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Set-based state estimation of nonlinear systems using constrained zonotopes and interval arithmetic*

Brenner S. Rego[1], Davide M. Raimondo[2], Guilherme V. Raffo[1,3]

***Abstract*— This paper proposes a novel set-valued state estimation algorithm for nonlinear discrete-time systems with unknown-but-bounded disturbances. The problem is often addressed through conservative linearization, leading to severe overestimation. By combining important properties from interval arithmetic and the recently proposed constrained zonotopes, a highly tunable and accurate state estimation algorithm is developed, capable of providing tight bounds for the set-valued state estimation problem. A numerical experiment is presented to demonstrate the performance of the proposed strategy.**

I. INTRODUCTION

The problem of state estimation often arises in several fields, such as state-feedback control and fault detection and isolation (FDI). In the literature, state estimators are commonly based on stochastic approaches, such as Kalman filtering or particle filtering. However, when only bounds of external disturbances and uncertainties are known, set-valued state estimation comes into play [1]–[3].

The set-based state estimation problem has been extensively studied for linear discrete-time systems. The pioneering methods make use of ellipsoids to bound the trajectories of the system [4]. Other classical methods propose recursive state estimation algorithms based on parallelotopes [1] and also zonotopes [5]. More recent works combine different class of sets for set-valued estimation, such as ellipsoids and zonotopes [6], and even combinations of zonotopic methods with Kalman filtering can be found [7]. However, since these sets are not closed in every set operation that arises in the state estimation problem, recently, in [3], a generalization of zonotopes, the *constrained zonotopes*, is used to overcome most of the difficulties encountered by the former algorithms.

While the state estimation problem is already consolidated for linear systems, nonlinear state estimation is still an open field, both for stochastic or set-based approaches. An important tool for reachability analysis of nonlinear systems is *interval arithmetic*. With a wide range of applications, such as global optimization, parameter estimation and robust control [8], interval arithmetic can be used to generate guaranteed bounds on the range of real valued functions, through interval extensions [9]. However, severe overestimation often occurs due to interval dependency and the wrapping effect [10]. Such problems are mitigated by computing unions of interval extensions over a partitioned domain (*refinements*).

Alternatives for reachability analysis of nonlinear systems include conservative linearization [11], although still result in substantial overestimation for systems with highly nonlinear behavior. An attempt to bound the trajectories of nonlinear systems by zonotopes is found in [12], by means of interval arithmetic and over-approximated intersections of zonotopes and strips. Nevertheless, such algorithm is shown in [13] to diverge in the presence of process disturbances. The former approach is improved in [13] by means of DC programming. However, the symmetry of zonotopes still imposes severe limitations to set-valued estimation of nonlinear dynamic systems, since the reachable sets are not even convex in general [14]. A conservative polynomialization approach is then proposed in [14], to bound the trajectories of nonlinear systems by polynomial zonotopes, which are a generalization of zonotopes to a class of non-convex sets. However, intersection algorithms has not been yet developed for polynomial zonotopes, in order to take into account measurement data.

In view of the problem, this work proposes a new method to bound the trajectories of nonlinear discrete-time systems with unknown-but-bounded disturbances, by combining the advantages of interval arithmetic and the recently proposed constrained zonotopes. The latter ones can describe arbitrary convex polytopes, and have been already employed for linear state estimation with substantial accuracy and effectiveness [3]. On the other hand, by using refinements [9], the reachable sets of nonlinear discrete-time systems can be obtained with desired accuracy. The major contribution of this manuscript is a highly tunable algorithm capable of computing accurate bounds for trajectories of nonlinear systems. Outer approximation methods of constrained zonotopes by *unions of intervals* are also presented. Finally, a numerical example is described in order to demonstrate the effectiveness of the proposed algorithm. This paper corresponds to a first step in the design of a nonlinear state estimator that makes use of the constrained zonotopes properties.

A. Problem Formulation

In this work, we consider a class of nonlinear discrete-time systems described by the dynamic equations

$$\begin{aligned} \boldsymbol{x}_k &= \boldsymbol{f}(\boldsymbol{x}_{k-1}, \boldsymbol{u}_{k-1}) + \boldsymbol{D}_w \boldsymbol{w}_{k-1}, \\ \boldsymbol{y}_k &= \boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{D}_v \boldsymbol{v}_k, \end{aligned} \tag{1}$$

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[1]Brenner S. Rego and Guilherme V. Raffo (corresponding author) are with the Graduate Program in Electrical Engineering - Federal University of Minas Gerais (UFMG), 31270-901, Belo Horizonte, MG, Brazil. Brenner S. Rego and Guilherme V. Raffo are Members of the National Institute of Science and Technology (INCT) for Cooperative Autonomous Systems Applied to Security and Environment. {brennersr7, raffo}@ufmg.br

[2]Davide M. Raimondo is with the Identification and Control of Dynamic Systems Laboratory, University of Pavia, Italy. davide.raimondo@unipv.it

[3]Guilherme V. Raffo is with the Department of Electronics Engineering, Federal University of Minas Gerais (UFMG), Belo Horizonte, MG, Brazil.

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where $\boldsymbol{x}_k \in \mathbb{R}^{n_x}$ are the system states, $\boldsymbol{u}_k \in \mathbb{R}^{n_u}$ are known inputs, $\boldsymbol{w}_k \in \mathbb{R}^{n_x}$ denotes process disturbances, $\boldsymbol{y}_k \in \mathbb{R}^{n_y}$ are the measured outputs, and $\boldsymbol{v_k} \in \mathbb{R}^{n_y}$ are measurement disturbances. We assume that the disturbances are bounded by $\boldsymbol{w}_k \in W_k$ and $\boldsymbol{v}_k \in V_k$, where W_k and V_k are known compact sets.

