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Traffic Flow Inspired Analysis and Boundary Control for a Class of 2x2 Hyperbolic Systems

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***Abstract*— The paper presents results for a class of 2x2 systems of nonlinear hyperbolic PDEs on a 1-D bounded domain, inspired by second-order traffic flow models. The model consists of two first-order hyperbolic PDEs with a dynamic boundary condition that involves the time derivative of the velocity. The developed model has features that are important from a traffic-theoretic point of view: is completely anisotropic and information travels forward exactly at the same speed as traffic. It is shown that, for all physically meaningful initial conditions, the model admits a globally defined, unique, classical solution that remains positive and bounded for all times. Furthermore, a nonlinear, explicit boundary feedback law is developed, which achieves global stabilization of arbitrary equilibria. The stabilizing feedback law depends only on the inlet velocity and consequently, the measurement requirements for the implementation of the proposed boundary feedback law are minimal. The efficiency of the proposed boundary feedback law is demonstrated by means of a numerical example.**

I. Introduction

The study of vehicular traffic flow utilizing hyperbolic Partial Differential Equations (PDEs) goes back to the 1950s with the appearance of the LWR first-order model (see [27,32]). In order to describe more accurately the velocity dynamics, second-order models were later studied (see [1,29,37]). All 1-D traffic flow models were developed for unbounded domains (usually the whole real axis). Researchers working on second-order models as well as critics of second-order models (see [11]) have agreed that a valid traffic flow model must: (i) include the vehicle conservation equation, (ii) admit bounded solutions which predict positive values for both density and velocity, (iii) obey the so-called anisotropy principle, i.e., the fact that a vehicle is influenced only by the traffic dynamics ahead of it, (iv) not allow waves traveling forward with a speed greater than the traffic speed. Recently, researchers have developed two phase models (see [7,25]), which agree with experimental results that report strong differences between the free and congested vehicular flow.

Recent advances in the boundary feedback control of hyperbolic systems of PDEs (see for instance [2,3,6,8,9,10,12,13,18,22,23,30,31,35,36]) as well as advances in the control of discrete-time, finite-dimensional traffic flow models (see [16,17,19,28] and references therein) have motivated the study of well-posedness and control of traffic flow models on bounded domains. Both issues (well-posedness and control) for first-order models in bounded domains were studied in [4,5,33] by means of boundary conditions at the inlet and outlet that may or may not become active at certain time instants. The stabilization of equilibrium profiles for second-order models in bounded domains by means of boundary feedback was also studied in [24,38].

In the present work, a novel, hyperbolic, nonlinear, second-order, 1-D traffic flow model on a bounded domain is proposed. The arguments leading to the derivation of the model are based on the assumption that the road is relatively crowded. It consists of two quasilinear first-order PDEs with a dynamic nonlinear boundary condition that involves the time derivative of the velocity, which may be viewed as boundary relaxation, analogously to in-domain relaxation in second-order traffic flow models [1,37]. The presence of this dynamic boundary condition makes the model non-standard, and thus, the existence and uniqueness of its solutions cannot be guaranteed by using standard results (see [2,20,25]). The existence and uniqueness issues are studied in the present work. Specifically, it is shown that for all physically meaningful initial conditions the model admits a globally defined, unique, classical solution that remains positive and bounded for all times. As a result, we can guarantee that the proposed model has all of the four features mentioned in the first paragraph that are important from a traffic-theoretic point of view. The second contribution of the present work is the study of the control problem for the proposed model. Specifically, we design a simple, nonlinear, boundary feedback law, adjusting the inlet flow (via, e.g., ramp metering). The boundary feedback law employs only measurements of the inlet velocity, and consequently, the measurement requirements for implementation of the proposed controller are minimal. Moreover, it is shown that the developed control design achieves global asymptotic stabilization of arbitrary equilibria, in the sup-norm of the logarithmic deviation of the state from its equilibrium point. The efficiency of the proposed feedback law is demonstrated by means of a numerical example.

The structure of the present work is as follows: Section II is devoted to the presentation of the model and the statement of the first main result (Theorem 2.1) which guarantees, for all physically meaningful initial conditions, the existence of a globally defined, unique, classical solution that remains positive and bounded for all times. The control design and the statement of the second main result, which guarantees global stabilization of arbitrary equilibria of the model (Theorem 3.1) are given in Section III. A simple illustrating example is presented in Section IV. The proofs of the main results are omitted due to space

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limitations, but they can be found in [21] (Section 5). The concluding remarks are provided in Section IV.

Notation. Throughout this paper, we adopt the following notation.

* $\Re_+ := [0,+\infty)$. For a real number $x \in \Re$, $[x]$ denotes the integer part of x, i.e., the greatest integer which is less or equal to x.

* Let $U \subseteq \Re^n$ be a set with non-empty interior and let $\Omega \subseteq \Re$ be a set. By $C^0(U\,;\Omega)$, we denote the class of continuous mappings on U, which take values in Ω. By $C^k(U\,;\Omega)$, where $k \ge 1$, we denote the class of continuous functions on U, which have continuous derivatives of order k on U and take values in Ω. When Ω is omitted, i.e., when we write $C^k(U)$, it is implied that $\Omega = \Re$.

