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Maximum Conditional Probability Stochastic Controller for Scalar Linear Systems with Additive Cauchy Noises

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***Abstract*— In this work a stochastic controller, motivated by the sliding mode control methodology, is proposed for linear, single-state system with additive Cauchy distributed noises. The control law utilizes the time propagated probability density function (pdf) of the system state given measurements that has been derived in recent studies addressing the Cauchy estimation problem. The motivation for the proposed approach is mainly the high numerical complexity of the currently available methods for such systems. The controller performance is evaluated numerically and compared to an alternative approach presented recently and to a Gaussian approximation to the problem. A fundamental difference between the Cauchy and the Gaussian controllers is their response to noise outliers. While all controllers respond to process noises, even to the outliers, the Cauchy controllers drive the state faster towards zero after those events. On the other hand, the Cauchy controllers do not respond to measurement noise outliers, while the Gaussian does. The newly proposed Cauchy controller exhibits similar performance to the previously proposed one, while requiring lower computational effort.**

I. Introduction

The majority of stochastic control methodologies assume that the dynamical system is driven by additive Gaussian process and measurement noises [1]. Many solutions for the control problem for linear systems with Gaussian noise distributions can be readily found in the literature, e.g., the commonly used Linear Quadratic Gaussian (LQG), Linear Exponential Gaussian (LEG), $\mathcal{H}_2$ and $\mathcal{H}_\infty$, and more [1]–[3]. However the Gaussian assumption is not compatible for systems with heavy tailed distributed noises exhibiting significant measurement and process noise outliers caused by, e.g., radar and sonar sensors, atmospheric disturbances and air turbulence, and more [4], [5]. Using Gaussian approximations for such cases was found insufficient showing significant performance degradation in the presence of outliers [6], [7].

To properly address the control problem for system with heavy tailed distributed noises they have to be accounted for in the design process. In recent studies it was demonstrated that the Cauchy distribution is an attractive candidate to better represent process and measurement noise outliers in both estimation and control problems [6]–[10]. The Cauchy distribution is challenging because its first moment is not well defined and its higher moments are infinite. Consequently, when posing the control problem for such systems, commonly used performance cost (e.g., the quadratic cost of LQG) are not properly defined.

In estimation, it was shown that the conditional probability density function (pdf) of the state given the measurement history has finite first and second moments, yielding the conditional minimum variance Cauchy estimator [8]–[10]. The algorithm that propagated the conditional pdf was found to be a nonlinear function of the measurement sequence. This clearly complicates the control design task. Nonetheless, the conditional pdf was incorporated in an optimal predictive control (OPC) methodology that uses a special computable cost criterion which is well defined and finite when conditioned on the available measurements [6], [7]. The main drawback of those controllers is their high complexity and computational load. The goal of the current work is to address those aspects hence proposing a real-time implementable controller without affecting its performance.

The controller design presented in the current study is motivated by the Sliding Mode Control (SMC) method [11], [12]. In SMC, the desired closed-loop dynamic characteristics of the system is specified by defining a sliding variable. Nullifying the sliding variable, thus driving the system into a sliding mode, guarantees the desired performance. The control method ensures that a system with bounded disturbances will attain the sliding mode after a finite time. However, when addressing systems with unbounded or stochastic inputs, sliding mode behaviour cannot be obtained even when the time frame is not limited. As an alternative, it was proposed regulating the sliding variable within a predefined bound around the sliding mode [13]–[15]. This method was successfully applied for system with Gaussian noises and relies on the second moment of the system state. Since those moments are not finite in the Cauchy noise case, the method proposed in [14] was modified and expanded to address the heavy tailed noise case considered here.

This paper is organized as follows. The problem is formulated in section II. Section III presents the rationale of the proposed stochastic SMC motivated control design method, detailed next in sections IV and V. Section VI addresses several numerical simulation results, demonstrating the performance of the suggested control methodology while comparing it to alternative controllers. The paper is concluded in section VII.

