2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Cooperative Joint Estimation and Localization using Mobile Multi-agent Systems: A Minimum Energy Estimator Approach

T. Ngoc Ha   A. Pedro Aguiar

Abstract— **This paper investigates the problem of joint localization and target tracking using mobile multi-agent networks. We consider the challenge situation that each agent only has partially observation of the target system and also needs to estimate its relative location in order to utilize meaningfully the target observation. Using a minimum-energy filtering approach, we propose optimal distributed filtering algorithms for local state and target state estimation for network agents that utilize local, relative and communicated measurements that can be obtained at general event times. Numerical examples illustrate the performance of the algorithms for the task of estimating the evolution of a gas concentration in a field due to a gas leak using multi-agent systems.**

I. Introduction

Distributed estimation problems along with information exchange among sensor nodes are central issues in sensor and multi-agent networks. In this paper we refer to a multi-agent network as a set of dynamical systems called agents, possibly heterogeneous, that are linked through a communication network. Typical examples include the family of Wireless Sensor Networks (WSN), which are spatially distributed sensors with the capacity of processing information that they exchange with each other to monitor some specific feature or target. Another example is the family of cooperative autonomous robotic vehicles that compared with WSN, they have the additional freedom of being mobile. An important challenging problem that arises in distributed estimation for multi-agent systems is the fact that the individual agents usually only receive partially informative measurements, which are insufficient to estimate the target state. More precisely, in many cases the target-single agent system is not observable (only a subset of the space is observable), but the overall system is (joint) observable assuming that there is an adequate communication between the agents. Further, there are circumstances where it is required that the agent must have knowledge of their own location with respect to a common reference in order to utilize the target measurements meaningfully. These two connected problems are referred as a joint estimation and localization problem. Numerous works have been addressed each problem separately. See for example [1]–[5] for localization, and [6]–[8] for target state estimation. A particular work that provides a solution for the localization problem for networked agents using distributed algorithms is the one proposed in [1], where a minimum-energy filtering approach was derived for continuous-time agents that receives continuously relative state measurements and communicated state estimates to improve their individual state estimates. This work as well as [9], that proposes a minimum energy estimator solution applied to a continuous state model with event time discrete measurements, are key for the derivation of the proposed solution presented in this paper. With regard to target state estimation, examples include the fully decentralized Kalman filter, as referred by the authors in [10], [11], but that require all-to-all communication among the agents of the network; and the works in [8], [12], that propose strategies to avoid the requirement of all-to-all communication by adding a consensus step. Also, different consensus algorithms that are appropriate for distributed Kalman-like filtering approaches have been proposed and their stability properties have been analyzed, e.g., [13]–[16]. Despite a rich body of literature on distributed target state estimation, most of the works assume implicitly that the measurements about the target do not depend on the agent location, or it is assumed that the localization problem is perfectly solved, or the agents know their own locations. A notably exception is the distributed strategy proposed in [17], where the joint estimation and localization is considered for agents that observe the targets following linear Gaussian models. Our work uses a deterministic approach (and in that sense it also relaxes this Gaussian assumption) by considering that the unknown noise and disturbance signals are deterministic L_∞ (bounded) or L_2 (finite energy) signals. In this paper, we formulate the joint estimation and localization problem as a distributed filtering problem for a network of agents using a deterministic optimal minimum energy estimation approach, e.g., [5], [9], [18], [19], where the agents use their estimated position resulted from a localization procedure to estimate the state of the target. We also considered that each agent can obtain local, relative, communicated, and target measurements at general (not necessarily periodic) discrete event times. Due to space limitations, the explicitly derivations of the filters are not described. In this paper, to illustrate the performance of the proposed filters, a numerical example is provided that emulates the task of estimating the evolution of a gas concentration in a field due to a gas leak using multi-agent systems.

*Research Center for Systems and Technologies, Faculty of Engineering, University of Porto (FEUP), Porto, Portugal. `tran.habk0605@gmail.com`, `pedro.aguiar@fe.up.pt`.

This work was partially supported by FCT R&D Unit SYSTEC - POCI-01-0145-FEDER-006933/SYSTEC funded by ERDF—COMPETE2020—FCT/MEC—PT2020 and project STRIDE - NORTE-01-0145-FEDER-000033, funded by ERDF—NORTE 2020.

II. Problem Formulation

Consider a network of $N^{[A]}$ dynamical agents sharing a common task of tracking a target that evolves according to the following dynamical target model

$$\dot{x}^{[T]} = A^{[T]}x^{[T]} + B^{[T]}u^{[T]} + R^{[T]}w^{[T]} \tag{1}$$

where $x^{[T]} \in \mathbb{R}^{n^{[T]}}$ is the state[1] of the target initialized at $x_0^{[T]} := x^{[T]}(0)$, $A^{[T]} \in \mathbb{R}^{n^{[T]}\times n^{[T]}}$ is the dynamic matrix, $u^{[T]}$ is the input signal, $w^{[T]}$ is considered to be an unknown bounded disturbance signal, and $R_i^{[T]} \in \mathbb{R}^{n^{[T]}\times n^{[T]}}$ is a known weighted matrix. Let $\mathcal{T} = \{t_k\} := \{t_0, t_1, t_2, \ldots\}$ be a sequence of increasing event times in $[t_0, t_f)$, with $t_0 = 0$, $t_f \geq t_0$, which can be infinity, and consider that each agent only obtain measurements and communicates at the event times $\{t_k\}$. In particularly, to track the target, each agent $i \in \mathcal{I}^{[A]} := \{1, 2, ..., N^{[A]}\}$ measures the target at discrete event times in the sequence $\{t_k^{[r]}\} \subset \mathcal{T}$ according to target measurement model

