2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

H_2 State-Feedback Synthesis under Positivity Constraint: Upper and Lower Bounds Computation of the Achievable Performance

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***Abstract*— This paper is concerned with the H_2 state-feedback synthesis problem under positivity constraint on the closed-loop system. This problem is believed to be a non-convex problem and hence exact treatment is not known to this date. With this difficulty in mind, in this paper, we first derive several semidefinite programs (SDPs) for the computation of the upper bounds of the achievable performance as well as suboptimal gains. However, if we rely only on the upper bound computation, we cannot say anything quantitatively on the quality of the computed suboptimal gains. For such quantitative evaluation, we next derive an SDP for the lower bound computation of the achievable performance. The key idea in deriving such an SDP is that, if the closed-loop system is positive, then the Lyapunov variable in the standard SDP for the H_2 state-feedback synthesis should be (elementwise) nonnegative. By numerical examples, we illustrate the effectiveness and limitation of the proposed strategy with upper and lower bounds computation.**

Keywords: H_2 state-Feedback synthesis, positivity constraint, upper and lower bound computation.

I. Introduction

A dynamical system is said to be (internally) positive if its state and output are nonnegative for any nonnegative initial state and nonnegative input [9]. The theory of linear time-invariant (LTI) positive systems is deeply rooted in the theory of nonnegative matrices [1], [12], and celebrated Perron-Frobenius theorem [12] has played a central role. Recently, however, it has been recognized that convex optimization particularly works fine for positive system analysis and synthesis, and intensive research efforts have been made along this direction, see, e.g., by Rantzer [14], [15], Mason and Shorten et al. [11], [13], [16], Tanaka and Langbort [18], Briat [2], and Ebihara et al. [7], [8].

For the *analysis* of positive systems, we can derive "strong" results by actively using the positivity property. Seeking such strong results has been a central issue in the research field of positive systems. On the other hand, the *synthesis* of positive systems often becomes surprisingly hard because of the natural constraint that the closed-loop system remains to be positive. Such a difficulty, however, can be circumvented for some problem instances with "diagonal stability" results. For instance, it is well known that a stable positive system allows a diagonal Lyapunov matrix in the Lyapunov inequality for its stability certificate. This fact facilitates the stabilizing state-feedback synthesis under the positivity constraint on the closed-loop system. Very recently, such a diagonal stability result has been successfully extended to the KYP-type linear matrix inequality (LMI) characterizing the H_∞ performance of positive systems [18], [15]. By employing such a diagonal Lyapunov matrix, we can enforce the closed-loop positivity without losing convexity and hence we can cast the H_∞ state-feedback synthesis problem under positivity constraint as a semidefinite programming problem (SDP). In stark contrast, we cannot employ such a diagonal Lyapunov matrix in the H_2 case. This is because in the H_2 case the Lyapunov matrix plays the role as the controllability (or observability) Gramian of the closed-loop system. Due to this reason, the H_2 state-feedback synthesis problem under positivity constraint is believed to be a non-convex problem and exact treatment is not known to this date. Possibly due to this difficulty, the study on this issue has been scarce, but recently Deaecto and Geromel proposed an effective local search algorithm for suboptimal state-feedback gain computation [5].

Aiming at SDP-based effective treatment of the H_2 state-feedback synthesis problem under positivity constraint, in this paper, we first derive several SDPs for the computation of the upper bounds of the best achievable H_2 performance as well as suboptimal gains. These SDPs are obtained by simply restricting the Lyapunov matrices to be diagonal in the standard LMIs for the H_2 state-feedback synthesis [17], and again restricting an auxiliary variable to be diagonal in a "dilated" LMI shown in [6]. Qualitatively, the latter SDP is promising since it allows us to employ an unconstrained Lyapunov matrix. If we rely only on the upper bound computation, however, we cannot say anything quantitatively on the quality of the computed suboptimal gains. For such quantitative evaluation, we next derive an SDP for the lower bound computation. The key idea in deriving such an SDP is that, if the closed-loop system is positive, then the Lyapunov variable in the standard H_2 synthesis LMI should be (elementwise) nonnegative. Finally, by numerical examples, we illustrate the effectiveness and limitation of the proposed strategy with upper and lower bounds computation.

We use the following notation. For given two matrices A and B of the same size, we write $A > B$ ($A \geq B$) if $A_{ij} > B_{ij}$ ($A_{ij} \geq B_{ij}$) holds for all (i,j), where A_{ij} stands for the (i,j)-entry of A. We define $\mathbb{R}_+^{n\times m} := \{X \in \mathbb{R}^{n\times m} : X \geq 0\}$. The notation $\mathbb{S}_{++}^n$ ($\mathbb{S}_+^n$) denotes the set of positive (semi)definite matrices of size n. For $X = X^T \in \mathbb{R}^{n\times n}$, we write $X \succ 0$ ($X \prec 0$) to denote $X \in \mathbb{S}_{++}^n$ ($-X \in \mathbb{S}_{++}^n$). The sets of Metzler, Hurwitz stable, and diagonal matrices of size n are denoted by $\mathbb{M}^n$, $\mathbb{H}^n$, and $\mathbb{D}^n$, respectively. Finally, for $A \in \mathbb{R}^{n\times n}$, we define $\mathrm{He}\{A\} := A + A^T$.

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II. Preliminaries and Problem Setting

A. LMIs for H_2 Norm Characterization of LTI Systems

Let us consider the LTI system $\mathcal{P}$ described by

$$\mathcal{P}: \left\{ \begin{array}{rcl} \dot{x}(t) &=& \mathcal{A}x(t) + \mathcal{B}w(t), \\ z(t) &=& \mathcal{C}x(t), \end{array} \right. \quad (1)$$
$$\mathcal{A} \in \mathbb{R}^{n\times n},\ \mathcal{B} \in \mathbb{R}^{n\times n_w},\ \mathcal{C} \in \mathbb{R}^{n_z\times n}.$$

The transfer function matrix of $\mathcal{P}$ is defined by $\mathcal{P}(s) = \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B}$. For the case where $\mathcal{A} \in \mathbb{H}^n$, we denote by $\|\mathcal{P}\|_2$ the H_2 norm of the system $\mathcal{P}$. The next lemma characterizes the H_2 norm of the system $\mathcal{P}$ by LMIs.

Lemma 1: [17], [6] Let us consider the system $\mathcal{P}$ described by (1). Then, for given scalars $\gamma > 0$ and $b\ (= a^{-1}) > 0$, the next four conditions are equivalent.

(i) $\mathcal{A} \in \mathbb{H}^n$ and $\|\mathcal{P}\|_2 < \gamma$.

