115 & 0.987 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.883 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.883 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.744 & -0.291 \\ 0 & 0 & 0 & 0 & 0.218 & 0.962 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0.115 & 0.007 & 0.029 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.029 & 0.109 & 0.014 \end{pmatrix}^T.$$

The state and input constraints are $-15 \leq x_i \leq 15$ for $i = 1, \ldots, 6$ and $-3 \leq u_j \leq 3$ for $j = 1, 2$. The weighting matrices on the states and inputs read $Q = I^{6\times 6}$ and $R = 0.01 I^{2\times 2}$. The terminal state weighting matrix P and the terminal set $\mathscr{T}$ are determined as in (3). The horizon is set to $N = 40$. This example results in a QP with 674 inequalities and 80 optimization variables.

Figure 5 illustrates the results for the SISO system graphically first. The optimal approach from [13] is compared to the proposed new approach. The figure depicts the trajectories of the states $x(k)$, the inputs $u(k)$ and the indicator function $e(k)$ with $e(k) = 1$ if a QP is solved and $e(k) = 0$ otherwise, as well as the regions of validity in state-space. In the

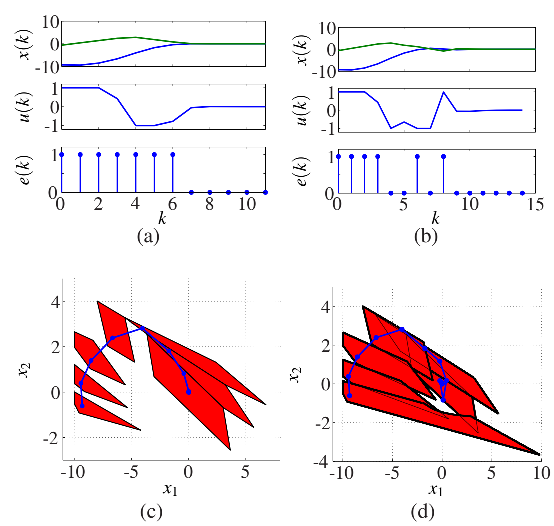


Fig. 5: Regulation of a second order SISO system from a random initial state to the origin. The optimal approach (left) is compared to the new suboptimal approach (right).

optimal case (left) seven QPs and thus seven control laws and polytopes must be calculated to regulate the system to the origin. The control law can be reused for more than one time step in the last polytope only. The enlarged regions introduced in Section III (see Fig. 2 and Fig. 3) are illustrated on the right. For time step $k = 8$ a QP has to be solved that has not to be solved in the optimal case. The reason for that is the slightly different course of the trajectory in state-space compared to the optimal case. It is apparent from the figure that the regions are enlarged and the number of QPs is reduced compared to event-triggered MPC with the exact regions.

Since results on any single trajectory are only anecdotal, we compare the optimal and suboptimal approaches proposed in [13] to the new one presented here with 10^3 random initial conditions for both examples. The third column of Table I shows the number of solved QPs in the optimal approach. The number of saved QPs is denoted ΔQPs and given in the fifth and seventh column. Table I also states the computational effort in terms of flops. More precisely, these numbers are the numbers that result from adding (21), (22) and (23) for all time steps $k > 0$. Note that the flops for the calculation of the regions are not considered, since these calculations are carried out on the central node. In both examples the new approach saves as many QPs as in the suboptimal approach from [13]. Moreover, the new approach has a computational effort on the local node similar to the effort in the optimal approach. The computational effort compared to the suboptimal approach in [13] can be reduced by around 74 percent for the SISO system and 93 percent for the MIMO system. In summary the approach proposed here saves as many QPs as the previously suggested suboptimal approach in [13], but, in contrast to the previous one, the new approach does not increase the computational effort on the local node. For more information concerning the amount of transmitted data in regional mpc the reader is referred to [18].

V. CONCLUSIONS AND FUTURE WORKS

We improved an event-triggered MPC setup, in which a computational powerful central node determines locally optimal affine feedback laws, and a local node simply evaluates these control laws as long as feasibility and stability of the closed-loop system can be guaranteed. A considerable improvement was achieved by characterizing an enlarged, nonlinearly bounded region of validity of the current affine law with simple linear and nonlinear inequalities. We showed fewer QPs need to be solved than in the optimal case, and the computational effort for evaluating the control law can be decreased compared to a first approach in [13]. Future research has to investigate if the regions of validity can be enlarged. Furthermore, the approach needs to be extended to robust MPC.

TABLE I: Computational effort on central node (number of solved QPs) and local node (10^8 flops). Results are shown for 100 and 1000 random initial conditions to corroborate that they do not depend on the number of analysed runs.

