2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Comparing Loewner and Krylov based model order reduction for time delay systems

Pieter Lietaert[1] and Karl Meerbergen[1]

Abstract— **We use model order reduction to set up a reduced, linear model for high dimensional, time-delay problems. Three approaches are compared with respect to computational efficiency for setting up the reduced model, and, with respect to accuracy. The first approach uses a linear pencil formulation that is spectrally equivalent to a rational matrix-valued interpolant of the non-linear time delay formulation in the Laplace domain. The second approach is based on the Loewner framework. The Loewner framework is easier to use and is suitable for a black box problem. However, for high dimensional, white box models, the linear pencil formulation outperforms the Loewner framework. This is done by exploiting the structure of the problem and making use of a rational Krylov method with only a few distinct interpolation points, and, consequently, a small number of expensive sparse LU factorizations. In contrast, setting up the Loewner matrices requires many large scale linear system solves which, as our experiments show, can be the most important factor in the computation time. The third approach is a combination of both ideas, where the computationally more expensive Loewner approach is applied to the reduced model obtained by the Krylov approach.**

I. Introduction

We look at problems that are described by delay differential equations and we reformulate them as reduced order, linear systems. We consider two options for this approximation and compare both in terms of error as well as computation time.

As a first option, the time-delay system matrix is approximated in the Laplace domain by rational interpolation. We use the interpolant based on the AAA algorithm, used for solving the nonlinear eigenvalue problem [7]. The resulting matrix-valued interpolant is used to construct a linear pencil. This pencil has the same spectrum as the rational matrix function. The dimension of this system is larger than that of the original system. However, Krylov-based model reduction can be used to reduce the pencil to small size. A compact formulation of the iteration vectors (CORK) allows to compute a Krylov basis and project the pencil with low memory cost, and, only requires the solution of linear systems of the size of the non-linear problem [13]. Additionally, using rational Krylov iterations with a small number of distinct shifts, i.e. between 1 and 5, we can use higher order moment matching around these points to accurately interpolate the high-order linear pencil formulation. This approach avoids using many interpolation points and requires only a small number of sparse LU factorizations of the size of the original non-linear problem. We call this the *Krylov approach*.

*This work was supported by KU Leuven Research Council grant OT14/074

[1]Department of Computer Science, KU Leuven, University of Leuven, 3001 Heverlee, Belgium. pieter.lietaert@cs.kuleuven.be

A related method uses the spectral discretization, which also leads to a rational interpolant [10], [12]. The linear model is reduced by the Arnoldi method. The reason why we did not select this approach is that, to date, this method does not support more than one interpolation point. We have observed in the numerical experiments that the use of more than one interpolation point is beneficial for the reduction and approximation by the reduced model.

The second option uses simple interpolation of the system's output leading to Loewner matrices [8]. We therefore call this the *Loewner approach*. In this framework, a reduced, linear system is set up directly from the rational function that interpolates the transfer function of the delay system. Only samples of the transfer function of the original large scale delay system are used. In the case of a black box model, Loewner is the only possible method among the two options presented here. The Krylov approach requires more detailed knowledge of the non-linear formulation of the system, i.e. the expression of the system matrix as a sum of matrix coefficients weighted with non-linear scalar functions:

$$A(s) = f_0(s)A_0 + f_1(s)A_1 + \cdots + f_g(s)A_g. \qquad (1)$$

At first sight, the Loewner approach seems more appealing than the Krylov approach. Numerical experiments also show that, for the same reduced dimension k, a lower error can be reached over the frequency range of interest, in comparison to matching higher order moments at zero. However, to set up the Loewner matrix, $2k$ distinct sample points of the original non-linear transfer function are needed, requiring many large scale system solves. Numerical experiments show that, for our setup, this is the most important factor in the computational cost. This leads to a much higher computation time for the Loewner approach compared to the Krylov approach for the same desired error level.

Note that both the Krylov approach with AAA and the Loewner approach use a barycentric form to construct a rational interpolant. The important difference, however, is that for Loewner the interpolant approximates the transfer function directly, while for the Krylov approach, the non-linear scalar functions $f_i(s)$ of system matrix (1) are approximated.

