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Controllability Degree of Directed Line Networks: Nodal Energy and Asymptotic Bounds

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Abstract— **This paper studies the controllability degree of dynamical networks with continuous-time dynamics. To quantify the controllability degree of a network, we introduce a new notion termed *nodal energy*, which is the control energy required to change the state of a single node while keeping the final states of the other nodes unchanged. Since it is extremely challenging to analyze the nodal energy of general networks, this paper focuses on a special class of directed line networks with single control nodes. This choice allows us to derive the explicit expression of the nodal energy of different network nodes, and hence, reveal novel controllability properties of complex networks. Our analysis shows that (i) differently from their discrete-time counterpart, directed line networks with continuous-time dynamics are always difficult to control, as the control energy grows linearly or even exponentially with the network cardinality, (ii) the numerical investigation of the controllability degree of line networks is unreliable, because the condition number of the controllability Gramian grows exponentially fast as the network cardinality increases, and (iii) nodal energies may be inversely related to the graphical distance from the control node because, depending on the network weights, distant nodes may require far less nodal energy than immediate neighbors.**

I. Introduction

Controllability is a fundamental property of dynamical complex networks, which characterizes whether or not a network can be driven from any initial state to any other desired state. The study of controllability of complex networks has attracted extensive attention in recent years [1]–[6]. While most of the existing work focuses on the binary problem whether or not a network is controllable, less effort has been devoted to studying the controllability degree of complex networks [7]–[12]. The controllability degree of a network can be quantitatively measured by the energy required to control the state of the network.

It has been shown in [12] that undirected networks with symmetric adjacency matrices are difficult to control, in the sense that the control energy may grow exponentially fast as the number of the nodes increases. This result implies that certain biological, social, and technological networks may be particularly robust to localized interventions, because of the prohibitively large energy required to arbitrarily modify the network state. However, it is still unclear whether this property also holds for other networks such as directed networks with asymmetric adjacency matrices.

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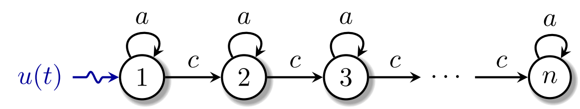


Fig. 1: The directed line network studied in this paper. The scalars a and c are edge weights.

While the controllability degree of discrete-time dynamical networks has been studied by many researchers [7]–[10], the networks with continuous-time dynamics have received much less attention, notable exceptions being [11], [12]. One reason is that the controllability Gramian of a continuous-time network involves integral calculation and is more difficult to analyze compared to the discrete case. The previous studies on continuous-time networks adopt numerical simulation or mathematical approximation to reveal the controllability properties. This paper aims at deriving rigorous mathematical expressions of the control energy of continuous-time networks. Since it is extremely challenging to do that for general networks, we focus a class of directed line networks with single control nodes (see Figure 1). Due to the special structure of the line network, we are able to explicitly calculate the controllability Gramian, which is difficult to obtain for general networks.

Most of the existing studies on controllability degree of networks adopt the notion of *eigen energy*, which is the energy required to change the state of a network along an eigenvector of the controllability Gramian [12]. The notion of eigen energy was originally proposed for general linear dynamical systems instead of networks. In this paper, we introduce a new notion termed *nodal energy*, which is the energy required to change the state of a single node while maintaining the final states of the other nodes unchanged. Nodal energy has a much clearer graphical interpretation than eigen energy.

The contribution of this paper is threefold. First, we introduce the notion of nodal energy, and derive the analytical expressions of the nodal energies of continuous-time directed line networks. Our analysis shows that line networks with continuous-time dynamics are always difficult to control, because the worst-case nodal energy grows linearly or exponentially fast as the cardinality of the network increases. Second, we quantify the condition number of the controllability Gramian of line networks, and show that it increases exponentially fast as the network cardinality increases. This result implies that that numerical investigation of the controllability properties of line networks is unreliable. Third, we show a counterintuitive property that the pattern of nodal energies may be inversely related to the graphical distance to the control node. In particular, nodes that are distant from the control node may require far less nodal energy than immediate neighbors. Finally, as a minor result, we show that the notion of nodal energy is also not directly related to the steady-state gain of the network transfer function.

