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Pareto suboptimal robust controllers in multi-objective generalized H_2 problem

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Abstract— **A novel multi-objective robust control problem is studied for systems with structured norm-bounded uncertainty and robust generalized H_2 norms as criteria. Necessary conditions for Pareto optimality are formulated. Pareto optimal solutions turn out to be among optimal solutions for multi-objective costs in the form of Germeyer convolution. Pareto suboptimal controllers are defined as the optimal solutions for the upper bounds of the multi-objective costs and characterized in terms of LMIs. The upper and lower bounds of the multi-objective cost are used to compute a suboptimality measure which allows to estimate a "difference" between Pareto suboptimal and unavailable Pareto optimal controllers. Two-criteria robust control problem for a mathematical model of the rotor rotating in active magnetic bearings is considered as an application of this theory.**

Keywords: Multi-objective problem; Pareto optimal set; robust control; structured norm-bounded uncertainty; generalized H_2 norm; LMIs; active magnetic bearings.

I. Introduction

Most practical design control problems very often involve multiple performance objectives and uncertainty. Multi-objective design is critical in problems, where meaningful trade-off have to be made between conflicting performance objectives and, on the other hand, mathematical models of real process often contain unknown parameters and disturbances. Consideration of these two important aspects simultaneously results in extremely difficult theoretical schemes. As a rule, multi-objective control problems are stated when there is no uncertainty in mathematical model of a plant.

For the first time, the multi-objective approach has been applied to synthesizing Pareto optimal control design for LQG problem [1]. In paper [2], the Pareto optimal set for the problem with multiple H_2 criteria has been characterized in terms of Riccati equation solutions by using Youla parametrization and scalar multi-objective cost in the form of a weighted sum of criteria. For the last decades, there were obtained valuable results in the H_∞ and H_2 optimal control designs. The H_∞ and H_2 norms have clear physical interpretations as disturbance attenuation levels for various classes of deterministic and stochastic disturbances. The multi-objective control problems with such criteria are well known to be very complex because of difficulties of finding a scalar multi-objective cost which would characterize the Pareto optimal set (see, for example, [3]). There are a lot of papers (see, for example, [4–7] and the references therein) in which the optimization problems are to minimize one criterion subject to multiple constraints on other criteria. Such problems are referred to multi-objective ones as well, although Pareto optimal solutions are not sought therein.

In the recent papers [8, 9], Pareto suboptimal controllers have been synthesized in multi-objective control problems with H_∞ and γ_0 criteria. The relative losses of these Pareto suboptimal solutions compared to the Pareto optimal ones do not exceed $1-\sqrt{N}/N$, where N is number of criteria. The paper [10] deals with the multi-objective control problems with generalized H_2-norms as the criteria, where necessary conditions for Pareto optimality are derived and Pareto optimal controllers are characterized in terms of LMIs. The generalized H_2-norm arises as an induced norm of the system operator when the input is a signal of the bounded L_2-norm and the output signal is measured by its peak value. The various definitions of the generalized H_2 norm were firstly introduced in [11] and considered in [12, 10].

The literature devoted to theoretical aspects of multi-objective control design for uncertain systems is rather limited. In the paper [13], multi-objective quadratic guaranteed-cost problem for linear uncertain system was considered. The plant uncertainty was modeled as a linear weighted sum of given matrices with unknown bounded non-stationary weights. Instead of a maximum of the quadratic costs over admissible uncertainties, the authors used their upper bounds as criteria and derived Pareto suboptimal controllers in terms of Riccati equations. A relationship between the controllers synthesized and Pareto optimal ones for the initial problem left unclarified.

The present paper deals with the multi-objective robust control design for systems with structured norm-bounded uncertainty and robust generalized H_2 norms as criteria. The robust generalized H_2 norm is understood as the maximum of the generalized H_2 norm over all admissible uncertainties. Necessary conditions for Pareto optimality are formulated. It is established that Pareto optimal solutions turn out to be among optimal solutions to multi-objective costs in the form of Germeyer convolution. The upper and lower bounds of these multi-objective costs are derived and characterized in terms of LMIs. Then Pareto suboptimal controllers are defined as the optimal solutions for the upper bounds of the multi-objective costs. The upper and lower bounds of the multi-objective cost are used to compute a suboptimality measure which allows to estimate a "difference" between Pareto suboptimal and unavailable Pareto optimal controllers.

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As application of this theory two-criteria robust control problem is considered for a mathematical model of the rotor rotating in active magnetic bearings.

