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Estimation of the basin of attraction of a practical high-gain observer

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***Abstract*— In this paper, a technique is given to estimate the basin of attraction of a "practical" high–gain observer for polynomial autonomous systems. Such a goal is pursued by determining a set such that the restriction of the observability map of the system to such a set is a diffeomorphism. Differently from other techniques available in the literature, which are based on the numeric computation of the trajectories of a system corresponding to the observability map, the proposed procedure provides exact certificates of invertibility of the observability map. The application of the proposed method to the design of a nonlinear observer is discussed.**

I. INTRODUCTION

In several control and monitoring processes, unmeasurable state variables have to be estimated only on the basis of the measured outputs [1], [2], [3], [4], [5], [6], [7]. A rather standard solution to such a problem for linear systems is to replicate the dynamics of the system feeding back the output error though a linear gain (the Luenberger observer [8]).

The problem is much more challenging when one deals with nonlinear systems. Many different approaches have been proposed in the literature to address this problem. For instance, in [9], [10], necessary and sufficient conditions are given for the existence of a transformation that maps the system into a linear one up to a change of variables and an output injection. Another technique consists in feeding back to a replica of the system the output error through a switching nonlinear term (the sliding mode observer [11], [12]).

One of the aspects that makes the problem more daunting is that, in order to being able to design an observer, the system has to satisfy some observability properties. For linear systems, such properties reduce to the requirement that the observability matrix has full rank [13]. On the other hand, for nonlinear systems, such properties essentially corresponds to injectivity of the observability map, which is a function that links the current state of the system with the value of the time derivatives of the output [14], [15], [16], [17].

The main objective of this paper is to design a procedure that provides an exact estimate of the domain of invertibility of the observability map. Differently from other techniques available in the literature [18], [19], which are based on computing numerically the solutions to a nonlinear system, the given method provides an exact estimate of the domain of invertibility. Finally, it is shown how the proposed procedure can be coupled with the techniques given in [20], [21], [22] to design 'practical" observers for nonlinear systems.

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II. DESIGN OF PRACTICAL OBSERVERS

For the notation used in this paper, the reader is referred to [23], [24]. Consider the following polynomial system:

$$\dot{x} = f(x), \quad y = h(x), \tag{1}$$

where $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, $h(\cdot) : \mathbb{R}^n \to \mathbb{R}^q$, $x = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^\top$, q is the number of the *output variables*, $y = \begin{bmatrix} y_1 & \dots & y_q \end{bmatrix}^\top$, $f_i(x) \in \mathbb{Q}[x]$, $i = 1, \dots, n$, and $h_i(x) \in \mathbb{Q}[x]$, $i = 1, \dots, q$, (*i.e.*, the entries of f and h are polynomials in x, with rational coefficients). Let $\Phi_f(t, x)$, $\Phi_f(\cdot, \cdot) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, be the *flow* associated with the *vector field* $f(x)$ (see [25]), which satisfies the following relations for all the times t for which it is defined (possibly, up to $+\infty$), so that $x(t) = \Phi_f(t, x^0)$ is the *unique maximal solution* of (1) from the initial state $x(0) = x^0$:

$$\tfrac{\partial \Phi_f(t,x)}{\partial t} = f \circ \Phi_f(t, x), \quad \Phi_f(0, x) = x, \quad \forall x \in \mathbb{R}^n.$$

For simplicity, assume that, for each $x \in \mathbb{R}^n$, the maximal solution of system (1) is defined for all $t \in [0, +\infty)$. Let $L_f h_i^0(x) := h_i(x)$, $L_f h_i^{j+1}(x) := \frac{\partial L_f h_i^j(x)}{\partial x} f(x)$, $j = 0, 1, \dots$, be the *directional derivatives* of $h_i(x)$ along $f(x)$, $i = 1, \dots, q$. Let $\chi_1, \dots, \chi_q$ be q non-negative integers such that $\chi_1 + \dots + \chi_q = n$; in the following, they will be referred to as the *observability indices*. Define the following vectors:

$$y_e = \begin{bmatrix} y_1^{(0)} \cdots y_1^{(\chi_1-1)} \cdots y_q^{(0)} \cdots y_q^{(\chi_q-1)} \end{bmatrix}^\top,$$
$$O(x) = \begin{bmatrix} L_f h_1^0(x) \cdots L_f h_1^{\chi_1-1}(x) \cdots L_f h_q^0(x) \cdots L_f h_q^{\chi_q-1}(x) \end{bmatrix}^\top,$$

where $y_j^{(i)}(t) := \frac{\mathrm{d}^i y_j(t)}{\mathrm{d}t^i}$, $i \geq 0$, $j = 0, \dots, \chi_i - 1$; y_e is referred to as the *extended output vector* and $O(x)$ as the *observability map* associated with the observability indices $\chi_1, \dots, \chi_q$. Clearly, one has $y_e(t) = O(\Phi_f(t, x))$, $\forall t \geq 0$, $\forall x \in \mathbb{R}^n$. Let $x^o \in \mathbb{R}^n$ and for any $r \in \mathbb{R}$, $r > 0$, let $\mathcal{B}_r(x^o)$ be the ball of $\mathbb{R}^n$ centered in x^o with radius r,

$$\mathcal{B}_r(x^o) := \{x \in \mathbb{R}^n : (x - x^o)^\top (x - x^o) \leq r^2\}.$$

A *bijective map* (also called a *one-to-one and onto* correspondence) $\eta = \varphi(\xi)$, with domain $\mathcal{X}$ and co-domain $\mathcal{Y}$, is a map such that each element ξ in $\mathcal{X}$ is paired with exactly one element η in $\mathcal{Y}$ ($\eta = \varphi(\xi)$) and, *viceversa*, each element η in $\mathcal{Y}$ is paired with exactly one element ξ in $\mathcal{X}$ (*i.e.*, there exists a function $\varphi^{-1}(\cdot)$ from $\mathcal{Y}$ to $\mathcal{X}$, which is called the *inverse* map of $\eta = \varphi(\xi)$, such that $\xi = \varphi^{-1}(\eta)$). A map, with domain $\mathcal{X}$ and co-domain $\mathcal{Y}$, that is one-to-one and onto is a *homeomorphism* from $\mathcal{X}$ to $\mathcal{Y}$ if both $\eta = \varphi(\xi)$ and its inverse $\xi = \varphi^{-1}(\eta)$ are C^0 on $\mathcal{X}$ and $\mathcal{Y}$, respectively. A map, with domain $\mathcal{X}$ and co-domain $\mathcal{Y}$, that is one-to-one and onto is a *diffeomorphism* from $\mathcal{X}$ to $\mathcal{Y}$ if both $\eta = \varphi(\xi)$ and its inverse $\xi = \varphi^{-1}(\eta)$ are C^1 on $\mathcal{X}$ and $\mathcal{Y}$, respectively.

