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Regulation of the downside angular velocity of a drilling string with a P-I controller

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Abstract— **In this paper, we demonstrate that a simple Proportional-Integral (P-I) controller allows to regulate the angular velocity of a drill-string despite unknown friction torque and by only using measurement of angular velocity at the surface. Our model is an one dimensional damped wave equation coupled at one boundary to dynamics which are subject to uncertainties. The control and the measured output are located at the other side. Writing this balance law system into Riemann coordinates, we design a novel Lyapunov functional in order to prove the exponential stability of the equilibrium in closed loop and show how it implies the regulation of the downside angular velocity.**

I. Introduction

In order to find and exploit oil, one needs to dig deeper and deeper into the earth's surface. The first consequence of raising the length of the pipe is the increase of several phenomena causing damage until the break of the device. These unwanted phenomena are mainly characterized by oscillations inside the pipe. These oscillations may appear in axial, radial and lateral directions. According to several studies (i.e [7], [4]), the radial oscillation, namely Stick-Slip phenomena, is the most disturbing one. Indeed, it results angular deformations traveling along the pipe, leading to important damage and furthermore and it is the source of other oscillating phenomena (Bit-Bounced and lateral oscillation).

From a mathematical point of view, first studies on this topic were based on lumped parameter model as in [4]. Nevertheless, the increase of the length of the pipe forces one to consider a distributed parameter model in order to deal with all the possible oscillations frequencies.

The control theory for such mathematical model is still an active domain of research. Recently, several works exploiting a PDE-based model to stabilize oscillating phenomena have been made. For example, by using a backstepping approach as in ([3]) or a flatness one in ([10]). Another method transforms the whole system in an equivalent time-delay system before ensuring its stability ([7]). Note however, that this transformation is impossible when taking into account a distributed damping along the drill pipe.

The main contribution of this article is the analysis of the closed loop stability when the control is provided in the form of a proportional integral (P-I) feedback depending on the topside angular velocity measurement. This shows that this control regulates the angular velocity of the drill bit to a given reference.

Since the seminal paper of S.A. Pohjolainen in 1982 [8], the problem of output regulation for PDE systems have received a huge interest from the control community. Following this paper, an important effort has been made to consider more general class of PDE and also to relax some crucial assumptions. For instance, it has been shown in [14] that it is possible to relax the compactness requirement on the operator. Moreover, it has been shown in [5] or [15] that it was possible to design a P-I for boundary control for different classes of hyperbollic systems. Following the route of a recent contributions in [13], [11], [2], we prove the regulation and the stabilization employing a Lyapunov approach.

In [3] and [9] the regulation problem of the angular velocity for drilling is also considered. In these works, an observer is built in order to perform a full state feedback and design a backstepping transformation. To compare, our control design is fairly simple since it is a P-I control law which only needs the surface angular velocity measurement. Moreover, our design methodology employs a novel Lyapunov design which should allow to consider nonlinear terms in the future.

II. Problem statement

A. Regulation of the angular velocity

We use the following PDE-ODE model describing the mechanical oscillations evolving along the drill string as given in [7], $t > 0$:

$$\rho J\theta_{tt}(x,t) = GJ\theta_{xx}(x,t) - \beta\theta_t(x,t), x \in]0, L[\quad (1)$$
$$GJ\theta_x(0,t) = c_a(\theta_t(0,t) - \Omega(t)) \quad (2)$$
$$I_b\theta_{tt}(L,t) = -GJ\theta_x(L,t) - T_{fr}\left(\theta_t(L,t)\right) \quad (3)$$

where for all x in $(0, L)$, $\theta(x,t)$ is the angular position of the drill string at point x and time t with respect to a given reference frame. Also, subscripts $t, x, tt, ..$ denote the first or second derivative w.r.t variables t or x. G, J, ρ, β, c_a and I_b are positive mechanical parameters and $\Omega(t)$ is the velocity of the rotatory table which is used as control. The different parameters are given in the following table (II-A). $T_{fr} : \mathbb{R} \mapsto \mathbb{R}$ is a function which describes the frictions between the drill bit and the earth. As Stribeck effect occurs at low velocity when the lubrication is not totally effective [7], this function is highly nonlinear for small values of $\theta_t(L,t)$. However, when the drill pipe is rotating fast enough, this frictions function becomes linear. Hence, in the sequel

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	Name	Values
G	Shear modulus	79.6×10^9 $N.m^{-2}$
ρ	Mass density	7850 $kg.m^{-3}$
J	Eq. mass moment of inertia	1.19×10^{-5} m^4
β	Distributed damping	0.05 $kg.m.s^{-1}$
c_a	Torque transmitted	2000 $N.m.s.rad^{-1}$
L	Length of the pipe	2000 m
I_b	BHA inertia	311 $kg.m^{-2}$
c	Propagation speed	3184.3 $m.s^{-1}$
c_b	Dynamic friction	0.03 $kg.s^{-1}$
T_0	Coulomb friction	7500 N

