Necessary and sufficient conditions for the existence of cycles in evolutionary dynamics of two-strategy games on networks

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Abstract— We study the convergence of evolutionary games on networks, in which the agents can choose between two strategies, by modeling the dynamics as a discrete time Markov process with a finite state space. Based on the transition matrix associated with the Markov process we construct a necessary and sufficient condition for the existence of cycles to evolutionary game dynamics under synchronous updating governed by an arbitrary deterministic update rule. We are able to identify the equilibrium states and cycles and show that under any initial condition, the dynamics converge to either an equilibrium state or a cycle in finite time. A similar result is shown to apply for a general class of asynchronous update rules. For stochastic update rules, we derive a property that is sufficient for the existence of a *unique* limiting matrix, which characterizes the stochastic game dynamics. Consequently, we formulate a necessary and sufficient condition for the existence of cycles that holds for all levels of synchrony in the updating process. We illustrate how our results can be applied in two ways: first, for a given game, one can always calculate the required payoffs to prevent a trajectory to converge to a cycle; second, the effect of network structures on the fixation probability is explored numerically. Since the results hold for arbitrary payoff functions, they also apply to multiplayer games that in general cannot be reduced to an equivalent two-player game.

I. INTRODUCTION

Even in the case when agents can only choose between two strategies, evolutionary games played by a collective of such agents on networks can have a rich set of equilibrium states at which both strategies co-exist. In particular, convergence to an equilibrium state is not a given fact, and in some cases complex *cycles* of adjustments can occur [1], [2]. Studying these complex dynamics has been of interest to scientists in a variety of research fields. Evolutionary biologists use these models to predict whether a mutant or invading species, via natural selection, survives or even takes over an entire population of incumbents [3]. The dynamics of evolutionary games can not only represent a genetic process, but also model social learning or cultural evolution, which, although a simplification of reality, captures crucial aspects of (human) interaction [4]. Economists for instance use evolutionary games to study institutions that emerge over time from the cumulative experience of many individuals that are not fully rational and have incomplete knowledge [5]. Sociologists study how micro decisions in a social networks result in outcomes with certain global social phenomena. When the

macro-level outcome is not favorable, an important corresponding question is how the evolutionary game dynamics can be *influenced* or *designed* to achieve a more desirable outcome for the social system [6]. In this context systems and control theory becomes a powerful tool. In addition to finding equilibrium states, the existence of cycles in the evolutionary dynamics is a key question one may ask, since, especially in many social and engineering applications, it is useful to know whether agents converge to some equilibrium or converge to a never ending cycle of adjustments.

For finite networks the evolutionary game dynamics, usually described by discrete-time nonlinear systems, cannot be approximated faithfully by using a mean field approach, which usually assumes the number of agents is infinite [7]–[9]. Then, for specific initial conditions and network structures, it is possible to obtain analytic expressions for the probability that a single invading strategy takes over the entire network [10], [11]. However, generalizing these results for any initial condition or network structure is a challenging problem [12]. In order to show convergence from any initial condition on an arbitrary graph, a Lypaunov like argument may be employed [13], [14]; however, because there typically exist a rich set of equilibrium states and possibly multiple cycles, finding a suitable potential function is very difficult.

In contrast to the above literature, we model the evolutionary game dynamics as a discrete time Markov-process that allows the analysis of both deterministic and stochastic game dynamics [15]. This method is typically used to model stochastic Moran processes that admits only two homogeneous equilibrium states. Here we apply this analysis method to *arbitrary networks and update rules*. Consequently, we are able to identify all (nontrivial) equilibrium states for both synchronous and asynchronous dynamics. And, because the proposed method is independent of the payoff function, in contrast to other existing analysis methods, our method applies to multiplayer games as well. For synchronous deterministic dynamics, we construct a necessary and sufficient condition for the existence of cycles to the evolutionary game dynamics that applies to an arbitrary update rule. For the synchronous stochastic case we determine a general property of the update rule that is sufficient for the existence of a *unique* limiting matrix describing the stochastic game dynamics. Based on this matrix, a necessary and sufficient condition for the existence of cycles is given. We show that this also applies to the asynchronous stochastic game dynamics. Finally, we study deterministic asynchronous dynamics by formulating a condition on the activation sequence and update rule that again ensures the existence of a unique

limiting matrix describing the asymptotic behavior of the evolutionary game dynamics. Although the computational complexity of constructing the transition matrix of the Markov process is a clear limitation, the approach used in this paper is complementary to other analysis methods in the literature; note in particular that the mean-field approach assumes infinite well-mixed populations and the potential function approach applies only to specific payoff functions and (deterministic) dynamics. Our main contributions can be summarized as follows: First, we apply a powerful analysis method to evolutionary games that is able to characterize all equilibrium states of two-strategy evolutionary games on networks, and the initial conditions leading to convergence to them. Second, by providing sufficient conditions for the existence of a *unique* limiting matrix we identify when this method can also be used to effectively determine if there exist cycles in the evolutionary game dynamics. Third, by studying both synchronous and asynchronous game dynamics we show under which conditions the existence of cycles for both types of dynamics can be studied using the same framework.

