2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Estimating the Wigner distribution of linear time-invariant dynamical systems

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***Abstract*— An estimation algorithm for the Wigner distribution (time-frequency representation) of the unmeasured states of a linear time-invariant system is presented. Given that the inputs and outputs are measured, the algorithm involves designing a Luenberger-like observer for each frequency of interest. Under noise-free conditions, we show that the estimates converge to the true Wigner distribution under a detectability assumption on the time-frequency representation. The estimation algorithm provides estimates which converge to a neighbourhood of the true Wigner distribution where its norm is dependent on the norm of the measurement noise. We also illustrate the efficacy of the estimation algorithm on an academic example and a model of neuron populations.**

I. Introduction

Time-frequency representation of signals is the transformation of signals whose domain is the single dimensional real line to a domain that is a two dimensional real plane with time and frequency, respectively on each dimension [4], [11]. Several time-frequency representations have been developed, namely the short-time Fourier transform, wavelet transform and a class of quadratic time-frequency transformations which includes the Wigner transform [11], [4]. Both time-frequency and frequency based analyses have proved useful for non-stationary signals (time and frequency-varying signals) in system identification [1], [7], [9] and controller design [3].

Motivated by applications in the field of neuroscience where the analysis of electroencephalographic (EEG) signals is useful in identifying brain states [10], [12], [5], we focus on the Wigner distribution of signals generated by linear time-invariant (LTI) dynamical systems. Our aim is to design an algorithm to estimate the Wigner distribution of the unmeasured states, given that the Wigner distribution of its input and output can be measured. While the Wigner distribution of the unmeasured states can be taken after estimating the states in the time-domain via conventional methods such as the Luenberger observer, our algorithm provides an alternative way by obtaining an estimate of the Wigner distribution of the states directly.

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The first and third authors are members of the LCCC Linnaeus Center and the ELLIIT Strategic Research Area at Lund University. The third author is funded by the Swedish Research Council, grant 2016-04764. The authors acknowledge the Crafoord Foundation and the eSSENCE strategic research programme for supporting this work.

The time-frequency transformation of LTI systems has been studied in [6], where its distinctive feature lies in that the resulting transformation of the LTI system is also a linear dynamical system with respect to time in the time-frequency domain, although the transformation is non-linear. Hence, our estimation algorithm involves designing a Luenberger-like estimator for estimating the Wigner distribution of the unmeasured states for each frequency of interest, under the assumption that the Wigner distribution of its inputs and outputs are given. We provide convergence guarantees in that the estimates will converge to the true Wigner distribution in the absence of measurement noise. Additionally, the estimates will converge to a neighbourhood of the true Wigner distribution, where the size of the neighbourhood is dependent on a gain tunable by the user and the norm of the Wigner distribution of the measurement noise and its cross-terms with the unmeasured state.

To the best of our knowledge, this is the first work which addresses the estimation of the time-frequency distribution of the unmeasured states of an LTI system. Moreover, this work is novel in the following manner:

- No distribution of the measurement noise is assumed, i.e. non-Gaussian noise can be considered.
- The attenuation of measurement noise and the artefacts can be designed by the user, but is a tradeoff with the convergence speed of the estimates.

This paper is organised as follows. We start by providing a preliminary on notations in Section II. Next, we formulate the problem in Section III and Section IV provides a method of estimating the Wigner distribution of the unmeasured states of a LTI system. We illustrate the estimation algorithm on an academic example and a model of brain dynamics in Section V and conclude the paper in Section VI.

II. Preliminaries

- Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_{\geq 0} = [0, \infty)$, $\mathbb{R}_{>0} = (0, \infty)$, $\mathbb{N} = \{0, 1, \dots\}$, $\mathbb{N}_{\geq 1} = \{1, 2, \dots\}$ and $i = \sqrt{-1}$.
- Let (u, v) where $u \in \mathbb{R}^{n_u}$ and $v \in \mathbb{R}^{n_v}$ denote the vector $(u^T, v^T)^T$.
- For a matrix $A \in \mathbb{R}^{m \times n}$, let A^T and A^* denote the transpose and Hermitian of A, respectively and $\text{vec}(A)$ denote the vectorisation of A, i.e. $\text{vec}(A) := (a_{11}, a_{21}, \dots, a_{n1}, a_{12}, \dots, a_{nn})^T$. Given matrices B and x of the appropriate dimensions, the following transformation holds

$$\text{vec}(AxB) = (B^T \otimes A)\text{vec}(x),$$

where $\otimes$ denotes the Kronecker product.

