Output Regulation of Linear Stochastic Systems: the Full-Information Case

Giordano Scarciotti

Abstract— The full information output regulation problem for linear stochastic systems is addressed. A general class of linear systems is considered, namely systems in which the state, control variable and exogenous variable may appear simultaneously in the drift term and in the diffusion term of the differential equation. Similarly, we consider a stochastic signal generator, thus allowing tracking and/or rejecting Brownian motions in addition to deterministic trajectories. In the paper we first characterize the steady-state response of the interconnection of the system with the signal generator and then we solve the full information output regulation problem. The results of the paper are illustrated by means of two examples. Finally a short discussion of the error feedback regulator problem concludes the paper.

I. INTRODUCTION

The problem of output regulation is a fundamental problem in control theory. This problem is concerned with the design of a control law achieving two objectives: first, the closed-loop system must be asymptotically stable; second, the output must asymptotically track reference signals and reject unmeasured disturbances, under the assumption that both references and disturbances are produced by a known signal generator (also known as exogenous system). A complete solution of the problem for linear deterministic systems has been given in the works of Francis, Wonham and Davison in the 1970s, see [1], [2], [3], [4]. Starting in 1990 with the works of Isidori, Byrnes and Huang [5], [6], the problem has been solved for several classes of nonlinear systems, see *e.g.* [7], [8], [9], [10]. In recent years, the problem has been generalized in various directions. For instance, the output regulation problem is posed and solved for some classes of hybrid systems in [11], [12] and for linear time-varying systems in [13].

In this paper we focus on the problem of output regulation of linear stochastic systems. Stochastic differential equations are a class of systems in which uncertainty is modeled by means of stochastic processes, *i.e.* time sequences representing the evolution of variables subjected to random variations. Stochastic systems arise in a multitude of different applications and problems such as the production planning problem [14], [15], the investment versus consumption problem [16], [17], [15], reinsurance and dividend management [18], [15], the technology diffusion problem [19], [20], [15] and the optimal stopping problem [21], [15]. While several results have been produced for the problem of control of stochastic

systems, see *e.g.* [15], [22] and references therein, the problem of output regulation of stochastic systems has not been systematically studied. One of the rare results in this direction is the work of [23], [24] in which Markovian jumping linear systems are considered. However, the underlying systems are linear deterministic systems.

In this paper we address the problem of output regulation for a general class of linear stochastic systems. For general class, we mean systems in which the state, the control input and the exogenous input may appear simultaneously in both the drift term and diffusion term of the equation, *i.e.* they may multiply both the dt term and the $d\mathcal{W}$ term, where $\mathcal{W}$ is a Wiener process. Moreover, also the signal generator is selected as a stochastic system. In this setup, the regulation objective may include tracking and rejection of stochastic signals by means of a control variable that may be stochastic and acting on a state that may be stochastic. The paper is focused on the full information regulator problem and one of the main results obtained is a stochastic differential generalization of the classical regulator equations for linear deterministic systems. We provide two examples (a randomly generated system and an inverted pendulum on a cart on a vibrating platform) and we conclude the paper with a brief discussion of the error feedback output regulation problem, showing that in the stochastic framework the problem presents great challenges.

The rest of the paper is organized as follows. In Section II we introduce formally the problem of full information output regulation in the present setting. In Section III we characterize the steady-state response of the system in terms of a generalized (stochastic and differential) Sylvester equation. In Section IV we solve the full information output regulation problem. Two examples illustrate the theory in Section V. Finally, Section VI contains a short discussion of the error feedback case and some concluding remarks.

Notation. We use standard notation. $\mathbb{C}_{<0}$ ($\mathbb{C}_{\geq 0}$) denotes the set of complex numbers with negative (non-negative) real part. The symbol $\sigma(A)$ denotes the spectrum of the matrix $A \in \mathbb{R}^{n \times n}$. The symbol $\otimes$ indicates the Kronecker product. The superscript $\top$ denotes the transposition operator. $(\nabla, \mathcal{A}, \mathcal{P})$ indicates a probability space with a given set ∇, a σ-algebra $\mathcal{A}$ on ∇ and a probability measure $\mathcal{P}$ on the measurable space $(\nabla, \mathcal{A})$. Unless otherwise stated, all the stochastic integrals in this paper are intended as Itô integrals.

II. PROBLEM FORMULATION

In this section we formulate the full information output regulation problem for linear stochastic systems.

G. Scarciotti is with the Dept. of Electrical and Electronic Engineering, Imperial College London, London, SW7 2AZ, UK, E-mail: g.scarciotti@ic.ac.uk.

