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Robust Norm-bounded H_∞ static Output-feedback Control of retarded Stochastic Systems

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***Abstract*— An input-output approach for designing a constant output feedback controller for a linear time-invariant retarded system with stochastic state-multiplicative white noise sequences, that achieves a minimum bound on the H_∞ performance level is introduced. The stochastic uncertainties appear in the dynamic matrices, which correspond to the delayed and non-delayed states of the system.**

The solution of the robust norm-bounded H_∞ static output-feedback control problem is solved, for the stationary case, via the input-output approach where the system is replaced by a non-retarded system that contain, instead, deterministic norm-bounded uncertainties. In this problem, a cost function is defined which is the expected value of the standard H_∞ performance cost with respect to the stochastic parameters. We extend the results achieved for the nominal case, to the case where the system matrices contain norm bounded uncertainties.

I. Introduction

In the present paper we address the problem of robust H_∞ static output-feedback control of state-delayed, discrete-time, state-multiplicative linear systems via the *input-output* approach by applying modified stability and Bounded Real Lemma (BRL) results of [1], [2]. The multiplicative noise appears in both the delayed and the non delayed states of the system.

The stability analysis and control design for delayed-free systems with stochastic uncertainties have received much attention in the past (see [1] and the references therein), where mainly continuous-time non retarded systems were considered. In the late 80's, a renewed interest in the control and estimation designs of these systems has been encountered and solutions to the stochastic control and filtering problems of both: continuous-time and discrete-time systems, have been derived that ensure a worst case performance bound in the H_∞ sense [3]- [12]. Systems whose parameter uncertainties are modeled as white noise processes in a linear setting have been treated in [3] -[6], [12], for the continuous-time case and in [7] -[11] for the discrete-time case. Such models of uncertainties are encountered in many areas of applications (see [3] and the references therein) such as: nuclear fission and heat transfer, population models and immunology. In control theory such models are encountered in gain scheduling when the scheduling parameters are corrupted with measurement noise.

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Following the development of the theory of control and estimation for delayed-free systems, the stability and control of stochastic delayed systems of various types (i.e constant time-delay, slow and fast varying delay) have been a central issue in the theory of stochastic state-multiplicative systems over the two decade for both continuous-time systems [13]-[18] and discrete-time systems [19]-[23] (see also [2] for a comprehensive study).

We note that many results that have been obtained for the stability of deterministic retarded systems, since the 90's [24] - [32], have been extended also to the stochastic case, for both continuous and discrete-time systems.

In the discrete-time setting, the mean square exponential stability and the control and filtering problems of these systems were treated by several groups [19]-[23].

In the continuous-time stochastic setting, for example, the Lyapunov-Krasovskii (L-K) approach was applied in [16] and [17], to systems with constant delays, and stability criteria are derived for cases with norm-bounded uncertainties. The H_∞ state-feedback control for systems with time-varying delay is treated in [16] for restricted LKFs that provide delay-independent, rate dependent results. Also [15] considers H_∞ control (both state and output feedback) and estimation of time delay systems.

In [2] the various control and estimation problems for the delayed stochastic systems were obtained by applying the input-output approach. This approach was originally adopted by [30], [31] for deterministic systems and it was extended to delay-dependent solutions of both continuous and discrete-time counterpart stochastic problems [2]. This approach is based on the representation of the system's delay action by linear operators, with no delay, which in turn allows one to replace the underlying system with an equivalent one which possesses a norm-bounded uncertainty, and therefore may be treated by the theory of norm bounded uncertain, non-retarded systems with state-multiplicative noise ([2], see also [20] for the discrete-time case).

Static output-feedback control is applied in many areas of control engineering including process and flight control. In the latter, designing flight control systems, engineers prefer the simple and physically sound controllers that are recommended as cooked structures[33],[34] (see also [35]-[40] for various solutions). Only gains are included in these simple structures and the closed-loop poles are thus obtained by migration of the open-loop poles that have a clear physical meaning. In the present paper, we consider the problem of static output-feedback for delayed norm-bounded stochastic

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systems where the solution for the nominal case (i.e. with no norm-bounded uncertainties) has been solved in [40]. In our systems we allow for a time-varying delay where the uncertain stochastic parameters multiply both the delayed and the non delayed states in the state space model of the system. We note that in our systems both the dynamic matrix and the matrix of the delayed part are uncertain and multiply different stochastic white noise sequences.

