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On the construction of Jordan chains in the eigenstructure assignment for output-nulling subspaces

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Abstract— **This paper investigates several aspects related with the eigenstructure assignment problem for output-nulling subspaces. In particular, we deliver an alternative method for the computation of the feedback matrix that renders these subspaces invariant with respect to the closed-loop system matrix and assigns a defective eigenstructure in the closed-loop eigenstructure restricted to these subspaces. We show that this method, which consists in building the Jordan chains starting from the generalized eigenspace to the null-space of the Rosenbrock matrix, provides additional freedom in the resulting basis matrix.**

I. INTRODUCTION

A fundamental building block of the so-called geometric approach to control theory [1], [13], [14] is the so-called output-nulling subspace, which is a particular kind of controlled invariant subspace where the state trajectory can be maintained with a corresponding output that is identically equal to zero. The three most important output-nulling subspaces used in virtually all control problems for which a geometric solution is available (including disturbance decoupling, model matching and noninteracting control problems) are the following:

- $\mathcal{V}^\star$, which represents the initial states of an LTI system for which there exists a control function that maintains the output identically at zero;
- $\mathcal{R}^\star$, which is the set of states that are reachable from the origin by means of a suitable control function that simultaneously maintains the output at zero;
- $\mathcal{V}_g^\star$, which represents the initial states for which there exists a control function that maintains the output identically at zero and at the same time ensures that the state of the system converges asymptotically to the origin.

Since all these subspaces are controlled invariant for the system, the control functions that achieve the corresponding defining properties can always be expressed in terms of a static state feedback $u = Fx$, where the matrix F is usually referred to as a *friend* of the associated output-nulling subspace. The choice of the friend F allows some freedom in the assignment of the closed-loop spectrum.

The classical methodologies for the calculation of $\mathcal{V}^\star$ and $\mathcal{R}^\star$ are based on sequences of subspaces of the state space, which converge in a finite number of iterations to the desired output nulling subspace. The feedback matrices that render these subspaces invariant for the closed loop are typically computed separately from the computation of a basis for the subspaces, by using changes of coordinate which isolate the assignable part of the spectrum.

On the contrary, the procedure given in [5, Proposition 4] to address the same problem does not use a sequence of subspaces, but hinges on the calculation of null-spaces of the system matrix pencil evaluated at the frequencies that one wants to assign in the free part of the spectrum restricted to this subspace. This algorithm delivers, as a by-product, the friend which guarantees the desired eigenstructure properties of the closed loop.

In [10], a parametric form was derived for all the friends that assign any desired inner and outer closed-loop free spectrum using the method of [5]. The consequent degrees of freedom were exploited to derive an algorithm that allows to obtain a friend which places the desired assignable closed-loop eigenvalues at arbitrary locations and also minimizes the Frobenius condition number of the matrix of closed-loop eigenvectors, which is a commonly used robustness measure, or that minimizes the Frobenius norm of the friend itself.

It was also shown in [10] that the method based on the parametric form derived from [5] dramatically improves eigenvalue insensitivity to parameter uncertainties, with significantly smaller gain and vastly improved accuracy (by some orders of magnitude) than the friends obtained from the only other two publicly available MATLAB® toolboxes *GA* and *Linear Systems Toolkit* of [1] and [2], respectively.

The result of [5], which was given under a set of unnecessarily stringent assumptions, has been recently revisited in [9], where it was proved under virtually no hypotheses on the system data. The method itself has been extended in [9] to also tackle possible defective closed-loop eigenstructures. In this case, the calculation of the null-spaces of the Rosenbrock matrix pencil at the desired frequencies is not sufficient to span all the linearly independent directions of the state space which are needed for the computation of a basis for the output nulling and of the corresponding friend. In [9], a sequence that parallels the classical one for the calculation of the generalized eigenspace in the Jordan canonical decomposition of a matrix/endomorphism was established, and an explicit method was offered for the construction of the Jordan chains. This method consists in choosing directions from the null-space of the Rosenbrock matrix and using essentially the sequence of subspaces mentioned above to construct the entire chain of vectors. This generalization of [5, Proposition 4] to the defective case yields an algorithm for the computation of a friend of $\mathcal{R}^\star$ (and, considering also the contribution of the invariant zeros, of $\mathcal{V}^\star$ and $\mathcal{V}_g^\star$) which

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assigns the eigenvalues of the closed-loop matrix restricted to $\mathcal{R}^\star$ with any desired multiplicity and any admissible Jordan structure.

In this paper we introduce a different method for the construction of the chains, which consists in choosing the initial vector of each Jordan chain from the counterpart of the generalized eigenspace (i.e., the last subspace of the sequence mentioned above), and builds all the other vectors simultaneously. The method presented in this paper has the important feature that the resulting basis is constructed in such a way that the initial vector is, loosely speaking, arbitrary on the output nulling subspace at hand, modulo choices on a set of zero Lebesgue measure. In other words, it is possible to solve problems, which appear to be useful in the solution of noninteracting and input-output decoupling problems where it is required to compute the smallest output nulling subspace containing a specific direction on $\mathcal{R}^\star$.
The same approach is adapted to the computation of a basis for $\mathcal{V}^\star$. In this case the existing techniques do not allow to build a friend that deliver the invariant zero structure in the defective case in a recursive fashion. On the contrary, we propose a new approach based on a different subspace recursion which ensure that the problem can be solved iteratively, providing, as a by-product, the associated feedback matrix. Our method also generates a sequence of indices which contain the full information on the eigenstructure of the invariant zeros of the system. In other words, unlike the other approaches available in the literature, no *a priori* knowledge on the multiplicities of the invariant zeros is necessary in order to build the Jordan chains.

