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Prescribed Time Scale Robot Navigation in Dynamic Environments

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***Abstract*— In this work, we consider the problem of prescribed time scale robot navigation in dynamic environments. Initially, we treat the problem for a special class of configuration spaces, namely sphere worlds, proposing a time-varying control scheme that drives the robot from (almost) all initial configurations to an arbitrary neighborhood of any desired configuration within a predetermined time span, and at the same time prevents any collisions with static and moving obstacles as well as with the workspace boundary along the way. The introduction of a novel vector field allows us to establish the safety of the system and simultaneously apply the Prescribed Performance Control technique to guarantee any predefined transient behavior. Subsequently, we leverage well-established transformations to apply the proposed scheme to the far more practical class of generalized sphere worlds. Finally, we validate the theoretical findings via a non-trivial numerical simulation.**

I. Introduction

Motion planning constitutes a fundamental problem in robotics [1], [2]. One particularly appealing problem is that of motion planning in *dynamic environments*, that is, environments that are time-varying. Evidently, motion planning algorithms that accommodate dynamic environments are far more versatile than those limited to *static environments* since in real-life application the latter are rarely met.

In [3] the problem of robot motion planning amongst static and moving obstacles is addressed through the velocity obstacles paradigm by utilizing velocity information to identify potential collisions. In [4] the Rapidly-exploring Random Tree (RRT) methodology is extended to the case of dynamic environments.

Among the many approaches to the static motion planning problem, methods based on the closed-loop evaluation of vector fields have received considerable attention owing to their low complexity and their ability to simultaneously tackle the motion planning and control problems. This approach has been based on the construction of an underlying *artificial potential function*[1] [5] and culminated to the provably correct — if properly tuned — *Navigation Functions* construct [6]–[8].

In [9] artificial potential fields are extended to the case of dynamic obstacles, and in [10] non-smooth navigation functions are introduced to handle dynamic environments.

While the aforementioned schemes solve the problem of safely driving the robot to a desired configuration, they provide no *a priori* guarantees on the required time span for task completion thus hindering integration with high-level, time-constrained task planning modules (*e.g.,* planning under MITL specifications [11]). To the best of our knowledge, the only treatment of the aforementioned issue for the static case lies in [12] where Loizou achieves (for almost all initial conditions) finite time convergence to the desired configuration by diffeomorphically transforming the configuration space to the metrically "trivial" point world (a closed ball with a finite number of points removed). Using the "pulled back" vector that points to the image of the desired configuration the aforementioned result is obtained.

In this work we address the same problem for the case of dynamic environments. Motivated by the fact that in real-world applications, merely convergence to some neighborhood of a point might be of interest, we recast the aforementioned problem by requiring that an arbitrary initial condition[2] is driven to a neighborhood of the desired configuration in predetermined time — henceforth referred to as *prescribed time scale behavior* — while avoiding collision with both static and moving obstacles as well as with the workspace boundary. Initially, we consider the case of configuration spaces that are *sphere worlds* [6] and propose a novel vector field that ensures obstacle avoidance and facilitates the use of the Prescribed Performance Control technique [13] to impose predetermined convergence to the desired configuration, resulting in a time-varying vector field planner. We also propose an extension of our methodology to the wider class of configuration spaces that are diffeomorphic to a sphere world, called *generalized sphere worlds*.

The rest of the paper is organized as follows. Section II formally defines the problem addressed in this work. Section III contains the main contribution of this work, developing the control scheme for the case of a sphere world. Section IV discusses the extension of the proposed methodology to a wider class of workspaces. Section V provides a numerical simulation validating the theoretical results. Section VI concludes this work by summarizing our contribution and discussing future research directions.

II. Problem Formulation

We consider a point robot[3] operating in a bounded workspace $\mathcal{W} \subset \mathbb{R}^n$ with $n \in \mathbb{N}_{\geq 2}$ and denote its position by $\mathrm{x} \in \mathcal{W}$. The workspace is assumed to be an open ball centered at the origin $\mathcal{W} \triangleq \{q \in \mathbb{R}^n : \|q\| < r_{\mathcal{W}}\}$, where $r_{\mathcal{W}} \in \mathbb{R}_{>0}$ is the workspace radius. The workspace is populated with $m \in \mathbb{N}$ closed sets O_i, $i \in \mathcal{J}_s \triangleq \{1, \ldots, m\}$, corresponding to *static obstacles*. In particular, each static obstacle $i \in \mathcal{J}_s$ is a ball centered at p_i with radius $r_i \in \mathbb{R}_{\geq 0}$, $O_i \triangleq O_i(t) \triangleq \{q \in \mathcal{W} : \|q - \mathrm{p}_i\| \leq r_i\}$. We can equivalently write $O_i = d_i^{-1}(\mathbb{R}_{\leq 0})$ and $\mathcal{W} = d_{\mathcal{W}}^{-1}(\mathbb{R}_{>0})$ where $d_i, d_{\mathcal{W}} : \mathbb{R}^n \to \mathbb{R}$ are defined by $d_i(\mathrm{x}) = \|\mathrm{x} - \mathrm{p}_i\|^2 - r_i^2$, $i \in \mathcal{J}$ and $d_{\mathcal{W}}(\mathrm{x}) = r_{\mathcal{W}}^2 - \|\mathrm{x}\|^2$, respectively.

The workspace also contains $k \in \mathbb{N}$ *moving obstacles*. The set occupied at time instant $t \in \mathbb{R}_{\geq 0}$ by each moving obstacle $i \in \mathcal{J}_m \triangleq \{m+1, \ldots, m+k\}$ is, analogously to the static case, $O_i(t) \triangleq \{q \in \mathcal{W} : \|q - \mathrm{p}_i(t)\| \leq r_i\}$,

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[1] A scalar-valued function on the configuration space, whose negated gradient vector field is used to drive the robot

[2] A well known topological obstruction dictates that merely almost global convergence is attainable through continuous time-invariant control laws [6].

