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Quadratic Stabilization of Discrete-Time Bilinear Control Systems*

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Abstract— **A stabilization problem for discrete-time bilinear control systems is considered. Using the linear matrix inequality technique and the concept of quadratic Lyapunov functions, an approach is proposed to the construction of the so-called stabilizability ellipsoid such that the trajectories of the closed-loop system starting from any point inside this ellipsoid asymptotically tend to the origin.**

The approach allows for an efficient construction of nonconvex approximations to stabilizability domains of bilinear control systems. The obtained results can be extended to various robust statements of the problem, to bilinear systems with many-dimensional control, and to bilinear control systems subjected to exogenous disturbances.

I. Introduction

Problems concerned with the stabilization of bilinear control systems are considered in numerous papers. Many of them appeared after the publication of the famous monograph [1], also see [2], [3], [4]. There exist various approaches to the solution; e.g., linear transformations are constructed that convert the original bilinear system into a linear one [5], [6]; another direction is associated with the use of observers [7]; in some papers a linear stabilizing control for bilinear systems is found from certain sufficient conditions of stability of quadratic differential equations [8], [9]. In some of the publications, the ellipsoidal approach is used for the problem of interest [7]; there exist numerous papers devoted to the design of nonlinear stabilizing control laws for bilinear systems [10], [11], [12], [13], etc.

Some of recent publications are devoted to discrete-time bilinear control systems, see [14], [15]; most of them are restricted to the consideration of the controllability problem. In [16], [17], quadratic Lyapunov functions are applied to bilinear control systems via the linear matrix inequalitie technique.

However the current paper provides essentially different results. Apart form a somewhat different statement of the problem, the problem of maximization of the stabilizability ellipsoid via the volumetric is formulated. Moreover, in contrast to the above mentioned papers, we formulate and solve a new problem of finding the so-called stabilizability domain. Finally, a special technique based on a modification of Petersen's lemma is exploited.

In this paper, using the linear matrix inequality technique [18] and a modification of the well-known Petersen's lemma, a regular approach to the stabilization of discrete-time bilinear control systems via static linear state feedbacks is proposed. In the state space of the system, an ellipsoid (a so-called stabilizability ellipsoid) is constructed, such that any trajectory starting inside this ellipsoid asymptotically tends to the origin. A natural extension of this approach allows for a simple and efficient construction of inner approximations to the stabilizability domains of discrete-time bilinear systems. We note that the continuous-time results concerned with the quadratic stabilization of the bilinear systems were obtained in [19].

We stress that, although the proposed approach is based on solving convex optimization problems, it allows for the construction of nonconvex approximations to stabilizability domains of bilinear control systems.

In this paper, we consider the discrete-time bilinear control systems with scalar control input only; however, the proposed approach can be generalized to the case of many-dimensional control input. The only difference is that the derivations get more cumbersome, while the substantial part remains the same.

From now onwards, $^\top$ denotes the transposition sign, $\|\cdot\|_2$ is the spectral form of a matrix, I is the identity matrix of appropriate dimension, and all matrix inequalities are understood in the sense of sign-definiteness.

II. Statement of the Problem

Consider the discrete-time bilinear control system

$$x_{\ell+1} = Ax_\ell + bu_\ell + Dx_\ell u_\ell, \tag{1}$$

with the initial point x_0, phase state $x_\ell \in \mathbb{R}^n$, and scalar control input $u_\ell \in \mathbb{R}$, where $A, D \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$.

We will seek for a linear static state feedback

$$u_\ell = k^\top x_\ell, \quad k \in \mathbb{R}^n, \tag{2}$$

which quadratically stabilizes the discrete-time bilinear system (1) inside a certain ellipsoid

$$\mathscr{E} = \left\{ x_\ell \in \mathbb{R}^n : \; x_\ell^\top P^{-1} x_\ell \le 1 \right\}, \quad P \succ 0,$$

with the center at the origin. In other words, the trajectory of the system (1) embraced with feedback (2) asymptotically tends to zeros for any initial condition x_0 inside the ellipsoid $\mathscr{E}$.