This paper is organized as follows: Based on the previous solution of the BRL for nominal systems, the BRL for norm-bounded uncertain systems is brought Section III. In Section IV-A, the solution of the nominal static output-feedback control problem is derived followed, in Section IV-B, by the solution of the robust norm-bounded static output-feedback control problem. The latter solution is achieved via a single LMI that yields the controller gain matrix.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, $\mathcal{N}$ is the set of natural numbers and the notation $P>0$, (respectively, $P\geq 0$) for $P\in\mathcal{R}^{n\times n}$ means that P is symmetric and positive definite (respectively, semi-definite). We denote by $L^2(\Omega,\mathcal{R}^n)$ the space of square-integrable $\mathcal{R}^n-$ valued functions on the probability space $(\Omega,\mathcal{F},\mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ is a σ algebra of a subset of Ω called events and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. By $(\mathcal{F}_k)_{k\in\mathcal{N}}$ we denote an increasing family of σ-algebras $\mathcal{F}_k\subset\mathcal{F}$. We also denote by $\tilde{l}^2(\mathcal{N};\mathcal{R}^n)$ the n-dimensional space of nonanticipative stochastic processes $\{f_k\}_{k\in\mathcal{N}}$ with respect to $(\mathcal{F}_k)_{k\in\mathcal{N}}$ where $f_k\in L^2(\Omega,\mathcal{R}^n)$. On the latter space the following l^2-norm is defined:

$$||\{f_k\}||^2_{\tilde{l}_2}=E\{\textstyle\sum_0^\infty ||f_k||^2\}=\sum_0^\infty E\{||f_k||^2\}<\infty,\quad \{f_k\}\in\tilde{l}_2(\mathcal{N};\mathcal{R}^n), \tag{1}$$

where $||\cdot||$ is the standard Euclidean norm. We denote by δ_{ij} the Kronecker delta function. Throughout the manuscript we refer to the notation of exponential l^2 stability, or internal stability, in the sense of [7] (see Definition 2.1, page 927, there).

II. PROBLEM FORMULATION

We consider the following linear retarded system:

$$\begin{aligned} &x_{k+1}=(\tilde{A}_0+D\nu_k)x_k+(\tilde{A}_1+F\mu_k)x_{k-\tau(k)}\\ &+B_1w_k+\tilde{B}_2u_k \quad x_l=0,\ l\leq 0,\\ &y_k=C_2x_k+D_{21}n_k \end{aligned} \tag{2a,b}$$

with the objective vector

$$z_k=C_1x_k+D_{12}u_k, \tag{3}$$

where the matrices $\tilde{A}_0,\ \tilde{A}_1$ and $\tilde{B}_2$ contain the following norm-bounded uncertainties:

$$\tilde{A}_0=A_0+E_0F_0\bar{H}_0,\ \tilde{A}_1=A_1+E_1F_1\bar{H}_1,\quad \tilde{B}_2=B_2+E_0F_0\bar{H}_2, \tag{4}$$

where

$$F_i^TF_i\leq I,\quad E_i\in\mathcal{R}^{n\times r_i},\ F_i\in\mathcal{R}^{r_i\times\bar{r}_i},\ \bar{H}_i\in\mathcal{R}^{\bar{r}_i\times n},\quad i=0,1,\quad \bar{H}_2\in\mathcal{R}^{\bar{r}_0\times l} \tag{5}$$

and where the matrices $E_0,\ E_1,\ \bar{H}_i,\ i=0,1,2$ are constant matrices. The system state vector is $x_k\in\mathcal{R}^n$, $u_k\in\mathcal{R}^l$ is the control signal, $w_k\in\mathcal{R}^q$ is the exogenous disturbance signal, $n_k\in\mathcal{R}^p$ is the measurement noise signal and $y_k\in\mathcal{R}^m$ is the measured output and $z_k\in\mathcal{R}^r$ is the state combination (objective function signal) to be regulated where the time delay bound is denoted by h. The variables $\{\mu_k\}$ and $\{\nu_k\}$ are zero-mean real scalar white-noise sequences that satisfy:

$$E\{\nu_k\nu_j\}=\delta_{kj},\ E\{\mu_k\mu_j\}=\delta_{kj},$$
$$E\{\mu_k\nu_j\}=0,\ \forall k,j\quad\geq 0.$$

The matrices $A_0,\ A_1,\ B_1,\ B_2,\ C_2,\ D_{21}$ and $D_{12},\ D,\ F$ in (2a,b), (3) are constant matrices of appropriate dimensions.

