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Rational Approximation of Distributed-Delay Control Laws via Moment-Matching

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Abstract—**This paper presents a novel method for approximating distributed-delay (DD) control laws by rational transfer functions. It does so via Moment-Matching (MM). Unlike existing methods, the inherent degrees of freedom in this method offer the designer the ability to preserve closed-loop (CL) properties and assure stability of approximation regardless of its order. A formula for the approximation of the modified Smith predictor is suggested and simulations are presented and compared with some known methods from the literature.**

Index Terms—**Time-delay Systems, distributed-delay elements, rational approximations, dead-time compensation, Moment-Matching**

I. Introduction

Time delay systems (TDS) are feedback control systems containing loop delays. Loop delays are common phenomena in many control applications, they impose strict limitations on the achievable performances of the CL. Otto.J.Smith suggested in [20] to introduce an infinite-dimensional internal feedback of the form $\Pi(s) = P_r - P$ (where $P = P_r e^{-sh}$, $P_r = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$) in the controller. This element is designed to compensate for the delay so that the remaining part of the controller, called the primary controller, can be designed to stabilize only the rational plant P_r and by doing so it guarantees that the overall controller stabilizes the CL. The structure proposed by Smith is called the Smith-Predictor (SP) or dead-time compensator (DTC). A clear drawback of this controller is its inability to treat unstable plants due to unstable pole-zero cancellation. A possible remedy to this is the modified Smith predictor (MSP) that replaces the first term in the predictor with yet another rational function $\tilde{P} = \left[\begin{array}{c|c} A & B \\ \hline Ce^{-Ah} & 0 \end{array}\right]$ that enforces the predictor to be a DD element. The MSP is actually the finite impulse response (FIR) completion of the delayed plant and it takes the form $\Pi(s) = \pi_h \left\{P_r\left(s\right)e^{-sh}\right\} \doteq \left[\begin{array}{c|c} A & B \\ \hline Ce^{-Ah} & 0 \end{array}\right] - \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] e^{-sh} = Ce^{-Ah}\int_0^h e^{(A-sI)t}dtB$ (thus called DD element).

Control laws involving DD elements are used extensively in control of TDS. They appear in various configurations of DTC [27], [6], [26] (most of which are rearrangements of the MSP), they arise in the solution to the H^2 [11], L^1 [14] and H^∞ [7] optimization problems, they enable finite spectrum assignment (FSA) [4], [25], preview tracking [10], etc.

Although always stable, the implementation of the DD element is not always trivial. When the plant is stable so must be the part $\tilde{P}$, in this case (like in the SP) the DD element can be implemented as the difference between two stable transfer functions, whereas when the plant is unstable, the DD contains unstable pole-zero cancellations between the two transfer functions and must be implemented as one stable block (see [27]).

The literature offers three approaches to cope with DD element implementation. The straightforward approach is to use the so-called lumped-delay-approximation (LDA), this is a non-rational approach in which the DD element is approximated by a system of commensurate lumped delays, similarly to the Reinmann sum approximation of integrals. Van-Assche et al. addressed this method in [23] through numerical example, the example demonstrated numerical instability irrespective of the discretization step or integration method (rectangular, trapezoidal or Simpson rules). The approximation in [23] so as in [24], [16] and [5] introduced unstable poles to the CL even for an arbitrary small interval partitions. Following these results, the problem of safe implementation of DD controllers was referred in [8] as one of the open problems in the control of TDS. The problem was considered open until L.Mirkin in [12] argued that the reported implementation problems are not intrinsic and that DD element is indeed approximable using LDA. In his paper Mirkin stated the reasons for the instability and offered a solution for safe implementation of DD elements using LDA. Another approach, this time non-rational and non-linear was suggested by Mondié et al. in [17], that is a non-linear resetting mechanism to assure the stability of the implementation. The underlying idea is to introduce a periodic reset of the state vector x in order to prevent the hidden modes from giving rise to unbounded growth in x.

This paper deals with another class of approximations, that is the approximation of the infinite dimensional DD element using a rational transfer function. Methods for rational approximation based on the δ operator and on bi-linear (BL) transformations were suggested in [22] in chapters 12 and 13 respectively. J. R. Partington & P. M. Mäkilä suggested other rational methods (see [9] and references therein) including a class of shift-based methods that are said to approximate a DD element with the fastest H^∞ convergence rate possible.

