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Energy Tracking for the Sine-Gordon Equation with Dissipation via Boundary Control*

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Abstract— **The energy tracking problem for the one-dimensional sine-Gordon equation with dissipation via boundary control is posed. With the use of the Speed-Pseudogradient method we design a family of control laws for solving this problem. An estimate of the tracking error via the derivative of a prespecified time-varying energy level is obtained. The convergence of the tracking error to zero is proved under the assumptions that the dissipation is absent and the derivative of the prespecified energy level eventually vanishes.**

I. INTRODUCTION

The energy control problems have attracted an interest of control theorists and engineers since mid-1990s. They were systematically studied in [1], [2] for finite-dimensional Hamiltonian systems. In a number of papers, energy control of pendulum systems was investigated [3], [4], [5]. At first glance, the energy control seems simple compared to regulation or tracking problems, since it means the stabilization of a hypersurface rather than the stabilization of 0- or 1-dimensional sets. In actuality, the problem is more complex due to complicated geometry of the constant energy surface that does not permit global linearization. The problem has numerous applications: stabilization of unstable pendulums [5], control of vibration set ups [6], escape from a potential well [7], controlled dissociation of molecules [8], etc. However, to the best of our knowledge the existing papers deal only with the case of finite-dimensional controlled system models.

Recently, the dynamical properties of the sine-Gordon equation have attracted a lot of attention of researchers, see, e.g., the monograph [9] and the references therein. The sine-Gordon model provides a number of interesting examples of complex nonlinear behaviour: solitons, kinks, antikinks, breathers, etc. The sine-Gordon model also serves for modelling various physical processes in nonlinear optics (propagation of an optical pulse in fibre waveguide [10]), in mechanics (transition from a static to a dynamic frictional regime [11]), etc.

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A number of results were obtained in control of sine-Gordon systems. In the first papers only stabilization problem was studied [12], [13], [14]. In [8] the energy control problem was first introduced and the speed-gradient based energy control algorithm was proposed.

Recently, the dynamics of the sine-Gordon energy control system was studied rigorously [15]. In [16] the problem of boundary energy control was examined. However, up to authors' knowledge, there are no papers dedicated to rigorous analysis of energy tracking problem, when the desired energy level is time-varying.

This paper is, perhaps, the first attempt to attack the energy tracking problem for the sine-Gordon equation.

II. PROBLEM FORMULATION

In this paper we study the one-dimensional sine-Gordon equation with dissipation and with the following initial and boundary conditions:

$$z_{tt}(t,x) + \rho z_t(t,x) - k z_{xx}(t,x) + \beta \sin z(t,x) = 0, \tag{1}$$

$$z(0,x) = z^0(x), \quad z_t(0,x) = z^1(x), \tag{2}$$

$$z(t,0) = 0, \quad z_x(t,1) = u(t), \tag{3}$$

$$y(t) = \big(z_t(t,1), H(z(t))\big). \tag{4}$$

Here $\beta \ge 0$, $\rho \ge 0$, and $k > 0$ are given parameters, $x \in [0,1]$, $t \ge 0$, $u(t)$ is a control input, $y(t)$ is the output, $z^0, z^1 \colon [0,1] \to \mathbb{R}$ are given functions, and

$$H(z) = \int_0^1 \left(\frac{z_t^2}{2} + k\frac{z_x^2}{2} + \beta(1 - \cos z) \right) dx$$

is the Hamiltonian for the sine-Gordon equation. One can easily verify that the Hamiltonian $H(z)$ is preserved along solutions of the unforced system, provided $\rho = 0$. Furthermore, $H(z)$ is a non-negative function, and $H(z) = 0$ if and only if $z = 0$. Therefore $H(z)$ can be viewed as *the energy* of a solution $z(t,x)$ of equation (1) (or *system's energy*) at time t.

Let a bounded, continuously differentiable function $H_* \colon \mathbb{R}_+ \to \mathbb{R}_+$ with bounded derivative be given (here, as usual, $\mathbb{R}_+ = [0, +\infty)$). Denote $h(\tau) = \sup_{t \ge \tau} \big| H_*'(t) \big|$ for all $\tau \ge 0$. We pose the following control problem: find a control law $u(t)$ such that for any $T \ge 0$ there exists $\tau \ge T$ for which

$$\big| H(z(t)) - H_*(t) \big| \le \theta \cdot h(\tau) \quad \forall t \ge \tau, \tag{5}$$

where $\theta > 0$ is a positive constant, and $z(t)$ is a solution of (1)–(4). We will also consider the stronger control objective of the form

$$\lim_{t \to +\infty} \big| H(z(t)) - H_*(t) \big| = 0.$$

Thus, the control objective is to track the time-varying energy level $H_*(t)$ in the system (1)–(4). Note that the set $\mathscr{M}(\tau) = \{z(t,x)\colon H(z) = H_*(\tau)\}$ is an infinite dimensional manifold. Therefore our aim is to track not a given reference trajectory $z_*(t,x)$, but a given reference manifold that is changing in time.