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II. Problem Formulation

In this work we consider a linear, discrete-time, single-state stochastic system

$$x_{k+1} = \Phi x_k + u_k + w_k, \quad (1a)$$

$$z_k = H x_k + v_k, \quad (1b)$$

where x_k is the system state, u_k is the control signal used to regulate x_k, z_k is the measurement, and k is the time index. Φ and H are assumed to be known. w_k and v_k are process and measurement noise sequences, respectively, that are assumed to be white, independent of each other, and Cauchy distributed with pdf-s given by

$$f_{W_k}(w_k) = \frac{\beta/\pi}{w_k^2 + \beta^2}, \ \beta > 0, \quad (2a)$$

$$f_{V_k}(v_k) = \frac{\gamma/\pi}{v_k^2 + \gamma^2}, \ \gamma > 0. \quad (2b)$$

These pdf-s have a zero median and known scaling parameters $\beta > 0$ and $\gamma > 0$. The initial state is assumed to be independent of w and v and Cauchy distributed, i.e.,

$$f_{X_1}(x_1) = \frac{\alpha/\pi}{(x_1 - \bar{x}_1)^2 + \alpha^2}, \ \alpha > 0, \quad (3)$$

with a known scaling parameter $\alpha > 0$ and median $\bar{x}_1$. The goal is to design an output feedback controller, i.e., u_k, being a function of $y_k = \{z_1, \ldots, z_k\}$, to regulate the state x_k. The tracking problem can be addressed using a similar method.

III. Stochastic Control Design Concept

The challenge in addressing the control design problem of linear systems with Cauchy distributed noises stems from the fact that notions like control-estimation separation and certainty equivalence do not hold in this case. Specifically, it was shown in previous studies that the minimum conditional variance estimator of such systems is nonlinear in the measurement sequence [8]–[10]. Moreover, the unconditional moments of the system state are either not defined or infinite. Thus the commonly used control design criteria have to be modified when addressing such system, as was suggested in [6], [7]. The resulting controller were shown to be a highly nonlinear function of the measurement sequence, entailing high computational load.

Due to the nonlinear nature of the anticipated controller, in order to address the computational challenge of the known solutions [6], [7], this study suggests a design methodology that is motivated by the SMC method that addresses mainly systems with bounded uncertainties and noises [11], [12], and is extended to stochastic systems [13]–[15]. We first briefly reviews those results.

Consider the linear multivariate discrete-time system:

$$x_{k+1} = \Phi x_k + \Lambda u_k + \Gamma w_k, \quad (4a)$$

$$z_k = H x_k + v_k, \quad (4b)$$

where $x_k \in \mathbb{R}^n$ is the system state, u_k and z_k are the scalar control and measurement signals, respectively, w_k and v_k are scalar process and measurement noises, respectively, and Φ, Λ, Γ and H are known matrices with appropriate dimensions. First, assuming full state information and bounded disturbance w_k for all k, the task is to regulate x_k.

In SMC we define a sliding variable

$$s_k = c x_k, \quad (5)$$

where for a scalar control input u_k, $s_k : \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}^{1\times n}$ is a design parameter such that $c\Lambda \neq 0$. The sliding surface is defined as

$$\mathcal{S} = \left\{ x_k \middle| \, s_k = c x_k = 0 \right\}. \quad (6)$$

When $\mathcal{S}$ is attained, the system is said to be in a sliding mode. c is chosen such that when the system is in the sliding mode it complies with the desired closed loop dynamics requirements (see for details [11], [12].)

The discrete sliding mode control signal is given by

$$u_k = u_{(dis)_k} + u_{(eq)_k} = -\rho \, \mathrm{sign}(s_k) - (c\Lambda)^{-1} c\Phi x_k. \quad (7)$$

$u_{(dis)_k}$ is a discontinuous controller designed to drive the system into a sliding mode in a finite time, while $u_{(eq)_k}$ ensures that the system remains on $\mathcal{S}$. The parameter ρ depends on the bound of the disturbance and regulates the finite time for the system to reach the sliding mode. SMC solutions are available also for the output feedback setting, where only the measurement z_k is available [11].