Notation. For convenience, a linear mapping between finite-dimensional spaces and a matrix representation with respect to a particular basis are not distinguished notationally. The image and the kernel of matrix A are denoted by $\operatorname{im} A$ and $\ker A$, respectively. The Moore-Penrose pseudo-inverse of A is denoted by $A^\dagger$. The spectrum of a square matrix A, denoted by $\sigma(A)$, is the multi-set of the eigenvalues of A counting multiplicities. Given a linear map $A : \mathcal{X} \longrightarrow \mathcal{Y}$ and a subspace $\mathcal{S}$ of $\mathcal{Y}$, the symbol $A^{-1}\mathcal{S}$ stands for the inverse image of $\mathcal{S}$ with respect to the linear map A, i.e., $A^{-1}\mathcal{S} = \{x \in \mathcal{X} \,|\, Ax \in \mathcal{S}\}$. If $\mathcal{J} \subseteq \mathcal{X}$, the restriction of the map A to $\mathcal{J}$ is denoted by $A|\mathcal{J}$. If $\mathcal{X} = \mathcal{Y}$ and $\mathcal{J}$ is A-invariant, the eigenvalues of A restricted to $\mathcal{J}$ are denoted by $\sigma(A|\mathcal{J})$. If $\mathcal{J}_1$ and $\mathcal{J}_2$ are A-invariant subspaces and $\mathcal{J}_1 \subseteq \mathcal{J}_2$, the mapping induced by A on the quotient space $\mathcal{J}_2/\mathcal{J}_1$ is denoted by $A|\mathcal{J}_2/\mathcal{J}_1$, and its spectrum is denoted by $\sigma(A|\mathcal{J}_2/\mathcal{J}_1)$. The symbol $\oplus$ stands for the direct sum of subspaces. The symbol α^* denotes the complex conjugate of $\alpha \in \mathbb{C}$.

II. GEOMETRIC PRELIMINARIES

In what follows, whether the underlying system evolves in continuous or discrete time is irrelevant. Accordingly, we denote by $\mathbb{T}$ the time index set of any signal, on the understanding that this represents either $\mathbb{R}^+$ in the continuous time or $\mathbb{N}$ in the discrete time. The symbol $\mathbb{C}_g$ denotes either the open left-half complex plane or the open unit disc in the continuous or discrete time, respectively. Consider a linear time-invariant system Σ governed by

$$\Sigma : \begin{cases} \rho\, x(t) = A\, x(t) + B\, u(t), & x(0) = x_0 \\ y(t) = C\, x(t) + D\, u(t), \end{cases} \tag{1}$$

where, for all $t \in \mathbb{T}$, $x(t) \in \mathcal{X} = \mathbb{R}^n$ is the state, $u(t) \in \mathcal{U} = \mathbb{R}^m$ is the control input, $y(t) \in \mathcal{Y} = \mathbb{R}^p$ is the output, and A, B, C and D are appropriate dimensional constant real-valued matrices. The operator ρ denotes either the time derivative in the continuous time, i.e., $\rho x(t) = \dot{x}(t)$, or the unit time shift in the discrete time, i.e., $\rho x(t) = x(t+1)$. Let the system Σ described by (1) be identified with the quadruple (A, B, C, D). We define the Rosenbrock system matrix pencil in the indeterminate $\lambda \in \mathbb{C}$ as

$$P_\Sigma(\lambda) \stackrel{\text{def}}{=} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}, \tag{2}$$

[11]. The invariant zero structure of the system in (1) is given by the zeros, multiplicity included, of the greatest common divisor of the minors of order $n + \min\{m, p\}$ of $P_\Sigma(\lambda)$, see [3]. Given an invariant zero $\lambda = z \in \mathbb{C}$, the rank deficiency of $P_\Sigma(\lambda)$ at $\lambda = z$, with respect to its normal rank, is the geometric multiplicity of the invariant zero z, and is equal to the number of elementary divisors of $P_\Sigma(\lambda)$ associated with $\lambda = z$. The degree of the product of the elementary divisors of $P_\Sigma(\lambda)$ corresponding to the invariant zero z is the algebraic multiplicity of z.

If the pair (A, B) is reachable, there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that all the eigenvalues of $A + BF$ are assignable along with their multiplicities. However, there is no complete freedom in assigning the Jordan structure of $A + BF$. The problem of determining the set of all possible Jordan structures that are obtainable by state feedback is completely solved by the so-called Rosenbrock Theorem [11, Thm 4.2, p. 190].

We now recall some concepts of classical geometric control theory that will be used in the sequel. More details can be found e.g. in [13]. A *controlled invariant subspace* $\mathcal{V}$ of Σ is a subspace of $\mathcal{X}$ satisfying $A\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$. An *output-nulling subspace* $\mathcal{V}$ of Σ is a controlled invariant subspace which satisfies the inclusion $\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V} \oplus 0_p) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix}$, which is equivalent to the existence of a feedback matrix $F \in \mathbb{R}^{m\times n}$ such that $(A + BF)\mathcal{V} \subseteq \mathcal{V} \subseteq \ker(C + DF)$. Any matrix F satisfying these two inclusions is referred to as a *friend* of $\mathcal{V}$. We denote by $\mathfrak{F}(\mathcal{V})$ the set of friends of $\mathcal{V}$. The set of output-nulling subspaces of Σ is closed under subspace addition. We denote by $\mathcal{V}^\star$ the largest output-nulling subspace of Σ.

The so-called *output-nulling reachability subspace* $\mathcal{R}_\mathcal{V}$ on an output-nulling subspace $\mathcal{V}$ is the smallest $(A + BF)$-invariant subspace of $\mathcal{X}$ containing the subspace $\mathcal{V} \cap B \ker D$, where $F \in \mathfrak{F}(\mathcal{V})$. Let $\mathcal{R}^\star \stackrel{\text{def}}{=} \mathcal{R}_{\mathcal{V}^\star}$: this subspace represents the points of the state-space that are reachable from the origin with trajectories that correspond to identically zero output.

