2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Laplace One-Step Controller for Linear Scalar Systems*

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Abstract— **Uncertainties in many physical systems have impulsive properties poorly modeled by Gaussian distributions. Building on work to develop Cauchy controllers, a Laplace controller is explored as a heavier-tailed alternative to the Gaussian. Whereas the Cauchy density has no moments, the Laplace density has finite moments of all orders as the Gaussian density. For a scalar discrete linear system with additive Laplace process and measurement noises, the one-step optimal control problem is considered, where the conditional expectation of the cost criterion is determined as a function of the measurements and the control in closed form. The optimal control is determined numerically for different values of noise parameters and cost criterion weightings, and its properties are examined.**

I. INTRODUCTION

In many engineering applications, random processes or noises have volatility that are not well-modeled by Gaussian distributions. We suggest in this paper that Laplace distributions are an appropriate distribution to model volatile processes. The Laplace distribution finds application in a variety of disciplines that range from image and speech recognition to finance [1]. Past efforts have used Cauchy distributions, whose heavy tails better capture these phenomena [2],[3]. The Cauchy probability density function (pdf) does not have well-defined moments, such that the mean is not defined and the variance is infinite. However, like the Gaussian pdf, the Laplace pdf has finite moments of all orders. In spite of the finiteness of moments, the tails of the Laplace pdf are substantially longer than those of the Gaussian pdf, although not as long as those of the Cauchy pdf. Because developing an estimator using the Cauchy pdfs *directly* became intractable beyond the scalar case, the multivariate estimator was developed using characteristic functions [2], from which the conditional mean and conditional variance were obtained. Fortunately, the form of the characteristic function for the Cauchy estimation problem is similar to the Laplace density function, since they are Fourier transformations of each other, and this similarity was used to apply an integral formula in [2] to determine the time propagation of the conditional pdf for a scalar linear system with additive Laplace noise in closed form.

In this paper, the derivation of the one-step scalar Laplace controller is described. For scaler linear dynamic system, the conditional expectation of the cost criterion is defined consistent with the Laplace conditional pdf so that the conditional expectation associated with the performance index can be taken in closed form. Therefore, this leads to a deterministic maximization problem, where the control can be found as a function of the measurement. The character of this new performance measure will be discussed. It appears that this structure can be generalized to any stage time and vector dynamic systems.

In Section II, we state the scalar Laplace controller problem, define the conditional expectation of the cost criterion, and derive the a posteriori conditional density function after one measurement update. This is then used in Section III, where we evaluate the conditional expectation of the cost criterion to be maximized. In section IV, we provide a numerical example and show the relationship between the optimal control and the measurements for various system parameters. In section V, we discuss some interesting properties and numerical challenges. Finally, we offer some concluding remarks in section VI.

II. SCALAR LAPLACE CONTROLLER

The Laplace one-step controller system and the conditional expectation of the cost criterion is presented.

A. Problem Statement

Consider the one-step scalar dynamic system

$$\begin{aligned} x_2 &= \Phi x_1 + u_1 + w_1 \\ z_1 &= H x_1 + v_1 \end{aligned} \tag{1}$$

where x_2 is a scalar state, u_1 is the control that is to be found as a function of the measurement history, and z_1 is a scalar measurement. The scalar parameters Φ and H are given constants. The independent random variable x_1, v_1, and w_1 are Laplace distributed as

$$f_{X_1}(x_1) = \frac{\alpha}{2} e^{-\alpha|x_1 - \bar{x}_1|} \tag{2}$$

$$f_W(w_k) = \frac{\beta}{2} e^{-\beta|w_k|} \tag{3}$$

$$f_V(v_k) = \frac{\gamma}{2} e^{-\gamma|v_k|} \tag{4}$$

where $\bar{x}_1$ is the mean.

To simplify the control problem, the state variable is decomposed as

$$x_1 = \tilde{x}_1 + \bar{x}_1 \tag{5}$$

*This material is based on work supported by the National Science Foundation under grant number NSF/ENG/ECCS-BSF 1607502

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where $\tilde{x}_1$ is a stochastic part and $\bar{x}_1$ is the deterministic part with dynamics and measurement functions

$$\begin{aligned} \tilde{x}_2 &= \Phi\tilde{x}_1 + w_1, \quad \tilde{z}_1 = H\tilde{x}_1 + v_1 \\ \bar{x}_2 &= \Phi\bar{x}_1 + u_1, \quad \bar{z}_1 = H\bar{x}_1 \end{aligned} \tag{6}$$

Using this decomposition, the density of $\tilde{x}_1$ is

$$f_{\tilde{X}_1}(\tilde{x}_1) = \frac{\alpha}{2} e^{-\alpha|\tilde{x}_1|} \tag{7}$$