$$r_i^{[T]}(t) = C_i^{[T]}(x_i^{[A]}(t))x^{[T]}(t) + D_i^{[T]}v_i^{[T]}(t),\ t \in \{t_k^{[r]}\} \subset \mathcal{T} \tag{2}$$

where $r_i^{[T]} \in \mathbb{R}^{n^{[r]}}$, and $v_i^{[T]} \in \mathbb{R}^{n^{[r]}}$ denotes target measurement noise. Note that the output equation (2) that models how agent i gets measurements of the target depends on its state value $x_i^{[A]}$. We consider that each agent is governed by a dynamical system of the form

$$\dot{x}_i^{[A]} = A_i^{[A]}x_i^{[A]} + B_i^{[A]}u_i^{[A]} + R_i^{[A]}w_i^{[A]} \tag{3}$$

that is initialized at $x_{i_0}^{[A]} := x_i^{[A]}(0)$, $A_i^{[A]} \in \mathbb{R}^{n^{[A]}\times n^{[A]}}$, $R_i^{[A]} \in \mathbb{R}^{n^{[A]}\times n^{[A]}}$, $u_i^{[A]}$ in the input signal and $w_i^{[A]} \in \mathbb{R}^{n^{[A]}}$ is an unknown disturbance signal. The output equation associated to (3), which corresponds to local measurements, is given by

$$y_i^{[A]}(t) = C_i^{[A]}x_i^{[A]}(t) + D_i^{[A]}v_i^{[A]}(t), \quad t \in \{t_k^{[y]}\} \subset \mathcal{T} \tag{4}$$

In this setup, we also consider that each agent can get relative measurements with respect to their neighbors[2] according to

$$z_{ij}^{[A]}(t) = F_{ij}(x_j^{[A]}(t) - x_i^{[A]}(t)) + E_{ij}\epsilon_{ij}(t)$$
$$t \in \{t_k^{[z]}\} \subset \mathcal{T}, \quad j \in \mathcal{N}_i \tag{5}$$

and data related to the state estimates that are transmitted by the neighbors given by

$$s_{ij}^{[A]}(t) = H_{ij}^{[A]}\hat{x}_j^{[A]}(t) + G_{ij}^{[A]}\eta_{ij}^{[A]}(t), \tag{6a}$$
$$s_{ij}^{[T]}(t) = H_{ij}^{[T]}\hat{x}_j^{[T]}(t) + G_{ij}^{[T]}\eta_{ij}^{[T]}(t), \tag{6b}$$
$$t \in \{t_k^{[s]}\} \subset \mathcal{T}, \quad j \in \mathcal{N}_i$$

where $\hat{x}_i^{[A]}$ is the local state estimation of agent i, and $\hat{x}_j^{[T]}$ is the state estimated by agent j of the target. In (4)–(6), $C_i^{[A]} \in \mathbb{R}^{n^{[y]}\times n^{[A]}}$, $F_{ij} \in \mathbb{R}^{n^{[z]}\times n^{[A]}}$, $H_{ij} \in \mathbb{R}^{n^{[s]}\times n^{[A]}}$, $D_i \in \mathbb{R}^{n^{[y]}\times n^{[y]}}$, $E_{ij} \in \mathbb{R}^{n^{[z]}\times n^{[z]}}$, $G_{ij} \in \mathbb{R}^{n^{[s]}\times n^{[s]}}$ are known weighted matrices. The signals $v_i^{[A]} \in \mathbb{R}^{n^{[y]}}$, $\eta_{ij} \in \mathbb{R}^{n^{[s]}}$, and $\epsilon_{ij} \in \mathbb{R}^{n^{[z]}}$ are considered to be unknown deterministic bounded signals that model measurement noise, disturbances due to errors in the relative measurements, and communication errors, respectively.

We can now describe the problem addressed in this paper. Consider a given network formed by multiple (possible heterogeneous) dynamic agents, where each agent i obeys the individual dynamics (3) and acquire local, relative and communication data $\{y_i^{[A]}, z_{ij}^{[A]}, s_{ij}^{[A]}, s_{ij}^{[T]}\}_{t\in\{t_k\}}$ at discrete event times according to (4)-(6). Consider also a dynamic target, governed by the state equation (1) that is observed at discrete event times by the agents as stated in (2). Given the described setup, the goal is to solve in a distributed manner the joint estimation of the target and the agents. We call it as the joint estimation and localization problem. Under feasibility conditions this can be precisely formulated as following.

Problem 1: Derive a filter that run in each agent $i \in \mathcal{I}^{[A]}$ and outputs:

1) an estimate $\hat{x}_i^{[A]}$ of the local state $x_i^{[A]}$ such that the *localization estimation error* $e_i^{[A]}(t) := \hat{x}_i^{[A]}(t) - x_i^{[A]}(t)$ converges to zero as time t goes to infinity or in the presence of bounded disturbances, should be ultimately bounded with a bounded value that depends on the size of the disturbances and noise bounds;
2) an estimate $\hat{x}_i^{[T]}$ of the target state $x^{[T]}$ such that, the *target tracking error* $e_i^{[T]}(t) := \hat{x}_i^{[T]}(t) - x^{[T]}(t)$ remains bounded and converge to an ultimate bound with size relative to the value of the bounds of the disturbance signals as well as the localization error.

