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A Limit Cycle Control Method for Multi-modal and 2-dimensional Piecewise Affine Systems - State Feedback Control Case

Tatsuya Kai

Abstract— **This study considers the limit cycle control problem for multi-modal and 2-dimensional piecewise affine control systems with control input terms. The main purpose of the limit cycle control problem is to design gains of a state feedback controller such that a solution trajectory of the closed loop system generates a desired limit cycle. First, a synthesis method of a multi-modal and 2-dimensional reference piecewise affine system and some characteristics of the system are presented. Next, solving a matching condition such that a closed loop system coincides with a reference system, we derive analytic solutions and their existence conditions for three cases. Then, numerical simulations are performed in order to confirm the effectiveness of the proposed method.**

I. INTRODUCTION

A limit cycle is one of the most famous and important phenomena in nonlinear systems, along with chaos, fractals, and solitons. In the simplest word, a stable limit cycle is defined as a closed curve in a phase space that attracts other solution trajectories as time approaches infinity, and its behavior shows a self-excited oscillation. Indeed, we can easily find examples of limit cycles in the real world [1]. For examples, stable gaits of humanoid robots [2], [3], [4] in robotics, periodic motions of machines [5], [6] in mechanical engineering, oscillators [7], [8] in electrical engineering, catalytic hypercycles [9], [10] and the Belousov-Zhabotinsky reaction [11], [12] in chemistry, circadian rhythms [13], [14] and firefly flashing [15], [16] in biology, boom-bust cycles [17], [18] in economics, and so on.

Various researches on limit cycles have been done from the mathematical and control engineering viewpoints so far [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36]. In particular, some conditions for nonlinear systems that generate periodic solutions and some applications are shown in [19], and a synthesis method of hybrid systems whose solution trajectories converge to desired trajectories is developed in [24]. In these studies, it is guaranteed that solution trajectories of the systems converges to a desired closed curve, and the existence of limit cycles was confirmed by numerical simulations, however, the mathematical guarantee of the existence of limit cycles was not proven. On the other hand, the authors proposed a synthesis method of multi-modal and 2-dimensional piecewise affine systems that generate desired limit cycles in [31], [32], [33], [34], the authors proposed a synthesis method of a multi-modal and 2-dimensional piecewise affine system that generates a desired limit cycle and proved existence and uniqueness of the limit cycle for the proposed system. In [35], [36], the authors considered a control problem for a piecewise affine system with a control input term, and showed matching conditions such that the closed-loop system is equivalent to the reference system that generates a desired limit cycle. However, the matching conditions were solved only for the specific case, and an analytic solution and its existence conditions have not been obtained for the general case.

The goal of this work is to derive an analytic solution to the matching conditions and its existence conditions. This paper is organized as follows. First, Section 2 gives a summary on the synthesis method of a piecewise affine system that generates a desired limit cycle. Next, Section 3 formulates the limit cycle control problem and shows matching conditions. Then, an analytic solution and existence conditions of the matching conditions are derived. Finally, a numerical simulation is shown to check the effectiveness of the proposed method in Section 4.

II. LIMIT CYCLE SYNTHESIS OF PIECEWISE AFFINE SYSTEMS

First, this section summarizes a synthesis method of piecewise affine systems that generate desired limit cycles. We will utilize the synthesis method in order to derive the main result in Section 3. See [31], [32], [33], [34] for details. Consider the 2-dimensional Euclidian space: $\mathbf{R}^2$, its coordinate: $x = [\, x_1 \;\; x_2 \,]^\mathsf{T} \in \mathbf{R}^2$, and the origin of $\mathbf{R}^2$: O. Let us set N $(N \geq 3)$ points $P_i \neq O$ $(i = 1, \cdots, N)$ in $\mathbf{R}^2$ and denote the vector from O to P_i by $p_i = [\, p_i^1 \;\; p_i^2 \,]^\mathsf{T}$. We also denote the angle between the half line OP_i and the x_1-axis by θ_i. Now, without loss of generality, we assume that the points $P_1 \cdots, P_N$ are located in the counterclockwise rotation from the x_1-axis, that is, $0 \leq \theta_1 < \cdots < \theta_N$ holds. Next, we define the semi-infinite region D_i which is sandwiched by the half lines OP_i and OP_{i+1} and the line segment C_i joining P_i and P_{i+1}, where $P_{N+1} = P_1$. Set a polygon as a union of C_i:

$$C := \bigcup_{i=1}^{N} C_i. \tag{1}$$

Fig. 1 shows an example of a polygonal closed curve for $N = 5$. Now, consider the affine system defined in D_i:

$$\dot{x} = \tilde{a}_i + \tilde{A}_i x, \;\; x \in D_i \tag{2}$$

where x is the state variable, and $\tilde{a}_i \in \mathbf{R}^2$, $\tilde{A}_i = \mathbf{R}^{2\times 2}$ are the affine term and the coefficient matrix, respectively. Hence, we treat the N-modal and 2-dimensional piecewise affine system (2) in $\mathbf{R}^2$.

