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Numerical solution of a third-kind Volterra integral equation using an operational matrix technique

Somayeh Nemati[1] and Pedro M. Lima[2*]

Abstract

In this work, an operational matrix of fractional integration based on an adjustment of hat functions is used for solving a class of third-kind Volterra integral equations. We show that the application of this numerical technique reduces the problem to a linear system of equations that can be efficiently solved. Two examples are considered to demonstrate the accuracy and efficiency of the proposed method.

1. Introduction

In this paper we consider the following linear third-kind Volterra integral equation (VIE)

$$t^{\beta}u(t)=t^{\beta}g(t)+\int_0^t (t-x)^{-\alpha}k(t,x)u(x)dx, \quad t\in I:=[0,T], \tag{1}$$

where $\beta>0,\ \alpha\in[0,1),\ \alpha+\beta>0,\ g(t)$ is a continuous function on I and $k(t,x)$ is continuous on $\Delta:=\{(t,x):0\le x\le t\le T\}$, and has the form

$$k(t,x)=x^{\alpha+\beta-1}k_1(t,x), \tag{2}$$

where $k_1\in C(\Delta)$. This class of equations has attracted the attention of researchers in the last years. The existence, uniqueness and regularity of solutions to (1) were discussed in [1], where the authors have derived necessary conditions for converting (1) into a so-called cordial VIE (see [2, 3]). The case where $\alpha+\beta\ge 1$ is of special interest because in this case if $k_1(0,0)\ne 0$ the integral operator associated to (1) is not compact and it is not possible to assure the solvability of the equation by classical numerical methods. In [4] the authors have introduced a modified graded mesh to overcome the solvability problem and they have shown that with this kind of mesh the collocation method has the optimal order of global convergence. In this article we follow a different approach, which is based on expanding the solution over a basis of adjusted hat functions (AHF) and approximating the integral operator by an operational matrix. Basis of hat functions have been successfully applied to the numerical solution of different kinds of equations [5], [6]. The advantage of this technique is that it reduces the problem to a system of linear equations with a simple structure, which splits into sub-systems of 3 equations. We recall that the same numerical method has been successfully applied to the solution of integral and integro-differential equations with different kinds of singularities, in particular, with fractional order derivatives [7], [8].

The rest of the paper is organized as follows: Section 2 is devoted to some basic definitions and concepts. In Section 3, we introduce the operational matrix of fractional integration for AHFs basis. The operational matrix technique is proposed to solve (1) in Section 4. Numerical examples are presented in Section 5 to demonstrate the applicability and accuracy of the new scheme. Finally, concluding remarks are given in Section 6.

2. Preliminaries

In this section, we give some useful definitions and properties which will be used further in this paper.

Definition 1 *Suppose that n be a positive integer number of multiple three and $h=\frac{T}{n}$. A set of an AHFs is defined on $[0,T]$ as [9]*

$$\psi_0(t)=\begin{cases}\frac{-1}{6h^3}(t-h)(t-2h)(t-3h), & 0\le t\le 3h,\\ 0, & \text{otherwise},\end{cases} \tag{3}$$

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†1The first author is with the Department of Mathematics of the University of Mazandaran, Babolsar, Iran s.nemati@umz.ac.ir; 2 the second author is with CEMAT (Center for Computational and Stochastic Mathematics), Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1, 1049-001 LISBOA, Portugal plima@math.ist.utl.pt

for $k = 0,1,\ldots,\frac{n}{3}-1$, if $i = 3k+1$,

$$\psi_i(t) = \begin{cases} \frac{1}{2h^3}(t-(i-1)h)(t-(i+1)h)(t-(i+2)h), \\ \qquad\qquad (i-1)h \le t \le (i+2)h, \\ 0, \qquad \text{otherwise}, \end{cases} \tag{4}$$

for $k = 0,1,\ldots,\frac{n}{3}-1$, if $i = 3k+2$,

$$\psi_i(t) = \begin{cases} \frac{-1}{2h^3}(t-(i-2)h)(t-(i-1)h)(t-(i+1)h), \\ \qquad\qquad (i-2)h \le t \le (i+1)h, \\ 0, \qquad \text{otherwise}, \end{cases} \tag{5}$$