Remark 1: It should be noted that all results below can be easily extended to the case

$$z_t(t,1) = u(t), \quad y(t) = \big(z_x(t,1), H(z(t))\big) \quad t \geq 0,$$

i.e. one can "swap" $z_t(t,1)$ and $z_x(t,1)$.

III. Speed-Gradient Algorithm

Let us design the required control law with the use of the Speed-Pseudogradient algorithm [17], [8] (or, equivalently, the nonsmooth Speed-Gradient algorithm [18], [19], [20]). To this end, introduce the goal function

$$Q(z,t) = \frac{1}{2}\big(H(z) - H_*(t)\big)^2 \tag{6}$$

that measures the difference between the current energy level $H(z)$ and the reference energy level $H_*(t)$. Formally differentiating (6) we obtain that the speed of change of the goal function $Q(z,t)$ along solutions of the system (1)–(4) has the form

$$\begin{aligned}
\frac{d}{dt}Q(z,t) &= \big(H(z) - H_*(t)\big) \\
&\quad \left[\int_0^1 \big(z_t z_{tt} + k z_x z_{xt} + \beta \sin z z_t\big)\,dx - \frac{d}{dt}H_*(t)\right] \\
&= \big(H(z) - H_*(t)\big) \\
&\quad \left[\int_0^1 \Big(z_t\big(-\rho z_t + k z_{xx} - \beta \sin z\big) - k z_{xx} z_t + \beta \sin z z_t\Big)\,dx \right.\\
&\quad \left. + k z_x(t,1) z_t(t,1) - k z_x(t,0) z_t(t,0) - \frac{d}{dt}H_*(t)\right] \\
&= \big(H(z) - H_*(t)\big)\left[-\rho\int_0^1 z_t^2\,dx + k u(t) z_t(t,1) - \frac{d}{dt}H_*(t)\right].
\end{aligned}$$

According to the Speed-Pseudogradient algorithm one defines the control law as follows

$$u(t) = -\gamma\psi(z,t),$$

where $\gamma > 0$ is a scalar gain, and $\psi(z,t)$ is a function satisfying the so-called *pseudogradient* (or *sharp angle*) condition

$$\psi(z,t)\cdot\frac{\partial}{\partial u}\frac{d}{dt}Q(z,t) \geq 0.$$

Observe that

$$\frac{\partial}{\partial u}\frac{d}{dt}Q(z,t) = k\big(H(z) - H_*(t)\big)z_t(t,1).$$

Therefore we define $\psi(z,t) = \operatorname{sign}(H(z) - H_*(t))z_t(t,1)$, and obtain the following control law

$$u(t) = -\gamma \operatorname{sign}\big(H(z) - H_*(t)\big)z_t(t,1). \tag{7}$$

Let us point out that this control law can be viewed as the nonsmooth Speed-Gradient algorithm corresponding to the nonsmooth goal function $Q(z,t) = |H(z) - H_*(t)|$.

IV. Performance of the Control Law

Let us study the performance of the system (1)–(4) with control law (7). Hereinafter, we suppose that the following assumption on the well-posedness of this system holds true.

Assumption 1: There exists a nonempty set $\mathscr{H}_0 \subseteq H^2[0,1] \times H^1[0,1]$ of "sufficiently smooth" initial conditions, such that for any $(z^0, z^1) \in \mathscr{H}_0$ there exists a unique "sufficiently regular" solution $z(t,x)$ of the system (1)–(4), (7) such that

1) $z(t)$ is defined on the maximal interval of existence $[0, T_{\max})$, and if $T_{\max} < +\infty$, then $H(z(t)) \to +\infty$ as $t \to T_{\max}$,
2) the function $H(z(\cdot))$ is locally absolutely continuous.

Remark 2: Let us note that the question whether a solution of (1)–(4), (7) exists is particularly challenging, since (1) is a nonlinear hyperbolic equation, and the boundary condition (7) is nondissipative, and nonlinearly depends on the derivative $z_t(t,1)$. As far as the authors could check, all standard method for proving the existence of solutions of such equations fail in this case. Therefore we pose the above assumption as a challenging problem for future research.

Let, as above, $h(\tau) = \sup_{t\geq\tau}|H_*'(t)|$ for any $\tau \geq 0$. The following result holds true.

Theorem 1. *Let $0 < \underline{H} \leq H_*(t) \leq \overline{H} < +\infty$ for all $t \geq 0$, and let $(z^0, z^1) \in \mathscr{H}^0$ be such that $H(z(0)) \neq 0$ (i.e. the initial conditions are nonzero). Suppose also that $0 \leq \beta < k\pi^2/4$, and*

$$\eta := \frac{k\pi^2 - 4\beta}{k\pi^2 + 4\beta} > \max\left\{\frac{\rho}{k}, \frac{2\rho}{\min\{1,k\}} + \rho\right\}. \tag{8}$$