As mentioned, the reduced linear pencil from the Krylov approach might not have a sufficiently low dimension for the desired accuracy. Therefore, we propose a hybrid approach, combining both methods. First, a reduced pencil of size K is set up, where $K \ll n$, but still considerably large. Second, the Loewner approach is applied to the linear system represented by this K-dimensional pencil, to further reduce its dimension to a comparable size that can be reached by, for

example, applying Loewner directly to the high dimensional problem.

One important question is the choice of the interpolation points. The iterative rational Krylov algorithm (IRKA) is an elegant way to compute optimal poles for the Loewner approach. IRKA minimizes the $\mathcal{H}_2$-norm of the approximation error [4]. This has also led to the Loewner based iterative procedure named TF-IRKA [2]. TF-IRKA can also be used for (non-linear) delay equations. Alternatively, for optimal model order reduction in the $\mathcal{H}_\infty$-norm, one can use balanced truncation [1]. Note that, following from the definition of the $\mathcal{H}_2$-norm, the (semi)-optimal set of interpolation points minimizes the approximation error for frequencies up to infinity. For this paper, we consider frequencies within a limited interval. Modifications of IRKA and balanced truncation are available that deal with frequency boundaries [14], [3]. To our knowledge, TF-IRKA can not be directly adapted to deal with a limited interval. Therefore, and because we reach a sufficiently small error without using optimal points, we only use equidistant interpolation points in the frequency range.

Note that the goal is to reach a linear reduced model, as apposed to, for example, forming a non-linear reduced order model. This allows to apply any of the well known techniques that have been developed in linear model theory, like, for example, design of controllers for linear systems [9], as well as time domain simulations of the reduced model.

We consider in this paper the following time delay differential equation and output function:

$$\begin{cases} \dot{v}(X,t) = \nabla^2 v(X,t) + \sum_{j=1}^{m} a_j(X) v(\ell_j - X, t-\tau_j) + b(X)u(X,t), \\ y(X,t) = c(X)v(X,t), \end{cases}$$

with delays τ_j, shifts ℓ_j, $j = 1, \ldots, m$ and $X = [x, y, z]$ a 3-dimensional domain. Here, u is the input signal and y the output. After discretization with central differences, using n discretization points, and after transforming to the frequency domain (see, e.g., [10]), we get

$$\begin{cases} (A_0 - sI + \sum_{j=1}^{m} A_j e^{-\tau_j s})\mathbf{x}(s) = -\mathbf{b}u(s), \\ y(s) = \mathbf{c}^T \mathbf{x}(s), \end{cases} \tag{2}$$

where $s = i\omega = i2\pi f$, with ω the angular frequency and f the frequency, $A_j \in \mathbb{R}^{n\times n}$, $\mathbf{x}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$. We presume a high dimensional system, i.e. $n \gg 1$. This example is based on a system from [5], [9]. The system contains the non-linear matrix-valued function

$$A(s) = A_0 - sI + \sum_{j=1}^{m} A_j e^{-\tau_j s}. \tag{3}$$

In the following, we consider model order reduction techniques to approximate (2) by a linear, smaller model

$$\begin{aligned} (\widehat{A} - s\widehat{E})\hat{x}(s) &= \hat{b}u(s), \\ \hat{y}(s) &= \hat{c}^* \hat{x}(s), \end{aligned} \tag{4}$$

where $\hat{A}, \hat{E} \in \mathbb{C}^{k\times k}$ and $\hat{x}, \hat{b}, \hat{c} \in \mathbb{C}^k$. Model order reduction is successful if $k \ll n$.

II. Krylov approach

In the Krylov approach, we first interpolate the non-linear matrix function (3) by a rational polynomial expressed as

$$A(s) \approx R(s) = A_0 - sI + \sum_{i=1}^{d} \phi_i(s) P_i,$$

for some rational polynomial basis functions $\phi_i(s), i = 1, \ldots, d$. This formulation is then used to set up a linear pencil $\mathbb{A} - s\mathbb{E}$, that has the same eigenvalues as the rational matrix polynomial $R(s)$. We use the following formulation:

$$\mathbb{A} = \left[\begin{array}{cccc|} A_0 & P_1 & \cdots & P_d \\ \hline \multicolumn{4}{c|}{M \otimes I_n} \end{array}\right], \quad \mathbb{E} = \left[\begin{array}{cccc|} I & 0 & \cdots & 0 \\ \hline \multicolumn{4}{c|}{N \otimes I_n} \end{array}\right],$$

where $\mathbb{A}, \mathbb{E} \in \mathbb{C}^{n(d+1)\times n(d+1)}$ and $M, N \in \mathbb{R}^{d\times(d+1)}$. Matrices M and N express the linear relation between the basis functions so that

$$(M - sN) \begin{bmatrix} 1 \\ \phi_1(s) \\ \phi_2(s) \\ \vdots \\ \phi_d(s) \end{bmatrix} = 0. \tag{5}$$