II. Preliminary

A. Generalized H_2 norm

Let vector $a \in R^{n_a}$ be partitioned as $a = \operatorname{col}(a_1, \dots, a_m)$, where $a_i \in R^{n_i}$, $\sum_{i=1}^m n_i = n_a$. Denote a generalized ∞-norm of the vector a through $|a|_{g\infty} = \max_{1\le i\le m} |a_i|_2$, where $|a_i|_2$ is the Euclidean norm of the vector a_i. In particular, if $m = 1$, then $|a|_{g\infty} = |a|_2$, and if $m = n_a$, then $|a|_{g\infty} = |a|_\infty = \max_{1\le i\le n_a} |a_i|$, where a_i are the components of the vector a. Denote

$$\|a\|_2 = \left(\int_0^\infty |a(t)|^2 \, dt \right)^{1/2}, \quad \|a\|_{g\infty} = \sup_{t\ge 0} |a(t)|_{g\infty}$$

for vector-function $a(t)$. Let a symmetric $(n \times n)$-matrix $M = M^{\mathrm{T}} \ge 0$ be partitioned on blocks $M = (M_{ij})$, $M_{ij} \in R^{n_i \times n_j}$, $i, j = 1, \dots, m$, $\sum_{i=1}^m n_i = n$. Denote its *generalized maximum eigenvalue* as $\lambda_{\mathrm{gmax}}(M) = \max_{1\le i\le m} \lambda_{\max}(M_{ii})$, where $\lambda_{\max}(M_{ii})$ is the maximum eigenvalue of the matrix M_{ii}. In particular, if $m = 1$, then $\lambda_{\mathrm{gmax}}(M) = \lambda_{\max}(M)$, and if $m = n$, then $\lambda_{\mathrm{gmax}}(M) = d_{\max}(M)$, where $d_{\max}(M)$ is the maximum diagonal entry of the matrix M. Obviously, $d_{\max}(M) \le \lambda_{\mathrm{gmax}}(M) \le \lambda_{\max}(M)$.

Consider an internally stable linear dynamic system

$$\dot{x} = Ax + Bv, \quad z = Cx, \quad x(0) = 0, \tag{1}$$

where $x \in R^{n_x}$ is the state, $z \in R^{n_z}$ is the objective output. The transfer matrix of this systems from input v to output z is denoted as $H_{zv}(s) = C(sI - A)^{-1}B$. The generalized H_2 norm has been defined in [8] and a more general concept of the generalized H_2 norm has been introduced in [10] to have additional possibility of partitioning the objective output on a number of vectors of smaller dimensions. And namely, let the output of the system (1) be represented in the form $z = \operatorname{col}(z_1, \dots, z_m)$, $z_i \in R^{n_i}$, $\sum_{i=1}^m n_i = n_z$ and, respectively, $C = \operatorname{col}(C_1, \dots, C_m)$. Define the generalized H_2 norm of the transfer function matrix of this system as follows:

$$\|H_{zv}\|_{g2} = \sup_{v\in L_2} \frac{\|z(t)\|_{g\infty}}{\|v\|_2} = \sup_{v\in L_2} \frac{\sup_{t\ge 0} |z(t)|_{g\infty}}{\|v\|_2}. \tag{2}$$

In particular, if $m = 1$ or $m = n_z$, then $\|H_{zv}\|_{g2}$ is the same as defined in [11].

Theorem 2.1 [10]

$$\|H_{zv}\|_{g2} = \lambda_{gmax}^{1/2}(CY_* C^{\mathrm{T}}), \tag{3}$$

where $Y_* = Y_*^{\mathrm{T}} > 0$ is a solution to Lyapunov equation

$$AY + YA^{\mathrm{T}} + BB^{\mathrm{T}} = 0.$$

Corollary 2.1. *The generalized H_2 norm can be computed as $(\min \gamma^2)^{1/2}$ subject to LMIs*

$$\begin{pmatrix} AY + YA^{\mathrm{T}} & \star \\ B^{\mathrm{T}} & -I \end{pmatrix} \le 0, \quad \begin{pmatrix} Y & \star \\ C_i Y & \gamma^2 I_{n_i} \end{pmatrix} \ge 0,$$

$$i = 1, \dots, m.$$

B. Germeyer scalarization in multi-objective problems

A key concept in multi-objective optimization for given criteria $J_k(\Theta)$, $k = 1, \dots, N$ is the Pareto optimal set. The set $\mathcal{P} = \{\Theta_P\}$, where

$$\Theta_P = \arg\min_{\Theta} \{J_k(\Theta), \, k = 1, \dots, N\}, \tag{4}$$

is the Pareto optimal if there is not a matrix Θ such that the inequalities $J_k(\Theta) \le J_k(\Theta_P)$, $k = 1, \dots, N$, with at least one of the inequalities being strict, be valid. The multi-objective cost

$$J_\alpha(\Theta) = \max_{1\le k\le N} \{J_k(\Theta)/\alpha_k\}, \tag{5}$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$, $\alpha_k > 0$, $k = 1, \dots, N$, is called Germeyer convolution [14]. Necessary conditions for Pareto optimality in the problem under consideration are formulated as follows.

