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# Structured linear fractional parametric controller $\mathcal{H}_\infty$ design and its applications

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***Abstract*— This paper proposes a simple but yet effective approach to structured parametric controller design in a linear fractional form. The main contribution consists in using structured $\mathcal{H}_\infty$ oriented optimization tools in an original manner to either construct a parametric controller or a family of controllers with varying performances. Practical and numerical issues are also discussed to provide practitioners a simple way to deploy the proposed process. The overall approach is illustrated through two numerical examples: first, a controller parametrized by the model characteristics applied on a clamped beam model and second, a parametric performance controller applied on a very complex fluid flow control setup.**

## I. INTRODUCTION

### A. Motivating context and problem formulation

In numerous industrial and research applications, the $n$-th order $n_u$ inputs $n_y$ outputs linear dynamical model describing a system can either be given in an invariant form as $\mathbf{H}(s) = C(sI_n - A)^{-1}B + D \in \mathcal{H}_\infty^{n_y \times n_u}$, equipped with realization $\mathcal{S} : (A, B, C, D)$ defined as

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t), \quad (1)$$

or in a parametric form $\mathbf{H}(s, \mathbf{p}) = C(\mathbf{p})\big(sI_n - A(\mathbf{p})\big)^{-1}B(\mathbf{p}) + D(\mathbf{p}) \in \mathcal{H}_\infty^{n_y \times n_u}$, equipped with realization $\mathcal{S}(\mathbf{p}) : \big(A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p}), D(\mathbf{p})\big)$ defined as

$$\dot{\mathbf{x}}(t) = A(\mathbf{p})\mathbf{x}(t)+B(\mathbf{p})\mathbf{u}(t), \mathbf{y}(t) = C(\mathbf{p})\mathbf{x}(t)+D(\mathbf{p})\mathbf{u}(t), \quad (2)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^{n_u}$, $\mathbf{y}(t) \in \mathbb{R}^{n_y}$ and $\mathbf{p} \in \mathcal{P} \subseteq \mathbb{R}^{n_p}$ represent the state, input, output and parameter vectors, respectively. Moreover, the $\mathcal{P}$ subspace is closed, the Laplace variable is denoted $s$ and the $A$, $A(\mathbf{p})$, $B$, $B(\mathbf{p})$, $C$, $C(\mathbf{p})$, $D$ and $D(\mathbf{p})$ matrices are of appropriate dimension[1].

*Remark 1 (**LPV** vs. parametric):* It is noteworthy to distinguish the **parametric** form with the Linear Parameter Varying (**LPV**) one. Indeed while in the latter case the parameter $\mathbf{p}$ is considered as varying, in the former one, which we consider in this paper, the parameter simply is a frozen physical system coefficient (*e.g.* the geometrical parameters of an aircraft wing [1], or the section of an open-channel [2]) or controller tuning coefficient. Parametric models can appear when the system's model is $\mathbf{p}$ dependent or when the $\mathbf{p}$ parameter is an artificial one that characterizes the closed-loop performances (see the example section).

In the parametric case $\mathbf{H}(s, \mathbf{p})$ (2), it is interesting, in a pre-design phase, to be able to construct a controller as a function of the parameter $\mathbf{p}$ value, that reaches a given performance level. Similarly, in the non-parametric one $\mathbf{H}(s)$ (1), for practical reason, one can be interested in constructing a $\mathbf{p}$ parametric dependent controller, achieving varying performances, which can be tested and directly adjusted on the real system during tests validations[2]. Parametric controller design is then clearly a challenging task for many industrial applications as it provides the possibility to tune the control performance according to the plant configuration or to provide practitioners the ability to test a family of control laws in a simpler manner.

Mathematically, in the considered framework, given a model as in (1) or (2), we aim at synthesizing a $n_K$-th order $\mathbf{p}$ dependent controller $\mathbf{K}^\star(s, \mathbf{p}) \in \mathcal{H}_\infty^{n_u \times n_y}$ described as in the following a Linear Fractional (**LF**) structure:

$$\mathbf{K}^\star(s, \mathbf{p}) = \mathcal{F}_u\big(\mathbf{K}(s), \Delta\big), \quad (3)$$

that ensures closed-loop stability and achieves some $\mathcal{H}_\infty$ performances. With reference to (3), we denote $\mathbf{K} \in \mathcal{K} \subseteq \mathcal{H}_\infty^{n_u \times n_y}$ the controller rational function, $\Delta = \mathbf{p}I_{n_\Delta} \in \mathbb{R}^{n_\Delta \times n_\Delta}$[3] the parametric matrix block and $\mathcal{F}_u(.,.)$ denotes the upper linear fractional operator defined as (for appropriate partitions of $M$ and $\Delta$) by $\mathcal{F}_u(M, \Delta) = M_{22} + M_{21}\Delta(I - M_{11})^{-1}M_{12}$ [4] [4].

Obviously, many solutions have been derived in the literature to design such a parametric controller (3). Among the methods, the so-called **LPV** community did provide a lot of very interesting tools and procedures mostly oriented to varying parameters, see *e.g.* [5]. In addition, the robust control community also introduced a set of mathematical results in this sense. Among them one should mention the Linear Fractional Representation (**LFR**) framework and the associated control set-up (see *e.g.* the approaches addressing the $\mathcal{H}_\infty$ norm [6], [7], [8] or the $\mathcal{H}_2$ one [9]). The Youla parametrization also provides a framework to deal with these issues (see *e.g.* [10] for more details). For further details, reader is invited to refer to the many results of *e.g.* C. Scherer [11], [12], P. Apkarian [13], [14], [8], G. Balas [15], [16] and co-workers.

