2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Roesser form of (wave) linear repetitive processes and structural stability

Olivier Bachelier
Univ. Poitiers, LIAS - ENSIP
2 rue Pierre Brousse, TSA 41105, F-86073 Poitiers, France
Email: olivier.bachelier@univ-poitiers.fr

Thomas Cluzeau
Univ. Limoges, CNRS, XLIM, UMR 7252
123 avenue Albert Thomas, F-87060 Limoges, France
Email: thomas.cluzeau@unilim.fr

***Abstract*—We use the algebraic analysis approach to multidimensional systems theory to study (wave) linear 2D discrete repetitive processes. In a previous work, we have proved that every linear 2D discrete repetitive process can be transformed into an equivalent (in the sense of algebraic analysis) explicit Roesser model. In the present paper we first investigate the conservation of the important notion of structural stability via this equivalence transformation. Then we extend the previous results to *wave* linear repetitive processes: we prove that such a general model can always be transformed into an equivalent implicit Roesser model which may be used to study stability properties of wave linear repetitive processes.**

I. Introduction

For a wide variety of applications studied in systems theory, one has to consider that the system dynamics propagate in two (or more) independent directions such as time and space variables or several distinct space variables. Consequently, people have been interested in generalizing tools that have been developed for 1D models to handle 2D or even multidimensional (nD) models. The present paper is concerned with systems of linear functional equations whose unknowns are bivariate sequences depending on two discrete independent variables.

Given a linear 2D discrete system, one can always perform changes of (dependent) variables in order to modify the form of the equations of the system. A convenient state-space representation of linear 2D discrete models is the so-called Roesser model [25]. The main underlying reason of the success of the Roesser model is that it is very close to a 1D model which has interesting advantages when generalizing tools designed for 1D models to the 2D case. For instance, one can construct state feedback control laws to structurally stabilize Roesser models [3]. Therefore, given any linear 2D discrete system, one is interested in the possibility to transform the system into a Roesser model. If it is possible, then this transformation can be used to study stability and stabilization properties of the system through the Roesser model. This technique has been successfully applied in [2] to the case of another classical model appearing in the study of linear 2D discrete systems, namely, the Fornasini-Marchesini model [15]. For a transformation to be useful, one has to require that some built-in properties of the system are preserved. This is the idea behind the distinct notions of *equivalence* transformations ([18], [29], [13] and references therein) that have been introduced.

This work was supported by the ANR-13-BS03-0005 (MSDOS)

The present article deals with linear repetitive processes [26] and *wave* linear repetitive processes [17] which are used in the description of iterative learning control schemes [21]. The main contribution is to explicit equivalence transformations between such models and Roesser models (Theorems 1, 2, and 5). The notion of equivalence that we consider here is that of the algebraic analysis approach to linear systems theory (see Section II-A and Lemma 1). Moreover, in the case of linear repetitive processes, we prove that these transformations preserve structural stability (Theorems 3 and 4). In our previous paper [1], we have pointed out the important differences between the notion of equivalence that we use and the input / output equivalence studied in [4], [5] while both notions can be useful. Note that in [16], the authors study the transformation of a wave linear repetitive process into an input / output equivalent Roesser model.

The paper is organized as follows. Section II provides a very brief introduction to the algebraic analysis approach to linear systems theory and introduces Roesser models. Section III concerns linear repetitive processes. We first give two ways of constructing an explicit Roesser model equivalent to a given linear repetitive process. Then we prove that the notion of structural stability is preserved through these equivalence transformations. Finally, Section IV deals with wave linear repetitive processes. We provide an implicit Roesser model that is equivalent to a given wave linear repetitive process.

II. Preliminaries

A. Algebraic analysis

Algebraic analysis is a mathematical theory that has been first developed by Malgrange for studying linear systems of partial differential equations with constant coefficients [19]. It was then generalized by the Japanese school of Sato to handle linear systems of partial differential equations with varying coefficients (see, e.g., [20]) and, nowadays, this approach is used to study built-in properties of a wide class of systems of linear functional equations (see [8], [11], [22], [28], [24] and references therein). Within the algebraic analysis framework, a linear system is studied by means of an associated left module over a ring of functional operators that is in general non-commutative. Indeed, every linear system of functional

978-3-9524-2699-9 ©2018 EUCA

equations can be written as $R\eta = 0$, where R is a $q \times p$ matrix with entries in a ring $\mathbb{D}$ of functional operators and η is a vector of p unknown functions. We can then associate to the matrix $R \in \mathbb{D}^{q\times p}$ the finitely presented left $\mathbb{D}$-module $M = \mathbb{D}^{1\times p}/(\mathbb{D}^{1\times q} R)$. If $\mathcal{F}$ is a left $\mathbb{D}$-module, then the properties of the linear system (also called behavior) $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$ can be studied by means of the module properties of M since Malgrange's isomorphism [19] yields $\ker_{\mathcal{F}}(R.) \cong \hom_{\mathbb{D}}(M, \mathcal{F})$.

