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Information-Constrained Optimal Control of Distributed Systems with Power Constraints

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Abstract— **In this paper we address the problem of information-constrained optimal control for an interconnected system subject to one-step communication delays and power constraints. The goal is to minimize a finite-horizon quadratic cost by optimally choosing the control inputs for the subsystems, accounting for power constraints in the overall system and different information available at the decision makers. To this purpose, due to the quadratic nature of the power constraints, the LQG problem is reformulated as a linear problem in the covariance of state-input aggregated vector. The zero-duality gap allows us to equivalently consider the dual problem, and decompose it into several sub-problems according to the information structure present in the system. Finally, the optimal control inputs are found in a form that allows for offline computation of the control gains.**

I. INTRODUCTION

Technological advances in computation and communication, and societal needs have revived the research interest in control of interconnected systems [5]. Some examples include smart grids, communication networks, and transportation systems. Traditionally, arguments in favor of distributed control (compared to centralized) are geographically distributed sensors, limited local computational power at the plant side, robustness against single-node failure and information privacy.

In general, the design of distributed control is difficult because it imposes information constraints on individual decision makers. Such constraints arise due to either partial information exchange between decision makers or communication delay. In the problem we address herein, decision makers are able to communicate the full information they receive - either due to own measurements or from other decision makers, however, with delay. In other words, information constraints are due to communication delays between decision makers. The information constraints, sometimes referred to as information structure, play a key role in determining the optimal control and decide on its computational tractability. Indeed, in [6] a linear quadratic Gaussian team problem is constructed with a non-classical information pattern and it is shown that a linear controller is not necessarily optimal. This problem is addressed in [7] where it is shown that the so-called partially nested information structure guarantees existence of optimal control laws that are linear in the associated information. Finally, a strong result characterizing the class of all information-constrained problems which may be cast as a convex program is given in [9].
Inspiration for our approach is given by the work in [4] which suggests that the information hierarchy existing between the decision makers can be exploited to obtain the optimal solution. First explicit solutions to linear quadratic Gaussian team problems that adopt similar approach are given in [10]. The authors however, consider a typical unconstrained linear quadratic team problem. But in reality, e.g. actuation capabilities are limited and thus must be accounted for in the design procedure.

The main contribution of this paper is a method to compute optimal control laws, for a power-constrained system with given information structure. We assume the latter to be induced by a one-step communication delays between the decision makers. To this end, the problem is reformulated in its dual Lagrangian form, where the covariance of the state-input aggregated vector is defined as decision variable. The information structure is then exploited to split the optimization problem into simpler sub-problems that have alike structure. Indeed, in-network control [2] is seen as the decomposition of a complex task into smaller subtasks resulting in computationally inexpensive local control actions. From an application point of view, the goal is to implement and analyze the developed approach within a network infrastructure, exploiting the possibility of existing (but limited) in-network processing, in order to improve control performance.

The remainder of the paper is outlined as follows. We start with problem setup in II. The method to decouple problem into several sub-problems via covariance decomposition is presented in section III. In section IV we provide structural characterization of the solution to the problem and finally conclusions are given in V.

II. PROBLEM SETTING

Consider a large-scale interconnected dynamical system composed of N physically-coupled linear time-invariant (LTI) subsystems. Formally, the physical interconnections are described through a graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$. We will refer to it as the physical interconnection graph. Each node $i \in \mathscr{V}$ corresponds to one of the subsystems $i \in \{1, \dots, N\}$. An edge $(j,i) \in \mathscr{E}$ if dynamics of node i is directly affected by node j. We assume that $\mathscr{G}$ is connected and undirected, i.e., $(i,j) \in \mathscr{E}$ if and only if $(j,i) \in \mathscr{E}$. The set of direct neighbors of

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decision maker i is defined as $\mathcal{N}_i = \{j \,|\, (j,i) \in \mathcal{E}\}$. The length of the shortest path between nodes i and j will be denoted by d_{ij}. Clearly, if $j \in \mathcal{N}_i$ then $d_{ij} = 1$. The dynamics of the i-th subsystem is given by a first order stochastic difference equation

$$x_i(k+1) = A_i x_i(k) + B_i u_i(k) + \sum_{j \in \mathcal{N}_i} A_{ij} x_j(k) + w_i(k), \quad (1)$$

where $A_i \in \mathbb{R}^{n_i \times n_i}$, $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, $x_i(k) \in \mathbb{R}^{n_i}$ is the state and $u_i(k) \in \mathbb{R}^{m_i}$ is the control signal of the i-th subsystem. The noise process $w_i(k) \in \mathbb{R}^{n_i}$ is zero-mean i.i.d. Gaussian noise with covariance matrix Σ_w. The initial state $x_i(0)$ is a random variable with zero-mean and finite covariance Σ_x. Moreover, $x_i(0)$ and $w_i(k)$ are assumed to be pair-wise independent at each time instant k and every i. For a more compact notation, equation (1) can be rewritten as

