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A strategy to accelerate consensus in leader-follower networks

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***Abstract*— In this paper we present a new strategy to accelerate convergence in consensus networks. We consider a multi-agent system with a fixed communication structure: each agent updates its state value based on the states of the neighbouring agents, with the common goal of achieving consensus. In this set-up, we select p leaders among the agents whose task is to accelerate the convergence speed, by exchanging information among each other and by elaborating an additional control signal, based only on the state evolution of the p leaders. We first provide a complete description of the system, then investigate under what conditions we can freely allocate the overall system dynamics, and finally address the main objective: accelerating the consensus speed.**

I. INTRODUCTION

In the last years a significative thrust of research has been devoted to the subjects of multi-agent systems and interconnected networked devices. Consensus algorithms and synchronization techniques have attracted great interest because they allow to achieve a desired collective behavior by iterating local computations among neighbouring agents/nodes [1], [2], [3]. Consensus is a basic and simple concept used as a key tool in a wide spectrum of applications; it refers to the ability of a group of nodes to reach an agreement on a value, by means of distributed elaboration of the local information [1], [2], [4]. While necessary and sufficient conditions for consensus have been widely studied, for a variety of different set-ups and assumptions on the communication networks and the agents' models, the search for algorithms that can improve the convergence speed is still an active research topic [5], also motivated by the fact that the convergence speed of some graph topologies, commonly used to model connectivity in wireless networks, is significantly slow [6], [7]. The research techniques aiming to improve the convergence of consensus algorithms can be split into two main groups: those focusing on the best possible choice of the communication structure and weights, possibly using constrained optimization techniques [8], [9], [10], and those that introduce memory elements and additional elaboration in the distributed algorithms of some or all network nodes [11], [12], [13], [14].

In this paper, we report some preliminary results on the design of a multiple-leader-based protocol aiming to accelerate consensus, by making use of a set of variables that are shared among the leaders. The results reported in this paper extend those recently achieved for the single leader case in [14], while the concept of leader-controlled distributed consensus and the first results on the single leader control protocol appeared in [15]. The control protocol provided in this paper represents an extension of those proposed in [14], [15] not only because it addresses the case of an arbitrary number of leaders, but mostly because it has a much more general structure. The literature on the use of multiple leaders to control a network is relatively recent. Stemming from the research activity on distributed tracking problem in leader-follower networks, target destination and containment control of a group of mobile autonomous agents with multiple stationary or dynamic leaders was first studied in [16], [17], [18]. Recently, another line of research emerged with a different design goal. In the papers [19], [20] interesting cluster synchronization and cluster consensus problems have been considered by assuming the presence of multiple leaders.

The paper is organized as follows. After some preliminaries, the problem set-up is described in Section II. In Section III the characteristic polynomial of the overall controlled multi-agent system is derived (Section III-A), and the problem of freely allocating the system eigenvalues and hence dynamics is investigated (Section III-B) by resorting to the polynomial matrix approach, whose fundamental concepts are briefly recalled in Section I-B. Leaders-controlled distributed consensus is thoroughly studied in Section IV.

A. General notation

$\mathbb{Z}_+$ and $\mathbb{R}$ denote the set of nonnegative integers and the set of real numbers, respectively. We let $\mathbf{e}_i$ be the ith element of the canonical basis in $\mathbb{R}^n$ (n being clear from the context), with all entries equal to zero except for the ith one which has unitary value. We let $\mathbf{1}_k$ and $\mathbf{0}_k$ denote the k-dimensional real vectors whose entries are all 1 or all 0, respectively. Given a matrix A, with entries in any ring or field, the (i, j)th entry of A is denoted either by a_{ij} or by $[A]_{ij}$, and its *transpose* by $A^\top$. The *spectrum* of A, denoted by $\sigma(A)$, is the set of its eigenvalues and the *spectral radius* is the maximum modulus of the elements of $\sigma(A)$. For a *positive matrix* $A \in \mathbb{R}^{n\times n}$, i.e., a matrix whose entries are nonnegative real numbers, the spectral radius is always an eigenvalue. Given two positive matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$ the inequality $A_1 > A_2$ means that each entry of the matrix on the left hand-side is greater than or equal to the corresponding entry of the matrix on the right hand-side, and the inequality is strict in at least one case. A polynomial $a(z) \in \mathbb{R}[z]$ is said to be *Schur* if $a(\lambda) = 0$ for some $\lambda \in \mathbb{C}$ implies $|\lambda| < 1$. Given $\alpha_1, \dots, \alpha_n$, $\mathrm{diag}\{\alpha_1, \dots, \alpha_n\}$ denotes the $n \times n$ diagonal matrix with

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(i,i)-th entry equal to α_i.

B. Polynomial matrices and matrix fraction descriptions

We denote by $\mathbb{R}[z]$ and by $\mathbb{R}(z)$ the ring of polynomials and the field of rational functions, respectively, in the indeterminate z with coefficients in $\mathbb{R}$, and by $\mathbb{R}[z]^{p\times m}$ (by $\mathbb{R}(z)^{p\times m}$) the set of $p\times m$ matrices with entries in $\mathbb{R}[z]$ (in $\mathbb{R}(z)$). If $a(z)\in\mathbb{R}[z]$, the *degree* of $a(z)$ is denoted by $\deg[a(z)]$. Given $A(z)\in\mathbb{R}[z]^{p\times m}$, we define its ith *column index (or degree)*, with $i\in\{1,2...m\}$, as the maximum of the degrees of the polynomial entries of the ith column of $A(z)$, namely $\deg[\mathrm{col}_i(A(z))] := \max_{j=1,2,...,p}\deg[A(z)]_{ji}$, and we denote them by $\mu_1,\mu_2,..,\mu_m$. *Row indices (degrees)* are analogously defined and we denote them by $\nu_1,\nu_2,..,\nu_p$.