Consider an output-nulling subspace $\mathcal{V}$ of Σ and denote by $\mathcal{R}_\mathcal{V}$ the output-nulling reachable subspace on $\mathcal{V}$. Let

$F \in \mathfrak{F}(\mathcal{V})$. The eigenstructure $\sigma(A + BF\,|\,\mathcal{V})$ can be split into the two parts $\sigma(A + BF|\mathcal{R}_v)$ and $\sigma(A + BF|\mathcal{V}/\mathcal{R}_v)$: the eigenvalues in $\sigma(A + BF|\mathcal{R}_v)$ and their multiplicities are all freely assignable[1] with a suitable choice of $F \in \mathfrak{F}(\mathcal{V})$, whereas $\Gamma_i(\mathcal{V}) \stackrel{\text{def}}{=} \sigma(A + BF|\mathcal{V}/\mathcal{R}_v)$, is fixed for all the choices of $F \in \mathfrak{F}(\mathcal{V})$. If all the eigenvalues of $\Gamma_i(\mathcal{V})$ are in $\mathbb{C}_g$, then $\mathcal{V}$ is said to be *inner stabilizable*. The eigenstructure $\mathcal{Z} \stackrel{\text{def}}{=} \Gamma_i(\mathcal{V}^\star)$ is the invariant zero structure of Σ. Similarly, recalling that we denoted by $\mathcal{R}$ the reachable subspace of the pair (A, B), the eigenstructure $\sigma(A + BF|\mathcal{X}/\mathcal{V})$ can be split into the two parts $\sigma(A + BF|\mathcal{V} + \mathcal{R}/\mathcal{V})$ and $\sigma(A + BF|\mathcal{X}/\mathcal{V} + \mathcal{R})$: the eigenvalues in $\sigma(A + BF|\mathcal{V} + \mathcal{R}/\mathcal{V})$ are all freely assignable along with their multiplicity with a suitable choice of F in $\mathfrak{F}(\mathcal{V})$, whereas the eigenstructure $\Gamma_o(\mathcal{V}) \stackrel{\text{def}}{=} \sigma(A + BF|\mathcal{X}/\mathcal{V} + \mathcal{R})$ is fixed, [12]. If $\Gamma_o(\mathcal{V}) \subset \mathbb{C}_g$, then $\mathcal{V}$ is said to be *outer stabilizable*. As for the case of pole assignment with state feedback for a reachable pair (A, B), while it is true that the eigenvalues of $\sigma(A + BF|\mathcal{R}_v)$ and $\sigma(A + BF|\mathcal{V} + \mathcal{R}_0/\mathcal{V})$ are completely assignable along with their multiplicities by a suitable friend $F \in \mathfrak{F}(\mathcal{V})$, their Jordan structure is not completely arbitrary, but it is constrained by two independent sets of Kronecker invariants corresponding to two suitably defined pairs of matrices that we now characterize. The following result, which is a more detailed version of [13, Theorem 4.18], holds.

Theorem 1: Consider an output-nulling subspace $\mathcal{V}$ and an associated friend $F^\circ \in \mathfrak{F}(\mathcal{V})$.[2] Consider an $n \times n$ non-singular matrix $T = [\ T_1 \quad T_2 \quad T_3 \quad T_4\ ]$ such that **(i)** T_1 is a basis matrix for $\mathcal{R}_v$; **(ii)** $[\ T_1 \quad T_2\ ]$ is a basis matrix for $\mathcal{V}$; **(iii)** $[\ T_1 \quad T_2 \quad T_3\ ]$ is a basis matrix for $\mathcal{V} + \mathcal{R}$. Moreover, consider an $m \times m$ non-singular matrix $\Omega = [\ \Omega_1 \quad \Omega_2\ ]$ such that Ω_1 is a basis matrix for $B^{-1}\mathcal{V} \cap \ker D$. Then, we have

$$\tilde{A}^\circ \stackrel{\text{def}}{=} T^{-1}(A + BF^\circ)T = \begin{bmatrix} A^\circ_{1,1} & A^\circ_{1,2} & A^\circ_{1,3} & A^\circ_{1,4} \\ 0 & A^\circ_{2,2} & A^\circ_{2,3} & A^\circ_{2,4} \\ 0 & 0 & A^\circ_{3,3} & A^\circ_{3,4} \\ 0 & 0 & 0 & A^\circ_{4,4} \end{bmatrix}, \quad (3)$$

$$\tilde{B} \stackrel{\text{def}}{=} T^{-1}B\Omega = \begin{bmatrix} B_{1,1} & B_{1,2} \\ 0 & B_{2,2} \\ 0 & B_{3,2} \\ 0 & 0 \end{bmatrix}, \quad (4)$$

where the pairs $(A^\circ_{1,1}, B_{1,1})$ and $(A^\circ_{3,3}, B_{3,2})$ are reachable. Moreover, the set of all friends of $\mathcal{V}$ are given by

$$F \stackrel{\text{def}}{=} F^\circ + \Omega\tilde{F}T^{-1}, \quad \text{with} \quad \tilde{F} \stackrel{\text{def}}{=} \begin{bmatrix} F_{1,1} & \star & \star & \star \\ 0 & 0 & F_{2,3} & \star \end{bmatrix}, \quad (5)$$

where $F_{1,1}$, $F_{2,3}$, as well as the entries indicated with $\star$, are arbitrary.

The structure of F in (5) shows that the eigenstructures of $A^\circ_{2,2}$ and $A^\circ_{4,4}$ are fixed for any friend of $\mathcal{V}$. Clearly, if $\mathcal{V}$ is the subspace $\mathcal{V}^\star$, then $\mathcal{Z} = \sigma(A^\circ_{2,2})$. Furthermore, the set of all possible Jordan structures that are obtainable for $\sigma(A + BF|\mathcal{R}_v)$ are those, and only those, which satisfy the conditions of the Rosenbrock Theorem for the pair $(A^\circ_{1,1}, B_{1,1})$. Likewise, the set of all possible Jordan structures that are obtainable for $\sigma(A + BF|\mathcal{V} + \mathcal{R}/\mathcal{V})$ are those, and only those, which satisfy the conditions of the Rosenbrock Theorem for the pair $(A^\circ_{3,3}, B_{3,2})$.

[1] An assignable set of eigenvalues here is always intended to be a set of complex numbers which is mirrored with respect to the real axis.

[2] As detailed earlier, F° can be obtained for example as $F^\circ = -UV^\dagger$, where U is obtained from $\begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} V & B \\ 0 & D \end{bmatrix}^\dagger \begin{bmatrix} A \\ C \end{bmatrix} V$.