The ellipsoid $\mathscr{E}$ is referred to as a *stabilizability ellipsoid*, the linear static state feedbacks (2) are assumed.

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III. Auxiliary Result: Petersen Lemma

The so-called *Petersen's lemma* [20] gives an efficient tool for solving various problems in robust stabilization and robust control.

Lemma 1 (Petersen): Let $G = G^\top \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n\times p}$, and $N \in \mathbb{R}^{n\times q}$. The inequality

$$G + M\Delta N^\top + N\Delta^\top M^\top \prec 0$$

holds for all

$$\Delta \in \mathbb{R}^{p\times q}: \ \|\Delta\|_2 \le 1$$

if and only if there exists a real number ε such that

$$\begin{pmatrix} G+\varepsilon MM^\top & N \\ N^\top & -\varepsilon I \end{pmatrix} \prec 0.$$

From now and onward, all matrix inequalities are understood in the sign-definiteness sense.

Hence, Petersen's lemma allows to reduce checking the sign-definiteness of the matrix family

$$G + M\Delta N^\top + N\Delta^\top M^\top$$

with matrix uncertainty Δ to the feasibility of a matrix inequality with one scalar variable ε.

Certain generalizations of Petersen's lemma were obtained in [21], [22]; the modification below relates to the vector uncertainty satisfying an ellipsoidal constraint.

Lemma 2: Let $G = G^\top \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n\times q}$, $N \in \mathbb{R}^{n}$, and $0 \prec Q = Q^\top \in \mathbb{R}^{q\times q}$. The matrix inequality

$$G + M\delta N^\top + N\delta^\top M^\top \prec 0$$

holds for all

$$\delta \in \mathbb{R}^{q}: \quad \delta^\top Q\delta \le 1$$

if and only if there exists a real number ε such that

$$\begin{pmatrix} G & M & N \\ M^\top & -\varepsilon Q & 0 \\ N^\top & 0 & -\frac{1}{\varepsilon} I \end{pmatrix} \prec 0.$$

This result will be essentially used in the exposition of this paper. It is important to note that the estimate

$$G + M\delta N^\top + N\delta^\top M^\top \preceq G + \frac{1}{\varepsilon} MQ^{-1}M^\top + \varepsilon NN^\top \tag{3}$$

holds for all admissible uncertainties δ and for all $\varepsilon > 0$.

IV. Main Result

Embracing the system (1) with the linear static state feedback (2), we obtain the closed-loop system

$$x_{\ell+1} = A_c x_\ell + Dx_\ell k^\top x_\ell \tag{4}$$

with the closed-loop matrix $A_c = A + bk^\top$.

We introduce the quadratic form

$$V(x) = x^\top Qx, \quad Q \succ 0,$$

which is the Lyapunov function for the closed-loop system (4) if

$$V(x_{\ell+1}) < V(x_\ell), \quad \ell = 1, 2, \dots$$

Calculating the value

$$\begin{aligned} V(x_{\ell+1}) &= x_{\ell+1}^\top Q x_{\ell+1} \\ &= (A_c x_\ell + Dx_\ell k^\top x_\ell)^\top Q (A_c x_\ell + Dx_\ell k^\top x_\ell) \\ &= x_\ell^\top A_c^\top QA_c x_\ell + x_\ell^\top A_c^\top QDx_\ell k^\top x_\ell \\ &\quad + x_\ell^\top k x_\ell^\top D^\top QA_c x_\ell + x_\ell^\top k x_\ell^\top D^\top QDx_\ell k^\top x_\ell \\ &= x_\ell^\top (A_c^\top QA_c + A_c^\top QDx_\ell k^\top \\ &\quad + kx_\ell^\top D^\top QA_c + kx_\ell^\top D^\top QDx_\ell k^\top) x_\ell \end{aligned}$$

along the trajectories of the system, we arrive at the condition

$$\begin{aligned} x_\ell^\top (A_c^\top QA_c + A_c^\top QDx_\ell k^\top + kx_\ell^\top D^\top QA_c \\ + kx_\ell^\top D^\top QDx_\ell k^\top) x_\ell < x_\ell^\top Q x_\ell. \end{aligned}$$

Thus, the quadratic form $V(x)$ serves as a Lyapunov function for the closed-loop discrete-time system (4) if the condition

$$\begin{aligned} A_c^\top QA_c + A_c^\top QDx_\ell k^\top + kx_\ell^\top D^\top QA_c \\ + kx_\ell^\top D^\top QDx_\ell k^\top \prec Q \end{aligned}$$

holds.

