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State Feedback Regulator Design for Coupled Linear Wave Equations

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Abstract— **In this paper the state feedback regulator design is considered for a system of coupled linear wave equations with distinct constant wave speeds and a spatially-varying coupling. To this end, recent results for the backstepping control of coupled linear parabolic systems are extended to coupled linear wave equations. Subsequently, an additional feedforward controller is determined to ensure output regulation for reference inputs and disturbances described by a finite-dimensional signal model. The design of the resulting state feedback regulator is based on explicitly solving the regulator equations, for which a simple existence condition is provided. Exponential stability is verified for the closed-loop system. Two coupled wave equations subject to destabilizing boundary conditions are utilized to illustrate the results of the paper.**

I. Introduction

The *backstepping method* has become an important tool to solve stabilization problems for boundary controlled distributed-parameter systems (DPSs) (for an overview see, e. g., [13]). The controller is obtained from mapping the DPS into a suitable stable target system by means of a Volterra-type integral transformation. The latter is inherently invertible and can be obtained from solving the so-called *kernel equations*, which results in a systematic approach for the controller design.

Currently, the backstepping control of coupled parabolic systems attracts the attention of many researchers. Starting with the first systematic solution for the case of constant coefficients in [2], the recent result [14] extends the backstepping method to coupled parabolic systems with spatially-varying coefficients and Dirichlet boundary conditions (BCs). It is well-known that for wave equations (see [12]) and for second-order hyperbolic partial integro-differential equations (PIDEs) (see [11]) with a destabilizing BC at the free end the same type of backstepping transformation as for parabolic systems can be utilized in the backstepping design. Thereby, the resulting kernel equations share the same form as in the parabolic case. This suggests to extend the current results for coupled parabolic systems to systems of coupled wave equations and hyperbolic PIDEs with destabilizing BCs at the free end.

In this paper first results towards this goal are presented. In particular, linear coupled wave equations with distinct constant wave velocities and a spatially-varying in-domain coupling are considered. This class of systems arise in applications, for instance if rubber-like materials for absorbing vibrations have to be modeled. It is shown that the backstepping methods for coupled parabolic systems can also be applied to solve stabilization problems for this system class. As Robin BCs are imposed at both ends of the wave equations, the recent results [6], [7] for coupled parabolic systems with Neumann BCs is utilized to determine a state feedback controller for the DPSs in question. Thereby, a different target system has to be assigned compared to the parabolic case. More specifically, it is given by a cascade of wave equations, where ”stiffness BCs” and ”damping BCs” are imposed at the boundaries in order to achieve exponential stability of the closed-loop system. This extends the results in [12] to the considered multivariable case.

The *output regulation problem* for the considered class of wave equations in the case of a single PDE is still an active research topic (see, e. g., [1] for other results on output regulation for DPSs). This amounts to determine a stabilizing regulator that ensures the reference tracking despite of unmeasurable disturbances. Typically, the disturbances are assumed to be harmonic with unknown amplitudes but known frequencies. For example, the case of wave equations with a disturbance of this type acting at the output is considered in [8]. The collocated case is treated in [9], while the recent work [10] deals with anticollocated disturbances. In [5] a robust state feedback regulator was designed for second-order hyperbolic PIDEs subject to disturbances defined in-domain and at both boundaries. This regulator was determined on the basis of the *internal model principle* in order to ensure output regulation in the presence of model uncertainties, which do not destabilize the closed-loop system.

In this paper the backstepping stabilizer for coupled wave equations is also extended to a state feedback regulator in order to take reference inputs and unmeasurable disturbances into account. For this, it is assumed that the exogenous signals can be generated by an autonomous signal model. Thereby, the disturbances can act in-domain, at both boundaries and at the output to be controlled. The latter can be defined at the boundary, pointwise in-domain or distributed. The corresponding feedforward controller ensuring output regulation follows from solving the so-called *regulator equations*. Their explicit solution is determined and an easily verifiable existence condition is derived. Finally, exponential stability is verified for the resulting closed-loop system leading to a systematic solution of the posed output regulation problem.

The next section presents a formulation of the considered output regulation problem. Subsequently, Section III is devoted to its solution. In particular, a state feedback is determined with the backstepping approach to ensure closed-loop stability and a feedforward controller is designed by solving the regulator equations. The results of this paper are

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illustrated for two coupled wave equations with destabilizing boundary conditions in simulations.

II. Problem formulation

Consider the *system of coupled linear wave equations*

$$\partial_t^2 x(z,t) = \Lambda \partial_z^2 x(z,t) + A(z)x(z,t) + G_1(z)d(t) \quad (1a)$$
$$\partial_z x(0,t) = Q_0 x(0,t) + G_2 d(t), \qquad t>0 \quad (1b)$$
$$\partial_z x(1,t) = Q_1 x(1,t) + u(t) + G_3 d(t), \qquad t>0 \quad (1c)$$
$$y(t) = \mathcal{C}[x(t)] + G_4 d(t), \qquad t \geq 0 \quad (1d)$$

with (1a) defined on the domain $(z,t) \in (0,1) \times \mathbb{R}^+$, the state $x(z,t) \in \mathbb{R}^n$ and the input $u(t) \in \mathbb{R}^n$. The system (1) is subject to the unmeasurable disturbance $d(t) \in \mathbb{R}^q$ and has the initial conditions (ICs) $x(z,0) = x_{0,1}(z), \dot{x}(z,0) = x_{0,2}(z)$, $z \in [0,1]$. Furthermore, the matrix Λ has the form