Then for all $\varepsilon > 0$ such that $2\rho/(\eta - \rho) < \varepsilon < \min\{1,k\}$, and for any

$$\gamma \in \left[\frac{1 - \sqrt{1 - \varepsilon^2/k}}{\varepsilon}, \frac{1 + \sqrt{1 - \varepsilon^2/k}}{\varepsilon}\right] \tag{9}$$

a solution of (1)–(4), (7) is defined on $\mathbb{R}_+$, and the following statements hold true:

1) *for any $\tau \geq 0$ one has $H(z(t)) = H_*(t)$ for some $t \geq \tau$;*
2) *there exists $\theta > 0$ such that for any $\tau \geq 0$ one has*

$$\begin{aligned}
&\min\big\{0, H(z(\tau)) - H_*(\tau)\big\} - \theta\cdot h(\tau) \\
&\quad - \big(e^{-2\rho\theta} - 1\big)H(z(\tau)) \leq H(z(t)) - H_*(t) \\
&\quad \leq \max\{0, H(z(\tau)) - H_*(\tau)\} + \theta\cdot h(\tau)
\end{aligned} \tag{10}$$

for all $t \geq \tau$. In particular, for any $T \geq 0$ there exists $\tau \geq T$ such that

$$\big|H(z(t)) - H_*(t)\big| \leq \theta\cdot h(\tau) + \big(e^{-2\rho\theta} - 1\big)H_{\max}$$

for all $t \geq \tau$, where $H_{\max} = \max\{H(z(0)), \overline{H}\}$.

Furthermore, for any $\varepsilon \in (0, \min\{1,k\})$ there exists $\theta > 0$ such that inequality (10) holds true for all $\gamma > 0$ satisfying (9) and for all $\rho \geq 0$ satisfying the inequality

$$\rho \leq \min\left\{k\eta_0, \frac{\varepsilon\eta_0}{2+\varepsilon}\right\} \tag{11}$$

for some $\eta_0 \in (0, \eta)$.

In particular, if $\rho = 0$, then for any $\gamma > 0$ there exists $\theta > 0$ such that for all $T \geq 0$ one has

$$\big|H(z(t)) - H_*(t)\big| \leq \theta \cdot h(\tau) \quad \forall t \geq \tau \tag{12}$$

for some $\tau \geq T$. If, additionally, $h(\tau) \to 0$ as $\tau \to +\infty$, then the control goal

$$\lim_{t \to +\infty} \big|H(z(t)) - H_*(t)\big| = 0 \tag{13}$$

is achieved.

Remark 3: Let us note that $\theta > 0$ in (10) does not depend on the function $H_*(t)$ itself. This constant depends only on the parameters of the system ($k > 0$, $\rho \geq 0$ and $\beta \geq 0$), upper and lower bounds $\underline{H}$ and $\overline{H}$, initial conditions and the control gain $\gamma > 0$.

Let us informally discuss the assumptions of Theorem 1. Note that if $H(z(\tau)) = 0$ (or, equivalently, if $z(\tau) = 0$) for some $\tau \geq 0$, then $z(t) = 0$ for all $t \geq 0$, provided a solution of (1)–(4), (7) is unique. Therefore, we use the assumption that the reference energy $H_*(t)$ is bounded away from zero ($H_*(\cdot) \geq \underline{H}$) in order to guarantee that the energy $H(z(t))$ does not vanish. The upper bound $H_*(t) \leq \overline{H} < +\infty$ is essential for obtaining estimates of the tracking error (10). Note that the independence of θ in this estimate on the dissipation coefficient $\rho \geq 0$ satisfying (11) for any fixed $\varepsilon \in (0, \min\{1, k\})$ means that the smaller is the dissipation in the system (i.e. the dissipation coefficient $\rho \geq 0$), the better is the lower estimate in (10).

The inequality $0 \leq \beta < k\pi^2/4$ is utilized in order to ensure that the nonlinearity $\beta \sin z$ does not affect the linear dynamics too much, while inequality (8) guarantees that the dissipation, i.e. the parameter $\rho \geq 0$, is not too great so that the control action can (partly) compensate the effect of dissipation in the case when the energy $H(z(t))$ must be increased. Finally, bounds (9) on the gain $\gamma > 0$ seem to be artificial, since they arise more as a deficiency of the proof than a natural assumption. Moreover, it should be pointed out that all assumptions on the parameters of the system (1)–(4), (7) arise as a result of the use of a particular technique (Lyapunov function) in the proof of Theorem 1, and a physical meaning of these assumptions is unclear.

Let us also informally explain why the condition $\lim_{t \to +\infty} H'_*(t) = 0$ is, in essence, necessary for the achievement of control goal (13). The necessity of this assumption follows from the structure of the boundary control law (7) itself. Indeed, the sine-Gordon equation describes the propagation of nonlinear waves. Every time a peak of a wave reaches the boundary $x = 1$, the control vanishes, since it is proportional to $z_t(t, 1)$ (see (7)). If at this point the reference energy level $H_*(t)$ starts changing rapidly (i.e. $H'_*(t)$ is large), then the error $|H(z(t)) - H_*(t)|$ inevitably increases. When the peak of a wave passes the boundary $x = 1$, the control action increases allowing system's energy $H(z(t))$ to catch the reference energy level $H_*(t)$. This control lag results in the fact that one can only prove the achievement of weaker control goal (5) in the general case, i.e. one can only obtain an upper estimate for the tracking error via the upper estimate of the derivative of the reference energy level $H_*(t)$. In order to ensure the achievement of original control goal (13) one has to suppose that the speed of change of the reference energy level $H_*(t)$ slows down with time and tends to zero as $t \to +\infty$.

