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Optimal actuator design for linear systems with multiplicative noise.

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***Abstract*—This paper addresses optimal actuator design for linear systems with process noise. It is well-known that the control that minimizes a quadratic cost in the state and control for a system with linear dynamics corrupted by additive Gaussian noise is of feedback type and its design depends on the solution of an associated Riccati equation. We consider here the case where the noise is multiplicative, by which we mean that its intensity is dependent on the state. We show how to derive the actuator that minimizes a linear quadratic cost. The solution requires to optimize a function defined on a manifold of low rank matrices; we provide a gradient descent algorithm to perform this optimization and show that this gradient descent converges to the global minimum almost surely.**

I. Introduction

Linear stochastic systems are a very widely used model in engineering, biology and physics, due to the breadth of the situations they can describe [11], [19] . The most commonly used such model is the linear dynamics with additive Gaussian noise model, which can be described by the stochastic differential equation (SDE)

$$dx_t = Ax_t dt + budt + Gdw_t,$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n,\times m}$ and $G \in \mathbb{R}^{n\times p}$ and w_t is a standard vector-valued Wiener process [21]. It is well known, and we come back to this below, that control minimizing the expected cost $\lim_{T\to\infty} \mathbb{E}\left(\frac{1}{T}\int_0^T (x^\top Qx + u^2)dt\right)$ is of feedback type, and its explicit form is known. Now it is clear that the value of the optimal (with respect to u) cost will be dependent on the actuator b. The problem of finding the b that minimizes this cost is called the *optimal actuator placement*. This problem is in general difficult, and easily seen to be non-convex. We provide in this paper a solution to the optimal actuator placement problem for the dynamics corrupted with *multiplicative noise*

$$dx_t = Ax_t dt + budt + Gdw_t + G_1 x dv_t, \tag{1}$$

where $G_1 \in \mathbb{R}^{n\times n}$ and v_t is a Wiener process independent from w_t.

The problem of optimal actuator design, and the dual problem of optimal sensor design, have a long history. In the infinite-dimensional case, we mention [8], [12], [20], in which issues of existence of an optimal design and its relation to an optimal finite-dimensional design are explored. In the finite-dimensional case, choosing a set of sensors c_i out of a *finite* family of potential sensors, has been investigated by several authors. In [24], the authors assign a cost to each sensor and show that optimally choosing a subset of sensors meeting cost constraints is an NP-hard problem, and furthermore exhibit a class of dynamics for which greedy algorithms yield a provably good approximation to the optimal selection. A relaxation of the previous problem is studied in [22]. The problem is also related to problems of optimal design for minimum energy control [6]. Such problem has also been extensively studied in the case in which there is a discrete set of sensors/actuators to choose from [23], [10], [16]. For a study of the application of such problems to sensor networks, and stochastic algorithm to perform the selection, we refer to [14]. Finally, the closest work to the work here is [1], which we refer to more extensively below.

A. Terminology and notation

First we will recall some basic definitions and results that are needed in the paper. A matrix $M \in \mathbb{R}^{n\times n}$ is called *convergent* if all of its eigenvalues have norm less than 1 or, said otherwise, if its spectral radius is less than 1. A matrix is called *Hurwitz*, if all of its eigenvalues have negative real parts. It is not hard to see that a matrix M is convergent if and only if the matrix M^k approaches 0 as k approaches infinity, and also M is Hurwitz if and only if $\exp(Mt)$ approaches 0 as t approaches infinity. Hence convergent and Hurwitz matrices describe stable linear dynamics in discrete and continuous time respectively. We denote by $[A, B] := AB - BA$ the commutator of matrices A and B. We also write

$$[B, \Omega] =: \operatorname{ad}_B \Omega = B\Omega - \Omega B.$$

We let Sym_n the set of real symmetric $n \times n$ matrices. For $A, G \in \mathbb{R}^{n\times n}$, we set

$$\mathcal{L}_{A,G} : \mathrm{Sym}_n \to \mathrm{Sym}_n : X \mapsto A^\top X + XA + G^T XG.$$

B. Problem statement and background

Background and preliminary results. We consider the LTI control system

$$\dot{x} = Ax + bu, \tag{2}$$

where $A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^{n\times m}$, and introduce the quadratic cost

$$C(u, x_0) = \int_0^\infty (x^\top(t)Qx(t) + u^\top(t,x)u(t,x))dt, \quad (3)$$

where $Q > 0$ is given positive definite matrix. We recall that the pair (A, b) is called stabilizable if the uncontrollable modes of (A, b) are stable. It is known that if (A, b) is stabilizable [4], then the control u that minimizes the above cost, which we denote as $u_{\min}(x_0)$, is given by $u_{\min}(x) = -b^\top Px$, where the matrix P is the unique positive-definite solution of the algebraic Riccati equation

$$A^\top P + PA - Pbb^\top P + Q = 0.$$

Furthermore, we can show that $C(u_{\min}(x_0), x_0) = \operatorname{tr}(PL)$ with $L = x_0x_0^\top$.