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Another method (mentioned in [13]) that is proved to be effective is the so-called Padé approximation in which the pure delay element inside the MSP is being replaced with its $[n, n]$ rational Padé approximation[1].

This paper offers yet another rational approximation method, completely different from those mentioned before and with focus on the practical need of the approximation - which is to maintain the designed close-loop properties using a low order rational transfer function.

The paper is organized as follows, first a preliminary section is brought in which the notion of moment is explained followed by an explanation about model reduction using moment matching. Third section deals with the main idea - how to exploit moment matching in order to build a rational, thus finite dimensional, approximation to the infinite dimensional DD element - The MSP. In the forth section the advantages of the method are discussed, guidelines for using the method are provided and a comparison is being held with regards to other known methods from the literature, simulations are included.

II. Preliminaries: Moment-Matching

A. The notion of moment

Given a matrix valued function of time $h : \mathbb{R} \mapsto \mathbb{R}^{\text{pxm}}$, its k^{th} moment at an arbitrary point $s_0 \in \mathbb{C}$ is defined as:

$$\eta_k(s_0) = \int_0^{\infty} t^k h(t) e^{-s_0 t} dt$$

Or similarly by the k derivative of its Laplace transform $H(s)$ at $s = s_0$:

$$\eta_k(s_0) = \frac{(-1)^k}{k!} \frac{d^k}{ds^k} H(s)|_{s=s_0} \in \mathbb{R}^{\text{pxm}}$$

For a linear systems of the form $\mathcal{H} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ the impulse response and its Laplace transform are given by:

$$h(t) = Ce^{At}B + \delta(t)D \ , t \geq 0$$

and

$$H(s) = C(sI - A)^{-1}B + D$$

Therefore its moments are:

$$\eta_0(s_0) = C(s_0 I - A)^{-1}B + D$$

$$\eta_k(s_0) = C(s_0 I - A)^{-(k+1)}B \ \ , k > 0$$

A key observation - the moments of a system H(s) at a point $s_0 \in \mathbb{C}$ are the coefficients of the Laurent series of H(s) around this point:

$H(s) = H(s_0) + H^{(1)}(s_0)\frac{(s-s_0)^1}{1!} + \cdots + H^{(k)}(s_0)\frac{(s-s_0)^k}{k!} + \cdots = \eta_0(s_0) + \eta_1(s_0)(s - s_0) + \cdots + \eta_k(s_0)(s - s_0)^k + \cdots$

[1] A formula for $[n, n]$ Padé approximation of delay: $e^{-sh} \approx \frac{R_{[n,n]}(-sh)}{R_{[n,n]}(sh)} = \frac{\sum_{i=0}^{n} \frac{(2n-i)!n!}{(2n)!(n-i)!i!}(-sh)^i}{\sum_{i=0}^{n} \frac{(2n-i)!n!}{(2n)!(n-i)!i!}(sh)^i}$

B. Model reduction by MM

Consider the system: (X^* stands for the complex conjugate of X)

$$\mathcal{H} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \ A \in \mathbb{R}^{n \times n} \ B, C^* \in \mathbb{R}^n$$

The problem of approximation via MM consists of finding a low order system $(r < n)$ of the form

$$\hat{\mathcal{H}} = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] \ \hat{A} \in \mathbb{R}^{r \times r} \ \hat{B}, \hat{C}^* \in \mathbb{R}^r$$

so that its k_i order expansions around different m points $s_i \in \mathbb{C}$, $i = 1...m$ i.e.:

$$\hat{H}(s) = \hat{\eta}_0(s_1) + \hat{\eta}_1(s_1)(s - s_1) + \cdots + \hat{\eta}_{k_1}(s_1)(s - s_1)^{k_1} + \cdots$$

$$\vdots$$

$$\hat{H}(s) = \hat{\eta}_0(s_i) + \hat{\eta}_1(s_i)(s - s_i) + \cdots + \hat{\eta}_{k_i}(s_i)(s - s_i)^{k_i} + \cdots$$

$$\vdots$$

$$\hat{H}(s) = \hat{\eta}_0(s_m) + \hat{\eta}_1(s_m)(s - s_m) + \cdots + \hat{\eta}_{k_m}(s_m)(s - s_m)^{k_m} + \cdots$$

are such that the first k_i moments are matched at each point i.e. $\eta_{1,\ldots,k_i}(s_i) = \hat{\eta}_{1,\ldots,k_i}(s_i) \ \forall i = 1...m$

For more information on MM see [2] and references therein.

