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On Connection between Two Conventional Types of Impulsive Control Systems in Respect of Sensitivity and Relaxation

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***Abstract*— In the first part of the paper, we establish a connection between the two mainstream frameworks of impulsive control — systems with explicit jump conditions, and measure-driven systems: we show how to reformulate impulsive system of the first type as a complementarity problem for a measure differential equation. In the second part, the point of our interest is the limit behavior of ill-posed impulse control systems after infinitesimal perturbations of impulsive actions. The reduction, proposed in the first part, is applied to design i) a trajectory relaxation — the compactification of the trajectory tube of the original system in the weak* topology of the space BV of functions with bounded variation and ii) the desired set of limit solutions to the perturbed complementarity problem. The main result is a constructive representation of the limit solutions by a special discontinuous space-time transformation.**

I. INTRODUCTION

Impulsive control is a branch of mathematical control theory with discontinuous states. It draws inspiration from real-life applications to robotics, telecommunication, economics and management [5], [11], [18], [41], [43], [58], where modeled objects or processes may change their current state in very short time periods, so that these changes are naturally idealized as instantaneous jumps occuring at prescribed (or desired) time instants.

In the bibliography, related to impulsive systems, we meet two popular frameworks, which are rather resembling in spirit, but different in their mathematical background.

A. Impulsive systems with explicit specification of jumps

The first (and, somehow, a simpler) setting is presented, e.g., by [5], [11], [23], [30], [41], [56], [59], where impulsive transitions are defined through explicit jump conditions. Models of this sort look as a combination of an ordinary differential equation

$$\frac{dx}{dt} \doteq \dot{x} = f(x) \quad \lambda\text{-a.e. on } \mathcal{T}, \tag{1}$$

and a series of discrete events

$$x(\tau) \in \Phi\big(x(\tau^-)\big), \quad \tau \in \mathcal{I}, \tag{2}$$

where $f : \mathrm{X} \doteq \mathbb{R}^n \to \mathrm{X}$ and $\Phi : \mathrm{X} \rightrightarrows \mathrm{X}$ are given single- and multivalued functions of sufficient regularity; $\mathcal{I} \subset \mathcal{T} \doteq [0, T]$, $T > 0$, is an at most countable set playing the part of a control input; $x(\tau^-)$ denotes the left one-sided limit of a state trajectory $x : \mathcal{T} \to \mathrm{X}$ at a point τ (hereinafter, we agree to operate with right continuous trajectories); λ denotes the usual Lebesgue measure on the real line $\mathbb{R}$, and "a.e." abbreviates "almost everywhere with respect to (w.r.t.) the associated measure" (below, we will deal with different measures).

In some works, e.g., [31], we also meet a more general form of jump condition (2):

$$\big(x(\tau^-), x(\tau)\big) \in \mathcal{Z}, \quad \tau \in \mathcal{I}, \tag{3}$$

with a given closed set $\mathcal{Z} \subseteq \mathrm{X}^2$ called the reset map. In the context of control theory, the selection of points $\big(x(\tau^-), x(\tau)\big)$, $\tau \in \mathcal{I}$, within $\mathcal{Z}$ is due to the guide, and together with the set $\mathcal{I}$ of jump points it represents an extra control input.

In practice, systems of the raised class may describe an object drifting in the vector field (1), whose proper motion can be corrected at instants $\tau \in \mathcal{I}$, chosen by the controller, while the diapason of admissible corrections is specified through inclusion (3).

Though model (1), (3) is simple and useful for applications, its mathematically accurate statement typically requires specific and rather restrictive assumptions on the input data, ensuring the compactness of the tube of solutions, and, say, an a priori prescribed separation of adjacent jump instants. Without such assumptions, the model generically becomes ill-posed, and related variational problems lack the existence of solution, even in the simplest cases [21], [43].

In what follows, we accept the two technical assumptions with respect to system (1), (3): the function f is locally Lipschitz continuous and satisfies the sublinear growth condition, and there exists a real $M > 0$ such that $\mathrm{Var}_{\mathcal{T}}\, x \le M$, where $\mathrm{Var}_A\, \phi$ denotes the total variation of a function ϕ on a set A. The latter assumption can be regarded as an extra constraint for (1), (3).