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \quad (2)$$

with $\lambda_1 > \ldots > \lambda_n > 0$. This means that constant but mutually different *wave velocities* λ_i, $i = 1,2,\ldots,n$, are considered. Furthermore, it is assumed that the spatially varying coupling $A \in (C^1[0,1])^{n\times n}$. The matrix Q_0 is diagonal, i. e., no coupled boundary conditions (BCs) at $z=0$ are allowed. The output to be controlled $y(t) \in \mathbb{R}^n$ can be defined at the boundaries, pointwise or distributed in-domain. This is modeled by the formal *output operator*

$$\mathcal{C}[h] = \sum_{i=1}^{l} C_i h(z_i) + \textstyle\int_0^1 C(z)h(z)\mathrm{d}z \quad (3)$$

with $h(z) \in \mathbb{C}^n$ and $C_i \in \mathbb{R}^{n\times n}$, $z_i \in [0,1]$, $i = 1,2,\ldots,l$, as well as $C(z) = [c_{ij}(z)] \in \mathbb{R}^{n\times n}$, in which c_{ij} are piecewise continuous functions.

Remark 1: The plant (1), thus introduced, may be seen as a set of coupled strings, where the free ends at $z=0$ are subject to possibly destabilizing forces. The actuation at $z=1$ is a stabilizing force, that is provided by a suitable actuator. ◁

It is assumed that the disturbance d and the reference input r can be described as the solution of the *signal model*

$$\dot{v}(t) = Sv(t), \qquad t>0, \qquad v(0) = v_0 \in \mathbb{R}^{n_v} \quad (4a)$$
$$d(t) = P_d v(t), \qquad t \geq 0 \quad (4b)$$
$$r(t) = P_r v(t), \qquad t \geq 0 \quad (4c)$$

with $P_d \in \mathbb{R}^{q\times n_v}$ and $P_r \in \mathbb{R}^{n\times n_v}$. Therein, the *spectrum* $\sigma(S)$ of the diagonalizable matrix S only contains eigenvalues on the imaginary axis. This allows the modeling of bounded and persistently acting exogenous signals. In particular, the exogenous signals can be constant or trigonometric functions of time as well as linear combinations of both signal forms. For the design it is required that (4) as well as the disturbance input locations characterized by G_i, $i = 1,2,\ldots,4$, in (1) are known and that $G_1 \in (C^1[0,1])^{n\times q}$.

In this paper, the *state feedback regulator problem* is solved for (1). This amounts to designing a *state feedback regulator*

$$\begin{aligned} u(t) &= -K_v v(t) - K_1 x(1,t) - K_2 \dot{x}(1,t) \\ &\quad - \textstyle\int_0^1 K_x(z)x(z,t)\mathrm{d}z - \int_0^1 K_{\dot{x}}(z)\dot{x}(z,t)\mathrm{d}z \\ &= \mathcal{K}[v(t), x(t), \dot{x}(t)] \end{aligned} \quad (5)$$

with the *feedback gains* $K_v \in \mathbb{R}^{n\times n_v}$ and K_1, K_2, K_x, $K_{\dot{x}} \in \mathbb{R}^{n\times n}$ such that the closed-loop system is exponentially stable and

$$\lim_{t\to\infty} e_y(t) = \lim_{t\to\infty} (y(t) - r(t)) = 0 \quad (6)$$

holds independent from the initial values of the plant (1) and of the signal model (4).

III. State Feedback Regulator Design

A. Backstepping Stabilization

By applying (5) to (1) and taking (4), one obtains the closed-loop system

$$\dot{v}(t) = Sv(t) \quad (7a)$$
$$\partial_t^2 x(z,t) = \Lambda \partial_z^2 x(z,t) + A(z)x(z,t) + G_1(z)P_d v(t) \quad (7b)$$
$$\partial_z x(0,t) = Q_0 x(0,t) + G_2 P_d v(t) \quad (7c)$$
$$\partial_z x(1,t) = Q_1 x(1,t) + \mathcal{K}[v(t), x(t), \dot{x}(t)] + G_3 P_d v(t) \quad (7d)$$
$$e_y(t) = \mathcal{C}[x(t)] + (G_4 P_d - P_r)v(t). \quad (7e)$$