III. DD Approximation via MM

A. Signal Interpretation and its stabilization problem counterpart

A. Astolfi in [1] offered a different interpretation to the problem of model reduction by moment matching for linear systems, the following proposition is inspired by his paper:

Proposition 1. *Given a system $G(s) \in H^{\infty}$, an elegant way to match its $\sum_{i=1}^{m} k_i$ moments at m different locations (frequencies) to the moments of an approximated system is to match the steady state gains of their responses to an impulse response of another system whose poles are located (on the $j\omega$ axis) where matching is desired. (multiplicity of a pole implies higher order of the moment)*

Definition 2. Stabilization Problem: Let $G(s)$ be stable, construct a stable $R(s)$ such that

$$G_e(s) = (G(s) - R(s)) W(s) \quad (1)$$

is stable for a given unti-stable $W(s)$.

Claim 3. The problem of matching steady-state gains is equivalent to the stabilization problem.

Proof: If $R(s)$ stabilizes $G_e(s)$ then $G(s_i) - R(s_i) \equiv 0$ at all poles s_i of $W(s)$. In case of multiplicity of 'k' order, the following must hold $\frac{d^k}{ds^k}(G(s) - R(s))|_{s=s_i} \equiv 0$. In other words, in order for $R(s)$ to stabilize $G_e(s)$, all first k moments of $G(s)$ and $R(s)$ must match at the poles of $W(s)$. ■

Assume that $G(s)W(s)$ can be decomposed into $W_G(s)+\widetilde{G}(s)$ so that the poles of $W_G(s)$ coincide with those of $W(s)$ and so that $\widetilde{G}(s)$ is stable. Applying this assumption to (1) yields $G_e(s)=W_G(s)+\widetilde{G}(s)-R(s)W(s)$. Subtracting from both sides of the equation the stable part $\widetilde{G}(s)$ results in an equivalent stabilization problem of the form:

$$\tilde{G}_e(s)=W_G(s)-R(s)W(s) \tag{2}$$

which has the same structure as the well-known general estimation setup (GES) [15] . With proper adjustments to the GES one can derive the solvability conditions and the solution:

Proposition 4. *[15] A stabilizing controller $R(s)\in H^\infty$ exists iff $\mathcal{H}:=\begin{bmatrix} W_G(s)\\ W(s)\end{bmatrix}$ admits a left coprime factorization of the form:*

$$H(s)=\begin{bmatrix} I & M_v(s)\\ 0 & M_y(s)\end{bmatrix}^{-1}\begin{bmatrix} N_v(s)\\ N_y(s)\end{bmatrix} \tag{3}$$

for some $M_y(s), M_v(s), N_y(s), N_v(s)\in H^\infty$. (proved in [15], proposition 3)

Lemma 5. *[15] If a factorization of the form (3) exists, the set of all stable and stabilizing $R(s)$ is given by*

$$R(s)=-M_v(s)+Q(s)M_y(s) \tag{4}$$

for an arbitrary $Q(s)\in H^\infty$. The set of all stable error systems is

$$\tilde{G}_e(s)=N_v(s)-Q(s)N_y(s) \tag{5}$$

proof is provided in [15], Lemma 1.

Return to (2) and consider the minimal realization

$$\begin{bmatrix} W_G(s)\\ W(s)\end{bmatrix}=\left[\begin{array}{c|c} \bar{A} & \bar{B}\\ \hline C_g & D_g\\ C_w & D_w\end{array}\right] \tag{6}$$

If $(C_w,\bar{A})$ is detectable (necessity was shown in [18] Lemma 3.6) then

$$\begin{bmatrix} M_v(s) & N_v(s)\\ M_y(s) & N_y(s)\end{bmatrix}=\left[\begin{array}{c|cc} \bar{A}+LC_w & L & \bar{B}+LD_w\\ \hline C_g & 0 & D_g\\ C_w & I & D_w\end{array}\right] \tag{7}$$

for any L such that $\bar{A}+LC_w$ is Hurwitz.
substituting (7) into (4) yields the set of all stable and stabilizing $R(s)$:

$$R(s)=-\left[\begin{array}{c|c} \bar{A}+LC_w & L\\ \hline C_g & 0\end{array}\right]+Q(s)\left[\begin{array}{c|c} \bar{A}+LC_w & L\\ \hline C_w & I\end{array}\right] \tag{8}$$

and the set of all proper controllers whose order is the same as the order of $\mathcal{W}$ is obtained by choosing $Q(s)$ to be static (Q_∞):

$$R(s)=-\left[\begin{array}{c|c} \bar{A}+LC_w & L\\ \hline C_g & 0\end{array}\right]+Q_\infty\left[\begin{array}{c|c} \bar{A}+LC_w & L\\ \hline C_w & I\end{array}\right] \tag{9}$$

