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Computing the distance to instability for delay systems with uncertainties in the system matrices and in the delay terms

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Abstract— **In this paper we propose an algorithm to compute the distance to instability of a linear system of delay differential equations (DDEs) containing uncertainties in the delay terms as well as in matrices coefficients. For what regards the system matrices, any structure on the perturbation can be considered in order to allow only specific parameters to change; moreover, real-valued matrix perturbations are taken into account. The algorithm relies on the computation of the pseudospectral abscissa of the system and performs a bisection-Newton's method to find the minimum size of the perturbation that generates instability. A few illustrative examples, including a model for a rotating cutting machine, finally show the correctness and the efficiency of the method.**

I. INTRODUCTION

In the investigation of a dynamical system's asymptotical behaviour, the computation of the distance to instability (or stability radius) plays a fundamental role in the assessment of its stability robustness. Many existing algorithms allow to compute the stability radius of dynamical systems described by first or higher order ODEs as, among others, the ones presented in [1], [2] and [3]. However, when we come to delay differential equations (DDEs), the analysis of stability and its robustness becomes a much more complex problem. In this paper we deal with systems of DDEs as follows

$$\dot{x}(t) = \sum_{i=1}^{m} A_i x(t-\tau_i), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $A_1, \ldots, A_m \in \mathbb{R}^{n\times n}$, $n \geq 2$ and $0 \leq \tau_1 < \cdots < \tau_m$ are the time-delays. The solution of this system is asymptotically stable if and only if all the roots of the characteristic equation of the associated delay eigenvalue problem (DEP) lie in the open left-half of the complex plane, namely they all have a negative real part. In literature, many attempts to characterize the asymptotical behaviour of a dynamical system in the delay-parameter space have been made: without being exhaustive, several characterizations of delay-dependent stability (see [4] and [5]) and delay-independent stability (see [6] and [7]) have been proposed. We refer the reader interested in a complete overview of the methods to the monograph [8]. Less results are available when we consider the robustness of the stability w.r.t. potential uncertainties in our mathematical model: in [9], the robustness of the delay-independent stability is assessed w.r.t. uncertainties in system matrices; in [10] a lower-bound on the stability radius of a time-delay system simultaneously subject to real-valued perturbations on the delay terms and to complex-valued perturbations on the system matrices is presented.

In this paper, we consider and fully exploit structured real-valued perturbations of the system matrices and real-valued perturbations on the delay terms: this is the most realistic framework to investigate the robustness of stability. Here we propose a method to exactly compute the distance to instability of a time-delay system as in (1) w.r.t. to the aforementioned uncertainties on the system. The distance to instability is defined as the size of the smallest perturbation that pushes one (or more) eigenvalues of a stable system on the imaginary axis, thus generating instability; therefore, by definition, the distance to instability is strictly connected with the pseudospectral abscissa, that is the real part of the rightmost eigenvalue of the systems arisen from all the possible ε-bounded perturbations on the original system. As we shall see in Section II, the distance to instability as the unique root of equation can also be defined as the minimum perturbation for which the system is not robustly stable.

In literature, many existing algorithms are available to compute the pseudospectral abscissa, both for linear and nonlinear eigenvalue problems (e.g. polynomial or delay eigenvalue problem); among the others, the method developed in [11] accounts for real-valued perturbations in the standard eigenvalue problem, whereas in [12] complex-valued uncertainties on the matrices coefficients of nonlinear eigenvalue problems are considered. The real-valued nature of uncertainties and the nonlinearity of the eigenvalue problem are simultaneously included in the method developed in [13]. In the latter work, the method also accounts for perturbations with any structure, namely only single coefficients or blocks of the matrices can be perturbed while the others remain constant; in this paper we extend this algorithm in order to account for *combined* perturbations on the system matrices and the delay parameters.

The paper is organized as follows: in Section II we introduce the pseudospectral approach to the stability robustness analysis. The algorithm for the computation of the pseudospectral abscissa is described in Section III. In Section IV we illustrate the bisection-Newton's method implemented to find the distance to instability and in Section V we provide some numerical results.

II. THE PSEUDOSPECTRAL APPROACH

The scope of this work is to evaluate the distance to instability of a dynamical system affected by real-valued structured perturbations on the coefficient matrices and by real-valued perturbations on the delay terms; considering

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system (1), the associated perturbed DEP reads as follows:

$$M(\lambda)y = \left(\lambda I_n - \sum_{i=1}^{m}\big(A_i + B_i\delta A_i C_i\big)e^{-\lambda(\tau_i+\delta\tau_i)}\right)y = 0, \tag{2}$$

where $y \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$, I_n is the identity matrix, $A_i \in \mathbb{R}^{n\times n}$, B_i, δA_i, $\mathbb{C}_i$ are real-valued shape matrices of appropriate dimensions for all $i = 1,\dots,m$, $0 \le \tau_1 < \cdots < \tau_m$ are the time-delays and $\delta\tau_i \in \mathbb{R}$ are such that $|\delta\tau_i| < |\tau_i|$ for all $i = 1,\dots,m$. Matrices B_i, C_i define the structure of the perturbation on each matrix A_i of the system; this allows us to perturb only a part of each matrix, e.g. a block or a single coefficient. Therefore the size of each matrix δA_i may differ, i.e. $\delta A_i \in \mathbb{R}^{p_i\times q_i}$, $i = 1,\dots,m$. In the following, we will indicate the domain of matrices δA_i as $\mathbb{R}^* := \mathbb{R}^{p_1\times q_1} \times \cdots \times \mathbb{R}^{p_m\times q_m}$.

