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On the state space realization of separable periodic 2D systems

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***Abstract*— This paper is a preliminary contribution to the study of the realization problem for periodic 2D input/output systems. More concretely, separable periodic 2D SISO systems are considered and state space representations by means of separable periodic *Roesser* models are obtained using suitable invariant formulations.**

I. Introduction

Periodic systems and their state space representations play an important role in several domains. For instance, in the 1D case they have recently become relevant in the context of the construction and implementation of 1D convolutional codes, as it has been shown that introducing periodicity can lead to better error correcting properties, [1], [2].

With the increasing importance of 2D convolutional codes, [3]–[5], the state space realization of periodic 2D codes – viewed as 2D dynamical systems – as also gained importance. This is, among others, a motivation for our work.

A possible method for the realization of a periodic 2D input/output system in periodic state space form is to construct an invariant formulation of the periodic input/output system, apply standard realization techniques for invariant 2D systems in order to obtain an invariant 2D state space representation, and finally try obtain a periodic 2D state space representation from the invariant one. This method encounters several difficulties, mainly due to the well-known problems (such as the issue of minimality) inherent to the realization of invariant 2D systems, [6].

Separable 2D systems do not present the same difficulties as the general ones concerning the question of minimality, [7], but are nonetheless relevant in several applications, as for instance (again) in 2D convolutional coding (in particular in the study of composition codes, obtained by the series concatenation of two 1D codes each of them acting on a different direction). Therefore, here we focus on this class of systems, and apply the aforementioned method to obtain (when possible) realizations by periodic 2D separable *Roesser* state space models.

II. Preliminaries

We consider separable periodic single–input/single–output (SISO) systems defined over $\mathbb{N}^2$. Such systems are described by equations of the form:

$$\big(p_{(i,j)}(\sigma_1,\sigma_2)y\big)(i,j)=\big(q_{(i,j)}(\sigma_1,\sigma_2)u\big)(i,j)\,,(i,j)\in\mathbb{N}^2,$$

where, for $(i,j)\in\mathbb{N}^2$,

$$\begin{aligned}(\sigma_1 v)\,(i,j) &= v\,(i+1,j)\\ (\sigma_2 v)\,(i,j) &= v\,(i,j+1)\,,\end{aligned}$$

i.e., σ_1 and σ_2 represent the usual 2D shifts; $p_{(i,j)}\,(z_1,z_2)\in\mathbb{R}\,[z_1,z_2]\backslash\{0\}$ and $q_{(i,j)}\,(z_1,z_2)\in\mathbb{R}\,[z_1,z_2]$ are 2D periodically varying polynomials of period P and Q, i.e.:

$$\begin{aligned}p_{(i,j)} &= p_{(i+P,j)} = p_{(i,j+Q)}\\ q_{(i,j)} &= q_{(i+P,j)} = q_{(i,j+Q)},\end{aligned} \tag{1}$$

where P and Q are the smallest integers for which equations (1) hold. Furthermore, $p_{(i,j)}\,(z_1,z_2)$ is assumed to be separable, i.e.,

$$p_{(i,j)}\,(z_1,z_2) = p_{1_{(i,j)}}\,(z_1)\,p_{2_{(i,j)}}\,(z_2)\ \ .$$

From now on, a system satisfying these conditions is referred to as 2D *separable* (P,Q)–periodic system with horizontal period P and vertical period Q.

Separable invariant 2D systems are well-known to allow a representation by means of 2D separable *Roesser* state space models, see [8], given by[1]

$$\begin{bmatrix}x^h(i{+}1,j)\\x^v(i,j{+}1)\end{bmatrix}=\begin{bmatrix}A^{hh}&0\\A^{vh}&A^{vv}\end{bmatrix}\begin{bmatrix}x^h(i,j)\\x^v(i,j)\end{bmatrix}+\begin{bmatrix}B^h\\B^v\end{bmatrix}u\,(i,j)$$

$$y\,(i,j)=\begin{bmatrix}C^h & C^v\end{bmatrix}\begin{bmatrix}x^h(i,j)\\x^v(i,j)\end{bmatrix}+Du\,(i,j)$$

where $x^h\in\mathbb{R}^{n_h}$ is the horizontal state vector, $x^v\in\mathbb{R}^{n_v}$ is the vertical state vector, u is the input, y is the output, and A^{hh}, A^{vh}, A^{vv}, B^h, B^v, C^h, C^v and D are constant

[1] This is one of the two possible forms; the alternative form corresponds to considering a matrix A^{hv} instead of the right–upper zero block and replacing the matrix A^{vh} by a zero block. However, for our purposes it is enough to consider just one of the forms, as the reasonings to be presented can easily be translated in terms of the other one.

matrices of appropriate sizes.