A matrix $U(z)\in\mathbb{R}[z]^{m\times m}$ is unimodular if its determinant is a zero degree polynomial, i.e. $\det U(z) = c \neq 0, c\in\mathbb{R}$. Unimodular matrices have polynomial inverses. $A(z)\in\mathbb{R}[z]^{p\times m}$ is said to be *right prime* if $\mathrm{rank}A(\lambda)=m$ for every $\lambda\in\mathbb{C}$. This is the case if and only if condition $A(z)=\bar{A}(z)\Delta(z)$, with $\bar{A}(z)\in\mathbb{R}[z]^{p\times m}$ and $\Delta(z)\in\mathbb{R}[z]^{m\times m}$, implies that $\Delta(z)\in\mathbb{R}[z]^{m\times m}$ is unimodular, namely the only *right divisors* of $A(z)$ are unimodular. Left prime matrices are similarly defined and characterized.

Given $A(z)\in\mathbb{R}[z]^{p\times m}$, with column indices $\mu_1,\mu_2,\ldots,\mu_m$, its *leading column coefficient matrix* $A_{hc}\in\mathbb{R}^{p\times m}$ is the matrix whose ith column consists of the coefficients of z^{μ_i} in the ith column of $A(z)$, and $A(z)$ is *column reduced* [21] if A_{hc} is of full column rank. Similarly, we can define the leading row coefficient matrix and the notion of row reduced matrix.

Given $A(z)\in\mathbb{R}[z]^{p\times m}$, $B(z)\in\mathbb{R}[z]^{q\times m}$ e $C(z)\in\mathbb{R}[z]^{\ell\times m}$, the equation

$$X(z)A(z)+Y(z)B(z)=C(z), \tag{1}$$

in the unknown matrices $X(z)\in\mathbb{R}[z]^{\ell\times p}$ and $Y(z)\in\mathbb{R}[z]^{\ell\times q}$, is called *(polynomial matrix) Diophantine equation*. A solution of (1) exists if and only if every right divisor $\Delta(z)$ of $\begin{bmatrix}A(z)\\B(z)\end{bmatrix}$ is a right divisor of $C(z)$; if $\begin{bmatrix}A(z)\\B(z)\end{bmatrix}$ is right prime, equation (1) has a solution for every $C(z)\in\mathbb{R}[z]^{\ell\times m}$.

Given $G(z)\in\mathbb{R}(z)^{p\times m}$, we say that a pair $(N_R(z),D_R(z))\in\mathbb{R}[z]^{p\times m}\times\mathbb{R}[z]^{m\times m}$ provides a *right matrix fraction description (rMFD)* of $G(z)$ if $\det D_R(z)\neq 0$ and $N_R(z)D_R^{-1}(z)=G(z)$. If $\begin{bmatrix}N_R(z)\\D_R(z)\end{bmatrix}$ is right prime, $N_R(z)D_R^{-1}(z)$ is a *right coprime matrix fraction description (rcMFD)* of $G(z)$. Left matrix fraction descriptions and left coprime matrix fraction descriptions (lMFD and lcMFD, respectively) are analogously defined.

A matrix $G(z)\in\mathbb{R}(z)^{p\times m}$ whose entries are all proper (strictly proper) is called *proper (strictly proper)*. If $G(z)\in\mathbb{R}(z)^{p\times m}$ is proper (strictly proper) then any of its rMFDs $N_R(z)D_R(z)^{-1}$ satisfies $\deg[\mathrm{col}_iN_R(z)]\leq\deg[\mathrm{col}_iD_R(z)]$ ($\deg[\mathrm{col}_iN_R(z)]<\deg[\mathrm{col}_iD_R(z)]$) for every $i=1,2...m$. The converse is not necessarily true. However, if $D_R(z)$ is column reduced then the previous inequalities on the column indices ensure properness (strict properness).

II. Problem setup

In this paper we assume to have a group of N interacting agents: to each ith agent, $i\in\{1,..,N\}$, we associate a local quantity, $x_i(t)$, that the agent uses to infer some global information by means of local interactions. In detail, each agent updates its own "state" based on a linear combination of the state values of its own *neighbors*, so that if we denote by $\mathbf{x}(t)$ the N-dimensional vector obtained by stacking the states of the N agents (at time t), it updates according to the following discrete-time linear state-space model:

$$\mathbf{x}(t+1)=(I_N-\kappa L)\mathbf{x}(t), \tag{2}$$

where $\kappa>0$ is a given real parameter known as *coupling strength* and L is the *Laplacian matrix* encoding the communication structure among agents. As well known [22], the Laplacian matrix $L\in\mathbb{R}^{N\times N}$ associated with a weighted and directed graph is a matrix with nonpositive off-diagonal entries, and diagonal entries determined according to the rule $[L]_{ii}=-\sum_{j\neq i}[L]_{ij}$. As a consequence, $A:=I_N-\kappa L$ is a matrix with positive off-diagonal entries and it satisfies $A\mathbf{1}_N=\mathbf{1}_N$. Systems evolving according to (2) have been extensively studied in the last years because they describe a wide number of applications in cooperative multi-agent robotics, wireless sensor networks, distributed decision making [1], [2], [4]. System (2) is a *consensus network* if for every choice of the initial conditions $x_i(0), i=1,2,\ldots,N$, there exists $\alpha\in\mathbb{R}$ such that