Since a distance measure is scalar, we need to define a scalar measure of the overall uncertainty, i.e., which takes into account all the perturbations from which the system is affected. Let us express $\Delta := (\Delta A, \Delta\tau) = \big(\delta A_1,\dots,\delta A_m,\delta\tau_1,\dots,\delta\tau_m\big) \in \mathbb{R}^* \times \mathbb{R}^m$ as the tuple containing the matrices perturbations δA_i and the delays perturbations $\delta\tau_i$. Introducing the weights w_i, $v_i \in \mathbb{R}_0^+ \cup \{+\infty\}$ with $i = 1,\dots,m$, we define the following global norm

$$\|\Delta\|_{\text{glob}} := \left\| \begin{bmatrix} w_1\|\delta A_1\|_F \\ \vdots \\ w_m\|\delta A_m\|_F \\ v_1|\delta\tau_1| \\ \vdots \\ v_m|\delta\tau_m| \end{bmatrix} \right\|_\infty, \tag{3}$$

where $\|\cdot\|_F$ stands for the matricial Frobenius norm. Therefore, an ε-bounded set of perturbation Δ is such that

$$\|\delta A_i\|_F \le \frac{\varepsilon}{w_i}, \quad |\delta\tau_i| \le \frac{\varepsilon}{v_i},\ i = 1,\dots,m.$$

Therefore, setting a weight $w_i = +\infty$ (or $v_i = +\infty$) implies that $\delta A_i = 0$ (or $\delta\tau_i = 0$) and that the relative nominal matrix (or delay) is assumed free from uncertainties. Given the above definition, for a fixed ε, we define the structured pseudospectrum of the perturbed DEP (2) as

$$\Lambda_\varepsilon := \bigcup_{\substack{\Delta\in\mathbb{R}^*\times\mathbb{R}^m \\ \|\Delta\|_{\text{glob}}\le\varepsilon}} \left\{\lambda\in\mathbb{C} : \det M(\lambda) = 0\right\}. \tag{4}$$

Given the notion of pseudospectrum, we introduce the pseudospectral abscissa function α_ε, whose general definition reads as follows:

$$\alpha_\varepsilon := \sup\{\Re(\lambda) : \lambda \in \Lambda_\varepsilon\}.$$

The expert reader, however, will observe that, from the well known properties of a DEP spectrum (see for instance Chapter 1 of [14]), the pseudospectral abscissa can also be defined as a maximum. From the definition of the pseudospectral abscissa α_ε, the distance to instability easily follows as

$$\inf\{\varepsilon : \alpha_\varepsilon \ge 0\}.$$

Intuitively, we look for the minimum combined perturbation as defined in (3) that pushes one (or more) eigenvalues in the closed right-half of the complex plane. The method that we are going to illustrate in the following strongly relies on the next result about the continuity of the spectral abscissa, which is the real part of the rightmost point of a DEP (see again [14] for a proof of the result).

Theorem 1: Let α be the spectral abscissa of the DEP associated with system (1). Then the function $\alpha : (\mathbb{R}^{n\times n})^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$

$$\big(A_1,\dots,A_m,\tau_1,\dots,\tau_m\big) \longrightarrow \alpha\Big(\big(A_1,\dots,A_m,\tau_1,\dots,\tau_m\big)\Big)$$

is continuous.

This result may seem trivial, but continuity w.r.t. the delay parameters does not hold when switching to neutral or algebraic delay differential equations: in these latter cases, the generated DEP is allowed to have a singular leading matrix and the spectral abscissa may not be continuous w.r.t. delay variations (see [15]).

As the pseudospectral abscissa is the maximum of a set of continuous function, it is also continuous w.r.t. to the system matrices $A_1,\dots,A_m$ and the delay terms $\tau_1,\dots,\tau_m$. Moreover, $\varepsilon_1 < \varepsilon_2$ gives that $\Lambda_{\varepsilon_1} \subseteq \Lambda_{\varepsilon_2}$ (the set of allowable perturbations is enlarged); from the property of supremum function, the pseudospectral abscissa α_ε is therefore nondecreasing w.r.t. ε (and in the generic case strictly increasing and differentiable around the zero crossing). We now have all the ingredients to redefine the distance to instability as the (unique) root of the equation

$$\alpha_\varepsilon = 0.$$

Therefore, our method aims at computing the real stability radius of system (1) as the root of this equation; to this purpose, we exploit an algorithm that computes the pseudospectral abscissa of DEP (2).

III. An iterative method to compute the Pseudospectral Abscissa

The main idea of our algorithm is to generate a sequence of perturbed ε-bounded DEPs as in (2) such that the spectral abscissa of these problems is a monotonically increasing function, and the corresponding eigenvalue converges to the rightmost point of the pseudospectrum; as a matter of fact, the algorithm can be intepreted as a discretization of a gradient flow in the space of matrices perturbations and of delay terms pertubations induced by maximizing the real part of the rightmost eigenvalue. For what regards the perturbations on the system matrices, we exploit some low-rank properties of the *optimal* matrices perturbations, to which we will refer in the following as the matrices that generate the globally rightmost point of the pseudospectrum; analogously, we define the *optimal* time-delay perturbations. As we will see in the following, the behaviour of the derivative of the spectral

abscissa w.r.t. these two different kinds of perturbations is comparable. In this work, we will make large use of the derivatives of eigenvalues w.r.t. different terms contained in the DEP; here we provide an easy reformulation of Lemma 2.7 in [16], where a general formula for these derivatives is given. In the following, we will then adapt this formula to our problem.