B. Roesser models

In the present paper, we focus on linear 2D discrete systems so that the dependent variables (e.g., sequences) considered are functions of two discrete independent variables denoted by i and j. Certain particular forms of the equations modeling a linear 2D discrete system are more suited than the others for studying some properties of the system. One model that has been widely used in the literature is the so-called Roesser model [25]. Its main advantage relies on the fact that its defining equations resemble as much as possible those of a linear 1D discrete system, namely, $x(i+1) = F\,x(i) + H\,u(i)$, $y(i) = K\,x(i) + L\,u(i)$. Consequently, some techniques previously developed for the 1D case can sometimes be mimiced to the case of 2D Roesser models. The state-space representation of a Roesser model [25] can be written as: for all $i,\, j \in \mathbb{N}$,

$$E\begin{pmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{pmatrix} = \underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{A}\begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix} + \underbrace{\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}}_{B} u(i,j),$$

$$y(i,j) = \underbrace{\begin{pmatrix} C_1 & C_2 \end{pmatrix}}_{C}\begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix} + D\,u(i,j), \tag{1}$$

where x^h (resp. x^v) is the horizontal (resp. vertical) state vector of dimension d_h (resp. d_v), u is the input vector of dimension d_u, y is the output vector of dimension d_y, and E, A_{ij}, B_i, C_i (for $i, j = 1, 2$), D are matrices of appropriate dimensions with constant entries in a field $\mathbb{K}$ (e.g., $\mathbb{K} = \mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$). When the square matrix E is non-singular, the model (1) is said to be *regular* or *explicit* and we can always assume w.l.o.g. that $E = I_d$, where $d = d_h + d_v$. Otherwise the Roesser model is said to be *singular* or *implicit*.

Within the algebraic analysis framework, if $\mathbb{D} = \mathbb{Q}[\sigma_1, \sigma_2]$ denotes the commutative polynomial ring in two shift operators σ_1 (w.r.t. i) and σ_2 (w.r.t. j) with coefficients in $\mathbb{Q}$, then an explicit Roesser model (1) is written as $R\,\eta = 0$, with:

$$R = \begin{pmatrix} \sigma_1\, I_{d_h} - A_{11} & -A_{12} & -B_1 & 0 \\ -A_{21} & \sigma_2\, I_{d_v} - A_{22} & -B_2 & 0 \\ -C_1 & -C_2 & -D & I_{d_y} \end{pmatrix}. \tag{2}$$

III. Linear repetitive processes

Modeling a linear 2D discrete phenomenon does not always provide the form (1) of a Roesser model. For instance, the description of iterative learning control schemes [21] rather provides the model of a linear repetitive process [26]. Using the same notation as for the definition of Roesser models, the state-space representation of a linear repetitive process [26] is generally written as: for all $i \in \mathbb{N}$, for all $0 \le j \le \alpha$, with $\alpha \in \mathbb{N}^*$,

$$\begin{aligned} x(i+1, j+1) &= \mathcal{A}\, x(i+1, j) + \mathcal{B}_0\, y(i,j) + \mathcal{B}\, u(i+1, j), \\ y(i+1, j) &= \mathcal{C}\, x(i+1, j) + \mathcal{D}_0\, y(i,j) + \mathcal{D}\, u(i+1, j), \end{aligned} \tag{3}$$

where x is the state vector of dimension d_x, u is the input vector of dimension d_u, y is the pass profile vector of dimension d_y which serves as the output vector, and $\mathcal{A}$, $\mathcal{B}_0$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}_0$, $\mathcal{D}$ are matrices of appropriate dimensions with constant entries in a field $\mathbb{K}$ (e.g., $\mathbb{K} = \mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$).

One should remark that one difference with Roesser models is that, here, the independent variable j is bounded. This has important consequences in particular in the way initial conditions have to be chosen to get a proper system description: see [23], [26] for more details. However, for many issues concerning linear repetitive processes (see, for instance, [27], [4], [5]), this distinction can be omitted and in the sequel we shall not take it into account and we thus always consider i and j as formal independent variables.

Using the algebraic analysis framework over $\mathbb{D} = \mathbb{Q}[\sigma_1, \sigma_2]$, the model of a linear repetitive process (3) to which we add the output equation $z = y$ is written as $\mathcal{R}\,\eta = 0$, where:

$$\mathcal{R} = \begin{pmatrix} \sigma_1\,\sigma_2\, I_{d_x} - \mathcal{A}\,\sigma_1 & -\mathcal{B}_0 & -\mathcal{B}\,\sigma_1 & 0 \\ -\mathcal{C}\,\sigma_1 & \sigma_1\, I_{d_y} - \mathcal{D}_0 & -\mathcal{D}\,\sigma_1 & 0 \\ 0 & -I_{d_y} & 0 & I_{d_y} \end{pmatrix}. \tag{4}$$

A. Equivalence with an explicit Roesser model

As previously explained, the Roesser model (1) has many advantages when one wants to adapt 1D tools for studying systems properties to the 2D case. Consequently, people have been interested in the possibility to rewrite the equations of other forms of 2D models under the Roesser form. To achieve this, a natural idea consists in performing a change of variables in the original model so that the equations written on the new variables yield a Roesser model. Moreover, if one wants some built-in properties of the system to be preserved through this transformation then not all changes of variables are admissible. This is the reason why distinct notions of *equivalence* transformations have been defined: see, for instance, [18], [29], [13] and references therein. Within the algebraic analysis approach to linear systems theory used in the present paper, two linear systems are said to be equivalent if their associated $\mathbb{D}$-modules are isomorphic and we have the following explicit characterization in terms of matrices:

Lemma 1 ([11], [13]). *Let $R \in \mathbb{D}^{q\times p}$, $R' \in \mathbb{D}^{q'\times p'}$ and consider the associated $\mathbb{D}$-modules $M = \mathbb{D}^{1\times p}/(\mathbb{D}^{1\times q} R)$ and $M' = \mathbb{D}^{1\times p'}/(\mathbb{D}^{1\times q'} R')$. Then, the linear systems $R\,\eta = 0$ and $R'\,\eta' = 0$ are equivalent if and only if there exist six* *matrices* $P \in \mathbb{D}^{p \times p'}$, $Q \in \mathbb{D}^{q \times q'}$, $P' \in \mathbb{D}^{p' \times p}$, $Q' \in \mathbb{D}^{q' \times q}$, $Z \in \mathbb{D}^{p \times q}$, *and* $Z' \in \mathbb{D}^{p' \times q'}$ *satisfying:*

$$\begin{aligned} R\,P = Q\,R', \qquad & R'\,P' = Q'\,R, \\ P\,P' + Z\,R = I_p, \quad & P'\,P + Z'\,R' = I_{p'}. \end{aligned} \tag{5}$$

Note that implementations of the algorithms developed in [11] to compute the matrices appearing in Lemma 1 are included both in the Maple package OREMORPHISMS [12] based on OREMODULES [9] and in the Mathematica package OREALGEBRAICANALYSIS [14]. With the notations of Lemma 1, this algebraic notion of equivalence yields a one-to-one correspondence between $\mathcal{F}$-solutions of $R\,\eta = 0$ and $\mathcal{F}$-solutions of $R'\,\eta' = 0$ given by the invertible changes of variables $\eta = P\,\eta'$ and $\eta' = P'\,\eta$. Moreover this notion of equivalence clearly preserved the (homological) invariants of the $\mathbb{D}$-modules which has interested consequences for the systems properties [8].

In our previous paper [1], we proved that a linear repetitive process (3) is equivalent to an explicit Roesser model (1):

Theorem 1 ([1], Theorem 1)**.** *The linear repetitive process model (3) is equivalent to the explicit Roesser model given by (1) with:*

$$A = \left(\begin{array}{ccc|c} 0 & \mathcal{B}_0 & 0 & \mathcal{B}_0\,\mathcal{C} \\ 0 & \mathcal{D}_0 & 0 & \mathcal{D}_0\,\mathcal{C} \\ 0 & 0 & 0 & 0 \\ \hline I_{d_x} & 0 & 0 & \mathcal{A} \end{array}\right), \quad B = \left(\begin{array}{c} \mathcal{B} \\ \mathcal{D} \\ I_{d_u} \\ \hline 0 \end{array}\right), \tag{6}$$
$$C = \left(\begin{array}{ccc|c} 0 & I_{d_y} & 0 & \mathcal{C} \end{array}\right), \qquad D = 0.$$

The change of variables allowing to rewrite the equations of a given linear repetitive process under the form of a Roesser model is not unique. For instance, to obtain (6), we have introduced the new formal variables:

$$\begin{aligned} & x^h = \begin{pmatrix} x_1^h \\ x_2^h \\ x_3^h \end{pmatrix}, \begin{cases} x_1^h(i,j) = x(i,j+1) - \mathcal{A}\,x(i,j), \\ x_2^h(i,j) = y(i,j) - \mathcal{C}\,x(i,j), \\ x_3^h(i,j) = u(i,j), \end{cases} \\ & x^v(i,j) = x(i,j), \\ & u'(i,j) = u(i+1,j), \quad y'(i,j) = y(i,j). \end{aligned} \tag{7}$$

But using the other choice of formal variables given by

$$\begin{aligned} & x^h(i,j) = \begin{pmatrix} x_1^h \\ x_2^h \end{pmatrix}, \begin{cases} x_1^h(i,j) = x(i,j+1) - \mathcal{A}\,x(i,j) - \mathcal{B}\,u(i,j), \\ x_2^h(i,j) = y(i,j) - \mathcal{C}\,x(i,j) - \mathcal{D}\,u(i,j), \end{cases} \\ & x^v = \begin{pmatrix} x_1^v \\ x_2^v \end{pmatrix}, \quad \begin{cases} x_1^v(i,j) = x(i,j), \\ x_2^v(i,j) = u(i,j), \end{cases} \\ & u'(i,j) = u(i,j+1), \quad y'(i,j) = y(i,j), \end{aligned} \tag{8}$$

provides the following alternative of Theorem 1:

Theorem 2. *The linear repetitive process model (3) is equivalent to the explicit Roesser model given by (1) with:*

$$A = \left(\begin{array}{cc|cc} 0 & \mathcal{B}_0 & \mathcal{B}_0\,\mathcal{C} & \mathcal{B}_0\,\mathcal{D} \\ 0 & \mathcal{D}_0 & \mathcal{D}_0\,\mathcal{C} & \mathcal{D}_0\,\mathcal{D} \\ \hline I_{d_x} & 0 & \mathcal{A} & \mathcal{B} \\ 0 & 0 & 0 & 0 \end{array}\right), \quad B = \left(\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ I_{d_u} \end{array}\right), \tag{9}$$
$$C = \left(\begin{array}{cc|cc} 0 & I_{d_y} & \mathcal{C} & \mathcal{D} \end{array}\right), \qquad D = 0.$$

Proof. Let $R = \mathcal{R}$ be given by (4) and let R' be given by (2) with the matrices A, B, C and D defined by (9). Then, we can check that the relations (5) are satisfied by the matrices

$$P = \begin{pmatrix} 0 & 0 & I_{d_x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{d_y} \\ 0 & 0 & 0 & I_{d_u} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{d_y} \end{pmatrix},$$

$$Q = \begin{pmatrix} I_{d_x} & 0 & \sigma_1\,I_{d_x} & 0 & \mathcal{B}_0 \\ 0 & I_{d_y} & 0 & 0 & -\sigma_1\,I_{d_y} + \mathcal{D}_0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P' = \begin{pmatrix} \sigma_2\,I_{d_x} - \mathcal{A} & 0 & -\mathcal{B} & 0 \\ -\mathcal{C} & 0 & -\mathcal{D} & I_{d_y} \\ I_{d_x} & 0 & 0 & 0 \\ 0 & 0 & I_{d_u} & 0 \\ 0 & 0 & \sigma_2\,I_{d_u} & 0 \\ 0 & 0 & 0 & I_{d_y} \end{pmatrix}, \quad Q' = \begin{pmatrix} I_{d_x} & 0 & -\mathcal{B}_0 \\ 0 & I_{d_u} & \sigma_1\,I_{d_u} - \mathcal{D}_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -I_{d_y} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z' = \begin{pmatrix} 0 & 0 & -I_{d_x} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{d_y} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{d_u} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\square$