System	x_0	Opt. [13]		Subopt. [13]		New ap.	
		QPs	fl.	ΔQPs	fl.	ΔQPs	fl.
SISO	100	426	0.001	63	0.006	63	0.001
	1000	4396	0.012	608	0.056	608	0.015
MIMO	100	1781	0.597	602	9.673	602	0.606
	1000	17441	6.023	6145	96.76	6145	6.063

REFERENCES

[1] H. Ferreau, H. Bock, and M. Diehl, "An online active set strategy to overcome the limitations of explicit MPC," *International Journal of Robust and Nonlinear Control*, vol. 18, no. 8, pp. 816–830, 2008.

[2] G. Pannocchia, J. B. Rawlings, and S. J. Wright, "Fast, large-scale model predictive control by partial enumeration," *Automatica*, vol. 43, no. 5, pp. 852 – 860, 2007.

[3] A. Bemporad and C. Filippi, "Suboptimal explicit MPC via approximate multiparametric quadratic programming," in *Proceedings of the 40th IEEE Conference on Decision and Control*, vol. 5, 2001, pp. 4851–4856.

[4] A. Bemporad, M. Morari, V. Dua, and E. N. Pistikopoulos, "The explicit linear quadratic regulator for constrained systems," *Automatica*, vol. 38, pp. 3–20, 2002.

[5] P. Tøndel, T. Johansen, and A. Bemporad, "An algorithm for multiparametric quadratic programming and explicit MPC solutions," in *Proc. of the 40th IEEE Conference on Decision and Control*, vol. 2, 2001, pp. 1199–1204.

[6] H. Li and Y. Shi, "Event-triggered robust model predictive control of continuous-time nonlinear systems," *Automatica*, vol. 50, no. 5, pp. 1507–1513, 2014.

[7] A. Ferrara, G. P. Incremona, and L. Magni, "Model-based event-triggered robust MPC/ISM," in *European Control Conference (ECC)*, 2014, pp. 2931–2936.

[8] G. P. Incremona, A. Ferrara, and L. Magni, "Hierarchical model predictive/sliding mode control of nonlinear constrained uncertain systems," *IFAC-PapersOnLine*, vol. 48, no. 23, pp. 102 – 109, 2015.

[9] D. Bernardini and A. Bemporad, "Energy-aware robust model predictive control based on noisy wireless sensors," *Automatica*, vol. 48, no. 1, pp. 36–44, Jan. 2012.

[10] D. Lehmann, E. Henriksson, and K. H. Johansson, "Event-triggered model predictive control of discrete-time linear systems subject to disturbances," in *European Control Conference (ECC)*, 2013, pp. 1156–1161.

[11] A. Eqtami, D. V. Dimarogonas, and K. J. Kyriakopoulos, "Event-triggered control for discrete-time systems," in *Proc. of the 2010 American Control Conference*, 2010, pp. 4719–4724.

[12] ——, "Novel event-triggered strategies for model predictive controllers," in *Proc. of the 50th IEEE Conference on Decision and Control and European Control Conference*, 2011, pp. 3392–3397.

[13] M. Jost, M. Schulze Darup, and M. Mönnigmann, "Optimal and suboptimal event-triggering in linear model predictive control," in *Proceedings of the 2015 European Control Conference*, 2015, pp. 1147–1152.

[14] M. Mönnigmann, J. Otten, and M. Jost, "Nonlinear MPC defines implicit regional optimal control laws," in *Proceedings of the 5th IFAC Conference on Nonlinear Model Predictive Control (NMPC)*, 2015, pp. 134–139.

[15] K. König and M. Mönnigmann, "Regional mpc with active set updates," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 11 859 – 11 864, 2017.

[16] ——, "Regionale prädiktive Regelung - Modellprädiktive Regelung mittels stückweise definiertem Riccati-Regler," *at-Automatisierungstechnik*, vol. 65, no. 12, pp. 808–821, 2017.

[17] M. Schulze Darup, R. Dyrska, K. König, and M. Mönnigmann, "A novel event-triggered robust mpc scheme for linear systems," in *Proceedings of the 56th IEEE Conference on Decision and Control*, 2017, pp. 2941–2946.

[18] P. Berner and M. Mönnigmann, "A comparison of four variants of event-triggered networked MPC," in *Proceedings of the 2016 IEEE Multi-Conference on Systems and Control*, 2016, pp. 1519–1524.

[19] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, pp. 789 – 814, 2000.

[20] E. G. Gilbert and K. T. Tan, "Linear systems with state and control constraints: the theory and application of maximal output admissible sets," *IEEE Transactions on Automatic Control*, vol. 36, no. 9, pp. 1008–1020, 1991.