In principle, the *regulator equations* for determining the gain K_v in (5) can directly be formulated for the representation (1). The result, however, cannot be solved explicitly, as the system has spatially-varying coefficients. Consequently, also the solvability of these regulator equations is not easy to analyse. Therefore, the *backstepping transformation*

$$\tilde{x}(z,t) = x(z,t) - \textstyle\int_0^z K(z,\zeta)x(\zeta,t)\mathrm{d}\zeta = \mathcal{T}_c[x(t)](z) \quad (8)$$

with the kernel $K(z,\zeta) \in \mathbb{R}^{n\times n}$ is introduced to map (7) into the *target system*

$$\dot{v}(t) = Sv(t) \quad (9a)$$
$$\partial_t^2 \tilde{x}(z,t) = \Lambda \partial_z^2 \tilde{x}(z,t) - \widetilde{A}_0(z)\tilde{x}(0,t) + H_1(z)P_d v(t) \quad (9b)$$
$$\partial_z \tilde{x}(0,t) = \widetilde{Q}_0 \tilde{x}(0,t) + G_2 P_d v(t) \quad (9c)$$
$$\partial_z \tilde{x}(1,t) = -\widetilde{C}_1 \partial_t \tilde{x}(1,t) + (G_3 P_d - K_v)v(t) \quad (9d)$$
$$e_y(t) = \mathcal{C}\mathcal{T}_c^{-1}[\tilde{x}(t)] + (G_4 P_d - P_r)v(t). \quad (9e)$$

with

$$\widetilde{A}_0(z) = \begin{bmatrix} 0 & \ldots & \ldots & 0 \\ \widetilde{A}_{0,21}(z) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \widetilde{A}_{0,n1}(z) & \ldots & \widetilde{A}_{0,n\,n-1}(z) & 0 \end{bmatrix}, \quad (10)$$

which is a cascade of wave equations and thus has a simpler structure. In (9e) the *inverse backstepping transformation*

$$x(z,t) = \tilde{x}(z,t) + \textstyle\int_0^z L(z,\zeta)\tilde{x}(\zeta,t)\mathrm{d}\zeta = \mathcal{T}_c^{-1}[\tilde{x}(t)](z) \quad (11)$$

is used. The elements $\widetilde{A}_{0,ij}(z)$, $i > j$, in $\widetilde{A}_0(z)$ are determined by the kernel $K(z,\zeta)$. These couplings have to be introduced, because a complete decoupling of the PDEs in (7b) leads to an overdetermined set of *kernel equations* to be solved for determining $K(z,\zeta)$. Moreover, both the matrices $\widetilde{Q}_0$ and $\widetilde{C}_1$ are diagonal and positive definite, i. e.,

$$\widetilde{Q}_0 = \operatorname{diag}(\tilde{q}_1,\ldots,\tilde{q}_n) \;\text{ and }\; \widetilde{C}_1 = \operatorname{diag}(\tilde{c}_1,\ldots,\tilde{c}_n) \quad (12)$$

hold.

Remark 2: This choice of the target system has the intuitive meaning that "stiffness BCs" are imposed at $z=0$ and "damping BCs" are assigned at $z=1$ in order to stabilize the coupled wave equations. ◁

For the derivation of the kernel equations, differentiate (8) w.r.t. time, insert (7b) and utilize (9b) to get

$$\begin{aligned}
\partial_t^2\tilde{x}(z,t) &= \Lambda\partial_z^2\tilde{x}(z,t) - \widetilde{A}_0(z)\tilde{x}(0,t) + H_1(z)P_d v(t)\\
&\quad + \Lambda\partial_z^2\textstyle\int_0^z K(z,\zeta)x(\zeta,t)\mathrm{d}\zeta + \widetilde{A}_0(z)x(0,t)\\
&\quad - H_1(z)P_d v(t) + A(z)x(z,t) + G_1(z)P_d v(t)\\
&\quad - \textstyle\int_0^z K(z,\zeta)A(\zeta)x(\zeta,t)\mathrm{d}\zeta - \int_0^z K(z,\zeta)\Lambda\partial_\zeta^2 x(\zeta,t)\mathrm{d}\zeta\\
&\quad - \textstyle\int_0^z K(z,\zeta)G_1(z)P_d\mathrm{d}\zeta v(t). \qquad (13)
\end{aligned}$$

Integration by parts and the Leibniz differentiation rule yield after simple calculations in view of (7c)

$$\begin{aligned}
\partial_t^2\tilde{x}(z,t) &= \Lambda\partial_z^2\tilde{x}(z,t) - \widetilde{A}(z)\tilde{x}(0,t) + H_1(z)P_d v(t)\\
&\quad + \textstyle\int_0^z(\Lambda K_{zz}(z,\zeta) - K_{\zeta\zeta}(z,\zeta)\Lambda - K(z,\zeta)A(\zeta))x(\zeta,t)\mathrm{d}\zeta\\
&\quad + (\widetilde{A}_0(z) + K(z,0)\Lambda Q_0 - K_\zeta(z,0)\Lambda)x(0,t)\\
&\quad + (\Lambda K'(z,z) + \Lambda K_z(z,z) + K_\zeta(z,z)\Lambda + A(z))x(z,t)\\
&\quad + (\Lambda K(z,z) - K(z,z)\Lambda)\partial_z x(z,t)\\
&\quad + (\mathcal{T}_c[G_1](z) - H_1(z) + K(z,0)\Lambda G_2)P_d v(t). \qquad (14)
\end{aligned}$$