The Roesser models given by Theorems 1 and 2 are equivalent to the original linear repetitive process in the sense of algebraic analysis. We shall alert the reader on the fact that this does not imply that the systems are necessarily equivalent if one considers other notions of equivalence. For instance, the last two equations of (7) and (8) imply that the Roesser models given in Theorems 1 and 2 are not input / output equivalent to the original linear repetitive process. In [1], we have illustrated the fact that algebraic analysis equivalence does not imply input / output equivalence and conversely, input / output equivalence does not imply equivalence in the sense of algebraic analysis. Note that the problem of constructing a Roesser model input / output equivalent to a given linear repetitive process has been studied in [4], [5].

B. Structural stability

Let us consider a linear 2D discrete system $R\,\eta = 0$, where R is a matrix with entries in $\mathbb{D} = \mathbb{Q}[\sigma_1, \sigma_2]$ and η is a vector of unknown bivariate sequences. As soon as stability issues are concerned, one has to make the distinction between state x, input u and output y variables. Consequently, the vector η is split as $\eta = (x^T \quad u^T \quad y^T)^T$ and the matrix R is split accordingly[1]:

$$R = \begin{pmatrix} T & U & 0 \\ V & W & -I_{d_y} \end{pmatrix}, \tag{10}$$

[1]In the general case, T can be a non-square matrix: see, e.g., [2].

with $T \in \mathbb{D}^{d_x \times d_x}$, $U \in \mathbb{D}^{d_x \times d_u}$, $V \in \mathbb{D}^{d_y \times d_x}$, $W \in \mathbb{D}^{d_y \times d_u}$. Then, introducing the region

$$\overline{\mathbb{S}}^2 = \left\{(z_1, z_2) \in \overline{\mathbb{C}}^2 \,|\, \forall i = 1,\, 2,\, |z_i| \geq 1\right\},$$

where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, we give the following definition: we refer to [2] and the references therein for more explanations and justifications concerning Definition 1.

Definition 1 ([2])**.** *The linear 2D discrete system given by (10) is said to be* structurally stable *if, for all $(\lambda_1, \lambda_2) \in \overline{\mathbb{S}}^2$, the constant matrix $T(\lambda_1, \lambda_2)$ satisfies $\det(T(\lambda_1, \lambda_2)) \neq 0$.*

In practice structural stability can be effectively checked using the algorithms in [6], [7] that have been implemented in MAPLE.

We shall now prove that this important notion of structural stability is preserved via the equivalence transformation given by Theorem 1. Note that [2, Theorem 1] asserts that a sufficient condition for this to be true is that the linear systems defined by the matrices T_{LRP} and T_R, corresponding respectively to the matrix T of the representation (10) of the linear repetitive process (3) and of the Roesser model given by (1) with (6), are equivalent in the sense of algebraic analysis. We refer to [2] for more details. Here, this sufficient condition does not seem to hold but we can still prove the following result:

Theorem 3. *The linear repetitive process (3) is structurally stable if and only if the Roesser model given by (1) with (6) is structurally stable.*

Proof. From Definition 1, the Roesser model given by (1) with (6) is structurally stable if for all $(\lambda_1, \lambda_2) \in \overline{\mathbb{S}}^2$,

$$a = \det \begin{pmatrix} \lambda_1 I_{d_x} & -\mathcal{B}_0 & 0 & -\mathcal{B}_0\, \mathcal{C} \\ 0 & \lambda_1 I_{d_y} - \mathcal{D}_0 & 0 & -\mathcal{D}_0\, \mathcal{C} \\ 0 & 0 & \lambda_1 I_{d_u} & 0 \\ -I_{d_x} & 0 & 0 & \lambda_2 I_{d_x} - \mathcal{A} \end{pmatrix} \neq 0.$$

Permuting first the last two block rows and then the last two block columns, we get

$$a = \det \begin{pmatrix} \lambda_1 I_{d_x} & -\mathcal{B}_0 & -\mathcal{B}_0\, \mathcal{C} & 0 \\ 0 & \lambda_1 I_{d_y} - \mathcal{D}_0 & -\mathcal{D}_0\, \mathcal{C} & 0 \\ -I_{d_x} & 0 & \lambda_2 I_{d_x} - \mathcal{A} & 0 \\ 0 & 0 & 0 & \lambda_1 I_{d_u} \end{pmatrix}$$

so that

$$a = \lambda_1^{d_u} \det \begin{pmatrix} \lambda_1 I_{d_x} & -\mathcal{B}_0 & -\mathcal{B}_0\, \mathcal{C} \\ 0 & \lambda_1 I_{d_y} - \mathcal{D}_0 & -\mathcal{D}_0\, \mathcal{C} \\ -I_{d_x} & 0 & \lambda_2 I_{d_x} - \mathcal{A} \end{pmatrix}.$$

