2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

A certificate of global asymptotic stability for planar polynomial systems

Laura Menini, Corrado Possieri and Antonio Tornambe

***Abstract*— An algorithmic procedure is given for obtaining a certificate of global asymptotic stability for planar polynomial systems having the origin as equilibrium point. The procedure is not Lyapunov based and uses methods from Algebraic Geometry to study the sign of polynomial functions.**

I. INTRODUCTION

The Global Asymptotic Stability (GAS) of the equilibrium point of a planar non-linear system $\dot{x} = f(x)$, having just one equilibrium point, can be deduced by Lyapunov techniques (see, *e.g.*, Theorem 56 in Sec. 5.3 of [1]); however, as well known, building a Lyapunov function, and, in particular, a "global" one, it is not an easy task. Starting from some past contributions [2], [3], for planar systems, conditions based on the positivity of some functions (see [4], [5], [6]) have been discovered. Here, the sufficient conditions reported in [4], known as the Markus-Yamabe conjecture, are used in view of their simplicity. Such a result guarantees GAS if the eigenvalues of the Jacobian matrix $\frac{\partial f}{\partial x}$ have negative real part for all $x \in \mathbb{R}^2$. The objective of this paper is the application of algebraic geometry techniques to give an exact certificate of GAS, when the vector field $f(x)$ is polynomial.

Extensions to more general classes of systems, like rational systems, can be derived, and also many of the more powerful sufficient conditions extending the Markus-Yamabe conjecture can be dealt with by using the methods developed in Sections III and IV.

II. GAS OF PLANAR SYSTEMS AND NOTATION

Consider a planar non-linear system

$$\dot{x} = f(x), \tag{1}$$

where $x \in \mathbb{R}^2$, $x = [\begin{array}{cc} x_1 & x_2 \end{array}]^\top$, and $f(\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ is of class C^1 on $\mathbb{R}^2$. Assume $f(0) = 0$, and let $J_f(x) := \frac{\partial f(x)}{\partial x}$.

Theorem 1 (see Theorem B of [4])**.** *If the two eigenvalues of $J_f(x)$ have negative real part for all $x \in \mathbb{R}^2$, then the equilibrium point $x = 0$ of (1) is GAS.*

Remark 1. *For each $x \in \mathbb{R}^2$, the two eigenvalues of $J_f(x)$ have negative real part if and only if $\det(J_f(x)) > 0$ and $\mathrm{trace}(J_f(x)) < 0$.*

Let $g(x) : \mathbb{R}^2 \to \mathbb{R}$ be any function having the whole $\mathbb{R}^2$ as domain, and $g(\mathbb{R}^2)$ as co-domain.

(i) If any, the *infimum value* (or *infimum* or *greatest lower bound*) of $g(x)$ over $\mathbb{R}^2$ is the greatest element of $\mathbb{R}$ that is less than or equal to all the elements of $g(\mathbb{R}^2)$. If $g(x)$ does not admit an infimum over $\mathbb{R}^2$, then it is said to be *unbounded from below* over $\mathbb{R}^2$.

(ii) If any, the *minimum value* of $g(x)$ over $\mathbb{R}^2$ is the minimum element of $g(\mathbb{R}^2)$.

Corollary 1. *The equilibrium point $x = 0$ of (1) is GAS if the following three conditions hold:*
(i) $\det(J_f(x))$ and $-\mathrm{trace}(J_f(x))$ both have finite infimum values over $\mathbb{R}^2$,
(ii) such infimum values are greater than or equal to zero,
(iii) zero is not the minimum value of either $\det(J_f(x))$ or $-\mathrm{trace}(J_f(x))$.

Note that the conditions for GAS in Corollary 1 are only sufficient, and are quite conservative; however, they can be easily used to study classes of planar systems.

Example 1. *Let $f(x) = [\begin{array}{cc} \phi(x_1, x_2) & -x_2 \end{array}]^\top$, where $\phi(x_1, x_2)$ is a bivariate polynomial. It can be seen that $\det(J_f(x)) = -\frac{\partial \phi}{\partial x_1}$ and $-\mathrm{trace}(J_f(x)) = -\frac{\partial \phi}{\partial x_1} + 1$; therefore, the system is GAS if $-\frac{\partial \phi}{\partial x_1} > 0$ for all $x \in \mathbb{R}^2$.*

Example 2. *Let $f(x) = [\begin{array}{cc} \phi(x_1, x_2) & \pm x_1 \end{array}]^\top$, where $\phi(x_1, x_2)$ is a bivariate polynomial. It can be seen that $\det(J_f(x)) = \mp\frac{\partial \phi}{\partial x_2}$ and $-\mathrm{trace}(J_f(x)) = -\frac{\partial \phi}{\partial x_1}$; the system is GAS if $\mp\frac{\partial \phi}{\partial x_2} > 0$ and $-\frac{\partial \phi}{\partial x_1} > 0$ for all $x \in \mathbb{R}^2$.*

Example 3. *Let $\phi(x_1, x_2)$ be a bivariate polynomial and $f(x) = [\begin{array}{cc} -x_1 + \phi(x_1, x_2) & -x_2 + \phi(x_1, x_2) \end{array}]^\top$. It can be seen that $\det(J_f(x)) = 1 - \left(\frac{\partial \phi}{\partial x_1} + \frac{\partial \phi}{\partial x_2}\right)$ and $-\mathrm{trace}(J_f(x)) = 2 - \left(\frac{\partial \phi}{\partial x_1} + \frac{\partial \phi}{\partial x_2}\right)$; therefore, the system is GAS if $1 - \left(\frac{\partial \phi}{\partial x_1} + \frac{\partial \phi}{\partial x_2}\right) > 0$ for all $x \in \mathbb{R}^2$.*

Example 4. *Let $\phi(x_1, x_2)$ be a bivariate polynomial, homogeneous of degree $d > 0$,* i.e.*, such that $\left(x_1\frac{\partial \phi}{\partial x_1} + x_2\frac{\partial \phi}{\partial x_2}\right) = d\,\phi$. Let $f(x) = [\begin{array}{cc} -x_1\phi(x_1, x_2) & -x_2\phi(x_1, x_2) \end{array}]^\top$. It can be seen that $\det(J_f(x)) = (1+d^2)\phi^2$ and $-\mathrm{trace}(J_f(x)) = (2 + d)\phi$; the system is GAS if $\phi > 0$ for all $x \in \mathbb{R}^2$.*

To give an *exact* certificate of GAS using Corollary 1 it is assumed that the entries of f are polynomials. Basic notions from Algebraic Geometry, including Gröbner bases and the Elimination Theorem, are taken from [7] and recalled in [8]-[13]; more advanced concepts are posponed to Section IV.