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[1]Throughout this paper, we denote $\mathcal{H}_2^{n_y \times n_u}$ (resp. $\mathcal{H}_\infty^{n_y \times n_u}$) or simply $\mathcal{H}_2$ (resp. $\mathcal{H}_\infty$), the open subspace of $\mathcal{L}_2$ (resp. $\mathcal{L}_\infty$) with matrix-valued function $\mathbf{H}(s)$ with $n_y$ outputs, $n_u$ inputs, $\forall s \in \mathbb{C}$, which are analytic in $\mathbf{Re}(s) > 0$ (resp. $\mathbf{Re}(s) \geq 0$). Moreover $\mathcal{H}_2$ functions integral along the imaginary axis are bounded.

[2]This last case is particularly interesting when real tests are costly and engineers cannot stop the process, re-tune the law and re-start the tests. An illustration of this situation can be found in aeronautics, *e.g.* for aircraft flight and ground tests, as in [3].

[3]Note that here the assumption of $\Delta = \mathbf{p}I_{n_\Delta}$ is made to stick with the robust analysis framework, but $\Delta$ can be any square matrix.

[4]In this work we denote the controller $\mathbf{K}^\star(s, \mathbf{p})$ as fractional since it admits a linear fractional decomposition.

## B. Contributions and outlines

The result provided in this paper aims at addressing the problem of structured parametric controller design (3) in the linear framework for (1) and (2) models. More specifically, a simple but yet very effective methodology to design such $\mathbf{p}$ dependent controller (or controller family) achieving $\mathcal{H}_\infty$ performances, is detailed in the rest of the paper. In addition, using the structured $\mathcal{H}_\infty$ oriented optimization tools made available in MATLAB through the `hinfstruct` method [8], we also provide a detailed approach with numerical issues to deal with this problem in order to give practitioners the key steps to solve this kind of problem. The proposed approach is applied on a simple numerical benchmark problem a simple and on a quite challenging open-cavity fluid dynamical model, extracted from [17].

The remaining of the paper is organized as follows: the main result, *i.e.* the synthesis of a structured parametric linear fractional controller achieving $\mathcal{H}_\infty$ performances is detailed in Section II. Then, Section III, provides practitioners some numerical and practical issues to easily optimize such a controller. Numerical examples are given in Section IV detailing the design of a structured parametric controller in a linear fractional form for two interesting cases: first, based on a parametric model of a clamped beam, and secondly, based on a non-parametric model of a fluid flow open cavity geometry, including parametric closed-loop performances. Discussions close the paper in Section V.

# II. MAIN RESULT: STRUCTURED LINEAR FRACTIONAL PARAMETRIC CONTROLLER SYNTHESIS

## A. Problem formulation with $\mathcal{H}_\infty$ performances

Let us consider a linear dynamical model of the form (1) or (2). As evoked in the introductory part, we aim at designing a $\mathbf{p}$ dependent parametric controller in linear fractional form that ensures some $\mathcal{H}_\infty$ performances. As is standard in the robust framework, let us first define the following generalized plant $\mathbf{T}(\mathbf{p}) = \mathbf{W}_i(s)\mathbf{H}(s,\mathbf{p})\mathbf{W}_o(s,\mathbf{p})$, where, $\mathbf{W}_i(s)$ and $\mathbf{W}_o(s,\mathbf{p})$ are the weighting filters defining the input and parametric (or not) output signals. Both $\mathbf{W}_i(s)$ and $\mathbf{W}_o(s,\mathbf{p})$ are constructed by the user to define the desired performances attenuation and its bandwidth. The associated state-space realization is then given by[5],

$$\left\{\begin{array}{rcl} \dot{\mathbf{x}}(t) & = & A(\mathbf{p})\mathbf{x}(t) + B_1(\mathbf{p})\mathbf{w}(t) + B_2(\mathbf{p})\mathbf{u}(t) \\ \mathbf{z}(t) & = & C_1(\mathbf{p})\mathbf{x}(t) + D_{11}(\mathbf{p})\mathbf{w}(t) + D_{12}(\mathbf{p})\mathbf{u}(t) \\ \mathbf{y}(t) & = & C_2(\mathbf{p})\mathbf{x}(t) + D_{21}(\mathbf{p})\mathbf{w}(t) + D_{22}(\mathbf{p})\mathbf{u}(t) \end{array}\right. \tag{4}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{w}(t) \in \mathbb{R}^{n_w}$, $\mathbf{u}(t) \in \mathbb{R}^{n_u}$, $\mathbf{z}(t) \in \mathbb{R}^{n_z}$ and $\mathbf{y}(t) \in \mathbb{R}^{n_y}$ are the states, exogenous input, control input, performance output and measurement signals, respectively. Then, the associated performance transfer from $\mathbf{w}(t)$ to $\mathbf{z}(t)$, parametrized by $\mathbf{p}$, is defined as,

$$\mathbf{T}(s,\mathbf{p}) = \mathbf{W}_i(s)\mathbf{H}(s,\mathbf{p})\mathbf{W}_o(s,\mathbf{p}). \tag{5}$$

[5]Note that according to the original plant, the parameter dependency can either come from the system model itself, if described by (2), or from the performance weighting filters $\mathbf{W}_o(s,\mathbf{p})$, or both.

