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Stable input-output inversion for nondecoupable nonminimum-phase linear systems

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Abstract— **Feedforward control can enhance the performances in the control and regulation of dynamic systems. With this aim, a new inversion formula to solve the stable input-output inversion problem is presented for multivariable nonminimum-phase linear systems. It is based on the computation of the system transfer function inverse and the splitting of the zero dynamics transfer function into stable and unstable parts. Differently from the known alternative state-space methods, the presented approach is applicable to systems that cannot be input-output decoupled by state feedback.**

I. Introduction

Performances in the control and regulation of dynamic systems can be improved by the adoption of feedforward control techniques [19]. Among these, the input-output inversion (or inversion-based control) method allows to choose the desired output and an inverse input that causes the desired output can be determined by means of an inversion procedure. For minimum-phase systems, the procedure uses directly the inverse system to obtain a so-called standard inversion [18], [8]. However, this inversion fails in the nonminimum-phase case because it leads to an unbounded inverse input regardless of the boundedness of the desired output.

A breakthrough leading to a bounded (albeit noncausal) inverse input for nonminimum-phase systems was presented in [2], [4] and [9] for the nonlinear and linear cases respectively. In these works, the (stable) input-output inversion relies on the construction of the *normal form* in a state-space setting [10]. Then, a bounded noncausal solution of the zero dynamics driven by the desired output can be determined by a convolution integral (linear case) or by Picard iterations (nonlinear case). Eventually, a bounded noncausal inverse input is determined. However, the normal form can only be determined when the corresponding *decoupling matrix* [10] is nonsingular. Hence, only for (input-output) decoupable systems these procedures are effective [5]. Alternative approaches similar to the stable input-output inversion are those in [7] and [20]. The former approach is based on modifying the desired output to solve a two-point boundary value problem whereas the latter one uses differential flatness. However, since both approaches cannot ensure the actual desired output — designed according to the control application — they may fail in satisfying relevant control specifications such as e.g. those on undershooting and overshooting.

Focusing on multivariable nonminimum-phase linear systems, a new solution to the stable input-output inversion problem is presented. This solution is based on the inverse of the system transfer function and the splitting of the transfer function of the zero dynamics into stable and unstable parts. An advantage of the presented approach is the possibility to apply it to nondecoupable systems, i.e. systems that cannot be decoupled by state feedback (cf. (9), Definition 10 and Theorem 1).

Paper organization: Section II introduces C_{p}^{∞}, the set of vector piecewise C^{∞}-functions as the signal space for the input, output, and state time-functions. The system behavior [17] is presented in Section III. Weak solutions of the state equation are introduced and relations between continuity orders of the input and the output are established (Proposition 2 and Theorem 2). In Section IV, matrix fraction descriptions [11] are used to characterize the transfer function and its inverse. Especially relevant are the results on the input's zero dynamics (Theorem 4) and the input's particular solution (Proposition 4). In a behavioral setting, the stable input-output inversion problem is posed and solved in Section V (Problem 1 and Theorem 7). The partial fraction expansion of the zero dynamics transfer function (cf. (17), (18)) leads to the inversion formula (22) which is a direct generalization of an analogous formula for scalar systems [14], [3]. Section VI presents an example of feedforward regulation for a nondecoupable system. Concluding remarks end the paper in Section VII.

Notation: The set of natural numbers, including 0 is $\mathbb{N}$. Scalars and real-valued functions are denoted by lower-case letters. Vectors and vector-valued functions are denoted by bold lower-case letter, whereas matrices are denoted by capital letters with a few exceptions carefully remarked.

Given a matrix $C \in \mathbb{R}^{m\times n}$, its entries are denoted by $c_{i,j}$ or $(C)_{i,j}$, $i = 1,\dots,m$, $j = 1,\dots,n$, so that $C \equiv [c_{i,j}]$ or $C \equiv [(C)_{i,j}]$. The i-th row and the j-th column of C are denoted by $\boldsymbol{c}^i \in \mathbb{R}^{1\times n}$ and $\boldsymbol{c}_j \in \mathbb{R}^{m\times 1} \equiv \mathbb{R}^m$ respectively.

A function $\boldsymbol{f} \equiv (f_1,\dots,f_m) : \mathbb{R} \to \mathbb{R}^m$ belongs to C^n or more precisely $C^n(\mathbb{R},\mathbb{R}^m)$ if its components f_i are continuous with continuous derivatives until the n-th order over $\mathbb{R}$. When $\boldsymbol{f} \in C^n$ we say that $\boldsymbol{f}$ has *continuity order* n. If there exist derivatives of all order for the components f_i then $\boldsymbol{f}$ belongs to C^{∞}. The n-th order derivative of $\boldsymbol{f}$ is denoted by $\boldsymbol{f}^{(n)} := (f_1^{(n)},\dots,f_m^{(n)})$ or $D^n\boldsymbol{f} := (D^n f_1,\dots,D^n f_m)$. The right-hand and left-hand limits of $\boldsymbol{f}$ in $t \in \mathbb{R}$ are denoted as $\boldsymbol{f}(t^+) := \lim_{v\to t^+} \boldsymbol{f}(v)$ and $\boldsymbol{f}(t^-) := \lim_{v\to t^-} \boldsymbol{f}(v)$, with component-wise limits. Function $\boldsymbol{f}$ is said to be *causal* is $\boldsymbol{f}(t) = \boldsymbol{0}$, $t < 0$. When $m = 1$ we are reduced to the scalar case.

Denote by $\mathcal{L}[\cdot]$ and $\mathcal{L}^{-1}[\cdot]$ the Laplace and the inverse

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Laplace transform respectively. The unit step function is denoted by $1(t)$: $1(t) = 0$ if $t < 0$ and $1(t) = 1$ if $t \geq 0$. The empty set is denoted by $\emptyset$.

II. THE SPACE OF SIGNALS C^∞_{p}

The proposed space of time-signals is the set of piecewise C^∞-functions which is a simplified version of the more general $\mathcal{L}_1^{\mathrm{loc}}$ used in [17]. This choice that is fully justified for the needs of the inversion-based control [3] is instrumental in deriving many new results herein presented (e.g. Theorem 7). Let us consider the following definitions [3].

***Definition** 1 (Sparse sets):* A set $S \subset \mathbb{R}$ is said to be sparse if for any real interval $[a, b]$, $S \cap [a, b]$ has finite cardinality or it is the empty set.