When addressing stochastic system with unbounded noises it is not feasible to drive the system into a sliding mode or maintain it over time: the stochastic signal will continuously divert the system from the sliding surface. Hence, to apply the SMC ideas for such systems, concepts like the sliding surface and finite time convergence have to be modified to yield a stochastic version of the SMC approach. Such ideas were proposed in [13]–[15] where the disturbances were assumed to be Gaussian white noises.

The main idea in these studies is to construct an output feedback control low u_k such that the system (4) is driven to a vicinity of the sliding surface $\mathcal{S}$ of (6) with a given probability and ensure that it stays in this region. Specifically, for a bound $L > 0$ and $0 < \varepsilon < 1$, the goal is to attain

$$P\left(|s_k| < L\right) = 1 - \varepsilon \ \text{ a.s.} \quad \text{for} \quad k > N \quad (8)$$

for some integer N. In [14] it is shown that for the Gaussian case, by incorporating a Kalman filter estimator, a stochastic controller can be designed to ensure (8), where, for a given ε, the ultimate bound L can be determined analytically as a function of the measurement-independent second moment of the state estimation error.

This is not the case for systems with Cauchy distributed noises, for which estimator and the estimation error variance are highly nonlinear functions of the measurement sequence [8]–[10]. Moreover, due to the heavy tail characteristics of the process noise, there is no time propagated estimate of the state needed to apply the methodology in [14]. Hence, a modified approach is proposed here to address systems with Cauchy distributed noises.

Normally, in stochastic control, the control signal is determined by the unconditional first and second moments of the system state. In the Cauchy case the unconditional first moment of the state is undefined while its higher moments are infinite. However, the *conditional* pdf of the system state given the measurements or its characteristic function can be determined analytically [8]–[10]. Specifically, for a known measurement history y_k and a control input u_k, the conditional pdf $f_{X_{k+1}|Y_k}(x_{k+1}|y_k)$ can be expressed in a closed form. This pdf does not have a well defined first moment and has infinite second and higher moments.

Therefore, instead of designing a controller for a prescribed probability $(1-\varepsilon)$ in (8) and computing the bound L for it as suggested in [14], for the Cauchy case it is proposed to set the bound L a-priori and determine the control signal u_k such that the a-priori conditional probability of the sliding variable s_{k+1} to be in the band $\pm L$ around zero is maximized, i.e.,

$$u_k^* = \arg\max_{u_k} P\left(|s_{k+1}| < L \Big| y_k \right). \tag{9}$$

The above probability can be computed using the conditional pdf $f_{X_{k+1}|Y_k}(x_{k+1}|y_k)$. Fortunately, the previously derived Cauchy estimators determine analytically those pdf-s or their characteristic functions, that can be used to compute u_k^* in (9), which will be referred to as the maximum-conditional-probability-controller (MCPC).

Next we show how the maximization problem can be solved using the conditional pdf-s or alternatively their characteristic functions for the single state, scalar systems. Since the state is scalar, in this case the sliding variable of (5) is chosen as the state, i.e., $c = 1$.

IV. Probability Density Function Implementation

The proposed control synthesis method relies on the conditional pdf $f_{X_{k+1}|Y_k}(x_{k+1}|y_k)$ derived as part of the minimum conditional variance estimator for the system described in (1) [8]. Although in the current study the controller synthesis utilizes only $f_{X_{k+1}|Y_k}(x_{k+1}|y_k)$, also the measurement updated $f_{X_k|Y_k}(x_k|y_k)$ is required to attain a recursive analytical representation of $f_{X_{k+1}|Y_k}(x_{k+1}|y_k)$. In [8] it was shown that those two pdfs can be expressed as

$$f_{X_k|Y_k}(x_k|y_k) = \sum_{i=1}^{k+1} \frac{a_i(k|k)x_k + b_i(k|k)}{\big(x_k - \sigma_i(k|k)\big)^2 + \omega_i^2(k|k)}, \tag{10a}$$

$$f_{X_{k+1}|Y_k}(x_{k+1}|y_k) = \sum_{i=1}^{k+1} \frac{a_i(k+1|k)x_{k+1} + b_i(k+1|k)}{\big(x_{k+1} - \sigma_i(k+1|k)\big)^2 + \omega_i^2(k+1|k)}. \tag{10b}$$

Recursive, closed form expressions for all the parameters in Eqs. (10) can be found in reference [8].