This decomposition is very useful in the development of the cost criterion. Theorem 1 of [4] shows that the conditional expectation of the cost criterion need only be conditioned on the $\tilde{\sigma}$-algebra generated by the measurement $\tilde{z}_1$, since u_1 need only be adapted to $\tilde{\sigma}$. This is used in constructing the cost criterion

$$J_{\tilde{Z}_1}(u_1) = E\left[e^{-c|u_1|-S|x_2|}|\tilde{z}_1\right] \tag{8}$$

where the conditional expectation is taken with respect to the density

$$f_{\tilde{X}_2|\tilde{Z}_1} = \frac{\bar{f}_{\tilde{X}_2|\tilde{Z}_1}}{f_{\tilde{Z}_1}} \tag{9}$$

B. Measurement Update at $k = 1$

We now construct the conditional probability density function (cpdf) $f_{\tilde{X}_2|\tilde{Y}_1}$. At the first update, the cpdf is

$$f_{\tilde{X}_1|\tilde{Y}_1}(\tilde{x}_1|\tilde{z}_1) = \frac{f_{\tilde{X}_1,\tilde{Y}_1}(\tilde{x}_1,\tilde{z}_1)}{f_{\tilde{Y}_1}(\tilde{z}_1)} = \frac{f_{\tilde{Y}_1|\tilde{X}_1}(\tilde{z}_1|\tilde{x}_1)f_{\tilde{X}_1}(\tilde{x}_1)}{f_{\tilde{Y}_1}(\tilde{z}_1)} \tag{10}$$

where the measurement history is defined as $\tilde{Y}_k = \{\tilde{z}_1, \cdots, \tilde{z}_k\}$ and, therefore, $\tilde{Y}_1 = \tilde{z}_1$ is the measurement history up to $k = 1$.

Let us define the unnormalized conditional pdf as

$$\begin{aligned} \bar{f}_{\tilde{X}_1|\tilde{Y}_1}(\tilde{x}_1|\tilde{z}_1) &= f_{\tilde{X}_1,\tilde{Y}_1}(\tilde{x}_1,\tilde{z}_1) = f_{\tilde{Y}_1|\tilde{X}_1}(\tilde{z}_1|\tilde{x}_1)f_{\tilde{X}_1}(\tilde{x}_1) \\ &= f_V(\tilde{z}_1 - H\tilde{x}_1)f_{\tilde{X}_1}(\tilde{x}_1) \end{aligned} \tag{11}$$

Using (2) and (4), (11) becomes

$$\bar{f}_{\tilde{X}_1|\tilde{Y}_1}(\tilde{x}_1|\tilde{z}_1) = \frac{\alpha\gamma}{4} e^{-\gamma|\tilde{z}_1 - H\tilde{x}_1|-\alpha|\tilde{x}_1|} \tag{12}$$

The normalization factor $f_{\tilde{Y}_1}(\tilde{z}_1)$ is given by

$$\begin{aligned} f_{\tilde{Y}_1}(\tilde{z}_1) &= \int_{-\infty}^{\infty} f_{\tilde{X}_1,\tilde{Y}_1}(\tilde{x}_1,\tilde{z}_1)\,d\tilde{x}_1 \\ &= \frac{\alpha\gamma}{4}\int_{-\infty}^{\infty} e^{-\gamma|\tilde{z}_1 - H\tilde{x}_1|-\alpha|\tilde{x}_1|}\,d\tilde{x}_1 \end{aligned} \tag{13}$$

which, using the integral formula of Appendix B of [2], becomes

$$\begin{aligned} f_{\tilde{Y}_1}(\tilde{z}_1) &= \frac{\alpha\gamma}{2(\gamma^2H^2 - \alpha^2)} \\ &\quad \times \left(\gamma\,|H|\,e^{-\frac{\alpha}{|H|}|z_1|} - \alpha e^{-\gamma|z_1|}\right) \end{aligned} \tag{14}$$

This integral is straight forward, but integrals in the forthcoming sections are more challenging and, therefore, the integral formula in Appendix B of [2] is important.