Now multiplying the last block row by λ_1 and adding to it the first block row leads to:

$$a = \lambda_1^{d_u - d_x} \det \begin{pmatrix} \lambda_1 I_{d_x} & -\mathcal{B}_0 & -\mathcal{B}_0\, \mathcal{C} \\ 0 & \lambda_1 I_{d_y} - \mathcal{D}_0 & -\mathcal{D}_0\, \mathcal{C} \\ 0 & -\mathcal{B}_0 & \lambda_1 (\lambda_2 I_{d_x} - \mathcal{A}) - \mathcal{B}_0\, \mathcal{C} \end{pmatrix},$$

so that

$$a = \lambda_1^{d_u} \det \begin{pmatrix} \lambda_1 I_{d_y} - \mathcal{D}_0 & -\mathcal{D}_0\, \mathcal{C} \\ -\mathcal{B}_0 & \lambda_1 \lambda_2 I_{d_x} - \lambda_1 \mathcal{A} - \mathcal{B}_0\, \mathcal{C} \end{pmatrix}.$$

We then replace the last block column by itself minus the first block column multiplied on the right by $\mathcal{C}$ to get:

$$a = \lambda_1^{d_u} \det \begin{pmatrix} \lambda_1 I_{d_y} - \mathcal{D}_0 & -\lambda_1\, \mathcal{C} \\ -\mathcal{B}_0 & \lambda_1 \lambda_2 I_{d_x} - \lambda_1 \mathcal{A} \end{pmatrix}.$$

Finally permuting the block rows and then the block columns, we obtain

$$a = \lambda_1^{d_u} \det \begin{pmatrix} \lambda_1 \lambda_2 I_{d_x} - \lambda_1 \mathcal{A} & -\mathcal{B}_0 \\ -\lambda_1\, \mathcal{C} & \lambda_1 I_{d_y} - \mathcal{D}_0 \end{pmatrix}.$$

We thus have: for all $(\lambda_1, \lambda_2) \in \overline{\mathbb{S}}^2$

$$a \neq 0 \Leftrightarrow b = \det \begin{pmatrix} \lambda_1 \lambda_2 I_{d_x} - \lambda_1 \mathcal{A} & -\mathcal{B}_0 \\ -\lambda_1\, \mathcal{C} & \lambda_1 I_{d_y} - \mathcal{D}_0 \end{pmatrix} \neq 0,$$

which ends the proof since, from Definition 1, the linear repetitive process (3) is structurally stable if, for all $(\lambda_1, \lambda_2) \in \overline{\mathbb{S}}^2$, $b \neq 0$. □

Note that the proof of Theorem 3 given above can be easily adapted to prove the analogous result for the Roesser model given by (1) with (9):

Theorem 4. *The linear repetitive process (3) is structurally stable if and only if the Roesser model given by (1) with (9) is structurally stable.*

IV. WAVE LINEAR REPETITIVE PROCESSES

A. Definition

The model (3) of a linear repetitive process can be generalized so that the previous i pass contribution to the dynamics at a given sample j on the current $(i+1)$ pass does not only depend on the same instance j but on a window of points $j - \omega_l \leq j \leq j + \omega_h$, for given $\omega_l,\, \omega_h \in \mathbb{N}$. This generalized model which appears in iterative learning control design is then called *wave* linear repetitive process [17]. The state-space representation of a wave linear repetitive process is written as: for all $i \in \mathbb{N}$, for all $0 \leq j \leq \alpha$, with $\alpha \in \mathbb{N}^*$,

$$\begin{aligned} x(i+1, j+1) &= \mathcal{A}\, x(i+1, j) + \sum_{k=-\omega_l}^{\omega_h} \mathcal{B}_k\, y(i, j+k) + \mathcal{B}\, u(i+1, j), \\ y(i+1, j) &= \mathcal{C}\, x(i+1, j) + \sum_{k=-\omega_l}^{\omega_h} \mathcal{D}_k\, y(i, j+k) + \mathcal{D}\, u(i+1, j), \end{aligned} \tag{11}$$

where $\mathcal{A}$, the $\mathcal{B}_k$'s, $\mathcal{B}$, $\mathcal{C}$, the $\mathcal{D}_k$'s, and $\mathcal{D}$ are matrices of appropriate dimensions with constant entries in a field $\mathbb{K}$ (e.g., $\mathbb{K} = \mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$). Note that setting $\omega_l = \omega_h = 0$, we find again the linear repetitive process (3).

In order to use the algebraic analysis framework to study wave linear repetitive processes, one should first remark that, as soon as $\omega_l > 0$, the ring $\mathbb{D} = \mathbb{Q}[\sigma_1, \sigma_2]$ does not allow to write the wave linear repetitive process (11) under the form $R\, \eta = 0$, with the entries of R in $\mathbb{D}$. Indeed one needs to introduce the inverse shift σ_2^{-1} which acts on a bivariate sequence $f(i,j)$ as $\sigma_2^{-1}.f(i,j) = f(i, j-1)$. Consequently we introduce the new commutative polynomial ring $\mathbb{E} = \mathbb{Q}[\sigma_1, \sigma_2, \sigma_2^{-1}]$, where $\sigma_2 \circ \sigma_2^{-1} = \sigma_2^{-1} \circ \sigma_2 = \mathrm{id}_{\mathbb{E}}$ and we have $\mathbb{D} \subset \mathbb{E}$. The model of a linear repetitive process (11) to which we add the output equation $z = y$ is written as $\mathcal{R}_w \, \eta = 0$, where:

$$\mathcal{R}_w = \begin{pmatrix} \sigma_1 \, \sigma_2 \, I_{d_x} - \mathcal{A} \, \sigma_1 & -\sum_{k=-\omega_l}^{\omega_h} \mathcal{B}_k \, \sigma_2^k & -\mathcal{B} \, \sigma_1 & 0 \\ -\mathcal{C} \, \sigma_1 & \sigma_1 \, I_{d_y} - \sum_{k=-\omega_l}^{\omega_h} \mathcal{D}_k \, \sigma_2^k & -\mathcal{D} \, \sigma_1 & 0 \\ 0 & -I_{d_y} & 0 & I_{d_y} \end{pmatrix}. \tag{12}$$

In [16], the authors show that a given wave linear repetitive process of the form (11) can be explicitly transformed into an input / output equivalent singular Roesser model (1). The purpose of the next section is to show that we can also construct a singular Roesser model which is equivalent to (11) in the sense of algebraic analysis.

