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Development of a Control Theoretic Based Simulation Model of a Supply and Distribution System with Reusable Items

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***Abstract*— Control theoretic modelling and control approaches provide a systematic way to address the problem associated with inventory control of supply and distribution systems. Supply chain systems usually contain a number of lead times among supply nodes and are susceptible to demand variation. In this work, we developed a linear uncertain discrete time model of a distribution system with multiple supply nodes and pure delays by considering reusable supply items and multiple supply chains. The model allows for a three-level distribution system which include customer, distributor and plant, where customers can also supply items to other requesting customers. The model presented here paves the way towards developing robust control policies for distribution of items under state and input constraints. In order to determine robust resources a discrete event simulation model is developed based on the proposed discrete model using Simulink® toolbox SimEvents. Finally a robust distribution policy and item return policy is obtained considering operational cost and customer satisfaction.**

I. Introduction

The inventory and production system is a dynamical system which invariably comprises various lead times and several delays among supply nodes. In real life scenarios, the nominal demands are usually known, however, difficulty arises when the stochastic nature of system makes it difficult to maintain the stability of the system when demand and supply vary beyond the nominal values. In the control theory based methods, ordering policy is determined through feedback which is a function of the current state of the echelons (levels) and demand. Detailed reviews of the application of control theory to supply chain investigation at different stages are provided in [1]. A model based on control theoretic approaches can be efficiently used to study the effect of lead times and various ordering policies. The control literature contains several methods which can be applied to the supply chain system using continuous time or discrete time deterministic or stochastic models. A first attempt to use continuous time linear control systems theory to investigate the supply chain system behavior is presented in [2]. Also, frequency domain techniques to understand the demand amplification dynamics commonly known as the bullwhip effect is used by [3]. In this work control theoretical aspects have been considered to study the demand amplification. Since cost minimization is inherent in the supply chain problem, researchers turn their attention towards optimal control theory. The optimal control problem has been addressed in [4].

The discrete time model represents a natural representation of a supply chain due to a finite review period and an ability to incorporate pure delays in the system model. A recent work on using state feedback control under unit delay with controllability and observability condition is presented in [5]. In [6] it is concluded that demand amplification is only caused if delay in the system is combined with the demand forecasting. In the last two decades researchers have shown some interest in using Model Predictive Control (MPC) for inventory and production control. MPC is an optimization based multi-variable control algorithm which deals with interaction and constraints. The goal of the MPC design is to predict the future evolution of the system and find a control policy that brings the system to some desired stock level satisfying the relevant constraints and the minimization of an objective (cost) function (see e.g. [7]). Due to the in-built online optimization algorithm, this method also accommodates the scheduling problem where other control methods fail to perform. MPC was first applied to inventory and production management by [8] and later extended by other researchers – see, for example [9]. In this work the MPC scheme is used to manage a multi-echelon multi-product supply chain with deterministic demand.

In this paper, our approach is to develop simulation model based on the control theoretic discrete time model of a distribution system. The reason of development of control theoretic model is twofold. Firstly the model facilitate the development of robust policy using control systems approaches and secondly we show that using the simulation model, a robust distribution policy can also be obtained based on specific scenarios. In this paper we consider an example of a three-level supply chain distribution system of multiple re-usable items with multiple supply chains. Following is the main contribution of the paper:

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1) we develop a linear uncertain discrete time control theoratic model of a distribution system with multiple supply nodes and multiple pure delays by considering reusable supply items.
2) Finally we present a discrete simulation model developed using Simulink® toolbox SimEvents to determine the robust control policy considering a three customer-three distributor-one plant system.

II. System Definition

The supply chain distribution system consists of multiple customers at several locations managed by a Prime Service Contractor (PSC) which also has multiple distributor locations. It is assumed that there are m distributor locations, n customer locations, and L of usable items which are available again after use. A customer requests items and the PSC chooses a distribution policy considering the cost of delivery and items availability at other locations. Supply of items requires some lead time due to transportation and potential item unavailability time. Each location has different lead times for the delivery and has different unavailability times. Limited storage is also available at each customer location to store the items if require. We assume that if an item is damaged after use it is discarded. The complete system and the dynamics of stocked items are illustrated in the Fig.1.

