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Computing the admissible reference state-trajectories for differentially non-flat kinematics of non-Standard N-Trailers

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Abstract— **Computing the admissible reference state-trajectory for the non-flat dynamical systems is generally a non-trivial problem. If the system is additionally nonminimum-phase the problem may become especially difficult. Kinematics of the N-trailer vehicles equipped with off-axle hitching belong to such a class of systems. We show how to compute (reconstruct) the non-jackknifing reference trajectories for joint angles of the so-called non-Standard-N-Trailers in the case when the reference output-trajectory is prescribed for a posture of a last trailer. Presentation and discussion of computational algorithms is complemented by selected numerical results.**

I. Introduction

Computing the admissible reference trajectories for a dynamical system belongs to fundamental and practically important problems in control theory. Let $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ represent dynamics of a general system with an associated output $\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x})$, where $\boldsymbol{x} \in \mathbb{R}^n$ is a state, $\boldsymbol{u} \in \mathbb{R}^m$ is a control input, while $\boldsymbol{y} \in \mathbb{R}^\chi$, $\chi < n$, denotes a controlled output. An *admissible* reference state-trajectory $\boldsymbol{x}_r(t)$ can be generally considered as a response of the *exosystem*, [8], i.e. $\dot{\boldsymbol{x}}_r(t) = \boldsymbol{f}(\boldsymbol{x}_r(t), \boldsymbol{u}_r(t))$, for some prescribed initial condition $\boldsymbol{x}_r(0)$, where $\boldsymbol{u}_r(t)$ is a bounded reference input of appropriate properties (imposed by a requirement of admissibility). In many applications, however, one is rather interested in the so-called *inversion problem* [3], where the reference output-trajectory $\boldsymbol{y}_r(t)$ is prescribed a priori as a function of time, while the corresponding reference state-trajectory $\boldsymbol{x}_r(t)$ and the reference input $\boldsymbol{u}_r(t)$ have to be found (reconstructed) upon the knowledge of $\boldsymbol{y}_r(t)$. If dynamics of the considered minimum-phase system is differentially flat, see e.g. [14], [10], and $\boldsymbol{y}$ is the so-called flat output (for $\chi = m$), then computation of $\boldsymbol{x}_r(t)$ reduces to a relatively straightforward algebraic problem since for a given output-trajectory $\boldsymbol{y}_r(t)$ one can find the corresponding reference signals $\boldsymbol{x}_r(t)$ and $\boldsymbol{u}_r(t)$ as functions of $\boldsymbol{y}_r(t)$ and its successive time derivatives. On the other hand, if the system dynamics is not differentially flat (shortly: is non-flat, [5]), finding the reference state $\boldsymbol{x}_r(t)$ corresponding to the prescribed output-trajectory $\boldsymbol{y}_r(t)$ is (much) more difficult [9]. For the special class of non-flat *Liouvillian systems* [2], an additional difficulty comes from the need of time-integration of some exosystem's component equations which do not belong to the largest flat subsystem of the original system dynamics (see [2], [14]).

If the system is not only non-flat but also nonminimum-phase, that is, if its internal dynamics is unstable [7], [3], [6], the problem becomes especially difficult. In this context, a particularly interesting example is the N-trailer system equipped with the off-axle hitches – the structure widely used in transportation applications. It is a well known fact, see [1], [12], that kinematic model of the tractor-trailer vehicles comprising two or more off-axle hitches in a kinematic chain make the N-trailer kinematics a non-flat system (in contrast to the so-called Standard N-Trailers equipped solely with on-axle hitches [13]). Moreover, the off-axle interconnections are a source of the nonminimum-phase property of N-trailer kinematics under specific motion conditions [4]. As a consequence, by choosing a posture of a selected vehicle's segment (usually the last trailer) as a controlled output $\boldsymbol{y}$, it is generally a non-trivial task to compute the *admissible* reference trajectory for the whole state (including the joint angles) corresponding to the prescribed reference output-trajectory $\boldsymbol{y}_r(t)$ which avoids a jackknife effect in both forward and backward motion conditions [11].

Surprisingly, the above problem has not been addressed so far with enough attention in the literature for the non-flat N-trailer kinematics, although it is related to the essential reference-generation and reference-planning problems for these practical systems. The problem becomes especially important at the current times when one observes a rapidly advancing development of highly-automated and unmanned articulated ground vehicles for the commercial and industrial applications.

Our intension in this paper is to fill, to some extent, the gap mentioned above by showing how to compute the admissible (i.e., non-jackknifing) reference state-trajectories for the N-trailers equipped solely with off-axle hitches, that is, for the so-called non-Standard N-Trailer (nSNT) vehicles [4]. It is shown that solution to the reference-generation problem for the nSNT kinematics is not unique, even for simple (i.e., of constant-curvature) output-trajectories. Next, we introduce a strict characterization of the jackknife effect, avoidance of which will be an underlying admissibility criterion for the reference state-trajectories. We explain how the signs of hitching offsets (kinematic parameters of a vehicle) in relation to the reference motion strategy (forward or backward motion), influence a behavior of internal dynamics of the reference exosystem and, as a consequence, affects a difficulty of a solution to the reference-state generation problem. Finally, we propose a numerical computational pro-

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cedure which allows approximating the admissible reference state-trajectories for the nSNT vehicles corresponding even to complex (i.e., of varying-curvature) prescribed reference output-trajectories. Formal considerations included in the paper are illustrated by selected numerical results.

