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Learning multi-step prediction models for receding horizon control

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Abstract— **In this paper, the derivation of multi-step-ahead prediction models from sampled input-output data of a linear system is considered. Specifically, a dedicated prediction model is built for each future time step of interest. Each model is linearly parametrized in a suitable regressor vector, composed of past output values and past and future input values. In addition to a nominal model, the set of all models consistent with data and prior information is derived as well, making the approach suitable for robust control design within a Model Predictive Control framework. The resulting parameter identification problem is solved through a sequence of convex programs. Convergence of the identified error bounds to their theoretical minimum is demonstrated, under suitable assumptions on the measured data, and features like worst-case accuracy computation are illustrated in a numerical example.**

I. Introduction

Model Predictive Control (MPC, [11]) exploits a model of the plant to predict and optimize the cost function of interest over a finite horizon, while enforcing operational constraints. The prediction model is either based on physical principles, or it exploits one of the many available black-box system identification techniques [18]. From the theoretical standpoint the number of contributions dealing with the inter-connections between model identification and predictive control has been traditionally rather limited, especially when it comes to deriving robust MPC approaches, i.e. related to uncertain systems. Examples of articles studying the interactions between model identification and MPC are [23], [10], [21], where however only nominal prediction models are considered, without an attempt to also derive a model of the associated uncertainty, that is a fundamental ingredient to guarantee robustness, see e.g. [2], [12], [14], [15] for deterministic (i.e. worst-case) approaches and [4], [6], [7], [16], [22] for stochastic ones, as well as the surveys [19], [8].

The increasing popularity of machine learning techniques to model the behavior of complex phenomena today fosters a renewed interest in the study of identification techniques tailored to MPC [1], [28], [5], [24], [13], [29], [17], where attempts to derive suitable uncertainty models are also made. Along this line of research, Set Membership (SM) methodologies are being adopted by several researchers for the model identification phase, since they are conceived to provide, in addition to a nominal model of the plant, also a quantification of the associated uncertainty, which can be exploited for robust control design. Examples of recent contributions exploiting SM techniques are [5], [17], and [24]. In all these contributions, a one-step ahead prediction error minimization is employed to identify the model parameters and the associated uncertainty set. However, this approach may yield poor nominal accuracy [9] and large guaranteed uncertainty bounds when the one-step model is iterated to generate predictions for the whole considered horizon. When employing a multi-step error minimization in the identification phase, the resulting optimization problem is generally non-convex even in the case of linear dynamics, and still the problem of deriving a guaranteed uncertainty model is generally intractable. In this paper, in order to overcome these problems we propose the use of multi-step prediction models that are affine in their parameters, and we adopt a SM approach to identify these models and the related uncertainty. Specifically, a dedicated prediction model is built for each future time step of interest. In addition to a nominal model, the set of all models consistent with data and prior information is derived as well. The resulting parameter identification problem is solved through a sequence of convex programs that provide also the guaranteed prediction error bounds. Finally, we illustrate the proposed approach in a numerical example.

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II. Problem formulation

Let us consider a single-input, single-output (SISO), open-loop stable, discrete-time, strictly proper linear time invariant (LTI) system with n states, input $u(k) \in \mathbb{R}$ and output $z(k) \in \mathbb{R}$, where $k \in \mathbb{Z}$ is the discrete time variable. The measurement of the system output, $y(k)$, is affected by a bounded additive disturbance $d(k)$:

$$y(k) = z(k) + d(k). \tag{1}$$

Assumption 1: (Measurements)

- $|d(k)| \leq \overline{d}, \forall k \in \mathbb{Z}$.
- The input $u(k)$ is measured with no noise. □

Assumption 2: (Input bounds) The input u belongs to a compact set: $u(k) \in \mathbb{U} \subset \mathbb{R}, \forall k \in \mathbb{Z}$. □

Let us denote with $p \in \mathbb{N}$ a finite number of time steps in the future. We are interested in deriving a prediction model of the future output $z(k+p)$, exploiting the input and output measurements collected in the time interval $[k-o+1, k]$, where $o \in \mathbb{N}$ is the chosen order of the model, and the future (planned) inputs in the interval $[k, k+p-1]$. Specifically, at any instant k let us define the regressor $\varphi_p(k) \in \mathbb{R}^{2o-1+p}$ as:

$$\varphi_p(k) \doteq \begin{bmatrix} Y_o^T(k) & U_o^T(k) & U_p^T(k) \end{bmatrix}^T \tag{2}$$

where ^{T} is the matrix transpose operation and

$$\begin{aligned} Y_o(k) &\doteq \begin{bmatrix} y(k) & y(k-1) & \dots & y(k-o+1) & \end{bmatrix}^T \\ U_o(k) &\doteq \begin{bmatrix} u(k-1) & \dots & u(k-o+1) & \end{bmatrix}^T \\ U_p(k) &\doteq \begin{bmatrix} u(k) & u(k+1) & \dots & u(k+p-1) & \end{bmatrix}^T \end{aligned} \quad (3)$$

Then, we consider the following linear model structure:

$$\hat{z}(k+p) = \varphi_p(k)^T \theta_p \quad (4)$$

where $\hat{z}(k+p)$ is the predicted output and $\theta_p \in \mathbb{R}^{2o-1+p}$ is the model parameter vector. We refer to models of the form (4) as "multi-step", since for each $p \in \mathbb{N}$ the corresponding prediction model provides directly an estimate of $z(k+p)$.