$$x(k+1) = Ax(k) + Bu(k) + w(k) \quad (2)$$

where the stacked vectors are $x(k) = (x_1^\top(k), \ldots, x_N^\top(k))^\top \in \mathbb{R}^n$, $w(k) = (w_1^\top(k), \ldots, w_N^\top(k))^\top \in \mathbb{R}^n$, $u(k) = (u_1^\top(k), \ldots, u_N^\top(k))^\top \in \mathbb{R}^m$, $n = \sum_{i=1}^N n_i$ and $m = \sum_{i=1}^N m_i$. The admissible control policies at time instant k are measurable functions of the information available to each decision maker i (sometimes also referred to as player i)

$$u_i(k) = \gamma_k^i(\mathcal{I}_k^i) \quad (3)$$

where $\mathcal{I}_k^i$, $k = 0, \ldots, T-1$, is defined as

$$\mathcal{I}_k^i = \{\mathcal{I}_{k-1}^i, x_k^i, u_{k-1}^i\} \bigcup_{j \in \mathcal{N}_i} \{\mathcal{I}_{k-1}^j\}, \quad k > 0, \quad (4)$$

and $\mathcal{I}_0^i = \{x_0^i\}$. In other words, the information set of each decision maker i is updated at time instant k by the current state and the one-step delayed information from the direct neighbors $\mathcal{N}_i$. The objective is to minimize the following global control cost

$$J_{\mathcal{C}} = \mathrm{E}\left[\sum_{k=0}^{T-1} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^\top Q \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + x(T)^\top Q_T x(T)\right] \quad (5)$$

where the matrix Q is partitioned according to the vector $z(k) = \begin{bmatrix} x(k)^\top u(k)^\top \end{bmatrix}^\top$ i.e.

$$Q = \begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{bmatrix}. \quad (6)$$

The matrix Q_{uu} is assumed to be positive-definite matrix, while Q and Q_T are assumed to be semi-definite positive. We also assume controllability of pair (A,B) as well as detectability of $(Q^{\frac{1}{2}}, A)$. Moreover, it is assumed that each decision maker knows the parameters of the overall system. The cost (5) is to be minimized under power constraints, which are defined as

$$\mathrm{E}\left[z(k)^\top W_i z(k)\right] \leq p_k^i, \quad \forall i = 1, \ldots, M \quad (7)$$

where $W_i \in \mathbb{R}^{(n+m)\times(n+m)}$, $i = 1, \ldots, M$, is a positive semi-definite weighting matrix. By appropriate choice of W_i, the set of constraints in (7) captures either constraints present in the power of the overall system, or those related to the individual subystems. Ultimately, the problem is formally stated as

$$\begin{aligned} \min_{\gamma_{0:T-1}} \quad & \mathrm{E}\left[\sum_{k=0}^{T-1} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^\top Q \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + x(T)^\top Q_T x(T)\right] \quad (8) \\ \text{s.t.} \quad & (2), (3), (7) \end{aligned}$$

where $\gamma_k = [\gamma_k^1, \ldots, \gamma_k^N]$ is composed of all players control policies. Before stating the main result of this section we define the notion of partial nestedness [11].

Definition 1: The information structure $\mathcal{I}_k = \{\mathcal{I}_k^1, \ldots, \mathcal{I}_k^N\}$ is partially nested if, for every admissible policy (3), whenever $u_i(\tau)$ affects $\mathcal{I}_k^j$, then $\mathcal{I}_\tau^i \subset \mathcal{I}_k^j$.

Lemma 1 (Partial nestedness): The information structure defined by (4) is partially nested.

Proof: Let d_{ji} be the length of shortest path $j \to i$ in the physical interconnection graph. Considering (4), the information set $\mathcal{I}_k^i$ is influenced by measurement $x_j(k - d_{ji})$, or equivalently by $u_j(k - d_{ji} - 1)$. Thus, to check if information structure (4) is partially nested, one should verify the condition: $\mathcal{I}_{k-d_{ji}-1}^j \subset \mathcal{I}_k^i$. Recalling the assumption that graph $\mathcal{G}$ is connected and undirected, the information sets of decision makers i and j are explicitly written as

$$\mathcal{I}_k^i = \bigcup_{n=1,\ldots,N} \{x_n(0 : k - d_{ni})\},$$
$$\mathcal{I}_{k-d_{ji}-1}^j = \bigcup_{n=1,\ldots,N} \{x_n(0 : k - d_{nj} - d_{ji} - 1)\},$$

which reduces the partial nestedness condition to: $d_{nj} + d_{ji} + 1 \geq d_{ni}$. Since d_{ni} is the length of the shortest path between nodes n and i in $\mathcal{G}$, one can write: $d_{ni} \leq d_{nj} + d_{ji} < d_{nj} + d_{ji} + 1$ which concludes the proof. ∎