T. Kai is with Faculty of Industrial Science and Technology, Tokyo University of Science, JAPAN kai@rs.tus.ac.jp

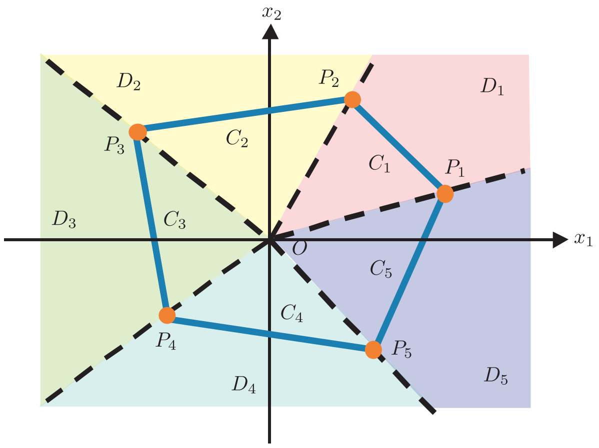


Fig. 1 : Example of Polygonal Closed Curve ($N = 5$)

The limit cycle synthesis problem for (2) is as follows.

Problem 1 [Limit Cycle Synthesis Problem] : For the N-modal and 2-dimensional piecewise affine system (2), design $\tilde{a}_i$, $\tilde{A}_i$ $(i = 1, \cdots, N)$ such that a given polygonal closed curve C (1) is a unique and stable limit cycle of the system. ∎

A solution to Problem 1 has been derived by the authors [32], [34], and $\tilde{a}_i$ and $\tilde{A}_i$ in (2) are given by

$$\begin{aligned}
\tilde{a}_i &= \\
&\begin{bmatrix} -\lambda_i(p_i^2 - p_{i+1}^2)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1) - \omega_i(p_i^1 - p_{i+1}^1) \\ \lambda_i(p_i^1 - p_{i+1}^1)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1) - \omega_i(p_i^2 - p_{i+1}^2) \end{bmatrix}, \\
\tilde{A}_i &= \\
&\begin{bmatrix} -\lambda_i(p_i^2 - p_{i+1}^2)^2 & \lambda_i(p_i^2 - p_{i+1}^2)(p_i^1 - p_{i+1}^1) \\ \lambda_i(p_i^2 - p_{i+1}^2)(p_i^1 - p_{i+1}^1) & -\lambda_i(p_i^1 - p_{i+1}^1)^2 \end{bmatrix},
\end{aligned} \tag{3}$$

where $\omega_i \neq 0$ and $\lambda_i > 0$ are parameters to be set freely. Existence and Uniqueness of the limit cycle for the system (2), (3) is guaranteed by the next theorem [32], [34].

Theorem 1 : For the N-modal and 2-dimensional piecewise affine system (2), (3), assume that

$$\omega_i > 0, \ \ \forall i \in \{1, \cdots, N\}. \tag{4}$$

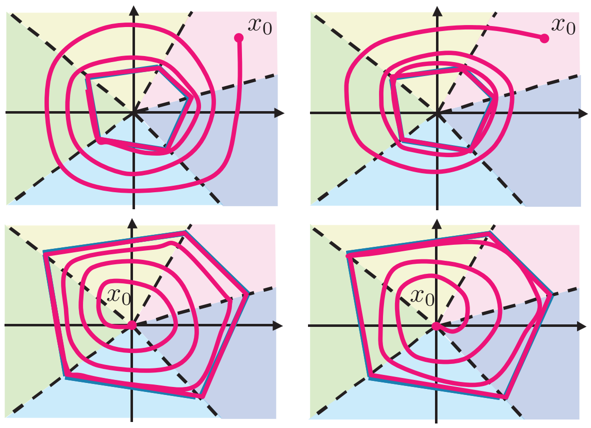
or

$$\omega_i < 0, \ \ \forall i \in \{1, \cdots, N\}. \tag{5}$$

holds. Then, the unique and stable limit cycle of the system (2), (3) is equivalent to C.

It is shown that the system (2), (3) has two important properties. Now, rotational directions of solution trajectories of the system (2), (3) is defined by the next [32], [34].

Definition 1 : For limit cycle solution trajectories of the N-modal and 2-dimensional piecewise affine system (2), (3), one that rotates in the clockwise/counterclockwise and direction is called *a limit cycle solution trajectory in the clockwise/counterclockwise rotation* (see Fig. 2).



[a] Crockwise Rotation [b] Countercrockwise Rotation

Fig. 2 : Clockwise and Counterclockwise Rotations of Limit Cycle Solution Trajectories

The relationship between rotational directions of limit cycles and the parameters in (3) is shown in the following proposition [32], [34].