II. EVOLUTIONARY GAMES ON NETWORKS

Consider a simple undirected graph $\mathbb{G} = (\mathcal{N}, \mathcal{E})$ where the set of nodes $\mathcal{N} = \{1, \ldots, N\}$ represents agents. Let $\boldsymbol{A}(\mathbb{G}) \in \mathbb{R}^{N \times N}$ denote the adjacency matrix associated with $\mathbb{G}$. On the other hand, for a given adjacency matrix $\boldsymbol{A}$, the associated graph is referred to as $\mathbb{G}(\boldsymbol{A})$. For each agent $n \in \mathcal{N}$, let $s_n \in S_n$ be their strategy, where S_n denotes the finite set of pure strategies. We write all the players' strategies on the network into a vector, $s = [s_1, s_2 \ldots, s_N]^\top$, where $s_n \in S_n$, denotes a *pure strategy profile* or *state vector*. The set of pure strategy profiles or the state space is then given by $\mathcal{S} = S_1 \times S_2 \times \cdots \times S_N$. For any $s \in \mathcal{S}$ and agent n, let $\pi_n(s) \in \mathbb{R}$ be the *payoff* of agent n, given the state s . The payoff vector is then given by $\pi(s) = [\pi_1(s), \ldots, \pi_N(s)]^\top$. We denote agent n's *payoff function* by $\pi_n : \mathcal{S} \to \mathbb{R}$ and the combined payoff function by $\pi : \mathcal{S} \to \mathbb{R}^n$. A *game on a network* is then defined by the triplet $\mathbf{G} = (\mathbb{G}, \mathcal{S}, \pi)$, where $\mathbb{G}$ is the graph describing the topology of the network. The analysis in this paper is restricted to games on networks in which for all agents their sets of pure strategies are the same, and containing only two strategies, i.e. $S_n = \{A, B\}$ for all $n \in \mathcal{N}$, and thus $\mathcal{S} = \{A, B\}^N$.

Now, denote agent n's state at time t by $s_n(t)$. Assume the state vector evolves in discrete time steps according to the (local) states and payoffs in the game:

$$s(t+1) = \rho[s(t), \pi(t)], \ t \in \mathbb{Z}_{\geq 0} \tag{1}$$

where $\rho : \mathcal{S} \times \mathbb{R}^n \to \mathcal{S}$, is the *update rule* that governs the dynamics, and $\mathbb{Z}_{\geq 0}$ denotes the set of nonnegative integers. We denote an evolutionary game $\mathbf{G}$, governed by the update rule ρ as $(\mathbf{G}, \rho)$. For simplicity we assume that all agents update according to the same update rule but the theory developed in this paper is applicable also for more general cases. Agents may update their states simultaneously, and form *synchronous* dynamics. On the opposite, if at each time step only one agent updates her state, it results in

asynchronous dynamics. We study both cases, and show that the resulting dynamics may differ considerably.

The local property of the update rule ρ is defined by the adjacency matrix $\boldsymbol{A}(\mathbb{G})$. When interactions are pairwise the set of edges $\mathcal{E}$ naturally defines local interactions between each neighboring pair in the network. For agents n with degree d_n, this results in d_n two-player, two-strategy games. We also allow for multiplayer games, in which the graph structure defines a group of interacting neighboring agents $\bar{\mathbf{N}}_n = \mathbf{N}_n \cup \{n\}$ where $\mathbf{N}_n = \{m \in \mathcal{N} : a_{nm} > 0\}$. We call $\bar{\mathbf{N}}_n$ the *neighborhood* of agent n in this paper.

Example 1 (Spatial Linear Public Goods Game): Let $A = 1$ and $B = 0$ and $r \geq 1$. A typical multiplayer game is the public goods game with the payoff function [16]

$$\pi_n(t) = \sum_{m \in \bar{\mathbf{N}}_n} \frac{\left(\sum_{l=1}^{N} a_{ml} s_l(t) + s_m(t) \right) r}{1 + d_m} - (d_n + 1) s_n(t). \tag{2}$$

III. PROBLEM FORMULATION

Now that we have formally introduced evolutionary games on networks we continue with defining the concepts of equilibrium states and cycles in the context of evolutionary game theory.

