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A moment matching approach for systems of delayed differential equations

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Abstract— **In this paper we propose a moment matching approach for approximating time-delay systems with delays in the states. This approach is based on the unique solution of an associated Sylvester-like equation. It results in a family of parametrized, finite-dimensional, reduced order models that match a set of prescribed moments of the given delay-system. We show that, by properly choosing the free parameters, we obtain a finite-dimensional approximation that achieves moment matching at both the zero and first order derivatives of the transfer function of the given delay-system. This approach does not require the explicit computation of the transfer function and its derivatives. We illustrate the potential of this method with a simple example of a state delay-system. The constructed reduced order model for this delay-system is more accurate than approximations obtained using the well-known Padé approximation.**

I. Introduction

Time-delay systems are a wide family of infinite-dimensional systems yielded by the modelling of phenomena and processes, see, e.g., [18], [23]. Approximation of time-delay systems involves finding a finite-dimensional model of the original system with state delays, satisfying appropriate prescribed constraints. For example, one can use the well-known Padé approximation ([10]), which was introduced by Frobenius ([8]) who developed the basic algorithmic aspects of the theory. The attribution to Padé, seems to be due to its study of particular cases which may occur ([19]). It can lead to a rational model of a time-delay system that can be implemented in control design, see, e.g., [17]. Here finite-order models are obtained utilizing Padé type approximations of the function $e^{-\tau s}$. The results therein may possess stability or approximation error properties. During the past decades, considerable advances have been made in this direction. For instance, a family of systems that achieve moment matching is characterized in [21].

In this paper we propose a moment matching approach to the approximation of time-delay systems along the lines of [15], [3], [13]. This approach yields a family of parametrized, reduced order finite-dimensional approximations different from the recent approach taken in [21]. There, the approximations are time-delay systems, with reduced order delay-free (finite-dimensional) part and a reduced number of delays. One attractive feature of the approach of this paper is that (similar to the finite-dimensional case) the reduced order models match a set of prescribed moments of the given delay-system and leaves degrees of freedom useful for meeting further constraints. Moreover, we show that in the family of parametrized models that achieve moment matching there exists a (unique) model that achieves moment matching at the first order derivatives of the associated transfer functions, thus yielding improved accuracy, without employing additional delay elements as in [21]. Note that achieving moment matching at the zero and first order derivatives at a specific set of interpolation points is a first order optimality condition for the solution of the problem of model reduction with optimal H_2 norm of the approximation error, see, e.g., [1], [7], [11].

The paper is organized as follows. In Section II we present the goals of the paper and formulate the model order reduction problem. In Section III we recall the definition of moment and relate it to the (unique) solution of a Sylvester-like equation. In Section IV, we first recall the property of moment matching. In Subsection IV-A we obtain the family of finite-dimensional, parametrized, reduced order models that achieve moment matching for the given delay-system with delays in the states. In Subsection IV-B we present a method which computes the finite-dimensional approximation that achieves moment matching at both the zero and the first order derivatives of the transfer function of the given system. We also prove that this model is a member of the family of reduced order finite-dimensional approximations calculated in Subsection IV-A. Consequently, the computation of the transfer functions and their derivatives is not needed in this approach. The theory is illustrated with a simple example of a state-delay system in Section IV. The paper ends with a concluding section.

Notation: (A, B, C) denotes the state-space realization of a linear, dynamical system. $\sigma(A)$ denotes the spectrum of the matrix A or the operator A. A^* is the transpose conjugate of the matrix A. $\mathrm{ran}(\Pi)$ denotes the range (image) of the operator Π. Let V be a linear space. Then $\dim(V)$ denotes the dimension of the space V. $\mathbb{C}$ is the set of complex numbers and $\mathbb{N}$ is the set of non-negative integers. $L_2((a, b); U)$ denotes the set of Lebesgue measurable $U-$valued function $f : (a, b) \to U$ such that $\int_a^b \|f(t)\| dt \leq \infty$. $\mathcal{L}(\mathbb{C}^n)$ is the space of linear operators from $\mathbb{C}^n$ to $\mathbb{C}^n$.