Taking into consideration that problem (8) is subject to power constraints, it is convenient to reformulate it in terms of covariance as the new decision variable

$$V(k) = \mathrm{E}\left[z(k) z(k)^\top\right] = \mathrm{E}\left[\begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^\top\right]$$

With the additional constraint given by (3), problem (8) is posed as a covariance selection problem

$$\begin{aligned} \min_{V(0:T-1)\succeq 0} \quad & tr(Q_T V_{xx}(T)) + \sum_{k=0}^{T-1} tr(QV(k)) \quad (9) \\ \text{s.t.} \quad & FV(0)F^\top = \Sigma_x \\ & \begin{bmatrix} A & B \end{bmatrix} V(k) \begin{bmatrix} A & B \end{bmatrix}^\top + \Sigma_w = FV(k+1)F^\top \\ & tr(W_i V(k)) \leq p_k^i, \quad \forall i = 1, \ldots, M \end{aligned}$$

where $F = \begin{bmatrix} I & 0 \end{bmatrix}$. Part of the result above is derived from the fact that, for a generic matrix Θ the following identity holds

$$\mathrm{E}\left[z(k)^\top \Theta z(k)\right] = tr(\Theta V(k)).$$

Additionally, rewriting the system dynamics equation (2) in terms of a covariance variable V

$$FV(k+1)F^\top = V_{xx}(k+1) = \mathrm{E}\left[x(k+1)x(k+1)^\top\right]$$
$$= \begin{bmatrix}A & B\end{bmatrix} V(k) \begin{bmatrix}A & B\end{bmatrix}^\top + \Sigma_w$$

and translating the initial condition $\mathrm{E}\left[x(0)x(0)^\top\right] = \Sigma_x$ into

$$V_{xx}(0) = \mathrm{E}\left[x(0)x(0)^\top\right] = FV(0)F^\top = \Sigma_x,$$

the form in (9) is obtained.

III. Information decomposition

A. Covariance Decomposition

For the sake of simplicity of derivation we demonstrate the method on a two-player system. Considering the state equation (2), each decision maker at each time instant k is able to compute the estimate of the state x based on the common information $\mathscr{I}_k^0$ the two players have at time instant k, i.e.

$$\mathscr{I}_k^0 = \mathscr{I}_k^1 \cap \mathscr{I}_k^2 = \{x(0:k-1), u(0:k-1)\}, \tag{10}$$

later referred to as the coordinator's information set. The estimate is given by

$$\hat{x}(k) = \mathrm{E}\left[x(k)|\mathscr{I}_k^0\right] = Ax(k-1) + Bu(k-1), \tag{11}$$

since $\mathrm{E}\left[w(k-1)|\mathscr{I}_k^0\right] = 0$. Locally, after measuring its own state x_i each decision maker can compute the local noise value at the previous time step as

$$\omega_i(k) = w_i(k-1) = x_i(k) - M_i^\top \hat{x}(k) \tag{12}$$

where $M_1^\top = \begin{bmatrix}I & 0\end{bmatrix}^\top$, $M_2^\top = \begin{bmatrix}0 & I\end{bmatrix}^\top$.

The quantities $\hat{x}, \omega_1, \omega_2$ form a pair-wise independent components of state. Due to linearity of the state decomposition given by (11), (12) and partial nestedness of the information structure (4) one can represent the optimal control input in the form

$$u(k) = \hat{\phi}(k) + \begin{bmatrix}\phi_1(k) \\ \phi_2(k)\end{bmatrix} \tag{13}$$

where $\hat{\phi}(k) = -L_0(k)\hat{x}(k)$, $\phi_1(k) = -L_1(k)\omega_1(k)$ and $\phi_2(k) = -L_2(k)\omega_2(k)$, for some gains L_0, L_1, L_2. Aiming for the decomposition of problem (9), we define a vector $\bar{z}$ of state components $\hat{x}, \omega_1, \omega_2$ and input components $\hat{\phi}, \phi_1, \phi_2$

$$\bar{z}(k) = \begin{bmatrix}\hat{x}(k)^\top & \hat{\phi}(k)^\top | \omega_1(k)^\top & \phi_1(k)^\top | \omega_2(k)^\top & \phi_2(k)^\top\end{bmatrix}^\top \tag{14}$$

whose blocks are independent. Additionally, denoting the state decomposition (11) - (12) and input decomposition in (13) as

$$(x^0(k), x^1(k), x^2(k)) = (\hat{x}(k), \omega_1(k), \omega_2(k)),$$
$$(u^0(k), u^1(k), u^2(k)) = (\hat{\phi}(k), \phi_1(k), \phi_2(k)).$$

