2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

Multimoment Matching Analysis of One-Sided Krylov Subspace Model Order Reduction for Nonlinear and Bilinear Systems

Oluwaleke Agbaje, Dina Laila Shona and Olivier Haas

***Abstract*— Modelling of complex systems comes with higher complexity and computational costs. Model order reduction enables better models to be exploited by control, diagnosis and prognosis algorithms. Effective model order reduction requires efficient methods to generate reduced order models. This is where the use of Krylov subspaces for model order reduction is of great advantage, especially when associated with high order bilinear model approximations of high order nonlinear models. This paper demonstrates the use of an Improved Phillips type projection for a one-sided Krylov subspace projection for reducing bilinear models. A new multimoment matching analysis of the proposed model reduction scheme is provided, and compared to some existing results in the literature.**

Keywords: Model Order Reduction, Krylov Subspaces, Improved Phillips Projection, Bilinear System, Nonlinear System.

I. INTRODUCTION

Bilinear models are commonly used to represent the dynamic of a system due to their ability to approximate nonlinear systems whilst being closely related to a linear system structure. Approximating nonlinear dynamics using bilinear structures often results in sparse matrices with large dimensions [1]. This creates the need for model order reduction (MOR). Reduced dimension models are more suitable for control [2], design [3] and reducing simulation time cost [4]. The aim of model order reduction techniques is to reduce the dimension of the system while preserving the dynamics of the input and output relationship as much as possible, and retaining acceptable level of accuracy. Linear model order reduction approaches such as Krylov subspace projection [5], balanced truncation [5] and H_2 model reduction [6] have been extended to bilinear models [1], [7]–[11]. Balanced truncation and H_2-norm model reduction approaches are beyond the scope of this paper. Bilinear models are particularly important because they are suitable for approximating nonlinear systems and models whilst retaining a well structured mathematical framework within which linear systems coexist. They have been used to approximate a wide range of physical/electrical [1], chemical [12], biological [13], social [8] and engineering systems [14], as well as manufacturing processes [15].

The first one-sided Krylov subspace projection technique for bilinear model order reduction was proposed by J.R Phillips [7]. His work has greatly influenced the papers written by Z. Bai and D. Skoogh [1] and L. Feng and P. Benner [16]. The work done by Z. Bai and D. Skoogh shows superior input-output preservation capacity but cannot be described as one-sided projection due to the awkward computation of the reduced order model which requires more computational effort. In [16] it has been stated that using the same Krylov subspace bases as in [1], the same moments are matched without utilising the awkward computation of the reduced order model. In [9] a slightly different approach is used. A constant input condition has been imposed in order to match the moments of the resulting linear model. There exists other methods which match moments at multiple expansion points. Other variants of this approach also ensure that stability is preserved [17].

This paper shows a unique multimoment matching analysis for bilinear models at the expansion point of zero. It proposes a new set of Krylov subspaces for matching more multimoments of a bilinear model when compared to [7], using a one-sided projection technique. A case study of a transmission line model is used to compare the developed methods with those presented in [7], [9] and [16]. A special case of a bilinearised nonlinear system has been used for the case study.

The next section describes bilinear systems. Section III describes Krylov subspace model order reduction for bilinear models. Section IV analyses, for their multimoments matching capacity, the one-sided Krylov subspace methods as presented in [7] and [16]. This is followed by a presentation and analysis of the improved method in Section V. A case study applying the proposed method to a nonlinear transmission line is presented in Section VI.

II. BILINEAR SYSTEMS

Bilinear models can be represented in various forms [18]. In this paper, we focus on bilinear systems that take the form:

$$\dot{x} = Ax + \sum_{i=1}^{m} N_i x u_i + Bu, \tag{1}$$

$$y = Cx, \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $N_i \in \mathbb{R}^{n\times n}$ for $i = 1, 2, \dots, m$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $x \in \mathbb{R}^n$ are the states, with $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ being the system input and output respectively. For a single-input-single-output (SISO) model the input and output dimensions are $m = p = 1$. Otherwise when they are greater than one, the bilinear system is of multi-input-multi-output (MIMO). Other variations of this configuration exist such as multi-input-single-output (MISO) and single-input-multi-output (SIMO). This paper focuses on the SISO bilinear model. Generally, the bilinearity is defined by a product of

The authors are with the School of Mechanical, Aerospace and Automotive Engineering, Coventry University, Coventry CV1 5FB, UK. oluwaleke.agbaje@coventry.ac.uk

the system states and inputs [14]. Therefore, for a fixed input, the bilinear model is linear in state and for a fixed state, it is linear in the input.