Definition 1: Given $(\mathbf{G}, \rho)$, a state s^* is called an *equilibrium state* of the evolutionary game dynamics governed by update rule (1) if $\rho(s^*, \pi^*) = s^*$. If additionally there exists a pair of agents $n, m \in \mathcal{N}$ such that $s_n^* \neq s_m^*$, then we refer to s^* as a *non-trivial* equilibrium point.

Definition 2: A *cycle* of length T in the evolutionary game, is defined to be a set of states such that $s(t + T) = s(t)$, for any $t \geq 0$.

Definition 3: A *stochastic cycle* in the evolutionary game governed by a stochastic update rule is a set of states $\Gamma = \{s_1, \ldots, s_T\}$ for which the following conditions hold

  i. any trajectory starting from s' in Γ can reach any state in Γ;

  ii. a trajectory starting from any state $s' \in \Gamma \subset \mathcal{S}$ will stay in Γ. Formally, $\Pr[s(t+1) \notin \Gamma : s(t) = s'] = 0$, for all $t \in \mathbb{N}_{\geq 1}$;

For deterministic dynamics, a cycle corresponds to a *periodic solution* of the evolutionary dynamics (1). On the other hand, a stochastic cycle, corresponds to a *non-trivial recurrent class* (introduced in the section IV), in the stochastic evolutionary dynamics in which (1) assigns a certain *probability* to switch the state.
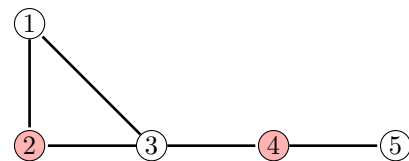

Fig. 1. A simple graph $\mathbb{G}$, with $N = 5$. Red color indicates the state A, and white indicates the state B.

We are interested in characterizing the asymptotic behavior of the two-strategy evolutionary game governed by some deterministic or stochastic update rule ρ. Specifically, given some $(\mathbf{G}, \rho)$ we want to answer the following questions: (i) Are there cycles in the evolutionary game dynamics? (ii) Which states belong to a cycle, and what initial conditions lead to convergence to the cycle (in probability)? (iii) Are there non-trivial equilibrium states, and how can one characterize them? (iv) What is the set of initial conditions that leads to the convergence to some equilibrium state s^* (in probability)?

IV. METHODS

We model the two-strategy evolutionary game on a network as a discrete-time *homogeneous* Markov process on the finite state space $\mathcal{S}$. Let $\mathbf{T} = [t_{ss'}] \in \mathbb{R}^{2^n \times 2^n}$ be the transition matrix, in which $t_{ss'}$ denotes the probability of the transition from state s to s' Formally, the values of the elements in $\mathbf{T}$ are described by the following conditional probabilities $t_{ss'} = \Pr\left[s(t+1) = s' \mid s(t) = s\right]$. Given $(\mathbf{G}, \rho)$, based on the local game interactions and the update rule, one can construct the transition matrix $\mathbf{T} \in \mathbb{R}^{2^n \times 2^n}$, in which each row and column index of $\mathbf{T}$ is associated to a state $s(t) \in \mathcal{S}$ and $s(t+1) \in \mathcal{S}$, respectively. For notational purposes, we denote the transition matrix of the evolutionary game $(\mathbf{G}, \rho)$ as $\mathbf{T}(\mathbf{G}, \rho)$. Naturally, $\sum_{s' \in \mathcal{S}} t_{ss'} = 1$ and hence $\mathbf{T}$ is row stochastic. A state s is called an *absorbing state* when $t_{ss} = 1$ and $t_{ss'} = 0$ for all $s' \in \mathcal{S} \backslash \{s\}$. We say a state s is *accessible* from state s' if, in some finite time, the probability of moving from state s to state s' is positive. A *recurrent class* [5], Ξ of the Markov process is a set of states such that all states in Ξ are accessible from one another, and no state outside Ξ is accessible from any state inside it. A state is called *recurrent* when it belongs to a recurrent class. Obviously, when a recurrent class contains only one state, that state is an absorbing state. When the recurrent class contains more than one state we call it a *nontrivial recurrent class*. All states that do not belong to a recurrent class are *transient states*.