III. System Model

A. Model for Individual Customer

The stocked dynamics of the p^{th} customer for item l can be represented using the following equation:

$$\begin{aligned} S_{p,l}(k) &= S_{p,l}(k-1) + \sum_{\substack{j=1\\ j\neq p}}^{n} P_{jp,l}(k - T_{u_{j,l}} - T_{d_{jp,l}} - 1) \\ &+ \sum_{i=1}^{m} \bar{P}_{ip,l}(k - D_{ip,l} - 1) - \sum_{j=1}^{n} d_{pj,l}(k-1) \\ &- \sum_{i=1}^{m} \bar{d}_{pi,l}(k-1) - d_{ap,l}(k-1), \end{aligned} \tag{1}$$

for all $p = 1, 2, \cdots, n$, and $l = 1, 2, \cdots, L$ where $S_{p,l}(k)$ is the amount of stocked item l at the p^{th} customer, $P_{jp,l}$ is the item l arrived from other customers, $P_{ip,l}$ is the item l arrived from each q^{th} distributor, $d_{pj,l}$ is the item going out from the p^{th} customer to other customer, $\bar{d}_{pi,l}$ are the items returning to the requested distributor, $d_{pj,l}(k)$ is the demand received from each customers and $d_{ap,l}(k)$ is the item damaged during use. $T_{u_{j,l}}$ is the use time of item l at the p^{th} customer location and $T_{d_{jp,l}}$ is delivery time of the item. Equation (1) can be written in a standard form by splitting the equation into a series of first-order difference equations [10] and by introducing a new variable as follows: Let us suppose $\lambda_{pj,l} = T_{u_{j,l}} + T_{d_{jp,l}}$ and:

$$z_{jp,l,(\lambda_{pj,l}-1)}(k) = P_{jp,l}(k-1) = u_{jp,l}(k-1), \tag{2}$$

for all $p = 1, 2, \cdots, n$ and $j \neq p$. Also;

$$w_{ipl,(D_{ip,l}-1)}(k) = \bar{P}_{ip,l}(k-1) = \bar{u}_{ip,l}(k-1), \tag{3}$$

where $u_{jp,l}$ and $\bar{u}_{ip,l}$ are control inputs. Hence the stocked good dynamics equation for the p^{th} customer (1) can be written as follows:

$$\begin{aligned} &S_{p,l}(k) = S_{p,l}(k-1) + \sum_{\substack{j=1\\ j\neq p}}^{n} z_{jpl,1}(k-1) + \\ &\sum_{i=1}^{m} w_{ipl,1}(k-1) - \sum_{j=1}^{n} d_{pj,l}(k-1) - \sum_{i=1}^{m} \bar{d}_{pi,l}(k-1) \\ &- d_{ap,l}(k-1). \end{aligned} \tag{4}$$

Equations (2), (3) and (4) can be written in the state space form as follows:

$$\begin{aligned} Z_{p,l}(k+1) &= A_{p,l} Z_{p,l}(k) + B_{p,l} U_{p,l}(k) + D_{p,l} \xi_{p,l}(k) \\ Y_{p,l}(k) &= C_{p,l} Z_{p,l}(k); \end{aligned} \tag{5}$$

where, $Z_{p,l}(k) \in \Re^{r_{p,l}}$, $r_{p,l} = \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_{pj,l} + \sum_{i=1}^{m} D_{ip,l} - m - n + 2$ is the state vector, the $U_{p,l}(k) \in \Re^{n+m-1}$ is a control vector and $\xi_{p,l}(k) \in \Re^{n+m+1}$ considered as a disturbance vector for the customer p which include external demands and discarded items. Also, $Y_{p,l}(k) \in \Re$ is the output of the system i.e. stock level. The definition of other variables are given in the appendix. The system matrix $A_{p,l}$ in (5) has one eigenvalue of 1 and others eigenvalues are zero. It can be seen that the states of the system can be changed from initial state to another state through system input vector $U_{p,l}$ and it is assumed that the stock is measurable, hence the system is controllable and observable.

