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Tailored MPC for mobile robots with very short prediction horizons*

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Abstract— **We propose a novel predictive control algorithm tailored for the kinematic model of a mobile robot. The scheme exploits two properties of the robot: finite time controllability and the fact that any given path can be traversed with arbitrarily chosen time parametrization if input constraints are ignored. Combining these two properties allows us to design a terminal cost that guarantees recursive feasibility and set point convergence of the MPC closed-loop — independently of the chosen prediction horizon. We illustrate the efficiency of the proposed controller with some numerical experiments.**

I. INTRODUCTION

Model predictive control (MPC) for nonholonomic mobile robots has gained considerable attention during the past decades, see, e.g., [1] and [2]. In MPC the future behavior of the system is predicted and optimized based on the most recent state measurement. Then, the first portion of the computed input signal is applied before the procedure is repeated, see, e.g., the textbooks [3] for details. However, the stability analysis of predictive controllers is far from being trivial, cp. [4] and [5]. Hence, either terminal ingredients, i.e. a terminal region and/or terminal costs are added [6], or a growth condition on the value function w.r.t. the employed running costs has to be verified for a rigorous stability analysis, see [7], [8], [9]. In this context, the example of the nonholonomic mobile robot (which is locally equivalent to Brockett's nonholonomic integrator) is extensively studied, see, e.g. [10] or [11] and the references therein. Here, the linearization is not stabilizable. Hence, various control strategies like, e.g. sliding mode [12] or adaptive tracking control [13] were applied. Structurally, a discontinuous control [14] is obtained and the incorporation of input saturation limits [1] causes additional difficulties.

We present a new MPC algorithm designed particularly for kinematic models of wheeled mobile robots. In contrast to other control strategies [15], [16] we seek to exploit structural properties of these mobile robots [17]. Namely, that the control effort spent on a particular sampling interval can be arbitrarily divided on this and the upcoming sampling intervals with only changing the speed, but not the path. This property is the key feature needed in the proposed approach, which allows to exploit general motion planning in the considered setting. This may allow to deal with static and/or moving obstacles in the future, see [18] and [19], respectively. Based on this property, we construct a terminal cost such that the MPC closed-loop is recursively feasible and convergence to a desired set point is ensured. Doing so yields a compromise between the (classical) methodologies based on terminal costs and constraints [20] and the results waiving these additional stipulations, see, e.g. [21], [11].

The paper is organized as follows. First, the problem formulation is introduced in Section II before we solve the regulation problem step by step. In Section III, we propose a sufficient condition to ensure satisfaction of the state constraints. In Section IV, a terminal cost is constructed, which overestimates the cost-to-go. In Section V, we combine both ingredients to design an MPC scheme tailored to the kinematic model of the mobile robot. Finally, simulations are conducted in Section VI to demonstrate the effectiveness of the proposed control scheme before conclusions are drawn in VII.

Notation. We denote natural numbers (including 0), integers, real numbers, and non-negative real numbers with $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{R}_{\geq 0}$, respectively. We further define $\mathbb{N}_{[i,j]}$ as $\mathbb{N} \cap [i,j]$ for $i,j \in \mathbb{N}$. Moreover, $\mathcal{L}^1_{\mathrm{loc}}([0,\infty),\mathbb{R}^2)$ denotes the space of all Lebesgue-measurable functions $h : [0,\infty) \to \mathbb{R}^2$ such that $\int_{\mathcal{B}} \|h(\tau)\| \mathrm{d}\tau < \infty$ holds for each compact set $\mathcal{B} \subset [0,\infty)$. Finally, we occasionally use the function $\mathrm{atan2}(x_2,x_1)$ with its standard definition and $\mathrm{tanc} : (-\pi/2,\pi/2) \to [1,\infty)$ with

$$\mathrm{tanc}(t) := \begin{cases} 1 & t = 0, \\ \tan(t)\, t^{-1} & \text{otherwise.} \end{cases}$$

Note that tanc is continuous on its domain.

II. PROBLEM FORMULATION

We consider the system

$$\dot{x}(t) = \begin{pmatrix} \cos(x_3(t)) \\ \sin(x_3(t)) \\ 0 \end{pmatrix} u_1(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2(t) \qquad (1)$$

with $x(0) = x^0$, which describes the kinematics of a nonholonomic two-wheeled mobile robot. The first two states of this driftless input-affine system describe the position of the robot in the x_1-x_2-plane while the third component x_3 represents its orientation. The inputs u_1 and u_2 refer to the longitudinal and rotational velocity of the robot. Note that u_1 and u_2 can be easily transformed to angular velocities

$$\omega_1 = \frac{1}{d}\,(2u_1 - bu_2) \quad \text{and} \quad \omega_2 = \frac{1}{d}\,(2u_1 + bu_2), \qquad (2)$$

of the left and right wheel, respectively, where d is the diameter of the wheels and b is the track width. The

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movement of the robot is subject to constraints. The motion in the x_1-x_2-plane is limited by a convex and compact set $\mathcal{C}$. Moreover, the angular velocities of the wheels are limited by $\omega_{\max}$. Formally, this results in the state and input constraints $x(t) \in \mathcal{X}$ and $u(t) \in \mathcal{U}$ with the convex sets

$$\mathcal{X} \;:=\; \left\{ x \in \mathbb{R}^3 \;\middle|\; (x_1 \, x_2)^\top \in \mathcal{C} \right\} \quad \text{and} \tag{3}$$

$$\mathcal{U} \;:=\; \left\{ u \in \mathbb{R}^2 \;\middle|\; 2\,|u_1| + b\,|u_2| \leq d\,\omega_{\max} \right\}. \tag{4}$$

Starting from an initial position $x^0 \in \mathbb{R}^3$, we are interested to steer the robot to a final position $x^\star$ without violating the constraints. In addition, we want to minimize the quadratic performance criterion

$$J_\infty(x^0, u) := \int_0^\infty \|x(t) - x^\star\|_Q^2 + \|u(t)\|_R^2 \,\mathrm{d}t \tag{5}$$

w.r.t. the control $u \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), \mathbb{R}^2)$. Here, $x(t)$ denotes the state trajectory $x(t; x^0, u)$ while $Q \in \mathbb{R}^{3\times 3}$ and $R \in \mathbb{R}^{2\times 2}$ are positive definite, symmetric matrices.

