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Robust Model Predictive Control Based on Stabilizing Parameter Space Calculus

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***Abstract*— The aim of this study is to introduce a novel approach for robust model predictive control (MPC) design based on stabilizing parameter spaces. In order to determine the stabilizing parameter regions, a Lyapunov equation based approach is proposed for nominal systems. In addition to the free controller parameters, it is also possible to determine boundaries of uncertain parameters in the present approach. The precomputed stability conditions on controller parameters are inserted to the MPC problem formulation as constraints. By this way, the stability of the closed-loop system is ensured. The proposed approach allows to design the nominal MPC, instead of the robust one. Using the predetermined constraints, the MPC is implemented to optimize the controller parameters over this stabilizing set. This paper introduces three particular control scenarios that tune the basic properties of the novel approach, e.g., runtime and computational effort. Two illustrative case studies are presented to demonstrate the efficiency of the proposed robust MPC strategy.**

I. INTRODUCTION

Model predictive control represents the current state-of-art in optimal operating of plants. Significant progress has been achieved in MPC development and implementation in past three decades, see [1]. Compared to the conventional control strategies, e.g., PID controllers for single-input and single-output (SISO) systems, and linear-quadratic-optimal controller (LQ-optimal controller) for multi-input and multi-output systems (MIMO), the main advantage of MPC design is its ability to handle the constraints in an optimal manner. The overall control performance of MPC is highly affected by the imperfect model used for the future prediction of the plant behavior, i.e., so-called *process-model mismatch*. The implementation of MPC in *receding horizon fashion*, i.e., the closed-loop MPC, is able to reduce this phenomena. If the model is highly effected by uncertain parameters, then the control performance can be significantly decreased, and, in the worst case, the closed-loop system behavior can lead to an unstable response.

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The influence of uncertain parameters can be reduced by using *robust* MPC. The main limitation of practical robust MPC implementation is the complexity of the solved optimization problem in each control step. Conventional robust MPC design procedures directly introduce the effect of the uncertain parameters into the optimization problem. In the case of *non-parametric* uncertainties, e.g., additive disturbances, measurement noise, etc., a norm-based formulation is introduced. If the system has *parametric uncertainties*, then a vertex representation is considered. It leads to an exponential number of uncertain system vertices that have to be taken into account, while designing the robust MPC.

It was shown that parametric uncertainties can be handled using the linear matrix inequalities (LMIs) [2]. The pioneer work of LMI-based robust MPC design [3] formulates the optimization problem in the form of semidefinite programming (SDP) [4]. This approach was improved by many later works, e.g., [5] improved control performance by designing two controllers for infinity prediction horizon, [6] reduced conservativeness of constraints on control inputs, in [7] the runtime was reduced by smart fixing of the selected control inputs, etc. The robust MPC for a nonlinear system with input-to-state stability was designed in [8]. The output-feedback robust MPC was designed in [9]. The implementation of LMI-based robust MPC is limited to the systems with slow dynamics or small number of uncertain parameters. Another perspective to handle the uncertain parameters is to design so-called *tubes*, see [10]. With respect to the study of admissible variations in the controller gains, the linear framework was extended to a piecewise affine one in [11].

The first studies aiming the determination of the stabilizing parameter spaces for discrete-time systems date back to early 1960s [12]. After this pioneering work, several approaches have been published based on transformations and Tchebyshev representation. Additionally, several frequency domain based approaches were proposed to calculate the stabilizing parameters of nominal discrete time systems [13]. Frequency sweeping, decoupling at singular frequencies are required in frequency based approaches and the accuracy of the results depends on the griding step sizes. Moreover, in such approaches results generally depend on the controller type. This means that derived results are valid only for the specified controller type. If the controller type changes the calculations should be renewed.

In order to overcome the mentioned drawbacks, a Lyapunov equation based stability mapping technique is presented in this study. Furthermore, the required number of equations that should be solved to determine the stability boundaries are significantly reduced. Additionally, the proposed approach is also applicable for different type of controllers [14]. For the case of higher number of uncertain parameters, discrete Kharitonov Theorem can also be used in accordance with the proposed Lyapunov equation based approach in order to reduce computational complexity.

The main contribution of this paper is proposal of a robust MPC design strategy in a *non-robust* manner. Technically speaking, the designed MPC is evaluated over the robust subset of feasible domain of the control problem. Then, the robust stability of the closed-loop control performance is guaranteed for slow variations of controller parameters. The main advantage of this strategy is the effective design of an optimal controller in each control step subject to the system with uncertainties. The main idea is to shift the computationally demanding procedure of uncertainty handling to the *off-line* phase of controller design, and precompute the robust stabilizing space of controller parameters in advance. Then, in the *on-line* phase of closed loop control, MPC solves the optimization problem over this stabilizing domain to evaluate the optimal controller parameters.