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II. PRELIMINARIES

In this paper, we consider the single-input single-output system of delayed differential equations

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}_0 x(t) + \sum_{i=1}^{p} \mathbf{A}_i x(t-h_i) + \mathbf{B}u(t), \quad t \geq 0 \\ x(0) &= r, \\ x(\theta) &= f(\theta), \quad -h_p \leq \theta < 0, \\ y(t) &= \mathbf{C}x(t), \end{aligned} \tag{1}$$

where $0 \leq h_1 < \cdots < h_p$ and $h_u \geq 0$ represent time-delays, $\mathbf{A}_i \in \mathcal{L}(\mathbb{C}^n), i = 0, ..., p$, $\mathbf{B} \in \mathbb{C}^n$, $\mathbf{C} \in \mathbb{C}^{1\times n}$, $r, x(t) \in \mathbb{C}^n$, $f \in L_2([-h_p, 0]; \mathbb{C})$ and $u \in L_2([0, h_u]; \mathbb{C})$ for all $h_u > 0$. We would like to approximate the retarded system (1) by a finite-dimensional, reduced order model without delays (F, G, H), given by

$$\begin{aligned} \dot{\xi}(t) &= F\xi(t) + Gu(t), \ t \geq 0, \ \ \xi(0) = \xi_0, \\ \eta(t) &= H\xi(t), \end{aligned} \tag{2}$$

where $\xi_0 \in \mathbb{C}^\nu$, $\xi(t) \in \mathbb{C}^\nu$, $F : \mathbb{C}^\nu \to \mathbb{C}^\nu$, $G : \mathbb{C} \to \mathbb{C}^\nu$, and $H : \mathbb{C}^\nu \to \mathbb{C}$, with $\nu \in \mathbb{N}$, such that additional requirements are met. The reduced order model is obtained using moment matching techniques.
The system (1), has the transfer function

$$\mathbf{G}(s) = \mathbf{C} \left(\Delta(s)\right)^{-1} \mathbf{B},$$

where

$$\Delta(s) = sI - \mathbf{A}_0 - \sum_{i=1}^{p} \mathbf{A}_i e^{-sh_i}, \quad \text{for } s \in \mathbb{C}. \tag{3}$$

One can reformulate system (1) as an abstract differential equation (see [5, Chapter 2]) on the so-called M2 space model ([6]).

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t), \ t \geq 0, \ \ z(0) = z_0, \\ y(t) &= Cz(t), \end{aligned} \tag{4}$$

on the state space $M_2([-h_p, 0]; \mathbb{C})$ with the state vector

$$z(t) = \begin{pmatrix} x(t) \\ x(t+\cdot) \end{pmatrix}$$

and A, the infinitesimal generator of the corresponding C_0-semigroup given by

$$A \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} A_0 r + \sum_{i=1}^{p} A_i f(-h_i) \\ \frac{df}{d\theta}(\cdot) \end{pmatrix},$$

with domain

$$D(A) = \left\{ \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \in M_2([-h_p, 0]; \mathbb{C}) \mid f \text{abs. cont.}, \right. \\ \left. \frac{df}{d\theta}(\cdot) \in L_2([-h_p, 0]; \mathbb{C}) \text{ and } f(0) = r \right\}.$$

The spectrum of A is discrete and it is given by $\sigma_p(A) = \{s \in \mathbb{C} \mid \Delta(s) = 0\}$. Let $\rho(A)$ be the resolvent of A. For this family of delay-systems, $\rho_\infty(A)$, the component of $\rho(A)$ that contains an interval $[r, \infty)$, is equal to $\rho(A)$. Furthermore, B is a bounded operator.