$$\lim_{t\to+\infty}x_i(t)=\alpha, \qquad \forall\ i\in\{1,\ldots,N\}, \tag{3}$$

or, equivalently, $\lim_{t\to+\infty}\mathbf{x}(t)=\alpha\mathbf{1}_N, \exists\alpha\in\mathbb{R}$. The constant α is called the *consensus value* [1] for system (2), corresponding to the given initial conditions. Under some well-studied conditions on graph topology and κ [1], that we will steadily assume in the sequel of the paper, A is an irreducible positive matrix with a simple and strictly dominant eigenvalue in 1, and system (2) is a consensus network. If so, the consensus value is equal to

$$\alpha=\frac{\mathbf{w}_A^\top\mathbf{x}(0)}{\mathbf{w}_A^\top\mathbf{1}_N}, \tag{4}$$

where $\mathbf{w}_A$ is a left eigenvector of A corresponding to the unitary eigenvalue. It is important to note that the final value on which the agents agree is a linear function of the initial conditions $x_i(0), i\in\{1,2,\ldots,N\}$, of the agents.

Assume now that, in the current set-up (that we assume fixed, and hence not the target of our control design problem), p of the N agents take the role of leaders, and each of them can influence the group dynamics by making use of a control signal u_i that is elaborated based on the past leaders' dynamics. The purpose of the leaders' action is to improve the performance of the group dynamics, in particular to accelerate consensus or to achieve finite-time consensus. If we assume, without loss of generality (w.l.o.g.), that the p leaders are the first p agents, the multi-agent system with consensus protocol (2) and p leaders exerting an additional control action becomes:

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + \begin{bmatrix} I_p \\ 0 \end{bmatrix} \mathbf{u}(t), \tag{5}$$

where $\mathbf{u} \in \mathbb{R}^p$ is the additional control action. Assume that the p leaders generate the control input $\mathbf{u}(t)$ by elaborating the leaders' states $x_i(t), i = 1, 2, \ldots, p$, according to the following discrete-time μ-dimensional state model, with state variable $\varepsilon(t)$:

$$\begin{aligned} \varepsilon(t+1) &= F\varepsilon(t) + \begin{bmatrix} G & 0 \end{bmatrix} \mathbf{x}(t), & (6) \\ \mathbf{u}(t) &= H\varepsilon(t) + \begin{bmatrix} J & 0 \end{bmatrix} \mathbf{x}(t), & (7) \end{aligned}$$

where $F \in \mathbb{R}^{\mu\times\mu}$, $G \in \mathbb{R}^{\mu\times p}$, $H \in \mathbb{R}^{p\times\mu}$ and $J \in \mathbb{R}^{p\times p}$. If we block-partition the vector $\mathbf{x}(t)$ and the matrix A as:

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{x}_L(t) \\ \mathbf{x}_F(t) \end{bmatrix}, \qquad A = \begin{bmatrix} A_{LL} & A_{LF} \\ A_{FL} & A_{FF} \end{bmatrix}, \tag{8}$$

where $\mathbf{x}_L(t) \in \mathbb{R}^p, \mathbf{x}_F(t) \in \mathbb{R}^{N-p}$, $A_{LL} \in \mathbb{R}^{p\times p}$ and $A_{FF} \in \mathbb{R}^{(N-p)\times(N-p)}$, upon introducing the augmented state vector

$$\boldsymbol{\chi}(t) := \begin{bmatrix} \mathbf{x}(t) \\ \varepsilon(t) \end{bmatrix} = \left[\begin{array}{c} \mathbf{x}_L(t) \\ \mathbf{x}_F(t) \\ \hline \varepsilon(t) \end{array}\right] \in \mathbb{R}^{N+\mu}, \tag{9}$$

system (5) under the leader control action (6)-(7) can be compactly described by:

$$\boldsymbol{\chi}(t+1) = \left[\begin{array}{cc|c} A_{LL}+J & A_{LF} & H \\ A_{FL} & A_{FF} & 0 \\ \hline G & 0 & F \end{array}\right] \left[\begin{array}{c} \mathbf{x}_L(t) \\ \mathbf{x}_F(t) \\ \hline \varepsilon(t) \end{array}\right] =: \mathcal{M}\boldsymbol{\chi}(t). \tag{10}$$

Remark 1. *Note that the nonzero entries of the matrix J correspond to direct links among the leaders. So, unless we impose $J = 0$ or that J has a nonzero pattern included in the nonzero pattern of the matrix A_{FF}, this protocol may introduce additional communication links among the leaders. The case when we constrain the matrix J to have a nonzero pattern included in the nonzero pattern of A_{FF} will be the object of future investigation.*

III. Shaping the Evolution of the Group: Leaders-Controlled Dynamics

As a first step of our analysis, we quantify the impact of the leaders' action on the overall group in terms of dynamic behavior. To this goal, we define the proper rational transfer matrices involved in the description of the original system (5) and of the control procotol (6)-(7), introduce some fundamental assumptions, and then represent these transfer matrices by means of either left or right MFDs (see Subection I-B).