C. Propagation from $k = 1$ to $k = 2$

For the propagation, we start with the joint density

$$\begin{aligned} &\bar{f}_{\tilde{X}_1,\tilde{X}_2|Y_1}(\tilde{x}_2,\tilde{x}_1|\tilde{z}_1) \\ &\quad = \bar{f}_{\tilde{X}_1|\tilde{Y}_1}(\tilde{x}_1|\tilde{z}_1)f_{\tilde{X}_2|\tilde{X}_1}(\tilde{x}_2|\tilde{x}_1,\tilde{z}_1) \\ &\quad = \bar{f}_{\tilde{X}_1|\tilde{Y}_1}(\tilde{x}_1|\tilde{z}_1)f_{\tilde{X}_2|\tilde{X}_1}(\tilde{x}_2|\tilde{x}_1) \\ &\quad = \bar{f}_{\tilde{X}_1|\tilde{Y}_1}(\tilde{x}_1|\tilde{z}_1)f_W(\tilde{x}_2 - \Phi\tilde{x}_1) \\ &\quad = \frac{\alpha\gamma}{4}e^{-\alpha|\tilde{x}_1|-\gamma|\tilde{z}_1 - H\tilde{x}_1|}\frac{\beta}{2}e^{-\beta|\tilde{x}_2-\Phi\tilde{x}_1|}. \end{aligned} \tag{15}$$

To obtain the marginal density, we integrate to obtain

$$\begin{aligned} &\bar{f}_{\tilde{X}_2|\tilde{Y}_1}(\tilde{x}_2,\tilde{z}_1) \\ &\quad = \frac{\alpha\beta\gamma}{8}\int_{-\infty}^{\infty} e^{-\alpha|\tilde{x}_1|-\beta|\tilde{x}_2-\Phi\tilde{x}_1|-\gamma|\tilde{z}_1-H\tilde{x}_1|}\,d\tilde{x}_1 \\ &\quad = \frac{\alpha\beta\gamma}{8}\times \\ &\quad\quad \int_{-\infty}^{\infty} e^{-\alpha|\tilde{x}_1|-\beta|\Phi|\left|\frac{\tilde{x}_2}{\Phi}-\tilde{x}_1\right|-\gamma|H|\left|\frac{\tilde{z}_1}{H}-\tilde{x}_1\right|}\,d\tilde{x}_1 \end{aligned} \tag{16}$$

which, again using Appendix B of [2], becomes

$$\begin{aligned} &\bar{f}_{X_2|Y_1}(x_2,z_1) \\ &\quad = \frac{\alpha\beta\gamma}{8}\Big[\bar{g}_1 e^{-\beta|\tilde{x}_2|-\gamma|\tilde{z}_1|} \\ &\quad\quad + \bar{g}_2 e^{-\frac{\alpha}{|\Phi|}|\tilde{x}_2|-\frac{\gamma}{\Phi}|H\tilde{x}_2-\Phi\tilde{z}_1|} \\ &\quad\quad + \bar{g}_3 e^{-\frac{\alpha}{|H|}|\tilde{z}_1|-\frac{\beta}{|H|}|H\tilde{x}_2-\Phi\tilde{z}_1|}\Big] \end{aligned} \tag{17}$$

where

$$\begin{aligned} \bar{g}_1 &= \frac{1}{\alpha+\beta\,|\Phi|\,\mathrm{sgn}\frac{\tilde{x}_2}{\Phi}+\gamma\,|H|\,\mathrm{sgn}\frac{\tilde{z}_1}{H}} \\ &\quad - \frac{1}{-\alpha+\beta\,|\Phi|\,\mathrm{sgn}\frac{\tilde{x}_2}{\Phi}+\gamma\,|H|\,\mathrm{sgn}\frac{\tilde{z}_1}{H}} \\ &= \frac{2\alpha}{\alpha^2-\left(-\beta\Phi\mathrm{sgn}\tilde{x}_2+\gamma\,|H|\,\mathrm{sgn}\tilde{z}_1\right)^2} \\ \bar{g}_2 &= \frac{1}{\beta\,|\Phi|+\alpha\mathrm{sgn}\left(-\frac{\tilde{x}_2}{\Phi}\right)+\gamma\,|H|\,\mathrm{sgn}\left(\frac{\tilde{z}_1}{H}-\frac{\tilde{x}_2}{\Phi}\right)} \\ &\quad - \frac{1}{-\beta\,|\Phi|+\alpha\mathrm{sgn}\left(-\frac{\tilde{x}_2}{\Phi}\right)+\gamma\,|H|\,\mathrm{sgn}\left(\frac{\tilde{z}_1}{H}-\frac{\tilde{x}_2}{\Phi}\right)} \\ &= \frac{2\beta\,|\Phi|}{\beta^2\Phi^2-\left(\alpha\mathrm{sgn}\tilde{x}_2+\gamma H\mathrm{sgn}\left(\tilde{x}_2-\frac{\Phi\tilde{z}_1}{H}\right)\right)^2} \\ \bar{g}_3 &= \frac{1}{\gamma\,|H|+\alpha\mathrm{sgn}\left(-\frac{\tilde{z}_1}{H}\right)+\beta\,|\Phi|\,\mathrm{sgn}\left(\frac{\tilde{x}_2}{\Phi}-\frac{\tilde{z}_1}{H}\right)} \\ &\quad - \frac{1}{-\gamma\,|H|+\alpha\mathrm{sgn}\left(-\frac{\tilde{z}_1}{H}\right)+\beta\,|\Phi|\,\mathrm{sgn}\left(\frac{\tilde{x}_2}{\Phi}-\frac{\tilde{z}_1}{H}\right)} \\ &= \frac{2\gamma\,|H|}{\gamma^2H^2-\left(-\frac{\alpha}{\mathrm{sgn}H}\mathrm{sgn}\tilde{z}_1+\beta\Phi\mathrm{sgn}\left(\tilde{x}_2-\frac{\Phi\tilde{z}_1}{H}\right)\right)^2} \end{aligned} \tag{18}$$