(ii) There exist $W \in \mathbb{S}^n_{++}$ and $Q \in \mathbb{S}^{n_z}_{++}$ such that
$$\mathcal{A}W + W\mathcal{A}^T + \mathcal{B}\mathcal{B}^T \prec 0, \quad (2a)$$
$$\begin{bmatrix} Q & \mathcal{C}W \\ W\mathcal{C}^T & W \end{bmatrix} \succ 0, \quad (2b)$$
$$\mathrm{trace}(Q) < \gamma^2. \quad (2c)$$

(iii) There exist $X \in \mathbb{S}^n_{++}$ and $Z \in \mathbb{S}^{n_w}_{++}$ such that
$$\begin{bmatrix} \mathcal{A}X + X\mathcal{A}^T & X\mathcal{C}^T \\ \mathcal{C}X & -I \end{bmatrix} \prec 0, \quad (3a)$$
$$L(Z, X, \mathcal{B}) := \begin{bmatrix} Z & \mathcal{B}^T \\ \mathcal{B} & X \end{bmatrix} \succ 0, \quad (3b)$$
$$\mathrm{trace}(Z) < \gamma^2. \quad (3c)$$

(iv) There exist $X \in \mathbb{S}^n_{++}$, $Z \in \mathbb{S}^{n_w}_{++}$ and $G \in \mathbb{R}^{n\times n}$ such that
$$\begin{bmatrix} 0 & -X & 0 \\ -X & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A} \\ I \\ \mathcal{C} \end{bmatrix} G \begin{bmatrix} I & -bI & 0 \end{bmatrix} \right\} \prec 0, \quad (4a)$$
$$L(Z, X, \mathcal{B}) \succ 0, \quad (4b)$$
$$\mathrm{trace}(Z) < \gamma^2. \quad (4c)$$

Moreover, for every feasible solution $(X, Z) = (\mathcal{X}, \mathcal{Z})$ of (3),
$$(X, Z, G) = (\mathcal{X}, \mathcal{Z}, -a(A - aI)^{-1}\mathcal{X}) \quad (5)$$
is a feasible solution of (4). Conversely, for every feasible solution $(X, Z) = (\mathcal{X}, \mathcal{Z})$ of (4) with some G, $(X, Z) = (\mathcal{X}, \mathcal{Z})$ is a feasible solution of (3).

In this lemma, the LMIs (2) and (3) are well known and they are based on the Gramian-based H_2 norm characterization [17], [19]. In fact the Lyapunov matrix W in (2) corresponds to the controllability Gramian of the system $\mathcal{P}$ and the Lyapunov matrix X in (3) corresponds to the inverse of the observability Gramian of the system $\mathcal{P}$. On the other hand, the LMI (4) is known as a "dilated" LMI corresponding to (3). This LMI has the striking "decoupling" property such that the Lyapunov matrix X becomes free from the multiplications between the coefficient matrices $\mathcal{A}$ and $\mathcal{C}$ due to the introduction of the additional variable G. Similarly to more elegant discrete-time system counterparts [3], [4], the LMI (4) is shown to be effective in dealing with multi-objective control problems [6]. The "decoupling" property of the LMI (4) is shown to be useful also in obtaining suboptimal gains for the H_2 state-feedback control problem under positivity constraint on the closed-loop system.

B. Problem Setting

For firm setting of the problem to be considered in this paper, we begin with the very basic definition and result for the positivity of LTI systems.

Definition 1: [9] The LTI system (1) is said to be *positive* if its state and output are both nonnegative for any nonnegative initial state and nonnegative input.

Proposition 1: [9] The LTI system (1) is positive if and only if
$$\mathcal{A} \in \mathbb{M}^{n\times n},\ \mathcal{B} \in \mathbb{R}^{n\times n_w}_+,\ \mathcal{C} \in \mathbb{R}^{n_z\times n}_+. \quad (6)$$

We next move on to the problem setting. Let us consider the LTI system P described by

$$P: \left\{ \begin{array}{rcl} \dot{x}(t) &=& Ax(t) + B_1w(t) + B_2u(t), \\ z(t) &=& C_1x(t) + D_{12}u(t), \end{array} \right. \quad (7)$$
$$A \in \mathbb{R}^{n\times n},\ B_1 \in \mathbb{R}^{n\times n_w}_+,\ B_2 \in \mathbb{R}^{n\times n_u},$$
$$C_1 \in \mathbb{R}^{n_z\times n},\ D_{12} \in \mathbb{R}^{n_z\times n_u}.$$

Note that we have assumed that B_1 is nonnegative. For this system, we apply a state-feedback control $u = Fx$ with a state-feedback gain $F \in \mathbb{R}^{n_u\times n}$. Then, the closed-loop system, denoted by P_F, can be described by

$$P_F: \left\{ \begin{array}{rcl} \dot{x}(t) &=& (A + B_2F)x(t) + B_1w(t), \\ z(t) &=& (C_1 + D_{12}F)x(t). \end{array} \right. \quad (8)$$

The problem in this paper is to design a state-feedback gain $F \in \mathbb{R}^{n_u\times n}$ such that $\|P_F\|_2$ is minimized under the constraint that P_F is positive. If we define

$$\begin{array}{l} \mathcal{F}_{\mathrm{s}} := \left\{ F \in \mathbb{R}^{n_u\times n} :\ A + B_2F \in \mathbb{H}^n \right\}, \\ \mathcal{F}_{\mathrm{p}} := \left\{ F \in \mathbb{R}^{n_u\times n} :\ A + B_2F \in \mathbb{M}^n, C_1 + D_{12}F \in \mathbb{R}^{n_z\times n}_+ \right\}, \end{array} \quad (9)$$

the problem can be stated formally as follows:

$$\gamma^\star = \inf_{F\in\mathcal{F}_{\mathrm{s}}\cap\mathcal{F}_{\mathrm{p}}} \|P_F\|_2. \quad (10)$$

Remark 1: For the closed-loop system P_F to be positive, it is necessary from Proposition 1 that $B_1 \in \mathbb{R}^{n\times n_w}_+$. This is the reason why we have assumed this from the outset in (7). In the LQ problem for positive systems, B_1 corresponds to the initial condition and hence is intrinsically nonnegative. It is also true that retaining positivity is a natural requirement. It follows that this is a typical case of the H_2 state-feedback synthesis problem under positivity constraint on the closed-loop system treated in this paper.

Remark 2: In (10) we enforce the positivity to the closed-loop system P_F. Therefore, it might be natural to assume that the open-loop system P is also positive as in
$$A \in \mathbb{M}^n, B_2 \in \mathbb{R}^{n\times n_u}_+,\ C_1 \in \mathbb{R}^{n_z\times n}_+,\ D_{12} \in \mathbb{R}^{n_z\times n_u}_+. \quad (11)$$
However, we prefer general problem setting and hence proceed without assuming (11). Our results in this paper are of course valid even in the case of (11).