Then, mathematically, the $\mathcal{H}_\infty$ parametric control design objective consists in finding the optimal controller $\mathbf{K}^\star(s,\mathbf{p})$, mapping $\mathbf{y}(t)$ to $\mathbf{u}(t)$, such that,

$$\mathbf{K}^\star(s,\mathbf{p}) := \arg\min_{\mathbf{K}\in\mathcal{K}}\max_{\mathbf{p}\in\mathcal{D}} \left\| \mathcal{F}_l\Big(\mathbf{T}(s,\mathbf{p}), \mathcal{F}_u\big(\mathbf{K}(s),\Delta\big)\Big)\right\|_{\mathcal{H}_\infty} \tag{6}$$

where $\mathcal{F}_l(.,.)$ denotes the lower linear fractional operator defined as $\mathcal{F}_l(P,\Delta) = P_{11}+P_{12}\Delta(I-P_{22})^{-1}P_{21}$, $\mathbf{T}(.,\mathbf{p}) \in \mathcal{H}_\infty$ is the parameter dependent generalized plant performance transfer, $\Delta \in \mathbb{R}^{n_\Delta \times n_\Delta}$ is a user-defined diagonal structure gathering the parametric variation of $\mathbf{p} \in \mathcal{P} \subseteq \mathbb{R}^{n_p}$. Finally, $\mathbf{K} \in \mathcal{K} \subseteq \mathcal{H}_\infty$ is the controller to be optimized. This last dynamical system might then be structured as *(i)* a full block or *(ii)* a sparse matrix, affine or not (see Section II-C for details).

## B. Solution as a linear fractional form

Now one aims at solving (6) with a controller in a linear fractional form, recast as:

$$\mathcal{F}_u\big(\mathbf{K}(s),\Delta\big) := \mathcal{F}_u\Big(\mathcal{F}_u\big(K, \frac{1}{s}I_{n_K}\big),\Delta\Big), \tag{7}$$

where $K =$

$$\begin{bmatrix} A_K & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix} \in \mathbb{R}^{(n_K+n_\Delta+n_u)\times(n_K+n_\Delta+n_y)}. \tag{8}$$

Problem (6) consists now in seeking for $K^\star$ solving (9). It is now clear that if a solution of (9) is found, then one can reconstruct the parametric controller $\mathbf{K}^\star(s,\mathbf{p})$ through relation (7). Nevertheless (9) still requires to solve for all $\mathbf{p} \in \mathcal{P}$, which leads in practice to a infinite number of $\mathcal{H}_\infty$ problems to solve. As in the existing robust control framework, an alternative consists in solving the above problem for a finite number $M \in \mathbb{N}$ of parameter values $\mathbf{p}_j$ ($j = 1,\dots,M$), as exposed in (10). If a solution is found, then, following (7), the parametric controller $\mathbf{K}^\star(s,\mathbf{p})$ in linear fractional form is obtained. Still, optimality is now dependent on the problem and the number $M$. A trade-off between the complexity (high $M$) and the numerical reliability (low $M$) has to be observed. Before entering into numerical considerations, crucial for successful application, let us now be more specific on the controller subset $\mathcal{K}$, defining the controller structure and on its implications on the matrix $K$ (8).

## C. Considerations about $\mathcal{K}$ and parameter dependency

Some insight on the parametrization of the solution through the $\mathcal{K}$ subspace, the $\Delta$ block and especially the $K$ matrix (8) are described.

*1) The $\mathcal{K}$ subspace and $K$ structure:* with reference to the original optimisation problem (6), the decision variable is $\mathbf{K}$ which belongs to $\mathcal{K}$. Let us derive some specific $\mathcal{K}$ sets and their implication on the problem solved in (10):

- if $\mathcal{K} = \mathcal{H}_\infty^{(n_u\times n_y)}$, then, $K$ (8) is a full block matrix. This stands as the most generic case where all variables in $K$ are adjustable and the controller obtained might be proper and the dependency with $\mathbf{p}$ is rational.

$$K^\star := \arg \min_{K \in \mathbb{R}^{n_K n_\Delta n_u \times n_K n_\Delta n_y}} \max_{\mathbf{p} \in \mathcal{P}} \left\| \mathcal{F}_l\Big( \mathbf{T}(s,\mathbf{p}), \mathcal{F}_u\big( \mathcal{F}_u(K, \frac{1}{s} I_{n_K}), \Delta \big) \Big) \right\|_{\mathcal{H}_\infty} \tag{9}$$

$$K^\star := \arg \min_{K \in \mathbb{R}^{(n_K + n_\Delta + n_u) \times (n_K + n_\Delta + n_y)}} \max_{\mathbf{p}_j \in \mathbb{R} \ j=1,\dots,M} \left\| \mathcal{F}_l\Big( \mathbf{T}(s,\mathbf{p}_j), \mathcal{F}_u\big( \mathcal{F}_u(K, \frac{1}{s} I_{n_K}), \Delta_j \big) \Big) \right\|_{\mathcal{H}_\infty} \tag{10}$$

- if $\mathcal{K} = \mathcal{H}_2^{(n_u \times n_y)}$, then, $K$ (8) is a full block matrix except for the $D_{yu}$ term and $D_{zu}$ and/or $D_{yw}$ which are null. The obtained controller is then strictly proper and the dependency with $\mathbf{p}$ might be rational too.