***Definition** 2 (C^∞_{p}, set of piecewise C^∞-functions):* A function $\boldsymbol{f} = (f_1, \dots, f_m)$ belongs to C^∞_{p}, called the set of (vector) piecewise C^∞-functions, if there exist a sparse set S such that $\boldsymbol{f} \in C^\infty(\mathbb{R} \backslash S, \mathbb{R}^m)$ and for any $n \in \mathbb{N}$ and $t \in S$ the limits $\boldsymbol{f}^{(n)}(t^-)$ and $\boldsymbol{f}^{(n)}(t^+)$ exist and are finite. When $\boldsymbol{f}$ is defined in $t \in S$, i.e. all the components of $\boldsymbol{f}$ are defined in t, conventionally $\boldsymbol{f}(t) := \boldsymbol{f}(t^+)$.

***Definition** 3:* $C^{-1}(\mathbb{R}, \mathbb{R}^m) := C^\infty_{\mathrm{p}}(\mathbb{R}, \mathbb{R}^m)$ denotes the set of (vector) piecewise C^∞-functions defined over the whole set of reals.

***Definition** 4 (Discontinuity sets):* Given $\boldsymbol{f} \in C^\infty_{\mathrm{p}}$ with $\boldsymbol{f} = (f_1, \dots, f_m)$, the zero-order discontinuity set is $S^{(0)}_{\boldsymbol{f}} := \{t \in \mathbb{R} : \boldsymbol{f}$ is not defined in t, i.e. at least one component of $\boldsymbol{f}$ is not defined in t, or $\boldsymbol{f}(t^-) \neq \boldsymbol{f}(t^+)\}$; the n-th order discontinuity set is $S^{(n)}_{\boldsymbol{f}} = \{t \in \mathbb{R} : \boldsymbol{f}^{(n)}$ does not exist in t, i.e. al least one componet of $\boldsymbol{f}^{(n)}$ does not exist in $t\}$.

A straightforward result on discontinuity sets is the following.

***Lemma** 1:* Let $\boldsymbol{f} \in C^{-1}(\mathbb{R}, \mathbb{R}^m)$ and $k \in \mathbb{N}$. Then $\boldsymbol{f} \in C^n(\mathbb{R}, \mathbb{R}^m)$ if and only if $S^{(n)}_{\boldsymbol{f}} = \emptyset$.

***Lemma** 2:* Let $\boldsymbol{f} \in C^{-1}(\mathbb{R}, \mathbb{R}^m)$ and $L \in \mathbb{R}^{n \times m}$ be a matrix with $\mathrm{rank}(L) = m$. Then $S^{(0)}_{\boldsymbol{f}} = S^{(0)}_{L\boldsymbol{f}}$.

Proof: Taking into account that $\boldsymbol{f} \in C^\infty_{\mathrm{p}}(\mathbb{R}, \mathbb{R}^m)$ (cf. Definition 3) we have $S^{(0)}_{\boldsymbol{f}} = \{t \in \mathbb{R} : \ \boldsymbol{f}(t^-) \neq \boldsymbol{f}(t^+)\}$ and $S^{(0)}_{L\boldsymbol{f}} = \{t \in \mathbb{R} : \ L\boldsymbol{f}(t^-) \neq L\boldsymbol{f}(t^+)\}$. Relation $L\boldsymbol{f}(t^-) \neq L\boldsymbol{f}(t^+)$ is equivalent to $L(\boldsymbol{f}(t^-) - \boldsymbol{f}(t^+)) \neq 0$, i.e. $\boldsymbol{f}(t^-) - \boldsymbol{f}(t^+) \notin \ker L$. But $\ker L = \{\boldsymbol{0}\}$ so that $\boldsymbol{f}(t^-) - \boldsymbol{f}(t^+) \neq 0$. □

***Definition** 5 (Smoothness degree):* A signal $\boldsymbol{f} \in C^\infty_{\mathrm{p}}$ with $\boldsymbol{f} = (f_1, \dots, f_m)$ is said to have smoothness degree -1 if $\boldsymbol{f} \notin C^0(\mathbb{R}, \mathbb{R}^m)$. Signal $\boldsymbol{f}$ has smoothness degree $p \in \mathbb{N}$ if $\boldsymbol{f} \in C^p(\mathbb{R}, \mathbb{R}^m)$ and $\boldsymbol{f} \notin C^{p+1}(\mathbb{R}, \mathbb{R}^m)$.

III. THE BEHAVIOUR OF SQUARE MULTIVARIABLE LINEAR SYSTEMS

Let Σ be a linear time-invariant continuous-time system satisfying

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \tag{1}$$

$$\boldsymbol{y} = C\boldsymbol{x} \tag{2}$$

where $\boldsymbol{x} \in C^\infty_{\mathrm{p}}(\mathbb{R}, \mathbb{R}^n)$ is the state, $\boldsymbol{u}, \boldsymbol{y} \in C^\infty_{\mathrm{p}}(\mathbb{R}, \mathbb{R}^m)$ are the input and the output respectively, and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$ with $\mathrm{rank}(B) = \mathrm{rank}(C) = m$. The following assumption are made: 1) Σ is controllable and observable; 2) Σ is nonminimum phase and the zero dynamics [10] is hyperbolic (i.e. there are no zeros on the imaginary axis of $\mathbb{C}$); 3) Σ is invertible, i.e. there exists the inverse of its transfer function $H(s) := C(sI - A)^{-1}B$.

Solutions of the state equation (1) are introduced as *weak solutions* [17], according to the following definition.

***Definition** 6 (Weak solutions):* Given an input $\boldsymbol{u} \in C^\infty_{\mathrm{p}}(\mathbb{R}, \mathbb{R}^m)$, the state $\boldsymbol{x} \in C^\infty_{\mathrm{p}}(\mathbb{R}, \mathbb{R}^n)$ is a (weak) solution of equation (1) if there exists a constant $\boldsymbol{c} \in \mathbb{R}^n$ such that:

$$\boldsymbol{x}(t) = \int_0^t A\boldsymbol{x}(v)dv + \int_0^t B\boldsymbol{u}(v)dv + \boldsymbol{c}, \quad t \in \mathbb{R}. \tag{3}$$

Weak solutions can be characterized by the following useful result.