Proposition 1 : For the N-modal and 2-dimensional piecewise affine system (2), (3), if (4) holds, its limit cycle solution trajectory moves in the counterclockwise rotation. Conversely, if (5) holds, it moves in the clockwise rotation.

Moreover, periods of limit cycles of the system (2), (3) can be characterized by the next proposition [32], [34].

Proposition 2 : When a limit cycle solution trajectory of the N-modal and 2-dimensional piecewise affine system (2), (3) is sufficiently close to C, the period with which it rotates around C is given by

$$T \approx \sum_{i=1}^{N} \frac{1}{|\omega_i|}. \tag{6}$$

From Propositions 1 and 2, it is confirmed that the rotating direction and the period of a limit cycle solution trajectory of the system (2), (3) can be decided by tuning the values of ω_i.

III. ANALYTIC SOLUTION TO LIMIT CYCLE CONTROL PROBLEM

A. *Formulation of Limit Cycle Control Problem*

In this section, we shall consider a controller design problem on generation of limit cycles for piecewise affine control systems. First, this subsection gives the problem formulation. Consider the piecewise affine control system defined in D_i:

$$\dot{x} = a_i + A_i x + b_i u, \ \ x \in D_i, \tag{7}$$

where $u \in \mathbf{R}$ is the control input and $b_i \in \mathbf{R}^2$ is the coefficient vector for the control input. We also consider the state feedback law in D_i:

$$u = k_i x + l_i, \ \ x \in D_i, \tag{8}$$

where $k_i \in \mathbf{R}^2$ and $l_i \in \mathbf{R}$. Substituting (8) into (7), we obtain *the closed-loop system*:

$$\dot{x} = a_i + b_i l_i + (A_i + b_i k_i)x, \ \ x \in D_i. \tag{9}$$

For the closed-loop system (9), we deal with the next problem on generating desired limit cycles.

Problem 2 [Limit Cycle Control Problem] : For the closed-loop system (9) that consists of a piecewise affine control system (7) and a state feedback law (8), design the gains of (8): $k_i, l_i \ (i = 1, \cdots, N)$ such that a given polygonal closed curve C is a unique and stable limit cycle of (9).

B. Matching Condition and Analytic Solution

This subsection derives an analytic solution to Problem 2. To do this, we use the method explained in Section 2 and call the system (2), (3) *the reference system*. Set the following notations for the system (9):

$$\begin{aligned} a_i &= \begin{bmatrix} a_i^1 \\ a_i^2 \end{bmatrix}, \ A_i = \begin{bmatrix} A_i^{11} & A_i^{12} \\ A_i^{21} & A_i^{22} \end{bmatrix}, \\ b_i &= \begin{bmatrix} b_i^1 \\ b_i^2 \end{bmatrix}, \ k_i = \begin{bmatrix} k_i^1 & k_i^2 \end{bmatrix}. \end{aligned} \tag{10}$$

Conditions such that the closed-loop system (9) is consistent with the reference system (2), (3) can be obtained by the following theorem [35].

Theorem 2 : The closed-loop system (9) is equivalent to the reference system (2), (3) if and only if *the matching conditions*:

$$a_i^1 + b_i^1 l_i = -\lambda_i (p_i^2 - p_{i+1}^2)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1) -\omega_i (p_i^1 - p_{i+1}^1), \tag{11}$$

$$a_i^2 + b_i^2 l_i = \lambda_i (p_i^1 - p_{i+1}^1)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1) -\omega_i (p_i^2 - p_{i+1}^2), \tag{12}$$

$$A_i^{11} + b_i^1 k_i^1 = -\lambda_i (p_i^2 - p_{i+1}^2)^2, \tag{13}$$

$$A_i^{12} + b_i^1 k_i^2 = \lambda_i (p_i^2 - p_{i+1}^2)(p_i^1 - p_{i+1}^1), \tag{14}$$

$$A_i^{21} + b_i^2 k_i^1 = \lambda_i (p_i^2 - p_{i+1}^2)(p_i^1 - p_{i+1}^1), \tag{15}$$

$$A_i^{22} + b_i^2 k_i^2 = -\lambda_i (p_i^1 - p_{i+1}^1)^2, \tag{16}$$

$$(i = 1, \cdots, N)$$

hold. ∎

The matching conditions in the i-mode (11)–(16) consist of 6 algebraic linear equations, and 5 unknown variables: $k_i^1, \ k_i^2, \ l_i, \ \omega_i, \ \lambda_i$. Regarding (11)–(16) as a simultaneous linear equation and considering its existence condition, we can derive an analytic solution and existence conditions of the matching conditions as the main theorem.