B. Equivalence with an implicit Roesser model

In the case of a linear repetitive process (3), the pass profile vector y is only shifted with respect to the independent variable i so that introducing y in the horizontal state vector x^h allows to construct an equivalent explicit Roesser model. See the changes of variables (7) and (8) given in Section III-A. In the more general case of a wave linear repetitive process (11), the pass profile vector y is both shifted with respect to i and j. Consequently, in order to get a Roesser form of the model, y should both appear in the horizontal x^h and vertical x^v state vectors. But, if one wants an equivalence transformation, then one has to provide a link between these new two formal variables (i.e., one has to encode the fact that they are equal) since they both stand for the pass profile vector y of the wave linear repetitive process. The fact that, in a Roesser model, such a link (without shifts) between x^h and x^v can only be obtained using an implicit equation, leads to an *implicit* equivalent Roesser model. Generalizing the change of variables (7) and taking into account the latter fact, we shall prove that introducing the following new formal variables

$$x^h = \begin{pmatrix} x_1^h \\ x_2^h \\ x_3^h \end{pmatrix}, \quad \begin{cases} x_1^h(i,j) = x(i,j+1) - \mathcal{A} \, x(i,j), \\ x_2^h(i,j) = y(i,j) - \mathcal{C} \, x(i,j), \\ x_3^h(i,j) = u(i,j), \end{cases}$$

$$x^v = \begin{pmatrix} x_1^v \\ x_2^v \\ \vdots \\ x_{\omega_l+\omega_h+2}^v \end{pmatrix}, \quad \begin{cases} x_1^v(i,j) = x(i,j), \\ x_2^v(i,j) = y(i,j-\omega_l), \\ \quad \vdots \\ x_{\omega_l+2}^v(i,j) = y(i,j), \\ \quad \vdots \\ x_{\omega_l+\omega_h+2}^v(i,j) = y(i,j+\omega_h), \end{cases}$$

$$u'(i,j) = u(i+1,j), \quad y'(i,j) = y(i,j), \tag{13}$$

will provide an implicit Roesser model which is equivalent to the wave linear repetitive process (11).

In the rest of this section, for the ease of presentation, we shall assume that $\omega_l = 2$ and $\omega_h = 1$. Note that Theorem 5 below can be easily generalized to any value of ω_l and ω_h.

Theorem 5. *The wave linear repetitive process model (11) with $\omega_l = 2$ and $\omega_h = 1$ is equivalent to the implicit Roesser model given by (1) where:*

$$E = \mathrm{diag}(I_{d_x}, I_{d_y}, I_{d_u}, I_{d_x}, I_{d_y}, I_{d_y}, I_{d_y}, 0_{d_y}),$$

$$A = \left(\begin{array}{ccc|ccccc} 0 & 0 & 0 & 0 & \mathcal{B}_{-2} & \mathcal{B}_{-1} & \mathcal{B}_0 & \mathcal{B}_1 \\ 0 & 0 & 0 & 0 & \mathcal{D}_{-2} & \mathcal{D}_{-1} & \mathcal{D}_0 & \mathcal{D}_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline I_{d_x} & 0 & 0 & \mathcal{A} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{d_y} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{d_y} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{d_y} \\ 0 & -I_{d_y} & 0 & -\mathcal{C} & 0 & 0 & I_{d_y} & 0 \end{array} \right),$$

$$B = \left(\begin{array}{c} \mathcal{B} \\ \mathcal{D} \\ I_{d_u} \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right), \quad \begin{cases} C = \left(\begin{array}{ccc|ccccc} 0 & I_{d_y} & 0 & \mathcal{C} & 0 & 0 & 0 & 0 \end{array} \right), \\ D = 0. \end{cases} \tag{14}$$

Proof. Let $R = \mathcal{R}_w$ be the matrix with entries in $\mathbb{E}$ given by (12) with $\omega_l = 2$ and $\omega_h = 1$ and let R' be the matrix with entries in $\mathbb{D} \subset \mathbb{E}$ given by

$$R' = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 & -\mathcal{B}_{-2} & -\mathcal{B}_{-1} & -\mathcal{B}_0 & -\mathcal{B}_1 & -\mathcal{B} & 0 \\ 0 & \sigma_1 & 0 & 0 & -\mathcal{D}_{-2} & -\mathcal{D}_{-1} & -\mathcal{D}_0 & -\mathcal{D}_1 & -\mathcal{D} & 0 \\ 0 & 0 & \sigma_1 & 0 & 0 & 0 & 0 & 0 & -I_{d_u} & 0 \\ -I_{d_x} & 0 & 0 & \sigma_2 - \mathcal{A} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_2 & -I_{d_y} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_2 & -I_{d_y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_2 & -I_{d_y} & 0 & 0 \\ 0 & -I_{d_y} & 0 & -\mathcal{C} & 0 & 0 & I_{d_y} & 0 & 0 & 0 \\ 0 & -I_{d_y} & 0 & -\mathcal{C} & 0 & 0 & 0 & 0 & 0 & I_{d_y} \end{pmatrix}.$$