the covariance matrix of $\bar{z}(k)$ is given by

$$\bar{V}(k) = \mathrm{E}\left[\bar{z}(k)\bar{z}(k)^\top\right] = \begin{bmatrix} V^0(k) & 0 & 0 \\ 0 & V^1(k) & 0 \\ 0 & 0 & V^2(k) \end{bmatrix} \tag{15}$$

where covariance matrices $V^l, l \in \{0,1,2\}$, of the individual blocks are

$$V^l(k) = \mathrm{E}\left[\begin{bmatrix}x^l(k) \\ u^l(k)\end{bmatrix}\begin{bmatrix}x(k) \\ u(k)\end{bmatrix}^\top\right] = \begin{bmatrix}V_{x^lx^l}(k) & V_{x^lu^l}(k) \\ V_{u^lx^l}(k) & V_{u^lu^l}(k)\end{bmatrix}.$$

The sparsity of $\bar{V}$ is due to block-independency of the vector $\bar{z}$ and due to presence of zero-mean Gaussian noise.
Finally, recalling (1), for the sake of compactness, A and B are partitioned as

$$A = \begin{bmatrix}A_1 | A_2\end{bmatrix}, \quad B = \begin{bmatrix}B_1 | B_2\end{bmatrix}.$$

where $A_1 \in \mathbb{R}^{n\times n_1}$, $A_2 \in \mathbb{R}^{n\times n_2}$, $B_1 \in \mathbb{R}^{n\times m_1}$ and $B_2 \in \mathbb{R}^{n\times m_2}$. Similarly, referring to (6), matrix Q is partitioned as

$$Q = [Q_{x_1} | Q_{x_2} | Q_{u_1} | Q_{u_2}]$$

where $Q_{x_1} \in \mathbb{R}^{(n+m)\times n_1}$, $Q_{x_2} \in \mathbb{R}^{(n+m)\times n_2}$, $Q_{u_1} \in \mathbb{R}^{(n+m)\times m_1}$, and $Q_{u_2} \in \mathbb{R}^{(n+m)\times m_2}$. Furthermore, we define the following two matrices

$$Q^1 = [Q_{x_1} | Q_{u_1}], \quad Q^2 = [Q_{x_2} | Q_{u_2}].$$

B. Equivalent Problem Formulation

In order to rewrite the constraints appearing in equation (9) as a function of $\bar{V}$, vectors $x(k)$, $u(k)$, $z(k)$ are obtained pre-multiplying the new variable $\bar{z}(k)$ according to

$$x(k) = C_x\bar{z}(k), \qquad u(k) = C_u\bar{z}(k), \qquad z(k) = C\bar{z}(k). \tag{16}$$

where

$$C = \begin{bmatrix}C_x \\ C_u\end{bmatrix} = \left[\begin{array}{cc|cc|cc} I & 0 & I & 0 & 0 & 0 \\ & & 0 & 0 & I & 0 \\ \hline 0 & I & 0 & I & 0 & 0 \\ & & 0 & 0 & 0 & I \end{array}\right].$$

The evolution of the original state $x(k)$ expressed as a function of $\bar{z}(k)$ is now

$$x(k+1) = \begin{bmatrix}A & B\end{bmatrix} C\bar{z}(k) + w(k). \tag{17}$$

Combining the expressions in equations (15), (16) and (17) the variance of the state x can be written as

$$V_{xx}(k) = \mathrm{E}\left[x(k)x(k)^\top\right] \tag{18}$$
$$= \mathrm{E}\left[(C_x\bar{z}(k))(C_x\bar{z}(k))^\top\right] = C_x\bar{V}(k)C_x^\top$$

In the same way the variance of input $u(k)$ equals

$$V_{uu}(k) = \mathrm{E}\left[u(k)u(k)^\top\right] = C_u\bar{V}(k)C_u^\top \tag{19}$$

Finally, from (2) and (18), the evolution of the system's state imposes the following recursive covariance equation

$$C_x\bar{V}(k+1)C_x^\top = V_{xx}(k+1) = \mathrm{E}\left[x(k+1)x(k+1)^\top\right] \tag{20}$$
$$= \begin{bmatrix}A & B\end{bmatrix} C\,\mathrm{E}\left[\bar{z}(k)\bar{z}(k)^\top\right] C^\top \begin{bmatrix}A & B\end{bmatrix}^\top + \mathrm{E}\left[w(k)w(k)^\top\right]$$
$$= \begin{bmatrix}A & B\end{bmatrix} C\bar{V}(k)C^\top \begin{bmatrix}A & B\end{bmatrix}^\top + \Sigma_w.$$

Similarly from the assumption on the state initial condition, the equivalent condition for the covariance is written as

$$V_{xx}(0) = \mathrm{E}\left[x(0)x(0)^\top\right] = C_x\bar{V}(0)C_x^\top = \Sigma_x. \tag{21}$$

We then have the following proposition which is the main achievement of this subsection.