***Proposition** 1:* Given an input $\boldsymbol{u} \in C^\infty_{\mathrm{p}}(\mathbb{R}, \mathbb{R}^m)$, a function $\boldsymbol{x} \in C^\infty_{\mathrm{p}}(\mathbb{R}, \mathbb{R}^n)$ is a (weak) solution of the state equation (1) if and only if all the following conditions hold:

$$a)\ \boldsymbol{x} \in C^0(\mathbb{R}, \mathbb{R}^n), \tag{4}$$

$$b)\ S^{(1)}_{\boldsymbol{x}} = S^{(0)}_{\boldsymbol{u}}, \tag{5}$$

$$c)\ \dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t), \quad t \in \mathbb{R} \setminus S^{(0)}_{\boldsymbol{u}}. \tag{6}$$

Proof: ($\Longrightarrow$) Relation (3) implies that $\boldsymbol{x}$ is a continuous function on $\mathbb{R}$, i.e. condition (4) holds. By taking the derivative of the terms of (3) we note that: $\dot{\boldsymbol{x}}(t)$ is defined only on $\mathbb{R} \setminus S^{(1)}_{\boldsymbol{x}}$, $D(\int_0^t A\boldsymbol{x}(v)dv) = A\boldsymbol{x}(t), \quad t \in \mathbb{R}$, $D(\int_0^t B\boldsymbol{u}(v)dv) = B\boldsymbol{u}(t)$ is defined only on $\mathbb{R} \setminus S^{(0)}_{B\boldsymbol{u}}$. Since $\mathrm{rank}\, B = m$, by Lemma 2, we get $S^{(0)}_{B\boldsymbol{u}} = S^{(0)}_{\boldsymbol{u}}$. The derivatives of the right-hand and left-hand sides of (3) must coincide both as domain of existence and associated values. Hence we deduce conditions (5) and (6).

($\Longleftarrow$) By integration of (6) we obtain $\int_0^t \dot{\boldsymbol{x}}(v)dv = \int_0^t A\boldsymbol{x}(v)dv + \int_0^t B\boldsymbol{u}(v)dv, \quad t \in \mathbb{R}$. From $\boldsymbol{x} \in C^0(\mathbb{R}, \mathbb{R}^n)$, $\int_0^t \dot{\boldsymbol{x}}(v)dv = \boldsymbol{x}(t) - \boldsymbol{x}(0)$ so that equation (3) is obtained having set $\boldsymbol{c} := \boldsymbol{x}(0)$. □

In our development, a central role is played by the behaviour of Σ [17].

***Definition** 7 (Behavior set):* The behavior set of Σ is defined as $\mathcal{B} := \{(\boldsymbol{u}, \boldsymbol{y}) \in C^\infty_{\mathrm{p}}(\mathbb{R}, \mathbb{R}^m) \times C^\infty_{\mathrm{p}}(\mathbb{R}, \mathbb{R}^m) : \boldsymbol{y}(t) = C\boldsymbol{x}(t),\ t \in \mathbb{R}$ being $\boldsymbol{x} \in C^\infty_{\mathrm{p}}(\mathbb{R}, \mathbb{R}^n)$ a weak solution of $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$, cf. Definition 6 $\}$.

***Definition** 8 (Relative degree):* Under the current assumptions, Σ has (vector) relative degree $\boldsymbol{r} = [r_1\ r_2\ \cdots\ r_m]^T$, with $r_i := \min\{j : \boldsymbol{c}^i A^{j-1} B \neq \boldsymbol{0}, j = 1, \dots, n\}$, $i = 1, \dots, m$.

The i-th component of the relative degree is the minimum derivation order necessary for the input $\boldsymbol{u}$ to appear in a derivative of the i-th output's component. Indeed, from (2) and (6) it follows that for $i = 1, \dots, m$:

$$y_i^{(r_i)}(t) = \boldsymbol{c}^i A^{r_i} \boldsymbol{x}(t) + \boldsymbol{c}^i A^{r_i - 1} B\boldsymbol{u}(t), \quad t \in \mathbb{R} \backslash S^{(0)}_{\boldsymbol{u}} \tag{7}$$

with $\boldsymbol{c}^i A^{r_i - 1} B \neq \boldsymbol{0}$.

***Definition** 9 (Vector derivative):* Given $\boldsymbol{f} \in C^\infty_{\mathrm{p}}(\mathbb{R}, \mathbb{R}^m)$ and a vector $\boldsymbol{k} \in \mathbb{N}^m$, the $\boldsymbol{k}$-th order derivative of $\boldsymbol{f}$ is denoted by $\boldsymbol{f}^{(\boldsymbol{k})}$ and is defined as $\boldsymbol{f}^{(\boldsymbol{k})} := (f_1^{(k_1)}, \dots, f_m^{(k_m)})$.

With the help of the above vector derivative notation, the m scalar equation (7) can be joined into the following vector equation:

$$\boldsymbol{y}^{(\boldsymbol{r})}(t) = \Psi \boldsymbol{x}(t) + \Gamma \boldsymbol{u}(t), \quad t \in \mathbb{R} \setminus S_{\boldsymbol{u}}^{(0)}, \tag{8}$$

where:

$$\Psi := \begin{bmatrix} \boldsymbol{c}^1 A^{r_1} \\ \boldsymbol{c}^2 A^{r_2} \\ \vdots \\ \boldsymbol{c}^m A^{r_m} \end{bmatrix} \in \mathbb{R}^{m \times n}, \ \Gamma := \begin{bmatrix} \boldsymbol{c}^1 A^{r_1 - 1} B \\ \boldsymbol{c}^2 A^{r_2 - 1} B \\ \vdots \\ \boldsymbol{c}^m A^{r_m - 1} B \end{bmatrix} \in \mathbb{R}^{m \times m}. \tag{9}$$

The introduced matrix Γ, called the *decoupling matrix*, has a significant role in the control of square multivariable systems. Indeed, Γ must be nonsingular when: 1) the construction of the normal form of Σ [13] is required such as e.g. in solving the stable input-output inversion problem in a state-space setting [4], [9]; 2) input-output decoupling by static state feedback is sought [5].

In particular a systems is said to be decoupable according the following definition.

***Definition** 10 (Decoupable systems):* Σ is said to be (input-output) decoupable (by static state feedback) if there exist constant matrices $F_x \in \mathbb{R}^{m \times n}$ and $F_v \in \mathbb{R}^{m \times m}$ such that $\boldsymbol{u} = F_x \boldsymbol{x} + F_v \boldsymbol{v}$, (with $\boldsymbol{v} \in \mathbb{R}^m$ being the new input vector), determines a closed-loop transfer function $C(sI - A - BF_x)^{-1} BF_v$ that is diagonal.

***Theorem** 1 ([5]):* Σ is decoupable if and only if the decoupling matrix Γ is nonsingular.

***Remark** 1:* Our approach can allow that Γ be singular. Hence, stable inversion (cf. Theorem 7) is extended to nondecoupable systems.

The relative degree concept dictates a first result on the output continuity orders.