III. MAIN RESULT

In this section, we consider the problem of obtaining basis matrices for the subspaces $\mathcal{R}^\star$ and $\mathcal{V}^\star$. The result presented in the following theorem generalizes [5, Proposition 4] in several directions: **1)** the system is not necessarily strictly proper; **2)** the number of values in the set $\mathcal{L}$ of desired closed-loop eigenvalues of $A+BF\,|\,\mathcal{R}^\star$ that we wish to assign does not need to be greater or equal to the dimension of $\mathcal{R}^\star$; **3)** it is not necessary to assume that no member of $\mathcal{L}$ has an invariant zero as its real part; **4)** we do not need to assume that $\begin{bmatrix} B \\ D \end{bmatrix}$ is of full column-rank. In the next section, we will show how to also weaken the assumption that the eigenvalues are non-defective.

Theorem 2: Let $r = \dim \mathcal{R}^\star$. Let $\mathcal{L} = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ be an s-conformably ordered set of self-conjugate distinct complex numbers disjoint from the invariant zeros, i.e., $\mathcal{L} \cap \mathcal{Z} = \emptyset$. Let for all $k \in \{1, \ldots, r\}$ denote by $\begin{bmatrix} X_k \\ Y_k \end{bmatrix}$ a basis matrix for $\ker P_\Sigma(\lambda_k)$ partitioned conformably with $P_\Sigma(\lambda_k)$. Let this basis be chosen in such a way that $\begin{bmatrix} X_{k+1} \\ Y_{k+1} \end{bmatrix} = \begin{bmatrix} X_k^* \\ Y_k^* \end{bmatrix}$ when $k \leq 2s$ is odd. Let

$$V_k \stackrel{\text{def}}{=} \begin{cases} \Re\mathrm{e}\{X_k\} & k \leq 2s \text{ odd} \\ \Im\mathrm{m}\{X_k\} & k \leq 2s \text{ even} \\ X_k & \text{if } k > 2s, \end{cases} \quad W_k \stackrel{\text{def}}{=} \begin{cases} \Re\mathrm{e}\{Y_k\} & k \leq 2s \text{ odd} \\ \Im\mathrm{m}\{Y_k\} & k \leq 2s \text{ even} \\ Y_k & \text{if } k > 2s. \end{cases}$$

Then, $\mathcal{R}^\star = \mathrm{im}[\ V_1 \quad V_2 \quad \ldots \quad V_r\ ]$.

The following result shows that the characterization of $\mathcal{R}^\star$ given in Theorem 2 can be used as an algorithm that determines a basis for $\mathcal{R}^\star$ and, simultaneously, a friend of $\mathcal{R}^\star$ that delivers a certain desired closed-loop eigenstructure.

Corollary 1: Consider a spanning matrix for $\mathcal{R}^\star$ as constructed in Theorem 2, i.e., $\mathcal{R}^\star = \mathrm{im}[\ V_1 \quad V_2 \quad \ldots \quad V_r\ ]$. Let $\{v_1, v_2, \ldots, v_r\}$ be a set of columns extracted from $[\ V_1 \quad V_2 \quad \ldots \quad V_r\ ]$ to form a basis for $\mathcal{R}^\star$, and let $\{w_1, w_2, \ldots, w_r\}$ denote the corresponding columns of $[\ W_1 \quad W_2 \quad \ldots \quad W_r\ ]$ constructed as in Theorem 2. If v_k is a column of V_j, let us denote by μ_k the eigenvalue λ_j. Let $\{v_1, v_2, \ldots, v_r\}$ be constructed in such a way that the multi-set $\{\mu_1, \mu_2, \ldots, \mu_r\}$ is self-conjugate. Then,

$$F = [\ w_1 \quad w_2 \quad \ldots \quad w_r\ ][\ v_1 \quad v_2 \quad \ldots \quad v_r\ ]^\dagger \quad (6)$$

is an output-nulling friend of $\mathcal{R}^\star$, and $\sigma(A + BF\,|\,\mathcal{R}^\star) = \{\mu_1, \ldots, \mu_r\}$.

IV. THE DEFECTIVE CASE

In this section, we develop a method to extend [5, Proposition 4] to the case of repeated eigenvalues. In particular, we show how to deal with the case when the multiplicity is such that the closed-loop matrix is defective i.e., it has a non-diagonal Jordan structure.

A. Computation of $\mathcal{R}^\star$

The following theorem shows how a basis matrix can be computed for $\mathcal{R}^\star$ using a sequence of subspaces, see [9] for a proof.

Lemma 1: Let $r = \dim \mathcal{R}^\star$. Let $\lambda \in \mathbb{R} \setminus \mathcal{Z}$. Denote by $\begin{bmatrix} V_1 \\ W_1 \end{bmatrix}$ a basis matrix for $\ker P_\Sigma(\lambda)$. Consider the sequence of matrices V_k and W_k defined by the recursion

$$\begin{bmatrix} V_k \\ W_k \end{bmatrix} = \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}^\dagger \begin{bmatrix} V_{k-1} \\ 0 \end{bmatrix} + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} K, \quad k \geq 2, \qquad (7)$$

where the columns of $\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$ form a basis matrix for $\ker P_\Sigma(\lambda)$ and K is an arbitrary matrix of suitable size. Then, $\mathcal{R}^\star = \text{im}[\ V_1 \quad V_2 \quad \dots \quad V_r\ ]$.

We recall that when computing a Jordan chain of generalized eigenvectors for a square matrix A

$$\begin{aligned} A\,v_1 &= \lambda\, v_1 \\ A\,v_2 &= \lambda\, v_2 + v_1 \\ &\vdots \\ A\,v_h &= \lambda\, v_h + v_{h-1}, \end{aligned}$$

where h denotes the smallest index such that $\ker(A - \lambda I)^h = \ker(A - \lambda I)^{h+1}$, there are two strategies for determining a corresponding linearly independent set of generalized eigenvectors. The first consists in choosing the last element of the chain $v_h \in \ker(A - \lambda I)^h \setminus \ker(A - \lambda I)^{h-1}$; the computation of $v_{h-1} = (A - \lambda I)\, v_h$, $v_{h-2} = (A - \lambda I)\, v_{h-1}$, …, $v_1 = (A - \lambda I)\, v_2$ yields a basis for the Jordan chain starting from v_h.

