2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

# Merging Vehicles at Junctions using Mixed-Integer Model Predictive Control

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***Abstract*— A method is proposed for vehicle merging scenarios in junctions with relative cost prioritization. The method is based on Model Predictive Control, employing Mixed Integer Quadratic Program optimization. The scheme provides optimal control properties while maintaining safety and recursive feasibility. The latter properties are ensured through positive control invariance of simple time headway constraints. For examples with two vehicles, tunable prioritization and gap acceptance are verified and presented on a decision graph. Priorities are then demonstrated to be respected in an example with four vehicles.**

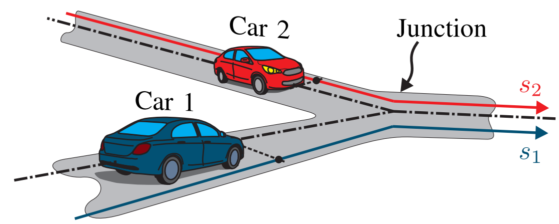


Fig. 1. A schematic of a two-car merging scenario. $s_1$, $s_2$ are positions along route.

## I. Introduction

This paper addresses the control of vehicles merging at an uncontrolled junction, *i.e.* one not equipped traffic lights or signals. This is a small but important part of the great challenge of autonomous driving [1], especially in urban settings [2]. It is a hybrid problem, combining discrete choice of sequencing between cars in the merged stream with continuous choice of speed, subject to constraints. (Only the longitudinal control is considered here.) The problem is therefore combinatorial, non-convex, and NP-hard. We adopt the framework of Model Predictive Control (MPC) to handle the constraints [3], with Mixed-Integer Quadratic Programming as the optimizer to handle the non-convexity [4].

Set invariance [5] is known to play a key role in assuring the feasibility of MPC [6], including cases using MIQP to embed logic [7]. Invariance has also been employed to provide results on safety [8], [9]. A key contribution of this manuscript is a set of conditions for the invariance of a simple time headway constraint, *i.e.* the requirement to keep a certain time separation between the controlled vehicle and a vehicle or junction ahead. This advice is found in the UK Highway Code as the "two second rule" [10, Rule 126], for example. We employ it in a centralized control scheme for now, noting that solution of the centralized problem is a route to cooperative behaviour in a more scalable distributed decision-making framework [11].

The recent survey by Chen and Englund [12] provides a broad overview of research into junction control. Colombo and Del Vecchio [13] studied the full hybrid problem of crossing at junctions, noting the importance of invariance, in terms of complexity of the scheduling. Others have used continuous MPC or similar optimal control approaches for junction crossing, solving sequencing by either discrete scheduling protocols [14], [15], heuristics [16], [17], first-in-first-out [18], a putative higher-level control [19] or searching through all combinations [20]. Müller *et al.* [21] tackled the scheduling problem using MILP. Altché *et al.* [22] developed mixed-integer coordination constraints for longitudinal control, including both scheduling and motion planning, which we also employ. Those authors later extended their work into a receding horizon MPC framework [23], including a guarantee of feasibility, but based not in invariance but on a sufficiently long horizon. Qian *et al.* [17] proposed an MPC framework with pre-defined sequencing. Their 'brake-safe' concept for collision avoidance is a form of set invariance. Hafner *et al.* [24] used a similar constraint representation for merging with backward reachable sets but a different control approach, including vehicle-to-vehicle communications and experimental validation.

The key contribution of this paper is the use of simple headway constraint to guarantee recursive feasibility when using an arbitrary length horizon.

## II. Problem Statement

Consider a set of vehicles $\mathcal{N} := \{1, \dots, N\}$ with kinematics model represented by the following linear state space system:

$$\forall n\colon\ x_n^{k+1} = \begin{bmatrix} 1 & \delta\mathrm{t} \\ 0 & 1 \end{bmatrix} x_n^k + \begin{bmatrix} \frac{\delta\mathrm{t}^2}{2} \\ \delta\mathrm{t} \end{bmatrix} u_n^k, \qquad n \in \mathcal{N}, \quad (1)$$

where the discrete time period is $\delta$t, superscripts signify time at discrete time *e.g.* $k$ while agents distinguished by subscripts *e.g.* $n$. System states are $x^k = [s^k, v^k]^T$ with $s^k$ position along a predefined, fixed route and $v^k$ speed. Furthermore, control input $u^k = a^k$ is the acceleration. The vehicles have their respective limits defined as:

$$0 \leq \quad v_n^k \ \leq v_{n\,\max} \quad (2)$$
$$a_{n\,\min} \leq \quad a_n^k \ \leq a_{n\,\max}. \quad (3)$$