B. Model for Individual Distributor

The stocked items dynamics for item l for the q^{th} distributor can be modeled using the following equations:

$$\begin{aligned} \bar{S}_{q,l} &= \bar{S}_{q,l}(k-1) + \sum_{j=1}^{n} \hat{P}_{jq,l}(k - T_{u_{j,l}} - D_{jq,l} - 1) + \\ &\tilde{P}_{q,l}(k - T_{L_{q,l}} - 1) - \sum_{j=1}^{n} d_{qj,l}(k-1), \end{aligned} \tag{6}$$

for all $q = 1, 2, \cdots, m$ and $l = 1, 2, \cdots, L$, where $\bar{S}_{q,l}$ is the amount of stocked items l at the q^{th} distributor, $\tilde{P}_{q,l}$ are the items arrived at q^{th} distributor location from a supplier with transport time of $T_{L_{q,l}}$, $\hat{P}_{jq,l}$ are the items arrived back from each customer after use with delivery time of $D_{jq,l}$, and $d_{q,j,l}$ are the demand received from each customer. The transform difference equations for (6) can be written in a standard form similar to the way obtained in the case of individual customer

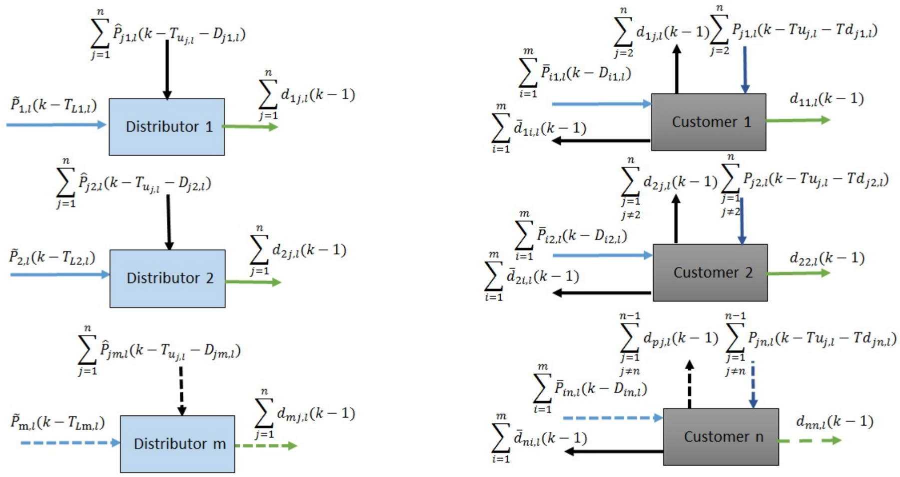

Fig. 1: Supply chain dynamics.

by introducing new variable as follows: Let us suppose $\bar{\lambda}_{qj,l} = T_{u_{j,l}} + D_{jq,l}$ and:

$$\bar{z}_{jql,(\bar{\lambda}_{qj,l}-1)}(k) = \hat{P}_{jq,l}(k-1) = u_{d_{jq,l}(k-1)} \tag{7}$$

for all $j = 1, 2, \cdots, n$. Also,

$$\bar{w}_{q,l,(T_{L_{q,l}}-1)}(k) = \tilde{P}_{q,l}(k-1) = \bar{u}_{dq,l}(k-1) \tag{8}$$

for $i = 1, 2, \cdots, m$ where, $u_{d_{jq,l}}$ and $\bar{u}_{d_{q,l}}$ are control inputs. Hence the stocked good dynamics equation (6) for the q^{th} distributor can be written as follows:

$$\bar{S}_{q,l}(k) = \bar{S}_{q,l}(k-1) + \bar{w}_{ql,1}(k-1) + \sum_{j=1}^{n} \bar{z}_{jql,1}(k-1) - \sum_{j=1}^{n} \bar{d}_{qj,l}(k-1). \tag{9}$$