- The identity matrix of dimension r is denoted by $\mathbb{I}_r$ and the zero matrix of dimension r by s is denoted by $0_{r\times s}$.
- A block diagonal matrix with matrices A_1, A_2, …, A_n is denoted by $\text{diag}(A_1, A_2, \dots, A_n)$.
- For a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, the ∞-norm of x is denoted $|x| := \max\{|x_1|, |x_2|, \dots, |x_n|\}$.

III. PROBLEM FORMULATION

Consider the following linear time-invariant (LTI) dynamical system

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + d,\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, $y \in \mathbb{R}^{n_y}$ and $d \in \mathbb{R}^{n_y}$ are the state, input, output and measurement noise, respectively. We assume that the input u and output y can be measured. We define the time-frequency (Wigner) distribution $W_{\xi,\psi}(t,\omega)$ of signals $\xi : [0,\infty] \to \mathbb{R}^m$, $\psi : [0,\infty] \to \mathbb{R}^m$ according to [6, Eq. (6)], i.e. for all $t \geq 0$ and $\omega \in \mathbb{R}$

$$W_{\xi,\psi}(t,\omega) := \int_{-\infty}^{+\infty} \xi\left(t - \frac{\tau}{2}\right)\psi^T\left(t + \frac{\tau}{2}\right)e^{-i\tau w}d\tau.$$

When the two signals considered are identical, i.e. $\xi = \psi$, we use the short form notation W_ξ. Hence, $W_x(t,\omega)$, $W_u(t,\omega)$ and $W_y(t,\omega)$ denote the time-frequency distributions of x, u and y, respectively.

According to [6], system (1) is represented in the time-frequency domain as follows

$$\begin{aligned}&\frac{1}{4}\frac{\partial^2}{\partial t^2}W_x(t,\omega) + \frac{1}{2}\frac{\partial}{\partial t}W_x(t,\omega)A_\omega + \frac{1}{2}A^*_\omega\frac{\partial}{\partial t}W_x(t,\omega)\\ &\qquad\qquad + A^*_\omega W_x(t,\omega)A_\omega = BW_u(t,\omega)B^T\\ &W_y(t,\omega) = CW_x(t,\omega)C^T + \eta(t,\omega),\end{aligned} \tag{2}$$

where $A_\omega := i\omega\mathbb{I}_{n_x} - A^T$, $A^*_\omega := -i\omega\mathbb{I}_{n_x} - A$ and $\eta(t,\omega) := W_d(t,\omega) + CW_{x,d}(t,\omega) + W_{d,x}(t,\omega)C^T$.

Our objective is to estimate the time-frequency distribution of the unmeasured state $W_x(t,\omega)$ under the assumption that the time-frequency distributions of the input $W_u(t,\omega)$ and the output $W_y(t,\omega)$ respectively, are known.

IV. ESTIMATING THE WIGNER DISTRIBUTION OF THE UNMEASURED STATES

In this section, the time-frequency representation (2) is written compactly in state space form and an estimation algorithm is designed with convergence guarantees. To this end, let $z(t,\omega) := (z_1(t,\omega), z_2(t,\omega))$, where $z_1(t,\omega) := W_x(t,\omega)$ and $z_2(t,\omega) := \frac{\partial z_1(t,\omega)}{\partial t} + 2z_1(t,\omega)A_\omega + 2A^*_\omega z_1(t,\omega)$. We obtain the following system

$$\begin{aligned}\frac{\partial z_1(t,\omega)}{\partial t} &= z_2 - 2z_1A_\omega - 2A^*_\omega z_1\\ \frac{\partial z_2(t,\omega)}{\partial t} &= -4A^*_\omega z_1 A_\omega + 4BW_u(t,\omega)B^T\\ W_y(t,\omega) &= Cz_1(t,\omega)C^T + \eta(t,\omega).\end{aligned} \tag{3}$$