for $k = 1,2,\ldots,\frac{n}{3}-1$, if $i = 3k$,

$$\psi_i(t) = \begin{cases} \frac{1}{6h^3}(t-(i-3)h)(t-(i-2)h)(t-(i-1)h), \\ \qquad\qquad (i-3)h \le t \le ih, \\ \frac{-1}{6h^3}(t-(i+1)h)(t-(i+2)h)(t-(i+3)h), \\ \qquad\qquad ih \le t \le (i+3)h, \\ 0, \qquad \text{otherwise}, \end{cases} \tag{6}$$

and

$$\psi_n(t) = \begin{cases} \frac{1}{6h^3}(t-(T-h))(t-(T-2h))(t-(T-3h)), \\ \qquad\qquad T-3h \le t \le T, \\ 0, \qquad \text{otherwise}. \end{cases} \tag{7}$$

These functions are linearly independent in $L^2(I)$ and satisfy the following properties

$$\psi_i(jh) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \tag{8}$$

$$\sum_{i=0}^{n} \psi_i(t) = 1.$$

An arbitrary function $y(t) \in L^2(I)$ can be approximated by a series of the AHFs as

$$y(t) \simeq y_n(t) = \sum_{i=0}^{n} a_i \psi_i(t) = A^T \Psi(t) = \Psi^T(t) A, \tag{9}$$

where

$$\Psi(t) = [\psi_0(t), \psi_1(t), \ldots, \psi_n(t)]^T, \tag{10}$$

and

$$A = [a_0, a_1, \ldots, a_n]^T, \tag{11}$$

in which $a_i = y(ih)$. Also an arbitrary bivariate function $k(t,x) \in L^2(I \times I)$ can be approximated by a series of AHFs as

$$\begin{aligned} k(t,x) \simeq k_n(t,x) = &\sum_{i=0}^{n} \sum_{j=0}^{n} k(ih, jh) \psi_i(t) \psi_j(x) \\ =&\Psi^T(t) K \Psi(x), \end{aligned} \tag{12}$$

where

$$K = [k(ih, jh)]_{(n+1)\times(n+1)}.$$

Theorem 1 *[9] Suppose that $y(t) \in C^4([0,T])$ and $y_n(t) = \sum_{i=0}^{n} y(ih)\psi_i(t)$ is the AHFs expansion of $y(t)$, then $|y(t) - y_n(t)| = O(h^4)$.*

Using (9) and the properties (8), it is possible to show that

$$\Psi(t)\Psi^T(t) A \simeq \tilde{A}\Psi(t), \tag{13}$$

where A is defined by (11) and $\tilde{A} = diag(a_0, a_1, \ldots, a_n)$ (for details see [9]). Furthermore, for a $(n+1)\times(n+1)$ matrix $K = [k_{i,j}]$, $i, j = 0,1,\ldots,n$, we have

$$\Psi^T(t) K \Psi(t) \simeq \hat{K}\Psi(t), \tag{14}$$

where $\hat{K}$ is a $1 \times (n+1)$ vector defined by

$$\hat{K} = [k_{0,0}, k_{1,1}, k_{2,2}, \ldots, k_{n,n}].$$

There are some different definitions of fractional integrals in the literature. In the present paper, we consider the Riemann-Liouville fractional integral operator I_t^α which will help us to obtain a numerical solution of (1).

Definition 2 *The Riemann-Liouville integral operator I_t^α of order $\alpha \ge 0$ is defined by*

$$I_t^\alpha y(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} y(x) dx, & \alpha > 0, \\ y(t), & \alpha = 0, \end{cases}$$

where $\Gamma(\alpha)$ is the gamma function which is defined as

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

3. Fractional order operational matrix of integration

The purpose of this section is to introduce the fractional order operational matrix of integration for the AHFs basis functions. To this aim, we present the following theorem.