Theorem 2.2 [10] *Let $(\gamma_1, \dots, \gamma_N)$ be a Pareto optimal point in the space of criteria and Θ_α provide a minimal value of the multi-objective cost $J_\alpha(\Theta)$ at $\alpha_k = \gamma_k / \max_{1\le i\le N} \gamma_i$. Then $\Theta_\alpha \in \mathcal{P}$ and $J_k(\Theta_\alpha) = \gamma_k$, $k = 1, \dots, N$.*

III. Robust generalized H_2 norm and its bounds

Consider an uncertain system described by the equations

$$\begin{aligned} \dot{x} &= Ax + B_\Delta v_\Delta + B_v v, \\ z_\Delta &= C_\Delta x + D_\Delta v_\Delta + D_v v, \\ v_\Delta &= \Delta(t) z_\Delta, \\ z &= \operatorname{col}(z^{(1)}, \dots, z^{(m)}) = Cx, \end{aligned} \tag{6}$$

where $C = \operatorname{col}(C^{(1)}, \dots, C^{(m)})$, x is the state, v_Δ is the uncertainty input, z_Δ is the uncertainty output, Δ is the uncertainty, z is the objective output consisting of the vectors $z^{(i)}$, $i = 1, \dots, m$, $v \in L_2$ is the disturbance, A is a stable matrix. The uncertainty $\Delta(t)$ is assumed to belong to the following set of norm-bounded time-varying structured uncertainties:

$$\begin{aligned} \Upsilon_{\Delta_t} = \{ & \operatorname{bdiag}(\delta_1(t) I_{l_1}, \dots, \delta_r(t) I_{l_r}, \\ & \Delta_1(t), \dots, \Delta_f(t)), \, \|\Delta(t)\| \le 1\}. \end{aligned}$$

Suppose also the system (6) is well-posed, i.e. $\det(I - D_\Delta \Delta(t)) \ne 0$.

For such a class of uncertain systems, a robust H_2 norm was introduced by Iwasaki in [15] and characterized in terms of LMIs. In this paper, we consider a more general concept of the robust generalized H_2 norm defined as

$$\mathcal{L} = \sup_{\Delta(t)\in\Upsilon_{\Delta_t}} \sup_{v\in L_2} \frac{\|z(t)\|_{g\infty}}{\|v\|_2}. \tag{7}$$

Theorem 3.1 *The upper bound $\mathcal{L}^+$ of the robust generalized H_2 norm can be found as $(\min \gamma^2)^{1/2}$:*

$$\begin{pmatrix} AY + YA^{\mathrm{T}} & \star & \star & \star \\ B_v^{\mathrm{T}} & -I & \star & \star \\ RB_\Delta^{\mathrm{T}} & 0 & -R & \star \\ C_\Delta Y & D_v & D_\Delta R & -R \end{pmatrix} \le 0, \tag{8}$$

$$C^{(i)} Y C^{(i)\mathrm{T}} \le \gamma^2 I, \quad i = 1, \dots, m, \quad Y = Y^{\mathrm{T}} > 0,$$

where $R \in \mathcal{K} = \{\mathrm{bdiag}\,(K_1, \dots, K_r, k_1 I_{m_1}, \dots, k_f I_{m_f})\}$, $K_i = K_i^{\mathrm{T}} > 0$, $k_j > 0$.

Proof. Equations (6) can be rewritten as follows:

$$\begin{aligned} \dot{x} &= Ax + (B_\Delta R^{1/2} \quad B_v) \begin{pmatrix} \bar{v}_\Delta \\ v \end{pmatrix}, \\ \bar{z}_\Delta &= R^{-1/2} C_\Delta x + \hat{D}_v \begin{pmatrix} \bar{v}_\Delta \\ v \end{pmatrix}, \\ z &= Cx, \end{aligned} \tag{9}$$

where $\hat{D}_v = (R^{-1/2} D_\Delta R^{1/2} \quad R^{-1/2} D_v)$, $\bar{v}_\Delta = R^{-1/2} v_\Delta$, $\bar{z}_\Delta = R^{-1/2} z_\Delta$, $R \in \mathcal{K}$. Since $R^{-1/2}\Delta = \Delta R^{-1/2}$, then $\|\bar{v}_\Delta\|_2 \le \|\bar{z}_\Delta\|_2$.

Let $V(x) = x^{\mathrm{T}} X x$ with $X = X^{\mathrm{T}} > 0$ satisfy inequality

$$\dot{V} + |\bar{z}_\Delta|^2 - (|\bar{v}_\Delta|^2 + |v|^2) < 0. \tag{10}$$

Integrate this inequality over $[0, t]$ to obtain $x^{\mathrm{T}}(t) X x(t) < \|v\|_2^2$ for all admissible $\Delta(t)$. Writing the last inequality in the form

$$\begin{pmatrix} X^{-1} & x(t) \\ x^{\mathrm{T}}(t) & \|v\|_2^2 \end{pmatrix} > 0$$

and using Schur lemma, we get

$$Y = X^{-1} > x(t) x^{\mathrm{T}}(t) / \|v\|_2^2$$

and, hence,

$$\frac{z_i(t) z_i^{\mathrm{T}}(t)}{\|v\|_2^2} \le C^{(i)} Y C^{(i)\mathrm{T}} \le \lambda_{\max}(C^{(i)} Y C^{(i)\mathrm{T}}) I. \tag{11}$$

The inequalities (10) and (11) immediately imply (8).