For ease of computation, we further transform (3) into the system below. To this end, let $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)$, $\tilde{z}_1 := \text{vec}(z_1)$, $\tilde{z}_2 := \text{vec}(z_2)$, $\tilde{u} := \text{vec}(W_u)$, $\tilde{y} := \text{vec}(W_y)$ and $\tilde{\eta} := \text{vec}(\eta)$.

$$\begin{aligned}\frac{\partial \tilde{z}(t,\omega)}{\partial t} &= F(w)\tilde{z}(t,\omega) + G\tilde{u}(t,\omega)\\ \tilde{y}(t,\omega) &= H\tilde{z}(t,\omega) + \tilde{\eta}(t,\omega).\end{aligned} \tag{4}$$

where $F(w) = \begin{bmatrix} -2A^T_\omega \otimes \mathbb{I}_{n_x} - \mathbb{I}_{n_x} \otimes 2A^*_\omega & \mathbb{I}_{n_x^2}\\ -4A^T_\omega \otimes A^*_\omega & 0_{n_x^2\times n_x^2}\end{bmatrix}$, $G = \begin{bmatrix} 0_{n_x^2\times n_u^2}\\ 4B\otimes B\end{bmatrix}$ and $H = \begin{bmatrix} C\otimes C & 0_{n_y^2\times n_x^2}\end{bmatrix}$.

For each $w \in \mathbb{R}$, we design an estimator with the following dynamics

$$\begin{aligned}\frac{\partial \hat{z}(t,\omega)}{\partial t} &= F(w)\hat{z}(t,\omega) + G\tilde{u}(t,\omega)\\ &\quad + L(w)(\hat{y}(t,\omega) - \tilde{y}(t,\omega))\\ \hat{y}(t,\omega) &= H\hat{z}(t,\omega),\end{aligned} \tag{5}$$

where $\hat{z}(t,\omega)$ is the estimate of the time-frequency distribution $\tilde{z}(t,\omega)$. The estimator gain $L(\omega)$ is designed such that $F(\omega) - L(\omega)H$ is Hurwitz, which can always be achieved under the following assumption.

Assumption 1: The pair $(F(\omega), H)$ is detectable for all $\omega \in \Omega \subseteq \mathbb{R}$. □

In general, detectability of the time-domain LTI system (1) does not imply that Assumption 1 is satisfied. However, the assumption holds under some conditions on the time-domain LTI system (1), which we state in the proposition below. Its proof can be found in the appendix.

Proposition 1: Consider the LTI system (1) and its time-frequency representation (4). $F(\omega)$ from (4) is Hurwitz for all $\omega \in \mathbb{R}$ if and only if the conditions below hold:

(i) A from (1) is Hurwitz,
(ii) $F(\omega)$ is Hurwitz for $\omega = 0$.

□

The proposition allows the detectability of the time-frequency representation (4) to be verified from the LTI system (1), since a stable linear system is also detectable.

We now proceed with providing convergence guarantees for the estimates of the states' Wigner distribution. Let the estimation error be $e(t,\omega) := \hat{z}(t,\omega) - z(t,\omega)$. The error system has the dynamics

$$\frac{\partial e(t,\omega)}{\partial t} = (F(\omega) - L(\omega)H)e(t,\omega) + L(\omega)\tilde{\eta}(t,\omega). \tag{6}$$

We provide the following convergence guarantee.

Theorem 1: Consider system (4), the time-frequency representation of system (1) and the estimator (5). Suppose Assumption 1 holds. Then, for all $\omega \in \Omega \subseteq \mathbb{R}$, there exist $k = k(\omega) > 0$, $\lambda = \lambda(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ such that the solutions of the estimation error system (6) satisfies the following:

$$|e(t,\omega)| \leq k\exp(-\lambda t)|e(0,\omega)| + \gamma \sup_{s\in[0,t]}|\tilde{\eta}(s,\omega)|,\ \forall t \geq 0, \tag{7}$$

for all $e(0,\omega) \in \mathbb{R}^{2n_x^2}$. □

Proof: For any $\Omega \subseteq \mathbb{R}$, suppose that the pair $(F(\omega), H)$ is detectable for all $\omega \in \Omega$. According to Theorem 2.1 in