In [1], [7], the Carleman bilinearisation process [19] has been used to approximate input affine nonlinear models of the form

$$\dot{x} = f(x) + Bu, \tag{3}$$
$$y = Cx. \tag{4}$$

The nonlinear function $f(x) : \mathbb{R}^n \to \mathbb{R}^n$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$. These often result in very high order bilinear models which require model order reduction.

A. Transfer function and multimoments of bilinear systems

The input-output relationship of nonlinear and bilinear systems are often represented using the convolution theorem [1]. Consider a SISO bilinear model (1)-(2). The input-output characteristics of the system can be described using an infinite sum of convolution integrals, such that

$$y(t) = \sum_{k=1}^{\infty} y_k(t), \tag{5}$$

where $y_k(t)$ is the output of the k^{th} subsystem and can be represented as

$$y_k(t) = \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} h(t_1, t_2, \dots, t_k) u(t - t_1 - t_2 - \dots - t_k) \dots \times u(t - t_k) dt_k \dots dt_1, \tag{6}$$

where $h(t_1, t_2, \dots, t_k)$ is the kernel or also known as the impulse response, which can be represented as

$$h(t_1, t_2, \dots, t_k) = Ce^{At_k - 1} N \dots e^{At_2} N e^{At_1} B. \tag{7}$$

A multivariable Laplace transform of the kernels can be used to define a transfer function for $h(t_1, t_2, \dots, t_k)$ given as

$$H(s_1, s_2, \dots, s_k) = C(s_k I - A)^{-1} N (s_{k-1} I - A)^{-1} N \dots (s_2 I - A)^{-1} N (s_1 I - A)^{-1} B. \tag{8}$$

The concept of transfer functions for time-invariant bilinear systems has also been discussed in [1]. $H(s_1, s_2, \dots, s_k)$ is referred to as the transfer function for the k^{th} subsystem, which can be expanded using a multivariable Maclaurin series such that

$$H(s_1, \dots, s_k) = \sum_{l_k=1}^{\infty} \dots \sum_{l_1=1}^{\infty} m(l_1, l_2, \dots, l_k) s_1^{l_1-1} s_2^{l_1-1} \dots s_k^{l_k-1}, \tag{9}$$

with

$$m(l_1, \dots, l_k) = (-1)^k C A^{-l_k} N \dots A^{l_2} N A^{-l_1} B \tag{10}$$

being the multimoments of the k^{th} subsystem.

Consider the first subsystem transfer function, with its Maclaurin series expansion

$$H(s_1) = C(s_1 I - A)^{-1} B = \sum_{l_1=1}^{\infty} m(l_1) s_1^{l_1-1}, \tag{11}$$

and its moments

$$m(l_1) = -CA^{-l_1} B. \tag{12}$$

Also consider the second subsystem transfer function, with its Maclaurin series expansion

$$\begin{aligned} H(s_1, s_2) &= C(s_2 I - A)^{-1} N (s_1 I - A)^{-1} B, \\ &= \sum_{l_2=1}^{\infty} \sum_{l_1=1}^{\infty} m(l_1, l_2) s_1^{l_1-1} s_2^{l_2-1} \end{aligned} \tag{13}$$

and the associated multimoments

$$m(l_1, l_2) = CA^{-l_2} N A^{-l_1} B. \tag{14}$$

The aim of Krylov subspaces model order reduction for bilinear systems/models is to match as many moments and multimoments, i.e. $m(l_1) = \hat{m}(l_1)$ and $m(l_1, l_2) = \hat{m}(l_1, l_2)$, of the original model in a reduced order model, where $\hat{m}(l_1)$ and $\hat{m}(l_1, l_2)$ are the moments and multimoments of a reduced order model. As will be shown in subsequent sections, the order of the reduced order model increases with the amount of moments matched.