Proposition 1: Let $\bar{V}$ be the covariance of the extended vector $\bar{z}$. Problem (8) is equivalent to

$$\begin{aligned}\min_{\bar{V}(0:T)\succeq 0} \quad & tr(C_x^\top Q_T C_x \bar{V}(T)) + \sum_{k=0}^{T-1} tr(C^\top QC\bar{V}(k)) \qquad (22)\\ \text{s.t.} \quad & C_x\bar{V}(0)C_x^\top = \Sigma_x \\ & C_x\bar{V}(k+1)C_x^\top = \begin{bmatrix}A & B\end{bmatrix} C\bar{V}(k)C^\top \begin{bmatrix}A & B\end{bmatrix}^\top + \Sigma_w \\ & tr(C^\top W_i C\bar{V}(k)) \leq p_k^i\end{aligned}$$

Proof: The proof follows from problem in (8) and equations (20) and (21). ■

Remark 1: Although the methodology is presented for the case of 2-player system, it can be extended to a system of N players using an algorithmic approach for state decomposition [8].

IV. Information-oriented Optimization via Dual Decomposition

In this section we proceed to define the dual problem to (22), which allows to transform the original constrained problem (8) into an unconstrained one. To this end, we introduce dual variables $S(k) \in \mathbb{R}^{n\times n}, k = 0,\dots,T$, to account for constraints on the evolution of $\bar{V}(k)$, as defined in (20) and (21) . Additionally, dual scalar variables $\tau_i(k) \in \mathbb{R}^+, k = 0,\dots,T-1$, are defined to account for power constraints in the overall system.

A. Computation of Dual Variables

Introducing the Langrange multipliers $S(0),\dots,S(T)$ and $\tau_i(0),\dots,\tau_i(T-1)$ the primal problem (22) is equivalent to

$$\begin{aligned}\max_{S(0:T),\tau_i(0:T-1)} \min_{\bar{V}(0:T)} \quad & tr\left(S(0)(\Sigma_x - C_x\bar{V}(0)C_x^\top)\right) \\ & + tr(C_x^\top Q_T C_x\bar{V}(T)) + \sum_{k=0}^{T-1} tr(QC\bar{V}(k)C^\top) \\ & + \sum_{k=0}^{T-1} tr\left(S(k+1)\begin{bmatrix}A & B\end{bmatrix} C\bar{V}(k)C^\top \begin{bmatrix}A & B\end{bmatrix}^\top\right) \\ & + \sum_{k=0}^{T-1} tr\left(S(k+1)(\Sigma_w - C_x\bar{V}(k+1)C_x^\top)\right) \\ & + \sum_{k=0}^{T-1}\sum_{i=1}^{M} tr\left(\tau_i(k)(C^\top W_i C\bar{V}(k) - p_k^i\right) \qquad (23)\end{aligned}$$

where the constraints in (22) now appear as part of the cost in form of linear operators on covariance matrix $\bar{V}(k)$. Defining the Hamiltonian of the system

$$\begin{aligned}H(T) &= tr\{C_x^T(Q_T - S(T))C_x\bar{V}(T)\} \\ H(k) &= tr\{C^\top(Q + \begin{bmatrix}A & B\end{bmatrix}^\top S(k+1)\begin{bmatrix}A & B\end{bmatrix} + \\ &- \begin{bmatrix}S(k) & 0\\ 0 & 0\end{bmatrix} + \sum_{i=1}^{M}\tau_i(k)W_i)C\bar{V}(k)\} \quad \text{for } k = 0,\dots,T-1\end{aligned}$$

the dual problem in (23) is rewritten as

$$\begin{aligned}\max_{S(0:T),\tau_i(0:T-1)} \min_{\bar{V}(0:T)} \quad & H(T) + \sum_{k=0}^{T-1}\{H(k) + \Sigma_w trS(k+1)\} + \\ & + \Sigma_x trS(0) - \sum_{k=0}^{T-1}\sum_{i=1}^{M}\tau_i(k)p_k^i.\end{aligned}$$

With the boundary condition on the Hamiltonian it follows $H(T) = 0$, hence $S(T) = Q_T$. The dual function is finite if and only if

$$Q + \begin{bmatrix}A & B\end{bmatrix}^\top S(k+1)\begin{bmatrix}A & B\end{bmatrix} - \begin{bmatrix}S(k) & 0\\ 0 & 0\end{bmatrix} + \sum_{i=1}^{M}\tau_i(k)W_i \succeq 0. \tag{24}$$