III. Evaluating the Cost Criterion

The cost criterion (8) will now be evaluated by taking the expectation with respect to the conditional pdf (17)

$$\begin{aligned} J_{\tilde{Z}_1}(u_1) &= E\left[e^{-c|u_1|-S|\tilde{x}_2|}|\tilde{z}_1\right] \\ &= \frac{\alpha\beta\gamma}{8f_{\tilde{Z}_1}(\tilde{z}_1)}\exp\left(-c\,|u_1|\right)\int_{-\infty}^{\infty}\tilde{F}(\tilde{x}_2)\,d\tilde{x}_2 \end{aligned} \tag{19}$$

where

$$
\begin{aligned}
\tilde{F}(\tilde{x}_2) &= \bar{g}_1 e^{-\beta|\tilde{x}_2|-\gamma|\tilde{z}_1|+S|\tilde{x}_2+\Phi\bar{x}_1+u_1|} \\
&+ \bar{g}_2 e^{-\frac{\alpha}{|\Phi|}|\tilde{x}_2|+\frac{\gamma}{|\Phi|}|H\tilde{x}_2-\Phi\tilde{z}_1|-S|\tilde{x}_2+\Phi\bar{x}_1+u_1|} \\
&+ \bar{g}_3 e^{-\frac{\alpha}{|H|}|\tilde{z}_1|-\frac{\beta}{|H|}|H\tilde{x}_2-\Phi\tilde{z}_1|-S|\tilde{x}_2+\Phi\bar{x}_1+u_1|}
\end{aligned} \tag{20}
$$

The applicable integral equation in Appendix B of [2] requires modification of the g term for each exponential term to match the function form of the argument of the exponential for convenience. These additions to the g terms are added with an associated zero coefficient for consistency [2]. Evaluating the integral yields

$$
\begin{aligned}
J_{\tilde{Z}_1}(u_1) = \frac{\alpha\beta\gamma}{8 f_{\tilde{Z}_1}(\tilde{z}_1)} e^{-c|u_1|} \times \\
\Big[G_1^1 e^{-S|\Phi\bar{x}_1+u_1|-\gamma|\tilde{z}_1|} \\
+ G_1^2 e^{-\beta|\Phi\bar{x}_1+u_1|-\gamma|\tilde{z}_1|} \\
+ G_2^1 e^{-S|\Phi\bar{x}_1+u_1|-\frac{\gamma|H|}{|\Phi|}|\tilde{z}_1|} \\
+ G_2^2 e^{-S|\Phi\bar{x}_1+u_1|-\frac{\alpha}{|\Phi|}|\tilde{z}_1|} \\
+ G_2^3 e^{-\frac{\alpha}{|\Phi|}|\Phi\bar{x}_1+u_1|-\frac{\gamma|H|}{|\Phi|}\left|\frac{\Phi\tilde{z}_1}{H}+\Phi\bar{x}_1+u_1\right|} \\
+ G_3^1 e^{-\beta\left|\frac{\Phi\tilde{z}_1}{H}+\Phi\bar{x}_1+u_1\right|-\frac{\alpha}{|H|}|\tilde{z}_1|} \\
+ G_3^2 e^{-S\left|\frac{\Phi\tilde{z}_1}{H}+\Phi\bar{x}_1+u_1\right|-\frac{\alpha}{|H|}|\tilde{z}_1|} \Big]
\end{aligned} \tag{21}
$$

where $f_{\tilde{Z}_1}(\tilde{z}_1)$ is given in (14) and the G's, constructed using the integral formula in Appendix B.2 of [2] are

