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Approximated solutions to the nonlinear $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control approaches formulated in the Sobolev space*

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***Abstract*— Two important paradigms in control theory are the classical nonlinear $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control approaches. Their efficiency have already been demonstrated in several applications and the background theory is well developed. Despite their many advantages, they suffer from deficiencies such as minimum settling-time and minimum overshoot. An interesting approach to solve these lacks is the formulation of both controllers in the Sobolev space. Thanks to the nature of the $\mathcal{W}_{1,2}$ − norm, the cost variable and its time derivative are taken into account in the cost functional, leading to improved transient and steady-state performance. Nevertheless, the HJB and HJBI equations that arises from the problem formulation in the Sobolev space are very hard to solve analytically. This work proposes an approach to approximate their solutions by adapting the classical Successive Galerkin Approximation Algorithms (SGAA). Numerical experiments are used to corroborate the proposed approach capacity to deal with underactuated systems when controlling the two-wheeled self-balanced vehicle.**

I. INTRODUCTION

Two of the most important paradigms in control theory are the classical $\mathcal{H}_2$ [1] and $\mathcal{H}_\infty$ [2] controllers. They have already been used for controlling several systems, and their efficiency was verified in many practical experiments. Despite their many advantages, such controllers suffers from deficiencies such as minimum settling-time and minimum overshoot. Many approaches have been proposed to deal with these lacks, as for example: weighting functions [3], loop-shaping [4], linear programming and Lawson's algorithm [5], and modified cost functions [6]. An interesting approach for controlling the transient and steady-state closed-loop responses is the formulation of both controllers in the Sobolev space. Thanks to the nature of $\mathcal{W}_{1,2}$-norm of functions in the Sobolev space[1], the cost variable and its time derivative are taken into account in the cost functional, which results in controllers providing better transient and steady-state performance than the ones corresponding to the Lebesgue $\mathcal{L}_2$ space.

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[1]The Sobolev $\mathcal{W}_{m,p}$ − norm of a signal is defined as $||\boldsymbol{z}||_{\mathcal{W}_{m,p}} = \left(\int_{t_0}^{\infty}(||\boldsymbol{z}||^p + ||\frac{d\boldsymbol{z}}{dt}||^p + ... + ||\frac{d^m\boldsymbol{z}}{dt^m}||^p)dt\right)^{\frac{1}{p}}$.

In [7] the nonlinear $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control approaches are formulated in the Sobolev space via dynamic-programming. Both problems result in solving nonlinear first-order partial differential equations (PDEs), the well-known Hamilton-Jacobi-Bellman (HJB) and Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations, respectively. For full-state feedback linear systems, the linear $\mathcal{W}_{1,2}$-$\mathcal{H}_2$ and $\mathcal{W}_{1,2}$-$\mathcal{H}_\infty$ control problems are solved using a matrix Riccati equation [8], for which efficient numerical techniques can be used to achieve the solution. Nevertheless, for nonlinear systems, the resulting HJB and HJBI equations are hard to solve analytically. In particular, according to [7], the HJBI is considered to be "horrendous and impossible to compute the solution". In the present work, solutions for both nonlinear $\mathcal{W}_{1,2}$-$\mathcal{H}_2$ and $\mathcal{W}_{1,2}$-$\mathcal{H}_\infty$ control problems, formulated in the Sobolev space, are proposed using the Galerkin method [9] by extending the classical Successive Galerkin Approximation Algorithms (SGAA).

The SGAA was proposed in [10] with the purpose to approximate the solution of the HJB equation. Thereafter, it was extended to solve the HJBI equation in [11]. These works solved both HJB and HJBI equations numerically, which resulted from the nonlinear $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems in the Lebesgue $\mathcal{L}_2$ space.

When dealing with the nonlinear $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers of mechanical systems, usually, the control design explores the mechanical properties of the system to obtain a solution of the HJB and HJBI equations [12], [13], [14]. In particular, the control design for underactuated mechanical systems often requires the use of the pseudo-inverse operation to obtain the applied control signal. On one hand, the control design considers only the dynamics of *controlled* DOF and assumes the *remaining* ones have stable zero dynamics or stabilized through an additional control strategy [15], [16], [17]. On the other hand, the controller is designed taking into account underactuated mechanical systems with input coupling [18]. Moreover, as stated in [19], the standard formulation of the nonlinear $\mathcal{H}_\infty$ control and, consequently, of the nonlinear $\mathcal{H}_2$ control for Euler-Lagrange mechanical systems presents a limitation in the way to weigh the cost variable. For its appropriate formulation, the weighting matrices must be considered like positive real scalars multiplied by the identity matrix, limiting the adjustment of the control law for systems with multiple DOF with different dynamics. Therefore, in this work, the proposed approach is designed in order to deal with underactuated systems, although it can also be directly applied to control fully actuated systems. Besides, it does not require a fixed weighting structure of the cost variable.

Aiming to explore the proposed approach capacity to deal with underactuated systems, the nonlinear $\mathcal{W}_{1,2}$-$\mathcal{H}_2$ and $\mathcal{W}_{1,2}$-$\mathcal{H}_\infty$ controllers are corroborated by numerical experiments with a two-wheeled self-balanced vehicle. These type of vehicles can be found in different variations as the pendulum on an autonomous two-wheeled vehicle [20] and the Segway [21]. Since it is a benchmark in the control area, there are many works in literature proposing linear and nonlinear controllers for it, as for example: LQR [22], PID [23], forwarding [20], nonlinear $\mathcal{H}_\infty$ [18], among others.