III. Distributed Localization

This section addresses the first part of the Problem 1 by reformulating into a minimum-energy optimal estimation problem. In what follows, for simplicity of notation, the label $[A]$ is omitted. To tackle the distributed localization problem, we first introduce an additional model (used only for the filter design) that relates the true state with its estimate that is communicated between the agents. In particular, note that some of the data provided at each agent i is given via the communication model (6a) that includes the information about the estimates (but not the true state) of the neighboring states. Since a distributed solution is sought, the filter at each agent i must not be coupled to the solution x_j of the filter at agent j. To address this point, we use the approach in [1] by proposing the following model at node i regarding the unknown signal x_j

$$F_{ij}\hat{x}_j(t) = F_{ij}x_j(t) + W_{ij}\delta_{ij} \tag{7}$$

where δ_{ij} is viewed as an unknown disturbance signal, $W_{ij} \in R^{n^{[z]}\times n^{[z]}}$ is the weight that agent i considers for the estimation error $(\hat{x}_j - x_j)$. Another point to take into consideration is the fact that when there is a relative measurement (see equation (5)), in order to update the local state estimate at the event time $t = t_k^{[z]}$, agent i

[1]The variables associated with agent or target are distinguished by labels $^{[A]}$ and $^{[T]}$. For simplicity, we assume only one target, but the methodology can be easly generalized for a finite number of targets.

[2]The network is presented as a directed graph $\mathcal{G}\{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V}$ is the set of vertex and $\mathcal{E}$ is the set of edges. The set of neighbors of agent i is denoted by $\mathcal{N}_i = \{j, \{i, j\} \in \mathcal{E}\}$, where i is the head and j is the tail of the edge.

needs to have the information about $x_j(t_k^{[z]})$. This can be obtained from the most recent communication $s_{ij}(t_{\bar{k}}^{[s]}), \bar{k} = \arg\max_{l\in\mathbb{N}_{\geq 0}} \{t_l^{[s]} \in \mathcal{T} : t_l^{[s]} \leq t\}$.

Algorithm 1 Local State Estimator (Agent i)

Input: $y_i(t_k^{[y]})$, $z_{ij}(t_k^{[z]})$, $s_{ij}(t_k^{[s]})$
Output: $\hat{x}_i(t)$, $Q_i(t)$
Initialization: $t = t_0 := 0$, $k = 0$, $Q_i(0) = Q_{i0}, \hat{x}_i(0) = \hat{x}_{i0}$

while $t < t_f$ **do**
  if $t_k \leq t < t_{k+1}$ **then**

$$\dot{Q}_i(t) = -Q_i(t)A_i - A_i^T Q_i(t) - Q_i(t)R_iR_i^T Q_i(t), \quad Q_i(t_k) = Q_{ik} \tag{8a}$$
$$\dot{\hat{x}}_i(t) = A_i\hat{x}_i(t) + B_iu_i, \quad \hat{x}_i(t_k) = \hat{x}_{ik} \tag{8b}$$

  end if
  if $t = t_{k+1}^{[y]}$ **then**

$$Q_i(t) = Q_i(t^-) + C_i^T(D_iD_i^T)^{-1}C_i \tag{9a}$$
$$\hat{x}_i(t) = \hat{x}_i(t^-) - Q_i^{-1}(t)C_i^T(D_iD_i^T)^{-1}(C_i\hat{x}_i(t^-) - y_i(t)) \tag{9b}$$

  end if
  if $t = t_{k+1}^{[z]}$ **then**
    $q_{ij}(t) = z_{ij}(t) - M_{ij}s_{ij}(t_{\bar{k}}^{[s]})$

$$Q_i(t) = Q_i(t^-) + \sum_{j\in\mathcal{N}_i} {F_{ij}}^T U_{ij}^{-1} F_{ij} \tag{10a}$$
$$\hat{x}_i(t) = \hat{x}_i(t^-) - Q_i(t)^{-1} \sum_{j\in\mathcal{N}_i} {F_{ij}}^T U_{ij}^{-1}(F_{ij}\hat{x}_i(t^-) + q_{ij}(t)) \tag{10b}$$

  end if
end while

To this end, rewrite (5) as

$$z_{ij}(t) = F_{ij}\hat{x}_j(t) - F_{ij}x_i(t) - W_{ij}\delta_{ij} + E_{ij}\epsilon_{ij}$$

Notice that the term $F_{ij}\hat{x}_j(t)$ is unknown but can be partially recovered from $H_{ij}\hat{x}_j$ by using the pseudo-inverse matrix H_{ij}^+, that is

$$z_{ij}(t) = F_{ij}H_{ij}^+H_{ij}\hat{x}_j(t) - F_{ij}x_i(t) - W_{ij}\delta_{ij} + E_{ij}\epsilon_{ij} + F_{ij}[I - H_{ij}^+H_{ij}]\hat{x}_j(t) \tag{11}$$

We also need to predict the term $H_{ij}\hat{x}_j(t)$ at $t = t_k^{[z]}$ from the most recent communication $s_{ij}(t_{\bar{k}}^{[s]})$ recovered from j. To this end, let $\vartheta_{ij} := H_{ij}\hat{x}_j$ and compute its dynamics

$$\dot{\vartheta}_{ij} = [H_{ij}A_jH_{ij}^+]\vartheta_{ij} + H_{ij}A_j[I - H_{ij}^+H_{ij}]\hat{x}_j \tag{12}$$

where we used $\dot{\hat{x}}_j = A_j\hat{x}_j$, and added and subtracted the term $H_{ij}A_jH^+\vartheta_{ij}$. Then, from (12) we can predict $\vartheta_{ij}(t)$ at the time $t = t_k^{[z]}$, which yields

$$\vartheta_{ij}(t) = \phi(t, t_{\bar{k}}^{[s]})\vartheta_{ij}(t_{\bar{k}}^{[s]}) + \int_{t_{\bar{k}}^{[s]})}^{t} \phi(t,\tau)H_{ij}A_j[I - H_{ij}^+H_{ij}](\hat{x}_j(\tau))d\tau \tag{13}$$