Namely, all that is needed in order to approximate a system $G(s)\in H^\infty$ by a rational transfer function via MM is to derive (6) and to use (8) or (9). Equation (6) consists of the state-space formulation of two systems, $\mathcal{W}$ (the system that dictates the moments locations) and $\mathcal{W}_\mathcal{G}$ (the system whose impulse response is the steady-state response of $\mathcal{G}$ to the impulse response of $\mathcal{W}$).

When the approximated system ($G(s)$) is rational, $\mathcal{W}_\mathcal{G}$ can be constructed using straightforward state-space machinery as follows:
Let $G(s)$ and $W(s)$ have the minimal realizations $\left[\begin{array}{c|c} A & B\\ \hline C & 0\end{array}\right]$ and $\left[\begin{array}{c|c} A_w & B_w\\ \hline C_w & 0\end{array}\right]$ accordingly with $spec(A)\cap spec(A_w)\in\emptyset$. Their product can be written as $G(s)W(s)=$

$$\left[\begin{array}{cc|c} A & BC_w & 0\\ 0 & A_w & B_w\\ \hline C & 0 & 0\end{array}\right]=\left[\begin{array}{cc|c} A & 0 & -S_wB_w\\ 0 & A_w & B_w\\ \hline C & CS_w & 0\end{array}\right]=\underbrace{\left[\begin{array}{c|c} A_w & B_w\\ \hline CS_w & 0\end{array}\right]}_{W_G}+\underbrace{\left[\begin{array}{c|c} A & -S_wB\\ \hline C & 0\end{array}\right]}_{\widetilde{G}}$$

where the transformation matrix $\begin{bmatrix} I & -S_w\\ 0 & I\end{bmatrix}$ is used to derive the second realization and S_w is the solution to the Sylvester equation

$$AS_w-S_wA_w+BC_w=0 \tag{10}$$

Hence, the state space representation of $\mathcal{H}:=\begin{bmatrix} W_G(s)\\ W(s)\end{bmatrix}$ is given by

$$\begin{bmatrix} W_G(s)\\ W(s)\end{bmatrix}=\left[\begin{array}{c|c} A_w & B_w\\ \hline CS_w & 0\\ C_w & 0\end{array}\right] \tag{11}$$

When the approximated system is irrational one needs to calculate the steady-state response using other arguments as axplained next.

B. $\Pi(s)$ Approximation

In the case of the MSP the goal is to use MM in order to construct a rational approximation to an infinite-dimensional DD-based control law of the form $\Pi(s)=\left[\begin{array}{c|c} A & B\\ \hline Ce^{-Ah} & 0\end{array}\right]-\left[\begin{array}{c|c} A & B\\ \hline C & 0\end{array}\right]e^{-sh}=Ce^{-Ah}\int_0^h e^{(A-sI)t}dtB$. Based on the results in (III-A) all that is needed is to construct a state-space formulation of the steady state gains of the output of $\Pi(s)$ to the input which is the impulse response of an arbitrary unstable transfer function $W(s)$. The idea is straightforward, write an expression to the output signal (a simple convolution of $w(t)$ and $\pi(t)$) then apply the Laplace transform and then isolate the part whose poles coincide with those of $W(s)$.
$\pi(t)=C\cdot e^{-A(h-t)}B\cdot 1_{[0,h]}(t)$ - Impulse response of $\Pi(s)$.

$w(t) = C_w \int_0^t e^{A_w(t-\theta)} B_w \delta(\theta)\, d\theta = C_w e^{A_w t} B_w$ - impulse response of $W(s)$.