We care in this paper about actuator design, and hence b is considered to be a free parameter. Note that if the matrix A is stable, then for any $b \in \mathbb{R}^{n\times m}$, the pair (A, b) is stabilizable, and hence the optimal cost is well-defined over $\mathbb{R}^{n\times m}$. Thus, for a stable matrix A, a positive-definite matrix Q, and an initial state x_0 given, we can ask the question:

How should we design the matrix b, such that the optimal cost $C(u_{\min}(x_0), x_0)$ is as small as possible?

First, we must place restriction on the matrices b. Indeed, it is not too difficult to see intuitively that if $\|b\|$ increases, all other things equal, then $C(u_{\min}(x_0), x_0)$ decreases. A proof of this fact is essentially reduced to results about the monotonicity of the Ricatti equation such as the ones in [13]. We thus constraint the norm of b by considering the set so that

$$b^\top b = \gamma^2 I.$$

This also adds the requirement that the actuators are orthogonal to each other, an assumption we will discuss below. Now noting that the cost C depends on the product $bb^\top$, we can rephrase the problem as follows. Let γ be a real parameter, and A, Q, L be such that A is Hurwitz, and Q, L are positive-definite. Minimize the function $J_\gamma(B) = \operatorname{tr}(LP)$, where $P = P(B)$ is the solution of

$$A^\top P + PA - \gamma^2 PBP + Q = 0,$$

over the set

$$\Gamma := \{B = bb^\top | b^\top b = I\}. \quad (4)$$

We can furthermore remove the dependence of the optimal design from the initial state x_0 by averaging over an "isotropic" initial state as follows: assuming the initial state is distributed according to a rotationally invariant distribution (about the origin), such as a multivariate normal distribution center at the origin, then

$$\mathbb{E}C(u_{\min}(x_0)x_0)) = k \operatorname{tr} P,$$

for some positive constant k and where $\mathbb{E}$ is the expectation operator. This is the deterministic actuator placement problem.

C. Statement of the results

We explore in this paper the actuator placement problem for control systems which are corrupted by additive and multiplicative noise. To be more precise, consider the control system described by the stochastic differential equation (1). We introduce the cost

$$C = \lim_{T\to\infty} \mathbb{E}\left(\frac{1}{T}\int_0^\top (x^\top Qx + u^2)dt\right), \quad (5)$$

where $Q > 0$ is given positive-definite matrix. It can be shown that when $G_2 = 0$, the optimal control $u_{\min}$ in steady state is given again by the equations (3). Hence the addition of additive noise does not change the methods to solve the problem, nor the properties of the solution set in a meaningful way.

We will hence focus on the multiplicative noise case

$$dx = Axdt + budt + Gxdw, \quad (6)$$

with associated cost as in Eq. (5).

Throughout the paper, we will assume that the matrices A and G satisfy the following technical condition:

$$\left|\int_0^\infty e^{tA^\top} G^\top G e^{tA} dt\right| < 1. \quad (7)$$

Equivalently, we require that the unique positive semi-definite solution X of the Lyapunov equation

$$A^\top X + XA + G^\top G = 0$$

is a convergent matrix. Under these assumptions, the control minimizing the cost in this case can be seen to be $u_{\min} = -b^\top Px$, where P is the unique positive-definite solution [7] of

$$A^\top P + PA + Q + G^\top PG - PBP = 0. \quad (8)$$

The minimum expected cost is equal to $C_{\min} = \operatorname{tr}(PL), L = x_0x_0^\top$. Thus, the problem we will be solving is:

Problem 1. *Let $\gamma \in \mathbb{R}$ and A, Q, L be given matrices, such that A is Hurwitz, and Q is positive-definite, and L is positive semi-definite of rank 1. Minimize the function $J_\gamma(B) = tr(LP)$, where $P(B)$ is the solution of*

$$A^\top P + PA + Q + G^\top PG - \gamma^2 Pbb^\top P = 0,$$

over the set $\Gamma = \{B = bb^\top | b^\top b = I\}$.

As mentioned earlier, it is well-known that the control which minimizes C is given by $u_{opt} = -b^\top Px_t$ where P satisfies the stochastic Riccati equation (8). The minimal cost is equal to $C(u_{opt}) = \operatorname{tr}(PL), L = x_0x_0^\top$. We thus define

$$J_\gamma(B) : \Gamma \to \mathbb{R} : B \mapsto \operatorname{tr}(LP),$$

where L is a positive semi-definite matrix and P satisfies the Ricatti equation. Recall that γ enters in the definition of J_γ through its appearance in the Riccati equation (8).