*2) The parametric dependency (rational vs. affine) and $K$ structure:* with reference to (7)-(8), the controller realization $\mathcal{S}_K(\Delta_j)$ at $\mathbf{p}_j$, associated with the linear fractional is

$$\begin{aligned} \mathcal{S}_K(\Delta_j): \quad & \big(A_K + B_w \Delta_j M_j C_z, B_u + B_w \Delta_j M_j D_{zu}, \\ & C_y + D_{yw} \Delta_j M_j C_z, D_{yu} + D_{yw} \Delta_j M_j D_{zu}\big) \end{aligned}$$

where $M_j = (I_{n_\Delta} - D_{zw} \Delta_j)^{-1}$. Then,

- if $D_{zw} \neq 0$, $\mathbf{K}(s,\mathbf{p})$ parameter dependency is rational,
- if $D_{zw} = 0$, $\mathbf{K}(s,\mathbf{p})$ parameter dependency is affine.

As illustrated in the examples, this selection impacts the solution and complexity. Still reader should keep in mind that the rational case theoretically provides a less conservative solution than the affine one. However, this is balanced by a more complex parametrization, which in practice can be a brake in the optimization procedure.

*3) The parametric dependency order $n_\Delta$:* in problem (10), we consider the $\Delta$ block to be known. Obviously, in the complete problem, this is not true and additional research should be conducted to consider it as a tuning variable. Moreover, it is convenient to consider it as a diagonal block to fit the robust control analysis tools. In this preliminary study we simply focus on the case where $\Delta = \mathbf{p} I_{n_\Delta} \in \mathbb{R}^{n_\Delta \times n_\Delta}$. Then, the only tuning variable that a user has to deal with is the dimension $n_\Delta$. So far, as illustrated later in the examples, no clear solution on the mechanism is known and the optimal choice is dependent on both the complexity and representativeness of this structure. However, it is to be kept in mind that a $n_\Delta = 0$ implies a non parametric controller (*i.e.* classical non parametric **LTI** controller) and $n_\Delta > 0$ leads to a parametrized control accompanied by an increasing complexity.

## III. Numerical discussions

### A. Problem parametrization

As detailed above, the considered optimization problem (10) is then function of $K \in \mathbb{R}^{n_K n_\Delta n_u \times n_K n_\Delta n_y}$ and $M \in \mathbb{N}$. The optimization problem contains $n_K^2 n_\Delta^2 n_u n_y$ real variables to solve. Obviously, $n_u$, $n_y$ are imposed by the sensors and actuators set-up of the control problem, one still has to choose $n_\Delta$ (the parametric complexity) and $n_K$ (the controller order). While the $n_\Delta$ has already been evoked in the above section, the remaining coefficient to deal with is the order $n_K$ of the controller. This latter is generally taken low to obtain a reduced order complexity. However, even with a low $n_K$, the parameter number increases with a square complexity. Consequently it is consistent to simplify the number of parameters. With reference to (8), one way to deal with this, is to consider the following tridiagonal $A_K$ matrix structure:

$$A_K = \begin{pmatrix} \times & \times & 0 & \dots & & \dots & 0 \\ \times & \times & \times & 0 & & \dots & 0 \\ 0 & \times & \times & \times & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \ddots & & \times & 0 \\ 0 & \dots & & 0 & \times & \times & \times \\ 0 & \dots & & & 0 & \times & \times \end{pmatrix}. \tag{11}$$

Such a representation is non conservative in theory (see Lanczos [] and provides a considerable few variable to tune. It has show an effectiveness in many numerical applications (from model identification, approximation and control).

### B. Initialization of the optimization problem

Still, problem (10) is NP complex and no global optimal solution can be guaranteed [8]. Consequently, from a practical point of view, the initialization phase plays a crucial role. This is why, after having parametrized the above problem, the authors suggest to initialize the problem of parametric controller design $K$ (8) as follows: set $A_K$, $B_u$, $C_y$ and $D_{yu}$ to gain values obtained with *e.g.* an un-parametrized $\mathcal{H}_\infty$ optimization problem solved at the nominal $\mathbf{p}$ value. Then, set $B_w$, $C_z$, $D_{zw}$, $D_{zu}$ and $D_{yw}$ to null values.

### C. Parametric controller stability issue

As a practical remark, one might look for a parametric controller which is also stable (this kind of requirement is often requested in applications such as aeronautics or hydraulic systems). To address this constrain, following the linear fractional formulation of the problem, in addition to problem (10), the following constrain might also be simultaneously solved:

$$\left\| \mathbf{W}_K \mathcal{F}_u\big( \mathcal{F}_u(K, \frac{1}{s} I_{n_K}), \Delta \big) \right\|_{\mathcal{H}_\infty} < \gamma \tag{12}$$

where $\mathbf{W}_K \in \mathcal{H}_\infty$ is a weighting function and $\gamma$ is the $\mathcal{H}_\infty$ performance of the control problem, *i.e.* (10) $< \gamma$.