TABLE I
PARAMETERS VALUES

we consider that

$$T_{fr}(\theta_t(L,t)) = c_b\theta_t(L,t) + T_0 \tag{4}$$

where c_b is a real number and T_0 is also an unknown real number which is assumed to be constant. The measured output of this system is the angular position of the pipe at the top. In other words, we measure the following quantity

$$y(t) = \theta_t(0,t).$$

Our control objective is to regulate the velocity at the bottom which is denoted by

$$\underline{y}(t) = \theta_t(L,t),$$

to a prescribed constant reference velocity.

The control to solve is the following: we wish to find a control input $\Omega(t)$ depending only on the measured output such that for every (unknown) constant value of T_0, the *to be regulated* output $\underline{y}$ is enslaved to a given constant reference value denoted y_{ref}, speed at which the bottom velocity of the pipe is regulated. The structure of the control law is simple P-I control law. More precisely, the control input is provided by dynamic error feedback modeled as :

$$\Omega(t) = -k_p[y(t)-y_{ref}]-k_i\eta \ , \ \dot{\eta} = y(t)-y_{ref} \ \forall t \geq 0. \tag{5}$$

In the following, we give sufficient conditions on the parameters G, J, ρ, β, c_a and I_b such that there exist k_p and k_i which ensure the exponential stability of the equilibrium and the regulation. In other words, along the solutions of the system (1) with the control input (5) the equilibrium has to be exponentially stable and also we need that:

$$\lim_{t\to+\infty} |\underline{y}(t) - y_{ref}| = 0. \tag{6}$$

In a first part of the paper, the model is written in Riemannian coordinates. Defining the state space solution and its topology we give our main result which shows that regulation is obtained by our P-I control law. The remaining part of the paper is devoted to show this result. The proof starts by showing that the regulation property is implied by the exponential stability of the equilibrium state of the closed loop systems. To show that the equilibrium state is exponentially stable, we construct a Lyapunov functional.

B. Riemannian coordinates:

The first step of the study is to write the drilling model given in mechanical coordinates in system (1)-(2)-(3)-(4) in closed loop with the control law (5) into a normalized Riemann coordinates. In [2], authors gave a general method to reduce linear hyperbolic systems of conservation law with ODE at its boundaries in first order transport equation coupled with ODE via theirs boundary conditions. Note however, that in eq. (1), $\beta \neq 0$, hence, we are dealing with systems of balance law. If the method remains similar, the resulting transport equations are coupled with each other. For x in $(0,1)$ and $t \geq 0$ let

$$\varphi^-(x,t) = \left[\theta(Lx, \frac{Lt}{c})\right]_t - \left[\theta(Lx, \frac{Lt}{c})\right]_x, \tag{7}$$

$$\varphi^+(x,t) = \left[\theta(Lx, \frac{Lt}{c})\right]_t + \left[\theta(Lx, \frac{Lt}{c})\right]_x, \tag{8}$$

$$z(t) = \left[\theta(L, \frac{Lt}{c})\right]_t, \tag{9}$$

$$\xi(t) = \frac{2L}{c}\eta(t), \tag{10}$$

with $c^2 = \frac{G}{\rho}$. Employing equations (1)-(2)-(3)-(4) and (5), this implies for all $t \geq 0$

$$\varphi_t(x,t) = \begin{bmatrix} -\partial_x - \frac{\lambda}{2} & -\frac{\lambda}{2} \\ -\frac{\lambda}{2} & \partial_x - \frac{\lambda}{2} \end{bmatrix} \varphi(x,t), \ \forall x \in (0,1), \tag{11}$$

$$z_t(t) = -(a+b)z(t) + a\varphi^-(1,t) + d, \tag{12}$$

$$\xi_t(t) = \varphi^-(0,t) + \varphi^+(0,t) - \tilde{y}_{ref} \tag{13}$$

with the boundary conditions

$$\begin{aligned} \varphi^-(0,t) &= \alpha_0\varphi^+(0,t) \\ &\quad + K_p(\varphi^-(0,t) + \varphi^+(0,t) - \tilde{y}_{ref}) + K_i\xi(t), \end{aligned} \tag{14}$$

$$\varphi^+(1,t) = -\varphi^-(1,t) + 2z(t), \tag{15}$$

with $\varphi(x,t) = \begin{pmatrix} \varphi^-(x,t) \\ \varphi^+(x,t) \end{pmatrix}$ and the normalized parameters are given as