with $\phi(t, t_0) = e^{H_{ij}A_{ij}H_{ij}^+(t-t_0)}$. Substituting (13) into (11) and using (6b) we finally obtain

$$z_{ij}(t) = M_{ij}s_{ij}(t_{\bar{k}}^{[s]}) - F_{ij}x_i(t) - M_{ij}G_{ij}\eta_{ij}(t) - W_{ij}\delta_{ij}(t) + E_{ij}\epsilon'_{ij}(t) \tag{14}$$

where $M_{ij} = F_{ij}H_{ij}^+\phi(t, t_{\bar{k}}^{[s]})$ and $\epsilon'_{ij}(t) = \epsilon_{ij}(t) + E_{ij}^{-1}F_{ij}[I - H_{ij}^+H_{ij}]\hat{x}_j + F_{ij}H_{ij}^+\int_{t_n^{[s]})}^{t}\phi(t,\tau)H_{ij}A_j[I - H_{ij}^+H_{ij}]\hat{x}_j(\tau)d\tau$ that can be viewed as an unknown disturbance signal.

We are now ready to state the main result of this section.

Theorem 1: The output signal $\hat{x}_i(t)$ given by Algorithm 1 is the solution of the minimum-energy estimation problem formulated for each agent i as follows: Given the measurements of the local state y_i, relative measurements z_{ij} described as (14) and communication data s_{ij}, compute the optimal estimate $\hat{x}_i(t)$ at time t defined by

$$\hat{x}_i(t) := \arg\min_{\xi\in\mathbb{R}^n} J_i(\xi, t) \tag{15}$$

with

$$\begin{aligned}
J_i(\xi, t) = \min_{w_i, v_i, \eta_{ij}, \epsilon'_{ij}, \delta_{ij}} & (x_i(0) - \hat{x}_{i0})^T Q_i(0)(x_i(0) - \hat{x}_{i0}) \\
& + \int_0^t \|w_i(\tau)\|d\tau + \sum_{t'\in\left\{t_k^{[y]}\right\}_0^{\mathbf{k}}} \|v_i(t')\|^2 \\
& + \sum_{j\in\mathcal{N}_i}\Big\{ \sum_{t'\in\left\{t_k^{[z]}\right\}_0^{\mathbf{k}}} \left(\left\|\epsilon'_{ij}(t')\right\|^2 + \left\|\left(\delta_{ij}(t')\right\|^2\right.\right. \\
& + \left.\left.\left\|\left(\eta_{ij}(t')\right\|^2\right) + \sum_{t'\in\left\{t_k^{[s]}\right\}_0^{\mathbf{k}}} \|\eta_{ij}(t')\|^2 \right\}
\end{aligned}$$

$$\begin{aligned}
\text{s.t} \quad & \dot{x}_i = A_ix_i + B_iu_i + R_iw_i, \quad x_i(t) = \xi, \\
& y_i(t') = C_ix_i(t') + D_iv_i(t'), \quad t' \in \left\{t_k^{[y]}\right\}_0^{\mathbf{k}} \\
& z_{ij}(t') = M_{ij}s_{ij}(t_{\bar{k}}^{[s]}) - F_{ij}x_i(t') - M_{ij}G_{ij}\eta_{ij}(t') \\
& \qquad - W_{ij}\delta_{ij}(t') + E_{ij}\epsilon'_{ij}(t'), \quad t' \in \left\{t_k^{[z]}\right\}_0^{\mathbf{k}}, \\
& s_{ij}(t') = H_{ij}\hat{x}_j(t') + G_{ij}\eta_{ij}(t'), \quad t' \in \left\{t_k^{[s]}\right\}_0^{\mathbf{k}} \\
& F_{ij}\hat{x}_j(t') = F_{ij}x_j(t') + W_{ij}\delta_{ij}(t')
\end{aligned} \tag{16}$$

where $\bar{k} = \arg\max_{l\in\mathbb{N}_{\geq 0}}\{t_l^{[s]} \in \mathcal{T} : t_l^{[s]} \leq t'\}$, $\mathbf{k} = \arg\max_{k\in\mathbb{N}_{\geq 0}}\{t_k \in \mathcal{T} : t_k \leq t\}$, the matrix $Q_i(0) > 0$, and the state initial condition $\hat{x}_{i0} \in R^{n^{[A]}}$ encode a-priori information about the location of agent i. □

Algorithm 1 describes the proposed distributed estimator to solve the localization problem, where $q_{ij}(t_{k+1}^{[z]}) = z_{ij}(t_{k+1}^{[z]}) - M_{ij}s_{ij}(t_{\bar{k}}^{[s]}), \bar{k} = \arg\max_{l\in\mathbb{N}_{\geq 0}}\{t_l^{[s]} \in \mathcal{T} : t_l^{[s]} \leq t_{k+1}^{[z]}\}$ and $U_{ij} = E_{ij}E_{ij}^T + M_{ij}G_{ij}G_{ij}^T{M_{ij}}^T + W_{ij}W_{ij}^T$. Note that it is a filtering-like and iterative implementation that continuously improves the estimates as more data is acquired, and it is an optimal filter.

IV. Distributed filtering for target tracking

This section makes uses of the previous results and investigates the distributed filter for target tracking to solve the second part of Problem 1.