Notation. Let $\mathbb{K}$ be any field of characteristic zero (*i.e.*, including the field of rational numbers $\mathbb{Q}$), and let $\overline{\mathbb{K}}$ be its algebraic closure; if $\mathbb{K}$ is either $\mathbb{Q}$, or one of its algebraic extensions $\mathbb{Q}_e$ (to include the case of algebraic parameters), or the field of rational functions, with rational coefficients, of

L. Menini and A. Tornambe are with the Dipartimento di Ingegneria Civile e Ingegneria Informatica, Univ. Roma "Tor Vergata", Roma, Italy.
C. Possieri is with the Dipartimento di Elettronica e Telecomunicazioni, Politecnico di Torino, Torino, Italy.
[menini,tornambe]@disp.uniroma2.it, possieri@ing.uniroma2.it

some real, non-algebraic, parameters, then `Macaulay` [14] can be used for the computations needed. The variables x_1, x_2, and auxiliary variables, belong to the *ground field* denoted by $\mathbb{H}$, which will be either $\mathbb{H} = \mathbb{R}$ or $\mathbb{H} = \mathbb{C}$ Some derivations will need auxiliary variables in addition to x_1 and x_2; therefore, to recall the notation, $x = [\begin{array}{ccc} x_1 & \ldots & x_n \end{array}]^\top$. Let $\mathbb{K}[x]$ be the *ring* of all the polynomials in x, and $\mathbb{K}(x)$ be the *field* of all the rational functions in x, both with coefficients in $\mathbb{K}$. Given $p_1, \ldots, p_m \in \mathbb{K}[x]$, the set

$$\langle p_1, \ldots, p_m \rangle := \{\textstyle\sum_{i=1}^{m} q_i p_i, \quad q_i \in \mathbb{K}[x],\, i = 1, \ldots, m\}$$

is the *ideal* in $\mathbb{K}[x]$ *generated* by $p_1, \ldots, p_m$, and the set

$$\mathbb{V}_{\mathbb{H}^n}(p_1, \ldots, p_m) := \{x \in \mathbb{H}^n : p_i(x) = 0,\, i = 1, \ldots, m\},$$

is the *affine variety* in $\mathbb{H}^n$ generated by $p_1, \ldots, p_m$. If $\mathcal{I} = \langle p_1, \ldots, p_m \rangle$, then $\mathbb{V}_{\mathbb{H}^n}(\mathcal{I})$ stands for $\mathbb{V}_{\mathbb{H}^n}(p_1, \ldots, p_m)$; the points in $\mathbb{V}_{\mathbb{H}^n}(\mathcal{I})$ are referred to as the roots of the ideal $\mathcal{I}$.

For $\mathcal{I}_a := \langle p_1, \ldots, p_{m_a} \rangle$ and $\mathcal{I}_b := \langle q_1, \ldots, q_{m_b} \rangle$ in $\mathbb{K}[x]$,

$$\mathcal{I}_a + \mathcal{I}_b := \langle p_1, \ldots, p_{m_a}, q_1, \ldots, q_{m_b} \rangle.$$

An ideal that can be generated by one element is *principal*; if s is scalar, each ideal in $\mathbb{K}[s]$ is principal. If s is scalar and $p \in \mathbb{K}[s]$, its *square-free part* is $\mathrm{SF}(p(s)) := p(s)/\gcd\big(p(s), \partial p(s)/\partial s\big)$.

Theorem 2 (Elimination Theorem, see Theorem 2 at page 122 of [7])**.** *Let $j \in \{1, \ldots, n-1\}$. Let $\mathcal{I}$ be an ideal in $\mathbb{K}[x]$ and let $\mathcal{G}_{\mathcal{I}}$ be the reduced Gröbner basis of $\mathcal{I}$ with respect to the Lex monomial order $\succ$, with $x_1 \succ x_2 \succ \ldots \succ x_n$. Then, $\mathcal{G}_{\mathcal{I}} \cap \mathbb{K}[x_{j+1}, \ldots, x_n]$ is the reduced Gröbner basis of $\mathcal{I}_j$ with respect to the Lex monomial order $\succ$, with $x_{j+1} \succ \ldots \succ x_n$.*

III. Computation of the infimum/minimum

Consider the problem of computing the minimum value of $p \in \mathbb{K}[x_1, x_2]$ over $\mathbb{R}^2$. A *critical point* (candidate to be a minimum value) of p is any pair $(x_1^\star, x_2^\star)$ such that $\frac{\partial p}{\partial x_i}\Big|_{(x_1,x_2)=(x_1^\star,x_2^\star)} = 0$, $i = 1, 2$; the image $p^\star = p(x_1^\star, x_2^\star)$ of a critical point through p is called a *critical value* of p. The critical values can be computed through algebraic geometry, by the Elimination Theorem [7]. Define the ideal $\overline{\mathcal{I}} = \langle \frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, P - p \rangle$ in $\mathbb{K}[x_1, x_2, P]$, where P is an additional variable. Since P is scalar, the elimination ideal $\mathcal{I} \cap \mathbb{K}[P]$ is principal, whence there is $q_C \in \mathbb{K}[P]$ such that $\mathcal{I} \cap \mathbb{K}[P] = \langle q_C(P) \rangle$; the critical values, if any, are real roots of $q_C(P)$, and if p admits a minimum value p°, then $q_C(p^\circ) = 0$, necessarily. The following example shows that the smallest critical value need not be the minimum of p.

Example 5. *Consider the following polynomial in $\mathbb{Q}[x_1, x_2]$: $p(x_1, x_2) = x_1^2 x_2^2 + x_1^2 - 2x_1 x_2^2 - 4x_1 x_2 - 2x_1 + x_2^2 + 4x_2 + 5$. The reduced Gröbner basis of $\overline{\mathcal{I}} = \langle \frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, P - p \rangle$ with respect to the Lex monomial order $\succ$, with $x_1 \succ x_2 \succ P$, is constituted by $p_1 = P - 4$, $p_2 = x_2$, $p_3 = x_1 - 1$. Then $q(P) = P - 4$, $P_1^\star = 4$ is the only critical value of p and $(x_1^\star, x_2^\star) = (1, 0)$ is the corresponding critical point. Since p can be rewritten as $p(x_1, x_2) = (x_1 - 1)^2 + (2 - (x_1 - 1)x_2)^2$, one has that p is bounded from below; in addition, taking $(x_1^\ell, x_2^\ell) = (\frac{1+\ell}{\ell}, 2\ell)$, one has $p(x_1^\ell, x_2^\ell) = \frac{1}{\ell^2}$, which tends to 0 as $\ell \to +\infty$. The greatest lower bound of p is $p^\star = \lim_{\ell \to +\infty} p(x_1^\ell, x_2^\ell) = 0$, which shows that $P_1^\star = 4$ is not the minimum value of p over $\mathbb{R}^2$; in particular, p does not take a minimum over $\mathbb{R}^2$, but only an* infimum value.

The following definition is borrowed from [15], where it is shown that the critical values are the candidates to be minimum values, whereas the generalized critical values are the candidates to be infimum values.