The following sections are structured as: Section II presents the formulation of nonlinear $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers in the Sobolev space; Section III proposes the extended sucessive Galerkin approximation algorithms; Section IV presents a brief background about the Galerkin's method and how it is applied to solve the nonlinear $\mathcal{W}_{1,2}$-$\mathcal{H}_2$ and $\mathcal{W}_{1,2}$-$\mathcal{H}_\infty$ control problems; Section V designs both controllers for a two-wheeled self-balanced vehicle and presents numerical results; Section VI summarizes the contributions and future works.

II. Nonlinear $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control formulated in the Sobolev space

In this section the nonlinear $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers are formulated in the Sobolev space. The control laws are designed considering the following system

$$P: \begin{cases} \dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} + k(\boldsymbol{x})\boldsymbol{w}; \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ \boldsymbol{z} = h(\boldsymbol{x}), \end{cases} \tag{1}$$

such that $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector, $\boldsymbol{u} \in \mathbb{R}^m$ is the input vector, $\boldsymbol{w} \in \mathbb{R}^p$ is the disturbance vector, and $\boldsymbol{z} \in \mathbb{R}^q$ is the cost variable. It is assumed $f(\boldsymbol{0}) = \boldsymbol{0}$, $h(\boldsymbol{0}) = \boldsymbol{0}$, and all states are measurable.

A. The nonlinear $\mathcal{W}_{1,2}$-$\mathcal{H}_2$ control

Considering no disturbances affecting the system (1) ($\boldsymbol{w} = \boldsymbol{0}$), the nonlinear $\mathcal{H}_2$ controller, formulated in the Sobolev space, is designed in order to obtain the control law $\boldsymbol{u} \in \mathcal{U}$ that minimizes the cost functional $J_2 = \frac{1}{2}||\boldsymbol{z}||^2_{W_{1,2}}$. Therefore, the optimization problem is stated as

$$V_2 = \min_{u \in \mathcal{U}} J_2 = \min_{u \in \mathcal{U}} \frac{1}{2} \int_{t_0}^{\infty} \left(||\boldsymbol{z}||^2 + ||\dot{\boldsymbol{z}}||^2\right) dt,$$

in which $||\cdot||$ denotes the Euclidean norm. In order to solve this problem, it is formulated via dynamic-programming, from which the associated Hamiltonian is given by

$$\mathbb{H}_2 = \left(\frac{\partial V_2}{\partial \boldsymbol{x}}\right)' [f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}] + \frac{1}{2}\left(||\boldsymbol{z}||^2 + ||\dot{\boldsymbol{z}}||^2\right), \tag{2}$$

with boundary condition $V(\boldsymbol{0}) = 0$. Equation (2) in the expanded form yields

$$\begin{aligned} \mathbb{H}_2 = \left(\frac{\partial V_2}{\partial \boldsymbol{x}}\right)' [f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}] + \frac{1}{2}h'(\boldsymbol{x})h(\boldsymbol{x}) + \\ \frac{1}{2}f'(\boldsymbol{x})\nabla h'(\boldsymbol{x})\nabla h(\boldsymbol{x})f(\boldsymbol{x}) + \boldsymbol{u}'g'(\boldsymbol{x})\nabla h'(\boldsymbol{x})\nabla h(\boldsymbol{x})f(\boldsymbol{x}) \\ + \frac{1}{2}\boldsymbol{u}'g(\boldsymbol{x})\nabla h'(\boldsymbol{x})\nabla h(\boldsymbol{x})g(\boldsymbol{x})\boldsymbol{u}. \end{aligned} \tag{3}$$

The optimal control law $\boldsymbol{u}^*$ is obtained by minimizing the Hamiltonian (3) with respect to $\boldsymbol{u}$ as follows[2]

$$\begin{aligned} \frac{\partial \mathbb{H}_2}{\partial \boldsymbol{u}} = g'(\boldsymbol{x})\frac{\partial V_2}{\partial \boldsymbol{x}} + g'(\boldsymbol{x})\nabla h'(\boldsymbol{x})\nabla h(\boldsymbol{x})f(\boldsymbol{x}) + \\ g(\boldsymbol{x})'\nabla h'(\boldsymbol{x})\nabla h(\boldsymbol{x})g(\boldsymbol{x})\boldsymbol{u}^* = 0, \end{aligned} \tag{4}$$

which leads to

$$\boldsymbol{u}^* = -\boldsymbol{R}^{-1}\left(g'(\boldsymbol{x})\frac{\partial V_2}{\partial \boldsymbol{x}} + g'(\boldsymbol{x})\boldsymbol{B}f(\boldsymbol{x})\right), \tag{5}$$

where $\boldsymbol{R} \triangleq g'(\boldsymbol{x})\boldsymbol{B}g(\boldsymbol{x})$ with $\boldsymbol{B} \triangleq \nabla h'(\boldsymbol{x})\nabla h(\boldsymbol{x}) > 0$, and $g(\boldsymbol{x})$ has full column rank. The HJB equation associated to the problem is obtained by substituting the optimal control law (5) in the Hamiltonian (3).

Note that, different from the classical formulation in Lebesgue $\mathcal{L}_2$ space, in the nonlinear $\mathcal{W}_{1,2}$-$\mathcal{H}_2$ control approach the transient and steady-state performance are reached by the presence of the time derivative of the cost variable in the cost functional, being this variable independent of the control inputs.