In general, the resulting optimal control problem (OCP) is hard to solve due to the nonlinear system dynamics (1), the infinite time span considered in the performance criterion (5), and the considered class of control functions. To simplify the solution of the OCP, we subsequently consider sampled-data systems with zero order hold, i.e., for some fixed sampling period $\Delta t > 0$, u is constant on each interval $[t_k, t_{k+1})$, $k \in \mathbb{N}$, where $t_k := k\,\Delta t$. This restriction allows to exactly discretize the system dynamics (1). To see this, note that a system trajectory emanating from a state $\bar{x}$ under a constant input $\bar{u}$ describes a segment of a circle in the x_1-x_2-plane and a linear variation in the x_3-direction. More precisely, the system state ξ at time $t \geq 0$, depending on $\bar{x}$ and $\bar{u}$, is

$$\xi(t, \bar{x}, \bar{u}) = \bar{x} + \begin{pmatrix} \frac{\bar{u}_1}{\bar{u}_2} \left(\sin(\bar{x}_3 + t\,\bar{u}_2) - \sin(\bar{x}_3)\right) \\ \frac{\bar{u}_1}{\bar{u}_2} \left(\cos(\bar{x}_3) - \cos(\bar{x}_3 + t\,\bar{u}_2)\right) \\ t\,\bar{u}_2 \end{pmatrix}. \tag{6}$$

Note that (6) is well-defined even for $\bar{u}_2 = 0$. In this case, the robot moves in a straight line, which can be inferred from

$$\lim_{\bar{u}_2 \to 0} \xi(t, \bar{x}, \bar{u}) = \bar{x} + t\,\bar{u}_1 \begin{pmatrix} \cos(\bar{x}_3) & \sin(\bar{x}_3) & 0 \end{pmatrix}^\top. \tag{7}$$

Now, using (6), we find the discrete time system

$$x(t_{k+1}) = f(x(t_k), u(t_k)) \quad \text{with} \quad f(\bar{x}, \bar{u}) := \xi(\Delta t, \bar{x}, \bar{u}).$$

Then, using the (infinite) sequence $\hat{\boldsymbol{u}}_\infty = (\hat{u}(t_0), \hat{u}(t_1), \ldots)$ of decision variables and the stage cost

$$\ell(\bar{x}, \bar{u}) := \int_0^{\Delta t} \|\xi(t, \bar{x}, \bar{u}) - x^\star\|_Q^2 \,\mathrm{d}t + \Delta t \|\bar{u}\|_R^2, \tag{8}$$

the original OCP simplifies to

$$\text{minimize} \quad \sum_{k=0}^{\infty} \ell(\hat{x}(t_k), \hat{u}(t_k)) \tag{9}$$

$$\begin{aligned} &\text{w.r.t.} \quad \hat{\boldsymbol{u}}_\infty = (\hat{u}(t_k))_{k=0}^\infty \quad \text{s.t.} \quad \hat{x}(t_0) = x^0 \quad \text{and} \\ &\hat{x}(t_{k+1}) = f(\hat{x}(t_k), \hat{u}(t_k)) \qquad \forall\, k \in \mathbb{N}, \\ &\hat{u}(t_k) \in \mathcal{U} \qquad \forall\, k \in \mathbb{N}, \\ &\xi(\tau, \hat{x}(t_k), \hat{u}(t_k)) \in \mathcal{X} \qquad \forall\, k \in \mathbb{N},\ \tau \in [0, \Delta t]. \end{aligned}$$

Although simplified, the OCP (9) is still hard to solve due to the infinite number of decision variables and the demanding state constraints. We address both issues in the two following sections.

III. Satisfying the State Constraints

Next, we derive a sufficient condition for the satisfaction of the state constraints on the interval $[t_k, t_{k+1}]$ while checking only a finite number of (auxiliary) states instead of

$$\xi(\tau, \hat{x}(t_k), \hat{u}(t_k)) \in \mathcal{X} \qquad \text{for all } \tau \in [0, \Delta t]. \tag{10}$$

As a preparation, note that $\hat{x}(t_k) \in \mathcal{X}$ and $\hat{x}(t_{k+1}) \in \mathcal{X}$ do not necessarily imply $\xi(\tau, \hat{x}(t_k), \hat{u}(t_k)) \in \mathcal{X}$, $\tau \in (0, \Delta t)$. However, the latter condition can be guaranteed if one additional auxiliary state is checked. This approach is summarized in Lemma 1, which is shown in the appendix. The underlying idea of the proof is illustrated in Fig. 1.