### D. A MATLAB *based solution*

Let us now provide an insight on a practical implementation, through a few lines of MATLAB code, illustrating how to implement such an approach in a very simple way (note that here, a full block $K$ structure is considered). Let us define the integral operator and $K \in \mathbb{R}^{n_K n_\Delta n_u \times n_K n_\Delta n_y}$ matrix as:

```
Is = tf(1,[1 0]);
```

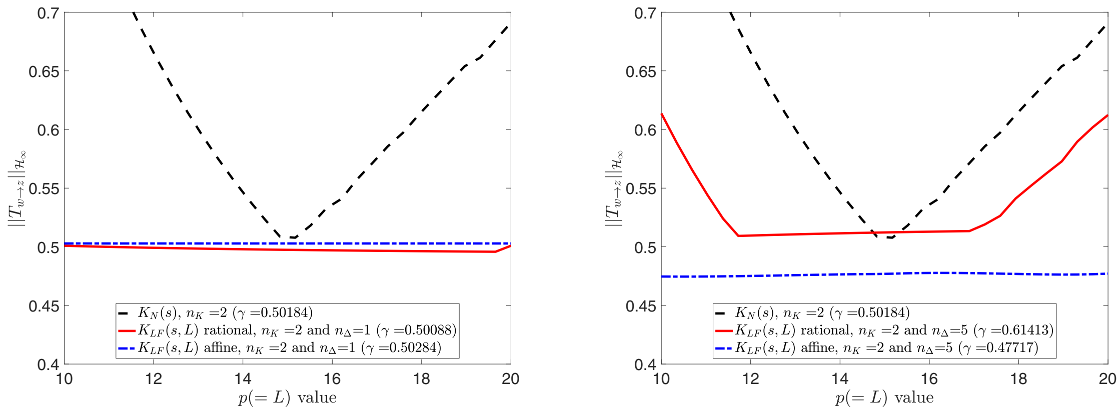


Fig. 1. Left frame $n_\Delta = 1$ and $n_K = 2$, right frame $n_\Delta = 5$ and $n_K = 2$. $\mathcal{H}_\infty$ performances for varying length $L$ values of the beam when looped with *(i)* $\mathbf{K}_N(s)$, a non parametric controller synthesized on the nominal case $L = 15$ (black dashed), *(ii)* $\mathbf{K}_{LF}(s, L)$ rational parametric and *(iii)* $\mathbf{K}_{LF}(s, L)$ affine parametric controller synthesized using $M$ configurations (red solid, resp. blue dash dotted).

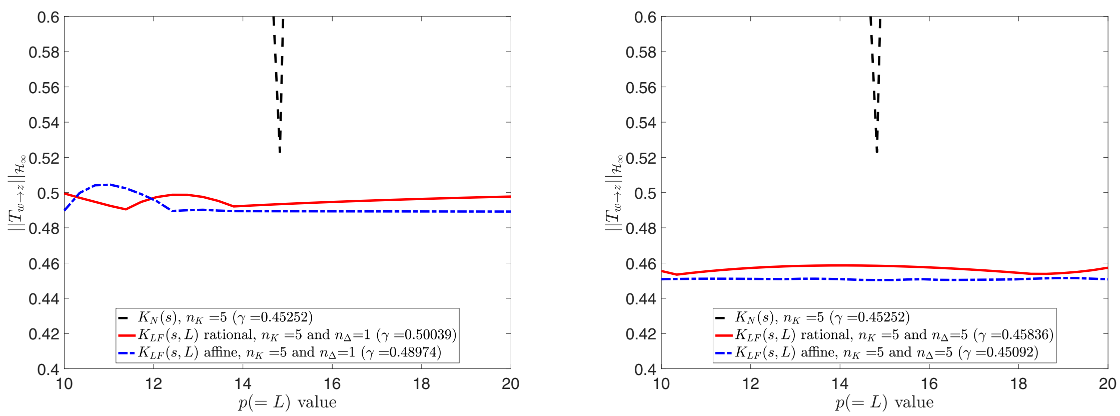


Fig. 2. Left frame $n_\Delta = 1$ and $n_K = 5$, right frame $n_\Delta = 5$ and $n_K = 5$. $\mathcal{H}_\infty$ performances for varying length $L$ values of the beam when looped with *(i)* $\mathbf{K}_N(s)$, a non parametric controller synthesized on the nominal case $L = 15$ (black dashed), *(ii)* $\mathbf{K}_{LF}(s, L)$ rational parametric and *(iii)* $\mathbf{K}_{LF}(s, L)$ affine parametric controller synthesized using $M$ configurations (red solid, resp. blue dash dotted).

```
K0 = [A Bw Bu; Cz Dzw Dzu; Cy Dyw Dyu];
K = realp('K',K0);
```

Then, by denoting the generalized plants $\mathbf{T}(s, \mathbf{p}_j)$ =`T{j}` (for `j=1,...,M`), construct the optimization problem (10) as (where $n_K$ =`nk`, $n_\Delta$ =`ndelt`)

```
Ttot = [];
for j = 1:M
Deltaj = eye(ndelt)*p(j);
Klfj = lft(Deltaj,lft(Is*eye(nk,nk),K));
Ttot = append(Ttot,lft(T{j},Klfj));
Ttot = append(Ttot,Klfj*Wk);
end
```

With reference to the above loop, the first line corresponds to the evaluation of the $\Delta$ block at $\mathbf{p}_j$, the second, to the evaluation of $\mathbf{K}(s, \mathbf{p}_j)$ and the third/fourth to the concatenation of the $\mathcal{F}_l(\mathbf{T}(s, \mathbf{p}_j), \mathcal{F}_u(\mathcal{F}_u(K, \frac{1}{s} I_{n_K}), \Delta_j))$ in the structure `Ttot`. Note also that `Klfj*Wk` is added to ensure controller stability and a given roll off dictated by $\mathbf{W}_K$, as in (12). Finally, the above problem is solved through the `hinfstruct` function [8] as:

```
[Kopt,gamma,info] = hinfstruct(Ttot);
```

Leading to the $K$ matrix which cans then be used to easily construct $\mathbf{K}(s, \mathbf{p})$.