Another method consists in computing the generalized eigenvectors from the eigenspace $\ker(A - \lambda I)$ to the generalized eigenspace $\ker(A - \lambda I)^h$, by selecting a vector $v_1 \in \ker(A - \lambda I)$, and then solving — for example using the Gaussian Elimination — the linear equations $(A - \lambda I)\, v_2 = v_1$, $(A - \lambda I)\, v_3 = v_2$, and so on, until we get $(A - \lambda I)\, v_h = v_{h-1}$. In this way we obtain $v_2 \in \ker(A - \lambda I)^2$, …, $v_h \in \ker(A - \lambda I)^h$.

The same two strategies can be generalized for the construction of the Jordan chains of $A + BF\,|\,\mathcal{R}^\star$, exploiting the sequence of subspaces introduced in Lemma 1. The counterpart of the second strategy can be carried out by using directly the sequence of Lemma 1, and yields Jordan chains that are arbitrarily long. Indeed, using the fact that the eigenstructure of $A + BF\,|\,\mathcal{R}^\star$ consisting of a single Jordan mini-block of size r relative to $\lambda \in \mathbb{R} \setminus \mathcal{Z}$ always satisfies the conditions of the Rosenbrock Theorem, we can obtain the feedback matrix as detailed in the following result.

Corollary 2: Let $r = \dim \mathcal{R}^\star$. Let $\lambda \in \mathbb{R} \setminus \mathcal{Z}$. Let $\begin{bmatrix} v_1 \\ w_1 \end{bmatrix} \in \ker P_\Sigma(\lambda)$. Consider the sequence

$$\begin{bmatrix} v_k \\ w_k \end{bmatrix} = \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}^\dagger \begin{bmatrix} v_{k-1} \\ 0 \end{bmatrix} + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} K, \quad k \geq 2, \qquad (8)$$

where $\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$ is a basis matrix for $\ker P_\Sigma(\lambda)$ and K is an arbitrary matrix. Then, $\mathcal{R}^\star = \text{span}\{v_1, \dots, v_r\}$. Moreover, $F = [\,w_1 \quad w_2 \quad \dots \quad w_r\,][\,v_1 \quad v_2 \quad \dots \quad v_r\,]^\dagger$ is a friend of $\mathcal{R}^\star$, and $A + BF\,|\,\mathcal{R}^\star$ is similar to a Jordan mini-block of dimension r corresponding to the eigenvalue λ.

Corollary 2 shows how a friend F can be computed for $\mathcal{R}^\star$ so that $A + BF\,|\,\mathcal{R}^\star$ is a single Jordan mini-block if we start from a single column of a basis matrix $\begin{bmatrix} V_1 \\ W_1 \end{bmatrix}$ of $\ker P_\Sigma(\lambda)$. If the rank of V_1 is s and $v_{1,1}, v_{1,2}, \dots, v_{1,s}$ are linearly independent columns of V_1 and $w_{1,1}, w_{1,2}, \dots, w_{1,s}$ are the corresponding columns of W_1, starting from each $v_{1,j}$ we can build a chain as

$$\begin{bmatrix} v_{k,j} \\ w_{k,j} \end{bmatrix} = \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}^\dagger \begin{bmatrix} v_{k-1,j} \\ 0 \end{bmatrix} + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} K, \quad k \geq 2. \qquad (9)$$

In this way, we build the chains

$$\begin{aligned} &v_{1,1} \to v_{2,1} \to v_{3,1} \to \dots \\ &v_{1,2} \to v_{2,2} \to v_{3,2} \to \dots \\ &\vdots \\ &v_{1,s} \to v_{2,s} \to v_{3,s} \to \dots, \end{aligned}$$

where all these vectors are in $\mathcal{R}^\star$. If $v_{1,1}, \dots, v_{\nu_1,1}, \dots, v_{1,h}, \dots, v_{\nu_h,h}$, with $h \leq s$, are linearly independent, then computing F as

$$\begin{aligned} F = [\,&w_{1,1} \dots w_{\nu_1,1}\,|\,\dots\,|\,w_{1,h} \dots w_{\nu_h,h}\,] \\ &\times [\,v_{1,1} \dots v_{\nu_1,1}\,|\,\dots\,|\,v_{1,h} \dots v_{\nu_h,h}\,]^\dagger \end{aligned}$$

yields another friend of $\mathcal{R}^\star$ such that $A + BF\,|\,\mathcal{R}^\star$ consists of h Jordan mini-blocks associated with the closed-loop eigenvalue λ with sizes $\nu_1, \dots, \nu_h$. It is obvious at this point that the set $\{v_{1,1}, \dots, v_{\nu_1,1}, v_{1,2}, \dots, v_{\nu_2,2}, \dots, v_{1,h}, \dots, v_{\nu_h,h}\}$ is linearly independent if and only if this is an admissible Jordan structure for the pair $(A^\circ_{1,1}, B_{1,1})$ in Theorem 1. It follows that s is the maximum number of Jordan blocks that can be associated with the same eigenvalue λ. For example, if the null-space of $\ker P_\Sigma(\lambda)$ is 1-dimensional but $\dim \mathcal{R}^\star = r > 1$, then any friend F of $\mathcal{R}^\star$ will be such that $A + BF\,|\,\mathcal{R}^\star$ is a single Jordan block of size r.

Consider the real eigenvalues $\lambda_1, \dots, \lambda_\nu$, and a Jordan structure $J = \text{diag}\{J(\lambda_1), \dots, J(\lambda_\nu)\}$, where each Jordan block $J(\lambda_i)$ is composed of g_i Jordan mini-blocks $J(\lambda_i) = \text{diag}\{J_1(\lambda_i), \dots, J_{g_i}(\lambda_i)\}$, where for all $i \in \{1, \dots, \nu\}$ and $k \in \{1, \dots, g_i\}$ the dimension of the mini-block $J_k(\lambda_i)$ is denoted by $p_{i,k}$. Thus, $p_{i,1} + \dots + p_{i,g_i}$ is equal to the algebraic multiplicity of the eigenvalue λ_i in the Jordan form. Constructing the sequence

$$\begin{aligned} &\begin{bmatrix} v_{i,k,1} \\ w_{i,k,1} \end{bmatrix} \in \ker P_\Sigma(\lambda_i) \\ &\begin{bmatrix} v_{i,k,\ell} \\ w_{i,k,\ell} \end{bmatrix} = P_\Sigma(\lambda_i)^\dagger \begin{bmatrix} v_{i,k,\ell-1} \\ 0 \end{bmatrix}, \quad \ell \in \{2, \dots, p_{i,k}\} \end{aligned}$$

for all $i \in \{1, \dots, \nu\}$ and $k \in \{1, \dots, g_i\}$, we obtain the following result (see [9] for the proof).