The paper is organized as follows. Section 2 states the problem of robust MPC design considering the stabilizing parameter space. Section 3 presents in detail the main contributions of the paper, i.e., the calculus of the stabilizing parameter space. We show, how to construct the set of controller parameters that guarantees the closed-loop robust stability in discrete-time domain. A novel robust MPC design procedure based on the optimization over the stabilizing parameter space is introduced in Section 4. Two simulation case studies demonstrate the efficiency of the proposed strategy in Section 5, followed by the conclusions and open problems in Section 6.

II. Notation and Problem Statement

The following notation is used in the paper:

1) $\mathbb{R}^n$ denotes the n-dimensional space of real-valued vectors, $\mathbb{R}^{n\times m}$ represents the $(n \times m)$-dimensional space of real-valued matrices.
2) For a real-valued matrix A, $A^\top$ denotes its transposition, $A^\dagger$ is its pseudo-inverse, and A^{-1} is its inverse, if exists.
3) I denotes the square identity matrix of appropriate dimensions.
4) For matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, $A \otimes B = [a_{i,j}B]_{i,j-1}^{m,n} \in \mathbb{R}^{mp\times nq}$ denotes a Kronecker product.
5) Operator vec : $\mathbb{R}^{m\times n} \to \mathbb{R}^{mn}$ maps a matrix into a vector of its columns.
6) Operator $\overline{\text{vec}}$: $\mathbb{R}^{n\times n} \to \mathbb{R}^{n(n+1)/2}$ maps the unique entries of a given n-dimensional symmetric matrix into a vector of its columns.

State space representation of a linear uncertain system in discrete-time domain can be represented as:

$$x(\kappa+1) = A(q)\,x(\kappa) + B(q)\,u(\kappa), \quad x_0 = x(0), \quad (1a)$$
$$y(\kappa) = C\,x(\kappa), \quad (1b)$$

where κ represents an element of the discrete-time domain and $x \in \mathbb{R}^{n_\mathrm{x}}$ are system states, $u \in \mathbb{R}^{n_\mathrm{u}}$ are control inputs, $q \in \mathbb{Q} \subset \mathbb{R}^{n_\mathrm{q}}$ are uncertain parameters, $A(q) \in \mathbb{R}^{n_\mathrm{x}\times n_\mathrm{x}}$ is the uncertain system state matrix, $B(q) \in \mathbb{R}^{n_\mathrm{x}\times n_\mathrm{u}}$ is the uncertain input matrix, and $C \in \mathbb{R}^{n_\mathrm{y}\times n_\mathrm{x}}$ is the output matrix. The input constraints of an uncertain system in (1) are: $u(\kappa) \in \mathbb{U}$, where $\mathbb{U}$ is a set of admissible values.

For an uncertain system in (1), the problem is to design the controller $K \in \mathbb{R}^{n_\mathrm{u}\times n_\mathrm{x}}$ of the feedback control law

$$u(\kappa) = -K\,x(\kappa), \quad (2)$$

that guarantees the robust stability $\forall q \in \mathbb{Q}$, and optimizes the overall control performance.

III. Stabilizing Parameter Space Calculation

In this section, the details of the Lyapunov equation based stability mapping technique that is proposed within the scope of this study are discussed. This approach is independent from the controller structure and the type of the free parameters. Due to the flexible structure of the proposed stability mapping approach it is also compatible with the currently existing approaches like discrete Kharitonov polynomials.

A. Lyapunov Equation Based Method

A discrete-time closed-loop system can be represented in the form of the autonomous system

$$x(\kappa+1) = A_\mathrm{cl}(K)\,x(\kappa), \quad x(0) = x_0, \quad (3)$$

where $A_\mathrm{cl}(K) \in \mathbb{R}^{n_\mathrm{x}\times n_\mathrm{x}}$ is the closed-loop system matrix and K represents the controller parameters. The corresponding Lyapunov equation of system (3) is

$$A_\mathrm{cl}^\top(K)P(K)A_\mathrm{cl}(K) - P(K) = -Q, \quad (4)$$

where the positive definiteness of matrices P and Q is a necessary and sufficient condition for stability. In the case of linear systems, Q can be selected as any positive definite matrix (for instance I) [2]. By this way, the stability problem is transformed to ensuring the positive definiteness of P. The Kronecker product as well as the "vec" operator can be used to transform this matrix equality problem to the standard linear set of equations

$$(A_\mathrm{cl}^\top(K) \otimes A_\mathrm{cl}^\top(K) - I \otimes I)\mathrm{vec}(P(K)) = -\mathrm{vec}(Q). \quad (5)$$