[3] Treating a robot with volume can be achieved by "transferring" its volume to the other workspace entities and considering it as a point.

where $\mathrm{p}_i(t) : \mathbb{R}_{\geq 0} \to \mathcal{W}$ is the trajectory of the center and $r_i \in \mathbb{R}_{\geq 0}$ the radius. In this respect, the static free space $\mathcal{F}_s$ is defined by $\mathcal{F}_s \triangleq \mathcal{W} \setminus \bigcup_{i \in \mathcal{J}_s} O_i$ and the free space of the robot at each time instant $t \in \mathbb{R}_{\geq 0}$ is $\mathcal{F}_t \triangleq \mathcal{F}_s \setminus \bigcup_{i \in \mathcal{J}_m} O_i(t)$. For any $\tau \in \mathbb{R}_{\geq 0}$ let us also define $\mathcal{F}_{>\tau} \triangleq \bigcap_{t \in \mathbb{R}_{\geq \tau}} \mathcal{F}_t$. Following [6], we say that the *sphere world* assumption holds if the following geometrical conditions are met:

Assumption 1 (Sphere World Assumption) *Each obstacle O_i is contained in workspace $\mathcal{W}$ and obstacle sets are pairwise disjoint, i.e., $O_i(t) \cap O_j(t) = \emptyset$, for every $t \in \mathbb{R}_{\geq 0}, i, j \in \mathcal{J} \triangleq \mathcal{J}_s \cup \mathcal{J}_m$, $i \neq j$.*

Remark 1. Assumption 1 implies the existence of some $\bar{r} \in \mathbb{R}_{>0}$ such that $\|\mathrm{p}_i(t) - \mathrm{p}_j(t)\| > r_i + r_j + 2\bar{r}$, $\forall t \in \mathbb{R}_{\geq 0}$, $i, j \in \mathcal{J}$, $i \neq j$, and $\inf_{q \in \partial \mathcal{W}} \|q - \mathrm{p}_i(t)\| > r_i + 2\bar{r}$, $\forall t \in \mathbb{R}_{\geq 0}$, $i \in \mathcal{J}$. It should also be noted that the case of a robot with its volume contained in a ball of radius $r \in \mathbb{R}_{>0}$ reduces to the point case for $r'_i = r_i + r$, $i \in \mathcal{J}$ and $r'_{\mathcal{W}} = r_{\mathcal{W}} - r$.

Assumption 2 *For every $i \in \mathcal{J}_m$, the function $p_i : \mathbb{R}_{\geq 0} \to \mathcal{W}$ is continuously differentiable and there exists $M \in \mathbb{R}_{\geq 0}$ such that $M = \sup \{\|\dot{p}_i(t)\| : t \in \mathbb{R}_{\geq 0}\}$.*

We now proceed with the statement of the problem addressed in this work:

Problem 1 (Prescribed Time Scale Navigation) *Assuming single integrator robot kinematics,*

$$\dot{\mathrm{x}} = \mathrm{u} \tag{1}$$

for any pair $(\varrho, \tau) \in \mathbb{R}^2_{>0}$ and any desired configuration $\mathrm{x}_d \in \mathcal{F}_{>\tau}$, determine a smooth time-varying controller $\mathrm{u} : \mathbb{R}_{\geq 0} \times \mathcal{F}_s \to \mathbb{R}^n$ such that for almost all initial configurations $\mathrm{x}_0 \in \mathcal{F}_0$, the closed-loop system has a unique solution $\mathrm{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ with

$$\mathrm{x}(t) \in \mathcal{F}_t, \quad \forall t \in \mathbb{R}_{\geq 0} \tag{2}$$

and

$$\|\mathrm{x}(t) - \mathrm{x}_d\| \leq \varrho, \quad \forall t \geq \tau. \tag{3}$$

Remark 2. Intuitively, equation (3) means that by time τ the robot will have entered a ball of radius ϱ centered at the desired configuration, and remain inside it thereafter.

III. METHODOLOGY

In this section we introduce a control scheme that provably solves the Prescribed Time Scale Navigation Problem (Problem 1). Initially, we consider the distance of the line segment, with endpoints the robot position and the desired configuration, from an obstacle and show that its gradient with respect to the robot position is a well-defined vector field. This family of vector fields, one per obstacle, serves as a basis for establishing the safety of the system. Subsequently, we utilize the aforementioned vector fields to design a controller based on the Prescribed Performance Control methodology, that solves Problem 1.

A. Segment-to-Point-Distance-Derived Vector Field

Given a robot configuration $\mathrm{x} \in \mathcal{F}_t$ and for a given desired configuration $\mathrm{x}_d \in \mathcal{F}_t$, we define the closed, convex set $\mathcal{S}(\mathrm{x})$ of points lying in the *line segment* between x and x_d by

$$\mathcal{S}(\mathrm{x}) \triangleq \{q \in \mathbb{R}^n : q = (1 - \lambda)\mathrm{x} + \lambda \mathrm{x}_d,\ \lambda \in [0, 1]\}. \tag{4}$$

As a result of the convexity of the set $\mathcal{S}(\mathrm{x})$, and the inclusion relation $\mathcal{F} \subset \mathcal{W}$, it follows that $\mathcal{S}(x)$ is contained in $\mathcal{W}$. Additionally, we define the function $\beta_i : \mathcal{W} \setminus \{\mathrm{x}_d\} \to \mathbb{R}_{\geq 0}$ by $\beta_i(\mathrm{x}) = \inf \left\{ \|q - \mathrm{p}_i\|^2 : q \in \mathcal{S}(\mathrm{x}) \right\}$, $i \in \mathcal{J}$ which is the squared euclidean distance of the set $\mathcal{S}(\mathrm{x})$ from the obstacle center p_i. Since $\mathcal{S}(x)$ is closed and convex it follows that there exists a *unique point* $y_i(\mathrm{x}) \in \mathcal{S}(\mathrm{x})$ such that $\beta_i(\mathrm{x}) = \|y_i(\mathrm{x}) - \mathrm{p}_i\|^2$ and by (4) that there is a unique $\lambda_i(\mathrm{x}) \in [0, 1]$, such that $y_i(\mathrm{x}) = \mathrm{x} - \lambda_i(\mathrm{x})(\mathrm{x} - \mathrm{x}_d)$. Defining the continuously differentiable function $\tilde{\lambda}_i : \mathcal{W} \setminus \{\mathrm{x}_d\} \to \mathbb{R}$ as

$$\tilde{\lambda}_i(\mathrm{x}) = \frac{(\mathrm{x} - \mathrm{p}_i)^\top (\mathrm{x} - \mathrm{x}_d)}{\|\mathrm{x} - \mathrm{x}_d\|^2} \tag{5}$$

it is readily verifiable that $\lambda_i : \mathcal{W} \setminus \{\mathrm{x}_d\} \to [0, 1]$ is given by:

$$\lambda_i(\mathrm{x}) = \begin{cases} 0, & \tilde{\lambda}(\mathrm{x}) \in \mathbb{R}_{\leq 0} \\ \tilde{\lambda}_i(\mathrm{x}), & \tilde{\lambda}(\mathrm{x}) \in (0, 1) \\ 1, & \tilde{\lambda}(\mathrm{x}) \in \mathbb{R}_{\geq 1} \end{cases} \tag{6}$$

and is continuous as a composition of continuous functions. In particular, $\lambda_i = \mathrm{s} \circ \tilde{\lambda}_i$ where $\mathrm{s} : \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathrm{s}(x) = \begin{cases} 0, & x \in \mathbb{R}_{\leq 0} \\ x, & x \in (0, 1) \\ 1, & x \in \mathbb{R}_{\geq 1} \end{cases}. \tag{7}$$

For notational brevity we will occasionally refer to $\lambda_i(\mathrm{x})$ by λ_i, without stating the dependence on x.