B. Measure-driven systems

Another setup [1]–[4], [6]–[8], [12]–[15], [17], [18], [20], [26]–[29], [33]–[39], [42]–[51], [53], [57], [58], [60]–[63], which is also referred in the literature to as "impulsive control systems" is based on trajectory relaxation of ordinary control-affine equations

$$\dot{x} = f(x) + G(x)\, u \text{ for } \lambda\text{-a.a. } t \in \mathcal{T} \tag{4}$$

("a.a." abbreviates "almost all") with a "soft" (energetic) constraint on the input signals being measurable functions $u : \mathcal{T} \to \mathbb{R}^m$:

$$\int_{\mathcal{T}} \|u(t)\|_{\mathbb{R}^m}\, dt \le M.$$

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Here, $G : \mathrm{X} \to \mathbb{R}^{n\times m}$ is a given matrix-valued function representing the "singular dynamics".

Impulsive trajectory relaxation of dynamics (4) in a relevant weak topology (customary, the topology of pointwise convergence or the weak* topology of the space $BV = BV(\mathcal{T}, \mathrm{X})$ of functions with bounded variation) leads to a measure differential equation (MDE) of the form

$$dx = f(x)\,dt + G(x)\,\mu(dt), \tag{5}$$

whose solutions yet can be discontinuous BV-functions. Here, the part of control is played by the vector-valued Borel measure $\mu = \mu(dt) \in C^*(\mathcal{T}, \mathbb{R}^m) \simeq BV(\mathcal{T}, \mathbb{R}^m)$,[1] that is a generalization of signal $u\,dt$.

In this setup, jump exit points are not specified by an a priori condition, but should be computed as solutions to a limit system [29], which describes the evolution of a state inside the phase of jump.

In the next section, we throw a bridge between the raised two concepts of impulsive control systems, and show how impulsive system (1), (3) can be incorporated into the framework of measure-driven equations.

II. CONNECTION BETWEEN TWO FRAMEWORKS

Consider system (1), (3) with a given initial position $x(0^-) = x_0 \in \mathrm{X}$.

First notice that, in contrast to (5), model (1), (3) does not contain impulsive actions explicitly, which would be in charge of state discontinuities. Meanwhile, such actions are implicitly involved and can be formally introduced through a "slack variable" being a singular atomic n-dimensional measure

$$\mu \doteq \sum_{\tau\in\mathcal{I}} w_\tau\,\delta_\tau,$$

where $w_\tau \in \mathrm{X}$ and δ_τ are Dirac point-mass measures (distributions) supported at points τ. This implies rewriting (1) as an MDE similar to (5):

$$x(0^-) = x_0; \quad dx = f(x)\,dt + \mu(dt), \quad t \in \mathcal{T},$$

whose solution is defined by the integral equivalent

$$x(t) = x_0 + \int_0^t f\big(x(\theta)\big)\,d\theta + F_\mu(t), \quad t \in \mathcal{T}, \tag{6}$$

while (3) can be reformulated as follows:

$$\big(x(\tau^-), x(\tau)\big) \in \mathcal{Z}, \quad \tau \in D_\mu. \tag{7}$$

Here, F_μ is the distribution function of μ with $F_\mu(0^-) = 0$, and D_μ denotes the discrete support of μ:

$$D_\mu \doteq \{t \in \mathcal{T} : \mu(\{\tau\}) \neq 0\}.$$

Jumps $\Delta x(\tau) \doteq x(\tau) - x(\tau^-)$ of a trajectory x are computed as

$$\Delta x(\tau) = \mu(\{\tau\}).$$

[1] In fact, this is the case, only if the vector fields defined by the columns of the matrix G satisfy the Frobenius commutativity assumption [43], [50], [54]. In general, the notion of impulse control is more complicated: it is a measure together with additional objects [2], [3], [33], which are required to perform a unique selection of the multivalued input-output map.