Definition 1. *Let $p \in \mathbb{K}[x_1, x_2]$ and $J_p(x) := \frac{\partial p(x)}{\partial x}$.*

(1.1) $p^\star$ is a critical value *of p if there is a* critical point *$x^\star \in \mathbb{R}^n$ such that $p^\star = p(x^\star)$ and $J_p(x^\star) = 0$;*

(1.2) $p^{\star\star}$ is a generalized critical value *of p if there is a sequence of points $x^\ell \in \mathbb{R}^2$ such that $\lim_{\ell \to +\infty} p(x^\ell) = p^{\star\star}$, $\lim_{\ell \to +\infty} \|x^\ell\| = +\infty$ and $\lim_{\ell \to +\infty} \|x^\ell\| \cdot \|J_p(x^\ell)\| = 0$;*

(1.3) let $\mathcal{C}_v^\star$ and $\mathcal{C}_v^{\star\star}$ be the sets of all $p^\star$ and $p^{\star\star}$, respectively, and let $\mathcal{E}_v^* = \mathcal{C}_v^\star \cup \mathcal{C}_v^{\star\star}$. The elements of $\mathcal{E}_v^*$ are referred to as generalized extremal values *and, if not distinguished between critical values and generalized critical values, they are represented as p^*.*

By the results in [15], and the elimination theory, a polynomial q_{gc} can be computed such that the finite variety $\mathbb{V}_{\mathbb{R}}(q_{gc})$ contains $\mathcal{E}_v^*$. This can be done as follows.

Algorithm 1 (Computation of q_{gc})**.**

Step 1 *Define the following ideal in $\mathbb{K}[x_1, x_2, P_1, P_2, Q_{1,1}, Q_{1,2}, Q_{2,1}, Q_{2,2}, P]$, where the P_i's, the $Q_{i,j}$'s and P are additional variables:*

$$\mathcal{I} = \langle \tfrac{\partial p(x)}{\partial x_1} - P_1, \tfrac{\partial p(x)}{\partial x_2} - P_2, x_1 \tfrac{\partial p(x)}{\partial x_1} - Q_{1,1},$$
$$x_1 \tfrac{\partial p(x)}{\partial x_2} - Q_{1,2}, x_2 \tfrac{\partial p(x)}{\partial x_1} - Q_{2,1}, x_2 \tfrac{\partial p(x)}{\partial x_2} - Q_{2,2}, P - p(x) \rangle.$$

Step 2 *Compute the elimination ideal:*

$$\mathcal{I}_a = \mathcal{I} \cap \mathbb{K}[P_1, P_2, Q_{1,1}, Q_{1,2}, Q_{2,1}, Q_{2,2}, P].$$

Step 3 *Define $\mathcal{I}_b = \mathcal{I}_a + \langle P_1, P_2, Q_{1,1}, Q_{1,2}, Q_{2,1}, Q_{2,2} \rangle$.*

Step 4 *Compute the elimination ideal $\mathcal{I}_{gc} = \mathcal{I}_b \cap \mathbb{K}[P]$; since it is principal, there is $q_{gc} \in \mathbb{K}[P]$ such that $\mathcal{I}_{gc} = \langle q_{gc}(P) \rangle$.*

Note that $\mathcal{I}_{gc}$ need not coincide with $\mathcal{I}_a \cap \mathbb{K}[P]$. It is shown in [15] that the generalized critical values of p are roots of q_{gc}, *i.e.*, $\mathcal{E}_v^* \subseteq \mathbb{V}_{\mathbb{R}}(q_{gc})$. The elements of the finite set $\mathbb{V}_{\mathbb{R}}(q_{gc})$, *i.e.*, the real roots of q_{gc}, are called candidate generalized extremal values. Here, an approximate computation of such roots will be sufficient.

Example 6. *Consider again Example 5. By Algorithm 1, it can be computed that $\mathcal{I}_{gc} = \langle P(P-4) \rangle$; hence, $q_{gc}(P) = P(P-4)$. As expected, the critical value $p^\star = 4$ and the generalized critical value $p^{\star\star} = 0$ are both in $\mathbb{V}_{\mathbb{R}}(q_{gc})$.*

Let

$$\mathbb{V}_{\mathbb{R}}(q_{gc}) = \{p_1^*, p_2^*, \ldots, p_\sigma^*\}, \tag{2}$$

where $p_1^* < p_2^* < \ldots < p_\sigma^*$, and the p_i^*'s are the candidate generalized extremal values. Let $p_0^* = -\infty$ and $p_{\sigma+1}^* = +\infty$.

Since $\mathbb{R}^2$ is path-connected, the set $p(\mathbb{R}^2)$ (*i.e.*, the set of values that $p(x)$ takes over $\mathbb{R}^2$) is an interval of $\mathbb{R}$, as more formally stated in the following Facts (F.1)-(F.3).

(F.1) $p(\mathbb{R}^2)$ is an interval of $\mathbb{R}$, and the two boundary points of the interval belong to $\mathbb{V}_{\mathbb{R}}(q_{gc}) \cup \{p_0^*, p_{\sigma+1}^*\}$. This means that there are $i_1, i_2 \in \{0, 1, \ldots, \sigma+1\}$ such that

$$(p_{i_1}^*, p_{i_2}^*) \subseteq f(\mathbb{R}^2) \quad \text{and} \quad p(\mathbb{R}^2) \subseteq [p_{i_1}^*, p_{i_2}^*].$$

(F.2) If the index i_1 of Fact (F.1) is $i_1 = 0$, then $p(x)$ is unbounded from below over $\mathbb{R}^2$. Otherwise, if $p_{i_1}^*$ is a critical value, then $p^\circ = p_{i_1}^*$ is the minimum value of p over $\mathbb{R}^2$, whereas if $p_{i_1}^*$ is a generalized critical value, then $p_{i_1}^*$ is the infimum of $p(x)$ over $\mathbb{R}^2$.

(F.3) If the equation

$$p(x) = P. \tag{3}$$

has at least one real solution in x for $P = V_j$, being V_j any real number such that $V_j \in (p_{j-1}^*, p_j^*)$ for some $j \in \{1, \ldots, \sigma\}$, then it has at least one solution $\forall P \in (p_{j-1}^*, p_j^*)$.

Lemma 1. *Let* (2) *hold with q_{gc} computed as in Algorithm 1.*
(i) $p(x)$ is unbounded from below over $\mathbb{R}^2$ if and only if, for an arbitrary real number $V_1 < p_1^*$, equation (3) *with $P = V_1$ has at least one real solution in x.*
(ii) If j is such that, for two arbitrary real numbers $V_j \in (p_{j-1}^*, p_j^*)$ and $V_{j+1} \in (p_j^*, p_{j+1}^*)$, equation (3) *with $P = V_j$ has no real solution in x, whereas equation* (3) *with $P = V_{j+1}$ has at least one real solution in x, then p_j^* is the infimum of $p(x)$ over $\mathbb{R}^2$; in addition, if* (3) *has at least one real solution in x for $P = p_j^*$, then $p^\circ = p_j^*$ is the minimum value of $p(x)$ over $\mathbb{R}^2$.*

The lemma above can be used to obtain a certificate that a polynomial $p(x)$ satisfies $p(x) > 0$ for all $x \in \mathbb{R}^2$, as needed to apply Corollary 1, by means of the following procedure.