Let the observability map $O(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ be given. Let $x^o \in \mathbb{R}^n$ and $\mathcal{X}_{x^o}$ be a subset of $\mathbb{R}^n$ having x^o as an interior point; let $\mathcal{Y}_{O(x^o)} = O(\mathcal{X}_{x^o})$. The *restriction* of $O(x)$ to the domain $\mathcal{X}_{x^o}$ and to the co-domain $\mathcal{Y}_{O(x^o)}$ (briefly, to $(\mathcal{X}_{x^o}, \mathcal{Y}_{O(x^o)})$) is a new function, denoted $O(\cdot)|\mathcal{X}_{x^o}$, from $\mathcal{X}_{x^o}$ to $\mathcal{Y}_{O(x^o)}$ such that $O(x)|\mathcal{X}_{x^o} = O(x)$, for all $x \in \mathcal{X}_{x^o}$. Under some conditions on the point x^o, the domain $\mathcal{X}_{x^o}$ can be chosen as a neighborhood $\mathcal{B}_r(x^o)$ of x^o being sufficiently small so that $O(x)|\mathcal{B}_r(x^o)$ is one-to-one and onto (see the subsequent Lemma 1). For instance, for $n = 1$, the function $x^2|\mathcal{B}_r(1)$ is one-to-one and onto for all $1 \geq r > 0$ and it is not one-to-one if $r > 1$, whereas $x^2|\mathcal{B}_r(0)$ is not one-to-one for any $r > 0$. If $y_e = O(x)|\mathcal{X}_{x^o}$ is one-to-one and onto, its inverse is denoted $x = O^{-1}(y_e)|\mathcal{X}_{x^o}$.

Definition 1. *Let $x^o \in \mathbb{R}^n$ (possibly, $f(x^o) \neq 0$);*

(1.1) system (1) is (respectively, weakly*) locally observable about x^o if there is a neighborhood $\mathcal{B}_r(x^o)$, $r \in \mathbb{R}$, $r > 0$, of x^o such that the restriction $O(x)|\mathcal{B}_r(x^o)$ of $O(x)$ to $(\mathcal{B}_r(x^o), O(\mathcal{B}_r(x^o)))$ is a (respectively, homeomorphism) diffeomorphism from $\mathcal{B}_r(x^o)$ to $O(\mathcal{B}_r(x^o))$;*

(1.2) system (1) is (respectively, weakly*) globally observable if the restriction $O(x)|\mathcal{B}_r(x^o)$ of $O(x)$ to $(\mathcal{B}_r(x^o), O(\mathcal{B}_r(x^o)))$ is a (respectively, homeomorphism) diffeomorphism from $\mathcal{B}_r(x^o)$ to $O(\mathcal{B}_r(x^o))$, $\forall r \in \mathbb{R}$, $r > 0$.*

The following lemma gives a simple sufficient condition for the local observability of the polynomial system (1).

Lemma 1. *Let $x^o \in \mathbb{R}^n$ and define $J_O(x) := \frac{\partial O(x)}{\partial x}$. If $\det(J_O(x^o)) \neq 0$, then there exists a (sufficiently, small) $r \in \mathbb{R}$, $r > 0$, such that $O(x)|\mathcal{B}_r(x^o)$ is a diffeomorphism from $\mathcal{B}_r(x^o)$ to $O(\mathcal{B}_r(x^o))$, whence the non-linear polynomial system (1) is locally observable about x^o.*

Note that the invertibility of $J_O(x)$ at x^o is not necessary for the non-linear polynomial system (1) to be weakly locally observable about x^o. In fact, the scalar system $\dot{x} = f(x), y = x^3$, with $n = 1$, is weakly locally observable about the origin because $x^3|\mathcal{B}_r(0)$ is one-to-one and onto for any $r > 0$, although its Jacobian $3x^2$ is zero at $x^o = 0$.

If $r^\star \in \mathbb{R}$, $r^\star > 0$, is such that $O(x)|\mathcal{B}_{r^\star}(x^o)$ is one-to-one and onto, and $\det(J_O(x)) \neq 0$, for all $x \in \mathcal{B}_{r^\star}(x^o)$, then for any $r \in [0, r^\star]$ one has that $O(x)|\mathcal{B}_r(x^o)$ is one-to-one and onto, and $\det(J_O(x)) \neq 0$, for all $x \in \mathcal{B}_r(x^o)$. This implies that, under the assumptions and conditions of Lemma 1, there exists a supremum value $r_{\sup} \in \mathbb{R}$, $r_{\sup} > 0$, (possibly, $r_{\sup} = +\infty$) such that for any $r \in [0, r_{\sup})$ one has that $O(x)|\mathcal{B}_r(x^o)$ is a diffeomorphism, for all $x \in \mathcal{B}_r(x^o)$.