Our goal is to obtain accurate bounds on the evolution of the system states, through the well-known prediction-correction paradigm, in which the set-valued estimation for (1) consists in obtaining sets $\bar{X}_k$ and $\hat{X}_k$ such that [12]:

$$\bar{X}_k \supseteq \{\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_{k-1}) + \boldsymbol{D}_w \boldsymbol{w} : \boldsymbol{x} \in \hat{X}_{k-1}, \boldsymbol{w} \in W_{k-1}\}, \quad (2)$$

$$\hat{X}_k \supseteq \{\boldsymbol{x} \in \bar{X}_k : \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}_v \boldsymbol{v} = \boldsymbol{y}_k, \boldsymbol{v} \in V_k\}, \quad (3)$$

in which (2) is called *prediction step*, and (3) is referred to as *correction step*.

II. Preliminaries

A. Operations with sets

In the problem of set-valued state estimation of linear systems, some common operations with sets arise. Let $Z, W \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, and $\boldsymbol{R} \in \mathbb{R}^{m \times n}$, then

$$\boldsymbol{R}Z \triangleq \{\boldsymbol{R}\boldsymbol{z} : \boldsymbol{z} \in Z\}, \quad (4)$$

$$Z \oplus W \triangleq \{\boldsymbol{z} + \boldsymbol{w} : \boldsymbol{z} \in Z, \boldsymbol{w} \in W\}, \quad (5)$$

$$Z \cap_{\boldsymbol{R}} Y \triangleq \{\boldsymbol{z} \in Z : \boldsymbol{R}\boldsymbol{z} \in Y\}, \quad (6)$$

in which (4) corresponds to the linear image of Z, (5) denotes the Minkowski sum of sets, and (6) is a generalized intersection [3]. Such operations are known to be very costly and numerically unstable when using convex polytopes, either in half-space representation (H-rep) or vertex representation (V-rep) in high dimensions.

B. Constrained zonotopes

Constrained zonotopes are an extension of zonotopes (see Appendix), recently introduced in [3].

Definition 1: A set $Z \subset \mathbb{R}^n$ is a *constrained zonotope* if there exists $(\boldsymbol{G}, \boldsymbol{c}, \boldsymbol{A}, \boldsymbol{b}) \in \mathbb{R}^{n \times n_g} \times \mathbb{R}^n \times \mathbb{R}^{n_c \times n_g} \times \mathbb{R}^{n_c}$ such that

$$Z = \{\boldsymbol{c} + \boldsymbol{G}\boldsymbol{\xi} : \|\boldsymbol{\xi}\|_\infty \leq 1, \boldsymbol{A}\boldsymbol{\xi} = \boldsymbol{b}\}. \quad (7)$$

Equation (7) is referred as *constrained generator representation* (CG-rep). For compactness, we use the shorthand notation $Z = \{\boldsymbol{G}, \boldsymbol{c}, \boldsymbol{A}, \boldsymbol{b}\}$ for constrained zonotopes, and $Z = \{\boldsymbol{G}, \boldsymbol{c}\}$ for zonotopes, as in [3].

The equality constraints $\boldsymbol{A}\boldsymbol{\xi} = \boldsymbol{b}$ enable several properties. For instance, while zonotopes are centrally symmetric sets, constrained zonotopes can represent *arbitrary* convex polytopes. Moreover, constrained zonotopes are closed under (4), (5) and (6), which are computed *exactly* through the following identities [3]:

$$\boldsymbol{R}Z = \{\boldsymbol{R}\boldsymbol{G}_z, \boldsymbol{R}\boldsymbol{c}_z, \boldsymbol{A}_z, \boldsymbol{b}_z\}, \quad (8)$$

$$Z \oplus W = \left\{ \begin{bmatrix} \boldsymbol{G}_z & \boldsymbol{G}_w \end{bmatrix}, \boldsymbol{c}_z + \boldsymbol{c}_w, \begin{bmatrix} \boldsymbol{A}_z & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_w \end{bmatrix}, \begin{bmatrix} \boldsymbol{b}_z \\ \boldsymbol{b}_w \end{bmatrix} \right\}, \quad (9)$$

$$Z \cap_{\boldsymbol{R}} Y = \left\{ \begin{bmatrix} \boldsymbol{G}_z & \boldsymbol{0} \end{bmatrix}, \boldsymbol{c}_z, \begin{bmatrix} \boldsymbol{A}_z & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_y \\ \boldsymbol{R}\boldsymbol{G}_z & -\boldsymbol{G}_y \end{bmatrix}, \begin{bmatrix} \boldsymbol{b}_z \\ \boldsymbol{b}_y \\ \boldsymbol{c}_y - \boldsymbol{R}\boldsymbol{c}_z \end{bmatrix} \right\}. \quad (10)$$

In what follows, some essential results on constrained zonotopes from [3] are recalled, which are required by the proposed state estimation algorithm.

Property 1: For every $Z = \{\boldsymbol{G}, \boldsymbol{c}, \boldsymbol{A}, \boldsymbol{b}\} \subset \mathbb{R}^n$, $Z \neq \emptyset$ iff $\min_{\boldsymbol{\xi}} \{\|\boldsymbol{\xi}\|_\infty : \boldsymbol{A}\boldsymbol{\xi} = \boldsymbol{b}\} \leq 1$.

Property 2: Let $P = \{\boldsymbol{z} : \boldsymbol{H}\boldsymbol{z} \leq \boldsymbol{k}\} \subset \mathbb{R}^n$ be a convex polytope in H-rep, and choose $Z = \{\boldsymbol{G}, \boldsymbol{c}\} \subset \mathbb{R}^n$ and $\boldsymbol{\sigma} \in \mathbb{R}^n$ such that $P \subseteq Z$ and $\boldsymbol{H}\boldsymbol{z} \in [\boldsymbol{\sigma}, \boldsymbol{k}]$, $\forall \boldsymbol{z} \in P$. Then, the convex polytope P can be written in CG-rep as

$$P = \left\{ [\boldsymbol{G} \ \boldsymbol{0}], \boldsymbol{c}, \left[\boldsymbol{H}\boldsymbol{G} \ \ \mathrm{diag}\left(\frac{\boldsymbol{\sigma} - \boldsymbol{k}}{2} \right) \right], \frac{\boldsymbol{k} + \boldsymbol{\sigma}}{2} - \boldsymbol{H}\boldsymbol{c} \right\}. \quad (11)$$

C. Interval arithmetic

Interval arithmetic is based on computation with sets, by regarding real compact intervals as a new number system [9]. An *interval* is defined by $\{\beta \in \mathbb{R} : \beta^{\mathrm{L}} \leq \beta \leq \beta^{\mathrm{U}}\}$, where $\beta^{\mathrm{L}}, \beta^{\mathrm{U}} \in \mathbb{R}$ are called its *endpoints*. A shorthand notation for such interval is $[\beta^{\mathrm{L}}, \beta^{\mathrm{U}}]$.