Theorem 2 *Let $\Psi(t)$ be the AHFs vector given by (10) and $\alpha > 0$, then*

$$I_t^{\alpha}\Psi(t) \simeq P^{(\alpha)}\Psi(t), \tag{15}$$

where $P^{(\alpha)}$ is the $(n+1)\times(n+1)$ operational matrix of fractional integration of order α in the Riemann-Liouville sense and is given by

$$P^{(\alpha)} = \chi_{\alpha,h}\begin{bmatrix} 0 & \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{n-2} & \gamma_{n-1} & \gamma_n \\ 0 & \eta_0 & \eta_1 & \eta_2 & \cdots & \eta_{n-3} & \eta_{n-2} & \eta_{n-1} \\ 0 & \xi_{-1} & \xi_0 & \xi_1 & \cdots & \xi_{n-4} & \xi_{n-3} & \xi_{n-2} \\ 0 & \beta_{-2} & \beta_{-1} & \beta_0 & \cdots & \beta_{n-5} & \beta_{n-4} & \beta_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \eta_0 & \eta_1 & \eta_2 \\ 0 & 0 & 0 & 0 & \cdots & \xi_{-1} & \xi_0 & \xi_1 \\ 0 & 0 & 0 & 0 & \cdots & \beta_{-2} & \beta_{-1} & \beta_0 \end{bmatrix}, \tag{16}$$

where $\chi_{\alpha,h} = \frac{h^{\alpha}}{\Gamma(\alpha+4)}$ and

$$\gamma_q = \begin{cases} q^{\alpha}\left[-q^3 + 2q^2(3+\alpha) - \frac{11}{6}q(2+a)(3+a) \right. \\ \quad \left. +(1+\alpha)(2+\alpha)(3+\alpha)\right], \qquad 1 \le q \le 3, \\ q^{\alpha}\left[-q^3 + 2q^2(3+\alpha) - \frac{11}{6}q(2+a)(3+a) \right. \\ \quad \left. +(1+\alpha)(2+\alpha)(3+\alpha)\right] + (q-3)^{\alpha+1}\left[q^2 \right. \\ \quad \left. +q(\alpha-3) + 2 + \frac{1}{3}\alpha(\alpha-4)\right], \qquad 4 \le q \le n, \end{cases} \tag{17}$$

$$\eta_q = \begin{cases} (q+1)^{\alpha+1}\left[3q^2 - 9q - 5q\alpha + 6 + 3\alpha^2 + 10\alpha\right], \\ \qquad 0 \le q \le 2, \\ (q+1)^{\alpha+1}\left[3q^2 - 9q - 5q\alpha + 6 + 3\alpha^2 + 10\alpha\right] \\ \quad + (q-2)^{\alpha+1}\left[-3q^2 - 4q\alpha + 3 - \frac{3}{2}\alpha^2 + \frac{1}{2}\alpha\right], \\ \qquad 3 \le q \le n-1, \end{cases} \tag{18}$$

$$\xi_q = \begin{cases} (q+2)^{\alpha+1}\left[-3q^2 + 4q\alpha + 3 - \frac{3}{2}\alpha^2 + \frac{1}{2}\alpha\right], \\ \qquad -1 \le q \le 1, \\ (q+2)^{\alpha+1}\left[-3q^2 + 4q\alpha + 3 - \frac{3}{2}\alpha^2 + \frac{1}{2}\alpha\right] \\ \quad + (q-1)^{\alpha+1}\left[3q^2 + 9q + 5q\alpha + 6 + 3\alpha^2 + 10\alpha\right], \\ \qquad 2 \le q \le n-2, \end{cases} \tag{19}$$