Theorem 3 : Assume that the N-modal and 2-dimensional piecewise affine control system (7) and the polygonal closed curve C (1) satisfy

$$b_i^1 \neq 0, \ \ \forall i \in \{1, \cdots, N\}, \tag{17}$$

$$\det[\, b_i \ \ p_i - p_{i+1} \,] \neq 0, \ \ \forall i \in \{1, \cdots, N\}, \tag{18}$$

$$b_i^{\mathsf{T}} (p_i - p_{i+1}) \neq 0, \ \ \forall i \in \{1, \cdots, N\}, \tag{19}$$

$$\det[\, A_i (p_i - p_{i+1}) \ \ b_i \,] = 0, \ \ \forall i \in \{1, \cdots, N\}. \tag{20}$$

In addition, consider the following three cases:

(a) $p_i^1 - p_{i+1}^1 \neq 0, \ p_i^2 - p_{i+1}^2 \neq 0$,

(b) $p_i^1 - p_{i+1}^1 \neq 0, \ p_i^2 - p_{i+1}^2 = 0$,

(c) $p_i^1 - p_{i+1}^1 = 0, \ p_i^2 - p_{i+1}^2 \neq 0$,

for each mode $i \, (i = 1, \cdots, N)$ and assume that for $\omega_i \, (i = 1, \cdots, N)$ calculated by (21), (4) or (5) holds, and for $\lambda_i (i = 1, \cdots, N)$ calculated by (22),

$$\lambda_i > 0, \ \ \forall i \in \{1, \cdots, N\} \tag{23}$$

holds. Then, the gains of the state feedback law (8): $k_i, l_i (i = 1, \cdots, N)$ such that the unique and stable limit cycle of (9) is equivalent to C are given by (24), (25).

(Proof) Regarding $k_i^1, \ k_i^2, \ l_i, \ \omega_i, \ \lambda_i$ as unknown variables, we can represent the matching conditions for the i-mode (11)-(16) as a simultaneous linear equation in the matrix form (26). The necessary and sufficient condition such that (26) has a unique solution can be given by the rank condition:

$$\operatorname{rank} \mathcal{A} = \operatorname{rank} [\, \mathcal{A} \ \ \mathcal{B} \,], \tag{27}$$

where the augmented coefficient matrix is defined as (28). By applying row basic deformation to (28) under the assumptions of (17)–(19), we can transform (28) into (29). Thus, it turns out the necessary and sufficient condition such that (31) holds is

$$\begin{aligned} &\frac{A_i^{21} b_i^1 - A_i^{11} b_i^2}{(p_i^2 - p_{i+1}^2)\{b_i^1 (p_i^1 - p_{i+1}^1) + b_i^2 (p_i^2 - p_{i+1}^2)\}} \\ &\quad + \frac{A_i^{22} b_i^1 - A_i^{11} b_i^2}{(p_i^1 - p_{i+1}^1)\{b_i^1 (p_i^1 - p_{i+1}^1) + b_i^2 (p_i^2 - p_{i+1}^2)\}} = 0, \end{aligned} \tag{30}$$

and we can see that (34) is equivalent to (20). Therefore, (17)–(20) are the existence conditions of the unique solution for the matching conditions (11)-(16). Next, under the assumption of the conditions (17)–(20), by solving the simultaneous linear equation (26), we can obtain the analytic solution (21)–(25) for the cases (a), (b), and (c). In addition, we can derive the conditions such that the unique and stable limit cycle of (9) is C for the calculated ω_i and λ_i as is the case with Theorem 1. ∎

The merits of Theorem 3 are as follows; (i) the conditions (17)–(20) are represented in simple forms and are very easy to check, (ii) the conditions (17)–(20) are common to the three cases (a), (b), and (c), (iii) the analytic solution (21)-(25) are obtained in the explicit forms (we do not have to solve any differential/difference equations) and are also easy to calculate, (iv) existence and uniqueness of the limit cycle can be easily checked with calculated values of ω_i and λ_i. For a computational example of Theorem 3, see the next section.

IV. SIMULATIONS

In this section, a numerical simulation is performed to verify the proposed method. Consider the case where $N = 5$ and $P_1 = (2, \ 0)$, $P_2 = (0, \ 2)$, $P_3 = (-2, \ 2)$, $P_4 = (-2, \ -2)$, $P_5 = (1, \ -1)$. The polygonal closed curve C

$$\omega_i = \begin{cases} \text{Case (a)} : \dfrac{(a_i^1 b_i^2 - a_i^2 b_i^1)(p_i^1 - p_{i+1}^1) + (-A_i^{22} b_i^1 + A_i^{12} b_i^2)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1)}{(p_i^1 - p_{i+1}^1)\{b_i^1(p_i^2 - p_{i+1}^2) - b_i^2(p_i^1 - p_{i+1}^1)\}} \\ \text{Case (b)} : \dfrac{(a_i^2 b_i^1 - a_i^1 b_i^2)(p_i^1 - p_{i+1}^1) + (A_i^{22} b_i^1 - A_i^{21} b_i^2)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1)}{b_i^2 (p_i^1 - p_{i+1}^1)^2} \\ \text{Case (c)} : \dfrac{(a_i^1 b_i^2 - a_i^2 b_i^1)(p_i^2 - p_{i+1}^2) + (A_i^{21} b_i^1 - A_i^{11} b_i^2)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1)}{b_i^1 (p_i^2 - p_{i+1}^2)^2} \end{cases} \tag{21}$$