Algorithm 2 Target State Estimator (Agent i)

Input: $r_i(t_k^{[r]})$, $s_{ij}(t_k^{[s]})$
Output: $\hat{x}_i(t)$, $Q_i(t)$
Initialization: $t = t_0 := 0$, $k = 0$, $Q_i(0) = Q_{i0}, \hat{x}_i(0) = \hat{x}_{i0}$

while $t < t_f$ **do**
  if $t_k^{[r,s]} \le t < t_{k+1}^{[r,s]}$ **then**

$$\dot{Q}_i(t) = -Q_i(t)A - A^T Q_i(t) \\ - Q_i(t)RR^T Q_i(t) \quad Q_i(t_k) = Q_{ik} \tag{17a}$$

$$\dot{\hat{x}}_i(t) = A\hat{x}_i(t) + Bu \quad \hat{x}_i(t_k) = \hat{x}_{ik} \tag{17b}$$

  end if
  if $t = t_{k+1}^{[r]}$ **then**

$$Q_i(t) = (Q_i(t^-) + S_i^{[r]}) \tag{18a}$$

$$\hat{x}_i(t) = \hat{x}_i(t^-) - Q_i^{-1}(t)(S_i^{[r]}\hat{x}_i(t^-) - y_i^{[r]}) \tag{18b}$$

  end if
  if $t = t_{k+1}^{[s]}$ **then**

$$Q_i(t) = (Q_i(t^-) + S_i^{[s]}) \tag{19a}$$

$$\hat{x}_i(t) = \hat{x}_i(t^-) - Q_i^{-1}(t)(S_i\hat{x}_i(t^-) - y_i^{[s]}) \tag{19b}$$

  end if
end while

Note that the observation matrix $C_i^{[T]}(x_i^{[A]})$ in (2) can be decomposed into

$$C_i^{[T]}(x_i^{[A]}) = C_i^{[T]}(\hat{x}_i^{[A]}) + \bar{C}_i^{[T]}(\hat{x}_i^{[A]}, e_i^{[A]}) \tag{20}$$

From the target measurement (2) we obtain

$$r_i(t) = \hat{C}_i^{[T]} x^{[T]}(t) + D_i v_i'(t), \quad t \in \{t_k^{[r]}\} \subset \mathcal{T} \tag{21}$$

where $\hat{C}_i^{[T]} := C_i^{[T]}(\hat{x}_i^{[A]})$ and $v_i' = v_i + D_i^{-1}\bar{C}_i^{[T]}(\hat{x}_i^{[A]}, e_i^{[A]})$. As in the previous section, an additional model is needed at agent i in order to avoid coupling to the solution with $\hat{x}_j^{[T]}$ received from agent j via communications. Thus, we use

$$H_{ij}^{[T]}\hat{x}_j^{[T]}(t) = H_{ij}^{[T]}x^{[T]}(t) + W_{ij}^{[T]}\delta_{ij}^{[T]}(t) \tag{22}$$

Before the derivation of target state estimator at each agent, several manipulations associated to $s_{ij}^{[T]}$ are needed. For the sake of simplicity, label $[T]$ is omitted. Using (22), the communication from the neighbors at $t = t_k^{[s]}$ presented as (6b) can be rewritten as

$$s_{ij}(t) = H_{ij}x(t) + P_{ij}^{[s]}\rho_{ij}(t) \tag{23}$$

where $P_{ij}^{[s]} := [G_{ij} \quad W_{ij}]$ and $\rho_{ij}(t) = [\eta_{ij}^T(t) \quad \delta_{ij}^T(t)]^T$. At $t = t_k^{[r]}$, agent i can use the communication from its neighbors to compensate the lack of information about x due to the fact that the target may be locally unobservable. Following the same type of algebraic manipulations that were used to derive (14), this compensating information can be accessed through $s_{ij}(t_{\bar{k}}^{[s]}), \bar{k} = \arg\max_{l \in \mathbb{N}_{\ge 0}} \{t_l^{[s]} \in \mathcal{T} : t_l^{[s]} \le t\}$, and therefore

$$K_{ij}s_{ij}(t_{\bar{k}}^{[s]}) = H_{ij}x + P_{ij}^{[r]}\rho_{ij}' \tag{24}$$

where $P_{ij}^{[r]} := [K_{ij}G_{ij} \ W_{ij}]$, $\rho_{ij}' = \rho_{ij} + {P^{[r]}}_{ij}^{-1}\nu_{ij}$, $\nu_{ij} = -\int_{t_{\bar{k}}^{[s]}}^{t} \Psi(t,\tau)H_{ij}A[I - H_{ij}^{+}H_{ij}]\hat{x}_j(\tau)d\tau$ and $\Psi(t,\tau)$ is the transition matrix of the dynamic system

$$\dot{\vartheta}_{ij} = [H_{ij}AH_{ij}^{+}]\vartheta_{ij} + H_{ij}A[I - H_{ij}^{+}H_{ij}]\hat{x}_j,$$

that is, $\Psi(t, t_0) = e^{H_{ij}AH_{ij}^{+}(t-t_0)}$. In (24), $K_{ij} = \Psi(t, t_{\bar{k}}^{[s]})$. Given the local measurement of the target state $r_i(t_k^{[r]})$, the communication of the target state from the neighbors $s_j(t_k^{[s]})$, Algorithm 2 defines the solution for the target state estimation problem, where

$$U_{ij}^{[r]} = H_{ij}^T \left(P_{ij}^{[r]} {P^{[r]}}_{ij}^T\right)^{-1} H_{ij} \\ U_i^{[r]} = \hat{C}_i^T \left(D_i D_i^T\right)^{-1} \hat{C}_i, \quad S_i^{[r]} = U_i^{[r]} + \sum_{j \in \mathcal{N}_i} U_{ij}^{[r]} \tag{25a}$$