Using (5) each element of the P matrix can be determined from

$$\mathrm{vec}(P) = M^{-1}(K)\,\mathrm{vec}(-Q), \quad (6)$$

where

$$M(K) = A_\mathrm{cl}^\top(K) \otimes A_\mathrm{cl}^\top(K) - I \otimes I. \quad (7)$$

Considering the numerators and denominators of all leading principal minors of P, the approach requires the solution of $2n$ parametric equations. In our previous study [15], it was shown that this number can be reduced significantly by further analysis. Using the properties of the Kronecker product, determinant $\mid M(K) \mid$ can be evaluated as:

$$\mid M(K) \mid = \prod_{i=1}^{n} \prod_{j=1}^{n} (\lambda_i \lambda_j - 1), \qquad (8)$$

where λ_1,...,λ_n are the eigenvalues of $A_{\mathrm{cl}}(K)$. Using this relationship between the eigenvalues of the system matrix A_{cl} and the roots of $\mid M(K) \mid$, it was shown in our previous study [16] that it is sufficient to check only $\mid M(K) \mid = 0$ to determine the stability boundaries of a given linear discrete-time system.

The dimension of the matrix $M(K)$ in (7) is $n^2 \times n^2$. Some duplicated products like $(\lambda_1\lambda_2 - 1)$ and $(\lambda_2\lambda_1 - 1)$, are included in (8). There is at most $n(n+1)/2$ unique elements in a $n \times n$ symmetric matrix P. Thus, the repeated elements can be eliminated using elimination and duplication matrices. As a result, the equation (5) can be re-formulated to

$$T^{\dagger} M(K) T \overline{\mathrm{vec}}(P(K)) = -\overline{\mathrm{vec}}(Q), \qquad (9)$$

where T is the full column rank *duplication matrix*, and *elimination matrix* $T^{\dagger}$ is its pseudo inverse. The duplication matrix T and the elimination matrix $T^{\dagger}$ do not depend on the free controller parameters. Additionally, $\overline{\mathrm{vec}}(P(K))$ only includes the unique elements of the original P matrix as:

$$\overline{\mathrm{vec}}(P(K)) = \begin{bmatrix} P_{11} & \dots & P_{n1} & P_{22} & \dots \end{bmatrix}^{\top}. \qquad (10)$$

Using this approach, all unique elements of the original P matrix are determined by

$$\overline{\mathrm{vec}}(P(K)) = M_{\mathrm{T}}^{-1}(K) \overline{\mathrm{vec}}(-Q). \qquad (11)$$

where

$$M_{\mathrm{T}}(K) = T^{\dagger} M(K) T. \qquad (12)$$

The corresponding relation between the eigenvalues of $\mid M_{\mathrm{T}}(K) \mid$ and the matrix $A_{\mathrm{cl}}(K)$ is

$$|M_{\mathrm{T}}(K)| = \prod_{i=1}^{n} \prod_{j \geq i}^{n} (\lambda_i \lambda_j - 1). \qquad (13)$$

It is clear from (13) that the duplicated eigenvalues are eliminated in this case. This reduces the dimension of the matrix that maps the stability boundaries of the system in (3) from $n^2 \times n^2$ to $(n(n+1)/2) \times (n(n+1)/2)$.

This time-domain based procedure avoids the main drawbacks of the frequency-domain-based approaches. That is there is no need of frequency sweeping, decoupling at singular frequencies and griding. Moreover, there are no linear transformations needed in this approach, since the problem is directly defined on discrete-time domain. In addition to the determination of the stabilizing controller parameters, the proposed approach is also applicable to determine the bounds of the uncertain parameters.

Otherwise, the convexity of the derived stabilizing regions plays a crucial role in terms of guaranteeing the solution time of the robust MPC problem since it is aimed to insert the analytical stability boundaries to the MPC problem formulation. The problem can still be solved with non-convex constrains. However, in such cases can the solver times not be guaranteed. If the resulting stabilizing region is not convex for the considered system, then the inner approximation can be used to derive a convex region inside the stabilizing parameter region. Additionally, the nature of the MPC creates a switching behavior since the controller parameters are changed in each time instance. Such parameter variations may also effect the stability characteristics. For this reason, in the next section, a conservative approach that is based on common Lyapunov functions will be proposed to overcome this switching effect.