To shorten the notation, in the matrix R' above and in the matrices below, $\sigma_i \, I_d$ has been replaced by σ_i for $i = 1$ or 2 and $d = d_x$, d_y, or d_u. We have to prove that the $\mathbb{E}$-modules respectively associated to R and R' are isomorphic over $\mathbb{E}$. To achieve this it suffices to check that the four relations (5) of Lemma 1 are satisfied by the six matrices

$$P = \begin{pmatrix} 0 & 0 & 0 & I_{d_x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{d_y} \\ 0 & 0 & I_{d_u} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{d_y} \end{pmatrix},$$

$$Q = \begin{pmatrix} I_{d_x} & 0 & -\mathcal{B} & \sigma_1 & \sigma_2^{-1} \mathcal{B}_{-2} & \sigma_2^{-1} \mathcal{B}_{-1} + \sigma_2^{-2} \mathcal{B}_{-2} & -\mathcal{B}_1 \\ 0 & I_{d_y} & -\mathcal{D} & 0 & \sigma_2^{-1} \mathcal{D}_{-2} & \sigma_2^{-1} \mathcal{D}_{-1} + \sigma_2^{-2} \mathcal{D}_{-2} & -\mathcal{D}_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix.$$

$$\begin{pmatrix} \sigma_2 \, \mathcal{B}_1 + \sigma_2^{-1} \mathcal{B}_{-1} + \sigma_2^{-2} \mathcal{B}_{-2} + \mathcal{B}_0 & -\sigma_2 \, \mathcal{B}_1 - \sigma_2^{-1} \mathcal{B}_{-1} - \sigma_2^{-2} \mathcal{B}_{-2} - \mathcal{B}_0 \\ \mathcal{D}_1 \sigma_2 + \sigma_2^{-1} \mathcal{D}_{-1} + \sigma_2^{-2} \mathcal{D}_{-2} + \mathcal{D}_0 & -\mathcal{D}_1 \sigma_2 - \sigma_2^{-1} \mathcal{D}_{-1} - \sigma_2^{-2} \mathcal{D}_{-2} + \sigma_1 - \mathcal{D}_0 \\ 0 & 0 \end{pmatrix},$$

$$P' = \begin{pmatrix} \sigma_2 - \mathcal{A} & 0 & 0 & 0 \\ -\mathcal{C} & I_{d_y} & 0 & 0 \\ 0 & 0 & I_{d_u} & 0 \\ I_{d_x} & 0 & 0 & 0 \\ 0 & \sigma_2^{-2} & 0 & 0 \\ 0 & \sigma_2^{-1} & 0 & 0 \\ 0 & I_{d_y} & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_1 & 0 \\ 0 & 0 & 0 & I_{d_y} \end{pmatrix}, \quad Q' = \begin{pmatrix} I_{d_x} & 0 & 0 \\ 0 & I_{d_y} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{d_y} \end{pmatrix},$$

$$Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -I_{d_y} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Z' = \begin{pmatrix} 0 & 0 & 0 & -I_{d_x} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_{d_y} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_2^{-1} & \sigma_2^{-2} & 0 & \sigma_2^{-2} & -\sigma_2^{-2} \\ 0 & 0 & 0 & 0 & 0 & \sigma_2^{-1} & 0 & \sigma_2^{-1} & -\sigma_2^{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{d_y} & -I_{d_y} \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_{d_y} & \sigma_2 & -\sigma_2 \\ 0 & 0 & -I_{d_u} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

□

As it was the case in Section III-A, Theorem 5 does not imply that the implicit Roesser model obtained is input / output equivalent to the original wave linear repetitive process. Indeed, Equation (13) shows that this is surely not the case.

In future work, we would be interested in investigating the structural stability of a wave linear repetitive process using the fact that the structural stability of the equivalent Roesser model might be preserved by the equivalence transformation. However, several preliminary questions have to be answered: (i) What should be the definition of structural stability for linear 2D system over the ring $\mathbb{E}$? (ii) If a definition was found and if the property was proved to be preserved, what would be its exact meaning for the wave linear repetitive process? Would it make sense? The answers seem to be far from obvious.

ACKNOWLEDGMENT

The authors would like to thank all the members of the ANR project MSDOS for discussions about this subject. We are also grateful to Prof. K. Gałkowski for interactions concerning this work and for encouraging us to generalize the results of [1] to the general case of *wave* linear repetitive processes.

REFERENCES

[1] O. Bachelier, T. Cluzeau. *Digression on the Equivalence between Linear 2D Discrete Repetitive Processes and Roesser Models.* In Proceedings of the 10th international Workshop on Multidimensional (nD) Systems (nDS'17), Zielona Góra (Poland), 2017.

[2] O. Bachelier, T. Cluzeau, R. David, and N. Yeganefar. *Structural Stabilization of Linear 2D Discrete Systems Using Equivalence Transformations.* Multidimensional Systems and Signal Processing, Volume 28, Issue 4, pp 1629-1652, 2017.

[3] O. Bachelier, N. Yeganefar, D. Mehdi, and W. Paszke. *State feedback structural stabilization of 2D discrete Roesser models.* Proceedings of the 9th International Workshop on Multidimensional (nD) Systems (nDS'15), Vila Real (Portugal), 07-09/09/2015.