Centralized coordination fashion is employed for the vehicles with perfect information sharing in the vicinity of the junction. Such a merging scenario is pictured on Fig. 1 with two single lane junction arms with one automated car on

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each, where from safe initial states of vehicles controlled safe merging has to be executed.

## III. Control System

### A. Model Predictive Control

This section reviews the MPC problem formulation. At each time step, the MPC solves the following weighted multi-objective optimal control problem in a receding-horizon fashion with $T$ time steps prediction:

$$\min_{\{u,x\}} \sum_{j=0}^{T-1} \sum_{n=1}^{N} w_n l\left(x_n^{k+j+1}, u_n^{k+j}\right) \tag{4a}$$

subject to

$$x_n^{k+0} = x_n^k \tag{4b}$$
$$x_n^{k+j+1} = A x_n^{k+j} + B u_n^{k+j} \tag{4c}$$
$$x_n^{k+j} \in \mathcal{P}_n \tag{4d}$$
$$u_n^{k+j} \in \mathcal{U}_n, \tag{4e}$$

where $l\left(x_n, u_n\right)$ is the stage cost for agent $n$ weighted by $w_n$, where $\sum_{n=1}^{N} w_n = 1$. Vehicle kinematics (1) is represented as equality constraints (4c). Sets $\mathcal{P}_n$ and $\mathcal{U}_n$ represent the admissible states, especially speed limits, and control inputs from (3). Each vehicle has its respective policy presented as a quadratic objective:

$$l\left(x_n, u_n\right) = Q\left(v_n - v_{\mathrm{des}n}\right)^2 + R\left(u_n\right)^2, \tag{5}$$

where $Q$ and $R$ are positive weighting constants and $v_{\mathrm{des}n}$ is the desired speed.

### B. Collision Set

Collision sets $C_{pq}$ are defined between each conflicting vehicle pairs $p, q \in \mathcal{N}$ $(p \neq q)$ over the respective $s_p$ and $s_q$ position planes. $C_{pq}$ is convex bounding polytope of the collection of undesired configurations *e.g.* physical contact between vehicles. A convex bounding polygon approximation approach is used in [22] with shown cases for simple traffic situations. Our main interest, the merging scenario in *e.g.* a simplistic urban junction Fig. 1 results, Fig. 2 characterized by four inequalities.

$$\begin{aligned} \mathcal{C}_{pq} = \{ s_p, s_q \in \mathbb{R}^2 \mid s_p > L_{pq}^1, \quad s_q > L_{pq}^2, \\ s_p - s_q > L_{pq}^3, \quad s_q - s_p > L_{pq}^4 \ \}, \end{aligned} \tag{6}$$

The right hand side of the inequalities $L_{pq}^1 := s_{\mathrm{off}p} - l_{pq}^1$, $L_{pq}^2 := s_{\mathrm{off}q} - l_{pq}^2$, $L_{pq}^3 := s_{\mathrm{off}p} - s_{\mathrm{off}q} - l_{pq}^1$, $L_{pq}^4 := s_{\mathrm{off}q} - s_{\mathrm{off}p} - l_{pq}^2$, are describing the shape of collision set with $l_{pq}^1, l_{pq}^2$ constants and its offset on the plane with $s_{\mathrm{off}p}$ and $s_{\mathrm{off}q}$ according Fig. 2. The offset constants are chosen for the positions where transition to car-following is happening.