Lemma 2: Let $M(\lambda,\theta) : \mathbb{C}\times\mathbb{C}^d \longrightarrow \mathbb{C}^{n\times n}$ be continuously differentiable, with λ a simple eigenvalue whose corresponding left and right eigenvectors with unit norm are x and y, and θ a set of parameters, then

$$\frac{\partial\lambda}{\partial\theta_i} = -\frac{x^*\frac{\partial M(\lambda,\theta)}{\partial\theta_i}y}{x^*\frac{\partial M(\lambda,\theta)}{\partial\lambda}y}, \qquad i=1,\dots,d, \tag{5}$$

where x^* is the conjugate transpose vector of x.

Our algorithm to compute the pseudospectral abscissa is based on the next fundamental result; this characterizes the optimal matrices perturbations and the optimal delay-terms perturbations of problem (2). Note that here and in the following, due to the symmetry of the (pseudo)spectrum of a DEP, we will refer to the globally rightmost point as the one in the upper half of complex plane.

Theorem 3: Let $\lambda_{\rm RM}$ be a globally rightmost point of the structured ε-pseudospectrum and assume it is a simple eigenvalue for some ε-bounded perturbation Δ. Then

(i) There always exists a set of perturbations $\widetilde{\Delta A} = (\widetilde{\delta A_1},\dots,\widetilde{\delta A_m})$, where $\widetilde{\delta A_i}$ has rank at most two and $w_i\left\|\widetilde{\delta A_i}\right\|_F \le \varepsilon$ for all $i=1,\dots,m$, and for which the rightmost eigenvalue is equal to $\lambda_{\rm RM}$;

(ii) Let x,y be left and right eigenvectors of $\lambda_{\rm RM}$ respectively normalized such that $\|x\| = \|y\| = 1$ and

$$\xi := x^*\Big(I_n + \sum_{i=1}^{m}\big(A_i + B_i\delta A_i C_i\big)\tau_i e^{-\lambda(\tau_i+\delta\tau_i)}\Big)y > 0 \tag{6}$$

and let us define

$$X = [\Re(x)\ \Im(x)],\ Y = [\Re(y)\ \Im(y)],$$

and

$$\Gamma_i = \begin{bmatrix} \Re(e^{-(\tau_i+\delta\tau_i)\lambda_{\rm RM}}) & -\Im(e^{-(\tau_i+\delta\tau_i)\lambda_{\rm RM}}) \\ \Im(e^{-(\tau_i+\delta\tau_i)\lambda_{\rm RM}}) & \Re(e^{-(\tau_i+\delta\tau_i)\lambda_{\rm RM}}) \end{bmatrix}$$

for $i=1,\dots,m$. Then for each i, $B_i^TX\Gamma_iY^TC_i^T$ can be either zero or nonzero. In the latter case, a particular set of optimal perturbations can be expressed as

$$\widetilde{\delta A_i} = -\frac{\varepsilon}{w_i}\frac{B_i^TX\Gamma_iY^TC_i^T}{\left\|B_i^TX\Gamma_iY^TC_i^T\right\|_F},\ i=1,\dots,m. \tag{7}$$

(iii) Let x,y be defined as before, then

$$\frac{\partial\Re(\lambda_{\rm RM})}{\partial\delta\tau_i} = \frac{1}{\xi}\Re\Big(x^*\big(A_i+B_i\delta A_iC_i\big)\lambda e^{-\lambda(\tau_i+\delta\tau_i)}y\Big) \tag{8}$$

can also be either zero or nonzero. In the latter case, the optimal time-delay perturbations are such that

$$|\widetilde{\delta\tau_i}| = \frac{\varepsilon}{v_i}.$$

Proof: We do not report here the proof of the first part of the theorem; we refer the interested reader to Proposition 6.2 of [13]. The proof of the second part is the natural extension of the proof of Theorem 3.3 in [13] from unstructured to structured perturbations on system matrices where we included the uncertainties on the delay terms. Let us assume that $B_i^TX\Gamma_iY^TC_i^T$ is a non-zero matrix for $i=1,\dots,m$. Let us indicate with $(\delta A_i)_{s,t}$ the coefficient in position (s,t) of matrix δA_i, and with $B_i^{(s)}, C_{i(t)}$ respectively the s-th column of B_i and the t-th row of C_i, for $s=1,\dots,p_i,\ t=1,\dots,q_i$. From Lemma 2, using some algebra manipulations we obtain that

$$\begin{aligned}
\frac{\partial\Re(\lambda_{\rm RM})}{\partial\delta A_i} &= \left(\frac{\partial\Re(\lambda_{\rm RM})}{\partial(\delta A_i)_{s,t}}\right)_{\substack{s=1,\dots,p_i\\t=1,\dots,q_i}} = \\
&= -\tfrac{1}{\xi}\Re\left(x^*B_i^{(s)}C_{i(t)}ye^{-\lambda(\tau_i+\delta\tau_i)}\right)_{\substack{s=1,\dots,p_i\\t=1,\dots,q_i}} = \\
&= -\tfrac{1}{\xi}\left(B_i^{(s)T}\Re\Big(xy^*e^{-\lambda(\tau_i+\delta\tau_i)}\Big)C_{i(t)}^T\right)_{\substack{s=1,\dots,p_i\\t=1,\dots,q_i}} = \\
&= -\tfrac{1}{\xi}B_i^TX\Gamma_iY^TC_i^T \neq 0.
\end{aligned}$$

We want each δA_i to be ε-bounded, so we impose the following constraints

$$g_i := \sum_{\substack{s=1,\dots,p_i\\t=1,\dots,q_i}}\big(\delta A_i\big)_{s,t}^2 - \frac{\varepsilon^2}{w_i^2} \le 0, \quad i=1,\dots,m.$$