Let $\mathcal{W}_t$ be a standard Wiener process defined on a probability space $(\nabla, \mathcal{A}, \mathcal{P})$. A stochastic process x_t is a function of two variables such that for each $t \in \mathbb{R}$, $x(t, \cdot)$ is a random variable and for each $w \in \nabla$, $x(\cdot, w)$ is called path of x. For ease of notation, we indicate the paths as just functions of t, *e.g.* the path of x_t as $x : t \mapsto x(t)$ and the path of $\mathcal{W}_t$ as $\mathcal{W} : t \mapsto \mathcal{W}(t)$ (this is common in the literature, see *e.g.* [25]). Consider a stochastic linear, single-input, single-output system described by the equations

$$
\begin{aligned}
dx &= [Ax + Bu + P\omega]dt + [Fx + Gu + R\omega]d\mathcal{W}, \\
y &= Cx + Du, \\
e &= y + Q\omega, \\
x(0) &= x_0,
\end{aligned}
\tag{1}
$$

with $x(t) \in \mathbb{R}^n$, $\omega(t) \in \mathbb{R}^\nu$, $u(t) \in \mathbb{R}$, $e(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $P \in \mathbb{R}^{n \times \nu}$, $F \in \mathbb{R}^{n \times n}$, $G \in \mathbb{R}^{n \times 1}$, $R \in \mathbb{R}^{n \times \nu}$ $C \in \mathbb{R}^{1 \times n}$, $Q \in \mathbb{R}^{1 \times \nu}$ and $D \in \mathbb{R}$. Assume that the initial condition $x_0 \in \mathbb{R}^n$ is independent of $\mathcal{W}(t)$ for all $t > 0$. Under this assumption the initial value problem associated with (1) has a unique solution, see *e.g.* [25]. The exogenous variable ω is generated by the signal generator

$$
\begin{aligned}
d\omega &= S\omega dt + J\omega d\mathcal{W}, \\
\omega(0) &= \omega_0,
\end{aligned}
\tag{2}
$$

with $S \in \mathbb{R}^{\nu \times \nu}$, $J \in \mathbb{R}^{\nu \times \nu}$ and $\omega_0 \in \mathbb{R}^\nu$ independent of $\mathcal{W}(t)$ for all $t > 0$. Assume also that S and J are commuting matrices (this assumption is discussed in Remark 1). The problem that we want to solve is the stochastic version of the classical full information output regulation problem, see *e.g.* [1], [2], [3], [4]. Since the output regulation problem can be partitioned in the two subproblems of achieving stability and achieving zero steady-state error, before precisely formulating the output regulation problem, we need to clarify the notion of stability which is used in the paper. In particular, in the remaining we use the notions of *almost surely asymptotic stability in probability* (which in the following we call some times *asymptotic stochastic stability*) and *asymptotic mean-square stability* (see [26, Chapter 1.5] for the two definitions).

Lemma 1 (See [27, Section 11.4]). System (1) is *asymptotically mean-square stable* if and only if $\sigma(I \otimes A + A \otimes I + F \otimes F) \subset \mathbb{C}_{<0}$.

Corollary 1 (See [28, Corollary 1.5.7]). If system (1) is asymptotically mean-square stable, then it is asymptotically stochastically stable.

Asymptotic mean-square stability has been often preferred in the problem of stabilization of linear stochastic systems. A large body of results has been proposed to characterize this type of stability (see the references in [28]) and a stabilizing controller can be determined by solving a Lyapunov-type linear matrix inequality, see [28, Lemma 1.7.3]. On the other hand, since almost surely asymptotic stability in probability is implied by asymptotic mean-square stability, in this paper we use the former notion to develop tighter results. These

results will be implied when working with the mean-square notion by virtue of Corollary 1. We can now formulate the output regulation problem.

Problem 1 (Full information Output Regulation Problem). Consider system (1), driven by the signal generator (2). The full information regulator problem consists in determining a static regulator

$$
u = Kx + \Gamma(t, w)\omega,
\tag{3}
$$

where $K \in \mathbb{R}^{1 \times n}$ and $\Gamma(t, w) \in \mathbb{R}^{1 \times \nu}$ such that the following two conditions hold.

(S) The close-loop system obtained by interconnecting system (1) and (3) with $\omega \equiv 0$ is almost surely asymptotically stable in probability.

(R) The close-loop system obtained by interconnecting system (1), the signal generator (2) and the control law (3) satisfies

$$
\lim_{t \to \infty} e(t) = 0
$$

almost surely, for any $(x(0), \omega(0)) \in \mathbb{R}^n \times \mathbb{R}^\nu$.