In addition to the prediction model, we also want to derive guaranteed bounds on its accuracy. In particular, we aim for the following *global* guaranteed accuracy bound:

$$|z(k+p) - \hat{z}(k+p)| \leq \tau_p(\theta_p). \quad (5)$$

The value $\tau_p(\theta_p)$ is termed "global" since it holds for any value of the regressor φ_p within a specified set, as we further detail in the remainder of this paper.

A collection of multi-step models derived for all $p \in [1, \overline{p}]$ provides an estimated sequence of future system outputs, up to the prediction horizon $\overline{p} < \infty$, together with an associated sequence of guaranteed uncertainty intervals $\tau_p(\theta_p)$. These models can be then embedded in a robust finite-horizon optimal control problem to be solved in a receding-horizon approach, thus realizing a MPC law based on multi-step predictions [26]. Three important reasons to consider multi-step models are:

1) the model identification procedure results in convex optimization problems, as opposed to the nonlinear programs arising when identifying one-step-ahead models with an output-error criterion [18];
2) the guaranteed bound $\tau_p(\theta_p)$ pertaining to a multi-step model is less conservative than the one pertaining to a one-step-ahead model iterated p times;
3) by considering an independent model for each value of p, there is no need for trade-offs between model accuracy at high-frequency (i.e. short prediction horizon) vs. low-frequency (i.e. long prediction horizon) that arise when choosing the simulation horizon in the identification of one-step-ahead models with an output-error criterion [23].

The use of a model structure that is linear in the parameters is justified both by point 1) above and by the observation that, in the case of zero measurement noise, the true output $z(k+p)$ is indeed a linear function of the regressor, provided that the following assumption holds.

Assumption 3: (Observability, reachability and model order) The system at hand is completely observable and reachable, and $o \geq n$. □

Under Assumption 3, with straightforward manipulations one can show that the state at time k is in general a linear function of the past n input-output values, and that the output at time $k+p$ is linear in the state and in the planned input values, hence resulting in the model structure (4). From the practical standpoint, Assumption 3 can be relaxed to account only for the observable and controllable sub-space of the system state, and it can be satisfied by estimating the system order n.

Assumption 2, combined with the asymptotic stability of the system under study, results in bounded sets where the regressor φ_p evolves in time. Specifically, for a given horizon p we consider a compact set Φ_p containing the regressor values of interest:

$$\varphi_p(k) \in \Phi_p \subset R^{2o-1+p}, \Phi_p \text{ compact}, \forall p \in \mathbb{N}, \forall k \in \mathbb{Z}.$$

Φ_p needs not to be known explicitly and is in general a complicated set that depends on the system input-output trajectories. Rather, we assume that for a given value of p a finite batch of experimental data is available:

$$(\tilde{\varphi}_p(i), \tilde{y}_p(i)), i = 1, \dots, N_p. \quad (6)$$

In (6), the notation $\tilde{\cdot}$ indicates specific measured values of a quantity. $\tilde{\varphi}_p(i)$ are the measured instances of the regressor $\varphi_p \in \Phi_p$, and $\tilde{y}_p(i)$ the corresponding measured values of noise-corrupted outputs p steps in the future, i.e. $\tilde{y}_p(i) = z(i+p) + d(i+p)$. $N_p < \infty$ is the total number of available pairs $(\tilde{\varphi}_p(i), \tilde{y}_p(i))$ composing the data-set. These pairs can be easily built from a given data-set of measured input-output values, collected e.g. in a preliminary experiment on the system. We further define the vectors of sampled variables $\tilde{v}_p(i) \doteq [\tilde{\varphi}_p(i)^T \ \tilde{y}_p(i)]^T, i = 1, \dots, N_p$ and the corresponding countable set containing them:

$$\tilde{\mathscr{V}}_p^{N_p} = \left\{ \tilde{v}_p(i) = \begin{bmatrix} \tilde{\varphi}_p(i) \\ \tilde{y}_p(i) \end{bmatrix}, i = 1, \dots N_p \right\} \subset \mathbb{R}^{2o+p}. \quad (7)$$