As $\lambda_{\rm RM}$ is the globally rightmost point of the pseudospectrum, from the theory of Lagrange multipliers we have that

$$\frac{\partial\Re(\lambda_{\rm RM})}{\partial\delta A_i} - 2\mu_i\delta A_i = 0, \quad i=1,\dots,m$$

where $\mu_i \ge 0$ are the multipliers associated with the g_i inequality constraints. Since $\frac{\partial\Re(\lambda_{\rm RM})}{\partial\delta A_i}$ is a nonzero matrix, then μ_i is positive and g_i is an active constraint; thus δA_i is a positive multiple of $\frac{\partial\Re(\lambda_{\rm RM})}{\partial\delta A_i}$ and in particular $\|\delta A_i\|_F = \frac{\varepsilon}{w_i}$ for $i=1,\dots,m$, from which we derive the thesis. The proof of point (iii) can be easily deduced from the proof of point (ii) by applying the same reasoning to the perturbation $\delta\tau_i$ in the interval $[\tau_i-\varepsilon/v_i,\tau_i+\varepsilon/v_i],\ i=1,\dots,m$; here we used the derivative in (8) which is obtained from Lemma 2. ∎

This theorem provides useful information about the optimal matrices perturbations and the optimal delay-terms perturbations; in particular, we exploit the aforementioned low-rank properties for the matrices perturbations, so that we can restrict their research to the manifold

$$\begin{aligned}
\widehat{\mathcal{S}}_F := \{(\delta A_1,\dots,\delta A_m)\in\mathbb{R}^* : &\text{rank}(\delta A_i)\le 2,\\
&w_i\|\delta A_i\|_F \le \varepsilon,\ i=1,\dots,m\}.
\end{aligned} \tag{9}$$

In order to find the set of optimal perturbations $\widetilde{\Delta} := (\widetilde{\Delta A},\widetilde{\Delta\tau})$, we define a continuous path in manifold $\widehat{\mathcal{S}}_F$ and one in the space of delay perturbations $\widehat{\mathcal{S}}_\tau := [-\frac{\varepsilon}{v_1},+\frac{\varepsilon}{v_1}]\times\cdots\times[-\frac{\varepsilon}{v_m},+\frac{\varepsilon}{v_m}]$; then our algorithm will perform a gradient

flow discretization along these paths to converge to $\widetilde{\Delta}$. Let us first define the paths in a general way as follows

$$\begin{cases} \delta A_i(t) = -\frac{\varepsilon}{w_i} U_i(t) Q_i(t) V_i(t)^T, \quad t \in \mathbb{R}^+ \\ \delta\tau_i(t) = \frac{\varepsilon}{v_i} q_i(t), \quad t \in \mathbb{R}^+ \end{cases} \tag{10}$$

where the following properties need to be satisfied:

$$\begin{cases} U_i(t)^T \dot{U}_i(t) = 0, \qquad \forall\ t \geq 0 \\ V_i(t)^T \dot{V}_i(t) = 0, \qquad \forall\ t \geq 0 \\ \|Q_i(t)\|_F \leq 1, \qquad \forall\ t \geq 0 \\ |q_i(t)| \leq 1, \qquad \forall\ t \geq 0 \end{cases} \tag{11}$$

and $U_i(t) \in \mathbb{R}^{p_i \times 2}, V_i(t) \in \mathbb{R}^{q_i \times 2}, Q_i(t) \in \mathbb{R}^{2\times 2}$. The interested reader is referred to [13] for a detailed argumentation about the choice of the matrix decomposition. We now need to build matrices and functions with such properties. Let us introduce some additional matrices $R_i(t) \in \mathbb{R}^{p_i \times 2}$, $S_i(t) \in \mathbb{R}^{q_i \times 2}$ $M_i(t) \in \mathbb{R}^{2\times 2}$ and functions $r_i(t)$ for $i = 1, \ldots, m$ and let us indicate with $\langle A, B\rangle = \text{Trace}(A^T B)$ the Frobenius inner product of two matrices: it is easy to prove that the solutions of the following differential equations (where we omit the parameter t to simplify the notation)

$$\begin{cases} \dot{U}_i = \Big(I_n - U_i U_i^T\Big) R_i, \\ \dot{V}_i = \Big(I_n - V_i V_i^T\Big) S_i, \\ \dot{Q}_i = \begin{cases} M_i - \langle M_i, Q_i\rangle Q_i, & \text{if } \|Q_i\|_F = 1,\ \langle M_i, Q_i\rangle > 0 \\ M_i, & \text{otherwise,} \end{cases} \\ \dot{q}_i = \begin{cases} 0 & \text{if } |q_i| = 1,\ r_i q_i > 0 \\ r_i, & \text{otherwise,} \end{cases} \end{cases} \tag{12}$$

satisfy the properties spelled out in (11) without losing of generality for any choice of $R_i(t), S_i(t), M_i(t), r_i(t)$; in practice, we look for matrices and functions with the desired properties as solutions of differential equations (12). Now, in order to construct some specific paths using equations (10), we need to make a choice on the arbitrary matrices $R_i(t)$, $S_i(t)$, $M_i(t)$ and functions $r_i(t)$. To this purpose, we use the derivative of an eigenvalue w.r.t. t adapted from Lemma 2, where the eigenvalue problem $M(\lambda, t)$ is

$$\left(\lambda I_n - \sum_{i=1}^{m} \big(A_i + B_i \delta A_i(t) C_i\big) e^{-\lambda\left(\tau_i + \delta\tau_i(t)\right)}\right) y = 0, \tag{13}$$