Following the rationale presented in the previous section and expressed implicitly in equations (9), the optimal control signal u_k is determined by maximizing the cost function

$$J_k^* = \max_{u_k} \int_{-L}^{L} f_{X_{k+1}|Y_k}(x_{k+1}|y_k)\, dx_{k+1}, \tag{11}$$

or alternatively

$$J_k^* = \max_{u_k} \int_{-L-u_k}^{L-u_k} f^{u_k=0}_{X_{k+1}|Y_k}(x_{k+1}|y_k)\, dx_{k+1}. \tag{12}$$

where $f^{u_k=0}_{X_{k+1}|Y_k}(x_{k+1}|y_k)$ is the pdf of x_{k+1} given y_k while setting $u_k = 0$. Its analytical form is identical to that given in (10b) with appropriate changes in the u_k-dependant pdf parameters. The advantage of using (12) is that $f^{u_k=0}_{X_{k+1}|Y_k}(x_{k+1}|y_k)$ has to be computed only for $u_k = 0$ and not repeatedly for every candidate u_k when solving (11). Hence, the form in (12) will be used in the sequel to determine numerically the optimal cost J_k^* to yield the optimal control signal given by

$$u_k^* = \arg\max_{u_k} \int_{-L-u_k}^{L-u_k} f^{u_k=0}_{X_{k+1}|Y_k}(x_{k+1}|y_k)\, dx_{k+1}. \tag{13}$$

Using the explicit form of $f^{u_k=0}_{X_{k+1}|Y_k}(x_{k+1}|y_k)$ given in (10b), the integral in (12) can be solved analytically to yield

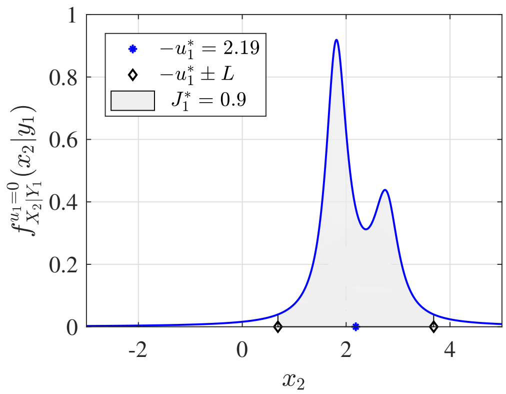
$$\begin{aligned} J_k(u_k) = \sum_{i=1}^{k+1} \Bigg\{ & \frac{a_i}{2} \ln \frac{\Big(L - u_k - \sigma_i\Big)^2 + \omega_i^2}{\Big(-L - u_k - \sigma_i\Big)^2 + \omega_i^2} \\ & + \frac{a_i\sigma_i + b_i}{\omega_i} \Bigg[\arctan\left(\frac{L - u_k - \sigma_i}{\omega_i}\right) \\ & - \arctan\left(\frac{-L - u_k - \sigma_i}{\omega_i}\right) \Bigg] \Bigg\}, \end{aligned} \tag{14}$$

where, for simplicity, the time indexes $(k+1|k)$ were removed from the pdf parameters. The cost in (14) is a highly nonlinear function of the optimization parameter u_k. Hence, $J_k(u_k)$ is maximized using standard numerical optimization tools available, e.g., in MATLAB. Once the optimal u_k^* is determined, it is used as an input to the system and to compute the actual time propagated $f_{X_{k+1}|Y_k}(x_{k+1}|y_k)$.