A. *Model order reduction problem formulation*

Our goal is to approximate the infinite-dimensional state linear system (4) by a finite-dimensional, reduced order model such that additional constraints as

- preservation of poles of the original system,
- matching first order derivatives of the transfer function,
- small approximation error

are satisfied. The reduced order model is obtained by projecting onto a particular finite dimensional subspace of the state-space Z and using moment matching techniques.
Thus, to the retarded system (A, B, C) there corresponds a finite-dimensional system

$$\begin{aligned} &\dot{\xi}(t) = F\xi(t) + Gu(t), \ t \geq 0, \ \ \xi(0) = \xi_0, \\ &\psi(t) = H\xi(t), \end{aligned} \tag{5}$$

denoted by (F, G, H), where $\xi_0 \in \mathbb{C}^\nu$, $\xi(t) \in \mathbb{C}^\nu$, $F : \mathbb{C}^\nu \to \mathbb{C}^\nu$, $G : \mathbb{C} \to \mathbb{C}^\nu$, and $H : \mathbb{C}^\nu \to \mathbb{C}$, respectively.
The reduced order model is computed using moment matching techniques. The definition of moment is presented in Section III and is related to the solution of a Sylvester-like equation. The property of moment matching is given in Section IV.

III. THE NOTION OF MOMENT FOR SYSTEMS OF DELAYED DIFFERENTIAL EQUATIONS

The moments at a given point s^* in $\rho_\infty(A)$ are the coefficients of the Taylor expansion of $\mathbf{G}(s)$ about s^*, similar to the finite dimensional case (see, e.g., [2], [3]).

Definition 1. Let $s^* \in \rho_\infty(A)$ and $k \in \mathbb{N}$. The k-moment of (4) at s^* is $\eta_k(s^*) \in \mathbb{C}$ defined by

$$\eta_k(s^*) = \frac{(-1)^k}{k!} \frac{d^k \mathbf{G}}{ds^k}(s^*), \tag{6}$$

with the integer $k > 1$.

The definition is consistent with the notion of moment for time-delay systems described in [21].
In the sequel, we present a particular Sylvester-like equation. Let $s^* \in \rho_\infty(A)$ $(s^* \neq \infty)$, and consider $L_\nu : \mathbb{C}^\nu \to \mathbb{C}$ and $\Sigma_\nu : \mathbb{C}^\nu \to \mathbb{C}^\nu$ given by

$$L_\nu = [1 \ 0 \ ... \ 0], \tag{7}$$

$$\Sigma_\nu = \begin{bmatrix} s^* & 1 & 0 & \dots & 0 \\ 0 & s^* & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & s^* & 1 \\ 0 & \dots & \dots & 0 & s^* \end{bmatrix}. \tag{8}$$

One can write the following Sylvester-like equation,

$$\Pi \Sigma_\nu - A\Pi = BL_\nu, \tag{9}$$

where $\Pi : \mathbb{C}^\nu \to \mathbb{C}^n$. We make the following working assumption.

Assumption 1. $\sigma(\Sigma_\nu) \cap \sigma(A) = \emptyset$.

By the definition of A in (4), equation (9) can be written as

$$A_0\Pi + \sum_{i=1}^{p} A_i \Pi e^{-\Sigma_\nu h_i} + BL_\nu = \Pi\Sigma_\nu. \tag{10}$$

which has a unique solution (see [21]). For $s^* \in \rho_\infty(A)$, Assumption 1 is satisfied.

Lemma 1. *The unique solution* Π*, of equation (9) has finite rank. Moreover,*

$$\dim(\mathrm{ran}(\Pi)) \leq \nu.$$

Remark 1. Note that $\{v_k\}_k \subset D(A)$ are not necessarily linearly independent. The space

$$\mathrm{span}\{v_k \mid k = 0, ..., \nu - 1\}$$

is known as an input Krylov subspace of the state space.

We make the following standing assumption, throughout the rest of the paper.

Assumption 2. $\dim(\mathrm{ran}(\Pi)) = \nu$.

The following result (which is an extension of a similar result obtained for the finite-dimensional case [3]) provides a relation of the k-moments of (4), at $s^* \in \rho_\infty(A)$, with Π.