II. Kinematic model of the N-Trailer

Let us consider the N-trailer vehicle (see Fig. 1) comprising a unicycle-like tractor and arbitrary number of N trailers, all interconnected by the off-axle hitched passive rotary joints (the nSNT structure). Kinematic parameters of the vehicle

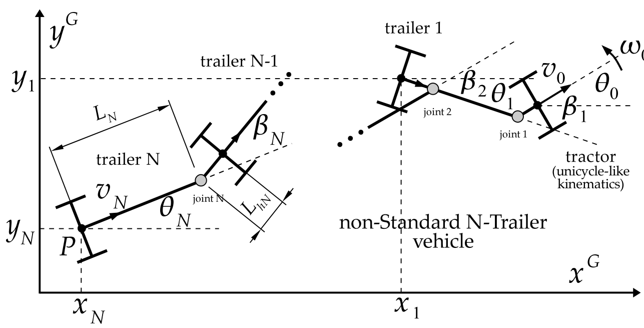

Fig. 1. Kinematic structure of the non-Standard N-Trailer (nSNT) vehicle.

are: the trailer lengths $L_i > 0$, and the hitching offsets $L_{hi} \neq 0$, $i = 1, \ldots, N$. We admit all possible combinations of signs of the hitching offsets, that is, either all positive (i.e., with the hitching point located *behind* the preceding wheels' axle; cf. joint 1 in Fig. 1), all negative (i.e., with the hitching point located *in front of* the preceding wheels' axle; cf. joint 2 in Fig. 1), or mixed signs of offsets (positive and negative) in a vehicle structure (as in Fig. 1).

Configuration of the N-trailer and kinematic control input of the vehicle will be described, respectively, by

$$\boldsymbol{q} \triangleq \begin{bmatrix} \boldsymbol{\beta} \\ \boldsymbol{q}_N \end{bmatrix} \in \mathbb{T}^N \times \mathbb{R}^3 \quad \text{and} \quad \boldsymbol{u}_0 = \begin{bmatrix} \omega_0 \\ v_0 \end{bmatrix} \in \mathbb{R}^2, \qquad (1)$$

where $\boldsymbol{\beta} = [\beta_1 \ \ldots \ \beta_N]^\top$ is a vector of joint-angles which determines a vehicle *shape*, $\boldsymbol{q}_N = [\theta_N \ x_N \ y_N]^\top$ describes a posture of the distinguished segment (the *guidance segment*), while ω_0 and v_0 are, respectively, an angular and longitudinal velocities of a tractor.

Assuming the absence of the skid-slip phenomena in motion of the vehicles' wheels, every ith vehicle segment ($i = 0, \ldots, N$) can be treated as a unicycle (we write $\mathrm{s}\alpha = \sin\alpha$ and $\mathrm{c}\alpha = \cos\alpha$)

$$\dot{\boldsymbol{q}}_i = \boldsymbol{G}(\boldsymbol{q}_i)\boldsymbol{u}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\theta_i & \mathrm{s}\theta_i \end{bmatrix}^\top \boldsymbol{u}_i, \quad i = 0, \ldots, N, \quad (2)$$

with posture $\boldsymbol{q}_i = [\theta_i \ x_i \ y_i]^\top \in \mathbb{R}^3$ and velocity $\boldsymbol{u}_i = [\omega_i \ v_i]^\top \in \mathbb{R}^2$. As a consequence, a kinematic model of the N-trailer can be written as a driftless system (see, e.g., [4])

$$\underbrace{\begin{bmatrix} \dot{\boldsymbol{\beta}} \\ \dot{\boldsymbol{q}}_N \end{bmatrix}}_{\dot{\boldsymbol{q}}} = \underbrace{\begin{bmatrix} \boldsymbol{S}_\beta(\boldsymbol{\beta}) \\ ----- \\ \boldsymbol{S}_N(\boldsymbol{\beta}, \boldsymbol{q}_N) \end{bmatrix}}_{\boldsymbol{S}(\boldsymbol{q})} \boldsymbol{u}_0 = \begin{bmatrix} \boldsymbol{c}^\top \boldsymbol{\Gamma}_1(\beta_1) \\ \boldsymbol{c}^\top \boldsymbol{\Gamma}_2(\beta_2)\boldsymbol{J}_1(\beta_1) \\ \vdots \\ \boldsymbol{c}^\top \boldsymbol{\Gamma}_N(\beta_N)\boldsymbol{J}_{N-1}^1(\boldsymbol{\beta}) \\ -------- \\ \boldsymbol{c}^\top \boldsymbol{J}_N^1(\boldsymbol{\beta}) \\ \boldsymbol{d}^\top \boldsymbol{J}_N^1(\boldsymbol{\beta})\mathrm{c}\theta_N \\ \boldsymbol{d}^\top \boldsymbol{J}_N^1(\boldsymbol{\beta})\mathrm{s}\theta_N \end{bmatrix} \boldsymbol{u}_0, \qquad (3)$$

where $\boldsymbol{\Gamma}_i(\beta_i) \triangleq \boldsymbol{I}_{2\times2} - \boldsymbol{J}_i(\beta_i)$, $\boldsymbol{I}_{2\times2} \in \mathbb{R}^{2\times2}$ is an identity matrix, $\boldsymbol{J}_i^1(\boldsymbol{\beta}) \triangleq \boldsymbol{J}_i(\beta_i) \ldots \boldsymbol{J}_1(\beta_1)$, $\boldsymbol{c}^\top \triangleq [1 \ 0]$, $\boldsymbol{d}^\top \triangleq [0 \ 1]$,