and the derivative reads as follows:

$$\begin{aligned} &\dot{\lambda}(t) := \tfrac{d\lambda(t)}{dt} = \\ &= \frac{-x^*\left(\sum_{i=1}^m B_i \delta\dot{A}_i(t) C_i e^{-\lambda\left(\tau_i+\delta\tau_i(t)\right)}\right) y}{x^*\left(I_n + \sum_{i=1}^m \left(A_i + B_i\delta A_i(t) C_i\right)\tau_i e^{-\lambda(\tau_i+\delta\tau_i(t))}\right) y} + \\ &+ \frac{x^*\left(\sum_{i=1}^m \left(A_i + B_i\delta A_i(t) C_i\right)\lambda\delta\dot{\tau}_i(t) e^{-\lambda\left(\tau_i+\delta\tau_i(t)\right)}\right) y}{x^*\left(I_n + \sum_{i=1}^m \left(A_i + B_i\delta A_i(t) C_i\right)\tau_i e^{-\lambda(\tau_i+\delta\tau_i(t))}\right) y}. \end{aligned} \tag{14}$$

By definition, each point in $\widehat{\mathcal{S}}_F \times \widehat{\mathcal{S}}_\tau$ corresponds to a perturbed DEP as in equation (13); we would like to build paths in $\widehat{\mathcal{S}}_F$ and $\widehat{\mathcal{S}}_\tau$ such that the spectral abscissa function of the corresponding DEPs is monotonically increasing. By means of the same normalization used in equation (6), we set the denominator in equation (14) equal to a positive real number ξ. Let us now consider for the sake of simplicity the case when $\dot{Q}_i(t) = M_i$ and $\dot{q}_i(t) = r_i$, and let us omit the dependence on t in the notation, for sake of clarity, then we write

$$\begin{aligned} \mathfrak{R}(\dot{\lambda}) &= \textstyle\sum_{i=1}^m \frac{\varepsilon}{\xi w_i} \mathfrak{R}\big(x^*(B_i \dot{U}_i Q_i V_i^T C_i + B_i U_i \dot{Q}_i V_i^T C_i \\ &\quad + B_i U_i Q_i \dot{V}_i^T C_i) e^{-\lambda(\tau_i+\delta\tau_i)} y\big) + \\ &+ \textstyle\sum_{i=1}^m \frac{\varepsilon}{\xi v_i} \mathfrak{R}\big(x^*(A_i + B_i\delta A_i C_i)\lambda\dot{q}_i e^{-\lambda(\tau_i+\delta\tau_i)} y\big) = \\ &= \textstyle\sum_{i=1}^m \frac{\varepsilon}{\xi w_i} \big\langle X, B_i (I_n - U_i U_i^T) R_i Q_i V_i^T C_i Y \Gamma_i^T \big\rangle + \\ &+ \textstyle\sum_{i=1}^m \frac{\varepsilon}{\xi w_i} \big\langle X, B_i U_i M_i V_i^T C_i Y \Gamma_i^T \big\rangle + \\ &+ \textstyle\sum_{i=1}^m \frac{\varepsilon}{\xi w_i} \big\langle X, B_i U_i Q_i S_i^T (I_n - V_i V_i^T) C_i Y \Gamma_i^T \big\rangle + \\ &+ \textstyle\sum_{i=1}^m \frac{\varepsilon}{\xi v_i} \underbrace{\mathfrak{R}\big(x^*(A_i + B_i\delta A_i C_i)\lambda e^{-\lambda(\tau_i+\delta\tau_i)} y\big)}_{:=\beta_i} \dot{q}_i = \\ &= \textstyle\sum_{i=1}^m \frac{\varepsilon}{\xi w_i} \big\langle B_i^T X \Gamma_i Y^T C_i^T V_i Q_i^T, (I_n - U_i U_i^T) R_i \big\rangle + \\ &+ \textstyle\sum_{i=1}^m \frac{\varepsilon}{\xi w_i} \big\langle U_i^T B_i^T X \Gamma_i Y^T C_i^T V_i, M_i \big\rangle + \\ &+ \textstyle\sum_{i=1}^m \frac{\varepsilon}{\xi w_i} \big\langle S_i, (I_n - V_i V_i^T) C_i Y \Gamma_i^T X^T B_i U_i Q_i \big\rangle + \\ &+ \textstyle\sum_{i=1}^m \frac{\varepsilon}{\xi v_i} \beta_i \dot{q}_i, \end{aligned}$$

and analogous expressions hold for the other cases in (12). Now, since $(I_n - U_i U_i^T)$, $(I_n - V_i V_i^T)$ are positive semi-definite, the increasing monotonicity is guaranteed by the following choices

$$\begin{cases} \dot{U}_i = \Big(I_n - U_i U_i^T\Big) B_i^T X \Gamma_i Y^T C_i^T V_i Q_i^T, \\ \dot{V}_i = \Big(I_n - V_i V_i^T\Big) C_i Y \Gamma_i^T X^T B_i U_i Q_i, \\ \dot{Q}_i = \begin{cases} M_i - \langle M_i, Q_i\rangle Q_i, & \text{if } \|Q_i\|_F = 1,\ \langle M_i, Q_i\rangle > 0, \\ M_i, & \text{otherwise,} \end{cases} \\ \dot{q}_i = \begin{cases} 0 & \text{if } |q_i| = 1,\ \beta_i q_i > 0 \\ \beta_i, & \text{otherwise,} \end{cases} \end{cases} \tag{15}$$

with $M_i = U_i^T B_i^T X \Gamma_i Y^T C_i^T V_i$. Given these choices for $\dot{U}_i, \dot{V}_i, \dot{Q}_i$ and $\dot{q}_i$, at each iteration of the algorithm we build a new perturbation of the original matrices and delay terms by moving in the ascent direction of the spectral abscissa w.r.t. parameter t; this is carried out by means of a forward Euler discretization of the differential equations (15) and then a projection of the new quantities onto the paths described in equation (10). Therefore, each iteration of our algorithm builds a new perturbed DEP like in (2) such that the sequence of the corresponding spectral abscissae is monotonically increasing. For the boundedness of the pseudospectrum, this sequence converges. We report here a short description in Algorithm 1.