The above linear pencil is spectrally equivalent to the polynomial formulation $R(s)$ [13]: for any eigenpair $(\lambda, x \neq 0)$ of R, i.e., $R(\lambda)x = 0$ we have that $(\mathbb{A} - \lambda\mathbb{E})\mathbf{x} = 0$, where

$$\mathbf{x} = [1, \phi_1(\lambda), \ldots, \phi_d(\lambda)]^T \otimes x.$$

A. AAA approximation

To find the rational approximation, i.e., the rational basis functions ϕ_i and factors P_i, we use the AAA algorithm [11]. For the AAA algorithm, the non-linear function $f(s) = e^{-\tau s}$ is approximated in barycentric form, i.e.,

$$f(s) \approx \sum_{i=1}^{d} w_i f_i \phi_i(s), \tag{6}$$

where

$$\phi_i(s) = \frac{1}{s - z_i} \Bigg/ \sum_{j=1}^{d} \frac{w_j}{s - z_j}, \quad i = 1, \ldots, d,$$

with weights w_i, function values $f_i = f(z_i)$ and interpolation points z_i. In the AAA algorithm, the interpolation points are chosen to minimize the error over a given test set and the weights follow from a least squares approximation [11]. It is possible to find common weights w_i and interpolation points z_i for all functions $f^{(j)}(s) = e^{-\tau_j s}$, $j = 1, \ldots, m$ of the delay problem, so that we have:

$$P_i = \sum_{j=1}^{m} A_j e^{-\tau_j z_i}.$$

The basis functions $\phi_i(s)$ satisfy (5) with

$$M - sN = \begin{bmatrix} -1 & w_1 & w_2 & \cdots & w_{d-1} & w_d \\ & s - z_1 & z_2 - s & & & \\ & & s - z_2 & \ddots & & \\ & & & \ddots & z_{d-1} - s & \\ & & & & s - z_{d-1} & z_d - s \end{bmatrix}.$$

See [7] for the details.

System (2) is now approximated by

$$\begin{cases} R(s)\mathbf{x}(s) = -\mathbf{b}u(s), \\ y(s) = \mathbf{c}^*\mathbf{x}(s), \end{cases}$$

which has the same input/output map as the linear system

$$\begin{cases} (\mathbb{A} - s\mathbb{E})x(s) = -bu(s), \\ y(s) = c^*x(s), \end{cases}$$

with $c = e_1 \otimes \mathbf{c}$, $b = e_1 \otimes \mathbf{b}$ and $x = (1, \phi_0(s), \ldots, \phi_d(s))^T \otimes \mathbf{x}$.

B. Compact Krylov method

The linear pencil $\mathbb{A} - s\mathbb{E}$ is of very high dimension ($n(d+1)$), which is a multiple of the high dimension n. However, methods have been developed that exploit the structure of the pencil so that computing a Krylov space from $\mathbb{A}$ and $\mathbb{E}$ involves only computations in the original high dimension n. This is accomplished by expressing the iteration vectors in the Krylov method in a low-rank format. We refer to [13] for the technical details on efficiently building a Krylov space and representing the iteration vectors in a compact form for $\mathbb{A} - s\mathbb{E}$, and to [6] for building Krylov spaces for $\mathbb{A} - s\mathbb{E}$ and its transpose, $\mathbb{A}^T - s\mathbb{E}^T$.