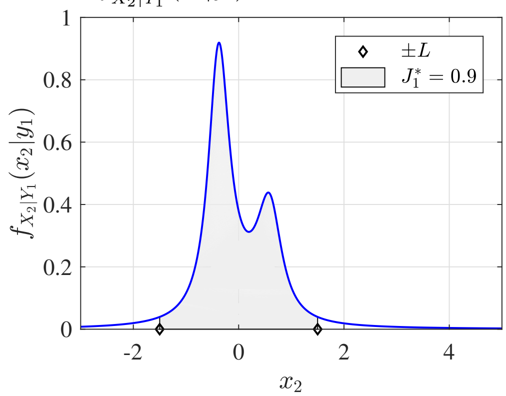
To demonstrate the shifted relation between the two pdfs mentioned above together with the maximization process, we consider a sample system with parameters $\Phi = 0.9, H = 1, \bar{x}_1 = 2, \alpha = 0.02, \beta = 0.2$, and $\gamma = 0.1$. At time $k = 1$, the initial state was drawn from its Cauchy distribution to be $x_1 = 2.01$ while the measurement was drawn to be $z_1 = 3.1$. The time propagated $f^{u_1=0}_{X_2|Y_1}(x_2|y_1)$ is depicted in Fig. 1a. Choosing $L = 1.5$ and solving (12) it was determined numerically that $J_1^* = 0.9$, depicted in Fig. 1a by the grayed out area, with $u_1^* = -2.19$ also shown in this figure. Finally, $f_{X_2|Y_1}(x_2|y_1)$ computed with the optimal control signal is depicted in Fig. 1b. Since $u_1^* = -2.19$ is negative, it clearly shows that the latter pdf is shifted by 2.19 to the left compared to $f^{u_1=0}_{X_2|Y_1}(x_2|y_1)$ in Fig. 1a.

V. Characteristic Function Implementation

An alternative approach for the optimal control strategy discussed in this work is to utilize the characteristic function of the un-normalized conditional pdf (cf-ucpdf)



(a) $f_{X_2|Y_1}^{u_1=0}(x_2|y_1)$, J_1^* of (12) and u_1^*.



(b) $f_{X_2|Y_1}(x_2|y_1)$ and J_1^* of (11).

Fig. 1: $f_{X_2|Y_1}^{u_1=0}(x_2|y_1)$, $f_{X_2|Y_1}(x_2|y_1)$, J_1^* and u_1^*.

$f_{X_{k+1}|Y_k}(x_{k+1}|y_k)$ addressed in the alternative Cauchy estimator design [9]. This approach will be in particular useful when addressing in the future vector-state systems, for which a pdf estimation solution does not exist [10]. The unnormalized measurement-updated and time-propagated cf-ucpdf-s mentioned above are expressed as

$$\bar{\phi}_{X_k|Y_k}(\nu) = \sum_{i=1}^{k+1} \Big(c_i(k|k) + j d_i(k|k)\,\mathrm{sign}(\nu) \Big) \times e^{-\omega_i(k|k)|\nu| + j\sigma_i(k|k)\nu}, \quad (15a)$$

$$\bar{\phi}_{X_{k+1}|Y_k}(\nu) = \sum_{i=1}^{k+1} \Big(c_i(k+1|k) + j d_i(k+1|k)\,\mathrm{sign}(\nu) \Big) \times e^{-\omega_i(k+1|k)|\nu| + j\sigma_i(k+1|k)\nu}, \quad (15b)$$

where j is the pure imaginary number. Recursive, closed form expressions for all the parameters in Eqs. (15) can be found in reference [9]. To obtain the characteristic functions of the *normalized* conditional pdf, the above characteristic functions have to be divided by

$$f_{Y_k}(y_k) = \sum_{i=1}^{k+1} c_i(k|k), \quad (16)$$

to yield

$$\phi_{X_k|Y_k}(\nu) = \frac{\bar{\phi}_{X_k|Y_k}(\nu)}{f_{Y_k}(y_k)}, \quad \phi_{X_{k+1}|Y_k}(\nu) = \frac{\bar{\phi}_{X_{k+1}|Y_k}(\nu)}{f_{Y_k}(y_k)}. \quad (17)$$