Lemma 1: For given $\eta \in (0,1)$ and $\Delta t \in (0, \infty)$, let the set $\mathcal{D}$ be defined as

$$\mathcal{D} := \left\{ u \in \mathbb{R}^2 \;\middle|\; |u_2| \leq \frac{\eta\,\pi}{\Delta t} \right\}. \tag{11}$$

Further, let $\bar{x} \in \mathcal{X}$ and $\bar{u} \in \mathcal{D}$ be such that $f(\bar{x}, \bar{u}) \in \mathcal{X}$ and assume $g(\bar{x}, \bar{u}) \in \mathcal{C}$, where

$$g(\bar{x}, \bar{u}) := \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} + \frac{\Delta t\,\bar{u}_1}{2} \operatorname{tanc}\left(\frac{\Delta t\,\bar{u}_2}{2}\right) \begin{pmatrix} \cos(\bar{x}_3) \\ \sin(\bar{x}_3) \end{pmatrix}. \tag{12}$$

Then, $\xi(t, \bar{x}, \bar{u}) \in \mathcal{X}$ holds for every $t \in [0, \Delta t]$.

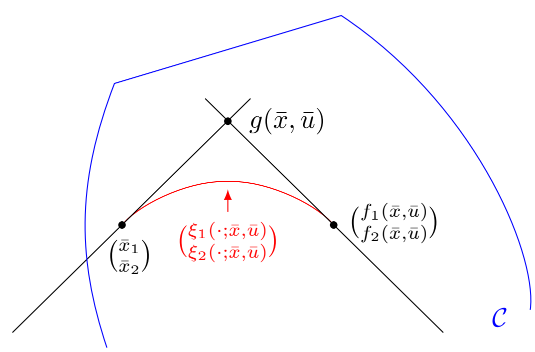

Fig. 1. Idea of Lemma 1. The projections of $\bar{x}$ and $f(\bar{x}, \bar{u})$ and $g(\bar{x}, \bar{u})$ form a triangle containing the projected trajectory $\xi(t, \bar{x}, \bar{u})$, $t \in [0, \Delta t]$.

Lemma 1 provides a sufficient condition to ensure that no intermittent violation of the (intractable) state constraint (10) can occur if the easily checkable conditions

$$\hat{x}(t_k) \in \mathcal{X},\ \hat{u}(t_k) \in \mathcal{D}, \text{ and } g(\hat{x}(t_k), \hat{u}(t_k)) \in \mathcal{C}$$

for all $k \in \mathbb{N}_0$ are incorporated in the OCP (9). Later, we see that the resulting conservatism neither affects the feasibility of the OCP nor the performance of the MPC closed-loop.

IV. Bypassing the Infinite Horizon

The presented approach to handle the state constraints is obviously tailored for the mobile robot. In contrast, the following approach to handle the infinite horizon in (9) initially seems to be standard. In fact, we simply truncate the infinite sum in the OCP (9) after a finite prediction horizon N and add a terminal cost function $\varphi : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$. The resulting OCP is given by

$$\text{minimize} \quad \sum_{k=0}^{N-1} \ell(\hat{x}(t_k), \hat{u}(t_k)) + \varphi(\hat{x}(t_N)) \tag{13}$$

w.r.t. $\hat{\boldsymbol{u}}_N = (\hat{u}(t_k))_{k=0}^{N-1}$ s.t. $\hat{x}(t_0) = x^0$ and

$$\begin{aligned} \hat{x}(t_{k+1}) &= f(\hat{x}(t_k), \hat{u}(t_k)) && \forall\, k \in \mathbb{N}_{[0,N-1]}, \\ \hat{u}(t_k) &\in \mathcal{U} \cap \mathcal{D} && \forall\, k \in \mathbb{N}_{[0,N-1]}, \\ \hat{x}(t_{k+1}) &\in \mathcal{X} && \forall\, k \in \mathbb{N}_{[0,N-1]}, \\ g(\hat{x}(t_k), \hat{u}(t_k)) &\in \mathcal{C} && \forall\, k \in \mathbb{N}_{[0,N-1]}. \end{aligned}$$

Note that the OCP (13) already uses Lemma 1 to handle the state constraints. In the remaining section, we deduce suitable terminal costs φ, which guarantees both recursive feasibility of the OCP (13) and convergence to the desired set point of the MPC closed-loop. However, the construction of φ, which exploits two particular properties of the mobile robot, is non-standard.

The first property is uniform controllability in finite time. More precisely, we make use of the following claim, which is proven in the appendix (see Fig. 2 for the main idea).

Lemma 2: For given $x^\star \in \mathcal{X}$, $\Delta t \in (0, \infty)$, and $\eta \in [0.5, 1)$, let $\mathcal{D}$ be defined as in (11) and let $\mathcal{E}$ be given by

$$\mathcal{E} := \{x \in \mathbb{R}^3 \,|\, |x_3 - x_3^\star| \leq \pi\}. \tag{14}$$

Then, for every $\tilde{x} \in \mathcal{X} \cap \mathcal{E}$, there exist $\hat{u}(t_0), \ldots, \hat{u}(t_3) \in \mathcal{D}$ such that

$$\hat{x}(t_j) \in \mathcal{X} \cap \mathcal{E} \qquad \forall\, j \in \mathbb{N}_{[1,3]}, \tag{15a}$$

$$g(\hat{x}(t_j), \hat{u}(t_j)) \in \mathcal{C} \qquad \forall\, j \in \mathbb{N}_{[0,2]}, \tag{15b}$$

$$\hat{x}(t_4) = x^\star, \tag{15c}$$

where $\hat{x}(t_0) = \tilde{x}$ and $\hat{x}(t_{j+1}) = f(\hat{x}(t_j), \hat{u}(t_j))$, $j \in \mathbb{N}_{[0,3]}$.