A standard state space dynamics equation for the q^{th} distributor can be written using (7), (8) and (9) as follows:

$$\begin{aligned} \bar{Z}_{q,l}(k+1) &= \bar{A}_{q,l}\bar{Z}_{q,l}(k) + \bar{B}_{q,l}\bar{U}_{q,l}(k) + \bar{D}_{q,l}\bar{\xi}_{q,l}(k); \\ \bar{Y}_{q,l}(k) &= \bar{C}_{q,l}\bar{Z}_{q,l}(k), \end{aligned} \tag{10}$$

where, $\bar{Z}_{q,l}(k) \in \Re^{\bar{r}_{q,l}}$, $\bar{r}_{q,l} = \sum_{j=1}^{n} \bar{\lambda}_{qj,l} + T_{L_{q,l}} - n$ is the state vector, the $\bar{U}_{q,l}(k) \in \Re^{n+1}$ is a control vector and $\bar{\xi}_{q,l}(k) \in \Re^{n}$ considered a disturbance vector for the distributor q, $\bar{Y}_{q,l}(k) \in \Re$ is the output of the system i.e. stock level and

$$\bar{Z}_{q,l}(k) = [\bar{S}_{p,l}(k), \bar{z}_{jql,1}(k), \cdots \bar{z}_{jql,\bar{\lambda}_{qj,l}-1}, \bar{w}_{ql,1}(k), \cdots, \bar{w}_{ql,T_{L_{q,l}}-1}(k)]^{T}.$$

Matrices $\bar{A}_{q,l}$, $\bar{B}_{q,l}$, $\bar{D}_{q,l}$ and $\bar{C}_{q,l}$ have similar definitions as for customer stocked good dynamics and omitted here for the sake of brevity. The system (10) shares the same characteristics as system (5) and is thus controllable and observable.

IV. Model for Complete System

For a centralized control, customer and distributor systems must be considered as a single system. It is assumed that information flows freely among all the nodes (customer, distributor and plant locations) at the same and different levels. In the combined system dynamics it can be seen that the demand received (disturbance) at distributors' location is the control input to the respective customers. Also, the disturbance to the system is reduced to only number of customers present in the system i.e. demand from each customer. Let us define the following variables by augmenting all customer dynamics: $\tilde{A}_l = diag(A_{1,l}, A_{2,l}, \cdots, A_{n,l})$,

$$\tilde{Z}_l(k) = \begin{bmatrix} Z_{1,l}(k) \\ Z_{2,l}(k) \\ \vdots \\ Z_{n,l}(k) \end{bmatrix}, \quad \tilde{U}_l(k) = \begin{bmatrix} U_{1,l}(k) \\ U_{2,l}(k) \\ \vdots \\ U_{n,l}(k) \end{bmatrix},$$

$$\tilde{B}_l = \begin{bmatrix} B_{1,l} & \breve{D}_{12,l} & \cdots & \breve{D}_{1n,l} \\ \breve{D}_{21,l} & B_{2,l} & \cdots & \breve{D}_{2n,l} \\ \vdots & \vdots & \vdots & \vdots \\ \breve{D}_{n1,l} & \cdots & \breve{D}_{n(n-1),l} & B_{n,l}, \end{bmatrix},$$