$$
\begin{aligned}
G_1^1 &= \frac{\frac{2\alpha}{\alpha^2-(-\beta\Phi+\gamma|H|\mathrm{sgn}\tilde{z}_1)^2}}{\beta - S\mathrm{sgn}\bar{x}_2} - \frac{\frac{2\alpha}{\alpha^2-(\beta\Phi+\gamma|H|\mathrm{sgn}\tilde{z}_1)^2}}{-\beta - S\mathrm{sgn}\bar{x}_2} \\
G_1^2 &= \frac{\frac{2\alpha}{\alpha^2-(-\beta\Phi\mathrm{sgn}\bar{x}_2+\gamma|H|\mathrm{sgn}\tilde{z}_1)^2}}{S + \beta\mathrm{sgn}\bar{x}_2} - \frac{\frac{2\alpha}{\alpha^2-(-\beta\Phi\mathrm{sgn}\bar{x}_2+\gamma|H|\mathrm{sgn}\tilde{z}_1)^2}}{-S + \beta\mathrm{sgn}\bar{x}_2} \\
G_2^1 &= \frac{\frac{2\beta|\Phi|}{(\beta\Phi)^2-\left(\alpha+\gamma+|H|\mathrm{sgn}\frac{\Phi\tilde{z}_1}{H}\right)^2}}{\frac{\alpha}{|\Phi|}+\frac{\gamma|H|}{|\Phi|}\mathrm{sgn}\frac{\Phi\tilde{z}_1}{H} - S\mathrm{sgn}\bar{x}_2} \\
&- \frac{\frac{2\beta|\Phi|}{(\beta\Phi)^2-\left(-\alpha+\gamma+|H|\mathrm{sgn}\frac{\Phi\tilde{z}_1}{H}\right)^2}}{-\frac{\alpha}{|\Phi|}+\frac{\gamma|H|}{|\Phi|}\mathrm{sgn}\frac{\Phi\tilde{z}_1}{H} - S\mathrm{sgn}\bar{x}_2} \\
G_2^2 &= \frac{\frac{2\beta|\Phi|}{(\beta\Phi)^2-\left(\gamma H+\alpha\mathrm{sgn}\left(-\frac{\Phi\tilde{z}_1}{H}\right)\right)^2}}{\frac{\gamma|H|}{|\Phi|}+\frac{\alpha}{|\Phi|}\mathrm{sgn}\left(-\frac{\Phi\tilde{z}_1}{H}\right) - S\mathrm{sgn}\left(\Phi\bar{x}_1+u_1+\frac{\Phi\tilde{z}_1}{H}\right)} \\
&- \frac{\frac{2\beta|\Phi|}{(\beta\Phi)^2-\left(\gamma H+\alpha\mathrm{sgn}\left(-\frac{\Phi\tilde{z}_1}{H}\right)\right)^2}}{-\frac{\gamma|H|}{|\Phi|}+\frac{\alpha}{|\Phi|}\mathrm{sgn}\left(-\frac{\Phi\tilde{z}_1}{H}\right) - S\mathrm{sgn}\left(\Phi\bar{x}_1+u_1+\frac{\Phi\tilde{z}_1}{H}\right)} \\
G_2^3 &= \frac{\frac{2\beta|\Phi|}{(\beta\Phi)^2-\left(\alpha\mathrm{sgn}\bar{x}_2+\gamma H\mathrm{sgn}\left(\frac{\Phi\tilde{z}_1}{H}+\bar{x}_2\right)\right)^2}}{S+\frac{\alpha}{|\Phi|}\mathrm{sgn}\bar{x}_2+\frac{\gamma|H|}{|\Phi|}\mathrm{sgn}\left(\frac{\Phi\tilde{z}_1}{H}+\bar{x}_2\right)} \\
&- \frac{\frac{2\beta|\Phi|}{(\beta\Phi)^2-\left(\alpha\mathrm{sgn}\bar{x}_2+\gamma H\mathrm{sgn}\left(\frac{\Phi\tilde{z}_1}{H}+\bar{x}_2\right)\right)^2}}{-S+\frac{\alpha}{|\Phi|}\mathrm{sgn}\bar{x}_2+\frac{\gamma|H|}{|\Phi|}\mathrm{sgn}\left(\frac{\Phi\tilde{z}_1}{H}+\bar{x}_2\right)} \\
G_3^1 &= \frac{\frac{2\gamma|H|}{(\gamma H)^2-\left(\beta\Phi+\frac{\alpha H}{|H|}\mathrm{sgn}\tilde{z}_1\right)^2}}{\beta - S\mathrm{sgn}\left(\bar{x}_2+\frac{\Phi\tilde{z}_1}{H}\right)} - \frac{\frac{2\gamma|H|}{(\gamma H)^2-\left(\beta\Phi+\frac{\alpha H}{|H|}\mathrm{sgn}\tilde{z}_1\right)^2}}{-\beta - S\mathrm{sgn}\left(\bar{x}_2+\frac{\Phi\tilde{z}_1}{H}\right)} \\
G_3^2 &= \frac{\frac{2\gamma|H|}{(\gamma H)^2-\left(\beta\Phi\mathrm{sgn}\left(\frac{\Phi\tilde{z}_1}{H}+\bar{x}_2\right)+\frac{\alpha H}{|H|}\mathrm{sgn}\tilde{z}_1\right)^2}}{S+\beta\mathrm{sgn}\left(\bar{x}_2+\frac{\Phi\tilde{z}_1}{H}\right)} \\
&- \frac{\frac{2\gamma|H|}{(\gamma H)^2-\left(\beta\Phi\mathrm{sgn}\left(\frac{\Phi\tilde{z}_1}{H}+\bar{x}_2\right)+\frac{\alpha H}{|H|}\mathrm{sgn}\tilde{z}_1\right)^2}}{-S+\beta\mathrm{sgn}\left(\bar{x}_2+\frac{\Phi\tilde{z}_1}{H}\right)}
\end{aligned}
$$