B. Calculation of Common Quadratic Lyapunov Functions

The proposed Lyapunov equation based stability mapping technique is an efficient way of determining the analytical expressions of stability boundaries of a given linear time-invariant (LTI) system. However, it should be noted that due to the structure of MPC, the resulting controller parameters may vary at each time instance. Such a case will lead to parameter switching which may possibly have an effect on the overall stability of closed loop system [17]. Systems with discrete changing dynamics, where each dynamic is time-invariant and can be modeled separately, are called *switching systems* in literature, see [18]. Such systems can be described as:

$$x(\kappa+1) = A_{\mathrm{cl}}(K)\, x(\kappa), \quad \text{where: } A_{\mathrm{cl}}(K) \in \{A_{\mathrm{cl}_1} ... A_{\mathrm{cl}_l}\}, \qquad (14)$$

If the dwell time of the discussed system is lower than the sampling time it can be stated that no additional effort is required [17]. For the other case an alternative approach could be proposed using the common quadratic Lyapunov functions. If the same Lyapunov matrix P solves equation (4) for each subsystem then it can be stated that the switching system is stable as well. The detailed proof can be found in [17]. The stability condition can be represented as

$$A_{\mathrm{cl}_i}^{\top}(K) P A_{\mathrm{cl}_i}(K) - P = -Q_i, \quad i = 1, \dots l, \qquad (15)$$

where the matrices P and Q_i are positive definite. It is also possible to generalize the proposed approach in a conservative way by fixing matrix P to a specific predetermined value. In this case (5) can be rewritten as:

$$\mathrm{vec}(Q(K)) = -(A_{\mathrm{cl}}^{\top}(K) \otimes A_{\mathrm{cl}}^{\top}(K) - I \otimes I)\mathrm{vec}(P), \qquad (16)$$

where Q matrix depends on the controller parameters, since P is fixed from the very beginning. By fixing the matrix P, positive definiteness of P and stability of switching system is guaranteed (conservatively). The controller parameters that make the Q matrix positive definite should be searched for stability. However, the results will be conservative, since we are searching for the positive definite Q matrices that results in the same P, i.e., a common quadratic Lyapunov function (CQLF). In [19] we optimized the choice of P by an numerical optimization algorithm.

Derived results in terms of exact stability boundaries and switching system stability are meaningful from the practical point of view, since it is aimed to guarantee the stability of a given MPC controlled system in the off-line phase. The MPC which is used for the calculations in the off-line phase is discussed the following section. Effectiveness of the derived theoretical results will be discussed in Section V in detail.

IV. Robust MPC Design Based on a Stabilizing Space Calculus

This section introduces the design of a robust MPC strategy in an indirect way. MPC is evaluated over the robustly stabilizing set of controller parameters $\mathbb{K}$. As a consequence, the robust stability of the closed-loop control system is achieved. The main benefit of this approach is to design an optimal controller in receding control fashion subject to system uncertainties in an effective way.

An indirect implementation of robust stability is ensured by the calculus of the stabilizing parameter space $\mathbb{K}$ in off-line phase, see Section III. Then, it becomes possible to achieve robust stability of the closed loop system in (1) using an arbitrary controller $K \in \mathbb{K}$. Next, in the on-line phase, it is sufficient to evaluate the parameters of $K \in \mathbb{K}$ to optimize the overall control performance. Moreover, the constraints on control inputs u are satisfied. Finally, the optimization problem of MPC has the form

$$\min_{K} \left(x_N^\top W_\mathrm{N} x_N + \sum_{\kappa=0}^{N-1} \left(x_\kappa^\top W_\mathrm{x} x_\kappa + u_\kappa^\top W_\mathrm{u} u_\kappa \right) \right), \quad (17a)$$

$$\text{s.t.}: \quad x_{\kappa+1} = A\, x_\kappa + B\, u_\kappa, \quad (17b)$$

$$u_\kappa = -K_\kappa\, x_\kappa, \quad (17c)$$

$$u_\kappa \in \mathbb{U}, \forall \kappa \geq 0, \quad (17d)$$

$$K \in \mathbb{K}, \quad (17e)$$

$$x_N \in \mathbb{X}_\mathrm{N}, \quad (17f)$$

$$x_0 = x(0), \quad (17g)$$

where $\mathbb{K} \subset \mathbb{R}^{n_\mathrm{u} \times n_\mathrm{x}}$ is the set of robustly stabilizing controller parameters, and $W_\mathrm{N} \succeq 0$, $W_\mathrm{x} \succeq 0$, $W_\mathrm{u} \succ 0$ are weighting matrices of appropriate dimensions. In (17), the quadratic objective function (17a) is minimized subject to a future system behavior predicted using a nominal system (17b), i.e., an idealized system with neglected or mean values of uncertain parameters. Simultaneously, the constraints on control inputs in (17d) have to be satisfied, where $\mathbb{U}$ is a polytopic set containing the origin in its interior. The terminal set $\mathbb{X}_\mathrm{N}$ in (17f) contains origin and is considered to be robust positive invariant under the feedback law in (17c). The control input is evaluated using linear control law in (17c) for the precomputed set of robustly stabilizing parameters $\mathbb{K}$ in (17e). The control problem in (17) is evaluated for given initial conditions in (17g).