## IV. Numerical Examples

Let us now, illustrate the efficiency of the proposed approach. To this aim, two use-cases are considered. The first one is a linear model of a clamped beam parametrized by its length, while the second one comes from fluid dynamics and represents the open cavity flow problem [17].

### A. Clamped beam parametric model

The first considered example is the Timoshenko clamped beam, described in [18]. This model is a single input single output model ($n_u = 1$, $n_y = 1$), and its dynamical matrices are obtained by finite element meshing. In the considered case, we select a meshing of 6 nodes. The resulting model is then of dimension $n = 60$. In addition, this parametrized with the length $L(= \mathbf{p})$ of the beam. In our case we consider this length varying between 10 and 20m. this results in a model as

(2), defined as $\mathbf{H}(s,L) = C\big(sI_n - A(L)\big)^{-1}B + D \in \mathcal{H}_\infty^{1\times 1}$. The objective considered in this case is to minimize the $\mathcal{H}_\infty$ norm of the only input/output transfer (the extremity vertical force to the vertical displacement) with a parametric controller $\mathbf{K}^\star(s,L)$. More specifically, the following previous notations, generalized plant $\mathbf{T}(L)$ is considered:

$$\left\{\begin{array}{rcl} \dot{\mathbf{x}}(t) & = & A(L)\mathbf{x}(t) + B\mathbf{w}(t) + B\mathbf{u}(t) \\ \mathbf{z}(t) & = & C\mathbf{x}(t) \\ \mathbf{y}(t) & = & C\mathbf{x}(t) \end{array}\right. . \quad (13)$$

To be complete, a stability (and bandwidth) constrain is also added to the problem with $W_K = \frac{10^{-1}}{s/100+1}$, in (12). Then, the procedure exposed in Section II is applied for $L = \{10, 12.5, 15, 17.5, 20\}$ (*i.e.* $M = 5$) and for different $n_K$ and $n_\Delta$ values. Then, the $\mathcal{H}_\infty$ norm of the single input single output transfer $\mathbf{T}(s,L)$ is evaluated for varying frozen values of $L \in [10\ 20]$. Some results are shown on Figures 1 and 2.

With reference to Figures 1 and 2, multiple comments can be done. First, as expected, a nominal controller $\mathbf{K}_N^\star$, synthesized for the $L$ parameter mean value (black dashed curves) cannot perform well over the entire range of parametric variation while $\mathbf{K}_{LF}^\star$, the linear fractional ones does (rational: red solid curves and affine: blue sh dotted). Then, by comparing the two parametric controllers $\mathbf{K}_{LF}^\star$, both provide a performance level below the one obtained during the synthesis (see $\gamma$ values in the legend), which confirms the effectiveness and consistency of the proposed approach. Interestingly, on Figure 1, the increase of $n_\Delta$ does not necessarily lead to a better attenuation in the rational case. This can be justified by the variable number increase, and due to the non-optimality of the approach. This is not the case on Figure 2, where $\mathcal{H}_\infty$ performances are enhanced. Still the fractional form does not seems to enhance the efficiency. However, so far, it is hard to conclude on the reason, and additional work should be conducted.

### B. Parametric control of an open cavity fluid flow model

This second use case describes a fluid dynamical behavior over a two-dimensional open square cavity. Details of this setting are first described in [17]. With reference to Figure 3, the model under consideration consists of the transfer from the actuator $\mathbf{u}(t)$ to the the sensor $\mathbf{y}(t)$ when air flows along the cavity from left to right, at $U_\infty$ velocity.

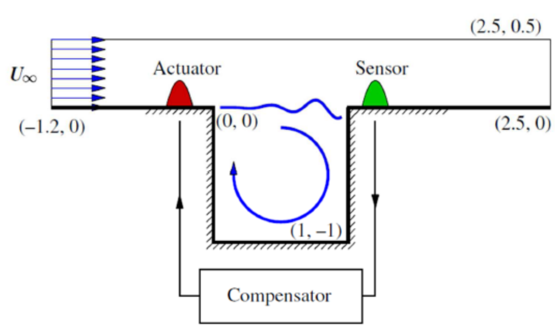


Fig. 3. Open cavity control setting. Numbers at the extremity are the coordinate of the boundary and $U_\infty$ stand as the uniform flow velocity.

With reference to Figure 4, the model is obtained through a dedicated software allowing obtaining the frequency response at given frequencies (black dot). Then, thanks to the Lowener interpolatory method [19], the **LTI** exact model $\mathbf{H}_L = C\big(sI_n - A\big)^{-1}B \in \mathcal{L}_2^{1\times 1}$ (of order $n = 176$, dashed blue) and approximate $\mathbf{H}_r = C_r\big(sI_r - A_r\big)^{-1}B_r \in \mathcal{L}_2^{1\times 1}$ model (of order $r = 16$, solid red), are obtained[6]. The obtained model $\mathbf{H}_r$ exhibits six unstable modes which is consistent with the physics.