Since the primal problem (22) is convex and constraints are affine, Slater's condition can be relaxed. Indeed, the constraints in (22) are composed of linear equalities and inequalities and domain of the defined cost function is open, the Slater's condition reduces to feasibility. To this end, it is easy to verify that the set of constraints in (22) defines a non-empty region. Hence, the dual problem is equivalent to the primal and is stated as

$$\max_{S(0:T),\tau_i(0:T-1)} \quad tr(S(0))\Sigma_x + \Sigma_w\sum_{k=1}^{T} trS(k) - \sum_{k=0}^{T-1}\sum_{i=1}^{M}\tau_i(k)p_k^i \tag{25}$$

$$\begin{aligned}\text{s.t.} \quad & Q(k) \\ & + \begin{bmatrix}A^\top S(k+1)A - S(k) & A^\top S(k+1)B \\ B^\top S(k+1)A & B^\top S(k+1)B\end{bmatrix} \succeq 0 \\ & S(T+1) = 0\end{aligned}$$

where the constraint in (25) is obtained from (24) by defining

$$Q(k) = \begin{cases} Q + \sum_{i=1}^{M}\tau_i(k)W_i, & k = 1,\dots,T-1 \\ \begin{bmatrix}Q_T & 0\\ 0 & 0\end{bmatrix}, & k = T.\end{cases} \tag{26}$$

With fixed values of τ_i, the previous equation is maximized for every time-instant k with

$$\begin{aligned}S(k) &= A^\top S(k+1)A + Q_{xx}(k) - L(k)^\top Y(k)L(k) \qquad (27)\\ Y(k) &= (B^\top S(k+1)B + Q_{uu}(k)) \\ L(k) &= Y(k)^{-1}(B^\top S(k+1)A + Q_{xu}^\top(k))\end{aligned}$$

which can be proved by analogously to [3]. Indeed, the choice of $S(k)$ should be made such that $trS(k)$ is maximized and at the same time constraint in (24) is satisfied, under the condition that the optimal value of $S(k+1)$ is known. To this end, since any choice of $S(k)$ with trace greater than the trace of (27) violates the constraint in (24), the choice in (27) is optimal. The variables τ_i have to be computed numerically from (25) accounting for (27).

B. Optimal Information-constrained Control

In this subsection we show how to obtain the solution via information decomposition. In paragraph III-A we introduced state, input and covariance decomposition. In the 2-player's case, we obtain three information sets: $\mathscr{I}_0, \mathscr{I}_1, \mathscr{I}_2$, that are defined by (4), (10) and referred herein as the coordinator, first subsystem and second subsystem respectively. Moreover, the coordinator is assumed to have the following information about the overall system

$$\left(A_0, B_0, Q^0, x^0(k)\right) \triangleq (A, B, Q, \hat{x}(k)).$$

Before stating the main result of this paper, we define the expression for $J^l,\ l = 0,1,2$ as

$$\begin{aligned} J^l(V^l,S,\tau) &= tr\left(Q_T F_l^\top V^l(T) F_l\right) + \sum_{k=0}^{T-1} tr\left(Q^l V^l(k)\right) \quad (28)\\ &+ tr\left\{S(k+1)\left(\left[A_l|B_l\right] V^l(k) \left[A_l|B_l\right]^\top\right)\right\}\\ &- tr\left\{S(k+1)\left(F_l^\top V^l(k+1) F + \frac{\Sigma_w}{3}\right)\right\}\\ &+ tr\left\{S(0)\left(F_l^\top V^l(0) F_l - \frac{\Sigma_x}{3}\right)\right\}\\ &+ \sum_{k=0}^{T-1}\sum_{i=1}^{M} tr\left(\tau_i(k) W_i V^l(k) - q_k^i\right) \end{aligned}$$

where F_0, F_1 and F_2 are such that

$$\begin{aligned} F_0 V^0(k) F_0{}^\top &= V_{\hat{x}\hat{x}}(k), \quad (29)\\ F_1 V^1(k) F_1{}^\top &= \begin{bmatrix} V_{\omega_1\omega_1}(k) & 0\\ 0 & 0\end{bmatrix},\\ F_2 V^2(k) F_2{}^\top &= \begin{bmatrix} 0 & 0\\ 0 & V_{\omega_2\omega_2}(k)\end{bmatrix}. \end{aligned}$$

Moreover, the definition of q_k^i is given by identity: $p_k^i = 3q_k^i$. We can now state the main result of this paper.