Conditions (6), (7) provide a useful reformulation of system (1), (3) as a specific complementarity problem for a particular measure differential equation. To clarify the term "complementarity", one can assume (in fact, without loss of generality) that the set $\mathcal{Z}$ is represented as

$$\mathcal{Z} = \{(x, y) \in \mathrm{X}^2 : W(x, y) = 0\},$$

where $W : \mathrm{X}^2 \to \mathbb{R}$ is a certain continuous nonnegative function. Setting

$$y(t) = W\big(x(t^-), x(t)\big), \quad t \in \mathcal{T},$$

(7) turns into the "orthogonality" condition

$$y \perp |\mu|$$

($|\mu|$ stands for the total variation of μ), which says that

$$y(\tau)\,\|w_\tau\| = 0 \text{ for all } \tau \in D_\mu.$$

In what follows, we will need to operate with general (in particular, continuous) measures. In view of this, we remark here that (7) can be also equivalently written as

$$W\big(x(t^-), x(t)\big) = 0 \text{ for } |\mu|\text{-a.a. } t \in \mathcal{T}, \tag{8}$$

or, alternatively, in terms of the support of a measure:

$$\operatorname{supp}|\mu| \in \big\{t \in \mathcal{T} : W\big(x(t^-), x(t)\big) = 0\big\},$$

Condition (8) represents a more convenient form of the complementarity, which makes sense for continuous measures as well.

Thus, any impulsive system (1), (3) can be formally reduced to a complementarity problem for an MDE. However, as we will observe below, one has to be accurate when performing such a reduction to obtain a meaningful, well-posed model.

III. ILL-POSEDNESS AND RELAXATION

In the forthcoming sections, we discover that even the simplest impulsive systems (1), (3) can be ill-posed because of i) the lack of compactness of the trajectory tube, and/or ii) sensitivity with respect to small perturbations of impulsive signals, and we look for an efficient way to overcome these difficulties.

A. Relaxation of impulsive dynamics

One of the reasons for considering (6), (8) instead of (1), (3) is due to the fact that (6), (8) suggests a straight way to design a trajectory relaxation of (1), (3). By the trajectory relaxation we mean a compactification of the tube of solutions to impulsive systems (1), (3) in a relevant weak topology, which, in our option, is the weak* topology of BV. In our case, to perform such a compactification, one first needs to admit in (6), (8) general type Borel measures with a nontrivial continuous component μ_c.

To illustrate that this is needed, if we search for a well-posed counterpart of (1), (3), consider the following simple planar case:

$$x_1(0^-) = x_2(0^-) = 0; \ \dot{x}_1 = \dot{x}_2 = 1 \ \lambda\text{-a.e. on } [0, 1], \tag{9}$$

$$x_2(\tau) = 0, \ \tau \in \mathcal{I}. \tag{10}$$

Take a natural number n, and place impulses at instants $\tau = \frac{k}{n}$, $k = \overline{1,n}$. By increasing n, we observe that the respective solution $(x_1, x_2)^n(t) = (t, t - \frac{k-1}{n})$, $t \in [\frac{k-1}{n}, \frac{k}{n})$, $(x_1, x_2)^n(1) = (1, 0)$, tends uniformly on $[0, 1]$ to a continuous function $t \mapsto (t, 0)$. The latter is not admitted by the impulsive system, but satisfies its counterpart of the type (6), (8):

$$x_1(0^-) = x_2(0^-) = 0;\ d(x_1, x_2) = (1, 1)\,dt + \mu(dt), \quad (11)$$

$$x_2(t) = 0 \text{ for } |\mu|\text{-a.a. } t \in [0, 1], \quad (12)$$

with the continuous measure $\mu = (0, -dt)$.

This example illustrates one of the effects, which cause the ill-posedness of impulsive systems (1), (3), namely, the accumulation of jumps up to a continuous motion (so called impulsive sliding modes). Another phenomenon, which also has to be taken into account by a proper relaxation, is due to occurrence of multiple jumps at a time instant. This is to be discussed in Section III-D.