The problem addressed in this paper is to find a procedure to represent (when possible) a given SISO (P,Q)–periodic 2D *separable* input/output system by means of a 2D *Roesser separable* model with (P,Q)–periodically varying coefficients:

$$\begin{aligned}\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1)\end{bmatrix} &= A(i,j)\begin{bmatrix} x^h(i,j) \\ x^v(i,j)\end{bmatrix} + B(i,j)\,u(i,j) \\ y(i,j) &= C(i,j)\begin{bmatrix} x^h(i,j) \\ x^v(i,j)\end{bmatrix} + D(i,j)\,u(i,j)\end{aligned} \tag{2}$$

i.e., where matrices

$$\begin{aligned} A(i,j) &:= \begin{bmatrix} A^{hh}(i,j) & 0 \\ A^{vh}(i,j) & A^{vv}(i,j)\end{bmatrix},\ B(i,j) := \begin{bmatrix} B^h(i,j) \\ B^v(i,j)\end{bmatrix} \\ C(i,j) &:= \begin{bmatrix} C^h(i,j) & C^v(i,j)\end{bmatrix}\,, \end{aligned} \tag{3}$$

vary periodically with period (P,Q), meaning that, for all possible values of the horizontal and vertical discrete variables (i,j),

$$\begin{aligned} A(i,j) &= A(i+P,j) = A(i,j+Q) \\ B(i,j) &= B(i+P,j) = B(i,j+Q) \\ C(i,j) &= C(i+P,j) = C(i,j+Q) \\ D(i,j) &= D(i+P,j) = D(i,j+Q)\,. \end{aligned}$$

This periodic 2D *Roesser* state space model will be denoted by

$$\Sigma(\cdot,\cdot) = (A(\cdot,\cdot), B(\cdot,\cdot), C(\cdot,\cdot), D(\cdot,\cdot))\,.$$

We say that $\Sigma(\cdot,\cdot)$ represents (or is a *representation* or a *realization* of) a given input/output system if the possible input/output trajectories of $\Sigma(\cdot,\cdot)$ coincide with the ones of the original input/output system. In order to solve the proposed periodic realization problem we first consider invariant formulations for the given periodic input/output system, use invariant realization techniques to obtain invariant 2D *Roesser* state space models, and finally try to construct a periodic realization from the invariant one. This method has proved to be successful in the 1D case, see [9]. Moreover, preliminary results on the nonseparable 2D case have been presented in [10] using the same method, but many questions were left unanswered.

III. Invariant formulations

For the sake of simplicity we consider only the case $P = 2 = Q$. In this case, letting $t_i = 0,1$, with $i = 1,2$, the periodic input/output equations defining a separable $(2,2)$–periodic input/output system can be rewritten as

$$\begin{aligned}&\left(p_{1_{(t_1,t_2)}}(\sigma_1)\, p_{2_{(t_1,t_2)}}(\sigma_2)\, y\right)(2k+t_1, 2\ell+t_2) \\ &\qquad = \left(q_{(t_1,t_2)}(\sigma_1,\sigma_2)\, u\right)(2k+t_1, 2\ell+t_2)\,,\end{aligned}$$

with $k,\ell \in \mathbb{N}$. This is equivalent to

$$\begin{aligned}&\underbrace{\begin{bmatrix} p_{1_{(0,0)}}(\sigma_1)\, p_{2_{(0,0)}}(\sigma_2) \\ p_{1_{(1,0)}}(\sigma_1)\, p_{2_{(1,0)}}(\sigma_2)\,\sigma_1 \\ p_{1_{(0,1)}}(\sigma_1)\, p_{2_{(0,1)}}(\sigma_2)\,\sigma_2 \\ p_{1_{(1,1)}}(\sigma_1)\, p_{2_{(1,1)}}(\sigma_2)\,\sigma_1\sigma_2 \end{bmatrix}}_{\mathcal{P}(\sigma_1,\sigma_2)} y(2k,2\ell) \\ &\qquad = \underbrace{\begin{bmatrix} q_{(0,0)}(\sigma_1,\sigma_2) \\ q_{(1,0)}(\sigma_1,\sigma_2)\,\sigma_1 \\ q_{(0,1)}(\sigma_1,\sigma_2)\,\sigma_2 \\ q_{(1,1)}(\sigma_1,\sigma_2)\,\sigma_1\sigma_2 \end{bmatrix}}_{\mathcal{Q}(\sigma_1,\sigma_2)} u(2k,2\ell)\,, \end{aligned} \tag{4}$$

with $k,\ell \in \mathbb{N}$. Decomposing the polynomials columns $\mathcal{P}(z_1,z_2)$ and $\mathcal{Q}(z_1,z_2)$ as

$$\mathcal{P}(z_1,z_2) = \mathcal{P}^L\left(z_1^2,z_2^2\right)\begin{bmatrix} 1 \\ z_1 \\ z_2 \\ z_1z_2\end{bmatrix}$$

and

$$\mathcal{Q}(z_1,z_2) = \mathcal{Q}^L\left(z_1^2,z_2^2\right)\begin{bmatrix} 1 \\ z_1 \\ z_2 \\ z_1z_2\end{bmatrix},$$