Assumptions. *(1). $\Sigma_{LF} = (A_{FF}, A_{FL}, A_{LF}, A_{LL})$ is a minimal realization of its transfer matrix*

$$W_{LF}(z) := A_{LF}(zI_{N-p} - A_{FF})^{-1}A_{FL} + A_{LL}, \tag{11}$$

namely it is reachable and observable. We refer to $W_{LF}(z) \in \mathbb{R}(z)^{p\times p}$ as to the leaders-followers transfer matrix.
(2). $N_R(z)D_R(z)^{-1}$ is an rcMFD of $W_{LF}(z)$, and by the previous assumption (1) it entails no loss of generality assuming that $\det D_R(z) = \det(zI_{N-p} - A_{FF})$.
(3). We assume, without loss of generality, that $D_R(z)$ is column reduced with column indices $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$. Indeed, we can always reduce ourselves to this situation by means of a suitable unimodular matrix $V(z)$ (with $\det V(z) = 1$) acting on the columns of $D_R(z)$. Clearly, $[N_R(z)V(z)][D_R(z)V(z)]^{-1}$ is another rcMFD of $W_{LF}(z)$ (satisfying (2)). As a result, due to the fact that $N_R(z)D_R(z)^{-1}$ is a proper rational matrix, $\Phi(z) := \begin{bmatrix} D_R(z) \\ N_R(z) \end{bmatrix}$ is also column reduced with column indices $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$ and can be expressed as:

$$\Phi(z) = \begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix} \begin{bmatrix} z^{\mu_1} & & \\ & \ddots & \\ & & z^{\mu_p} \end{bmatrix} + \begin{bmatrix} D_{lc}(z) \\ N_{lc}(z) \end{bmatrix}, \tag{12}$$

where the degree of the ith column of $\begin{bmatrix} D_{lc}(z) \\ N_{lc}(z) \end{bmatrix}$ is strictly smaller than μ_i for every $i \in \{1, 2, \ldots, p\}$, and D_{hc} is a nonsingular square matrix.
(4). Let $W_{aux}(z) := H(zI_\mu - F)^{-1}G + J$ be the transfer matrix of the state-space model $\Sigma_{aux} = (F, G, H, J)$, describing the p leaders' control protocol. We let $X(z)^{-1}Y(z)$ denote a left MFD of $W_{aux}(z)$, and we assume w.l.o.g. that $\det X(z) = \det(zI_\mu - F)$.

Remark 2. *Among all the previous assumptions, the only restrictive one is assumption (1). It admits a quite immediate interpretation: leaders must be able to fully influence the dynamics of the followers, as well as to observe the followers' states, in order to be able to implement some feedback control action that freely modifies the overall dynamics of the multi-agent system. This hypothesis allows to simplify the subsequent analysis, but it is not strictly necessary, in the sense that a control action that improves the system performance could still be implemented but it would be somewhat constrained. Indeed, if (5) is a consensus network, then A is a positive and irreducible matrix. This ensures that*

$$A = \begin{bmatrix} A_{LL} & A_{LF} \\ A_{FL} & A_{FF} \end{bmatrix} > \begin{bmatrix} 0 & 0 \\ 0 & A_{FF} \end{bmatrix},$$

and the monotonicity properties of the spectral radius [23] guarantees that $1 = \rho(A) > \rho(A_{FF})$. So, even if Σ_{LF} is not a minimal realization of its transfer matrix, it is nonetheless always stabilizable and detectable, which means that the uncontrollable or unobservable dynamics always converge to zero. However, this would also mean that when trying to allocate the eigenvalues of the overall system dynamics or, in particular, to accelerate the consensus speed, the uncontrollable or unobservable dynamics could not be modified and hence would limit the final performance. So, if possible, the number p and the specific selection of the p agents should be chosen in a way to satisfy this assumption.

A. Characteristic polynomial

We now derive the characteristic polynomial of the multi-agent system under protocol (6)-(7). To this end, we recall a technical lemma.

Lemma 3. *[24] Given a square matrix* $\mathcal{M} = \begin{bmatrix} R & S \\ P & Q \end{bmatrix}$, *with* $R \in \mathbb{R}^{N\times N}$, $S \in \mathbb{R}^{N\times\mu}$, $P \in \mathbb{R}^{\mu\times N}$, $Q \in \mathbb{R}^{\mu\times\mu}$ *and* Q *nonsingular, its determinant can be expressed as*

$$\det \mathcal{M} = \det Q \cdot \det\left(R - SQ^{-1}P\right). \tag{13}$$

Proposition 4. *Consider the multi-agent system (5) with* p *leaders adopting the control protocol (6)-(7). Under the previous Assumptions (1)-(4), the characteristic polynomial* $p_{\mathcal{M}}(z) = \det(zI_{N+\mu} - \mathcal{M})$ *of the state matrix* $\mathcal{M}$ *of system (10) is equal to*

$$p_{\mathcal{M}}(z) = \det\left(\begin{bmatrix}(zX(z) - Y(z)) & -X(z)\end{bmatrix}\begin{bmatrix} D_R(z) \\ N_R(z)\end{bmatrix}\right). \tag{14}$$