III. STEADY-STATE OF LINEAR STOCHASTIC SYSTEMS

Since the regulator requirement (R) is a condition on the steady-state behavior of the system, it is instrumental for the solution of the problem to provide a description of the steady-state response of the system. In this section we characterize the steady-state of system (1) in terms of a generalized (dynamic and stochastic) Sylvester equation (see [29] for a preliminary result). Before providing the main result of this section, we introduce some preliminary definitions and properties. Let $\Phi(t) \in \mathbb{R}^{n \times n}$ be the fundamental matrix of the homogeneous equation corresponding to (1), *i.e.*

$$
d\Phi(t) = \Phi(t)\left(Adt + Fd\mathcal{W}\right),
\tag{4}
$$

with $\Phi(0) = I$. Let $\Sigma(t) \in \mathbb{R}^{\nu \times \nu}$ be the fundamental matrix corresponding to (2). Since S and J are assumed to be commuting matrices, $\Sigma(t)$ commutes with S and J, *i.e.*

$$
d\Sigma = (Sdt + Jd\mathcal{W})\Sigma(t),
\tag{5}
$$

holds with $\Sigma(0) = I$. Finally, we introduce the following assumption.

Assumption 1. The matrix $S - \frac{1}{2}J^2$ has all the eigenvalues with non-negative real part, *i.e* $\sigma\left(S - \frac{1}{2}J^2\right) \subset \mathbb{C}_{\geq 0}$.

Assumption 1 implies, in virtue of the commutativity of S and J, that $\omega(t) \equiv 0$ is almost surely stable in probability. We are now ready to give a characterization of the steady-state response of system (1).

Lemma 2. Consider the interconnection of system (1) and the signal generator (2) with $u = 0$. Suppose that Assumption 1 holds and that system (1) is almost surely asymptotically stable in probability. Then the steady-state response of the output of such interconnection is

$$
y_{ss}(t) = C\Pi(t, w)\omega(t)
$$

almost surely, where $\Pi(t, w) \in \mathbb{R}^{n \times \nu}$ solves the stochastic differential matrix equation

$$d\Pi = \left(A\Pi - \Pi \left(S - J^2 \right) - F\Pi J + P - RJ \right) dt$$
$$+ \left(F\Pi - \Pi J + R \right) d\mathcal{W}, \qquad (6)$$

where the initial condition $\Pi(0)$ is the unique solution of the Sylvester equation

$$A\Pi(0) - \Pi(0)S = -P.$$

The solution of the stochastic differential equation is given by

$$\Pi = \Phi(t) \left[\int_0^t \Phi(\tau)^{-1}(P - FR)\Sigma(\tau)d\tau \right.$$
$$\left. + \int_0^t \Phi(\tau)^{-1}R\Sigma(\tau)d\mathcal{W} + \Pi(0) \right] \Sigma(t)^{-1}. \qquad (7)$$

Remark 1. The assumption that S and J are commuting matrices is needed in the proof of the previous result. Simulations show that the commuting assumption is not required, *i.e.* Lemma 2 holds for non-commuting matrices S and J. However, we have not been able yet to provide a proof of Lemma 2 without this assumption. Note also that this is anyway a mild assumption because the signal generator is selected by the designer.

IV. THE FULL INFORMATION REGULATOR PROBLEM

In this section we solve the full information output regulator problem. We begin with providing a preliminary result in which the stability requirement (S) is assumed to hold. We then provide a result on the placement of the closed-loop eigenvalues and we solve the regulator problem.

Lemma 3. Consider the full information regulator problem 1. Suppose Assumption 1 holds. Suppose in addition that there exist a matrix K such that condition (S) holds. Then condition (R) holds if and only if there exist matrices Π and Γ such that the equations

$$d\Pi = [(A + BK)\Pi - \Pi(S - J^2) - (F + GK)\Pi J + P$$
$$+ B\Gamma - (G\Gamma + R)J]dt$$
$$+ [(F + GK)\Pi - \Pi J + R + G\Gamma]d\mathcal{W},$$
$$0 = (C + DK)\Pi + Q + D\Gamma, \qquad (8)$$

with $(A + BK)\Pi(0) - \Pi(0)S = -(P + B\Gamma(0))$ holds.

Lemma 3 solves part of Problem 1. Before developing a solution for the whole problem, we need to introduce the concept of stabilizability in the stochastic framework.

Definition 1. System (1) is stochastically stabilizable if there exists a matrix K such that the closed-loop system obtained interconnecting $u = Kx$ and system (1) with $\omega \equiv 0$ is almost surely asymptotically stable in probability.