To estimate the lower bound of the robust generalized H_2 norm consider the system (6) with time-invariant norm-bounded structured uncertainty $\Delta \in \Upsilon_\Delta$ described by the equations

$$\begin{aligned} \dot{x} &= A_\Delta x + B_{v\Delta} v, \\ z &= Cx, \end{aligned} \tag{12}$$

where

$$\begin{gathered} A_\Delta = A + B_\Delta \Delta (I - D_\Delta \Delta)^{-1} C_\Delta, \\ B_{v\Delta} = B_v + B_\Delta \Delta (I - D_\Delta \Delta)^{-1} D_v, \\ \Upsilon_\Delta = \{\mathrm{bdiag}\,(\delta_1 I_{l_1}, \dots, \delta_r I_{l_r}, \Delta_1, \dots, \Delta_f),\ \|\Delta\| \le 1\}. \end{gathered}$$

Define the lower bound $\mathcal{L}^-$ of $\mathcal{L}$ as follows

$$\mathcal{L}^- = \sup_{\Delta \in \Upsilon_\Delta} \sup_{v \in L_2} \frac{\|z(t)\|_{g\infty}}{\|v\|_2} = \sup_{\Delta \in \Upsilon_\Delta} \|H_{zv}(\Delta)\|_{g2}, \tag{13}$$

where $H_{zv}(\Delta)$ is the transfer matrix of the system (12). Thus, the robust generalized H_2 norm satisfies the inequalities

$$\mathcal{L}^- \le \mathcal{L} \le \mathcal{L}^+, \tag{14}$$

where the upper bound is computed according to Theorem 3.1, while the lower bound can be found via an optimization procedure over parameters Δ in view of Theorem 2.1.

IV. Multi-objective robust generalized H_2 control problem

A. Problem statement

Consider the uncertain plant with multiple objective outputs

$$\begin{aligned} \dot{x} &= Ax + B_\Delta v_\Delta + B_v v + B_u u, \\ z_\Delta &= C_\Delta x + D_\Delta v_\Delta + D_v v + D_{\Delta u} u, \\ v_\Delta &= \Delta(t) z_\Delta, \\ z_1 &= C_1 x + D_1 u, \ \cdots, \ z_N = C_N x + D_N u. \end{aligned} \tag{15}$$

Each vector objective output consists of the collection of vectors

$$\begin{gathered} z_k = \mathrm{col}\,(z_k^{(1)}, \dots, z_k^{(m_k)}), \quad z_k^{(i)} = C_k^{(i)} x + D_k^{(i)} u, \\ i = 1, \dots, m_k, \quad k = 1, \dots, N. \end{gathered}$$

Suppose control is in the state-feedback form $u = \Theta x$. The closed-loop system has the form

$$\begin{aligned} \dot{x}_c &= A_c(\Theta) x_c + B_\Delta v_\Delta + B_v v, \\ z_\Delta &= \widehat{C}_\Delta(\Theta) x_c + D_\Delta v_\Delta + D_v v, \\ v_\Delta &= \Delta(t) z_\Delta, \\ z_1 &= C_1(\Theta) x_c, \ \cdots, \ z_N = C_N(\Theta) x_c, \end{aligned} \tag{16}$$

where

$$\begin{gathered} A_c(\Theta) = A + B_u \Theta, \quad \widehat{C}_\Delta(\Theta) = C_\Delta + D_{\Delta u} \Theta, \\ C_k(\Theta) = C_k + D_k \Theta = \\ \mathrm{col}\,(C_k^{(1)} + D_k^{(1)} \Theta, \dots, C_k^{(m_k)} + D_k^{(m_k)} \Theta), \ k = 1, \dots, N. \end{gathered}$$

Let us introduce objective costs $\mathcal{L}_k(\Theta)$, $k = 1, \dots, N$ as follows

$$\mathcal{L}_k(\Theta) = \sup_{\Delta(t) \in \Upsilon_{\Delta_t}} \sup_{v \in L_2} \frac{\|z_k(t)\|_{g\infty}}{\|v\|_2}. \tag{17}$$

Optimal control with respect to each cost is referred to the robust generalized H_2 control, while Pareto optimal controls in the multi-objective control problem

$$\Theta_P = \arg\min_\Theta \{\mathcal{L}_k(\Theta), \ k = 1, \dots, N\} \tag{18}$$

are referred to the multi-objective robust generalized H_2 controls.

According to Theorem 2.2 Pareto optimal solutions should be sought among the optimal solutions Θ_α for the scalar multi-objective cost

$$J_\alpha(\Theta) = \max_{1 \le k \le N} \{\mathcal{L}_k(\Theta) / \alpha_k\}, \quad J_\alpha(\Theta_\alpha) = \min_\Theta J_\alpha(\Theta).$$

B. The lower and upper bounds for the multi-objective cost

Now let us utilize the lower and upper bounds of $\mathcal{L}_k$ given in (14) to obtain

$$J_\alpha^-(\Theta) \le J_\alpha(\Theta) \le J_\alpha^+(\Theta), \tag{19}$$

where

$$\begin{aligned} J_\alpha^-(\Theta) &= \max_{1 \le k \le N} \{\mathcal{L}_k^-(\Theta) / \alpha_k\}, \\ J_\alpha^+(\Theta) &= \max_{1 \le k \le N} \{\mathcal{L}_k^+(\Theta) / \alpha_k\}. \end{aligned} \tag{20}$$