***Lemma** 3:* Let $(\boldsymbol{u}, \boldsymbol{y}) \in \mathcal{B}$. Then $y_i \in C^{r_{i-1}}$, $i = 1, \ldots, m$.

A relation between the continuity orders of the input and output signals can be expressed as follows.

***Proposition** 2:* Let $(\boldsymbol{u}, \boldsymbol{y}) \in \mathcal{B}$ and $p \in \mathbb{N} \cup \{-1\}$. If $\boldsymbol{u} \in C^p(\mathbb{R}, \mathbb{R}^m)$ then $y_i \in C^{p + r_i}(\mathbb{R}, \mathbb{R})$, $i = 1, \ldots, m$.

For decoupable systems, a stronger result on input-output continuity orders is the following.

***Theorem** 2:* Let Σ be input-output decoupable. Consider $(\boldsymbol{u}, \boldsymbol{y}) \in \mathcal{B}$ and $p \in \mathbb{N} \cup \{-1\}$. Then $\boldsymbol{u}$ has smoothness degree p if and only if $\boldsymbol{y}^{(\boldsymbol{r})}$ has smoothness degree p.

For brevity proofs of Lemma 3, Proposition 2, and Theorem 2 are omitted.

IV. Input-output properties

The transfer function of Σ is $H(s) = C(sI - A)^{-1} B$. As known [1], [11], it can be rewritten as a coprime left Matrix Fraction Description (MFD) $H(s) = P^{-1}(s) Q(s)$ where $P(s)$ and $Q(s)$ are suitable polynomial matrices. By extension from the scalar case [3], the behavior of Σ can also be expressed as the set of weak solutions of the differential equation

$$P(D) \boldsymbol{y}(t) = Q(D) \boldsymbol{u}(t), \ t \in \mathbb{R} \tag{10}$$

so that $\mathcal{B} = \{(\boldsymbol{u}, \boldsymbol{y}) \in C_{\mathrm{p}}^{\infty}(\mathbb{R}, \mathbb{R}^m) \times C_{\mathrm{p}}^{\infty}(\mathbb{R}, \mathbb{R}^m) : P(D) \boldsymbol{y}(t) = Q(D) \boldsymbol{u}(t) \text{ weakly}\}$.

The poles of Σ are introduced as the roots of the *pole polynomial* according to the following definition.

***Definition** 11 (pole polynomial [1]):* The *pole polynomial* $p_H(s)$ of Σ is defined as the monic least common denominator of all nonzero minors of $H(s)$.

Under the current assumptions $p_H(s) = det(sI - A)$ and also $p_H(s) = c \det P(s)$ with a suitable $c \neq 0$.

***Definition** 12 (minimal pole polynomial [1]):* The *minimal pole polynomial* $m^P(s)$ of Σ is defined as the monic least common denominator of all nonzero entries of $H(s)$.

The zeros of Σ are the roots of the *zero polynomial* according to this definition.

***Definition** 13 (zero polynomial [1]):* The *zero polynomial* $z_H(s)$ of Σ is defined as the monic greatest common divisor of the numerators of all the highest-order nonzero minors of $H(s)$ after all their denominators have been set equal to $p_H(s)$.

Remark that $z_H(s) = c_1 \det \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix}$ and also $z_H = c_2 \det Q(s)$ with suitable scalars $c_1, c_2 \neq 0$.

By assumption, Σ is invertible. Hence, $H^{-1}(s)$ is well defined and

$$H^{-1}(s) = Q^{-1}(s) P(s) = \frac{\operatorname{adj}[Q(s)] P(s)}{\det Q(s)}. \tag{11}$$

The polynomial division of every entry of the product $\operatorname{adj}[Q(s)] P(s)$ leads to

$$(\operatorname{adj}[Q(s)] P(s))_{ij} = q_{0,ij}(s) \det Q(s) + p_{0,ij}(s) \tag{12}$$

where $q_{0,ij}(s)$ and $p_{0,ij}(s)$ are polynomials for which $\deg p_{0,ij}(s) < \deg \det Q(s)$. By defying $Q_0(s) := [q_{0,ij}(s)]$, $P_0(s) := [p_{0,ij}(s)]$, and $H_0 := P_0(s) / \det Q(s)$ it follows that

$$H^{-1}(s) = Q_0(s) + H_0(s) \tag{13}$$

where $H_0(s)$ is a strictly proper rational matrix that represents the so-called zero dynamics [10].

***Lemma** 4:* There exists a polynomial matrix $P_1(s)$ such that

$$H_0(s) = Q^{-1}(s) P_1(s). \tag{14}$$

Proof: From (11) and (13) we obtain $Q^{-1}(s) P(s) = Q_0(s) + H_0(s)$. By multiplying this relation by $Q(s)$ from the left $P(s) = Q(s) Q_0(s) + Q(s) H_0(s)$. Here $Q(s) H_0(s)$ is necessarily a polynomial matrix because it is the difference of two polynomial matrices. Hence, define $P_1(s) := Q(s) H_0(s)$ and multiply this relation by $Q^{-1}(s)$ from the left to obtain (14), the left MFD of $H_0(s)$. □

The dual concept of minimal pole polynomial, i.e. the *minimal zero polynomial* is then introduced.

***Definition** 14 (minimal zero polynomial):* The *minimal zero polynomial* $m^Z(s)$ of Σ is defined as the monic least common denominator of all nonzero entries of $H_0(s)$.

***Remark** 2:* From the previous definitions and Lemma 4 it follows that the (minimal) zero polynomial of Σ is equal to the (minimal) pole polynomial associated to $H_0(s)$.

Pole and zero modes are crucial notions which are introduced as follows.

***Definition** 15 (pole and zero modes):* Given a real or complex pole (zero) $p \in \mathbb{R}$ or $p = \sigma \pm j\omega \in \mathbb{C}$ ($z \in \mathbb{R}$ or $z = \rho \pm j\psi \in \mathbb{C}$) with multiplicity μ (ν) as a root of the minimal pole (zero) polynomial $m^P(s)$ ($m^Z(s)$), the associated *pole (zero) modes* are the time-functions $e^{pt}, te^{pt}, \ldots, t^{\mu-1}e^{pt}$ or $e^{\sigma t}\cos(\omega t), e^{\sigma t}\sin(\omega t), \ldots, t^{\mu-1}e^{\sigma t}\cos(\omega t), t^{\mu-1}e^{\sigma t}\sin(\omega t)$ ($e^{zt}, te^{zt}, \ldots, t^{\nu-1}e^{zt}$ or $e^{\rho t}\cos(\psi t), e^{\rho t}\sin(\psi t), \ldots, t^{\nu-1}e^{\rho t}\cos(\psi t), t^{\nu-1}e^{\psi t}\sin(\psi t)$) respectively. All the pole (zero) modes are denoted by $m_i^P(t)$ ($m_i^Z(t)$), $i = 1, \ldots, m_P(m_Z)$ with $m_P := \deg m^P(s)$ ($m_Z := \deg m^Z(s)$).