$$\lambda = \frac{\beta L}{c\rho J} \ , \alpha_0 = \frac{GJ - c_a c}{GJ + c_a c}, \tag{16}$$

$$a = \frac{GJL}{I_b c^2} \ , b = \frac{c_b L}{c I_b} \ , d = -\frac{T_0 L^2}{I_b c^2}, \tag{17}$$

the normalized reference and the normalized output to regulate are respectively

$$\tilde{y}_{ref} = \frac{c y_{ref}}{2L} \ , \tilde{\underline{y}}(t) = \varphi^-(1,t) + \varphi^+(1,t) = 2z(t) \tag{18}$$

and the normalized P-I gains K_p, K_i are given as

$$K_p = \frac{-c_a c}{GJ + c_a c} k_p \ , \quad K_i = \frac{-c_a c}{GJ + c_a c} k_i. \tag{19}$$

Equations (11), (12), (13) with boundary conditions (14) (15) define a hyperbolic partial differential equation coupled at the boundaries with two external ordinary differential equations. The state space denoted by $\mathbb{X}$ is the Hilbert space defined as

$$\mathbb{X} = (L^2(0,1))^2 \times \mathbb{R}^2,$$

with the norm defined for $v = (\varphi^-, \varphi^+, z, \xi) \in \mathbb{X}$ as

$$\|v\|_{\mathbb{X}} = \|\varphi^-\|_{L^2(0,1)} + \|\varphi^+\|_{L^2(0,1)} + |z| + |\xi|.$$

We introduce also a smoother state space defined as :

$$\mathbb{X}_1 = (H^1(0,1))^2 \times \mathbb{R}^2,$$

As it has been shown in [2], for each initial condition v_0 in $\mathbb{X}$ which satisfies the boundary conditions (14) and (15), there exists a unique weak solution that we denoted v and which belongs to $C^0([0,+\infty);\mathbb{X})$. Moreover, if the initial condition v_0 satisfies also the C^1-compatibility condition (see [2] for more details) and lies in $\mathbb{X}_1$ then the solution is strong, and belongs to :

$$v \in C^0([0,+\infty);\mathbb{X}_1) \cap C^1([0,+\infty);\mathbb{X}). \tag{20}$$

C. Main result

With all these preliminaries, we are now able to state our main result.

Theorem 1 (Regulation and stabilization): For all (α_0,λ, a, b) in $\mathbb{R}^4$ with $0 \leq \lambda < 2$, $a > 0$, $b \geq 0$ there exist real numbers K_i, K_p and positive real numbers k and ν such that for all constant references $\tilde{y}_{ref}$, all unknown d and all initial conditions in $\mathbb{X}$ the following holds.

1) There exists an equilibrium state denoted v_∞ which is globally exponentially stable in $\mathbb{X}$ for system (11)-(15). More precisely, we have :

$$\|v(t) - v_\infty\|_{\mathbb{X}} \leq k \exp(-\nu t)\|v_0 - v_\infty\|_{\mathbb{X}}; \tag{21}$$

2) Moreover, if v_0 satisfies the C^1-compatibility condition and it is in $\mathbb{X}_1$, the regulation is achieved, i.e.

$$\lim_{t\to+\infty} |\underline{\tilde{y}}(t) - \tilde{y}_{ref}| = 0. \tag{22}$$

As in [3] or [9], we solve the regulation problem around the equilibrium state by acting at the opposed boundary. The advantages of our approach is that we control the rotatory table and not directly the quantity $\theta_x(0,t)$ and that we only use $\theta_t(0,t)$ for designing our controller.

The stability analysis of this kind of models has been considered in [11], [2]. The dynamics at the top side boundary is due to the integral action of the control law. The stability analysis of PDE coupled with integral action has been initiated by [8] for parabolic systems (see also [15] for hyperbolic systems) following a spectral analysis. We follow the methodology of [12] taking into account dynamics at the boundary and a system of balance laws, not a conservation one (see [2]).

It has been shown in [1], that canceling the reflexion when $|\alpha_0| \leq 1$ with a proportional gain K_p, would not be an interesting approach due to the lack of robustness with respect to input delays. In the following corollary[1], we show that if $|\alpha_0| < 1$ it is not necessary to have a proportional part for obtaining regulation.

Corollary 1 (Regulation for stable systems): Assume $|\alpha_0| < 1$. For all (λ, a, b) in $\mathbb{R}^3$ with $0 \leq \lambda < 2$, $a > 0$, $b \geq 0$, there exist K_i in $\mathbb{R}$ and positive real numbers, k and ν such that the conclusion of the Theorem 1 holds with $K_p = 0$.

[1]The proof of this corollary is deduced from the following Lyapunov analysis picking $\alpha_p = \alpha_0$.