By reordering the rows and columns of $\mathbf{T}$ such that the absorbing states are at the end, the canonical form of the transition matrix is given by $\mathbf{T}_c = \begin{pmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{0}_\eta & \mathbf{I}_\eta \end{pmatrix}$, where η is the number of absorbing states in the discrete time Markov process. Each element in sub-matrix $\mathbf{Q} \in \mathbb{R}^{|\mathcal{S}-\eta| \times |\mathcal{S}-\eta|}$ describes the probability of reaching an absorbing state from the corresponding transient state in one time-step. When each agent $i \in \mathcal{V}$, throughout the course of the game dynamics, updates her state according to one update rule ρ, the transition probabilities will be constant, resulting in a time-homogeneous Markov process. In this case, the probability to reach a state s from state s' in $k > 0$ time steps is given by the ss'th element in the matrix

$$\mathbf{T}_c^k = \begin{pmatrix} \mathbf{P}^k & \sum_{p=0}^{k-1} \mathbf{P}^p \mathbf{Q} \\ \mathbf{0}_\eta & \mathbf{I}_\eta \end{pmatrix}. \qquad (3)$$

Notice that when there are no cycles in the evolutionary game dynamics, all transient states will eventually go to an absorbing state, thus it holds that $\lim_{k \to \infty} \mathbf{P}^k = 0$. Therefore, $|\lambda_i| \leq 1$ for each eigenvalue λ_i of $\mathbf{P}$ and $(\mathbf{I} - \mathbf{P})$ is invertible. In the limit $k \to \infty$ the right upper block of the matrix in (3) is then equal to $\mathbf{R} = (\mathbf{I} - \mathbf{P})^{-1}\mathbf{Q}$, and thus the elements in $\mathbf{R}$ describe the probabilities of transient states to converge to one of the absorbing states of the Markov process. Note that this procedure may not apply when cycles in the evolutionary game dynamics exist. In what follows we will study the characteristics of $\mathbf{T}$ and $\mathbf{T}^\infty$, when such cycles may exist.

V. SYNCHRONOUS DYNAMICS

When the updating process of the evolutionary game is synchronous, at each time step all agents simultaneously update their state. For deterministic update rules this results in a very specific transition matrix, which can be taken advantage of when determining the convergent behavior of the evolutionary game dynamics. Simulation based research has shown that this type of dynamics can exhibit complex periodic behavior, which depends highly on the initial state vector [2]. We start with examining the evolutionary game dynamics governed by an arbitrary *deterministic* update rule that given the current state, payoffs and the set of active agents in the game, determines uniquely the state vector at the next time step. We then continue with *stochastic* update rules in which the update rule is typically a set-valued function.

A. Deterministic update rules

Let η, γ be the number of equilibrium states and the number of cycles that exist in the synchronous deterministic dynamics, respectively. The following theorem can be formulated.

Theorem 1: Given $(\mathbb{G}, \rho)$, for the evolutionary game dynamics (1), governed by the deterministic update rule ρ, the sum of the number equilibrium states and the number of cycles is the same as the algebraic multiplicity of eigenvalue 1, i.e., $\eta + \gamma = |\{\lambda \in \lambda(\mathbf{T}) \mid \lambda = 1\}|$.

Before continuing to the proof of Theorem 1 let us define the following notions from graph theory.

Definition 4: A path $\mathbf{P}$ in graph $\mathbb{G}$ is a nonempty subgraph of $\mathbb{G}$ such that the set of nodes $\mathcal{V}(\mathbf{P}) = \{v_0, v_1, \ldots, v_k\}$ and the set of edges $\mathcal{E}(\mathbf{P}) = \{(v_0, v_1), \ldots (v_{k-1}, v_k)\}$ where each v_i is *distinct*.

Definition 5: A *weakly connected component* of a graph is a *maximum* subgraph of a directed graph in which, ignoring the direction of the edges, there exist a path between any two nodes.

Note that we take a graph that contains merely one node with a self-arc as a weakly connected component.