Although the sets $\mathbb{U}$, $\mathbb{K}$ in (17d)–(17e) can be convex, the optimization problem in (17) is not convex, in general. The non-convex formulation originates in (17c), where the prediction horizon $N > 1$ introduces the multi-linear terms into (17b). The solution of the optimization problem in general form (17) subject to $N > 1$, leads to *a non-convex scenario*.

If the considered runtime prevents implementation of NP-hard optimization problem in (17), then the one-step-ahead prediction horizon, i.e., *a convex scenario*, should be considered.

Otherwise, if a limited hardware computational power prevents solving even a convex optimization problem, then an optimal time-invariant controller K can be designed for $N > 1$ in off-line phase, i.e., *fast scenario* is implemented. However, it should be noted that if a time-invariant controller is implemented then the advantage of receding horizon control strategy vanishes.

V. Case Studies

Within the scope of this study, two illustrative case studies are included to demonstrate the efficiency of the presented approach in terms of stabilizing parameter space calculations and robust MPC design. A nominal system that was used in the literature is considered in the Case Study I as a benchmark example. In addition to the nominal results, in the Case Study II, a benchmark uncertain system is considered. The stability of parameter uncertain system and the problem of guaranteeing the stable performance are discussed in detail.

In both case studies, Wolfram Mathematica 10.3 software was used to evaluate symbolic calculations for the determination of stabilizing parameter regions. The robust MPC in (17) was evaluated using CPU i5 1.7 GHz, 6 GB memory, in MATLAB R2014b using YALMIP toolbox [20].

A. Case Study I: Nominal System

To demonstrate the evaluation of the stabilizing parameter spaces, Case Study I considers a benchmark system presented in [21]. Here, the discrete-time system transfer function is

$$G_\mathrm{P}(z) = \frac{z+1}{z^2 - 1.5z + 0.5}, \quad (18)$$

which corresponds to the state-space representation

$$\begin{aligned} x(\kappa+1) &= \begin{bmatrix} 0 & 1 \\ -0.5 & 1.5 \end{bmatrix} x(\kappa) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\kappa), \\ y(\kappa) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(\kappa). \end{aligned} \quad (19)$$

The closed-loop system matrix $A_\mathrm{cl}(K)$ for system (19) and the state feedback controller $u(\kappa) = -Kx(\kappa)$ is

$$A_\mathrm{cl}(K) = \begin{bmatrix} 0 & 1 \\ -0.5 - K_1 & 1.5 - K_2 \end{bmatrix}, \quad (20)$$

that results in the following $\mid M_\mathrm{T}(K) \mid$

$$\mid M_\mathrm{T}(K) \mid = (K_1 - 0.5)(K_2 + K_1)(K_2 - (3 + K_1)) \quad (21)$$

as the determinant of matrix $M_\mathrm{T}(K)$. The parametric solution of $\mid M_\mathrm{T}(K) \mid = 0$ leads to the following stability boundaries:

$$K_1 = 0.5, \quad K_1 = K_2 - 3, \quad K_1 = -K_2. \quad (22)$$

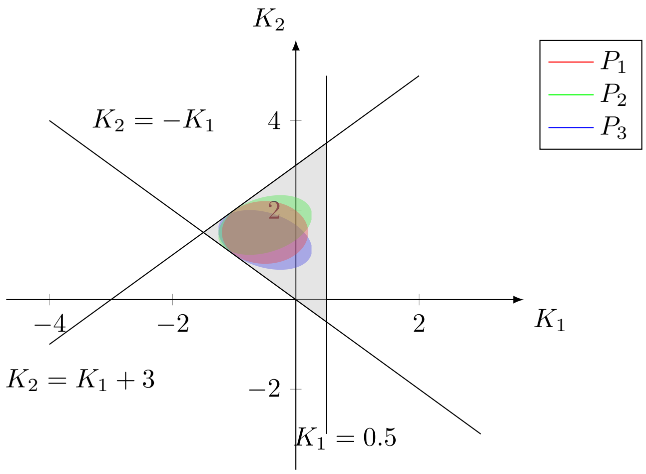


Fig. 1. Stabilizing parameter space in Case Study I.