The control performance criterion of (12) can be expressed directly using the characteristic function of the associated pdf, i.e.,

$$\begin{aligned} J_k^* &= \max_{u_k} \int_{-L-u_k}^{L-u_k} f_{X_{k+1}|Y_k}^{u_k=0}(x_{k+1}|y_k)\, dx_{k+1} \\ &= \max_{u_k} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(L\nu)}{\nu} e^{j u_k \nu} \phi_{X_{k+1}|Y_k}^{u_k=0}(\nu) d\nu, \quad (18) \end{aligned}$$

where $\phi_{X_{k+1}|Y_k}^{u_k=0}(\nu)$ is the characteristic function of $f_{X_{k+1}|Y_k}^{u_k=0}(x_{k+1}|y_k)$. It should be noted that since the normalization in (17) involves a real and positive value $f_{Y_k}(y_k)$, an equivalent cost function can be defined using the unnormalized characteristic function, i.e.,

$$\bar{J}_k(u_k) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(L\nu)}{\nu} e^{j u_k \nu} \bar{\phi}_{X_{k+1}|Y_k}^{u_k=0}(\nu) d\nu, \quad (19)$$

the maximization of which with respect to u_k will yield the same optimal u_k^* as in (18).

The integral in (19) can be solved analytically for $\bar{\phi}_{X_{k+1}|Y_k}^{u_k=0}(\nu)$ given in (15b). It is expressed as

$$\begin{aligned} \bar{J}_k(u_k) = \frac{1}{\pi} \sum_{i=1}^{k+1} \Bigg\{ & \frac{d_i}{2} \ln \frac{\big(L - u_k - \sigma_i\big)^2 + \omega_i^2}{\big(-L - u_k - \sigma_i\big)^2 + \omega_i^2} \\ & + c_i \Bigg[\arctan\left(\frac{L - u_k - \sigma_i}{\omega_i} \right) \\ & - \arctan\left(\frac{-L - u_k - \sigma_i}{\omega_i} \right) \Bigg] \Bigg\}. \quad (20) \end{aligned}$$

For simplicity, in (20) the time indexes $(k+1|k)$ were removed from the parameters of the characteristic function. As in the previous section addressing the pdf implementation of the control strategy, the optimal control signal that maximizes the cost in (20) is determined numerically. It is then used to both drive the system and determined the time propagated $\bar{\phi}_{X_{k+1}|Y_k}(\nu)$.

VI. Numerical Simulation Results

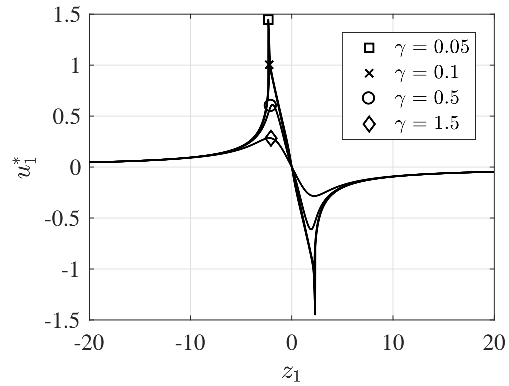
In this section, the performance of the proposed controller is evaluated using a numerical simulation and compared to the performance of an alternative OPC method and a Gaussian approximation, LEG, both fully discussed in [6]. First, in subsection VI-A, the control signal at the first time step is evaluated as a function of the first measurement for different system parameters. This reveals some important characteristics of the proposed control method. Then the time response characteristics of the three evaluated controllers are evaluated in subsection VI-B. Both stable and unstable open loops dynamics are examined here.