Lemma 2. *Consider the system* (A, B, C)*. Let* Π *be the unique solution of (9). If Assumption 1 holds, then the first* ν *moments of system (4), at* $s^* \in \rho_\infty(A)$ *satisfy*

$$[\eta_0(s^*) \; ... \; \eta_{\nu-1}(s^*)] = C\Pi \,\mathrm{diag}\{1, -1, ..., (-1)^{\nu-1}\}.$$

Remark 2. Note that Lemma 2 corresponds to Σ_ν and L_ν defined by (7). However, one may choose any non-derogatory matrices $S \in \mathbb{C}^{\nu\times\nu}$ and $L \in \mathbb{C}^{1\times\nu}$, such that $\sigma(S) = \{s^*, ..., s^*\}$ and the pair (L, S) is observable, see [21, Lemma 3].

$$\Pi S - A\Pi = BL, \tag{11}$$

which can be rewritten as

$$A_0\Pi + \sum_{i=1}^{p} A_i \Pi e^{-Sh_i} + BL = \Pi S, \tag{12}$$

which has a unique solution if $\sigma(S) \cap \sigma(A) = \emptyset$.

IV. Moment matching

A. Family of finite-dimensional approximations that achieve moment matching

Consider the systems (A, B, C) and (F, G, H) given by (4) and (5), respectively. Consider $l \geq 0$ and let $s_i \in \rho_\infty(A) \cap \rho(F)$, $i = 0, ..., l$ be $l + 1$ given points. Take $j_i \geq 0$ such that

$$\sum_{i=0}^{l} (j_i - 1) = \nu, \tag{13}$$

where ν is the dimension of the system (F, G, H). For each $i = 0, ..., l$, denote the first $j_i + 1$ moments at s_i of the systems (A, B, C) and (F, G, H) (see Definition 1) by $\eta_0(s_i), ..., \eta_{j_i}(s_i)$ and $\hat\eta_0(s_i), ..., \hat\eta_{j_i}(s_i)$, respectively.

Definition 2 (Moment matching)**.** A system (F, G, H) matches ν moments of a given system (A, B, C) at $\{s_0, ..., s_l\}$, if

$$\eta_k(s_i) = \hat\eta_k(s_i), \tag{14}$$

for all $k = 0, ..., j_i$, $i = 0, ..., l$, with ν satisfying (13).

In this section, we obtain a family of reduced order models (F, G, H) of the given system (A, B, C), that achieve moment matching.
Consider $S : \mathbb{C}^\nu \to \mathbb{C}^\nu$, with $\sigma(S) = \{s_0, ..., s_l\}$, where s_i has multiplicity j_i, such that

$$\sigma(S) \cap \sigma(A) = \emptyset. \tag{15}$$

Let $L : \mathbb{C}^\nu \to \mathbb{C}$ be such that (L, S) is observable.
A system (F, G, H) is a *reduced order model of* (A, B, C) *at* $\sigma(S)$ if the following two conditions are satisfied

$$\sigma(F) \cap \sigma(S) = \emptyset, \tag{16a}$$

$$C\Pi = HP, \tag{16b}$$

where $P : \mathbb{C}^\nu \to \mathbb{C}^\nu$ is the unique solution of the Sylvester-like equation

$$PS - FP = GL, \tag{17}$$

and $\Pi : \mathbb{C}^\nu \to \mathbb{C}^n$ is the unique solution of (11), which exists according to Remark 2.
Note that $G : \mathbb{C} \to \mathbb{C}^\nu$ is a free parameter. Hence, we have a family of reduced order models (2) parametrized in G, which we denote by $\{(F, G, H)\}_G$. The following result states that the family of reduced order models $\{(F, G, H)\}_G$ at $\sigma(S)$ achieves moment matching.

Theorem 1. *Consider a time-delay system* (A, B, C)*. Let* S *satisfy (15) and* L *be such that the pair* (L, S) *is observable. There exists a family of reduced order models of* (A, B, C) *at* $\sigma(S)$*, denoted by* $\{(F, G, H)\}_G$*, parametrized in* G *such that, for all* $(F, G, H) \in \{(F, G, H)\}_G$*, equalities (14) are satisfied, described by*

$$\begin{aligned} \dot\xi &= (S - GL)\xi + Gu, \;\; t \geq 0, \;\; \xi(0) = \xi_0, \\ \psi &= C\Pi\xi, \end{aligned} \tag{18}$$

with $\xi_0 \in \mathbb{C}^\nu$*,* $\xi(t) \in \mathbb{C}^\nu$ *and* $\psi(t) \in \mathbb{C}$*.*

From a computational point of view, one may proceed as follows.