II. Problem Setup and Preliminaries

A. Network Dynamics

The directed line network studied in this paper is illustrated in Figure 1. The network consists of n nodes and one single control input. The dynamics of the network are described by the following linear time-invariant continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathbb{R}$ is the single control input. The adjacency matrix $A \in \mathbb{R}^{n\times n}$ and the input matrix $B \in \mathbb{R}^{n\times 1}$ have the specific form of

$$A = \begin{bmatrix} a & 0 & \ldots & 0 \\ c & a & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & c & a \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n\times 1},$$

where $a < 0$ is the weight of the self-loops and $c > 0$ is the weight of each directed edge between two adjacent nodes. Notice that the network is stable for any $a < 0$, since the eigenvalues of A are all equal to a.

B. Network Controllability and Controllability Degree

The dynamical system (1) is controllable if there exists a control input that can steer the state of the network from any initial value $x(0)$ to any final desired value $x(t_f)$ where $t_f > 0$. Let $K \triangleq [B, AB, A^2B, \ldots, A^{n-1}B] \in \mathbb{R}^{n\times n}$ be the controllability matrix. It is well-known that system (1) is controllable if and only if the rank of K equals n [13], [14]. For the line network, the controllability matrix K is

$$\begin{aligned} K &= [B, AB, A^2B, \ldots, A^{n-1}B] \\ &= \begin{bmatrix} 1 & * & * & \ldots & * \\ 0 & c & * & \ldots & * \\ 0 & 0 & c^2 & \ldots & * \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & c^{n-1} \end{bmatrix}, \end{aligned}$$

where $*$ represents the value of entries that do not contribute to the rank of the matrix. Since K is upper-triangular, the network is controllable if and only if $c \neq 0$.

An alternative criterion for controllability is based on the controllability Gramian, which is defined as

$$G \triangleq \int_0^\infty e^{At}BB^Te^{A^Tt}\mathrm{d}t. \tag{2}$$

It is known that the Gramian is always positive semi-definite, and it is positive definite if and only if system (1) is controllable [14, Section 3.4]. It is worth mentioning that the controllability matrix K and the controllability Gramian G do not have evident connections for continuous-time networks. Instead, for discrete-time networks, the matrix KK^T equals the n-step controllability Gramian [7], [8].

The controllability Gramian can be used to quantitatively characterize the controllability degree of a network, as measured by the energy required for a control task. To be precise, let $x(0) = 0$ and $x(t_f) = x_f$ be the initial and final network states, respectively. Then, the least energy that is required to control the state from 0 to x_f is [12]

$$E(x_f) = x_f^TG^{-1}x_f.$$

When $c > 0$, since system (1) is controllable as analyzed above, the Gramian is positive definite and all its eigenvalues are real and positive. Let $\{\lambda_i(G)\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ be the eigenvalues and unit-norm eigenvectors of G, respectively. The energy required to control the network state to v_i is $E(v_i) = v_i^TG^{-1}v_i = 1/\lambda_i(G)$, which is referred to the ith *eigen energy* [12]. Let $\lambda_{\min}(G) = \min_i \lambda_i(G)$. Then, for any x_f,

$$E(x_f) \leq E_{\max} = \frac{1}{\lambda_{\min}(G)},$$

where $E_{\max}$ is the energy required to steer the network state to the unit-norm eigenvector associated with $\lambda_{\min}(G)$.