We use a rational Krylov approach to set up the two projection bases V and $W \in \mathbb{C}^{n(d+1)\times K}$, both in low rank format. Their columns span

$$\mathcal{K}_V = \bigoplus_{i=1}^{r} \text{span}\{v_i, \mathbb{G}_i v_i, \ldots, \mathbb{G}_i^{t-1} v_i\},$$

and

$$\mathcal{K}_W = \bigoplus_{i=1}^{r} \text{span}\{w_i, \widetilde{\mathbb{G}}_i w_i, \ldots, \widetilde{\mathbb{G}}_i^{t-1} w_i\},$$

respectively, where

$$\begin{aligned} v_i &= (\mathbb{A} - s_i\mathbb{E})^{-1}(e_1 \otimes \mathbf{b}), \\ w_i &= (\mathbb{A} - s_i\mathbb{E})^{-*}(e_1 \otimes \mathbf{c}), \end{aligned}$$

and

$$\begin{aligned} \mathbb{G}_i &= (\mathbb{A} - s_i\mathbb{E})^{-1}\mathbb{E}, \\ \widetilde{\mathbb{G}}_i &= (\mathbb{A} - s_i\mathbb{E})^{-*}\mathbb{E}^*, \end{aligned}$$

for $i = 1, \ldots, r$ and shifts s_i, $i = 1, \ldots, r$. It follows that $K = tr$. Using the low-rank formulation of the iteration vectors allows to compute V and W using only r n-dimensional LU-factorizations, which, as numerical experiments show, is for very large n the most expensive part.

The reduced linear pencil is the result of the Galerkin projection of the linear pencil:

$$\begin{aligned} \widetilde{A} = W^*\mathbb{A}V, \quad \widetilde{E} = W^*\mathbb{E}V, \\ \tilde{b} = W^*(e_1 \otimes \mathbf{b}), \quad \tilde{c} = V^*(e_1 \otimes \mathbf{c}), \end{aligned} \tag{7}$$

where $\widetilde{A}, \widetilde{E} \in \mathbb{C}^{K\times K}$ and $\tilde{b}, \tilde{c} \in \mathbb{C}^K$. We use a tilde and capital letter K here, because, as we will see in Section IV, we can further reduce the dimension of this linearization to a lower value k.

Also note that we can exploit the low-rank structure of V and W to compute the projected system (7) efficiently [6]. In a practical implementation, an orthonormal basis of the Krylov subspace is built using the rational Krylov method.

The method has the following interpolation properties. The transfer function $\widetilde{H} = \tilde{c}^*(\widetilde{A} - s\widetilde{E})^{-1}\tilde{b}$ of the reduced model Hermite interpolates the transfer function $H(s) = y(s)/u(s)$ in the r interpolation points, i.e.,

$$\frac{d^j H}{ds_i^j} = \frac{d^j \widetilde{H}}{ds_i^j}$$

for $j = 0, \ldots, 2t-1$ and $i = 1, \ldots, r$.

III. Loewner approach

In contrast to the Krylov approach, we can also use the Loewner framework [8]. Here, a (reduced) linear model (4) is set up using

$$\widehat{A} = \mathbb{L}_s, \quad \widehat{E} = \mathbb{L},$$

where $\mathbb{L}, \mathbb{L}_s \in \mathbb{R}^{k\times k}$ are the Loewner and shifted Loewner matrices, based on two sets of interpolation points $s_i^{(1)}$ and $s_i^{(2)}$, $i = 1, \ldots, k$:

$$\mathbb{L} = \begin{bmatrix} \frac{y(s_1^{(1)}) - y(s_1^{(2)})}{s_1^{(1)} - s_1^{(2)}} & \cdots & \frac{y(s_k^{(1)}) - y(s_1^{(2)})}{s_k^{(1)} - s_1^{(2)}} \\ \vdots & \ddots & \vdots \\ \frac{y(s_1^{(1)}) - y(s_k^{(2)})}{s_1^{(1)} - s_k^{(2)}} & \cdots & \frac{y(s_k^{(1)}) - y(s_k^{(2)})}{s_k^{(1)} - s_k^{(2)}} \end{bmatrix}$$

$$\mathbb{L}_s = \begin{bmatrix} \frac{s_1^{(1)} y(s_1^{(1)}) - s_1^{(2)} y(s_1^{(2)})}{s_1^{(1)} - s_1^{(2)}} & \cdots & \frac{s_k^{(1)} y(s_k^{(1)}) - s_1^{(2)} y(s_1^{(2)})}{s_k^{(1)} - s_1^{(2)}} \\ \vdots & \ddots & \vdots \\ \frac{s_1^{(1)} y(s_1^{(1)}) - s_k^{(2)} y(s_k^{(2)})}{s_1^{(1)} - s_k^{(2)}} & \cdots & \frac{s_k^{(1)} y(s_k^{(1)}) - s_k^{(2)} y(s_k^{(2)})}{s_k^{(1)} - s_k^{(2)}} \end{bmatrix}.$$