For any given value of $\varphi_p \in \Phi_p$, the corresponding measured system output y_p is not uniquely determined a priori, rather it belongs to a set $Y_p(\varphi_p) \subset R$, due to the measurement noise d (both in the regressor and in the corresponding output). In view of Assumptions 1 and 3 and of the compactness of Φ_p, the set $Y_p(\varphi_p)$ is also compact for any $\varphi_p \in \Phi_p$. We can then define the following continuous counterpart of the set $\tilde{\mathscr{V}}_p^{N_p}$:

$$\mathscr{V}_p = \left\{ v_p \doteq \begin{bmatrix} \varphi_p \\ y_p \end{bmatrix} : y_p \in Y_p(\varphi_p), \forall \varphi_p \in \Phi_p \right\} \subset \mathbb{R}^{2o+p}. \quad (8)$$

Namely, the set $\mathscr{V}_p$ contains all possible regressors φ_p in the compact Φ_p and, for each value of φ_p , all possible output values in the corresponding compact set $Y_p(\varphi_p)$. We consider the following assumption linking the sets $\tilde{\mathscr{V}}_p^{N_p}$ and $\mathscr{V}_p$.

Assumption 4: (Data-set) For any $\beta > 0$, there exists a value of $N_p < \infty$ such that:

$$d_2\left(\mathscr{V}_p, \tilde{\mathscr{V}}_p^{N_p}\right) \leq \beta$$

where $d_2\left(\mathscr{V}_p, \tilde{\mathscr{V}}_p^{N_p}\right) \doteq \max_{v \in \mathscr{V}_p} \min_{u \in \tilde{\mathscr{V}}_p^{N_p}} \|u - v\|_2$ is the distance between sets $\mathscr{V}_p$ and $\tilde{\mathscr{V}}_p^{N_p}$. □

Assumption 4 is equivalent to assuming that $\lim_{N_p \to \infty} d_2\left(\mathscr{V}_p, \tilde{\mathscr{V}}_p^{N_p}\right) = 0$, i.e. that by adding more points to the measured data set, the underlying set of all trajectories of

interest is densely covered. This is essentially an assumption on the persistence of excitation of the inputs used for the preliminary experiments, together with an assumption of bound-exploring property of the additive disturbance d, such that the bound $\overline{d}$ in Assumption 1 is actually tight.

We are now in position to formulate the problem that we will address in the remainder of this paper:

Problem 1 Under Assumptions 1-4, for a given $\overline{p} < \infty$, $p \in \mathbb{N}$ and any $p = 1, \ldots, \overline{p}$, use the available data (6) to identify a multi-step model of the form (4) and estimate the associated guaranteed bounds (5). □

We propose in the next section an approach to solve **Problem 1**, based on a Set Membership (SM) identification methodology [20], [27], which guarantees convergence of the derived error bounds to suitably defined optimal values.

III. Learning multi-step prediction models: a set membership approach

For the sake of simplicity, in the following we consider a generic value of $p \in [1, \overline{p}]$, the extension to any other value is straightforward. The proposed approach consists of the following steps.

1) Define an optimality criterion to evaluate the model estimates, corresponding to an optimal (i.e. minimal) error bound.
2) Derive a procedure to estimate the optimal error bound.
3) Based on the available data and the error bound estimate, build the set of all parameters that are consistent with this information (Feasible Parameter Set, FPS).
4) Using the information summarized in the FPS, for any given model of the form (4) compute the related guaranteed error bound τ_p.
5) Select a nominal model with minimal guaranteed error bound.

We describe next each step in detail, followed by a discussion on tuning and extensions of the approach.

A. Optimal parameters and optimal error bound

Assume that a value of θ_p has been fixed. Then, under the considered working assumptions, $\forall k \in \mathbb{Z}$:

$$\begin{aligned} y(k+p) &= z(k+p) + d(k+p) \\ &= \hat{z}(k+p) + \varepsilon_p(\theta_p, \varphi_p(k)) + d(k+p) \\ &= \varphi_p(k)^T \theta_p + \varepsilon_p(\theta_p, \varphi_p(k)) + d(k+p) \end{aligned} \tag{9}$$

where $\varepsilon_p(\theta_p, \varphi_p(k))$ is the error between the true system output and the estimated one:

$$\varepsilon_p(\theta_p, \varphi_p(k)) = z(k+p) - \varphi_p(k)^T \theta_p \tag{10}$$

The quantity $\varepsilon_p(\theta_p, \varphi_p(k))$ accounts for the quality of the chosen parameter values, for the model order mismatch ($o > n$, see Assumption 3) and for the noise in the regressor measurements. Since the underlying system dynamics are time-invariant, ε_p depends inherently on the model parameter vector θ_p and on the specific regressor $\varphi_p(k)$. $\varepsilon_p(\theta_p, \varphi_p(k))$ is bounded because both $y(k+p)$ and $\theta_p^T \varphi_p(k)$ are, due to the stability of the system and compactness of the set containing the input values. From (10) and Assumption 1 we have:

$$\begin{aligned} |y(k+p) - \varphi_p(k)^T \theta_p| &\leq |\varepsilon_p(\theta_p, \varphi_p(k))| + \overline{d} \\ &\leq \bar{\varepsilon}_p(\theta_p) + \overline{d} \end{aligned} \tag{11}$$

where $\bar{\varepsilon}_p(\theta_p)$ is the global error bound with respect to all possible regressors of interest in the compact Φ_p:

$$\begin{aligned} \bar{\varepsilon}_p(\theta_p) = \; & \min_{\varepsilon \in \mathbb{R}} \; \varepsilon \\ & \text{subject to} \\ & |y_p - \varphi_p^T \theta_p| \leq \varepsilon + \overline{d}, \forall (\varphi_p, y_p) : \begin{bmatrix} \varphi_p \\ y_p \end{bmatrix} \in \mathscr{V}_p \end{aligned} \tag{12}$$