* Let $T \in (0,+\infty)$ and $u:[0,T]\times[0,1] \to \Re$ be given. We use the notation $u[t]$ to denote the profile at certain $t \in [0,T]$, i.e., $(u[t])(x) = u(t,x)$ for all $x \in [0,1]$. For a bounded $w:[0,1] \to \Re$ the sup-norm is given by $\|w\|_\infty := \sup_{0 \le x \le 1}\left(|w(x)|\right)$.

* $W^{2,\infty}([0,1])$ is the Sobolev space of C^1 functions on $[0,1]$ with Lipschitz derivative.

* By K we denote the class of strictly increasing continuous functions $a:\Re_+ \to \Re_+$ with $a(0)=0$. By K_∞ we denote the class of functions $a \in K$ with $\lim_{s\to+\infty} a(s) = +\infty$. By KL we denote the set of all functions $\sigma \in C^0(\Re_+ \times \Re_+\,;\Re_+)$ with the properties: (i) for each $t \ge 0$, $\sigma(\cdot,t)$ is of class K; (ii) for each $s \ge 0$, $\sigma(s,\cdot)$ is non-increasing with $\lim_{t\to+\infty}\sigma(s,t) = 0$.

II. A Non-Standard Traffic Flow Model

A. Model Description

Second-order traffic flow models involve a system of hyperbolic PDEs on the positive semiaxis. The state variables are the vehicle density $\rho(t,x)$ and the vehicle velocity $v(t,x)$, where $t \ge 0$ is time and x is the spatial variable. All traffic flow models involve the conservation equation

$$\frac{\partial \rho}{\partial t}(t,x) + v(t,x)\frac{\partial \rho}{\partial x}(t,x) + \rho(t,x)\frac{\partial v}{\partial x}(t,x) = 0 \qquad (1)$$

and an additional PDE for the velocity. In a relatively crowded road, the vehicle velocity depends heavily on the velocity of downstream vehicles. Therefore, the following equation may be appropriate for the description of the evolution of the velocity profile:

$$\frac{\partial v}{\partial t}(t,x) - c\frac{\partial v}{\partial x}(t,x) = 0 \qquad (2)$$

where $c > 0$ is a constant related to the drivers' speed of adjusting their velocity. Equation (2) may also arise as a linearization of the equation of the Aw-Rascle-Zhang model (see [1,37]) without an in-domain relaxation term. Here, we consider the model (1), (2) on a bounded domain, i.e., we assume that $x \in [0,1]$. The full model requires the specification of two boundary conditions. One boundary condition describes the inlet conditions and more particularly the effect of the inlet demand $q(t) > 0$ and takes the form

$$\rho(t,0) = h\left(\frac{q(t)}{v(t,0)}\right), \text{ for } t \ge 0 \qquad (3)$$

where $h \in C^2(\Re_+)$ is a non-decreasing function that satisfies

$$h(s) = s \text{ for } s \in [0, \rho_{\max} - \varepsilon] \text{ and } h(s) = \rho_{\max} \text{ for } s \ge \rho_{\max} \qquad (4)$$

where $\rho_{\max} > 0$ is a constant related to the physical upper bound of density in the particular road and $\varepsilon \in (0,\rho_{\max})$ is a sufficiently small constant. Notice that (3) implies that the inlet demand $q(t) > 0$ is equal to the vehicle inflow $\rho(t,0)v(t,0)$, provided that $q(t) \le \left(\rho_{\max} - \varepsilon\right)v(t,0)$. The boundary condition (3) as well as the rest of the model (1), (2) comes together with the following requirement:

$$\rho(t,x) > 0 \text{ and } v(t,x) > 0, \text{ for all } (t,x) \in \Re_+ \times [0,1] \qquad (5)$$

Condition (5) is an essential requirement for traffic flow models and it should be noticed here that some second-order traffic flow models do not meet this requirement. In what follows, we show that the proposed model meets this requirement.

In order to have a well-posed hyperbolic system, we also need a boundary condition at the outlet $x = 1$. Assuming that the flow downstream the outlet is uncongested (free), it is reasonable to assume that the relaxation term becomes dominant. So, we get

$$\frac{\partial v}{\partial t}(t,1) = -\mu\left(v(t,1) - f\left(\rho(t,1)\right)\right), \text{ for } t \ge 0 \qquad (6)$$

where $\mu > 0$ is a constant and $f \in C^1(\Re_+)$ is a positive, bounded, non-increasing function that expresses the fundamental diagram relation between density and velocity.