According to the chosen observability indices $\chi_1, \ldots, \chi_q$, the entries of ξ and its estimate $\hat{\xi}$ can be indexed as:

$$\xi = \begin{bmatrix} \xi_{1,0} & \cdots & \xi_{1,\chi_1-1} & \cdots & \xi_{q,\chi_q-1} \end{bmatrix}^\top,$$
$$\hat{\xi} = \begin{bmatrix} \hat{\xi}_{1,0} & \cdots & \hat{\xi}_{1,\chi_1-1} & \cdots & \hat{\xi}_{q,\chi_q-1} \end{bmatrix}^\top.$$

In particular, there exist q functions $\phi_1(\xi), \ldots, \phi_q(\xi)$ (which need not be polynomial and need not be expressed in closed-form through elementary functions) such that the non-linear polynomial system (1) can be expressed locally in $O(\mathcal{B}_{r_{\sup}}(x^o))$ as follows, for $i = 1, \ldots, q$:

$$\dot{\xi}_{i,j} = \xi_{i,j+1}, \qquad i = 0, \ldots, \chi_i - 2,$$
$$\dot{\xi}_{i,\chi_i-1} = \phi_i(\xi), \qquad y_i = \xi_{i,0};$$

in particular, one has

$$\phi_i(\xi) = L_f h_i^{\chi_1} \circ O^{-1}(\xi)|\mathcal{B}_r(x^o), \quad i = 1, \ldots, q.$$

It is noteworthy that the knowledge of the functions $\phi_1(\xi), \ldots, \phi_q(\xi)$ is not needed in the design of a practical high-gain observer, as proposed in [20].

Hence, by [20], a high-gain observer (which, under some conditions, ensures practical stability of the error dynamics) can be easily designed as follows, for $i = 1, \ldots, q$:

$$\dot{\hat{\xi}}_{i,0} = \hat{\xi}_{i,1} + \tfrac{k_{i,0}}{\varepsilon}(y_i - \hat{\xi}_{i,0}), \quad i = 0, \ldots, \chi_i - 2, \tag{2a}$$
$$\dot{\hat{\xi}}_{i,\chi_i-1} = \tfrac{k_{i,\chi_i-1}}{\varepsilon^{\chi_i}}(y_i - \hat{\xi}_{i,0}), \tag{2b}$$

where $\hat{\xi}_{i,j}$ is the estimate of $\xi_{i,j}$, the polynomials $p_i(s) = s^{\chi_i} + s^{\chi_i-1}k_{i,0} + \ldots + k_{i,\chi_i-1}$, $i = 1, \ldots, q$, are Hurwitz and $\varepsilon > 0$ is a small parameter.

Assumption 1. *Let $x(0)$ be the initial condition of the non-linear polynomial system (1) from the initial time $t = 0$. Let $x(t) = \Phi_f(t, x(0))$ be the corresponding solution; assume that $\Phi_f(t, x(0))$ is defined for all times $t \geq 0$. Let x^o be any point in $\mathbb{R}^n$. Assume that there exists a supremum value $r_{\sup} > 0$ such that the restriction $O(x)|\mathcal{B}_{r_{\sup}}(x^o)$ of $O(x)$ to $(\mathcal{B}_{r_{\sup}}(x^o), O(\mathcal{B}_{r_{\sup}}(x^o)))$ is a diffeomorphism. Let $r^\star > 0$ be any real number less than $r_{\sup}$.*

(1.1) If $r_{\sup}$ is finite, assume that for any $x(0) \in \mathcal{B}_{r^\star}(x^o)$ one has $x(t) \in \mathcal{B}_{r_{\sup}}(x^o)$ for all times $t \geq 0$;

(1.2) if $r_{\sup}$ is not finite, letting $\xi(t) = O(x(t))$, $t \geq 0$, assume that for any $x(0) \in \mathcal{B}_{r^\star}(x^o)$ one has

$$\sup_{i \in \{1,\ldots,q\}} \sup_{t \in [0,+\infty]} (\phi_i(\xi(t))) < +\infty.$$

Under Assumption 1, one has the following result about the practical stability of the resulting error dynamics.

Theorem 1 (see [20])**.** *Let $r^\star > 0$ be any real number less than $r_{\sup}$ such that Assumption 1 holds. For any (arbitrarily small) $d > 0$ and for any (arbitrarily large) $T > 0$, there is $\varepsilon^* > 0$ such that if $0 < \varepsilon < \varepsilon^*$, then, letting $\tilde{\xi} = \xi - \hat{\xi}$,*

$$x(0) \in \mathcal{B}_{r^\star}(x^o) \quad \Longrightarrow \quad \|\tilde{\xi}(t)\| \leq d, \quad \forall t \geq T.$$

Given an estimate $\hat{\xi}$ of ξ, the estimate $\hat{x}$ of x can be constructed as $\hat{x} = O^{-1}(\hat{\xi})|\mathcal{B}_r(x^o)$ if the inverse function $x = O^{-1}(\xi)|\mathcal{B}_r(x^o)$ is known; otherwise, the approach proposed in [21], [22] can be adopted, as detailed below.

For simplicity, omit the dependence of the restriction $O(x)|\mathcal{B}_{r_{\sup}}(x^o)$ on its domain $\mathcal{B}_{r_{\sup}}(x^o)$ of definition. According to [21], [26], proceed as follows. Let $\hat{x}$ be an estimate of x, and let $\tilde{x} := x - \hat{x}$ be the respective estimation error. The corresponding estimation error in the ξ-coordinates is $\tilde{\xi} := \xi - \hat{\xi} = \xi - O(\hat{x})$. Define the function $V(\tilde{\xi}) = (\xi - O(\hat{x}))^\top(\xi - O(\hat{x}))$, which is a positive definite function of $\tilde{\xi}$, as well as of $\tilde{x} = x - \hat{x}$. Let $J_O(x) = \frac{\partial O(x)}{\partial x}$. The time derivate of V is $\dot{V} = 2(\xi - O(\hat{x}))^\top(\dot{\xi} - J_O(\hat{x})\dot{\hat{x}})$.