The reduced input and output vector are formed by function evaluations:

$$\hat{b} = \begin{bmatrix} y(s_1^{(1)}) \\ y(s_2^{(1)}) \\ \vdots \\ y(s_k^{(1)}) \end{bmatrix}, \quad \hat{c} = \begin{bmatrix} \bar{y}(s_1^{(2)}) \\ \bar{y}(s_2^{(2)}) \\ \vdots \\ \bar{y}(s_k^{(2)}) \end{bmatrix}.$$

Note that the Loewner framework requires evaluations of the transfer function in $2k$ different values of s, and thus $2k$ LU-factorization of size n. The number of different shifts r in the Krylov approach is usually much smaller. Note as well, that the Loewner matrices result from a barycentric approximation of the transfer function. This should be compared to

the Krylov approach with AAA, where a barycentric form is used to approximate the non-linear functions of the system matrix (6), instead of approximating the transfer function directly.

We will see in the numerical experiments that the Loewner framework is more accurate for a lower reduced dimension. However, because the original non-linear model needs to be evaluated in $2k$ points, its computational cost can be significantly higher.

IV. Hybrid approach

The linear system based on pencil (7), i.e.

$$\begin{aligned}(s\widetilde{E} - \widetilde{A})\widetilde{x}(s) &= \widetilde{b} \\ \widetilde{y}(s) &= \widetilde{c}^*\widetilde{x}(s),\end{aligned} \tag{8}$$

can still have a relatively high dimension K, for a certain desired accuracy. Therefore, it is useful to further reduce its dimension to size k, comparable to the size obtained from the Loewner approach on the non-linear system. Since (8) is linear, we can apply any linear model order reduction technique. For a fair comparison, we decided to use the Loewner approach again, but this time for linear model reduction of (8), instead of non-linear model reduction of the original problem. In this case, we need to evaluate the reduced output values $\widetilde{y}(s_i) = \widetilde{c}^*(\widetilde{A} - s\widetilde{B})^{-1}\widetilde{b}$. Although the dimension K can still be large, it is much lower than the original dimension n. As a consequence, applying Loewner on (8) requires significantly less computation time than the Loewner approach for the non-linear model directly. The result is a hybrid approach, where first a linear model (8) of order K is built by a computationally more attractive method, and, second, where the Loewner approach is applied to this linear, smaller model with significantly cheaper LU-factorizations.

Alternative to Loewner, as (8) is a linear model, we could also apply, for example, frequency limited balanced truncation [3].

V. Numerical experiments

We apply the above methods for a specific problem. We set the domain $[x, y, z]$ to $[0, \pi] \times [0, \pi] \times [0, \pi]$ with $m = 3$, and choose $a_1(x, y, z) = a_2(x, y, z) = a_3(x, y, z) = 2\sin(x)\sin(y)\sin(z)$, shifts $\ell_1 = [\pi, 0, 0]$, $\ell_2 = [0, \pi, 0]$, $\ell_3 = [0, 0, \pi]$ and delays $\tau_1 = 1,\ \tau_2 = 2,\ \tau_3 = 4$.

Note that, with these values, the system matrices are symmetric. We are interested in solutions in the frequency domain close to zero. Specifically, we consider $s \in [0.01i, 10i]$. Furthermore, $\mathbf{b}$ is chosen as a vector containing random values $\mathbf{b}_i \in [0, 1]$ and $\mathbf{c} = \mathbf{b}$. Each dimension, x, y, z, is discretized with an equal amount $\hat{n}$ of equidistant points, leading to a total dimension of $n = \hat{n}^3$. In the following we show the relative error of the value of the transfer function of the reduced system compared to original one, computed as:

$$E(s) = \frac{|H(s) - \widehat{H}(s)|}{|H(s)|},$$

where $u(s)$ is taken to be 1, so that

$$H(s) = \mathbf{c}^T A(s)^{-1}\mathbf{b}, \quad \text{and} \quad \widehat{H}(s) = \hat{c}^*(\widehat{A} - s\widehat{B})^{-1}\hat{b},$$

with $A(s)$ as in (3) and the reduced values as in (4). The error is computed over the test set $s_j,\ j = 1, \ldots, 500$, with logarithmically spaced points between 10^{-2} and 10, since we are most interested in values close to zero.