B. Traffic-Theoretic Features of the Model

Equations (1), (2), (3), (6) form a non-standard system of nonlinear hyperbolic PDEs. The reason that system (1), (2), (3), (6) cannot be studied by existing results in hyperbolic systems (see [2,20,25]) is the non-standard boundary condition (6). However, in what follows, we show that system (1), (2), (3), (6) exhibits unique, positive, globally defined C^1 solutions for all positive initial conditions. Moreover, we show that density and velocity are bounded from above by certain bounds that depend only on the initial conditions and the physical upper bounds of the density and velocity, i.e., $\rho_{\max}$ and $v_{\max} = f(0)$, respectively. Before we show this, it is important to emphasize that (1), (2), (3), (6):

- is a traffic flow model that can be applied to bounded domains, i.e., $x \in [0,1]$, without imposing a boundary condition with no physical meaning or assuming knowledge of the density/velocity out of the domain,
- is completely anisotropic, i.e., the velocity depends only on the velocity of downstream vehicles,
- is a hyperbolic model with two eigenvalues v and $-c$; consequently, information travels forward exactly in the same speed as traffic,
- allows only equilibria which satisfy the fundamental diagram law $v = f(\rho)$, i.e., when $q(t) \equiv q_{eq} > 0$ then the equilibrium profiles are given by $\rho(x) \equiv \rho_{eq}$, $v(x) \equiv f(\rho_{eq})$, where $\rho_{eq} > 0$ is a solution of $\rho_{eq} = h\left(\frac{q_{eq}}{f(\rho_{eq})}\right)$.

All the above features are important for a traffic flow model.

C. Characteristic Form of the System

Let $\rho_{eq} \in (0, \rho_{\max})$ be a given constant. The nonlinear transformation of the density variable

$$\rho(t,x) = \rho_{eq} \exp(w(t,x)) \frac{c + f(\rho_{eq})}{c + v(t,x)} \tag{7}$$

gives the equation

$$\frac{\partial w}{\partial t}(t,x) + v(t,x) \frac{\partial w}{\partial x}(t,x) = 0 \tag{8}$$

with the boundary conditions

$$w(t,0) = \ln\left(\rho_{eq}^{-1} h\left(\frac{q(t)}{v(t,0)} \right) \frac{c + v(t,0)}{c + f(\rho_{eq})} \right),$$

$$\frac{\partial v}{\partial t}(t,1) = -\mu \left(v(t,1) - f\left(\rho_{eq} \exp(w(t,1)) \frac{c + f(\rho_{eq})}{c + v(t,1)} \right) \right) \tag{9}$$

The hyperbolic system (2), (8), (9) is nothing else but the hyperbolic system (1), (2), (3), (6) in Riemann coordinates. Provided that the initial conditions are positive, i.e., $\rho(0,x) > 0$, $v(0,x) > 0$, for $x \in [0,1]$, we are in a position to construct a unique solution to (1), (2), (3), (6) by constructing a unique solution to (2), (8), (9) and employing the nonlinear transformation (7).

D. First Main Result

The solution of (2), (8), (9) is constructed by the following theorem. Its proof can be found in [21] (Section 5).

Theorem 2.1: *Let $a \in C^2(\Re_+ \times \Re_+)$ be a given function and let $c > 0$, $\mu \ge 0$ be given constants. Let $g \in C^1(\Re_+ \times \Re)$ be a given function for which there exists a constant $v_{\max} > 0$ such that the following inequality holds*

$$0 < g(0,w) \le g(v,w) \le v_{\max} \text{, for all } v \in \Re_+, w \in \Re \tag{10}$$

Let $\theta, \varphi \in W^{2,\infty}([0,1])$ be given functions with $\varphi(x) > 0$ for all $x \in [0,1]$, for which the equalities

$$\theta(0) = a(0, \varphi(0)),$$

$$\frac{\partial a}{\partial t}(0,\varphi(0)) + c \frac{\partial a}{\partial v}(0,\varphi(0)) \varphi'(0) = -\varphi(0)\theta'(0),$$

$$\varphi'(1) = -\mu c^{-1} \left(\varphi(1) - g(\varphi(1), \theta(1)) \right)$$

hold. Then the initial-boundary value problem

$$\frac{\partial w}{\partial t}(t,x) + v(t,x) \frac{\partial w}{\partial x}(t,x) = \frac{\partial v}{\partial t}(t,x) - c \frac{\partial v}{\partial x}(t,x) = 0,$$
$$\text{for all } (t,x) \in \Re_+ \times [0,1] \tag{11}$$

$$w(t,0) - a(t, v(t,0)) = \frac{\partial v}{\partial t}(t,1) + \mu\left(v(t,1) - g\left(v(t,1), w(t,1) \right) \right) = 0$$
$$\text{for all } t \ge 0 \tag{12}$$

$$w(0,x) - \theta(x) = v(0,x) - \varphi(x) = 0 \text{, for all } x \in [0,1] \tag{13}$$

admits a unique solution $w, v \in C^1(\Re_+ \times [0,1])$. Moreover, the solution $w, v \in C^1(\Re_+ \times [0,1])$ has Lipschitz derivatives on every compact $S \subset \Re_+ \times [0,1]$ and satisfies the following inequalities for all $(t,x) \in \Re_+ \times [0,1]$:

$$\| w[t] \|_\infty \le \max\left(B_t, \|\theta\|_\infty \right) \tag{14}$$

$$\min\left(\min_{0 \le x \le 1} (\varphi(x)), \min\left\{ g(0,w) : |w| \le \max\left(B_t, \|\theta\|_\infty \right) \right\} \right) \le v(t,x)$$
$$\le \max\left(\max_{0 \le x \le 1} (\varphi(x)), v_{\max} \right) \tag{15}$$

where

$$B_t := \max\left\{ |a(s,v)| : s \in [0,t], 0 \le v \le \max\left(\max_{0 \le x \le 1} (\varphi(x)), v_{\max} \right) \right\}$$