$$u_{ij}^{[r]} = H_{ij}^T \left(P_{ij}^{[r]} {P^{[r]}}_{ij}^T\right)^{-1} K_{ij}s_{ij}(t_{\bar{k}}^{[s]}) \\ u_i^{[r]} = \hat{C}_i^T \left(D_i D_i^T\right)^{-1} r_i, \quad y_i^{[r]} = u_i^{[r]} + \sum_{j \in \mathcal{N}_i} u_{ij}^{[r]} \tag{25b}$$

$$U_{ij}^{[s]} = H_{ij}^T \left(P_{ij}^{[s]} {P^{[s]}}_{ij}^T\right)^{-1} H_{ij}, \quad S_i^{[s]} = \sum_{j \in \mathcal{N}_i} U_{ij}^{[s]} \tag{26a}$$

$$u_{ij}^{[s]} = H_{ij}^T \left(P_{ij}^{[s]} {P^{[s]}}_{ij}^T\right)^{-1} s_{ij}, \quad y_i^{[s]} = \sum_{j \in \mathcal{N}_i} u_{ij}^{[s]} \tag{26b}$$

Theorem 2: The output signal of Algorithm 2 solves the minimum-energy estimation problem for each agent i formulated as follows: Given the local measurement of the target state r_i and the communication s_{ij} from the neighbors compute the optimal estimate $\hat{x}_i(t)$ of the state of the target provided by agent i at time t defined by

$$\hat{x}_i(t) := \arg\min_{\xi \in \mathbb{R}^n} J_i(\xi, t) \tag{27}$$

with

$$J_i(\xi,t) = \min_{w, v_i', \rho_{ij}', \rho_{ij}} (x_i(0) - \hat{x}_{i0})^T Q_i(0)(x_i(0) - \hat{x}_{i0})) \\ + \int_0^t \|w(\tau)\| d\tau + \sum_{t' \in \left\{t_k^{[r]}\right\}_0^{\mathbf{k}}} \|v_i(t')\|^2 \\ + \sum_{j \in \mathcal{N}_i} \Big\{ \sum_{t' \in \left\{t_k^{[r]}\right\}_0^{\mathbf{k}}} \left\|\rho_{ij}'(t')\right\|^2 + \sum_{t' \in \left\{t_k^{[s]}\right\}_0^{\mathbf{k}}} \left\|\rho_{ij}(t')\right\|^2 \Big\}$$

$$\text{s.t} \quad \dot{x} = Ax + Bu + Rw, \quad x = \xi \\ r_i(t') = \hat{C}_i(t') + D_i v_i'(t)), \ t' \in \left\{t_k^{[r]}\right\}_0^{\mathbf{k}}, \\ K_{ij}s_{ij}(t') = H_{ij}x(t') + P_{ij}^{[r]}\rho_{ij}'(t'), \ t' \in \left\{t_k^{[r]}\right\}_0^{\mathbf{k}}, \\ s_{ij}(t') = H_{ij}x(t') + P_{ij}^{[s]}\rho_{ij}(t'), t' \in \left\{t_k^{[s]}\right\}_0^{\mathbf{k}} \tag{28}$$

where $\bar{k} = \arg\max_{l \in \mathbb{N}_{\ge 0}} \{t_l^{[s]} \in \mathcal{T} : t_l^{[s]} \le t'\}$, the matrix $Q_i(0) > 0$, and the state initial condition $\hat{x}_{i0} \in R^{n^{[T]}}$ encode a-priori information about the target state estimate given by agent i. □

V. Numerical Example

In this section, the proposed solutions for the joint estimation and localization problem are illustrated through numerical results. An example of a small network of agents is studied to make it easier to highlight the key concepts behind the proposed algorithms. In this example, the agents have the task to estimate the evolution of a gas concentration in a field due to a gas leak. We consider a target model given by (1) that results from a discretization in space (method of lines) using finite central-difference approximations of the advection-diffusion PDE equation

$$\frac{\partial u}{\partial t} = -v_{\mathbf{x}}\frac{\partial u}{\partial \mathbf{x}} - v_{\mathbf{y}}\frac{\partial u}{\partial \mathbf{y}} + D_{\mathbf{x}}\frac{\partial^2 u}{\partial \mathbf{x}^2} + D_{\mathbf{y}}\frac{\partial^2 u}{\partial \mathbf{y}^2} + S(t,\mathbf{x},\mathbf{y})$$

that describes the evolution in time and space (2D) of the gas concentration of the field $u(t,\mathbf{x},\mathbf{y})$. The coefficients $v_{\mathbf{x}}$, $v_{\mathbf{y}}$ are the velocity field, and $D_{\mathbf{x}}$, $D_{\mathbf{y}}$ are the diffusion rates, in the $\mathbf{x}$, $\mathbf{y}$ direction, respectively. $S(t,\mathbf{x},\mathbf{x})$ is an external source that emulates a gas leak, which in this particular numerical example is given by

$$S(t,\mathbf{x},\mathbf{y}) = \left[\begin{array}{cc} 5e^{(-0.01\mathbf{x})}e^{(-0.01\mathbf{y})} & 0 \le \mathbf{x} \le 4, 8 \le \mathbf{y} \le 10 \\ 0 & otherwise \end{array} \right.$$