An interval is said to be *degenerate* if $\beta^{\mathrm{L}} = \beta^{\mathrm{U}}$. Arithmetic operations are defined for intervals as $[\beta^{\mathrm{L}}, \beta^{\mathrm{U}}] \odot [\gamma^{\mathrm{L}}, \gamma^{\mathrm{U}}] \triangleq \{\beta \odot \gamma : \beta \in [\beta^{\mathrm{L}}, \beta^{\mathrm{U}}], \gamma \in [\gamma^{\mathrm{L}}, \gamma^{\mathrm{U}}]\}$, with $\odot$ denoting any of the four basic arithmetic operations[1]. Note that the interval addition is equivalent to (5). Endpoint expressions for interval arithmetic operations are presented in [9]. Elementary functions, such as $\{\sin, \cos, \tan, \ln\}$, are defined for intervals through their ranges over them. Moreover, the *midpoint* and *diameter* of an interval are defined by $\mathrm{mid}\big([\beta^{\mathrm{L}}, \beta^{\mathrm{U}}]\big) \triangleq (1/2)(\beta^{\mathrm{L}} + \beta^{\mathrm{U}})$ and $\mathrm{diam}\big([\beta^{\mathrm{L}}, \beta^{\mathrm{U}}]\big) \triangleq \beta^{\mathrm{U}} - \beta^{\mathrm{L}}$.

Definition 2: A set $B \subset \mathbb{R}^n$ is an *interval vector* if there exists $(\boldsymbol{\beta}^{\mathrm{L}}, \boldsymbol{\beta}^{\mathrm{U}}) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $B = \{\boldsymbol{\beta} \in \mathbb{R}^n : \beta_j^{\mathrm{L}} \leq \beta_j \leq \beta_j^{\mathrm{U}}, j = 1, 2, \ldots, n\}$.

The interval addition and interval subtraction are defined for interval vectors component-wise. In this work, we say that an interval vector is *degenerate* if, for any j, $\beta_j^{\mathrm{L}} = \beta_j^{\mathrm{U}}$. Otherwise, it is said to be *non-degenerate*. The midpoint and diameter are also defined component-wise.

Guaranteed bounds on the range of real valued functions can be obtained by means of *interval extensions* (see Appendix). However, overestimation is recurrent due to *interval dependency* and the *wrapping effect*. The latter is inherent to multivariable mappings of interval variables, which results in overestimation and is bypassed by the use of zonotopes [10]. The former results from the multi-occurrence of a same interval variable, and can be mitigated by rearranging the algebraic expression or by using different interval extensions.

Another way to reduce overestimation of interval extensions is the use of refinements [9]. In order to proceed, we first define *interval bundles* and n_d*-partitions*.

Definition 3: Given n_b interval vectors $B_j \subset \mathbb{R}^n$, an *interval bundle* is defined by $\mathcal{B} \triangleq \bigcup_{j=1}^{n_b} B_j$.

It is clear that an interval bundle is not a interval vector in general (with exception of the case $n = 1$). Thus, an interval bundle cannot be defined through endpoints. However, it can be regarded as a *list of interval vectors*, without computing the union explicitly. Moreover, the concept of an interval bundle is closely related to subpavings [8].

[1] The interval division is defined provided that $0 \notin [\gamma^{\mathrm{L}}, \gamma^{\mathrm{U}}]$.

Definition 4: Let B be a non-degenerate interval vector in $\mathbb{R}^n$. The n_d-*partition* of B is an interval bundle $\mathcal{B} = \cup_{j=1}^{n_b} B_j$ such that $\mathcal{B} = B$, $\text{diam}(B_j) = (1/n_d)\text{diam}(B)$, and the intersection of two intervals B_j and B_i produces a degenerate interval vector in $\mathbb{R}^n$ for every $i \neq j$, with $i, j = 1, 2, \ldots, n_b$, and $n_d \in \mathbb{N}\backslash 0$.

The n_d-partition of an interval vector can be seen as a "multi-dimensional grid" with n_d divisions per dimension. By definition, the diameter of each interval vector composing the grid is inversely proportional to the number of divisions, and proportional to the diameter of the original interval vector. Note that $n_b = n_d$ for the unidimensional case.

For a real valued function $f(\cdot)$ and a n_d-partition $\mathcal{B} = \cup_{j=1}^{n_b} B_j$ of an interval vector B, a *refinement* of an interval extension $F(\cdot)$ over $\mathcal{B}$ is defined by $\cup_{j=1}^{n_b} F(B_j)$. The conservatism of a refinement can be substantially reduced if compared with single interval extensions over B (see Appendix), and since each $F(B_j)$ results in an interval vector, a refinement is also an interval bundle.