C. Nodal Energy

The eigen-properties of the Gramian play key roles in the analysis of the controllability degree, but they are typically difficult to characterize analytically. To overcome this issue, in this paper we introduce the *nodal energy*, which is defined as

$$E_i \triangleq E(e_i) = e_i^TG^{-1}e_i = [G^{-1}]_{ii}, \tag{3}$$

where e_i is the ith column of the identity matrix, and $[G^{-1}]_{ii}$ is the ith diagonal entry of G^{-1}. The notion of nodal energy has several advantages compared to eigen energy. First, nodal energy has a clear physical interpretation: it is the energy required to change the state from $x(0) = 0$ to $x(t_f) = e_i$, i.e., the state of node i from 0 to 1 while maintaining the final states of the other nodes as 0. As a comparison, the eigen energy is the energy required to change the state of the network to an eigenvector which is usually difficult to obtain analytically and may not be of practical interest. Second, nodal energy is easier to compute than eigen energy. In fact, E_i equals the ith diagonal entry of G^{-1}, which we are able to compute as we show later. Further, the closed-form expression that we obtain provides more insights into how the network properties affect its controllability degree. Third, the study of nodal energy can provide useful information about the eigen energy. For instance, since $E_i \leq E_{\max}$ for all i, we have $E_{\max} \to \infty$ if $E_i \to \infty$ as $n \to \infty$. Finally, with the nodal energy, the controllability degree of a network can be conveniently visualized (see, for example, Figure 2).

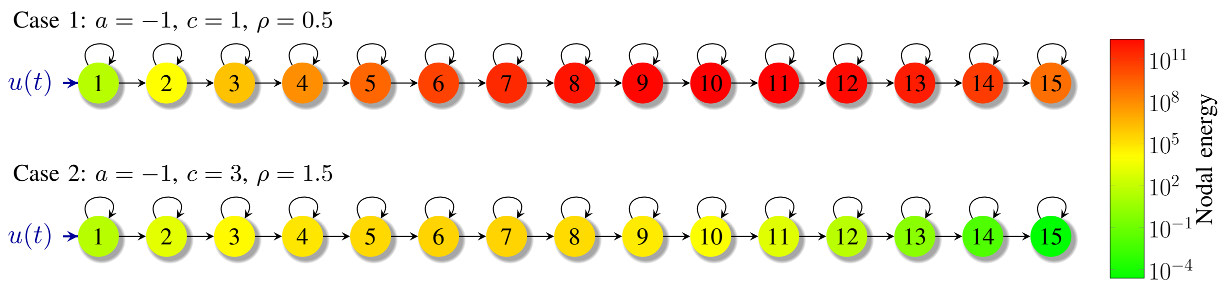


Fig. 2: Visualization of the nodal energy as given in Proposition 2 in the case of $n = 15$. The color of node i indicates the value of E_i.

III. Main Results

In this section we first calculate the explicit expression of the Gramian as defined in (2) and then compute the expression of the nodal energy as defined in (3). A series of conclusions about the controllability degree of line networks are finally drawn.

For a directed line network, let

$$\rho \triangleq \frac{c}{-2a} > 0, \quad \alpha \triangleq \frac{1}{-2a} > 0.$$

The constant ρ indicates the ratio between the interaction weight and the self-loop weight. The value of ρ significantly affects the controllability degree of the network as will be shown later. The constant α indicates the self-loop dynamics and a large α implies that the self-dynamics is weak.

The explicit expression of the controllability Gramian of a line network is given below.

Proposition 1. ***(Controllability Gramian)*** *The controllability Gramian is*

$$G = \alpha DPD,$$

where $D = \mathrm{diag}(1, \rho, \rho^2, \ldots, \rho^{n-1}) \in \mathbb{R}^{n\times n}$ *and* $P \in \mathbb{R}^{n\times n}$ *is the symmetric Pascal matrix with entries*

$$[P]_{ij} = \binom{i+j-2}{i-1},$$

for $i, j \in \{1, \ldots, n\}$.

Proof. See Appendix A. □

The Pascal matrix P in Proposition 1 has a special structure. To illustrate, in the case of $n = 5$, the Pascal matrix reads as

$$P_{5\times 5} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix}.$$

One feature of the Pascal matrix is that its inverse has an known expression [15]. Due to this property, we are able to obtain the explicit expression of G^{-1} and, consequently, the nodal energy.