$$\beta_q = \begin{cases} (q+3)^{\alpha+1}\left[q^2 + 3q - q\alpha + 2 + \frac{1}{3}\alpha^2 - \frac{4}{3}\alpha\right], \\ \qquad -2 \le q \le 0, \\ (q+3)^{\alpha+1}\left[q^2 + 3q - q\alpha + 2 + \frac{1}{3}\alpha^2 - \frac{4}{3}\alpha\right] \\ \quad - q^{\alpha}\left[2q^3 + 22q + \frac{55}{3}q\alpha + \frac{11}{3}\alpha^2 q\right], \\ \qquad 1 \le q \le 3, \\ (q+3)^{\alpha+1}\left[q^2 + 3q - q\alpha + 2 + \frac{1}{3}\alpha^2 - \frac{4}{3}\alpha\right] \\ \quad - q^{\alpha}\left[2q^3 + 22q + \frac{55}{3}q\alpha + \frac{11}{3}\alpha^2 q\right] \\ \quad + (q-3)^{\alpha+1}\left[q^2 - 3q + q\alpha + 2 + \frac{1}{3}\alpha^2 - \frac{4}{3}\alpha\right], \\ \qquad 4 \le q \le n-3. \end{cases} \tag{20}$$

Proof. It can be proved in a similar way as the proof of the corresponding theorem (Theorem 3.1) in [7].□

4. Numerical solution of the third-kind Volterra integral equations

In this section, a numerical scheme is proposed for obtaining a numerical solution of (1). To do this, we suppose $f(t) = t^{\beta}g(t)$, and then expand the functions t^{β}, $u(t)$, $f(t)$ and $k(t,x)$ by series of AHFs as follows:

$$t^{\beta} \simeq A^T\Psi(t), \tag{21}$$

$$u(t) \simeq U^T\Psi(t), \tag{22}$$

$$f(t) \simeq F^T\Psi(t), \tag{23}$$

$$k(t,x) \simeq \Psi^T(t)K\Psi(x), \tag{24}$$

where the vector U is an unknown vector with entries u_i, $i = 0,1,\ldots,n$, to be determined. By substituting (21)–(24) into (1), we get

$$\begin{aligned} U^T\Psi(t)\Psi^T(t)A \simeq & F^T\Psi(t) \\ & + \Psi^T(t)K\int_0^t (t-x)^{-\alpha}\Psi(x)\Psi^T(x)U\,dx. \end{aligned} \tag{25}$$

Using (25) and taking into consideration (13) and (15), we obtain

$$\begin{aligned} U^T\tilde{A}\Psi(t) &\simeq F^T\Psi(t) + \Psi^T(t)K\tilde{U}\int_0^t (t-x)^{-\alpha}\Psi(x)dx \\ &= F^T\Psi(t) + \Psi^T(t)K\tilde{U}\int_0^t (t-x)^{(1-\alpha)-1}\Psi(x)dx \\ &= F^T\Psi(t) + \Gamma(1-\alpha)\Psi^T(t)K\tilde{U}I_t^{1-\alpha}\Psi(t) \\ &\simeq F^T\Psi(t) + \Gamma(1-\alpha)\Psi^T(t)K\tilde{U}P^{(1-\alpha)}\Psi(t), \end{aligned} \tag{26}$$

where $\tilde{A} = diag(0, h^\beta, (2h)^\beta, ..., T^\beta)$. Let us define

$$Q = K\tilde{U}P^{(1-\alpha)},$$

then using (14) and the linear independence of the AHFs, and, finally replacing $\simeq$ with $=$, we have the following system of linear algebraic equations in the unknown entries of the vector U,

$$U^T\tilde{A} - \Gamma(1-\alpha)\hat{Q} = F^T, \tag{27}$$

where $\hat{Q}$ is a vector with the diagonal entries of Q. Since the first equation of the above system is a degenerate equation, we have to impose the condition $u(0) = g(0)$ for its solvability. With this additional condition, we obtain $u_0 = g(0)$. In order to determine the remaining unknown entries of U, we solve one by one the equations of system (27). To this aim, suppose that

$$P^{(1-\alpha)} = [\lambda_{i,j}], \ \ i,j = 0,1,2,\ldots,n,$$

then it follows from (16) that

$$\lambda_{i,j} = \begin{cases} 0, & j = 0, \\ 0, & \text{if } j = 3k+1 \text{ and } (j+3) \leq i \leq n, \\ 0, & \text{if } j = 3k+1 \text{ and } (j+2) \leq i \leq n, \\ 0, & \text{if } j = 3k \text{ and } (j+1) \leq i \leq n. \end{cases}$$