Algorithm 2 (An exact certificate: is $p(x) > 0$ over $\mathbb{R}^2$?)**.**
Step 1. *Compute the polynomial q_{gc} by using Algorithm 1.*
Step 2. *Compute the square free part of q_{gc},*

$$\overline{q}_{gc}(P) = \mathrm{SF}(q_{gc}(P));$$

$\overline{q}_{gc}$ has the same roots of q_{gc} but with multiplicity one (thus avoiding numeric difficulties due to multiple roots).
Step 3. *Isolate the σ roots of $\overline{q}_{gc}$,* i.e., *if $\overline{q}_{gc}(0) \neq 0$, find σ disjoint closed intervals $[\underline{p}_j, \overline{p}_j]$, $\underline{p}_j, \overline{p}_j \in \mathbb{Q}$, $j = 1, \ldots, \sigma$, such that $p_j^* \in (\underline{p}_j, \overline{p}_j)$ for any $j = 1, \ldots, \sigma$. If $\overline{q}_{gc}(0) = 0$, then one root of $\overline{q}_{gc}$ is zero, and is always known exactly, whence the corresponding interval collapses into the single point 0 and the remaining intervals are $\sigma - 1$ in number.*
Step 4. *Let $\overline{p}_0 = -\infty$ and $\underline{p}_{\sigma+1} = +\infty$. Starting with $j = 1$, check if equation* (3) *has at least one real solution in x for $P = V_j$, for some $V_j \in \mathbb{Q}$, $V_j \in (\overline{p}_{j-1}, \underline{p}_j)$. Increase j by one and repeat Step 4 until the first value of j for which the equation admits a real solution. Let $\overline{j} := j$.*
Step 5. *By Lemma 1, if $\overline{j} = 1$, then $p(x)$ is unbounded from below, and therefore the answer is NO. If $\overline{j} > 1$, then $p_{\overline{j}-1}^*$ is the infimum of $p(x)$. If $p_{\overline{j}-1}^*$ is negative, the answer is NO, whereas if $p_{\overline{j}-1}^*$ is positive then the answer is YES ($p(x) > 0$ over $\mathbb{R}^2$). If $p_{\overline{j}-1}^* = 0$, then a further test is needed: if equation* (3) *has at least one real solution in x for $P = 0$, the answer is NO (0 is a minimum of $p(x)$) whereas if there is no real solution, the answer is YES (0 is the infimum but not the minimum of $p(x)$ over $\mathbb{R}^2$).*

Step 3 can be performed by standard algorithms like `polylib::realroots(p, eps)` in MATLAB or by the Sturm test (Theorem 1.4 in [16]), which is also useful to count how many of the roots of $\overline{q}_{gc}(P)$ are negative/positive.

To complete Steps 4 and 5, it is necessary to have a certificate about the existence of a real solution of a polynomial scalar equation in two variables. This is not an easy task in general and its solution is developed in Section IV-B.

IV. Certificate for the existence of real roots

A. The Shape Lemma for zero-dimensional ideals

In this section, the n-dimensional variable x is considered.

The *radical* of $\mathcal{I}$, denoted $\sqrt{\mathcal{I}}$, is the set $\sqrt{\mathcal{I}} = \{p \in \mathbb{K}[x] : p^j \in \mathcal{I}$ for some $j \in \mathbb{Z}, j \geq 1\}$; $\sqrt{\mathcal{I}}$ is an ideal in $\mathbb{K}[x]$. Clearly, $\mathcal{I} \subset \sqrt{\mathcal{I}}$, but $\mathbb{V}_{\mathbb{H}^n}(\mathcal{I}) = \mathbb{V}_{\mathbb{H}^n}(\sqrt{\mathcal{I}})$. An ideal $\mathcal{I}$ is said to be a *radical ideal* if it coincides with $\sqrt{\mathcal{I}}$.

The ideal $\mathcal{I} = \langle p_1, \ldots, p_m \rangle$ is *zero-dimensional* if the set $\mathbb{V}_{\mathbb{C}^n}(p_1, \ldots, p_m)$ is finite; if $\mathcal{I}$ is radical and $\mathbb{V}_{\mathbb{C}^n}(p_1, \ldots, p_m)$ is constituted by d points, then $\mathcal{I}$ is said to have *degree d*. If $\mathcal{I}$ is not radical, the definition of degree must be amended to take into account the root multiplicities; roughly speaking one can imagine a radical ideal as an ideal without multiple roots. The ideal $\mathcal{I}$ is zero-dimensional if and only if the elimination ideals $\mathcal{I} \cap \mathbb{K}[x_i]$, $i = 1, \ldots, n$, are non-empty (and necessarily principal, being x_i scalar), $\mathcal{I} \cap \mathbb{K}[x_i] = \langle G_i(x_i) \rangle$, with $G_i(x_i)$ being non-constant, $i = 1, \ldots, n$. If the ideal $\mathcal{I}$ is zero-dimensional, then it is radical if and only if the polynomials $G_i(x_i)$ are *square-free*.

If the ideal $\mathcal{I}$ is zero-dimensional, then its radical can be computed as $\sqrt{\mathcal{I}} = \mathcal{I} + \langle \mathrm{SF}(G_1(x_1)), \ldots, \mathrm{SF}(G_n(x_n)) \rangle$.

Lemma 2 (Shape Lemma, Proposition 2.3 at page 15 of [16])**.** *Let $\mathcal{I} = \langle p_1, \ldots, p_m \rangle$ be a radical zero-dimensional ideal of degree d, in $\mathbb{K}[x_1, \ldots, x_n]$. If the d points in $\mathbb{V}_{\mathbb{C}^n}(\mathcal{I})$ have distinct x_n-coordinates, then the reduced Gröbner basis $\mathcal{G}_{\mathcal{I}}$ of $\mathcal{I}$ w.r.t. the Lex monomial order $\succ$, with $x_1 \succ \ldots \succ x_n$, is constituted by n polynomials, with the special* shape*:*

$$\mathcal{G}_{\mathcal{I}} = \{g_n(x), g_{n-1}(x), \ldots, g_1(x)\}, \tag{4a}$$

$$g_n(x) = h_n(x_n), \tag{4b}$$

$$g_{n-1}(x) = x_{n-1} - h_{n-1}(x_n), \ldots, g_1(x) = x_1 - h_1(x_n), \tag{4c}$$

where $h_i \in \mathbb{K}[x_n]$, $i = 1, \ldots, n$. In addition, $h_n(x_n)$ has degree d and $h_i(x_n)$, $i = 1, \ldots, n-1$, have degrees $< d$.

The reduced Gröbner basis (4) is said to be in the *Shape Lemma form* (4). Lemma 2 is very useful for computing (an approximation of) the elements of $\mathbb{V}_{\mathbb{R}^n}(p_1, \ldots, p_m)$, when $\mathbb{K}$ is a field contained in $\mathbb{R}$. The reduced Gröbner basis of a zero-dimensional ideal $\mathcal{I}$ need not be in the Shape Lemma form (4) if either $\mathcal{I}$ is not radical or there are two points in $\mathbb{V}_{\mathbb{C}^n}(\mathcal{I})$ with the same x_n-coordinate. The second obstruction can be eliminated by using an additional variable as described in the following.