The dynamics of the estimate $\hat{x}$ are chosen as [22], [27]

$$\dot{\hat{x}} = \tfrac{\mu}{2} J_O^{-1}(\hat{x})(\xi - O(\hat{x})), \tag{3}$$

where $\mu \in \mathbb{R}$, $\mu > 0$. Therefore, if $\dot{\xi} = 0$, one has $\dot{V} = -\mu V$, which shows that the dynamics of the estimation error $\tilde{x} = x - \hat{x}$ are exponentially stable; instead, if $\dot{\xi}$ is not zero but bounded as implied by Assumption 1, the dynamics of the estimation error $\tilde{x}$ are practically stable (see [21], [26], [22]), whence the ultimate estimation error can be made arbitrarily small by taking μ sufficiently large.

In the actual application of Theorem 1, it is fundamental the "certified" computation of the supremum value $r_{\sup}$, so to know an estimate $\mathcal{B}_{r_{\sup}}(x^o)$ of the domain of invertibility of the observability map about the chosen point x^o (*i.e.*, the domain where the observer (2) can be actually used). This is the objective of the following section.

III. Computation of a "certified" estimate of the domain of invertibility of the observability map about a given point x^o

Let $n = 1$; hence, $O(x) = h(x)$. By the Rolle Theorem (see [28]), if $h(x)$ is C^0 on the closed interval $[a, b]$, C^1 on the open interval (a, b), and $h(a) = h(b)$, then there exists $c \in (a, b)$ such that $\left.\frac{\partial h(x)}{\partial x}\right|_{x=c} = 0$. This implies that if $\frac{\partial h(x)}{\partial x} \neq 0$, $\forall x \in [a, b]$, then $h(x)|[a, b]$ is a diffeomorphism. This strong property is no longer true if $n \geq 2$. In particular, if $n \geq 2$, given $x^o \in \mathbb{R}^n$, it is difficult to determine the largest subset $\mathcal{X}_{x^o}$ of $\mathbb{R}^n$ having x^o as an interior point such that $O(x)|\mathcal{X}_{x^o}$ is a diffeomorphism and the largest $\mathcal{Y}_{O(x^o)}$ such that $\mathcal{Y}_{O(x^o)} = O(\mathcal{X}_{x^o})$, for some $\mathcal{X}_{x^o} \subset \mathbb{R}^n$ having x^o as an interior point, and such that $O(x)|\mathcal{X}_{x^o}$ is a diffeomorphism. In the following, two results will be reviewed:

(i) it is possible to characterize (see [18]) the largest domain $\mathcal{Y}_{O(x^o)}$, star-shaped with respect to $O(x^o)$, such that there exists $\mathcal{X}_{x^o} \subset \mathbb{R}^n$ having x^o as an interior point such that $\mathcal{Y}_{O(x^o)} = O(\mathcal{X}_{x^o})$ and $O(x)|\mathcal{X}_{x^o}$ is a diffeomorphism;

(ii) it is possible to characterize (see [29]) the largest ball $\mathcal{B}_{\sup}(x^o)$, centered at x^o, such that $O(\mathcal{B}_{\sup}(x^o))$ is star-shaped and $O(x)|\mathcal{X}_{x^o}$ is a diffeomorphism.

Finally, it will be shown, as an original contribution, that the above largest ball $\mathcal{B}_{\sup}(x^o)$ can be determined through algebraic geometry, thus allowing one to obtain certified estimates of the domain of invertibility of $O(x)$.

Let the entries of $f(x)$ and $h(x)$ be elements of $\mathbb{Q}[x]$, so that the entries of the observability map $O(x)$ are in $\mathbb{Q}[x]$. Compute the Jacobian matrix $J_O(x) = \frac{\partial O(x)}{\partial x}$ and its inverse $J_O^{-1}(x)$, whose entries are in $\mathbb{Q}(x)$. Let x^o be the point of $\mathbb{R}^n$ about which one is interested to find an estimate $\mathcal{B}_r(x^o)$ of the domain of invertibility of the observability map $O(x)$; assume that $\det(J_O(x^o)) \neq 0$, so that it is different from zero for all $x \in \mathbb{R}^n$ about x^o. A set $\mathcal{S}$ in $\mathbb{R}^n$ is *star-shaped* with respect to $\xi^o = O(x^o) \in \mathcal{S}$ if for all ξ in $\mathcal{S}$ the line segment from ξ^o to ξ is in $\mathcal{S}$.

Let $\mathcal{Y}_{O(x^o)} = O(\mathcal{X}_{x^o})$ be the largest domain, star-shaped with respect to $O(x^o)$, such that $O(x)|\mathcal{X}_{x^o}$ is a diffeomorphism. Let $v \in \mathbb{R}^n$ be arbitrary and consider the system

$$\tfrac{\mathrm{d}x(\ell)}{\mathrm{d}\ell} = J_O^{-1}(x(\ell))v. \tag{4}$$

By the Chauchy Theorem (see [30]), there exists a maximal interval $[0, L_v)$ (possibly, $L_v = +\infty$) and a unique solution $x(\ell)$ of the differential equation (4) from the initial condition $x(0) = x^o$. By construction, the set of points

$$\mathcal{R}_v = \{\xi \in \mathbb{R}^n : \xi = O(x(\ell)),\, \ell \in [0, L_v)\}$$

is a segment (a ray, if $L_v = +\infty$) starting from $O(x^o)$ along the direction identified by v, contained in $\mathcal{Y}_{O(x^o)}$. In particular, it is possible to show (see [18]) that,

$$\mathcal{Y}_{O(x^o)} = \textstyle\bigcup_{v \in \mathbb{R}^n} \mathcal{R}_v.$$

Similarly, by defining the set of points

$$\mathcal{S}_v = \{x \in \mathbb{R}^n : x = x(\ell), \ell \in [0, L_v)\},$$

it can be shown (see [18]) that $\mathcal{Y}_{O(x^o)} = O(\mathcal{X}_{x^o})$, where

$$\mathcal{X}_{x^o} = \textstyle\bigcup_{v \in \mathbb{R}^n} \mathcal{S}_v.$$

Example 1. *Consider the polynomial system (1), with*

$$\begin{aligned} f(x) &= \begin{bmatrix} -x_1^3 + x_1^2 x_2 - x_1 x_2^2 + x_1 + x_2^3 - x_2 \\ -x_1^3 - x_1^2 x_2 - x_1 x_2^2 + x_1 - x_2^3 + x_2 \end{bmatrix}, \\ h(x) &= x_1. \end{aligned}$$