The quantity $\bar{\varepsilon}_p(\theta_p)$ is the tightest bound on the global (i.e. worst-case) estimation error that a given parameter vector θ_p can produce. We can now define the optimal parameter values (i.e. optimal models) as those that minimize such a bound. As a technical assumption, we consider all the parameters within a compact set $\Omega \subset \mathbb{R}^{2o-1+p}$. This assumption allows us to use maximum and minimum operators instead of supremum and infimum. The set Θ_p^0 of optimal parameter values is the following:

$$\Theta_p^0 = \left\{ \theta_p^0 : \theta_p^0 = \arg\min_{\theta_p \in \Omega} \; \bar{\varepsilon}_p(\theta_p) \right\}, \tag{13}$$

and we denote with $\bar{\varepsilon}_p^0$ the corresponding optimal error bound:

$$\bar{\varepsilon}_p^0 = \min_{\theta_p \in \Omega} \; \bar{\varepsilon}_p(\theta_p). \tag{14}$$

Considering (12)-(14) we can alternatively write:

$$\Theta_p^0 = \left\{ \theta_p : |y_p - \varphi_p^T \theta_p| \leq \bar{\varepsilon}_p^0 + \overline{d}, \forall (\varphi_p, y_p) : \begin{bmatrix} \varphi_p \\ y_p \end{bmatrix} \in \mathscr{V}_p \right\}. \tag{15}$$

B. Estimating the optimal error bound

The optimal models and optimal error bound cannot be computed in practice, since the solution to (13) would imply the availability of an infinite number of data and the solution to an infinite-dimensional optimization program. However, we can compute an estimate $\underline{\lambda}_p \approx \bar{\varepsilon}_p^0$ from the available experimental data, by solving the following linear program (LP):

$$\begin{aligned} \underline{\lambda}_p = \; & \min_{\theta_p, \lambda > 0} \; \lambda \\ & \text{subject to} \\ & |\tilde{y}_p - \tilde{\varphi}_p^T \theta_p| \leq \lambda + \overline{d}, \forall (\varphi_p, y) : \begin{bmatrix} \tilde{\varphi}_p \\ \tilde{y}_p \end{bmatrix} \in \tilde{\mathscr{V}}_p^{N_p} \end{aligned} \tag{16}$$

The inequality constraint in (16) on λ is required to enforce a positive estimate of the error bound: without this constraint, the obtained estimate could result to be negative, especially in presence of small amount of data and output disturbance realizations with much smaller magnitude than the considered bound $\overline{d}$.

The following result shows that, under the considered working assumptions, the estimate (16) converges to the optimal one, $\bar{\varepsilon}_p^0$.

Theorem 1: Let Assumptions 1-4 hold. Then:

1) $\underline{\lambda}_p \leq \bar{\varepsilon}_p^0$;
2) $\forall \rho \in (0, \bar{\varepsilon}_p^0] \; \exists \, N_p < \infty \; : \; \underline{\lambda}_p \geq \bar{\varepsilon}_p^0 - \rho$ □

Proof: See [25] ■

Theorem 1 implies that $\lim_{N_p \to \infty} (\bar{\varepsilon}_p^0 - \underline{\lambda}_p) = 0^+$, i.e. that the estimate (16) converges to the optimum from below. This is unavoidable with the considered problem settings, where limited information is available. In practice, one can increase the number N_p of experimental data and observe the behavior of $\underline{\lambda}_p$, which typically (if the preliminary experiments are informative enough) quickly converges to a limit. Then, a practical approach to compensate for the uncertainty caused by the use of a finite number of measurements is to inflate the value $\underline{\lambda}_p$:

$$\hat{\bar{\varepsilon}}_p = \alpha \underline{\lambda}_p, \; \alpha > 1. \tag{17}$$

With sufficiently large N_p, the coefficient α can be chosen very close to 1. We show an example of such a procedure in section IV, and provide more comments on the tuning of α in section III-F. We consider the following assumption in the remainder of this paper:

Assumption 5: (Estimate of the optimal error bound) $\hat{\bar{\varepsilon}}_p \geq \bar{\varepsilon}_p^0$. □