B. Sensitivity with respect to perturbations and the risk of hyper-relaxation

Consider system (6), (8) and suppose that a control signal μ, sent by the guide, is translated into the model with an error of a small total magnitude: Let $x = x[\mu]$ be the exact, expected solution. Assume that we apply the control w_τ at an instant τ and try to calculate the jump $\Delta x(\tau) \doteq x(\tau) - x(\tau^-)$. Prior to jump – due to errors in measurement – instead of a nominal value $x(\tau^-)$, we actually deal with a vector $\underline{x}(\tau^-)$, where $\underline{x}$ is a perturbation of x in a certain natural topology (to be specified below). Analogously, the actual state after the jump is not $x(\tau)$ but a vector $\overline{x}(\tau)$, where $\overline{x}$ is another perturbation of x. Since the pair $\big(\underline{x}(\tau^-), \overline{x}(\tau)\big)$ may be out of the relation $\mathcal{Z}$, we are to assume that condition (8) is satisfied with some accuracy. We ask what happens with the tube of solutions to the perturbed system as the parameter of perturbation tends to zero. One can expect that, as regular, this set somehow tends to the trajectory tube of (6), (8). However, this is not the case, even for the simplest models, and this is another side of ill-posedness.

For example, consider a trivial one-dimensional impulsive system

$$x(0^-) = 0; \quad \dot{x} = 0 \ \lambda\text{-a.e. on } [0, 1], \quad (13)$$

$$x(\tau) = x(\tau^-),\ \tau \in \mathcal{I}, \quad (14)$$

with an obvious unique solution $x \equiv 0$ for any set $\mathcal{I}$. A formal equivalent of (13), (14) is the following complementarity problem:

$$x(0^-) = 0; \quad dx = \mu(dt), \quad t \in [0, 1], \quad (15)$$

$$\Delta x(t) = 0 \text{ for } \mu\text{-a.a. } t \in [0, 1], \quad (16)$$

with $\mu(dt) \doteq \sum_{\tau \in \mathcal{I}} w_\tau\, \delta_\tau$, $w_\tau \in \mathbb{R}$.

We are going to study the sensitivity of the set of solutions to system (15), (16) with respect to small perturbations of condition (16) under an extra constraint

$$|\mu|([0, 1]) \leq 1/3,$$

which trivially holds for the exact solution. Given $\varepsilon > 0$, consider the following perturbation of (16):

$$\sum_{\tau \in D_\mu} |\overline{x}(\tau) - \underline{x}(\tau^-)|\, |w_\tau| \leq \varepsilon, \quad (17)$$

where $(\underline{x}, \underline{\mu})$, $\underline{x} \doteq x[\underline{\mu}]$, and $(\overline{x}, \overline{\mu})$, $\overline{x} \doteq x[\overline{\mu}]$, are such that $|\underline{\mu}| \preceq |\mu| \preceq |\overline{\mu}|$ ($\preceq$ denotes the standard order on the set of measures; this is just a technical assumption, for convenience), and

$$\|\underline{x} - x\|_C + \|x - \overline{x}\|_C \leq \varepsilon. \quad (18)$$

In this example, we consider uniform perturbations of x. In general, one can admit perturbations in the weak* topology of BV, which is a natural topology for the context of measure-driven systems [43], see Definition 3.1 below. One can observe that arbitrarily small (in the proposed sense) errors in the input signal can essentially change the expected solution. Indeed, take $\varepsilon = 1/n$, and consider the measures

$$(\overline{\mu}^n, \mu^n, \underline{\mu}^n) = \frac{1}{3n}\left(\sum_{k=1}^{n} \delta_{\frac{k-1}{n}},\ \sum_{k=1}^{n} \delta_{\frac{3k-2}{3n}},\ \sum_{k=1}^{n} \delta_{\frac{3k-1}{3n}}\right).$$

As is simply checked, the pair (x^n, μ^n), $x^n = x[\mu^n]$, satisfies (17), (18) with $\underline{x}^n = x[\underline{\mu}^n]$ and $\overline{x}^n = x[\overline{\mu}^n]$, while the trajectory

$$x^n(t) = \begin{cases} 0, & t \in \big[0, \frac{1}{3n}\big), \\ \frac{k}{3n}, & t \in \big[\frac{3k-2}{3n}, \frac{3k+1}{3n}\big), \\ 1/3, & t \in \big[1 - \frac{2}{3n}, 1\big) \end{cases}$$

uniformly tends to the continuous function $1/3\, t$ as $n \to \infty$.