Differentiating (8) w.r.t. z, evaluating the result for $z=0$ and inserting (7c) results in

$$\begin{aligned}
\partial_z\tilde{x}(0,t) &= \partial_z x(0,t) - K(0,0)x(0,t) + G_2 P_d v(t)\\
&= (Q_0 - K(0,0))x(0,t) + G_2 P_d v(t). \qquad (15)
\end{aligned}$$

Hence, the BC (9c) is obtained for

$$K(0,0) = Q_0 - \widetilde{Q}_0. \qquad (16)$$

The latter result and requiring that (14) equals (9) yield the the *kernel equations*

$$\begin{aligned}
\Lambda K_{zz}(z,\zeta) - K_{\zeta\zeta}(z,\zeta)\Lambda &= K(z,\zeta)A(\zeta) && (17a)\\
K_\zeta(z,0)\Lambda - K(z,0)\Lambda Q_0 &= \widetilde{A}_0(z) && (17b)\\
\Lambda K'(z,z) + \Lambda K_z(z,z) + K_\zeta(z,z)\Lambda &= -A(z) && (17c)\\
K(z,z)\Lambda - \Lambda K(z,z) &= 0 && (17d)\\
K(0,0) &= Q_0 - \widetilde{Q}_0 && (17e)
\end{aligned}$$

with (17a) defined on $0<\zeta<z<1$ and

$$H_1(z) = \mathcal{T}_c[G_1](z) + K(z,0)\Lambda G_2. \qquad (18)$$

It is shown in [7] that the kernel equations (17) have a piecewise C^2-solution and can be determined with method of successive approximations. Furthermore, differentiating (8) w.r.t. z, evaluating for $z=1$ and comparing the result with (9d) gives the feedback gains

$$\begin{aligned}
K_1 &= Q_1 - K(1,1) && (19a)\\
K_2 &= \widetilde{C}_1 && (19b)\\
K_x(z) &= -K_z(1,z) && (19c)\\
K_{\dot{x}}(z) &= -\widetilde{C}_1 K(1,z). && (19d)
\end{aligned}$$

Similar to the previous determination of (17), the kernel equations

$$\begin{aligned}
\Lambda L_{zz}(z,\zeta) - L_{\zeta\zeta}(z,\zeta)\Lambda &= -A(z)L(z,\zeta) && (20a)\\
L_\zeta(z,0)\Lambda - L(z,0)\Lambda\widetilde{Q}_0 &= \textstyle\int_0^z L(z,\zeta)\widetilde{A}_0(\zeta)\mathrm{d}\zeta + \widetilde{A}_0(z) && (20b)\\
\Lambda L'(z,z) + \Lambda L_z(z,z) + L_\zeta(z,z)\Lambda &= -A(z) && (20c)\\
L(z,z)\Lambda - \Lambda L(z,z) &= 0 && (20d)\\
L(0,0) &= Q_0 - \widetilde{Q}_0 && (20e)
\end{aligned}$$

of the *inverse backstepping transformation* (11) follow after differentiating (11) as well as using (7) and (9). Therein, (20a) is defined on $0<\zeta<z<1$. In [7] it is shown that (20) has a piecewise C^2-solution.

B. Regulator Equations

In what follows the change of coordinates

$$\tilde{\varepsilon}(z,t) = \tilde{x}(z,t) - \Pi(z)v(t) \qquad (21)$$

with $\Pi(z)\in\mathbb{R}^{n\times n_v}$ is determined, that maps (9) into the *error system*

$$\begin{aligned}
\dot{v}(t) &= Sv(t) && (22a)\\
\partial_t^2\tilde{\varepsilon}(z,t) &= \Lambda\partial_z^2\tilde{\varepsilon}(z,t) - \widetilde{A}_0(z)\tilde{\varepsilon}(0,t) && (22b)\\
\partial_z\tilde{\varepsilon}(0,t) &= \widetilde{Q}_0\tilde{\varepsilon}(0,t) && (22c)\\
\partial_z\tilde{\varepsilon}(1,t) &= -\widetilde{C}_1\partial_t\tilde{\varepsilon}(1,t) && (22d)\\
e_y(t) &= \mathcal{C}\mathcal{T}_c^{-1}[\tilde{\varepsilon}(t)], && (22e)
\end{aligned}$$

in which the PDE subsystem is decoupled from the ODE (22a). For this, differentiate (21) twice w.r.t. time and insert (9). Then, a simple calculation yields the *regulator equations*

$$\begin{aligned}
\Pi(z)S^2 - \Lambda\Pi''(z) &= -\widetilde{A}_0(z)\Pi(0) + H_1(z)P_d && (23a)\\
\Pi'(0) - \widetilde{Q}_0\Pi(0) &= G_2 P_d && (23b)\\
\mathcal{C}\mathcal{T}_c^{-1}[\Pi] &= P_r - G_4 P_d && (23c)
\end{aligned}$$

with (23a) defined on $z\in(0,1)$, which has to be fulfilled by $\Pi(z)$ in order to obtain (22). With, this the feedback gain

$$K_v = -\widetilde{C}_1\Pi(1)S + G_3 P_d - \Pi'(1) \qquad (24)$$

follows from taking the spatial derivative of (21), evaluating the result for $z=1$ and inserting (9d) to ensure (22d). The next lemma presents the condition for the solvability of the regulator equations by determining the explicit solution.