Theorem 3: Let $\lambda_1, \dots, \lambda_\nu$ be real and different from the invariant zeros. Then, J is an admissible Jordan structure for $A + BF\,|\,\mathcal{R}^\star$ if and only if $\{v_{i,k,\ell}\,|\,i \in \{1, \dots, \nu\}, k \in \{1, \dots, g_i\}, \ell \in \{1, \dots, p_{i,k}\}\}$ is a linearly independent set of vectors. If J is an admissible Jordan structure for $A + BF\,|\,\mathcal{R}^\star$, we have

$$\mathcal{R}^\star = \text{span}\,\{v_{i,k,\ell}\,|\,i = 1, \dots, \nu,\ k = 1, \dots, g_i,\ \ell = 1, \dots, p_{i,k}\}.$$

Moreover, any F such that $F\,v_{i,k,\ell} = w_{i,k,\ell}$ for all $i \in \{1,\dots,\nu\}, k \in \{1,\dots,g_i\}, \ell \in \{1,\dots,p_{i,k}\}$ is such that $A + B\,F\,|\,\mathcal{R}^\star$ is similar to J.

We now consider the counterpart of the first strategy for the construction of the Jordan chains of $A + B\,F\,|\,\mathcal{R}^\star$. In this case, instead of starting from arbitrary vectors of $\ker P_\Sigma(\lambda)$, we begin from arbitrary vectors of the largest generalized eigenspace obtained in Lemma 1. More precisely, consider the sequence of subspaces in (7) of Lemma 1. We recall that $\mathcal{R}^\star = \mathrm{im}[\ V_1 \quad V_2 \quad \dots \quad V_r\ ]$. On the other hand, we can define by η the smallest index for which $\mathcal{R}^\star = \mathrm{im}[\ V_1 \quad V_2 \quad \dots \quad V_\eta\ ]$. In other words, we know that it is always possible for this sequence of subspaces to generate the entire subspace $\mathcal{R}^\star$ by iterating it $r = \dim \mathcal{R}^\star$ times, but it may very well be possible to obtain $\mathcal{R}^\star$ with a smaller number of iterations. We define the subspaces

$$\begin{aligned}
\mathcal{S}_1 &= \mathrm{im}\, V_1 \\
\mathcal{S}_2 &= \mathrm{im}[\ V_1 \quad V_2\ ] \\
&\vdots \\
\mathcal{S}_\eta &= \mathrm{im}[\ V_1 \quad V_2 \quad \dots \quad V_\eta\ ].
\end{aligned}$$

By definition of η we have $\mathcal{S}_\eta = \mathcal{R}^\star$. Let us now consider an arbitrary vector $v \in \mathcal{S}_\eta \setminus \mathcal{S}_{\eta-1}$. In view of Lemma 1, there exists $w_\eta \in \mathbb{R}^m$ such that

$$\begin{aligned}
(A - \lambda\, I)\, v + B\, w_\eta &= v_{\eta-1} \\
C\, v + D\, w_\eta &= 0
\end{aligned}$$

where $v_{\eta-1} \in \mathcal{S}_{\eta-1}$. Hence, applying again Lemma 1 we find that there exists $w_{\eta-1} \in \mathbb{R}^m$ such that

$$\begin{aligned}
(A - \lambda\, I)\, v_{\eta-1} + B\, w_{\eta-1} &= v_{\eta-2} \\
C\, v_{\eta-1} + D\, w_{\eta-1} &= 0
\end{aligned}$$

where $v_{\eta-2} \in \mathcal{S}_{\eta-2}$. Continuing in this fashion, Lemma 1 guarantees the existence of vectors $v_1 \in \mathcal{S}_1$ and $w_1 \in \mathbb{R}^m$ such that

$$\begin{aligned}
(A - \lambda\, I)\, v_1 + B\, w_1 &= 0 \\
C\, v_1 + D\, w_1 &= 0
\end{aligned}$$

These equations can be re-written as

$$\begin{bmatrix}
A-\lambda I & B & 0 & 0 & \dots & 0 & 0 & 0 \\
C & D & 0 & 0 & \dots & 0 & 0 & 0 \\
-I & 0 & A-\lambda I & B & \dots & 0 & 0 & 0 \\
0 & 0 & C & D & \dots & 0 & 0 & 0 \\
0 & 0 & -I & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & A-\lambda I & B & 0 \\
0 & 0 & 0 & 0 & \dots & C & D & 0 \\
0 & 0 & 0 & 0 & \dots & -I & 0 & B \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & D
\end{bmatrix}
\begin{bmatrix}
v_1 \\ w_1 \\ v_2 \\ w_2 \\ \vdots \\ v_{\delta-1} \\ w_{\delta-1} \\ w_\delta
\end{bmatrix}
= -
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ A-\lambda I \\ C
\end{bmatrix} v, \qquad (10)$$

by choosing $\delta = \eta$. The previous equation always admits solutions if $v \in \mathcal{S}_\eta$, and choosing $v \in \mathcal{S}_\eta \setminus \mathcal{S}_{\eta-1}$, delivers the vectors of a Jordan chain of size η which can be obtained as the Jordan structure of $A + B\,F\,|\,\mathcal{R}_v$ using the feedback matrix

$$F = [\ w_1 \dots w_\eta\ ][\ v_1 \dots v_\eta\ ]^\dagger,$$

where $\mathcal{R}^\star \supseteq \mathcal{R}_v \stackrel{\text{def}}{=} \mathrm{im}[\ v_1 \dots v_\eta\ ]$. Note that, since $\mathcal{S}_\eta \supset \mathcal{S}_{\eta-1}$, the previous method works for almost all the choices of $v \in \mathcal{S}_\eta = \mathcal{R}^\star$.