III. Proof Of Theorem 1

A. Stabilization implies regulation

In this first subsection, we explicitly give the equilibrium state of the system (11), (12), (13) with the boundary conditions (14) and (15). We show also that if we assume that K_p and K_i are selected such that this equilibrium point is exponentially stable along the closed loop, the regulation is achieved.

If we denote $v_\infty = (\varphi_\infty, z_\infty, \xi_\infty)$ the equilibrium states of (11)-(15), we obtain

$$\begin{aligned}
z_\infty &= \frac{\tilde{y}_{ref}}{2}, \\
\xi_\infty &= \frac{(1+\alpha_0)(a(1+\lambda)+b) - 2a\alpha_0}{2aK_i}\tilde{y}_{ref} - \frac{1+\alpha_0}{aK_i}d, \\
\varphi_\infty^-(x) &= -\frac{\lambda}{2}\tilde{y}_{ref}x + \frac{1}{1+\alpha_0}\left(\alpha_0\tilde{y}_{ref} + K_i\xi_\infty\right), \\
\varphi_\infty^+(x) &= \frac{\lambda}{2}\tilde{y}_{ref}x + \frac{\tilde{y}_{ref}}{1+\alpha_0} - \frac{K_i}{1+\alpha_0}\xi_\infty.
\end{aligned}$$

In the following, we first show that this regulation problem can be rephrased as a stabilization of an equilibrium state.

Proposition 1: Assume that there exists a functional $W : \mathbb{X} \to \mathbb{R}_+$, and positive real numbers ω and L such that

$$\frac{\|v_\infty - v\|_{\mathbb{X}}^2}{L} \leq W(v) \leq L\|v_\infty - v\|_{\mathbb{X}}^2. \tag{23}$$

Moreover, assuming that for all v_0 in $\mathbb{X}$ and all t_0 in $\mathbb{R}_+$ such that the solution v of (11)-(15) initiated from v_0 is C^1 at $t = t_0$, we have :

$$\dot{W}(v(t)) \leq -\omega W(v(t)). \tag{24}$$

Then points 1) and 2) of Theorem 1 hold.

Proof: The proof of point 1) is standard. Let v_0 be in $\mathbb{X}_1$ and satisfies the C^0 and C^1-compatibility conditions. It yields that v is smooth for all t. Consequently, (24) is satisfied for all $t \geq 0$. With Grönwall lemma, this implies that

$$W(v(t)) \leq e^{-\omega t}W(v_0).$$

Hence with (23), this implies that (21) holds with $k = L$ and $\nu = \frac{\omega}{2}$ for initial conditions in $\mathbb{X}_1$. $\mathbb{X}_1$ being dense in $\mathbb{X}$, the result holds also with initial condition in $\mathbb{X}$ and point 1) is satisfied.

Let us show point 2). Note that along solutions of the system (11)-(15), one has

$$(\varphi^-(x,t) - \varphi^+(x,t))_t = (\varphi^-(x,t) + \varphi^+(x,t))_x.$$

Since, at the equilibrium, $(\varphi^-(x,t) - \varphi^+(x,t))_t = 0$, it yields

$$(\varphi_\infty^-(x) + \varphi_\infty^+(x))_x = 0\ .$$

This implies that $\varphi_\infty^-(x)+\varphi_\infty^+(x)$ is constant along the string.

In one hand, with the definition of $\underline{\tilde{y}}(t)$ in (18)-(15) and the definition of the equilibrium, one has

$$\underline{\tilde{y}}(t) - \tilde{y}_{ref} = \varphi^-(1,t) + \varphi^+(1,t) - \varphi_\infty^-(1) - \varphi_\infty^+(1). \tag{25}$$

To show that equation (22) holds, we need to show that the right hand side of the former equation tends to zero. This may be obtained provided the initial condition is in $\mathbb{X}_1$. Indeed, let v_0 be in $\mathbb{X}_1$ and satisfies C^1 compatibility conditions. With (20), we know that $v_t \in C([0,\infty);\mathbb{X})$. Moreover, v_t satisfies the dynamics (11)-(15) with $d = 0$ and $y_{ref} = 0$. Hence, $\|v_t(t)\|_{\mathbb{X}}$ converges exponentially toward 0 and in particular

$\|\varphi_t^-(\cdot,t)\|_{L^2(0,1)} + \|\varphi_t^+(\cdot,t)\|_{L^2(0,1)} \leq ke^{-\nu t}\|v_t(0)\|_{\mathbb{X}}$.