[2], for all $\omega \in \Omega$, there exists $L(\omega) \in \mathbb{R}^{2n_x \times n_y}$ such that $F(\omega) - L(\omega)H$ is Hurwitz. Therefore, by the variation of constants formula, we have from (6) that

$$\begin{array}{rcl} |e(t,\omega)| & \leq & k(w)e^{-\lambda(\omega)t}|e(0,\omega)| \\ & & +|L(\omega)| \int_0^t e^{-\lambda(\omega)(t-s)} |\tilde{\eta}(s,\omega)| ds \\ & \leq & k(\omega)e^{-\lambda(\omega)t}|e(0,\omega)| \\ & & +|L(\omega)| \int_0^t e^{-\lambda(\omega)(t-s)} ds \left(\sup_{s\in[0,t]} |\tilde{\eta}(s,\omega)| \right). \end{array}$$

Since $\int_0^t e^{-\lambda(\omega)(t-s)} ds = \frac{1}{\lambda(\omega)}\left(1 - e^{\lambda(\omega)t}\right) \leq \frac{1}{\lambda(\omega)}$, we obtain (7) as desired with $\gamma(\omega) := \frac{|L(\omega)|}{\lambda(\omega)}$. ■

The estimate $\hat{z}(t,\omega)$ is guaranteed to converge exponentially in t, for all ω, to a neighbourhood of the time-frequency distribution $z(t,\omega)$ for all initial conditions $z(0,\omega)$ and $\hat{z}(0,\omega)$. When the system (1) is noise-free, we guarantee exponential convergence of the estimate $\hat{z}(t,\omega)$ to the true time-frequency distribution $z(t,\omega)$ in t, for all ω.

V. Case studies

Two examples are considered in this section. We start with an academic scalar example to showcase the application of the estimation algorithm. Then, we proceed to the main motivation behind this work, which is a nonlinear model from the neuroscience community. We first linearise around its equilibrium point before applying the estimator developed in Section IV.

A. An academic example

Consider system (1) with $a = -1$, $b = 1$ and $c = 2$, with input u

$$u(t) = \begin{cases} \sin(2\pi f_0 t), & t \geq 10, \\ 0, & t \in [0, 10), \end{cases} \tag{8}$$

where $f_0 = 0.15$ and the measurement noise d is a Gaussian noise with 0 mean and variance 0.01. The system is initialised at $x(0) = 2$.

Our set of frequencies of interest is $\Omega := \{\ 2\pi \cdot 0.010,\ 2\pi \cdot 0.011,\ \ldots,\ 2\pi \cdot 0.250\ \}$. The resulting initialisation of the time-frequency representation (2) of system (1) at $x(0) = 2$ is $C_x(0,\omega) = 4$ and we intialised the estimator with $\hat{z}(0,\omega) = 0$ for all $\omega \in \Omega$. We choose $L(\omega) = (-0.1, -0.1)$ such that $F(\omega) - L(\omega)H$ is Hurwitz for all $\omega \in \Omega$. As guaranteed by Theorem 1, the estimation error converges to a neighbourhood of the origin, whereby the size of the neighbourhood is dictated by the measurement noise d. Figures 1-4 show the norm of the estimation error, the absolute value of the Wigner distribution of x and its estimate provided by (5), respectively.

B. A model of neuron populations

We consider a model of neuron populations, also known as a neural mass model, from [8]. This model describes the dynamics of the mean membrane potential of neuron populations (unmeasurable states), which produces patterns in the electroencephalogram, also known as the EEG (measurement), related to various brain activities categorised by