$$\boldsymbol{J}_i(\beta_i) = \begin{bmatrix} -(L_{hi}/L_i)\mathrm{c}\beta_i & (1/L_i)\mathrm{s}\beta_i \\ L_{hi}\,\mathrm{s}\beta_i & \mathrm{c}\beta_i \end{bmatrix} \qquad (4)$$

is a transformation matrix, having a well defined inverse $\boldsymbol{J}_i^{-1}(\beta_i)$ in the case of nSNT kinematics. Matrix (4) maps velocity $\boldsymbol{u}_{i-1}$ into $\boldsymbol{u}_i$ between any two neighboring vehicle segments, that is, $\boldsymbol{u}_i = \boldsymbol{J}_i(\beta_i)\boldsymbol{u}_{i-1}$. By iterative application of the latter mapping, and its inverse, one gets two key relations (valid for $i \in \{1, \ldots, N\}$)

$$\boldsymbol{u}_i = \prod_{j=i}^{1} \boldsymbol{J}_j(\beta_j)\boldsymbol{u}_0, \qquad \boldsymbol{u}_{i-1} = \prod_{j=i}^{N} \boldsymbol{J}_j^{-1}(\beta_j)\boldsymbol{u}_N, \qquad (5)$$

which will be useful in further considerations.

III. Admissible reference trajectories

The reference configuration trajectory for kinematics (3)

$$\boldsymbol{q}_r(t) \triangleq [\boldsymbol{\beta}_r^\top(t) \ \boldsymbol{q}_{Nr}^\top(t)]^\top \in \mathbb{T}^N \times \mathbb{R}^3 \qquad (6)$$

consists of the reference *shape-trajectory* $\boldsymbol{\beta}_r(t)$ and the reference *guidance-trajectory* $\boldsymbol{q}_{Nr}(t)$. Any *admissible* reference trajectory (6) should be *compatible* with kinematics (3) by satisfying dynamics of the exosystem

$$\dot{\boldsymbol{q}}_r(t) = \boldsymbol{S}(\boldsymbol{q}_r(t))\boldsymbol{u}_{0r}(t), \qquad (7)$$

with $\boldsymbol{u}_{0r}(t)$ being a reference velocity determined for the tractor segment.

In most cases of tracking control problems defined for the N-trailers a primary objective is to guide the distinguished vehicle's segment (the guidance segment) along a time-parametrized reference path prescribed a priori. In this case, $\boldsymbol{q}_{Nr}(t)$ becomes the reference output-trajectory for the N-trailer system. A subordinate reference shape-trajectory $\boldsymbol{\beta}_r(t)$ should correspond to the prescribed guidance-trajectory through equation (7). Therefore, it is useful to rewrite (7) in an alternative form which better reflects the above hierarchical formulation of the desired motion for the N-trailer kinematics. Namely, by recalling (5) one may write

$$\begin{aligned} \dot{\boldsymbol{q}}_r(t) &= \boldsymbol{S}(\boldsymbol{q}_r(t)) \prod_{j=i}^{N} \boldsymbol{J}_j^{-1}(\beta_{jr}(t))\boldsymbol{u}_{Nr}(t) \\ &= \begin{bmatrix} \boldsymbol{S}_\beta(\boldsymbol{\beta}_r(t)) \prod_{j=i}^{N} \boldsymbol{J}_j^{-1}(\beta_{jr}(t)) \\ \boldsymbol{G}(\boldsymbol{q}_{Nr}(t)) \end{bmatrix} \boldsymbol{u}_{Nr}(t), \end{aligned} \qquad (8)$$

where $\boldsymbol{u}_{Nr}(t) = [\omega_{Nr}(t) \ v_{Nr}(t)]^\top \in \mathbb{R}^2$ is a reference *guiding-velocity* prescribed along the guidance-trajectory $\boldsymbol{q}_{Nr}(t)$, while the form of matrix $\boldsymbol{G}$ in (8) results from (2) taken for $i = N$.

A. The admissible reference guidance-trajectory

Let the reference guidance-trajectory

$$\boldsymbol{q}_{Nr}(t) = [\theta_{Nr}(t)\ x_{Nr}(t)\ y_{Nr}(t)]^\top \in \mathbb{R}^3 \tag{9}$$

satisfy the following admissibility conditions:

C1. $\forall\, t \geq 0\ \dot{\boldsymbol{q}}_{Nr}(t) = \boldsymbol{G}(\boldsymbol{q}_{Nr}(t))\boldsymbol{u}_{Nr}(t)$,
C2. $\forall\, t \geq 0\ \|\boldsymbol{u}_{Nr}(t)\| \neq 0$,
C3. $\forall\, t \geq 0\ \|\boldsymbol{u}_{Nr}(t)\| < \infty$, $\|\dot{\boldsymbol{u}}_{Nr}(t)\| < \infty$.

Condition C1 (being a consequence of (8)) guarantees satisfaction of the nonholonomic constraints imposed due to the unicycle-like kinematics (2), C2 represents a general *persistent excitation* condition for the guidance-trajectory (9), while C3 ensures boundedness and some minimal degree of smoothness of the reference guiding-velocity along $\boldsymbol{q}_{Nr}(t)$. In other words, the reference guidance-trajectory (9) can be any sufficiently smooth and persistently exciting reference trajectory feasible for the unicycle kinematics.