Let us also note that control goal (12) cannot be achieved in the case of non-zero dissipation for a similar reason. Namely, every time a peak of a wave reaches the boundary $x = 1$, the control vanishes, while the energy dissipates continuously, which results in the increase of the error $|H(z(t)) - H_*(t)|$, provided $H_*(t) > H(z(t))$.

Control law (7) is, obviously, nonsmooth, which may result in chattering (see the results of numerical simulation in the case when $H_*(t)$ is constant in [16]). In order to avoid this problem one can utilize the continuous Speed-Pseudogradient control law of the form

$$u(t) = -\gamma\psi\big(H(z) - H_*(t)\big) z_t(t, 1), \tag{14}$$

where $\psi\colon \mathbb{R} \to \mathbb{R}$ is an odd continuous nondecreasing function such that $\psi(0) = 0$ and $\psi(s)s > 0$ for all $s \neq 0$. However, observe that this control vanishes not only when $z_t(t, 1)$ tends to zero, but also as $H(z(t))$ approaches $H_*(t)$. Therefore it is natural to expect that control law (14) performs worse than control law (7) in the presence of dissipation due to its inability to compensate the effect of dissipation as $H(z(t))$ tends to $H_*(t)$.

Let the same assumption on the well-posedness of the system (1)–(4), (14) as Assumption 1 be valid.

Theorem 2. *Let $0 < \underline{H} \leq H_*(t) \leq \overline{H} < +\infty$ for all $t \geq 0$, and let $(z^0, z^1) \in \mathscr{H}^0$ be such that $H(z(0)) \neq 0$. Suppose also that $0 \leq \beta < k\pi^2/4$, and*

$$\eta := \frac{k\pi^2 - 4\beta}{k\pi^2 + 4\beta} > \max\left\{\frac{\rho}{k}, \frac{2\rho}{\min\{1, k\}} + \rho\right\}.$$

Then for any $\gamma > 0$ a solution of (1)–(4), (14) is defined on $\mathbb{R}_+$, and $H(z(\cdot))$ is bounded. Moreover, for any $\Delta \in (0, \underline{H})$ such that

$$\frac{2\rho}{\eta - \rho} < \min\left\{1, k, \frac{\psi(\Delta)}{\psi(H_{\max})}\sqrt{k}\right\}, \tag{15}$$

there exist $0 < \gamma_1(\Delta) < \gamma_2(\Delta) < +\infty$ such that for all $\gamma \in (\gamma_1(\Delta), \gamma_2(\Delta))$ the following statements hold true:

1) *for any $\tau \geq 0$ one has $|H(z(t)) - H_*(t)| < \Delta$ for some $t \geq \tau$;*
2) *there exists $\theta > 0$ such that for any $\tau \geq 0$ one has*

$$\begin{aligned} \min\big\{-\Delta, H(z(\tau)) - H_*(\tau)\big\} - \theta \cdot h(\tau) \\ - \big(e^{-2\rho\theta} - 1\big)H_{\max} \leq H(z(t)) - H_*(t) \\ \leq \max\big\{\Delta, H(z(\tau)) - H_*(\tau)\big\} + \theta \cdot h(\tau) \end{aligned} \tag{16}$$

for all $t \geq \tau$, where $H_{\max} = \max\{H(z(0)), \overline{H}\}$. In particular, for any $T \geq 0$ there exist $\tau \geq T$ such that

$$\big|H(z(t)) - H_*(t)\big| \leq \Delta + \theta \cdot h(\tau) + \big(e^{-2\rho\theta} - 1\big)H_{\max}$$

for all $t \geq \tau$.

If $\rho = 0$, then for any $\Delta > 0$ and $\gamma > 0$ there exists $\theta > 0$ such that for all $T \geq 0$ one has

$$\left|H(z(t)) - H_*(t)\right| \leq \Delta + \theta \cdot h(\tau) \quad \forall t \geq \tau$$

for some $\tau \geq T$. If, additionally, $h(\tau) \to 0$ as $\tau \to +\infty$, then the control goal

$$\lim_{t\to+\infty} \left|H(z(t)) - H_*(t)\right| = 0$$

is achieved.

Remark 4: As in Theorem 1, the constant $\theta > 0$ from (16) does not depend on $\gamma \in (\gamma_1(\Delta), \gamma_2(\Delta))$ and any dissipation coefficient $\rho \geq 0$ smaller than some $\rho_0 > 0$. Furthermore, the constant θ depends only on the parameters of the problem and the control gain $\gamma > 0$, and does not depend on the function $H_*(t)$ itself. Additionally, θ from the theorem above tends to θ from Theorem 1 as $\Delta \to 0$.