Now let us associate a *directed* adjacency matrix $\mathbf{A}(\mathbf{T}) \in \mathbb{R}^{2^N \times 2^N}$ to $\mathbf{T}$ such that $a_{ij} = 1$ if $t_{ij} > 0$. In this case, each node in the set $\mathcal{V} = \{1 \ldots 2^N\}$ represents a *state* in $\mathcal{S}$, hence, from now, each index will refer to some *state* $s \in \mathcal{S}$, unless stated otherwise. Because of the fully deterministic

nature of the dynamics it holds that $A(T) = T$. More importantly, at each time step the state is updated to a *unique* consecutive state, resulting in a row stochastic binary transition matrix. And hence $\mathbb{G}(T)$ is a 1-regular graph with *outdegree* $d_i^- = 1 \; \forall i \in \bar{V}$. According to the connectivity, we can decompose the graph $\mathbb{G}(T)$ into g isolated weakly-connected components, among which there is no connection. It is clear that $g \geq 1$, where the equal sign holds only when $\mathbb{G}(T)$ is weakly-connected itself. The following Lemma can now be formulated, of which proof is omitted due to the page limit.

Lemma 1: Each of the g weakly-connected components contains exactly 1 equilibrium state *or* 1 cycle (i.e., 1 recurrent class).

Since the union of the weakly connected components makes up the entire graph $A(T)$, and thus the full state space from Lemma 1, one can conclude that it is only possible to converge to either a cycle or an equilibrium point.

Lemma 2: Each weakly connected component $\mathbb{G}(H)$ has a transpose graph, whose adjacency matrix is $H^\top$, contains a spanning tree.

The proof of Lemma 2 is omitted for brevity. We are now equipped to prove Theorem 1.

Proof.(Proof of Theorem 1) From Lemma 1 and 2 it follows that $\mathbb{G}(T)$ can be composed into weakly connected components $\mathbb{G}(H)$ for which the transpose graph $\mathbb{G}(H^\top)$ contains a spanning tree. Hence, the Laplacian matrix associated to $\mathbb{G}(H^\top)$ denoted by $L(H^\top) = D(H)^\top - A(H)^\top$ has a simple zero eigenvalue (Lemma 1.1 in [17]). Since the *outdegree* (defined in [17]) of all nodes in $\mathbb{G}(H)$ are equal to one, the same can be said for the *indegree* of $\mathbb{G}(H^\top)$. It follows that $L(H^\top) = 1I - A(H^\top)$. Then, the eigenvalues of L are the solutions of $\det(A(H^\top) - (1-\lambda)I) = 0$. Now, denote the spectrum of $A(H^\top)$ by μ, it follows that $\mu = 1 - \lambda$. Then, $A(H^\top)$ must have a simple eigenvalue equal to 1, corresponding to the simple eigenvalue 0 of $L(H^\top)$. In addition, $A(H) = H$ and of course $A(H^\top) = H^\top$. Because the matrices $H^\top$ and H have the same set of eigenvalues, it holds that $\mu(H^\top) = \mu(H)$. Since, the eigenvalues of the weakly connected components make up the eigenvalues of the (super)graph $A(T)$, the number of eigenvalues of $A(T)$ equal to 1 is simply the sum of equilibrium states and cycles in the evolutionary game

Fig. 2. The transition graph of the spatial public goods game (2) (with $6 < r < 12$) on graph 1. The dynamics evolve under unconditional imitation [10].Three weakly connected components exist. The equilibrium states are labeled by 1 and 32, representing the states $(0, 0, 0, 0, 0)$ and $(1, 1, 1, 1, 1)$, respectively. The states labeled by 3 and 5 form a cycle of length 2.

dynamics. The proof is complete. □

Denote the cardinality of a set by $|\cdot|$. Using Lemma 1 and 2 we are able to formulate the following necessary and sufficient condition for the existence of cycles in synchronous deterministic evolutionary dynamics, whose proof is omitted due to the page limit.

Corollary 1: Given $(\mathbb{G}, \rho)$, for an arbitrary deterministic update rule with synchronous updating, cycles to the evolutionary dynamics exist if and only if $\gamma = |\{\lambda \in \lambda(T) : \lambda = 1\}| - \text{trace}(T) > 0$. Moreover, the number of cycles is exactly γ.

Let us consider the general case in which both equilibrium states and cycles exist in the evolutionary game dynamics. The following two proposition answer fundamental questions on the convergence time and domains of attraction of the equilibrium states and cycles. For brevity, the proofs are omitted.

Proposition 1: Starting from any initial condition $s \in \mathcal{S}$, the trajectory of a deterministic and synchronous evolutionary game dynamics converges to an equilibrium or a cycle in a finite time $k(s) \leq 2^n - \eta - 2\gamma$, where η, γ are the numbers of equilibria and cycles, respectively.