Firstly, note that the following inequalities hold:

$$\min_{\Theta} J_\alpha^-(\Theta) = \min_{\Theta} \sup_{\Delta\in\Upsilon_\Delta} \max_{1\le k\le N}\{\|H_{z_k v}(\Delta,\Theta)\|_{g2}/\alpha_k\} \ge \sup_{\Delta\in\Upsilon_\Delta} \min_{\Theta} \max_{1\le k\le N}\{\|H_{z_k v}(\Delta,\Theta)\|_{g2}/\alpha_k\} = \sup_{\Delta\in\Upsilon_\Delta} \min_{\Theta} \|H_{zv}(\Delta,\Theta)\|_{g2},$$

where $H_{zv}(\Delta,\Theta)$ is the transfer matrix of the closed-loop system of the form (12) with $A = A_c(\Theta)$, $C_\Delta = \widehat{C}_\Delta(\Theta)$, and $C^{\mathrm{T}} = C_c^{\mathrm{T}}(\Theta) = (\alpha_1^{-1}C_1^{\mathrm{T}}(\Theta)\ldots\alpha_N^{-1}C_N^{\mathrm{T}}(\Theta))$. Thus, we get the following lower bound for the multi-objective cost:

$$\hat{J}_\alpha^- = \sup_{\Delta\in\Upsilon_\Delta} \min_{\Theta} \|H_{zv}(\Delta,\Theta)\|_{g2} \le \min_{\Theta} J_\alpha(\Theta), \tag{21}$$

where $\min\limits_{\Theta} \|H_{zv}(\Delta,\Theta)\|_{g2}$ is computed in accordance with Corollary 2.1 as $(\min\gamma^2)^{1/2}$ under the LMIs

$$\begin{pmatrix} \Psi & \star \\ B_{v\Delta}^{\mathrm{T}} & -I \end{pmatrix} \le 0, \begin{pmatrix} Y & \star \\ C_k^{(i)}Y + D_k^{(i)}Z & \alpha_k^2\gamma^2 I \end{pmatrix} \ge 0,$$
$$Y > 0, \quad i = 1,\ldots,m_k, \quad k = 1,\ldots,N,$$

where

$$\Psi = A_\Delta Y + Y A_\Delta^{\mathrm{T}} + B_{\Delta u}Z + Z^{\mathrm{T}}B_{\Delta u}^{\mathrm{T}}, \quad Z = \Theta Y,$$
$$B_{\Delta u} = B_u + B_\Delta\Delta(I - D_\Delta\Delta)^{-1}D_{\Delta u}.$$

According to Theorem 3.1, the upper bound in (19) for a given matrix Θ is computed as $(\min\gamma^2)^{1/2}$ under the LMIs

$$\begin{pmatrix} \Phi & \star & \star & \star \\ B_v^{\mathrm{T}} & -I & \star & \star \\ RB_\Delta^{\mathrm{T}} & 0 & -R & \star \\ (C_\Delta + D_{\Delta u}\Theta)Y & D_v & D_\Delta R & -R \end{pmatrix} \le 0,$$
$$\begin{pmatrix} Y & \star \\ (C_k^{(i)} + D_k^{(i)}\Theta)Y & \alpha_k^2\gamma^2 I \end{pmatrix} \ge 0, \quad Y > 0, \tag{22}$$

where $\Phi = (A + B_u\Theta)Y + Y(A + B_u\Theta)^{\mathrm{T}}$, $i = 1,\ldots,m_k$, $k = 1,\ldots,N$.

C. Synthesis of Pareto suboptimal controllers

To get an exact solution to the multi-objective problem (18) is impossible, therefore we consider the following auxiliary problem:

$$\Theta_P^+ = \arg\min_{\Theta}\{\mathcal{L}_k^+(\Theta),\, k = 1,\ldots,N\}. \tag{23}$$

The desired solutions to this problem will be referred as Pareto suboptimal solutions for the initial multi-objective problem. The gain matrices Θ_P^+ are among the optimal solutions Θ_α^+ for the scalar multi-objective cost $J_\alpha^+(\Theta)$ given in (20). For minimizing $J_\alpha^+(\Theta)$ with respect to Θ, we need to find $(\inf\gamma^2)^{1/2}$ subject to the LMIs (22), where $\Theta Y = Z$, with respect to the variables Y, Z and γ^2, which results in $\Theta_\alpha^+ = ZY^{-1}$.

It is reasonable to estimate a "difference" between the gain matrices Θ_α and Θ_α^+. To characterize this difference implicitly let us introduce a suboptimality measure expressed by

$$\eta(\alpha) = \frac{\hat{J}_\alpha^+ - \hat{J}_\alpha^-}{\hat{J}_\alpha^+}, \tag{24}$$

where $\hat{J}_\alpha^+ = \min_\Theta J_\alpha^+(\Theta) = J_\alpha^+(\Theta_\alpha^+)$ and $\hat{J}_\alpha^-$ is given in (21). This characteristics varies on the segment $[0,1]$ and depends on $\alpha_1,\ldots,\alpha_N$. The less the value of η, the nearer the Pareto suboptimal solution to the Pareto optimal one.