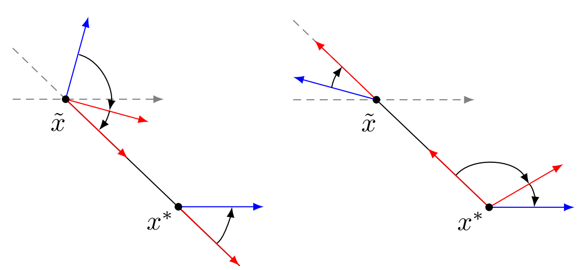

Fig. 2. Illustration of the simple manoeuvre used in the proof of Lemma 2 with two turning steps at the beginning (left) or at the end (right).

Lemma 2 shows that the robot can be steered from any state $\tilde{x} \in \mathcal{X} \cap \mathcal{E}$ to $x^\star$ in at most four steps if the only input constraint is given by $\mathcal{D}$ (instead of $\mathcal{U} \cap \mathcal{D}$). Roughly speaking, we first orientate the robot such that it either points to or away from the final position in the x_1-x_2-plane. Then, the robot drives to the final position $x^\star$ on a straight line (either forward or backward) before adjusting its orientation. The described maneuver may require four time steps since the constraint $\mathcal{D}$ may require that the initial or final rotation is split up into two steps. Later, we associate the state $\tilde{x}$ from Lemma 2 with the state $\hat{x}(t_N)$ of the OCP (13). A terminal trajectory satisfying the condition (15) will then be used to evaluate the terminal cost $\varphi(\hat{x}(t_N))$.

Remark 3: The set $\mathcal{E}$ from (14) acts as a terminal set. It is, however, important to note that $\mathcal{E}$ is structurally different to conventional terminal sets used in MPC, see, e.g., [3]. In fact, there are two remedies in order to alleviate this stipulation if needed. Firstly, from a practitioner's point of view, the orientations $x_3 + 2\pi j$, $j \in \mathbb{Z}$, all coincide. Thus, adapting the desired orientation such that $x^0 \in \mathcal{E}$ suffices. Alternatively, one could easily enlarge $\mathcal{E}$. In this case, however, additional steps might be required in Lemma 2 to allow longer turning phases.

We continue with the construction of the terminal cost φ. It remains to compute a meaningful cost for a terminal trajectory of the from (15). In this context, the main problem is that the four terminal steps may not satisfy the actual input constraints $\mathcal{U}$. As a consequence, simply using the stage cost (8) may not be meaningful. Instead we exploit the following property.

Lemma 4: Let $\bar{x} \in \mathbb{R}^3$, $\bar{u} \in \mathbb{R}^2$, and $\alpha \geq 0$. Then

$$\xi(t, \bar{x}, \alpha\bar{u}) = \xi(\alpha t, \bar{x}, \bar{u}) \tag{16}$$

holds for every $t \geq 0$.

Proof: Relation (16) immediately follows from (6) or, for the special case $\bar{u}_2 = 0$, from (7). ∎

Roughly speaking, property (16) says that we can drive through a given segment with arbitrary velocity without changing the resulting path. Now, let us assume that (at least) one move of the maneuvers from Lemma 2 violates the input constraints, i.e., there exists $j \in \mathbb{N}_{[0,3]}$ such that $\bar{u} := \hat{u}(t_j) \notin \mathcal{U}$ holds. Then, relation (16) tells us that we can drive the same path with a reduced velocity if we allow a longer time span. We will use this observation to define a stage cost for steps that may not satisfy the input constraints. To this end, we introduce the adapted stage cost

$$\ell_\varphi(\bar{x}, \bar{u}) := \ell(\bar{x}, \bar{u}) + \beta \Delta t \|\bar{u}\|_R^2$$

that contains an additional penalization of the input $\bar{u}$ based on some weighting factor $\beta > 0$. We will now use ℓ_φ to define the terminal cost φ based on the OCP

$$\varphi(\tilde{x}) := \min_{\boldsymbol{u}_4} \sum_{k=0}^{3} \ell_\varphi(\hat{x}(t_k), \hat{u}(t_k)) \tag{17}$$

$$\begin{aligned} \text{s.t.} \quad \hat{x}(t_0) &= \tilde{x} \\ \hat{x}(t_{k+1}) &= f(\hat{x}(t_k), \hat{u}(t_k)) && \forall\, k \in \mathbb{N}_{[0,3]}, \\ \hat{u}(t_k) &\in \mathcal{D} && \forall\, k \in \mathbb{N}_{[0,3]}, \\ \hat{x}(t_k) &\in \mathcal{X} \cap \mathcal{E} && \forall\, k \in \mathbb{N}_{[0,3]}, \\ g(\hat{x}(t_k), \hat{u}(t_k)) &\in \mathcal{C} && \forall\, k \in \mathbb{N}_{[0,3]}, \\ \hat{x}(t_4) &= x^\star. \end{aligned}$$

Clearly, this OCP can easily be integrated in (13), which leads to

$$
\begin{aligned}
V_N(x^0) = \min_{\boldsymbol{u}_{N+4}} & \sum_{k=0}^{N-1} \ell(\hat{x}(t_k), \hat{u}(t_k)) + \sum_{k=N}^{N+3} \ell_\varphi(\hat{x}(t_k), \hat{u}(t_k)) \\
\text{s.t.} \quad & \hat{x}(t_0) = x^0 \qquad (18)\\
& \hat{x}(t_{k+1}) = f(\hat{x}(t_k), \hat{u}(t_k)) \quad \forall k \in \mathbb{N}_{[0,N+3]}, \\
& \hat{u}(t_k) \in \mathcal{U} \cap \mathcal{D} \quad \forall k \in \mathbb{N}_{[0,N-1]}, \\
& \hat{u}(t_k) \in \mathcal{D} \quad \forall k \in \mathbb{N}_{[N,N+3]}, \\
& \hat{x}(t_k) \in \mathcal{X} \cap \mathcal{E} \quad \forall k \in \mathbb{N}_{[0,N+3]}, \\
& g(\hat{x}(t_k), \hat{u}(t_k)) \in \mathcal{C} \quad \forall k \in \mathbb{N}_{[0,N+3]}, \\
& \hat{x}(t_{N+4}) = x^\star.
\end{aligned}
$$

Note that the additional state constraints $\mathcal{E}$ are applied to every step k in order to prepare the stability proof in the next section.