Letting $I(x) = x_1^2 + x_2^2 - 1$, one has $L_f I = -2I - 2I^2$ (I is a semi-invariant associated with f), which implies (by looking at the solutions of the auxiliary scalar equation $\dot{I} = -2I - 2I^2$) that the circumference centered at the origin of $\mathbb{R}^2$, with unitary radius, described by the polynomial equation $I(x) = 0$, is asymptotically stable, but not globally. One has

$$J_O(x) = \begin{bmatrix} 1 & 0 \\ -3x_1^2 + 2x_1 x_2 - x_2^2 + 1 & x_1^2 - 2x_1 x_2 + 3x_2^2 - 1 \end{bmatrix}.$$

Thus, one can compute

$$J_O^{-1}(x) = \begin{bmatrix} 1 & 0 \\ \frac{3x_1^2 - 2x_1 x_2 + x_2^2 - 1}{x_1^2 - 2x_1 x_2 + 3x_2^2 - 1} & \frac{1}{x_1^2 - 2x_1 x_2 + 3x_2^2 - 1} \end{bmatrix}.$$

Since $\det(J_O(x)) = (x_1 - x_2)^2 + 2x_2^2 - 1$, there exists a closed curve in $\mathbb{R}^2$ such that $\det(J_O(x)) = 0$; its image through $\xi = O(x)$ is the variety associated with the polynomial $16\xi_1^6 - 32\xi_1^4 + 40\xi_1^3\xi_2 + 20\xi_1^2 - 36\xi_1\xi_2 + 27\xi_2^2 - 4$. Choose $x^o = 0$. The Ważewski equation (4),

$$\begin{aligned} \tfrac{\mathrm{d}x_1(\ell)}{\mathrm{d}\ell} &= v_1, \\ \tfrac{\mathrm{d}x_2(\ell)}{\mathrm{d}\ell} &= \tfrac{(3x_1^2 - 2x_1 x_2 + x_2^2 - 1)v_1 + v_2}{x_1^2 - 2x_1 x_2 + 3x_2^2 - 1}, \end{aligned}$$

can be solved numerically (along its maximal interval $[0, L_v)$, where, in this simple case, L_v is such that $x(L_v)$ belongs to the curve $\mathbb{V}_{\mathbb{R}^2}((x_1 - x_2)^2 + 2x_2^2 - 1)$) by letting $v_1 = \cos(\theta)$ and $v_2 = \sin(\theta)$ and choosing 150 equally spaced values of θ from 0 to 2π. Such trajectories have been used to compute the corresponding trajectories $\xi(\ell) = O(x(\ell))$. The image of the curve described by the equation

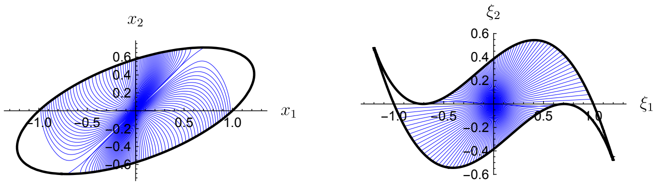
$(x_1-x_2)^2+2x_2^2-1=0$ through $\xi=O(x)$ can be computed easily: first, one has to consider the ideal of $\mathbb{Q}[x_1,x_2]$,

$$\mathcal{I}_s = \langle (x_1-x_2)^2+2x_2^2-1, \\ \xi_1-x_1, \xi_2-(-x_1^3+x_1^2x_2-x_1x_2^2+x_1+x_2^3-x_2)\rangle,$$

and, secondly, one has to compute the elimination ideal $\mathcal{I}_s \cap \mathbb{Q}[\xi_1,\xi_2]$, which results to be principal and generated by

$$\xi_1^6-2\xi_1^4+\tfrac{5}{2}\xi_1^3\xi_2+\tfrac{5}{4}\xi_1^2-\tfrac{9}{4}\xi_1\xi_2+\tfrac{27}{16}\xi_2^2-\tfrac{1}{4}.$$

In this case, one has $\mathcal{X}_{x^o}$ (the blue curve in Figure 1(a)) is contained into the region of $\mathbb{R}^2$ delimited by the curve described by $(x_1-x_2)^2+2x_2^2=1$, and $\mathcal{Y}_{O(x^o)}$ is contained into the region of $\mathbb{R}^2$ delimited by the curve described by $\xi_1^6-2\xi_1^4+\frac{5}{2}\xi_1^3\xi_2+\frac{5}{4}\xi_1^2-\frac{9}{4}\xi_1\xi_2+\frac{27}{16}\xi_2^2=\frac{1}{4}$. In Figure 1, the plots of such curves are depicted in black.



(a) $x(\ell)$ (b) $\xi(\ell)$

Fig. 1. Solutions of the Ważewski equation (4).

If one allows $v\in\mathbb{R}^n$ appearing in equation (4) to be a function of $x(\ell)$, one can take $v=(\xi^o-O(x(\ell)))$, where $\xi^o=O(x^o)$, thus obtaining an equation similar to (3),

$$\frac{\mathrm{d}x(\ell)}{\mathrm{d}\ell}=J_O^{-1}(x(\ell))(\xi^o-O(x(\ell))). \tag{5}$$

Clearly, if $\det(J_O(x^o))\neq 0$, by using the Lyapunov function $V=(\xi^o-O(x(\ell)))^\top(\xi^o-O(x(\ell)))$, it is easy to show that x^o is asymptotically stable. Let Ω_{x^o} be the *basin of attraction* of x^o, then $O(x)|\Omega_{x^o}$ is a diffeomorphism [19].