Let $\mathcal{I} = \langle p_1, \dots, p_m \rangle$ be a radical zero-dimensional ideal of degree d, in $\mathbb{K}[x_1, \dots, x_n]$ and let $\mathbb{V}_{\mathbb{C}^n}(p_1, \dots, p_m) = \{x^1, \dots, x^d\}$, where $x^j = [\ x_1^j \ \ \dots \ \ x_n^j\ ]^\top$, $j = 1, \dots, d$, and d is the degree of $\mathcal{I}$; $q \in \mathbb{K}[x]$ is *separating for* $\mathcal{I}$ if $q(x^i) \neq q(x^j), \quad i, j \in \{1, \dots, n\}, i \neq j$. Almost all polynomials in $\mathbb{K}[x]$ are separating for $\mathcal{I}$; a "candidate" separating polynomial q can be simply obtained by randomly taking q in $\mathbb{K}[x]$. Let s be another variable and define the following extended ideal in $\mathbb{K}[x_1, \dots, x_n, s]$:

$$\mathcal{I}_e = \mathcal{I} + \langle s - q(x_1, \dots, x_n) \rangle.$$

Since q is separating for $\mathcal{I}$, the reduced Gröbner basis of $\mathcal{I}_e$ with respect to the Lex monomial order $\succ$, with $x_1 \succ \dots \succ x_n \succ s$, is in the Shape Lemma form (4):

$$\mathcal{G}_{\mathcal{I}_e} = \{h_{n+1}(s), x_n - h_n(s), \dots, x_1 - h_1(s)\}.$$

B. Computation of a real root of a bivariate polynomial

Consider the problem of computing at least one real root of a polynomial $p \in \mathbb{K}[x_1, x_2]$, *i.e.*, $\hat{x} \in \mathbb{R}^2$, $\hat{x} = [\ \hat{x}_1 \ \ \hat{x}_2\ ]^\top$, such that $p(\hat{x}) = 0$. W.l.g. , assume that the ideal $\langle p \rangle$ is radical (*i.e.*, $\langle p \rangle = \sqrt{\langle p \rangle}$) (otherwise $\langle p \rangle$ can be substituted by its radical). Here, it is needed only to guarantee that a real solution $\hat{x}$ exists or does not exist, but, instrumentally, it will be explained how to compute an approximation of $\hat{x}$.

Lemma 3. *Consider the radical ideal $\langle p \rangle$ in $\mathbb{K}[x_1, x_2]$. If $\langle p \rangle \neq \langle 1 \rangle$, then the equation $p(x_1, x_2) = 0$ has an infinite number of roots over $\mathbb{C}^2$.*

The Shape Lemma given in Section IV-A works well for zero-dimensional ideals; therefore, two methods will be given to compute a real root of p by using a related problem that has a finite number of solutions, if any. The variety $\mathbb{V}_{\mathbb{R}^2}(p)$, if not trivial, can be seen as a curve in $\mathbb{R}^2$; the first method finds its singular points, which, if any, are a finite set. The second method (to be used if there are not singular points, which is the generic case) is based on the solution of an ancillary minimization problem, whose candidate critical points are generically a finite subset of the roots of p.

The first method is based on the following definition of *singularity ideal*, based on the *critical ideal*.

Definition 2. *Let $\mathbb{H}$ be either $\mathbb{R}$ or its algebraic closure $\mathbb{C}$. The* critical ideal *of $p \in \mathbb{K}[x_1, \dots, x_n]$ is the following polynomial ideal in $\mathbb{K}[x_1, \dots, x_n]$:*

$$\mathcal{I}_C(p) = \langle \tfrac{\partial p}{\partial x_1}, \dots, \tfrac{\partial p}{\partial x_n} \rangle. \tag{5}$$

Each point in the critical variety *of p, $\mathcal{V}_C(p) = \mathbb{V}_{\mathbb{H}^n}(\frac{\partial p}{\partial x_1}, \dots, \frac{\partial p}{\partial x_n})$, if not empty, is a (real if $\mathbb{H} = \mathbb{R}$ or complex if $\mathbb{H} = \mathbb{C}$)* critical point *of the polynomial p.*

The singularity ideal *of $p \in \mathbb{K}[x_1, \dots, x_n]$ is the following ideal in $\mathbb{K}[x_1, \dots, x_n]$:*

$$\mathcal{I}_S(p) = \langle p \rangle + \mathcal{I}_C(p). \tag{6}$$

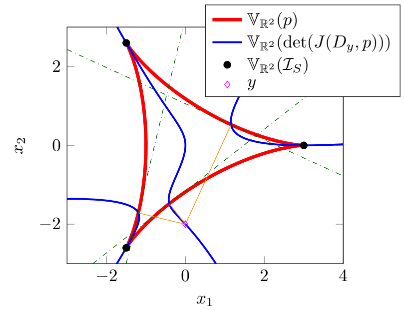

Fig. 1. The tricuspoid of Examples 7 and 8. The dashed lines are the lines tangent to the tricuspoid at the intersections with $\mathbb{V}_{\mathbb{R}^2}(\det(J(D_y, p)))$.

Each point in the singularity variety *of p, $\mathcal{V}_S(p) = \mathbb{V}_{\mathbb{H}^n}(p, \frac{\partial p}{\partial x_1}, \dots, \frac{\partial p}{\partial x_n})$, if not empty, is a* singularity point *of the polynomial p.*

Lemma 4. *Consider the radical ideal $\langle p \rangle$ in $\mathbb{K}[x_1, x_2]$ and assume that $\langle p \rangle \neq \langle 1 \rangle$. Since $\langle p \rangle$ is radical, one has that $\mathcal{I}_S(p)$ is zero-dimensional. Let $q \in \mathbb{K}[x_1, x_2]$ be a separating polynomial for the singularity ideal $\mathcal{I}_S(p)$. Consider the ideal $\mathcal{I}_e = \sqrt{\mathcal{I}_S(p)} + \langle s - q \rangle$ in $\mathbb{K}[x_1, x_2, s]$.*

(4.1) $\mathcal{I}_e$ is a zero-dimensional radical ideal in $\mathbb{K}[x_1, x_2, s]$;

(4.2) there exist three univariate polynomials $h_3, h_2, h_1 \in \mathbb{K}[s]$, with $\deg(h_3) > \deg(h_i)$, $i = 1, 2$, such that

$$\mathcal{G}_{\mathcal{I}_e} = \{h_3(s), x_2 - h_2(s), x_1 - h_1(s)\}$$

is the reduced Gröbner basis of $\mathcal{I}_e$ with respect to the Lex monomial order $\succ$, with $x_1 \succ x_2 \succ s$;

(4.3) if, according to the Sturm Theorem, $n_{(-\infty,+\infty)}$ is the number of real roots of $h_3(s)$, then $p(x_1, x_2) = 0$ has at least $n_{(-\infty,+\infty)}$ real solutions.

Lemma 4 can be actually used as a certificate of the existence of real roots of a scalar bivariate polynomial, when it presents real singularity points, as shown hereafter.