Remark 2.2: Theorem 2.1 shows that the appropriate space (state space) for studying the hyperbolic system (1), (2), (3), (6) is the space

$$X=\Big\{(\rho,v)\in\left(W^{2,\infty}([0,1])\right)^2 :$$
$$\left.\begin{array}{c}\min(\rho(x),v(x))>0 \; for\; all\;\; x\in[0,1],\\ cv'(1)=-\mu\left(v(1)-f(\rho(1))\right),\\ \exists a_1>0,a_2\in\Re\;\; such\;\; that\;\; \rho(0)=h(a_1),\\ v(0)\rho'(0)+\rho(0)v'(0)=a_2h'(a_1)\end{array}\right\}. \qquad (16)$$

In order to construct a solution $\left(\rho[t],v[t]\right)\in X$ of (1), (2), (3), (6) with initial conditions in $(\rho_0,v_0)\in X$, we apply Theorem 2.1 with

$$a(t,v):=\begin{cases}\ln\left(\rho_{eq}^{-1}h\left(\dfrac{q(t)}{v}\right)\dfrac{c+v}{c+f(\rho_{eq})}\right) & if\;\; v>0\\ \ln\left(\rho_{eq}^{-1}\rho_{\max}\dfrac{c+v}{c+f(\rho_{eq})}\right) & if \quad v=0\end{cases},$$

$$g(v,w):=f\left(\rho_{eq}\exp(w)\frac{c+f(\rho_{eq})}{c+v}\right),\; v_{\max}:=f(0),$$

$$\theta(x)=\ln\left(\frac{\rho_0(x)\left(c+v_0(x)\right)}{\left(c+f(\rho_{eq})\right)\rho_{eq}}\right),\; \varphi(x)=v_0(x) \text{ for all } x\in[0,1]$$

and we consider $q\in C^2\left(\Re_+;(0,+\infty)\right)$ to be the input of the model. The set of admissible inputs consists of all functions $q\in C^2\left(\Re_+;(0,+\infty)\right)$ that satisfy the compatibility conditions

$$\rho_0(0)=h\left(\frac{q(0)}{v_0(0)}\right)$$

$$v_0(0)\rho_0'(0)+\rho_0(0)v_0'(0)+h'\left(\frac{q(0)}{v_0(0)}\right)\frac{\dot q(0)}{v_0(0)}=$$
$$ch'\left(\frac{q(0)}{v_0(0)}\right)\frac{q(0)}{v_0^2(0)}v_0'(0)$$

The solution $\left(\rho[t],v[t]\right)\in X$ of (1), (2), (3), (6) is found by using the solution $\left(w[t],v[t]\right)$ of (11), (12), (13) in conjunction with formula (7). Notice that if $v_0(x)\le v_{\max}$ for all $x\in[0,1]$, then estimate (15) implies that $0<v(t,x)\le v_{\max}$ for all $(t,x)\in\Re_+\times[0,1]$ and for all admissible $q\in C^2\left(\Re_+;(0,+\infty)\right)$. Similarly, by performing more detailed calculations than those in the proof of Theorem 2.1, we are in a position to verify that if $\rho_0(x)\le\rho_{\max}\dfrac{c+v_{\max}}{c}$ for all $x\in[0,1]$, then the estimate $0<\rho(t,x)\le\rho_{\max}\dfrac{c+v_{\max}}{c}$ holds for all $(t,x)\in\Re_+\times[0,1]$ and for all admissible $q\in C^2\left(\Re_+;(0,+\infty)\right)$. When $c>>v_{\max}$, the previous estimate implies that the upper bound for density is approximately $\rho_{\max}$.

III. Controlling the Traffic Flow Model

A. Motivation for Control Design

The fact that for the case $q(t)\equiv q_{eq}>0$ the equilibrium profiles for (1), (2), (3), (6) are given by $\rho(x)\equiv\rho_{eq}$, $v(x)\equiv f(\rho_{eq})$, where $\rho_{eq}>0$ is a solution of $\rho_{eq}=h\left(\dfrac{q_{eq}}{f(\rho_{eq})}\right)$, implies that there may be multiple equilibria. For example, for the case $f(\rho)=A\exp(-b\rho)$, where $A,b>0$ are constants (that corresponds to the so-called Underwood model; see for instance [34]) if $q_{eq}\in\left[A\rho_{\max}\exp\left(-b\rho_{\max}\right),\dfrac{A}{b}\exp(-1)\right]$ and $1<b\rho_{\max}$ then there are (at least) two solutions of the equation $\rho_{eq}=h\left(\dfrac{q_{eq}}{f(\rho_{eq})}\right)$: one solution in the interval $\left(0,b^{-1}\right]$ and $\rho_{\max}$. Consequently, it is not possible to guarantee that for every initial condition $(\rho_0,v_0)\in X$ with

$$\rho_0(0)=h\left(\frac{q_{eq}}{v_0(0)}\right),$$
$$v_0(0)\rho_0'(0)+\rho_0(0)v_0'(0)=ch'\left(\frac{q(0)}{v_0(0)}\right)\frac{q_{eq}}{v_0^2(0)}v_0'(0),$$