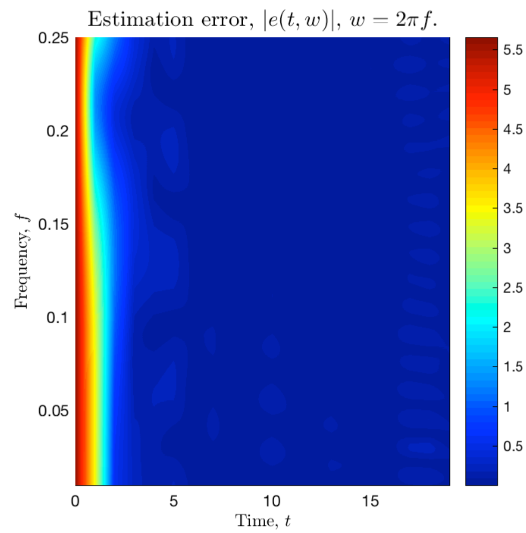

Fig. 1. Scalar example: Norm of estimation error, $|e(t,\omega)|$ converges to a neighbourhood of 0 as $t \to \infty$, for all $\omega \in \Omega$.

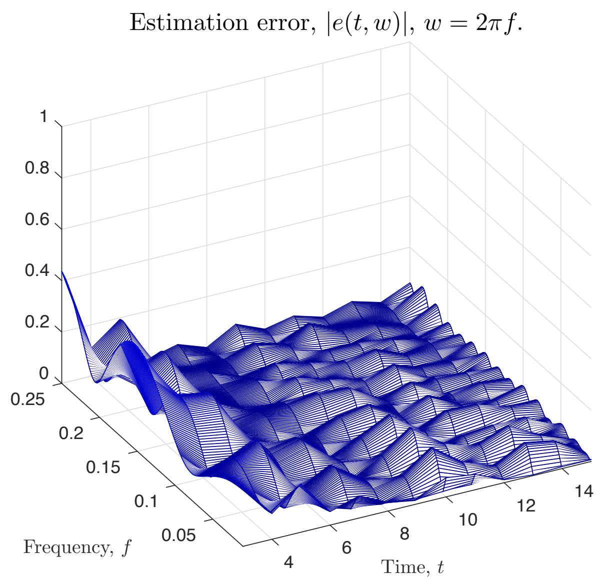

Fig. 2. Scalar example: Norm of estimation error, $|e(t,\omega)|$ for $t \in [3, 15]$ and $\omega \in \Omega$.

frequency range. For example, an EEG signal in the $0-3$ Hz range is known as the delta wave and is associated with the brain being in deep relaxation. Therefore, tracking the time evolution and the frequency content of the mean membrane potential of neuron populations (the unmeasurable states) can provide insights into the underlying mechanisms of various brain activities.

To this end, the neural model from [8] is a nonlinear dynamical system and can be written as follows below by taking the states to be $x = (x_1, x_2, \ldots, x_6) \in \mathbb{R}^6$, where x_1, x_3, x_5 are the mean membrane potential of each of the three interconnected neuron populations, respectively; x_2, x_4

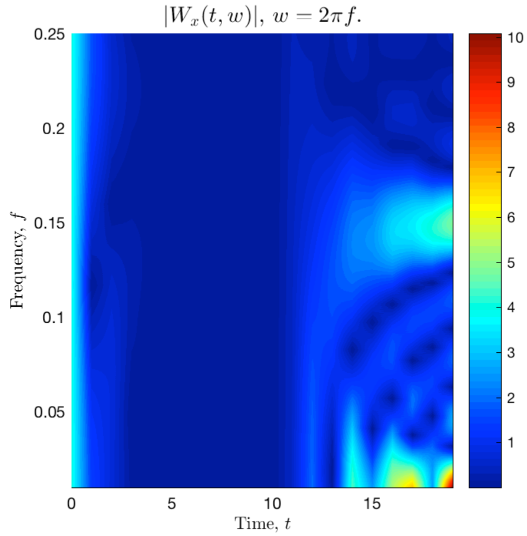

Fig. 3. Scalar example: Absolute value of the Wigner distribution of x, $|W_x(t,\omega)|$.