$y(t) = \int_0^t \pi(t-\tau) w(\tau)\, d\tau$ - Convolution of $\pi(t)$ with $w(t)$.

$$y(t) = \int_0^t \underbrace{C \cdot e^{-A(h-t+\tau)} B \cdot 1_{[0,h]}(t-\tau)}_{\pi(t-\tau)} \underbrace{C_w e^{A_w \tau} B_w}_{w(\tau)} d\tau$$

the following change of variables $\tilde{\tau} = t - \tau$ yields

$y(t) = \int_t^0 C \cdot e^{-A(h-\tilde{\tau})} B \cdot 1_{[0,h]}(\tilde{\tau}) C_w e^{A_w(t-\tilde{\tau})} B_w (-1)\, d\tilde{\tau} = \int_0^t C \cdot e^{-A(h-\tilde{\tau})} B C_w e^{A_w(t-\tilde{\tau})} B_w 1_{[0,h]}(\tilde{\tau})\, d\tilde{\tau}$

it can be shown that (derivation excluded due to length limitation):

$$\mathcal{L}\{y(t)\} = Y(s) = \underbrace{\left[\begin{array}{c|c} A_w & B_w \\ \hline -C \cdot \Sigma_{12} & 0 \end{array}\right]}_{W_\Pi(s)} + \tag{12}$$

$$\underbrace{C \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \end{bmatrix} \int_0^h \exp\left(\begin{bmatrix} A - sI & BC_w \\ 0 & A_w - sI \end{bmatrix} t \right) dt \begin{bmatrix} 0 \\ B_w \end{bmatrix}}_{\tilde{\Pi}(s)}$$

where $\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ 0 & \Sigma_{22} \end{bmatrix} := \exp\left(- \begin{bmatrix} A & BC_w \\ 0 & A_w \end{bmatrix} h \right) = \begin{bmatrix} e^{-Ah} & -\int_0^h e^{-A(h-\zeta)} B C_w e^{-A_w \zeta} d\zeta \\ 0 & e^{-A_w h} \end{bmatrix}$

so that $\tilde{\Pi}(s)$ is stable and $W_\Pi(s)$ is the part whose poles coincide with those of $W(s)$ and that represents the steady-state response.

Substituting the first term in the right hand side of (12) into (9) yields the k^{th} order rational approximation of the MSP:

$$\Pi_a(s) = \left[\begin{array}{c|c} A_w + LC_w & L \\ \hline C\Sigma_{12} - C_w Q_\infty & Q_\infty \end{array}\right] \tag{13}$$

The only question is how to construct $\mathcal{W}$ (namely, where to locate the moments) and how to utilize the degrees of freedom L, Q_∞ to the practical need of preserving the original design of the CL.

IV. Advantages and Examples

The ability to set the approximation error to zero at specific points of interest assists in the preservation of many desirable CL qualities, for example:

- Keeping the CL stable under maximal order constraint.
- Maintaining the designed stability margins and transient behavior by setting moments around the crossover frequency. This quality is of great importance since it is known that when the controller is designed to enlarge the CL bandwidth it leads to the proliferation phenomenon (introduces more crossover frequencies to the open loop) which leads, in turn, to the rapid deterioration of the delay-margins. Thus one needs to devote special attention to the crossover frequencies in order to maintain the designed robustness of the CL to uncertainties in the delay.
- Addressing specific frequencies of interest one intends to attenuate or amplify (for example cases of well-known disturbances).
- Keeping the designed steady state behavior by setting moments at low frequencies.

Throughout this section a comparison will be held with regards to other known methods from the literature (all are rational approximations) such as: Padé [13] , P1S [9], Delta Operator and Bi-linear Transformation (see [22] chapters 12 and 13 respectively).

A. Stability of approximation

The first obvious advantage with respect to other rational methods is the ability to guarantee the stability of the approximation regardless of its order. Consider the following plant (from [21]) $P = P_r e^{-sh}$ where $P_r = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2500 & -2525 & 26 & 1 \\ \hline 808 & 80 & 0 & 0 \end{array}\right]$; $h = 1[sec]$. The MSP is of the form $\Pi(s) = C \int_0^h e^{A(t-h)} e^{-st} dt B$ and its frequency response is depicted in figure 1:

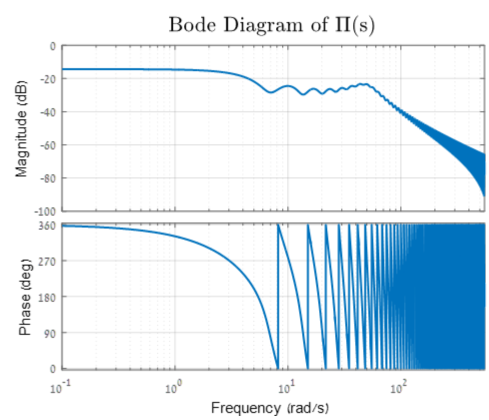

Figure 1. Bode diagram of $\Pi(s)$

L can be exploited in order to locate the poles of the approximation (Ackerman's formula [3]), therefore the choice of stable poles renders the approximation stable regardless of its order. For the other rational methods mentioned above there is a minimal order requirement for stability.