We reformulate this condition in the following matrix form:

$$\begin{pmatrix} A_c^\top QA_c - Q \\ +A_c^\top QDx_\ell k^\top + kx_\ell^\top D^\top QA_c & kx_\ell^\top D^\top Q \\ QDx_\ell k^\top & -Q \end{pmatrix} \prec 0,$$

or

$$\begin{aligned} \begin{pmatrix} A_c^\top QA_c - Q & 0 \\ 0 & -Q \end{pmatrix} + \begin{pmatrix} A_c^\top QD \\ QD \end{pmatrix} x_\ell \begin{pmatrix} k^\top & 0 \end{pmatrix} \\ + \begin{pmatrix} k \\ 0 \end{pmatrix} x_\ell^\top \begin{pmatrix} D^\top QA_c & D^\top Q \end{pmatrix} \prec 0. \end{aligned} \tag{5}$$

We require that the matrix inequality (5) hold for all x_ℓ from the ellipsoid

$$\begin{aligned} \mathscr{E} &= \{x_\ell \in \mathbb{R}^n: \quad V(x_\ell) \le 1\} \\ &= \{x_\ell \in \mathbb{R}^n: \quad x_\ell^\top Q x_\ell \le 1\}. \end{aligned}$$

At that, inside the ellipsoid $\mathscr{E}$, the quadratic form $V(x_\ell)$ is a Lyapunov function for the closed-loop system.

Using Lemma 2, we obtain the equivalent matrix inequality

$$\begin{pmatrix} A_c^\top QA_c - Q & 0 & A_c^\top QD & k \\ 0 & -Q & QD & 0 \\ D^\top QA_c & D^\top Q & -\varepsilon Q & 0 \\ k^\top & 0 & 0 & -\frac{1}{\varepsilon} I \end{pmatrix} \prec 0. \tag{6}$$

Note that the feasibility of the matrix inequality (6) implies (see [23]) the Schur stability of the matrix A_c. It means that the desired feedback (2) stabilizes the linear discrete-time control system

$$x_{\ell+1} = Ax_\ell + bu_\ell. \tag{7}$$

Applying Schur lemma [24] to (6) we obtain the matrix inequality

$$\begin{pmatrix} A_c^\top QA_c - Q & A_c^\top QD & k \\ D^\top QA_c & D^\top QD - \varepsilon Q & 0 \\ k^\top & 0 & -\frac{1}{\varepsilon} I \end{pmatrix} \prec 0$$

which can be represented in the form

$$\begin{pmatrix}-Q & 0 & k\\ 0 & -\varepsilon Q & 0\\ k^\top & 0 & -\frac{1}{\varepsilon}I\end{pmatrix} + \begin{pmatrix}A_c^\top\\ D^\top\\ 0\end{pmatrix} Q \begin{pmatrix}A_c & D & 0\end{pmatrix} \prec 0.$$

Hence, by Schur lemma we arrive at

$$\begin{pmatrix}-Q & 0 & k & A_c^\top\\ 0 & -\varepsilon Q & 0 & D^\top\\ k^\top & 0 & -\frac{1}{\varepsilon}I & 0\\ A_c & D & 0 & -Q^{-1}\end{pmatrix} \prec 0.$$

Denoting $P = Q^{-1}$ and pre- and post-multiplying the inequality above by the matrix

$$\begin{pmatrix}P & 0 & 0 & 0\\ 0 & P & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{pmatrix},$$

we obtain the condition

$$\begin{pmatrix}-P & 0 & Pk & P(A+bk^\top)^\top\\ 0 & -\varepsilon P & 0 & PD^\top\\ k^\top P & 0 & -\frac{1}{\varepsilon}I & 0\\ (A+bk^\top)P & DP & 0 & -P\end{pmatrix} \prec 0.$$