Note the explicit use of the terms in (18) in constructing the G terms. With some algebraic manipulations, the G terms can be simplified to

$$
\begin{aligned}
G_1^1 &= \frac{4\alpha\left(\beta\delta_1 - 2S\beta\Phi\gamma\left|H\right|\mathrm{sgn}\bar{x}_1\mathrm{sgn}\tilde{z}_1\right)}{\left(\beta^2-S^2\right)\left[\delta_1^2-4\left(\beta\Phi\right)^2\left(\gamma H\right)^2\right]} \\
G_1^2 &= \frac{-4\alpha S}{\left(\beta^2-S^2\right)\left(\delta_1+2\beta\Phi\gamma\left|H\right|\mathrm{sgn}\tilde{z}_1\mathrm{sgn}\bar{x}_2\right)} \\
G_2^1 &= \frac{-4\alpha\beta\Phi^2}{\left(\delta_2^2-4\alpha^2\left(\gamma H\right)^2\right)} \times \\
&\quad \frac{\delta_2 - 2\gamma\left|H\right|\mathrm{sgn}\frac{\Phi\tilde{z}_1}{H}\left(\gamma\left|H\right|\mathrm{sgn}\frac{\Phi\tilde{z}_1}{H} - S\left|\Phi\right|\mathrm{sgn}\bar{x}_2\right)}{-\alpha^2+\left(\gamma H\right)^2+\left(S\Phi\right)^2-2\gamma\left|H\right|S\left|\Phi\right|\mathrm{sgn}\frac{\Phi\tilde{z}_1}{H}\mathrm{sgn}\bar{x}_2}
\end{aligned}
$$

$$
\begin{aligned}
G_2^2 &= \frac{4\beta\Phi^2}{\delta_2^2-4\alpha^2\left(\gamma H\right)^2} \times \\
&\quad \frac{\delta_2\gamma\left|H\right| - 2\alpha\left(\gamma\left|H\right|\right)\mathrm{sgn}\frac{\Phi\tilde{z}_1}{H}\Omega}{\left(\gamma H\right)^2-\left(\alpha\mathrm{sgn}\frac{\Phi\tilde{z}_1}{H}+S\left|\Phi\right|\mathrm{sgn}\left(\frac{\Phi\tilde{z}_1}{H}+\bar{x}_2\right)\right)^2} \\
\Omega &= \alpha\mathrm{sgn}\frac{\Phi\tilde{z}_1}{H}+S\left|\Phi\right|\mathrm{sgn}\left(\frac{\Phi\tilde{z}_1}{H}+\bar{x}_2\right) \\
G_2^3 &= \frac{4\beta S\left|\Phi\right|^3}{\left(\delta_2-2\alpha\gamma\left|H\right|\mathrm{sgn}\bar{x}_2\mathrm{sgn}\left(\frac{\Phi\tilde{z}_1}{H}+u_1\right)\right)} \times \\
&\quad \frac{1}{\left(S\Phi\right)^2-\left(\alpha\mathrm{sgn}\bar{x}_2+\gamma\left|H\right|\mathrm{sgn}\left(\frac{\Phi\tilde{z}_1}{H}+\bar{x}_2\right)\right)^2}
\end{aligned}
$$