On another hand, denoting $\tilde{\varphi}(x,t) = \varphi(x,t) - \varphi_\infty(x)$, employing (11), it yields

$$\|\varphi_t^+(\cdot,t)\|_{L^2(0,1)} \geqslant \|\tilde{\varphi}_x^+(\cdot,t)\|_{L^2(0,1)} \\ -\lambda\left(\|\tilde{\varphi}^+(\cdot,t)\|_{L^2(0,1)} + \|\tilde{\varphi}^+(\cdot,t)\|_{L^2(0,1)}\right), \quad (26)$$

$$\text{and } \|\varphi_t^-(\cdot,t)\|_{L^2(0,1)} \geqslant \|\tilde{\varphi}_x^-(\cdot,t)\|_{L^2(0,1)} \\ -\lambda\left(\|\tilde{\varphi}^+(\cdot,t)\|_{L^2(0,1)} + \|\tilde{\varphi}^+(\cdot,t)\|_{L^2(0,1)}\right). \quad (27)$$

Consequently $\|\tilde{\varphi}_x^-(\cdot,t)\|_{L^2(0,1)}$ and $\|\tilde{\varphi}_x^+(\cdot,t)\|_{L^2(0,1)}$ converge also to zero. With Sobolev embedding

$\sup_{x\in[0,1]}|\varphi(x,t) - \varphi_\infty(x)| \leqslant C\|\varphi(\cdot,t) - \varphi_\infty(\cdot)\|_{H^1(0,1)}$,

where C is a positive real number. It implies that :

$\lim_{t\to+\infty}|\varphi^-(1,t) + \varphi^+(1,t) - \varphi_\infty^-(1) - \varphi_\infty^+(1)| = 0$.

Consequently, with (25), it yields that (22) holds and point 2) is satisfied. Consequently, (22) holds and point 2) is satisfied. ■

Considering this proposition and the linearity of the system, it turns out that to prove Theorem 1, it is sufficient to construct a Lyapunov functional. This property doesn't depend on the value of y_{ref} and the unknown parameter d. So in the following, it is assumed that $y_{ref} = d = 0$ and we design a Lyapunov functional.

B. Lyapunov functional construction

1) Step 1: Let us forget the ξ dynamic: In this subsection we construct a Lyapunov function for the system (11)-(15) without taking into account the dynamic of ξ into the integral part which will be added in the next subsection. So here, we consider this PDE system

$$\varphi_t(x,t) = \begin{bmatrix} -\partial_x - \frac{\lambda}{2} & -\frac{\lambda}{2} \\ -\frac{\lambda}{2} & \partial_x - \frac{\lambda}{2} \end{bmatrix}\varphi(x,t),\ \forall x\in(0,1), \\ z_t(t) = -(a+b)z(t) + a\varphi^-(1,t), \quad (28)$$

with the boundary conditions

$$\varphi^-(0,t) = \alpha_0\varphi^+(0,t) + K_p(\varphi^-(0,t)+\varphi^+(0,t)) + K_i\xi(t), \\ \varphi^+(1,t) = -\varphi^-(1,t) + 2z(t). \quad (29)$$

Let $V : L^2(0,1)^2 \times \mathbb{R} \mapsto \mathbb{R}_+$ be the function defined by :

$$V(\varphi,z) = qz^2 + \int_0^1 (\varphi^-)^2 e^{-\mu x}dx + \int_0^1 p(\varphi^+)^2 e^{\mu x}dx \quad (30)$$

which is well defined. With a slight abuse of notation, we write $V(t) = V(\varphi(\cdot,t),z(t))$ and we denote by $\dot{V}(t)$ the time derivative of the Lyapunov function along the solutions which are C^1 in time.

Proposition 2: For all (λ,a,b,α_0) with $2 > \lambda > 0$, $a > 0$ and $b \geq 0$ there exist positive real numbers K_p, p, μ, q, ω_1 and ω_2 such that along the C^1 solution of the system (28) and (29) :

$$\dot{V}(t) \leq -\omega_1 V(t) + \nu|\xi(t)|^2. \quad (31)$$

Proof: Let

$$V(t) = V_1(t) + V_2(t), \quad (32)$$

with $V_1(t) = qz^2(t)$,

$$V_2(t) = \int_0^1 (\varphi^-(x,t))^2 e^{-\mu x}dx + \int_0^1 p(\varphi^+(x,t))^2 e^{\mu x}dx.$$