For the AAA approximation, a test set of size 10000 is chosen over the interval of interest, i.e. $s \in [0.01i, 10i]$. The AAA algorithm results in an approximation of the 3 non-linear functions of the time delay system with $d = 22$ poles. The Krylov approach is then used to set up three linear pencils of size $K = 50, 100$ and 150, using three shifts at $0i, 5i, 10i$ and matching an approximately equal number of moments for each shift. For this example, we use a one-sided Krylov method, i.e. $\mathbb{W} = \mathbb{V}$. For the Loewner approach, k equally spaced interpolation points $s_i \in [0.01i, 10i]$ are used. We compute the approximation for different values $k = 10, 20, 40$. The hybrid approach, finally, uses the most accurate linear pencil of size $K = 150$, and applies the Loewner approach to reduce this pencil, using the same interpolation points as the Loewner approach directly on the non-linear model, for different $k = 10, 20, 40$.

For $\hat{n} = 30$, i.e., $n = 27000$, the relative error for the reduced linear pencil of size $K = 150$ and for the Loewner pencil for $k = 40$, is plotted in Figure 1. From the figure it is clear that the Loewner pencil leads to a lower reduced dimension for the same level of error. Therefore, as described in the previous section, Loewner model reduction is also applied to the linear pencil of size $K = 150$, reducing its size to $k = 40$. Its error can also be seen in Figure 1 and is limited by the errors of both the Loewner approximation of the non-linear model and the Krylov approach.

Figure 2 shows the error, averaged over the frequency domain:

$$E_m = \frac{\sum_{i=1}^{R} E(s_i)}{R},$$

for the test set $s_i, i = 1, \ldots, 500$ between 10^{-2} and 10. The figure shows the mean error for different reduced dimensions k for the three approaches. Note that the hybrid approach reaches the same error as the Loewner approach for the same reduced dimension k. This follows from the fact that the hybrid approach uses the Loewner framework on the K-dimensional model, resulting in an error at the same level as that of the Loewner approach on the original model, as long the error of the K-dimensional Krylov model is lower. See also Figure 1. In our example, the Loewner error supersedes the Krylov error in the low frequency range. The mean error is dominated by this error at the low frequencies.

Finally, Figure 3 compares the mean error as a function of computation time[1]. There are three values for each method, corresponding to the reduced dimensions $K = 50, 100, 150$ (Krylov approach) and $k = 10, 20, 40$ (Loewner and Hybrid

[1]Using MATLAB 2017b, averaged over 3 runs, computations performed on machine with 64-bit Intel processor, 28 cores, 2.6 Ghz processors and 128 GB RAM.

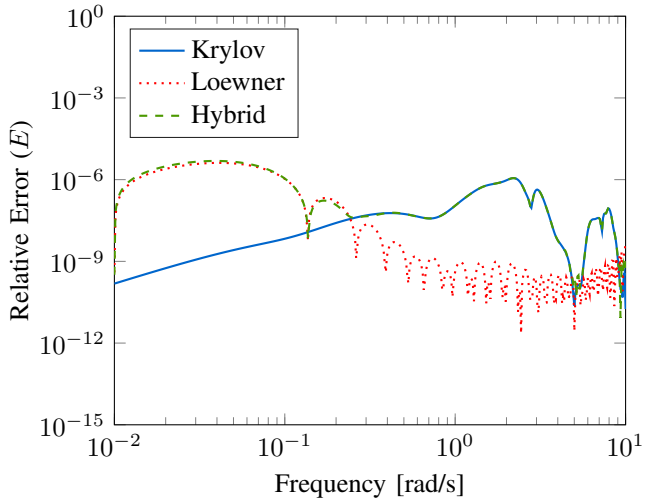

Fig. 1. Relative error as a function of the frequency $f = \mathrm{Re}(s)$ for $\hat{n} = 30$. Linear pencil of reduced dimension $K = 150$ (——), Loewner with $k = 40$ (·····) and IRKA applied to the reduced pencil with $k = 40$ (- - -).