Lemma 1: (Regulator equations). The numerator $N(s)$ of the transfer matrix $F(s) = N(s)D^{-1}(s)$ of (1) from u to y is $N(s) = \mathcal{C}\mathcal{T}_c^{-1}[E_1^T M(\cdot, s)]$ with

$$\begin{aligned} M(z,s) &= \mathrm{e}^{\Upsilon(s)z}(E_1 + E_2\widetilde{Q}_0) \\ &\quad + \textstyle\int_0^z \mathrm{e}^{\Upsilon(s)(z-\zeta)} E_2 \Lambda^{-1} \widetilde{A}_0(\zeta)\mathrm{d}\zeta, \end{aligned} \tag{25}$$

in which

$$\Upsilon(s) = \begin{bmatrix} 0 & I \\ s^2\Lambda^{-1} & 0 \end{bmatrix}$$

as well as

$$E_1 = \begin{bmatrix} I_n \\ 0 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 \\ I_n \end{bmatrix} \tag{26}$$

were utilized. The regulator equations (23) have a unique piecewise twice differentiable solution $\Pi(z) \in \mathbb{R}^{n \times n_v}$, iff

$$\det N(\lambda) \neq 0, \quad \forall \lambda \in \sigma(S). \tag{27}$$

Proof: Postmultiply (23) by the linearly independent eigenvectors φ_i, $i = 1, 2, \ldots, n_v$, of S w.r.t. the eigenvalues μ_i. Furthermore, introduce the abbreviations $\pi_i = \Pi\varphi_i$, $h_{1,i} = -H_1 P_d \varphi_i$, $g_{j,i} = G_j P_d \varphi_i$, $j = 2, 4$ and $p_{r,i} = P_r \varphi_i$. This leads to the BVPs

$$\begin{aligned} \pi_i''(z) &= \mu_i^2 \Lambda^{-1}(z)\pi_i(z) \\ &\quad + \Lambda^{-1}(\widetilde{A}_0(z)\pi_i(0) + h_{1,i}(z)) \end{aligned} \tag{28a}$$

$$\pi_i'(0) - \widetilde{Q}_0\pi_i(0) = g_{2,i} \tag{28b}$$

$$\mathcal{C}\mathcal{T}_c^{-1}[\pi_i] = p_{r,i} - g_{4,i} \tag{28c}$$

with (28a) defined on $z \in (0,1)$ for $i = 1, 2, \ldots, n_v$. Observe that with the matrices in (26) one can write

$$\begin{bmatrix} \pi_i(0) \\ \pi_i'(0) \end{bmatrix} = E_1\pi_i(0) + E_2\pi_i'(0) = (E_1 + E_2\widetilde{Q}_0)\pi_i(0) + E_2 g_{2,i}, \tag{29}$$

in which (28b) was taken into account. Hence, the solution of (28a) and (28b) is

$$\pi_i(z) = E_1^T M(z, \mu_i)\pi_i(0) + m(z, \mu_i) \tag{30}$$

with $M(z,s)$ defined in the lemma and

$$\begin{aligned} m(z,\mu_i) &= E_1^T(\mathrm{e}^{\Upsilon(\mu_i)(z-\zeta)} E_2 g_{2,i} \\ &\quad - \textstyle\int_0^z \mathrm{e}^{\Upsilon(\mu_i)(z-\zeta)} E_2 \Lambda^{-1} h_{1,i}(\zeta)\mathrm{d}\zeta). \end{aligned} \tag{31}$$

Inserting this in (28c) gives

$$\mathcal{C}\mathcal{T}_c^{-1}[E_1^T M(\cdot, \mu_i)]\pi_i(0) = -\mathcal{C}\mathcal{T}_c^{-1}[m(\cdot, \mu_i)] + p_{r,i} - g_{4,i}. \tag{32}$$

Thus, iff the condition (27) of the lemma holds, then the latter result is uniquely solvable for $\pi_i(0)$. Hence, the solution

$$\Pi(z) = [\pi_1(z) \ \ldots \ \pi_{n_v}(z)][\varphi_1 \ \ldots \ \varphi_{n_v}]^{-1} \tag{33}$$

of (23) can be obtained. Since the kernel equations (17) have a piecewise C^2-solution (see [7]) and $G_1 \in (C^1[0,1])^{n \times q}$, the elements of $H_1(z)$ and thus $h_{1,i}(z)$ are piecewise C^1 (see (18)). Consequently, the result $\Pi(z)$ is the piecewise C^2-solution of (23).