Now, let $k = 2^n - \eta - 2\gamma$. Define the important sets: $\mathcal{S}_e \triangleq \{i : T_{ii} = 1\}$, $\mathcal{S}_{te}^i \triangleq \{p : T_{pi}^k = 1, i \in \mathcal{S}_e\}$. Denote $\mathcal{S}_{te} = \cup_{i \in \mathcal{S}_e} \mathcal{S}_{te}^i$. Define $\mathcal{S}_c \triangleq \{j \notin \mathcal{S}_{te} : \exists i : T_{ij}^k = 1\}$, and $\mathcal{S}_{tc}^j \triangleq \{m : T_{mj}^k = 1, j \in \mathcal{S}_c\}$.

Proposition 2: The trajectory converges to the equilibrium $i \in \mathcal{S}_e$, if it starts from any state in the set $\mathcal{S}_{te}^i$; it converges to the cycle that contains j, $j \in \mathcal{S}_c$, if it starts from any state in the set $\mathcal{S}_{tc}^j$.

B. Stochastic update rules

The result in the previous section in general does not apply when the update rule is stochastic. Under stochastic dynamics, since agents update their states in a probabilistic way, given a current state, the update rule does not determine uniquely the state at the next time step. Hence, some states can *in probability* converge to more than one equilibrium point or cycle, and the arguments used in the deterministic case fall apart. We use the following formal expression for a stochastic update rule

$$\tilde{\rho} : \mathcal{S} \times \mathbb{R}^N \to [0, 1]^N, \tag{4}$$

where $[0, 1]^N$ is a vector in which each element takes values from the range $[0, 1]$, that describes the probability for the corresponding agent to play a certain strategy.

It is worth to mention the following important property that indeed holds for many stochastic update rules studied in evolutionary game theory.

Resistance to change property: We say an evolutionary stochastic update rule satisfies the resistance to change property if and only if for all $t \in \mathbb{Z}_{\geq 0}$ and all $n \in \mathcal{N}$, it holds that $\Pr[s_n(t+1) = s_n(t)] \geq \epsilon$. Where ϵ is positive and bounded from below. This property of a stochastic update rule reflects a certain inertia in the decision making process that, as we will show next, can be taken advantage when characterizing
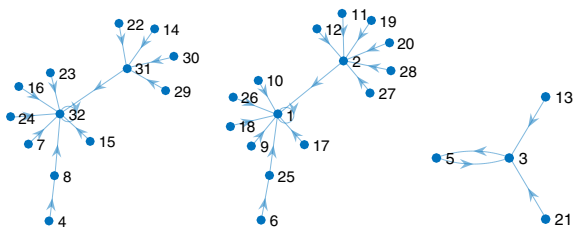
the asymptotic behavior of the evolutionary game dynamics. Using the method introduced in Section IV, one can construct a stochastic matrix $\boldsymbol{T} \in \mathbb{R}^{2^N \times 2^N}$ describing the transitions of the 2^N states in $\mathcal{S}$. The values of the elements in $\boldsymbol{T}$ can be calculated by the updating rule. The following theorem can be formulated:

Theorem 2: For any evolutionary game governed by a stochastic update rule of the form (4) that satisfies the resistance to change property, it is possible to converge to a stochastic cycle *if and only if* there exists an index i, such that $0 < t_{ii}^{\infty} < 1$.

Proof. First observe that because the resistance to change condition is satisfied, for any state $s(t)$, each player has a positive probability to stay at the current state, i.e., $\forall n \in \mathcal{N}$, $\Pr[s_n(t+1) = s_n(t)] \geq \epsilon$, which implies that $\Pr[s(t+1) = s(t)] \geq 0$. This results in an important property of $\boldsymbol{T}$: all the diagonal elements are strictly positive, i.e., $t_{ii} > 0$. A stochastic matrix with positive diagonal entries is aperiodic [18], which implies that $\lim_{k \to \infty} \boldsymbol{T}^k$ exists and is denoted by $\boldsymbol{T}^{\infty}$. By looking at this limiting matrix $\boldsymbol{T}^{\infty}$, we obtain the necessary and sufficient condition for the existence of cycles in the stochastic game. In order to prove the statement, suppose the state i is in a cycle, there must be at least one agent that has some incentive or probability to change her state. The probabilities for this agent to switch states or to remain in the current one are both positive. This claim holds still if there are more than one agents which are possibly switching their states. It follows that the state i has positive probability to remain the same and also positive probability to change. In other words, $t_{ii} > 0$ and $t_{ij} > 0$ for some $j \in \mathcal{S}$. If the state i is in the cycle, we observe that j is certainly in this cycle, which means that after some time steps (depending on the period of the cycle) the trajectory starting from state j returns to i. Without loss of generality, we assume after one step j can go back to i, which implies that $t_{ji} > 0$. It follows that $t_{ii}^{\infty} > 0, t_{ij}^{\infty} > 0$.