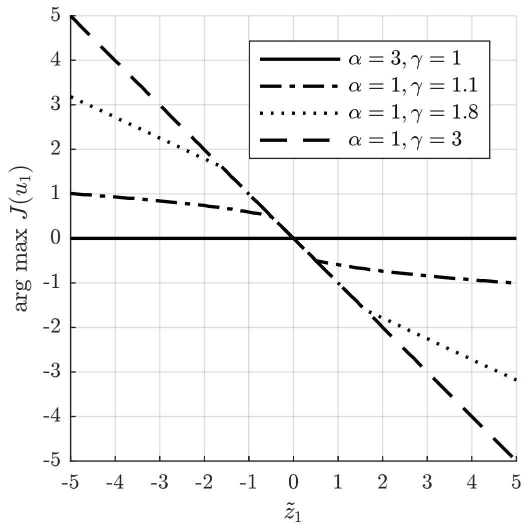

Fig. 1. Optimal u_1 versus measurement z_1 for $\beta = 1, S = 1.01$

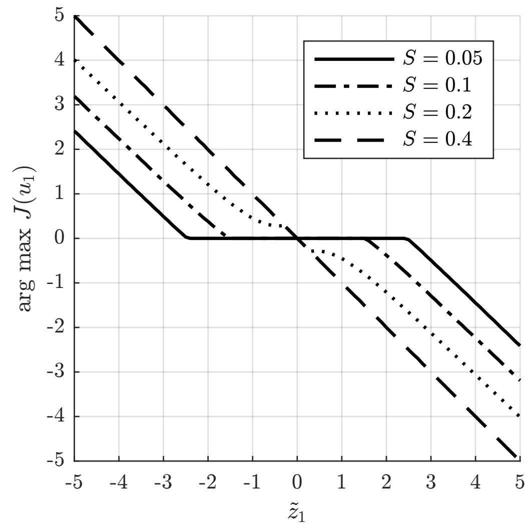

Fig. 2. Optimal u_1 versus measurement $\tilde{z}_1$ for $\alpha = 1, \beta = 0.5, \gamma = 3$

$$G_3^1 = \frac{4\gamma\,|H|\left[\delta_3\beta + 2\beta\Phi\alpha\frac{H}{|H|}S\mathrm{sgn}\tilde{z}_1\mathrm{sgn}\left(\frac{\Phi\tilde{z}_1}{H} + \bar{x}_2\right)\right]}{\left(\beta^2 - S^2\right)\left(\delta_3^2 - 4\left(\beta\Phi\right)^2\alpha^2\right)}$$

$$G_3^2 = \frac{-4\gamma\,|H|\,S}{\left(\beta^2 - S^2\right)\left(\delta_3 - 2\beta\Phi\alpha\frac{H}{|H|}\mathrm{sgn}\tilde{z}_1\mathrm{sgn}\left(\frac{\Phi\tilde{z}_1}{H} + \bar{x}_2\right)\right)}$$

where

$$\begin{aligned}\delta_1 &= \alpha^2 - \beta^2 - \gamma^2 \\ \delta_2 &= -\alpha^2 + \beta^2 - \gamma^2 \\ \delta_3 &= -\alpha^2 - \beta^2 + \gamma^2\end{aligned} \tag{22}$$

The cost criterion (21) is to be maximized with respect to u_1. This will make u_1 an explicit function of the measurement z_1. That is,

$$J^*_{\tilde{Z}_1} = \max_{u_1} J_{\tilde{Z}_1}(u_1) \implies u_1 = u_1(\tilde{z}_1) \tag{23}$$

IV. Numerical Example

Let $\Phi = H = 1, \bar{x}_1 = 0, \bar{x}_2 = \Phi\bar{x}_1 + u_1 = u_1$. Maximizing over u_1 given $\tilde{z}_1$, we obtain plots of $u_1^* = \mathrm{argmax} J(u_1)$ versus $\tilde{z}_1$ for different parameters α, β, γ, S and c. u_1^* is found by evaluating $J(u_1)$ over a grid of $\tilde{z}_1, u_1$ values and finding the maximum for each $\tilde{z}_1$. For each case below, the weighting on the control is held at $c = 1.0$. Figure 1 shows the optimal control u_1^* versus the measurement $\tilde{z}_1$ when $\beta = 1.0$ and $S = 1.01$. α is the uncertainty parameter of the initial condition, $\tilde{x}_1$, and γ is the measurement noise parameter, z_1. Since the form of the pdf of $\tilde{x}_1$ is $\frac{\alpha}{2}e^{-\alpha|\tilde{x}_1 - \bar{x}_1|}$, a large value of α corresponds to less uncertainty in $\tilde{x}_1$. Likewise, a large value of γ corresponds to less uncertainty in $\tilde{z}_1$. Looking at Figure 1, we can see that when the initial condition is much more certain than the measurement ($\alpha = 3, \gamma = 1$), then the $u^* = 0$, as there is no need to move. Conversely, when the initial condition is much less certain than the measurement ($\alpha = 1, \gamma = 3$), then $u_1^* = -\tilde{z}_1$. The intermediate cases show an interesting split, where for small $\tilde{z}_1$, the $u_1^* \approx -\tilde{z}_1$ but then breaks away at larger values of $\tilde{z}_1$, recognizing the reduced relative certainty of $\tilde{z}_1$.