III. Krylov subspace model order reduction for bilinear models

A. Galerkin projection

For a bilinear system (1)-(2), where zero initial condition $x_0 = 0$ is assumed, approximating the system states will result in a system of lower dimension. Projection methods try to achieve this by using the approximation $x \approx V\hat{x}$, where $\hat{x}$ is the new set of states and V is the projection matrix. Hence, (1)-(2) can be rewritten as

$$V\dot{\hat{x}} = AV\hat{x} + NV\hat{x}u + Bu, \tag{15}$$
$$\hat{y} = CV\hat{x}. \tag{16}$$

Premultiplying (15) by V^T, results in a set comprising of a new system matrix, input and output vectors

$$V^T V\dot{\hat{x}} = V^T AV\hat{x} + V^T NV\hat{x}u + V^T Bu, \tag{17}$$
$$\hat{y} = CV\hat{x}, \tag{18}$$

that due to the orthogonality of V, implying $V^T V = I$, gives

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{N}\hat{x}u + \hat{B}u, \tag{19}$$
$$\hat{y} = \hat{C}\hat{x}, \tag{20}$$

of lower dimensions. The reduced system is of q states, $q << n$, $q \in \mathbb{R}$, with system matrices, input and output vectors labeled as $\hat{A}, \hat{N}, \hat{B}$ and $\hat{C}$, with

$$\hat{A} = V^T AV \in \mathbb{R}^{q\times q}, \tag{21}$$
$$\hat{N} = V^T NV \in \mathbb{R}^{q\times q}, \tag{22}$$
$$\hat{B} = V^T B \in \mathbb{R}^{q\times 1}, \tag{23}$$
$$\hat{C} = CV \in \mathbb{R}^{1\times q}. \tag{24}$$

This transformation is often referred to as the Galerkin projection [7]. The appropriate projection matrix V needs to be found using one-sided Krylov subspace techniques.

B. Krylov subspaces for MOR of bilinear models

Definition 3.1 (Krylov subspace): The q^{th} Krylov subspace is defined as

$$K_q(\mathbb{N},\mathbb{M}) = span\{\mathbb{N}^0\mathbb{M}, \mathbb{N}^1\mathbb{M}, ..., \mathbb{N}^{q-1}\mathbb{M}\}, \tag{25}$$

where $\mathbb{N} \in \mathbb{R}^{n\times n}$, $\mathbb{M} \in \mathbb{R}^{n\times 1}$ and $q, n \in \mathbb{Z}$. $\mathbb{N}$ and $\mathbb{M}$ are known as the starting matrix and vector, respectively and they form the basis of the Krylov subspace.

In [7], a multimoment matching approach has been proposed using the Krylov subspaces

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, B), \tag{26}$$

$$span\{V^{\{k\}}\} = K_{q_k}(A^{-1}, NV^{\{k-1\}}), \tag{27}$$

$$span\{V\} = \{\bigcup_{k=1}^{j} V^{\{k\}}\}, \tag{28}$$

where $V^{\{k\}}$ is the basis of the q_k^{th} Krylov subspace $K_{q_k}(\mathbb{N},\mathbb{M})$ and j is the number of Krylov subspaces used. The Krylov subspace (26), as defined, matches $q_1 - 1$ moments of the first subsystem of the bilinear model.

In the multimoment analysis presented in Sections IV and V, only $V^{\{1\}}$ and $V^{\{2\}}$ are used to compute V, i.e. the Krylov subspaces and projection matrices are defined as

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, B), \tag{29}$$

$$span\{V^{\{2\}}\} = K_{q_2}(A^{-1}, NV^{\{1\}}), \tag{30}$$

$$span\{V\} = \{\bigcup_{k=1}^{2} V^{\{k\}}\}. \tag{31}$$

The result of using V^k where k is greater than 2 can readily be extrapolated from the work done here.

The projection matrix V is computed as a union of $V^{\{1\}}$ and $V^{\{2\}}$. The dimension of V is therefore $n \times (q_2 q_1 + q_1)$, where n is the dimension of x, as A is $n \times n$, q_1 and q_2 refer to the Krylov subspaces $K_{q_1}(A^{-1}, B)$ and $K_{q_2}(A^{-1}, NV^{\{1\}})$ respectively. The formulation of V and its dimension are the same for the other types of Krylov subspace approaches to be discussed in this section.

In the work influenced by [7] and [1], Feng and Benner have proposed the Krylov subspaces

$$span\{V^{\{1\}}\} = K_{q1}(A^{-1}, A^{-1}B), \tag{32}$$

$$span\{V^{\{k\}}\} = K_{q_k}(A^{-1}, A^{-1}NV^{\{k-1\}}), \tag{33}$$

$$span\{V\} = \{\bigcup_{k=1}^{j} V^{\{k\}}\}, \tag{34}$$

for matching the maximum amount of multimoments. In this case, the Krylov subspaces $span\{V^{\{1\}}\}$ and $span\{V^{\{2\}}\}$ are

$$span\{V^{\{1\}}\} = K_{q1}(A^{-1}, A^{-1}B), \tag{35}$$

$$span\{V^{\{2\}}\} = K_{q2}(A^{-1}, A^{-1}NV^{\{1\}}) \tag{36}$$

and the corresponding projection matrix is formed as

$$span\{V\} = \{\bigcup_{k=1}^{2} V^{\{k\}}\}. \tag{37}$$

Note that the major difference between the projection types discussed in this section is the definition of the Krylov subspaces. The slight differences in the definition will be shown in subsequent sections to have a significant impact on the input-output relationship preservation for the reduced order model.