III. Nonlinear state estimation

This section presents the proposed algorithm for set-valued state estimation of the class of nonlinear systems described in Section I-A. The steps to be solved are restated here, for clarity:

a) Prediction step: Given $\hat{X}_{k-1} \subset \mathbb{R}^{n_x}$ and $W_{k-1} \subset \mathbb{R}^{n_x}$, compute a set $\bar{X}_k \subset \mathbb{R}^{n_x}$ bounding the possible trajectories of the system (1). In other words, obtain

$$\bar{X}_k \supseteq \{\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_{k-1}) + \boldsymbol{D}_w \boldsymbol{w} : \boldsymbol{x} \in \hat{X}_{k-1}, \, \boldsymbol{w} \in W_{k-1}\}. \quad (12)$$

b) Correction step: Given $\bar{X}_k \subset \mathbb{R}^{n_x}$, $V_k \subset \mathbb{R}^{n_y}$, and the measured output $y_k \in \mathbb{R}^{n_y}$, compute a set $\hat{X}_k$ bounding the trajectories of the system (1) that are consistent with the current measurement. In other words, obtain

$$\hat{X}_k \supseteq \{\boldsymbol{x} \in \bar{X}_k : \boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}_v \boldsymbol{v}, \, \boldsymbol{v} \in V_k\}. \quad (13)$$

A. The main ideas

We first present the underlying ideas of the proposed algorithm. In this work, some assumptions are made with respect to the set-valued state estimation problem described in Section I-A:

- ***A1)*** *The bounds on the process and measurement disturbances can be described by constrained zonotopes (Definition 1);*
- ***A2)*** *A initial guess $\boldsymbol{x}_0 \in \bar{X}_0$ is available, in which $\bar{X}_0$ is a constrained zonotope.*

Since constrained zonotopes can represent arbitrary convex polytopes, assumptions *A1* and *A2* impose very little restrictions to the problem. Moreover, interval vectors, parallelotopes, and zonotopes are particular constrained zonotopes [3], thus these classes of sets are covered by such assumptions.

For simplicity, we first present a solution to the correction step. The problem of computing a set $\hat{X}_k$ satisfying (13) can be restated in terms of the generalized intersection (6) [3], [15]. Define a constrained zonotope $Y_k \subset \mathbb{R}^{n_y}$ such that $Y_k = \boldsymbol{y}_k \oplus (-\boldsymbol{D}_v V_k)$. Then, (13) is restated as

$$\hat{X}_k \supseteq \{\boldsymbol{x} \in \bar{X}_k : \boldsymbol{C}\boldsymbol{x} \in Y_k\} = \bar{X}_k \cap_{\boldsymbol{C}} Y_k. \quad (14)$$

Since Y_k is a constrained zonotope, if $\bar{X}_k$ is also a constrained zonotope, an exact bound for (14) can be obtained through (10). However, even if $\bar{X}_0$ and W_k are constrained zonotopes, the *reachable sets* (see Appendix) of the nonlinear system (1) are not even convex in general [14]. Thus, the main challenge is to solve the prediction step (12) such that the obtained bound $\bar{X}_k$ is a constrained zonotope, which is the main contribution of this work.

An early attempt to obtain bounds $\bar{X}_k$ as zonotopes can be found in [12]. However, since zonotopes are centrally symmetric, severe conservativeness can occur in over-approximating a non-convex set by a zonotope. Since constrained zonotopes can describe at least any convex polytope, the associated conservativeness is substantially reduced. However, for the knowledge of the authors, no methods for approximating the reachable sets of nonlinear systems using interval bundles and exploiting constrained zonotopes have been proposed so far.

In order to bound the possible trajectories of the nonlinear system (1) using constrained zonotopes, we subdivide the prediction step into three main problems:

- ***P1)*** *Compute an interval bundle $\mathcal{B} = \cup_{j=1}^{n_b} B_j$ such that $\hat{X}_{k-1} \subseteq \mathcal{B}$;*
- ***P2)*** *Compute a refinement $\cup_{j=1}^{n_b} F(B_j, \boldsymbol{u}_{k-1})$, where $F(\cdot)$ is an interval extension of $\boldsymbol{f}(\cdot)$;*
- ***P3)*** *Compute a constrained zonotope $\bar{X}_k$ such that $\cup_{j=1}^{n_b} F(B_j) \oplus \boldsymbol{D}_w W_{k-1} \subseteq \bar{X}_k$.*

While problem *P2* can be solved through interval arithmetic, for the best knowledge of the authors, no methods have been proposed to solve the problems *P1* and *P3* so far. Our solution to these problems is presented in the next subsection, along with the complete state estimation algorithm.

B. The algorithm

In order to obtain tight bounds on the possible trajectories in (12), we formalize a method for computing the smallest interval vector containing a constrained zonotope, briefly mentioned in [3].

Proposition 1: Let $Z = \{\boldsymbol{G}, \boldsymbol{c}, \boldsymbol{A}, \boldsymbol{b}\} \subset \mathbb{R}^n$. The *interval hull* of Z is obtained component-wise by solving the pair of linear programs $\zeta_j^{\text{L}} = \min\{\boldsymbol{c}_j + \boldsymbol{G}_j \boldsymbol{\xi} : \|\boldsymbol{\xi}\|_\infty \leq 1, \, \boldsymbol{A}\boldsymbol{\xi} = \boldsymbol{b}\}$, $\zeta_j^{\text{U}} = \max\{\boldsymbol{c}_j + \boldsymbol{G}_j \boldsymbol{\xi} : \|\boldsymbol{\xi}\|_\infty \leq 1, \, \boldsymbol{A}\boldsymbol{\xi} = \boldsymbol{b}\}$, for each $j = 1, 2, \ldots, n$, with $\boldsymbol{G}_j$ denoting the j-th line of $\boldsymbol{G}$, and $Z \subseteq [\boldsymbol{\zeta}^{\text{L}}, \, \boldsymbol{\zeta}^{\text{U}}]$.

Proof: The proof follows from Definition 2. □

The ability to compute the interval hull of a constrained zonotope is required in what follows. In order to solve *P1* with as less overestimation as possible, our goal is to obtain the *least interval bundle* containing $\hat{X}_{k-1}$. For such end, obtaining the interval hull of $\hat{X}_{k-1}$ is of utmost importance. The next proposition shows how to generate the least interval bundle containing $\hat{X}_{k-1}$ from a given n_d-partition of the interval hull of $\hat{X}_{k-1}$.