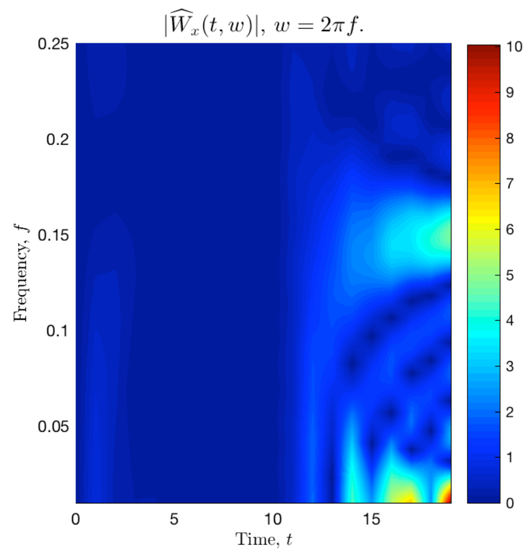

Fig. 4. Scalar example: Absolute value of the estimate of the Wigner distribution of x, $|\widehat{W}_x(t,\omega)|$.

and x_6 are its corresponding time derivative of the mean membrane potential; the input u models the aggregated input from surrounding neuron populations and is modelled by a uniformly distributed signal between 120 and 320 Hz, the measurement y is the electroencephalogram (EEG) and d is the measurement noise, which we model to be a Gaussian signal with mean 0 and variance 0.01^2. The nonlinear model is as follows

$$\begin{aligned} \dot{x} &= A_J x + \phi(x,u), \\ y &= Cx + d, \end{aligned} \tag{9}$$

where

$$\begin{aligned} A_J &= \begin{pmatrix} 0_{3\times 3} & \mathbb{I}_3 \\ -\mathrm{diag}(a^2,a^2,b^2) & -\mathrm{diag}(2a,2a,2b) \end{pmatrix}, \\ C &= \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}, \\ \phi(x,u) &= \begin{pmatrix} 0_{3\times 1} \\ \theta_A a S(y) \\ \theta_A a c_2 S(c_1 x_1) \\ \theta_B b c_4 S(c_3 x_1) \end{pmatrix}, \\ S(v) &:= 2e_0/(1+e^{r(v_0-v)}). \end{aligned}$$

The known parameters are $a = 100$, $b = 50$, $\theta_A = 3.25$, $\theta_B = 22$, $c_1 = 135$, $c_2 = 108$, $c_3 = 33.75$, $c_4 = 33.75$, $e_0 = 2.5$, $r = 0.56$, $v_0 = 6$.

The neural model (9) is linearised about its equilibrium point $x^* = (x_1^*, x_2^*, x_3^*, 0, 0, 0)$, where

$$\begin{aligned} x_1^* &= \theta_A a^{-1} S(y^*), \\ x_2^* &= \theta_A a^{-1}\left(u^* + c_2 S\left(c_1 \theta_A a^{-1} S(y^*)\right)\right), \\ x_3^* &= \theta_B b^{-1} c_4 S\left(c_3 \theta_A a^{-1} S(y^*)\right), \end{aligned}$$

y^* and u^* satisfy

$$\begin{aligned} y^* =& \theta_A a^{-1} u^* + \theta_A a^{-1} c_2 S\left(\theta_A a^{-1} c_1 S(y^*)\right) \\ & - \theta_B b^{-1} c_4 S\left(\theta_A a^{-1} c_3 S(y^*)\right). \end{aligned}$$

The resulting linearised system is in the form of (1) with

$$\begin{aligned} A &= \begin{pmatrix} 0_{3\times 3} & \mathbb{I}_3 \\ \mathrm{diag}(2a,2a,2b)M(x^*) & -\mathrm{diag}(2a,2a,2b) \end{pmatrix}, \\ M(x^*) &= \frac{1}{2}\begin{pmatrix} -a & m_2(x_2^*,x_3^*) & -m_2(x_2^*,x_3^*) \\ m_1(x_1^*) & -a & 0 \\ m_3(x_1^*) & 0 & -b \end{pmatrix}, \end{aligned}$$

$m_1(x_1^*) = \theta_A c_1 c_2 S'(c_1 x_1^*)$, $m_2(x_2^*, x_3^*) = \theta_A S'(x_2^* - x_3^*)$, $m_3(x_1^*) = \theta_B c_3 c_4 S'(c_3 x_1^*)$,

$B = (0,0,0,1,0,0)$ and C is as defined for system (9), where $S'(v)$ denotes the derivative of the sigmoid function S.