Proposition 2. ***(Nodal Energy)*** *The ith nodal energy is*

$$E_i = [G^{-1}]_{ii} = \frac{1}{\alpha\rho^{2(i-1)}} \sum_{k=i}^{n} \binom{k-1}{i-1}^2.$$

Proof. See Appendix B. □

The nodal energy as given in Proposition 2 is visualized in Figure 2. As it can be seen, even for a network of 15 nodes, the control energy may reach extremely large values. Figure 2 also illustrates that for different values of ρ the energy required to control node n may differ significantly. The following theorem studies the nodal energies more closely.

Theorem 1. ***(Asymptotic Behavior)*** *The nodal energies of nodes* 1 *and* n *are, respectively,*

$$E_1 = \frac{n}{\alpha}, \quad \textit{and} \quad E_n = \frac{1}{\alpha\rho^{2(n-1)}}. \tag{4}$$

Moreover,

$$E_i \geq E_1, \quad \textit{for } i \leq (n-\rho)/(\rho+1),$$
$$E_i \geq E_n, \quad \textit{for } i \geq (n-\rho)/(\rho+1).$$

Proof. When $i = 1$, it follows from Proposition 2 that

$$E_1 = \frac{1}{\alpha\rho^0} \sum_{k=1}^{n} \binom{k-1}{0}^2 = \frac{1}{\alpha}\sum_{k=1}^{n} 1 = \frac{n}{\alpha}.$$

When $i = n$, it follows from Proposition 2 that

$$E_n = \frac{1}{\alpha\rho^{2(n-1)}} \binom{n-1}{n-1}^2 = \frac{1}{\alpha\rho^{2(n-1)}}.$$

The asymptotic behavior of E_1 or E_n immediately follows.

For any $i \in \{1, \ldots, n\}$, we have

$$E_i \geq \frac{1}{\alpha\rho^{2(i-1)}} \frac{1}{n-i+1} \left(\sum_{k=i}^{n} \binom{k-1}{i-1} \right)^2$$
$$= \frac{1}{\alpha\rho^{2(i-1)}} \frac{1}{n-i+1} \binom{n}{i}^2 \triangleq r_i,$$

where the equality is due to the identity about the sum of binomial coefficients over upper index that $\sum_{k=i}^{n} \binom{k-1}{i-1} = \binom{n}{i}$. We next analyze the lower bound r_i. First of all, it is easy to see

$$r_1 = \frac{n}{\alpha} = E_1, \quad r_n = \frac{1}{\alpha\rho^{2(n-1)}} = E_n.$$

The expression of r_{i+1} is

$$r_{i+1} = \frac{1}{\alpha\rho^{2i}} \frac{1}{n-i} \binom{n}{i+1}^2 = \frac{1}{\alpha\rho^{2i}} \frac{1}{n-i} \binom{n}{i}^2 \left(\frac{n-i}{i+1}\right)^2.$$

Then, we have

$$\frac{r_{i+1}}{r_i} = \frac{1}{\rho^2} \frac{n-i+1}{n-i} \left(\frac{n-i}{i+1}\right)^2 \geq \frac{1}{\rho^2} \left(\frac{n-i}{i+1}\right)^2 \triangleq \gamma_i.$$

Case 1: when $i \leq (n-\rho)/(\rho+1)$, we have $\gamma_i \geq 1$ and hence $r_{i+1} \geq r_i \Rightarrow r_i \geq r_1$. It then follows that $E_i \geq r_i \geq r_1 = E_1$. Case 2: when $i \geq (n-\rho)/(\rho+1)$, we have $\gamma_i \leq 1$ and hence $r_{i+1} \leq r_i \Rightarrow r_n \leq r_i$. It then follows that $E_i \geq r_i \geq r_n = E_n$. □

Theorem 1 has some important implications. First, although node 1 is directly affected by the control input, the nodal energy E_1 grows linearly with the network cardinality. The intuitive reason is that changing the state of node 1 does not require too much energy, but steer the other nodes' states to zero will require much more energy. Second, although node n is the farthest from the control input, the nth nodal energy can remain constant, increase, or even decrease with the network cardinality depending on the network parameter ρ. In particular, as n grows,

1) $E_n = \frac{1}{\alpha}$ for any n if $\rho = 1$;
2) E_n decreases to zero exponentially if $\rho > 1$;
3) E_n increases to infinity exponentially if $\rho < 1$.