Theorem 1 (Information-constrained optimal control): Let the system dynamics be given by equation (2). Considering the optimization problem defined in (8) and denoting by $S(k)$ and $\tau_i(k)$ the optimal values of the dual variables introduced in (23) we state the following.

i. The problem (8) is decoupled into the sum of independent sub-problems that are linear in the respective decision variables, i.e., it is equivalent to

$$\sum_{l=0}^{2} \min_{V^l(0:T)} J^l(V^l(0:T), S(0:T), \tau_{1:M}(0:T-1)) \quad (30)$$

where J^l is defined in (28) and V^l, $l = 0,1,2$ are defined in (15).

ii. The optimal covariances V^l, $l = 0,1,2$ of (30) are computed according to

$$\begin{aligned} V^l(k) &= \begin{bmatrix} V_{xx}^l(k) & V_{xu}^l(k)\\ V_{ux}^l(k) & V_{uu}^l(k)\end{bmatrix}, \quad (31)\\ V_{xx}^l(0) &= \frac{\Sigma_x}{3},\\ V_{ux}^l(k) &= -L_l(k) V_{xx}^l(k),\\ V_{uu}^l(k) &= V_{ux}^l(k)\left(V_{xx}^l(k)\right)^{-1} V_{xu}^l(k),\\ V_{xx}^l(k+1) &= \begin{bmatrix} A_l & B_l\end{bmatrix} V^l(k) \begin{bmatrix} A_l & B_l\end{bmatrix}^\top + \Sigma_w. \end{aligned}$$

where $L_l(k)$ is

$$L_l(k) = \left(B_l^\top S(k+1) B_l + Q_{uu}^l\right)^{-1}\left(A_l^\top S(k+1) B_l + Q_{xu}^l\right)^\top. \quad (32)$$

Proof: [of *i*.] From Proposition (22), problem (8) and (22) are equivalent. Furthermore, from equations (12), (15) and (23) accounting for the specific structure of matrix C_x one gets

$$\begin{aligned} C_x \bar{V}(k) C_x^\top &= V_{xx}(k) = V_{\hat{x}\hat{x}}(k) + \begin{bmatrix} V_{\omega_1\omega_1}(k) & 0\\ 0 & V_{\omega_2\omega_2}(k)\end{bmatrix}\\ &= F_0 V^0(k) F_0{}^\top + F_1 V^1(k) F_1^\top + F_2 V^2(k) F_2^\top \end{aligned}$$

where F_0, F_1 and F_2 are extraction matrices since $V_{\hat{x}\hat{x}}(k)$, $V_{\omega_1\omega_1}(k)$ and $V_{\omega_2\omega_2}(k)$ are square submatrices of $V^0(k), V^1(k)$ and $V^2(k)$ respectively. On the other hand, from the block-diagonal structure of $\bar{V}(k)$ and sparsity of C, one obtains

$$tr(QC\bar{V}(k)C^\top) = tr\left(QV^0(k)\right) + tr\left(Q^1 V^1(k)\right) + tr\left(Q^2 V^2(k)\right)$$

Analogously, we obtain

$$\begin{aligned} \begin{bmatrix} A & B\end{bmatrix} C\bar{V}(k)C^\top \begin{bmatrix} A & B\end{bmatrix}^\top &= \begin{bmatrix} A & B\end{bmatrix} V^0(k) \begin{bmatrix} A & B\end{bmatrix}^\top\\ &+ \begin{bmatrix} A_1 & B_1\end{bmatrix} V^1(k) \begin{bmatrix} A_1 & B_1\end{bmatrix}^\top\\ &+ \begin{bmatrix} A_2 & B_2\end{bmatrix} V^2(k) \begin{bmatrix} A_2 & B_2\end{bmatrix}^\top \end{aligned}$$

With algebraic reordering the proof of the first part is completed. ■

Proof: [of *ii*.] The second and fifth equation of (31) stated follow respectively from the condition on the variance of the initial state and equation (20). To prove the second and third equation, observe that the decoupled problems in (30) have a similar structure. Therefore, with the optimal values of $S(k)$ and $\tau_i(k)$, each problem in equation (30) is written as

$$\min_{V^l(0:T-1)\succeq 0} \sum_{k=0}^{T-1} tr(Z^l(k) V^l(k)) + \sum_{k=0}^{T} tr(S(k)) - \sum_{k=0}^{T-1}\sum_{i=1}^{M} \tau_i(k) q_k^i$$

where $Z^l(k)$ is given by

$$Z^l(k) = \begin{bmatrix} X_l Y_l{}^{-1} X_l{}^\top & X_l\\ X_l{}^\top & Y_l\end{bmatrix}$$

and the values of matrices X_l and Y_l are computed recursively

$$\begin{aligned} X_l &= A_l{}^\top S(k+1) B_l + Q_{xu}^l\\ Y_l &= B_l{}^\top S(k+1) B_l + Q_{uu}^l. \end{aligned}$$

To conclude the proof, exploiting the linearity of the subproblems, in order to compute the optimal covariances V_l it is sufficient to verify if the condition $tr(Z^l(k)V^l(k)) = 0$ is satisfied for a certain choice of the covariance matrix V_l. Indeed

$$tr(Z^l(k)V^l(k)) = tr\begin{bmatrix} X_l Y_l{}^{-1} X_l{}^\top V_{xx}^l + X_l V_{ux}^l & *\\ * & X_l^\top V_{xu}^l + Y_l V_{uu}^l\end{bmatrix}$$