(4) can be written as:

$$\left(\mathcal{P}^L(\sigma_1,\sigma_2)Y\right)(k,\ell) = \left(\mathcal{Q}^L(\sigma_1,\sigma_2)U\right)(k,\ell)\,,\ k,\ell\in\mathbb{N}, \tag{6}$$

where

$$U(k,\ell) = \begin{bmatrix} u(2k,2\ell) \\ u(2k+1,2\ell) \\ u(2k,2\ell+1) \\ u(2k+1,2\ell+1)\end{bmatrix} \tag{7a}$$

and

$$Y(k,\ell) = \begin{bmatrix} y(2k,2\ell) \\ y(2k+1,2\ell) \\ y(2k,2\ell+1) \\ y(2k+1,2\ell+1)\end{bmatrix} \tag{7b}$$

are the *lifted* trajectories corresponding to u and y, respectively, (notice the replacement of the shifts σ_i^2 (i=1,2) by σ_i (i=1,2), due to the change of independent variable). This defines an invariant 2D system which is called the *invariant formulation*, or the *lifted version*, of the original periodic system. Clearly, an input/output trajectory (u,y) belongs to the periodic system if and only if the corresponding lifted trajectory (U,Y) belongs to its lifted version.

As is well-known a 2D invariant system

$$P(\sigma_1,\sigma_2)\,Y = Q(\sigma_1,\sigma_2)\,U$$

is representable by a 2D separable *Roesser* model with input U and output Y if and only if the square matrix $P(z_1,z_2)$ is invertible and the transfer function $G(z_1,z_2) = P^{-1}(z_1,z_2)\,Q(z_1,z_2)$ is quarter-plane causal (see [11]) and has separable denominator, i.e., it can be written as

$$G(z_1,z_2) = \frac{N(z_1,z_2)}{d_1(z_1)\,d_2(z_2)}$$

where $N(z_1,z_2)$ is a 2D polynomial matrix and $d_i(z_i)$ ($i = 1,2$) are 1D polynomials, [8].

It turns out that the invariant formulation (6) may not satisfy the aforementioned requirements. However, in several interesting situations (such as, for instance, for some periodic 2D input/output systems arising from the series interconnection of two periodic 1D input/output systems each of which evolves in a different direction), the invariant formulation (6) does correspond to a quarter-plane causal separable denominator transfer function. Without entering into further details, we here assume that this is the case.

In order to obtain an invariant formulation for $(2,2)$–periodic 2D *Roesser* state space model, note that, for $P = Q = 2$, the equations (2) can be rewritten as:

$$\begin{aligned}
\begin{bmatrix} x^h(2k+t_1+1,2\ell+t_2) \\ x^v(2k+t_1,2\ell+t_2+1)\end{bmatrix} &= A(t_1,t_2)\begin{bmatrix} x^h(2k+t_1,2\ell+t_2) \\ x^v(2k+t_1,2\ell+t_2)\end{bmatrix} \\
&\quad + B(t_1,t_2)\,u(2k+t_1,2\ell+t_2) \\
y(2k+t_1,2\ell+t_2) &= C(t_1,t_2)\begin{bmatrix} x^h(2k+t_1,2\ell+t_2) \\ x^v(2k+t_1,2\ell+t_2)\end{bmatrix} \\
&\quad + D(t_1,t_2)\,u(2k+t_1,2\ell+t_2)
\end{aligned} \tag{8}$$

where $k,\ell \in \mathbb{N}$, and $t_i = 0,1$ ($i = 1,2$), and the matrices A, B, C and D are decomposed as in (3). Denote

$$A(0,0) =: A_1 =: \begin{bmatrix} A_1^{hh} & 0 \\ A_1^{vh} & A_1^{vv}\end{bmatrix};\ A(1,0) =: A_2 =: \begin{bmatrix} A_2^{hh} & 0 \\ A_2^{vh} & A_2^{vv}\end{bmatrix}$$

$$A(0,1) =: A_3 =: \begin{bmatrix} A_3^{hh} & 0 \\ A_3^{vh} & A_3^{vv}\end{bmatrix};\ A(1,1) =: A_4 =: \begin{bmatrix} A_4^{hh} & 0 \\ A_4^{vh} & A_4^{vv}\end{bmatrix}$$

and likewise for all the other matrices.