B. The nonlinear $\mathcal{W}_{1,2}$-$\mathcal{H}_\infty$ control

Consider again system (1), the nonlinear $\mathcal{H}_\infty$ controller, formulated in the Sobolev space, is designed in order to achieve the control law $\boldsymbol{u} \in \mathcal{U}$ that minimizes the cost functional $J_\infty = \frac{1}{2}||\boldsymbol{z}||^2_{W_{1,2}} - \frac{1}{2}\gamma^2||\boldsymbol{w}||^2$, for the worst case of the disturbances $\boldsymbol{w} \in \mathcal{W}$. Therefore, the optimization problem is stated as

$$V_\infty = \min_{\boldsymbol{u} \in \mathcal{U}} \max_{\boldsymbol{w} \in \mathcal{W}} \int_0^\infty \frac{1}{2}\left(||\boldsymbol{z}||^2 + ||\dot{\boldsymbol{z}}||^2 - \gamma^2||\boldsymbol{w}||^2\right) dt,$$

The control design is derived by solving a Min-Max optimization problem, which can be formulated via Game Theory. The associated Hamiltonian is obtained as

$$\begin{aligned} \mathbb{H}_\infty = \left(\frac{\partial V_\infty}{\partial \boldsymbol{x}}\right)' [f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} + k(\boldsymbol{x})\boldsymbol{w}] \\ + \frac{1}{2}\left(||\boldsymbol{z}||^2 + ||\dot{\boldsymbol{z}}||^2 - \gamma^2||\boldsymbol{w}||^2\right), \end{aligned} \tag{6}$$

which is given in its expanded form by

$$\begin{aligned} \mathbb{H}_\infty = &\left(\frac{\partial V_\infty}{\partial \boldsymbol{x}}\right)' [f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} + k(\boldsymbol{x})\boldsymbol{w}] \\ &+ \frac{1}{2}h'h + \frac{1}{2}f'(\boldsymbol{x})\nabla h'\nabla h f(\boldsymbol{x}) - \gamma^2\boldsymbol{w}'\boldsymbol{w} \\ &+ \frac{1}{2}f'(\boldsymbol{x})\nabla h'\nabla h g(\boldsymbol{x})\boldsymbol{u} + \frac{1}{2}f'(\boldsymbol{x})\nabla h'\nabla h k(\boldsymbol{x})\boldsymbol{w} \\ &+ \frac{1}{2}\boldsymbol{u}'g'(\boldsymbol{x})\nabla h'\nabla h g(\boldsymbol{x})\boldsymbol{u} + \frac{1}{2}\boldsymbol{u}'g'(\boldsymbol{x})\nabla h'\nabla h f(\boldsymbol{x}) \\ &+ \boldsymbol{u}'g'(\boldsymbol{x})\nabla h'\nabla h k(\boldsymbol{x})\boldsymbol{w} + \frac{1}{2}\boldsymbol{w}'k'(\boldsymbol{x})\nabla h'\nabla h f(\boldsymbol{x}) \\ &+ \frac{1}{2}\boldsymbol{w}'k'(\boldsymbol{x})\nabla h'\nabla h g(\boldsymbol{x})\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}'k'(\boldsymbol{x})\nabla h'\nabla h k(\boldsymbol{x})\boldsymbol{w}. \end{aligned} \tag{7}$$

In order to obtain the worst case of the disturbance, $\boldsymbol{w}^*$, and the optimal control law, $\boldsymbol{u}^*$, the partial derivatives of (7)

[2]For the sake of simplicity, throughout the text some function dependencies are omitted.

with respect to these variables are computed as follows

$$\frac{\partial \mathbb{H}_\infty}{\partial \boldsymbol{u}} = g'(\boldsymbol{x})\frac{\partial V_\infty}{\partial \boldsymbol{x}} + g'(\boldsymbol{x})\nabla h'\nabla h f(\boldsymbol{x}) \tag{8}$$
$$+g'(\boldsymbol{x})\nabla h'\nabla h g(\boldsymbol{x})\boldsymbol{u}^* + g'(\boldsymbol{x})\nabla h'\nabla h k(\boldsymbol{x})\boldsymbol{w}^* = 0,$$

$$\frac{\partial \mathbb{H}_\infty}{\partial \boldsymbol{w}} = k'(\boldsymbol{x})\frac{\partial V_\infty}{\partial \boldsymbol{x}} + k'(\boldsymbol{x})\nabla h'\nabla h f(\boldsymbol{x}) \tag{9}$$
$$+k'(\boldsymbol{x})\nabla h'\nabla h g(\boldsymbol{x})\boldsymbol{u}^* + k'(\boldsymbol{x})\nabla h'\nabla h k(\boldsymbol{x})\boldsymbol{w}^* - \gamma^2\boldsymbol{w}^* = 0.$$

Therefore, by manipulating (8) and (9) leads to

$$\boldsymbol{w}^* = \left(\gamma^2\boldsymbol{I} - k'\boldsymbol{B}k + k'\boldsymbol{B}g\boldsymbol{R}^{-1}g'\boldsymbol{B}k\right)^{-1} \tag{10}$$
$$\times\left[k'\frac{\partial V_\infty}{\partial \boldsymbol{x}} + k'\boldsymbol{B}f - k'\boldsymbol{B}g\boldsymbol{R}^{-1}\left(g'\frac{\partial V_\infty}{\partial \boldsymbol{x}} + g'\boldsymbol{B}f\right)\right],$$

$$\boldsymbol{u}^* = -\boldsymbol{R}^{-1}\left(g'\frac{\partial V_\infty}{\partial \boldsymbol{x}} + g'(\boldsymbol{x})\boldsymbol{B}f(\boldsymbol{x}) + g'(\boldsymbol{x})\boldsymbol{B}k(\boldsymbol{x})\boldsymbol{w}^*\right), \tag{11}$$

in which $(\boldsymbol{u}^*, \boldsymbol{w}^*)$ is the saddle-point solution of the problem.