Proposition 2: Let $Z = \{\boldsymbol{G}, \boldsymbol{c}, \boldsymbol{A}, \boldsymbol{b}\} \subset \mathbb{R}^n$. For a given n_d, the *least interval bundle* $\mathcal{B} = \cup_{j=1}^{n_b} B_j$ containing Z is

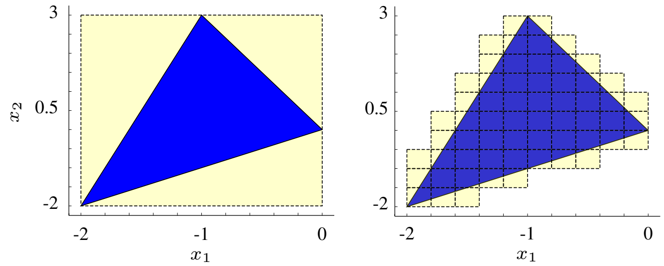

Fig. 1. Solid lines (blue) depicts a constrained zonotope $Z = \{\boldsymbol{G}, \boldsymbol{c}, \boldsymbol{A}, \boldsymbol{b}\}$, in which $\boldsymbol{G} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}$, $\boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\boldsymbol{A} = \begin{bmatrix} -2 & 1 & -1 \end{bmatrix}$, and $\boldsymbol{b} = 2$. Left: the interval hull of Z (dashed). Right: the least interval bundle containing Z, for a 10-partition (dashed).

obtained by the union of all intervals B_j, resulting from the n_d-partition of the interval hull of Z, such that

$$Z \cap_{\boldsymbol{I}} B_j \neq \emptyset, \tag{15}$$

where $\boldsymbol{I}$ denotes the identity matrix.

Proof: Let H denote the interval hull of Z. Clearly, for a given n_d, the n_d-partition $\bar{\mathcal{B}} = \cup_{j=1}^{\bar{n}_b} B_j$ of H is the least interval bundle containing H. Since for every B_j in $\bar{\mathcal{B}}$ such that $Z \cap_{\boldsymbol{I}} B_j = \emptyset$, $\nexists \boldsymbol{z} \in Z : \boldsymbol{z} \in B_j$, by removing all such B_j from $\bar{\mathcal{B}}$, yields $\mathcal{B} = \cup_{j=1}^{n_b} B_j \supset Z$, with $n_b \leq \bar{n}_b$. Since for every B_j in $\mathcal{B}$, $\exists \boldsymbol{z} \in Z : \boldsymbol{z} \in B_j$, by removing any such B_j from $\mathcal{B}$, yields $\tilde{\mathcal{B}} \not\supseteq Z$. □

Remark 1: Since every interval vector is a constrained zonotope (see Appendix), the relation (15) can be verified *exactly* through (10) and Property 1 for each B_j, from which we are able to compute the least interval bundle containing $\hat{X}_{k-1}$ effectively. Further reductions in overestimation can be achieved by increasing the number of divisions when computing the least interval bundle.

Fig. 1 illustrates the underlying ideas regarding Propositions 1 and 2. We are then able to state Algorithm 1, which solves problem *P1*.

Algorithm 1 Solver for P1

1: Compute the interval hull of $\hat{X}_{k-1}$ by means of Proposition 1
2: Compute the n_d-partition of the interval hull according to Definition 3
3: Compute the least interval bundle $\mathcal{B} = \cup_{j=1}^{n_b} B_j$ containing $\hat{X}_{k-1}$ through Proposition 2

Problem *P2* can be solved straightforwardly by using interval arithmetic. The associated algorithm is presented in Algorithm 2, which essentially generates a refinement of $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_{k-1})$ over $\mathcal{B} = \cup_{j=1}^{n_b} B_j$, thus with reduced overestimation (see Appendix).

Algorithm 2 Solver for P2

1: **for** $j = 1, ..., n_b$ **do**
2: Compute an interval extension of $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_{k-1})$ for B_j, from $\mathcal{B} = \cup_{j=1}^{n_b} B_j$ obtained through Algorithm 1
3: **end for**

Algorithm 2 allows different interval extensions to be used[2], which may be tested offline for choosing which one leads to less conservatism, according to the nonlinear expression $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_{k-1})$. Furthermore, as in the case of algorithms involving *zonotope bundles* [16], Algorithm 2 can be parallelized in order to improve computational performance.

In order to propose a solution to problem *P3*, we first define the concept of *interval vertices*.

Definition 5: Let $\mathcal{B} = \cup_{j=1}^{n_b} B_j$ be an interval bundle in $\mathbb{R}^n$. The j-th interval vector B_j is an *interval vertex* of $\mathcal{B}$ iff at least one component of any of its endpoints coincide with one component of any endpoint of the interval hull of $\mathcal{B}$.

Proposition 3: Let $\mathcal{B} = \cup_{j=1}^{n_b} B_j$ be an interval bundle in $\mathbb{R}^n$. The *interval hull* of $\mathcal{B}$ is obtained component-wise by searching for $\zeta_i^{\mathrm{L}} = \min\{\beta_{j,i}^{\mathrm{L}}\}$, $\zeta_i^{\mathrm{U}} = \max\{\beta_{j,i}^{\mathrm{U}}\}$ for each $i = 1, 2, \ldots, n$, and $\mathcal{B} \subseteq [\boldsymbol{\zeta}^{\mathrm{L}}, \boldsymbol{\zeta}^{\mathrm{U}}]$.

Proof: Similarly to Proposition 1, the proof follows from the definition of an interval vector. □

Ideally, to solve problem *P3* with reduced conservatism, a convex hull for $\cup_{j=1}^{n_b} F(B_j, \boldsymbol{u}_{k-1})$ must be obtained. However, due to the complexity of this operation, our solution consists in the following: **(i)** build an auxiliary convex polytope, using the *midpoints of the interval vertices* of $\cup_{j=1}^{n_b} F(B_j, \boldsymbol{u}_{k-1})$ as its vertices, and denote it by $\tilde{X}_k$; and **(ii)** move the hyperplanes of $\tilde{X}_k$ to obtain a constrained zonotope bounding $\cup_{j=1}^{n_b} F(B_j, \boldsymbol{u}_{k-1})$. Algorithm 3 clarifies how to perform the first step.