These boundaries divide the whole $K_1 - K_2$ space into 7 subspaces (Fig. 1). Stability characteristics of each region can be determined by selecting a controller parameter pair from each subspace, and checking the stability of the closed-loop system for the selected pair. In order to automatize that process, intersection points of the solutions are determined, and using the gradients of the solution functions at that point, a (K_1, K_2) pair from every subspace is determined. For this case study, analytical conditions on stabilizing parameter space can be calculated as

$$\begin{aligned}&(-0.5 < K_2 \leq 1.5 \wedge -K_2 < K_1 < 0.5) \vee \\ &(1.5 < K_2 < 3.5 \wedge -3 + K_2 < K_1 < 0.5).\end{aligned} \tag{23}$$

The stability boundaries (22) and the stabilizing set (23) is depicted in Fig. 1.

The conditions (23) correspond to a convex region, and they can be directly used in the proposed robust MPC design approach that was described in Section IV. Implementing the stability conditions into the MPC design problem as the constraints is the novelty of this case study. The MPC design approach considers the stabilizing set of controller parameters to ensure the optimal system response.

In this case, derived exact stability conditions can be directly applicable. However, to demonstrate the applicability of the proposed approach in Section III-B stabilizing parameter spaces for 3 different selections of CQLFs (fixed P matrices) are also calculated. The P matrices

$$P_1 = \begin{bmatrix} 1.133 & 0 \\ 0 & 2.133 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.75 & 0.5 \\ 0.5 & 3 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} 1.75 & -0.5 \\ -0.5 & 3 \end{bmatrix}, \tag{24}$$

are selected for the further analysis. Derived stabilizing parameter regions for the switching system stability are also included in Fig. 1 where each ellipsoid represents the corresponding stabilizing parameter region for the related selection of the P matrix. As it is seen in Fig. 1, the results are conservative as expected. Since the dwell time of the discussed system is adequate, initial approach can be used without parameter switching problems.

In our MPC problem (17), there are active constraints on control input $-1.4 \leq u(\kappa) \leq 0.7$, prediction horizon $N = 3$, $W_\mathrm{x} = \mathrm{diag}([1, 1]) \times 10^3$, $W_\mathrm{u} = 1 \times 10^{-3}$ and the initial conditions $x(0) = [1, 1]^\top$. The controller is designed subject to the set $\mathbb{K}$ defined in (23). The computed initial values of controller parameters are $K(1) = [-0.1669, 1.4994]^\top$. The other control gains $K(\kappa)$ evaluated for $\kappa > 1$ are also selected from the stabilizing set $\mathbb{K}$.

We have also investigated the closed-loop control performance subject to the parameter space defined by the set $P_1 \subset \mathbb{K}$. To make the optimization problem more tractable, we used maximum-volume inner approximation that corresponds to a square in this specific case. Particularly, we obtained a set $\tilde{P}_1$ limiting the values of the controller parameters as

$$\begin{bmatrix} -0.9950 \\ 1.0050 \end{bmatrix} \preceq K \preceq \begin{bmatrix} -0.0050 \\ 1.9950 \end{bmatrix}. \tag{25}$$

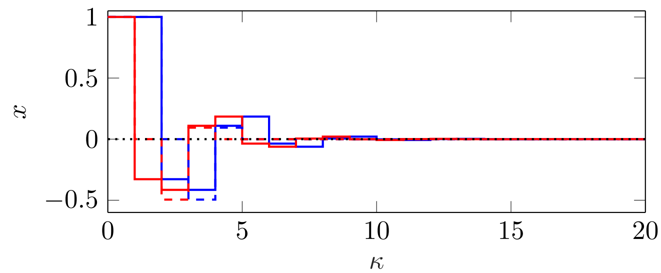
The approximation via set $\tilde{P}_1 \subset P_1$ represents a conservative approach and the control performance of the original nonlinear constraints describing the convex set P_1 will be investigated in our further research. The rest of the MPC design setup was the same as in the previous case to make the results comparable. The computed initial values of controller parameters are $K(1) = [-0.3820, 1.5356]^\top$. The other control gains $K(\kappa)$ evaluated for $\kappa > 1$ are also selected from the set $K(\kappa) \in \tilde{P}_1$. The control performance is depicted in Fig. 2, where we compare the results optimized subject to $K \in \mathbb{K}$ and subject to $K \in \tilde{P}_1$. As can be seen, the control trajectories of system states converge to the origin (Fig. 2(a)). The general behavior of both solutions is quite similar. However, the response of the system, which explicitly considers the switching behavior of the controller gains has a bigger undershoot. The associated optimal sequence of control actions is also plotted in Fig. 2(b). Likewise, the difference between the two control signals is not significant. In order to limit the undershoot relatively more control signal is applied at the beginning to the system that is designed using the exact parameter boundaries. We have also evaluated the following

$$J = \sum_{k=0}^{N-1} \left(x_\kappa^\top W_\mathrm{x} x_\kappa + u_\kappa^\top W_\mathrm{u} u_\kappa \right) \tag{26}$$