IV. Multimoment Matching analysis

A. Multimoment matching for Phillips type projection

Using the Krylov subspace $K_{q_1}(A^{-1}, B)$ for computing $V^{\{1\}}$, as has been done in [7], it can be shown that this only matches $q_1 - 1$ moments of the first transfer function of the bilinear model. This consequently affects the computing of $V^{\{2\}}$. However, this formulation of Krylov subspaces can be shown to match multimoments of the multivariable transfer function $H(s_1, s_2)$ of the bilinear model.

Theorem 4.1: The Krylov subspaces $K_{q_1}(A^{-1}, B)$ and $K_{q_2}(A^{-1}, NV^{\{1\}})$ when used to compute the projection matrix V as defined in (29)-(31) match the multimoments of a bilinear model (1)-(2) and a reduced order bilinear model such that $\hat{m}(l_1, l_2) = m(l_1, l_2)$, $l_1 = 1, \ldots, q_1 - 1$, $l_2 = 1, \ldots, q_2 - 1$, if the reduced order model matrices are computed as $\hat{A} = V^TAV$, $\hat{N} = V^TNV$, $\hat{B} = V^TB$ and $\hat{C} = CV$, where V spans the Krylov subspaces $K_{q_1}(A^{-1}, B)$ and $K_{q_2}(A^{-1}, NV^{\{1\}})$ and $V^TV = I$.

Proof: This multimoment matching property can be shown by first substituting the reduced order matrices (21)-(24) into the multimoments of the reduced order model, $\hat{m}(l_1, l_2)$. From (10) the multimoment of the reduced order bilinear model can be defined as

$$\hat{m}(l_1, l_2) = \hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B}. \tag{38}$$

Now, using the definition of the reduced order matrices, $\hat{A}$, $\hat{N}$, $\hat{B}$ and $\hat{C}$, the right hand side terms of (38) can be rewritten as

$$CV(V^TAV)^{-l_2}(V^TNV)(V^TAV)^{-l_1}V^TB. \tag{39}$$

Because $A^{-(q_1-1)}B$ belongs to the Krylov subspace K_{q_1}, therefore it can be written that $A^{-(q_1-1)}B = V^{\{1\}}r_{(q_1)}$. Then $B = A^{(q_1-1)}V^{\{1\}}r_{(q_1)}$. Also because $V^{\{1\}} \in V$, then $V^{\{1\}}r_{(q_1)} = Vp_{(q_1)}$, where $l_1 = 1, \ldots, q_1 - 1$ and $l_2 = 1, \ldots, q_2 - 1$ and $r_{(i)}$ and $p_{(i)}$ are appropriate parameters and dimensions, where $r_{(i)} \in \mathbb{R}^{q_1}$, $p_{(i)} \in \mathbb{R}^{q_1+q_1q_2}$ for $i \leq q_1$ and $p_{(i)} \in \mathbb{R}^{(q_1+q_1q_2)\times q_1}$ for $i > q_1$. For any value of $q_1 \in \mathbb{Z}, q_1 > 0$, when $l_1 = q_1 - 1$

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2}V^TNVp_{(q_1)}. \tag{40}$$

Since $Vp_{(q_1)} = V^{\{1\}}r_{(q_1)}$ then

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2}V^TNV^{\{1\}}r_{(q_1)}. \tag{41}$$

Moreover, from (30) - (31), $NV^{\{1\}} \in V$, therefore $NV^{\{1\}} = Vp_{(q_1+1)}$ and

$$\begin{aligned}\hat{m}(l_1, l_2) &= CV(V^TAV)^{-l_2}V^TVp_{(q_1+1)}r_{(q_1)} \\ &= CV(V^TAV)^{-l_2}V^TAA^{-1}Vp_{(q_1+1)}r_{(q_1)}.\end{aligned} \tag{42}$$