$\breve{D}_{pj,l} = D_{p,l}$; except entries corresponding to $d_{jp,l}$ with $j = p$ are omitted and entries which are not corresponding to $u_{pj,l}$ for $j = 1, 2, \cdots, n$ and $j = p$, are zero. Also;

$$\tilde{\xi}_l(k) = \begin{bmatrix} \xi_{1,l}(k) \\ \xi_{2,l}(k) \\ \vdots \\ \xi_{n,l}(k) \end{bmatrix}, \quad \tilde{D}_l = \begin{bmatrix} \hat{D}_{1,l} & \mathbf{0} & \cdots & \mathbf{0} \\ 0 & \hat{D}_{2,l} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \hat{D}_{n,l}, \end{bmatrix},$$

where, $\hat{D}_{p,l} = D_{p,l}$ except all entries corresponding to $d_{pj,l}$ with $j \neq p$ are zero. Also define following variables by augmenting all distributor dynamics: $\bar{A}_l = diag(\bar{A}_{1,l}, \bar{A}_{2,l}, \cdots, \bar{A}_{m,l})$, $\bar{B}_l = diag(\bar{B}_{1,l}, \bar{B}_{2,l}, \cdots, \bar{B}_{m,l})$, $\bar{D}_l = diag(\bar{D}_{1,l}, \bar{D}_{2,l}, \cdots, \bar{D}_{m,l})$, and $\bar{C}_l =$

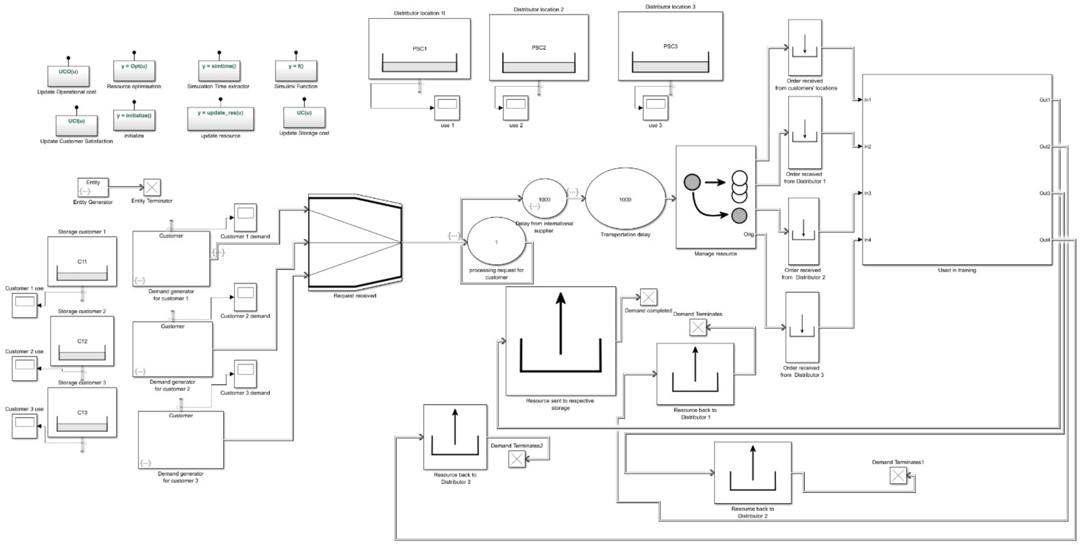

Fig. 2: Simulink® SimEvents discrete simulation model.

$diag(\bar{C}_{1,l}, \bar{C}_{2,l}, \cdots, \bar{C}_{m,l})$. Also,

$$\bar{U}_{q,l}(k) = \left[\begin{array}{cccc} u_{d1q,l}(k) & \cdots & u_{dnq,l}(k) & \bar{u}_{dq,l}(k) \end{array}\right]^T,$$
$$\bar{\xi}_{q,l}(k) = \left[\begin{array}{cccc} \bar{d}_{q1,l}(k) & \bar{d}_{q2,l}(k) & \cdots & \bar{d}_{qn,l}(k) \end{array}\right]^T,$$

$$\bar{Z}_l(k) = \begin{bmatrix} \bar{Z}_{1,l}(k) \\ \bar{Z}_{2,l}(k) \\ \vdots \\ \bar{Z}_{m,l}(k) \end{bmatrix},\ \bar{U}_l(k) = \begin{bmatrix} \bar{U}_{1,l}(k) \\ \bar{U}_{2,l}(k) \\ \vdots \\ \bar{U}_{m,l}(k) \end{bmatrix},$$