V. Robust control of rotating rotor in active magnetic bearings

A. Mathematical model

As an application of the theory developed, we consider the control problem for a vertical rotor rotating with a given angular speed in active magnetic bearings. The motion of the rotor is governed by the differential equations [16]

$$\begin{aligned} J\ddot{\varphi} &= l_0[(F_2^l - F_1^l) - (F_2^u - F_1^u)] - J_z\omega\dot{\psi}, \\ J\ddot{\psi} &= l_0[(F_3^u - F_4^u) - (F_3^l - F_4^l)] + J_z\omega\dot{\varphi}, \\ m\ddot{\xi} &= F_3^u - F_4^u + F_3^l - F_4^l, \\ m\ddot{\zeta} &= F_2^u - F_1^u + F_2^l - F_1^l, \end{aligned} \tag{25}$$

where ξ, ζ are the coordinates of the center of mass of the rotor, φ, ψ are the angles of the rotor rotations with respect to two axes which are perpendicular to the vertical axis, l_0 is the semi-length of the rotor, the superscripts u and l denote the electromagnetic forces acting on the rotor from the upper and lower magnetic bearings, respectively, m is the mass of the rotor, J, J_z are the main moments of inertia of the rotor, ω is the angle velocity of the rotating rotor. The pairs of the electromagnetic forces in (25) are given as follows

$$\begin{aligned} F_2^u - F_1^u &= \frac{L_0S_0}{2}\left\{\frac{I_{2u}^2}{(S_0 - \zeta_u)^2} - \frac{I_{1u}^2}{(S_0 + \zeta_u)^2}\right\}, \\ F_2^l - F_1^l &= \frac{L_0S_0}{2}\left\{\frac{I_{2l}^2}{(S_0 - \zeta_l)^2} - \frac{I_{1l}^2}{(S_0 + \zeta_l)^2}\right\}, \end{aligned} \tag{26}$$

$$\begin{aligned} F_3^u - F_4^u &= \frac{L_0S_0}{2}\left\{\frac{I_{3u}^2}{(S_0 - \xi_u)^2} - \frac{I_{4u}^2}{(S_0 + \xi_u)^2}\right\}, \\ F_3^l - F_4^l &= \frac{L_0S_0}{2}\left\{\frac{I_{3l}^2}{(S_0 - \xi_l)^2} - \frac{I_{4l}^2}{(S_0 + \xi_l)^2}\right\}, \end{aligned} \tag{27}$$

where I_{1u}, I_{2u}, I_{3u}, I_{4u}, I_{1l}, I_{2l}, I_{3l}, I_{4l} are the currents in the upper and lower magnetic bearings, L_0 is the value of a nominal inductance, S_0 is the nominal value of a gap in the magnetic bearings, ξ_u, ζ_u, ξ_l, ζ_l are the measurable displacements of the rotor in the upper and lower magnetic bearings. For sufficiently small angles φ and ψ, the variables ξ_u, ζ_u, ξ_l, ζ_l can be presented as

$$\xi_u = \xi + \psi l_0,\ \xi_l = \xi - \psi l_0,\ \zeta_u = \zeta - \varphi l_0,\ \zeta_l = \zeta + \varphi l_0. \tag{28}$$

Note that the system (25) with taken into account the formulae for electromagnetic forces (26) is nonlinear. Using the well-known feedback linearization approach we denote

$$\begin{aligned} F_2^u - F_1^u = u_1, \quad F_2^l - F_1^l = u_2, \\ F_3^u - F_4^u = u_3, \quad F_3^l - F_4^l = u_4. \end{aligned} \tag{29}$$

Substituting these relations in (25), we arrive at a linear controlled system

$$
\begin{aligned}
J\ddot{\varphi} &= l_0(u_2 - u_1) - J_z\omega\dot{\psi},\\
J\ddot{\psi} &= l_0(u_3 - u_4) + J_z\omega\dot{\varphi}, \qquad (30)\\
m\ddot{\xi} &= u_3 + u_4, \quad m\ddot{\zeta} = u_1 + u_2.
\end{aligned}
$$

Then the dependence of the currents (for example, I_{1u} and I_{2u}) on the state and control variables has the form

$$
I_{1u} = \begin{cases} 0 & u_1 \geq 0 \\ \sqrt{\dfrac{2|u_1|}{L_0 S_0}}(S_0 + \zeta_u) & u_1 < 0 \end{cases}, \qquad (31)
$$

$$
I_{2u} = \begin{cases} \sqrt{\dfrac{2|u_1|}{L_0 S_0}}(S_0 - \zeta_u) & u_1 \geq 0 \\ 0 & u_1 < 0 \end{cases}. \qquad (32)
$$