Proposition 1 *For each $\mathrm{i} \in \mathcal{J}$, the function β_i is continuously differentiable and thus $\nabla \beta_i : \mathcal{W} \setminus \{\mathrm{x}_d\} \to \mathbb{R}^n$ is a well defined locally Lipschitz continuous vector field.*

Proposition 2 *For each $i \in \mathcal{J}$, and each $\mathrm{x} \in \mathcal{W} \setminus \{\mathrm{x}_d\}$ it holds that $\nabla \beta_i(\mathrm{x})^\top (\mathrm{x} - \mathrm{x}_d) \leq 0$.*

Proposition 3 *For each $i \in \mathcal{J}$, the function β_i attains its minimum value in the set $\mathcal{M}_i \triangleq \{q \in \mathcal{W} \setminus \{\mathrm{x}_d\} : q = \mathrm{p}_i + \mu(\mathrm{p}_i - \mathrm{x}_d),\ \mu \in \mathbb{R}_{\geq 0}\}$ which has zero Lebesgue measure.*

Finally, the following lemma will be employed in the subsequent analysis.

Lemma 1 *For each $i \in \mathcal{J}$, and any $\mathrm{x} \in \mathcal{W} \setminus \{\mathrm{x}_d\}$ it holds that*

$$\nabla \beta_i(\mathrm{x})^\top \nabla d_i(\mathrm{x}) = \begin{cases} 4\|\mathrm{x} - \mathrm{p}_i\|^2, & \lambda_i(\mathrm{x}) = 0 \\ 4\left(1 - \lambda_i(\mathrm{x})\right) \beta_i(\mathrm{x}), & \lambda_i(\mathrm{x}) > 0 \end{cases}. \tag{8}$$

For the proofs of the preceding propositions please refer to [14].

B. Controller Design

In this work, prescribed performance control will be adopted in order to achieve practical convergence of the robot to the desired configuration in predefined time. Thus let us first define the squared euclidean distance of the robot from the desired configuration as $\gamma(\mathrm{x}) = \|\mathrm{x} - \mathrm{x}_d\|^2$. Following [13], prescribed performance is achieved when $\gamma(\mathrm{x}(t))$ is bounded above by a strictly decreasing function of time, called a performance function. The mathematical expression of prescribed performance is given by the following inequality

$$\gamma(\mathrm{x}(t)) < \rho(t), \quad \forall t \in \mathbb{R}_{\geq 0} \tag{9}$$

where

$$\rho(t) = (\rho_0 - \rho_\infty)\exp(-lt) + \rho_\infty \tag{10}$$

is a designer-specified, smooth, bounded and decreasing function of time such that: i) $\rho_0 > \gamma(\mathrm{x}_0)$ and ii) the parameters $l, \rho_\infty \in \mathbb{R}_{>0}$ incorporate the desired transient and steady state performance specifications, respectively. In particular, the decrease rate of $\rho(t)$, which is affected by the constant l, introduces a lower bound on the rate of convergence of γ. Furthermore, the constant ρ_∞ can be set arbitrarily small, thus achieving practical convergence of the system to the desired configuration.

Apparently, the appropriate selection of the aforementioned and the parameters and satisfaction of (9) imply (3), leading to prescribed time scale behavior. In particular, for any instance $(\varrho, \tau, \mathrm{x}_0, \mathrm{x}_d) \in (\mathbb{R}_{>0})^2 \times \mathcal{F}_0 \times \mathcal{F}_{>\tau}$ of the Prescribed Time Scale Navigation Problem it suffices that

$$\rho_0 > \gamma(\mathrm{x}_0) \tag{11a}$$

$$\rho_\infty < \varrho^2 \tag{11b}$$

$$l \geq \max\left\{0, -\frac{1}{\tau}\ln\left(\frac{\varrho^2 - \rho_\infty}{\rho_0 - \rho_\infty}\right)\right\}. \tag{11c}$$

Before we proceed, we shall define the normalized squared distance from the desired configuration $\xi(t, \mathrm{x})$ as follows:

$$\xi(t, \mathrm{x}) \triangleq \frac{\gamma(\mathrm{x})}{\rho(t)} \tag{12}$$

and the prescribed performance region $\Omega_\xi \triangleq \mathbb{R}_{<1}$. Defining the increasing bijective mapping $\mathrm{T} : \Omega_\xi \to \mathbb{R}$ of the performance domain as $T(\star) = \ln\left(\frac{1}{1-\star}\right)$, the transformed squared distance from the desired configuration $\varepsilon(\xi) \in \mathbb{R}$ is defined as

$$\varepsilon(\xi) \triangleq T(\xi). \tag{13}$$

Taking the time derivative of (13) yields

$$\dot{\varepsilon} = J_T(t, \xi)(\dot{\gamma} + \alpha(t)\gamma) \tag{14}$$

where $J_T(t, \xi)$ and $\alpha(t)$ are given by $J_T(t, \xi) \triangleq \frac{\partial T(\xi)}{\partial \xi}\frac{1}{\rho(t)} > 0,\ \alpha(t) \triangleq -\frac{\dot{\rho}(t)}{\rho(t)} > 0$.