- Select S and L such that the pair (L, S) is observable.
- Using any numerically efficient algorithm that does not require the computation of the moments, determine or estimate $H = C\Pi$ (e.g., Krylov methods).
- Write the family of finite-dimensional models

  $$\{(S - GL, G, C\Pi)\}_G,$$

  described by (18).

The free parameter G is used to identify (subfamilies of) model(s) which satisfy desired properties. In the following, we present a simple illustrative example. We and compare the models resulting from moment matching and with pole placement with the Padé approximation of the given system.

The poles are placed in the (approximated) poles of the given delay-system. Since (19) has no zeros, we do no place any zeros in our approximations.

Example 1. Consider the delayed differential system described by

$$\begin{aligned}\dot{x} &= x(t-1) + u(t), \\ y &= x(t),\end{aligned} \tag{19}$$

with the transfer function

$$\mathbf{G}(s) = \frac{1}{s - e^{-s}}. \tag{20}$$

Note that the first three poles of the system are given by $p_1 = 0.5671$ and $p_{2,3} = -1.7179 \pm 4.2348j$.
Let $s_1 = 0$, $s_2 = 1.5$ and $s_3 = 1.8$. Let $L = [1\ 1\ 1]$ and $S = \operatorname{diag}\{s_1, s_2, s_3\}$. Note that relation (15) is satisfied. Then, by Theorem 1, the family of reduced order models $(F, G, H) = (S - GL, G, C\Pi)$ that match the moments of (20) at s_1, s_2 and s_3 is described by

$$\begin{aligned} F &= \begin{bmatrix} -g_1 & -g_1 & -g_1 \\ -g_2 & 1.5 - g_2 & -g_2 \\ -g_3 & -g_3 & 1.8 - g_3 \end{bmatrix}, \ G = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}, \\ H &= [-1\ 0.7832\ 0.6117]. \end{aligned} \tag{21}$$

A third order model (21) of $\mathbf{G}(s)$, with the parameter G such that the model has the poles at $s_1 = 0.5671$ and $s_{2,3} = -1.7179 \pm 4.2348j$, is given by a system (21), with

$$\begin{aligned} F &= \begin{bmatrix} 4.3869 & 4.3869 & 4.3869 \\ 58.6423 & 60.1423 & 58.6423 \\ -69.1785 & -69.1785 & -67.3978 \end{bmatrix}, \\ G &= \begin{bmatrix} -4.3869 \\ -58.64231 \\ 69.1785 \end{bmatrix} \text{ and} \\ H &= [-1\ 0.7832\ 0.6117], \end{aligned} \tag{22}$$

with the transfer function

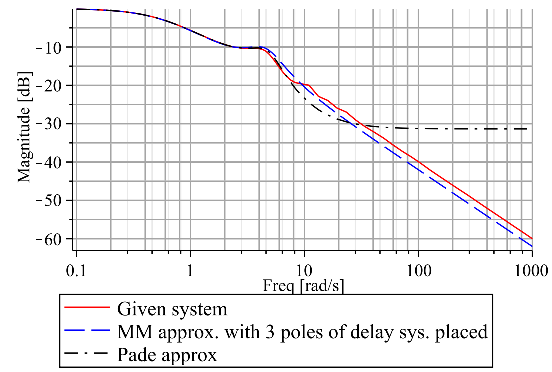
$$K_r(s) = \frac{0.7909s^2 + 4.6952s + 11.8846}{(s - 0.5671)(s + 1.7179 + 4.2348j)(s + 1.7179 - 4.2348j)}. \tag{23}$$

For comparison, one may choose the third order Padé approximation of $\mathbf{G}(s)$, given by