[4] M. S. Boudellioua, K. Gałkowski, and E. Rogers. *On the connection between discrete linear repetitive processes and 2D discrete linear systems*. Multidimensional Systems and Signal processing, (DOI 10.1007/s11045-016-0454-8), 2017.

[5] M. S. Boudellioua, K. Gałkowski, and E. Rogers. *Reduction of discrete linear repetitive processes to nonsingular Roesser models via elementary operations*. IFAC 2017 World Congress, Toulouse, France, 2017.

[6] Y. Bouzidi, A. Quadrat, and F. Rouillier. *Computer algebra methods for testing the structural stability of multidimensional systems.* Proceedings of the 9th international Workshop on Multidimensional (nD) Systems (nDS'15), Vila Real (Portugal), 07-09/09/2015.

[7] Y. Bouzidi, F. Rouillier. *Certified Algorithms for proving the structural stability of two dimensional systems possibly with parameters.* Proceeding of the 22nd International Symposium on Mathematical Theory of Networks and Systems (MTNS), 2016.

[8] F. Chyzak, A. Quadrat, and D. Robertz. *Effective algorithms for parametrizing linear control systems over Ore algebras.* Appl. Algebra Engrg. Comm. Comput., 16, 319-376, 2005.

[9] F. Chyzak, A. Quadrat, and D. Robertz. OREMODULES*: A symbolic package for the study of multidimensional linear systems.* Springer, Lecture Notes in Control and Inform. Sci., 352, 233-264, 2007. http://pages.saclay.inria.fr/alban.quadrat/OreModules.html

[10] T. Cluzeau. *A constructive algebraic analysis approach to the equivalence of multidimensional linear systems.* In Proceedings of the 9th international Workshop on Multidimensional (nD) Systems (nDS'15), Vila Real (Portugal), 2015.

[11] T. Cluzeau, A. Quadrat. *Factoring and decomposing a class of linear functional systems.* Linear Algebra Appl., 428, 324-381, 2008.

[12] T. Cluzeau, A. Quadrat. OREMORPHISMS*: A homological algebraic package for factoring, reducing and decomposing linear functional systems.* Springer, Lecture Notes in Control and Inform. Sci., 388, 179-194, 2009. http://pages.saclay.inria.fr/alban.quadrat/OreMorphisms.html

[13] T. Cluzeau, and A. Quadrat. *Equivalence of Linear Functional Systems.* Rapport de Recherche INRIA (RR-9000), 2016. To appear in the Special Issue "Algebraic and Symbolic Computation Methods in Dynamical Systems" in the Springer series Advances in Delays and Dynamics, 2017.

[14] T. Cluzeau, A. Quadrat, M. Tõnso. OREALGEBRAICANALYSIS*: A Mathematica package for the algorithmic study of linear functional systems.* OreAlgebraicAnalysis project, 2015. http://pages.saclay.inria.fr/alban.quadrat/OreAlgebraicAnalysis.html

[15] E. Fornasini, G. Marchesini. *Doubly indexed dynamical systems: state space models and structural properties.* Math. Systems Theory, 12, 59-72, 1970.

[16] K. Gałkowski, M. S. Boudellioua, and E. Rogers. *Reduction of wave linear repetitive processes to singular Roesser model form.* Private communication. To be published, 2017.

[17] K. Gałkowski, B. Cichy, E. Rogers, and J. Lam. *Stabilization of a class of uncertain "wave" discrete linear repetitive processes.* Proceedings of the 45th IEEE Conference on Decision and Control, San Diego (USA), 2006, pp. 1435-1440.

[18] D. Johnson. *Coprimeness in multidimensional system theory and symbolic computation.* PhD thesis, Loughborough University of Technology, Leicestershire, UK, 1993.

[19] B. Malgrange. *Systèmes différentiels à coefficients constants.* Séminaire Bourbaki 1962/63, 1-11, 1962.

[20] M. Kashiwara. *Algebraic Study of Systems of Partial Differential Equations.* Mémoires de la Société Mathématique de France, English translation (Kyoto 1970), 63, 1995.

[21] K. L. Moore. *Iterative Learning Control for Deterministic Systems*. Springer, London, 1993.

[22] U. Oberst. *Multidimensional constant linear systems.* Acta Appl. Math., 20, 1-175, 1990.

[23] W. Paszke. *Analysis and synthesis of multidimensional system classes using linear matrix inequality methods.* Lecture Notes in Control and Computer Science, Vol. 8, Ph.D. thesis, Univ. of Zielona Góra, 2005.

[24] A. Quadrat. *An introduction to constructive algebraic analysis and its applications.* Les cours du CIRM, Journées Nationales de Calcul Formel (2010), 1(2), 281-471, 2010.

[25] R. P. Roesser. *A discrete state-space model for linear image processing.* IEEE Trans. on Automatic Control, 20(1), 1-10,1975.

[26] E. Rogers, K. Gałkowski, and D. H. Owens. Control Systems Theory and Applications for Linear Repetitive Processes. Control and Information Sciences. Springer-Verlag, Berlin Heidelberg, 2007.

[27] B. Sulikowski, K. Gałkowski, E. Rogers, and D. H. Owens. *Output feedback control of discrete linear repetitive processes.* Automatica, 40, p. 2167-2173, 2004.

[28] J. Wood, E. Rogers, and D. H. Owens. *Behaviours, Modules, and Duality.* Multidimensional Signals, Circuits and Systems, K. Galkowski, J. Wood, Eds., Taylor & Francis, Ch. 3, 45-56, 2001.

[29] E. Zerz. *On strict system equivalence for multidimensional systems.* International Journal of Control, 73(6), 495-504, 2000.