In order to obtain the HJBI equation associated to the problem, it is necessary to replace (10) and (11) in (7), which leads to a complex partial differential equation, which is hard to solve analytically. In [7] the resulting HJBI is presented, which is assumed to be intractable, being proposed an alternative approach making use of the backstepping technique in order to deal with the problem. From our best knowledge there are no works in literature proposing analytical solutions for this equation. Therefore, in this work approximated solutions to the HJB and HJBI equations obtained in the Sobolev space are proposed in the next section by adapting the classical Sucessive Galerkin Approximation Algorithm from [24].

III. Successive Galerkin Approximation Algorithm

The resulting HJB and HJBI equations obtained from the nonlinear $\mathcal{W}_{1,2}$-$\mathcal{H}_2$ and $\mathcal{W}_{1,2}$-$\mathcal{H}_\infty$ control problems, respectively, are in a quadratic form, which is not suitable to apply directly the Galerkin's method. By using this method to solve the problem leads to multiple solutions, one of which generates an optimal control law. In order to ensure that the obtained solution corresponds to a stabilizing control law, the SGAA is applied. This algorithm decreases the problem's complexity to a non-quadratic form leading to a single solution.

Therefore, in order to solve the nonlinear $\mathcal{W}_{1,2}$-$\mathcal{H}_2$ approach, the proposed SGAA is described in Algorithm 1. Although the number of iterations presented in Algorithm 1 goes from 1 to ∞, the stopping criterion $V^{(i)} = V^{(i+1)}$ is used when seeking the optimal solution of the HJB equation. In particular, $\boldsymbol{u}^{(i)}$ will ensure stability of the system (1) on the same region of the state space as $\boldsymbol{u}^{(0)}$ does, even though in our tests the final optimal control law have presented an enlarged domain of attraction. In addition, as stated by [25], it is not possible to find an admissible control that can stabilize an initial condition that is unstable. The convergence proof of the proposed SGAA follows the same procedure as in [11].

Algorithm 1 SGAA to nonlinear $\mathcal{W}_{1,2}$-$\mathcal{H}_2$ approach.

1: Let $\boldsymbol{u}^{(0)}$ be any initial stabilizing control law for system (1) with $\boldsymbol{w} = \boldsymbol{0}$ and stability region Ω.
2: **for** $i = 0$ to ∞ **do**
3: Solve for $V^{(i)}$ from:
$$\left(\frac{\partial V_2^{(i)}}{\partial \boldsymbol{x}}\right)'\left[f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}^{(i)}\right] + \frac{1}{2}\left(||\boldsymbol{z}||^2 + ||\dot{\boldsymbol{z}}(\boldsymbol{u}^{(i)})||^2\right)$$
4: Update the Control:
$$\boldsymbol{u}^{i+1} = -\boldsymbol{R}^{-1}\left(g'(\boldsymbol{x})\frac{\partial V_2^{(i)}}{\partial \boldsymbol{x}} + g'(\boldsymbol{x})\boldsymbol{B}f(\boldsymbol{x})\right)$$
5: **end for**

The proposed SGAA procedure for the nonlinear $\mathcal{W}_{1,2}$-$\mathcal{H}_\infty$ control problem is described in Algorithm 2. As in Algorithm 1, the number of iterations goes from 1 to ∞, but the following stopping criteria is also used, $V^{(i,j)} = V^{(i,j+1)}$, leading to obtain $V^{(i,\infty)} = V^{(i+1,\infty)}$. The $\mathcal{H}_\infty$ index γ must be selected such that the problem is feasible, since if it is not true the algorithm does not converge.

In order to use the proposed algorithms and approximate the solution $V(\boldsymbol{x})$ of Hamiltonians $\mathbb{H}_2$ and $\mathbb{H}_\infty$, the Galerkin's method is applied to solve numerically the respective PDEs. In the next section, the general formulation of the Galerkin's method is briefly presented, followed by its design to solve the nonlinear $\mathcal{W}_{1,2}$-$\mathcal{H}_2$ and $\mathcal{W}_{1,2}$-$\mathcal{H}_\infty$ control problems.

Algorithm 2 SGAA to nonlinear $\mathcal{W}_{1,2}$-$\mathcal{H}_\infty$ approach.