This toy example illustrates a very pathological effect: though the trajectory tube of the system is compact (the one-point set consisting of a single solution $x \equiv 0$), the model is not well-posed yet, since infinitesimal perturbations of solution result in inflation of the integral funnel (here, it inflates up to the rectangle $[0, 1] \times [0, 1/3]$ in the extended state space). We call such an inflated funnel the hyper-relaxation of an impulsive system. The exact trajectory tube $\mathbb{X}$ of a unperturbed system is not dense in the hyper-relaxation $\widetilde{\mathbb{X}}$ (as the present example illustrates, $\mathbb{X}$ can be “almost nothing” inside $\widetilde{\mathbb{X}}$). The hyper-relaxation is a consequence of perturbations of “fictitious jumps” to be avoided by a correct model formulation. To this end, one has to weed out ambiguous, “inactive” components μ_j, $j \in \{1, \ldots, n\} \setminus \mathbb{J}$, of the slack variable μ, where the index set $\mathbb{J}$ contains the numbers of state coordinates x_j that can endure nontrivial jumps due to the relation $\mathcal{Z}$, i.e., $j \in \mathbb{J}$ iff there exists a pair of vectors $x, y \in \mathrm{X}$ with $(x, y) \in \mathcal{Z}$ and $\|y_j - x_j\| \neq 0$. This can be done, say, by considering, instead of (6), the equation

$$x(t) = x_0 + \int_0^t f\big(x(\theta)\big)\, d\theta + A\, F_\mu(t), \quad t \in \mathcal{T}, \quad (19)$$

with the matrix $A = (a_{ij})_{i,j=\overline{1,n}}$, $a_{ij} = 1$ for $i = j \in \mathbb{I}$ and $a_{ij} = 0$ otherwise.

Let us return to the case (9), (10). As one can observe, system (11), (12), is, in fact, a hyper-relaxation; among limit solutions of this system there are functions performing a single jump $(0,0) \to (1,0)$ at different time instants, which look quite surprising: jumps of such limit solutions are made in a "prohibited" direction, orthogonal to the direction of original jumps. In fact, these jumps are results of a continuous motion in the singular phase (revise theorem 3.2 below). Note that the hyper-relaxation naturally implies the appearance of the Lavrentiev phenomenon in related extremal problems. For instance, consider the following problem for system (11), (12): reach the point $(1,0)$ time-optimally, under the constraint $|\mu|([0,1]) \le 1$. Even for the relaxed system (with a general-type measure) the solution is $T^* = 1$. At the same time, in the hyper-relaxed model, the desired terminal state can be reached instantaneously, i.e., $T^* = 0$.

To avoid hyper-relaxation of (9), (10), one can deal with the following complementarity problem, instead of (11), (12):

$$x_1(0^-) = x_2(0^-) = 0; \ \dot{x}_1 = 1, \ dx_2 = dt + \mu(dt),$$
$$x_2(t) = 0 \text{ for } |\mu|\text{-a.a. } t \in [0,1],$$

with a scalar measure $\mu = \mu_1$.

C. Proper relaxation

Now, we give a formal general definition of ε-perturbation of the complementarity problem.

Let $Q: \mathbb{R}^3 \to \mathbb{R}$ be a nonnegative function vanishing only on the cone $\{(a,b,c) \in \mathbb{R}^3_+ : \ a \le b \le c\}$.