The time derivative of (32) along the system (28) with the boundary conditions (29) can be written $\forall\mu_1$ and μ_2 in $\mathbb{R}$:

$$\dot{V}(t) = -\mu_2 V_1(t) - (\mu-\mu_1)V_2(t) - w_0(t)^T\mathcal{P}w_0(t) \\ -\int_0^1 \varphi(x,t)^T\mathcal{N}(\mu_1)\varphi(x,t)dx - w_1(t)^T\mathcal{M}(\mu,\mu_2)w_1(t),$$

where

$${}_0(t) = \begin{pmatrix}\varphi^+(0,t) & K_i\xi(t)\end{pmatrix}^T, w_1(t) = \begin{pmatrix}\varphi^-(1,t) & z(t)\end{pmatrix}^T,$$

$$\text{and } \mathcal{P} = \begin{bmatrix} p-\alpha_p^2 & -\alpha_p \\ -\alpha_p & -\frac{1}{1-K_p}\end{bmatrix},$$

$$\mathcal{M} = \begin{bmatrix} e^{-\mu} - pe^{\mu} & 2pe^{\mu} - aq \\ 2pe^{\mu} - aq & (2(a+b)-\mu_2)q - 4pe^{\mu}\end{bmatrix},$$

$$\mathcal{N} = \begin{bmatrix} (\mu_1+\lambda)e^{-\mu x} & \lambda\frac{e^{-\mu x}+pe^{\mu x}}{2} \\ \lambda\frac{e^{-\mu x}+pe^{\mu x}}{2} & p(\mu_1+\lambda)e^{\mu x}\end{bmatrix},$$

where

$$\alpha_p = \frac{\alpha_0 + K_p}{1-K_p}. \quad (33)$$

First of all remark that if :

$$p - \alpha_p^2 > 0, \quad (34)$$

then there exists a positive real number ν such that

$$-w_0(t)^T\mathcal{P}w_0(t) \leq \nu|K_i\xi(t)|^2.$$

Besides, if :

$$\mu_2 > 0,\ (\mu-\mu_1) > 0,\ \mathcal{M} \geq 0,\ \mathcal{N} \geq 0 \quad (35)$$

the candidate Lyapunov functional satisfies (31).

Let us first show that we can find the parameters such that (35) is satisfied. Consider the mapping $(s_1,s_2,\mu) \mapsto F(s_1,s_2,)$ given by

$$F(s_1,s_2,\mu) = \sqrt{s_1}(\lambda+s_2) - \tfrac{\lambda}{2}\left(1 + s_1e^{\mu}\right).$$

Since $\lambda < 2$, picking μ sufficiently small, $0 < \mu < 1$, the following inequality is always true :

$$F(e^{-2\mu},\mu,\mu) = e^{-\mu}(\lambda+\mu) - \frac{\lambda}{2}\left(1+e^{-\mu}\right), \\ = \lambda\frac{e^{-\mu}-1}{2} + e^{-\mu}\mu > 0.$$

So, let p and μ_1 be positive real numbers such that

$$0 < p < e^{-2\mu},\ \mu_1 < \mu,$$

and $F(p,\mu_1,\mu) > 0$ which exists by continuity of the mapping F. Note that :

$$\det(\mathcal{N}) = p(\lambda+\mu_1)^2 - \frac{\lambda^2}{4}\left(e^{-\mu x} + pe^{\mu x}\right)^2, \\ > F(p,\mu_1,\mu)\left[\sqrt{p}(\lambda+\mu_1) + \frac{\lambda}{2}\left(e^{-\mu x}+pe^{\mu x}\right)\right] > 0.$$

So, $\mathcal{N} \geq 0$.

On the other side, $\mathcal{M} \geq 0$ if and only if

$$e^{-\mu} - pe^{\mu} \geq 0 \tag{36}$$
$$f(q) \geq 0 \tag{37}$$

where :

$$\begin{aligned} f(q) =&\ (e^{-\mu} - pe^{\mu})(q(2(a+b) - \mu_2) - 4pe^{\mu}) \\ &\qquad - (2pe^{\mu} - aq)^2 \\ =&\ -a'q^2 + b'q - c', \end{aligned}$$

and a', b' and c' are positive real numbers given as :

$$\begin{aligned} a' &= a^2 \\ b' &= (e^{-\mu} - pe^{\mu})(2(a+b) - \mu_2) + 4pae^{\mu} \\ c' &= (e^{-\mu} - pe^{\mu})4pe^{\mu} + 4p^2e^{2\mu} = 4p \end{aligned}$$

This function $f(q)$ is a polynomial , which is maximum for $q = \frac{b'}{2a'}$. In this case, $f(q)$ is strictly positive if and only if

$$b'^2 - 4a'c' = (b' - 2\sqrt{a'c'})(b' + 2\sqrt{a'c'}) > 0. \tag{38}$$

Since a', b' and c' are positive, it remains to verify that $b' - 2\sqrt{a'c'}$ is positive. Note that keeping in mind that $(e^{-\mu} - pe^{\mu}) > 0$, it yields

$$\frac{b' - 2\sqrt{a'c'}}{(e^{-\mu} - pe^{\mu})} = (2(a+b) - \mu_2) - \frac{4a(\sqrt{p} - pe^{\mu})}{e^{-\mu} - pe^{\mu}}, \tag{39}$$
$$= 2a\left(1 - \frac{2\sqrt{p}}{e^{-\mu} + \sqrt{p}}\right) + 2b - \mu_2. \tag{40}$$

Since $\frac{2\sqrt{p}}{e^{-\mu}+\sqrt{p}} < 1$, we can select $\mu_2 > 0$ sufficiently small such that $f(q) > 0$, so $\mathcal{M} \geq 0$.