III. Upper Bound and Suboptimal Gain Computation

A. Difficulty of Exact Treatment

In striking contrast with its simple formulation, the problem (10) is a hard problem. To explain the difficulty of the exact treatment of (10), let us consider its unconstrained (positivity-free) version given by

$$\gamma_{\mathrm{f}}^\star = \inf_{F\in\mathcal{F}_{\mathrm{s}}} \|P_F\|_2. \tag{12}$$

Then, it is well-known that this problem can be reduced to the following SDP:

$$\begin{array}{l}(\gamma_{\mathrm{f}}^\star)^2 = \displaystyle\inf_{\gamma_{\mathrm{sq}},W,Q,Y} \gamma_{\mathrm{sq}} \text{ subject to}\\ \mathrm{He}\{AW+B_2Y\}+B_1B_1^T \prec 0,\\ \begin{bmatrix} Q & C_1W+D_{12}Y\\ (C_1W+D_{12}Y)^T & W\end{bmatrix} \succ 0,\\ \mathrm{trace}(Q)<\gamma_{\mathrm{sq}}.\end{array} \tag{13}$$

This SDP is obtained from (2) and (8) via change of variables $Y := FW$. If this SDP is feasible, then the optimal gain $F^\star \in \mathcal{F}_{\mathrm{s}}$ with $\|P_{F^\star}\|_2 = \gamma_{\mathrm{f}}^\star$ can be reconstructed as $F^\star = Y^\star(W^\star)^{-1}$ where $Y^\star \in \mathbb{R}^{n_u\times n}$ and $W^\star \in \mathbb{S}_{++}^n$ are the solution resulting from the SDP.

In the above SDP, there is of course no guarantee that $F^\star \in \mathcal{F}_{\mathrm{p}}$. Therefore, to solve (10), we need to include additional constraints in (13) so that the positivity of the closed-loop system is ensured. However, if we want to preserve the SDP formulation, and if we want to preserve the necessary and sufficient treatment, such additional constraints are hardly available. This is the source of the difficulty in (10).

In the case where the performance index is the H_∞ norm, it has been shown recently that the Lyapunov matrix in the KYP-type LMI can be taken as diagonal without any conservatism [18], [15]. If we follow this idea and restrict W in (13) to be diagonal, we can obtain the next SDP that provides $F\in\mathcal{F}_s\cap\mathcal{F}_p$:

$$\begin{array}{l}(\gamma_{\mathrm{u},W_{\mathrm{d}}})^2 = \displaystyle\inf_{\gamma_{\mathrm{sq}},W_{\mathrm{d}},Q,Y} \gamma_{\mathrm{sq}} \text{ subject to}\\ \mathrm{He}\{AW_{\mathrm{d}}+B_2Y\}+B_1B_1^T \prec 0,\\ \begin{bmatrix} Q & C_1W_{\mathrm{d}}+D_{12}Y\\ (C_1W_{\mathrm{d}}+D_{12}Y)^T & W_{\mathrm{d}}\end{bmatrix} \succ 0,\\ \mathrm{trace}(Q)<\gamma_{\mathrm{sq}},\ W_{\mathrm{d}}\in\mathbb{S}_{++}^n\cap\mathbb{D}^n,\\ AW_{\mathrm{d}}+B_2Y\in\mathbb{M}^n,\ C_1W_{\mathrm{d}}+D_{12}Y\in\mathbb{R}_+^{n\times n}.\end{array} \tag{14}$$

If we denote by $(W_{\mathrm{d}}^\star, Q^\star, Y^\star)$ the solution resulting from the SDP (14), then a state-feedback gain $F_{\mathrm{u},W_{\mathrm{d}}}\in\mathcal{F}_{\mathrm{s}}$ with $\|P_{F_{\mathrm{u},W_{\mathrm{d}}}}\|_2 = \gamma_{\mathrm{u},W_{\mathrm{d}}}^\star \le \gamma_{\mathrm{u},W_{\mathrm{d}}}$ can be obtained by $F_{\mathrm{u},W_{\mathrm{d}}} = Y^\star(W_{\mathrm{d}}^\star)^{-1}$. Here, since $W_{\mathrm{d}}^\star\in\mathbb{S}_{++}^n\cap\mathbb{D}^n$ holds, we have

$$\begin{array}{l}AW_{\mathrm{d}}^\star+B_2Y^\star\in\mathbb{M}^n \Rightarrow A+B_2F_{\mathrm{u},W_{\mathrm{d}}}\in\mathbb{M}^n,\\ C_1W_{\mathrm{d}}^\star+D_{12}Y^\star\in\mathbb{R}_+^{n_z\times n}\Rightarrow C_1+D_{12}F_{\mathrm{u},W_{\mathrm{d}}}\in\mathbb{R}_+^{n_z\times n}.\end{array}$$

Namely, we can ensure that $F_{\mathrm{u},W_{\mathrm{d}}}\in\mathcal{F}_{\mathrm{p}}$ by restricting W to be diagonal in (13). This is the basic idea for the upper bound computation of $\gamma^\star$ in the present paper and this idea is described in more details in the next subsection.

It should be emphasized here that, in stark contrast with the H_∞ performance case, the approach with "diagonal W" given by (14) is conservative in general. Namely, there is certain gap between $\gamma^\star$ defined by (10) and $\gamma_{\mathrm{u},W_{\mathrm{d}}}$ given by (14). To explicate the reason for the gap in more convincing way, let us consider the next SDP that compute the genuine H_2 performance achieved by $F_{\mathrm{u},W_{\mathrm{d}}}$:

$$\begin{array}{l}(\gamma_{\mathrm{u},W_{\mathrm{d}}}^\star)^2 = \displaystyle\inf_{\gamma_{\mathrm{sq}},W,Q} \gamma_{\mathrm{sq}} \text{ subject to}\\ \mathrm{He}\{(A+B_2F_{\mathrm{u},W_{\mathrm{d}}})W\}+B_1B_1^T \prec 0,\\ \begin{bmatrix} Q & (C_1+D_{12}F_{\mathrm{u},W_{\mathrm{d}}})W\\ W(C_1+D_{12}F_{\mathrm{u},W_{\mathrm{d}}})^T & W\end{bmatrix} \succ 0,\\ \mathrm{trace}(Q)<\gamma_{\mathrm{sq}}.\end{array} \tag{15}$$

Then we can readily see that $\gamma^\star \le \gamma_{\mathrm{u},W_{\mathrm{d}}}^\star \le \gamma_{\mathrm{u},W_{\mathrm{d}}}$ holds. Moreover, if we include the additional constraint $W\in\mathbb{D}^n$ in the SDP (15), then the resulting value is equal to $\gamma_{\mathrm{u},W_{\mathrm{d}}}$. Here, if there is no gap between $\gamma^\star$ and $\gamma_{\mathrm{u},W_{\mathrm{d}}}$, it is necessary that $\gamma_{\mathrm{u},W_{\mathrm{d}}}^\star = \gamma_{\mathrm{u},W_{\mathrm{d}}}$ holds. However, we can hardly expect the satisfaction of this equality in general since the optimal value of W in the SDP (15) comes closer to the controllability Gramian $W_{\mathrm{cl}}\in\mathbb{S}_+^n$ of the closed-loop system that is nothing but the unique solution of the Lyapunov equation

$$\mathrm{He}\{(A+B_2F_{\mathrm{u},W_{\mathrm{d}}})W_{\mathrm{cl}}\}+B_1B_1^T=0.$$

From this form, we cannot expect $W_{\mathrm{cl}}\in\mathbb{D}^n$ in general, and hence there is certain gap between $\gamma_{\mathrm{u},W_{\mathrm{d}}}^\star$ and $\gamma_{\mathrm{u},W_{\mathrm{d}}}$. This implies that we cannot directly minimize (infimize) $\gamma_{\mathrm{u},W_{\mathrm{d}}}^\star$, and hence there is also certain gap between $\gamma^\star$ and $\gamma_{\mathrm{u},W_{\mathrm{d}}}^\star$.