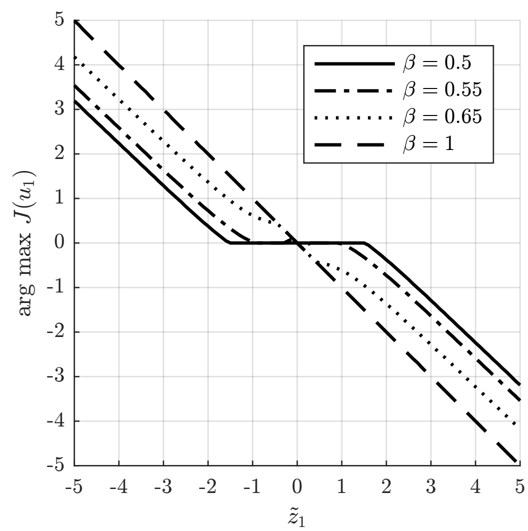

Fig. 3. Optimal u_1 versus measurement z_1 for $\alpha = 1$, $\gamma = 3$, $S = 0.1$

In Figure 2, we vary the weighting S on the final state $\tilde{x}_2$ when $\alpha = 1, \beta = 0.5$ and $\gamma = 3$. Here, the measurement is much more certain than the initial condition and the process noise, and the case were $S = 0.4$ is quite similar to the fourth plot in Figure 1. However, as we reduce S, we develop a dead zone in the optimal control for small $\tilde{z}_1$, since the cost of control is relatively high compared to that of the displacement from the desired final state. If the measurement is large, then most likely the state is large and the controller will move it back to the dead zone.

In Figure 3, we vary the process noise parameter when $\alpha = 1, \gamma = 3$ and $S = 0.1$. This is very similar to Figure 2.

The process noise affects the uncertainty of the final state, so as it decreases (β becomes large), the optimal control u_1^* approaches $-z_1$. However, if the final state is uncertain, then the controller develops a dead zone for small z_1 for the same reason as in Figure 2. The controller avoids using control to keep the state close to the origin when most likely the process noise will drive it away. However, if the measurement is large, pulling it back to the vicinity of the origin, determined by the dead zone, is worth the control effort.

V. DISCUSSION

The optimal control has a very complicated form which warrant further study. The complexity of the solutions arise from the various terms, and their relative importance lead to the interesting behavior we observe in the previous section. For example, we wish to identify the term or terms responsible for determining when the curves break away from the $u_1^* = -z_1$ line.

As can be seen in just a few cases in Figures 1-3, the relationships between the various parameters are rich, and the results are quite intricate. For example, we need to determine how these parameters affect the size of the control dead zones, the locations where the curve breaks away from $u_1^* = -z_1$, and the slopes of those curves.

Many numerical issues have arisen in the process of testing these parameters, whose values sometimes lead to division by zero. For example, when $\beta = S$, G_1^1, G_1^2, G_3^1 and G_3^2 all have zeros in the denominator. However, although there are singularities in the denominator, the sums might also be zero and cause a numerator denominator cancellation. Note that in (14), when $H = 1, \alpha = \gamma$, there appears to be a singularity in the denominator, which is canceled by the numerator. We also suspect that such singularities in the cost function lead to numerical errors in finding the optimal control. In certain cases, the optimization fails to find a solution or jumps from one minimum to another.

VI. CONCLUSIONS

This paper introduces the notion of using Laplace pdfs for constructing controllers that, because they are heavier tailed than Gaussian, may be more robust in the presence of impulse uncertainty. The results do show that the nonlinear controller structure acts differently than what would be expected from a Gaussian controller. For certain parameters, a linear controller exists for the Laplacian controller as would be the case for the Gaussian. However, for other parameter values, a dead zone occurs, producing a nonlinearity that enhances performance which would never occur for Gaussian controllers. It is shown that the Laplace estimator for the scalar system can be generalized to a multistage system [5]. Furthermore, this one-step controller can be extended to the m-step model predictive optimal controller or open-loop optimal feedback controller, at least for the scalar problem as given for Cauchy distributions in [4]

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