Proof: By making use of Lemma 3 we obtain

$$\begin{aligned} p_{\mathcal{M}}(z) \;=\; & \det(zI_p - A_{LL} - J)\det\left(\begin{bmatrix} zI_{N-p} - A_{FF} & 0 \\ 0 & zI_\mu - F\end{bmatrix} - \right. \\ & \left. - \begin{bmatrix} A_{FL} \\ G \end{bmatrix}\left[zI_p - A_{LL} - J\right]^{-1}\begin{bmatrix} A_{LF} & H\end{bmatrix}\right), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} p_{\mathcal{M}}(z) \;=\; & \det(zI_p - A_{LL} - J)\det(zI_{N-p} - A_{FF})\cdot \\ \cdot\; & \det(zI_\mu - F)\det\left(I - M_1M_2\right) \end{aligned}$$

where

$$M_1 = \begin{bmatrix} zI - A_{FF} & 0 \\ 0 & zI - F\end{bmatrix}^{-1}\begin{bmatrix} A_{FL} \\ G\end{bmatrix}\left[zI - A_{LL} - J\right]^{-1}$$

and $M_2 = \begin{bmatrix} A_{LF} & H\end{bmatrix}$. Using the identity $\det(I - M_1M_2) = \det(I - M_2M_1)$, it is possible to further elaborate the above expression, thus obtaining

$$\begin{aligned} & \det\left(zI_{N-p} - A_{FF}\right)\det(zI_\mu - F)\det\left(zI_p - A_{LL} - J - \right. \\ & \left. - \begin{bmatrix} A_{LF} & H\end{bmatrix}\begin{bmatrix} zI_{N-p} - A_{FF} & 0 \\ 0 & zI_\mu - F\end{bmatrix}^{-1}\begin{bmatrix} A_{FL} \\ G\end{bmatrix}\right) \\ = \;& \det(zI_{N-p} - A_{FF})\det(zI_\mu - F) \\ \cdot\; & \det\left(zI_p - W_{LF}(z) - W_{aux}(z)\right). \end{aligned}$$

Replacing the transfer matrices with their MFDs, namely by making use of the identities $W_{LF}(z) = N_R(z)D_R(z)^{-1}$ and $W_{aux}(z) = X(z)^{-1}Y(z)$, and taking into account the identities $\det D_R(z) = \det(zI_{N-p} - A_{FF})$ and $\det X(z) = \det(zI_\mu - F)$, one easily gets

$$\begin{aligned} p_{\mathcal{M}}(z) = & \det D_R(z)\det X(z)\cdot \\ & \cdot\det\left(zI_p - N_R(z)D_R(z)^{-1} - X(z)^{-1}Y(z)\right) \end{aligned}$$

leading to the final expression

$$\begin{aligned} p_{\mathcal{M}}(z) &= \det\left(zX(z)D_R(z) - X(z)N_R(z) - Y(z)D_R(z)\right) \\ &= \det\left(\begin{bmatrix}(zX(z) - Y(z)) & -X(z)\end{bmatrix}\begin{bmatrix} D_R(z) \\ N_R(z)\end{bmatrix}\right). \end{aligned}$$

□

B. Eigenvalue allocation

In this section we investigate the problem of determining under what conditions, given a monic polynomial $\delta(z) \in \mathbb{R}[z]$, a pair of polynomial matrices $(X(z), Y(z))$ can be found such that:
i) $X(z)$ is nonsingular and $\det X(z)$ is monic;
ii) $X(z)^{-1}Y(z)$ is a proper rational transfer matrix;
iii) the following identity holds:

$$\det\left(\begin{bmatrix}(zX(z) - Y(z)) & -X(z)\end{bmatrix}\begin{bmatrix} D_R(z) \\ N_R(z)\end{bmatrix}\right) = \delta(z). \tag{15}$$

To solve this problem we first start by choosing a nonsingular square matrix $\Delta(z) \in \mathbb{R}[z]^{p\times p}$, such that $\det\Delta(z) = \delta(z)$ and consider the (polynomial matrix) Diophantine equation:

$$\Delta(z) = \begin{bmatrix}(zX(z) - Y(z)) & -X(z)\end{bmatrix}\begin{bmatrix} D_R(z) \\ N_R(z)\end{bmatrix}, \tag{16}$$

equivalently written as

$$\begin{aligned} \Delta(z) &= \begin{bmatrix} X(z) & Y(z)\end{bmatrix}\begin{bmatrix} zI_p & -I_p \\ -I_p & 0\end{bmatrix}\begin{bmatrix} D_R(z) \\ N_R(z)\end{bmatrix} \\ &= \begin{bmatrix} X(z) & Y(z)\end{bmatrix}\begin{bmatrix} zD_R(z) - N_R(z) \\ -D_R(z)\end{bmatrix}. \end{aligned} \tag{17}$$

Lemma 5. *Assume that Assumptions (1)-(3) hold, and hence* $\Phi(z) = \begin{bmatrix} D_R(z) \\ N_R(z)\end{bmatrix}$ *is column reduced with column indices* $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$, *and described as in (12) with* $\det D_{hc} \neq 0$. *Then*

$$\Omega(z) := \begin{bmatrix} zD_R(z) - N_R(z) \\ -D_R(z)\end{bmatrix}$$

is column reduced with column indices $\mu_1 + 1 \geq \mu_2 + 1 \geq \cdots \geq \mu_p + 1$ *and* $D_R(z)[zD_R(z) - N_R(z)]^{-1}$ *is a right coprime MFD of a strictly proper rational matrix.*