To summarize, in general the SDP (14) provides only an upper bound $\gamma_{\mathrm{u},W_{\mathrm{d}}}^\star > \gamma^\star$ and suboptimal gain. As stated above, in the case of the H_2 performance, the matrix $W\in\mathbb{S}_+^n$ in (13) plays the role as the controllability Gramian of the closed-loop system. Due to this reason, we cannot restrict it to be diagonal without being conservative in general.

B. SDPs for Upper Bound Computation

By following the same idea as (14), we can obtain the next SDP from (3). The next SDP yields an upper bond of $\gamma^\star$ defined by (10) and a suboptimal state-feedback gain.

$$\begin{array}{l}(\gamma_{\mathrm{u},X_{\mathrm{d}}})^2 = \displaystyle\inf_{\gamma_{\mathrm{sq}},X_{\mathrm{d}},Z,Y} \gamma_{\mathrm{sq}} \text{ subject to}\\ \begin{bmatrix} \mathrm{He}\{AX_{\mathrm{d}}+B_2Y\} & (C_1X_{\mathrm{d}}+D_{12}Y)^T\\ C_1W_{\mathrm{d}}+D_{12}Y & -I\end{bmatrix} \prec 0,\\ L(Z,X_{\mathrm{d}},B_1)\succ 0,\ \mathrm{trace}(Z)<\gamma_{\mathrm{sq}},\ X_{\mathrm{d}}\in\mathbb{S}_{++}^n\cap\mathbb{D}^n,\\ AX_{\mathrm{d}}+B_2Y\in\mathbb{M}^n,\ C_1X_{\mathrm{d}}+D_{12}Y\in\mathbb{R}_+^{n\times n}.\end{array} \tag{16}$$

If we denote by $(X_{\mathrm{d}}^\star, Z^\star, Y^\star)$ the solution resulting from the SDP (16), then a state-feedback gain $F_{\mathrm{u},X_{\mathrm{d}}}\in\mathcal{F}_{\mathrm{s}}\cap\mathcal{F}_{\mathrm{p}}$ with $\|P_{F_{\mathrm{u},X_{\mathrm{d}}}}\|_2 = \gamma_{\mathrm{u},X_{\mathrm{d}}}^\star \le \gamma_{\mathrm{u},X_{\mathrm{d}}}$ can be obtained by $F_{\mathrm{u},X_{\mathrm{d}}} = Y^\star(X_{\mathrm{d}}^\star)^{-1}$. On the other hand, we can also obtain the next SDP on the basis of (4).

$$\begin{array}{l}(\gamma_{\mathrm{u},G_{\mathrm{d}},b})^2 = \displaystyle\inf_{\gamma_{\mathrm{sq}},X,G_{\mathrm{d}},Z,Y} \gamma_{\mathrm{sq}} \text{ subject to}\\ \begin{bmatrix} 0 & -X & 0\\ -X & 0 & 0\\ 0 & 0 & -I\end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} AG_{\mathrm{d}}+B_2Y\\ G_{\mathrm{d}}\\ C_1G_{\mathrm{d}}+D_{12}Y\end{bmatrix}\begin{bmatrix} I & -bI & 0\end{bmatrix}\right\}\prec 0,\\ L(Z,X,B_1)\succ 0,\ \mathrm{trace}(Z)<\gamma_{\mathrm{sq}},\ G_{\mathrm{d}}\in\mathbb{S}_{++}^n\cap\mathbb{D}^n,\\ AG_{\mathrm{d}}+B_2Y\in\mathbb{M}^n,\ C_1G_{\mathrm{d}}+D_{12}Y\in\mathbb{R}_+^{n\times n}.\end{array} \tag{17}$$

Again, if we denote by $(X^\star, G_{\mathrm{d}}^\star, Z^\star, Y^\star)$ the solution resulting from the SDP (17), then a state-feedback gain $F_{\mathrm{u},G_{\mathrm{d}},b}\in\mathcal{F}_{\mathrm{s}}\cap\mathcal{F}_{\mathrm{p}}$ with $\|P_{F_{\mathrm{u},G_{\mathrm{d}},b}}\|_2 = \gamma_{\mathrm{u},G_{\mathrm{d}},b}^\star \le \gamma_{\mathrm{u},G_{\mathrm{d}},b}$ can be obtained by $F_{\mathrm{u},G_{\mathrm{d}},b} = Y^\star(G_{\mathrm{d}}^\star)^{-1}$.

We now compare the SDPs (14), (16), and (17) for the upper bound computation. First, it is hard to draw any definite conclusions on which is the smaller (better) of the two values $\gamma_{\mathrm{u},W_{\mathrm{d}}}$ and $\gamma_{\mathrm{u},X_{\mathrm{d}}}$. This is because the SDP (14) yielding $\gamma_{\mathrm{u},W_{\mathrm{d}}}$ enforces diagonal structure to the Lyapunov matrix that corresponds to the controllability Gramian of the closed-loop system, whereas the SDP (16) yielding $\gamma_{\mathrm{u},X_{\mathrm{d}}}$ enforces diagonal structure to the Lyapunov matrix that corresponds to the inverse of the observability Gramian of of the closed-loop system. On the other hand, the SDP (17) yielding $\gamma_{\mathrm{u},G_{\mathrm{d}},b}$ is promising qualitatively since it does not enforce the diagonal structure to the Lyapunov matrix. It is nonetheless hard to compare quantitatively which is the smallest (best) among $\gamma_{\mathrm{u},W_{\mathrm{d}}}$, $\gamma_{\mathrm{u},X_{\mathrm{d}}}$, and $\gamma_{\mathrm{u},G_{\mathrm{d}},b}$ for arbitrarily chosen b, since the SDP (17) enforces the diagonal structure to the additional variable G. However, if we note the "recovery property" of the LMI (4) with respect to the LMI (3) shown in (5), we can state the advantage of the SDP (17) over the SDP (16) in the following way.