Following the ideas of [12], define lifted versions of the horizontal and vertical states as:

$$X^h(k,\ell) = \begin{bmatrix} x^h(2k,2\ell) \\ x^h(2k,2\ell+1)\end{bmatrix}$$

and

$$X^v(k,\ell) = \begin{bmatrix} x^v(2k,2\ell) \\ x^v(2k+1,2\ell)\end{bmatrix},$$

respectively, and consider $U(k,\ell)$ and $Y(k,\ell)$ as previously defined in eqs. (7a) and (7b). This yields the following linear 2D invariant separable *Roesser* model

$$\begin{aligned}
\begin{bmatrix} X^h(k+1,\ell) \\ X^v(k,\ell+1)\end{bmatrix} &= F\begin{bmatrix} X^h(k,\ell) \\ X^v(k,\ell)\end{bmatrix} + GU(k,\ell)\ , \\
Y(k,\ell) &= H\begin{bmatrix} X^h(k,\ell) \\ X^v(k,\ell)\end{bmatrix} + JU(k,\ell)
\end{aligned} \tag{10}$$

where matrices F, G, H and J are constant and can be decomposed as follows

$$\begin{aligned}
F &= \begin{bmatrix} F^{hh} & F^{hv} \\ F^{vh} & F^{vv}\end{bmatrix}\ ,\ G = \begin{bmatrix} G^h \\ G^v\end{bmatrix} \\
H &= \begin{bmatrix} H^h & H^v\end{bmatrix},
\end{aligned} \tag{11}$$

with the size of the blocks is determined by the sizes of X^h and X^v, and, moreover:

$$\begin{aligned}
F^{hh} &= \begin{bmatrix} A_2^{hh}A_1^{hh} & 0 \\ 0 & A_4^{hh}A_3^{hh}\end{bmatrix} \\
F^{hv} &= \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix} \\
F^{vh} &= \begin{bmatrix} A_3^{vv}A_1^{vh} & A_3^{vh} \\ A_4^{vv}A_2^{vh}A_1^{hh} & A_4^{vh}A_3^{hh}\end{bmatrix} \\
F^{vv} &= \begin{bmatrix} A_3^{vv}A_1^{vv} & 0 \\ 0 & A_4^{vv}A_2^{vv}\end{bmatrix} \\
G^h &= \begin{bmatrix} A_2^{hh}B_1^h & B_2^h & 0 & 0 \\ 0 & 0 & A_4^{hh}B_3^h & B_4^h\end{bmatrix} \\
G^v &= \begin{bmatrix} A_3^{vv}B_1^v & 0 & B_3^v & 0 \\ A_4^{vv}A_2^{vh}B_1^h & A_4^{vv}B_2^v & A_4^{vh}B_3^h & B_4^v\end{bmatrix} \\
H^h &= \begin{bmatrix} C_1^h & 0 \\ C_2^hA_1^{hh} & 0 \\ C_3^vA_1^{vh} & C_3^h \\ C_4^vA_2^{vh}A_1^{hh} & C_4^hA_3^{hh}\end{bmatrix} \\
H^v &= \begin{bmatrix} C_1^v & 0 \\ 0 & C_2^v \\ C_3^vA_1^{vv} & 0 \\ 0 & C_4^vA_2^{vv}\end{bmatrix}
\end{aligned} \tag{12}$$

and

$$J = \begin{bmatrix} D_1 & 0 & 0 & 0 \\ C_2^hB_1^h & D_2 & 0 & 0 \\ C_3^vB_1^v & 0 & D_3 & 0 \\ C_4^vA_2^{vh}B_1^h & C_4^vB_2^v & C_4^hB_3^h & D_4\end{bmatrix}$$

We denote this invariant lifted model by $\Sigma^L = (F,G,H,J)$, and say that Σ^L is *induced by* the original periodic model $\Sigma(\cdot,\cdot)$, or equivalently, that $\Sigma(\cdot,\cdot)$ *induces* Σ^L.

IV. $(2,2)$–PERIODIC SEPARABLE *Roesser* REPRESENTATIONS

In this section we investigate the questions of determining whether a given 2D invariant separable *Roesser* model is or not induced by a SISO $(2,2)$–periodic separable one,

and of obtaining a corresponding inducing $(2,2)$–periodic separable *Roesser* model in the case the answer to the previous question is positive.
For this purpose, we first analyse the structure of induced invariant 2D state space representations. As can be seen from (11) and (12), the corresponding matrices have a very particular form, based on which the factorization of certain suitably defined matrices can be performed. This is explained next.