Further, since $A^{-1}NV^{\{1\}} \in V$ and

$$A^{-1}NV^{\{1\}} = A^{-1}Vp_{(q_1+1)} = Vp_{(q_1+2)},$$

then

$$\begin{aligned}\hat{m}(l_1, l_2) &= CV(V^TAV)^{-l_2}V^TAVp_{(q_1+2)}r_{(q_1)} \\ &= CV(V^TAV)^{-l_2+1}p_{(q_1+2)}r_{(q_1)}.\end{aligned} \tag{43}$$

Using this routine iteratively until $q_2 = (l_2+1)$, (43) becomes

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CVp_{(q_1+l_2+2)}r_{(q_1)}. \tag{44}$$

Now $A^{-l_2}NV^{\{1\}} = Vp_{(q_1+l_2+2)}$, so

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CA^{-l_2}NV^{\{1\}}r_{(q_1)}. \tag{45}$$

Since $V^{\{1\}}r_{(q_1)} = A^{-(q_1-1)}B$ and $l_1 = q_1 - 1$, then,

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CA^{-l_2}NA^{-l_1}B. \tag{46}$$

Therefore,

$$\hat{m}(l_1, l_2) = m(l_1, l_2), \tag{47}$$

where $l_1 = 1, \ldots, q_1 - 1$ and $l_2 = 1, \ldots, q_2 - 1$. ■

B. Multimoment matching for Feng and Benner type projection

Theorem 4.2: Given the Krylov subspaces as proposed in [16], $K_{q_1}(A^{-1}, A^{-1}B)$ and $K_{q_2}(A^{-1}, A^{-1}NV^{\{1\}})$ match multimoments of the bilinear model (1)-(2) and a reduced order model such that $\hat{m}(l_1, l_2) = m(l_1, l_2)$, $l_1 = 1, \ldots, q_1$, $l_2 = 1, \ldots, q_2$, when the reduced order model matrices are computed as $\hat{A} = V^TAV$, $\hat{N} = V^TNV$, $\hat{B} = V^TB$ and $\hat{C} = CV$ with V spans the Krylov subspaces $K_{q_1}(A^{-1}, A^{-1}B)$ and $K_{q_2}(A^{-1}, A^{-1}NV^{\{1\}})$ and $V^TV = I$.

Proof: To match the multimoments of the multivariable transfer function, the Krylov subspaces proposed by Feng and Benner [16] can be used following a similar procedure as in Subsection IV-A. The multimoments of the reduced order model are defined as

$$\hat{m}(l_1, l_2) = \hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B}. \tag{48}$$

Using the definition of the reduced order matrices we have, $\hat{A} = V^TAV$, $\hat{N} = V^TAV$, $\hat{B} = V^TB$ and $\hat{C} = CV$. Substituting these matrices into (48), gives

$$\begin{aligned}\hat{m}(l_1, l_2) &= \hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} \qquad (49)\\ &= CV(V^TAV)^{-l_2}(V^TNV)(V^TAV)^{-l_1}V^TB.\end{aligned}$$

Because $A^{-q_1}B = V^{\{1\}}r_{(q_1)}$, then

$$\begin{aligned}\hat{m}(l_1, l_2) &= CV(V^TAV)^{-l_2}V^TNV^{\{1\}}r_{(q_1)} \\ &= CV(V^TAV)^{-l_2}V^TAA^{-1}NV^{\{1\}}r_{(q_1)}.\end{aligned} \tag{50}$$

From (36), we have $A^{-1}NV^{\{1\}} \in V$. $A^{-1}NV^{\{1\}} = Vp_{(q_1+1)}$ and

$$\begin{aligned}\hat{m}(l_1, l_2) &= CV(V^TAV)^{-l_2}V^TAVp_{(q_1+1)}r_{(q_1)} \\ &= CV(V^TAV)^{-l_2+1}p_{(q_1+1)}r_{(q_1)} \qquad (51)\\ &= CV(V^TAV)^{-l_2+1}V^TAA^{-1}Vp_{(q_1+1)}r_{(q_1)}.\end{aligned}$$

Moreover, $A^{-2}NV^{\{1\}} = Vp_{(q_1+2)} = A^{-1}Vp_{(q_1+1)}$ hence,

$$\begin{aligned}\hat{m}(l_1, l_2) &= CV(V^TAV)^{-l_2+1}V^TAVp_{(q_1+2)}r_{(q_1)} \\ &= CV(V^TAV)^{-l_2+2}p_{(q_1+2)}r_{(q_1)}.\end{aligned} \tag{52}$$