Theorem 1: Let us define

$$\gamma_{\mathrm{u},G_{\mathrm{d}}} := \inf_{b>0} \gamma_{\mathrm{u},G_{\mathrm{d}},b}. \tag{18}$$

Then, we have $\gamma_{\mathrm{u},G_{\mathrm{d}}} \le \gamma_{\mathrm{u},X_{\mathrm{d}}}$ where $\gamma_{\mathrm{u},X_{\mathrm{d}}}$ and $\gamma_{\mathrm{u},G_{\mathrm{d}},b}$ are defined by (16) and (17), respectively.

Proof: For the proof we focus on the "recovery property" of the LMI (4) with respect to the LMI (3) shown in (5). Since

$$\lim_{a\to\infty} -a(A-aI)^{-1}\mathcal{X} = \mathcal{X} \tag{19}$$

holds, we see that for every feasible solution $(X, Z) = (\mathcal{X}, \mathcal{Z})$ of (3), $(X, Z, G) = (\mathcal{X}, \mathcal{Z}, \mathcal{X})$ is a feasible solution of (4) if we take b sufficiently small. With this in mind, suppose the LMI in (16) for a given γ_{sq} is feasible with $(X_{\mathrm{d}}, Z, Y) = (\mathcal{X}_{\mathrm{d}}, \mathcal{Z}, \mathcal{Y})$. Then, from (19), we see that the LMI in (17) for the same γ_{sq} is feasible with $(X, G_{\mathrm{d}}, Z, Y) = (\mathcal{X}_{\mathrm{d}}, \mathcal{X}_{\mathrm{d}}, \mathcal{Z}, \mathcal{Y})$ if we take b sufficiently small. This clearly shows that $\gamma_{\mathrm{u},G_{\mathrm{d}}}$ defined by (18) satisfies $\gamma_{\mathrm{u},G_{\mathrm{d}}} \le \gamma_{\mathrm{u},X_{\mathrm{d}}}$. This completes the proof. ∎

This theorem basically states that, if we carry out a line-search over $b > 0$ in the SDP (17), then we can obtain a better (no worse) upper bound than that of the SDP (16). In practice, on the other hand, we are interested in the genuine H_2 performance achieved by computed state-feedback gains and hence we naturally focus on

$$\gamma^\star_{\mathrm{u},G_{\mathrm{d}}} := \inf_{b>0} \gamma^\star_{\mathrm{u},G_{\mathrm{d}},b}. \tag{20}$$

For the two values $\gamma^\star_{\mathrm{u},G_{\mathrm{d}}}$ and $\gamma^\star_{\mathrm{u},X_{\mathrm{d}}}$, however, there is no theoretical guarantee that $\gamma^\star_{\mathrm{u},G_{\mathrm{d}}} \le \gamma^\star_{\mathrm{u},X_{\mathrm{d}}}$ holds, even though this is in practice satisfied in most tested numerical examples (see Section V).

To summarize this section, the values $\gamma^\star_{\mathrm{u},W_{\mathrm{d}}}$, $\gamma^\star_{\mathrm{u},X_{\mathrm{d}}}$, $\gamma^\star_{\mathrm{u},G_{\mathrm{d}}}$ are in general merely upper bounds of $\gamma^\star$ and there is no theoretical guarantee that they are tight. Therefore, for quantitative evaluation on how these upper bounds are close to $\gamma^\star$, it is quite important to obtain its tight *lower* bounds. We focus on this issue in the next section.

IV. SDP for Lower Bound Computation

The most trivial and easily computable lower bound of $\gamma^\star$ is nothing but $\gamma^\star_{\mathrm{f}}$ defined by (12), i.e., the best achievable H_2 performance in the positivity-free counterpart problem. However, this lower bound must be loose since there is no care for the positivity constraint. For the computation of a tighter lower bound, the next lemma is useful.

Lemma 2: Suppose the system $\mathcal{P}$ in (1) is positive, i.e., $\mathcal{A} \in \mathbb{M}^n$, $\mathcal{B} \in \mathbb{R}_+^{n\times n_w}$, and $\mathcal{C} \in \mathbb{R}_+^{n_z\times n}$. Then the next condition is equivalent to (i)-(iv) of Lemma 1.

(ii') There exist $W \in \mathbb{S}^n_{++} \cap \mathbb{R}_+^{n\times n}$ and $Q \in \mathbb{S}^{n_z}_{++} \cap \mathbb{R}_+^{n_z\times n_z}$ such that (2) holds.

Namely, if the system $\mathcal{P}$ in (1) is positive, the we can restrict W and Q in (2) to be (elementwise) nonnegative.

Proof of Lemma 2: We prove that (i) ⇔ (ii') holds. The proof of (i)⇒(ii') is inspired in part from [10], [5].

(ii')⇒(i): This is obvious since (ii')⇒(ii)⇔(i).

(i)⇒(ii'): Suppose (i) holds, i.e., $\mathcal{A} \in \mathbb{H}^n$ and $\|\mathcal{P}\|_2 < \gamma$. It should be noted that $\|\mathcal{P}\|_2^2 = \mathrm{trace}(\mathcal{C}W_0\mathcal{C}^T)$ holds where $W_0 \in \mathbb{S}^n_+$ is the unique solution of the Lyapunov equation $\mathcal{A}W_0 + W_0\mathcal{A}^T + \mathcal{B}\mathcal{B}^T = 0$. It is also an elementary fact that $W_0 \in \mathbb{S}^n_+ \cap \mathbb{R}_+^{n\times n}$ holds [10], [5]. We also define $\mathcal{W} \in \mathbb{S}^n_{++}$ as the unique solution of $A\mathcal{W}+\mathcal{W}A^T+I = 0$. Then, we have $\mathcal{W} \in \mathbb{S}^n_{++}\cap\mathbb{R}_+^{n\times n}$. On the basis of these preliminary results, let us define $\nu > 0$ by $\nu := (\gamma^2 - \|\mathcal{P}\|_2^2)/(2\mathrm{trace}(C\mathcal{W}C^T))$ Then, we readily obtain

$$\mathcal{A}(W_0 + \nu\mathcal{W}) + (W_0 + \nu\mathcal{W})\mathcal{A}^T + \mathcal{B}\mathcal{B}^T = -\nu I \prec 0,$$

$$\begin{aligned}\mathrm{trace}\left(\mathcal{C}(W_0 + \nu\mathcal{W})\mathcal{C}^T\right) &= \|\mathcal{P}\|_2^2 + \frac{\gamma^2 - \|\mathcal{P}\|_2^2}{2} \\ &= \frac{\gamma^2 + \|\mathcal{P}\|_2^2}{2} \\ &< \gamma^2.\end{aligned}$$

Therefore, for any $\varepsilon \in (0, (\gamma^2 - \|\mathcal{P}\|_2^2)/2)$, it is clear that the pair (W, Q) defined by

$$W := W_0 + \nu\mathcal{W} \in \mathbb{S}^n_{++} \cap \mathbb{R}_+^{n\times n},$$
$$Q := \mathcal{C}(W_0 + \nu\mathcal{W})\mathcal{C}^T + \varepsilon I \in \mathbb{S}^{n_w}_{++} \cap \mathbb{R}_+^{n_w\times n_w}$$

satisfies (2). This completes the proof. ∎

On the basis of this lemma, we can obtain the next theorem for the lower bound computation of $\gamma^\star$ defined by (10).