We like to point out that the set $\mathcal{D}$ can be removed in the input constraints on $\hat{u}(t_k)$, $k \in \mathbb{N}_{[0,N-1]}$ for sufficiently fast sampling, which shows that no additional input constraints are imposed except for the construction of the terminal costs. Moreover, the value function V_∞ is finite for any given initial state $x^0 \in \mathcal{X} \cap \mathcal{E}$ since the robot may simply stay at x^0 until time t_N and, then, move to $x^\star$ ignoring the input constraints $\mathcal{U}$ using Lemma 2. Since $\mathcal{C}$ is compact, this results in $V_\infty(x^0) < \infty$. In conclusion, initial feasibility is satisfied for every $x^0 \in \mathcal{X}$.

V. Recursive feasibility and convergence

It remains to prove that the MPC closed-loop is recursively feasible and converges to the desired set point $x^\star$. To this end, let us first specify the MPC feedback law $\mu : \mathcal{X} \to \mathcal{U}$ as

$$\mu(\bar{x}) = \hat{u}^*(t_0), \tag{19}$$

where $\hat{u}^*(t_0)$ is the first element of the optimal control sequence $\boldsymbol{u}^*_{N+4}$ of the OCP (18) for $x^0 = \bar{x}$. We now study the closed-loop behaviour of the system

$$x(t_{k+1}) = f(x(t_k), \mu(x(t_k))). \tag{20}$$

As a preparation, we provide the following lemma.

Lemma 5: Let $\Delta t \in (0, \infty)$ and $\eta \in [0.5, 1)$ be given and let $\mathcal{D}$ be defined as in (11). Further, let $q, r \in (0, \infty)$ be defined by

$$r := \min_{u \in \partial\mathcal{U}} \|u\|_R^2 \quad \text{and} \quad q := \max_{x \in \mathcal{X} \cap \mathcal{E}} \|x - x^\star\|_Q^2 \tag{21}$$

and choose $\beta \geq q/r$. Then, for every $(\bar{x}, \bar{u}) \in \mathcal{X} \times \mathcal{D} \setminus \mathcal{U}$ satisfying $f(\bar{x}, \bar{u}) \in \mathcal{X}$ and $g(\bar{x}, \bar{u}) \in \mathcal{C}$, the inequality

$$\ell(\bar{x}, \alpha\bar{u}) + \ell_\varphi(f(\bar{x}, \alpha\bar{u}), (1-\alpha)\bar{u}) \leq \ell_\varphi(\bar{x}, \bar{u}) \tag{22}$$

holds with $\alpha \in (0, 1)$ satisfying $\alpha\bar{u} \in \partial\mathcal{U}$.

Proof: Since $\bar{u}$ is not contained in the compact set $\mathcal{U}$, which contains the origin in its interior, $\alpha \in (0, 1)$ holds. Next, we evaluate the three terms in (22). Using Lemma 4 for the first term, we find

$$
\begin{aligned}
\ell(\bar{x}, \alpha\bar{u}) &= \int_0^{\Delta t} \|\underbrace{\xi(t, \bar{x}, \alpha\bar{u})}_{=\xi(\alpha t, \bar{x}, \bar{u})} - x^\star\|_Q^2 \,\mathrm{d}t + \alpha^2 \Delta t \, \|\bar{u}\|_R^2 \\
&= \frac{1}{\alpha} \int_0^{\alpha\Delta t} \|\xi(\tau, \bar{x}, \bar{u}) - x^\star\|_Q^2 \,\mathrm{d}\tau + \alpha^2 \Delta t \, \|\bar{u}\|_R^2.
\end{aligned}
$$

Similarly, for the second term in (22), we obtain

$$
\begin{aligned}
&\ell_\varphi(f(\bar{x}, \alpha\bar{u}), (1-\alpha)\bar{u}) - (1+\beta)(1-\alpha)^2 \Delta t \, \|\bar{u}\|_R^2 \\
&= \int_0^{\Delta t} \|\xi(t, f(\bar{x}, \alpha\bar{u}), (1-\alpha)\bar{u}) - x^\star\|_Q^2 \,\mathrm{d}t \\
&= \frac{1}{1-\alpha} \int_0^{(1-\alpha)\Delta t} \|\xi(\tau, f(\bar{x}, \alpha\bar{u}), \bar{u}) - x^\star\|_Q^2 \,\mathrm{d}\tau \\
&= \frac{1}{1-\alpha} \int_{\alpha\Delta t}^{\Delta t} \|\xi(\tau, \bar{x}, \bar{u}) - x^\star\|_Q^2 \,\mathrm{d}\tau.
\end{aligned}
$$

Finally, the third term evaluates to

$$\ell_\varphi(\bar{x}, \bar{u}) = \int_0^{\Delta t} \|\xi(t, \bar{x}, \bar{u}) - x^\star\|_Q^2 \,\mathrm{d}t + (1+\beta) \, \Delta t \|\bar{u}\|_R^2.$$