We prove the necessity by contradiction. Suppose there does not exist an i such that $0 < t_{ii}^{\infty} < 1$ and the evolutionary game can converge to a cycle. For all $i \in \mathcal{S}$, it either holds that $t_{ii}^{\infty} = 0$ or $t_{ii}^{\infty} = 1$. For those states i that satisfy $t_{ii}^{\infty} = 0$, they are certainly not equilibria or in a cycle; for the rest states i' that satisfy $t_{i'i'}^{\infty} = 1$, they are equilibrium states. Thus we observe that there is no cycle, which results in a contradiction. The proof is complete. $\square$

Remark 1: Notice that the resistance to change property is a sufficient condition on the evolutionary update rule to have a *unique* limiting matrix T^{∞}. When the property is not satisfied, it is not guaranteed that this limiting matrix, describing the asymptotic behavior of the stochastic evolutionary game dynamics, exists because its associated directed graph may be periodic.

VI. ASYNCHRONOUS DYNAMICS

When the updating process is fully asynchronous, at each time step only one agent updates her state. Which agent gets activated can be chosen arbitrarily or proportional to the payoffs. In either case, the activation sequence brings about a

stochastic element in the dynamics that does not come from the update rules but results from the order of activations. This requires a slightly different approach when characterizing the convergence of the evolutionary game compared to the synchronous cases.

A. The activation sequence and reachable sets

We continue by formally introducing the activation sequence and show how to construct the transition matrix of asynchronous evolutionary dynamics. Let the activation sequence be denoted by $\mathcal{K} = \begin{bmatrix} k_1, & k_2, & \ldots \end{bmatrix}$, where $k_t = i$ when at time t, agent i is activated. Note that for full asynchronous dynamics all elements in $\mathcal{K}$ are scalar. Denote the probability that agent i is active at time t by $p_{t,i}$. We make the following assumptions related to the activation sequence $\mathcal{K}$:

Non-exclusive activation assumption: For any time t, and agent $i \in \mathcal{N}$, $p_{t,i} > 0$ and $\sum_{i \in \mathcal{N}} p_{t,i} = 1$.

Persistent activation assumption [13]: For any agent $i \in \mathcal{N}$ active at some time $t \in \mathbb{Z}_{\geq 0}$, there exists some finite time $t' \geq t$, at which agent i is active again.

These assumptions ensure that at each time step, an agent gets activated and all agents have a positive probability to become activated. Moreover, as time goes on, the probability of an agent not being active goes to zero.

Due to the asynchronous updating and the fact that $S_i \in \{0, 1\}$, for all t, it must hold that $||s(t) - s(t+1)||_1 \leq 1$. And for any state $s \in \mathcal{S}$, it must hold that the set of reachable states $\mathcal{F}_s$, resulting from an update rule obeys $\mathcal{F}_s \subseteq \{s^f \in \mathcal{S} : ||s - s^f||_1 \leq 1\}$. Now denote by $\mathcal{J}_{s\bar{s}}$ the set of agents that, given the current state s, if *one* of the agents in that set is active at time t, the state at the next time step is given by $s' \in \mathcal{F}_s$. Formally, $\mathcal{J}_{ss'} = \{i \in \mathcal{N} \,|\, k_t = i, s(t+1) = s', s(t) = s\}$ The probability of reaching state s' in one time step from state s is then given by: $t_{ss'} = \Pr[s(t+1) = s' \mid s(t) = s] = \sum_{w \in \mathcal{J}_{ss'}} p_{t,w} \Pr[s(t+1) = s' \mid s(t) = s, k_t = w] \leq 1$. This probability can be calculated using the update rule. As in the synchronous case by calculating $t_{ss'}$ for all 2^n states a transition matrix $\boldsymbol{T}_a$ can be derived using the method in section IV. It is worth noting that $\sum_{s' \in \mathcal{F}_s} t_{ss'} = \sum_{s' \in \mathcal{F}_s} \Pr[s(t+1) = s' \mid s(t) = s] = \Pr[s(t+1) \in \mathcal{F}_s \mid s(t) = s] = 1$. Hence, $\boldsymbol{T}_a$ is row stochastic.

B. Stochastic update rules

It turns out that the asynchronous case under stochastic update rules is very similar to the synchronous case and hence we are able to extend Theorem 2 to also apply for the asynchronous stochastic dynamics.