We prove the following result:

Theorem 1. *Generically for A, G, Q, for $\gamma > 0$ small enough, the function $J_\gamma(B)$ has $\binom{n}{m}$ critical points over the manifold Γ, exactly one of which is local minimum. Furthermore, the differential equation*

$$\dot{B} = -\gamma^2[B, [B, M]], \quad B(0) = B_0 \in \Gamma$$

where $M := PRP$, and P, R satisfy

$$A^\top P + PA + Q + G^\top PG - \gamma^2 PBP = 0,$$
$$(A - \gamma BP)R + R(A - \gamma BP)^\top + GRG^\top - L = 0.$$

converges to the global minimizer of $J_\gamma(B)$ from almost all initial state B_0.

This result in essence extends the results of [1] to the case of multiplicative noise, and show that one can also obtain an optimal design in this case, since the gradient flow of J, derived in this paper, will converge to the optimal design from a generic initial state.

We briefly sketch the proof. First, we will compute the gradient ∇J_γ of the function J_γ, with respect to an appropriately defined metric on the space Γ. Then we will show that as γ approaches 0, after well-chosen normalization, the function J_γ has a proper limit J_0^*. We will find the points at which ∇J_0^* vanishes, will show that their number is $\binom{n}{m}$, and that all of them are non-degenerate. We will compute the Hessian of J_0^* and thus find the signatures of the critical points. Since the number of critical points and their signatures are constant in the vicinity of 0, the theorem will follow.

II. Proof of the main result

A. Preliminary results

We now derive some preliminary results which may be of independent interest, and will be needed to prove the main result. They pertain to positive definite solutions of Lyapunov equations and the dependence of the Riccati equation with respect to its defining parameters.

The first result deals with the "generalized" Lyapunov equation

$$AX + XA^\top + G^\top XG - Q = 0,$$

which is a mix of the "discrete-time" Lyapunov equation $AXA^\top - X + Q = 0$ and "continuous-time" Lyapunov equation $AX + XA^\top + Q = 0$. It is also referred to as a Lyapunov Equation of mixed type [2]. In [2], [7], the following lemma is proved:

Lemma 1. *Let $A \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times n}$, where A is a Hurwitz matrix. We consider the generalized Lyapunov equation*

$$A^\top X + XA + G^\top XG + Q = 0. \tag{9}$$

The following statements are equivalent:

- *Equation (9) has a positive semi-definite solution $X \geq 0$ for some positive definite matrix $Q \in \mathbb{R}^{n\times n}$.*
- *The eigen-values of $\mathcal{L}_{A,G}$ have negative real parts.*

If any of the statements above are satisfied, equation (9) has has a unique symmetric (positive definite) solution X for any symmetric (positive definite) matrix Q. In this case, the solution X can be represented as the converging sum

$$X = \sum_{i=0}^{\infty} \mathcal{T}^i \left(\int_0^\infty e^{A^\top t} Q e^{At} dt \right),$$

where

$$\mathcal{T}(X) = \int_0^\infty e^{A^\top t} G^\top XG e^{At} dt.$$

The second preliminary result is to show that the positive definite solution of the Riccati equation (8) depends analytically on its parameters (under some assumptions to be listed). This result is an extension of [9], and the proof follows the same lines. We thus only sketch it.

Lemma 2. *Let $A, G, Q \in \mathbb{R}^{n\times n}$ be so that A is Hurwitz, Q is positive definite, and inequality (7) is satisfied. We introduce the function*

$$X : \Gamma \times \mathbb{R} \to \mathbb{R}^{n\times n} : (B, \gamma) \mapsto X(B, \gamma)$$

where $X(B, \gamma)$ is the unique positive definite solution of the Riccati equation

$$A^\top X + XA + Q + G^\top XG - \gamma^2 XBX = 0. \tag{10}$$

Then the map X is analytic.