Algorithm 3 Auxiliary convex polytope

1: Compute the interval vertices of $\cup_{j=1}^{n_b} F(B_j, \boldsymbol{u}_{k-1})$ according to Definition 5
2: Build a convex polytope in V-rep using the midpoints of the interval vertices
3: Convert the result from V-rep to H-rep

Fig. 2 demonstrates the essential stages to obtain the auxiliary convex polytope. Algorithm 3 requires the conversion from V-rep to H-rep, which can be expensive. However, in the proposed state estimation algorithm such conversion is performed only once per iteration. In addition, the second step is solved by means of Algorithm 4, which requires the computation of n_h linear programs.

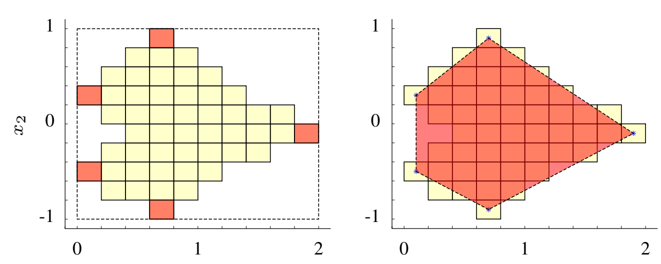

Fig. 2. Solid lines depicts an interval bundle $\mathcal{B}$. Left: the interval hull (dashed) and the interval vertices (red) of $\mathcal{B}$. Right: the auxiliary convex polytope, formed by the midpoints of the interval vertices (red, dashed).

[2]E.g., natural interval extension, mean value extension, and others [9].

In some cases, the refinement $\cup_{j=1}^{n_b} F(B_j, \boldsymbol{u}_{k-1})$ may not have enough interval vertices to form a simplex in $\mathbb{R}^{n_x}$. In such situation, a Minkowski sum can be performed over the associated vertices (midpoints) in V-rep using a "very small" zero-centered interval vector, to generate additional vertices before performing the conversion to H-rep.

Algorithm 4 Moving hyperplanes

Given $\tilde{X}_k$ in the H-rep $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{H}\boldsymbol{x} \leq \boldsymbol{k}\}, \boldsymbol{k} \in \mathbb{R}^{n_h}$
1: **for** $i = 1, 2, \dots n_h$ **do**
2: $k'_i = \min_{k_i}\{k_i : \boldsymbol{H}_i \boldsymbol{x}^{\{j,l\}} \leq k_i, \forall j, l\}$,
where $\boldsymbol{x}^{\{j,l\}}$ denotes the l-th vertex of the j-th interval vector in $\cup_{j=1}^{n_b} F(B_j, \boldsymbol{u}_{k-1})$
3: **end for**
4: Convert the resulting $\{\boldsymbol{H}, \boldsymbol{k}'\}$ to CG-rep using (11)

We now conclude our solution to *P3*. Given the constrained zonotope $\tilde{X}_k \supset \cup_{j=1}^{n_b} F(B_j, \boldsymbol{u}_{k-1})$ (obtained using Algorithms 3 and 4), a constrained zonotope $\bar{X}_k$ bounding the trajectories of the system can be obtained through

$$\bar{X}_k = (\tilde{X}_k \cap_{\boldsymbol{I}} H) \oplus \boldsymbol{D}_w W_{k-1}, \quad (16)$$

where H denotes the interval hull of $\cup_{j=1}^{n_b} F(B_j, \boldsymbol{u}_{k-1})$. Since the vertices of the constrained zonotope $\tilde{X}_k$ may extrapolate the original bounds of H, the intersection is introduced in order to obtain tighter bounds, and is computed using (6). The linear image and the Minkowski sum in (16) can be computed exactly using (8) and (9), respectively.

By solving problems *P1*, *P2* and *P3*, the prediction step (12) is solved. Moreover, since $\bar{X}_k$ is a constrained zonotope, the correction step (14) can be performed exactly using (10). The proposed state estimation algorithm can finally be stated, which we call as *constrained zonotope and interval bundle* (CZIB) algorithm.

Algorithm 5 CZIB algorithm

Prediction step
1: Compute $\mathcal{B} = \cup_{j=1}^{n_b} B_j \supset \hat{X}_{k-1}$ using Algorithm 1
2: Compute $\cup_{j=1}^{n_b} F(B_j, \boldsymbol{u}_{k-1})$ using Algorithm 2
3: Compute $\bar{X}_k$ through Algorithms 3, 4 and (16)
Correction step
4: Compute $\hat{X}_k$ using (14)

The CZIB algorithm is highly tunable. The compromise between conservatism and complexity is given by the number of divisions in Algorithm 1 and the interval extension used in Algorithm 2. Moreover, unlike known state estimation algorithms for linear systems [3] [15], due to the transformations performed in the prediction step (from CG-rep to interval bundle and vice-versa), the complexity of the constrained zonotope $\hat{X}_k$ *does not increase with time*. Such complexity is directly associated with the number of interval vertices (see Fig. 2), and can be limited by choosing only a given number of interval vertices among the existing ones, according to a desired criteria.

C. State estimation using interval bundles

It is possible to perform the prediction step (12) and the correction step (13), using only interval bundles, i.e., without performing transformations involving constrained zonotopes. However, assumptions *A1* and *A2* (see Section III-A) should be restricted to interval variables, or interval bundles containing the associated constrained zonotopes must be available. If intervals vectors are used as bounds for disturbances, the wrapping effect arises from the linear mappings in (12) and (13), which is avoided by the use of constrained zonotopes.

Moreover, in order to perform (14) using interval bundles, a set of consistent states must be obtained beforehand [12]. If the latter is also an interval bundle, the intersection must be computed *interval-by-interval*, yielding a worst-case factorial increase in the number of intervals composing the bundle. On the other hand, the intersection (14) is performed directly and exactly using constrained zonotopes.

IV. NUMERICAL EXAMPLE

This section presents a numerical experiment to demonstrate the performance of the CZIB algorithm.