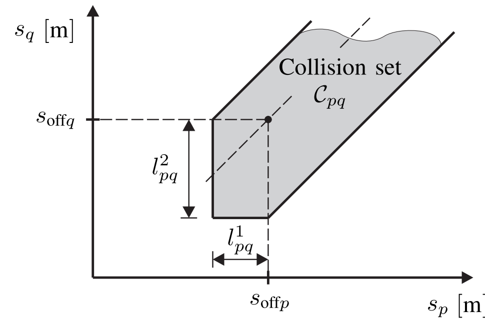


Fig. 2. $\mathcal{C}_{pq}$ collision set is indicated for 2 merging vehicles

### C. Mixed Integer Constraints

The separating hyperplane theorem is employed to exclude vehicle states from the convex $C_{pq}$ collision set. As well as remaining outside the collision set, additional headway margins $vt_{\mathrm{h}}$ are included, where $t_{\mathrm{h}} \geq 0$ is the headway time. These margins will later be interpreted in terms of constraint invariance. The set of inequalities presented act as separating hyperplanes for $v \geq 0$ condition from (2): $\forall i \in [1 \dots T]$ :

$$s_p^{k+i} \leq L_{pq}^1 - t_{\mathrm{h}p} v_p^{k+i} \tag{7a}$$
$$\text{or} \quad s_q^{k+i} \leq L_{pq}^2 - t_{\mathrm{h}q} v_q^{k+i} \tag{7b}$$
$$\text{or} \quad s_p^{k+i} - s_q^{k+i} \leq L_{pq}^3 - t_{\mathrm{h}p} v_p^{k+i} \tag{7c}$$
$$\text{or} \quad s_q^{k+i} - s_p^{k+i} \leq L_{pq}^4 - t_{\mathrm{h}q} v_q^{k+i}. \tag{7d}$$

Using the above set of inequalities, relaxation is formulated with "big-M" mixed integer representation [7], [4]:
$\forall i \in [0 \dots T-1]$ :

$$s_p^{k+i} \leq L_{pq}^1 - t_{\mathrm{h}p} v_p^{k+i} + M b_{pq,1}^{k+i}, \tag{8a}$$
$$s_q^{k+i} \leq L_{pq}^2 - t_{\mathrm{h}q} v_q^{k+i} + M b_{pq,2}^{k+i}, \tag{8b}$$
$$s_p^{k+i} - s_q^{k+i} \leq L_{pq}^3 - t_{\mathrm{h}p} v_p^{k+i} + M b_{pq,3}^{k+i}, \tag{8c}$$
$$s_q^{k+i} - s_p^{k+i} \leq L_{pq}^4 - t_{\mathrm{h}q} v_q^{k+i} + M b_{pq,4}^{k+i}, \tag{8d}$$
$$\sum_{j=1}^{4} b_{pq,j}^{k+i} \leq 3 \tag{8e}$$

where $b_{pq,j}^{k+i}$ are the binary decision variables and $M$ is a constant chosen sufficiently large. Furthermore (8e) provides the relaxation of no more than three from the total four hyperplanes per time step.

Corner cutting prevention is essential for these problem types [25]. However a stricter formulation is introduced for increased safety having no assumption on the dynamics of the leader vehicle in car-following phase *e.a.* (8c, 8d).

$$s_p^{k+i+1} \leq L_{pq}^1 - t_{\mathrm{h}p} v_p^{k+i+1} + M b_{pq,1}^{k+i}, \tag{9a}$$
$$s_q^{k+i+1} \leq L_{pq}^2 - t_{\mathrm{h}q} v_q^{k+i+1} + M b_{pq,2}^{k+i}, \tag{9b}$$
$$s_p^{k+i+1} - s_q^{k+i} \leq L_{pq}^3 - t_{\mathrm{h}p} v_p^{k+i+1} + M b_{pq,3}^{k+i}, \tag{9c}$$
$$s_q^{k+i+1} - s_p^{k+i} \leq L_{pq}^4 - t_{\mathrm{h}q} v_q^{k+i+1} + M b_{pq,4}^{k+i} \tag{9d}$$

bind the same binary variables with the same inequalities from (8) acting on the states at next time step, except for the leader vehicle. In contrary to the classical corner cutting