Remark 2. Differently from the deterministic case, there is no necessary and sufficient Hautus-like conditions to guarantee

stochastic stabilizability (as given in Definition 1) or mean-square stabilizability. Conditions for stochastic stabilizability are discussed in detail in [27, Section 11]. A necessary but not sufficient Hautus-like condition for mean-square stabilizability is given in [28, Lemma 1.7.2]. Mean-square stabilizability can be fully characterized by means of the solution of a matrix inequality, see [28, Lemma 1.7.3] and [204, 205, 191] for some special cases.

To the best of the author's knowledge the related problem of placing the eigenvalues of the closed-loop system has not been studied. While the solution of this problem is not necessary for this paper because we just need to stabilize the system (which can be achieved using the methods shown in, *e.g.* [28]), in the following we provide a result (an extension of Simon-Mitter's algorithm [30], [31]) for the placement of the eigenvalues of the matrix $A + BK - \frac{1}{2}(F + GK)^2$ for a special case in which this condition implies stochastic asymptotic stability. To this end let $v_\lambda^\top$ be a left eigenvector of $\bar{A} = A - \frac{1}{2}F^2$ associated to the eigenvalue $\lambda \in \mathbb{R}$. Let $\bar{g} = v_\lambda^\top G$, $\bar{f} = v_\lambda^\top F v_\lambda / (v_\lambda^\top v_\lambda)$ and $\bar{b} = v_\lambda^\top B - \bar{g}\bar{f}$.

Lemma 4. Let A and F be commuting matrices and consider $\gamma \in \mathbb{R}$, with $\gamma < \lambda$. Assume $\bar{b}$ and $\bar{g}$ are not simultaneously zero. Let $K = \bar{k}v_\lambda^\top$, where $\bar{k}$ is selected as follows.

- If $\bar{g} = 0$ and $\bar{b} \neq 0$,

$$\bar{k} = \frac{(\gamma - \lambda)}{\bar{b}}.$$

- If $\bar{g} \neq 0$,

$$\bar{k} = \frac{\bar{b} \pm \sqrt{\bar{b}^2 - 2\bar{g}^2(\gamma - \lambda)}}{\bar{g}^2}.$$

Then the matrix $A + BK - \frac{1}{2}(F + GK)^2$ has the same eigenvalues of $\bar{A}$ apart for λ that is replaced by γ.

Proof. We want to achieve

$$v_\lambda^\top \left(A + BK - \frac{1}{2}(F + GK)^2 \right) = v_\lambda^\top \gamma \qquad (9)$$

Since A and F commute, F commutes with $\bar{A}$. Thus $v_\lambda^\top$ is also a left eigenvector of F, *i.e.* $v_\lambda^\top F = \bar{f}v_\lambda^\top$. The left hand-side of equation (9) can be written as

$$v_\lambda^\top \left(\bar{A} + (B - \frac{1}{2}FG)\bar{k}v_\lambda^\top - \frac{1}{2}G\bar{k}v_\lambda^\top F - \frac{1}{2}G\bar{k}v_\lambda^\top G\bar{k}v_\lambda^\top \right)$$
$$= \lambda v_\lambda^\top + \left(v_\lambda^\top B - \frac{1}{2}\bar{f}\bar{g} \right) \bar{k}v_\lambda^\top - \frac{1}{2}\bar{g}\bar{k}\bar{f}v_\lambda^\top - \frac{1}{2}\bar{g}\bar{k}\bar{g}\bar{k}v_\lambda^\top$$
$$= \left(\lambda + \bar{b}\bar{k} - \frac{1}{2}\bar{g}^2\bar{k}^2 \right) v_\lambda^\top.$$

Hence, equation (9) holds only if

$$\bar{b}\bar{k} - \frac{1}{2}\bar{g}^2\bar{k}^2 = \gamma - \lambda.$$

The solution of this equation leads to the two cases stated in the lemma. $\qquad\square$

Remark 3. Similarly to the deterministic case, Lemma 4 can be adapted to place complex eigenvalues while still using a

real-valued feedback matrix (considering the simultaneous replacement of two eigenvalues). The assumption that $\gamma < \lambda$ is not restrictive when the objective is to stabilize the system. This assumption is sufficient (but not necessary) to guarantee that $\bar{k}$ is real. The hypothesis that the matrices A and F commute is restrictive. In addition, note that if more than one eigenvalue has to be replaced, then F and G must satisfy $F = \alpha A$ and $G = \alpha B$ for some $\alpha \in \mathbb{R}$. This property assures that after replacing the first eigenvalue, the resulting matrices are commuting as well (so that a new iteration can be carried out).

We now solve the full information output regulator problem.

Assumption 2. System (1) is stochastically stabilizable.