the solution $\left(\rho[t],v[t]\right)\in X$ of (1), (2), (3), (6) with $q(t)\equiv q_{eq}>0$ will converge to a specific equilibrium profile as $t\to+\infty$. This implies lack of global asymptotic stability. Moreover, such cases are the ones that ideally one would like to have: for the case $f(\rho)=A\exp(-b\rho)$, where $A,b>0$ are constants, the ideal operation of the freeway would be exactly where the flow becomes maximized, i.e., when $\rho=b^{-1}$. Notice that in this case and if $1\le b(\rho_{\max}-\varepsilon)$, where $\varepsilon\in(0,\rho_{\max})$ is the constant involved in (4), $q_{eq}=\dfrac{A}{b}\exp(-1)$ and we have (at least) two equilibria: b^{-1} and $\rho_{\max}$. In such cases, global stabilization of a specific equilibrium profile may be achieved by boundary feedback control.

B. Collocated Boundary Control Design and Stability Analysis

The following theorem, whose proof can be found in [21] (Section 5), shows that stabilization of the equilibrium profile for a given desired equilibrium density $\rho_{eq}>0$ can be achieved by controlling the inlet flow. It is important to notice that the stabilizing feedback law depends only on the inlet velocity. Therefore, the measurement requirements for the implementation of the proposed boundary feedback law are minimal.

Theorem 3.1: *Consider the nonlinear traffic flow model (1), (2), (3), (6) and let* $\rho_{eq} > 0$ *be the desired equilibrium density. Suppose that* $\rho_{eq} \le \frac{c}{c+f(\rho_{eq})}\left(\rho_{\max}-\varepsilon\right)$ *and that the following inequality holds:*

$$\left(v - f\left(\rho_{eq}\frac{c+f(\rho_{eq})}{c+v}\right)\right)\left(v - f\left(\rho_{eq}\right)\right) > 0 \text{, for all } v \ge 0,$$

$$v \ne f(\rho_{eq}) \tag{17}$$

Then there exists a function $Q \in KL$ *such that for every* $(\rho_0, v_0) \in X$ *for which the equalities* $\rho_0(0) = \rho_{eq}\frac{c+f(\rho_{eq})}{c+v_0(0)}$, $\rho_0'(0) = -\frac{-\rho_0(0)}{c+v_0(0)}v_0'(0)$ *hold, the initial-boundary value problem (1), (2), (3), (6) with*

$$q(t) = \rho_{eq} v(t,0)\frac{c+f(\rho_{eq})}{c+v(t,0)} \tag{18}$$

$$\rho(0,x) - \rho_0(x) = v(0,x) - v_0(x) = 0 \text{, for all } x \in [0,1] \tag{19}$$

admits a unique solution $\rho, v \in C^1\left(\Re_+ \times [0,1]\right)$, *with* $\left(\rho[t], v[t]\right) \in X$ *for all* $t \ge 0$ *satisfying the following estimate for all* $t \ge 0$:

$$\max_{0\le x\le 1}\left(\left|\ln\left(\frac{\rho(t,x)}{\rho_{eq}}\right)\right|\right) + \max_{0\le x\le 1}\left(\left|\ln\left(\frac{v(t,x)}{f(\rho_{eq})}\right)\right|\right) \le$$

$$Q\left(\max_{0\le x\le 1}\left(\left|\ln\left(\frac{\rho_0(x)}{\rho_{eq}}\right)\right|\right) + \max_{0\le x\le 1}\left(\left|\ln\left(\frac{v_0(x)}{f(\rho_{eq})}\right)\right|\right), t\right) \tag{20}$$

Remark 3.2: Notice that inequality (17) holds automatically for $v \ge f(0)$ and $v = 0$. Thus, inequality (17) is equivalent to the following implications

$$\rho_{eq} > \rho > \rho_{eq}\frac{c+f(\rho_{eq})}{c+f(0)} \Rightarrow \rho_{eq}\left(c+f(\rho_{eq})\right) > \rho(c+f(\rho))$$

$$\rho_{eq} < \rho < \rho_{eq}\frac{c+f(\rho_{eq})}{c} \Rightarrow \rho_{eq}\left(c+f(\rho_{eq})\right) < \rho(c+f(\rho))$$

Therefore, a sufficient condition for (17) is the assumption that the function $F(\rho) := \rho\left(c+f(\rho)\right)$ is increasing on the interval $\left(\rho_{eq}\frac{c+f(\rho_{eq})}{c+f(0)}, \rho_{eq}\frac{c+f(\rho_{eq})}{c}\right)$. Consequently, (17) holds automatically when $c + f(\rho) + \rho f'(\rho) > 0$ for all $\rho \in \left(\rho_{eq}\frac{c+f(\rho_{eq})}{c+f(0)}, \rho_{eq}\frac{c+f(\rho_{eq})}{c}\right)$. For example, when $f(\rho) = A\exp(-b\rho)$, where $A, b > 0$ are constants (Underwood model), we guarantee that (17) holds when the inequality $c\exp(b\rho) + A > Ab\rho$ holds for

$$\rho \in \left(\rho_{eq}\frac{c+A\exp(-b\rho_{eq})}{c+A}, \rho_{eq}\frac{c+A\exp(-b\rho_{eq})}{c}\right).$$

It should be noticed that in this case (17) holds automatically when the velocity ratio A/c is sufficiently small no matter what ρ_{eq} is: when $c\exp(2) \ge A$ the function $F(\rho) := \rho\left(c + A\exp(-b\rho)\right)$ is increasing on $\Re_+$.