The values of the moment of inertia J and mass m are assumed to be unknown exactly

$$
J = J_0(1 + \delta_1 w_1), \quad m = m_0(1 + \delta_2 w_2), \qquad (33)
$$

where J_0 and m_0 are the nominal values, w_1 and w_2 are the given parameters, δ_1 and δ_2 are unknown parameters satisfying the inequalities $|\delta_1| \leq 1$, $|\delta_2| \leq 1$. Let us introduce dimensionless variables and parameters as follows

$$
\begin{gathered}
\xi = S_0\xi', \quad \zeta = S_0\zeta', \quad t = Tt', \quad u_i = U_0 u_i',\\
\varphi = \varphi' S_0/l_0, \quad \psi = \psi' S_0/l_0,
\end{gathered}
$$

where $T = \sqrt{m_0 S_0/U_0}$. The equations (30) in the dimensionless form with additional disturbances $v = (v_1\, v_2\, v_3\, v_4)^T$ are rewritten as

$$
\begin{aligned}
(1 + \delta_1 w_1)\ddot{\varphi} &= \mu(u_2 - u_1) - \rho\dot{\psi} + v_1,\\
(1 + \delta_1 w_1)\ddot{\psi} &= \mu(u_3 - u_4) + \rho\dot{\varphi} + v_2,\\
(1 + \delta_2 w_2)\ddot{\xi} &= u_3 + u_4 + v_3,\\
(1 + \delta_2 w_2)\ddot{\zeta} &= u_1 + u_2 + v_4,
\end{aligned} \qquad (34)
$$

where $\mu = m_0 l_0^2/J_0,\ \rho = J_z T\omega/J_0$. Note also that

$$
\xi_u = \xi + \psi, \quad \xi_l = \xi - \psi, \quad \zeta_u = \zeta - \varphi, \quad \zeta_l = \zeta + \varphi. \qquad (35)
$$

B. Problem statement

For the system (34), we choose two performance indices

$$
\mathcal{L}_1 = \sup_{\substack{|\delta_i| \leq 1,\\ v \in L_2}} \frac{\max\{\sup_t \sqrt{\xi_l^2 + \zeta_l^2}, \sup_t \sqrt{\xi_u^2 + \zeta_u^2}\}}{\|v\|_2}.
$$

$$
\mathcal{L}_2 = \sup_{\substack{|\delta_i| \leq 1,\\ v \in L_2}} \frac{\max_i\{\sup_t |u_i|\}}{\|v\|_2}.
$$

The first of these performance indices is a maximal displacement of the rotor in the upper and lower magnetic bearings, while the second one is a maximal value of the control force. Denote

$$
\begin{aligned}
&x_1 = \varphi, \quad x_2 = \psi, \quad x_3 = \xi, \quad x_4 = \zeta,\\
&x_5 = \dot{\varphi}, \quad x_6 = \dot{\psi}, \quad x_7 = \dot{\xi}, \quad x_8 = \dot{\zeta},\\
&z_1 = \mathrm{col}(z_1^{(1)}, z_1^{(2)}), \quad z_2 = \mathrm{col}(z_2^{(1)}, z_2^{(2)}, z_2^{(3)}, z_2^{(4)}),\\
&z_1^{(1)} = \mathrm{col}(x_4 - x_1, x_3 + x_2),\ z_1^{(2)} = \mathrm{col}(x_4 + x_1, x_3 - x_2),\\
&z_2^{(i)} = u_i, \quad i = 1, \ldots, 4
\end{aligned}
$$

to rewrite the system (34) in the standard form

$$
\begin{aligned}
\dot{x} &= Ax + B_\Delta v_\Delta + B_v v + B_u u,\\
z_\Delta &= C_\Delta x + D_\Delta v_\Delta + D_v v + D_{\Delta u} u, \qquad (36)\\
z_1 &= C_1 x + D_1 u, \quad z_2 = C_2 x + D_2 u,
\end{aligned}
$$

where $v_\Delta = \Delta z_\Delta$,

$$
\Delta = \begin{pmatrix} \delta_1 I_2 & 0_2 \\ 0_2 & \delta_2 I_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0_4 & I_4 \\ 0_4 & G \end{pmatrix},
$$

$$
G = \begin{pmatrix} \widehat{G} & 0_2 \\ 0_2 & 0_2 \end{pmatrix}, \quad \widehat{G} = \rho \begin{pmatrix} \hat{0} & -1 \\ 1 & 0 \end{pmatrix},
$$

$$
B_\Delta = \begin{pmatrix} 0_4 \\ \widehat{B}_\Delta \end{pmatrix}, \quad \widehat{B}_\Delta = \begin{pmatrix} w_1 I_2 & 0_2 \\ 0_2 & w_2 I_2 \end{pmatrix},
$$

$$
B_v = \begin{pmatrix} 0_4 \\ \widehat{B}_v \end{pmatrix}, \quad B_u = \begin{pmatrix} 0_4 \\ \widehat{B}_u \end{pmatrix},
$$

$$
\widehat{B}_v = I_4, \quad C_\Delta = (0_4\ -G), \quad D_\Delta = -\widehat{B}_\Delta,
$$

$$
D_v = -I_4, \quad D_{\Delta u} = -\widehat{B}_u,
$$

$$
C_1 = (\widehat{C}_1\ 0_4), \quad D_1 = 0_4, \quad C_2 = (0_4\ 0_4), \quad D_2 = I_4,
$$

$$
\widehat{B}_u = \begin{pmatrix} -\mu & \mu & 0 & 0 \\ 0 & 0 & \mu & -\mu \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \ \widehat{C}_1 = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}.
$$

Consider two-criteria state feedback control problem for the system (36) with performance indices $\mathcal{L}_1$ and $\mathcal{L}_2$.