Note that Theorem 2 is very similar to Theorem 1 with the only difference being the additional constant error $\Delta > 0$ in the estimates of the tracking error $|H(z(t)) - H_*(t)|$. Moreover, note that the minimal value of this constant error $\Delta > 0$ depends on the dissipation parameter $\rho \geq 0$ due to (15). In other words, a sufficiently small tracking error can only be achieved in the case when the dissipation parameter $\rho \geq 0$ is small enough, which agrees with the discussion above.

Let us also point out that the choice of the function ψ significantly effects the tracking error predicted by the theorem. Namely, let $\psi(s) = \mathrm{sign}(s)|s|^\alpha$ for any $\alpha > 0$, and suppose that $\overline{H} \geq 1$. Then according to (15) one can improve the tracking error (namely, make $\Delta > 0$ smaller) by choosing a smaller value of the parameter $\alpha > 0$. In particular, the constant error $\Delta > 0$ can be made arbitrarily small by a proper choice of the parameter α. Note that in the limiting case $\alpha = 0$ one can set $\Delta = 0$ by virtue of Theorem 1.

V. The Proof of Theorem 1

The proof of Theorem 1 is very long, but straightforward. We divide it into several lemmas, and omit some details for the sake of shortness. The interested reader can easily fill in the gaps. Let us also note that the proof of Theorem 2 is, in essence, the same as the proof of Theorem 1.

Hereinafter, we suppose that $0 < \underline{H} \leq H_*(t) \leq \overline{H} < +\infty$ for all $t \geq 0$, and $H(z(0)) \neq 0$. Let $z(t)$ be a solution of (1)–(4), (7). It is easy to check that for any $t \in [0, T_{\max})$ such that $H(z(t)) \neq H_*(t)$ one has

$$\frac{d}{dt} H(z(t)) = -\rho \int_0^1 z_t^2 dx + u(t) z_t(t,1). \quad (17)$$

Utilizing this equality one can easily check that the following result holds true.

Lemma 1. *The the solution $z(t)$ of (1)–(4), (7) is defined for all $t \geq 0$, and for any $t_1 \geq 0$ and $t_2 > t_1$ one has $H(z(t)) \leq \max\{H(z(t_1)), \overline{H}\}$ for all $t \in [t_1, t_2]$.*

Being inspired by the ideas of Kobayashi [21], [12], [13], for any $\varepsilon > 0$ introduce two Lyapunov-like function for the system (1)–(4), (7) of the form

$$V_\varepsilon^+(t) = H(z(t)) + \varepsilon q(t), \quad V_\varepsilon^-(t) = H(z(t)) - \varepsilon q(t),$$

where $q(t) = \int_0^1 x z_t z_x \, dx$. One uses the function $V_\varepsilon^+(t)$ in the case $H(z(t)) > H_*(t)$, and the function $V_\varepsilon^-(t)$ in the case $H(z(t)) < H_*(t)$.

Lemma 2. *Let $0 \leq \beta < k\pi^2/4$ and*

$$\eta := \frac{k\pi^2 - 4\beta}{k\pi^2 + 4\beta} > \max\left\{\frac{\rho}{k}, \frac{2\rho}{\min\{1,k\}} + \rho\right\}. \quad (18)$$

Then for all $\varepsilon > 0$ such that $2\rho/(\eta - \rho) < \varepsilon < \min\{1,k\}$, and for all $t \geq 0$ one has

$$0 \leq (1 - \varepsilon k_0) H(z(t)) \leq V_\varepsilon^+(t) \leq (1 + \varepsilon k_0) H(z(t)), \quad (19)$$

$$0 \leq (1 - \varepsilon k_0) H(z(t)) \leq V_\varepsilon^-(t) \leq (1 + \varepsilon k_0) H(z(t)). \quad (20)$$

Moreover, for any $t \geq 0$ such that $H(z(t)) > H_*(t)$ one has

$$\frac{d}{dt} V_\varepsilon^+(t) \leq -\varepsilon \sigma_+ H(z(t)) + k u(t) z_t(t,1) + \frac{\varepsilon}{2} z_t^2(t,1) + \frac{\varepsilon k}{2} z_x^2(t,1), \quad (21)$$

while for any $t \geq 0$ such that $H(z(t)) < H_*(t)$ one has

$$\frac{d}{dt} V_\varepsilon^-(t) \geq \varepsilon \sigma_-(\varepsilon) H(z(t)) + k u(t) z_t(t,1) - \frac{\varepsilon}{2} z_t^2(t,1) - \frac{\varepsilon k}{2} z_x^2(t,1). \quad (22)$$

Here $k_0 = \max\{1, 1/k\}$ and

$$\sigma_+ = \eta - \frac{\rho}{k} > 0, \quad \sigma_-(\varepsilon) = \min\left\{\eta - \frac{2\rho}{\varepsilon} - \rho, \sigma_+\right\} > 0.$$

Proof: Taking into account the inequalities

$$|q(t)| \leq \frac{1}{2} \int_0^1 z_t^2 \, dx + \frac{1}{2} \int_0^1 z_x^2 \, dx \leq \max\left\{1, \frac{1}{k}\right\} H(z) \quad (23)$$

one obtains the validity of (19) and (20) for all $\varepsilon \in (0, k_0)$. Let us now prove inequalities (21) and (22).