Proof: We first observe that the matrix $\Omega(z)$ is right prime, because it can be written as

$$\Omega(z) = \begin{bmatrix} zI_p & -I_p \\ -I_p & 0\end{bmatrix}\begin{bmatrix} D_R(z) \\ N_R(z)\end{bmatrix},$$

and hence it is the product of a unimodular matrix and of a right prime matrix. So, if we show that $\det[zD_R(z) - N_R(z)] \neq 0$, then $D_R(z)[zD_R(z) - N_R(z)]^{-1}$ is a right coprime MFD. From (12) it is easy to see that

$$\Omega(z) = \begin{bmatrix} D_{hc} \\ 0\end{bmatrix}\begin{bmatrix} z^{\mu_1+1} & & \\ & \ddots & \\ & & z^{\mu_p+1}\end{bmatrix} + \begin{bmatrix} zD_{lc}(z) - N_R(z) \\ -D_R(z)\end{bmatrix},$$

with

$$\deg\left[\mathrm{col}_i\begin{bmatrix} zD_{lc}(z) - N_R(z) \\ -D_R(z)\end{bmatrix}\right] < \mu_i + 1$$

for every $i \in \{1, 2, \ldots, p\}$. This shows that $zD_R(z) - N_R(z)$ is column reduced and hence nonsingular. Moreover, $\Omega(z)$ is column reduced with column indices $\mu_1 + 1 \geq \mu_2 + 1 \geq \cdots \geq \mu_p + 1$, and its leading column coefficient matrix is

$\begin{bmatrix} D_{hc} \\ 0 \end{bmatrix}$. This implies that $D_R(z)[zD_R(z) - N_R(z)]^{-1}$ is a strictly proper matrix. $\square$

We have the following result.

Proposition 6. *Assume that Assumptions (1)-(3) hold, and hence the polynomial matrix $\Phi(z)$ is column reduced with column indices $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$. Let $D_L(z)^{-1}N_L(z)$ be a left coprime MFD of $W_{LF}(z)$, and assume w.l.o.g. that $\begin{bmatrix} -N_L(z) & D_L(z) \end{bmatrix}$ is row reduced with row indices $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_p$. If $\Delta(z) = \mathrm{diag}\{\delta_1(z), \ldots, \delta_p(z)\} \in \mathbb{R}[z]^{p\times p}$ is a diagonal matrix and its diagonal entries are monic polynomials satisfying*

$$\deg[\delta_i(z)] \geq \mu_i + \nu_i, \qquad i = 1, 2, \ldots, p, \tag{18}$$

then equation (16) has a solution $(X(z), Y(z))$ such that $\det X(z)$ is a nonzero monic polynomial, and $X(z)^{-1}Y(z)$ is a proper rational matrix.

Proof: Follows from Lemma 1 in [25] applied to (17) and from Lemma 5 (since Lemma 1 in [25] requires that the process transfer matrix is strictly proper and, as previously proved, $D_R(z)[zD_R(z) - N_R(z)]^{-1}$ is strictly proper). $\square$

IV. Leaders-controlled distributed consensus

In this section, we start by introducing the definition of leaders-controlled distributed consensus, by this meaning that the p leaders lead the group, both by imposing the control protocol (6)-(7) and by making use of the local interactions among neighbouring agents, to achieve consensus on some value that depends on the initial conditions of all the agents.

Definition 1 (Leaders-controlled distributed consensus)**.** *Consider a multi-agent system with p leaders described as in (5). We say that the control (6)-(7) leads the system to* leaders-controlled distributed consensus*, under the leaders' action $\mathbf{u}(t)$, if there exists a vector $\mathbf{c} \in \mathbb{R}^N$ such that for every choice of the agents' initial state $\mathbf{x}(0)$ and assuming that at $t = 0$ the auxiliary variables $\boldsymbol{\varepsilon}(0)$ are zero, one has*

$$\lim_{t\to+\infty} \mathbf{x}(t) = \mathbf{1}_N(\mathbf{c}^\top \mathbf{x}(0)). \tag{19}$$

We now determine conditions guaranteeing that a given state-space model $\Sigma_{aux} = (F, G, H, J)$, implementing the leaders' control protocol (6)-(7) leads the multi-agent system to leaders-controlled distributed consensus. The proof is similar to those for the two single-leader control protocols explored in [14], and hence it is omitted.

Proposition 7. *Consider a multi-agent system with p leaders described as in (5), exerting the control protocol (6)-(7), for given matrices (F, G, H, J), and suppose that Assumptions (1)-(4) hold. If the following two conditions hold:*

i) *$\mathcal{M}$ has a simple and strictly dominant eigenvalue in 1;*
ii) *the eigenvector of $\mathcal{M}$ associated with 1 takes the form $\begin{bmatrix} \mathbf{1}_N^\top & \mathbf{v}^\top \end{bmatrix}^\top$, for some $\mathbf{v} \in \mathbb{R}^\mu$,*

then the control action (6)-(7) leads the system to leaders-controlled distributed consensus, under the leaders' action $\mathbf{u}(t)$. Moreover, the multi-agent system state vector converges to

$$\lim_{t\to+\infty} \mathbf{x}(t) = \alpha \mathbf{1}_N\,, \qquad \text{with} \qquad \alpha = \frac{\mathbf{w}^\top \mathbf{x}(0)}{\mathbf{w}^\top \mathbf{1}_N + \mathbf{w}_{ext}^\top \mathbf{v}}$$

and hence condition (19) holds for

$$\mathbf{c}^\top = \frac{\mathbf{w}^\top}{\mathbf{w}^\top \mathbf{1}_N + \mathbf{w}_{ext}^\top \mathbf{v}},$$

where $\begin{bmatrix} \mathbf{w}^\top & \mathbf{w}_{ext}^\top \end{bmatrix}^\top$, with $\mathbf{w} \in \mathbb{R}^N$ and $\mathbf{w}_{ext} \in \mathbb{R}^\mu$, is a left eigenvector of $\mathcal{M}$ corresponding to the eigenvalue 1.