Consider the $(2,2)$–periodic *Roesser* model (8), with horizontal and vertical states of sizes n_h and n_v, respectively. Consider also the corresponding (induced) invariant representation (10) (with horizontal and vertical states of sizes $2n_h$ and $2n_v$, respectively). Define matrices $\mathcal{M}$ as follows:

$$\mathcal{M}_{11}^{h\star} := \begin{bmatrix} F_{11}^{h\star} & G_{11}^{h} \\ H_{21}^{\star} & J_{21} \end{bmatrix} \quad \mathcal{M}_{12}^{h\star} := \begin{bmatrix} F_{12}^{h\star} & G_{12}^{h} \\ H_{32}^{\star} & J_{32} \end{bmatrix}$$
$$\mathcal{M}_{11}^{v\star} := \begin{bmatrix} F_{11}^{v\star} & G_{11}^{v} \\ H_{31}^{\star} & J_{31} \end{bmatrix} \quad \mathcal{M}_{12}^{v\star} := \begin{bmatrix} F_{12}^{v\star} & G_{12}^{v} \\ H_{32}^{\star} & J_{32} \end{bmatrix}$$
$$\mathcal{M}_{21}^{\star h} := \begin{bmatrix} F_{21}^{\star h} & G_{21}^{\star} \\ H_{41}^{h} & J_{41} \end{bmatrix} \quad \mathcal{M}_{22}^{\star h} := \begin{bmatrix} F_{22}^{\star h} & G_{23}^{\star} \\ H_{42}^{h} & J_{43} \end{bmatrix}$$
$$\mathcal{M}_{21}^{\star v} := \begin{bmatrix} F_{21}^{\star v} & G_{21}^{\star} \\ H_{41}^{v} & J_{41} \end{bmatrix} \quad \mathcal{M}_{22}^{\star v} := \begin{bmatrix} F_{22}^{\star v} & G_{22}^{\star} \\ H_{42}^{v} & J_{42} \end{bmatrix}$$

where each symbol $\star$ represents either h or v and where each block–element is defined in the obvious way by the block–divisions in (12). Notice that, for $i = 1, 2$, matrices $\mathcal{M}_{1i}^{h\star}$ have size $(n_h+1) \times (n_\star+1)$, matrices $\mathcal{M}_{1i}^{v\star}$ have size $(n_v+1) \times (n_\star+1)$, matrices $\mathcal{M}_{2i}^{\star h}$ have size $(n_\star+1) \times (n_h+1)$ and, finally, matrices $\mathcal{M}_{2i}^{\star v}$ have size $(n_\star+1) \times (n_v+1)$. Moreover, define the following matrix, of size $(n_h+n_v+2) \times (n_h+1)$:

$$\mathcal{W}_1 := \left[\begin{array}{c} \mathcal{M}_{11}^{hh} \\ \hline \mathcal{M}_{21}^{vh} \end{array}\right] = \left[\begin{array}{cc} F_{11}^{hh} & G_{11}^{h} \\ H_{21}^{h} & J_{21} \\ \hline F_{21}^{vh} & G_{21}^{v} \\ H_{41}^{h} & J_{41} \end{array}\right]$$
$$= \left[\begin{array}{cc} A_2^{hh}A_1^{hh} & A_2^{hh}B_1^{h} \\ C_2^{h}A_1^{hh} & C_2^{h}B_1^{h} \\ \hline A_4^{vv}A_2^{vh}A_1^{hh} & A_4^{vv}A_2^{vh}B_1^{h} \\ C_4^{v}A_2^{vh}A_1^{hh} & C_4^{v}A_2^{vh}B_1^{h} \end{array}\right] . \tag{13}$$

Note that this matrix can be factored as

$$\mathcal{W}_1 = \underbrace{\left[\begin{array}{c} A_2^{hh} \\ C_2^{h} \\ \hline A_4^{vv}A_2^{vh} \\ C_4^{v}A_2^{vh} \end{array}\right]}_{n_h \text{ columns}} \left[\begin{array}{c|c} A_1^{hh} & B_1^{h} \end{array}\right],$$

implying that

$$\operatorname{rank} \mathcal{W}_1 \leqslant n_h . \tag{14}$$

Assuming that the previous factorization is of the form:

$$\mathcal{W}_1 = \left[\begin{array}{c} L_{1,1} \\ L_{1,2} \\ \hline L_{1,3} \\ L_{1,4} \end{array}\right] \left[\begin{array}{c|c} R_{1,1} & R_{1,2} \end{array}\right],$$

define another matrix, $\mathcal{W}_2$, of size $(n_v+1) \times (n_h+n_v+1)$, as follows:

$$\mathcal{W}_2 := \left[\begin{array}{c|c} \begin{array}{c} L_{1,3} \\ L_{1,4} \end{array} & \mathcal{M}_2^{vv} \end{array}\right] = \left[\begin{array}{c|cc} L_{1,3} & F_{22}^{vv} & G_{22}^{v} \\ L_{1,4} & H_{42}^{v} & J_{42} \end{array}\right] . \tag{15}$$