Example 2. *Consider the system given in Example 1 and let $x^o=0$. For such a system, the dynamics in (5) read as*

$$\dot{x}_1 = -x_1, \tag{6a}$$
$$\dot{x}_2 = \frac{-2x_1^3-x_2^3+(x_1^2+1)x_2}{x_1^2-2x_2x_1+3x_2^2-1}- \tag{6b}$$

The basin of attraction of x^o cannot be easily characterized through the Lyapunov function $V=O^\top(x)O(x)$. Therefore, such an analysis is carried out by inspecting the phase plot of system (6) (Figure 2). In such a plot, the trajectories of system (6) converging to x^o are depicted in blue, whereas the trajectories that do not converge to x^o are depicted in red. Thus, $\mathcal{X}_{x^o}$ is given by the portion of $\mathbb{R}^2$ depicted in blue. Note that the same analysis can be carried out by considering different values of x^o, leading to different results. For instance, Figure 3 depicts (with the same coloring used for Figure 2) the trajectories of system (5) for $x^o=[\,0.9 \quad 0.4\,]^\top$ and $x^o=[\,-1 \quad 0.5\,]^\top$. In both cases, the estimate of $\mathcal{X}_{x^o}$ is given by the subset of $\mathbb{R}^2$ depicted in blue.

Clearly, the differential equation (4) can be solved in closed-form only in rare case, whence the above approach can be actually applied only numerically.

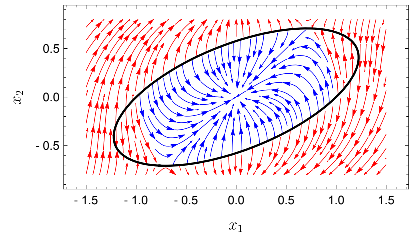
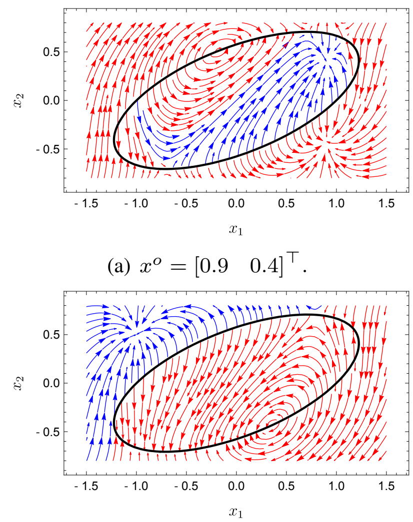


Fig. 2. Phase plot of system (6) with $x^o=0$.



(a) $x^o=[0.9 \quad 0.4]^\top$.

(b) $x^o=[-1 \quad 0.5]^\top$.

Fig. 3. Phase plot of system (5) with different x^o.

Compute the following rational function in $\mathbb{Q}(x)$:

$$q_{x^o}(x)=(x-x^o)^\top J_O^{-1}(x)(O(x)-O(x^o)),$$

which is analytic for all $x\in\mathbb{R}^n$ about x^o. Since $\det(J_O(x^o))\neq 0$, by taking the Taylor expansion of $q_{x^o}(x)$ at $x=x^o$, one has $q_{x^o}(x)=(x-x^o)^\top(x-x^o)+\ldots$, where $\ldots$ denotes higher order terms. This shows that, if $r\in\mathbb{R}$, $r>0$, is sufficiently small, then $q_{x^o}(x)\geq 0$, $\forall x\in\mathcal{B}_r(x^o)$.

Lemma 2 (see Proposition 1.3 of [29])**.** *Let $r^\star\in\mathbb{R}$, $r^\star>0$, be such that $\det(J_O(x))\neq 0$ for all $x\in\mathcal{B}_{r^\star}(x^o)$; hence, $\xi=O(x)|\mathcal{B}_r(x^o)$, for some $r\in\mathbb{R}$, $r^\star\geq r>0$, is a diffeomorphism and its co-domain $O(\mathcal{B}_r(x^o))$ is star-shaped with respect to ξ^o, $\xi^o=O(x^o)|\mathcal{B}_r(x^o)$, if and only if*

$$q_{x^o}(x)\geq 0, \quad \forall x\in\mathcal{B}_{r^\star}(x^o).$$

The boundary $\partial\mathcal{B}_r(x^o)$ of $\mathcal{B}_r(x^o)$ is the affine variety $\mathbb{V}_{\mathbb{R}^n}((x-x^o)^\top(x-x^o)-r^2)$; since $q_{x^o}(x)$ is continuous on $\partial\mathcal{B}_r(x^o)$ and $\partial\mathcal{B}_r(x^o)$ is compact for any $r\in\mathbb{R}$, $r^\star\geq r>0$, $q_{x^o}(x)$ has a minimum value over $\partial\mathcal{B}_r(x^o)$. Let

$$\bar{q}_{x^o}(x,\lambda):=q_{x^o}(x)+\lambda((x-x^o)^\top(x-x^o)-r^2),$$

where λ is the *Lagrange multiplier*; clearly, the minimum value of $q_{x^o}(x)$, under the constraint

$$c_r(x):=(x-x^o)^\top(x-x^o)-r^2=0,$$

coincides with the minimum value of $\bar{q}_{x^o}(x,\lambda)$. A *critical point* of $\bar{q}_{x^o}(x,\lambda)$ is any pair $(x^\star,\lambda^\star)\in\mathbb{R}^n\times\mathbb{R}$ such that:

$$\begin{aligned}\left.\frac{\partial\bar{q}_{x^o}(x,\lambda)}{\partial x_i}\right|_{(x,\lambda)=(x^\star,\lambda^\star)} &= 0, \quad i=1,\ldots,n,\\ \left.\frac{\partial\bar{q}_{x^o}(x,\lambda)}{\partial \lambda}\right|_{(x,\lambda)=(x^\star,\lambda^\star)} &= 0.\end{aligned}$$

The image $q^\star=q_{x^o}(x^\star)=\bar{q}_{x^o}(x^\star,\lambda^\star)$ is a *critical value* of $q_{x^o}(x)$ along $\partial\mathcal{B}_r(x^o)$; the minimum value of $q_{x^o}(x^\star)$ along $\partial\mathcal{B}_r(x^o)$ is necessarily a critical value of $q_{x^o}(x^\star)$ along $\partial\mathcal{B}_r(x^o)$. The critical values of $\bar{q}_{x^o}(x,\lambda)$ can be computed through algebraic geometry, as detailed hereafter.