Based on the previous relations, it remains to show

$$
\begin{aligned}
&\left(\frac{1-\alpha}{\alpha}\right) \int_0^{\alpha\Delta t} \|\xi(\tau, \bar{x}, \bar{u}) - x^\star\|_Q^2 \,\mathrm{d}\tau \\
&+ \left(\frac{\alpha}{1-\alpha}\right) \int_{\alpha\Delta t}^{\Delta t} \|\xi(\tau, \bar{x}, \bar{u}) - x^\star\|_Q^2 \,\mathrm{d}\tau \\
&\leq \left((1+\beta)(1-(1-\alpha)^2) - \alpha^2\right) \Delta t \|\bar{u}\|_R^2.
\end{aligned} \tag{23}
$$

This allows us to overestimate the l.h.s. of (23) by

$$\frac{1-\alpha}{\alpha} \int_0^{\alpha\Delta t} q \,\mathrm{d}\tau + \frac{\alpha}{1-\alpha} \int_{\alpha\Delta t}^{\Delta t} q \,\mathrm{d}\tau = \Delta t \, q.$$

and to underestimate the r.h.s. of (23) firstly using $\|\bar{u}\|_R^2 \geq r/\alpha^2$ and, then, $\alpha \in (0, 1)$, by

$$(\beta(2-\alpha) + 2(1-\alpha)) \frac{r\Delta t}{\alpha} \geq \beta r \Delta t.$$

In conclusion, Inequality (22) holds for $\beta \geq \frac{q}{r}$, which completes the proof. ∎

Based on Lemma 5, we can easily prove convergence of the closed-loop trajectory (20) to the desired target state with domain of attraction $\mathcal{X} \cap \mathcal{E}$.

Theorem 6: Let $\eta \in [0.5, 1)$, $\Delta t \in (0, \infty)$, and $N \in \mathbb{N}_{\geq 1}$ be given. Furthermore, let $\mathcal{D}$, $\mathcal{E}$, and β be defined as in (11), (14) and Lemma 5, respectively. Then, for each $x^0 \in \mathcal{X} \cap \mathcal{E}$ the trajectory of the controlled system (20) is feasible and converges to the equilibrium $x^\star$.

Proof: Firstly, we point out that, for every $x^0 \in \mathcal{X} \cap \mathcal{E}$, Lemma 2 implies feasibility of the input sequence

$$\boldsymbol{u}_{N+4} := (0, \ldots, 0, \hat{u}(t_N), \ldots, \hat{u}(t_{N+3})). \tag{24}$$

Then, the constraint $\hat{x}(t_k) \in \mathcal{X} \cap \mathcal{E}$, $k = 1$, implies that the successor state $f(x^0, \mu(x^0)) = \hat{x}^*(t_1)$ is contained in the set $\mathcal{X} \cap \mathcal{E}$ for every minimizer $\boldsymbol{u}^*_{N+4}$ of (18), which shows recursive feasibility of the MPC scheme.

It remains to prove convergence of the controlled state trajectory. Firstly, we show that the inequality

$$V_N(f(x^0, \mu(x^0))) \leq V_N(x^0) - \ell(x^0, \hat{u}^*(t_0)) \tag{25}$$

holds for every $x^0 \in \mathcal{X} \cap \mathcal{E}$. To this end, we construct a feasible input sequence $\tilde{\boldsymbol{u}}_{N+4}$ for the successor state $f(x^0, \mu(x^0))$ and show that the associated cost underestimates the r.h.s. of (25). For the construction of $\tilde{\boldsymbol{u}}_{N+4}$, we distinguish whether $\hat{u}^*(t_N)$ is contained in $\mathcal{U}$ or not. If $\hat{u}^*(t_N) \in \mathcal{U}$ holds, we simply choose

$$\tilde{\boldsymbol{u}}_{N+4} := (\hat{u}^*(t_1), \dots, \hat{u}^*(t_{N+3}), 0),$$

which yields $\ell_\varphi(\tilde{x}(t_{N+3}), \tilde{u}(t_{N+3})) = 0$ by construction. Since, in addition, $\ell(\hat{x}^*(t_N), \hat{u}^*(t_N)) \le \ell_\varphi(\hat{x}^*(t_N), \hat{u}^*(t_N))$ holds by definition of ℓ_φ, (25) is satisfied.

Now, let $\hat{u}^*(t_N) \notin \mathcal{U}$ hold. Then, we construct a feasible input sequence based on Lemma 5. We consequently choose $\alpha \in (0,1)$ such that $\alpha \hat{u}^*(t_N) \in \partial\mathcal{U}$ and define $\tilde{\boldsymbol{u}}_{N+4}$ by

$$(\hat{u}^*(t_1),\dots,\hat{u}^*(t_{N-1}),\alpha\hat{u}^*(t_N),(1-\alpha)\hat{u}^*(t_N),\hat{u}^*(t_{N+1}),\dots,\hat{u}^*(t_{N+3})).$$

Here, Inequality (25) follows from Inequality (22) of Lemma 5 applied with $\bar{x} = \hat{x}^*(t_N)$ and $\bar{u} = \hat{u}^*(t_N)$.