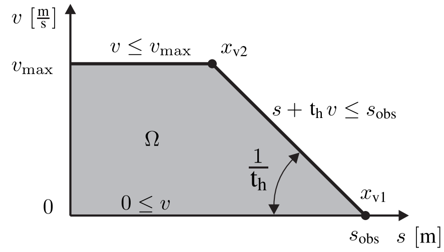


Fig. 3. Shaded area represents $\Omega$ set

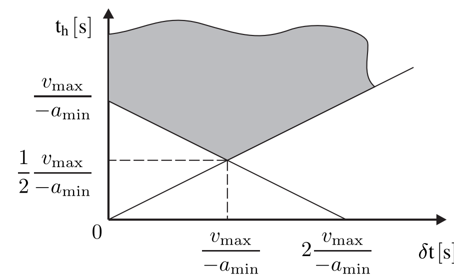


Fig. 4. Shaded region indicates $t_{\mathrm{h}}$-$\delta\mathrm{t}$ parameter choices that ensure $\Omega$ set to be positive control invariant

prevention, leader vehicle states are one time step behind in (9c) and (9d). Thus within a bundle of inequalities sharing the same binary variable the leader vehicle is treated as a static obstacle decoupling any next time step leader state predictions.

## IV. Invariance of Constraints

### A. Invariance of Headway Constraint

All of the constraints in (7) can be re-arranged into the form $s_p^{k+i} + t_{\mathrm{h}p} v_p^{k+i} \le s_{\mathrm{obs}}^{k+i}$, where $s_{\mathrm{obs}}^{k+i}$ represents some right-hand side limit corresponding to a preceding car or the entry to the junction. This section considers the invariance of such sets $\Omega$ in general. The aim is to prove that both vehicles are able to achieve a full stop by remaining in their respective invariant sets before reaching the collision set of the junction. For this section, the indices $k+i$ will be omitted and the limit position will be assumed to be a static obstacle at $s_{\mathrm{obs}}$.

*Theorem 1:* $\Omega := \{s, v \in \mathbb{R}^2,\ 0 \le v \le v_{\max},\ s + \mathrm{t_h} v \le s_{\mathrm{obs}}\}$ set is positive control invariant for the discrete system with kinematics (1) and $u \in [a_{\min}, 0]$, $a_{\min} < 0$ if $0 < \delta\mathrm{t} \le 2\mathrm{t_h}$, $\mathrm{t_h} \ge \frac{v_{\max}}{-a_{\min}} - \frac{\delta\mathrm{t}}{2}$ for $\mathrm{t_h} > 0$ headway time and $\delta\mathrm{t} > 0$ discrete time step. (See Fig. 3)

*Proof:* Let the initial states of a vehicle $x^k \in \Omega$ set, where $x_{\mathrm{v1}} = [s_{\mathrm{obs}}, 0]^T$ and $x_{\mathrm{v2}} = [s_{\mathrm{obs}} - \mathrm{t_h} v_{\max}, v_{\max}]^T$ are the vertices of convex polyhedral set. According to [26] it is a sufficient proof of invariance when the vertices of a polyhedral set are examined. Thus for invariance $x^{k+1} \in \Omega$ where the new states are obtained from (1),

**Case 1:** $x^k = x_{\mathrm{v1}} = [s_{\mathrm{obs}}, 0]^T$

$$s^{k+1} = s_{\mathrm{obs}} + \frac{1}{2} a^k \delta\mathrm{t}^2, \tag{10}$$

$$v^{k+1} = a^k \delta\mathrm{t}, \tag{11}$$

for the trivial solution of $a^k = 0$ the states remain unchanged satisfying $x^{k+1} \in \Omega$.

**Case 2:** $x^k = x_{\mathrm{v2}} = [s_{\mathrm{obs}} - \mathrm{t_h} v_{\max}, v_{\max}]^T$

$$s^{k+1} = s_{\mathrm{obs}} - \mathrm{t_h} v_{\max} + v_{\max}\delta\mathrm{t} + \frac{1}{2} a^k \delta\mathrm{t}^2, \tag{12}$$

$$v^{k+1} = v_{\max} + a^k \delta\mathrm{t}. \tag{13}$$

The new states have to satisfy each constraints defining $\Omega$. Inequality $0 \le v^{k+1}$ from (2) combined with (13) yields to:

$$\delta\mathrm{t} \le \frac{v_{\max}}{-a^k}, \quad \text{for} \quad a^k < 0. \tag{14}$$