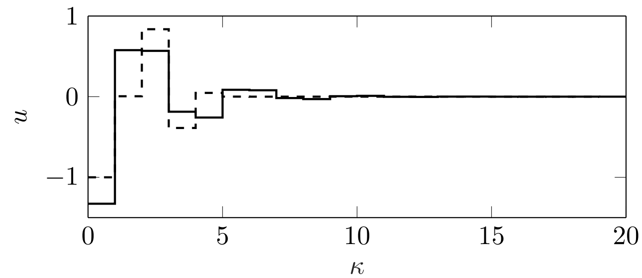
quadratic criterion of closed-loop control. Particularly, $J_\mathbb{K} = 3\,473$ and $J_{\tilde{P}} = 3\,508$. Here, $J_\mathbb{K}$ is lower than the $J_{\tilde{P}}$ as expected. However, it is also clear that $J_\mathbb{K}$ and $J_{\tilde{P}}$ and state trajectories in Fig. 2(a) are close to each other. It can be interpreted that for this specific case the actual optimal controller parameter set remains on the ellipsoidal region in Fig. 1. It is because of that there is not a drastic difference between the objective function values.

B. Case Study II: Uncertain System

In order to illustrate the results for the control scenario considering the system with the parametric uncertainties, the



(a) Controlled state trajectories generated subject to $\mathbb{K}$: x_1 (blue, solid), x_2 (red, solid); to $\tilde{P}_1$: x_1 (blue, dashed), x_2 (red, dashed), and reference (black dotted).



(b) Control inputs generated subject to $\mathbb{K}$: solid; to $\tilde{P}_1$: dashed.

Fig. 2. Closed-loop control performance assured by robust MPC in Case Study I.

following benchmark system was adopted from [22]

$$x(\kappa+1) = \begin{bmatrix} 1 & 0 \\ q & 1 \end{bmatrix} x(\kappa) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(\kappa), \qquad (27)$$

where the uncertain parameter is given as $0.5 \leq q \leq 1.5$. For a state-feedback controller the closed-loop system matrix A_{cl} has the form

$$A_{\mathrm{cl}}(K) = \begin{bmatrix} 1-K_1 & -K_2 \\ q & 1 \end{bmatrix}. \qquad (28)$$

Since the number of uncertain parameters is one and the number of free controller parameters are 2, the proposed Lyapunov equation based approach is directly applied for the closed-loop system in (28).

With respect to this strategy, the stability boundaries of the given system were determined using the parametric solution of $\mid M_{\mathrm{T}}(K) \mid = 0$. For this case $\mid M_{\mathrm{T}}(K) \mid$ and its roots can be respectively expressed as

$$\mid M_{\mathrm{T}}(K) \mid = K_2\ q\ (-K_1 + K_2\ q)(4 - 2K_1 + K_2\ q), \qquad (29)$$

$$K_2 = 0, \quad K_2 = (2K_1 - 4)/q, \quad K_2 = K_1/q. \qquad (30)$$

Using the parametric solutions of $\mid M_{\mathrm{T}}(K) \mid = 0$ given in (30), the stabilizing parameter region for the given system was determined as shown in Fig. 3. In this case, the intersections of stabilizing parameter spaces can be projected on the $K_1 - K_2$ plane. The stabilizing controller parameter space independent from the uncertain parameter can be determined as shown in Fig. 4.

The corresponding analytical conditions on controller parameters in terms of the stability of the closed loop system can be expressed as