We treat the following problem :

i) H_∞ Static output-feedback control :

We consider the uncertain system of (2a,b) and (3) and the following performance index:

$$J_{OF}\stackrel{\Delta}{=}||\bar{z}_k||^2_{\tilde{l}_2}-\gamma^2[||w_k||^2_{\tilde{l}_2}+||n_{k+1}||^2_{\tilde{l}_2}]. \tag{6}$$

Our objective is to find a controller of the following type

$$u_k=Ky_k \tag{7}$$

such that J_{OF} given by (6) is negative for all nonzero w_k,n_k where $w_k\in\tilde{l}^2_{\mathcal{F}_k}([0,\infty);\mathcal{R}^q),\ n_k\in\tilde{l}^2_{\mathcal{F}_k}([0,\infty];\mathcal{R}^p)$, for all the uncertainties described in (4)

III. THE BOUNDED REAL LEMMA

We first being the BRL result of [2] which was obtained for nominal systems:

Theorem 1 Consider the system (2a) and (3) with $\tilde{B}_2=0,\ G=0,\ D_{12}=0$ and where $\tilde{A}_0,\ \tilde{A}_1$ are replaced by A_0 and A_1, respectively. The system is exponentially stable in the mean square sense and, for a prescribed scalar $\gamma>0$ and a given scalar tuning parameter $\epsilon_b>0$, the requirement of $J_E<0$ is achieved for all nonzero $w\in\tilde{l}^2_{\mathcal{F}_k}([0,\infty);\mathcal{R}^q)$, if there exist $n\times n$ matrices $Q>0$, $R_1>0$ and a $n\times n$ matrix Q_m that satisfy $\tilde{\Gamma}<0$ where $\tilde{\Gamma}\stackrel{\Delta}{=}$

$$\begin{bmatrix} \tilde{\Gamma}_{11} & \tilde{\Gamma}_{12} & 0 & 0 & \tilde{\Gamma}_{15} & 0 & C_1^T\\ * & -Q & \tilde{\Gamma}_{23} & Q_m & 0 & QB_1 & 0\\ * & * & \tilde{\Gamma}_{33} & 0 & \tilde{\Gamma}_{35} & 0 & 0\\ * & * & * & -\epsilon_bQ & -h\epsilon_bQ_m^T & 0 & 0\\ * & * & * & * & -\epsilon_bQ & \epsilon_bhQB_1 & 0\\ * & * & * & * & * & -\gamma^2I_q & 0\\ * & * & * & * & * & * & -I_r \end{bmatrix}<0 \tag{8}$$

where

$$
\begin{aligned}
\tilde{\Gamma}_{11} &= -Q + D^T Q[1+\epsilon_b h^2]D + R_1,\\
\tilde{\Gamma}_{12} &= A_0^T Q + Q_m^T,\\
\tilde{\Gamma}_{15} &= \epsilon_b h[A_0^T Q + Q_m^T] - \epsilon_b h Q,\\
\tilde{\Gamma}_{23} &= QA_1 - Q_m,\\
\tilde{\Gamma}_{33} &= -R_1 + (1+\epsilon_b h^2)F^T Q F,\\
\tilde{\Gamma}_{35} &= \epsilon_b h[A_1^T Q - Q_m^T].
\end{aligned}
$$

In the uncertain case we obtain the following result:

Corollary 1 Consider the system (2a) and (3) with $\tilde{B}_2 = 0,\ G = 0,\ D_{12} = 0$. The system is exponentially stable in the mean square sense and, for a prescribed $\gamma > 0$ and a given tuning parameter $\epsilon_b > 0$, the requirement of $J_E < 0$ is achieved for all nonzero $w \in \tilde{l}^2_{\mathcal{F}_k}([0,\infty);\mathcal{R}^q)$, if there exist positive scalers $\bar{\epsilon}_1,\ \bar{\epsilon}_2,\ n \times n$ matrices $Q > 0,\ R_1 > 0$ and a $n \times n$ matrix Q_m that satisfy $\bar{\Gamma} < 0$ where $\bar{\Gamma} \stackrel{\Delta}{=}$