By imposing to the diagonal elements in latter equation to be zero and recalling the assumption on positive-definitness (and thus invertibility) of Q_{uu}^l it follows:

$$V_{ux}^l = -Y_l^{-1} X_l^\top V_{xx}^l = -L_l V_{xx}^l$$
$$V_{uu}^l = -Y_l^{-1} X_l^T V_{xu}^l = V_{ux}^l (V_{xx}^l)^{-1} V_{xu}^l$$

which concludes the proof. ■

Corollary 1: Consider the system (1) and the optimization problem defined in (8). For the 2-player system, the optimal control law is given by

$$u(k) = u^0(k) + \begin{bmatrix} u^1(k) \\ u^2(k) \end{bmatrix} \tag{33}$$

where $u^l(k), l = 0,1,2$ is computed as

$$u^l(k) = -L_l(k) x^l(k)$$

and L_l is defined by (32).

Proof: According to Proposition 1, the problem defined in (8) is equivalent to the covariance selection problem in (22). Since the latter is decomposed in Theorem 1 and optimal values of covariances are provided by (30), the optimal control law follows in straightforward manner. Indeed, the control inputs referring to the coordinator and two subsystems are given by

$$u_l(k) = -V_{ux}^l(k) {V_{xx}^l}^{-1}(k) x^l(k) = -L^l(k) x^l(k) \tag{34}$$

where

$$L_0(k) = (B^\top S(k+1) B + Q_{uu})^{-1} (A^\top S(k+1) B + Q_{xu})^\top$$
$$L_1(k) = (B_1^\top S(k+1) B_1 + Q_{uu}^1)^{-1} (A_1^\top S(k+1) B_1 + Q_{xu}^1)^\top$$
$$L_2(k) = (B_2^\top S(k+1) B_2 + Q_{uu}^2)^{-1} (A_2^\top S(k+1) B_2 + Q_{xu}^2)^\top .$$

■

C. Interpretation of Control Input Structure

Consider a 2-player network with one-step communication delay as depicted in Fig. 1. It can be transformed into an equivalent network by introducing a dummy node, herein referred to as coordinator $\mathscr{C}$ (this is illustrated in Fig. 2). The colocated control units of subsystems $\mathscr{S}_1$ and $\mathscr{S}_2$ are of limited computational power (e.g. they might be routers, switches etc.) and limited memory. The coordinating unit $\mathscr{C}$ is assumed to be able to perform more complex computations. However, as it can be noted, it also has access to limited information about the overall system - more precisely, it knows a one-step delayed information about both subsystems.

In our approach $\mathscr{C}$ computes and stores the sequences of $S(0:T)$ and $\tau_i(0:T-1)$ offline, based on equations (25) and (27). During the system execution, at time-instant k, the coordinator $\mathscr{C}$ sends the matrix $S(k+1)$ to the local units. Then, using equations (34), the calculation of the gains $L_1(k)$ and $L_2(k)$ is computed locally at the control units of $\mathscr{S}_1$ and $\mathscr{S}_2$ by matrix multiplications, thus avoiding additional memory requirements. Moreover, the coordinator $\mathscr{C}$, computes the estimate of the overall state based on delayed knowledge, and passes the command to units $\mathscr{S}_1$ and $\mathscr{S}_2$. Hence, the corresponding inputs to be applied to the plants are computed using local measurements and the control signal from the coordinator.

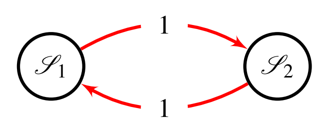

Fig. 1. 2-player problem

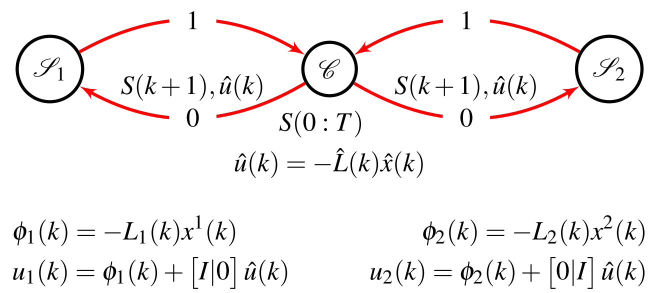

Fig. 2. Equivalent scheme at time instant k

V. CONCLUSIONS

In this paper a framework for power-constrained optimization based on information decomposition is introduced. The linear quadratic control problem with power constraints is decomposed accordingly through covariance decomposition and Lagrangian dual reformulation. As presented, the obtained equivalent problem is linear in the new decision variables and the control gains are computed offline. The approach adopted can be extended to a network of several players.

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