Theorem 2: Let us consider the SDP:

$$\gamma_{\mathrm{l}}^2 := \inf_{\gamma_{\mathrm{sq}},W,Q,Y} \gamma_{\mathrm{sq}} \text{ subject to}$$
$$\begin{aligned}&\mathrm{He}\{AW + B_2Y\} + B_1B_1^T \prec 0,\\ &\begin{bmatrix} Q & C_1W + D_{12}Y \\ (C_1W + D_{12}Y)^T & W \end{bmatrix} \succ 0,\\ &\mathrm{trace}(Q) < \gamma_{\mathrm{sq}},\end{aligned} \tag{21a}$$
$$\begin{aligned}&W \in \mathbb{S}^n_{++} \cap \mathbb{R}_+^{n\times n},\ Q \in \mathbb{S}^{n_z}_{++} \cap \mathbb{R}_+^{n_z\times n_z},\\ &AW + B_2Y + \alpha W \in \mathbb{R}_+^{n\times n},\\ &C_1W + D_{12}Y \in \mathbb{R}_+^{n_z\times n}.\end{aligned} \tag{21b}$$

Here, $\alpha \in \mathbb{R}_{++}$ is chosen to be sufficiently large. Then, we have

$$\gamma^\star_{\mathrm{f}} \le \gamma_{\mathrm{l}} \le \gamma^\star. \tag{22}$$

Moreover, if we define $F_{\mathrm{l}} = YW^{-1}$ using the solution W and Y resulting from the SDP (21) and if $F_{\mathrm{l}} \in \mathcal{F}_{\mathrm{p}}$ holds, then $\gamma_{\mathrm{l}} = \gamma^\star$ and F_{l} is an optimal feedback gain achieving $\|P_{F_{\mathrm{l}}}\|_2 = \gamma^\star$.

Proof: We first prove $\gamma^\star_{\mathrm{f}} \le \gamma_{\mathrm{l}}$ in (22). If we focus on the structure of the SDPs (13) and (21), we see that the SDP

(21) is nothing but the SDP (13) with additional constraint (21b). Therefore $\gamma_{\mathrm{f}}^\star \le \gamma_{\mathrm{l}}$ obviously holds.

We next prove $\gamma_{\mathrm{l}} \le \gamma^\star$ in (22). To this end, we note from Lemma 2 that $\gamma^\star$ defined by (10) can be characterized by the non-convex program given as follows:

$$(\gamma^\star)^2 := \inf_{\gamma_{\mathrm{sq}},W,Q,F} \gamma_{\mathrm{sq}} \text{ subject to}$$

$$\begin{aligned} & \mathrm{He}\{(A+B_2F)W\} + B_1B_1^T \prec 0, \\ & \begin{bmatrix} Q & (C_1+D_{12}F)W \\ W(C_1+D_{12}F)^T & W \end{bmatrix} \succ 0, \\ & \mathrm{trace}(Q) < \gamma_{\mathrm{sq}}, \end{aligned} \tag{23a}$$

$$\begin{aligned} & W \in \mathbb{S}_{++}^n \cap \mathbb{R}_+^{n\times n},\ Q \in \mathbb{S}_{++}^{n_z} \cap \mathbb{R}_+^{n_z\times n_z}, \\ & A + B_2F + \alpha I \in \mathbb{R}_+^{n\times n}, \\ & C_1 + D_{12}F \in \mathbb{R}_+^{n_z\times n}. \end{aligned} \tag{23b}$$

If the bilinear matrix inequality in (23) for a given γ_{sq} holds with $(W,Q,F) = (\mathcal{W},\mathcal{Q},\mathcal{F})$, then the LMI in (21) for the same γ_{sq} holds with $(W,Q,Y) = (\mathcal{W},\mathcal{Q},\mathcal{F}\mathcal{W})$. Namely, the SDP (21) is a convex relaxation of the non-convex program (23). It follows that $\gamma_{\mathrm{l}} \le \gamma^\star$ holds.

Finally, it is obvious from the SDP (21) that $F_{\mathrm{l}} \in \mathcal{F}_s$ and $\|P_{F_{\mathrm{l}}}\|_2 \le \gamma_{\mathrm{l}} \le \gamma^\star$. Therefore, if $F_{\mathrm{l}} \in \mathcal{F}_p$, we can readily see from the definition (10) that $\gamma_{\mathrm{l}} = \gamma^\star$ holds and F_{l} is an optimal state-feedback gain achieving $\|P_{F_{\mathrm{l}}}\|_2 = \gamma^\star$. This completes the proof. ∎

To summarize, we have derived the SDP (21) for the computation of the lower bound γ_{l}. If the best available upper bound $\min(\gamma_{\mathrm{u},W_{\mathrm{d}}}^\star, \gamma_{\mathrm{u},X_{\mathrm{d}}}^\star, \gamma_{\mathrm{u},G_{\mathrm{d}}}^\star)$ comes close to γ_{l}, we can conclude that the computed suboptimal gain is (nearly) optimal. We illustrate the effectiveness (and limitation) of this strategy by numerical examples in the next section.

V. Numerical Examples

A. Striking Examples

In this subsection we provide two examples under which our strategy works pretty well.

1) The case $\gamma_{\mathrm{l}} = \gamma_{\mathrm{u},W_{\mathrm{d}}}^\star$: Let us consider the case where $n=5$, $n_w=2$, $n_u=1$, and $n_z=2$ in (7) and

$$A = \begin{bmatrix} -2.34 & 0.77 & 0.96 & 0.31 & 0.65 \\ 0.24 & -2.24 & 0.74 & 0.70 & 0.73 \\ 0.92 & 0.85 & -2.83 & 0.90 & 0.79 \\ 0.25 & 0.34 & 0.57 & -2.17 & 0.37 \\ 0.14 & 0.44 & 0.57 & 0.75 & -2.67 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.43 & 0.50 \\ 0.42 & 0.76 \\ 0.36 & 0.72 \\ 0.39 & 0.04 \\ 0.50 & 0.49 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.43 \\ 0.31 \\ 0.77 \\ 0.12 \\ 0.42 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.66 & 0.15 & 0.83 & 0.77 & 0.57 \\ 0.14 & 0.19 & 0.89 & 0.47 & 0.35 \end{bmatrix},\ D_{12} = \begin{bmatrix} 0.61 \\ 0.90 \end{bmatrix}.$$