A. First Step Control

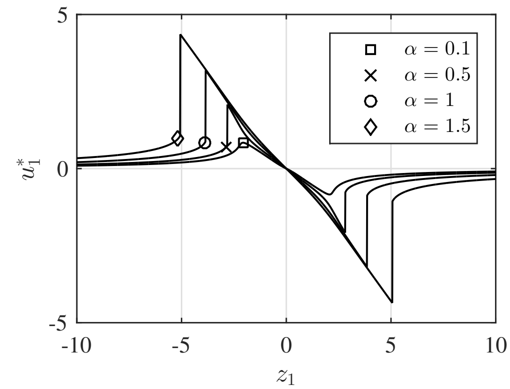
To better understand the underlying characteristics of the proposed controller, we explore first the optimal control value u_1^* as a function of the first measurement z_1. We consider a sample system with parameters $\Phi = 0.9$, $H = 1$, $\bar{x}_1 = 0$, and $\beta = 0.02$, and control design parameter $L = 1$. In the following case studies we will vary either the scaling parameter γ of the measurement noise pdf or the scaling parameter α of initial condition pdf.

Figure 2a depicts the optimal control signal u_1^* vs. z_1 for $\alpha = 0.1$ and different values of γ. Not surprisingly, the plot is symmetric around $z_1 = 0$. Moreover, it clearly shows the nonlinearity of u_1^* as a function of z_1, except of a nearly linear characteristics for small $|z_1|$ values. Interestingly, for large values of $|z_1|$ the optimal control signal tends to zero. This demonstrates the desirable notion that, since for heavy tail distributed measurement noises measurement outliers are more likely (than, e.g., for Gaussian distribution), a large valued measurement may imply a significant noise component and not a state deviation. Consequently, the control signal u_1^* should be restrained in such cases, as is shown in Fig. 2a.

In Fig. 2b the optimal control signal u_1^* vs. z_1 is shown for a fixed $\gamma = 0.2$ for different scaling parameters α. This plot demonstrates a similar nonlinear characteristics of the control signal as mentioned above with a sharper drop-off at high values of z_1.



(a) u_1^* vs. z_1 for different γ values.



(b) u_1^* vs. z_1 for different α values.

Fig. 2: MCPC control signal u_1^* vs. z_1 for different γ and α values.

B. Time Response

The performance of MCPC is compared next against the OPC and LEG controllers derived in [6] for a sequence of 100 time steps. The system parameters were set at $H = 1$, $\bar{x}_1 = 0$, $\alpha = 0.5$, $\beta = 0.02$, and $\gamma = 0.1$, and the controller parameter $L = 1$. Two cases were examined: a stable system with $\Phi = 0.95$, the results for which are shown in Fig. 3, and an unstable system with $\Phi = 1.05$ depicted in Fig. 4 that is clearly stabilized by the three controllers considered. These figures show clearly that MCPC and OPC exhibit similar performance when the process and measurement noises are small. However when the system faces significant noises, performance differences are observed. Firstly, when the system is exposed to a significant process noise, e.g., at time steps k=2 and k=14, all three controller respond to regulate the state back to zero. However, the transient responses of the three controller are very different. The LEG controller, although it drives the system state back to zero, its transient response is very slow compared to the other two controller. Among MCPC and OPC, the former has a much faster response obtained by a slightly increased control effort. Similar characteristics was observed for both open-loop stable and unstable systems.

A completely different behaviour is observed when the system is exposed to a significant measurement noise, like at $k = 52$. Here, the LEG ,being a linear controller, reacts to the deviated measurement, even though the state has remained close to its desired zero value. Favorably, both MCPC and OPC, that were designed to account for the heavy tail characteristics of the measurement noise, responded with a nearly zero control signal.

To conclude, MCPC utilizes more control effort than the OPC, yielding a faster response. The main advantage of MCPC is its lower computation burden, obtained due the much simpler scalar maximization problem that has to be solved to determine the optimal control signal.