Since the unicycle-like kinematics in condition C1 is a differentially flat system with flat outputs (x_{Nr}, y_{Nr}), designing the reference guidance-trajectory is easy, [14].

B. The admissible reference shape-trajectory

The main difficulty lies in computation of the corresponding reference trajectory for the shape variables, i.e., for the joint angles. According to (8), one observes that an admissible reference shape-trajectory $\boldsymbol{\beta}_r(t)$ is related with the reference guidance-trajectory $\boldsymbol{q}_{Nr}(t)$ only through the guiding-velocity $\boldsymbol{u}_{Nr}(t)$ satisfying the differential equation

$$\dot{\boldsymbol{\beta}}_r(t) = \boldsymbol{S}_\beta(\boldsymbol{\beta}_r(t)) \prod_{j=i}^{N} \boldsymbol{J}_j^{-1}(\beta_{jr}(t))\boldsymbol{u}_{Nr}(t). \tag{10}$$

Remark 1: From now on, by the *steady-solution* of (10) one has to understand solely its *steady-component*, disregarding any transient-component resulting from a possible selection of an incompatible initial value $\boldsymbol{\beta}_r(0)$.

Since kinematics (8) is a non-flat system, it is not known in general (except the simplest cases) how to analytically find a closed-form of the admissible reference shape-trajectory $\boldsymbol{\beta}_r(t) = [\beta_{1r}(t) \dots \beta_{Nr}(t)]^\top \in \mathbb{T}^N$ corresponding to the admissible reference guidance-trajectory (9). Moreover, in practical applications one usually wants to avoid a jackknife effect responsible for vehicle folding in the joints. Hence, avoidance of the jackknife should be imposed also on the reference shape-trajectory $\boldsymbol{\beta}_r(t)$ as an additional admissibility condition. In the literature, the jackknife effect is explained in various (usually only descriptive and non-strict) ways for the N-trailers. Following [11], let us formulate an additional (*anti-jackknife*) admissibility condition

C4. $\forall\, \boldsymbol{\beta}_r(t)\quad v_{i-1r}(\boldsymbol{\beta}_r(t), \boldsymbol{u}_{Nr}(t)) \cdot v_{ir}(\boldsymbol{\beta}_r(t), \boldsymbol{u}_{Nr}(t)) > 0$ for all $i = 1, \dots, N$,

where

$$v_{ir}(\boldsymbol{\beta}_r(t), \boldsymbol{u}_{Nr}(t)) = \boldsymbol{d}^\top \prod_{j=i+1}^{N} \boldsymbol{J}_j^{-1}(\beta_{jr}(t))\boldsymbol{u}_{Nr}(t) \tag{11}$$

is the reference longitudinal velocity for the ith vehicle's segment. Physical interpretation of condition C4 is simple and general: it admits only those reference shape-trajectories which lead to the same signs of the reference longitudinal velocities for all the vehicle's segments. Such a subset of reference trajectories has been called in [11] the Segment-Platooning (S-P) trajectories. Figure 2 explains interpretation of condition C4. On the picture, the tractor and first two trailers are moving backward along a circular reference guidance-trajectory, while the last trailer moves either forward (left subplot) or backward (right subplot). Both the reference motions are possible but only the reference trajectory on the right-hand side picture in Fig. 2 satisfies condition C4.

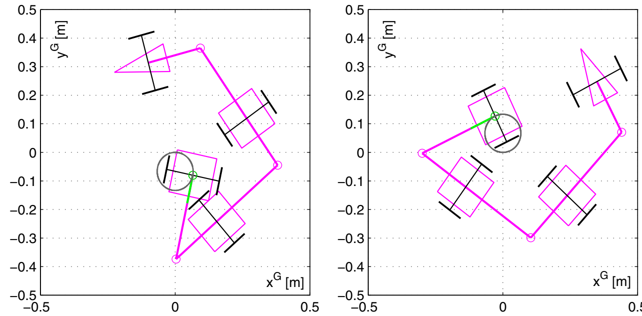

Fig. 2. Illustration of violation (left) and satisfaction (right) of the admissibility condition C4 for a nS3T vehicle in the case of an exemplary circular guidance-trajectory (denoted in grey).

It can be shown (see Section IV), that a steady-solution of equation (10) is not unique. There is a conjecture that equation (10) possesses (at least) 2^N distinct steady-solutions corresponding to the same velocity $\boldsymbol{u}_{Nr}(t)$. They result from two possible motions for every ith joint: one satisfying C4 and one violating C4. Thus, the imposition of condition C4 limits our interest only to a strict subset of all the possible shape-trajectories compatible with $\boldsymbol{q}_{Nr}(t)$. This fact additionally hinders the problem of finding an admissible shape-trajectory $\boldsymbol{\beta}_r(t)$. A possible approach to solving this non-trivial problem will be discussed in the next section.