This can be factored as

$$\mathcal{W}_2 = \left[\begin{array}{c|cc} A_4^{vv}A_2^{vh} & A_4^{vv}A_2^{vv} & A_4^{vv}B_2^{v} \\ C_4^{v}A_2^{vh} & C_4^{v}A_2^{vv} & C_4^{v}B_2^{v} \end{array}\right]$$
$$= \begin{bmatrix} A_4^{vv} \\ C_4^{v} \end{bmatrix} \begin{bmatrix} A_2^{vh} & A_2^{vv} & B_2^{v} \end{bmatrix},$$

allowing us to conclude that

$$\operatorname{rank} \mathcal{W}_2 \leqslant n_v . \tag{16}$$

Finally, consider the matrices $\mathcal{W}_3$ and $\mathcal{W}_4$ (of sizes $(n_v+1) \times (n_h+n_v+1)$ and $(n_h+n_v+1) \times (n_h+1)$, respectively), which are formed suppressing:

- the last column in $\left[\begin{array}{c|c} \mathcal{M}_{11}^{vh} & \mathcal{M}_{11}^{vv} \end{array}\right]$ (17a)

and

- the last row in $\left[\begin{array}{c} \mathcal{M}_{22}^{hh} \\ \hline \mathcal{M}_{22}^{vh} \end{array}\right]$, (17b)

respectively, i.e.:

$$\mathcal{W}_3 := \begin{bmatrix} F_{11}^{vh} & G_{11}^{v} & F_{11}^{vv} \\ H_{31}^{h} & J_{31} & H_{31}^{v} \end{bmatrix} \tag{18a}$$

and

$$\mathcal{W}_4 := \begin{bmatrix} F_{22}^{hh} & G_{23}^{h} \\ H_{42}^{h} & J_{43} \\ F_{22}^{vh} & G_{23}^{v} \end{bmatrix} . \tag{18b}$$

It is not difficult to see that these matrices can be factored as

$$\mathcal{W}_3 = \begin{bmatrix} A_3^{vv}A_1^{vh} & A_3^{vv}B_1^{v} & A_3^{vv}A_1^{vv} \\ C_3^{v}A_1^{vh} & C_3^{v}B_1^{v} & C_3^{v}A_1^{vv} \end{bmatrix}$$
$$= \begin{bmatrix} A_3^{vv} \\ C_3^{v} \end{bmatrix} \begin{bmatrix} A_1^{vh} & B_1^{v} & A_1^{vv} \end{bmatrix},$$

and

$$\mathcal{W}_4 = \begin{bmatrix} A_4^{hh} A_3^{hh} & A_4^{hh} B_3^h \\ C_4^h A_3^{hh} & C_4^h B_3^h \\ A_4^{vh} A_3^{hh} & A_4^{vh} B_3^h \end{bmatrix} = \begin{bmatrix} A_4^{hh} \\ C_4^h \\ A_4^{vh} \end{bmatrix} \begin{bmatrix} A_3^{hh} & B_3^h \end{bmatrix},$$

allowing us to conclude that

$$\operatorname{rank} \mathcal{W}_3 \leqslant n_v \quad \text{and} \quad \operatorname{rank} \mathcal{W}_4 \leqslant n_h \ . \tag{20}$$

Conversely, consider now a 2D separable $(2,2)$–periodic SISO system, construct its lifted invariant version as in (6), and assume that this is a quarter-plane causal separable 2D system with input U and output Y as in (7a) and (7b), respectively. Let $\Sigma^L = (F, G, H, J)$ be a corresponding (minimal) 2D invariant separable *Roesser* model representation, and suppose that the horizontal state vector X^h and the vertical state vector X^v have both an even number of components, say $2n_h$ and $2n_v$, respectively. Furthermore, decompose matrices F, G, H and J accordingly to what as been done in eqs. (10) to (12). Based on this decomposition, construct matrices $\widetilde{\mathcal{W}_1}$, $\widetilde{\mathcal{W}_2}$, $\widetilde{\mathcal{W}_3}$ and $\widetilde{\mathcal{W}_4}$, similar to what was done in (13), (15), (18a) and (18b), respectively. We shall show that if these matrices satisfy the rank conditions (14), (16) and (20), then it is possible to construct a periodic 2D *Roesser* Σ state space model that induces Σ^L. Such periodic model is then a periodic state space realization of the original periodic i/o system. Define the matrix $\widetilde{\mathcal{W}_1}$ as in equation (13), and assume that

$$\operatorname{rank} \widetilde{\mathcal{W}_1} \leqslant n_h \ .$$

Decompose this matrix as

$$\widetilde{\mathcal{W}_1} =: \left[\begin{array}{c} \widetilde{L}_{1,1} \\ \hline \widetilde{L}_{1,2} \end{array} \right] \left[\begin{array}{c|c} \widetilde{R}_{1,1} & \widetilde{R}_{1,2} \end{array} \right], \tag{21}$$

where the block–matrix $\widetilde{L}_{1,2}$ is of size $(n_v+1) \times n_h$ whilst matrix $\widetilde{R}_{1,2}$ is a column-matrix of size n_h.