In order to derive the transfer matrix $F(s) = N(s)D^{-1}(s)$ apply the backstepping transformation $\tilde{x}(z,t) = \mathcal{T}_c[x(t)](z)$ to (1) (see (8)). This leads for $d(t) \equiv 0$ and using the inverse backstepping transformation (11) to

$$\partial_t^2 \tilde{x}(z,t) = \Lambda \partial_z^2 \tilde{x}(z,t) - \widetilde{A}_0(z)\tilde{x}(0,t) \tag{34a}$$

$$\partial_z \tilde{x}(0,t) = \widetilde{Q}_0 \tilde{x}(0,t) \tag{34b}$$

$$\begin{aligned} \partial_z \tilde{x}(1,t) &= (Q_1 - K(1,1))\mathcal{T}_c^{-1}[\tilde{x}(t)](1) + u(t) \\ &\quad - \textstyle\int_0^1 K_z(1,z)\mathcal{T}_c^{-1}[\tilde{x}(t)](z)\mathrm{d}z \end{aligned} \tag{34c}$$

$$y(t) = \mathcal{C}\mathcal{T}_c^{-1}[\tilde{x}(t)]. \tag{34d}$$

Since Λ is diagonal, the transfer matrix $F(s)$ is readily determined for this system. As a result the numerator $N(s)$ of the lemma is obtained. ■

Remark 3: Notice that the transfer matrices introduced in this lemma and in the following are understood as mappings from an exponential input $u(t) = u_0\,\mathrm{e}^{st}$, $s \in \mathbb{C}$, $u_0 \in \mathbb{C}^n$, $t \geq 0$, to the unique exponential output for an appropriate IC (see [15]). If the condition of Lemma 1 holds, then the eigenmodes of (4) are not blocked by the transfer behavior of (1). Obviously, this is necessary for output regulation. ◃

C. State Feedback Regulator

If the condition of Lemma 1 is satisfied, then the closed-loop system (9) can be mapped into the error system (22). Hence, the exponential stability of (22b)–(22d) implies output regulation (6). This is the result of the following theorem.

Theorem 1 (State Feedback Regulator): Consider the controller (5) with the feedback gains (19) and (24). Introduce the Hilbert space $H_i = H^1(0,1) \times L_2(0,1)$ with the inner product $\langle x, y \rangle_{H_i} = \tilde{q}_i x_1(0)\overline{y_1(0)} + \langle x_1', y_1' \rangle_{L_2} + \langle x_2, y_2 \rangle_{L_2}$ and let $X = H_1 \times \cdots \times H_n$ be the state space. Then, the closed-loop system (22b)–(22d) has a unique solution $\varepsilon_{cl}(t) = \mathrm{col}(\tilde{\varepsilon}_1(\cdot,t), \partial_t\tilde{\varepsilon}_1(\cdot,t), \ldots, \tilde{\varepsilon}_n(\cdot,t), \partial_t\tilde{\varepsilon}_n(\cdot,t)) \in (C^1(\mathbb{R}^+; H))^{2n}$ for all $\varepsilon_{cl}(0) \in X$ compatible with the BCs. Furthermore, the origin of the closed-loop system (22b)–(22d) is exponentially stable in the induced norm $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$ and $\lim_{t\to\infty} e_y(t) = \lim_{t\to\infty} \mathcal{C}\mathcal{T}_c^{-1}[\tilde{\varepsilon}(t)] = 0$ holds.

Proof: The closed-loop system (22b)–(22d) can be written with $\tilde{\varepsilon} = \mathrm{col}(\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_n)$ componentwise in the form

$$\partial_t^2 \tilde{\varepsilon}_i(z,t) = \lambda_i \partial_z^2 \tilde{\varepsilon}_j(z,t) - \sum_{j=1}^{i-1} \widetilde{A}_{0,ij}(z)\tilde{\varepsilon}_i(0,t) \tag{35a}$$

for $i = 2, \ldots, n$ and $\partial_t^2 \tilde{\varepsilon}_1(z,t) = \lambda_1 \partial_z^2 \tilde{\varepsilon}_1(z,t)$ for $i = 1$, as well as

$$\partial_z \tilde{\varepsilon}_i(0,t) = \tilde{q}_i \tilde{\varepsilon}_i(0,t) \tag{35b}$$

$$\partial_z \tilde{\varepsilon}_i(1,t) = -\tilde{c}_i \partial_t \tilde{\varepsilon}_i(1,t) \tag{35c}$$

for $i = 1, \ldots, n$ (see (10), (12) and (22b)–(22d)). Define the operator

$$\mathcal{A}_i h = \begin{bmatrix} h_2 \\ h_1'' \end{bmatrix}, \quad i = 1, 2, \ldots, n \tag{36}$$

with $h_1(z), h_2(z) \in \mathbb{R}$ and the domain

$$\begin{aligned} &D(\mathcal{A}_i) \\ &= \{(h_1, h_2) \in H^2(0,1) \times H^1(0,1) \mid h_1'(0) = \tilde{q}_i h_1(0), \\ &\qquad h_1'(1) = -\tilde{c}_i h_2(1)\} \end{aligned} \tag{37}$$

in the Hilbert space H. In [5] it is shown that $\mathcal{A}_i$ is the generator of an exponentially stable C_0-semigroup on H. Obviously, this property is also inherited by $\mathcal{A} = \operatorname{col}(\mathcal{A}_1, \dots, \mathcal{A}_n)$ on X. With this, the system (35) has the system operator $\widetilde{\mathcal{A}} = \mathcal{A} + \Delta$, in which $\Delta : D(\mathcal{A}) \to X$ is a bounded linear operator since the point evaluation is a bounded operation in H^1. Consequently, $\widetilde{\mathcal{A}}$ is also the generator of an exponentially stable C_0-semigroup on X by [3, Lem. 3.2.2], because the operator Δ is, in addition, strictly lower triangular (see (35a)). Furthermore, also the output operator $\mathcal{C}\mathcal{T}_c^{-1}$ in (22e) is bounded on X (see (3)) so that $\lim_{t\to\infty} e_y(t)$ is implied by the previous derivations. ■