$$
\left[\begin{array}{ccccccccccccc}
\tilde{\Gamma}_{11} & \tilde{\Gamma}_{12} & 0 & 0 & \tilde{\Gamma}_{15} & 0 & C_1^T & \tilde{\Gamma}_{18} & 0 & \bar{\epsilon}_1 \bar{H}_0^T & QE_0 & 0 & 0\\
* & -Q & QA_1 - Q_m & Q_m & 0 & QB_1 & 0 & 0 & 0 & 0 & 0 & QE_1 & 0\\
* & * & -R_1 & 0 & \tilde{\Gamma}_{35} & 0 & 0 & 0 & \bar{\epsilon} F^T Q & 0 & 0 & 0 & \bar{\epsilon}_2 \bar{H}_1^T\\
* & * & * & -\epsilon_b Q & -h\epsilon_b Q_m^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & -\epsilon_b Q & \epsilon_b h Q B_1 & 0 & 0 & 0 & 0 & \tilde{\Gamma}_{5,11} & \tilde{\Gamma}_{5,12} & 0\\
* & * & * & * & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & -I_r & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & -Q & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & -Q & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & -\bar{\epsilon}_1 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & -\bar{\epsilon}_1 & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & * & -\bar{\epsilon}_2 & 0\\
* & * & * & * & * & * & * & * & * & * & * & * & -\bar{\epsilon}_2
\end{array}\right] < 0, \tag{9}
$$

where

$$
\begin{aligned}
&\tilde{\Gamma}_{12} = A_0^T Q + Q_m^T, \quad \tilde{\Gamma}_{18} = \sqrt{1+\epsilon_b h^2} D^T Q,\\
&\tilde{\Gamma}_{5,11} = \epsilon_b h Q E_0, \quad \tilde{\Gamma}_{5,12} = \epsilon_b h Q E_1
\end{aligned}
$$

and $\tilde{\Gamma}_{11},\ \tilde{\Gamma}_{15}$ and $\tilde{\Gamma}_{35}$ are given in the LMI of (8).

Proof:

Replacing $A_0,\ A_1$ by $\tilde{A}_0$ and $\tilde{A}_1$ respectively in (8), the following inequality is obtained:

$$
\begin{aligned}
&\tilde{\Gamma} + \Psi_1 F_0^T \Phi_1^T + [\Psi_1 F_0^T \Phi_1^T]^T + \Psi_2 F_1 \Phi_2^T\\
&+[\Psi_2 F_1 \Phi_2^T]^T < 0,\\
&\Psi_1^T = [\bar{H}_0\ 0\ 0\ 0\ 0\ 0\ 0\ ],\\
&\Phi_1^T = [0\ E_0^T\ 0\ 0\ \epsilon_b h E_0^T Q\ 0\ 0\ ],\\
&\Psi_2^T = [0\ E_1^T Q\ 0\ 0\ \epsilon_b h E_1^T Q\ 0\ 0\ ],\\
&\Phi_2^T = [0\ 0\ \bar{H}_1\ 0\ 0\ 0\ 0\ ],
\end{aligned} \tag{10}
$$

where $\tilde{\Gamma}$ is the LMI of (8). Noting that $[\bar{\epsilon}_1 \Psi_1 F_0 + \Phi_1]\bar{\epsilon}_1^{-1}[\bar{\epsilon}_1 \Psi_1 F_0 + \Phi_1]^T > 0,\ \bar{\epsilon}_1 > 0$ one obtains

$$
\bar{\epsilon}_1 \Psi_1 F_0 F_0^T \Psi_1^T + \Phi_1 \bar{\epsilon}_1^{-1} \Phi_1^T > \Psi_1 F_0^T \Phi_1^T + [\Psi_1 F_0^T \Phi_1^T]^T.
$$

Recalling that $F_0^T F_0 < I$ and applying similar argument for the fourth and fifth left size terms, involving $\Psi_2,\ F_1$ and Φ_2 in (20a), the inequality (20a) is guaranteed if the following LMI is guaranteed:

$$
\tilde{\Gamma} + \Psi_1 \bar{\epsilon}_1 \Psi_1^T + \Phi_1 \bar{\epsilon}_1^{-1} \Phi_1^T + \Psi_2 \bar{\epsilon}_2 \Psi_2^T + \Phi_2 \bar{\epsilon}_2^{-1} \Phi_2^T < 0.
$$

Replacing for $\Psi_1,\ \Psi_2,\ \Phi_1$ and Φ_2 (20b-f) and applying Schur complement, the LMI of (9) is obtained.