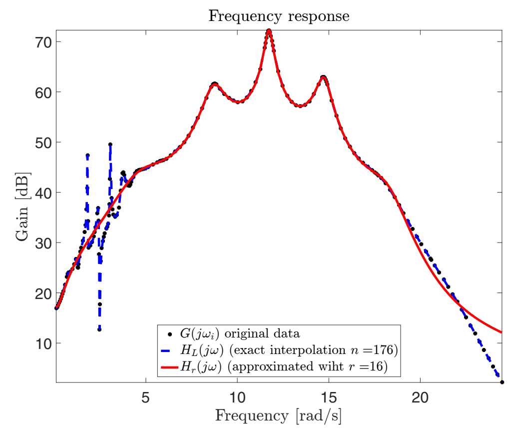


Fig. 4. Frequency response of the transfer from shear stress control signal $\mathbf{u}(t)$ and the pressure sensor $\mathbf{y}(t)$. Original simulated data (black dots), exact interpolated model using the Loewner framework [19] (dashed blue line) and its approximation of order $r = 16$ (solid red).

Then, with reference to the Figure 4, the control objective is to stabilize the system and minimize the $\mathcal{H}_\infty$ gain by using an output feedback (as illustrated on Figure 3). Since simulations are numerically costly and experimental tests are long and expensive, in order to simplify the experimental test matrix, physicists are interested in finding $\mathbf{K}^\star(s,\mathbf{p})$, a family of controllers achieving some varying performances according to an external parameter. More specifically, based on $\mathbf{H}_r$ one defines a generalized model $(\mathbf{p})$, which performances are parametrized though a parametric performance weight $\mathbf{W}_o(s,\mathbf{p})$ (here $\mathbf{p} \in [1\ 2]$, and $\mathbf{W}_o(s,\mathbf{p})$ for space restrictions). This weight affects the performance attenuation criteria on the main peak. Moreover, the controller structure is of dimension $n_K = 6$ and the parametric dependency is set to be affine and $M = 10$ points where used. Figure 5 illustrates the obtained performances and controller gains, as a function of the exogenous parameter $\mathbf{p}$.

Moreover, Figure 6 illustrates the impulse response of the controller (left frame) and of the closed-loop (right frame). Clearly, the proposed control laws, provides then varying performances a function of the $\mathbf{p}$ parameter, where the bigger $\mathbf{p}$ is, the stronger the attenuation is. Then, thanks to this family of controller physicists are now able to tune the controller on the real complete simulation in a simple way.

[6]Since this is out o the scope of the proposed contribution, details on the way to obtain these models are omitted for space limitation.

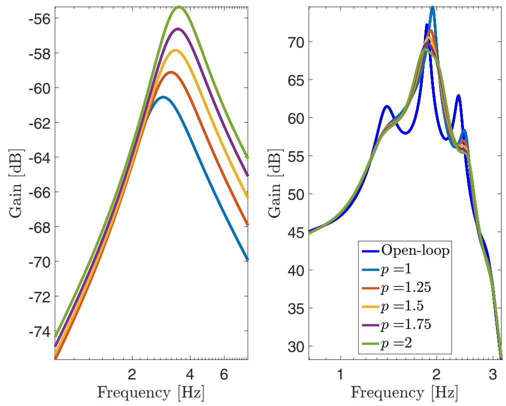


Fig. 5. Right: Bode diagram of the open-loop (blue), the closed-loop obtained with the parametric controller $\mathbf{K}(s, \mathbf{p})$ for varying $\mathbf{p}$ values (colored lines). Left: varying $\mathbf{K}(s, \mathbf{p})$ transfer as a function of $\mathbf{p}$.

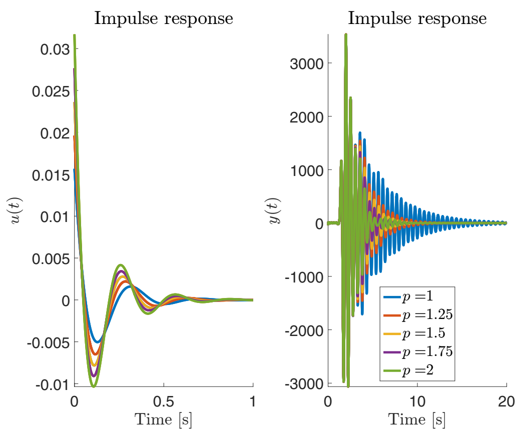


Fig. 6. Right: impulse response of the closed-loop obtained with the parametric controller $\mathbf{K}(s, \mathbf{p})$ for varying $\mathbf{p}$ values (colored lines). Left: varying $\mathbf{K}(s, \mathbf{p})$ impulse response as a function of $\mathbf{p}$.