IV. Computing the reference shape-trajectories

A. The case of constant-curvature guidance-trajectories

It is well known, see e.g. [11], that for the constant-curvature reference guidance-trajectories (corresponding to the rectilinear or circular trajectories for the positional coordinates $x_{Nr}(t), y_{Nr}(t)$) with $\boldsymbol{u}_{Nr} = \text{const}$ it is possible to find a closed-form steady-solution of (10). The steady-solution, however, is not unique; one gets 2^N sets of reference joint angles $\boldsymbol{\beta}_r$ corresponding to the same image of the reference guidance-trajectory. Figure 3 illustrates exemplary sets (denoted by S1 to S8) of 8 possible steady-solutions of (10) computed for the nS3T kinematics in the case of a circular reference guidance-trajectory with guiding-velocity $\boldsymbol{u}_{Nr} = [0.2\,\text{rad/s}\ \ 0.12\,\text{m/s}]^\top$. Note that only set S3 satisfies the admissibility condition C4 preventing occurrence of a jackknife effect. The admissible set of reference joint angles $\boldsymbol{\beta}_r = [\beta_{1r}\ \dots\ \beta_{Nr}]^\top$ for the constant-curvature guidance-trajectories can be computed by the iterative formula

$$\beta_{ir} = \begin{cases} 0 & \text{if} \quad \omega_{Nr} \equiv 0 \\ \text{Atan2}\,(z_{1i}, z_{2i}) & \text{if} \quad \omega_{Nr} \not\equiv 0 \end{cases}, \tag{12}$$

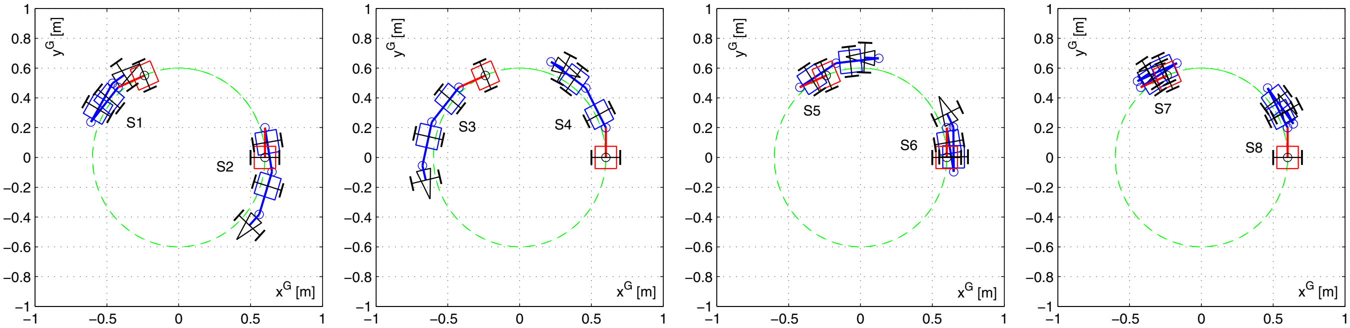

Fig. 3. Illustration of $2^3 = 8$ possible sets (S1 to S8) of reference joint angles $\boldsymbol{\beta}_r$ for the nS3T vehicle corresponding to the same circular reference guidance-trajectory denoted in green (only set S3 satisfies condition C4); the last trailer (being the guidance segment) has been highlighted in red.

where Atan2 $(,)$ is a four-quadrant inverse tangent function,

$$z_{1i} = L_i R_{i-1r} + L_{hi} R_{ir}, \;\; z_{2i} = R_{ir} R_{i-1r} - L_i L_{hi}, \quad (13)$$

$$R_{Nr} = \frac{v_{Nr}}{\omega_{Nr}}, \quad R_{ir} = \xi_i \sqrt{R_{i+1r}^2 + L_{i+1}^2 - L_{hi+1}^2}, \quad (14)$$

while $\xi_i \in \{-1, +1\}$ determines the signs of the reference curvature radii R_{ir} for particular vehicle's segments, and it should be selected as follows

$$\xi_i := \text{sgn}(v_{Nr}/\omega_{Nr}) \quad \text{for} \quad i \in \{0, \ldots, N-1\}. \quad (15)$$

Selection (15) is crucial, since it guarantees the same signs of steady curvature radii for all the vehicle segments and, as a consequence, satisfaction of admissibility condition C4.

B. The case of varying-curvature guidance-trajectories and the sign-homogeneous hitching offsets in the N-trailer

In conditions of varying-curvature guidance-trajectories with $\boldsymbol{u}_{Nr}(t) \neq$ const, it is generally not possible to find an analytical form of steady-solution to equation (10) which is time-varying in this case. In particular, for a periodic reference guiding-velocity $\boldsymbol{u}_{Nr}(t)$, the reference shape-trajectory can be a limit cycle of dynamics (10). Again, the limit cycle is not unique – there exist probably (at least) 2^N different limit cycles corresponding to the same velocity profile $\boldsymbol{u}_{Nr}(t)$. To find an admissible one, it is possible to numerically integrate equation (10), however this approach has some limitations. Upon the results of [11], [4], and numerous simulation investigations, one observes that numerical integration of (10) for $t \to \infty$ (forward time-flow), that is,

$$\boldsymbol{\beta}_r(t) = \boldsymbol{\beta}_r(0) + \int_0^t \boldsymbol{S}_\beta(\boldsymbol{\beta}_r(\tau)) \prod_{j=i}^{N} \boldsymbol{J}_j^{-1}(\beta_{jr}(\tau)) \boldsymbol{u}_{Nr}(\tau) d\tau$$

will converge, at least locally for some set of initial conditions $\boldsymbol{\beta}_r(0)$, to the steady-solution satisfying C4 if

$$\forall\, i \;\; (v_{Nr}(t)/L_{hi}) < 0. \quad (16)$$

On the other hand, numerical integration of (10) for $t \to -\infty$ (backward time-flow), will converge, at least locally for some set of initial conditions $\boldsymbol{\beta}_r(0)$, to the steady-solution satisfying C4 if

$$\forall\, i \;\; (v_{Nr}(t)/L_{hi}) > 0. \quad (17)$$

Both conditions, (16) and (17), impose a requirement of the common signs for all the hitching offsets in the N-trailer (the *sign-homogeneous* hitching). It is a direct consequence of the nonminimum-phase property of nSNT kinematics, see [4].