Proof. As already mentioned, the proof follows Delchamps' approach and consists of using the inverse function theorem on an appropriately defined map. Namely, consider the map

$$\phi(B, \gamma, X) = A^\top X + XA + Q + G^\top XG - \gamma XBX.$$

Its differential with respect to X is given by

$$d\phi = dX(A - \gamma BX) + (A^\top - \gamma XB)dX + G^\top dXG.$$

Introduce the map $M : \mathrm{Sym}_n \to \mathrm{Sym}_n$ defined as

$$M_{(B,\gamma)} : T \to T(A - \gamma BX) + (A^\top - \gamma XB)T + G^\top TG.$$

Note that equation (10) can be rewritten, adding and subtracting $\gamma^2 XBX$, as

$$X(A - \gamma^2 BX) + (A^\top - \gamma^2 XB)X + G^\top XG$$
$$+(Q + \gamma^2 XBX) = 0.$$

Therefore, $X(\gamma, B)$—defined as the unique psd solution of Eq. (10)—is also a solution of the equation

$$M_{(B,\gamma)}(T) + (Q + \gamma XBX) = 0,$$

i.e., setting $T = X$ solves the above equation. Thus, we can apply Lemma 1 and conclude that there exists a unique symmetric solution T to the equation $M_{(B,\gamma)}(T) = S$ for symmetric S. We conclude that $M_{(B,\gamma)}(T) : \mathrm{Sym}_n \to \mathrm{Sym}_n$ is surjective. Now, from the implicit function theorem applied to $\phi(B, \gamma, X)$, we conclude that every solution X of (10) for a given (B, γ) can be extended uniquely in a small enough neighborhood of (B, γ). Since the Riccati equation has a unique positive definite solution [3] for every $\gamma \in \mathbb{R}, B \in \Sigma$, the claim of the lemma follows. $\square$

B. Gradient of J_γ and its critical points

We now evaluate the gradient of the function J_γ defined over Γ. Recall that on a Riemannian manifold, the gradient ∇J is defined with respect to an inner product $\langle\cdot,\cdot\rangle$ on Γ—we will introduce an inner product below—as the unique solution of

$$D_\Delta J := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}J(B+\varepsilon\Delta) = \langle \nabla J, \Delta\rangle, \forall\, \Delta \in T_B\Gamma, \quad (11)$$

where we also introduce the notation $D_\Delta J$ for the directional derivative of J along Δ. In words, the variation of the function along the direction Δ is equal to the inner product of the gradient ∇J with Δ. Because J is defined on Γ, we need to first find the set of allowed variations around B, or the tangent space of Γ at B. We note that every $B \in \Gamma$ is so that $\operatorname{rank} B = m$ and $B^2 = B$, and thus is an orthogonal projection to the subspace spanned by the columns of $b \in \mathbb{R}^{n\times m}$, where $bb^\top = B$. Reciprocally, to each m-dimensional subspace of $\mathbb{R}^n$, we can assign a unique orthogonal projection matrix B onto that subspace. Hence elements in Γ are in one-to-one correspondence with m-dimensional subspaces of $\mathbb{R}^n$, i.e. with elements of the Grassmanian [18] of m-planes in $\mathbb{R}^n$. It is furthermore well-known that Γ is a differentiable manifold, and admits a well-defined tangent space at any $B \in \Gamma$ denoted by $T_B\Gamma$. It is given by

$$T_B\Gamma = \{[B,\Omega] | \Omega \in \operatorname{skew}(n)\}, \quad (12)$$

where we recall that $[A,B] := AB - BA$ is the commutator or Lie bracket of A and B, and $\operatorname{skew}(n) \subset \mathbb{R}^{n\times n}$ is the set of skew-symmetric matrices, i.e. $A \in \operatorname{skew}(n)$ if $A = -A^\top$.

An inner product on $T_B\Gamma$: We now introduce the inner product on $T\Gamma$ we will work with. We keep the introduction short, since the same inner product was used in [1], [15], [5]. We emphasize that the choice of inner product does not change the main results, but makes the analysis simpler.

Since every tangent vector $\Delta \in T_B\Gamma$ is of the form $\Delta = [B,\Omega]$ for some $\Omega \in \operatorname{skew}(n)$, a seemingly good choice $\langle\cdot,\cdot\rangle$ would be $\langle\Delta_1,\Delta_2\rangle_B = -\operatorname{tr}(\operatorname{ad}_B^{-1}(\Delta_1)\operatorname{ad}_B^{-1}(\Delta_2)) = -\operatorname{tr}(\Omega_1\Omega_2)$, where $\Delta_1 = [B,\Omega_1]$ and $\Delta_2 = [B,\Omega_2]$.

However, the choice of Ω_1 and Ω_2 is not unique, i.e., $\operatorname{ad}_B : \operatorname{skew}_n \to T_B\Gamma$ is not invertible. We thus define $\bar{\operatorname{ad}}_B(\cdot)$ as:

$$\bar{\operatorname{ad}}_B : \operatorname{skew}(n)/\ker\,(\operatorname{ad}_B) \to T_B\Gamma,$$

where we regard $\operatorname{skew}(n)/\ker\,(\operatorname{ad}_B)$ as the orthogonal of $\ker\,\operatorname{ad}_B$ in $\operatorname{skew}(n)$ for the well-defined inner product in $\operatorname{skew}(n)$ given by $\operatorname{tr}(\Omega_1\Omega_2^\top)$. Now $\bar{\operatorname{ad}}_B$ is invertible for every B by construction, and we can define the operator

$$\langle\Delta_1,\Delta_2\rangle_B = -\operatorname{tr}(\bar{\operatorname{ad}}_B^{-1}(\Delta_1)\bar{\operatorname{ad}}_B^{-1}(\Delta_2)). \quad (13)$$

One can show that it is a well-defined inner product on Γ.