Theorem 1. Consider the full information regulator problem. Suppose Assumptions 1 and 2 hold. Then there exists matrices K and Γ such that the control law (3) solves the full information regulator problem if and only if there exist two matrices Π and Δ such that the equations

$$d\Pi = [A\Pi - \Pi(S - J^2) - F\Pi J + P + B\Delta$$
$$-(G\Delta + R)J]dt + [F\Pi - \Pi J + R + G\Delta]d\mathcal{W},$$
$$0 = C\Pi + Q + D\Delta, \tag{10}$$

with $A\Pi(0) - \Pi(0)S = -(P + B\Delta(0))$ holds.

As a result of Theorem 1, a control law which solves the full information regulator problem is given by

$$u = Kx + (\Delta - K\Pi)\omega,$$

where K is any matrix such that the closed-loop system is almost surely asymptotically stable in probability and Δ and Π are such that the regulator equations (10) hold. Since the existence of K is guaranteed by Assumption 2, the solution of the full information problem relies on the solution of the equations (10).

V. EXAMPLES

In this section we illustrate the results of the paper by means of two simple examples.

A. A randomly generated system

Most of the quantities in this example have been randomly generated in MATLAB. To render the simulations reproducible, we have used the command *rng('default')* which sets the random generator of MATLAB to the Mersenne Twister with seed zero. The matrices of system (1) have been selected as

$$A = \begin{bmatrix} 0.8147 & 0.6324 & 0.9575 & 0.9572 \\ 0.9058 & 0.0975 & 0.9649 & 0.4854 \\ 0.1270 & 0.2785 & 0.1576 & 0.8003 \\ 0.9134 & 0.5469 & 0.9706 & 0.1419 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4218 \\ 0.9157 \\ 0.7922 \\ 0.9595 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.6557 & 0.0357 & 0.8491 & 0.9340 \end{bmatrix}^\top, \quad D = 0.4456,$$

$$Q = \begin{bmatrix} 0.1869 & 0.4898 \end{bmatrix}, \quad F = 0.1A, \quad G = 0.1B,$$
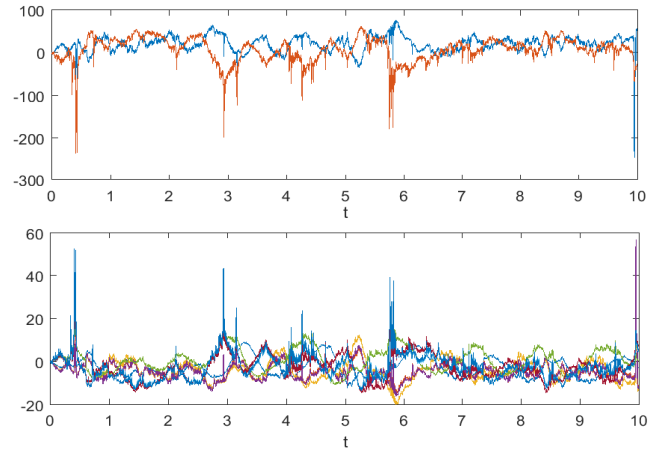

Fig. 1. Time history of the components of Γ (top) and of Π (bottom).

$$P = \begin{bmatrix} 0.6555 & 0.2769 \\ 0.1712 & 0.0462 \\ 0.7060 & 0.0971 \\ 0.0318 & 0.8235 \end{bmatrix}, \quad R = \begin{bmatrix} 0.6948 & 0.4387 \\ 0.3171 & 0.3816 \\ 0.9502 & 0.7655 \\ 0.0344 & 0.7952 \end{bmatrix}.$$

The matrices of the signal generator (2) have been selected as

$$J = \begin{bmatrix} 0.0679 & 0.0743 \\ 0.0758 & 0.0392 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} + \frac{1}{2}J^2.$$

The initial conditions of system (1) and of the signal generator (2) have been selected as

$$x(0) = \begin{bmatrix} 4.5699 & 3.9233 & 0.7591 & 2.1054 \end{bmatrix}^\top,$$
$$\omega(0) = \begin{bmatrix} 0.9548 & 0.1895 \end{bmatrix}^\top.$$