The constraint of $v^{k+1} \le v_{\max}$ from (2) with (13) is satisfied for $a^k \le 0$ control choice. And the final constraint gives:

$$s^{k+1} + \mathrm{t_h} v^{k+1} \le s_{\mathrm{obs}}. \tag{15}$$

With using (12), (13) and (15):

$$v_{\max} + a^k \left(\frac{1}{2}\delta\mathrm{t} + \mathrm{t_h}\right) \le 0 \tag{16}$$

concludes to the least constraining bound of

$$\frac{v_{\max}}{-a_{\min}} \le \frac{1}{2}\delta\mathrm{t} + \mathrm{t_h}. \tag{17}$$

Moreover by combining (14) and (16) it gives:

$$v_{\max} + a^k \left(\frac{1}{2}\delta\mathrm{t} + \mathrm{t_h}\right) \le 0 \le v_{\max} + a^k \delta\mathrm{t}, \tag{18}$$

$$\delta\mathrm{t} \le 2\mathrm{t_h}, \tag{19}$$

for $a^k < 0$. ■

On Fig. 4 $\delta\mathrm{t}$–$\mathrm{t_h}$ parameter regions are shown where $\Omega$ set takes positive control invariant property using (17) and (19).

*Remark:* $\Omega$ is not the maximal positive control invariant set in general. Such set could be obtained by computing the backward reachable set from $x_{\mathrm{v1}}$ vertex. However this would increase the number of linear inequalities for small values of $\delta\mathrm{t}$ and the computation complexity for the problem.

*Remark:* $\Omega$ can be closed by another side at arbitrary $s_{\mathrm{c}} < s^k$ resulting two more vertices ($x_{v3} = [s_{\mathrm{c}}, 0]^T, x_{v4} = [s_{\mathrm{c}}, v_{\max}]^T$). However these cases are covered by the trivial solutions of previous cases and $s_{\mathrm{c}} < s^k \le s^{k+1}$ thus can be disregarded from further problem formulation.

### B. Invariance of Full Constraints

In every case, $s_{\mathrm{obs}}^{(k+1)+i-1} \ge s_{\mathrm{obs}}^{k+i}$ , since the junction entry point does not move and the cars are assumed only to move forwards. Hence the invariance in the static obstacle case implies invariance in the complete junction case. This has two significant implications:

- The terminal constraint set is positively control invariant, implying recursive feasibility of the MPC [7], [6]. Note that this is for the nominal case only: uncertainty in the dynamics have not yet been considered.
- Since the constraint set is also control invariant, the existence of a safe abort trajectory is also guaranteed [8], [9]. Inequalities (9) were designed to provide safe abort trajectory for sudden stop of leader vehicle *e.g.* deceleration above the model parameters (an accident).

## V. Numerical Examples

In this section the previously described control system is examined through simulations.

### A. Recursive Feasibility

Feasibility tests for Y-junction merging were conducted with two identical vehicles for different $t_{\mathrm{h}}$–$\delta$t parameter pairs. Initial speeds were set to $v^0 = v_{\max} = v_{\mathrm{des}} = 10\ \mathrm{ms}^{-1}$. Initial positions were chosen randomly to provide conflicting arrivals at the junction and far to respect initial feasibility and non-interfering prediction (*e.a.* $s_p^0 \leq L_{pq}^1 - (t_{\mathrm{h}p} + T\delta t)v^0$ etc.). Furthermore, time horizon $T = 5$, $a_{\min} = -9.81/2\ \mathrm{ms}^{-2}$ and $a_{\max} = 3\ \mathrm{ms}^{-2}$, for simplicity $s_{\mathrm{off1}} = s_{\mathrm{off2}} = 0$ m moreover $l1_{12} = l2_{12} = 4$ m. The cost function to be minimized is set to quadratic speed error penalization without acceleration penalty:

$$J(v_1, v_2) = \gamma\, Q\,(v_1 - v_{\mathrm{des}})^2 + (1-\gamma) Q\,(v_2 - v_{\mathrm{des}})^2\ , \quad (20)$$

where a single relative weighting factor $\gamma \in (0,1)$ introduced for simplicity ($w_1 = \gamma$, $w_2 = 1-\gamma$). On Fig. 5 for each pair of parameters 150 simulations were run and indicated where problems remained feasible till the end of simulation or rendered infeasible due to constraint violation. The results