Example 7. *Consider the polynomial $p(x_1, x_2) = x_1^4 - 8x_1^3 + 2x_1^2x_2^2 + 18x_1^2 + 24x_1x_2^2 + x_2^4 + 18x_2^2 - 27$, which represents a* tricuspoid, i.e., *the red curve in Fig. 1. The singularity ideal $\mathcal{I}_S(p)$ is generated by p and by the following polynomials:*

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}x_1} &= 4x_1^3 - 24x_1^2 + 4x_1x_2^2 + 36x_1 + 24x_2^2, \\ \frac{\mathrm{d}p}{\mathrm{d}x_2} &= 4x_1^2x_2 + 48x_1x_2 + 4x_2^3 + 36x_2; \end{aligned}$$

since the reduced Gröbner basis of $\mathcal{I}_S(p)$ with respect to any monomial order is not $\{1\}$ and the ideal $\langle p \rangle$ is radical, the polynomial p has a finite number of singularities. The singularity ideal $\mathcal{I}_S(p)$ is not radical; it can be seen that $\sqrt{\mathcal{I}_S(p)}$ is generated by the following polynomials: $\{2x_1x_2 + 3x_2,\ 2x_1^2 - 3x_1 - 9,\ 8x_1x_2^2 - 54x_1 - 24x_2^2 + 162\}$. To compute the real singularities, the "candidate" separating polynomial for $\sqrt{\mathcal{I}_S(p)}$ is taken as $q(x_1, x_2) = x_1 + x_2$. Let $\mathcal{I}_e = \sqrt{\mathcal{I}_S(p)} + \langle s - q(x_1, x_2) \rangle$, where s is a new variable. The reduced Gröbner basis of $\mathcal{I}_e$ with respect to the Lex monomial order $\succ$, with $x_1 \succ x_2 \succ s$, is in the Shape Lemma form (4), $\mathcal{G}_{\mathcal{I}_e} = \{h_3(s), x_2 - h_2(s), x_1 - h_1(s)\}$, where: $h_3(s) = s^3 - \frac{27}{2}s + \frac{27}{2}$, $h_2(s) = -\frac{1}{3}s^2 + 3$, and

$h_1(s) = \frac{1}{3}s^2 + s - 3$. *By the Sturm Theorem, it can be seen that* h_3 *has three distinct real roots. The reals roots of* $h_3(s)$ *are* $r_1 = -\frac{3^{3/2}+3}{2}$, $r_2 = \frac{3^{3/2}-3}{2}$, $r_3 = 3$. *The corresponding singularity points of* p *are* $(-\frac{3}{2}, -\frac{3^{3/2}}{2})$, $(-\frac{3}{2}, \frac{3^{3/2}}{2})$ *and* $(3, 0)$, *respectively (marked by black points in Fig. 1). The equation* $p(x_1, x_2) = 0$ *has at least such three real roots.*

The case above is rare (almost all bivariate polynomials do not have real singular points), therefore a second method to find (a finite number of) real points in $\mathbb{V}_{\mathbb{R}^2}(p)$ is now given.

Let $x \in \mathbb{R}^2$ be any root of $p(x_1, x_2) = 0$. For any $y \in \mathbb{R}^2$, consider the half of the squared distance of y from x:

$$D_y(x_1, x_2) := \tfrac{1}{2}(x_1 - y_1)^2 + \tfrac{1}{2}(x_2 - y_2)^2. \tag{7}$$

The idea is to find, for fixed y, the real critical points of $D_y(x_1, x_2)$, along $p(x_1, x_2) = 0$. Let $\hat{D}_y(x_1, x_2, \lambda) := D_y(x_1, x_2) + \lambda p(x_1, x_2)$, where λ is the *Lagrange multiplier*. The *critical ideal* of $\hat{D}_y \in \mathbb{K}[x_1, x_2, \lambda]$ is

$$\begin{aligned} \mathcal{I}_C(\hat{D}_y) &= \langle \tfrac{\partial \hat{D}_y}{\partial \lambda}, \tfrac{\partial \hat{D}_y}{\partial x_1}, \tfrac{\partial \hat{D}_y}{\partial x_2} \rangle \\ &= \langle p, (x_1 - y_1) + \lambda \tfrac{\partial p}{\partial x_1}, (x_2 - y_2) + \lambda \tfrac{\partial p}{\partial x_2} \rangle. \end{aligned}$$

The two equations $(x_i - y_i) + \lambda \frac{\partial p}{\partial x_i} = 0, \quad i = 1, 2$, can be rewritten in matrix form as $b(\lambda) J(D_y, p) = 0$, where

$$\begin{aligned} b(\lambda) &= \begin{bmatrix} 1 & \lambda \end{bmatrix}, \\ J(D_y, p) &= \begin{bmatrix} x_1 - y_1 & x_2 - y_2 \\ \frac{\partial p}{\partial x_1} & \frac{\partial p}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial D_y}{\partial x_1} & \frac{\partial D_y}{\partial x_2} \\ \frac{\partial p}{\partial x_1} & \frac{\partial p}{\partial x_2} \end{bmatrix}; \end{aligned}$$

hence, such two equations have at least one solution over $\mathbb{C}$ only if $\det(J(D_y, p)) = 0$, and, in particular, one concludes

$$\begin{aligned} &\langle (x_1 - y_1) + \lambda \tfrac{\partial p}{\partial x_1}, (x_2 - y_2) + \lambda \tfrac{\partial p}{\partial x_2} \rangle \cap \mathbb{K}[x_1, x_2] \\ &= \langle (x_1 - y_1) \tfrac{\partial p}{\partial x_2} - (x_2 - y_2) \tfrac{\partial p}{\partial x_1} \rangle = \langle \det(J(D_y, p)) \rangle, \end{aligned}$$

whence $\mathcal{I}_C(\hat{D}_y) \cap \mathbb{K}[x_1, x_2] = \langle p, \det(J(D_y, p)) \rangle$. Note that, if p is not constant, then $\det(J(D_y, p))$ is not the zero polynomial, for almost all y. If y is considered fixed (so that, with probability zero, it can happen that $\det(J(D_y, p)) = 0$), one can take $\mathbb{K}$ as any field of characteristic zero (*e.g.*, $\mathbb{Q}$ or $\mathbb{Q}_e$); instead, if y is considered as a symbolic variable (so to consider "almost all" the values of y in $\mathbb{R}^2$), one can let $\mathbb{K} = \mathbb{Q}(y_1, y_2)$ or $\mathbb{K} = \mathbb{Q}_e(y_1, y_2)$.