A. Condition Number of the Controllability Gramian

Numerical simulation is a powerful tool for the empirical study of complex dynamical networks. However, we next show that numerical computation may be unreliable for the study of the controllability degree of large scale networks because the Gramian may become extremely ill-conditioned.

Theorem 2. ***(Condition Number of the Gramian)*** *The condition number of the Gramian satisfies*

$$\kappa(G) \geq \max\{\beta_1, \beta_2\},$$

where

$$\beta_1 = \max\left\{\frac{(2\rho)^{2(n-1)}}{\sqrt{\pi(n-1)}}, \frac{\sqrt{\pi(n-1)}}{(2\rho)^{2(n-1)}}\right\},$$
$$\beta_2 = \max\left\{n\rho^{2(n-1)}, \frac{1}{n\rho^{2(n-1)}}\right\}.$$

Proof. We first estimate the condition number based on the diagonal entries of G. Let $\sigma_{\max}$ and $\sigma_{\min}$ be the maximum and minimum singular values of a matrix, respectively. Because $\sigma_{\max}(G) \geq \max\{[G]_{11}, [G]_{nn}\}$ and $\sigma_{\min}(G) \leq \min\{[G]_{11}, [G]_{nn}\}$, the condition number of G satisfies

$$\kappa(G) = \frac{\sigma_{\max}(G)}{\sigma_{\min}(G)} \geq \max\left\{\frac{[G]_{nn}}{[G]_{11}}, \frac{[G]_{11}}{[G]_{nn}}\right\}. \tag{5}$$

It follows from Proposition 1 that $[G]_{11} = \alpha$ and

$$[G]_{nn} = \alpha\rho^{2(n-1)}\binom{2n-2}{n-1} \approx \alpha\rho^{2(n-1)}\frac{2^{2(n-1)}}{\sqrt{\pi(n-1)}}$$
$$= \alpha\frac{1}{\sqrt{\pi(n-1)}}(2\rho)^{2(n-1)}. \tag{6}$$

The approximation in the above equation is obtained from Stirling's formula [16]. If more accurate estimate is required, one may use the equality $\binom{2n-2}{n-1} = \frac{2^{2(n-1)}}{\sqrt{\pi(n-1)}}(1 - \frac{p_{n-1}}{n-1})$ where $1/9 < p_{n-1} < 1/8$. Substituting the expression of $[G]_{11}$ and $[G]_{nn}$ into (5) gives

$$\kappa(G) \geq \max\left\{\frac{1}{\sqrt{\pi(n-1)}}(2\rho)^{2(n-1)}, \frac{\sqrt{\pi(n-1)}}{(2\rho)^{2(n-1)}}\right\}. \tag{7}$$

As a result, when $2\rho \neq 1$, the condition number of G always increases with n exponentially.