C. Feasible Parameter Set

We can now exploit the estimated optimal error bound to construct the tightest set of parameter values that are consistent with all the prior information, i.e. the FPS Θ_p:

$$\Theta_p = \left\{ \theta_p : |\tilde{y}_p - \tilde{\varphi}_p^T \theta_p| \leq \hat{\bar{\varepsilon}}_p + \overline{d}, \forall (\tilde{\varphi}_p, \tilde{y}_p) : \begin{bmatrix} \tilde{\varphi}_p \\ \tilde{y}_p \end{bmatrix} \in \tilde{\mathscr{V}}_p^{N_p} \right\} \tag{18}$$

The set Θ_p is non-empty by construction, since under Assumption 5 we have (compare (15) and (18)):

$$\Theta_p^0 \subseteq \Theta_p. \tag{19}$$

If the FPS is bounded, it results in a polytope with at most N_p faces. If it is unbounded, then this is a sign that the employed measured data are not informative enough to derive a bound on the worst-case model error (as we show next) and that the number of available data N_p shall be increased until a bounded FPS is obtained. This situation usually occurs when very few data points are used (e.g. $N_p < 2o - 1 + p$) or the preliminary experiments are not informative enough (e.g. when only steady-state data are used).

D. Error bound computation for a generic model

Having defined the FPS, we can proceed to derive a bound on the prediction error achieved by any model of the form (4). Let us consider a generic value of θ_p to derive the prediction model (4). Then, considering any $\theta_p^0 \in \Theta_p^0$ and any $\varphi_p(k) \in \Phi_p$, using (10) and (15) we have:

$$\begin{aligned} & |z(k+p) - \hat{z}(k+p)| \\ = \; & |\varphi_p(k)^T \theta_p^0 - \varphi_p(k)^T \theta_p + \varepsilon(\theta_p^0, \varphi_p(k))| \\ \leq \; & |\varphi_p(k)^T (\theta_p^0 - \theta_p)| + \bar{\varepsilon}_p^0 \end{aligned} \tag{20}$$

The tightest bound we can derive on (20) is based on the knowledge that $\theta_p^0 \in \Theta_p$ (see (19)) and that $\bar{\varepsilon}_p^0 \leq \hat{\bar{\varepsilon}}_p$ (Assumption 5):

$$|\varphi_p(k)^T (\theta_p^0 - \theta_p)| + \bar{\varepsilon}_p^0 \leq \max_{\theta \in \Theta_p} |\varphi_p(k)^T (\theta - \theta_p)| + \hat{\bar{\varepsilon}}_p. \tag{21}$$

The latter bound is the tightest *local* error bound (i.e. valid for a given value of φ_p) for the model given by the considered parameter value θ_p. As stated in **Problem 1**, for the sake of using the model within a robust MPC framework, we want to derive a global bound $\tau_p(\theta_p)$ holding $\forall \varphi_p \in \Phi_p$. Considering (20)-(21) leads to:

$$\begin{aligned} & \max_{\varphi_p \in \Phi_p} |z(k+p) - \hat{z}(k+p)| \\ \leq \; & \max_{\varphi_p \in \Phi_p} \max_{\theta \in \Theta_p} |\varphi_p^T (\theta - \theta_p)| + \hat{\bar{\varepsilon}}_p \\ = \; & \tau_p(\theta_p) \end{aligned}$$

Such a bound cannot be derived exactly with finite data under the considered assumptions, and its computation would be intractable also if the set Φ_p were known precisely (unless some additional assumption is made, e.g. polytopic set Φ_p). However, we can approximate it by computing the maximum of (21) over the data-set $\tilde{\mathscr{V}}_p^{N_p}$:

$$\underline{\tau}_p(\theta_p) = \max_{i=1,\ldots,N_p} \max_{\theta \in \Theta_p} |\tilde{\varphi}_p(i)^T (\theta - \theta_p)| + \hat{\bar{\varepsilon}}_p \tag{22}$$

The following result shows convergence of $\underline{\tau}_p(\theta_p)$ to $\tau_p(\theta_p)$.

Lemma 1: Let Assumptions 1-4 hold. Then, for any $\theta_p \in \mathbb{R}^{2o-1+p}$:

1) $\underline{\tau}_p(\theta_p) \leq \tau_p(\theta_p)$;
2) $\forall \rho \in (0, \tau_p(\theta_p)] \; \exists \, N_p < \infty \; : \; \underline{\tau}_p(\theta_p) \geq \tau_p(\theta_p) - \rho$ □

Proof: Straightforward extension of the proof of Theorem 1. ■

Considerations similar to those of Theorem 1 hold also for the bound $\underline{\tau}_p(\theta_p)$: it is possible to monitor its behavior with increasing N_p in order to evaluate its convergence. As done in (17), we inflate this bound to account for the uncertainty deriving from our finite data-set:

$$\hat{\tau}_p(\theta_p) = \gamma \underline{\tau}_p(\theta_p), \; \gamma > 1, \tag{23}$$

and we assume that the resulting estimate is larger than the true bound:

Assumption 6: (Error bound estimate for a given θ_p) $\hat{\tau}_p(\theta_p) \geq \tau_p(\theta_p), \forall \theta_p \in \mathbb{R}^{2o-1+p}$. □