Next, introducing the auxiliary vector variable

$$y = Pk,$$

we eliminate k. Hence, since $P \succ 0$, the vector k can be recovered uniquely as

$$k = P^{-1}y.$$

As a result, we arrive at the matrix inequality

$$\begin{pmatrix}-P & 0 & y & PA^\top + yb^\top\\ 0 & -\varepsilon P & 0 & PD^\top\\ y^\top & 0 & -\frac{1}{\varepsilon}I & 0\\ AP+by^\top & DP & 0 & -P\end{pmatrix} \prec 0$$

linear in the matrix variable P and the vector variable y, with the scalar parameter ε.

Hence, we obtained the following result.

Theorem 1: Let the matrix P and the vector y satisfy the matrix inequality

$$\begin{pmatrix}-P & 0 & y & PA^\top + yb^\top\\ 0 & -\varepsilon P & 0 & PD^\top\\ y^\top & 0 & -\frac{1}{\varepsilon}I & 0\\ AP+by^\top & DP & 0 & -P\end{pmatrix} \prec 0$$

for a certain value of the parameter ε.

Then the linear static state feedback (2) with the gain matrix

$$k = P^{-1}y$$

stabilizes system (1) inside the ellipsoid

$$\mathscr{E} = \left\{x \in \mathbb{R}^n : \quad x^\top P^{-1} x \le 1\right\}.$$

Moreover, the quadratic form

$$V(x) = x^\top P^{-1} x$$

is a Lyapunov function for the closed-loop system (4) inside the ellipsoid $\mathscr{E}$.

It is natural to maximize the stabilizability ellipsoid via one or another criterion. In this paper we adopt the volumetric criteria, this leads us to the following statement.

Corollary 1: Let $\widehat{P}$ and $\widehat{y}$ be the solution of the convex optimization problem

$$\max \log\det P$$

subject to the constraint

$$\begin{pmatrix}-P & 0 & y & PA^\top + yb^\top\\ 0 & -\varepsilon P & 0 & PD^\top\\ y^\top & 0 & -\frac{1}{\varepsilon}I & 0\\ AP+by^\top & DP & 0 & -P\end{pmatrix} \prec 0 \qquad (8)$$

with respect to the matrix variable $P = P^\top \in \mathbb{R}^{n\times n}$, the vector variable $y \in \mathbb{R}^n$, and the scalar parameter ε.

Then the ellipsoid

$$\widehat{\mathscr{E}} = \left\{x \in \mathbb{R}^n : \quad x^\top \widehat{P}^{-1} x \le 1\right\}$$

is a stabilizability ellipsoid for the closed-loop system (1), (2) with the gain matrix

$$\widehat{k} = \widehat{P}^{-1}\widehat{y}.$$

We now make an important technical comment. In Corollary 1, the objective function is minimized under the *strict* constraint (8); in order to solve a respective well-posed problem, we consider the nonstrict matrix inequality

$$\begin{pmatrix}-\mu P & 0 & y & PA^\top + yb^\top\\ 0 & -\varepsilon P & 0 & PD^\top\\ y^\top & 0 & -\frac{1}{\varepsilon}I & 0\\ AP+by^\top & DP & 0 & -P\end{pmatrix} \preceq 0 \qquad (9)$$

for a certain $0 < \mu < 1$, and replace constraint (8) with this condition.