Now, define matrix $\widetilde{\mathcal{W}_2}$ similarly to what is done for $\mathcal{W}_2$ in equation (15), but using the matrix $\widetilde{L}_{1,2}$ obtained in (21) instead of $L_{1,2}$. Assume that

$$\operatorname{rank} \widetilde{\mathcal{W}_2} \leqslant n_v$$

and decompose

$$\widetilde{\mathcal{W}_2} =: \underbrace{\left[\begin{array}{c} \widetilde{L}_{2,1} \\ \hline \widetilde{L}_{2,2} \end{array} \right]}_{n_v \text{ columns}} \left[\begin{array}{c|c|c} \widetilde{R}_{2,1} & \widetilde{R}_{2,2} & \widetilde{R}_{2,3} \end{array} \right],$$

where $\widetilde{L}_{2,1}$ and $\widetilde{R}_{2,2}$ are (n_v)–square matrices, $\widetilde{R}_{2,1}$ is a $(n_v \times n_h)$ matrix while $\widetilde{L}_{2,2}$ is a row–matrix and $\widetilde{R}_{2,3}$ is a column–matrix.

Finally, define matrices $\widetilde{\mathcal{W}_3}$ and $\widetilde{\mathcal{W}_4}$ similarly to what is done for $\mathcal{W}_3$ and $\mathcal{W}_4$ in eqs. (17) and (18). Assume that

$$\operatorname{rank} \widetilde{\mathcal{W}_3} \leqslant n_v \quad \text{and} \quad \operatorname{rank} \widetilde{\mathcal{W}_4} \leqslant n_h$$

and decompose

$$\widetilde{\mathcal{W}_3} =: \underbrace{\left[\begin{array}{c} \widetilde{L}_{3,1} \\ \hline \widetilde{L}_{3,2} \end{array} \right]}_{n_v \text{ columns}} \left[\begin{array}{c|c|c} \widetilde{R}_{3,1} & \widetilde{R}_{3,2} & \widetilde{R}_{3,3} \end{array} \right]$$

and

$$\widetilde{\mathcal{W}_4} =: \underbrace{\left[\begin{array}{c} \widetilde{L}_{4,1} \\ \hline \widetilde{L}_{4,2} \\ \hline \widetilde{L}_{4,3} \end{array} \right]}_{n_h \text{ columns}} \left[\begin{array}{c|c} \widetilde{R}_{4,1} & \widetilde{R}_{4,2} \end{array} \right],$$

where $\widetilde{L}_{3,1}$ and $\widetilde{R}_{3,3}$ are (n_v)–square matrices, $\widetilde{L}_{4,1}$ and $\widetilde{R}_{4,1}$ are (n_h)–square matrices, $\widetilde{L}_{4,3}$ and $\widetilde{R}_{3,1}$ are $(n_v \times n_h)$ matrices while $\widetilde{L}_{3,2}$ and $\widetilde{L}_{4,2}$ are row–matrices and $\widetilde{R}_{3,2}$ and $\widetilde{R}_{4,2}$ are column–matrices.

Now, assume that, in the decomposition that emerges from eqs. (11) and (12), the blocks $F_{12}^{hh,vv}$, $F_{21}^{hh,vv}$, F^{hv}, G_{13}^h, G_{14}^h, G_{21}^h, G_{22}^h, G_{12}^v, G_{14}^v, H_{12}^h, H_{22}^h, H_{12}^v, H_{21}^v, H_{32}^v, H_{41}^v, J_{12}, J_{13}, J_{14}, J_{23}, J_{24}, J_{32} and J_{34} are null, and define a $(2,2)$–periodic separable SISO *Roesser* model of dimension $(n_h + n_v)$ $\Sigma(\cdot,\cdot) = (A(\cdot,\cdot), B(\cdot,\cdot), C(\cdot,\cdot), D(\cdot,\cdot))$, where the matrices in eqs. (2) and (3) are given by:

$$\left[\begin{array}{c|c} A(0,0) & B(0,0) \end{array} \right] = \left[\begin{array}{cc|c} \widetilde{R}_{1,1} & 0 & \widetilde{R}_{1,2} \\ \widetilde{R}_{3,1} & \widetilde{R}_{3,3} & \widetilde{R}_{3,2} \end{array} \right]$$

$$\left[\begin{array}{c|c} C(0,0) & D(0,0) \end{array} \right] = \left[\begin{array}{cc|c} H_{11}^h & H_{11}^v & J_{11} \end{array} \right]$$

$$\left[\begin{array}{c|c} A(1,0) & B(1,0) \end{array} \right] = \left[\begin{array}{cc|c} \widetilde{L}_{1,1} & 0 & G_{21}^h \\ \widetilde{R}_{2,1} & \widetilde{R}_{2,2} & \widetilde{R}_{2,3} \end{array} \right]$$

$$\left[\begin{array}{c|c} C(1,0) & D(1,0) \end{array} \right] = \left[\begin{array}{cc|c} \widetilde{L}_{1,2} & H_{22}^v & J_{22} \end{array} \right]$$