Algorithm 1: Set perturbation size ε, weights w_i, v_i, stepsize h, tolerance η. Initialize $(\lambda^{(1)}, x^{(1)}, y^{(1)})$, $(\tau_1^{(1)}, \ldots, \tau_m^{(1)})$ and $U^{(1)}, V^{(1)}, Q^{(1)}, q^{(1)}$ and repeat for $k = 1, 2, \ldots$

1. Compute X, Y, Γ_i as in Theorem 3;
2. Compute $\dot{U}_i^{(k)}, \dot{V}_i^{(k)}, \dot{Q}_i^{(k)}, \dot{q}_i^{(k)}$ using (15) for $i = 1, \ldots, m$;
3. Compute for $i = 1, \ldots, m$
$$\begin{cases} \widehat{U}_i^{(k+1)} = U_i^{(k)} + h\dot{U}_i^{(k)}, \\ \widehat{V}_i^{(k+1)} = V_i^{(k)} + h\dot{V}_i^{(k)}, \\ \widehat{Q}_i^{(k+1)} = Q_i^{(k)} + h\dot{Q}_i^{(k)}, \\ \widehat{\tau}_i^{(k+1)} = \tau_i^{(k)} + h\dot{q}_i^{(k)}; \end{cases}$$
4. Project $\widehat{U}_i^{(k+1)}, \widehat{V}_i^{(k+1)}, \widehat{Q}_i^{(k+1)}$ onto manifold $\widehat{\mathcal{S}}_F$: use a compact QR decomposition to normalize $\widehat{U}_i^{(k+1)}, \widehat{V}_i^{(k+1)}$, and, if $\left\|\widehat{Q}_i^{(k+1)}\right\|_F > 1$, also normalize it;
5. Possibly rescale $\widehat{\tau}_i^{(k+1)}$ to an ε-bounded set of delay terms:
$$\begin{cases} \tau_i^{(k+1)} = \min\left(\tau_i + \frac{\varepsilon}{v_i}, \widehat{\tau}_i^{(k+1)}\right), \text{ if } \dot{q}_i^{(k)} \geq 0 \\ \tau_i^{(k+1)} = \max\left(0, \tau_i - \frac{\varepsilon}{v_i}, \widehat{\tau}_i^{(k+1)}\right), \text{ if } \dot{q}_i^{(k)} < 0 \end{cases}$$
6. Define $\delta A_i = -\frac{\varepsilon}{w_i} U_i^{(k+1)} Q_i^{(k+1)} V_i^{(k+1)T}$ and $\tau_i + \delta\tau_i = \tau_i^{(k+1)}$ and compute the rightmost eigenvalue λ and the normalized eigenvectors x, y of the DEP (2);
7. Stop the algorithm if this condition holds
$$\Re(\dot{\lambda}) < \eta$$
otherwise start again from step 1;
8. Return the pseudospectral abscissa $\Re(\lambda)$.

In step 6 we need to solve DEPs: this is carried out using the algorithm and software described in [15]. For major details about the initialization we refer the reader to [13]: basically, δA_i are initialized performing one step of a fixed point iteration whose iterate function is defined in (7), while $\delta\tau_i$ are simply initialized following the derivative in Equation (8). Note that discretized gradient flow in Algorithm 1 can also be interpreted as the application of a first order optimization method (gradient search). Furthermore, due to space limitation we have only sketched the basic algorithm excluding the adaptive stepsize strategy in our implementation, which has an interpretation in terms of a globalization strategy. As a consequence, under standard conditions convergence to perturbations satisfying the first order necessary optimality conditions can be guaranteed. As we have to compute the *globally* rightmost point of the pseudospectrum, we use the strategy from [13], which is based on generating initial conditions from several rightmost eigenvalues. Finally, the previous algorithm also allows to include constraints on the dependence among matrices perturbations δA_i. For instance, we can impose some perturbations δA_i to be equal to each others; an example is given in the cutting process in Section V.

IV. A Bisection-Newton's method for the distance to instability

As we discussed in Section II, under mild conditions we can compute the distance to instability of system (2) as the value of ε corresponding to the unique root of the equation $\alpha_\varepsilon = 0$, where α_ε is the pseudospectral abscissa computed using the method in Section III. In order to use Newton's method to solve the equation, at least almost differentiability of the function is required. The pseudospectral abscissa function α_ε is by definition a real-valued increasing monotone function of ε and bounded on any interval $[\varepsilon_L, \varepsilon_R]$, thus it is a function of bounded variation; from the Lebesgue's differentiation theorem of functions of bounded variation (see for instance Theorem 3.4 of [17]) we can conclude that α_ε is almost everywhere differentiable w.r.t. ε; furthermore, it is differentiable whenever the rightmost eigenvalue corresponding to the optimal perturbations is simple. The next theorem then provides an explicit expression for the derivative.