$$\lambda_i = \begin{cases} \text{Case (a)} : \dfrac{-A_i^{22} b_i^1 + A_i^{12} b_i^2}{(p_i^1 - p_{i+1}^1)\{b_i^1(p_i^1 - p_{i+1}^1) + b_i^2(p_i^2 - p_{i+1}^2)\}} \\ \text{Case (b)} : \dfrac{-A_i^{22} b_i^1 + A_i^{12} b_i^2}{b_i^1 (p_i^1 - p_{i+1}^1)^2} \\ \text{Case (c)} : \dfrac{A_i^{21} b_i^1 - A_i^{11} b_i^2}{b_i^2 (p_i^2 - p_{i+1}^2)^2} \end{cases} \tag{22}$$

$$k_i = \begin{cases} \text{Case (a)} : \left[-\dfrac{A_i^{11}}{b_i^1} + \dfrac{(A_i^{22} b_i^1 - A_i^{12} b_i^2)(p_i^2 - p_{i+1}^2)^2}{b_i^1 (p_i^1 - p_{i+1}^1)\{b_i^1(p_i^1 - p_{i+1}^1) + b_i^2(p_i^2 - p_{i+1}^2)\}} \quad -\dfrac{A_i^{12}(p_i^1 - p_{i+1}^1) + A_i^{22}(p_i^2 - p_{i+1}^2)}{b_i^1(p_i^1 - p_{i+1}^1) + b_i^2(p_i^2 - p_{i+1}^2)} \right] \\ \text{Case (b)} : \left[-\dfrac{A_i^{11}}{b_i^1} \quad \dfrac{A_i^{12}}{b_i^1} \right] \\ \text{Case (c)} : \left[-\dfrac{A_i^{21}}{b_i^2} \quad -\dfrac{A_i^{12}}{b_i^1} \right] \end{cases} \tag{24}$$

$$l_i = \begin{cases} \text{Case (a)} : -\dfrac{a_i^1 (p_i^2 - p_{i+1}^2) - a_i^2 (p_i^1 - p_{i+1}^1)}{b_i^1 (p_i^2 - p_{i+1}^2) - b_i^2 (p_i^1 - p_{i+1}^1)} \\ \qquad + \dfrac{(A_i^{22} b_i^1 - A_i^{12} b_i^2)\{(p_i^1 - p_{i+1}^1)^2 - (p_i^2 - p_{i+1}^2)^2\}(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1)}{(p_i^1 - p_{i+1}^1)\{b_i^1(p_i^1 - p_{i+1}^1) + b_i^2(p_i^2 - p_{i+1}^2)\}\{b_i^1(p_i^2 - p_{i+1}^2) - b_i^2(p_i^1 - p_{i+1}^1)\}} \\ \text{Case (b)} : -\dfrac{a_i^2}{b_i^2} + \dfrac{(-A_i^{22} b_i^1 + A_i^{12} b_i^2)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^2)}{b_i^1 b_i^2 (p_i^1 - p_{i+1}^1)^2} \\ \text{Case (c)} : -\dfrac{a_i^1}{b_i^1} - \dfrac{(A_i^{21} b_i^1 - A_i^{11} b_i^2)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^2)}{b_i^1 b_i^2 (p_i^2 - p_{i+1}^2)} \end{cases} \tag{25}$$

is depicted in Fig. 3. We also consider a 5-modal piecewise affine control system:

$$\begin{aligned} D_1 &: \dot{x} = \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{a_1} + \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}}_{A_1} x + \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{b_1} u \\ D_2 &: \dot{x} = \underbrace{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}_{a_2} + \underbrace{\begin{bmatrix} -2 & 4 \\ -1 & 1 \end{bmatrix}}_{A_2} x + \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{b_2} u \\ D_3 &: \dot{x} = \underbrace{\begin{bmatrix} -4 \\ -6 \end{bmatrix}}_{a_3} + \underbrace{\begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix}}_{A_3} x + \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{b_3} u \\ D_4 &: \dot{x} = \underbrace{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}_{a_4} + \underbrace{\begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}}_{A_4} x + \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{b_4} u \\ D_5 &: \dot{x} = \underbrace{\begin{bmatrix} 1 \\ 4 \end{bmatrix}}_{a_5} + \underbrace{\begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix}}_{A_5} x + \underbrace{\begin{bmatrix} -1 \\ -2 \end{bmatrix}}_{b_5} u \end{aligned} \tag{31}$$

and it turns out that the system (30) satisfies the existence conditions (17)–(20).