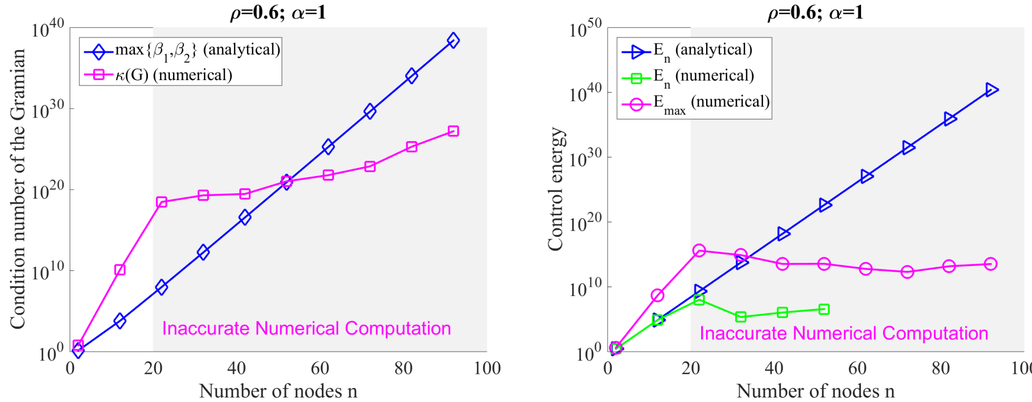
When $2\rho = 1$, the lower bound in (7) still increases with n but in a linear rate. In fact, we can show that $\kappa(G)$ still increases with n exponentially in the case of $2\rho = 1$ by examining the diagonal entries of G^{-1}. Since $[G^{-1}]_{11} = n/\alpha$ and $[G^{-1}]_{nn} = 1/(\alpha\rho^{2(n-1)})$ as indicated by Theorem 1, we have

$$\kappa(G) = \kappa(G^{-1}) = \frac{\sigma_{\max}(G^{-1})}{\sigma_{\min}(G^{-1})}$$
$$\geq \max\left\{\frac{[G^{-1}]_{nn}}{[G^{-1}]_{11}}, \frac{[G^{-1}]_{11}}{[G^{-1}]_{nn}}\right\}$$
$$= \max\left\{n\rho^{2(n-1)}, \frac{1}{n\rho^{2(n-1)}}\right\}. \tag{8}$$

As can be seen, when $\rho = 0.5$, $\kappa(G)$ still increases with n exponentially. □

An important implication of Theorem 2 is that the condition number $\kappa(G)$ increases exponentially fast with the network cardinality n for all values of ρ and α. As a result, empirical study based on numerical computation may be unreliable, thereby making a theoretical investigation necessary. It must be noted that the numerical unreliability does not depend on the selection of the notion of nodal energy or eigen energy. Instead, it is caused by the large condition number of G.

To illustrate, in Figure 3 we compare numerical results with our theoretical findings. Figure 3(a) shows that the condition number of the Gramian may reach at least 10^{40} when $n = 100$. Due to the extremely large condition number of the Gramian, the numerical computation becomes inaccurate and misleading when $n > 20$. This inaccurate numerical result may be even misleading. For example, as shown in Figure 3(b), the value of $E_{\max}$ increases with n at first and stops increasing after $n > 20$. This numerical result may lead to the conjecture that $n = 20$ might be a critical value and the controllability degree of the network may change significantly when $n > 20$. This conjecture is simply wrong because $E_{\max}$ should always increase with n.



(a) Condition number

(b) Control energy

Fig. 3: Numerical study on controllability degree may be unreliable when n is large. In the left figure (a), the numerical value of $\kappa(G)$ is inaccurate because it is supposed to be always greater than $\max\{\beta_1, \beta_2\}$ according to Theorem 2. In the right figure (b), the numerical values of E_n and $E_{\max}$ are also inaccurate because they are supposed to be equal to and greater than the analytical value of E_n, respectively. In this example, the numerical values were obtained by Matlab (2014b). The Gramian matrix was computed as the solution to the Lyapunov equation $AG + GA^T = -BB^T$ by the Matlab function lyap. The value of $E_{\max}$ was taken as the reciprocal of the minimum singular value of G which was computed by the Matlab function svd. The value of E_n was computed based on determinant of the adjugate matrix and it was computed as infinity for $n > 55$. The condition number of the Gramian was computed by the Matlab function cond. We avoided computing the inverse of the Gramian because warnings indicating that "the matrix is close to singular" were given when we tried to compute G^{-1} with $n > 20$.