$$\bar{\xi}_l(k) = \begin{bmatrix} \bar{\xi}_{1,l}(k) \\ \bar{\xi}_{2,l}(k) \\ \vdots \\ \bar{\xi}_{m,l}(k) \end{bmatrix}.$$

$$\begin{aligned} \bar{Z}_l(k+1) &= \bar{A}_l \bar{Z}_l(k) + \bar{B}_l \bar{U}_l(k) + \bar{D}_l \bar{\xi}_l(k) \\ \bar{Y}_l(k) &= \bar{C}_l \bar{Z}_l(k). \end{aligned} \tag{11}$$

Then the complete augmented system with customer and distributor dynamics combined together is given as:

$$\begin{aligned} Z_l(k+1) &= A_l Z_l(k) + B_l U_l(k) + D1_l \xi_l(k) \\ Y_l(k) &= C_l Z_l(k); \\ z_l(k) &= H_{1,l} x(k) + G_{1,l} u(k), \end{aligned} \tag{12}$$

where, $Z_l(k) = [\tilde{Z}^T(k) \quad \bar{Z}^T(k)]^T$, $A_l = diag(\tilde{A}_l, \bar{A}_l)$, $U_l = [\tilde{U}_l^T(k) \quad \bar{U}_l^T(k)]^T$, $C_l(k) = diag(\tilde{C}_l(k), \bar{C}_l(k))$,

$$\xi_l = [d_{11,l}(k), d_{a1,l}(k) \cdots, d_{nn,l}(k), d_{an,l}(k)]^T$$

$$D1_l = \begin{bmatrix} -1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & -1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}, \end{bmatrix}, B_l = \begin{bmatrix} \tilde{B}_l & \check{D}_l \\ \hat{D}_l & \bar{B}_{n,l}, \end{bmatrix},$$

$\check{D}_l = \bar{D}_l$, where all entries of $\bar{D}_{q,l}$ is zero except the entries corresponding to $u_{dq1,l}$ for $q = 1, 2, \ldots, m$ and for each $p = 1, 2, \ldots, n$, and $\hat{D}_l = \tilde{D}_l$ where all entries of $\tilde{D}_{p,l}$ is zero except the entries corresponding to $u_{1p,l}$ for $p = 1, 2, \ldots, n$, and for each $q = 1, 2, \ldots, m$. Also, $z_l(k)$ is the uncertainty output, $H_{1,l}(k)$ and $G_{1,l}(k)$ are uncertainty outputs matrices which are defined considering the level of uncertainty in the system.

A. System Constraints

The distribution system (12) subject to several constraints. The constraints are given below:

1) The states of stocked goods for customers and distributors are constrained as follows:
$$0 \le S_{p,l}(k) \le S_{p,l}^{max}, \quad \bar{S}_{q,l}(k) \ge 0, \tag{13}$$
where $S_{p,l}^{max}$ is the maximum items which can be stocked at any customer storage.
2) The control inputs are constrained as follows:
$$U_{p,l}(k) \ge 0,\ \bar{U}_{q,l}(k) \ge 0 \tag{14}$$
3) Delivery lead time should be less than or equal to the required delivery time
$$max\{\lambda_{pj,l}, D_{ip,l}\} \le T_{d_{p,l}}, \tag{15}$$
$\forall\ j = 1, \cdots, n$ where $j \ne p$ and $i = 1, \cdots, m$, and $T_{d_{p,l}}$ is the required delivery time:

The model (12) along with constraints (13) – (15) can be used to determine robust control policy using robust optimal control approaches, [11], [12]. The policies which are of interest could be initial allocation of items at each distributor location, items ordering time from the main supplier, items return policy after used by the customers etc. In the next section the model (12) is used to determine the control policies for the system utilizing a simulation model developed in SimEvents toolbox in the Simulink.