E. Nominal model selection

The last step in the proposed approach is to select a nominal model, that typically is found through least-squares estimation. Since the final goal is to employ the model in a MPC algorithm, we rather seek the model that minimizes the uncertainty bound. Specifically, considering that the tightest set that contains the optimal parameter values (i.e. with minimum error, see section (III-A)) is the FPS Θ_p, we search within this set for the parameter value that minimizes the resulting bound $\hat{\tau}_p(\theta_p)$:

$$\theta_p^* = \arg \min_{\theta_p \in \Theta_p} \hat{\tau}_p(\theta_p). \tag{24}$$

The resulting nominal model is $\hat{z}(k+p) = \varphi_p(k)^T \theta_p^*$, and the associated error bound is:

$$\hat{\tau}_p(\theta_p^*) = \gamma \left(\min_{\theta_p \in \Theta_p} \max_{i=1,\dots,N_p} \max_{\theta \in \Theta_p} |\tilde{\varphi}_p(i)^T (\theta - \theta_p)| \right) + \hat{\bar{\varepsilon}}_p. \tag{25}$$

Note that term $\hat{\bar{\varepsilon}}_p$ in (25) does not depend on θ_p^* and it converges to the optimal error bound $\bar{\varepsilon}_p^0$ as N_p increases (Theorem 1).

Remark 1: The optimization problem to be solved to compute θ_p^* (24) takes the form

$$\theta_p^* = \arg \min_{\theta_p \in \Theta_p} \max_{i=1,\dots,N_p} \max_{\theta \in \Theta_p} |\tilde{\varphi}_p(i)^T (\theta - \theta_p)|. \tag{26}$$

This problem can be solved by reformulating it as $2N_p + 1$ LPs [3], see [25]

F. Learning algorithm, tuning aspects and extensions

In the closing part of this section we report the overall identification algorithm, to be repeated for each prediction step $p = 1 \dots \bar{p}$:

1) Collect N_p measured regressor values and the corresponding measured output instances (see (6))
2) Solve the optimization problem (16) and compute $\hat{\bar{\varepsilon}}_p = \alpha \underline{\lambda}_p$, that is needed to define the FPS (18)
3) Solve (26) to derive the nominal model θ_p^* and select a value of $\gamma > 1$ to compute the worst case prediction error estimate $\hat{\tau}_p(\theta_p^*)$ (25).

The main tuning parameters in the approach are the model order o, the disturbance bound $\bar{d}$, and the scalars α and γ. In practical applications, a good estimate or even exact knowledge of the order of the plant is often available, otherwise o may be set conservatively high. Regarding the choice of $\bar{d}$, over- or under-estimating this value leads, respectively, to either a too optimistic estimate of the prediction errors or to a higher conservativeness. In a similar way, too large values of α and γ can lead to conservative error bounds, while values too close to 1 might give error bounds that are too tight and could be violated by new, previously unseen data. As a matter of fact, both conditions (over- and under-estimation) can be easily monitored on-line, and the derived tuning parameters $\bar{d}$, α and γ suitably adjusted. This feature can be exploited in view of employing the proposed approach in an adaptive framework, which is a planned extension of this work. Other planned extensions that can be developed relatively easily are to consider nonlinear systems, since we can still employ the same model parametrization and the described approach by embedding the effect of nonlinearities in the error term $\varepsilon_p(\theta_p, \varphi_p)$ (see (9)), or the extension to MISO or MIMO systems.

IV. Simulation results

The proposed algorithm has been tested on the benchmark example already considered in [23], i.e. on the system:

$$G(s) = 2\frac{229}{(s+1)(s^2+30s+229)} \tag{27}$$

The input-output samples are collected with a sampling time $T_s = 0,2$s according to the output equation $y(k) = z(kT_s) + d(kT_s)$, where $d(t)$ is a colored noise obtained by low-pass filtering a randomly generated number, with a first-order filter with time constant of 0.2 s. The disturbance $d(t)$ is bounded in the interval $[-0.2\ 0.2], \forall t$, and the prediction horizon is $\bar{p} = 10$.

The collected dataset consists of 500 input-output samples. The input is a three level signal taking values randomly each 4 s in the set $\{-1, 0, 1\}$.

The order of the model is $o = 3$ that matches the order of the system, consistently with Assumption 3, even though the system's dynamics are dominated by the pole in $\bar{s} = -1$.

Once $\underline{\lambda}_p$ is computed $\forall p = 1 \dots \bar{p}$, it is inflated according to (17) with the fixed coefficient $\alpha = 1.2$, i.e. with a 20% margin. The conservativeness of such a choice can be evaluated from the trend that $\underline{\lambda}_p$ exhibits with respect to the number of data. In view of Theorem 1, in fact, $\lim_{N_p \to \infty} (\bar{\varepsilon}_p^0 - \underline{\lambda}_p) = 0^+$ and the convergence rate gives a qualitative idea of how reliable $\hat{\bar{\varepsilon}}_p^0$ is with respect to the real unknown value ε_p^0. Fig. 1 presents the values of $\underline{\lambda}_p$ for the fixed steps $p = 3, 6, 9$ with an increasing percentage of the dataset used for the computation. Using the whole dataset, the curves approach an asymptotic value which presumably corresponds to the theoretical bound $\bar{\varepsilon}_p^0$.