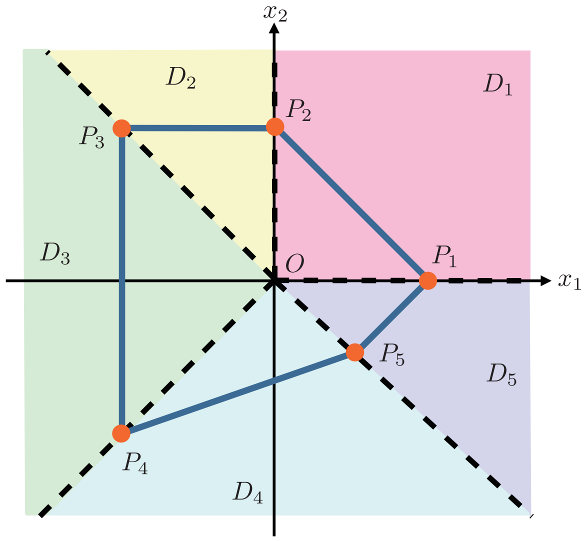


Figure 3 : 5 Modes and Polygonal Closed Trajectory in Simulation

$$
\underbrace{\begin{bmatrix} b_i^1 & 0 & 0 & 0 & (p_i^2 - p_{i+1}^2)^2 \\ 0 & b_i^1 & 0 & 0 & -(p_i^1 - p_{i+1}^1)(p_i^2 - p_{i+1}^2) \\ 0 & 0 & b_i^1 & p_i^1 - p_{i+1}^1 & (p_i^2 - p_{i+1}^2)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1) \\ b_i^2 & 0 & 0 & 0 & -(p_i^1 - p_{i+1}^1)(p_i^2 - p_{i+1}^2) \\ 0 & b_i^2 & 0 & 0 & -(p_i^1 - p_{i+1}^1)^2 \\ 0 & 0 & b_i^2 & p_i^2 - p_{i+1}^2 & -(p_i^1 - p_{i+1}^1)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1) \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} k_i^1 \\ k_i^2 \\ l_i \\ \omega_i \\ \lambda_i \end{bmatrix} = \underbrace{\begin{bmatrix} -A_i^{11} \\ -A_i^{12} \\ -a_i^1 \\ -A_i^{21} \\ -A_i^{22} \\ -a_i^2 \end{bmatrix}}_{\mathcal{B}} \tag{26}
$$

$$
[\,\mathcal{A}\ \mathcal{B}\,] = \left[\begin{array}{ccccc|c} b_i^1 & 0 & 0 & 0 & (p_i^2 - p_{i+1}^2)^2 & -A_i^{11} \\ 0 & b_i^1 & 0 & 0 & -(p_i^1 - p_{i+1}^1)(p_i^2 - p_{i+1}^2) & -A_i^{12} \\ 0 & 0 & b_i^1 & p_i^1 - p_{i+1}^1 & (p_i^2 - p_{i+1}^2)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1) & -a_i^1 \\ b_i^2 & 0 & 0 & 0 & -(p_i^1 - p_{i+1}^1)(p_i^2 - p_{i+1}^2) & -A_i^{21} \\ 0 & b_i^2 & 0 & 0 & -(p_i^1 - p_{i+1}^1)^2 & -A_i^{22} \\ 0 & 0 & b_i^2 & p_i^2 - p_{i+1}^2 & -(p_i^1 - p_{i+1}^1)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1) & -a_i^2 \end{array}\right] \tag{28}
$$

$$
\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & \dfrac{(p_i^2 - p_{i+1}^2)^2}{b_i^1} & -\dfrac{A_i^{11}}{b_i^1} \\ 0 & 1 & 0 & 0 & -\dfrac{(p_i^1 - p_{i+1}^1)(p_i^2 - p_{i+1}^2)}{b_i^1} & -\dfrac{A_i^{12}}{b_i^1} \\ 0 & 0 & 1 & \dfrac{p_i^1 - p_{i+1}^1}{b_i^1} & \dfrac{(p_i^2 - p_{i+1}^2)(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1)}{b_i^1} & -\dfrac{a_i^1}{b_i^1} \\ 0 & 0 & 0 & 1 & -\dfrac{\{b_i^1(p_i^1 - p_{i+1}^1) + b_i^2(p_i^2 - p_{i+1}^2)\}(p_i^1 p_{i+1}^2 - p_i^2 p_{i+1}^1)}{b_i^1(p_i^2 - p_{i+1}^2) - b_i^2(p_i^1 - p_{i+1}^1)} & \dfrac{a_i^1 b_i^2 - a_i^2 b_i^1}{b_i^1(p_i^2 - p_{i+1}^2) - b_i^2(p_i^1 - p_{i+1}^1)} \\ 0 & 0 & 0 & 0 & 1 & \dfrac{-A_i^{22} b_i^1 + A_i^{22} b_i^2}{(p_i^1 - p_{i+1}^1)\{b_i^1(p_i^1 - p_{i+1}^1) + b_i^2(p_i^2 - p_{i+1}^2)\}} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{A_i^{21} b_i^1 - A_i^{11} b_i^2}{(p_i^2 - p_{i+1}^2)\{b_i^1(p_i^1 - p_{i+1}^1) + b_i^2(p_i^2 - p_{i+1}^2)\}} + \dfrac{A_i^{22} b_i^1 - A_i^{11} b_i^2}{(p_i^1 - p_{i+1}^1)\{b_i^1(p_i^1 - p_{i+1}^1) + b_i^2(p_i^2 - p_{i+1}^2)\}} \end{array}\right] \tag{29}
$$