IV. STATIC OUTPUT-FEEDBACK CONTROL

A. static output-feedback control - the nominal case

In this section we address the static output-feedback control problem of the delayed state-multiplicative nominal noisy system. We consider the system of (2a,b), (3) and the constant controller of (7).

Denoting $\xi_k^T \stackrel{\Delta}{=} [x_k^T\ \ y_k^T]$ and $\bar{w}_k^T \stackrel{\Delta}{=} [w_k^T\ \ n_k^T]$, we obtain the following augmented system:

$$
\begin{aligned}
&\xi_{k+1} = \hat{A}_0 \xi_k + \tilde{B}\bar{w}_k + \hat{A}_1 \xi(k-\tau_k) + \tilde{D}\xi_k \nu_k +\\
&\tilde{F}\xi(k-\tau_k)\mu_k, \quad \xi(\theta) = 0,\ over[-h\ 0],\\
&\tilde{z}_k = \tilde{C}\xi_k,
\end{aligned} \tag{11}
$$

with the following matrices:

$$
\begin{aligned}
&\hat{A}_0 = \begin{bmatrix} A_0 & B_2 K\\ C_2 A_0 & C_2 B_2 K \end{bmatrix},\ \hat{A}_1 = \begin{bmatrix} A_1 & 0\\ C_2 A_1 & 0 \end{bmatrix},\\
&\tilde{F} = \begin{bmatrix} F & 0\\ C_2 F & 0 \end{bmatrix},\ \tilde{D} = \begin{bmatrix} D & 0\\ C_2 D & 0 \end{bmatrix},\\
&\tilde{C} = [C_1\ \ D_{12}K],\ \tilde{B} = \begin{bmatrix} B_1 & 0\\ C_2 B_1 & D_{21} \end{bmatrix}.
\end{aligned} \tag{12}
$$

Using the BRL result of Theorem 1, we obtain that (6) is satisfied if there exist matrices $\tilde{Q}$, $\tilde{R}_1$, $\tilde{R}_2$ and $\tilde{M}$ of the appropriate dimensions that satisfy the following inequality:

$$
\hat{\Gamma} \triangleq \begin{bmatrix}
\hat{\Gamma}_{11} & (\hat{A}_0+\tilde{M})^T\tilde{Q} & 0 & 0 & h(\hat{A}_0^T+\tilde{M}^T)\tilde{R}_2-\tilde{R}_2 h & 0 & \tilde{C}^T \\
* & -\tilde{Q} & \tilde{Q}(\hat{A}_1-\tilde{M}) & \tilde{Q}\tilde{M} & 0 & \tilde{Q}\tilde{B} & 0 \\
* & * & -\tilde{R}_1+\tilde{F}^T(\tilde{Q}+h^2\tilde{R}_2)\tilde{F} & 0 & h(\hat{A}_1^T-\tilde{M}^T)\tilde{R}_2 & 0 & 0 \\
* & * & * & -\tilde{R}_2 & -h\tilde{M}^T\tilde{R}_2 & 0 & 0 \\
* & * & * & * & -\tilde{R}_2 & h\tilde{R}_2\tilde{B} & 0 \\
* & * & * & * & * & -\gamma^2 I_{q+l} & 0 \\
* & * & * & * & * & * & -I_r
\end{bmatrix}, \tag{13}
$$

where $\hat{\Gamma}_{11} = -\tilde{Q} + \tilde{D}^T(\tilde{Q}+h^2\tilde{R}_2)\tilde{D} + \tilde{R}_1$.