Limitations (16)-(17) indicate that in the case of mixed signs of hitching offsets present in a vehicle chain (that is when $\exists\, i, j : \;\; L_{hi} L_{hj} < 0$), numerical integration of (10), regardless forward or backward in time, is no more helpful in finding an admissible reference shape-trajectory since the numerical solution of (10) is not convergent to a steady-solution satisfying C4 (i.e., it will converge to a jackknifing steady-solution satisfying (8) but violating C4).

C. The case of varying-curvature guidance-trajectories and the mixed-signs hitching offsets in the N-trailer

For the case of mixed-signs hitching offsets in a vehicle chain, there is a need of alternative computational method which will allow finding an approximation of the admissible steady-solution of (10). We are going to provide such a method by formulating the problem as an optimization task [15]. The method is valid under assumption that the prescribed reference guiding-velocity $\boldsymbol{u}_{Nr}(t)$ is periodic, i.e.,

$$\boldsymbol{u}_{Nr}(t) = \boldsymbol{u}_{Nr}(t + T), \quad (18)$$

where a finite number $T > 0$ denotes a period.

Remark 2: It is worth noting that assumption on periodicity of $\boldsymbol{u}_{Nr}(t)$, although limiting to some extent, can still be useful in practical applications. Periodicity of $\boldsymbol{u}_{Nr}(t)$ may naturally correspond to the closed (periodic) reference positional trajectory $(x_{Nr}(t), y_{Nr}(t))$, which has a closed-curve image on a motion plane. In this case, the admissible reference shape-trajectory $\boldsymbol{\beta}_r(t)$ should be obviously also periodic with the same period T (being a limit cycle of dynamics (10)). However, it is not the only possibility. One can also design non-periodic reference positional trajectories $(x_{Nr}(t), y_{Nr}(t))$ (e.g., by using splines) which have open-curve images on a motion plane, but impose the same values of the guiding-velocity on their ends, that is, when $(x_{Nr}(0), y_{Nr}(0)) \neq (x_{Nr}(T), y_{Nr}(T))$ but $\boldsymbol{u}_{Nr}(0) = \boldsymbol{u}_{Nr}(T)$. In this case, one can treat $\boldsymbol{u}_{Nr}(t)$ as *virtually* periodic.

Rewrite (10) in a shorter form

$$\dot{\boldsymbol{\beta}}_r(t) = \boldsymbol{f}\left(\boldsymbol{\beta}_r(t), \boldsymbol{u}_{Nr}(t)\right). \quad (19)$$

Let us express a numerical approximation $\hat{\boldsymbol{\beta}}_r(t) = [\hat{\beta}_{1r}(t)\ \dots\ \hat{\beta}_{Nr}(t)]^\top$ of the sought steady-solution of (19) in the parameterized form

$$\hat{\beta}_{ir}(t) = \boldsymbol{\Phi}^\top(t)\boldsymbol{p}_i, \qquad i \in \{1,\dots,N\}, \tag{20}$$

where $\boldsymbol{\Phi}(t) \in \mathbb{R}^p$ is a vector of some basis functions, and $\boldsymbol{p}_i \in \mathbb{R}^p$ is a set of parameters for the ith component of trajectory $\hat{\boldsymbol{\beta}}_r(t)$. Since the postulate (18) entails (virtual) periodicity of the sought steady-solution of (19), we select the Fourier basis of a basic frequency $2\pi/T$ to define the components of vector $\boldsymbol{\Phi}(t)$. Define the approximation error

$$\begin{aligned}\boldsymbol{\epsilon}(t,\boldsymbol{p}) &\triangleq \dot{\hat{\boldsymbol{\beta}}}_r(t) - \boldsymbol{f}(\hat{\boldsymbol{\beta}}_r(t), \boldsymbol{u}_{Nr}(t)) \qquad (21)\\ &\overset{(20)}{=} \begin{bmatrix}\dot{\boldsymbol{\Phi}}^\top(t)\boldsymbol{p}_1\\ \vdots \\ \dot{\boldsymbol{\Phi}}^\top(t)\boldsymbol{p}_N\end{bmatrix} - \boldsymbol{f}\left(\begin{bmatrix}\boldsymbol{\Phi}^\top(t)\boldsymbol{p}_1\\ \vdots \\ \boldsymbol{\Phi}^\top(t)\boldsymbol{p}_N\end{bmatrix}, \boldsymbol{u}_{Nr}(t)\right),\end{aligned}$$

which is the difference between the left- and right-hand side of equation (19) evaluated along the approximating trajectory $\hat{\boldsymbol{\beta}}_r(t)$. By using the elements of a Fourier basis in $\boldsymbol{\Phi}(t)$, a closed-form of the term $\dot{\boldsymbol{\Phi}}(t)$ can be derived analytically. For the sake of numerical computations, the error (21) is sampled with a sampling interval $T_s > 0$, being an integer multiplicity of period T. It leads to the approximation error samples $\boldsymbol{\epsilon}(kT_s, \boldsymbol{p})$, where $k = 0, 1, \dots, M$, and $MT_s = T$.