Thus, from Theorem 3, the unique solution of the matching conditions can be derive as

$$
\begin{aligned}
&D_1 : \omega_1 = \frac{1}{2},\ \lambda_1 = \frac{1}{4},\ k_1 = [\,-2\ \ -2\,],\ l_1 = 0, \\
&D_2 : \omega_2 = 1,\ \lambda_2 = \frac{1}{4},\ k_2 = [\,1\ \ -2\,],\ l_2 = -1, \\
&D_3 : \omega_3 = \frac{1}{4},\ \lambda_3 = \frac{7}{16}, k_3 = [\,-4\ \ 0\,],\ l_3 = -5, \\
&D_4 : \omega_4 = 2,\ \lambda_4 = \frac{1}{5},\ k_4 = \left[\,-\frac{7}{5}\ \ \frac{1}{5}\,\right],\ l_4 = -\frac{12}{5}, \\
&D_5 : \omega_5 = 4,\ \lambda_5 = 1,\ k_5 = [\,-1\ \ -1\,],\ l_5 = -1,
\end{aligned} \tag{32}
$$

where it must be noted that the modes 1, 4, 5 are classfied in the case (a), the mode 2 is in the case (b), and the mode 3 is in the case (c) in Theorem 3. We can easily confirm that (32) satisfies $\omega_i > 0,\ \lambda_i > 0,\ \forall i = \{1, \cdots, 5\}$, and hence the polygonal closed curve C is guaranteed as the unique and stable limit cycle of the closed-loop system from Theorem 3. The simulation results with the initial state $x_0 = [4\,4]^\mathsf{T}$ as an exterior point of C are depicted in Figs 4–6. Fig. 4 illustrates the solution trajectory on the x_1x_2-plane. In Figs. 5 and 6, the time histories of x_1 and x_2 are shown, respectively. The simulation results with the initial state $x_0 = [\,0\ \ 0\,]^\mathsf{T}$ as an interior point of C are also illsutrated in Figs 7–9. From these results, it turns out that the solution trajectory that starts from x_0 behaves as a limit cycle for the desired polygonal closed curve C. From Figs. 4 and 7, we can see that the solution trajectory moves in the counterclockwise rotation, and this result is coincident with Proposition 1 for the case where (4) holds. Moreover, for the period of the limit cycle trajectory, the estimated value in Proposition 2:

$$
T \approx \sum_{i=1}^{5} \frac{1}{|\omega_i|} = \frac{31}{4} \tag{33}
$$

completely agrees with the simulation results from Figs. 5, 6, 8, and 9. Consequently, these simulation results show the effectiveness of the proposed control method.

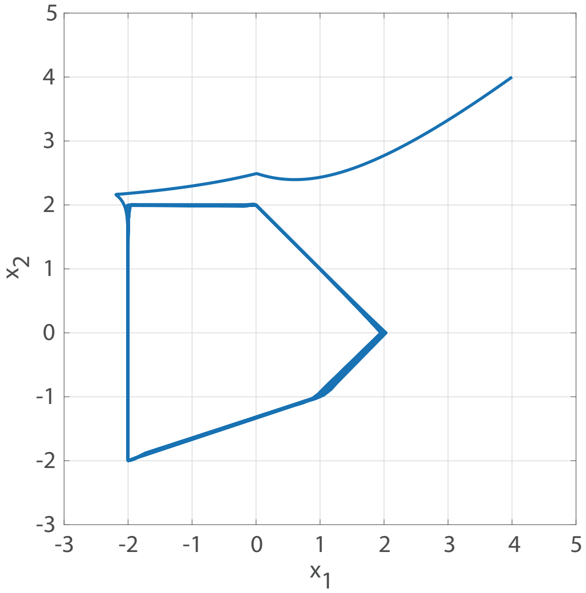


Figure 4 : Solution Trajectory on x_1x_2-Plane
($x_0 = [\,4\;\;4\,]^{\mathsf{T}}$)