Also, from (2) and (12) we conclude that

$$K = [k_{i,j}], \ \ i,j = 0,1,2,\ldots,n,$$

where

$$k_{i,j} = \begin{cases} 0, & j = 0, \\ k(ih, jh), & \text{otherwise.} \end{cases}$$

Therefore $\hat{Q}$ is given by

$$\begin{aligned}\hat{Q} = [0, \ &\sum_{l=1}^{3} k(h,lh)\lambda_{l,1}u_l, \sum_{l=1}^{3} k(2h,lh)\lambda_{l,2}u_l\lambda_{l,3}u_l, \\ &\sum_{l=1}^{6} k(4h,lh)\lambda_{l,4}u_l, \sum_{l=1}^{6} k(5h,lh)\lambda_{l,5}u_l, \\ &\sum_{l=1}^{6} k(6h,lh)\lambda_{l,6}u_l, \ldots, \sum_{l=1}^{n} k(T-2h,lh)\lambda_{l,n-2}u_l, \\ &\left.\sum_{l=1}^{n} k(T-h,lh)\lambda_{l,n-1}u_l, \sum_{l=1}^{n} k(T,lh)\lambda_{l,n}u_l\right].\end{aligned}$$

Furthermore, $U^T\tilde{A}$ and F^T are given, respectively, by

$$U^T\tilde{A} = \left[0, h^\beta u_1, (2h)^\beta u_2, (3h)^\beta u_3, (4h)^\beta u_4, \ldots, T^\beta u_n\right], \tag{28}$$

$$F^T = \left[0, h^\beta g(h), (2h)^\beta g(2h), (3h)^\beta g(3h), (4h)^\beta g(4h) \right. \\ \left. , \ldots, T^\beta g(T)\right]. \tag{29}$$

So the system (27) can be rewritten as

$$\begin{aligned} &E_i : (ih)^\beta u_i - \Gamma(1-\alpha)\sum_{l=1}^{3} k(ih,lh)\lambda_{l,i}u_l = (ih)^\beta g(ih), \\ &\qquad\qquad \text{if} \quad i = 1,2,3, \\ &E_i : (ih)^\beta u_i - \Gamma(1-\alpha)\sum_{l=1}^{6} k(ih,lh)\lambda_{l,i}u_l = (ih)^\beta g(ih), \\ &\qquad\qquad \text{if} \quad i = 4,5,6, \\ &\vdots \\ &E_i : (ih)^\beta u_i - \Gamma(1-\alpha)\sum_{l=1}^{n} k(ih,lh)\lambda_{l,i}u_l = (ih)^\beta g(ih), \\ &\qquad\qquad \text{if} \quad i = n-2, n-1, n. \end{aligned} \tag{30}$$

We observe that E_1, E_2 and E_3 form a system of three linear equations in three unknown parameters u_1, u_2 and u_3. We rewrite these equations as the following matrix form equation:

$$\begin{bmatrix} a_{1,1} & a_{2,1} & a_{3,1} \\ a_{1,2} & a_{2,2} & a_{3,2} \\ a_{1,3} & a_{2,3} & a_{3,3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} h^\beta g(h) \\ (2h)^\beta g(2h) \\ (3h)^\beta g(3h) \end{bmatrix}, \tag{31}$$

where for $i,j = 1,2,3$, we have

$$a_{i,j} = \begin{cases} -\Gamma(1-\alpha)k(ih,jh)\lambda_{j,i}, & i \neq j, \\ \\ (ih)^\beta - \Gamma(1-\alpha)k(ih,ih)\lambda_{i,i}, & i = j. \end{cases}$$