The proposed controller is then defined as

$$\mathrm{u}(t, \mathrm{x}) = u_\gamma(t, \mathrm{x}) + u_\beta(\mathrm{x}) \tag{15}$$

where

$$u_\gamma(t, \mathrm{x}) \triangleq -\left(\bar{\eta}\varepsilon(\eta\xi) + \frac{1}{2}\alpha(t)\right)(\mathrm{x} - \mathrm{x}_d) \tag{16}$$

and

$$u_\beta(t, \mathrm{x}) \triangleq \varepsilon(\xi)\sum_{i \in \mathcal{J}}\left(\frac{\sigma_{d_i}}{d_i}\nabla\beta_i\right) \tag{17}$$

where $\eta \triangleq \sum_{i \in \mathcal{J}} \eta_i$, $\bar{\eta} \triangleq \sum_{i \in \mathcal{J}} \frac{2}{1+\eta_i}$ and $\eta_i \triangleq (1 - \sigma_{\beta_i}) s_{d_i} + \sigma_{\beta_i}$. Also $\delta \in (0, \bar{r})$ (see Remark 1) is chosen so that the effect of obstacles is made local through the switches σ_{d_i}, s_{d_i} and σ_{β_i} which are defined in the Appendix.

Notice that the first term of the proposed control law (16) enforces the prescribed time scale behavior (9), whereas the second term (17) guarantees the forward invariance of the free space $\mathcal{F}$. The effect of each obstacle $i \in \mathcal{J}$ is made local through the switch $\sigma_{i,\delta}$. Choosing $\delta < \bar{r}$ implies that at each point $\mathrm{x} \in \mathcal{F}$ at most one obstacle is effective thus simplifying greatly the subsequent analysis of the qualitative properties of the control scheme.

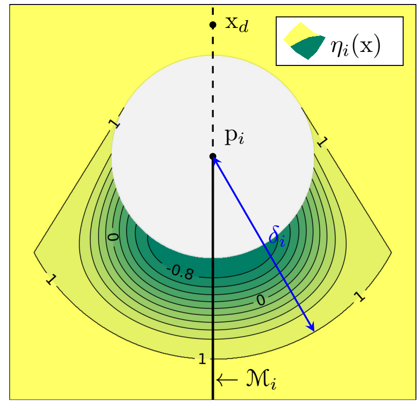


Fig. 1: Contour plot of η_i in the vicinity of an obstacle $i \in \mathcal{J}$. It is constructed so that $\eta_i(\mathrm{x}) = 1$ if and only if $\lambda_i(\mathrm{x}) \in (0, 1)$. This property is crucial for proving obstacle avoidance.

We now show that close to each obstacle $i \in \mathcal{J}$, β_i is strictly increasing along the trajectory of the system.

Lemma 2 *Let $\mathrm{x}(t; \mathrm{x}_0)$ be the solution of (1) under the control law (15) with initial condition $\mathrm{x}_0 \in \mathcal{F}_0$ defined for $t \in [0, \tau_{\max})$, $\tau_{\max} \in \mathbb{R}_{>0} \cup \{+\infty\}$. Then the set $\{\mathrm{x}_0 \in \mathcal{F}_0 : \exists t \in [0, \tau_{\max}), i \in \mathcal{J}_s$ such that $\mathrm{x}(t; \mathrm{x}_0) \in \mathcal{M}_i\}$ has zero Lebesgue measure.*

Proof (Sketch). Owing to Proposition 3 the set $\bigcup_{i \in \mathcal{J}_s} \mathcal{M}_i$ has zero Lebesgue measure. Assume that $\mathrm{x}_0 \notin \mathcal{M}_i, \forall i \in \mathcal{J}_s$. Then validity of the following facts can be easily established:

(i) For all $t \in [0, \tau_{\max})$, such that $\mathrm{x}(t) \in \{q : d_i(q) \geq \delta_i, \forall i \in \mathcal{J}\}$, it holds that $\dot{\beta}_i(t) = 0$.
(ii) For all $t \in [0, \tau_{\max})$ and $i \in \mathcal{J}_s$ such that $\mathrm{x}(t) \in \{q : d_i(q) < \delta_i \setminus \mathcal{M}_i\}$, it holds that $\dot{\beta}_i(t) < 0$.
(iii) By the uniqueness properties of the solution and the fact that finite unions of zero measure sets have zero measure, for each $i \in \mathcal{J}_s$ the set $\{\mathrm{x}_0 \in \mathcal{F}_0 : \exists t \in [0, \tau_{\max}), j \in \mathcal{J} \setminus \{i\}$ such that $\mathrm{x}(t; \mathrm{x}_0) \in \mathcal{M}_i \cap d_j^{-1}(\{\delta_j\})\}$ has zero Lebesgue measure.

Intuitively what the above mean is that if $\mathrm{x}_0 \notin \mathcal{M}_i$ then:

(i) If the solution stays away from the obstacles then $\beta_i(t) = \beta_i(0) \neq 0$ and therefore $\mathrm{x}(t) \notin \mathcal{M}_i$ for all $t \in [0, \tau_{\max})$.
(ii) If the solution approaches obstacle i, it will also move away from $\mathcal{M}_i$, and
(iii) although it may approach another set $\mathcal{M}_j$ for $j \neq i$, it will remain there for only a measure zero set of initial conditions.

Thus all cases are covered and the set under investigation is shown to have zero measure as a finite union of zero measure sets, namely $\bigcup_{i \in \mathcal{J}_s} \mathcal{M}_i$ and the set from case (iii). □

Finally, the main results of this work are summarized in the following theorem:

Theorem 1 *Any instance of (Problem 1) described by the tuple $(\varrho, \tau, \mathrm{x}_0, \mathrm{x}_d) \in (\mathbb{R}_{>0})^2 \times \mathcal{F}_0 \times \mathcal{F}_{>\tau}$ is solved by the control law (15) obeying (11a–11c).*

Proof. Owing to Assumption 1 and the particular choice of δ, we can afford, with no loss of generality, to only consider the case of a single obstacle.

Differentiating (12) with respect to time yields

$$\dot{\xi} = \frac{1}{\rho(t)}\left(\dot{\gamma} - \dot{\rho}(t)\xi\right). \tag{18}$$

The time derivative of γ is given by $\dot{\gamma} = \nabla\gamma^\top \dot{\mathrm{x}} = 2(\mathrm{x}-\mathrm{x}_d)^\top \dot{\mathrm{x}}$ which by substituting the control law (15) becomes

$$\dot{\gamma} = -2\bar{\eta}\varepsilon(\eta\xi)\gamma - \alpha(t)\gamma + 2(\mathrm{x}-\mathrm{x}_{\mathrm{d}})^\top u_\beta(\mathrm{x}). \tag{19}$$

Substituting (19) into (18) returns $\dot{\xi} = -2\bar{\eta}\varepsilon(\eta\xi)\xi + \frac{2}{\rho(t)}(\mathrm{x}-\mathrm{x}_d)^\top u_\beta(\mathrm{x})$. Letting, $y = (\xi, \mathrm{x})$, we consider the initial value problem

$$\dot{y} = F(t, y), \quad y(0) \in \Omega_\xi \times \mathcal{F} \tag{20}$$

where $F : \mathbb{R}_{\geq 0} \times \Omega_\xi \times \mathcal{F}_t \to \mathbb{R} \times \mathbb{R}^n$.