The matrix inequality (9) is equivalent to

$$\begin{pmatrix}A_c^\top Q A_c - \mu Q & 0 & A_c^\top Q D & k\\ 0 & -Q & QD & 0\\ D^\top Q A_c & D^\top Q & -\varepsilon Q & 0\\ k^\top & 0 & 0 & -\frac{1}{\varepsilon}I\end{pmatrix} \preceq 0.$$

Using (3), we obtain

$$\begin{aligned}A_c^\top Q A_c + A_c^\top Q D x_\ell k^\top + k x_\ell^\top D^\top Q A_c&\\ + k x_\ell^\top D^\top Q D x_\ell k^\top &\preceq \mu Q,\end{aligned}$$

therefore

$$\begin{aligned}V(x_{\ell+1}) = x_\ell^\top (A_c^\top Q A_c + A_c^\top Q D x_\ell k^\top&\\ + k x_\ell^\top D^\top Q A_c + k x_\ell^\top D^\top Q D x_\ell k^\top) x_\ell&\\ \le x_\ell^\top (\mu Q) x_\ell = \mu V(x_\ell).&\end{aligned}$$

We obtain a simple estimate for the decay rate of Lyapunov function:

$$V(x_\ell) \le \mu^\ell V(x_0).$$

Finally, from (9) it follows that

$$(A+bk^\top)P(A+bk^\top)^\top \prec \mu P,$$

that is,

$$\Big(\frac{A+bk^\top}{\sqrt{\mu}}\Big)P\Big(\frac{A+bk^\top}{\sqrt{\mu}}\Big)^\top \prec P.$$

Thus we conclude that the spectral radius of the matrix $A_c = A+bk^\top$ and therefore, of the closed-loop system (7) is knowingly less than $\sqrt{\mu}$.

Example 1: Consider the discrete-time bilinear system (1) with matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 0{,}5 & 0{,}3 \\ -0{,}3 & 0{,}5 \end{pmatrix}.$$

By Corollary 1, we obtain the matrix

$$\widehat{P} = \begin{pmatrix} 1{,}4062 & 0{,}5314 \\ 0{,}5314 & 0{,}4248 \end{pmatrix},$$

of the stabilizability ellipsoid, and the gain matrix

$$\widehat{k} = \begin{pmatrix} -1 \\ 1{,}6209 \end{pmatrix}.$$

the spectral radius of the closed-loop matrix is

$$\rho(A_c) = 0{,}6209,$$

whereas $\rho(A) = 1{,}6180$.

The associated stabilizability ellipse and a certain phase trajectory of the closed-loop system are depicted in Figure 1.

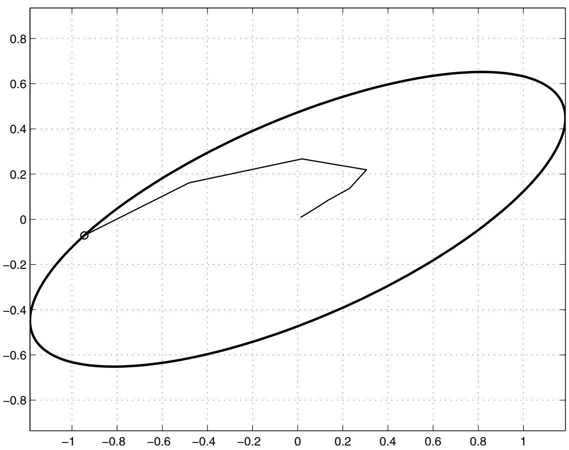

Fig. 1. Stabilizability ellipse from the Example 1.

All computations were performed in MATLAB by using the `cvx` toolbox [25].

V. Estimation of the Stabilizability Domain

In the previous section the stabilizability ellipsoid $\mathscr{E}$ for the discrete-time bilinear system (1) was obtained. Now our goal is to describe a much more complicated set $\mathscr{A}$ which, together with its every point x_0, contains a certain stabilizability ellipsoid for system (1):

$$x_0 \in \mathscr{E} \subset \mathscr{A}.$$

The set $\mathscr{A}$ is called the *stabilizability domain* of the bilinear system (1) associated with the linear static state feedback (2).

It is important to stress that the case with the stabilizability domain highly differs from the case of the stabilizability ellipsoid. Indeed, every point in the stabilizability ellipsoid is associated with a common stabilizing controller, while different points in the stabilizability domain may be associated with different stabilizing controllers for system (1).