Now, the problem of finding the approximating trajectory $\hat{\boldsymbol{\beta}}_r(t)$, with the components parameterized by (20), can be formulated as a 'curve fitting' optimization problem

$$\boldsymbol{p}^* = \min_{\boldsymbol{p}} J(\boldsymbol{p}), \quad \boldsymbol{p} = [\boldsymbol{p}_1^\top\ \dots\ \boldsymbol{p}_N^\top]^\top \in \mathbb{R}^{pN} \tag{22}$$

with the quadratic quality functional defined as

$$J(\boldsymbol{p}) \triangleq \boldsymbol{e}^\top(\boldsymbol{p})\,\boldsymbol{e}(\boldsymbol{p}), \tag{23}$$

where $\boldsymbol{e}(\boldsymbol{p}) \in \mathbb{R}^{(M+1)N}$ and

$$\boldsymbol{e}(\boldsymbol{p}) = [\boldsymbol{\epsilon}^\top(0,\boldsymbol{p})\ \boldsymbol{\epsilon}^\top(T_s,\boldsymbol{p})\ \dots\ \boldsymbol{\epsilon}^\top(MT_s,\boldsymbol{p})]^\top \tag{24}$$

includes the approximation errors' samples computed for the $M+1$ evenly distributed time instants along the whole single (virtual) period T of the sought steady-solution of (19).

To solve the problem (22)-(23), we propose to use the Newton's iterative algorithm

$$\boldsymbol{p}(l) = \boldsymbol{p}(l-1) - \kappa \boldsymbol{H}^\sharp(\boldsymbol{p}(l-1))\boldsymbol{e}(\boldsymbol{p}(l-1)), \tag{25}$$

where $l \in \mathbb{N}$ represents the iteration number, $\kappa > 0$ is a design coefficient, $(\cdot)^\sharp$ denotes the left-pseudoinverse operator, while the closed-form of matrix

$$\boldsymbol{H}(\boldsymbol{p}) \triangleq \frac{\partial \boldsymbol{e}(\boldsymbol{p})}{\partial \boldsymbol{p}} \in \mathbb{R}^{(M+1)N \times pN} \tag{26}$$

can be computed analytically thanks to the form of the approximation error (21).

Since we conjecture (by analogy to the case considered in Section IV-A) that equation (19) possesses (at least) 2^N (virtually) periodic steady-solutions, one has to expect that iteration (25) will be only locally convergent to the particular steady-solution which we are interested in. Being interested only in the admissible steady-solution satisfying condition C4, one has to properly initialize (25) finding a favorable initial condition $\check{\boldsymbol{p}} = \boldsymbol{p}(0)$.

To select a favorable initial vector $\check{\boldsymbol{p}} = [\check{\boldsymbol{p}}_1^\top\ \dots \check{\boldsymbol{p}}_N^\top]^\top \in \mathbb{R}^{pN}$ we propose to find the component vectors $\check{\boldsymbol{p}}_i$ for $i = 1, \dots, N$ by referring to the analytical form of the admissible steady-solution determined by (12)-(15). Let us define a quasi-stationary sampled trajectory $\check{\boldsymbol{\beta}}_r(kT_s)$ being a solution of the equation

$$\boldsymbol{f}\left(\check{\boldsymbol{\beta}}_r(kT_s), \boldsymbol{u}_{Nr}(kT_s)\right) = \boldsymbol{0} \tag{27}$$

for any fixed index k. Equation (27) should be solved with respect to $\check{\boldsymbol{\beta}}_r(kT_s)$ according to (12)-(15) for all $k \in \{0, 1, \dots, M\}$. Note that components of $\check{\boldsymbol{\beta}}_r(kT_s)$ computed in this way satisfy the admissibility condition C4. Next, by recalling the linear parameterization (20), one may compute

$$\check{\boldsymbol{p}}_i = \begin{bmatrix}\boldsymbol{\Phi}^\top(0)\\ \vdots \\ \boldsymbol{\Phi}^\top(MT_s)\end{bmatrix}^\sharp \begin{bmatrix}\check{\beta}_{ir}(0)\\ \vdots \\ \check{\beta}_{ir}(MT_s)\end{bmatrix} \tag{28}$$

for all $i = 1, \dots, N$, where $(\cdot)^\sharp$ is the left-pseudoinverse operator, and $\check{\beta}_{ir}$ is the ith component of $\check{\boldsymbol{\beta}}_r$. Since the initial guess (28) is computed upon the samples $\check{\beta}_{ir}(kT_s)$ which satisfy condition C4, one expects that the parameters $\boldsymbol{p}$ iterated by algorithm (25) started at $\check{\boldsymbol{p}}$ will be locally convergent to a point $\boldsymbol{p}^*$ corresponding to an optimal approximation of the sought admissible shape-trajectory $\boldsymbol{\beta}_r(t)$.

Two examples provided below illustrate quality of the above computational procedure obtained for the nS3T kinematics with trailer lengths $L_i = 0.25\,\text{m}$ and hitching offsets $L_{h1} = L_{h3} = +0.05\,\text{m}$, $L_{h2} = -0.05\,\text{m}$ (mixed-signs hitching). In the first example we have selected the reference guidance-trajectory $\boldsymbol{q}_{3r}(t)$ corresponding to a positional-image of the Cassini oval (a closed-curve path) in a motion plane, while for the second example we have chosen the reference guidance-trajectory corresponding to a positional-image of two smoothly-concatenated 7-degree polynomials (an open-curve path) linking three prescribed way-points P0-P2, see Fig. 4.