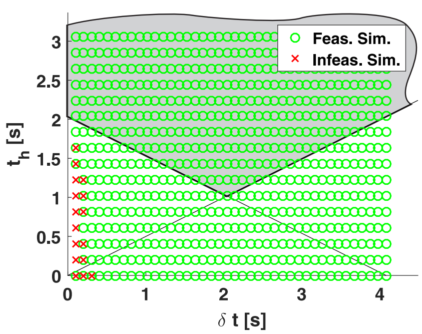


Fig. 5. MPC merging simulations for parameters $t_{\mathrm{h}}$–$\delta$t for $T = 5$ time horizon

show that for low $\delta$t choices the number of time horizon steps led to infeasible cases and for sufficiently long horizons all simulations turned out to be feasible, as predicted by Ref. [22].

Results for a second experiment are shown on Fig. 6 where a sudden stall of the leader vehicle was initiated. This disturbance was achieved by setting the speed of leader vehicles to 0 $\mathrm{ms}^{-1}$ at the previous position of it after passing the junction. Only parameters chosen according the conditions defined in *Theorem 1* were able to obtain positive control invariance for the vehicles and result in recursive feasibility. However MPC problems without positive control invariance sets fails to stop for harsh disturbances when the previous prediction is not followed perfectly.

### B. Prioritization

The effect of prioritization can be shown for conflicting vehicles approaching the junction and merging. For general-

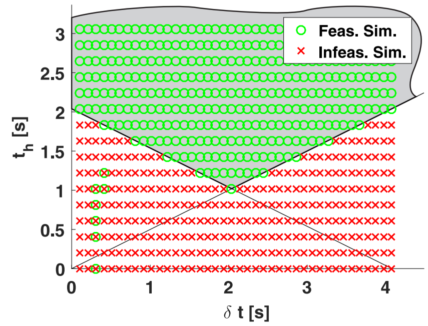


Fig. 6. MPC merging simulations for parameters $t_{\mathrm{h}}$–$\delta$t for $T = 5$ time horizon and sudden stop of the leader vehicle

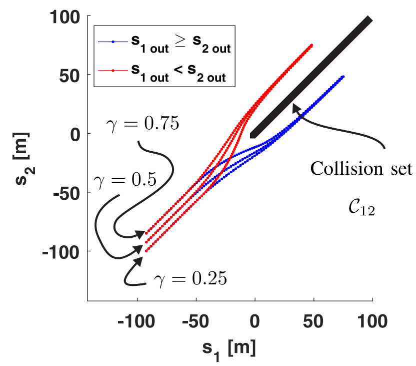


Fig. 7. Merging for three $\gamma$ and critical $\Delta d$ switching values with $\delta$t = 0.2 s, $t_{\mathrm{h}}$=2.1 s, $T$=25, $Q$=1, $R\simeq$5.1

ity in this experiment control penalty was defined as well:

$$J(v_1,u_1,v_2,u_2) = \gamma\left(Q\,(v_1 - v_{\mathrm{des}})^2 + R\,(u_1)^2\right) + (1-\gamma)\left(Q\,(v_2 - v_{\mathrm{des}})^2 + R\,(u_2)^2\right), \quad (21)$$

In this experiment for 3 different $\gamma$ values critical initial positions were identified, $s_1^0$ and $s_2^0$. To simplify the test $s_1^0$ initial position was kept constant in simulations while $\Delta d := s_2^0 - s_1^0$ parameter was controlling the relative initial position gap between the vehicles. Figure 7 presents the taken state trajectories showing with blue the cases when the first vehicle passed first the junction and with red when the second vehicle passed first. For a given $\gamma$ prioritization weight a critical $\Delta d$ value can be found where it is more beneficial to change the order of vehicles.