We address the second example presented in [12]. Consider a isothermal gas-phase reactor, charged with an initial amount of A and B, in which the species are allowed to react according to the reversible reaction $2A \rightleftharpoons B$. Define the system states as the partial pressure of each species in the reactor. The objective is to estimate all the system states while measuring the total pressure of the vessel as reaction proceeds. The dynamic equations of the system are given by (discretized through Euler)

$$x_{1,k} = x_{1,k-1} + T_s\left(-2k_1 x_{1,k-1}^2 + 2k_2 x_{2,k-1}\right) + k_3 w_{1,k-1},$$
$$x_{2,k} = x_{2,k-1} + T_s\left(k_1 x_{1,k-1}^2 - k_2 x_{2,k-1}\right) + k_3 w_{2,k-1},$$

where $k_1 = 0.16/60$ s^{-1} atm^{-1}, $k_2 = 0.0064/60$ s^{-1}, $k_3 = 0.0001$ and $T_s = 6$ s [12]. The measured output is given by $y_k = \begin{bmatrix} 1 & 1 \end{bmatrix} \boldsymbol{x}_k + v_k$. Existing disturbances are bounded by $\|\boldsymbol{w}_k\|_\infty \leq 1$, and $\|v_k\|_\infty \leq 0.3$. The initial states are assumed to belong to the constrained zonotope

$$\bar{X}_0 \triangleq \left\{ \begin{bmatrix} 2.5 & 0 & 1.0 \\ 0 & 0.5 & 1.0 \end{bmatrix}, \begin{bmatrix} 2.5 \\ 1.0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, 1 \right\}.$$

In the experiment, we use 12-partitions for Algorithm 1 and the *natural interval extension* [9] for Algorithm 2. An experiment using 3-partitions is also conducted to evaluate the resulting conservatism. The interval extensions were performed using the INTLAB toolbox [17]. The conversion from V-rep to H-rep in Algorithm 3 was performed using the MPT toolbox [18]. Moreover, we perform an initial correction step $\hat{X}_0 = \bar{X}_0 \cap_{\boldsymbol{C}} Y_0$ to obtain tighter initial bounds, where $Y_0 = \boldsymbol{y}_0 \oplus (-\boldsymbol{D}_v V_0)$. For comparison, a third experiment is conducted using the algorithm proposed in [19], based on first-order Taylor approximation by zonotopes, with complexity limited to 20 generators using the reduction algorithm of [5]. The real initial states are $\boldsymbol{x}_0 = \begin{bmatrix} 4.9 & 1.5 \end{bmatrix}^T$.

Fig. 3 shows the time evolution of the system states, and the estimated bounds provided by the CZIB algorithm[3], and the algorithm of [19]. Although reasonable for x_1, the bounds obtained using the latter are substantially conservative for x_2. Besides, the bounds obtained using the 12-partition CZIB are significantly less conservative in all cases, demonstrating the

[3]The plotted curves are the bounds of the interval hull of $\hat{X}_k$.

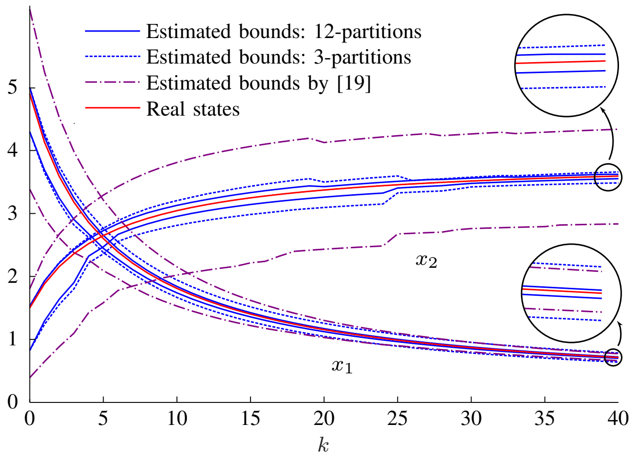

Fig. 3. Time evolution of the estimated bounds and the real states.

attained reduction in overestimation using the proposed method. In order to bypass the "simplex problem" (Section III-B), a Minkowski sum was performed using a zero-centered interval vector, with 10% of the total diameter, at every time it occurred. To conclude, the description of the constrained zonotope $\hat{X}_{40}$ in the 12-partition experiment was given by modest 13 generators and 9 constraints, without using any complexity reduction algorithm. The average execution times for the 3-partition and 12-partition experiments were 0.0855 and 0.6013 seconds per iteration, respectively, in MATLAB and CPLEX, with 8GB RAM and Intel Core i7 4510U 3.1 GHz processor, using only sequential computation.

V. Conclusions

This paper proposed a novel set-valued state estimation algorithm for a class of nonlinear discrete-time systems. By combining important properties from constrained zonotopes and interval bundles, a highly tunable state estimation algorithm was developed, capable of providing tight bounds for the set-valued estimation problem by constrained zonotopes. A numerical example has been presented to demonstrate the performance of the proposed strategy.

References

[1] L. Chisci, A. Garulli, and G. Zappa, "Recursive state bounding by parallelotopes," *Automatica*, vol. 32, no. 7, pp. 1049–1055, 1996.

[2] V. T. H. Le, C. Stoica, T. Alamo, E. F. Camacho, and D. Dumur, "Zonotopic guaranteed state estimation for uncertain systems," *Automatica*, vol. 49, no. 11, pp. 3418–3424, 2013.

[3] J. K. Scott, D. M. Raimondo, G. R. Marseglia, and R. D. Braatz, "Constrained zonotopes: a new tool for set-based estimation and fault detection," *Automatica*, vol. 69, pp. 126–136, 2016.

[4] F. Schweppe, "Recursive state estimation: Unknown but bounded errors and system inputs," *IEEE Transactions on Automatic Control*, vol. 13, no. 1, pp. 22–28, 1968.

[5] C. Combastel, "A state bounding observer based on zonotopes," in *Proc. of the 2003 ECC*, 2003, pp. 2589–2594.