Using the Mathematica function "RowReduce", it can be seen that the coefficient matrix of the system (31) can be reduced to the identity matrix by row transformations that transform a linear system into an equivalent one. Therefore it is a uniquely solvable system which can be easily solved using direct methods. After solving this system and substituting the obtained results for u_1, u_2 and u_3 into E_4, E_5 and E_6, a system of three linear equations in three unknown parameters u_4, u_5 and u_6 is obtained. This process continues until u_{n-2}, u_{n-1} and u_n are determined by solving E_{n-2}, E_{n-1} and E_n in which the results for $u_1, u_2, \ldots, u_{n-3}$ have been substituted. These results together with the result for u_0 are substituted in (22) in order to obtain an approximation of the solution $u(t)$ for (1). By this technique, the main problem is reduced to solving $\frac{n}{3}$ systems of three linear equations which in our implementation have been solved using the Mathematica function "Solve". This explains that the total number of operations required to apply this algorithm is $O(n)$, which makes this method more efficient than other methods previously described.

5. Illustrative examples

In this section, two examples are considered to demonstrate the applicability, effectiveness and high accuracy of the presented technique. Both examples were presented before in [4], so that we can compare the performance of our method with the methods used in that paper. In these examples we consider $T = 1$, therefore we have $h = \frac{1}{n}$. In order to show the error of the method, we introduce the notations:

$$e_n = \max_{0 \le i \le n} |y(t_i) - y_n(t_i)|,$$
$$p_n = \log_2 \left(\frac{e_n}{e_{2n}} \right),$$

where $t_i = ih$, and $y(t)$ and $y_n(t)$ are the exact solution, and the approximate solution given by the proposed method, respectively. In our implementation, the computations have been carried out in a personal computer with Intel(R) Pentium(R) CPU G620 @ 2.60 GHz with 4.00 GB random access memory and the codes were written in Mathematica 11.

Example 1 *As the first example, we consider equation (1) with* $\alpha = \frac{2}{3}$, $\beta = \frac{2}{3}$, $k(t,x) = \frac{\sqrt{3}}{3\pi} x^{1/3}$ *which yields an equation of Abel type:*

$$t^{2/3} u(t) = f(t) + \int_0^t \frac{\sqrt{3}}{3\pi} x^{1/3} (t-x)^{-2/3} u(x) dx, \quad t \in [0,1],$$

where

$$f(t) = t^{\frac{47}{12}} \left(1 - \frac{\Gamma(\frac{1}{3})\Gamma(\frac{55}{12})}{\pi\sqrt{3}\Gamma(\frac{59}{12})} \right).$$

The exact solution of this equation is $u(t) = t^{\frac{13}{4}}$. We have solved this equation with different values of n and reported the numerical results in Table 1 and Fig. 1. Table 1 presents the maximum error and the order of convergence together with the results of the collocation method [4]. Fig. 1 displays the plot of the absolute error with $n = 3, 6, 12, 24$. The numerical results suggest that our method has in this case convergence order 3.25. For the same example, the collocation method described in [4] has convergence order 3.

Example 2 *Consider now equation (1) with* $\beta = 1$, $\alpha = 0$, $k(t,x) = \frac{1}{2}$, *which is used in the modelling of some heat conduction problems with mixed-type boundary conditions:*

$$tu(t) = \frac{6}{7} t^3 \sqrt{t} + \int_0^t \frac{1}{2} u(x) dx, \quad t \in [0,1].$$

Table 1. Numerical results for Example 1.

Method	n	e_n	p_n
Present			
	3	3.82×10^{-3}	3.25
	6	4.02×10^{-4}	3.25
	12	4.22×10^{-5}	3.25
	24	4.44×10^{-6}	3.25
	48	4.67×10^{-7}	3.25
	96	4.91×10^{-8}	3.25
	192	5.16×10^{-9}	—
Method [4]			
N. points($m = 3$)	256	4.58×10^{-7}	2.96
Radau II ($m = 3$)	256	5.13×10^{-9}	3.28