The previous proposition has given conditions ensuring that a given quadruple of matrices (F, G, H, J) leads the multi-agent system to leaders-controlled distributed consensus. We want to explore now under what conditions on the original multi-agent system (5), such a quadruple of matrices can be found so that the overall system achieves leaders-controlled distributed consensus. To this end, we first explore equivalent characterizations of condition ii) of Proposition 7, assuming that all the matrices involved in the overall system description are known. The proof is omitted due to space constraints.

Lemma 8. *Suppose that Assumptions (1)-(4) hold and condition (16) holds for some given $\Delta(z) \in \mathbb{R}[z]^{p\times p}$. If $1 \notin \sigma(F)$, the following facts are equivalent:*

1) *There exists a vector $\mathbf{v} \in \mathbb{R}^\mu$ such that $\hat{\mathbf{v}} := \begin{bmatrix} \mathbf{1}_N^\top & \mathbf{v}^\top \end{bmatrix}^\top$ is an eigenvector of $\mathcal{M}$ corresponding to the unitary eigenvalue.*
2) [1] $\Delta(1)D_R^{-1}(1)\mathbf{1}_p = 0$.

As a result of the analysis of Section III-B and of the previous Lemma 8, we can make use of Proposition 7 to obtain a sufficient condition for a given multi-agent system (5) to admit a quadruple of matrices (F, G, H, J) such that the overall controlled system achieves leaders-controlled distributed consensus.

Corollary 9. *Consider a multi-agent system with p leaders described as in (5), and suppose that Assumptions (1)-(3) hold. If there exists a polynomial pair $(X(z), Y(z))$, with $\det X(z)$ a nonzero monic polynomial, such that $X^{-1}(z)Y(z) \in \mathbb{R}(z)^{p\times p}$ is an lMFD of a proper rational matrix, and*

i) *the determinant of $\Delta(z) := \begin{bmatrix} X(z) & Y(z) \end{bmatrix} \begin{bmatrix} zD_R(z) - N_R(z) \\ -D_R(z) \end{bmatrix}$ has a simple and strictly dominant zero in 1;*
ii) *$\Delta(1)D_R^{-1}(1)\mathbf{1}_p = 0$,*

then for any realization $\Sigma = (F, G, H, J)$ of $X^{-1}(z)Y(z)$ with $\det(zI - F) = \det X(z)$, the control (6)-(7) leads the system to leaders-controlled distributed consensus, under the leaders' action $\mathbf{u}(t)$.

[1]By Assumption (2), $\det D_R(z) = \det(zI_{N-p} - A_{FF})$. On the other hand, see Remark 2, A_{FF} is Schur, and hence $D_R(1)$ is nonsingular.

Proof: Follows from Proposition 7, by making use of Proposition 4 (see also (16)) for point i) and of Lemma 8 for point ii). $\square$

We are now in a position to derive our main result: it shows that under Assumptions (1)-(3) it is always possible to achieve arbitrary convergence speed for the overall multi-agent system, by making use of the control (6)-(7).

Theorem 10. *Consider a multi-agent system with p leaders described as in (5) and satisfying Assumptions (1)-(3). Then there exist $\mu \in \mathbb{Z}, \mu \geq 1$, and a μ-dimensional state-space model $\Sigma_{aux} = (F, G, H, J)$ such that the control (6)-(7) leads the system to leaders-controlled distributed consensus with any desired convergence rate, under the leaders' action $\mathbf{u}(t)$.*

Proof: We recall that, as a result of Assumptions (1)-(3), $\begin{bmatrix} D_R(z) \\ N_R(z) \end{bmatrix}$ is column reduced with column indices $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$, and $\det D_R(1) \neq 0$. We assume w.l.o.g. that $D_L(z)^{-1} N_L(z)$ is an lcMFD of $W_{LF}(z)$ and that $\begin{bmatrix} -N_L(z) & D_L(z) \end{bmatrix}$ is row reduced with row indices $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_p$. We can assume without loss of generality (details are omitted due to space constraints) that there exists an index $i \in \{1, \dots, p\}$ such that $D_R(1)\mathbf{e}_i = \mathbf{1}_p$. Indeed, we can always reduce ourselves to this case by simply multiplying $D_R(z)$ (and hence $N_R(z)$) on the right by a suitable nonsingular constant matrix U that does not affect any of the properties of $D_R(z)$ and of the pair $(N_R(z), D_R(z))$ (and hence preserves Assumptions (1)-(3)). Consider the diagonal matrix $\Delta(z) = \mathrm{diag}\{\delta_1(z), \dots, \delta_p(z)\} \in \mathbb{R}[z]^{p \times p}$ whose diagonal entries are chosen as follows:

- $\delta_i(z) = (z-1)\tilde{\delta}_i(z)$, where $\tilde{\delta}_i(z)$ is any monic Schur polynomial of degree $\mu_i + \nu_i - 1$.
- For every $j \neq i$, $\delta_j(z)$ is an arbitrary monic Schur polynomial of degree $\mu_j + \nu_j$.

This ensures that

1) $\det \Delta(z) = \prod_{j=1}^{p} \delta_j(z) = (z-1)p(z)$, for some Schur polynomial $p(z) \in \mathbb{R}[z]$;
2) $\deg \delta_j(z) = \mu_j + \nu_j, \ j = 1, 2, \dots, p$.