Here we construct an inner estimate

$$\widetilde{\mathscr{A}} \subset \mathscr{A}$$

of the domain $\mathscr{A}$. we show how to efficiently find the point on the boundary of $\widetilde{\mathscr{A}}$ in the arbitrary direction c. Finding such a point reduces to solving a convex optimization problem. Notably, since the set $\widetilde{\mathscr{A}}$ in essence is the union of the stabilizing ellipsoids, this set may happen to be *nonconvex*.

So we choose the arbitrary direction defined by the vector c of unit length, and require that the point γc belong to the stabilizability ellipsoid, and the value of the parameter γ is to be as large as possible. This can be easily doable in the framework of the proposed approach based on the linear matrix inequality technique.

Indeed, the condition

$$(\gamma c)^\top P^{-1}(\gamma c) \leq 1$$

that the point γc belongs to the ellipsoid with the matrix P can be rewritten by Schur lemma in the equivalent form

$$\begin{pmatrix} 1 & \gamma c^\top \\ \gamma c & P \end{pmatrix} \succeq 0,$$

linear in P and γ.

The following theorem provides a simple characterization of the set $\widetilde{\mathscr{A}}$.

Theorem 2: Let $c \in \mathbb{R}^n$ be the given vector, and $\widehat{\gamma}$ be the solution of the optimization problem

$$\max \gamma$$

subject to the constraints

$$\begin{pmatrix} -P & 0 & y & PA^\top + yb^\top \\ 0 & -\varepsilon P & 0 & PD^\top \\ y^\top & 0 & -\frac{1}{\varepsilon}I & 0 \\ AP+by^\top & DP & 0 & -P \end{pmatrix} \prec 0, \tag{10}$$

$$\begin{pmatrix} 1 & \gamma c \\ \gamma c^\top & P \end{pmatrix} \succeq 0,$$

with respect to the matrix variable $P = P^\top \in \mathbb{R}^{n\times n}$, the vector variable $y \in \mathbb{R}^n$, the scalar variable γ, and the scalar parameter ε.

Then the point $\widehat{\gamma}c$ belongs to the stabilizability domain $\widetilde{\mathscr{A}}$ for discrete-time bilinear system (1) in the direction c.

As in the previous section, for the well-posedness of the problem, the constraint (10) can be changed for the nonstrict linear matrix inequality (9).

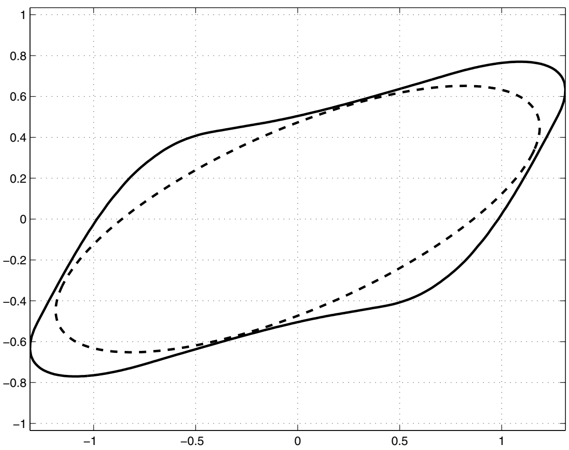

Fig. 2. Stabilizability domain $\widetilde{\mathscr{A}}$ from Example 1.

Figure 2 depicts an approximation of the set $\widetilde{\mathscr{A}}$ for the bilinear system from Example 1. To compare, the largest stabilizability ellipse for the closed-loop system is also shown (dotted line).

VI. CONCUSION

A stabilization problem for discrete-time bilinear control systems is considered. Using the linear matrix inequality technique and the concept of quadratic Lyapunov functions, we proposed an approach to the construction of the so-called stabilizability ellipsoid. It enables the construction of efficient estimates of nonconvex stabilizability domains of discrete-time bilinear control systems.

The proposed approach is easily implemented from the computational and technical points of view. The results obtained can be extended to various robust statements of the problem, in particular, for systems containing structured matrix uncertainty, to bilinear systems with multi-dimensional control, and to bilinear control systems subjected to exogenous disturbances.

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