Definition 3.1: Given $\varepsilon > 0$, a couple (x,μ), $x = x[\mu]$, is said to be an ε-solution of the complementarity problem (19), (8) if there exist two "comparison" control processes $(\underline{x}, \underline{\mu})$, $\underline{x} \doteq x[\underline{\mu}]$, and $(\overline{x}, \overline{\mu})$, $\overline{x} \doteq x[\overline{\mu}]$, with the following properties:

(i) Processes $(\underline{x}, \underline{\mu})$ and $(\overline{x}, \overline{\mu})$ belong to an ε-neighborhood of the process (x,μ) in the weak* topology of BV, i.e.,
$$\big\|(F_{|\underline{\mu}|}, \underline{x}) - (F_{|\mu|}, x)\big\| + \big\|(F_{|\mu|}, x) - (F_{|\overline{\mu}|}, \overline{x})\big\| \le \varepsilon$$
over $\big((0,T) \setminus D_{|\mu|}\big) \cup \{T\}$.

(ii) An ε-approximate "sandwich rule" is fulfilled:
$$\sum_{\tau \in D_{\check{\mu}}} Q\big(F_{|\underline{\mu}|}(\tau), F_{|\mu|}(\tau), F_{|\overline{\mu}|}(\tau)\big)\, \check{\mu}(\{\tau\}) \le \varepsilon.$$
Here, $\check{\mu} \doteq |\underline{\mu}| + |\mu| + |\overline{\mu}|$.

(iii) A perturbed version of the orthogonality condition holds:
$$\sum_{\tau \in D_\mu} W\big(\underline{x}(\tau^-), \overline{x}(\tau)\big)\ \mu(\{\tau\}) \le \varepsilon. \qquad (20)$$

Note that (20) is a relaxed version of a weak form of ordering $|\underline{\mu}| \preceq |\mu| \preceq |\overline{\mu}|$, which is a technical assumption performing a "sandwich rule" (see the example above), which serves to distinguish pairs (x,y) and (y,x) in respect of their belonging to the relation $\mathcal{Z}$.

Let $\overline{\mathbb{X}}$ denote the limit set for trajectory tube of the perturbed system, i.e., the set of all functions $x \in BV$ such that $x_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} x$ in the weak* topology of BV, where x_ε is an ε-solution of the complementarity problem (19), (8). We call $\overline{\mathbb{X}}$ the *proper* relaxation of system (1), (3), since it performs the complete description of the limit behavior of all impulsive solutions under infinitesimal perturbations. The points of $\overline{\mathbb{X}}$ are said to be generalized solutions of system (1), (3). Below, we effort to evaluate a constructive representation of $\overline{\mathbb{X}}$ by a singular time-spatial transform, in the spirit of [24], [25].

Note that, in the bibliography related to impulsive and hybrid control systems, one can meet other aspects of sensitivity, such as robustness (stability with respect to small perturbations of impulsive control, or sensitivity analysis of the system behavior at discrete events, see e.g. [9], [10], [19], [22], [32], [43]. Here one can also mention a rather different context, where the term "stability of impulsive systems" is more commonly used — the stability of the equilibrium position of an impulsive system, in the spirit of Lyapunov, — which is an important and popular area of impulsive control theory (see, e.g., [11], [15], [23], [30], [40], [41], [47]).

D. Space-time transform and limit behavior of generalized solutions

Consider an auxiliary *reduced* ordinary differential inclusion with a new time variable s, acting on an interval $\mathcal{S} \doteq [0,S]$, $S \ge T$:

$$\mathbf{x}(0) = \mathbf{x}_0; \quad \frac{d\mathbf{x}(s)}{ds} \in \operatorname{co}\big\{\mathbf{F}(\mathbf{x},\mathbf{u})\,|\ \mathbf{u} \in \mathbf{U}\big\}. \qquad (21)$$