Remark 4: It is intuitively clear that the elements of $\widetilde{Q}_0$ and $\widetilde{C}_1$ provide enough degrees of freedom in order to shape the closed-loop dynamics (see also Remark 2). ◁

IV. Example

The previous results are illustrated for an unstable 2×2 wave equation system. The wave velocities and the spatially-varying distributed coupling are given by

$$\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad A(z) = \begin{bmatrix} 0.5 & \sin(\pi z) \\ -0.5z & 1 - z^2 \end{bmatrix}.$$

Furthermore, the coefficients of the BC are

$$Q_0 = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad Q_1 = \begin{bmatrix} -1 & 0.7 \\ 0.3 & -1 \end{bmatrix}.$$

In order to consider boundary as well as in-domain outputs, the output-operator is defined as

$$\mathcal{C}[x(t)] = \begin{bmatrix} x_1(0,t) \\ \int_0^1 x_2(z,t)\mathrm{d}z \end{bmatrix}.$$

The disturbance $d(t) \in \mathbb{R}$ acts distributed in-domain, at the boundaries and at the output, hence

$$G_1(z) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; G_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; G_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } G_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In all simulations the plant is approximated with the Finite-Differences-Method utilizing an 804-dimensional ODE.

First, the stabilization is considered, i. e., the controller

$$\begin{aligned} u(t) &= -K_1 x(1,t) - K_2 \dot{x}(1,t) \\ &\quad - \int_0^1 K_x(z) x(z,t)\mathrm{d}z - \int_0^1 K_{\dot{x}}(z)\dot{x}(z,t)\mathrm{d}z \end{aligned}$$

is applied. Hereby, the stiffness BC (9c) of the target-system is set to $\widetilde{Q}_0 = 6I$. The kernel is calculated by solving the kernel equations (17) numerically in MATLAB by following [7]. Only a minor extension of this result is necessary, because of the different boundary condition (17e). Thereby, an artificial Dirichlet BC is added to establish well-posed kernel equations. This artificial BC is set to zero and the resulting integral equations are solved numerically with the method of successive approximations. An equidistant grid with 101 points for both z- and ζ-axis and trapezoidal integration are used. After 11 iteration steps, the maximum pointwise change of the absolute kernel values is less than 10^{-4} and the calculation is stopped, since the resulting approximation of the kernel is sufficiently accurate. In Fig. 1 the states of the stabilized system subject to an initial excitation are depicted. Thereby, the feedback was designed to prescribe $\widetilde{C}_1 = I$ for the target-system. Fig. 2 shows the solution of the closed-loop system in the L_2-norm for different target-system parameters. Clearly, the feedback ensures stability while the convergence can be adjusted by assigning boundary damping and stiffness due to the choice of $\widetilde{C}_1$ and $\widetilde{Q}_0$.

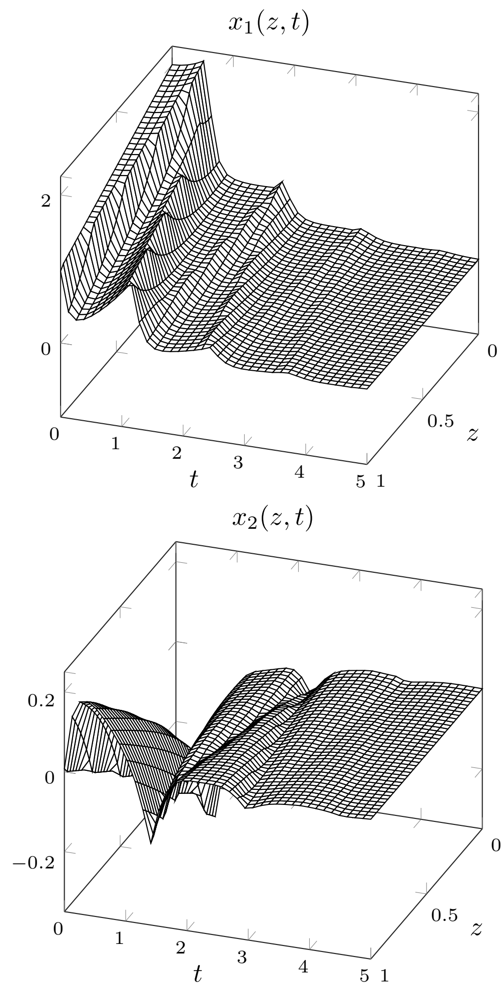

Fig. 1. Solution of the closed-loop system for the IC $x(z,0) = [2 - z \quad -0.3 + 0.3z]^\top$, $\dot{x}(z,0) = 0$ and the design parameters $\widetilde{C}_1 = I$, $\widetilde{Q}_1 = 6I$.