B. Controllability Degree and Steady State Gain

One may conjecture that if the magnitude of the control signal is amplified while propagating through the network, then the network may require little control energy (see also the notion of anisotropic network introduced in [8]). This conjecture motivates us to compare the controllability degree with the behavior of constant signals propagating over the line network. To this aim, we next derive the transfer function from the control input to any node in the network.

Proposition 3. ***(Transfer Function)*** *The transfer function from the control input to the kth node, with $k = 1, \ldots, n$, is*

$$F(s) = \frac{c^{k-1}}{(s-a)^k}.$$

Hence, the steady state gain is

$$F(0) = \frac{1}{c}\left(\frac{c}{|a|}\right)^k.$$

Proof. Let $C \in \mathbb{R}^n$ be the kth column of the identity matrix I_n. Then, the transfer function from the control input applied on the first node to the kth node is

$$F(s) = C^T(sI_n - A)^{-1}b = \frac{C^T \operatorname{adj}(sI_n - A)b}{\det(sI_n - A)},$$

where $\operatorname{adj}(sI_n - A)$ is the adjugate matrix of $sI_n - A$. Since $C^T \operatorname{adj}(sI_n - A)b$ is the entry in the kth row and first column of $\operatorname{adj}(sI_n - A)$, we have

$$C^T \operatorname{adj}(sI_n - A)b = (-1)^{k+1}(-c)^{k-1}(s-a)^{n-k}$$

It follows from $\det(sI_n - A) = (s-a)^n$ that

$$F(s) = \frac{(-1)^{k+1}(-c)^{k-1}(s-a)^{n-k}}{(s-a)^n} = \frac{c^{k-1}}{(s-a)^k}.$$

The steady-state gain can be obtained by setting $s = 0$. $\square$

The steady state gain quantifies the amplification of a constant (or low-frequency) signal. Notice that, when $c/|a| > 1$ and $c/(2|a|) = \rho < 1$, the steady state gain implies an amplification of the control signal. Yet, we showed earlier that the network remains difficult to control in this case, since the nodal energy E_n increases exponentially with n. We conclude that the controllability degree of a network may be significantly different from the propagation behavior of low-frequency signals.

IV. Conclusions

By introducing a new notion termed nodal energy, we reveal many new and counterintuitive controllability properties of a special class of directed line networks with continuous-time dynamics. Although the networks considered in this paper are special, the obtained results shed lights on and could be extended to study the controllability properties of more complex networks.

Appendix

A. Proof of Proposition 1

Our goal is to explicitly calculate the controllability Gramian. First, consider the special case of $c = 1$. Then A is a Jordan block and we have

$$e^{At}B = e^{at}\begin{bmatrix} 1 & & & & 0 \\ t & 1 & & & \\ \frac{t^2}{2!} & t & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ \frac{t^{n-1}}{(n-1)!} & \cdots & \frac{t^2}{2!} & t & 1 \end{bmatrix} B = e^{at}\begin{bmatrix} 1 \\ t \\ \frac{t^2}{2!} \\ \vdots \\ \frac{t^{n-1}}{(n-1)!} \end{bmatrix}.$$

As a result,

$$[e^{At}B]_i = e^{at}\frac{t^{i-1}}{(i-1)!}$$
$$\Rightarrow [e^{At}BB^Te^{A^Tt}]_{ij} = e^{2at}\frac{t^{i+j-2}}{(i-1)!(j-1)!}.$$

It follows that

$$\begin{aligned}[G]_{ij} &= \int_0^\infty [e^{At}BB^Te^{A^Tt}]_{ij}\mathrm{d}t \\ &= \int_0^\infty e^{2at}\frac{t^{i+j-2}}{(i-1)!(j-1)!}\mathrm{d}t \\ &= \frac{1}{(i-1)!(j-1)!}\int_0^\infty e^{2at}t^{i+j-2}\mathrm{d}t \\ &= \frac{1}{(i-1)!(j-1)!}\frac{(i+j-2)!}{(-2a)^{i+j-1}} \\ &= \frac{1}{(-2a)^{i+j-1}}\frac{(i+j-2)!}{(i-1)!(j-1)!} \\ &= \frac{1}{(-2a)^{i+j-1}}\binom{i+j-2}{i-1}. \end{aligned} \tag{9}$$

In the above derivation, we applied the identity that $\int_0^\infty x^n e^{-ax}\mathrm{d}x = n!/a^{n+1}$ for $n = 0, 1, 2, ...$ and $a > 0$. This identity can be verified using standard integral techniques. Details are omitted due to space limitation.