Finally, we pick K_p such that (34) holds (always possible since $p > 0$) which achieves the proof. ■

2) Step 2, adding the integral part: In this section, the integral parameter K_i is tuned based on the construction of a Lyapunov functional. Let : $W : \mathbb{X} \mapsto \mathbb{R}_+$ be the function defined by

$$W(\varphi, z, \xi) = V(\varphi, z) + rU(\xi, \varphi, z)^2,$$

where :

$$U(\xi, \varphi, z) = \xi + m^\top M(\varphi) + nz,$$

where $r > 0$, $m = (m_1, m_2)$ is a vector in $\mathbb{R}^2$ and n a real number that will be selected later and $M : L^1(0,1) \mapsto \mathbb{R}^2$ is an operator defined as :

$$M(\varphi) = \int_0^1 (I + Rx)\varphi(x)dx, \tag{41}$$

with :

$$R = \begin{bmatrix} \frac{\lambda}{2} & -\frac{\lambda}{2} \\ \frac{\lambda}{2} & -\frac{\lambda}{2} \end{bmatrix}.$$

Again, with a slight abuse of notation, we write $W(t) = W(\varphi(\cdot,t), z(t), \xi(t))$ and we denote by $\dot{W}(t)$ the time derivative of the Lyapunov function along solutions which are C^1 in time.

Proposition 3: Assume K_p, λ, a, b, α_0, p, μ, q and ω_1 are given such that inequality (31) holds, then there exist r, m_1, m_2, n and K_i such that :

$$\dot{W}(t) \leq -\omega_2 W(t).$$

Proof: First of all, note that

$$R \begin{bmatrix} -\frac{\lambda}{2} & -\frac{\lambda}{2} \\ -\frac{\lambda}{2} & -\frac{\lambda}{2} \end{bmatrix} = 0.$$

Hence, we get the property

$$\begin{aligned} \dot{M}(t) &= \int_0^1 (I + Rx)\varphi_t(x,t)dx \\ &= \int_0^1 (I + Rx) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \varphi_x(x,t) \\ &\qquad + \begin{bmatrix} -\frac{\lambda}{2} & -\frac{\lambda}{2} \\ -\frac{\lambda}{2} & -\frac{\lambda}{2} \end{bmatrix} \varphi(x,t)dx \end{aligned}$$

With an integration by parts, it yields :

$$\dot{M}(t) = \begin{bmatrix} -1 - \frac{\lambda}{2} & -\frac{\lambda}{2} \\ -\frac{\lambda}{2} & 1 - \frac{\lambda}{2} \end{bmatrix} \varphi(1,t) - \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \varphi(0,t),$$

and consequently,

$$\begin{aligned} \dot{U}(t) &= \varphi^-(0,t) + \varphi^+(0,t) \\ &+ m_1\left(\varphi^-(0,t) - \left(1 + \frac{\lambda}{2}\right)\varphi^-(1,t) - \frac{\lambda}{2}\varphi^+(1,t)\right) \\ &+ m_2\left(-\frac{\lambda}{2}\varphi^-(1,t) + \left(1 - \frac{\lambda}{2}\right)\varphi^+(1,t) - \varphi^+(0,t)\right) \\ &\qquad - n(a+b)z(t) + na\varphi^-(1,t) \end{aligned}$$

Employing the boundary conditions (29) and α_p defined in (33), it yields :

$$\begin{aligned} \dot{U}(t) &= \varphi^-(1,t)\left(an - m_1\left(1 + \frac{\lambda}{2}\right) - m_2\frac{\lambda}{2} - \frac{n(a+b)}{2}\right) \\ &+ \varphi^+(1,t)\left(m_2\left(1 - \frac{\lambda}{2}\right) - m_1\frac{\lambda}{2} - \frac{n(a+b)}{2}\right) \\ &+ \varphi^+(0,t)(\alpha_p + 1 + m_1\alpha_p - m_2) + \xi(t)\frac{K_i}{1 - K_p}(1 + m_1). \end{aligned}$$

Our aim is now to solve in m_1, m_2 and n the system

$$\begin{bmatrix} -1 - \frac{\lambda}{2} & -\frac{\lambda}{2} & \frac{a-b}{2} \\ -\frac{\lambda}{2} & 1 - \frac{\lambda}{2} & -\frac{a+b}{2} \\ \alpha_p & -1 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\alpha_p - 1 \end{bmatrix}$$