1: Let $\boldsymbol{u}^{(0)}$ be any initial stabilizing control law for the system (1) with $\boldsymbol{w} = \boldsymbol{0}$ and stability region Ω.
2: Set $\boldsymbol{w}^{(0,0)} = \boldsymbol{0}$
3: **for** $i = 0$ to ∞ **do**
4: **for** $j = 0$ to ∞ **do**
5: Solve for $V^{(i,j)}$ from:
$$\left(\frac{\partial V_\infty^{(i,j)}}{\partial \boldsymbol{x}}\right)'\left[f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}^{(i)} + k(\boldsymbol{x})\boldsymbol{w}^{(i,j)}\right]$$
$$+\frac{1}{2}\left(||\boldsymbol{z}||^2 + ||\dot{\boldsymbol{z}}(u^{(i)}, \boldsymbol{w}^{(i,j)})||^2 - \gamma^2||\boldsymbol{w}^{(i,j)}||^2\right) = 0$$
6: Update the Disturbance:
$$\boldsymbol{w}^{(i,j+1)} = \left(\gamma^2\boldsymbol{I} - k'\boldsymbol{B}k + k'\boldsymbol{B}g\boldsymbol{R}^{-1}g'\boldsymbol{B}k\right)^{-1}$$
$$\left[k'\frac{\partial V_\infty^{(i,j)}}{\partial \boldsymbol{x}} + k'\boldsymbol{B}f - k'\boldsymbol{B}g\boldsymbol{R}^{-1}\left(g'\frac{\partial V_\infty^{(i,j)}}{\partial \boldsymbol{x}} + g'\boldsymbol{B}f\right)\right]$$
7: **end for**
8: Update the Control:
$$\boldsymbol{u}^{(i+1)} = -\boldsymbol{R}^{-1}\Big(g'\frac{\partial V_\infty^{(i,\infty)}}{\partial \boldsymbol{x}} + g'(\boldsymbol{x})\boldsymbol{B}f(\boldsymbol{x}) + g'(\boldsymbol{x})\boldsymbol{B}k(\boldsymbol{x})\boldsymbol{w}^{(i,\infty)}\Big)$$
9: **end for**

IV. The Galerkin approximation

The Galerkin's method is commonly used to solve partial differential equations [26]. In this work, it is applied to achieve the solution $V(\boldsymbol{x})$ of Hamiltonians $\mathbb{H}_2$ and $\mathbb{H}_\infty$. Therefore, by rewriting these PDEs in a generic compact form

$$\mathcal{A}(V(\boldsymbol{x})) = \boldsymbol{0}, \tag{12}$$

the first step for applying the Galerkin's method is to place the solution of (12) in the Hilbert space, $V(\boldsymbol{x}) \in \mathcal{L}_2(\Omega)$. It is obtained by constraining this solution to a compact subset of the space Ω. The Galerkin approach assumes that one can select a set of functions $\Phi(\boldsymbol{x}) = [\phi_1(\boldsymbol{x})\ \phi_2(\boldsymbol{x})\ ...\ \phi_\infty(\boldsymbol{x})]$, which satisfies the problem's boundary condition, with $\Phi(\boldsymbol{x})$ being a complete basis of the space Ω. This implies that there exist coefficients c_j such that

$$||V(\boldsymbol{x}) - \sum_{j=1}^{\infty} c_j \phi_j(\boldsymbol{x})||_{\mathcal{L}_2(\Omega)} = 0.$$

Nevertheless, in practice the set of basis function is truncated with a finite number of terms

$$V_N(\boldsymbol{x}) = \sum_{j=1}^{N} c_j \phi_j(\boldsymbol{x}) = \boldsymbol{c}^T \Phi(\boldsymbol{x}), \tag{13}$$

which may not be a complete basis in the domain of interest[3].

Thus, by applying (13) in (12) generates the following error approach,

$$\mathcal{A}(V_N(\boldsymbol{x})) = \text{Error}(\boldsymbol{x}).$$

In the Galerkin's method the vector of coefficients $\boldsymbol{c}$ are determined by setting the projection of the error on the finite basis $\Phi(\boldsymbol{x})$ to zero, $\forall \boldsymbol{x} \in \Omega$, as follows

$$< \mathcal{A}(V_N(\boldsymbol{x})), \boldsymbol{\phi}_j(\boldsymbol{x}) >= \int_\Omega \mathcal{A}(V_N(\boldsymbol{x}))\boldsymbol{\phi}_j(\boldsymbol{x}) d\boldsymbol{\Omega} = 0, \tag{14}$$

with $j = 1,\ 2,\ ...\ N$.

Therefore, taking into account the Algorithm 1 proposed to solve the nonlinear $\mathcal{W}_{1,2}$-$\mathcal{H}_2$ control problem and the Hamiltonian (2), the procedure to achieve an approximated solution $V_2(\boldsymbol{x})$, $\forall \boldsymbol{x} \in \Omega$, is performed by

$$\int_\Omega \left[\left(\frac{\partial \boldsymbol{\Phi}' \boldsymbol{c}}{\partial \boldsymbol{x}} \right)' [f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}] + \frac{1}{2}\left(||\boldsymbol{z}||^2 + ||\dot{\boldsymbol{z}}||^2 \right) \right] \boldsymbol{\Phi}' d\Omega = 0,$$

in which, after some manipulations, leads to

$$\boldsymbol{c}' \int_\Omega \nabla \boldsymbol{\Phi}' [f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}] \boldsymbol{\Phi}' d\Omega = -\frac{1}{2} \int_\Omega \left(||\boldsymbol{z}||^2 + ||\dot{\boldsymbol{z}}||^2 \right) \boldsymbol{\Phi}' d\Omega,$$

and therefore, the vector of coefficients $\boldsymbol{c}$ can be obtained by

$$\boldsymbol{c}' = \left(-\frac{1}{2} \int_\Omega \left(||\boldsymbol{z}||^2 + ||\dot{\boldsymbol{z}}||^2 \right) \boldsymbol{\Phi}' d\Omega \right) \cdot \left(\int_\Omega \nabla \boldsymbol{\Phi}' \Big[f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} \Big] \boldsymbol{\Phi}' d\Omega \right)^{-1}. \tag{15}$$