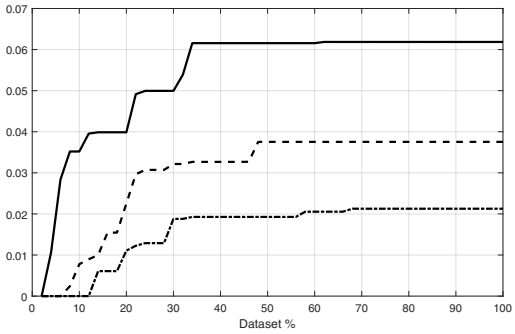

Fig. 1: Values of $\underline{\lambda}_p$ for different percentages of the dataset. Solid lines: $p = 3$; dashed black lines: $p = 6$; dash-dot line: $p = 9$

The multi-step predictors are eventually built by solving (26) to compute the nominal model for each step and the related guaranteed worst-case error bound $\hat{\tau}_p(\theta_p^*)$. The resulting bounds are dramatically less conservative then those coming from the iteration of the 1-step-ahead predictor, as shown in Fig. 2. We also carried out a comparison with the worst-case bounds associated to Least Square (LS) models, one for each p, reported in Fig. 2 too. The result confirms the optimality of our approach, that minimizes the uncertainty bounds. Note also that it may happen that a model computed with LS does not belong to the FPS. Finally, the descending trend of $\hat{\tau}_p(\theta_p^*)$ is due to the increasing number of noise-free input values $u(k)$ in the regressor as p increases, combined with the fading effect of past input and output data.

The final outcome of the algorithm consists in $\bar{p}$ nominal

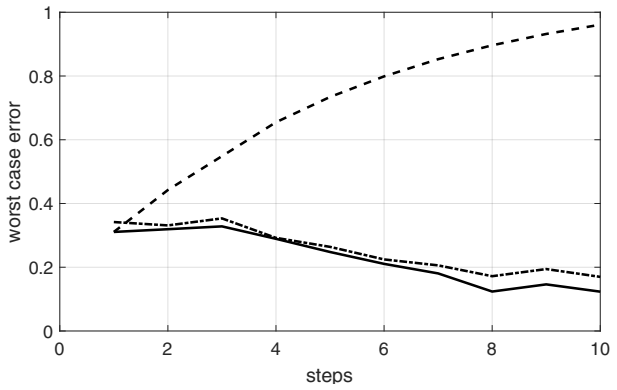

Fig. 2: Worst case error bounds. Solid lines: multistep approach ($\hat{\tau}_p(\theta_p^*)$); dashed black lines: iterated 1-step; dash-dot line: LS models $\hat{\tau}_p(\theta_p^{LS})$

models with the related guaranteed worst-case errors bounds, that are then tested with validation data. A detail of the simulation is reported in Fig. 3.

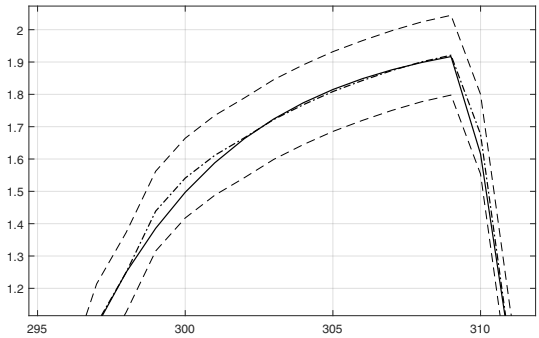

Fig. 3: Simulation with validation data for $p = \bar{p} = 10$. Solid lines: real samples $z(k)$; dashed black lines: guaranteed bounds; dash-dot line: prediction with nominal model

V. Conclusions

We presented an approach to derive multi-step prediction models, and the related uncertainty bounds, for an unknown linear system affected by additive measurement disturbance. Under suitable assumptions, we demonstrated convergence of the bounds to their theoretical minimum based on the available information. The approach requires the solution to linear programs only. The derived models are particularly suited to robust model predictive control design, since they can predict the future trajectory of the system on a finite horizon and the related uncertainty intervals.

References

[1] A. Aswani, H. Gonzalez, S. Sastry, and C. Tomlin. Provably safe and robust learning-based model predictive control. *Automatica*, 49:1216–1226, 2013.

[2] G. Betti, M. Farina, and R. Scattolini. A robust MPC algorithm for offset-free tracking of constant reference signals. *IEEE Transactions on Automatic Control*, 58(9):2394–2400, 2013.