Before applying the setup described in Section IV, we verify that the detectability condition stated in Theorem 1 is satisfied by applying Proposition 1 (the resulting matrix $F(\omega)$ in (4) is Hurwitz since A_J is is Hurwitz, as well as checking computationally that $F(0)$ is Hurwitz). Hence, the conditions of Theorem 1 are fulfilled. Our set of frequencies of interest is $\Omega = 2\pi \cdot \{0, 0.1, 0.2, \ldots, 2\}$. We initialise the estimators (5) at $\hat{z}(0,\omega) = 0_{2\cdot 6^2 \times 1}$ for all $\omega \in \Omega$. Figures 5 and 6 show the norm of the estimation error converges to a neighbourhood of 0 where the size of the neighbourhood is dictated by the measurement noise d, as guaranteed by Theorem 1.

VI. Conclusion

Our estimation algorithm for the Wigner distribution of the unmeasured states of a LTI system involves the design of Luenberger-like estimators for each frequency of interest. We proved that the estimates converge to their true values in the absence of measurement noise and to a neighbourhood of

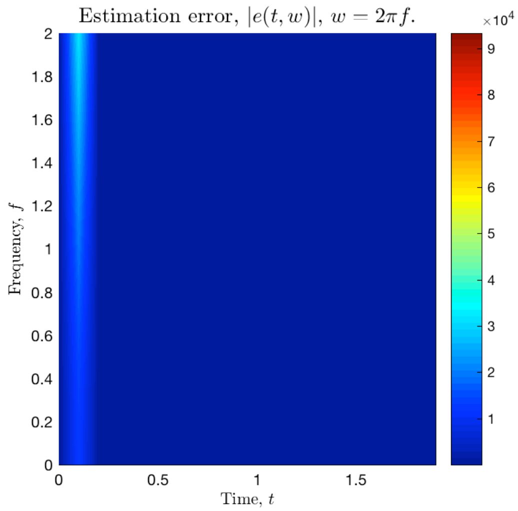

Fig. 5. Neural model: Norm of estimation error, $|e(t,\omega)|$ converges to a neighbourhood around 0 as $t \to \infty$ for all $\omega \in 2\pi \cdot [0,2]$.

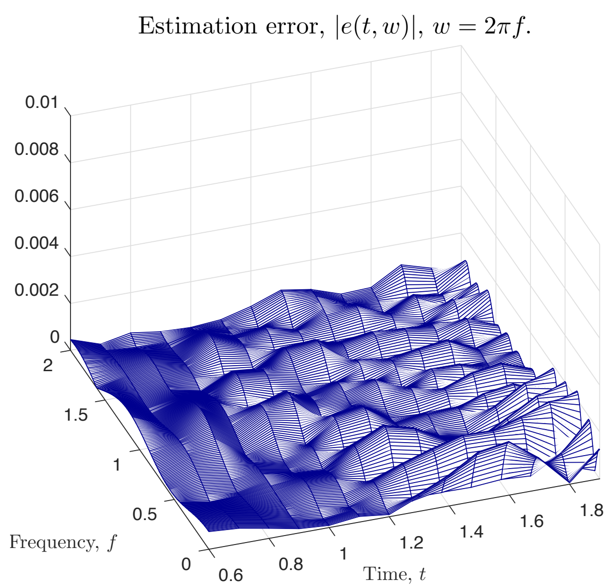

Fig. 6. Neural model: Norm of estimation error, $|e(t,\omega)|$ for $t \in [0.6, 1.9]$ and $\omega \in 2\pi \cdot [0,2]$.

the true value in the presence of bounded measurement noise. The size of the neighbourhood is tunable by the user and is dependent on the norm of the Wigner distribution of the measurement noise and its cross-terms with the unmeasured states.