Fix an arbitrary $t \geq 0$ such that $H(z(t)) \neq H_*(t)$. One has

$$\frac{d}{dt} q(t) = \int_0^1 \left(x z_{tx} z_t + x z_x z_{tt}\right) dx = \int_0^1 \left(x z_{tx} z_t - \rho x z_x z_t + k x z_x z_{xx} - \beta x z_x \sin z\right) dx. \quad (24)$$

Note that $d(0.5 x z_t^2)/dx = 0.5 z_t^2 + x z_t z_{tx}$. Therefore

$$\int_0^1 x z_t z_{tx} \, dx = -\frac{1}{2} \int_0^1 z_t^2 \, dx + \frac{1}{2} z_t^2(t,1). \quad (25)$$

Similarly, one has

$$\int_0^1 x z_x z_{xx} \, dx = -\frac{1}{2} \int_0^1 z_x^2 + \frac{1}{2} z_x^2(t,1). \quad (26)$$

Finally, integrating by parts one gets that

$$-\int_0^1 x z_x \sin z \, dx = \int_0^1 x (\cos z)_x \, dx = \cos z(t,1) - \int_0^1 \cos z \, dx \leq \int_0^1 (1 - \cos z) \, dx \leq \frac{1}{2} \int_0^1 z^2 \, dx \leq \frac{2}{\pi^2} \int_0^1 z_x^2 \, dx. \quad (27)$$

Here we used Wirtinger's inequality. Combining (24)–(27) one obtains that

$$\frac{d}{dt}q(t) \le -\rho q(t) - \frac{1}{2}\int_0^1 z_t^2\,dx - \sigma\beta\int_0^1 (1-\cos z)\,dx$$
$$-\frac{k}{2}\left(1-\frac{4\beta(1+\sigma)}{k\pi^2}\right)\int_0^1 z_x^2\,dx + \frac{1}{2}z_t^2(t,1) + \frac{k}{2}z_x^2(t,1)$$
$$\le -\rho q(t) - \min\left\{\sigma, 1-\frac{4\beta(1+\sigma)}{k\pi^2}\right\}H(z(t))$$
$$+\frac{1}{2}z_t^2(t,1) + \frac{k}{2}z_x^2(t,1) \quad (28)$$

for any $\sigma \in (0,1)$ such that

$$\min\left\{\sigma, 1-\frac{4\beta(1+\sigma)}{k\pi^2}\right\} > 0$$

(such σ exists due to our assumption that $0 \le \beta < k\pi^2/4$). Observe that

$$\max_{\sigma\in(0,1)}\min\left\{\sigma, 1-\frac{4\beta(1+\sigma)}{k\pi^2}\right\} = \frac{k\pi^2-4\beta}{k\pi^2+4\beta}.$$

Now, combining (28), (17) and the first inequality in (23) one gets that inequalities (21) and (22) are valid. It remains to note that assumption (18), and inequalities $0 \le \beta < k\pi^2/4$ and $2\rho/(\eta-\rho) < \varepsilon < \min\{1,k\}$ are necessary and sufficient to ensure that $\sigma_+ > 0$ and $\sigma_-(\varepsilon) > 0$. ■

Let $t_2 > t_1 \ge 0$ be such that $H(z(t)) < H_*(t)$ for all $t \in [t_1,t_2]$. From inequalities (22) and (20) it follows that

$$\frac{d}{dt}V_\varepsilon^-(t) \ge \frac{\varepsilon\sigma_-(\varepsilon)}{1+\varepsilon k_0}V_\varepsilon^-(t)$$
$$+k\gamma z_t^2(t,1) - \frac{\varepsilon}{2}z_t^2(t,1) - \frac{\varepsilon k\gamma^2}{2}z_t^2(t,1).$$

for any $t \in [t_1,t_2)$. Solving the quadratic inequality

$$-\frac{\varepsilon k}{2}\gamma^2 + k\gamma - \frac{\varepsilon}{2} \ge 0$$

one obtains that for any $t \in [t_1,t_2)$ and

$$\gamma \in \left[\frac{1-\sqrt{1-\varepsilon^2/k}}{\varepsilon}, \frac{1+\sqrt{1-\varepsilon^2/k}}{\varepsilon}\right] \quad (29)$$

(cf. (9)) the following inequality holds true:

$$\frac{d}{dt}V_\varepsilon^-(t) \ge C_\varepsilon^- V_\varepsilon^-(t), \quad C_\varepsilon^- = \frac{\varepsilon\sigma_-(\varepsilon)}{1+\varepsilon k_0} > 0$$

Hence applying (20) one finds that

$$H(z(t)) \ge \frac{1-\varepsilon k_0}{1+\varepsilon k_0}e^{C_\varepsilon^-(t-t_1)}H(z(t_1)), \quad \forall t \in [t_1,t_2). \quad (30)$$