[3] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[4] G.C. Calafiore and L. Fagiano. Robust model predictive control via scenario optimization. *IEEE Transactions on Automatic Control*, 58(1):219–224, 2013.

[5] M. Canale, L. Fagiano, and M.C. Signorile. Nonlinear model predictive control from data: a set membership approach. *International Journal of Robust and Nonlinear Control*, 24(1):123–139, 2014.

[6] M. Cannon, B. Kouvaritakis, and X. Wu. Probabilistic constrained MPC for multiplicative and additive stochastic uncertainty. *IEEE Transactions on Automatic Control*, 54(7):1626–1632, 2009.

[7] D. Chatterjee, P. Hokayem, and J. Lygeros. Stochastic receding horizon control with bounded control inputs: A vector space approach. *IEEE Transactions on Automatic Control*, 56(11):2704–2710, 2011.

[8] M. Farina, L. Giulioni, and R. Scattolini. Stochastic linear model predictive control with chance constraints – a review. *Journal of Process Control*, 44(Supplement C):53 – 67, 2016.

[9] M Farina and L Piroddi. Simulation error minimization identification based on multi-stage prediction. *International Journal of Adaptive Control and Signal Processing*, 25(5):389–406, 2011.

[10] W. Favoreel, B. De Moor, and M. Gevers. Spc: Subspace predictive control. *IFAC Proceedings Volumes*, 32(2):4004 – 4009, 1999. 14th IFAC World Congress 1999, Beijing, Chia, 5-9 July.

[11] G. C. Goodwin, M. M. Seron, and J. A. De Don. *Constrained control and estimation: an optimisation approach*. Springer, London, 2005.

[12] P.J. Goulart, E.C. Kerrigan, and J.M. Maciejowski. Optimization over state feedback policies for robust control with constraints. *Automatica*, 42:523–533, 2006.

[13] T.A. N. Heirung, E. Ydstie, and B. Foss. Dual adaptive model predictive control. *Automatica*, 80:340–348, 2017.

[14] M.V. Kothare, V. Balakhrisna, and M. Morari. Robust constrained model predictive control using linear matrix inequalities. *Automatica*, 32(10):1361–2378, 1996.

[15] W. Langson, I. Chryssochoos, S.V. Raković, and D.Q. Mayne. Robust model predictive control using tubes. *Automatica*, 40:125–133, 2004.

[16] P. Li, M. Wendt, and G. Wozny. A probabilistically constrained model predictive controller. *Automatica*, 38:1171–1176, 2002.

[17] D. Limon, J. Calliess, and J.M. Maciejowski. Learning-based nonlinear model predictive control. In *IFAC World Congress 2017*, pages 8035 – 8042, Toulouse, France, 2017.

[18] L. Ljung. *System Identification - Theory For the User (2nd ed.)*. PTR Prentice Hall, Upper Saddle River, N.J., 1999.

[19] D.Q. Mayne. Model predictive control: Recent developments and future promise. *Automatica*, 50:2967–2986, 2014.

[20] M. Milanese, J. Norton, H. Piet-Lahanier, and E. Walter (Eds.). *Bounding Approaches to System Identification*. Springer US, Boston, MA, 1996.

[21] F. Di Palma and L. Magni. A multi-model structure for model predictive control. *Annual Reviews in Control*, 28:47 – 52, 2004.

[22] G. Schildbach, L. Fagiano, C. Frei, and M. Morari. The scenario approach for stochastic model predictive control with bounds on closed-loop constraint violations. *Automatica*, 50(12):3009–3018, 2014.

[23] D. Shook, M. Coorous, and S.L. Shah. Identification for long-range predictive control. In *IEE Proceedings D (Control Theory and Applications)*, volume 138, pages 75–84. IET, 1991.

[24] M. Tanaskovic, L. Fagiano, R. Smith, and M. Morari. Adaptive receding horizon control for constrained MIMO systems. *Automatica*, 50(12):3019–3029, 2014.

[25] E. Terzi, M. Farina, L. Fagiano, and R. Scattolini. On multi-step prediction models for receding horizon control. IEEE, available at http://arxiv.org/abs/1802.09767.

[26] E. Terzi, M. Farina, L. Fagiano, and R. Scattolini. Robust predictive control with data-based multi-step prediction models. In *Submitted to European Control Conference 2018*. IEEE, available at http://home.deib.polimi.it/fagiano/docs/TR-TFFS-ECC18b.pdf.

[27] J.F. Traub and H. Woźniakowski. *A General Theory of Optimal Algorithms*. Academic Press, New York, 1980.

[28] X. Wang, Y. Sun, and K. Deng. Adaptive model predictive control of uncertain constrained systems. In *Proceedings of the 2014 American Control Conference*, pages 2857 – 2862, Portland, OR, 2014.

[29] J. Zhou, S. Di Cairano, and C. Danielson. Indirect adaptive MPC for output tracking of uncertain linear polytopic systems. In *Proceedings of the 2017 American Control Conference*, pages 3054 – 3059, Seattle, WA, 2017.