Taking $\tilde{R}_2 = \epsilon_b\tilde{Q}$, where $\epsilon_b > 0$ is a tuning scalar parameter, we obtain the following inequality:

$$
\begin{bmatrix}
\hat{\Gamma}_{11} & \hat{A}_0^T\tilde{Q}+\tilde{M}^T\tilde{Q} & 0 & 0 & \epsilon_b h[\hat{A}_0^T\tilde{Q}+\tilde{M}^T\tilde{Q}]-\epsilon_b h\tilde{Q} & 0 & \tilde{C}^T \\
* & -\tilde{Q} & \tilde{Q}\hat{A}_1-\tilde{Q}\tilde{M} & \tilde{Q}\tilde{M} & 0 & \tilde{Q}\tilde{B} & 0 \\
* & * & -\tilde{R}_1+(1+\epsilon_b h^2)\tilde{F}^T\tilde{Q}\tilde{F} & 0 & \epsilon_b h[\hat{A}_1^T\tilde{Q}-\tilde{M}^T\tilde{Q}] & 0 & 0 \\
* & * & * & -\epsilon_b\tilde{Q} & -h\epsilon_b\tilde{M}^T\tilde{Q} & 0 & 0 \\
* & * & * & * & -\epsilon_b\tilde{Q} & \epsilon_b h\tilde{Q}\tilde{B} & 0 \\
* & * & * & * & * & -\gamma^2 I_{q+l} & 0 \\
* & * & * & * & * & * & -I_r
\end{bmatrix} < 0, \tag{14}
$$

where $\hat{\Gamma}_{11} = -\tilde{Q} + \tilde{D}^T\tilde{Q}[1+\epsilon_b h^2]\tilde{D} + \tilde{R}_1$.

Denoting $\tilde{P} = Q^{-1}$ and taking the following partition:

$$
\tilde{P} = \tilde{Q}^{-1} = \begin{bmatrix} P & -\alpha^{-1}PC_2^T \\ -\alpha^{-1}C_2P & \hat{P} \end{bmatrix}, \tag{15}
$$

where $Q > 0, P > 0 \in \mathcal{R}^{n\times n}$ and $\hat{Q} > 0, \hat{P} > 0 \in \mathcal{R}^{m\times m}$, where α is a tuning scalar parameter, we multiply (14) by $diag\{\tilde{P},\tilde{P},\tilde{P},\tilde{P},\tilde{P},I_{q+l},I_r\}$ from the left and the right and we obtain the following inequality:

$$
\begin{bmatrix}
-\tilde{P}+\tilde{R}_p & \bar{\Upsilon}_{12} & 0 & 0 & \bar{\Upsilon}_{15} & 0 & \bar{\Upsilon}_{17} & 0 & \bar{\Upsilon}_{19} \\
* & -\tilde{P} & \bar{\Upsilon}_{23} & \tilde{M}_P & 0 & \tilde{B} & 0 & 0 & 0 \\
* & * & -\tilde{R}_P & 0 & \bar{\Upsilon}_{35} & 0 & 0 & \bar{\Upsilon}_{38} & 0 \\
* & * & * & -\epsilon_b\tilde{P} & -h\epsilon_b\tilde{M}_P^T & 0 & 0 & 0 & 0 \\
* & * & * & * & -\epsilon_b\tilde{P} & \epsilon_b h\tilde{B} & 0 & 0 & 0 \\
* & * & * & * & * & -\gamma^2 I_{q+l} & 0 & 0 & 0 \\
* & * & * & * & * & * & -I_r & 0 & 0 \\
* & * & * & * & * & * & * & -\tilde{P} & 0 \\
* & * & * & * & * & * & * & * & -\tilde{P}
\end{bmatrix} < 0, \tag{16}
$$