VII. Conclusions

A new control strategy has been developed for a scalar linear systems driven by Cauchy distributed process and measurement noises. The motivation for addressing this type of problems is that the commonly used Gaussian distribution assumption is inaccurate when modeling and controlling systems facing extreme conditions and noise outliers. The heavy tailed Cauchy distribution was found to better represent such system noises. A challenge when addressing Cauchy distributions results from the fact that they have an undefined first moment, and infinite higher moments. Consequently, commonly used control and estimation methods cannot be applied in this case. The proposed control strategy was derived using the principles of the deterministic sliding mode control method. Since our system is stochastic and thus continuously forced by unbounded noise input, sliding mode

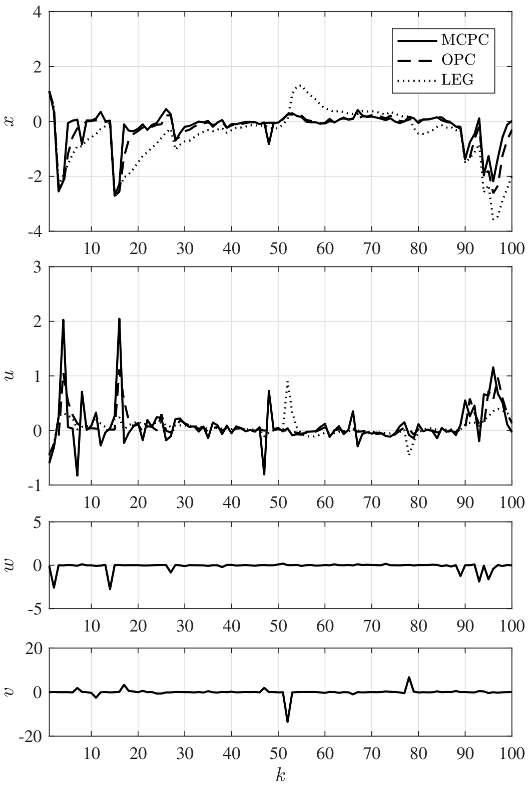


Fig. 3: Comparison between MCPC, OPC and LEG controllers for a stable system.

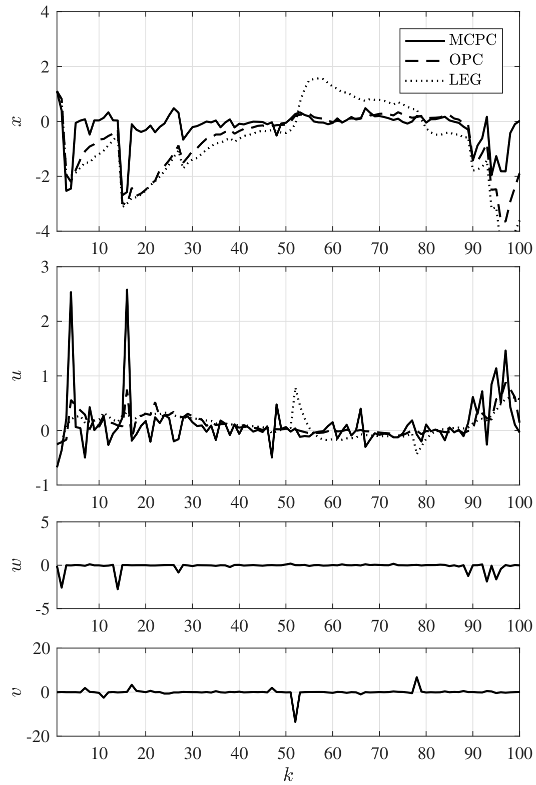


Fig. 4: Comparison between MCPC, OPC and LEG controllers for an unstable stable system.

cannot be attained. Alternatively, in this work we propose a control method that at each time step maximizes the conditional probably of the sliding variable being in a predefined bound around the sliding manifold. Two equivalent controller implementation were presented: one that uses the a priori conditional probability density function (pdf) of the sliding variable given the past measurement sequence and the second that uses the characteristic function of the above pdf. The performance of the proposed controller were tested numerically and compared to recently derived optimal predictive controller (OPC) and a Gaussian approximation (LEG). The proposed controller has demonstrated slightly better performance compared to OPC, while both outperformed the LEG controller. Although the newly proposed controller uses slightly higher control inputs, it is considerably simpler, requires less computing power and attains a quicker response than the OPC.

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