Here co means the convex hull of a set; $\mathbf{x} \in \mathbb{R}^{3n+5}$,

$$\mathbf{x} \doteq \big(\xi \in \mathbb{R}, \hat{y} \doteq (\underline{y}, y, \overline{y}) \in \mathrm{X}^3, \hat{\eta} \doteq (\underline{\eta}, \eta, \overline{\eta}) \in \mathbb{R}^3, \zeta \in \mathbb{R}\big),$$
$$\mathbf{u} \doteq \big(\alpha \in [0,1],\ \hat{\beta} \doteq (\underline{\beta}, \beta, \overline{\beta}) \in (B^n)^3\big) \in \mathbb{R}^{3n+1},$$

$$\mathbf{F} \doteq \begin{pmatrix} \alpha \\ \alpha\, f(\underline{y}) + A\,\underline{\beta} \\ \alpha\, f(y) + A\,\beta \\ \alpha\, f(\overline{y}) + A\,\overline{\beta} \\ \|\underline{\beta}\| \\ \|\beta\| \\ \|\overline{\beta}\| \\ \alpha\big(\Delta\hat{\eta} + \|\Delta\hat{y}\|\big) + (1-\alpha)\, Q(\eta) + \|\beta\|\, W(\underline{y}, \overline{y}) \end{pmatrix},$$

and

$$\mathbf{x}_0 \doteq \big(0,\ (x_0, x_0, x_0),\ (0,0,0),\ 0\big),$$

subject to the following terminal constraint:

$$\xi(S) = T,\ \Delta(y,\eta)(S) = 0 \in \mathbb{R}^{2(n+1)},\ \eta_+(S) \le M. \qquad (22)$$

Here, B^n stands for the unit ball in $\mathbb{R}^n$ with the respective norm; the set $\mathbf{U} \doteq \mathbf{U}(S)$ consists of collections $\mathbf{u}$ of Borel measurable vector functions $\mathcal{S} \to [0,1] \times (B^n)^3$, such that $\alpha(s) \ge 0$, $\alpha(s) + \|\underline{\beta}(s)\| + \|\beta(s)\| + \|\overline{\beta}(s)\| = 1$ λ-a.e. on $\mathcal{S}$. The operation Δ, applied to a vector $c = (\underline{c}, c, \overline{c})$, returns the vector $(\overline{c} - c, c - \underline{c})$.

The relation between $\overline{\mathbb{X}}$ and the trajectory tube of (21), (22) is established by the following

Theorem 3.1: 1) Denote $\tilde{\mu} \doteq \lambda + 3|\mu|$, and introduce a strictly increasing function $\Upsilon : \mathcal{T} \to [0, (\lambda + \tilde{\mu})(\mathcal{T})]$,

$$\Upsilon(t) = t + F_{\tilde{\mu}}(t),\ t \in \mathcal{T},$$

and its inverse υ : $[0, (\lambda + \tilde{\mu})(\mathcal{T})] \to \mathcal{T}$. For any $x \in \overline{\mathbb{X}}$, there exist $S \geq T$ and a solution $\mathbf{x} = (\xi, \hat{y} = (\underline{y}, y, \overline{y}), \hat{\eta} = (\underline{\eta}, \eta, \overline{\eta}), \zeta)$ on $\mathcal{S} \doteq [0, S]$ of the constrained differential inclusion (21), (22), such that the following relations hold:

$$\underline{y} \circ \Upsilon = y \circ \Upsilon = \overline{y} \circ \Upsilon = x, \tag{23}$$

$$\underline{\eta} \circ \Upsilon = \eta \circ \Upsilon = \overline{\eta} \circ \Upsilon = F_{|\mu|}, \tag{24}$$

$$\upsilon = \xi. \tag{25}$$

2) Let $\mathbf{x} = (\xi, \hat{y} = (\underline{y}, y, \overline{y}), \hat{\eta} = (\underline{\eta}, \eta, \overline{\eta}), \zeta)$ be a solution to the Cauchy problem for differential inclusion (21) on a time interval $\mathcal{S} = [0, S]$, $S \geq T$, such that conditions (22) hold. Define $x \in BV^+(\mathcal{T}, \mathrm{X})$ by the composition

$$x = y \circ \Xi \quad \text{on } \mathcal{T}, \tag{26}$$

where the map $\Xi :\ \mathcal{T} \to \mathcal{S}$ is defined by

$$\Xi(t) = \inf\{s \in \mathcal{S} |\ \xi(s) > t\},\ t \in [0, T),\ \Xi(T) = S. \tag{27}$$

Then, $x \in \overline{\mathbb{X}}$.