Using a rectangular grid and mapping the concentration of each point $p_{mn} = (\mathbf{x}_m, \mathbf{y}_n), m = 2,3,...,M, n = 2,3,...,N$ into an element of the target vector $x^{[T]} \in R^{(M-1)(N-1)}$ such that $u(t,\mathbf{x}_m,\mathbf{y}_n)$ correspond to the $(n-2)(M-1)+(m-1)$ element of $x^{[T]}(t)$, we obtain the target model (1) with $A^{[T]}$ given by a band matrix of size $(N-1)(M-1) \times (N-1)(M-1)$ and $B^{[T]}$ a vector of size $(M-1)(N-1)$ with only 2 non-zero elements. In this example, we used $v_{\mathbf{x}} = v_{\mathbf{y}} = 1$, $D_{\mathbf{x}} = D_{\mathbf{y}} = 2$, and $M = N = 5$. In the sequel, we consider two simulation scenarios. The first scenario, see Fig. 1, consists of a network of nine stationary agents distributed in the field with the communication topology described by an undirected graph with nodes located at $V = \{p_{(3,3)}, p_{(3,5)}, p_{(3,7)}, p_{(7,5)}, p_{(5,5)}, p_{(3,5)}, p_{(3,7)}, p_{(5,7)}, p_{(7,7)}\}$. In this case, $A_i^{[A]}$ is the zero matrix of dimension 2×2, $i = 1,2,...,9$ and each agent only communicates with the neighbor agents in distance less than 2m. The second scenario consists of a network of four agents, where three of them move along a circumference with origin at $(5,5)$ with radius 2m, and only communicates with the fixed agent located at $(5,5)$. In this scenario, the state of each agent has dimension four and for $i = 1,2,3$

$$A_i^{[A]} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_4^{[A]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The target observation model described by (2) is considered to depend on the distance between the position of agent i and the measured points p_{mn}, which we denote by Δ_{imn}. More precisely, in the numerical example the corresponding element of C_i is given by $\frac{1}{\Delta_{imn}}$ as long as Δ_{imn} is less

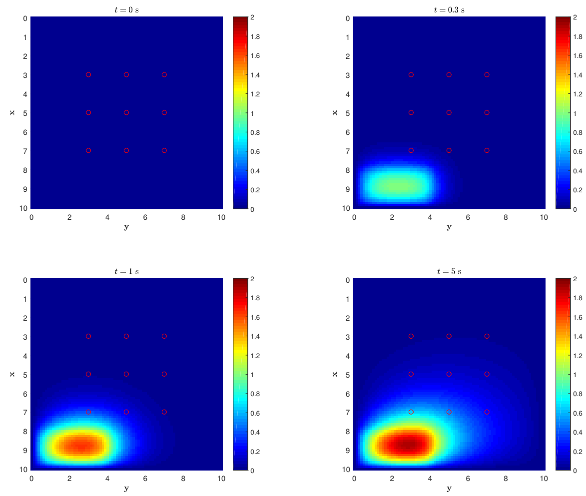

Fig. 1: Evolution of the gas concentration of the field and agent locations in Scenario 1.

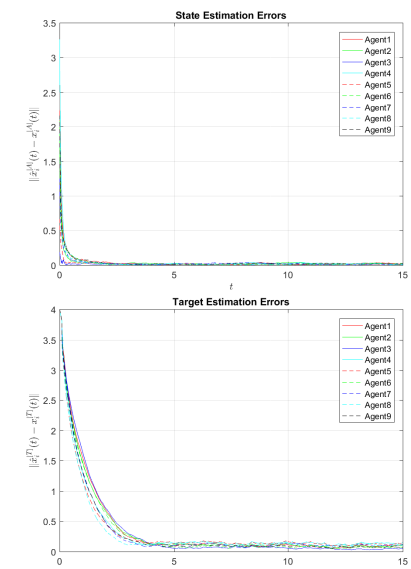

Fig. 2: Time evolution of the localization estimation errors $||e_i^{[A]}||$ (top) and the target tracking errors $||e_i^{[T]}||, i = 1,2,\dots,9$ (bottom) [Scenario 1].

than a threshold radius of 3m, otherwise is zero. All the disturbances are simulated as zero-mean Gaussian distributions and the local measurements are obtained every $0.01\,s$, the relative measurements every $0.02\,s$ and the communication events every $0.1\,s$.

Fig. 1-2 show the results of the first scenario with nine agents. In particular, Fig. 1 presents the evolution of the gas concentration at $t = 0s, 0.3s, 1s, 5s$, and Fig. 2 displays the time evolution of the localization estimation errors $||e_i^{[A]}||$ (top) and the target tracking errors $||e_i^{[T]}||, i = 1,2,\dots,9$, where it can be seen that the errors converge to almost zero. Note that using the estimated state provided by the localization procedure, each agent estimate the evolution of the gas concentration of the field.

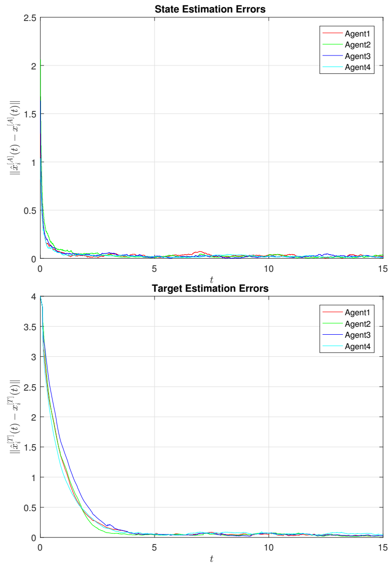

Fig. 3: Time evolution of the localization estimation errors $||e_i^{[A]}||$ (top) and the target tracking errors $||e_i^{[T]}||, i = 1, 2, \dots, 4$ (bottom) [Scenario 2].