- Padé approximation- This method does not guarantee a stable approximation because $R_{[n,n]}(\lambda_i) \neq e^{-\lambda_i h}$ at all eigenvalues λ_i of A. However, practice shows that in order to obtain a stable approximation one should increase the order n of the Padé approximation and generate more zeros until all unstable poles of the system are being effectively canceled. Next, one needs to manually remove the canceled zeros and poles in order to end up with a stable approximation. For the given example above, (using the default tolerance in Matlab which is $\sqrt{\epsilon} \approx 1.49e-8$) one needs to set the order of Padé to 31 in order to obtain a stable approximation.

- Padé-1-shift- Condition for stability is $n > R(A)h/2$ ($R(A) \doteq max\{Re\{eig(A)\}\}$). The order of the approximation is $O\left(\hat{A}\right) = n \cdot O(A)$, therefore the minimal order is $(\lfloor R(A) \cdot h/2 \rfloor + 1) \cdot O(A)$. In the example above $R(A) = 12.5$ so the minimal order of approximation needed to guarantee stability is 21.
- For both delta operator and bi-linear transformation methods: Condition for stability is $n > \lceil h/2.8max\{|eig(A)|\}\rceil$ and the order of the approximation is $O\left(\hat{A}\right) = n \cdot O(A)$, which leads in this example to a minimal order of 57.

B. Closed-Loop Considerations

First order plus delay systems (stable/unstable) are well studied as they appear commonly in many classes of systems (chemical processes, manufacturing chains, economy, reduced order of large scale systems, etc.). The following unstable first order plus dead time process $P(s) = \frac{1}{s-1}e^{-0.2s}$ is used in this section as a benchmark problem for understanding the CL considerations in the process of approximating a DD element. The control architecture suggested in order to satisfy some pre-defined performance specifications is depicted in figure 2:

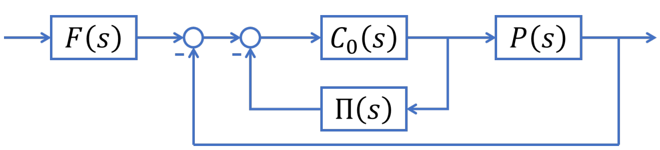

Figure 2. Control architecture

This control architecture consists of a PI primary controller $C_0(s) = K_p(1 + \frac{1}{T_i s})$, a MSP $\Pi(s) = \widetilde{P}(s) - P(s)$ with $\widetilde{P}(s) = Ce^{-Ah}(sI - A)^{-1}B - C\int_0^h e^{-A\tau}d\tau B$ (the constant $-C\int_0^h e^{-A\tau}d\tau B$ was added to the MSP in order to keep its static gain zero, this 'trick' is used in order to keep an integrator in the overall controller) and a first order pre-filter $F(s) = \frac{1}{\tau s+1}$. (full example with design criteria and detailed solution is brought in [19]). The goal here is to approximate the predictor with a rational transfer function of low order while maintaining the following designed properties of the CL:

1) Robust stability under $\pm 10\%$ variation in the time delay.
2) Reference command following for a unit step input: rise time$\leq 0.4[sec]$, settling to 2% window$\leq 1[sec]$, no overshoot, zero steady state error .
3) Disturbance attenuation for a unit step input: max error $\leq 1.15\times$ lowest possible max error $(d \cdot (1 - e^t))$, zero steady state error.
4) Noise rejection: $|T(\omega j)| < -20[dB]\ \forall\omega \geq 30[r/s]$
5) Control Signal: $|T_(r \to u)(\omega j)| < 15[dB]\ \forall\omega$

The first step of the approximation is to choose the moments for matching. This is equivalent to the question where to set the approximation error to zero. After taking a look at the open loop (fig.4) $L(s) = C(s)P(s)$ and at the controller's loop (fig.3) $C_0(s)\Pi(s)$ it is possible to deduce where matching is desired in order to preserve the CL stability.