By Proposition 6, we can claim that for this choice of $\Delta(z)$ equation (16) has a solution $(X(z), Y(z))$ such that $\det X(z)$ is a nonzero monic polynomial and $X(z)^{-1} Y(z)$ is a proper rational matrix. Moreover, we can always ensure (the details of the proof are omitted due to space constraints) that $X(1)$ is nonsingular. On the other hand, $\Delta(1)$ is a diagonal matrix whose diagonal entries are all nonzero except for the ith one, and $D_R(1)^{-1}\mathbf{1}_p = \mathbf{e}_i$. Therefore $\Delta(1) D_R(1)^{-1} \mathbf{1}_p = \Delta(1)\mathbf{e}_i = 0$. So, any realization $\Sigma_{aux} = (F, G, H, J)$ of the proper rational matrix $X(z)^{-1} Y(z)$, with $\det(zI - F) = \det X(z)$, allows to achieve consensus. $\square$

Remark 11. *It is worthwhile remarking that the Schur polynomials appearing in the diagonal entries of $\Delta(z)$ have been arbitrarily chosen, and hence one can guarantee any desired convergence speed. The price to pay is the size of the overall system that clearly coincides with $\deg \det \Delta(z)$ and hence can be quite high.*

REFERENCES

[1] R. Olfati-Saber, A. J. Fax, and R. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95 (1), pp. 215–233, 2007.
[2] W. Ren, R. Beard, and E. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Sys. Magazine*, vol. 27 (2), pp. 71–82, 2007.
[3] Y. Chen, J. Lu, X. Yu, and D. Hill, "Multi-agent systems with dynamical topologies: Consensus and applications," *IEEE Circuits Sys. Magazine*, vol. 13 (3), pp. 21–34, 2013.
[4] J. Tsitsiklis, "Problems in decentralized decision making and computation," Ph.D. dissertation, Department of EECS, MIT, 1984.
[5] W. Ren, R. W. Beard, and E. M. Atkins, "A survey of consensus problems in multi-agent coordination," in *Proc. 2005 American Control Conference*, Portland, OR, 2005, pp. 1859–1864.
[6] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Randomized gossip algorithms," *IEEE/ACM Trans. Networking*, vol. 14 (SI), pp. 2508–2530, 2006.
[7] S.-Y. Tu and A. H. Sayed, "Diffusion strategies outperform consensus strategies for distributed estimation over adaptive networks," *IEEE Trans. Signal Processing*, vol. 60 (12), pp. 6217–6234, 2012.
[8] D. Jakovetic, J. Xavier, and J. M. Moura, "Weight optimization for consensus algorithms with correlated switching topology," *IEEE Trans. Signal Processing*, vol. 58, no. 7, pp. 3788–3801, 2010.
[9] L. Xiao and S. Boyd, "Fast linear iterations for distributed averaging," *Systems & Control Letters*, vol. 53, no. 1, pp. 65–78, 2004.
[10] E. Ghadimi, M. Johansson, and I. Shames, "Accelerated gradient methods for networked optimization," in *Proc. 2011 American Control Conference*, San Francisco, CA, 2011, pp. 1668 – 1673.
[11] B. Johansson and M. Johansson, "Faster linear iterations for distributed averaging," in *Proc. 17th IFAC World Congr*, Prague, Czech Republic, 2008, pp. 2861–2866.
[12] H.-T. Zhang, M. Z. Chen, and G.-B. Stan, "Fast consensus via predictive pinning control," *IEEE Trans. Circuits and Systems I: Regular Papers*, vol. 58 (9), pp. 2247–2258, 2011.
[13] A. Sarlette, "Adding a single state memory optimally accelerates symmetric linear maps," *IEEE Trans. Aut. Contr.*, vol. 61, pp. 3533–3538, 2016.
[14] G. Parlangeli and M. Valcher, "Leader-controlled protocols to accelerate convergence in consensus networks," *submitted*, 2017.
[15] G. Parlangeli, "Enhancing convergence toward consensus in leader-follower networks," in *Proc. 20th IFAC World Congr.*, Toulouse, France, 2017, pp. 627–632.
[16] Z. Li, G. Wen, Z. Duan, and W. Ren, "Designing fully distributed consensus protocols for linear multi-agent systems with directed graphs," *IEEE Trans. Aut. Contr.*, vol. 60 (4), pp. 1152–1157, 2015.
[17] G. Shi and Y. Hong, "Global target aggregation and state agreement of nonlinear multi-agent systems with switching topologies," *Automatica*, vol. 45 (5), pp. 1165–1175, 2009.
[18] Y. Lou and Y. Hong, "Target containment control of multi-agent systems with random switching interconnection topologies," *Automatica*, vol. 48 (5), pp. 879–885, 2012.
[19] J. Qin and C. Yu, "Cluster consensus control of generic linear multi-agent systems under directed topology with acyclic partition," *Automatica*, vol. 49 (9), pp. 2898–2905, 2013.
[20] J. Qin, C. Yu, and B. D. Anderson, "On leaderless and leader-following consensus for interacting clusters of second-order multi-agent systems," *Automatica*, vol. 74, pp. 214–221, 2016.
[21] T. Kailath, *Linear Systems*. Prentice Hall, Inc., 1980.
[22] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslovak Math. J.*, vol. 23, pp. 298–305, 1973.
[23] N. Son and D. Hinrichsen, "Robust stability of positive continuous time systems," *Numer. Funct. Anal. Optimiz.*, vol. 17 (5 & 6), pp. 649–659, 1996.
[24] R. Horn and C. Johnson, *Matrix analysis*. Cambridge University Press, 2012.
[25] V. Kucera and P. Zagalak, "Proper solutions of polynomial equations," *IFAC Proceedings Volumes*, vol. 32, no. 2, pp. 1927 – 1932, 1999, 14th IFAC World Congress 1999, Beijing, China, 5-9 July.