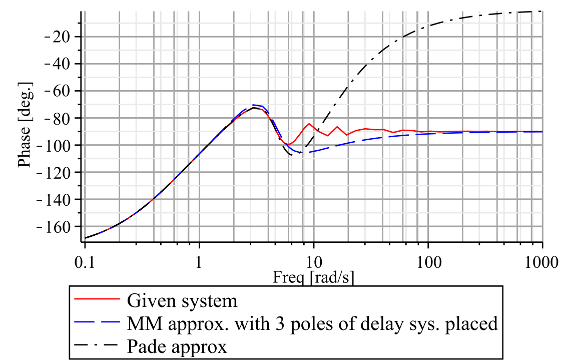
$$\mathbf{G}_a(s) = \frac{-0.0021s^3 - 0.05s^2 - 0.375s - 1}{0.0771s^3 - 0.2s^2 - 1.625s + 1} \approx \mathbf{G}(s).$$

Remark 3. Some important remarks are made at this point.

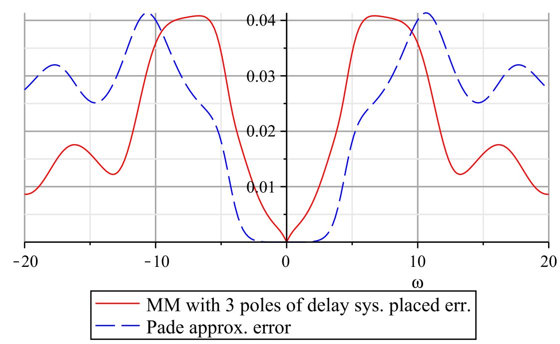
- From the family of reduced order models (21), yielded by Theorem 1, we have selected the reduced order model given by (22) by choosing three matching points and placing the three poles close to the first three poles of the delay-system to be approximated. Note that this model can also be obtained by using standard interpolation by rational functions (see [22, Ch. VIII]).
- The model model given by (22)performs (slightly) better than the Padé approximation of (19), in terms of the ∞-norm of the approximation error (see Table I and Figure 1(c)).
- The utilized moment matching approach does not require the computation of the transfer functions.
- To the authors' knowledge, a systematic general pole placement procedure which gives the reduced order model with the best approximation error in terms of the ∞-norm does not yet exist.



(a) Magnitude



(b) Phase



(c) Error

Fig. 1. Magnitude plot (a), phase plot (b) of (19), the third order model (22) (dashed blue line) and the third order order Padé approximation of (19) (dash-dotted black line). Figure (c) is the plot of the error between (19) and the third order approximation (22) (solid red line) and of the error between (19) and the third order Padé approximation (dashed blue line).

Instead of choosing the poles, we further propose (in Subsection IV-B) a moment matching approach based on imposing the condition of matching not only the moments of the system itself, but also the moments of the first order derivative of its transfer function.

B. Moment matching of transfer functions of time-delay systems and of their first order derivatives

In this section we present a method to obtain a reduced order model $(S - GL, G, C\Pi)$ that approximates (A, B, C) and satisfies matching properties at the first order derivative of $\mathbf{G}(s)$. The explicit computation of the derivatives is not required. We also prove that this model is a member of the family of ν order models $\{(S - GL, G, C\Pi)\}_G$ that achieve moment matching in the sense of Theorem 1.
Consider the Sylvester-like equation, in $\Upsilon : \mathbb{C}^\nu \to \mathbb{C}^n$,

$$S\Upsilon = \Upsilon A + L^* C, \tag{24}$$

with S such that (15) is satisfied. Equation (24) can be written as

$$S\Upsilon = \Upsilon A_0 + \sum_{i=1}^{p} \Upsilon A_i e^{-Sh_i} + L^* C.$$

Since (15) is satisfied, following arguments from [21, Section 3], it can be shown that (24) has a unique solution.

Assumption 3. The matrix $\Upsilon\Pi$ is invertible, where Π is the unique solution of (11).

Theorem 2. *Consider the system (A, B, C). Let $S : \mathbb{C}^\nu \to \mathbb{C}^\nu$ be such that condition (15) is satisfied. Let Υ be the solution of (24) and Π the solution of (11) and suppose that Assumption 3 holds. Let*

$$\widehat{\mathbf{G}}(s) = C\Pi[sI - (\Upsilon\Pi)^{-1}(\Upsilon(sI - \Delta(s))\Pi)]^{-1}(\Upsilon\Pi)^{-1}\Upsilon B, \tag{25}$$

with $\Delta(s)$ as in (3). Then $\widehat{\mathbf{G}}(s)$ satisfies the following properties.