Second, consider the general case where $c \neq 1$. The general case can be converted to the above special case as shown below:

$$A = \begin{bmatrix} a & & & 0 \\ c & a & & \\ & \ddots & \ddots & \\ 0 & & c & a \end{bmatrix} = c\begin{bmatrix} a/c & & & 0 \\ 1 & a/c & & \\ & \ddots & \ddots & \\ 0 & & 1 & a/c \end{bmatrix}$$
$$\triangleq c\bar{A}$$

Then,

$$G = \int_0^\infty e^{At}BB^Te^{A^Tt}\mathrm{d}t = \int_0^\infty e^{\bar{A}(ct)}BB^Te^{\bar{A}^T(ct)}\mathrm{d}t$$
$$= \frac{1}{c}\int_0^\infty e^{\bar{A}s}BB^Te^{\bar{A}^Ts}\mathrm{d}s.$$

By replacing a in (9) with a/c, we obtain

$$\begin{aligned}[G]_{ij} &= \frac{1}{c}\int_0^\infty [e^{\bar{A}s}BB^Te^{\bar{A}^Ts}]_{ij}\mathrm{d}s \\ &= \frac{1}{c}\frac{1}{(-2a/c)^{i+j-1}}\binom{i+j-2}{i-1} \\ &= \frac{1}{c}\left(\frac{c}{-2a}\right)^{i+j-1}\binom{i+j-2}{i-1} \\ &= \frac{1}{-2a}\left(\frac{c}{-2a}\right)^{i+j-2}\binom{i+j-2}{i-1} \\ &= \alpha\rho^{i+j-2}\binom{i+j-2}{i-1}, \end{aligned}$$

where $\alpha = 1/(-2a) > 0$ and $\rho = c/(-2a) > 0$.

B. Proof of Theorem 2

It follows from $G = \alpha DPD$ that $G^{-1} = \frac{1}{\alpha}D^{-1}P^{-1}D^{-1}$. The inverse of D is easy to calculate since D is diagonal. We only need to calculate P^{-1}. As shown in [15, Eq. (7)], the matrix P can be decomposed as $P = LL^T$, where $L \in \mathbb{R}^{n\times n}$ is a lower triangular matrix with

$$[L]_{ij} = \begin{cases} \binom{i-1}{j-1} & i \geq j, \\ 0 & i < j. \end{cases}$$

The inverse of P is $P^{-1} = JL^TLJ$ [15, Lemma 3], where $J = \mathrm{diag}\{1, -1, \dots, (-1)^{n-1}\}$. As a result, every element of P^{-1} can be calculated explicitly. Specifically, the ith diagonal entry of P^{-1} is

$$\begin{aligned}[P^{-1}]_{ii} &= [J]_{ii}^2[L^TL]_{ii} = [L^TL]_{ii} = \sum_{k=1}^n [L^T]_{ik}[L]_{ki} \\ &= \sum_{k=1}^n [L]_{ki}^2 = \sum_{k=i}^n [L]_{ki}^2 = \sum_{k=i}^n \binom{k-1}{i-1}^2. \end{aligned}$$

Thus, $[G^{-1}]_{ii} = \frac{1}{\alpha\rho^{2(i-1)}}[P^{-1}]_{ii} = \frac{1}{\alpha\rho^{2(i-1)}}\sum_{k=i}^n \binom{k-1}{i-1}^2$.

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