It is possible with :

$$\begin{aligned} n &= \frac{2(\alpha_p + 1)}{\alpha_p(a(\lambda - 1) + b) + a\lambda + a + b}, \\ m_2 &= \frac{+2a\alpha_p}{\alpha_p(a(\lambda - 1) + b) + a\lambda + a + b} + 1, \\ m_1 &= \frac{2a}{\alpha_p(a(\lambda - 1) + b) + a\lambda + a + b} - 1. \end{aligned}$$

It can be noticed that since $0 \leq \lambda < 2$ and $a + b > 0$, these parameters are well defined. In that case, it yields :

$$\dot{U}(t) = \xi(t)\frac{K_i}{1 - K_p}(1 + m_1).$$

So we select K_i such that $\frac{K_i}{1-K_p}(1+m_1) < 0$.
Hence, we get :

$$\frac{2U(t)\dot{U}(t)}{|K_i|} \leq -\xi(t)^2\left|\frac{1+m_1}{1-K_p}\right| + \xi(t)m^\top M(t) + \xi(t)nz, \\ \leq -c_1\xi(t)^2 + c_2V(t),$$

where c_1 and c_2 are obtained Cauchy Schwartz inequality and by completing the square. Finally, this yields

$$\dot{W}(t) \leq (c_2r|K_i| - \omega_1)V(t) + \nu|K_i\xi|^2 - rc_1|K_i|\xi(t)^2. \quad (42)$$

Picking $r|K_i|$ sufficiently small such that $c_2r|K_i| < \omega_1$ and $|K_i|$ sufficiently small, the result is yielded. ■

IV. Simulation

In this part, the PI controller is simulated on the model (1)-(3), with the non-linear friction law as in [7] :

$$T_{fr}(\theta_t(L,t)) = c_b\theta_t(L,t) + \\ \frac{2T_0}{\pi}\left(\alpha_1\theta_t(L,t)e^{|\alpha_2|\theta_t(L,t)} + \arctan(\alpha_3\theta_t(L,t))\right). \quad (43)$$

The numerical scheme used is a discretization in space of the equation (1) with 20 space nodes. The boundary conditions are then computed using spatial ghost nodes. Parameters are chosen following [6] and are given in the table II-A. Considering the case of deep-water drilling system as in [9], $\beta = 0.05N.s$ which corresponds to $\lambda \approx 0.3$. The following scenario is considered: initial values are taken close to the equilibrium state, at $t = 10s$ the values of T_0 in eq. (4) is increased by 50% during $1sec$ which produces the Stick-Slip phenomenon. In Fig 1 the gains are $k_p = 0.8$ and $k_i = 0.2$, leading to $\alpha_p \approx -0.15$. In the case where $|\alpha_p|$

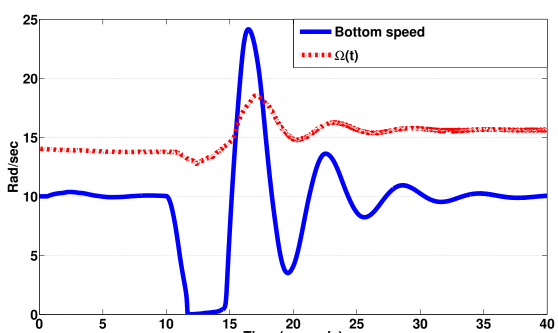

Fig. 1. Stabilization of the angular velocity

is small, regulation is achieved. Whereas, in the case of an integral controller, regulation can not be achieved due to the non-linearity. So the mechanical oscillation goes on, as what appears in Fig 2. In this figure, an integral controller with gains $k_p = 0$ and $k_i = 0.2$ is embedded ($\alpha_p = -0.74$).

V. Conclusion

In this paper we have presented an analysis of the a P-I controller to regulate the bottom velocity of a drill pipe. We have shown that exponential stability of the equilibrium could be achieved with this type of control laws. The result

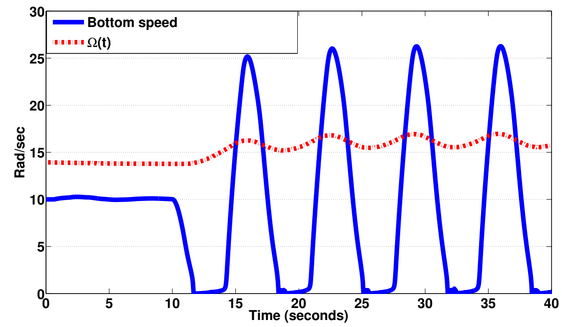

Fig. 2. Stabilization of the angular velocity

has been obtained employing a novel Lyapunov functional construction which is valid for all admissible mechanical parameters.

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