We now evaluate the left-hand side of Eq. (11), i.e. we compute the derivative of $J_\gamma(B)$, denoted by $D_\Delta J$ in the direction Δ. This derivative is well-defined from Lemma 2. From Eq. (12), it suffices to consider $\Delta = [B,\Omega]$ for $\Omega \in \operatorname{skew}(n)$. We sometimes write $D_\Omega J$ for $D_{[B,\Omega]}J$. Now assume Ω fixed and note that from that because the Riccati equation has a unique positive definite solution for all $B \in \Gamma$, the function $P(B)$ is well-defined as the solution of (8).

We introduce the short-hand notation $\dot B := [B,\Omega]$ and $\dot P := D_\Delta P$, for P defined as the positive definite of (8), and for $\Delta = [B,\Omega]$. Differentiating (8) in the direction Δ, we obtain

$$A^\top\dot P + \dot P A + G^\top \dot P G - \gamma^2\dot P BP - \gamma^2 P\dot B P - \gamma^2 PB\dot P = 0.$$

Gathering the terms multiplying $\dot P$ and $\dot B$, we obtain

$$(A-\gamma^2 BP)^\top\dot P + \dot P(A-\gamma^2BP) + G^\top\dot P G - \gamma^2 P\dot B P = 0$$

We can regard the equality above as a generalized Lyapunov equation in $\dot P$, similar to the one studied in Lemma 1.

Lemma 3. *Under the assumptions of Lemma 1, the derivative of J in the direction $\Delta = [B,\Omega]$ is given by*

$$D_\Omega(J) = -\gamma^2\operatorname{tr}([M,B]\Omega)$$

where $M := PR_iP$ and

$$R_i := \int_0^\infty\cdots\int_0^\infty e^{(A-\gamma^2BP)^\top t_1}G^\top e^{(A-\gamma^2BP)^\top t_2}G^\top \\ \ldots e^{(A-\gamma^2BP)^\top t_i}Le^{(A-\gamma^2BP)t_i}\ldots Ge^{(A-\gamma^2BP)t_1}dt_1\ldots dt_i$$

with P the positive definite solution of Eq. (8).

Proof. Applying Lemma (1) and the fact that $\operatorname{tr}([A,B]) = 0$, we get:

$$\begin{aligned}\dot P = &-\gamma^2\int_0^\infty e^{(A-\gamma^2BP)t_1}K\dot BPe^{(A-\gamma^2BP)^\top t_1}dt_1 \\ &-\gamma^2\int_0^\infty\int_0^\infty e^{(A-\gamma^2BP)t_2}Ge^{(A-\gamma^2BP)t_1}P\dot B \\ &\quad Pe^{(A-\gamma^2BP)^\top t_1}G^\top e^{(A-\gamma^2BP)^\top t_2}dt_1dt_2 - \ldots\end{aligned}$$

Using the above, we obtain

$$\begin{aligned}D_\Omega(J) &= \operatorname{tr}(L\dot P) \\ &= -\gamma^2\operatorname{tr}(\int_0^\infty Pe^{(A-\gamma^2BP)^\top t_1}Le^{(A-\gamma^2BP)t_1}Pdt_1\dot B) \\ &\quad -\gamma^2\operatorname{tr}(\int_0^\infty\int_0^\infty Pe^{(A-\gamma^2BL)^\top t_1}G^\top e^{(A-\gamma^2BP)^\top t_2}L \\ &\quad e^{(A-\gamma^2BP)t_2}Ge^{(A-\gamma^2BP)t_1}dt_1dt_2\dot B) - \ldots \\ &= -\gamma^2\operatorname{tr}([\sum_i M_i, B]\Omega),\end{aligned}$$

where $M_i := PR_iP$ and we set R_i as in the statement of the Lemma. Note that $\sum_i M_i$ converges since it is a linear transformation of a convergent series. Hence $D_\Omega(J)$ is well-defined. □

Next, we compute the gradient ∇J_γ of the function $J_\gamma(B)$.