C. Numerical solution for two-criteria control problem

The Pareto suboptimal controllers have been synthesized for given parameters $\mu = 1$, $\rho = 0.1$, $w_1 = w_2 = 0.5$. On Fig. 1, the coordinates of the points $(\mathcal{L}_1^+(\Theta_\alpha^+), \mathcal{L}_2^+(\Theta_\alpha^+))$ on the black curve correspond to the upper bounds of the robust generalized H_2 norms of the closed-loop system under Pareto suboptimal controllers synthesized. The blue and red solid curves present the sets of points corresponding to the values of the generalized H_2 norms under the above controllers when $\delta_1 = \delta_2 = -1$ and $\delta_1 = \delta_2 = 1$, respectively. The blue and red dashed curves present the sets of points corresponding to the values of the generalized H_2 norms under Pareto optimal controllers for the the systems without uncertainty when $\delta_1 = \delta_2 = -1$ and $\delta_1 = \delta_2 = 1$, respectively. On Fig. 2, the graph of the suboptimality measure $\eta = \eta(\alpha)$ is presented for $\alpha_1 = \alpha$, $\alpha_2 = 1 - \alpha$.

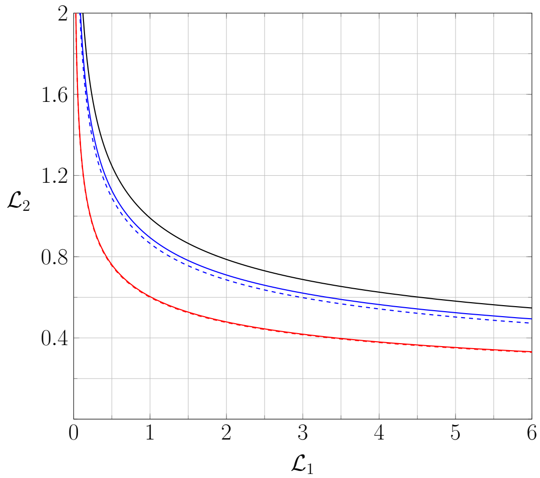

Fig. 1. The values of criteria for different controllers

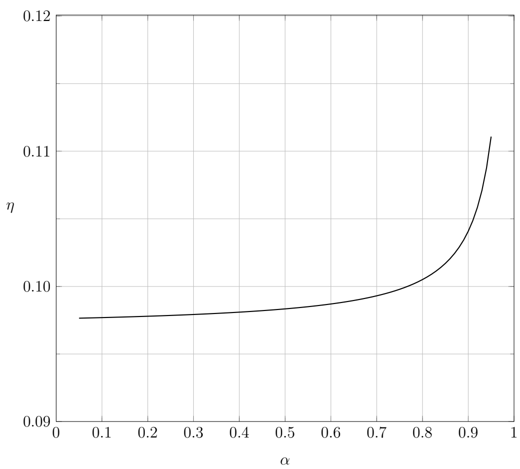

Fig. 2. The suboptimality measure versus α

From these figures it follows that the Pareto suboptimal controllers reveal some remarkable properties. Firstly, conservativeness of the proposed approach caused by using the upper bounds $\mathcal{L}_1^+$ and $\mathcal{L}_2^+$ instead of $\mathcal{L}_1$ and $\mathcal{L}_2$, is rather slight because the suboptimality measure is about 10 percents only. The Pareto optimal front corresponding to $\mathcal{L}_1$ and $\mathcal{L}_2$, which is impossible to characterize exactly, should lie over the blue dashed curve and under the black solid curve. Secondly, for the systems without uncertainty, the Pareto suboptimal controllers provide the generalized H_2 norms for concrete values of uncertainty close to ones under the Pareto optimal controllers (the blue and red dashed curves almost coincide with blue and red solid ones).

VI. Conclusion

A novel multi-objective robust control problem is studied for systems with structured norm-bounded uncertainty and robust generalized H_2 norms as criteria. The robust generalized H_2 norm is introduced as a worst-case generalized H_2 norm over structured norm-bounded uncertainties. A scalar multi-objective cost in the form of Germeyer convolution is used for solving the multi-objective problem. The upper and lower bounds of this cost are characterized in terms of LMIs. Pareto suboptimal controllers are determined as optimal solutions for the upper bounds of the multi-objective costs. A suboptimality measure is computed to estimate a gap between Pareto suboptimal and unavailable Pareto optimal controllers. Numerical experiments with active magnetic bearings show that the suboptimality measure is about 10 percents only.

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