$$0 < K_2 < 1.6 \wedge 1.5K_2 < K_1 < 0.25(8 + K_2). \qquad (31)$$

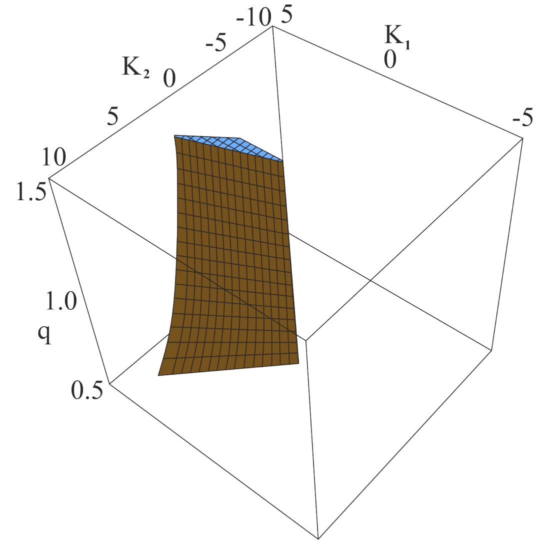


Fig. 3. Stabilizing Parameters for Case Study II

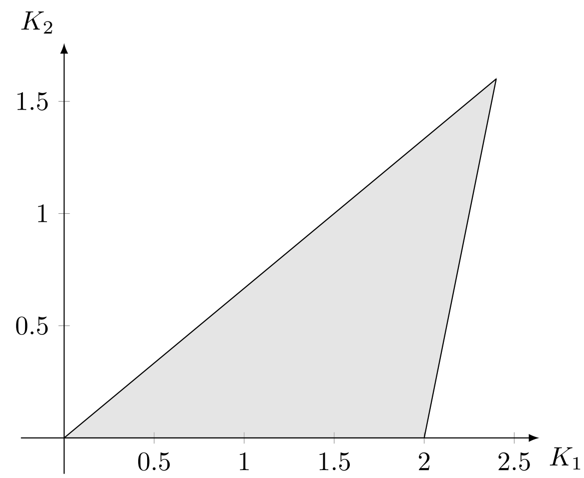


Fig. 4. Stabilizing Controller Parameters for Case Study II ($q = 1$)

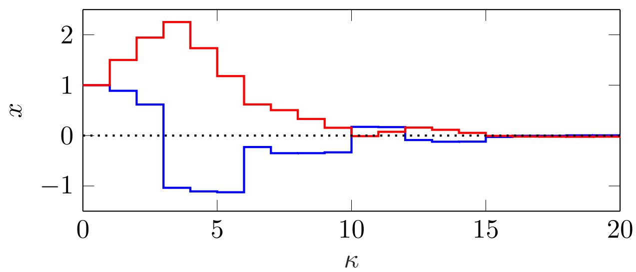
For the uncertain system (27) a robust MPC is designed using the derived stability conditions (31). These conditions are applied as constraints on controller parameters in the MPC algorithm to ensure the optimal closed-loop control performance.

For the simulation of the closed-loop control, a time-varying uncertainty $0.5 \leq q(\kappa) \leq 1.5$ is considered, i.e., $q(\kappa) \neq q(\kappa+1)$. The rest of the MPC setup is analogous to Section V-A to make the results comparable.

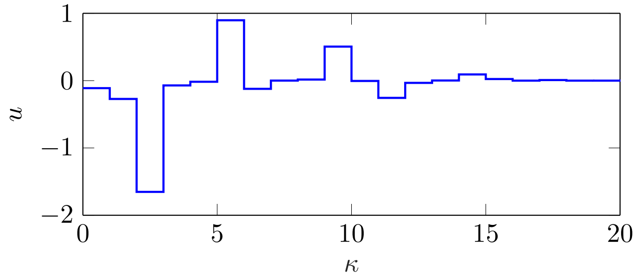
The computed initial values of controller parameters are $K(1) = [0.0893, 0.0224]^\top$. Fig. 5 shows simulation results of the closed-loop control performance. As can be seen, the control trajectories of the uncertain system converge into the origin (Fig. 5(a)), and the associated optimal sequence of control actions is depicted in Fig. 5(b).

VI. Conclusion

This paper presents a novel approach of robust MPC design based on evaluation of the stabilizing parameter



(a) Controlled state trajectories, x_1 (blue), x_2 (red), reference (black dotted).



(b) Control input.

Fig. 5. Closed-loop control performance assured by robust MPC in Case Study II.

space. The approach has two steps. Step I includes the determination of the stabilizing parameter space. This set is exactly evaluated using a Lyapunov equation based approach. Additionally, CQLF based conservative approach is also proposed to overcome the possibly destabilizing effects of switching. Then, in Step II, the derived stability conditions are inserted into the MPC problem as constraints on the controller parameters. MPC-based strategy is implemented to evaluate the optimal parameters of a controller subject to the constraints on control inputs and controller parameters. In this way, the robust closed-loop system stability and the optimal control performance in the presence of uncertain parameters are achieved.

The approach considers (i) *fast scenario*, i.e., an open-loop MPC that returns an optimal controller fixed during the control, (ii) *convex scenario*, i.e., the closed-loop MPC evaluated for the prediction horizon $N = 1$, and (iii) *non-convex scenario*, i.e., the closed-loop MPC evaluated considering $N > 1$. The advantage of the fast scenario is removing demanding evaluation of the optimization problem from on-line phase of controller design. The drawback of the non-convex scenario is NP-hard optimization problem. Therefore, the convex scenario (ii) is a trade off between the scenarios (i) and (iii), and represents an efficient way how to design robust MPC. Furthermore, effectiveness of the derived theoretical results are also verified over two benchmark case studies.

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