Two straightforward observations are missing to prove convergence. First, the feasible input sequence from (24) implies boundedness of $V_N(x^0)$ on $\mathcal{X} \cap \mathcal{E}$. Second, the stage cost $\ell(\bar{x}, \bar{u})$ can be underestimated by some class $\mathcal{K}$ function γ in the sense that $\ell(\bar{x}, \bar{u}) \ge \gamma(\|\bar{x} - x^\star\|_2)$ holds for all $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$. Now, using standard arguments, we can show that, for every $x^0 \in \mathcal{X} \cap \mathcal{E}$ and every $\epsilon > 0$, there exists a $k^\star$ (depending on x^0) such that $\|x(t_k) - x^\star\|_2 < \epsilon$ holds for every $k \ge k^\star$, which proves the assertion. ■

VI. Numerical Experiments

We illustrate the efficiency of our control scheme with some numerical simulations. In this context, we consider $b = d = 0.1\,\mathrm{m}$, $\omega_{\max} = 5\,\mathrm{s}^{-1}$, and $\Delta t = 0.5\,\mathrm{s}$. Moreover, we define the state constraints $\mathcal{C} = \{p \in \mathbb{R}^2 \,|\, \|p\|_\infty \le 2\,\mathrm{m}\}$, choose $\eta = 0.75$, and consider the weighting matrices

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1.5 & 0 \\ 0 & 0.5 \end{pmatrix}.$$

For these parameters, we easily compute $r \approx 0.0931$ and $q \approx 33.9739$ as in (21). The additional input weighting β for the terminal trajectory should thus be chosen larger than

$$\frac{q}{r} \approx 365.1064$$

according to Theorem 6. For our simulations, we choose $\beta = 10$ significantly smaller in order to illustrate that the condition on β in Theorem 6 is conservative.

Now, we consider the three initial states

$$x^0 \in \left\{ \begin{pmatrix} -1.5\,\mathrm{m} \\ 1.5\,\mathrm{m} \\ 0.5\,\pi \end{pmatrix}, \begin{pmatrix} -1.5\,\mathrm{m} \\ -1.5\,\mathrm{m} \\ \pi \end{pmatrix}, \begin{pmatrix} 1.5\,\mathrm{m} \\ -1.5\,\mathrm{m} \\ 0 \end{pmatrix} \right\}, \qquad (26)$$

that should all be steered to $x^\star = (1.5\,\mathrm{m}\ 1.5\,\mathrm{m}\ 0)^\top$. Clearly, the third initial state corresponds to a parallel-parking situation. We simulate the MPC closed-loop for the prediction horizons $N = 1$ and $N = 5$ and obtain the numerical results illustrated in Figures 3 and 4. The final position $x^\star$ is reached for every initial state and the optimal value function is decreasing as shown by Theorem 6. As apparent from Figure 3, the state trajectories of the MPC closed-loop are very similar for the horizons $N = 1$ and $N = 5$, which can be explained as follows. The comparison of the trajectories in Figures 3.(a) and 3.(b) shows that the predicted trajectories at $k = 0$ already provide good approximations of the final trajectories (at least for the first and third initial state). This observation implies that the designed cost function successfully captures the behavior of the system although some of the predicted inputs permissibly violate the input constraints $\mathcal{U}$. However, despite the similarity of the state trajectories, it is evident from Figure 4.(a) that the optimal value functions change with N (as expected).

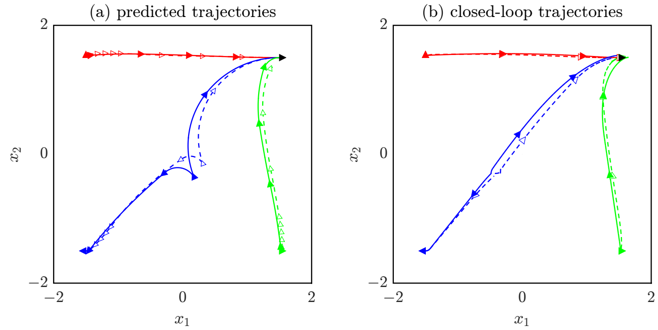

Fig. 3. Illustration of (a) the predicted trajectories at $k = 0$ and (b) the closed-loop trajectories from the three initial states (26). In (a), the robots position is illustrated at every of the $N + 4$ predicted steps. In (b), the robots position is illustrated every 10 time steps. In (a) and (b), trajectories resulting for $N = 1$ ($N = 5$) are depicted with solid (dashed) lines.

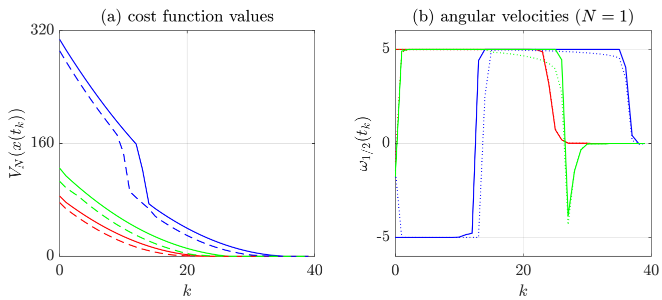

Fig. 4. Illustration of (a) optimal cost function values and (b) control inputs in terms of $\omega_{1/2}$ (as in (2)) along the trajectories in Figure 3.(b). In (a), solid and dashed lines again refer to the cases $N = 1$ and $N = 5$, respectively. In (b), solid and dotted lines refer to ω_1 and ω_2, respectively.

VII. Conclusions and Future Work

We presented a novel MPC scheme for a mobile robot. The tailored control scheme builds on a special terminal cost function. For its construction, we exploited finite time controllability of the unconstrained system and the particular property (16) that allows to drive along a path with different velocities. Although our approach is tailored for the studied mobile robot, we believe that the underlying concept can be generalized to other systems. In particular, systems that can be accurately described by kinematic models are likely to offer properties similar to (16). We stress, however, that Lemma 4 in its current form does not hold for, e.g., bicycle models. Nevertheless, the design of terminal cost functions similar to the one proposed here should be possible for chained systems (as in [22]) and future research will follow this direction.