### C. Decision graph

Simulation were conducted for a range of weightings $\gamma \in (0,1)$ and starting positions offsets $\Delta d$. Figure 8 presents the four quadrants from which the 2nd and 4th quadrant depicts the cases when first-come-first-served outcomes were observed. In these region prioritization was enforcing the initial order. This means that if a vehicle was ahead of the other it passed the junction first. However with $\gamma$ weighting in the 1st and 3rd quadrant the vehicle orders were able to be changed leading to consistent prioritization effect. By connecting the critical $\Delta d$ values on the decision graph a distinct switching line can be obtained separating the cases

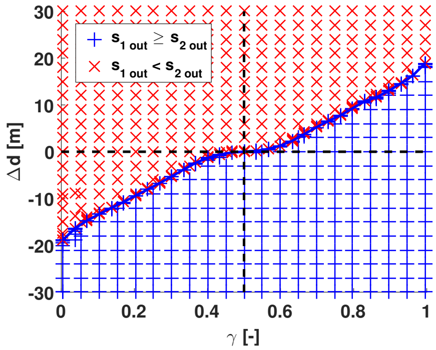


Fig. 8. Priority graph with FIFO regions in 2nd and 4th quadrant and shifted priority due to the $\gamma$ objective weighting in 1st and 3rd quadrants, $\delta t = 0.2$ s, $t_h = 2.1$ s, $T = 25$, $Q$=1, $R$≃5.1

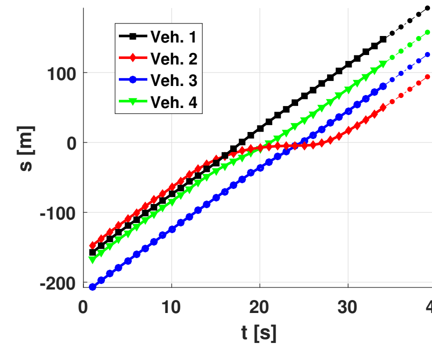


Fig. 10. Positions shown for 4 vehicles merging with taken trajectories as solid lines and predictions as dashed lines and circles, $s_{\mathrm{off}n=1\ldots4}$=0, $\delta t$=1 s, $t_h$=2.1 s, $T = 5$, $w_{n=1\ldots4}$=$[0.25, 0.05, 0.1, 0.6]$, $Q$=1, $R$≃5.1

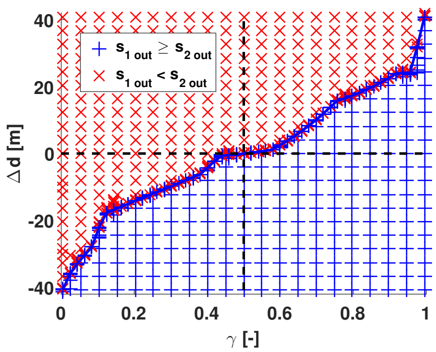


Fig. 9. Priority graph with artifact from discretization, $\delta t = 2.5$ s, $t_h$ = 2.1 s, $T = 2$, $Q$=1, $R$≃5.1

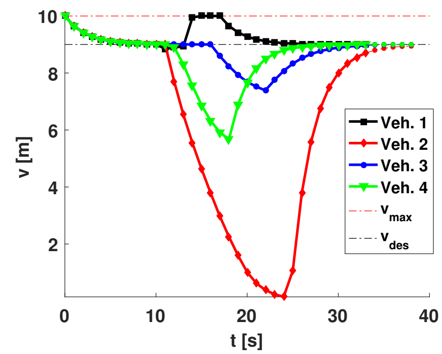


Fig. 11. Speeds shown for 4 vehicles merging with taken speed values as solid lines and predictions as dashed lines and circles, $\delta t = 1$ s, $t_h = 2.1$ s, $T = 5$, $w_{n=1\ldots4} = [0.25, 0.05, 0.1, 0.6]$, $Q$=1, $R$≃5.1

where one vehicle finished first from other finishing first. The switching line at no prioritization, $\gamma = 0.5$, was at $\Delta d = 0$ m. Figure 9 shows the effect of high $\delta t$ period times for the same $T\delta t$=5 s horizon time, resulting in a distorted, asymmetric decision graph due to the effect of discretization corner cutting prevention. The longer the time steps are the more some cases has to be constrained to satisfy the overlaying constraints.