Theorem 2. *The gradient ∇J_γ of the function J_γ with respect to the metric $\langle\cdot,\cdot\rangle$ defined above is*

$$\nabla(J_\gamma(B)) = \gamma^2[B,[B,M]],$$

where $M := PRP$, and P, R satisfy

$$A^\top P + PA + Q + G^\top PG - \gamma^2 PBP = 0,$$
$$(A - \gamma^2 BP)R + R(A - \gamma^2 BP)^\top + GRG^\top - L = 0.$$

Proof. The gradient ∇J_γ of J_γ satisfies

$$\langle \nabla J_\gamma, \Delta \rangle = D(J_\gamma)$$

for all vector fields $\Delta \in T\Gamma, \Delta = [B, \Omega], \Omega \in \text{skew}(n)$. Using the definition of the inner product given in Eq. (13), we obtain

$$\operatorname{tr}(\bar{\text{ad}}_B^{-1}(\nabla J_\gamma(B))\Omega) = \gamma^2 \operatorname{tr}([M, B](\Omega + \Theta)),$$

where $\Theta \in \ker \bar{\text{ad}}_B$ is arbitrary and M is as defined in the statement of the Theorem. Using the easily verified relation

$$\operatorname{tr}([A, B]C]) = \operatorname{tr}(A[B, C]),$$

we get $\operatorname{tr}([M, B]\Theta) = \operatorname{tr}(M[B, \Theta])$. Since $\Theta \in \ker \bar{\text{ad}}_B$,

$$\operatorname{tr}([M, B](\Omega + \Theta)) = \operatorname{tr}([M, B]\Omega) + \operatorname{tr}(M[B, \Theta]) = 0,$$

and therefore

$$\operatorname{tr}(\bar{\text{ad}}_B^{-1}(\nabla J_\gamma(B))\Omega) = -\gamma^2 \operatorname{tr}([M, B]\Omega)$$

for all $\Omega \in \text{skew}(n)$. Since $-\operatorname{tr}(\Omega_1\Omega_2)$ is a non-degenerate inner-product on $\text{skew}(n)$, this implies $\bar{\text{ad}}_B^{-1}(\nabla J_\gamma(B)) = \gamma^2 \operatorname{ad}_B(M)$ and

$$\nabla(J_\gamma(B)) = \gamma^2 \operatorname{ad}_B \operatorname{ad}_B M = \gamma^2 [B, [B, M]].$$

as announced. □

We record the immediate Corollary

Corollary 1. *The critical points of the function $J_\gamma(B)$ satisfy the equality*

$$[B, M] = 0,$$

where M is as defined in Theorem 2 .

Proof. The critical points of a function are exactly the points where its gradient vanishes. Since B is symmetric and $[B, M]$ is skew symmetric, $\gamma^2[B, [B, M]] = 0$ implies

$$[B, M] = 0.$$

as announced. □

III. Convergence of gradient descent

We aim to find an optimal actuator via a gradient descent

$$\dot{B} = -\gamma^2 [B, [B, M]] \tag{14}$$

with M defined in Theorem 2. It is not too difficult to see that the function $J_\gamma(B)$ is not convex, and hence we need to argue for the convergence of the method. We do so by showing that J_γ generically for the parameters A, G, Q has a *unique* minimum, and hence gradient descent will converge to that minimum from almost all initial value $B(0)$.

To this end, define the function

$$J_\gamma^* := \frac{1}{\gamma^2}(J_\gamma - J_0), \text{ with } J_0 := \operatorname{tr}(LP_0),$$

where P_0 is the positive definite solution of the equation

$$A^\top P_0 + P_0 A + Q + G^\top P_0 G = 0.$$

Furthermore, set

$$J_0^* := \lim_{\gamma \to 0} J_\gamma^*.$$

We know from Lemma 2 that $J_\gamma^*(B)$ is analytic in both γ and B and it clearly has the same critical points as $J_\gamma(B)$ for fixed $\gamma \neq 0$, since the two functions differ by a constant. Therefore, if we show that the critical points of the function J_0^* are *non-degenerate*, then it will follow that J_γ has the same number of critical points and the same corresponding signatures as J_0^* for small $\gamma \neq 0$.