***Theorem** 3:* The set of all solutions of the homogeneous differential equation $P(D)\boldsymbol{y}_{\mathtt{hom}}(t) = 0$ is given by

$$\boldsymbol{y}_{\mathtt{hom}}(t) = \sum_{i=1}^{m_P} \boldsymbol{f}_i m_i^P(t) \tag{15}$$

with $\boldsymbol{f}_i \in \mathcal{F}_i$ where $\mathcal{F}_i$, $i = 1, \ldots, m_P$ are suitable subspaces of $\mathbb{R}^m$ with $\sum_{i=1}^{m_P} \dim(\mathcal{F}_i) = \deg\det P(s)$ ($= \deg p_H(s) = n$.)

***Theorem** 4:* The set of all solutions of the homogeneous differential equation $Q(D)\boldsymbol{u}_{\mathtt{hom}}(t) = 0$ is given by

$$\boldsymbol{u}_{\mathtt{hom}}(t) = \sum_{i=1}^{m_Z} \boldsymbol{g}_i m_i^Z(t) \tag{16}$$

with $\boldsymbol{g}_i \in \mathcal{G}_i$ where $\mathcal{G}_i$, $i = 1, \ldots, m_Z$ are suitable subspaces of $\mathbb{R}^m$ with $\sum_{i=1}^{m_Z} \dim(\mathcal{G}_i) = \deg\det Q(s)$ ($= \deg z_H(s)$.)

Mathematically Theorems 3 and 4 are actually the same result but conjugated with respect to the $P(D)$ and $Q(D)$ matrix operators respectively. For brevity the proof is omitted. An alternative, equivalent formulation of this result is reported in [17, Theorem 3.2.16, p. 77].

***Remark** 3:* The time-function (15) is the free output response or output's pole dynamics. It is the system output when the input is kept to zero: $(0, \boldsymbol{y}_{\mathtt{hom}}) \in \mathcal{B}$. Dually, the time-function (16) is the input's zero dynamics. It is the system input when the output is kept to zero: $(\boldsymbol{u}_{\mathtt{hom}}, 0) \in \mathcal{B}$.

Define $h(t) \in \mathbb{R}^{m\times m}$ and $h_0(t) \in \mathbb{R}^{m\times m}$ as the analytical extensions over $\mathbb{R}$ of $\mathcal{L}^{-1}[H(s)]$ and $\mathcal{L}^{-1}[H_0(s)]$ respectively ($h(t)1(t)$ and $h_0(t)1(t)$ are the impulse matrix responses of Σ and of the zero dynamics system respectively). Also define q_i, $i = 1, \ldots, m$ as the degree of the i-th column of $Q_0(s)$, i.e. $q_i := \max_{j=1,\ldots,m} \deg q_{0,ji}(s)$, so that $\boldsymbol{q} := [q_1 \cdots q_m]^T$.

***Lemma** 5:* Given system Σ, then $\boldsymbol{q} \geq \boldsymbol{r}$ (component-wise inequality). If the decoupling matrix Γ is nonsingular, $\boldsymbol{q} = \boldsymbol{r}$.

Proof: For brevity the proof is omitted. $\square$

The following results emphasize relevant particular solutions of the differential equation (10) [3].

***Proposition** 3 (Output's particular solution):* Let $\boldsymbol{u} \in C_{\mathtt{p}}^{\infty}(\mathbb{R}, \mathbb{R}^m)$ and define $\boldsymbol{y}_{\mathtt{par}}(t) := \int_0^t h(t-v)\boldsymbol{u}(t)dv$, $t \in \mathbb{R}$. Then $(\boldsymbol{u}, \boldsymbol{y}_{\mathtt{par}}) \in \mathcal{B}$.

***Proposition** 4 (Input's particular solution):* Let $\boldsymbol{y} \in C_{\mathtt{p}}^{\infty}(\mathbb{R}, \mathbb{R}^m)$ and $y_i \in C^{q_i-1}$, $i = 1, \ldots, m$. Define $\boldsymbol{u}_{\mathtt{par}}(t) := Q_0(D)\boldsymbol{y}(t^+) + \int_0^t h_0(t-v)\boldsymbol{y}(t)dv$, $t \in \mathbb{R}$. Then $(\boldsymbol{u}_{\mathtt{par}}, \boldsymbol{y}) \in \mathcal{B}$.

Useful characterizations of the behavior $\mathcal{B}$ are the following.

***Theorem** 5:* Define $\mathcal{B}_{\mathtt{i/o}} := \{(\boldsymbol{u}, \boldsymbol{y}) \in C_{\mathtt{p}}^{\infty}(\mathbb{R}, \mathbb{R}^m) \times C_{\mathtt{p}}^{\infty}(\mathbb{R}, \mathbb{R}^m) : \boldsymbol{y}(t) = \int_0^t h(t-v)\boldsymbol{u}(v)dv + \sum_{i=1}^{m_P} \boldsymbol{f}_i m_i^P(t)$, $t \in \mathbb{R}$, $\boldsymbol{f}_i \in \mathcal{F}_i\}$. Then $\mathcal{B}_{\mathtt{i/o}} = \mathcal{B}$.

***Theorem** 6:* Define $\mathcal{B}_{\mathtt{o/i}} := \{(\boldsymbol{u}, \boldsymbol{y}) \in C_{\mathtt{p}}^{\infty}(\mathbb{R}, \mathbb{R}^m) \times C_{\mathtt{p}}^{\infty}(\mathbb{R}, \mathbb{R}^m) : y_i \in C^{q_i-1}$, $i = 1, \ldots, m$, $\boldsymbol{u}(t) = Q_0(t)\boldsymbol{y}(t^+) + \int_0^t h_0(t-v)\boldsymbol{y}(v)dv + \sum_{i=1}^{m_Z} \boldsymbol{g}_i m_i^Z(t)$, $t \in \mathbb{R}$, $\boldsymbol{g}_i \in \mathcal{G}_i\}$. Then $\mathcal{B}_{\mathtt{o/i}} \subset \mathcal{B}$.