Arguing in a similar way one can check that if $H(z(t)) > H^*(t)$ for all $t \in [t_1,t_2]$, then

$$H(z(t)) \le \frac{1+\varepsilon k_0}{1-\varepsilon k_0}e^{-C_\varepsilon^+(t-t_1)}H(z(t_1)), \quad \forall t \in [t_1,t_2), \quad (31)$$

where $C_\varepsilon^+ = \varepsilon\sigma_+/(1+\varepsilon k_0)$. Thus, roughly speaking, if $H(z(t)) > H_*(t)$, then $H(z(t))$ decreases exponentially until it reaches $H_*(t)$, while if $H(z(t)) < H_*(t)$, then $H(z(t))$ increases exponentially until it reaches $H_*(t)$. With the use of inequalities (30) and (31) one can easily verify that the following results hold true.

Lemma 3. *Suppose that $0 \le \beta < k\pi^2/4$, and inequality (18) is valid. Then for any $\varepsilon > 0$ such that $2\rho/(\eta-\rho) < \varepsilon < \min\{1,k\}$ and for all $\gamma > 0$ satisfying (29) one has*

$$H(z(t)) \ge \frac{1-\varepsilon k_0}{1+\varepsilon k_0}\min\{H(z(0)), \underline{H}\} \quad \forall t \ge 0.$$

Lemma 4. *Suppose that $0 \le \beta < k\pi^2/4$, and inequality (18) is valid. Then for any $\varepsilon > 0$ such that $2\rho/(\eta-\rho) < \varepsilon < \min\{1,k\}$, for all $\gamma > 0$ satisfying (29), and for any $\tau \ge 0$ there exists $t \ge \tau$ such that $H(z(t)) = H_*(t)$.*

Now, we can prove the main claim of Theorem 1.

Lemma 5. *Let $0 \le \beta < k\pi^2/4$, and inequality (18) be valid. Then for any $\varepsilon > 0$ such that $2\rho/(\eta-\rho) < \varepsilon < \min\{1,k\}$, for any $\gamma > 0$ satisfying (29), and for all $\tau \ge 0$ and $t \ge \tau$ estimates (10) hold true with*

$$\theta = \max\left\{-\frac{1+\varepsilon k_0}{\varepsilon\sigma_+}\ln\left(\frac{1-\varepsilon k_0}{1+\varepsilon k_0}\cdot\frac{\underline{H}}{H_{\max}}\right),\right.$$
$$\left.\frac{1+\varepsilon k_0}{\varepsilon\sigma_-(\varepsilon)}\ln\left(\frac{(1+\varepsilon k_0)^2}{(1-\varepsilon k_0)^2}\cdot\frac{\overline{H}}{H_{\min}}\right)\right\}, \quad (32)$$

where $H_{\max} = \max\{H(z(0)), \overline{H}\}$ *and* $H_{\min} = \min\{H(z(0)), \underline{H}\}$

Proof: Choose arbitrary $\varepsilon \in (2\rho/(\eta-\rho), \min\{1,k\})$, $\gamma > 0$ satisfying (29), $\tau \ge 0$ and $t \ge \tau$. We devide the proof into two parts.

Part I. Let us prove the upper estimate in (10). If $H(z(t)) \le H_*(t)$, then the upper estimate in (10) obviously holds true. Therefore, suppose that $H(z(t)) > H_*(t)$, and consider two cases.

Case I. Suppose that $H(z(\tau)) > H_*(\tau)$. Define

$$T = \inf\{s \ge \tau \mid H(z(s)) = H_*(s)\}. \quad (33)$$

Note that T is correctly defined by Lemma 4. Furthermore, $T > \tau$, and for any $s \in [\tau, T)$ one has $H(z(s)) > H_*(s)$. Hence and from Lemma 1 and inequality (31) one obtains that

$$H(z(s)) \le \frac{1+\varepsilon k_0}{1-\varepsilon k_0}e^{-C_\varepsilon^+(s-\tau)}H_{\max} \quad \forall s \in [\tau, T).$$

Recall that $H_*(\cdot) \ge \underline{H} > 0$. Denote

$$g_1(\theta) = \frac{1+\varepsilon k_0}{1-\varepsilon k_0}e^{-C_\varepsilon^+\theta}H_{\max},$$

and consider the equation $g_1(\theta) = \underline{H}$. A unique solution of this equation has the form

$$\theta_1 = -\frac{1+\varepsilon k_0}{\varepsilon\sigma_+}\cdot\ln\left(\frac{1-\varepsilon k_0}{1+\varepsilon k_0}\cdot\frac{\underline{H}}{H_{\max}}\right) > 0.$$

Note that θ_1 does not depend on τ, and from the definition of T and θ_1 it follows that $T - \tau \le \theta_1$.