In order to do this, first we first establish the following result

Proposition 1. *Let A and G be so that the assumption* (7) *is satisfied. Suppose also that there exists $x \in \mathbb{R}^n$, such that the pair (A, x) is controllable. Then, generically for all positive definite Q, and positive semi-definite L of rank* 1, *the function J_0^* has $\binom{n}{m}$ critical points.*

The derivation uses Theorem 3.6.1 in [7] and follows strictly the proof of Proposition 1 in [1], and we thus omit it here. We now evaluate the Hessian of J_0^*, that is the derivative of the gradient of J_0^*, to check that it is indeed non-degenerate. Recall that the Hessian is a symmetric bilinear form taking its argument in $T_B\Gamma$. We have the following result:

Proposition 2. *The Hessian $H_{J_0^*}$ of the function J_0^* satisfies the equality*

$$H_{J_0^*}(\Delta_1, \Delta_2) = \operatorname{tr}([M_0, \Omega_1][B, \Omega_2])$$

at critical points B of J_γ^*, where $\Delta_1 = [B, \Omega_1]$ and $\Delta_2 = [B, \Omega_2]$ for $\Omega_i \in \text{skew}(n)$ and the matrix $M_0 := P_0 R_0 P_0$ where P_0 positive definite solution of

$$A^\top P_0 + P_0 A + Q + G^\top P_0 G = 0,$$

and R_0 the positive definite solution of

$$AR_0 + R_0 A^\top + G^\top R_0 G - L = 0.$$

Proof. Let $F : \Gamma \to \mathbb{R}$ be a twice differentiable function. We have the general formula for the Hessian [17] H_F of F evaluated in the directions Δ_1, Δ_2:

$$H_F(\Delta_1, \Delta_2) = \Delta_1 \cdot \Delta_2 \cdot F + D_{\Delta_1}\Delta_2 \cdot F,$$

where Δ_1 and Δ_2 are arbitrary vector fields on $T\Gamma$ and $D_{\Delta_1}\Delta_2$ is the covariant derivative of Δ_2 along Δ_1. It is easy to see that second term on the right side of the formula above vanishes at the critical points of F, since $D_{\Delta_1}\Delta_2 \cdot F = \langle \nabla F, D_{\Delta_1}\Delta_2 \rangle = 0$ when $\nabla F = 0$. Hence we just need to evaluate

$$H_{J_0^*}^*(\Delta_1, \Delta_2) = \Delta_1 \cdot \Delta_2 \cdot J_0^*.$$

To proceed, we note that from Theorem 2 and the definition of J_0^* (recall that J_0 is constant) we get $\nabla J_0^* = [B, [B, M_0]$

where $M_0 := P_0 R_0 P_0$ and P_0, R_0 are as in the statement of the Proposition.

From the definition of the gradient and the inner product used, we have

$$\Delta_2 \cdot J_0^* = \langle \nabla J_0^*, \Delta_2 \rangle = \operatorname{tr}(\Omega_2 [B, M_0]).$$

Next we evaluate $D_1 \cdot D_2 \cdot J_0$, which is the derivative of $\operatorname{tr}(\Omega_2[B, M_0])$ in the direction D_1. This is easily seen to be :

$$\begin{aligned} D_1 \cdot D_2 \cdot J_0 &= \operatorname{tr}([M_0, [B, \Omega_1]]\Omega_2) \\ &= \operatorname{tr}([M_0, \Omega_1][B, \Omega_2]). \end{aligned}$$

This concludes the proof of the proposition. □

Recall that the signature of a bilinear form is a triplet of integers (n_+, n_-, n_0) with entries the number of positive, negative and zero eigenvalues of the bilinear form. The bilinear form is non-degenerate if $n_0 = 0$. The next step is to compute the signature of the bilinear form $H^* : (\Omega_1, \Omega_2) \to tr([M, \Omega_1][B, \Omega_2])$, which gives us the sign of the eigenvalues of the Hessian of J_0^0 at the critical points. We need to introduce the number of distinct partitions of an integer bounded by an integer: to this end, let n, k and l be positive integers. We call a partition of l into k parts a set of k (strictly) positive integers whose sum is l. We call the partition *distinct* if no integer in the sum is repeated. Finally, we say that the partition is bounded by n if not number in the sum is larger than n. We denote by $Q_n(k, l)$ the number of distinct partitions of k into l parts, bounded by n. For example $Q_4(9, 3) = 3$, since we have $9 = 3+3+3 = 4+4+1 = 4+3+2$. We have the following result:

Proposition 3. *The function J_0^* has $Q_n(m, n_+ + \frac{m(m+1)}{2})$ critical points with signatures $(n_+, n_-, 0)$, where (n_+, n_-) are all pairs for which $n_+ + n_- = d$, and $Q_n(k, l)$ is the number of ways to partition l into k parts no larger than n. Furthermore, exactly one of these critical points is a minimum. Furthermore, no critical points is degenerate generically for the parameters A, G, Q.*

We omit the proof here due to space constraints, but it follows the lines of the proof of Theorem 3 in [1]. The above proposition proves Theorem 1

IV. Summary and Outlook

We have shown that under some assumptions on the matrices A, G defining the dynamics, the problem of optimal actuator design of linear systems with multiplicative noise admit a well-defined unique solution. Furthermore, we have provided a gradient descent to obtain this solution and shown that it converges almost surely (with respect to the choice of initial state for the gradient descent equation).