The main result of the paper is a constructive representation of the hyper-extension:

Theorem 3.2: For any $x \in \overline{\mathbb{X}}$,

1) there exists a measure $\mu \in C^*(\mathcal{T}, \mathrm{X})$ such that x is a solutions of MDE (6) under control μ, i.e., $x = x[\mu]$;

2) for $|\mu|_c$-a.a. $t \in \mathcal{T}$ it holds

$$\big(x(t), x(t)\big) \in \mathcal{Z};$$

3) for any $\tau \in D_\mu$, the jump $\Delta x(\tau) \doteq \mu(\{\tau\})$ is divided into a series of subjumps, and, possibly, continuous motions in the singular mode: there exist a finite or countable set $\mathbb{I}_\tau$, and a set $\mathcal{R}_\tau \subseteq [0, 1]$, $\mathcal{R}_\tau = \cup_{i \in \mathbb{I}_\tau} \Omega^i_\tau$, where $\Omega^i_\tau \doteq (\underline{\theta}^i_\tau, \overline{\theta}^i_\tau)$, $\underline{\theta}^i_\tau < \overline{\theta}^i_\tau$, $\Omega^i_\tau \cap \Omega^j_\tau = \emptyset$ for all $i, j \in \mathbb{I}_\tau$, $i \neq j$, such that

$$\big(\varkappa_\tau(\underline{\theta}^i_\tau), \varkappa_\tau(\overline{\theta}^i_\tau)\big) \in \mathcal{Z} \quad \forall\, i \in \mathbb{I}_\tau,$$

$$\big(\varkappa_\tau(\theta), \varkappa_\tau(\theta)\big) \in \mathcal{Z} \quad \lambda\text{-a.e. on } [0, 1] \setminus \mathcal{R}_\tau,$$

where $\varkappa_\tau(\theta) = \mu(\{\tau\})\, \theta$, $\theta \in [0, 1]$.[2]

[2]In the most pathological case, the complement $[0, 1] \setminus \mathcal{R}_\tau$ may endure the topological structure of a so-called fat Cantor set. The separation of a jump into subjumps becomes plain, if the relation $\mathcal{Z}$ is anti-reflexive. In this case, part 3) of the theorem just says that, for each $\tau \in D_{|\mu|}$, there is an at most countable collection of vectors $\underline{\nu}^i_\tau, \overline{\nu}^i_\tau \in \mathrm{X}$, $i \in \mathbb{I}_\tau$, such that

$$\Delta x(\tau) = \sum_{i \in \mathbb{I}_\tau} (\overline{\nu}^i_\tau - \underline{\nu}^i_\tau),$$

$$\sum_{i \in \mathbb{I}_\tau} \|\overline{\nu}^i_\tau - \underline{\nu}^i_\tau\| = |\mu|(\{\tau\}),$$

and

$$(\underline{\nu}^i_\tau, \overline{\nu}^i_\tau) \in \mathcal{Z} \text{ for all } i \in \mathbb{I}_\tau.$$

Proofs of Theorems 3.1 and 3.2 are quite technical and cumbersome, and we omit them due to lack of space. We only note that, in principle, the arguments are similar to [24], [25], where analogous results have been derived for a simpler case, when the set $\mathcal{Z}$ takes the form $\mathcal{Z}_- \times \mathcal{Z}_+$ with $\mathcal{Z}_\pm \subseteq \mathrm{X}$ being prescribed jump permitting and jump destination domains. These arguments are due to the known properties [43] of a discontinuous time reparameterization and an accurate analysis of relations, implied by constraints (22). The proofs are supposed to appear in full in our forthcoming journal paper.

IV. CONCLUSION

We believe the paper contributes to the progress in the field of impulsive control and hybrid systems, particularly from the control-theoretical standpoint. The raised concept of sensitivity can be useful for stability analysis and trajectory tracking of technical systems of discrete-continuous type, which are naturally subject to errors in observation of a state and perturbations of discrete events. Another area of applications of the obtained theoretical results are systems with spontaneous transitions and estimated average value of perturbations.

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