1) For all $s_i \in \sigma(S)$, $i = 1, ..., \nu$,

$$\widehat{\mathbf{G}}(s_i) = \mathbf{G}(s_i),$$

i.e.,$\widehat{\mathbf{G}}$ matches the moments of $\mathbf{G}$ at $\sigma(S)$;

2) For all $s_i \in \sigma(S)$, $i = 1, ..., \nu$,

$$\frac{d\widehat{\mathbf{G}}(s_i)}{ds} = \frac{d\mathbf{G}(s_i)}{ds},$$

i.e., the derivatives of $\widehat{\mathbf{G}}$ and $\mathbf{G}$ match at $\sigma(S)$.

Finally, the reduced order model, from the family $\{(S - GL, G, C\Pi)\}_G$, which achieves moment matching at the zero and and first order derivatives of the transfer functions, simultaneously, is identified as a direct consequence of Theorem 2. The model $(S - GL, G, C\Pi)$ that matches the moments of $\mathbf{G}(s)$ and $d\mathbf{G}(s)/ds$ at $\sigma(S)$ is obtained by selecting

$$G = (\Upsilon\Pi)^{-1}\Upsilon B.$$

Example 2. Consider the system (19). Let $s_1 = 0$, $s_2 = 1.5399$ and $s_3 = 2.1$ be the interpolation points. A the third order approximations that matches the moments of $\mathbf{G}(s)$, and $d\mathbf{G}(s)/ds$ at s_1, s_2 and s_3 is given by a model (F, G, H), with

$$\begin{aligned} F &= \begin{bmatrix} 3.4056 & 3.4056 & 3.4056 \\ 32.625 & 34.1649 & 32.625 \\ -43.7371 & -43.7371 & -41.6371 \end{bmatrix}, \\ G &= \begin{bmatrix} -3.4056 \\ -32.625 \\ 43.7371 \end{bmatrix} \text{ and} \\ H &= [-1 \ 0.7544 \ 0.5057], \end{aligned} \tag{26}$$

with the transfer function

$$K_{rd}(s) = \frac{0.909s^2 + 5.2344s + 11.0129}{s^3 + 4.0667s^2 + 16.7914s - 11.0129}.$$

Reduced order model	L_∞ norm of the error
3^{rd} order Padé approximation of $\mathbf{G}$	0.04136
Moment matching with 3 poles placed close to first 3 poles of $\mathbf{G}$	0.04078
Moment matching of $\mathbf{G}(s)$ and $d\mathbf{G}(s)/ds$	0.03588

TABLE I

L_∞ NORMS OF THE APPROXIMATION ERROR

Remark 4. Using Theorem 2 and Krylov projection ideas to find Υ (see [4]), we have computed the reduced order model (26) that matches the moments of $\mathbf{G}(s)$ and $d\mathbf{G}(s)/ds$ at the given points s_1, s_2, s_3. Note that:

- this moment matching approach does not require the computation of the transfer functions or their first order derivatives;
- this model is more accurate (in the sense of ∞-norm) than the model yielded by the Padé approximation (see Table I).
- the poles are placed automatically by the method at $s_1 = 0.5671$ (which is the first real pole of (19)) and $s_{2,3} = -2.3169 \pm 3.7485j$.

V. Conclusions

In this paper we have presented a moment matching-based approach for approximating time-delay systems with delays in the states. It results in a family of parametrized, finite-dimensional, reduced order models that match a set of prescribed moments of the given delayed system. Furthermore, using moment matching at both the zero and first order derivatives of the transfer function of the given system, we can select a particular reduced order model from the parametrized class. Using a simple example of a state delayed system we illustrate that the reduced order model, obtained using the method proposed in this paper, can approximate the delay-system more accurate than reduced order models based on the Padé approximation.

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