Appendix

Proof of Lemma 1: Let us first note that the time-derivative of the trajectory $\xi(\cdot,\bar{x},\bar{u})$ at time t is

$$\frac{\partial \xi(t,\bar{x},\bar{u})}{\partial t} = \begin{pmatrix} \bar{u}_1 \cos(\bar{x}_3 + t\,\bar{u}_2) \\ \bar{u}_1 \sin(\bar{x}_3 + t\,\bar{u}_2) \\ \bar{u}_2 \end{pmatrix}. \qquad (27)$$

Hence, the tangents to the trajectory at times $t = 0$ and $t = \Delta t$ can be described as

$$\bar{x}+\lambda \begin{pmatrix} \bar{u}_1 \cos(\bar{x}_3) \\ \bar{u}_1 \sin(\bar{x}_3) \\ \bar{u}_2 \end{pmatrix} \text{ and } f(\bar{x},\bar{u})-\mu \begin{pmatrix} \bar{u}_1 \cos(\bar{x}_3+\Delta t\,\bar{u}_2) \\ \bar{u}_1 \sin(\bar{x}_3+\Delta t\,\bar{u}_2) \\ \bar{u}_2 \end{pmatrix}.$$

Next, we project both tangents to the x_1-x_2-plane and compute an intersection point of the projections. Clearly, an intersection can be derived from the equation

$$\bar{u}_1 \begin{pmatrix} \cos(\bar{x}_3) & \cos(\bar{x}_3 + \Delta t\,\bar{u}_2) \\ \sin(\bar{x}_3) & \sin(\bar{x}_3 + \Delta t\,\bar{u}_2) \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} f_1(\bar{x},\bar{u}) - \bar{x}_1 \\ f_2(\bar{x},\bar{u}) - \bar{x}_2 \end{pmatrix}.$$

Evaluating this linear system of equations yields

$$\lambda = \frac{\Delta t}{2} \operatorname{tanc}\left(\frac{\Delta t\,\bar{u}_2}{2}\right)$$

and, thus, that $g(\bar{x},\bar{u})$ as defined by (12) is an intersection point. Now, due to $\bar{u} \in \mathcal{D}$, the projected trajectory is such that the set $\{(\,\xi_1(t,\bar{x},\bar{u}) \quad \xi_2(t,\bar{x},\bar{u})\,)^\top \mid t \in [0,\Delta t]\}$ is contained in

$$\operatorname{conv}\left\{ \begin{pmatrix} \bar{x}_1 & \bar{x}_2 \end{pmatrix}^\top,\, \begin{pmatrix} f_1(\bar{x},\bar{u}) & f_2(\bar{x},\bar{u}) \end{pmatrix}^\top,\, g(\bar{x},\bar{u}) \right\}$$

(cf. Fig. 1). This observation proves the assertion since it shows that $\bar{x} \in \mathcal{X}$, $\bar{u} \in \mathcal{D}$, $f(\bar{x},\bar{u}) \in \mathcal{X}$, and $g(\bar{x},\bar{u}) \in \mathcal{C}$ implies $\xi(t,\bar{x},\bar{u}) \in \mathcal{X}$ for all $t \in [0,\Delta t]$. ■

Proof of Lemma 2: Firstly, let us define the angle $\gamma_0 := \operatorname{atan2}(x_2^\star - \tilde{x}_2, x_1^\star - \tilde{x}_1) \in (-\pi,\pi]$. Then, there exist $j \in \mathbb{Z}$ such that the conditions

$$\max\{|\gamma_j - x_3^\star|, |\gamma_j - \tilde{x}_3|\} \le \pi, \qquad (28a)$$

$$\min\{|\gamma_j - x_3^\star|, |\gamma_j - \tilde{x}_3|\} \le \pi/2 \le \eta\pi \qquad (28b)$$

are satisfied with $\gamma_j := \gamma_0 + j\pi$. Keep in mind that we have $|\tilde{x}_3 - x_3^\star| \le \pi$ by assumption. Next, assume that we have chosen an odd j such that

$$0 < \tilde{x}_3 - \gamma_j \le \pi/2 < \gamma_j - x_3^\star \le \pi$$

holds, see Fig. 2 (right) for an illustration. Then, we select the inputs

$$\hat{u}(t_0) = \frac{1}{\Delta t}\begin{pmatrix} 0 \\ \gamma_j - \tilde{x}_3 \end{pmatrix}, \quad \hat{u}(t_1) = \frac{1}{\Delta t}\begin{pmatrix} -\delta \\ 0 \end{pmatrix},$$

$$\hat{u}(t_2) = \frac{1}{\Delta t}\begin{pmatrix} 0 \\ -\pi/2 \end{pmatrix}, \qquad \hat{u}(t_3) = \frac{1}{\Delta t}\begin{pmatrix} 0 \\ x_3^\star - \gamma_j + \pi/2 \end{pmatrix},$$

with the distance δ defined as $\sqrt{(x_1^\star - \tilde{x}_1)^2 - (x_2^\star - \tilde{x}_2)^2}$. Note that all controls are taken from $\mathcal{D}$ since $\eta \in [0.5, 1)$ holds. In other words, we first rotate the robot such that it points in direction γ_j. Then, we drive backwards to the final state. There, we need two additional rotations to obtain the final orientation $x_3^\star$. Obviously, a similar sequence can be found for any $j \in \mathbb{Z}$ satisfying (28). Thereby, j being even or odd decides whether we drive to the final state in forward or backward direction (see the two cases in Fig. 2). ■

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