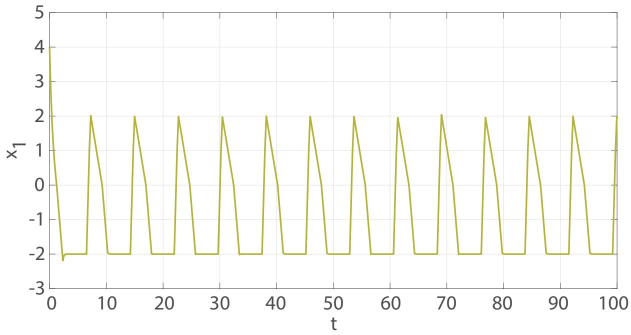


Figure 5 : Time History of x_1
($x_0 = [\,4\;\;4\,]^{\mathsf{T}}$)

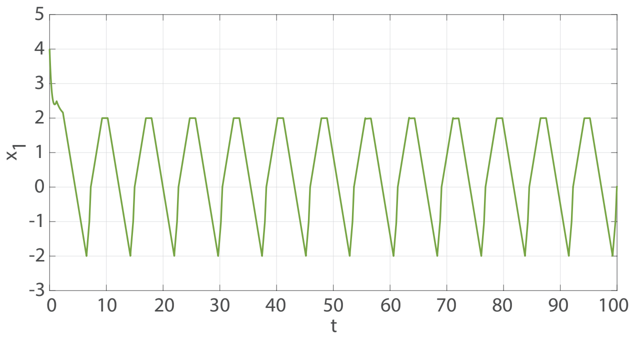


Figure 6 : Time History of x_2
($x_0 = [\,4\;\;4\,]^{\mathsf{T}}$)

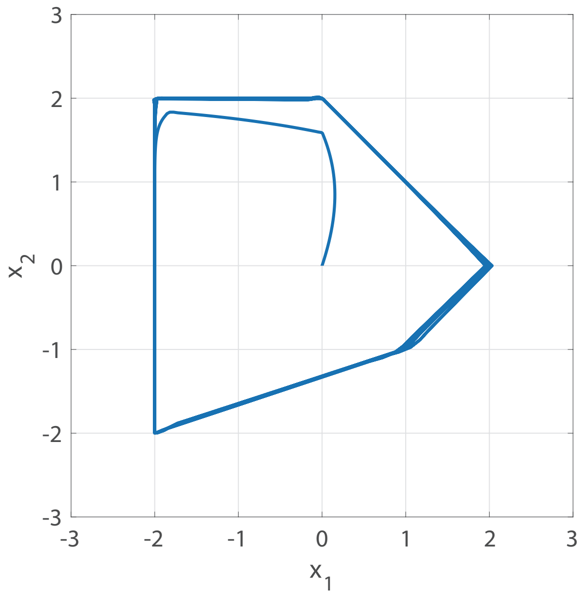


Figure 7 : Solution Trajectory on x_1x_2-Plane
($x_0 = [\,0\;\;0\,]^{\mathsf{T}}$)

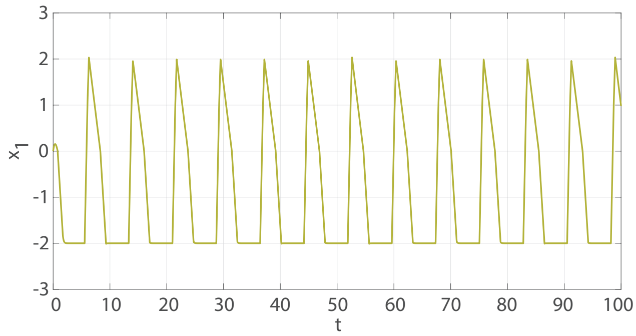


Figure 8 : Time History of x_1
($x_0 = [\,0\;\;0\,]^{\mathsf{T}}$)

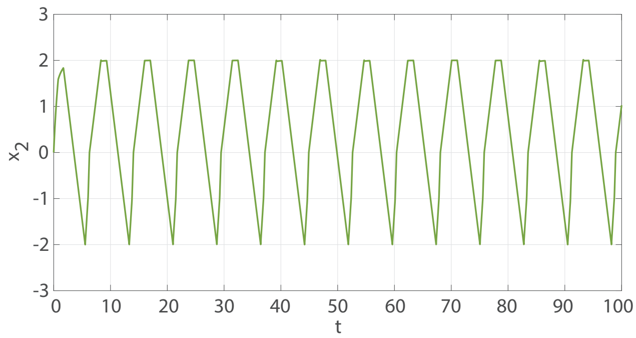


Figure 9 : Time History of x_2
($x_0 = [\,0\;\;0\,]^{\mathsf{T}}$)

V. CONCLUSIONS

This work has considered the limit cycle control problem for N-modal and 2-dimensional piecewise affine control systems, and has shown a controller design method of state feedback laws. Especially, for all the three cases, the analytic solutions and the existence conditions have been obtained by solving the matching conditions. Numerical simulations have demonstrated the effectiveness of the proposed method.

Our future work are as follows: extensions to multi-dimensional cases, nonlinear systems, and output feedback, relaxation of the existence conditions, and applications to real systems.

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