Remark 1: In the case of more than one item present in the system, the stocked dynamics of each individual item can be augmented in equation (12). The augmentation would increase the number of states in the system and may affect the capacity of each customer storage.

V. Simulation Model

In order to obtain control policies for a distribution system with three customers and three distributors and a plant, a discrete simulation model is developed using Simulink® SimEvents toolbox. The objective is to find the number of items required at each distributor location and item return policy so that operational cost is minimized and customer satisfaction is maximized. We suppose that we have a demand and stock uncertainty of 20% in the system and there is enough stock available at the plant. A snapshot of the SimEvents simulation model is shown in Fig. 2. The transport delay and various lead times between customers and distributors considered here are given in Table 1. The model starts with the random demand at each customer location. Here we introduce a satisfaction index variable which assigns value depending upon the time it takes to fulfill the complete order. For this purpose we define a backlog variable $b(k)$ for the customers:

$$b_{p,l}(k) = S_{p,l}(k) + \sum_{\substack{j=1\\ j\neq p}}^{n} P_{jp,l}(k - T_{u_{j,l}} - T_{d_{jp,l}}) + \sum_{i=1}^{m} \bar{P}_{ip,l}(k - D_{ip,l}) - d_{pp,l}(k - T_{d_{p,l}}) \tag{16}$$

for all $k \in K_p$ where K_p is a set of time steps at which demand is generated at the p^{th} customer. For each demand if $b_k < 0$ the satisfaction index goes down one unit each day until demand is fulfilled. If $b_k \geq 0$ then index goes up by one unit. Here we considered two return policies to analyze which are given below.

1) Items kept at customers' storage after use and only be returned to the distributor if it exceeds the customer storage capacity.
2) After use, all items are returned to the distributor from where it was delivered.

VI. Results

A full factorial simulation has been performed by considering nominal demand profile with demand variation in the range of entire uncertainty spectrum and given return policies for a full year. Three nominal demand profiles have been considered in the simulation i.e. low demand, average demand, and high demand. The simulation runs to simulate each possible scenarios. The return policy which results in lower cost and higher customer satisfaction over the entire range of uncertainty and demand profile is selected. Fig. 3 shows the comparison of running sum of satisfaction index for

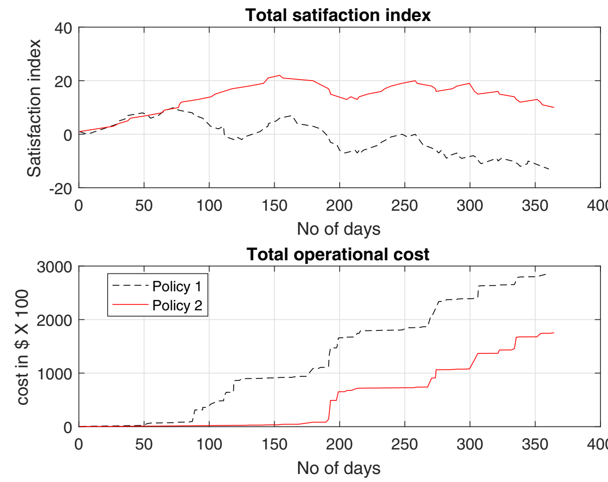

Fig. 3: Total satisfaction index and total operational cost for the return policies.

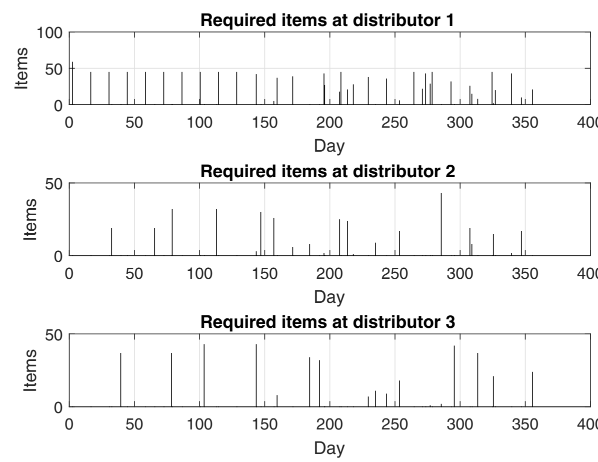

Fig. 4: Number of items required at any given day.

both the policies (i.e. taking the minimum of satisfaction index over all the simulation run at each time step). It is observed that return policy 2 is the best policy under the given condition. Also Fig. 4 shows the number of items required at each distributor at each time step for a stable operation.