Following [13, Section 2.2], the existence of a maximal solution $y(t)$ of (20) on a time interval $[0, \tau_{\max})$ such that $y(t) \in \Omega_\xi \times \mathcal{F}_t$ for all $t \in [0, \tau_{\max})$ is established. As an immediate consequence, it follows that

$$\xi(t) \in \Omega_\xi, \quad \forall t \in [0, \tau_{\max}) \tag{21}$$

and more specifically that $\xi(t) = \frac{\gamma(\mathrm{x}(t))}{\rho(t)} \in \mathbb{R}_{<1}$ for all $t \in [0, \tau_{\max})$, Since, by definition, $\rho(t) \in \mathbb{R}_{>0}$ it follows that $\|\mathrm{x}(t) - \mathrm{x}_d\|^2 < \rho(t),\ \forall t \in [0, \tau_{\max})$.

Furthermore, owing to (21), the transformed error $\varepsilon(\xi)$ is well-defined for all $t \in [0, \tau_{\max})$. We define the following two C^1, radially unbounded with respect to $\varepsilon(\xi)$ and positive definite functions,

$$V_1 = \varepsilon(\xi) \tag{22}$$

$$V_2 = V_1 - \varepsilon(\eta\xi). \tag{23}$$

Differentiating with respect to time and substituting (14) yields

$$\dot{V}_1 = J_T(t, \xi)(\dot{\gamma} + \alpha(t)\gamma) \tag{24}$$

$$\dot{V}_2 = J_T(t, \xi)\left(\dot{\gamma} + \alpha(t)\gamma\right) - J_T(t, \eta\xi)\left(\eta\dot{\gamma} + \alpha(t)\eta\gamma + \gamma\dot{\eta}\right) \tag{25}$$

which, by employing (19), become

$$\dot{V}_1 = -2J_T(t, \xi)\bar{\eta}\gamma\varepsilon(\eta\xi) + \overbrace{2J_T(t, \xi)(\mathrm{x}-\mathrm{x}_d)^\top u_\beta}^{\leq 0 \text{ (Proposition 2)}} \tag{26}$$

and

$$\begin{aligned}\dot{V}_2 = &- 2J_T(t, \xi)\bar{\eta}\gamma\varepsilon(\eta\xi) + 2J_T(t, \eta\xi)\bar{\eta}\gamma\eta\varepsilon(\eta\xi) - \\ &- J_T(t, \eta\xi)\gamma\dot{\eta} + T((\mathrm{x}-\mathrm{x}_{\mathrm{d}})^\top u_\beta)\end{aligned} \tag{27}$$

where, for brevity, $T((\mathrm{x}-\mathrm{x}_d)^\top u_\beta)$ denotes the terms proportional to $(\mathrm{x}-\mathrm{x}_d)^\top u_\beta$. The sign of the first term of (26) is determined by the sign of the term $\varepsilon(\eta\xi)$, which in turn is determined by the sign of the term η. In particular $\varepsilon(\eta\xi) \geq 0$ if and only if $\eta \geq 0$.

We consider the time derivative of η (please refer to the Appendix for the definitions of C_1 and C_2),

$$\begin{aligned}\dot{\eta} &= \nabla\eta^\top \dot{\mathrm{x}} + \frac{\partial \eta}{\partial t} \\ &= -C_1(\mathrm{x})\left(\bar{\eta}\varepsilon(\eta\xi) + 0.5\alpha(t)\right)\nabla d^\top(\mathrm{x}-\mathrm{x}_d) + \frac{1}{1-\xi}\frac{\sigma_{\cdot,\delta}}{d}\Big[\\ &\quad C_2(\mathrm{x})\|\nabla\beta\|^2 + C_1(\mathrm{x})\nabla d^\top\nabla\beta\Big] + \left(\frac{\partial\eta}{\partial p}\right)^\top \dot{p}.\end{aligned}$$

Note that $\left|\left(\frac{\partial\eta}{\partial p}\right)^\top \dot{p}\right| \leq \sup\left\{\|\frac{\partial\eta}{\partial p}\|\right\} M < +\infty$ (see Assumption 2).

Assuming that $\lim_{t\to\tau_{\max}} V_1(t) = +\infty$ we distinguish two cases and show that a contradiction arises.

Case 1: If $\lim_{t\to\tau_{\max}} \eta(t) = 1$, it follows by continuity of the solution that there exists an $\epsilon > 0$ independent of $\tau_{\max}$ such that $\eta(t) > 0$ for all $t \in (\tau_{\max} - \epsilon, \tau_{\max})$. Thus, by (26), $V_1(t) \leq \max\{V_1(\tau_{\max} - \epsilon), \max\{V_1(t) : [0, \tau_{\max} - \epsilon]\}\} < +\infty$, for all $t \in [0, \tau_{\max})$, which contradicts the hypothesis since $\sup\{V_1(t) : t \in [0, \tau_{\max})\}$ is bounded.

Case 2: If $\lim_{t\to\tau_{\max}} \eta(t) \neq 1$ (this includes the case that the limit does not exist). Then there exists $\epsilon > 0$ such that $\eta(t) < 1 - \epsilon$ for all $t \in (\tau_{\max} - \epsilon, \tau_{\max})$. It also holds that $\varepsilon(\xi) > \varepsilon(\eta\xi)$ and $|\varepsilon(\eta\xi)| \leq |\ln(\epsilon)|$. Therefore $\lim_{t\to\tau_{\max}} V_1(t) = +\infty$ implies that $\lim_{t\to\tau_{\max}} V_2(t) = +\infty$. By Lemma 2, we have that for almost all initial conditions $\lim_{t\to\tau_{\max}} \beta(t) \neq 0$ which in turn implies that $\lim_{t\to\tau_{\max}} \|\nabla\beta(t)\| > 0$ and owing to the fact that $\lim_{\xi\to 1} \frac{1/(1-\xi)}{\ln(1/(1-\xi))} = +\infty$ we have that $\lim_{t\to\tau_{\max}} \dot{\eta}(t) = -\infty$. Note also that in this case, by construction, $\mathrm{x} \in \lambda^{-1}((0,1))$ and therefore by Proposition 2 $T((\mathrm{x}-\mathrm{x}_d)^\top u_\beta) = 0$ (see also Fig. 1). Due to the fact that $\alpha(t) \leq \alpha(0)$, and that $C_1(\mathrm{x}), C_2(\mathrm{x})$ are positive and bounded for $\mathrm{x} \in \eta^{-1}([-1, 1-\epsilon])$ we can establish the existence of a positive $\epsilon' \leq \epsilon$ such that $\dot{V}_2(t) \leq 0$ for all $t \in (\tau_{\max} - \epsilon', \tau_{\max})$. As for the previous case, an upper bound for V_2 and therefore for V_1 is established. Note, however, that in the case of a moving obstacle the bound depends on $\tau_{\max}$. Fortunately, we can circumvent this difficulty owing to the fact that $\mathrm{x}_d \in \mathcal{F}_{>\tau}$ which implies that for a moving obstacle $\eta(t) = 1$ for all $t \in \mathbb{R}_{\geq\tau}$ allowing us to pick the maximum upper bound from the compact set $[0, \tau]$, thus decoupling our choice from $\tau_{\max}$.