We solved the SDPs (14), (16), and (17) and obtained upper bounds $\gamma_{\mathrm{u},W_{\mathrm{d}}}$, $\gamma_{\mathrm{u},W_{\mathrm{d}}}^\star$, $\gamma_{\mathrm{u},X_{\mathrm{d}}}$, $\gamma_{\mathrm{u},X_{\mathrm{d}}}^\star$, $\gamma_{\mathrm{u},G_{\mathrm{d}}}$, and $\gamma_{\mathrm{u},G_{\mathrm{d}}}^\star$. Here, to compute $\gamma_{\mathrm{u},G_{\mathrm{d}}}$ and $\gamma_{\mathrm{u},G_{\mathrm{d}}}^\star$ defined by (18) and (20), respectively, we carried out a line search with respect to b in (17) over $b \in [0.01, 10]$ with grid 0.01. We also solved the SDPs (21) with $\alpha = 100$ and (13) to compute lower bounds γ_{l} and $\gamma_{\mathrm{f}}^\star$. The results are shown in Table I and several remarks follow.

TABLE I

Computed lower and upper bounds: case 1.

SDP (14) $\gamma_{\mathrm{u},W_{\mathrm{d}}}$ ($\gamma_{\mathrm{u},W_{\mathrm{d}}}^\star$)	SDP (16) $\gamma_{\mathrm{u},X_{\mathrm{d}}}$ ($\gamma_{\mathrm{u},X_{\mathrm{d}}}^\star$)	SDP (17) $\gamma_{\mathrm{u},G_{\mathrm{d}}}$ ($\gamma_{\mathrm{u},G_{\mathrm{d}}}^\star$)	SDP (21) γ_{l}	SDP (13) $\gamma_{\mathrm{f}}^\star$
0.7909 (0.7037)	0.7544 (0.7037)	0.7155 (0.7037)	0.7037	0.4967

TABLE II

Computed lower and upper bounds: case 2.

SDP (14) $\gamma_{\mathrm{u},W_{\mathrm{d}}}$ ($\gamma_{\mathrm{u},W_{\mathrm{d}}}^\star$)	SDP (16) $\gamma_{\mathrm{u},X_{\mathrm{d}}}$ ($\gamma_{\mathrm{u},X_{\mathrm{d}}}^\star$)	SDP (17) $\gamma_{\mathrm{u},G_{\mathrm{d}}}$ ($\gamma_{\mathrm{u},G_{\mathrm{d}}}^\star$)	SDP (21) γ_{l}	SDP (13) $\gamma_{\mathrm{f}}^\star$
1.2220 (1.1351)	1.2564 (1.1351)	1.1639 (1.1351)	1.0893	0.8592

- As stated, the upper bounds $\gamma_{\mathrm{u},G_{\mathrm{d}}}$ and $\gamma_{\mathrm{u},G_{\mathrm{d}}}^\star$ are obtained by a line search with respect to b in (17) over $b \in [0.01, 10]$. Both values in Table I are obtained when $b = 2.38$.
- We see that $\gamma_{\mathrm{u},G_{\mathrm{d}}} \le \gamma_{\mathrm{u},X_{\mathrm{d}}}$ holds and this is consistent with Theorem 1. We also see that $\gamma_{\mathrm{f}}^\star \le \gamma_{\mathrm{l}}$ holds and this is consistent with Theorem 2.
- Those upper bounds $\gamma_{\mathrm{u},W_{\mathrm{d}}}^\star$, $\gamma_{\mathrm{u},X_{\mathrm{d}}}^\star$, and $\gamma_{\mathrm{u},G_{\mathrm{d}}}^\star$ take the same value 0.7037 since the gains $F_{\mathrm{u},W_{\mathrm{d}}}$, $F_{\mathrm{u},X_{\mathrm{d}}}$, $F_{\mathrm{u},G_{\mathrm{d}},b}$ resulting from the SDPs (14), (16), and (17) with $b = 2.38$ turn out to be the same and $F_{\mathrm{u},W_{\mathrm{d}}} = [\,-0.1556\ \ -0.2111\ \ -0.9889\ \ -0.5222\ \ -0.3889\,]$.
- If we rely only on the trivial lower bound $\gamma_{\mathrm{f}}^\star = 0.4967$, we cannot say anything quantitatively on the quality of $F_{\mathrm{u},W_{\mathrm{d}}}$ achieving $\|P_{F_{\mathrm{u},W_{\mathrm{d}}}}\|_2 = 0.7037$. However, due to the lower bound $\gamma_{\mathrm{l}} = 0.7037$, we can conclude that $F_{\mathrm{u},W_{\mathrm{d}}}$ is indeed an optimal state-feedback gain.
- In this example, it turns out that $F_{\mathrm{l}} \in \mathcal{F}_{\mathrm{p}}$ and $F_{\mathrm{l}} = F_{\mathrm{u},W_{\mathrm{d}}}$ hold where F_{l} is defined in Theorem 2. These facts in particular implies that the assumption $F_{\mathrm{l}} \in \mathcal{F}_{\mathrm{p}}$ in the last assertion in Theorem 2 holds. Therefore, only from this fact we can also conclude that $F_{\mathrm{l}} (= F_{\mathrm{u},W_{\mathrm{d}}})$ is an optimal state-feedback gain.

2) The case $\gamma_{\mathrm{l}} \approx \gamma_{\mathrm{u},W_{\mathrm{d}}}^\star$: Again let us consider the case where $n=5$, $n_w=2$, $n_u=1$, and $n_z=2$ in (7) and

$$A = \begin{bmatrix} -1.70 & 0.22 & 0.70 & 0.27 & 0.82 \\ 0.21 & -2.34 & 0.56 & 0.36 & 0.75 \\ 0.15 & 0.38 & -2.20 & 0.87 & 0.39 \\ 0.39 & 0.72 & 0.77 & -2.20 & 0.33 \\ 0.02 & 0.71 & 0.44 & 0.86 & -1.81 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.31 & 0.48 \\ 0.68 & 0.24 \\ 0.63 & 0.45 \\ 0.59 & 0.90 \\ 0.20 & 0.57 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.07 \\ 0.38 \\ 0.94 \\ 0.98 \\ 0.39 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.10 & 0.38 & 0.36 & 0.54 & 0.05 \\ 0.65 & 0.69 & 0.15 & 0.46 & 0.39 \end{bmatrix},\ D_{12} = \begin{bmatrix} 0.87 \\ 0.19 \end{bmatrix}.$$

We obtained the upper and lower bounds by following exactly the same procedure as in the preceding subsection. The results are shown in Table II and several remarks follow.