Theorem 4: Let $\big(\lambda_{\mathrm{RM}}(\varepsilon), x(\varepsilon), y(\varepsilon)\big)$ be respectively the globally rightmost point of Λ_ε and its left and right normalized eigenvectors; let also $\widetilde{\Delta A}(\varepsilon) = \big(\widetilde{\delta A_1}(\varepsilon), \ldots, \widetilde{\delta A_m}(\varepsilon)\big)$ and $\widetilde{\Delta\tau}(\varepsilon) = \big(\widetilde{\delta\tau_1}(\varepsilon), \ldots, \widetilde{\delta\tau_m}(\varepsilon)\big)$ be functions such that $\lambda_{\mathrm{RM}}(\varepsilon)$ is the unique rightmost eigenvalue of system (2) with $\delta A_i = \widetilde{\delta A_i}(\varepsilon)$ and $\delta\tau_i = \widetilde{\delta\tau_i}(\varepsilon)$ and

$$\widetilde{\delta A_i}(\varepsilon) = -\frac{\varepsilon}{w_i}\widetilde{U_i}(\varepsilon)\widetilde{Q_i}(\varepsilon)\widetilde{V_i}(\varepsilon)^T \text{ and } \widetilde{\delta\tau_i}(\varepsilon) = \frac{\varepsilon}{v_i}\widetilde{q_i}(\varepsilon)$$

for $i = 1, \ldots, m$; then denoting

$$\zeta := x^*\Big(I_n + \sum_{i=1}^{m}\big(A_i + B_i\widetilde{\delta A_i}C_i\big)e^{\left(-\lambda_{\mathrm{RM}}(\tau_i+\widetilde{\delta\tau_i})\right)}(\tau_i+\widetilde{\delta\tau_i})\Big)y$$

we can express the derivative of the pseudospectral abscissa w.r.t ε as follows

$$\begin{aligned} \frac{\partial\alpha_\varepsilon}{\partial\varepsilon} &= \frac{\partial\Re(\lambda_{\mathrm{RM}})}{\partial\varepsilon} = \\ &-\frac{1}{\zeta}\Re\left[x^*\left(\sum_{i=1}^{m} B_i\frac{\partial\widetilde{\delta A_i}}{\partial\varepsilon}C_i e^{\left(-\lambda_{\mathrm{RM}}(\tau_i+\widetilde{\delta\tau_i})\right)}\right)y + \right. \\ &\left. + x^*\left(\sum_{i=1}^{m}\big(A_i + B_i\widetilde{\delta A_i}C_i\big)e^{\left(-\lambda_{\mathrm{RM}}(\tau_i+\widetilde{\delta\tau_i})\right)}\lambda_{\mathrm{RM}}\frac{\partial\widetilde{\delta\tau_i}}{\partial\varepsilon}\right)y\right] = \\ &+\frac{1}{\zeta}\Re\left[x^*\left(\sum_{i=1}^{m}\frac{1}{w_i}B_i\widetilde{U_i}\widetilde{Q_i}\widetilde{V_i}^T C_i e^{\left(-\lambda_{\mathrm{RM}}(\tau_i+\frac{\varepsilon}{v_i}\widetilde{q_i})\right)} + \right.\right. \\ &\left.\left. + \sum_{i=1}^{m}\left(A_i - \frac{\varepsilon}{w_i}B_i\widetilde{U_i}\widetilde{Q_i}\widetilde{V_i}^T C_i\right)e^{\left(-\lambda_{\mathrm{RM}}(\tau_i+\frac{\varepsilon}{v_i}\widetilde{q_i})\right)}\lambda_{\mathrm{RM}}\frac{\widetilde{q_i}}{v_i}\right)y\right], \end{aligned}$$

where we omit the dependence on ε to simplify the notation. Observe that we obtain the last expression because we are dealing with *optimal* combined perturbations at every fixed ε.

This formula is again derived from Lemma 2, so we will not provide a proof. As Newton's method is a local method, we first look for a good starting value that guarantees its convergence. This is carried out by performing some iterations of the bisection method, that we repeat until we find an ε that generates a pseudospectral abscissa close enough to 0. We now have all the ingredients needed to apply Newton's method to our problem; the basic iteration is shortly described in the following Algorithm 2.

Algorithm 2: Initialize perturbation ε_1 and repeat for $k = 1, 2, \ldots$ until convergence

1. Compute the pseudospectral abscissa α_{ε_k} of DEP (2);
2. Update

$$\varepsilon_{k+1} = \varepsilon_k - \frac{\alpha_{\varepsilon_k}}{\frac{\partial \alpha_{\varepsilon_k}}{\partial \varepsilon}\,|_{\varepsilon=\varepsilon_k}}.$$

V. Numerical experiments

In this section we want to demonstrate the efficiency of both the algorithms here presented. First we use an example from [18] to show how the pseudospectral abscissa behaves by taking into account perturbations on the system matrices and on the delay terms, and in particular we compare the level sets of the pseudospectral abscissa and of the spectral abscissa; then we provide a benchmark of examples for which we computed the distance to instability.