The positive definiteness and radial unboundedness of V with respect to $\varepsilon(\xi)$ concludes that for every $t \in [0, \tau_{\max})$

$$\varepsilon(\xi(t)) \leq \bar{\varepsilon} = \max\{\epsilon(\xi(t)) : t \in [0, \tau_{\max})\} < +\infty. \tag{28}$$

Additionally, taking the inverse logarithmic function in (28) yields:

$$0 \leq \xi(t) \leq \bar{\xi} \triangleq 1 - \exp(-\bar{\varepsilon}) < 1, \quad \forall t \in [0, \tau_{\max}). \tag{29}$$

Thus, defining the non-empty compact subset $\Omega'_\xi \triangleq \left[0, \bar{\xi}\right] \subset \Omega_\xi$ (29) implies that $\xi(t) \in \Omega'_\xi,\ \forall t \in [0, \tau_{\max})$.

We now establish the existence of a compact forward invariant set $\mathcal{W}' \subset \mathcal{W}$. Note that, due to the fact that $\delta \in (0, \bar{r})$ and by Assumption 1, there exists $\epsilon \in \mathbb{R}_{>0}$ with $d_{\mathcal{W}}(\mathrm{x}_0) > \epsilon$ and $d_{\mathcal{W}}(\mathrm{x}_d) > \epsilon$ such that

$$\mathrm{u}(\cdot, \mathrm{x}) \equiv u_\gamma(\cdot, \mathrm{x}), \quad \forall \mathrm{x} \in \{q \in \mathcal{W} : d_{\mathcal{W}}(q) \in [0, \epsilon]\}. \tag{30}$$

For $\mathrm{x} \in \mathcal{W}$ such that $d_{\mathcal{W}}(\mathrm{x}) \leq \epsilon$,

$$\begin{aligned}\nabla d_{\mathcal{W}}(\mathrm{x})^\top(\mathrm{x}-\mathrm{x}_d) &= -2\mathrm{x}^\top(\mathrm{x}-\mathrm{x}_d) \\ &\leq -2\left(\|\mathrm{x}\|^2 - \|\mathrm{x}\|\|\mathrm{x}_d\|\right) \\ &< -2\left(\|\mathrm{x}\|^2 - \|\mathrm{x}\|^2\right) = 0.\end{aligned} \tag{31}$$

and thus $\nabla d_{\mathcal{W}}(\mathrm{x})^\top \mathrm{u}(\cdot, \mathrm{x}) > 0$ which in turn implies that

$$\mathrm{x}(t) \in \mathcal{W}' \triangleq \left\{q \in \mathcal{W} : d_{\mathcal{W}}(q) \in [\epsilon, r_{\mathcal{W}}^2]\right\}, \quad \forall t \in [0, \tau_{\max}) \tag{32}$$

establishing the existence of an invariant compact subset of the workspace $\mathcal{W}$.

Eventually, (32) in conjunction with the fact that[4] $\lim_{\mathrm{x}\to d^{-1}(\{0\})} \nabla d^\top \mathrm{u}(\cdot, \mathrm{x}) = +\infty$ imply the existence of a compact set $\mathcal{F}' \subset \bigcap_{[0,\tau_{\max})} \mathcal{F}_t$ such that $\mathrm{x}(t) \in \mathcal{F}',\quad \forall t \in [0, \tau_{\max})$. Moreover, $y(t) \in \Omega'_\xi \times \mathcal{F}' \subset \Omega_\xi \times \mathcal{F}_t,\ \forall t \in$

[4]This is due to the choice of η (see Fig. 1) in conjunction with Proposition 2.

$[0, t_{\max})$ and since $\Omega'_{\xi} \times \mathcal{F}'$ is a compact subset of $\Omega_{\xi} \times \mathcal{F}_t$, the solution $y(t)$ of (20) can be extended for $\tau_{\max} = +\infty$, so that for all $t \in \mathbb{R}_{\geq 0}$, $\|\mathrm{x}(t) - \mathrm{x}_d\|^2 \leq \rho(t)$, and $\mathrm{x}(t) \in \mathcal{F}_t$. $\square$

Remark 3. For the measure zero set of initial conditions that are not explicitly examined in the proof the solution has a finite escape time. Nevertheless safety is guaranteed for the whole interval of existence.

IV. Extension to Generalized Sphere Worlds

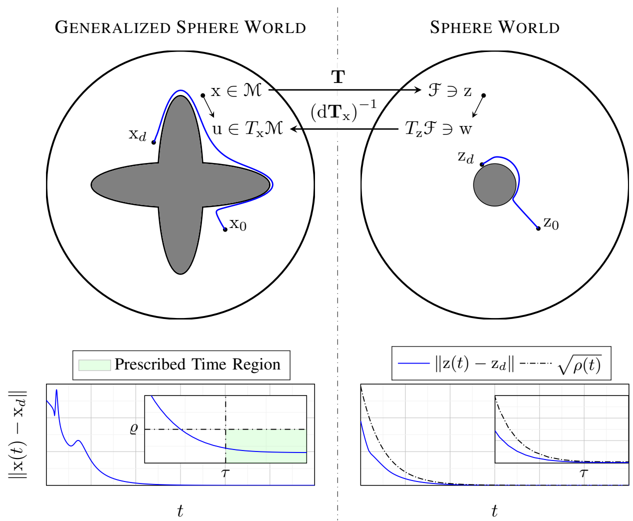


Fig. 2: Prescribed scale time navigation on a generalized sphere world: (top) Points $\mathrm{x} \in \mathcal{M}$ are mapped onto points $\mathrm{z} = \mathbf{T}(\mathrm{x})$. Velocity vectors w are calculated in $\mathcal{F}$ according to Theorem 1 and then "pulled-back" to M as $\mathrm{u} = (\mathrm{d}\mathbf{T}_{\mathrm{x}})^{-1}\,\mathrm{w}$. (bottom) Choosing ϱ' according to Theorem 2 guarantees that the robot exhibits the desired prescribed time scale behavior.