where

$$
\bar{\Upsilon}_{12} = \begin{bmatrix} PA_0^T + C_2^T\hat{P}_K^T B_2^T & PA_0^TC_2^T + C_2^T\hat{P}_K^TB_2^TC_2^T \\ \hat{P}C_2A_0^T + \alpha\hat{P}_K^TB_2^T & \hat{P}C_2A_0^TC_2^T + \alpha\hat{P}_K^TB_2^TC_2^T \end{bmatrix}
$$
$$
+\tilde{M}_P^T,
$$
$$
\bar{\Upsilon}_{17} = \begin{bmatrix} PC_1^T + C_2^T\hat{P}_K^TD_{12}^T \\ \hat{P}C_2C_1^T + \alpha\hat{P}_K^TD_{12}^T \end{bmatrix},
$$
$$
\bar{\Upsilon}_{15} = -\epsilon_b h\tilde{P} +
$$
$$
\epsilon_b h \begin{bmatrix} PA_0^T + C_2^T\hat{P}_K^TB_2^T & PA_0^TC_2^T + C_2^T\hat{P}_K^TB_2^TC_2^T \\ \hat{P}C_2A_0^T + \alpha\hat{P}_K^TB_2^T & \hat{P}C_2A_0^TC_2^T + \alpha\hat{P}_K^TB_2^TC_2^T \end{bmatrix},
$$
$$
\bar{\Upsilon}_{19} = \sqrt{1+\epsilon_b h^2}\begin{bmatrix} PD^T & PD^TC_2^T \\ \hat{P}C_2D^T & \hat{P}C_2D^TC_2^T \end{bmatrix},
$$
$$
\bar{\Upsilon}_{23} = \begin{bmatrix} A_1P & A_1C_2^T\hat{P} \\ C_2A_1P & C_2A_1C_2^T\hat{P} \end{bmatrix} - \tilde{M}_P,
$$
$$
\bar{\Upsilon}_{35} = \epsilon_b h\begin{bmatrix} PA_1^T & PA_1^TC_2^T \\ \hat{P}C_2A_1^T & \hat{P}C_2A_1^TC_2^T \end{bmatrix} - \epsilon_b h\tilde{M}_P^T,
$$
$$
\bar{\Upsilon}_{38} = \sqrt{1+\epsilon_b h^2}\begin{bmatrix} PF^T & PF^TC_2^T \\ \hat{P}C_2F^T & \hat{P}C_2F^TC_2^T \end{bmatrix}
$$

and where we denote

$$
\tilde{R}_p = \tilde{P}R_1\tilde{P}, \quad \tilde{M}_P = \tilde{M}\tilde{P}, \quad \hat{P}_K = \hat{P}K. \tag{17a,c}
$$

We arrive at the following theorem:

Theorem 2 Consider the system (2a,b) and (3), where $\tilde{A}_0,\ \tilde{A}_1$ and $\tilde{B}_2$ are replaced by $A_0,\ A_1$ and B_2, respectively. There exists a controller of the structure of (7) that achieves negative J_{OF} for all nonzero $w \in \tilde{L}^2_{\mathcal{F}_t}([0,\infty);\mathcal{R}^q)$, $n \in \tilde{L}^2_{\mathcal{F}_t}([0,\infty);\mathcal{R}^p)$, for a prescribed scalar $\gamma > 0$, a given upper bound h and positive tuning parameters α and ϵ_b if there exist matrices $P > 0,\ \hat{P} > 0,\ \tilde{M}_P,\ \tilde{R}_P > 0$ and $\hat{P}_K$, that satisfy (16). In the latter case the controller gain matrix is:

$$
K = \hat{P}^{-1}\hat{P}_K. \tag{18}
$$

B. static output-feedback control - the uncertain case

In the uncertain case, where the systems matrices $\tilde{A}_0,\ \tilde{A}_1$ and $\tilde{B}_2$ are given in (4), we obtain the following LMI condition, derived for simplicity for the case where $\bar{H}_2 = 0$,

$$\begin{bmatrix} \tilde{\Upsilon} & \bar{\epsilon}_1\Psi_1 & \Phi_1 & \bar{\epsilon}_2\Psi_2 & \Phi_2 \\ * & -\bar{\epsilon}_1 & 0 & 0 & 0 \\ * & * & -\bar{\epsilon}_1 & 0 & 0 \\ * & * & * & -\bar{\epsilon}_2 & 0 \\ * & * & * & * & -\bar{\epsilon}_2 \end{bmatrix} < 0, \quad (19)$$

where the $(1,1)$ block matrix $\tilde{\Upsilon}$ of the above inequality is the LMI of (16).

Proof: Replacing $A_0,\ A_1$ by $\tilde{A}_0$ and $\tilde{A}_1$ respectively in (16), the following inequality is obtained:

$$\begin{array}{l} \tilde{\Upsilon} + \Psi_1 F_0^T \Phi_1^T + [\Psi_1 F_0^T \Phi_1^T]^T + \Psi_2 F_1 \Phi_2^T \\ +[\Psi_2 F_1 \Phi_2^T]^T < 0, \\ \Psi_1^T = [[\bar{H}_0 P \ \ \bar{H}_0 C_2^T \hat{P}]\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ ], \\ \Phi_1^T = [0\ [E_0^T \ E_0^T C_2^T]\ 0\ 0\ \Phi_{1a}\ 0\ 0\ 0\ 0\ ], \\ \Phi_{1a} = \epsilon_b h [E_0^T \ E_0^T C_2^T], \\ \Psi_2^T = [0\ [E_1^T \ \ E_1^T C_2^T]\ 0\ 0\ \Psi_{2a}\ 0\ 0\ 0\ 0\ ], \\ \Psi_{2a} = \epsilon_b h [E_1^T \ \ E_1^T C_2^T], \\ \Phi_2^T = [0\ 0\ [\bar{H}_1 P \ \bar{H}_1 C_2^T \hat{P}]\ 0\ 0\ 0\ 0\ 0\ 0\ ], \end{array} \quad \text{(20a-g)}$$

where $\tilde{\Upsilon}$ is the LMI of (16).

Noting that

$$[\bar{\epsilon}_1 \Psi_1 F_0 + \Phi_1]\bar{\epsilon}_1^{-1}[\bar{\epsilon}_1 \Psi_1 F_0 + \Phi_1]^T > 0,\ \bar{\epsilon}_1 > 0$$

one obtains

$$\bar{\epsilon}_1 \Psi_1 F_0 F_0^T \Psi_1^T + \Phi_1 \bar{\epsilon}_1^{-1} \Phi_1^T > \Psi_1 F_0^T \Phi_1^T + [\Psi_1 F_0^T \Phi_1^T]^T.$$

Recalling that $F_0^T F_0 < I$ and applying similar argument for the fourth and fifth left size terms, involving $\Psi_2,\ F_1$ and Φ_2 in (20a), the inequality (20a) is guaranteed if the following LMI is guaranteed:

$$\tilde{\Gamma} + \Psi_1 \bar{\epsilon}_1 \Psi_1^T + \Phi_1 \bar{\epsilon}_1^{-1} \Phi_1^T + \Psi_2 \bar{\epsilon}_2 \Psi_2^T + \Phi_2 \bar{\epsilon}_2^{-1} \Phi_2^T < 0.$$

Replacing for $\Psi_1,\ \Psi_2,\ \Phi_1$ and Φ_2 (20b-g) and applying Schur complement, the LMI of (19) is obtained.

We thus arrive at the following theorem:

Theorem 3 Consider the system (2a,b) and (3), where $\bar{H}_2 = 0$. There exists a controller of the structure of (7) that achieves negative J_{OF} for all nonzero $w \in \tilde{L}^2_{\mathcal{F}_t}([0,\infty);\mathcal{R}^q)$, $n \in \tilde{L}^2_{\mathcal{F}_t}([0,\infty);\mathcal{R}^p)$, for a prescribed scalar $\gamma > 0$, a given upper bound h and positive tuning parameters α and ϵ_b if there exist matrices $P > 0,\ \hat{P} > 0,\ \tilde{M}_P,\ \tilde{R}_P > 0,\ \hat{P}_K$ and scalars $\bar{\epsilon}_1,\ \bar{\epsilon}_2$, that satisfy (19). In the latter case the controller gain matrix is given by (18).

V. Conclusions

In this paper the theory of linear H_∞ zero-order output-feedback control of state multiplicative noisy systems is extended to the norm-bounded uncertain discrete-time delayed systems, where the stochastic uncertainties are encountered in both the delayed and the non delayed states in the state space model of the system.

The delay is assumed to be unknown and time-varying where only the bound on its size is given. Delay dependent analysis and synthesis methods are extended to the norm-bounded uncertain case based on the input-output approach. This approach transforms the delayed system to a non-retarded system with norm-bounded operators. Based on the robust norm-bounded BRL derivation, where the additional norm bounded parameters appear in some of the system matrices, the robust static output-feedback control is formulated and solved. The solutions obtained for both the nominal systems and uncertain systems are expressed via a simple LMI conditions which can be readily solved without any iteration process.

An inherent over-design is admitted to our solution due to the initial use of the input-output approach which utilizes the use of bounded operators. These operators enable us to transform the retarded system to a norm-bounded one - as was previously done. However, the input-output approach enables for a simple and efficient solution of various control and filtering problems. While the solution of the BRL does not require a special structure assigned to the Lyapunov function that is applied, in the synthesis problem of either the nominal or uncertain case, an additional over design is admitted to the solution due to the structure imposed on the Lyapunov function that is used.

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