For Algorithm 2, considering the Hamiltonian (6), the procedure to obtain an approximated solution of $V_\infty(\boldsymbol{x})$, $\forall \boldsymbol{x} \in \Omega$, is conducted as

$$\int_\Omega \Big[\Big(\frac{\partial \boldsymbol{\Phi}' \boldsymbol{c}}{\partial \boldsymbol{x}} \Big)' [f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} + k(\boldsymbol{x})\boldsymbol{w}] + \frac{1}{2}\left(||\boldsymbol{z}||^2 + ||\dot{\boldsymbol{z}}||^2 - \gamma^2 ||\boldsymbol{w}||^2 \right) \Big] \boldsymbol{\Phi}' d\Omega = 0,$$

leading to

$$\boldsymbol{c}' \int_\Omega \nabla \boldsymbol{\Phi}' [f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} + k(\boldsymbol{x})\boldsymbol{w}] \boldsymbol{\Phi}' d\Omega = -\frac{1}{2} \int_\Omega \left(||\boldsymbol{z}||^2 + ||\dot{\boldsymbol{z}}||^2 - \gamma^2 ||\boldsymbol{w}||^2 \right) \boldsymbol{\Phi}' d\Omega,$$

and therefore, for this problem the vector of coefficients $\boldsymbol{c}$ is obtained by

$$\boldsymbol{c}' = \left(-\frac{1}{2} \int_\Omega \left(||\boldsymbol{z}||^2 + ||\dot{\boldsymbol{z}}||^2 - \gamma^2 ||\boldsymbol{w}||^2 \right) \boldsymbol{\Phi}' d\Omega \right) \cdot \left(\int_\Omega \nabla \boldsymbol{\Phi}' \big[f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} + k(\boldsymbol{x})\boldsymbol{w} \big] \boldsymbol{\Phi}' d\Omega \right)^{-1}. \tag{16}$$

In the next section, equations (15) and (16) are computed with the proposed Algorithms 1 and 2 in order to design the nonlinear $\mathcal{W}_{1,2}$-$\mathcal{H}_2$ and $\mathcal{W}_{1,2}$-$\mathcal{H}_\infty$ controllers for a two-wheeled self-balanced vehicle.

V. Results

To corroborate the controllers' efficiency, in this section numerical experiments are conducted with a two-wheeled self-balanced vehicle, illustrated in Figure 1.

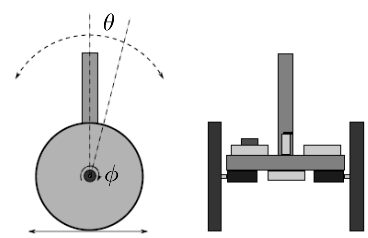

Fig. 1: The two-wheeled vehicle [18].

The vehicle's model was obtained from [18] and is given by

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{K}(\dot{\boldsymbol{q}}) + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{F}(\boldsymbol{q})u + \boldsymbol{w}, \tag{17}$$

in which

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} (M+m)r^2 + I_r & mlr\cos(\theta) \\ mlr\cos(\theta) & ml^2 + I_p \end{bmatrix}, \quad \boldsymbol{q} = \begin{bmatrix} \phi \\ \theta \end{bmatrix},$$

$$\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{bmatrix} 0 & -mlr\sin(\theta)\dot{\theta} \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{K}(\dot{\boldsymbol{q}}) = \begin{bmatrix} k\dot{\phi} \\ -k\dot{\phi} \end{bmatrix},$$

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} 0 \\ -mgl\sin(\theta) \end{bmatrix}, \quad \boldsymbol{F}(\boldsymbol{q}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

where $u \in \mathbb{R}$ is the torque applied on vehicles' wheels, $w_1, w_2 \in \mathcal{L}_2$ are disturbances applied to the system, m is the mass of the pendulum, M is the mass of the wheels, l is the distance from the axle to the pendulum center of mass, r is the wheel's radius, I_p is the pendulum moment of inertia, I_r is the inertia of the wheel, k is the static friction of the motor, and g is the gravity acceleration. The physical parameters used on numerical simulations are presented in Table I. The equations of motion (17) are represented in the state-space standard form (1), leading to

$$f(\boldsymbol{x}) = \begin{bmatrix} \dot{\theta} \\ -M^{-1}(\boldsymbol{q})\left[C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + K(\dot{\boldsymbol{q}}) + G(\boldsymbol{q})\right] \end{bmatrix},$$

$$g(\boldsymbol{x}) = \begin{bmatrix} 0 \\ M^{-1}(\boldsymbol{q})F(\boldsymbol{q}) \end{bmatrix}, \qquad k(\boldsymbol{x}) = \begin{bmatrix} 0 & 0 \\ M^{-1}(\boldsymbol{q}) \end{bmatrix},$$

[3] The finite set of basis functions must be selected to provide small approximation error in the domain of interest, ensuring the algorithms' convergence.