Since $\bar{q}_{x^o}\in\mathbb{Q}(x,\lambda)$, there exists two polynomials N_q, D_q in $\mathbb{Q}[x,\lambda]$ such that $\bar{q}_{x^o}(x,\lambda)=\frac{N_q(x,\lambda)}{D_q(x,\lambda)}$; note that, apart from possible cancellations, $D_q(x,\lambda)=\det(J_O(x))$, and therefore it is different from zero about x^o. Clearly,

$$\begin{aligned}\frac{\partial\bar{q}_{x^o}}{\partial x_i} &= D_q^{-2}\left(\frac{\partial N_q}{\partial x_i}D_q-N_q\frac{\partial D_q}{\partial x_i}\right), \quad i=1,\ldots,n,\\ \frac{\partial\bar{q}_{x^o}}{\partial\lambda} &= c_r,\\ \bar{q}_{x^o}-Q &= \frac{N_q-QD_q}{D_q},\end{aligned}$$

where Q is a variable representing the value attained by $\bar{q}_{x^o}$ at its critical points. Consider the ideal

$$\begin{aligned}\mathcal{I}=\langle&\frac{\partial N_q}{\partial x_1}D_q-N_q\frac{\partial D_q}{\partial x_1},\ldots,\frac{\partial N_q}{\partial x_n}D_q-N_q\frac{\partial D_q}{\partial x_n},\\ &c_r,N_q-QD_q\rangle\subset\mathbb{Q}(r)[x_1,\ldots,x_n,\lambda,Q].\end{aligned}\qquad(7)$$

It is worth pointing out that working with $\mathbb{Q}(r)$ as the ground field is equivalent to working symbolically with r.

Consider the elimination ideal $\mathcal{I}\cap\mathbb{Q}(r)[Q]$; since any ideal in $\mathbb{Q}(r)[Q]$ is *principal*, there exists a polynomial $p_r(Q)$ in $\mathbb{Q}(r)[Q]$ such that $\mathcal{I}\cap\mathbb{Q}(r)[Q]=\langle p_r(Q)\rangle$. In particular, all the real roots of $p_r(Q)$ are critical values of $q_{x^o}(x)$ along $\partial\mathcal{B}_r(x^o)$, whence the minimum value of $q_{x^o}(x)$ along $\partial\mathcal{B}_r(x^o)$ is necessarily a real root of $p_r(Q)$. The polynomial $p_r(Q)$ can be computed easily, by computing the reduced reduced Gröbner bases $\mathcal{G}_\mathcal{I}$ of $\mathcal{I}$ with respect to the Lex monomial order $\succ$, with $x_1\succ\ldots\succ x_n\succ\lambda\succ Q$; in particular, $p_r(Q)$ is the only element in $\mathcal{G}_\mathcal{I}$ that is independent of $x_1,\ldots,x_n,\lambda$.

Example 3. *Consider again Example 1. Let*

$$\begin{aligned}q_{x^o}(x) &= x^\top J_O^{-1}(x)O(x)=\frac{x_1^4+2x_1^2x_2^2+x_2^4-x_1^2-x_2^2}{x_1^2-2x_1x_2+3x_2^2-1},\\ c_r(x) &= x_1^2+x_2^2-r^2,\end{aligned}$$

and $\bar{q}_{x^o}(x,\lambda)=q_{x^o}(x)+\lambda\,c_r(x)=\frac{N_q(x,\lambda)}{D_q(x,\lambda)}$*, where*

$$\begin{aligned}N_q(x,\lambda) &= (x_1^4+2x_1^2x_2^2+x_2^4-x_1^2-x_2^2)\\ &\quad+\lambda(x_1^2+x_2^2-r^2)(x_1^2-2x_1x_2+3x_2^2-1),\\ D_q(x,\lambda) &= x_1^2-2x_1x_2+3x_2^2-1.\end{aligned}$$

Define the ideal $\mathcal{I}$ *as in* (7) *and fix the Lex monomial order with* $x_1>x_2>\lambda>Q$*. The elimination ideal* $\mathcal{I}\cap\mathbb{Q}(r)[Q]$ *is principal; in fact, one has* $I\cap\mathbb{Q}(r)[Q]=\langle p_r(Q)\rangle$*, where:*

$$p_r(Q) = Q^2+\frac{-2r^2(r-1)(r+1)(2r^2-1)}{2r^4-4r^2+1}Q+\frac{r^4(r-1)^2(r+1)^2}{2r^4-4r^2+1}.$$

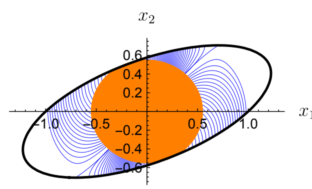
By using the Sturm Test, the roots in Q *of the polynomial* $p_r(Q)$ *are nonnegative if and only if* $r<r^\star=\frac{1}{\sqrt{2+\sqrt{2}}}$*. Since* $\mathcal{B}_r(x^o)\cap\mathbb{V}_{\mathbb{R}^2}(\det(J_O(x)))=\emptyset$ *for all* $r\in(0,r^\star)$*, this implies that* $O(x^o)|\mathcal{B}_{r^\star}(x^o)$ *is a diffeomorphism. The image of* $\mathbb{V}_{\mathbb{R}^2}(x_1^2+x_2^2-r^2)$ *through* $\xi=O(x)$ *can be computed easily: first, consider the ideal of* $\mathbb{Q}[x_1,x_2]$*,*

$$\begin{aligned}\mathcal{I}_s=\langle(&x_1^2+x_2^2-r^2,\xi_1-x_1,\\ &\xi_2-(-x_1^3+x_1^2x_2-x_1x_2^2+x_1+x_2^3-x_2)\rangle,\end{aligned}$$