Following this routine yields

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CVp_{(q_1+l_2)}r_{(q_1)} = CA^{-l_2}NV^{\{1\}}r_{(q_1)},$$

for any value of l_2. Note that $A^{-l_2}NV^{\{1\}} = Vp_{(q_1+l_2)}$. Since $V^{\{1\}}r_{(q_1)} = A^{-q_1}B$,

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CA^{-l_2}NA^{-l_1}B. \tag{53}$$

Therefore,

$$\hat{m}(l_1, l_2) = m(l_1, l_2), \tag{54}$$

where $l_1 = 1, \ldots, q_1$, $l_2 = 1, \ldots, q_2$ $p_{(i)}$, $i = 1, \ldots, (q_1 + q_2)$ and $r_{(i)}, i = 1, \ldots, q_1$ are appropriate parameters for achieving orthogonality. ■

V. Improved Phillips type projection

The method proposed by Phillips [7] can be readily improved because it has been shown that the Krylov subspace used to compute $V^{\{1\}}$ matches only $q_1 - 1$ moments of its linear approximation. Further observation shows that in order to match more moments, the Krylov subspace used to compute $V^{\{1\}}$ should be replaced by

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B). \tag{55}$$

This formulation for computing $V^{\{1\}}$ matches q_1 moments. This will improves the multimoment matching of the bilinear model. This approach is demonstrated numerically.

In the next subsection an analysis of multimoment matching for the Improved Phillips approach is shown.

A. Multimoment matching for Improved Phillips projection

For matching the multimoments of the multivariable transfer function, the following Krylov subspaces are proposed:

$$span\{V^{\{1\}}\} = K_{q_1}(A^{-1}, A^{-1}B), \tag{56}$$

$$span\{V^{\{2\}}\} = K_{q_2}(A^{-1}, NV^{\{1\}}), \tag{57}$$

$$span\{V\} = span\{\bigcup_{k=1}^{2} V^{\{k\}}\}. \tag{58}$$

Using these Krylov subspaces for MOR of bilinear models, the following theorem applies.

Theorem 5.1 (Improved Phillips type projection method): For a bilinear system/model (1)-(2), a reduced order model of dimensions $q_1q_2 + q_1$ can be constructed by using projection matrix V where $V^TV = I$, if V is computed using the Krylov subspaces (56)-(57). This Krylov subspaces formulation matches multimoments of the multivariable transfer function $H(s_1, s_2)$ of the bilinear model.

Proof: From (10) the multimoment of the reduced order bilinear model can be defined as

$$\hat{m}(l_1, l_2) = \hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B}. \tag{59}$$

From the definition of the reduced order matrices, the right hand side of (59) can be written as

$$CV(V^TAV)^{-l_2}(V^TNV)(V^TAV)^{-l_1}V^TB. \quad (60)$$

Because, $A^{-q_1}B = V^{\{1\}}r_{(q_1)}$, with $r_{(i)}$ being appropriate parameters for achieving orthogonality, then

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CV(V^TAV)^{-l_2}V^TNV^{\{1\}}r_{(q_1)}. \quad (61)$$

From (57), $NV^{\{1\}} \in V$, therefore $NV^{\{1\}} = Vp_{(q_1+1)}$, where $p_{(i)}$ are appropriate parameters for achieving orthogonality, thus

$$\begin{aligned}\hat{m}(l_1,l_2) &= CV(V^TAV)^{-l_2}V^TVp_{(q_1+1)}r_{(q_1)} \\ &= CV(V^TAV)^{-l_2}V^TAA^{-1}Vp_{(q_1+1)}r_{(q_1)}.\end{aligned} \quad (62)$$

Since $A^{-1}NV^{\{1\}} = Vp_{(q_1+2)} = A^{-1}Vp_{(q_1+1)}$,

$$\begin{aligned}\hat{m}(l_1,l_2) &= CV(V^TAV)^{-l_2}V^TAVp_{(q_1+2)}r_{(q_1)} \\ &= CV(V^TAV)^{-l_2+1}p_{(q_1+2)}r_{(q_1)} \\ &= CV(V^TAV)^{-l_2+1}V^TAA^{-1}Vp_{(q_1+2)}r_{(q_1)}.\end{aligned} \quad (63)$$

Moreover, as $A^{-2}NV^{\{1\}} = Vp_{(q_1+3)} = A^{-1}Vp_{(q_1+2)}$, then

$$\begin{aligned}\hat{m}(l_1,l_2) &= CV(V^TAV)^{-l_2+1}V^TAVp_{(q_1+3)}r_{(q_1)} \\ &= CV(V^TAV)^{-l_2+2}p_{(q_1+3)}r_{(q_1)}.\end{aligned} \quad (64)$$