***Corollary** 1:* Assume that the decoupling matrix Γ is nonsingular. Then $\mathcal{B}_{\mathtt{o/i}} = \mathcal{B}$.

Proof: A glimpse of the proofs of the above results is proposed. Theorems 5 and 6 follow from Theorems 3 and 4 (on the set of solutions of the homogeneous differential equations $P(D)\boldsymbol{y}_{\mathtt{hom}}(t) = 0$ and $Q(D)\boldsymbol{u}_{\mathtt{hom}}(t) = 0$) and Propositions 3 and 4 (on the output's and input's particular solutions of the differential equation $P(D)\boldsymbol{y}(t) = Q(D)\boldsymbol{u}(t)$) respectively. Indeed, as known [17], the set of (weak) solutions of (10) (i.e. the behavior $\mathcal{B}$) is given by a particular solution plus the set of all solutions of the associated homogeneous differential equation. Corollary 1 follows from Theorems 6 and 2. In the scalar case, complete proofs of Theorem 5 and Corollary 1 are reported in [3]. $\square$

V. Stable input-output inversion

The stable input-output inversion problem can be addressed as follows.

***Problem** 1:* Given a desired, bounded, sufficiently smooth output $\boldsymbol{y}_{\mathtt{d}} \in C_{\mathtt{p}}^{\infty}(\mathbb{R}, \mathbb{R}^m)$ find a bounded input $\boldsymbol{u}_{\mathtt{d}} \in C_{\mathtt{p}}^{\infty}(\mathbb{R}, \mathbb{R}^m)$ such that $(\boldsymbol{u}_{\mathtt{d}}, \boldsymbol{y}_{\mathtt{d}}) \in \mathcal{B}$.

By assumption, the zero dynamics of Σ is hyperbolic, i.e. the real parts of the system zeros are negative or positive. Hence, the zero polynomial (cf. Definition 13) can be factorized as $z_H(s) = z_H^-(s)z_H^+(s)$ where $z_H^-(s)$ and $z_H^+(s)$ are monic polynomials with root's real parts that are all negative and positive respectively. It follows that $\det Q(s) = cz_H^-(s)z_H^+(s)$ with a suitable scalar $c \neq 0$.

The transfer function of the zero dynamics of Σ is $H_0(s) = \frac{P_0(s)}{\det Q(s)} = \left[\frac{p_{0,ij}(s)}{\det Q(s)}\right]$ (cf. (12) and (13); $i, j = 1, \ldots, m$). By partial fraction expansion, the entries of $H_0(s)$ can be rewritten as

$$\frac{p_{0,ij}(s)}{\det Q(s)} = \frac{p_{0,ij}^-(s)}{z_H^-(s)} + \frac{p_{0,ij}^+(s)}{z_H^+(s)} \tag{17}$$

where $p_{0,ij}^-(s)$, $p_{0,ij}^+(s)$ are polynomials with $\deg p_{0,ij}^-(s) < \deg z_H^-(s)$, $\deg p_{0,ij}^+(s) < \deg z_H^+(s)$. Define $P_0^-(s) := [p_{0,ij}^-(s)]$, $P_0^+(s) := [p_{0,ij}^+(s)]$ and $H_0^- := \frac{P_0^-(s)}{z_H^-(s)}$, $H_0^+ := \frac{P_0^+(s)}{z_H^+(s)}$. Hence, $H_0(s)$ is split into stable and unstable parts:

$$H_0(s) = H_0^-(s) + H_0^+(s). \tag{18}$$

The strictly proper rational matrices $H_0^-(s)$ and $H_0^+(s)$ can be represented by left MFDs according to the following result.

***Lemma** 6:* There exist polynomial matrices $P_1^-(s)$ and $P_1^+(s)$ such that

$$H_0^-(s) = Q^{-1}(s)P_1^-(s), \quad H_0^+(s) = Q^{-1}(s)P_1^+(s). \quad (19)$$

Proof: It is omitted for brevity. □

Let $h_0^-(t) \in \mathbb{R}^{m\times m}$ and $h_0^+(t) \in \mathbb{R}^{m\times m}$ be the analytical extensions over $\mathbb{R}$ of $\mathcal{L}^{-1}[H_0^-(s)]$ and $\mathcal{L}^{-1}[H_0^+(s)]$ for which

$$h_0(t) = h_0^-(t) + h_0^+(t),\ t \in \mathbb{R}. \quad (20)$$

All the matrix functions $h_0(t)$, $h_0^-(t)$, $h_0^+(t)$ satisfy the homogeneous matrix differential equation associated to the operator $Q(D)$.

***Lemma** 7:* The following relations holds: $Q(D)h_0(t) = 0$, $Q(D)h_0^-(t) = 0$, $Q(D)h_0^+(t) = 0$, $t \in \mathbb{R}$.

Proof: It is based on the MFDs provided by Lemma 4 and Lemma 6 and on the concept of *impulse response matrix* [1]. Function $h_0(t)1(t)$ is the impulse response matrix of the zero system of Σ whose transfer function is $H_0(s) = Q^{-1}(s)P_1(s)$ (cf. (14)). This system is then described by the differential equation $Q(D)\boldsymbol{\eta}(t) = P_1(D)\boldsymbol{y}(t)$ where $\boldsymbol{y}(t)$ and $\boldsymbol{\eta}(t)$ are the input and the output respectively. Hence, it holds $Q(D)h_0(t) = 0$ for $t > 0$ and by analytical extension for all $t \in \mathbb{R}$. A similar reasoning leads to $Q(D)h_0^-(t) = 0$ and $Q(D)h_0^+(t) = 0$ for all $t \in \mathbb{R}$. □

Denote by $m_i^{Z-}(t)$, $i = 1,\dots m_Z^-$ and $m_i^{Z+}(t)$, $i = 1,\dots m_Z^+$ the stable and unstable zero modes ($m_Z^- + m_Z^+ = m_Z$; cf. Definition 15). Taking into account Lemma 7 and Theorem 4, there exist matrices $G_i^-, G_i^+ \in \mathbb{R}^{m\times m}$ such that $\operatorname{im} G_i^- \subseteq \mathcal{G}_i^-$, $\operatorname{im} G_i^+ \subseteq \mathcal{G}_i^+$ and

$$h_0^-(t) = \sum_{i=1}^{m_Z^-} G_i^- m_i^{Z-}(t), \quad h_0^+(t) = \sum_{i=1}^{m_Z^+} G_i^+ m_i^{Z+}(t) \quad (21)$$

where $\mathcal{G}_i^-$ and $\mathcal{G}_i^+$ are the (input) subspaces of $\mathbb{R}^m$ associated to the modes $m_i^{Z-}(t)$ and $m_i^{Z+}(t)$.