Repeating the aforementioned procedure we can build a basis for $\mathcal{R}^\star$ and the corresponding feedback matrix that delivers the maximum achievable number of Jordan blocks associated to the same eigenvalue. Let us start by defining the following indices

$$\begin{aligned}
\gamma_1 &= \dim \mathcal{S}_1 \\
\gamma_2 &= \dim \mathcal{S}_2 - \dim \mathcal{S}_1 \\
&\vdots \\
\gamma_\eta &= \dim \mathcal{S}_\eta - \dim \mathcal{S}_{\eta-1}.
\end{aligned}$$

We always have $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_\eta$ and we can always select γ_η vectors $\{v_{\eta,1}, \dots, v_{\eta,\gamma_\eta}\} \in \mathcal{S}_\eta$ that are not in $\mathcal{S}_{\eta-1}$. These vectors can be used create γ_η Jordan chains of dimension η via (10). We could potentially select $\gamma_{\eta-1}$ vectors from $\mathcal{S}_{\eta-1}$ that are not elements of $\mathcal{S}_{\eta-2}$. However, γ_η of these vectors will be already elements of the γ_η Jordan chains built at the previous step. Therefore, we can select $\gamma_{\eta-1} - \gamma_\eta$ vectors in $\mathcal{S}_{\eta-1} \setminus \mathcal{S}_{\eta-2}$ to build the same number of Jordan chains of length $\eta - 1$ by using (10) with $\delta = \eta - 1$. It is clear that the nature of the problem is iterative. Defining the following indices

$$\begin{aligned}
\psi_\eta &= \gamma_\eta \\
\psi_{\eta-1} &= \gamma_{\eta-1} - \gamma_\eta \\
&\vdots \\
\psi_1 &= \gamma_1 - \gamma_2
\end{aligned}$$

we can build the Jordan structure on $\mathcal{R}^\star$ with the maximum number of Jordan blocks by choosing ψ_i vectors from $\mathcal{S}_i \setminus \mathcal{S}_{i-1}$ to build ψ_i Jordan chains of the form $\{v_{1,\psi_i,j}, \dots, v_{i,\psi_i,j}\}$ using (10) with $\delta = i$, for all $i \in \{1,\dots,\eta\}$ and for all $j \in \{1,\dots,\psi_i\}$. The friend of $\mathcal{R}^\star$ that delivers such Jordan structure will be

$$\begin{aligned}
F = &[w_{1,\psi_\eta,1} \dots w_{\eta,\psi_\eta,1} | \dots | w_{1,\psi_\eta,\psi_\eta} \dots v_{\eta,\psi_\eta,\psi_\eta} | \dots | w_{1,\psi_1,1} | \dots | w_{1,\psi_1,\psi_1}] \\
&\times [v_{1,\psi_\eta,1} \dots v_{\eta,\psi_\eta,1} | \dots | v_{1,\psi_\eta,\psi_\eta} \dots v_{\eta,\psi_\eta,\psi_\eta} | \dots | v_{1,\psi_1,1} | \dots | v_{1,\psi_1,\psi_1}]^\dagger.
\end{aligned}$$

The generalization of Theorem 3 to the case of complex closed-loop eigenvalues requires the introduction of a sequence

$$\begin{aligned}
&\begin{bmatrix} x_{i,k,1} \\ y_{i,k,1} \end{bmatrix} \in \ker P_\Sigma(\lambda_i) \\
&\begin{bmatrix} x_{i,k,\ell} \\ y_{i,k,\ell} \end{bmatrix} = P_\Sigma(\lambda_i)^\dagger \begin{bmatrix} x_{i,k,\ell-1} \\ 0 \end{bmatrix} \qquad \ell \in \{2,\dots,p_{i,k}\}
\end{aligned}$$

for odd $i \in \{1,\dots,2\,s\}$ and $i \in \{2\,s+1,\dots,\nu\}$, where s represents the number of complex conjugate pairs and $\{\lambda_1,\dots,\lambda_\nu\}$ is assumed to be s-conformably ordered. Computing

$$v_{i,k,h} \stackrel{\text{def}}{=} \begin{cases} \Re\mathrm{e}\{x_{i,k,h}\} & \text{if } i \leq 2\,s \text{ is odd} \\ \Im\mathrm{m}\{x_{i,k,h}\} & \text{if } i \leq 2\,s \text{ is even} \\ x_{i,k,h} & \text{if } k > 2\,s, \end{cases}$$

and

$$w_{i,k,h} \stackrel{\text{def}}{=} \begin{cases} \Re\mathrm{e}\{y_{i,k,h}\} & \text{if } i \leq 2\,s \text{ is odd} \\ \Im\mathrm{m}\{y_{i,k,h}\} & \text{if } i \leq 2\,s \text{ is even} \\ y_{i,k,h} & \text{if } i > 2\,s \end{cases}$$

for $i \in \{1,\dots,\nu\}$, $k \in \{1,\dots,g_i\}$ and $h \in \{1,\dots,p_{i,k}\}$, the statement of Theorem 3 applies directly. The procedure based on the solution of (10) can be generalized to the complex case accordingly.

B. Computation of $\mathcal{V}^\star$ and $\mathcal{V}_g^\star$

In the non-defective case, the computation of a basis for $\mathcal{V}^\star$ does not present any issue: all that is required is to include the invariant zeros in the set of values λ to use when determining a basis for $\ker P_\Sigma(\lambda)$. The upper block of a basis matrix for such null-space has to be included in the spanning set for $\mathcal{V}^\star$. The situation is much more delicate in the defective case. Indeed, while for a generic λ Lemma 1 guarantees that the recursion (7) only produces matrices whose images are contained in $\mathcal{R}^\star$, this may not be the case when λ is an invariant zero. In other words, if we use (7) when $\lambda \in \mathcal{Z}$ to compute the subspaces $\mathcal{S}_i$ we may have that $\mathcal{S}_i \not\subseteq \mathcal{V}^\star$.