Definition 3. *For any* y *such that* $\det(J(D_y, p)) \neq 0$, *the* critical ideal *and the* critical variety *of* D_y along $p = 0$ *are:*

$$\begin{gathered} \mathcal{I}_C(D_y, p) := \langle p, \det(J(D_y, p)) \rangle \\ \mathcal{V}_C(D_y, p) := \mathbb{V}_{\mathbb{H}^2}(p, \det(J(D_y, p))). \end{gathered}$$

Lemma 5. *Consider the radical ideal* $\langle p \rangle$ *in* $\mathbb{K}[x_1, x_2]$, $\langle p \rangle \neq \langle 1 \rangle$. *Let* $y \in \mathbb{R}^2$ *be such that* $\mathcal{I}_C(D_y, p)$ *is zero-dimensional. Let* $q \in \mathbb{K}[x_1, x_2]$ *be a separating polynomial for* $\mathcal{I}_C(D_y, p)$. *Consider the following ideal in* $\mathbb{K}[x_1, x_2, s]$*:*

$$\mathcal{I}_e = \sqrt{\mathcal{I}_C(D_y, p)} + \langle s - q \rangle.$$

(5.1) $\mathcal{I}_e$ *is a zero-dimensional radical ideal in* $\mathbb{K}[x_1, x_2, s]$*;*

(5.2) there exist three univariate polynomials $h_3, h_2, h_1 \in \mathbb{K}[s]$, *with* $\deg(h_3) > \deg(h_i)$, $i = 1, 2$, *such that*

$$\mathcal{G}_{\mathcal{I}_e} = \{h_3(s), x_2 - h_2(s), x_1 - h_1(s)\} \tag{8}$$

is the reduced Gröbner basis of $\mathcal{I}_e$ *with respect to the Lex monomial order* $\succ$, *with* $x_1 \succ x_2 \succ s$*;*

(5.3) the bivariate polynomial $p(x_1, x_2)$ *has real roots if and only if the univariate polynomial* $h_3(s)$ *has real roots;*

(5.4) if, according to the Sturm Theorem, $n_{(-\infty, +\infty)}$ *is the number of real roots of* $h_3(s)$, *then* $p(x_1, x_2) = 0$ *has at least* $n_{(-\infty, +\infty)}$ *real solutions.*

Remark 2. *Lemma 4 gives a sufficient condition for the existence of a real root of* p *(*i.e.*, the existence of a real singularity point), whereas Lemma 5 gives a necessary and sufficient condition for the existence of a real root of* p *(*i.e.*, the existence of a real root of* $h_3(s)$*). The hypotheses of Lemma 5 are very mild, since* $\langle p \rangle$ *is radical and* $\mathcal{I}_C(D_y, p)$ *is zero-dimensional for almost all* $y \in \mathbb{R}^2$*; moreover, almost all* q *are separating for* $\mathcal{I}_C(D_y, p)$.

Remark 3. *If* $(\hat{x}_1, \hat{x}_2)$ *is a singularity point of* p, *then* $\left.\frac{\partial p}{\partial x}\right|_{(x_1, x_2) = (\hat{x}_1, \hat{x}_2)} = 0$, *whence* $\det(J(D_y, p))|_{(x_1, x_2) = (\hat{x}_1, \hat{x}_2)} = 0$, *which shows that* $\mathcal{V}_S(p) \subseteq \mathcal{V}_C(D_y, p)$. *By computing the primary decomposition [7, Ch. 4] of* $\mathcal{I}_C(D_y, p)$, *one can separate the contribution of the singularity points from the others.*

Example 8. *Let* p *be as in Example 7;* $D_y(x_1, x_2)$ *in* (7) *is a polynomial in* $\mathbb{Q}(y_1, y_2)[x_1, x_2]$, *and* $\det(J(D_y, p))$ *is not the zero polynomial for any* y. *The ideal* $\mathcal{I}_C(D_y, p)$ *is not radical; one can compute the radical* $\sqrt{\mathcal{I}_C(D_y, p)}$ *of* $\mathcal{I}_C(D_y, p)$, *and its primary decomposition [7, Ch. 4], which is* $\sqrt{\mathcal{I}_C(D_y, p)} = \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3$, *where*
$\mathcal{I}_1 = \langle x_2, x_1 - 3 \rangle$, $\quad \mathcal{I}_2 = \langle 4x_2^2 - 27, 2x_1 + 3 \rangle$,
$\mathcal{I}_3 = \langle g_1(x_1, x_2), g_2(x_1, x_2) \rangle$,
$g_1(x_1, x_2) = x_2^3 + a_{1,1}x_2^2 + a_{1,2}x_2 + a_{1,3}$, $g_2(x_1, x_2) = x_1 + a_{2,1}x_2^2 + a_{2,2}x_2 + a_{2,3}$, *where the* $a_{i,j}$*'s are rational functions of* y_1, y_2. $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *do not depend on* y *and they are the primary decomposition of* $\sqrt{\mathcal{I}_S(p)}$, $\sqrt{\mathcal{I}_S(p)} = \mathcal{I}_1 \cap \mathcal{I}_2$. *Taking* $q(x_1, x_2) = x_1 + x_2$, *and defining* $\mathcal{I}_e = \mathcal{I}_3 + \langle s - q(x_1, x_2) \rangle$ *as an ideal in* $\mathbb{Q}(y_1, y_2)[x_1, x_2, s]$, *one has that the reduced Gröbner basis of* $\mathcal{I}_e$ *with respect to the Lex monomial order* $\succ$, *with* $x_1 \succ x_2 \succ s$, *is in the Shape Lemma form (4),* $\mathcal{G}_{\mathcal{I}_e} = \{h_3(s), x_2 - h_2(s), x_1 - h_1(s)\}$, *where* $h_3(s)$ *is a polynomial in* s *of degree* 3, *whence necessarily with one real root; the specializations of* y *that must be avoided are only those for which some denominators of the rational coefficients of the* h_i*'s are zero. For* $y = [0 \ \ {-2}]^\top$ *as in Fig. 1, the real roots of* h_3 *are three, as the three real solutions of* $p(x_1, x_2) = 0$, *which are the intersections of the tricuspoid (bold red curve,* $\mathbb{V}_{\mathbb{R}^2}(p)$*) with the three blue curves (*$\mathbb{V}_{\mathbb{R}^2}(\det(J(D_y, p)))$*) that are not singular points.*

V. Example of application to GAS analysis

Consider Example 1 with

$$\begin{aligned} \phi \;=\; & -x_1 \left(6\,a^2 + 3\,a^3 + 6\,a^2 x_2 + 3\,a^2 {x_2}^2 - 3\,x_1 a - \right. \\ & \left. 3\,x_1 a x_2 - 3\,x_1 a {x_2}^2 + {x_1}^2 {x_2}^2 + {x_1}^2\right), \end{aligned}$$

where a is a scalar parameter. GAS can be guaranteed by Corollary 1 for all the values of a for which the following polynomial is globally positive: $p = 2\,a^2 + a^3 + 2\,a^2x_2 + a^2{x_2}^2 - 2\,x_1a - 2\,x_1ax_2 - 2\,x_1a{x_2}^2 + {x_1}^2{x_2}^2 + {x_1}^2$,being $-\frac{\partial\phi}{\partial x_1} = 3p$. By using Algorithm 1:

$$q_{gc}(P) = P^2 - \left(2a^3 + a^2\right)P + a^6 + a^5.$$

It is easy to check that $q_{gc}(P) = (P - a^3)(P - a^3 - a^2)$; hence, the square free part of q_{gc} differs from q_{gc} only if $a = 0$ (in such a case $\overline{q}_{gc}(P) = P$). Consider first $a = 0$; in such a case, $p = x_1^2x_2^2 + x_1^2$ is zero for $x = 0$ and therefore for $a = 0$ GAS cannot be guaranteed by Corollary 1. For $a > 0$, the two roots of $q_{gc}(P)$ are both strictly positive, whence to guarantee GAS it is sufficient to guarantee that p is not unbounded from below. The number V_1 mentioned at Step 4 of Algorithm 2 can be simply taken as $V_1 = 0$. To test if equation $p(x_1, x_2) = 0$ has real solutions, the method based on Lemma 5 is used. Choosing, *e.g.*, $a = 0.9$ and $y = [1\ 1]^\top$, it is first verified that the ideal $\mathcal{I}_C(D_y, p)$ is already radical, secondly the "candidate" separating polynomial $q = x_1 + x_2$ is chosen, and then the reduced[1] Gröbner basis $\mathcal{G}_{\mathcal{I}_e}$ is computed, which is in the Shape Lemma form (8) with

$$\begin{aligned}\tfrac{h_3(s)}{1E10} \approx\, & 1786 - 6743\,s + 11403\,s^2 - 10584\,s^3 + 6044\,s^4 - \\ & 2919\,s^5 + 1894\,s^6 - 1189\,s^7 + 473\,s^8 - 101\,s^9 + 9\,s^{10},\end{aligned}$$

where only the most significant digits are shown. By applying the Sturm test in $(-\infty, +\infty)$ to $h_3(s)$, it can be seen that it has no real roots, so that for $a = 0.9$ the polynomial p is not unbounded from below, it has a strictly positive infimum (one of the roots of $q_{gc}(P)$) then Corollary 1 guarantees GAS.

For $a < -1$, the two roots of $q_{gc}(P)$ are strictly negative: by Fact F.1, it is impossible that p is always positive.

For $a \in (-1, 0)$, $q_{gc}(P)$ has the positive root $a^3 + a^2$, and the negative root a^3, hence the tests at Step 4 of Algorithm 2 are needed in order to see if p assumes negative values or not. Choosing, *e.g.*, $a = -0.9$, one has $a^3 = -0.729$, taking $V_1 = -1$ and using Lemma 5 to see if equation $p(x_1, x_2) = -1$ has real solutions, repeating the steps above one guarantees that p is not unbounded from below. Choosing then $V_2 = 0$ (in between the two roots of $q_{gc}(P)$), and repeating all the steps with $y = [0\ 0]^\top$, $\mathcal{G}_{\mathcal{I}_e}$ is computed, which is in the Shape Lemma form (8) with

$$\begin{aligned}\tfrac{h_3(s)}{1e10} \approx\, & 36 + 236\,s + 743\,s^2 + 1530\,s^3 + 2272\,s^4 + \\ & 2436\,s^5 + 1829\,s^6 + 926\,s^7 + 305\,s^8 + 65\,s^9 + 9\,s^{10}.\end{aligned}$$

By the Sturm test it is seen that $h_3(s)$ has two real roots, so that for $a = -0.9$ the polynomial p has the strictly negative infimum $a^3 = -0.729$. Then Corollary 1 cannot be used to guarantee GAS. Note that the values of the real roots of $h_3(s)$ are not needed. The fact that there must be two real roots is confirmed by Fig. 2, where the two critical points of D_y are the intersections between the red and the blue curve.

[1] Apart from clearing the denominators, which is standard in Macaulay.

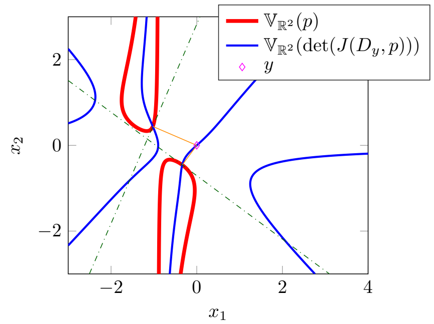

Fig. 2. Contour plot of p considered in Example V with $a = -0.9$.

VI. Conclusions

A certificate of global asymptotic stability is proposed for planar systems. A crucial step is a procedure to guarantee the existence (or the absence) of real solutions to a bivariate polynomial equation. The certificate is based on a known simple sufficient condition, but the methodology is more general and can be used to apply less conservative conditions, for higher order systems, to deal with rational systems or with systems with dynamics involving trigonometric functions.

References

[1] M. Vidyasagar, *Nonlinear systems analysis*. Englewood Cliffs,N.J.: Prentice-Hall, 1993.

[2] L. Markus, H. Yamabe, *et al.*, "Global stability criteria for differential systems," *Osaka Mathematical Journal*, vol. 12, no. 2, pp. 305–317, 1960.

[3] P. Hartman and C. Olech, "On global asymptotic stability of solutions of differential equations," *Transactions of the American Mathematical Society*, vol. 104, no. 1, pp. 154–178, 1962.

[4] C. Gutierrez, "A solution to the bidimensional global asymptotic stability conjecture," *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, vol. 12, no. 6, pp. 627 – 671, 1995.

[5] R. Feßler, "A proof of the two-dimensional markus-yamabe stability conjecture and a generalization," *Ann. Polon. Math*, vol. 62, no. 1, pp. 45–74, 1995.

[6] A. A. Glutsyuk, "A complete solution of the jacobian problem for vector fields on the plane," *Russian Mathematical Surveys*, vol. 49, no. 3, pp. 185–186, 1994.

[7] D. Cox, J. Little, and D. O'Shea, *Ideals, varieties, and algorithms*. New York: Springer Sci. Bus. Media, 2015.

[8] L. Menini and A. Tornambe, "On the use of algebraic geometry for the design of high-gain observers for continuous-time polynomial systems," vol. 19, 2014, pp. 43–48.

[9] ——, "On a Lyapunov equation for polynomial continuous-time systems," *International Journal of Control*, vol. 87, no. 2, pp. 393–403, 2014.

[10] ——, "Stabilization of polynomial nonlinear systems by algebraic geometry techniques," *IEEE Transactions on Automatic Control*, vol. 60, no. 9, pp. 2482–2487, 2015.

[11] L. Menini, C. Possieri, and A. Tornambe, "Switching signal estimator design for a class of elementary systems," *IEEE Trans. Autom. Control*, vol. 61, no. 5, pp. 1362–1367, 2016.

[12] ——, "Algebraic certificates of (semi)definiteness for polynomials over fields containing the rationals," *IEEE Tran. Autom. Control*, 2018.

[13] L. Menini, C. Possieri, and A. Tornambe, "Dead–beat regulation of mechanical juggling systems," *Asian J. Control*, vol. 20, no. 2, pp. 1–11, 2018.

[14] D. R. Grayson and M. E. Stillman, "Macaulay 2, a software system for research in algebraic geometry," 2017.

[15] K. Kurdyka, P. Orro, S. Simon, *et al.*, "Semialgebraic Sard theorem for generalized critical values," *Journal of differential geometry*, vol. 56, no. 1, pp. 67–92, 2000.

[16] B. Sturmfels, *Solving systems of polynomial equations*. Am. Math. Soc., 2002.