Using this iterative scheme, it can be derived that

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CVp_{(q_1+l_2+1)}r_{(q_1)} = CA^{-l_2}NV^{\{1\}}r_{(q_1)},$$

for any value of l_2, where it is true that $A^{-l_2}NV^{\{1\}} = Vp_{(q_1+l_2+1)}$. Since $V^{\{1\}}r_{(q_1)} = A^{-q_1}B$, then

$$\hat{C}\hat{A}^{-l_2}\hat{N}\hat{A}^{-l_1}\hat{B} = CA^{-l_2}NA^{-l_1}B. \quad (65)$$

Note that $p_{(i)}$ are matrices for values of i greater than q_1 otherwise they are vectors and $r_{(i)}$ are vectors for SISO bilinear models. Using this analysis, it can be observed that the Krylov subspaces (56)-(57) match the multi-moments, $\hat{m}(l_1,l_2)$ and $m(l_1,l_2)$, such that $l_1 = 1\ldots q_1$, and $l_2 = 1\ldots(q_2-1)$. ■

The advantage of using the Improved Phillips approach is that it is a compromise between the Phillips type projection [7] and the Feng and Benner [16] type projection. It improves the matched linear moments when computing $V^{\{1\}}$ and reduces the loss of information which occurs by multiplying the inverse of the system matrix A with N. This is for the case where N is a singular matrix.

VI. Case study: A nonlinear RC circuit

In this section, the proposed model reduction algorithm is applied to a transmission line model of 20^{th} order, i.e. $n = 20$ as illustrated in Fig. 1. The nonlinear circuit model is of the form (3)-(4), where $f(x)$=$f(v)$, the input and the output matrices are given as

$$f(v) = \begin{bmatrix} -g(v_1)-g(v_1-v_2) \\ g(v_1-v_2)-g(v_2-v_3) \\ \vdots \\ g(v)-g(v_{k-1}-v_k) \\ \vdots \\ g(v)-g(v_{n-1}-v_n) \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The model is composed of linear capacitors which are assumed to have unity capacitance, i.e. $C = 1$, and nonlinear resistors where the resistance $g(v)$ is a function of voltage

$$g(v) = \exp(40v) + v - 1. \quad (66)$$

The output of the nonlinear circuit is the voltage between node 1 and the ground. A state vector $x^{\otimes} = [x^{(1)}x^{(2)}x^{(3)}]$ has been defined for the Carleman bilinearisation of the nonlinear model. This results in bilinear system matrices, input and ouput vectors with the following dimensions. $A, N \in \mathbb{R}^{8420\times8420}$, $B \in \mathbb{R}^{8420\times1}$ and $C \in \mathbb{R}^{1\times8420}$.

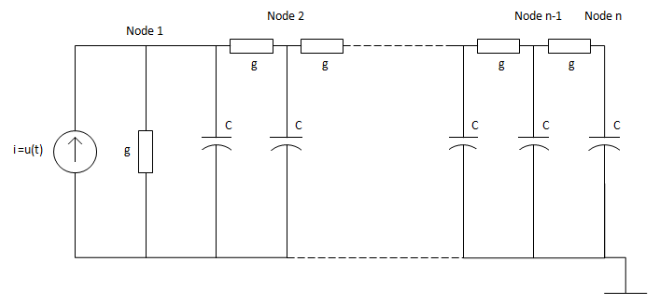

Fig. 1. A transmission line system represented by a circuit model with nonlinear R and C components.

A. MOR procedure

An algorithm proposed in [1] has been used for computing the projection matrix. The parameters q_1, q_2 and p_2 form part of the input to the algorithm where q_1 defines the order of the first Krylov subspace, $span\{V^{\{1\}}\}$ and q_2 defines the order of the second Krylov subspace, $span\{V^{\{2\}}\}$. p_2 defines the number of columns of $V^{\{1\}}$ which are used for computing $V^{\{2\}}$. An experiment has been carried out which shows the impact of these parameters on the reduced order models.