In this section, we discuss how to apply the aforementioned results to a wider class of configuration spaces than the sphere worlds described in Section II. In particular, we present a practical way for straightforwardly applying the proposed methodology to the class of configuration spaces called *generalized sphere worlds* (see Fig. 2).

Definition 1 *A configuration space $\mathcal{M}$ is called a generalized sphere world if there exists a diffeomorphism $\mathbf{T} : \mathcal{M} \to \mathcal{F}$ such that $\mathcal{F}$ is a sphere world.*

The construction of such transformations is beyond the scope of this work and the interested reader is referred to [7], [8] and [15] for a thorough treatment.

The results of this section are presented below (refer also to Fig. 2):

Theorem 2 *Let $\mathcal{P} = (\varrho, \tau, \mathrm{x}_0, \mathrm{x}_d) \in (\mathbb{R}_{>0})^2 \times \mathcal{M}_0 \times \mathcal{M}_{>\tau}$ be an instance of Problem 1 and $(\mathbf{T}_t)_{t \in \mathbb{R}_{\geq 0}} : \mathcal{M}_t \to \mathcal{F}_t$ be a C^1 family of diffeomorphisms, such that for every pair $t_1, t_2 \in \mathbb{R}_{\geq 0}$, $\mathbf{T}_{t_1}|_{\mathcal{M}_{>\tau}} \equiv \mathbf{T}_{t_2}|_{\mathcal{M}_{>\tau}}$, where $(\mathcal{F}_t)_{t \in \mathbb{R}_{\geq 0}}$ is a collection of sphere worlds. Furthermore, let $\mathrm{z}_0 \triangleq \mathbf{T}_0(\mathrm{x}_0)$, $\mathrm{z}_d \triangleq \mathbf{T}|_{\mathcal{M}_{>\tau}}(\mathrm{x}_d)$ and $\varrho' \triangleq \min\{\|\mathbf{T}(\mathrm{x}) - \mathrm{z}_d\| : \mathrm{x} \in \{q \in \mathcal{M}_{>\tau} : \|q - \mathrm{x}_d\| = \varrho\}\}$. Let w be the controller* (15) *that solves Problem 1 characterized by the tuple $(\varrho', \tau, \mathrm{z}_0, \mathrm{z}_d)$, then*

$$\mathrm{u}(t, \mathrm{x}) = (\mathrm{d}\mathbf{T}_{\mathrm{t}})_{\mathrm{x}}^{-1}\, \mathrm{w}(t, \mathbf{T}_{\mathrm{t}}(\mathrm{x})) \tag{33}$$

where $(\mathrm{d}\mathbf{T}_t)_{\mathrm{x}} : T_{\mathrm{x}}\mathcal{M}_t \to T_{\mathbf{T}_t(\mathrm{x})}\mathcal{F}_t$ is the Jacobianof $\mathbf{T}_t$ at $\mathrm{x} \in \mathcal{M}_t$, solves $\mathcal{P}$.

Proof. Let $\mathrm{z}(t) = \mathbf{T}_t(\mathrm{x})$. Then, $\dot{\mathrm{z}}(t) = (\mathrm{d}\mathbf{T}_{\mathrm{t}})_{\mathrm{x}}\,\dot{\mathrm{x}} + \frac{\partial \mathbf{T}_t}{\partial t}$. Substituting control law (33) yields $\dot{\mathrm{z}}(t) = \mathrm{w}\,(t, \mathrm{z}) + (\mathrm{d}\mathbf{T}_{\mathrm{t}})_{\mathrm{x}}^{-1} \frac{\partial \mathbf{T}_t}{\partial t}$. The second term of the previous equation is uniformly bounded owing to the bounded velocity of the obstacles and the fact that we must only examine the interval $[0, \tau]$. Thus Theorem 1 suggests that

$$\mathrm{z}(t) \in \mathcal{F}_t, \quad \forall t \in \mathbb{R}_{>0}, \text{ and} \tag{34}$$

$$\|\mathrm{z}(t) - \mathrm{z}_d\| \leq \varrho', \quad \forall t \in \mathbb{R}_{\geq \tau}. \tag{35}$$

Since $\mathbf{T}_{\mathrm{t}}$ are diffeomorphisms from $\mathcal{M}_t$ onto $\mathcal{F}_t$, ${\mathbf{T}_{\mathrm{t}}}^{-1}(\mathcal{F}_t) = \mathcal{M}_t$; thus (34) implies $\mathrm{x}(t) = {\mathbf{T}_{\mathrm{t}}}^{-1}(\mathrm{z}(t)) \in \mathcal{M}_t$, $\forall t \in \mathbb{R}_{\geq 0}$. Finally, by the definition of ϱ', it holds that

$$\{q \in \mathcal{F}_{>\tau} : \|q - \mathrm{z}_d\| \leq \varrho'\} \subset \\ \subset \mathbf{T}|_{\mathcal{M}_{>\tau}}(\{q \in \mathcal{M}_{>\tau} : \|q - \mathrm{x}_d\| \leq \rho\}) \tag{36}$$

which in conjunction with (35) implies that

$$\|\mathrm{x}(t) - \mathrm{z}_d\| \leq \varrho, \quad \forall t \in \mathbb{R}_{\geq \tau} \tag{37}$$

and concludes the proof. $\square$

Remark 4. The value ϱ' can be either calculated analytically or estimated — up to arbitrary precision granted by the smoothness properties of the diffeomorphism $\mathbf{T}|_{\mathcal{M}_{>\tau}}$— through sampling.