The results, obtained for the forward reference motion, have been presented in Figs. 5-6. In both cases of the prescribed guidance-trajectories, the sampling interval $T_s := T/M\,\text{s}$ was taken for computations with $M = 2000$, while the selected numbers of harmonics were equal to 100 and 1000 for the closed-curve case and the open-curve case, respectively. It is worth noting that both reference trajectories satisfy the admissibility condition C4 avoiding a jackknife effect in the reference vehicle's joints (all the reference velocities $v_{ir}(t)$ computed upon (11) posses the same signs). Periodicity of signals in Fig. 5 (with period $T = 60\,\text{s}$) is natural due to the closed-curve form of the reference guidance-trajectory in this case. On the other hand, periodicity of signals in Fig. 6 (with virtual period $T = 40\,\text{s}$) is not obvious since the reference guidance-trajectory is not in a closed-curve form in this case; periodicity is here a side-effect of using the Fourier basis in $\boldsymbol{\Phi}(t)$.

Fig. 4. The shapes of selected curves for the positional reference guidance-trajectory (left), and values of quality indexes of shape-trajectories' computations (middle and right) obtained for the two selected reference guidance-trajectories.

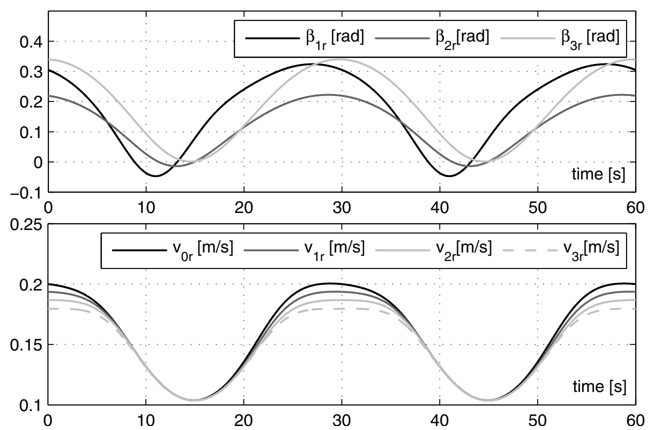

Fig. 5. Computed components of the admissible shape-trajectory $\boldsymbol{\beta}_r(t)$ and reference longitudinal velocities $v_{ir}(t)$ corresponding to the guidance-trajectory with the closed-curve image (Cassini oval) on a motion plane.

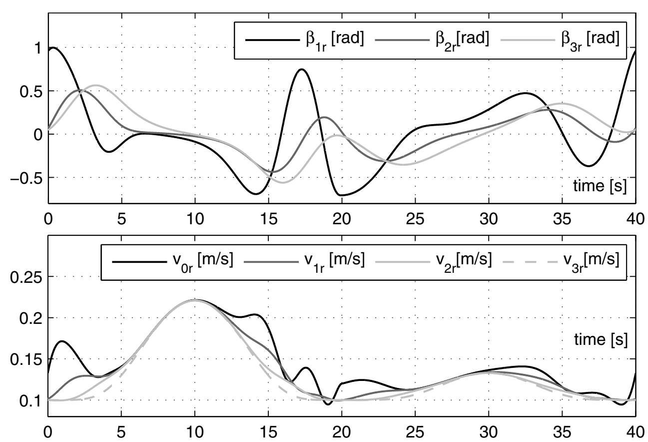

Fig. 6. Computed components of the admissible reference shape-trajectory $\boldsymbol{\beta}_r(t)$ and reference longitudinal velocities $v_{ir}(t)$ corresponding to the guidance-trajectory with the open-curve image (splines) on a motion plane.

Two additional graphs presented in Fig. 4 show the values of computational-quality indexes, $J_1 \triangleq T_s \cdot J(\boldsymbol{p}^*)$ and $J_2 \triangleq (\int_0^T \| \boldsymbol{q}_{3r}(\tau) - \hat{\boldsymbol{q}}_{3r}(\tau)\| \, d\tau)/T$, as functions of a number N_h of harmonics used in the vector $\boldsymbol{\Phi}(t)$. Index J_2 directly indicates how the computed approximation $\hat{\boldsymbol{\beta}}_r(t)$ affects accuracy of the approximate reference guidance-trajectory $\hat{\boldsymbol{q}}_{3r}(t)$, relative to the true one $\boldsymbol{q}_{3r}(t)$, when reconstructed by numerical integration of (7) taking $\boldsymbol{u}_{0r}(t) := \hat{\boldsymbol{u}}_{0r}(t) = \prod_{j=1}^{N} \boldsymbol{J}_j^{-1}(\hat{\beta}_{rj}(t))\boldsymbol{u}_{3r}(t)$. One observes that for a guidance-trajectory of the closed-curve form one requires less number of harmonics to get a good approximation accuracy when compared to the open-curve guidance-trajectory. The best values of index J_2, although biased by the cumulating integration errors, indicate acceptable quality of the computed reference trajectories for a reasonable number of harmonics.

V. Final remarks

In the paper, we have shown how the admissible reference state-trajectories can be computed for the non-flat kinematics of non-Standard N-Trailers which reveals a nonminimum-phase property in some motion conditions. The problem under consideration is non-trivial and still remains open to some extent. In particular, it is not clear how to find an admissible reference shape-trajectory for the case when the reference guiding-velocity does not satisfy assumption (18). Finally, it would be interesting to extend application of the proposed method to the N-trailers admitting mixed off-axle and on-axle hitches. It seems difficult, however, because for the on-axle hitching the equation (8) is not valid due to singularity of transformation matrix (4).

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