TABLE I: Vehicle parameters

Parameter	Value	Unit of Measure
I_r	0.0421	$kg \cdot m^2$
Ip	0.201	$kg \cdot m^2$
k	0.00215	$N \cdot m \cdot s/rad$
m	2.75	kg
M	3.75	kg
l	0.1435	m
r	0.25	m
g	9.8	m/s^2

with $\boldsymbol{x} = \begin{bmatrix}\theta & \dot{\phi} & \dot{\theta}\end{bmatrix}'$. The cost variable (also called penalty function) is selected as

$$\boldsymbol{z} = h(\boldsymbol{x}) = \begin{bmatrix} \frac{1}{\pi} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \boldsymbol{x}.$$

With the objective of regulating the states around their equilibrium point $\boldsymbol{x} = \mathbf{0}$, the nonlinear $\mathcal{W}_{1,2}$-$\mathcal{H}_2$ and $\mathcal{W}_{1,2}$-$\mathcal{H}_\infty$ controllers are designed by iterating Algorithms 1 and 2 and considering the Galerkin's approximation (15) and (16). A complete polynomial basis with degree six is used as basis functions, which is given by

$$\Phi(\boldsymbol{x}) = \begin{bmatrix}\theta & \dot{\phi} & \dot{\theta} & \dot{\phi}\theta & \theta\dot{\theta} & \dot{\phi}\dot{\theta} & \dot{\phi}^2 & \theta^2 & \dot{\theta}^2 & \cdots & \theta^6 & \dot{\phi}^6 & \dot{\theta}^6\end{bmatrix}.$$

The set Ω is the domain of interest in which the controller ensures stability to the system, and it must be selected as the region of the state-space in which the system works in. In this paper it is selected as $\Omega = \theta_\Omega \times \dot{\phi}_\Omega \times \dot{\theta}_\Omega = [-\frac{\pi}{4}, \frac{\pi}{4}] \times [-12, 12] \times [-1.2, 1.2]$. The $\mathcal{W}_{1,2}$-$\mathcal{H}_\infty$ index is set to $\gamma = 2$.

The integrals presented in (15) and (16) are performed using Gaussian quadrature with one point

$$\int_a^b \psi(\xi) d\xi = (b-a)\psi(\frac{b+a}{2}),$$

which gives the exact solution for polynomials with degree one, as illustrated in Figure 2. In order to apply the Gaussian quadrature, the domain Ω is split in several squares with dimension $\Delta = 0.1$. The integrated functions are assumed uncoupled, such that the following holds

$$\int_a^b \int_c^d \int_e^f \psi(\theta, \dot{\phi}, \dot{\theta}) d\theta \; d\dot{\phi} \; d\dot{\theta} = \int_a^b \psi(\theta) d\theta \int_c^d \psi(\dot{\phi}) d\dot{\phi} \int_e^f \psi(\dot{\theta}) d\dot{\theta}.$$

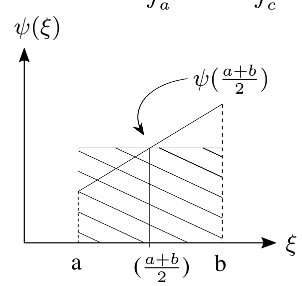

Fig. 2: Gaussian Quadrature with one point.

During the iterations of Algorithms 1 and 2, the coefficients $\boldsymbol{c}$ of the Galerkin's Method converge asymptotically to the solution. Therefore, in order to decrease the computational time taken to obtain the solution, it is used the stopping criteria $||\boldsymbol{c}^{i-1} - \boldsymbol{c}^i|| < 0.1$. In addition, a linear state feedback LQR controller was designed as the initial stabilizing control law, which resulted in

$$u^{(0)} = \begin{bmatrix}11.3132 & 1.0022 & 3.2589\end{bmatrix} \boldsymbol{x}.$$

By executing the modified SGAA, the following approximated solutions were obtained for HJB and HJBI equations, respectively,

$$\begin{aligned} V_2(\boldsymbol{x}) = & \frac{\dot{\phi}^2\theta^4}{1000} - \frac{\dot{\phi}^2\theta^2}{250} + \frac{\dot{\phi}^2\theta\dot{\theta}}{1000} + \frac{13\dot{\phi}^2}{500} + \frac{3\dot{\phi}\theta^5}{500} + \frac{\dot{\phi}\theta^4\dot{\theta}}{500} \\ & - \frac{49\dot{\phi}\theta^3}{1000} - \frac{3\dot{\phi}\theta^2\dot{\theta}}{250} + \frac{\dot{\phi}\theta\dot{\theta}^2}{500} + \frac{23\dot{\phi}\theta}{100} + \frac{79\dot{\phi}\dot{\theta}}{1000} + \frac{\theta^6}{125} \\ & + \frac{9\theta^5\dot{\theta}}{1000} + \frac{\theta^4\dot{\theta}^2}{500} - \frac{171\theta^4}{1000} - \frac{33\theta^3\dot{\theta}}{250} - \frac{7\theta^2\dot{\theta}^2}{200} + \frac{49\theta^2}{40} \\ & - \frac{\theta\dot{\theta}^3}{250} + \frac{109\theta\dot{\theta}}{125} + \frac{107\dot{\theta}^2}{500}, \end{aligned}$$

$$\begin{aligned} V_\infty(\boldsymbol{x}) = & \frac{\dot{\phi}^2\theta^4}{1000} - \frac{\dot{\phi}^2\theta^2}{250} + \frac{\dot{\phi}^2\theta\dot{\theta}}{1000} + \frac{7\dot{\phi}^2}{250} + \frac{7\dot{\phi}\theta^5}{1000} + \frac{\dot{\phi}\theta^4\dot{\theta}}{1000} \\ & - \frac{7\dot{\phi}\theta^3}{125} - \frac{3\dot{\phi}\theta^2\dot{\theta}}{250} + \frac{\dot{\phi}\theta\dot{\theta}^2}{500} + \frac{133\dot{\phi}\theta}{500} + \frac{91\dot{\phi}\dot{\theta}}{1000} \\ & + \frac{\theta^6}{125} + \frac{9\theta^5\dot{\theta}}{1000} + \frac{\theta^4\dot{\theta}^2}{1000} - \frac{189\theta^4}{1000} - \frac{71\theta^3\dot{\theta}}{500} - \frac{7\theta^2\dot{\theta}^2}{200} \\ & + \frac{1397\theta^2}{1000} - \frac{3\theta\dot{\theta}^3}{1000} + \frac{247\theta\dot{\theta}}{250} + \frac{117\dot{\theta}^2}{500}. \end{aligned}$$