## V. Conclusions and Perspectives

In this paper, a new simple but yet effective approach to design low order multiple inputs multiple outputs parametric linear fractional controller achieving $\mathcal{H}_\infty$ performances, has been introduced. The proposed framework is based on the developments in $\mathcal{H}_\infty$ oriented optimization, which are made available in [8]. The pivotal idea is based on the specific structure of the control operator, *i.e.* the fractional representation. To the author's feeling, this simple structure, linked with dedicated optimization tools, makes this approach both simple and mathematically well posed, and stands as a nice solution for many practitioners faced to parametric models and controller synthesis. Obviously, the results in this paper are not restricted single parameter dependency but yet, it is to be kept in mind that extension to multiple parameters will require dedicated attention due to the complexity increase in the optimization and in the selection of the $n_\Delta$ dimensions. The approach has been illustrated first on an academic benchmark, and then, validated on a very complex fluid flow dynamical model, for which simulation are very expensive and where users require an easy structure to implement in order to adjust on the complex phenomena. Future works might investigate convergence issues and selection of the $M$ parameter as well.

## References


[1] R. Gadient, E. Lavretsky, and K. Wise, "Very Flexible Aircraft Control Challenge Problem," in *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, Minneapolis, Minnesota, USA, August 2012.

[2] V. Dalmas, G. Robert, C. Poussot-Vassal, I. Pontes Duff, and C. Seren, "From infinite dimensional modelling to parametric reduced order approximation: Application to open-channel flow for hydroelectricity," in *Proceedings of the 15th European Control Conference*, Aalborg, Denmark, July 2016, pp. 1982–1987.

[3] C. Meyer, J. Prodigue, G. Broux, O. Cantinaud, and C. Poussot-Vassal, "Ground test for vibration control demonstrator," in *Proceedings of the 13th International Conference on Motion and Vibration Control*, Southampton, United Kingdom, July 2016, pp. 1–12.

[4] J.-F. Magni, "Linear fractional representation toolbox for use with matlab," Onera, Toulouse, France, Tech. Rep., 2006. [Online]. Available: http://w3.onera.fr/smac/

[5] S. Wang, H. Pfifer, and P. Seiler, "Robust Synthesis for Linear Parameter Varying Systems Using Integral Quadratic Constraints," *Automatica*, vol. 68, pp. 111–118, 2016.

[6] P. Gahinet and P. Apkarian, "An linear matrix inequality approach to $\mathcal{H}_\infty$ control," *International Journal of Robust and Nonlinear Control*, vol. 4, no. 4, pp. 421–448, January 1994.

[7] D. Henrion, D. Arzelier, D. Peaucelle, and J. B. Lasserre, "On parameter dependent Lyapunov functions for robust stability of linear systems," in *Proceedings of the 43rd IEEE Conference on Decision and Control*, Atlantis, Paradise Island, Bahamas, December 2004, pp. 887–892.

[8] P. Apkarian and D. Noll, "Nonsmooth $\mathcal{H}_\infty$ Synthesis," *IEEE Transaction on Automatic Control*, vol. 51, no. 1, pp. 71–86, January 2006.

[9] C. E. de Souza and A. Trofino, "Gain Scheduled $\mathcal{H}_2$ Controller Synthesis for Linear Parameter Varying systems via Parameter-dependent Lyapunov Functions," *International Journal of Robust and Nonlinear Control*, vol. 16, pp. 243–257, November 2005.

[10] J. Neering, R. Drai, M. Bordier, and N. Maiz, "Multiobjective robust control via youla parametrization," in *IEEE International Conference on Control Applications*, Munich, Germany, October 2006, pp. 1450–1456.

[11] C. W. Scherer, "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control for time-varying and linear parametrically-varying systems," *International Journal of Robust and Nonlinear Control*, vol. 6, no. 9-10, pp. 929–952, november 1996.

[12] C. Scherer, P. Gahinet, and M. Chilali, "Multiobjective Output-Feedback Control via LMI Optimization," *IEEE Transaction on Automatic Control*, vol. 42, no. 7, pp. 896–911, July 1997.

[13] P. Apkarian and P. Gahinet, "A Convex Characterization of Gain-Scheduled $\mathcal{H}_\infty$ Controllers," *IEEE Transaction on Automatic Control*, vol. 40, no. 5, pp. 853–864, May 1995.

[14] P. Gahinet, P. Apkarian, and M. Chilali, "Affine Parameter-Dependent Lyapunov Functions and Real Parametric Uncertainty," *IEEE Transaction on Automatic Control*, vol. 41, no. 3, pp. 436–442, March 1996.

[15] I. Fialho and G. Balas, "Road Adaptive Active Suspension Design using Linear Parameter Varying Gain-Scheduling," *IEEE Transaction on Control System Technology*, vol. 10, no. 1, pp. 43–54, January 2002.

[16] A. Hjartarson, P. Seiler, and G. Balas, "LPV Aeroservoelastic Control using the LPVTools Toolbox," in *AIAA Atmospheric Flight Mechanics Conference*, 2013.

[17] A. Barbagallo, D. Sipp, and P. Schmid, "Closed-loop control of an open cavity flow using reduced-order models," *Journal of Fluid Mechanics*, vol. 641, pp. 1–50, 2009.

[18] H. Panzer, J. Hubele, R. Eid, and B. Lohmann, "Generating a Parametric Finite Element Model of a 3D Cantilever Timoshenko Beam Using Matlab (Vol. TRAC-4, Nov. 2009)," Technical Reports on Automatic Control, Tech. Rep., November 2009.

[19] A. J. Mayo and A. C. Antoulas, "A framework for the solution of the generalized realization problem," *Linear Algebra and its Applications*, vol. 425, no. 2, pp. 634–662, 2007.