From (17) and (7) it follows that $dH(z(s))/ds \le 0$ for any $s \ge 0$ such that $H(z(s)) > H_*(s)$. Therefore $H(z(s)) \le H(z(\tau))$ for all $s \in [\tau, T)$. On the other hand, one has

$$H_*(s) \ge H_*(\tau) - h(\tau)(s-\tau) \ge H_*(\tau) - \theta_1 h(\tau)$$

for all $s \in [\tau, T)$. Consequently, one obtains that

$$H(z(s)) - H_*(s) \le |H(z(\tau)) - H_*(\tau)| + \theta_1 h(\tau) \qquad (34)$$

for all $s \in [\tau, T)$. Thus, if $t < T$, then the upper estimate in (10) holds true. Let, now, $t \ge T$. Denote

$$\begin{aligned} T_1 &= \sup\{s \le t \mid H(z(s)) = H_*(s)\}, \\ T_2 &= \inf\{s \ge t \mid H(z(s)) = H_*(s)\}. \end{aligned} \qquad (35)$$

Clearly, both T_1 and T_2 are correctly defined, $T \le T_1 < t < T_2$, and $H(z(s)) > H_*(s)$ for all $s \in (T_1, T_2)$. Repeating the same argument as above with τ replaced by T_1, and T replaced by T_2 one can easily verify that $H(z(t)) - H_*(t) \le \theta_1 h(\tau)$ (note that $H(z(T_1)) = H_*(T_1)$ by definition). Thus, the proof of the first case is complete.

Case II. Suppose that $H(z(\tau)) \le H_*(\tau)$. In this case the time instants

$$\begin{aligned} T_1 &= \sup\{s \in [\tau, t] \mid H(z(s)) = H_*(s)\}, \\ T_2 &= \inf\{s \ge t \mid H(z(s)) = H_*(s)\}. \end{aligned} \qquad (36)$$

are correctly defined, and $H(z(s)) > H_*(s)$ for all $s \in (T_1, T_2)$. Then repeating the same argument as in Case I with τ replaced by T_1, and T replaced by T_2 one can easily obtain estimate (34) with $|H(z(\tau)) - H_*(\tau)|$ replaced by zero.

Part II. Let us now prove the lower estimate in (10). If $H(z(t)) \ge H_*(t)$, then the lower estimate in (10) is obviously satisfied. Therefore, suppose that $H(z(t)) < H_*(t)$, and, as in the first part of the proof, consider two cases.

Case I. Suppose that $H(z(\tau)) < H_*(\tau)$. Define $T > \tau$ as in (33). Then $H(z(s)) < H_*(s)$ for all $s \in [\tau, T)$, and applying Lemma 3 and inequality (30) one obtains that

$$H(z(s)) \ge \frac{(1 - \varepsilon k_0)^2}{(1 + \varepsilon k_0)^2} e^{C_\varepsilon^+ (s - \tau)} H_{\min} \quad \forall s \in [\tau, T).$$

Recall that $H_*(\cdot) \le \overline{H}$ by our assumption. Define

$$g_2(\theta) = \frac{(1 - \varepsilon k_0)^2}{(1 + \varepsilon k_0)^2} e^{C_\varepsilon^+ \theta} H_{\min},$$

and consider the equation $g_2(\theta) = \overline{H}$. A unique solution of this equation has the form

$$\theta_2 = \frac{1 + \varepsilon k_0}{\varepsilon \sigma_-(\varepsilon)} \cdot \ln\left(\frac{(1 + \varepsilon k_0)^2}{(1 - \varepsilon k_0)^2} \cdot \frac{\overline{H}}{H_{\min}}\right).$$

Observe that $T - \tau \le \theta_2$, and θ_2 does not depend on τ.

Applying (17) and (7) one obtains that

$$\frac{d}{ds} H(z(s)) \ge -2\rho H(z(s)) \quad \forall s \in [\tau, T)$$

(recall that $H(z(s)) < H_*(s)$ for all $s \in [\tau, T)$). Hence

$$H(z(s)) \ge e^{-2\rho\theta_2} H(z(\tau)) \quad \forall s \in [\tau, T).$$

Observe also that for any $s \in [\tau, T)$ one has

$$H_*(s) \le H_*(\tau) + h(\tau)(s - \tau) \le H_*(\tau) + h(\tau)\theta_2.$$

Therefore

$$\begin{aligned} H(z(s)) - H_*(t) \ge &- \big|H(z(\tau)) - H_*(\tau)\big| \\ &- \theta_2 h(\tau) - \big(e^{-2\rho\theta_2} - 1\big) H_{\max} \quad \forall t \in [\tau, T). \end{aligned} \qquad (37)$$

Thus, if $t < T$, then the lower estimate in (10) holds true. Let, now, $t \ge T$. Defining T_1 and T_2 as in (35), and repeating the same argument as above with τ replaced by T_1, and T replaced by T_2 we complete the proof of Case I.

Case II. Suppose, now, that $H(z(\tau)) \ge H_*(\tau)$. Then defining T_1 and T_2 as in (36), and repeating the same argument as in the first case with τ replaced by T_1, and T replaced by T_2 one can easily obtain the validity of (37) with $|H(z(\tau)) - H_*(\tau)|$ replaced by zero. Thus, the proof is complete. ∎

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