For the second simulation scenario, Fig. 3 presents the evolution of the state estimation errors. It can be seen (for this particular example) that with a less number of agents (but moving), we obtain estimation performances close to the first scenario. Fig. 4 displays the evolution in time at a single point located at $(\mathbf{x}, \mathbf{y}) = (4, 4)$ of the "true" gas concentration (that is, the solution of the PDE), the target concentration (that is, the solution of the state target model corresponding to that point), and the estimated concentrations (that is, the estimated target state corresponding to that point) by each agent. One can see that there is a difference (steady-state error) between the true value of the gas concentration and the estimates from the filters, which are due to the pour approximately target model of the gas evolution, where we only have used 5 bins in each direction ($N = M = 5$). Note however, that the estimated states for each agent converges to a small neighborhood of the state of target model. Also, this steady-state error can be made arbitrarily small by increasing the number of bins as expense of increasing the computational complexity (size of matrix $A^{[T]}$).

VI. Conclusions

This work studied the problem of joint target estimation and agent localization in distributed manner for mobile multi-agent networks that utilize local, relative and communicated measurements that are obtained at discrete event times. The distributed estimators were derived as the solution of optimal minimum-energy problems. Numerical examples were implemented to illustrate the performance of the estimators. In the scope of this paper, only linear systems were studied. Future work will purse nonlinear systems and will also focus on more detailed simulations and real data experiments.

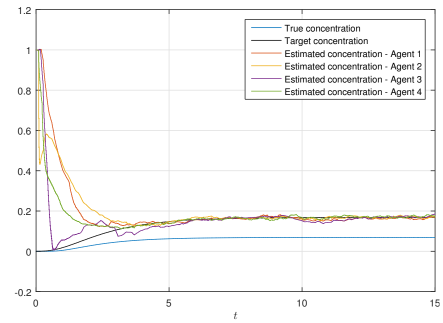

Fig. 4: Time evolution of the true, target, and estimated gas concentrations at the point $(\mathbf{x}, \mathbf{y}) = (4, 4)$ [Scenario 2].

References

[1] M. Zamani and A. P. Aguiar, "Distributed localization of heterogeneous agents with uncertain relative measurements and communications," in *Conf. Decision and Control (CDC)*, 2015, pp. 4702–4707.

[2] M. Zamani and V. Ugrinovskii, "Minimum-energy distributed filtering," in *Conf. Decision and Control (CDC)*, 2014, pp. 3370–3375.

[3] P. Barooah and J. P. Hespanha, "Estimation on graphs from relative measurements," *IEEE Control Systems*, vol. 27, no. 4, pp. 57–74, 2007.

[4] Y. Diao, Z. Lin, M. Fu, and H. Zhang, "A new distributed localization method for sensor networks," *in Asian Control Conf.(ASCC)*, 2013.

[5] M. Zamani, J. Trumpf, and R. Mahony, "Minimum-energy filtering for attitude estimation," *IEEE Transactions on Automatic Control*, vol. 58, no. 11, pp. 2917–2921, 2013.

[6] R. Olfati-Saber, "Distributed kalman filter with embedded consensus filters," in *Conf. Decision and Control (CDC) and European Control Conference (ECC).*, Dec 2005, pp. 8179–8184.

[7] U. A. Khan, S. Kar, A. Jadbabaie, and J. M. F. Moura, "On connectivity, observability, and stability in distributed estimation," in *Conf. Decision and Control (CDC)*, Dec 2010, pp. 6639–6644.

[8] R. Olfati-Saber, "Distributed kalman filtering for sensor networks," in *Conf. Decision and Control*, Dec 2007, pp. 5492–5498.

[9] A. P. Aguiar and J. P. Hespanha, "Minimum-energy state estimation for systems with perspective outputs," *IEEE Transactions on Automatic Control*, vol. 51, no. 2, pp. 226–241, Feb 2006.

[10] H. F. Durrant-Whyte, B. Rao, and H. Hu, "Toward a fully decentralized architecture for multi-sensor data fusion," in *Conf. Robotics and Automation,*, 1990, p. 13311336.

[11] B. Rao, H. Durrant-Whyte, and J. Sheen, "A fully decentralized multisensor system for tracking and surveillance," *The International Journal of Robotics Research*, vol. 12, no. 01, pp. 20–24, 1993.

[12] R. Olfati-Saber, "Kalman-consensus filter : Optimality, stability, and performance," in *Conf. Decision and Control (CDC) and Chinese Control Conference (CCC)*, Dec 2009, pp. 7036–7042.

[13] P. Yang, R. A. Freeman, and K. M. Lynch, "Distributed cooperative active sensing using consensus filters," in *Conf. on Robotics and Automation*, April 2007, pp. 405–410.

[14] R. A. Freeman, P. Yang, and K. M. Lynch, "Stability and convergence properties of dynamic average consensus estimators," in *Conf. Decision and Control*, 2006, pp. 338–343.

[15] H. Long, Zhihua, X. Fan, and S. Liu, "Improved average consensus scalable algorithm of target tracking for wireless sensor network," in *Conf. Chinese Control and Decision Conference (CCDC)*, May 2012, pp. 3336–3341.

[16] A. Petitti, D. D. Paola, A. Rizzo, and G. Cicirelli, "Consensus-based distributed estimation for target tracking in heterogeneous sensor networks," in *Conf. Decision and Control (CDC) and European Control Conference (ECC)*, Dec 2011, pp. 6648–6653.

[17] N. Atanasov, R. Tron, V. M. Preciado, and G. J. Pappas, "Joint estimation and localization in sensor networks," in *Conf. Decision and Control (CDC)*, 2014, pp. 6875–6882.

[18] R. E. Mortensen, "Maximum-likelihood recursive nonlinear filtering," *Journal of Optimization Theory and Applications*, vol. 2, no. 6, pp. 386–394, 1968.

[19] A. J. Krener, "The convergence of the minimum energy estimator," in *New Trends in Nonlinear Dynamics and Control and their Applications*. Springer, 2003, pp. 187–208.