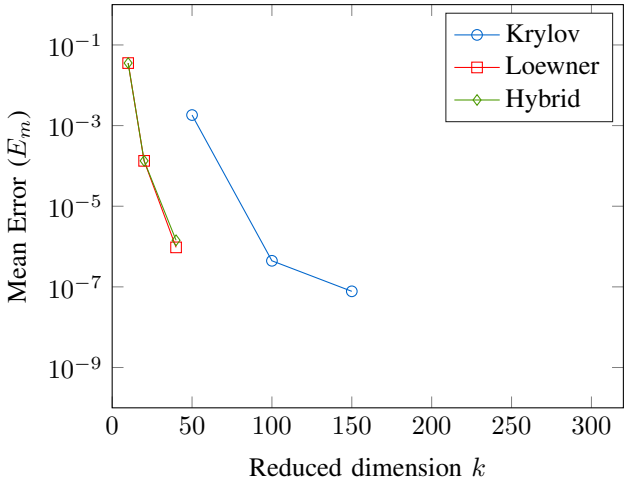

Fig. 2. Relative mean error as a function of the reduced dimension k.

approach). The Krylov approach is less expensive than the Loewner approach for the same level of accuracy. For $\hat{n} = 40$ ($n = 64000$), setting up the Loewner matrix is approximately 10 times more expensive than setting up the linear pencil. The hybrid approach, for the same reduced dimension k and for the same accuracy, results in a reduced linear pencil in significantly less time than the Loewner approach. Note that applying Loewner on the reduced pencil demands an insignificant computing time, resulting in an almost vertical line for the hybrid approach. The dominant computing time for the hybrid approach is the time for the Krylov approach to set up the sized $K = 150$ linear pencil.

VI. CONCLUSIONS

We have compared two techniques for model order reduction of time delay systems. The first approach consists of a rational Krylov method applied to a 'linearization' of the delay equation that performs high order Hermite interpolation