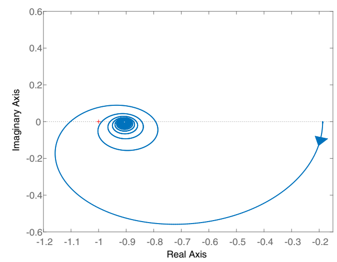

Figure 3. Polar plot of the controller's OL, $C_0(s)\Pi(s)$

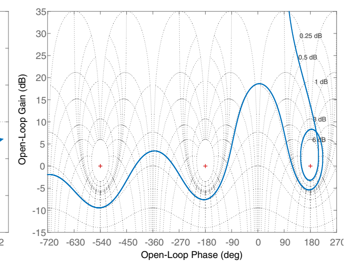

Figure 4. Nichols chart of OL, $C(s)P(s)$

It is clear from figure 3 that the inner loop of the controller is very close to the critical point and if the approximation won't be very good around these frequencies the inaccuracy can generate more unstable poles in the overall controller and in turn in the OL so that even what could easily seem as an accurate approximation of the OL can lead to an unstable CL. Furthermore, in order to keep the design robustly stable, it is crucial to achieve good enough approximation around the crossover frequencies and not to generate more crossovers in the approximation. Performance-wise low-frequency matching is needed to preserve the integrator behavior. Following the guidelines mentioned above, the moments' locations were chosen to be $\eta = [0, 0, 27.3i, -27.3i, 56.8i, -56.8i, 87i, -87i]$ (this defines the order of the approximation to be 8). Next step is to choose the parameters L, Q_∞. The best practice is to use an optimization algorithm to determine these parameters, in this example an ad-hock non-optimal method was used to choose the approximation's poles desired locations. Finally, construct $\Pi_{app}(s) = \left[\begin{array}{c|c} A_w + LC_w & L \\ \hline C \cdot \Sigma_{12} - C_w \cdot Q_\infty & Q_\infty \end{array}\right]$.

Results:

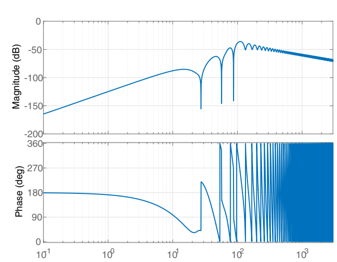

Figure 5. Bode of approximation error, $\Pi(s) - \Pi_{app}(s)$

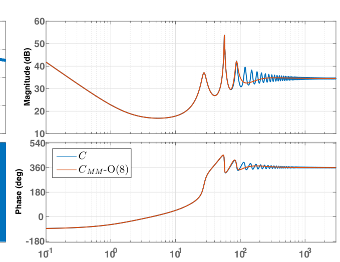

Figure 6. Bode of the controller $C(s)$

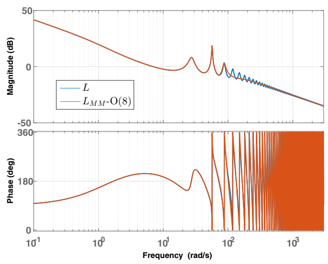

Figure 7. Bode of OL $L(s)$

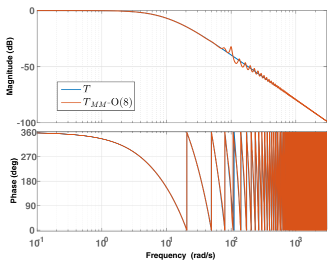

Figure 8. Bode of CL, $T(s)$

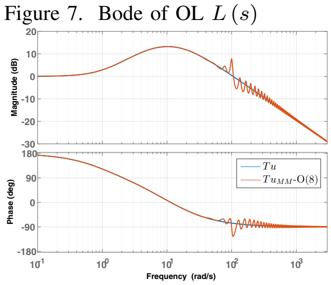

Figure 9. Bode of the transference from r to u $T_u(s)$

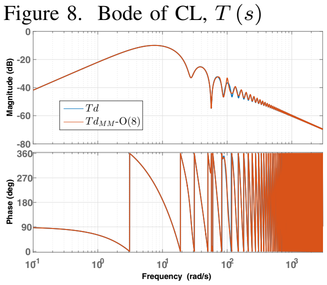

Figure 10. Bode of the transference from d to y $T_d(s)$

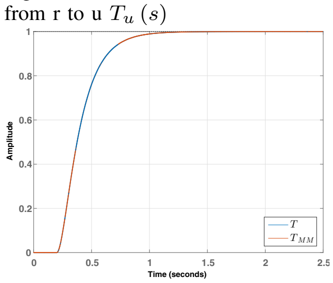

Figure 11. CL response for step reference

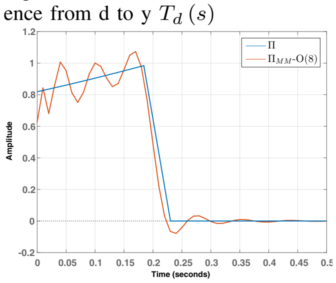

Figure 12. Impulse response of $\Pi(s)$

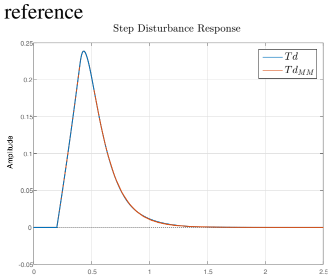

Figure 13. CL response for step disturbance

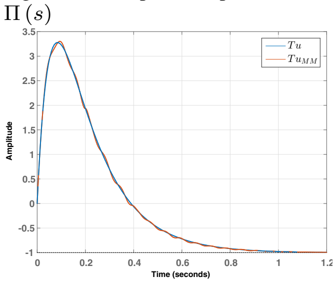

Figure 14. Control signal caused by step in reference

The eighth order approximation of $\Pi(s)$ is very accurate at all frequencies, the error is less than -36[dB] $\forall\omega$ (see figure 5). As can be seen in figures 6-10 there is a very good correlation between the original transfer functions and their approximated counterparts. Quality of approximation can also be seen in time domain (figures 11-14) where the command signal and the responses to reference and disturbance step inputs coincide with those of the original system. The results above were quantified in terms of relative error norms and summarized in table I alongside the results of the other methods.

Criteria*	MM	Padé	P1S	δ	BL**
$\Pi(s)\mid_2$	17.47%	23.98%	29.74%	31.59%	29.75%
$\Pi(s)\mid_\infty$	6.71%	8.22%	11.11%	13.38%	11.11%
$T(s)\mid_\infty$	1.48%	96.26%	9.35%	136.8%	9.37%
$T_d(s)\mid_\infty$	2.34%	151.8%	14.76%	217.5%	14.79%
$T_u(s)\mid_\infty$	32.41%	2141%	124.7%	404.9%	124.6%

Table I
COMPARISON BETWEEN DIFFERENT RATIONAL APPROXIMATIONS OF ORDER 8

* Notation: $X(s)\mid_n := 100 \cdot \frac{|X(s)-X_{app}(s)|_n}{|X(s))|_n}$
** The results of BL and P1S are very close in this specific example, this is not always the case.

Each method needs a different minimal order for the approximation in order to preserve the designed CL characteristics, the results are summarized in table II

Criteria	MM	Padé	P1S	δ	BL
CL stability	O(8)	O(10)	O(12)	O(19)	O(12)
Robust stability	O(8)	O(10)	O(44)	O(35)	O(44)
Reference following	O(8)	O(10)	O(12)	>O(100)	O(12)
Disturbance attenuation	O(8)	O(10)	O(12)	O(19)	O(12)
Noise rejection	O(8)	O(10)	O(18)	O(50)	O(18)
Control signal	O(8)	O(10)	O(18)	O(50)	O(18)

Table II
MINIMAL ORDER NEEDED FOR EACH METHOD

The results in table II show great advantage to MM and Padé. The minimal order needed to meet all the specifications with P1S and BL is 44. Keep in mind that although Padé demonstrated good performance and is simple to design, when high frequency accuracy is needed its order will increase accordingly (After all Padé is a special case of MM with $2n+1$ moments matched at the origin).

V. CONCLUSIONS AND FUTURE WORK

A novel method for rational approximation of DD element was presented and compared to other rational methods. The results show that while other methods are 'closed' and posses no DOF, proper utilization of the free parameters in this method can be exploited to preserve CL properties under maximal order constraint. Naturally, next step is to formulate a systematic procedure or to derive an analytic solution that yields the optimal parameters L, Q_∞ for a given cost function. Meanwhile, optimization algorithms such as genetic-algorithms proved to be effective.

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