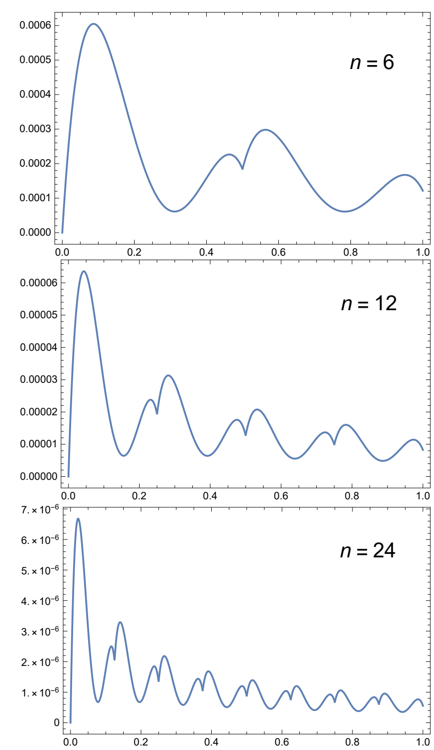

Figure 1. Plot of the absolute error with different values of n for Example 1.

This equation has the exact solution $u(t) = t^{\frac{5}{2}}$. We have applied the present method with different values of n to this equation and displayed the obtained result in Table 2 and Fig. 2. The maximum error and the order of convergence are shown and compared with the results obtained by the collocation method [4] in Table 2. In Fig. 2 the graphic of the absolute error is plotted. In this case, the estimate of the convergence order with our method is 2.5; concerning the method described in [4], the convergence order for this example is 2.

Table 2. Numerical results for Example 2.

Method	n	e_n	p_n
Present			
	3	1.13×10^{-3}	2.50
	6	2.00×10^{-4}	2.50
	12	3.54×10^{-5}	2.50
	24	6.26×10^{-6}	2.50
	48	1.11×10^{-6}	2.50
	96	1.96×10^{-7}	2.50
	192	3.46×10^{-8}	—
Method [4]			
Chebyshev ($m = 2$)	$N = 256$	1.30×10^{-5}	1.99

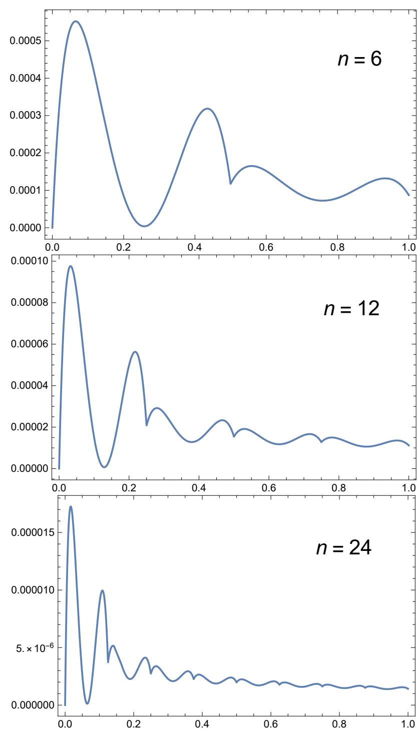

Figure 2. Plot of the absolute error with different values of n for Example 2.

Table 3 displays the computing time (in seconds) consumed for solving the final system with different values of n for Examples 1 and 2. These results show that for high values of n the computing time is approximately proportional to n, which is in agreement with what was written in section IV about the complexity of the algorithm.

6. Conclusion and Future Work

In this paper, a numerical technique based on an adjustment of hat functions is introduced for solving the

Table 3. Computing time in seconds.

n24	48	96	192
Example 1 0.015	0.016	0.031	0.063
Example 20.000	0.015	0.016	0.031

third-kind Volterra integral equations. The operational matrix of fractional integration of the basis function is presented and used to reduce the problem to a system of algebraic equations which can be easily solved. Two numerical examples are considered to confirm the applicability and accuracy of the method. The numerical results suggest that the convergence order of the method depends on the order of smoothness of the solution. More precisely if the solution is of the class $C^p([0,1])$, with $p = 2,3$, we observe that the convergence order is not less than p. As future work, we intend to investigate further the convergence of the method and to obtain theoretical results about its convergence order.

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