### *D. Multi-vehicle merging*

In this example four vehicles are approaching the junction, merging point on four different roads. Vehicle interactions were restricted by pair-wise, six collision sets for junction approaching and car following. The objective function was utilized with stage cost (5) and $v_{\mathrm{des}} = 0.9\, v_{\max} = 9\ \mathrm{ms}^{-1}$. From the initial states and with $w_{n=1\ldots4} = [0.25, 0.05, 0.1, 0.6]$ priorities a complex merging case is presented on Fig. 10. Vehicles change order according the non-trivial choices determined by minimal cost for the $T = 5$ horizon length and $\delta t = 1$ s. For example Veh. 2 which initially has the lead assigned the lowest priority and ends up passing the junction at last, in contrast with Veh. 4 assigned the highest priority thus amends the least of its course. Figure 11 shows the speed evolution of the vehicles where Veh. 1 has lower priority than Veh. 4 but it had an initial lead and pass the junction first. A cooperation between them can be observed as Veh. 1 attempts to contribute building up the required safe distance by accelerating to a higher speed than the desired speed.

### *E. Computational Time*

The simulations were run on a PC with Intel i7-4790 CPU and 16 GB memory. The MIQP problem was solved using Gurobi v7.5.1 via its Matlab interface. Computation times are shown on Fig. 12 for a set of simulations computed for state trajectories shown in Fig. 7. The MIQP is solved efficiently for non-conflicting part of the tests and for car-following phase as there only relatively few nodes must be explored in the problem. On the other hand an increased computation time can be observed for the part where decision making happens and many different binary values have to be explored to obtain the global optimum. For two vehicles with $T = 25$ steps time horizon which is $T\delta t = 5$ s, solution times were below the control period time. In the four vehicle example Fig. 13 shows the computation times with the same 5 s horizon but with $T = 5$ steps. It is remarked that more efficient solutions schemes can be obtained for the MIQP by exploiting structure. In the work [23] this was achieved through indicator binaries for collision set avoidance. This current work however does not attempt to optimize the method rather than show the proof of safe time-headway formulation.

## VI. Conclusion

A method was presented for vehicle prioritization and decision making in merging intersections retaining safety conditions through positive control invariant sets. A mixed integer model predictive control framework was presented with the derived parameter pairs for simple time headway formulation. Through these parameters positive control invariant and recursive feasibility properties were shown. Global optimal solution is obtained through the MIQP. Where vehicle prioritization and change of vehicle ordering was presented through multi-objective weighting parameters, potentially providing tunable gap acceptance with relative weight manipulation.

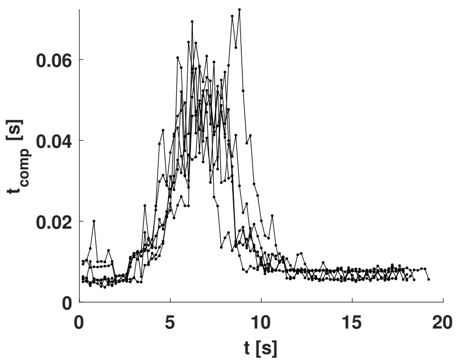


Fig. 12. Computation times shown for Fig. 7 simulations, $\delta t = 0.2$ s, $t_h = 2.1$ s, $T = 25$ simulations

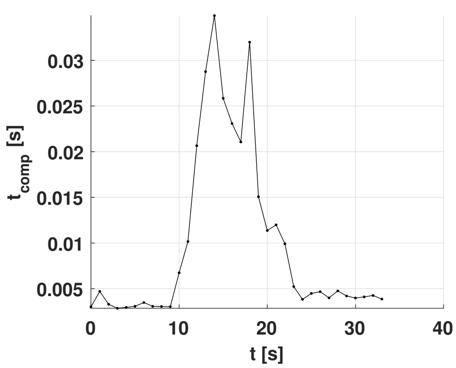


Fig. 13. Computation times shown for 4 vehicles merging $\delta t = 1$ s, $t_h = 2.1$ s, $T = 5$, $w_{n=1\ldots4} = [0.25, 0.05, 0.1, 0.6]$

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