We begin with determining a stabilizing feedback matrix K. This can be achieved using the results in *e.g.* [28] to determine a mean-square stabilizing feedback matrix. By Corollary 1 the resulting closed-loop system is asymptotically stable almost surely. Alternatively, we can apply Lemma 4 to place the eigenvalues of the matrix $\left(A + BK - \frac{1}{2}\left(F + GK\right)^2 \right)$. The desired eigenvalues are chosen to be -1.5, -2, -2.5 and -3. This is achieved by the feedback matrix

$$K = \begin{bmatrix} -6.6066 & 299.4586 & 38.2259 & -324.9929 \end{bmatrix}.$$

The matrices Γ and Π are computed via numerical implementation of the formulas in Theorem 1. Fig 1 shows the time history of the two components of the matrix Γ (top graph) and of the matrix Π (bottom graph). Fig. 2 (top graph) shows the time history of the output response of the system $y = Cx + Du$ (solid/blue line) and of the steady-state output response of the system $y_{ss} = (C\Pi + D\Gamma)\omega$ (dotted/red line). We observe that the steady-state output response of the system is described correctly by Lemma 2. Fig. 2 (bottom graph) shows the time history of the regulation error e. Note that the error is driven to zero. Hence, the obtained control law solves the full information output regulation problem.
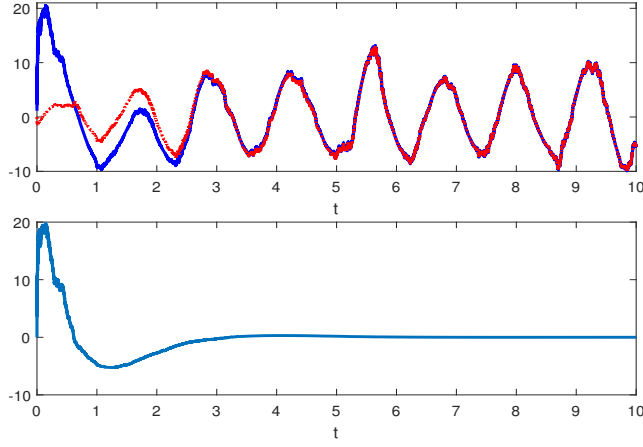
Fig. 2. Top graph: time history of the output response of the system $y = Cx + Du$ (solid/blue line) and of the steady-state output response of the system $y_{ss} = (C\Pi + D\Gamma)\omega$ (dotted/red line). Bottom graph: time history of the regulation error e.

Fig. 4. Top graph: time history of the output response of the system $y = Cx + Du$ (solid/blue line) and the steady-state output response of the system $y_{ss} = (C\Pi + D\Gamma)\omega$ (dotted/red line). Bottom graph: time history of the regulation error e.

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{m}{M} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{m+M}{lM} & 0 \end{bmatrix}, \qquad \begin{array}{l} C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \\[2mm] Q = \begin{bmatrix} -1 & 0 \end{bmatrix}, \quad D = 1, \end{array}$$

and zero matrices G, P and R. The reference is generated by the signal generator (2) with

$$S = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix},$$

zero matrix J and $\omega_0 = \begin{bmatrix} 0.2 & 0 \end{bmatrix}$. The initial condition of the pendulum is selected as $x = \begin{bmatrix} 0.1 & 0.1 & -\frac{\pi}{6} & 0 \end{bmatrix}^{\top}$. The values of the parameters are selected as $g = 9.8093\, ms^{-1}$, $l = 0.842\, m$, $m = 0.1\, kg$ and $M = 1\, kg$. Fig. 4 (top graph) shows the time history of the output response of the system $y = Cx + Du$ (solid/blue line) and of the steady-state output response of the system $y_{ss} = (C\Pi + D\Gamma)\omega$ (dotted/red line). The bottom graph shows the time history of the regulation error e. From the figures we see that the output of the system is regulated as desired.

VI. CONCLUSION AND FURTHER RESEARCH DIRECTIONS

In this paper we have considered the full information output regulation problem for linear stochastic systems. The class of systems considered is the one in which the state, the control variable and the disturbance appear in both the drift and diffusion terms. We have characterized the steady-state response of such a system interconnected with a stochastic signal generator obtaining a stochastic generalization of the deterministic Sylvester equation. We have then solved the output regulation problem. Finally, we have illustrated the results by means of two examples.