Lemma 3: Independent of the order of activation and the level of synchrony in the updating process, any evolutionary game governed by an evolutionary stochastic update rule of the form (4), that satisfies the resistance of change property can be associated to a unique limiting matrix $\boldsymbol{T}^{\infty}$.

The proof follows directly from the proof of theorem 2.

C. Deterministic update rules

For deterministic asynchronous games the resistance to change property obviously cannot be applied. Hence there is a need to define another property guarantee the existence of a non-periodic transition matrix:

Definition 6: A *success-based* update rule is any update rule in which for any t the following statement holds

$$\forall\, j \in \mathcal{N} \text{ such that } \pi_j(t) = \max_{k \in \mathcal{N}_j} \pi_k \Rightarrow s_j(t+1) = s_j(t).$$

We denote all succes-based update rules by ϕ.
For this general class of update rules, the following theorem can be obtained.

Theorem 3: For any evolutionary game $(\mathbb{G}, \phi)$ in which updating occurs asynchronously, it follows that $\boldsymbol{T}^\infty$ exists and is *unique*. Moreover, there exists a cycle in the evolutionary dynamics (1), satisfying (6), if and only if there exists an index i such that $0 < t_{ii}^\infty < 1$.

VII. NUMERICAL APPLICATIONS

A. Exact calculation of fixation probabilities

Consider a graph with N agents with $S_n = \{0, 1\}$ for all $n = 1 \ldots N$. Denote the set of initial condition under which there is exactly one agent with state 1, and $N-1$ agents with state 0 by $\mathcal{S}^0 \triangleq \{s(0) \in \mathcal{S} \mid \sum_{i=1}^{N} s_i(0) = 1\}$. Naturally $|\mathcal{S}^0| = N$. The fixation probability of state 1 is the probability that 1 takes over the whole network of 0-playing agents [10]. Let i^1 be the index of the state in which all individuals have state 1. For any initial condition $s(0) \in \mathcal{S}^0$, the fixation probability can be computed by the following equation $p_{s(0)}^{\text{fix}} = t_{s(0)i^1}^\infty$. What is more interesting, if we take network structures into account, is that the location of agent with state 1 in the network will change the fixation probability drastically (see Figure 3). Hence, when the dynamics are expected to be influenced via payoff manipulations it becomes very important which agent is targeted.

B. Escaping a cycle

For a certain range of the parameter values r in (2) (e.g. $0 < r < 12$), their exists a cycle of length two in the

deterministic version of the evolutionary game described in section VII-A (see also Figure 2). The set of states in the cycle is $\bar{s} = \{(0, 0, 1, 0, 0); (0, 0, 0, 1, 0)\}$. For ease of notation, we use $\bar{s} = \{s, s'\}$. Using (2) one can calculate $\pi(s') = (\frac{1}{4}r, \frac{1}{4}r, \frac{7}{12}r, \frac{13}{12}r - 3, \frac{5}{6}r)$. Suppose one wants to define some additional payoff δ that can be attributed to an agent in order to escape the cycle and converge to the equilibrium state $(1, 1, 1, 1, 1)$. From the transition matrix of the game, we know that the trajectory starting from state $(0, 0, 0, 1, 1)$ converges to this desired equilibrium state. From the unconditional imitation update rule given in [2] it follows that from state s', in order for the fifth agent to switch to state 1 it needs to hold that $\pi_4(s') + \delta > \pi_5(s')$ or equivalently $\frac{13}{12}r - 3 + \delta > \frac{5}{6}r \Rightarrow \delta > 3 - \frac{10}{12}r$. Hence, when the state is s' by assigning the fourth agent with an additional instantaneous payoff $\delta > 3 - \frac{10}{12}r$, the evolutionary game dynamics will converge to the desired equilibrium.

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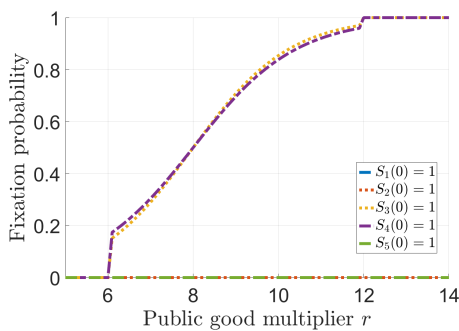
Fig. 3. Fixation probability on the graph in Figure 1 with payoff function (2). The dynamics of the evolutionary game on a graph are governed by synchronous proportional imitation[19]. When initially the 1-playing agent is located on nodes v_1, v_2, v_5, fixation is impossible.