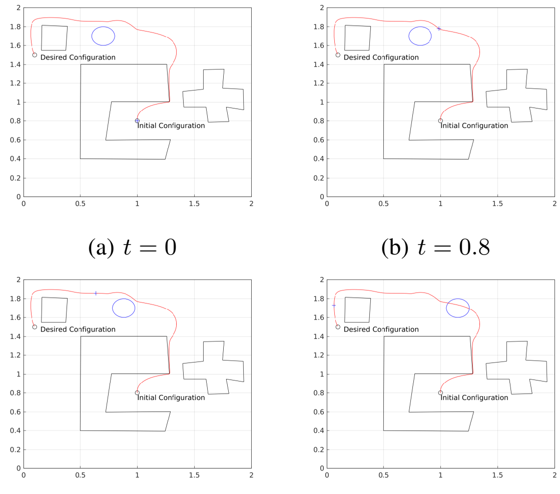
V. Numerical Simulation

In order to validate the efficacy of our approach, we present in this section simulation results of an indicative case study carried out using Matlab. In this scenario, the robot was instructed to navigate towards a desired configuration while avoiding collision with several static and a single moving obstacle occupying its workspace (see Fig. 3). To cope with the complexity of the workspace, a local transformation, described in [12], was used for mapping the moving disk-shaped obstacle into an isolated point, followed by a global transformation based on Harmonic Maps, introduced in [16], for mapping the static part of the workspace into a point world. Thus, designing our control inputs in the transformed workspace allows us to trivially extend the results of Section III to complex workspaces, such as the one considered here, validating the theoretical findings of Section IV.

As can be seen in Fig. 6, the proposed control scheme manages to successfully drive the robot to its goal configuration while satisfying the given specifications ($\tau = 4$ and $\varrho = 0.1$). Furthermore, Fig. 3 captures the configuration of the robot and its environment at four distinct time instances, showing that an otherwise imminent collision with the uncooperative moving obstacle was successfully prevented, a fact that one can also observe in Fig. 4. Finally, the control inputs required for accomplishing the given task under the corresponding specifications can be seen in Fig. 5.

VI. Conclusion & Future Work

In this paper we have proposed a control scheme that solves the Prescribed Time Scale Navigation Problem in spaces with moving obstacles. Namely, we have constructed a control algorithm that operates on sphere worlds and drives almost all initial configurations within any desired



(a) $t = 0$ (b) $t = 0.8$

(c) $t = 2$ (d) $t = 3$

Fig. 3: Configuration of the robot (marked with a blue cross) and its workspace at four distinct time instances. The robot's complete trajectory (up to $t = 6$) is plotted on all four subfigures (red line).

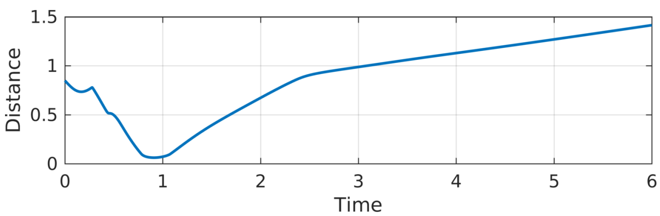


Fig. 4: Distance between robot and moving obstacle.

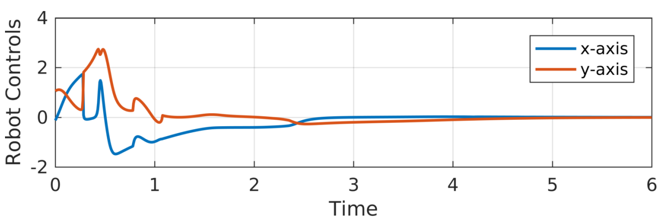


Fig. 5: Control effort.

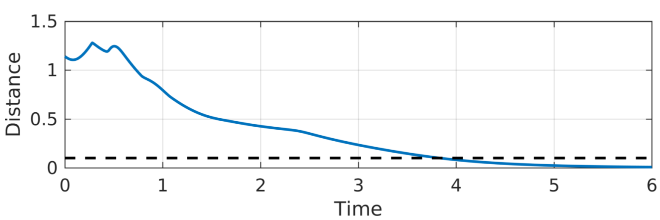


Fig. 6: Distance of the robot from the desired configuration (solid line). The dashed line corresponds to the value of the specified distance $\varrho = 0.1$ that must be attained by time $\tau = 4$.

distance from any configuration that is eventually sufficiently remote from moving obstacles in a predetermined time span while avoiding collisions. Furthermore, we have extended the applicability of the approach by showing how it can be applied to configuration spaces which can be mapped diffeomorphically and smoothly with respect to time to sphere worlds and verified the approach through a numerical simulation.

A particularly interesting question for future research efforts revolves around relaxing the smoothness (with respect to time) property of the diffeomorphisms which is an inevitable necessity towards addressing the same problem in spaces where topological changes occur (*e.g.,* merging of moving obstacles.)

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Appendix

For $\delta \in \mathbb{R}_{>0}$ we define the following two one-parameter families of switches $\sigma_\delta, s_\delta \in C^1(\mathbb{R}, \mathbb{R})$ by

$$\sigma_\delta(x) = \begin{cases} 1, & x \in \mathbb{R}_{\leq 0} \\ 2\left(\frac{x}{\delta}\right)^3 - 3\left(\frac{x}{\delta}\right)^2 + 1, & x \in (0, \delta) \\ 0, & x \in \mathbb{R}_{\geq \delta} \end{cases} \tag{38}$$

$$s_\delta(x) = \begin{cases} -1, & x \in \mathbb{R}_{\leq 0} \\ -4\left(\frac{x}{\delta}\right)^3 + 6\left(\frac{x}{\delta}\right)^2 - 1. & x \in (0, \delta) \; . \\ 1, & x \in \mathbb{R}_{\geq \delta} \end{cases} \tag{39}$$

Also, let us define $\sigma_{\beta_i}(\mathrm{x}) \triangleq (\sigma_{r_i^2} \circ \beta_i)(\mathrm{x})$, $\sigma_{d_i}(\mathrm{x}) \triangleq (\sigma_{\delta_i} \circ d_i)(\mathrm{x})$, $s_{d_i}(\mathrm{x}) \triangleq (s_{\delta_i} \circ d_i)(\mathrm{x})$, where $\delta_i \triangleq \delta(\delta + 2r_i)$. Finally, let us define $C_1(\mathrm{x}) \triangleq (1 - \sigma_{\beta_i}(\mathrm{x})) \frac{\partial s_{d_i}}{\partial d_i}(d_i(x))$ and $C_2(\mathrm{x}) \triangleq (1 - s_{d_i}(\mathrm{x})) \frac{\partial \sigma_{\beta_i}}{\partial \beta_i}(\beta_i(\mathrm{x}))$. We note that for every $\epsilon \in \mathbb{R}_{>0}$ both C_1 and C_2 are positive and bounded for $\mathrm{x} \in \eta^{-1}([-1, 1-\epsilon])$.