The next step to create a theory of output regulation for stochastic systems is to solve the error feedback problem, *i.e.* the case in which we can measure only to the error variable. We point out that the error feedback output regulation problem for stochastic systems presents additional challenges
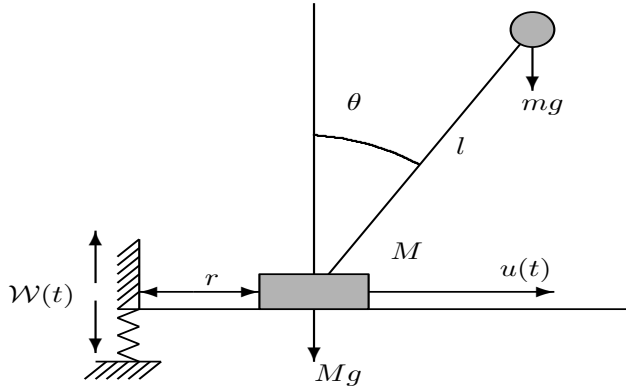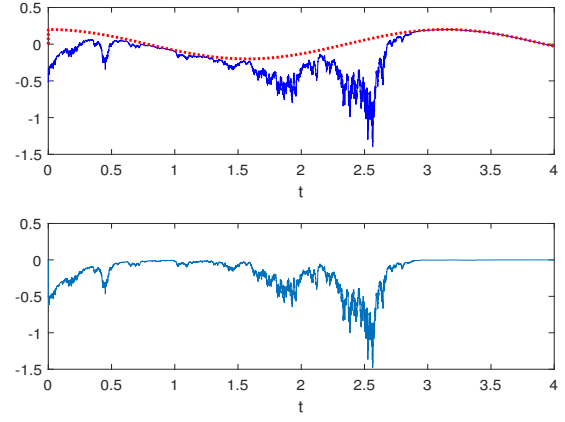

Fig. 3. Inverted pendulum on a cart on a randomly vibrating platform.

B. An inverted pendulum on a cart on a randomly vibrating platform

Consider an inverted pendulum on a cart on a randomly vibrating platform, see Figure 3. This example is inspired by [28, Section 1.9.2], see also [32], [33] where similar problems are studied. The cart is driven horizontally by an input u. The objective is to make the pendulum oscillate around the upper equilibrium position with period π by choosing the appropriate control u. In this scenario, it is assumed that the whole structure is subject to random vertical vibrations which affect the law of gravity in the system. The state of the system is $x = \begin{bmatrix} r & \dot{r} & \theta & \dot{\theta} \end{bmatrix}^{\top}$ and the linearized system is described by the equations (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{m}{M}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{m+M}{lM}g & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ -\dfrac{1}{lM} \end{bmatrix},$$

with respect to the deterministic version. To illustrate these difficulties, consider the system

$$dx = Ax dt + Fx d\mathcal{W}, \qquad y = Cx.$$

If we have access to measurements of the output but the Wiener process is not accessible (which would be the case in a realistic scenario), we can design only a deterministic observer, namely

$$d\xi = [A\xi + K(y - \eta)]\, dt, \qquad \eta = C\xi.$$

As a result, the observation error $x - \xi$ is described by

$$d(x - \xi) = (A - KC)(x - \xi)dt + Fx d\mathcal{W}.$$

As noted in *e.g.* [28], the estimated state ξ converges to x only if x converges to zero. Consequently, differently from the deterministic error feedback output regulation problem, we cannot separate the design of the feedback law from the design of the observer. Moreover, note that the regulator equations (10) cannot be solved in their present form without having access to $\mathcal{W}$. Further research directions include extending the results to multi-input, multi-output stochastic systems, to systems with multiple Wiener processes $\mathcal{W}_i$ and to nonlinear stochastic systems.

REFERENCES

[1] E. Davison, "The robust control of a servomechanism problem for linear time-invariant multivariable systems," *IEEE Transactions on Automatic Control*, vol. 21, no. 1, pp. 25–34, Feb 1976.

[2] B. A. Francis, "The linear multivariable regulator problem," *SIAM Journal on Control and Optimization*, vol. 15, no. 3, pp. 486–505, 1977.

[3] B. Francis and W. M. Wonham, "The internal model principle of control theory," *Automatica*, vol. 12, no. 5, pp. 457–465, 1976.

[4] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 3rd ed., ser. Stochastic Modelling and Applied Probability. Springer New York, 1985, vol. 10.

[5] A. Isidori and C. I. Byrnes, "Output regulation of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 35, no. 2, pp. 131–140, 1990.

[6] J. Huang and W. Rugh, "On a nonlinear multivariable servomechanism problem," *Automatica*, vol. 26, no. 6, pp. 963–972, 1990.

[7] C. I. Byrnes, F. Delli Priscoli, and A. Isidori, *Output Regulation of Uncertain Nonlinear Systems*, ser. Output Regulation of Uncertain Nonlinear Systems. Birkhäuser Boston, 1997.

[8] A. Serrani, A. Isidori, and L. Marconi, "Semi-global nonlinear output regulation with adaptive internal model," *IEEE Transactions on Automatic Control*, vol. 46, no. 8, pp. 1178–1194, Aug 2001.

[9] J. Huang, *Nonlinear Output Regulation: Theory and Applications*, ser. International series in pure and applied mathematics. Philadelphia, PA: SIAM Advances in Design and Control, 2004.