The system was simulated starting with initial conditions $\boldsymbol{x}(0) = [\frac{\pi}{4} \quad 0 \quad 0]'$. The obtained results are presented in Figure 3. As can be seen, the vehicle starts displaced from the desired upper vertical position and asymptotically converges to it, holding this position until external disturbances were applied. The applied external disturbances are also shown in Figure 3. They represents generalized torques and, due to the coupled dynamics of the system, their effects are perceptible in all states. Note that, when the disturbance w_2 affects the system, the wheels' velocity assumes values outside of the domain of interest Ω. However, the controller provided stability to the system and attenuated the disturbances' effects.

When comparing both controllers, since the $\mathcal{W}_{1,2}$-$\mathcal{H}_\infty$ is designed to attenuate disturbances, it generates control signals that provide better attenuation of external disturbances with faster transients. The control input signals were evaluated by means of the Integral of the Absolute Value of the Control Input's Time Derivative (IAVU) performance index. The results are shown in Table II. As can be seen, the $\mathcal{W}_{1,2}$-$\mathcal{H}_\infty$ controller spent less control effort.

TABLE II: Control Input's Performance Index

Performance Index	Computed by	$\mathcal{W}_{1,2}$-$\mathcal{H}_2$	$\mathcal{W}_{1,2}$-$\mathcal{H}_\infty$
IAVU	$\int_0^{t_f} \left\lvert \frac{d\boldsymbol{u(t)}}{dt} \right\rvert dt$	23.54	**21.94**

VI. CONCLUSIONS

In this work we proposed adapt the classical Successive Galerkin Approximation Algorithms to approximate solutions of the Hamilton-Jacobi-Bellman and Hamilton-Jacobi-Bellman-Isacs equations, that arises from the formulation of the nonlinear $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control approaches in the Sobolev space. To illustrate the ability of the proposed controllers to deal with underactuated systems, they were synthesized to control a two-wheeled self-balanced vehicle. Numerical experiments were conducted, which showed the efficiency of the designed control laws in the Sobolev space.

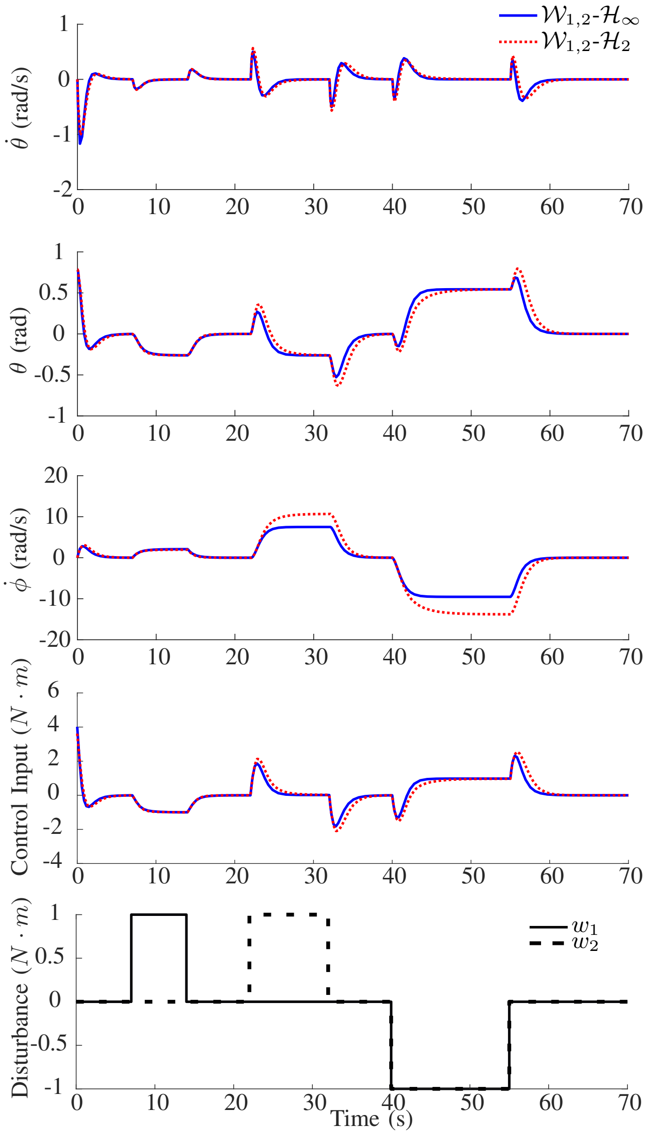

Fig. 3: Vehicle states, applied control input and external disturbances.

Future works include obtain analytical solutions and compare the steady-state and transient performance of nonlinear $\mathcal{W}_{1,2}$-$\mathcal{H}_2$ and $\mathcal{W}_{1,2}$-$\mathcal{H}_\infty$ controllers with the controllers formulated in the Lebesgue $\mathcal{L}_2$ space.

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