VII. Conclusion

In this work, a linear uncertain discrete time model of a three-level distribution system with multiple supply chains and pure delays are developed using the control theoretic modeling approach. The model facilitates the use of control systems approaches to obtain robust distribution and ordering policy for the system stability. The model incorporates pure delay in the system and allows for multiple items and multiple supply chains.

item use time	$T_{u_1,l}=1$	$T_{u_2,l}=2$	$T_{u_3,l}=1$
Transport time (customer)	$T_{d_{12,l}}=T_{d_{21,l}}=2$	$T_{d_{13,l}}=T_{d_{31,l}}=3$	$T_{d_{32,l}}=T_{d_{23,l}}=2$
Transport time (distributor)	$D_{11,l}=1$ $D_{22,l}=2$ $T_{L1,l}=3$	$D_{12,l}=D_{21,l}=3$ $D_{23,l}=D_{32,l}=2$ $T_{L2,l}=4$	$D_{13,l}=D_{31,l}=3$ $D_{33,l}=2$ $T_{L3,l}=5$

TABLE I: Lead time and Transport delays in units.

In addition a discrete event simulation model is also developed based on the proposed discrete time model using SimEvents toolbox in Simulink. The simulation model can be used to test various scenarios and can be run for short or long term planning. However in this work, the simulation is carried out for a year considering full range of uncertainty spectrum, and system stability, operational cost and customer satisfaction are observed. Finally the items required at each distributor for a stable operation is determined and the best policy for returning item after use is proposed.

VIII. Appendix A (system matrices in (5))

$$A_{p,l}=\begin{bmatrix} 1 & 1 & 0_{1\times(\lambda_{p1,l}-2)} & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots \\ 0 & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots \end{bmatrix} \in \Re^{r_{p,l}\times r_{p,l}},$$

$$B_{p,l}=\begin{bmatrix} \mathbf{0}_{1\times(n+m-1)} & \cdots \\ \mathbf{0}_{(\lambda_{p1,l}-2)\times(n+m-1)} & \cdots \\ 1 & \mathbf{0}_{1\times(n+m-2)} \\ \mathbf{0}_{(\lambda_{p2,l}-2)\times(n+m-1)} & \cdots \\ \vdots & \vdots \\ \mathbf{0}_{1\times(n+m-2)} & 1 \end{bmatrix} \in \Re^{N_p},$$

$$D_{p,l}=\begin{bmatrix} -1 & -1 & \cdots & -1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \in \Re^{r_{p,l}\times(n+m+1)},$$

$$\xi_{p,l}=\begin{bmatrix} d_{p1,l}(k) \\ \vdots \\ d_{pn,l}(k) \\ \bar{d}_{p1,l}(k) \\ \vdots \\ \bar{d}_{pm,l}(k) \\ d_{ap,l} \end{bmatrix},\ U_{p,l}=\begin{bmatrix} u_{1p,l}(k) \\ \vdots \\ u_{np,l}(k) \\ \bar{u}_{1p,l}(k) \\ \vdots \\ \bar{u}_{mp,l}(k)(k) \end{bmatrix},$$

$$Z_{p,l}(k)=\begin{bmatrix} S_{p,l}(k) \\ z_{jpl,1}(k) \\ \vdots z_{jpl,\lambda_{pj,l}-1} w_{ip,l,1}(k) \\ \vdots \\ w_{ipl,D_{ip,l}-1}(k) \end{bmatrix},$$

$$C_{p,l}=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $N_p = r_{p,l}\times(n+m-1)$.

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