[10] A. Pavlov, N. van de Wouw, and H. Nijmeijer, *Uniform Output Regulation of Nonlinear Systems: A Convergent Dynamics Approach*, ser. Systems & Control: Foundations & Applications. Birkhäuser Boston, 2006.

[11] L. Marconi and A. R. Teel, "Internal model principle for linear systems with periodic state jumps," *IEEE Transactions on Automatic Control*, vol. 58, no. 11, pp. 2788–2802, Nov 2013.

[12] D. Carnevale, S. Galeani, L. Menini, and M. Sassano, "Hybrid output regulation for linear systems with periodic jumps: Solvability conditions, structural implications and semi-classical solutions," *IEEE Transactions on Automatic Control*, vol. 61, no. 9, pp. 2416–2431, 2016.

[13] A. Ichikawa and H. Katayama, "Output regulation of time-varying systems," *Systems & Control Letters*, vol. 55, no. 12, pp. 999–1005, 2006.

[14] F. Modigliani and F. E. Hohn, "Production planning over time and the nature of the expectation and planning horizon," *Econometrica*, vol. 23, no. 1, pp. 46–66, 1955.

[15] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, ser. Stochastic Modelling and Applied Probability. Springer New York, 1999.

[16] R. C. Merton, "Lifetime portfolio selection under uncertainty: The continuous-time case," *The review of Economics and Statistics*, vol. 51, no. 3, pp. 247–257, 1969.

[17] ——, "Optimum consumption and portfolio rules in a continuous-time model," *Journal of economic theory*, vol. 3, no. 4, pp. 373–413, 1971.

[18] H. E. Leland and K. B. Toft, "Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads," *The Journal of Finance*, vol. 51, no. 3, pp. 987–1019, 1996.

[19] J. F. Reinganum, "On the diffusion of new technology: A game theoretic approach," *The Review of Economic Studies*, vol. 48, no. 3, pp. 395–405, 1981.

[20] K. Roberts and M. L. Weitzman, "Funding criteria for research, development, and exploration projects," *Econometrica: Journal of the Econometric Society*, pp. 1261–1288, 1981.

[21] H. Chernoff, "Optimal stochastic control," *Sankhyā: The Indian Journal of Statistics, Series A*, vol. 30, no. 3, pp. 221–252, 1968.

[22] M. Krstic and H. Deng, *Stabilization of Nonlinear Uncertain Systems*, ser. Communications and Control Engineering. Springer London, 1998.

[23] S. He, Z. Ding, and F. Liu, "Output regulation of a class of continuous-time Markovian jumping systems," *Signal Processing*, vol. 93, no. 2, pp. 411–419, 2013.

[24] ——, "Almost asymptotic regulation of Markovian jumping linear systems in discrete time," *Asian Journal of Control*, vol. 16, no. 6, pp. 1869–1879, 2014.

[25] T. C. Gard, *Introduction to Stochastic Differential Equations*, ser. Monographs and textbooks in pure and applied mathematics. M. Dekker, 1988.

[26] R. Khasminskii and G. N. Milstein, *Stochastic Stability of Differential Equations*, ser. Stochastic Modelling and Applied Probability. Springer Berlin Heidelberg, 2011.

[27] L. Arnold, *Stochastic differential equations*, ser. A Wiley-Interscience publication. Wiley, 1974.

[28] T. Damm, *Rational Matrix Equations in Stochastic Control*, ser. Lecture Notes in Control and Information Sciences. Springer Berlin Heidelberg, 2004.

[29] G. Scarciotti and A. R. Teel, "Model order reduction of stochastic linear systems by moment matching," in *20th IFAC World Congress, Toulouse, France, July 9-14*, 2017, pp. 6506–6511.

[30] J. D. Simon and S. Mitter, "A theory of modal control," *Information and Control*, vol. 13, no. 4, pp. 316–353, 1968.

[31] L. Gould, A. Murphy, and E. Berkman, "On the Simon-Mitter pole allocation algorithm–explicit gains for repeated eigenvalues," *IEEE Transactions on Automatic Control*, vol. 15, no. 2, pp. 259–260, Apr 1970.

[32] J. L. Bogdanoff and S. J. Citron, "Experiments with an inverted pendulum subject to random parametric excitation," *The Journal of the Acoustical Society of America*, vol. 38, no. 3, pp. 447–452, 1965.

[33] M. S. Howe, "The mean square stability of an inverted pendulum subject to random parametric excitation," *Journal of Sound and Vibration*, vol. 32, no. 3, pp. 407–421, 1974.