- The upper bounds $\gamma_{\mathrm{u},G_{\mathrm{d}}}$ and $\gamma_{\mathrm{u},G_{\mathrm{d}}}^\star$ in Table II are both obtained when $b = 3.14$.
- Again, we see that $\gamma_{\mathrm{u},G_{\mathrm{d}}} \le \gamma_{\mathrm{u},X_{\mathrm{d}}}$ holds and this is consistent with Theorem 1. We also see that $\gamma_{\mathrm{f}}^\star \le \gamma_{\mathrm{l}}$ holds and this is consistent with Theorem 2.
- Those upper bounds $\gamma_{\mathrm{u},W_{\mathrm{d}}}^\star$, $\gamma_{\mathrm{u},X_{\mathrm{d}}}^\star$, and $\gamma_{\mathrm{u},G_{\mathrm{d}}}^\star$ again take

the same value 1.1351 since the gains $F_{\mathrm{u},W_{\mathrm{d}}}$, $F_{\mathrm{u},X_{\mathrm{d}}}$, $F_{\mathrm{u},G_{\mathrm{d}},b}$ resulting from the SDPs (14), (16), and (17) with $b = 3.14$ turn out to be the same and
$F_{\mathrm{u},W_{\mathrm{d}}} = [\,-0.0513\;\; -0.4043\;\; -0.4138\;\; -0.6207\;\; -0.0575\,]$.

- If we rely only on the trivial lower bound $\gamma_{\mathrm{f}}^{\star} = 0.8592$, we cannot draw any strong conclusion on the quality of $F_{\mathrm{u},W_{\mathrm{d}}}$ achieving $\gamma_{\mathrm{u},W_{\mathrm{d}}}^{\star} = \|P_{F_{\mathrm{u},W_{\mathrm{d}}}}\|_2 = 1.1351$. However, due to the lower bound $\gamma_{\mathrm{l}} = 1.0839$, we can conclude that
$\gamma_{\mathrm{u},W_{\mathrm{d}}}^{\star} \le \dfrac{\gamma_{\mathrm{u},W_{\mathrm{d}}}^{\star}}{\gamma_{\mathrm{l}}}\gamma^{\star} \approx 1.042\gamma^{\star}$.
- In this example, it turns out that $F_{\mathrm{l}} \notin \mathcal{F}_p$. Namely, the SDP (21) does not yield an eligible feedback gain.

B. Test with Randomly Generated Systems

To evaluate more extensively the tightness of the upper bounds $\gamma_{\mathrm{u},W_{\mathrm{d}}}^{\star}$, $\gamma_{\mathrm{u},X_{\mathrm{d}}}^{\star}$, $\gamma_{\mathrm{u},G_{\mathrm{d}},b}^{\star}$ and the lower bound γ_{l} as well as the mutual relationship among the three upper bounds, we generated randomly $A \in \mathbb{M}^5 \cap \mathbb{H}^5$, $B_1 \in \mathbb{R}_+^{5\times 2}$, $B_2 \in \mathbb{R}_+^{5\times 1}$, $C_1 \in \mathbb{R}_+^{2\times 5}$, $D_{12} \in \mathbb{R}_+^{2\times 1}$ in (7) and computed the average of the next values.

- $|\gamma_{\mathrm{u},W_{\mathrm{d}}}^{\star} - \gamma_{\mathrm{u},G_{\mathrm{d}},1}^{\star}|/\gamma_{\mathrm{u},G_{\mathrm{d}},1}^{\star}$ and $|\gamma_{\mathrm{u},X_{\mathrm{d}}}^{\star} - \gamma_{\mathrm{u},G_{\mathrm{d}},1}^{\star}|/\gamma_{\mathrm{u},G_{\mathrm{d}},1}^{\star}$: to examine the difference between the actual H_2 performance achieved by the gains computed from the SDPs (14), (16), and (17) with $b = 1$.
- $\gamma_{\mathrm{u},W_{\mathrm{d}}}^{\star}/\gamma_{\mathrm{l}}$: to evaluate the gap between the upper bound $\gamma_{\mathrm{u},W_{\mathrm{d}}}^{\star}$ and the lower bound γ_{l},
- $\gamma_{\mathrm{l}}/\gamma_{\mathrm{f}}^{\star}$: to evaluate the quality of the lower bound γ_{l} in comparison with the trivial lower bound $\gamma_{\mathrm{f}}^{\star}$.

The average values obtained from 1000 randomly generated systems are as follows. First, it turns out that $|\gamma_{\mathrm{u},W_{\mathrm{d}}}^{\star} - \gamma_{\mathrm{u},G_{\mathrm{d}},1}^{\star}|/\gamma_{\mathrm{u},G_{\mathrm{d}},1}^{\star} \approx 0$ and $|\gamma_{\mathrm{u},X_{\mathrm{d}}}^{\star} - \gamma_{\mathrm{u},G_{\mathrm{d}},1}^{\star}|/\gamma_{\mathrm{u},G_{\mathrm{d}},1}^{\star} \approx 0$ in average and hence little difference exists among $\gamma_{\mathrm{u},W_{\mathrm{d}}}^{\star}$, $\gamma_{\mathrm{u},X_{\mathrm{d}}}^{\star}$, and $\gamma_{\mathrm{u},G_{\mathrm{d}},1}^{\star}$. In fact, it is observed that $F_{\mathrm{u},W_{\mathrm{d}}} = F_{\mathrm{u},W_{\mathrm{d}}} = F_{\mathrm{u},G_{\mathrm{d}},1}$ holds in almost all problem instances. Since these gains are computed from different SDPs, we conjecture that these gains are optimal in most problem instances. However, such a conjecture cannot be verified by this test since we obtain $\gamma_{\mathrm{u},W_{\mathrm{d}}}^{\star}/\gamma_{\mathrm{l}} = 5.35$ in average. Even though we also obtain $\gamma_{\mathrm{l}}/\gamma_{\mathrm{f}}^{\star} = 1.31$ in average and this shows in part the effectiveness of the lower bound computation, it is strongly expected that the lower bound γ_{l} is loose in general.

To summarize, in some problem instances the proposed strategy with upper and lower bounds computation works well, but the lower bounds seem to be loose in general. Therefore, it is quite important to develop more effective lower bound computation techniques.

VI. Conclusion

In this paper, we considered the H_2 state-feedback control problem under positivity constraint on the closed-loop system. We derived SDPs for the computation of the upper and lower bounds of the achievable H_2 performance. Numerical examples show that, in some problem instances, the upper and lower bounds come close and hence we can ensure the quality of the computed suboptimal gains. However, computed lower bounds seem to be loose in general. Currently, we have a prospect that we can develop a completely different technique for the lower bound computation if we work on discrete-time systems. In the discrete-time system setting, we can focus on FIR (finite impulse response) and a specific treatment of FIR paves the way for a novel and effective lower bound computation technique. This topic is currently under investigation.

VII. Acknowledgments

The author is grateful to Toshiki Shimasaki and Tomomichi Hagiwara, both from Kyoto University, for fruitful discussions.

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