Remark 3.3: When the compatibility conditions $\rho_0(0) = \rho_{eq}\frac{c+f(\rho_{eq})}{c+v_0(0)}$, $\rho_0'(0) = -\frac{-\rho_0(0)}{c+v_0(0)}v_0'(0)$ do not hold, then we satisfy the compatibility conditions implied by Theorem 2.1, namely we find $a > 0$ and $b \in \Re$ so that

$$\rho_0(0) = h\left(\frac{a}{v_0(0)}\right)$$

$$v_0(0)\rho_0'(0) + \rho_0(0)v_0'(0) = \frac{acv_0'(0) - bv_0(0)}{v_0^2(0)}h'\left(\frac{a}{v_0(0)}\right).$$

In such a case, we must modify the control input so that the compatibility conditions hold; the control input can be given by the formula

$$q(t) = \left(1 - g_T(t)\right)\left(a + bt\right) + g_T(t)\rho_{eq}v(t,0)\frac{c+f(\rho_{eq})}{c+v(t,0)}$$

where $T > 0$ is a small constant that satisfies $a + bT > 0$ and $g_T : \Re \to \Re$ is defined by $g_T(t) = 0$ for $t \le 0$, $g_T(t) = 1$ for $t \ge T$ and

$$g_T(t) = \frac{\exp(-t^{-1})}{\exp(-t^{-1}) + \exp(-(T-t)^{-1})}$$

for $t \in (0, T)$.

Remark 3.4: Estimate (20) is a stability estimate in the sup-norm of the logarithmic deviation of the state from its equilibrium values. The use of logarithmic deviation variables is customary for systems with positive state values (e.g., biological systems, see [18]).

IV. ILLUSTRATIVE EXAMPLE

We consider model (1), (2), (3), (6) with $f(\rho) = \frac{2}{5}\exp(1-\rho)$ (Underwood model), $c = 5$, $\mu = 10$, $\rho_{\max} = 27/10$, $\varepsilon = 10^{-6}$, $h(s) = s(1-g(s)) + \rho_{\max}g(s)$ for $s \ge 0$, where $g(s) = 0$, for $s \in [0, \rho_{\max} - \varepsilon]$, $g(s) = 1$, for $s \ge \rho_{\max}$ and

$$g(s) = \frac{\exp\left(-(s+\varepsilon-\rho_{\max})^{-1}\right)}{\exp\left(-(s+\varepsilon-\rho_{\max})^{-1}\right) + \exp\left(-(\rho_{\max}-s)^{-1}\right)},$$

for $s \in (\rho_{\max} - \varepsilon, \rho_{\max})$.

The objective is to stabilize the equilibrium point that maximizes the vehicle flow $\rho(x) \equiv \rho_{eq} = 1$, $v(x) \equiv f(\rho_{eq}) = 2/5$. It should be noticed that the open-loop system (1), (2), (3), (6) with $q(t) \equiv q_{eq} = 2/5$ has two equilibria: one is the desired equilibrium, and the other one is the fully congested equilibrium $\rho(x) \equiv \rho_{\max} = \frac{27}{10}$, $v(x) \equiv f(\rho_{\max}) = \frac{2}{5}\exp\left(-\frac{17}{10}\right)$. Numerical experiments show that the fully congested equilibrium attracts the solution of the open-loop system (1), (2), (3), (6) with $q(t) \equiv q_{eq} = \frac{2}{5}$ for many initial conditions. We choose the initial conditions $\rho_0(x) = 1$ for $x \in [0, 9/20]$, $\rho_0(x) = 2$, for $x \in [1/2, 1]$,

$$\rho_0(x) = 1 + \frac{\exp\left(-(x - 9/20)^{-1}\right)}{\exp\left(-(x - 9/20)^{-1}\right) + \exp\left((x - 1/2)^{-1}\right)},$$

for $x \in \left(\frac{9}{20}, \frac{1}{2}\right)$, and $v_0(x) = f(\rho_0(x))$, for $x \in [0,1]$. For this particular initial condition (but also for many others) the solution of the open-loop system (1), (2), (3), (6) with $q(t) \equiv q_{eq} = 2/5$ converges to the fully congested equilibrium

$$\rho(x) \equiv \rho_{\max} = \frac{27}{10}, \; v(x) \equiv f(\rho_{\max}) = \frac{2}{5}\exp\left(-\frac{17}{10}\right).$$

The deviation of the solution from the desired equilibrium is shown in Fig. 1, where the evolution of the sup-norm of the logarithmic deviation from the desired equilibrium $X(t) := \max_{0 \le x \le 1}\left(\left|\ln\left(\frac{\rho(t,x)}{\rho_{eq}}\right)\right|\right) + \max_{0 \le x \le 1}\left(\left|\ln\left(\frac{v(t,x)}{f(\rho_{eq})}\right)\right|\right)$ is shown for the open-loop system (1), (2), (3), (6) with $q(t) \equiv q_{eq} = 2/5$.