We believe that the next problem to tackle in this direction is the one of designing an optimal sensor and actuator in the output feedback case for the additive case. Indeed, it is well-known that combining a Kalman filter to estimate the state and using the estimate in an optimal LQ control yields an overall optimal performance: this is the celebrated separation principle. In this setting, does an optimal co-design of both the sensor c and the actuator b exist? And if so, how to obtain them?

References

[1] M-A Belabbas. Geometric methods for optimal sensor design. *Proceedings of the Royal Society, Series A Math Phys Eng Sci*, 472(2185):20150312, Jan 2016.

[2] Peter Benner and Tobias Damm. Lyapunov equations, energy functionals, and model order reduction of bilinear and stochastic systems. *SIAM journal on control and optimization*, pages 686–711, 2011.

[3] Sergio Bittanti, Alan J Laub, and Jan C Willems. *The Riccati Equation*. Springer Science & Business Media, 2012.

[4] R. Brockett. *Finite dimensional linear systems*. John Wiley & Sons, 1970.

[5] R Brockett. Dynamical systems that sort lists, solve linear programming problems and diagonalize symmetric matrices. *Linear Algebra Appl*, 146:79–91, 1991.

[6] X. Chen and M.-A. Belabbas. On optimal actuator placement for minimizing the worst-case control energy. In *Proceedings of the 20th World Congress of the International Federation of Automatic Control*, 2017.

[7] T. Damm. *Rational Matrix Equations in Stochastic Control*. Number 297 in Lecture Notes in Control and Information Sciences. Springer–Verlag, Berlin, 2004.

[8] Neda Darivandi, Kirsten Morris, and Amir Khajepour. An algorithm for LQ optimal actuator location. *Smart Materials and Structures*, 22(3):035001, 2013.

[9] D. Delchamps. Analytic feedback control and the algebraic Riccati equation. *IEEE Transactions on Automatic Control*, 29(11):1031–1033, 1984.

[10] Neil K Dhingra, Mihailo R Jovanović, and Zhi-Quan Luo. An admm algorithm for optimal sensor and actuator selection. In *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, pages 4039–4044. IEEE, 2014.

[11] Susanne Ditlevsen and Adeline Samson. Introduction to stochastic models in biology. In *Stochastic biomathematical models*, pages 3–35. Springer, 2013.

[12] Fariba Fahroo and Michael A Demetriou. Optimal actuator/sensor location for active noise regulator and tracking control problems. *Journal of Computational and Applied Mathematics*, 114(1):137–158, 2000.

[13] J.C. Geromel. Convex analysis and global optimization of joint actuator location and control problems. *IEEE Transactions on Automatic Control*, 34(7):711–720, July 1989.

[14] A.K. Gupta and D.K. Nagar. *Matrix Variate Distributions*. Monographs and Surveys in Pure and Applied Mathematics. Taylor & Francis, 1999.

[15] U. Helmke and J. B. Moore. *Optimization and dynamical systems*. Springer, London, 1994.

[16] Syed Talha Jawaid and Stephen L Smith. Submodularity and greedy algorithms in sensor scheduling for linear dynamical systems. *Automatica*, 61:282–288, 2015.

[17] J. Jost. *Riemannian Geometry and Geometric Analysis*. Springer, 2011.

[18] John M. Lee. *Introduction to smooth manifolds*. Graduate texts in mathematics. Springer, New York, NY [u.a.], 2. ed. edition, 2012.

[19] D.S. Lemons and P. Langevin. *An Introduction to Stochastic Processes in Physics*. Johns Hopkins Paperback. Johns Hopkins University Press, 2002.

[20] K. Morris. Linear-quadratic optimal actuator location. *IEEE Transactions on Automatic Control*, 56(1):113–124, January 2011.

[21] B. Øksendal. *Stochastic Differential Equations: An Introduction with Applications*. Hochschultext / Universitext. Springer, 2003.

[22] Guillaume Sagnol and Radoslav Harman. *Optimal Designs for Steady-State Kalman Filters*, pages 149–157. Springer International Publishing, Cham, 2015.

[23] Manohar Shamaiah, Siddhartha Banerjee, and Haris Vikalo. Greedy sensor selection: Leveraging submodularity. In *Decision and Control (CDC), 2010 49th IEEE Conference on*, pages 2572–2577. IEEE, 2010.

[24] Haotian Zhang, Raid Ayoub, and Shreyas Sundaram. Sensor selection for kalman filtering of linear dynamical systems: Complexity, limitations and greedy algorithms. *Automatica*, 78:202–210, 2017.