The next result gives an explicit closed-form expression of the inverse input $\boldsymbol{u}_{\mathtt{d}}$ solving Problem 1.

***Theorem** 7 (Stable inversion formula):* Let be given a desired output $\boldsymbol{y}_{\mathtt{d}} \in C^\infty_{\mathtt{p}}(\mathbb{R},\mathbb{R}^m)$ for which $y_{\mathtt{d},i} \in C^{q_i-1}$ and $y_{\mathtt{d},i}, y_{\mathtt{d},i}^{(1)}, \dots, y_{\mathtt{d},i}^{(q_i)}$ are all bounded ($i = 1,\dots,m$). Then a solution to Problem 1 can be expressed as follows:

$$\boldsymbol{u}_{\mathtt{d}}(t) = Q_0(D)\boldsymbol{y}_{\mathtt{d}}(t^+) + \int_{-\infty}^{t} h_0^-(t-v)\boldsymbol{y}_{\mathtt{d}}(v)dv - \int_{t}^{+\infty} h_0^+(t-v)\boldsymbol{y}_{\mathtt{d}}(v)dv,\ t \in \mathbb{R}. \quad (22)$$

Proof: From (21), the right side of (22) can be written as

$$Q_0(D)\boldsymbol{y}_{\mathtt{d}}(t^+) + \sum_{i=1}^{m_Z^-} G_i^- \int_{-\infty}^{t} \boldsymbol{y}_{\mathtt{d}}(v) m_i^{Z-}(t-v)dv - \sum_{i=1}^{m_Z^+} G_i^+ \int_{t}^{+\infty} \boldsymbol{y}_{\mathtt{d}}(v) m_i^{Z+}(t-v)dv. \quad (23)$$

All the addends of the above expression are bounded and so is the resulting $\boldsymbol{u}_{\mathtt{d}}(t)$. Indeed, $Q_0(D)\boldsymbol{y}_{\mathtt{d}}(t^+)$ involves derivatives of $\boldsymbol{y}_{\mathtt{d}}$ that are bounded by assumption and all the the integrals appearing in (23) are bounded too. For simplicity, in the following, we consider that the minimal zero polynomial $m^Z(s)$ (cf. Definition 14) has only simple real roots. For example, assume $m_i^{Z+}(t) = e^{z_i^+ t}$ with $z_i^+ > 0$ and define $\boldsymbol{y}_{\mathtt{d,sup}} := \sup_{t\in\mathbb{R}} |\boldsymbol{y}_{\mathtt{d}}(t)| \in \mathbb{R}^m$ (the absolute value and the supremum are applied component-wise). Hence, $|\int_t^{+\infty} \boldsymbol{y}_{\mathtt{d}}(v) m_i^{Z+}(t-v)dv| \le \boldsymbol{y}_{\mathtt{d,sup}} \int_t^{+\infty} e^{z_i^+(t-v)}dv = \frac{1}{z_i^+}\cdot \boldsymbol{y}_{\mathtt{d,sup}}$, $t \in \mathbb{R}$. Similarly, the boundedness of $\int_{-\infty}^t \boldsymbol{y}_{\mathtt{d}}(v) m_i^{Z-}(t-v)dv$ can be ascertained.

We will now show that $(\boldsymbol{u}_{\mathtt{d}}, \boldsymbol{y}_{\mathtt{d}}) \in \mathcal{B}_{o/i}$ (cf. Theorem 6). In (22), break the integrals into two parts at zero and rearrange them by taking into account (20) to obtain:

$$\boldsymbol{u}_{\mathtt{d}}(t) = Q_0(D)\boldsymbol{y}_{\mathtt{d}}(t^+) + \int_0^t h_0(t-v)\boldsymbol{y}_{\mathtt{d}}(v)dv + \int_{-\infty}^0 h_0^-(t-v)\boldsymbol{y}_{\mathtt{d}}(v)dv - \int_0^{+\infty} h_0^+(t-v)\boldsymbol{y}_{\mathtt{d}}(v)dv.$$

The last two integrals above are suitable linear combinations of the zero modes of Σ. Indeed, by (21) and still considering $m_i^{Z-}(t) = e^{z_i^- t}$, $m_i^{Z+}(t) = e^{z_i^+ t}$ ($z_i^- < 0$, $z_i^+ > 0$) these integrals can be expressed as

$$\begin{aligned}&\sum_{i=1}^{m_Z^-} G_i^- \int_{-\infty}^0 e^{z_i^-(t-v)}\boldsymbol{y}_{\mathtt{d}}(v)dv - \sum_{i=1}^{m_Z^+} G_i^+ \int_0^{+\infty} e^{z_i^+(t-v)}\boldsymbol{y}_{\mathtt{d}}(v)dv \\ &= \sum_{i=1}^{m_Z^-} G_i^- \boldsymbol{l}_i^- e^{z_i^- t} - \sum_{i=1}^{m_Z^+} G_i^+ \boldsymbol{l}_i^+ e^{z_i^+ t} \end{aligned} \quad (24)$$

where $\boldsymbol{l}_i^- := \int_{-\infty}^0 e^{-z_i^- t}\boldsymbol{y}_{\mathtt{d}}(v)dv \in \mathbb{R}^m$ and $\boldsymbol{l}_i^+ := \int_0^{+\infty} e^{-z_i^+ t}\boldsymbol{y}_{\mathtt{d}}(v)dv \in \mathbb{R}^m$ having taken into account that $\boldsymbol{y}_{\mathtt{d}}$ is bounded over $\mathbb{R}$. From $\operatorname{im} G_i^- \subseteq \mathcal{G}_i^-$ and $\operatorname{im} G_i^+ \subseteq \mathcal{G}_i^+$ we obtain $\operatorname{im} G_i^- \boldsymbol{l}_i^- \in \mathcal{G}_i^-$ and $\operatorname{im} G_i^+ \boldsymbol{l}_i^+ \in \mathcal{G}_i^+$ so that (24) is equal to $\sum_{i=1}^{m_Z} \boldsymbol{g}_i m_i^Z(t)$, $\boldsymbol{g}_i \in \mathcal{G}_i$ (cf. (21) and Theorem 4). Hence $\boldsymbol{u}_{\mathtt{d}}(t) = Q_0(D)\boldsymbol{y}_{\mathtt{d}}(t^+) + \int_0^t h_0(t-v)\boldsymbol{y}_{\mathtt{d}}(v)dv + \sum_{i=1}^{m_Z} \boldsymbol{g}_i m_i^Z(t)$ and along with $y_{\mathtt{d},i} \in C^{q_i-1}$, $i = 1,\dots,m$ this proves that $(\boldsymbol{u}_{\mathtt{d}}, \boldsymbol{y}_{\mathtt{d}}) \in \mathcal{B}_{\mathtt{o/i}}$. Theorem 6 states that $\mathcal{B}_{\mathtt{o/i}} \subseteq \mathcal{B}$ so that $(\boldsymbol{u}_{\mathtt{d}}, \boldsymbol{y}_{\mathtt{d}}) \in \mathcal{B}$. □

***Remark** 4:* Formula (22) can be applied to both decoupable and nondecoupable systems. In the former case, it appears that the inverse input (22) is equal to that obtained by the state-space approaches [4], [9].