For the sake of simplicity, let the invariant zeros $z_1,\dots,z_\mu$ be all real, and consider the invariant zero Jordan structure $J = \operatorname{diag}\{J(z_1), J(z_2),\dots,J(z_\mu)\}$, where each Jordan block $J(z_i)$ is composed of g_i Jordan mini-blocks $J(z_i) = \operatorname{diag}\{J_1(z_i), J_2(z_i),\dots,J_{g_i}(z_i)\}$, where for all $i \in \{1,\dots,\mu\}$ and $k \in \{1,\dots,g_i\}$ the dimension of the mini-block $J_k(z_i)$ is denoted by $p_{i,k}$. For all $i \in \{1,\dots,\mu\}$ and $k \in \{1,\dots,g_i\}$, there exists $\left[\begin{smallmatrix} v_{i,k,1} \\ w_{i,k,1}\end{smallmatrix}\right] \in \ker P_\Sigma(\lambda_i)$ such that the sequence $P_\Sigma(z_i)\left[\begin{smallmatrix} v_{i,k,\ell} \\ w_{i,k,\ell}\end{smallmatrix}\right] = \left[\begin{smallmatrix} v_{i,k,\ell-1} \\ 0\end{smallmatrix}\right]$ admits solutions for all $\ell \in \{2,\dots,p_{i,k}\}$. Constructing the sequence

$$\begin{bmatrix} v_{i,k,\ell} \\ w_{i,k,\ell}\end{bmatrix} = P_\Sigma(z_i)^\dagger \begin{bmatrix} v_{i,k,\ell-1} \\ 0\end{bmatrix}, \qquad \ell \in \{1,\dots,p_{i,k}\}$$

for all $i \in \{1,\dots,\mu\}$ and $k \in \{1,\dots,g_i\}$, the set $\{v_{i,k,\ell} \,|\, i \in \{1,\dots,\nu\}, k \in \{1,\dots,g_i\}, \ell \in \{2,\dots,p_{i,k}\}\}$ is linearly independent, and

$$\mathcal{V}^\star = \mathcal{R}^\star + \operatorname{span}\{v_{i,k,\ell} | i = 1,\dots,\nu,\ k = 1,\dots,g_i,\ \ell = 1,\dots,p_{i,k}\}.$$

Moreover, any F such that $F v_{i,k,\ell} = w_{i,k,\ell}$ for all $i \in \{1,\dots,\nu\}, k \in \{1,\dots,g_i\}, \ell \in \{1,\dots,p_{i,k}\}$ is such that $A + BF\,|\mathcal{V}^\star/\mathcal{R}^\star$ is similar to J. The main issue when building the Jordan chains associated with the invariant zeros in the defective case is to appropriately select the vectors $v_{i,k,1}$ in the eigenspace used to initiate the Jordan chains. We could build the Jordan chains by starting form the biggest generalized eigenspace and iterating backwards toward the eigenspace. Using (7), however, we obtain vectors that are not elements of $\mathcal{V}^\star$. This limitation can be overcome by redefining the iteration used to build the generalized eigenspace. Let $z_i \in \mathcal{Z}$ be an invariant zero with eigenstructure g_i, $p_{i,k}$ for all $k \in \{1,\dots,g_i\}$. We can define the following modified recursion of subspaces:

$$\begin{aligned} \mathcal{T}_1 &= \ker(P_\Sigma(z_i)) \qquad (11)\\ \mathcal{T}_k &= \mathcal{T}_{k-1} + \begin{bmatrix} A-\lambda I & B \\ C & D\end{bmatrix}^\dagger \left(\operatorname{im} P_\Sigma(z_i) \cap \begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}\mathcal{T}_{k-1}\right) \quad k \geq 2, \end{aligned}$$

Defining $\mathcal{H}_i = \begin{bmatrix} I & 0\end{bmatrix}\mathcal{T}_i$, we can stop iterating the sequence whenever $\dim \mathcal{H}_{\eta+1} = \dim \mathcal{H}_\eta$. We define the following indices

$$\begin{aligned} \gamma_1 &= \dim \mathcal{H}_1 - \dim(\mathcal{H}_1 \cap \mathcal{R}^\star)\\ \gamma_2 &= \dim \mathcal{H}_2 - \dim \mathcal{H}_1 + \dim(\mathcal{H}_1 \cap \mathcal{R}^\star) - \dim(\mathcal{H}_2 \cap \mathcal{R}^\star)\\ &\vdots\\ \gamma_i &= \dim \mathcal{H}_\eta - \dim \mathcal{H}_{\eta-1} + \dim(\mathcal{H}_{\eta-1} \cap \mathcal{R}^\star) - \dim(\mathcal{H}_\eta \cap \mathcal{R}^\star) \end{aligned}$$

from which we can compute straightforwardly the indices ψ_{i} as in the previous section. As in the previous case, ψ_j corresponds to the number Jordan chains of length j, i.e., we have ψ_j indices $\{k_1,\dots,k_j\}$ such that $j = p_{i,k}$. Moreover, the number of nonzero ψ_j, $j \in \{1,\dots\eta\}$ is the number of Jordan mini blocks g_i, i.e, $\sum_{j\in\{1,\dots,\eta\}} \operatorname{sign}(\psi_j) = g_i$. We can now easily apply the machinery already developed for $\mathcal{R}^\star$ by choosing ψ_j vectors from $\mathcal{H}_j$ to build ψ_j Jordan chains of length j via (10) by fixing $\delta = j$, for all $j \in \{1,\dots,\eta\}$ and for all the invariant zeros $z_i \in \mathcal{Z}$. The same procedure applies to the computation of $\mathcal{V}_g^\star$ by restricting the iteration over the zeros in $\mathcal{Z} \cap \mathbb{C}_g$.

V. CONCLUDING REMARKS

In this paper we have shown that [5, Proposition 4] holds under virtually no assumptions. This fact enables us to conclude that the dramatic computational improvements obtained in recent times in [10] in the computation of the fundamental output-nulling subspaces – which hinge on such result – are as general as the standard techniques based on subspace recursions.

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