In this case we can apply Theorem 3.1, since the condition $\rho_{eq} \le \frac{c}{c + f(\rho_{eq})}(\rho_{\max} - \varepsilon)$ as well as condition (17) hold (recall Remark 3.2). Fig. 1 and Fig. 2 show the evolution of the sup-norm of the logarithmic deviation from the desired equilibrium for the open- and closed-loop systems, respectively.

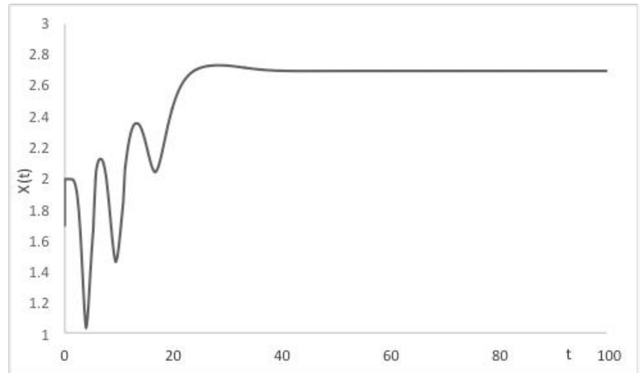

Fig. 1: Evolution of the sup-norm of the logarithmic deviation from the desired equilibrium

$$X(t) := \max_{0 \le x \le 1}\left(\left|\ln\left(\frac{\rho(t,x)}{\rho_{eq}}\right)\right|\right) + \max_{0 \le x \le 1}\left(\left|\ln\left(\frac{v(t,x)}{f(\rho_{eq})}\right)\right|\right)$$

for the open-loop system (1), (2), (3), (6) with $q(t) \equiv q_{eq} = 2/5$.

Fig. 3 and Fig. 4 show the convergence of the solution to the equilibrium profile $\rho(x) \equiv \rho_{eq} = 1$. It should be noted that at time $t = 6.58$, the solution has become identical (up to numerical accuracy) to the desired equilibrium. This is clear from Fig. 2, where it is shown the evolution of the sup-norm of the logarithmic deviation from the desired equilibrium

$$X(t) := \max_{0 \le x \le 1}\left(\left|\ln\left(\frac{\rho(t,x)}{\rho_{eq}}\right)\right|\right) + \max_{0 \le x \le 1}\left(\left|\ln\left(\frac{v(t,x)}{f(\rho_{eq})}\right)\right|\right)$$

for the closed-loop system (1), (2), (3), (6) with (18). Fig. 5 shows the time evolution of the control input $q(t)$. The control input tries to keep the inlet density close to 1, while the heavy congestion belt is "washed out" slowly (due to small vehicle velocity in the congestion belt).

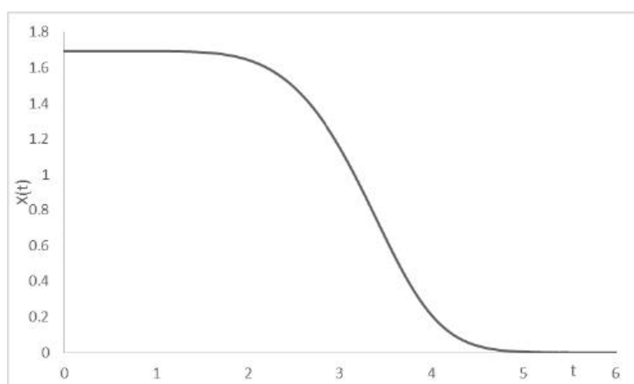

Fig. 2: Evolution of the sup-norm of the logarithmic deviation from the desired equilibrium

$$X(t) := \max_{0 \le x \le 1}\left(\left|\ln\left(\frac{\rho(t,x)}{\rho_{eq}}\right)\right|\right) + \max_{0 \le x \le 1}\left(\left|\ln\left(\frac{v(t,x)}{f(\rho_{eq})}\right)\right|\right)$$

for the closed-loop system (1), (2), (3), (6) with (18).