$$\left[\begin{array}{c|c} A(0,1) & B(0,1) \end{array} \right] = \left[\begin{array}{cc|c} \widetilde{R}_{4,1} & 0 & \widetilde{R}_{4,2} \\ F_{12}^{vh} & \widetilde{L}_{3,1} & G_{13}^v \end{array} \right]$$

$$\left[\begin{array}{c|c} C(0,1) & D(0,1) \end{array} \right] = \left[\begin{array}{cc|c} H_{32}^h & \widetilde{L}_{3,2} & J_{33} \end{array} \right]$$

$$\left[\begin{array}{c|c} A(1,1) & B(1,1) \end{array} \right] = \left[\begin{array}{cc|c} \widetilde{L}_{4,1} & 0 & G_{24}^h \\ \widetilde{L}_{4,3} & \widetilde{L}_{2,1} & G_{24}^v \end{array} \right]$$

and

$$\left[\begin{array}{c|c} C(1,1) & D(1,1) \end{array} \right] = \left[\begin{array}{cc|c} \widetilde{L}_{4,2} & \widetilde{L}_{2,2} & J_{44} \end{array} \right],$$

(where the matrices are suitably partitioned according to the sizes of the horizontal state (n_h), the vertical state (n_v), the input (1) and the output (1)). It is not difficult to check that the obtained $(2,2)$–periodic separable *Roesser* model $\Sigma(\cdot,\cdot)$ induces the invariant separable *Roesser* model Σ^L.

This leads to the following result.

Theorem 4.1: Let $\Sigma^L = (F, G, H, J)$ be a 2D invariant separable *Roesser* model. Then Σ^L is induced by a 2D $(2,2)$–periodic separable SISO *Roesser* model if and only if the following conditions are satisfied:

1) In Σ^L, the horizontal state has size $2n_h$ (for some $n_h \in \mathbb{N}$), the vertical state has size $2n_v$ (for some $n_v \in \mathbb{N}$); moreover the number of inputs and the number of outputs are equal to 4.
2) Considering the previously defined notations:
   - 2.1) $\operatorname{rank} \widetilde{\mathcal{W}}_1 \leqslant n_h$
   - 2.2) $\operatorname{rank} \widetilde{\mathcal{W}}_2 \leqslant n_v$
   - 2.3) $\operatorname{rank} \widetilde{\mathcal{W}}_3 \leqslant n_v$ and $\operatorname{rank} \widetilde{\mathcal{W}}_4 \leqslant n_h$
   - 2.4) F_{12}^{hh}, F_{21}^{hh}, F_{11}^{hv}, F_{12}^{hv}, F_{21}^{hv}, F_{22}^{hv}, F_{12}^{vv}, F_{21}^{vv}, G_{13}^{h}, G_{14}^{h}, G_{21}^{h}, G_{22}^{h}, G_{12}^{v}, G_{14}^{v}, H_{12}^{h}, H_{22}^{h}, H_{12}^{v}, H_{21}^{v}, H_{32}^{v}, H_{41}^{v}, J_{12}, J_{13}, J_{14}, J_{23}, J_{24}, J_{32} and J_{34} are null matrices. $\diamond$

Proof: The necessity of conditions (2) follows from the considerations that led to conditions (14), (16) and (20).

On the other hand, when the conditions of Theorem 4.1 are satisfied, a 2D $(2,2)$–periodic separable *Roesser* model $\Sigma(\cdot,\cdot)$ that induces Σ^L can be determined as explained in the considerations preceding the theorem.

Now, given a 2D separable $(2,2)$–periodic SISO system whose lifted version is realized by Σ^L, it is clear that the input/output trajectories generated by $\Sigma(\cdot,\cdot)$ coincide with the ones generated by the $(2,2)$–periodic input/output system. In other words, $\Sigma(\cdot,\cdot)$ is a $(2,2)$–periodic separable *Roesser* model realization of the original $(2,2)$–periodic input/output system, as desired. ■

It is worth noticing that if Σ^L is a minimal realization, so will be $\Sigma(\cdot,\cdot)$, as a lower dimensional periodic *Roesser* model realization $\widetilde{\Sigma}(\cdot,\cdot)$ would allow to construct a lower dimensional invariant *Roesser* model realization $\widetilde{\Sigma}^L$.

V. Conclusion

A procedure to realize separable periodic 2D SISO systems by means of separable periodic 2D *Roesser* state-space models was presented in this paper. Our method rests on the possibility of suitably decomposing certain matrices obtained from the invariant separable *Roesser* model realization of the lifted invariant formulation of the original periodic input/output system. This constitutes a drawback, since the decomposition of the aforementioned matrices is not independent from the particular invariant state space realization chosen for the invariant input/output system (even if that realization is minimal). One of the lines of future research is to investigate this situation, starting by looking for some particular cases where this issue may not arise.

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