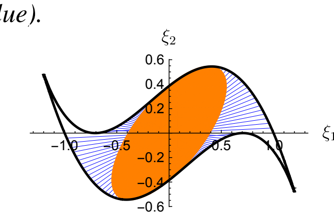
and, secondly, one has to compute the elimination ideal $\mathcal{I}\cap\mathbb{Q}[\xi_1,\xi_2]$*, which results to be principal and generated by*

$$\xi_1^2+\frac{1}{r^2-1}\xi_1\xi_2+\frac{1}{2r^4-4r^2+2}\xi_2^2-\frac{r^2}{2}.$$

The "exact" estimates $\mathcal{X}_{x^o}$ *and* $\mathcal{Y}_{O(x^o)}$ *are depicted (in orange) in Figure 4 together with the numerical estimates obtained in Example 1 (in blue).*



(a) $\mathcal{X}_{x^o}$



(b) $\mathcal{Y}_{O(x^o)}$

Fig. 4. Obtained estimates $\mathcal{X}_{x^o}$ and $\mathcal{Y}_{O(x^o)}$.

IV. A SIMULATIVE EXAMPLE

Consider the system

$$\dot{x}_1=x_2,\qquad\dot{x}_2=-x_1+x_1^3,\qquad y=x_2.\qquad(8)$$

Letting $I(x)=-x_1^4+2x_1^2+2x_2^2$, one has $L_fI(x)=0$, which implies that the level sets of $I(x)$ are invariant for system (8). Letting $x^o=0$, define the rational function

$$\begin{aligned}q_{x^o}(x) &= x^\top J_O^{-1}(x)O(x)=x_2^2-\frac{x_1(x_1^3-x_1)}{1-3x_1^2},\\ c_r &= x_1^2+x_2^2-r.\end{aligned}$$

Thus, one has $\bar{q}_{x^o}(x,\lambda)=q_{x^o}(x)+\lambda c_r(x)=\frac{N_q(x,\lambda)}{D_q(x,\lambda)}$, where

$$\begin{aligned}N_q(x,\lambda) &= \lambda r^2-x_1^2(\lambda+3\lambda r^2-3(\lambda+1)x_2^2+1)\\ &\quad+(3\lambda+1)x_1^4-(\lambda+1)x_2^2,\\ D_q(x,\lambda) &= 3x_1^2-1.\end{aligned}$$

Define the following ideal in $\mathbb{Q}(r)[x_1,x_2,\lambda,P]$

$$\mathcal{I}=\langle\frac{\partial N_d}{\partial x_1}D_d-N_d\frac{\partial D_d}{\partial x_1},\frac{\partial N_d}{\partial x_2}D_d-N_d\frac{\partial D_d}{\partial x_2},c_r,N_d-QD_d\rangle.$$

The (real) solutions to the system of equalities corresponding to the ideal $\mathcal{I}$ can be determined by using the tools given in [31], [32]. It turns out that the minimum value attained by $\bar{q}_{x^o}(x_1,x_2)$ is greater than $\min\{r^2,\frac{r^2(r^2-1)}{3r^2-1}\}$, that is nonnegative if $r<r^\star=\frac{1}{\sqrt{3}}$. Thus, since $\mathcal{B}_r(x^o)\cap\mathbb{V}_{\mathbb{R}^2}(\det(J_O(x)))=\emptyset$ for all $r\in(0,r^\star)$, $O(x^o)|\mathcal{B}_{r^\star}(x^o)$ is a diffeomorphism. Therefore, $\mathcal{X}_{x^o}$ is given by

$$\mathcal{X}_{x^o}=\{x\in\mathbb{R}^2:x_1^2+x_2^2-\tfrac{1}{3}<0\}.$$

Figure 5 depicts the obtained "exact" set $\mathcal{X}_{x^o}$ (in orange), the set $\mathbb{V}_{\mathbb{R}^2}(\det(J_O(x)))$ (in black), the numerical estimate obtained by inspecting the phase plot of system (5) (in blue), the greatest level set of $I(x)$ contained in $\mathcal{X}_{x^o}$ (in green), and the greatest circle that is contained in the level set of $I(x)$ (in purple). Therefore, if the initial condition of system (8) is in the latter set, then the trajectories of system (8) lie in the domain of invertibility of the observability map.

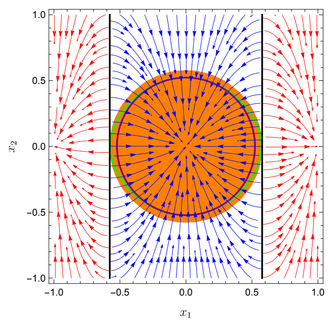
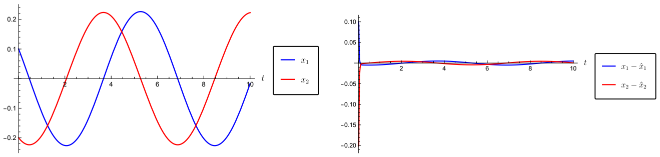


Fig. 5. Obtained estimate $\mathcal{X}_{x^o}$.

A numerical simulation has been carried out to test the observer given in (2). The parameters of the observer have been chosen as $k_1 = 2$, $k_2 = 1$, $\varepsilon = 10^{-3}$, $\mu = 10^2$, $\hat{\xi}(0) = 0$, the initial condition of system (8) have been chosen as $x(0) = [\begin{array}{cc} 0.1 & -0.2 \end{array}]^\top$ (that is in the purple set of Figure 5), and the initial condition of system (3) has been set to $\hat{x}(0) = 0$. Figure 6 depicts the results of such a simulation.



(a) State. (b) Estimation error.

Fig. 6. Simulation results.

As shown by such a figure, if the initial condition of system (8) is in an invariant set that is contained in $\mathcal{X}_{x^o}$, then the observer given in (2) can be used to "practically" estimate the current state of the system.

V. CONCLUSIONS

In this paper, a procedure to determine an estimate of the domain of invertibility of the observability map of a polynomial system has been proposed. Differently from other techniques, the given procedure is exact and does not require the computation of the trajectories of a nonlinear system.

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