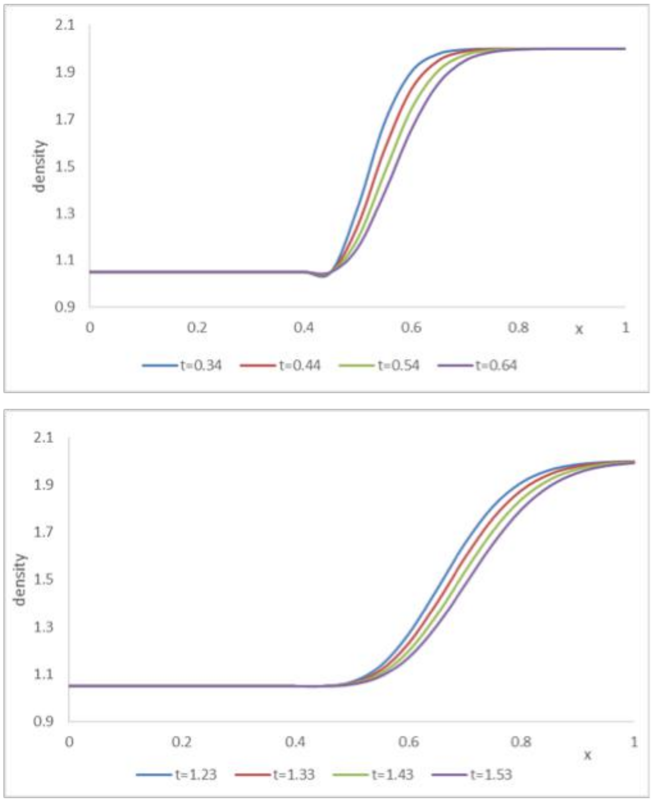

Fig. 3: Density profiles at various time instants for the closed-loop system (1), (2), (3), (6) with (18).

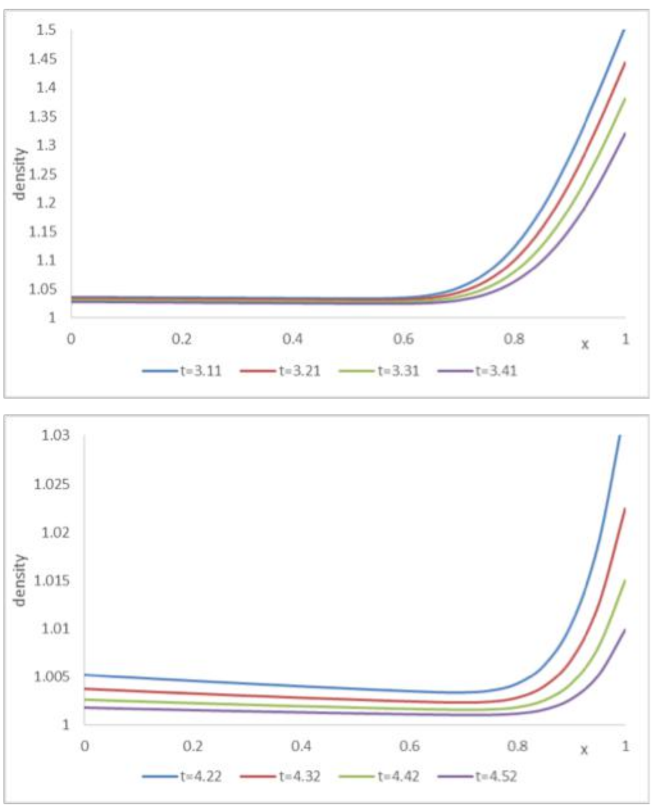

Fig. 4: Density profiles at various time instants for the closed-loop system (1), (2), (3), (6) with (18).

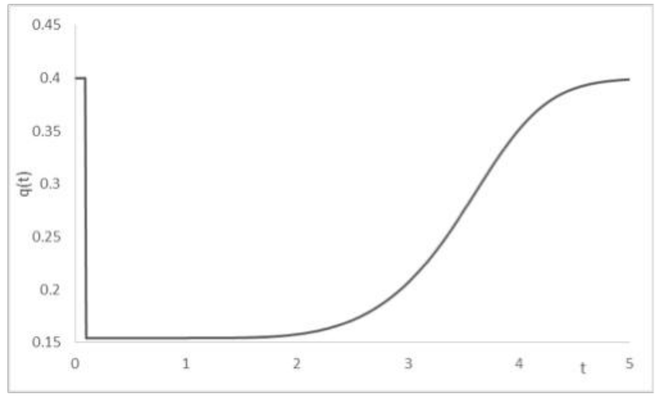

Fig. 5: The evolution of the control input $q(t)$ for the closed-loop system (1), (2), (3), (6) with (18).

V. CONCLUDING REMARKS

The paper provides results for a non-standard, hyperbolic traffic flow model on a bounded domain. The model has been developed for relatively crowded roads and consists of two first-order, hyperbolic PDEs with a dynamic boundary condition, which involves the time derivative of the velocity. Although simple, the proposed model has features that are important from a traffic-theoretic point of view: it is completely anisotropic, i.e., the velocity depends only on the velocity of downstream vehicles, and is a hyperbolic model for which information travels forward exactly at the same speed as traffic. It has been shown that for all physically meaningful initial conditions the model admits a globally defined, unique, classical solution that remains positive and bounded for all times (Theorem 2.1 and Remark 2.2). Moreover, it has been shown that global stabilization in the sup-norm of the logarithmic deviation of the state from its equilibrium point can be achieved for arbitrary equilibria by means of an explicit boundary feedback law which adjusts continuously the inlet flow (Theorem 3.1). It is important to notice that the stabilizing feedback law depends only on the inlet velocity. Therefore the measurement requirements for the implementation of the proposed boundary feedback law are minimal. The efficiency of the proposed boundary feedback was demonstrated by means of a numerical example.

Future work may involve the development of more complicated models, retaining the important characteristics of the proposed model, to capture secondary features of traffic flow dynamics. Another direction for future research is the use of sampled-data boundary feedback boundary for the stabilization of unstable equilibria.

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