Example 1: We consider a simplified version of the mechanical rotational cutting process presented in [18] and [19], where the delay term τ is constant in time; its characteristic equation reads

$$\lambda^2 + 2\xi\omega\lambda + \omega^2 + \frac{k}{m}(1 - e^{-\lambda\tau}) = 0$$

where ω is the natural frequency, ξ the damping ratio, m the modal mass and k the cutting force coefficient and it is associated with the DEP with matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\omega^2 - k/m & -2\xi\omega \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 \\ k/m & 0 \end{bmatrix},$$

and $\tau_1 = 0$, $\tau_2 := \tau$. We want to consider structured perturbations on the delay τ and on the coefficient k, so our perturbed DEP looks like system (2) where we set $B_1 = [0 \; 1/m]^T$, $B_2 = -B_1$, $C_1 = [1 \; 0]$, $C_2 = C_1$ and $\delta A_1 = \delta A_2$ is the scalar perturbation of parameter k. We want $|\delta k| \leq 1.5 * 10^6, |\delta\tau| \leq 1.5 * 10^{-3}$, thus we set $\varepsilon = 0.1$, $v_1 = +\infty$, $v_2 = 0.015^{-1}$, $w_1 = w_2 = (1.5 \cdot 10^7)^{-1}$. In Fig. 1 we plot on the k-τ plane the level sets equal to 0 of the spectral abscissa (blue line) and of the pseudospectral abscissa for $\varepsilon = 0.1$ (red line); we observe that each point on the pseudospectral abscissa level set is connected with a point on the spectral abscissa level set by means of an ε-bounded $\widetilde{\Delta}$ perturbation. The blocks built on a few points of the pseudospectral abscissa level set represent the area spanned in the k-τ plane by all the possible ε-bounded perturbations; it is easy to see that $\widetilde{\Delta}$ is often characterized by a maximum norm perturbation on the parameter k, while this does not always happen when considering perturbations on delay terms. For this reason we observe an occasionally flat nature of the pseudospectral abscissa level set. However, the latter behaviour can also be observed w.r.t. to matrices coefficients: as a matter of fact, the pseudospectral abscissa w.r.t. to structured system matrices perturbations is not always generated by a maximum-norm perturbation; the interested reader can refer to [13] for an example. Finally, note that if the above computations are repeated for $w_1 = w_2 = +\infty$, then the algorithms returs the *delay margin*.

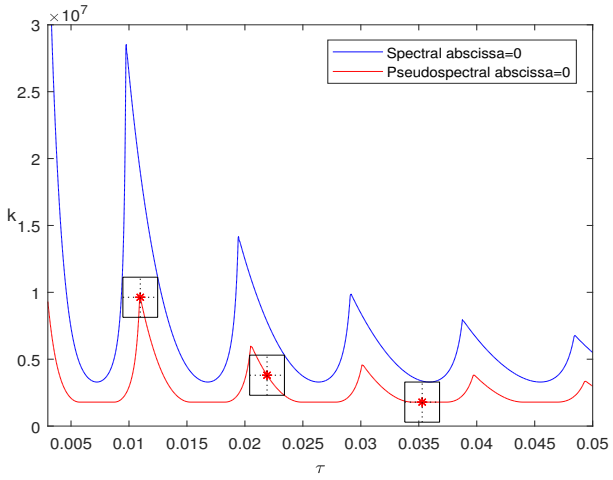

Fig. 1. Rectangular blocks represent an ε-bounded set of perturbations for the point marked with a red asterisk; the perturbation has maximum size $\frac{\varepsilon}{w_i}$ on the parameter k and $\frac{\varepsilon}{v_i}$ on the delay τ.

TABLE I

The table shows the distance to instability ε_r for problems in the benchmark

Problem	(n, m)	ε_r	α_{ε_r}	# it
1	(3,2)	1.112792276e-02	-2.663325696e-17	6
2	(3,2)	5.600882011e-01	2.753544234e-15	4
3	(5,6)	9.400004821e-01	6.365699553e-09	30
4	(5,6)	1.459273837e-01	4.107825191e-16	6
5	(10,3)	2.352096503e-03	-5.969697629e-16	4
6	(10,3)	9.139676624e-02	-5.520865458e-16	5
7	(20,3)	1.041499774e-01	1.000000000e-15	5
8	(30,3)	1.013715072e-01	-9.059148660e-16	5

Results of the bisection-Newton's method to compute the distance to instability are reported in Table I. We created a collection of problems from the literature where we have perturbed physical parameters appearing in system matrices and delay terms[1]. For all of these problems we ran the algorithm multiple times considering different perturbations, so they appear more than once in Table I; problems 1 and 2 are taken from [20], problems 3 and 4 are from [21]. Problems 5-8 are obtained from [22] as a discretization of a partial delay-differential equation, for which we adopt different refinements of the discretization. Details about the uncertainties considered in each problem are also included in the benchmark[1]. In Table I, the first column refers to the numbering of the problems in the benchmark, the second column displays the dimension of the problem and the number of delays, the third and fourth columns show respectively the distance to instability ε_r and the corresponding pseudospectral abscissa α_{ε_r} and the last column shows the number of Newton's method iterations needed for convergence.

[1]The collection is available at the webpage http://twr.cs.kuleuven.be/research/software/delay-control/benchmarks-distance-instability.zip

VI. Concluding remarks

We presented an algorithm that computes the pseudospectral abscissa of a delay eigenvalue problem with a nonsingular leading matrix, taking into account real-valued structured perturbations on the system matrices and perturbations on the delay terms. This is a local method, in the sense that the convergence conditions imposed on the algorithm also apply for locally but not globally rightmost points of the pseudospectrum; however, in practice, we have seldom observed this behaviour, and using different starting values has proved to be a good strategy to guarantee convergence to the globally rightmost point. This algorithm has then been extended to compute the distance to instability (the actual value, i.e. no possibly conservative lower or upper bounds) of delay eigenvalue problems using a bisection-Newton's method, which has turned out to be very fast in convergence. Thanks to the low-rank dynamics of optimal perturbations exploited in the computation of the pseudospectral abscissa and to its iterative nature, the method has also potential for sparse large-scale systems.

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