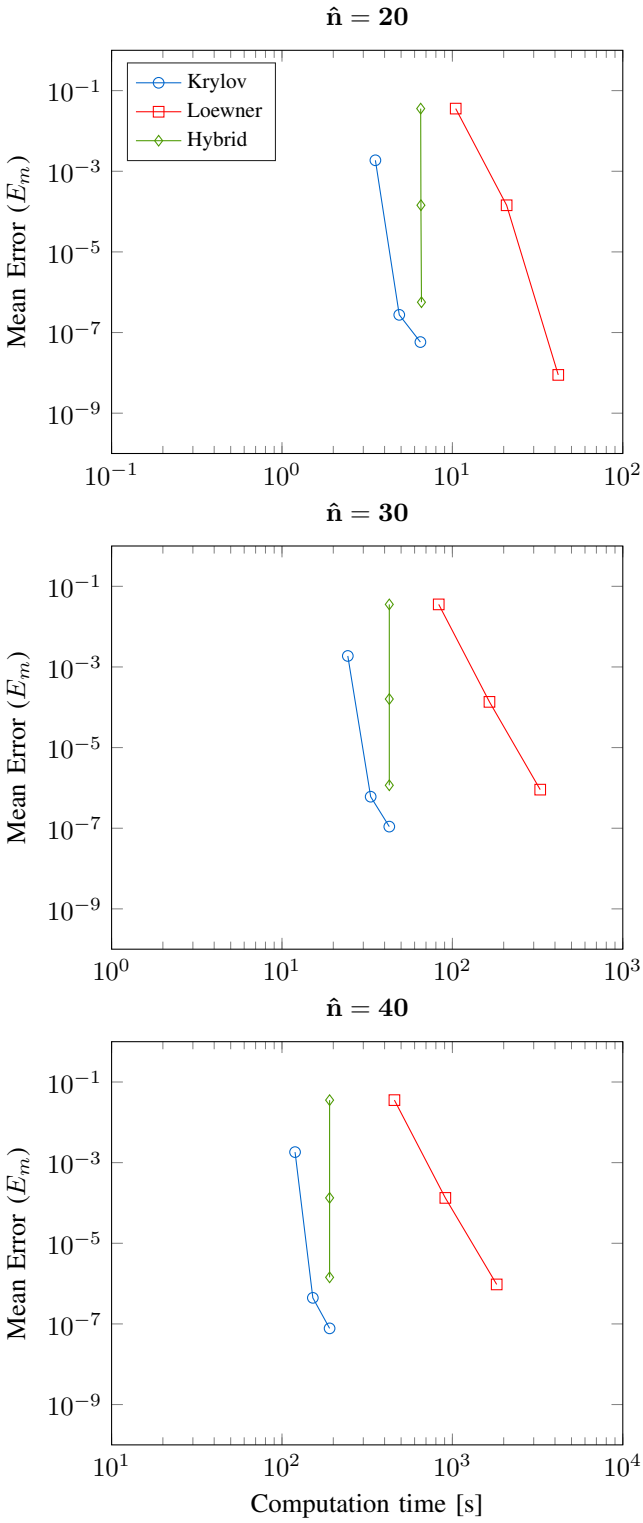

Fig. 3. Mean error in the interval $s \in [0.01, 10]$ as a function of computation time. (Top) $\hat{n} = 20, n = 8000$, (Middle) $\hat{n} = 30, n = 27000$, (Bottom) $\hat{n} = 40, n = 64000$.

of the transfer function in a small number of interpolation points, and then applying additional linear model order reduction to further reduce the dimension of the system. The second approach is the Loewner framework applied to the delay equation directly. Whereas the Loewner framework provides an elegant and simple way to set up a reduced, linear model, it appears to be very expensive for problems of high dimension, as it requires many evaluations of the non-linear matrix function, and, consequently, many high-dimensional system solves. The linearization approach combined with a second reduction step shows to be an efficient and reliable method for large scale problems. Its downside is the technical complexity to implement the method efficiently. The Krylov and hybrid approach presented here can also be adapted to result in a reduced model with real system matrices and to deal with multiple inputs and outputs. This is future work.

REFERENCES

[1] A. Antoulas. *Approximation of Large-Scale Dynamical Systems*, SIAM, 2005

[2] C. Beattie and S. Gugercin. *Realization-independent $\mathcal{H}_2$-approximation.* In 2012 IEEE 51st IEEE Conference on Decision and Control (CDC), pages 4953–4958, 2012.

[3] P. Benner, P. Kürschner and J. Saak. *Frequency-Limited Balanced Truncation with Low-Rank Approximations.* SIAM J. Sci. Comput., 38(1):A471–A499, 2016.

[4] S. Gugercin, A. C. Antoulas, and C. Beattie. *$\mathcal{H}_2$ model reduction for large-scale linear dynamical systems.* SIAM J. Matrix Anal. Appl., 2008.

[5] E. Jarlebring, K. Meerbergen, and W. Michiels. *A Krylov method for the delay eigenvalue problem.* SIAM J. Sci. Comput., 32(6):3278–3300, 2010.

[6] P. Lietaert, K. Meerbergen, and F. Tisseur. *Compact two-sided Krylov methods for nonlinear eigenvalue problems*, Technical Report, KULeuven, Departement of Computer Science, TW681, 2017.

[7] P. Lietaert, J. Pérez, B. Vandereycken, and K. Meerbergen. *Automatic rational approximation and linearization of nonlinear eigenvalue problems*. Preprint ArXiv:1801.08622, 2018.

[8] A.J. Mayo and A.C. Antoulas. *A framework for the solution of the generalized realization problem.* Linear Algebra Appl., 425(23):634 – 662, 2007.

[9] W. Michiels, G. Hilhorst, G. Pipeleers, T. Vyhldal and J. Swevers. *Reduced modelling and fixed-order control of delay systems applied to a heat exchanger.* IET Control Theory & Applications, September 2017.

[10] W. Michiels, E. Jarlebring and K. Meerbergen. *Krylov based model order reduction of time-delay systems.* SIAM J. Matrix Anal. Appl., 32(4):1399–1421, 2011.

[11] Y. Nakatsukasa, O. Sète, and L. N. Trefethen. *The AAA algorithm for rational approximation*, Preprint ArXiv 1612.00337, 2016.

[12] J. Peeters and W. Michiels. *Computing the H2 norm of large-scale time-delay systems*, Proceedings of the IFAC Joint conference, Grenoble, 2013, 11:114–119, 2013.

[13] R. Van Beeumen, K. Meerbergen, and W. Michiels *Compact rational Krylov methods for nonlinear eigenvalue problems*, SIAM J. on Matrix Anal. Appl., 36, pp. 820 – 838, 2015.

[14] P. Vuillemin. *Frequency-limited model approximation of large-scale dynamical models.* Space Physics, ISAE - Institut Superieur de l'Aéronautique et de l'Espace, 2014.