VI. An Example

Consider the system Σ with

$$A = \begin{bmatrix} -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \quad (25)$$

This system is controllable and observable. The zero polynomial and the zero minimal polynomial coincides, i.e.

$z_H(s) = m^Z(s) = s - 1$. Hence, Σ is nonminimum-phase and its zero dynamics is hyperbolic. The decoupling matrix $\Gamma = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is singular (cf. (9)) so that the system is nondecoupable. This precludes the possibility of applying the state-space inversion procedure of [4], [9]. Nevertheless, Σ is invertible and the stable input-output inversion provided by formula (22) can be applied.

The vector relative degree is $\boldsymbol{r} = [1\ 2]^T$ and the vector of the column degrees of $Q_0(s)$ is $\boldsymbol{q} = [3\ 4]^T$ (cf. (13) and Lemma 5). We desire a set-point transition on the outputs with a smooth planning given by the *transition polynomials* [15]. Specifically, outputs 1 and 2 should move from 0 to $y_{\mathrm{dc},1} := 2$, and $y_{\mathrm{dc},2} := 4$ with interval times $\tau_1 = 1$ s and $\tau_2 = 2$ s respectively. To obtain a continuous inverse input $\boldsymbol{u}_{\mathrm{d}}(t)$ (cf. (22)) we choose $y_{\mathrm{d},1}(t)$ and $y_{\mathrm{d},2}(t)$ with smoothness degrees (cf. Definition 5) equal to 3 and 4 respectively: $y_{\mathrm{d},i}(t) = 0$, $t < 0$, $y_{\mathrm{d},i}(t) = y_{\mathrm{dc},i}$, $t > \tau_i$, $i = 1, 2$; $y_{\mathrm{d},1}(t) = [-20(t/\tau_1)^7 + 70(t/\tau_1)^6 - 84(t/\tau_1)^5 + 35(t/\tau_1)^4] y_{\mathrm{dc},1}$, $t \in [0, \tau_1]$; $y_{\mathrm{d},2}(t) = [70(t/\tau_2)^9 - 315(t/\tau_2)^8 + 540(t/\tau_2)^7 - 420(t/\tau_2)^6 + 126(t/\tau_2)^5] y_{\mathrm{dc},2}$, $t \in [0, \tau_2]$. The inversion procedure (cf. Section V) requires to compute the differential operator $Q_0(D)$ whose entries are $q_{0,11}(D) = D + 1$, $q_{0,12}(D) = 1$, $q_{0,21}(D) = D^3 + 6\,D^2 + 14\,D + 19$, $q_{0,22}(D) = -D^4 - 6\,D^3 - 15\,D^2 - 25\,D - 32$, and $h_0^-(t) = \mathbf{0} \in \mathbb{R}^{2\times 2}$,

$$h_0^+(t) = \begin{bmatrix} 0 & 0 \\ 18\,\mathrm{e}^t & -36\,\mathrm{e}^t \end{bmatrix}, \quad t \in \mathbb{R}.$$

The inverse input $\boldsymbol{u}_{\mathrm{d}}(t)$, which is determined by (22), and output $\boldsymbol{y}_{\mathrm{d}}(t)$ are then plotted in Figure 1. Note that, in this case, $\boldsymbol{u}_{\mathrm{d}}(t)$ does not have *postaction* (or *postactuation*) [6] (at time $\max\{\tau_1, \tau_2\} = 2$ s the system is at the equilibrium) because there are no zeros with negative real part. On the other hand, input $\boldsymbol{u}_{\mathrm{d}}$ exhibits *preaction* (or *preactuation*) [12], [4] which is due to the positive real zero 1 (see $u_{\mathrm{d},2}(t)$, $t \in [-1, 0]$, in Figure 1).

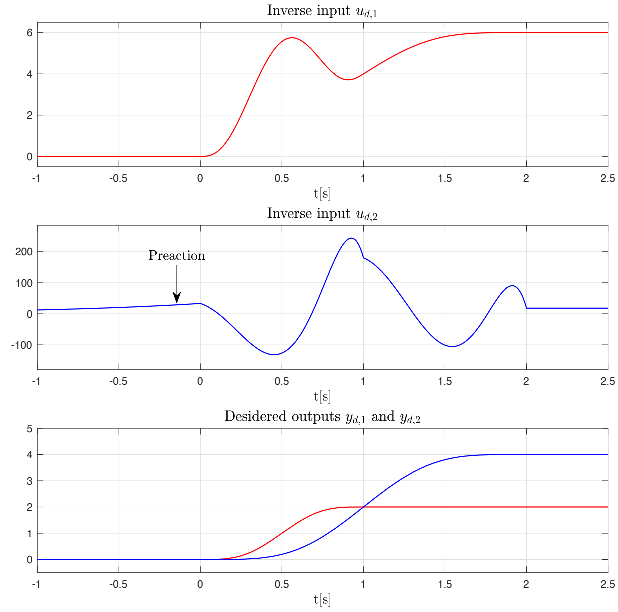

Fig. 1. Input and output components $u_{\mathrm{d},1}(t)$, $y_{\mathrm{d},1}(t)$ and $u_{\mathrm{d},2}(t)$, $y_{\mathrm{d},2}(t)$ are plotted with red and blue lines respectively.

VII. Conclusions

Input-output inversion allows *virtual* decoupling by feedforward control [16], i.e. by appropriately designing the desired (vector) output is possible to decouple all the scalar outputs from each other. Hence, the found inversion formula (22) that is based on the transfer function inverse $H^{-1}(s)$ (cf. (13)) allows input-output decoupling also for nonminimum-phase linear systems that are not decoupable by state feedback. To deal with these nondecoupable systems a possible state-space alternative approach may be based on the dynamic extension algorithm [10]. In such a way the decoupling matrix in the augmented state-space becomes nonsingular so that the stable inversion method of [4], [9] can be applied. However, the resulting overall inversion procedure may be cumbersome. This possible approach should be investigated in a future research.

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