[6] B. S. Chabane, C. S. Maniu, T. Alamo, E. Camacho, and D. Dumur, "Improved set-membership estimation approach based on zonotopes and ellipsoids," in *Proc. of the 2014 ECC*, 2014, pp. 993–998.

[7] C. Combastel, "Merging Kalman filtering and zonotopic state bounding for robust fault detection under noisy environment," in *Proc. of the 9th IFAC Symposium on Fault Detection, Supervision and Safety for Technical Processes*, 2015, pp. 289–295.

[8] L. Jaulin, M. Kieffer, O. Didrit, and Éric Walter, *Applied Interval Analysis*. Springer, 2001.

[9] R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*. Philadelphia, PA, USA: SIAM, 2009.

[10] W. Kühn, "Rigorously computed orbits of dynamical systems without the wrapping effect," *Computing*, vol. 61, no. 1, pp. 47–67, 1998.

[11] M. Althoff, O. Stursberg, and M. Buss, "Reachability analysis of nonlinear systems using conservative linearization," in *Proc. of the 47th IEEE CDC*, 2008, pp. 4042–4048.

[12] T. Alamo, J. Bravo, and E. Camacho, "Guaranteed state estimation by zonotopes," *Automatica*, vol. 41, no. 6, pp. 1035–1043, 2005.

[13] T. Alamo, J. M. Bravo, M. J. Redondo, and E. F. Camacho, "A set-membership state estimation algorithm based on DC programming," *Automatica*, vol. 44, no. 1, pp. 216–224, 2008.

[14] M. Althoff, "Reachability analysis of nonlinear systems using conservative polynomialization and non-convex sets," *Hybrid Systems: Computation and Control*, pp. 173–182, 2013.

[15] D. M. Raimondo, G. R. Marseglia, R. D. Braatz, and J. K. Scott, "Closed-loop input design for guaranteed fault diagnosis using set-valued observers," *Automatica*, vol. 74, pp. 107–117, 2016.

[16] M. Althoff and B. H. Krogh, "Zonotope bundles for the efficient computation of reachable sets," in *Proc. of the 50th IEEE CDC*, 2011, pp. 6814–6821.

[17] S. M. Rump, "INTLAB — INTerval LABoratory," in *Developments in Reliable Computing*, T. Csendes, Ed. Dordrecht: Springer Netherlands, 1999, pp. 77–104.

[18] M. Herceg, M. Kvasnica, C. N. Jones, and M. Morari, "Multi-parametric toolbox 3.0," in *Proc. of the 2013 ECC*, 2013, pp. 502–510.

[19] C. Combastel, "A state bounding observer for uncertain non-linear continuous-time systems based on zonotopes," in *Proc. of the 44th IEEE CDC, and 2005 ECC*, 2005, pp. 7228–7234.

Appendix

Zonotopes are a special class of convex sets that have become popular in set-based state estimation and fault detection of linear systems. A zonotope $Z \subset \mathbb{R}^n$ is defined by $Z \triangleq \{\boldsymbol{c} + \boldsymbol{G}\boldsymbol{\xi} : \|\boldsymbol{\xi}\|_\infty \leq 1\}$, which is referred as generator representation (G-rep), with $\boldsymbol{c}$ and $\boldsymbol{G}$ being the *center* and *generators*, respectively. Zonotopes are closed under (4) and (5), which can be computed exactly and efficiently [10]. However, the intersection (6) is not a zonotope in general, and tight bounds of the result are difficult to compute [12].

Every interval vector $B \subset \mathbb{R}^n$ can be written in G-rep as $B = \{(1/2)\text{diag}(\text{diam}(B)), \text{mid}(B)\}$ [10]. Therefore, interval vectors are zonotopes, and thus constrained zonotopes. Moreover, an interval valued function $F(\cdot)$ is an *interval extension* of a real valued function $f(\cdot)$, if for degenerate interval arguments, $F([\beta, \beta]) = f(\beta)$ [9].

Given system (1), initial states $\boldsymbol{x}_0 \in X_0$, inputs $\boldsymbol{u} \in U$, and disturbances $\boldsymbol{w} \in W$, the *reachable set* at a given $k = r$ is the set of all reachable states, given by $\{\boldsymbol{\phi}^r(\boldsymbol{x}_0, \boldsymbol{u}, \boldsymbol{w}) : \boldsymbol{x}_0 \in X_0,\, \boldsymbol{u} \in U, \boldsymbol{w} \in W\}$, where $\boldsymbol{\phi}(\cdot) \triangleq \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) + \boldsymbol{D}_w \boldsymbol{w}$, and $\boldsymbol{\phi}^r$ denotes r compositions of $\boldsymbol{\phi}(\cdot)$.

The motivation of the use of interval bundles comes from the following. Let $f(\cdot)$ be a real valued function, and $B = [\boldsymbol{\beta}^{\text{L}}, \boldsymbol{\beta}^{\text{U}}] \subset \mathbb{R}^n$ an interval vector. An interval extension $F(\cdot)$ is said to be *Lipschitz* if there exists a constant $l > 0$ such that $\|\text{diam}(F([\boldsymbol{\beta}^{\text{L}}, \boldsymbol{\beta}^{\text{U}}]))\|_\infty \leq l \|\text{diam}([\boldsymbol{\beta}^{\text{L}}, \boldsymbol{\beta}^{\text{U}}])\|_\infty$. Then, $\|\text{diam}(E)\|_\infty \leq (k/n_d)\|\text{diam}([\boldsymbol{\beta}^{\text{L}}, \boldsymbol{\beta}^{\text{U}}])\|_\infty$, for some constant k, where E denotes the *excess width* of the refinement $\cup_{j=1}^{n_b} F(B_j)$ for a n_d-partition $\cup_{j=1}^{n_b} B_j$ of B [9].

The excess width is a measure of the overestimation of an interval extension [9]. It is clear that the use of a refinement reduces the overestimation of a given interval extension, and that this reduction is proportional to the number of divisions within the n_d-partition.