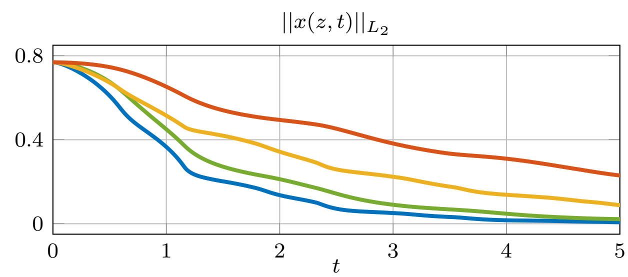

Fig. 2. L_2-norms of the closed-loop systems solution for the IC $x(z,0) = [2 - z \quad -0.3 + 0.3z]^\top$, $\dot{x}(z,0) = 0$ and different design parameters: $\widetilde{C}_1 = I$, $\widetilde{Q}_1 = 6I$ (blue); $\widetilde{C}_1 = \widetilde{Q}_1 = I$ (green); $\widetilde{C}_1 = 2I$, $\widetilde{Q}_1 = 6I$ (yellow); $\widetilde{C}_1 = 2I$, $\widetilde{Q}_1 = I$ (red).

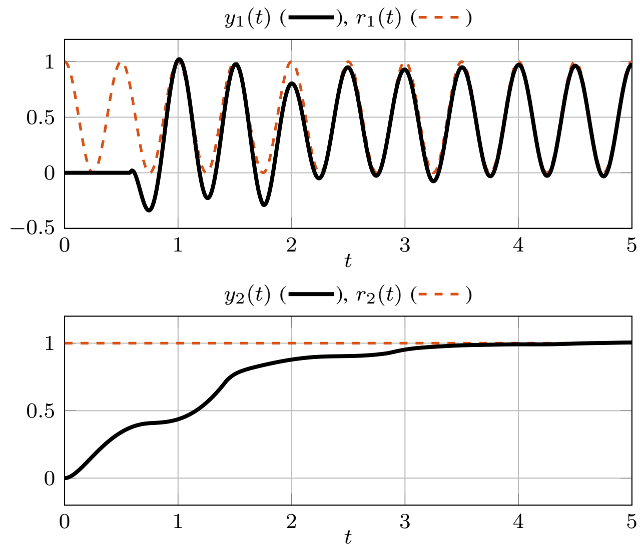

Fig. 3. Closed-loop reference tracking for $r_1(t) = \frac{1}{2} + \frac{1}{2}\cos(4\pi t)$ and $r_2(t) = 1$.

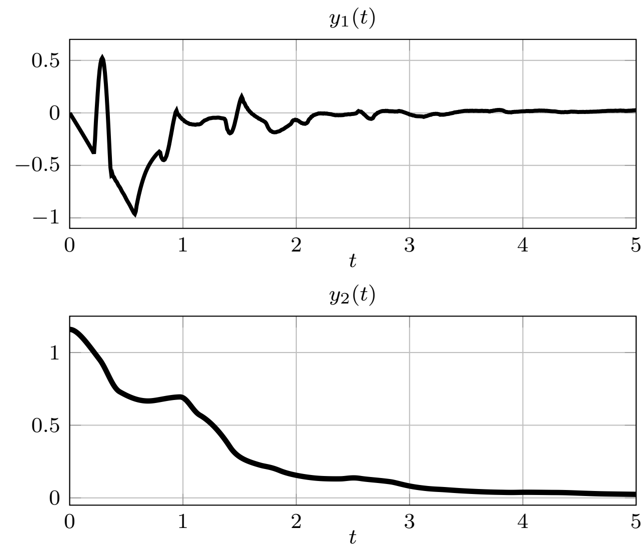

Fig. 4. Closed-loop disturbance rejection for $d(t) = 1$.

The related output regulation problem is solved for a constant disturbance $d(t) = 1$ and both sinusoidal as well as constant reference signals $r_1(t) = \frac{1}{2} + \frac{1}{2}\cos(4\pi t)$ and $r_2(t) = 1$. Therefore, the signal-model (4) with

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -4\pi & 0 \\ 0 & 4\pi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_r = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$P_d = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

is employed. The kernel is determined for $\widetilde{Q}_1 = 6I$ and the corresponding inverse kernel is calculated by using the reciprocity relation (see [13]). Then, the condition (27) for solving the regulator equations is verified. For all $\lambda \in \sigma(S) = \{0, \pm j4\pi\}$ holds $\det N(\lambda) \neq 0$, since $\det N(0) = 0.442$, $\det N(\pm j4\pi) = 0.0324$ so that the state feedback regulator exists. Hence, the state feedback (5) is computed after solving the regulator equations (23) and choosing $\widetilde{C}_1 = I$. Simulations confirm that reference tracking as well as disturbance rejection are ensured, as depicted in Fig. 3 and Fig. 4.

V. Concluding remarks

Future work considers the design of a disturbance observer in order to implement the state feedback regulator. For this, the results in [4] can be extended to the coupled wave equations. Furthermore, also the generalization of the considered systems to spatially-varying wave speeds, coupled BCs on both ends and additional local and Volterra integral terms in the PDEs would be interesting in order to widen the applicability of the obtained results.

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