Appendix

A. Proof of Proposition 1

Suppose A is diagonalisable without loss of generality, then A^T is also diagonalisable, i.e. there exists an invertible matrix S such that $A^T = SDS^{-1}$, where $D = \text{diag}(\lambda_1, \ldots, \lambda_{n_x})$ with λ_k, $k \in \{1, \ldots, n_x\}$ as the eigenvalues of A^T. With the following change of coordinates $\xi_j = S^T z_j S$, $j \in \{1,2\}$, we obtain the following system

$$\begin{aligned} \frac{\partial \xi_1}{\partial t} &= \xi_2 - 2\xi_1(i\omega\mathbb{I}_{n_x} - D) - 2(-i\omega\mathbb{I}_{n_x} - D)\xi_1 \\ \frac{\partial \xi_2}{\partial t} &= -4(-i\omega\mathbb{I}_{n_x} - D)\xi_1(i\omega\mathbb{I}_{n_x} - D) \\ &\quad +4S^T B W_u B^T S, \end{aligned}$$

by noting that $S^{-1}A_\omega S = i\omega\mathbb{I}_{n_x} - D$ and $S^T A_\omega^* S = -i\omega\mathbb{I}_{n_x} - D$. With $W_u = 0$, checking the stability of system (3) is equivalent to checking the stability of (10). Thus, we further transform system (10) according to the transformation performed in Section IV. Let $\tilde{\xi} = \left(\tilde{\xi}_1, \tilde{\xi}_2\right)$, $\tilde{\xi}_j := \text{vec}\,(\xi_j)$, $j \in \{1,2\}$ and $\tilde{u} := \text{vec}(W_u)$ to obtain

$$\frac{\partial \tilde{\xi}(t,\omega)}{\partial t} = \tilde{F}(\omega)\tilde{\xi}(t,\omega) + \tilde{G}\tilde{u}(t,\omega), \tag{10}$$

where $\tilde{F}(\omega) = \begin{bmatrix} F_1(\omega, D) & \mathbb{I}_{n_x^2} \\ F_2(\omega, D) & 0_{n_x^2 \times n_x^2} \end{bmatrix}$, $F_1(\omega, D) := -2(-i\omega\mathbb{I}_{n_x} - D) \otimes \mathbb{I}_{n_x} - \mathbb{I}_{n_x} \otimes 2(i\omega\mathbb{I}_{n_x} - D)$, $F_2(\omega, D) := -4(-i\omega\mathbb{I}_{n_x} - D) \otimes (i\omega\mathbb{I}_{n_x} - D)$ and $\tilde{G} = \begin{bmatrix} 0_{n_x^2 \times n_u^2} \\ 4S^T B \otimes BS \end{bmatrix}$.

Hence, we see that checking the stability of system (10) is equivalent to checking that $\tilde{F}(\omega)$ is Hurwitz. Due to $\tilde{F}(\omega)$ having block matrices $F_1(\omega, D)$ and $F_2(\omega, D)$ which are diagonal, the eigenvalues of $\tilde{F}(\omega)$ are generated by the roots of the following polynomial

$$s^2 - f_{1k}s - f_{2k} = 0, \; k \in \{1, \ldots, n_x\}, \tag{11}$$

where f_{1k} and f_{2k}, are the main diagonal entries of $F_1(\omega, D)$ and $F_2(\omega, D)$ respectively, $\text{Re}(f)$ and $\text{Im}(f)$ denotes the real and imaginary parts of f, respectively. Since we are concerned with the stability of the matrix $\tilde{F}(\omega)$, we require the real parts of the eigenvalues of $\tilde{F}(w)$ to be strictly negative, which are given by $\text{Re}(s) \in \{\text{Re}(\lambda_l) + \text{Re}(\lambda_j) \pm \sqrt{\Delta(\omega)} : l, j \in \{1, \ldots, n_x\}\}$, where $\Delta(\omega) = \text{Re}(\lambda_l + \lambda_j)^2 - 4\text{Im}(\lambda_l + \lambda_j)^2 - 16(\omega^2 + \text{Re}(\lambda_l\lambda_j)) \geq 0$.

Since $\text{Re}(\lambda_j) < 0$ for all $j \in \{1, \ldots, n_x\}$ (A is Hurwitz by assumption), the matrix $\tilde{F}(\omega)$ is Hurwitz if and only if

$$\sqrt{\Delta(\omega)} < -(\text{Re}(\lambda_l) + \text{Re}(\lambda_j)), \; \forall l, j \in \{1, \ldots, n_x\}. \tag{12}$$

We complete the proof by observing that condition (12) is satisfied for all $\omega \in \mathbb{R}$ if and only if (12) is satisfied for $\omega = 0$.

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