The results for the experiment suggest that choosing q_1 too high does not necessarily improve the output of the reduced order model. Likewise, selecting a high value of q_2 does not improve the results. However, values of p_2 higher than 1 is likely to increase the input-output preservation capability of the reduced order model. These experimental results suggest that the computation of a reduced order model which preserves input-output relationship which satisfies a set of performance criteria is highly dependent on $V^{\{1\}}$ therefore the correct selection of the parameters q_1 and p_2 is critical. This knowledge has been used to manually derive much lower reduced order models for the Phillips [7] type projection, the Feng and Benner [16] and the Improved

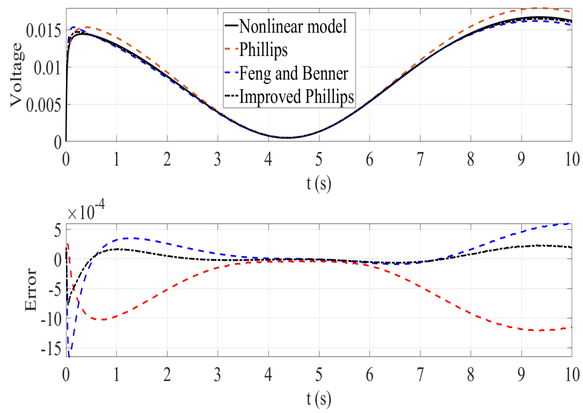

Fig. 2. Plots of outputs $\hat{y}$ and modelling error for the reduced order models (7^{th} order) and the nonlinear model output y (20^{th} order).

Phillips approaches. The results presented are for reduced order models of 7^{th} order where the parameters $q_1 = 5$, $p_2 = 2$, $q_2 = 1$, where the input u is

$$u = (\cos(2\pi t/10) + 1)/2. \tag{67}$$

B. Results

Fig. 2 shows a graphical comparison of the original nonlinear circuit and the simulated outputs and error for the Phillips [7], Feng and Benner [16] and Improved Phillips projections. This comparison shows a higher output preservation for the Improved Phillips approach. This trend can also be observed in TABLE I which summarises the RT^2, MSE, SSE and IAE values of the reduced order models. The Improved Phillips has a RT^2 value 1.4% and 0.28% higher than that of Phillips [7] and Fend and Benner [16] respectively.

TABLE I

PERFORMANCE CRITERIA FOR 7TH ORDER REDUCED ORDER MODELS.

	RT^2	MSE	IAE	SSE
Phillips	98.50	4.6991e-07	0.2846	2.4999e-04
Feng and Benner	99.66	1.0718e-07	0.1121	5.7020e-05
Improved Phillips	99.94	1.9686e-08	0.0482	1.0473e-05

VII. CONCLUSION

This study has provided a new analysis of different Krylov subspace methods proposed in the literature for matching the moments and multimoments of a bilinear model. The multimoment matching property of the Improved Phillips has been analysed. It has been shown that Feng and Benner matchtes the highest number of multi-moments followed by the Improved Phillips and then the Phillips methods. A simulation study has been used to compare the performance of the Krylov subspace approaches considered in this paper. For a 20^{th} order nonlinear model, the Carleman bilinearisation process has been used to bilinearise the system. The resulting structure enables Krylov subspaces to be applied efficiently. It is shown that the Improved Phillips leads to a better reduced order model in terms of RT^2, MSE, IAE and SSE. The quality of the reduced order model is dependent on the nature of system matrices being used for computing the Krylov subspaces. However, the high dependence of the reduced order model on the matched moments of the bilinear approximation can be used to improve the multimoment matching property of the J. R. Phillips type projection. The Krylov subspace approaches and their analysis have similar implications for MISO and MIMO bilinear models as described in [20].

REFERENCES

[1] Z. Bai and D. Skoogh, "A projection method for model reduction of bilinear dynamical systems," *Linear algebra and its applications*, vol. 415, no. 2, pp. 406–425, 2006.

[2] G. Schelfhout *et al.*, "Model reduction for control design," Ph.D. dissertation, Department of Electrical Engineering, Katholieke Universiteit Leuven, 1996, 1996.

[3] M. I. Younis, E. M. Abdel-Rahman, and A. Nayfeh, "A reduced-order model for electrically actuated microbeam-based MEMS," *Microelectromechanical Systems, Journal of*, vol. 12, no. 5, pp. 672–680, 2003.

[4] Z. Filipi, H. Fathy, J. Hagena, A. Knafl, R. Ahlawat, J. Liu, D. Jung, D. N. Assanis, H. Peng, and J. Stein, "Engine-in-the-loop testing for evaluating hybrid propulsion concepts and transient emissions-HMMWV case study," SAE Technical Paper, Tech. Rep., 2006.

[5] T. Aizad, M. Sumisławska, O. Maganga, O. Agbaje, N. Phillip, and K. J. Burnham, "Investigation of model order reduction techniques: A supercapacitor case study," in *Advances in Systems Science*. Springer, 2014, pp. 795–804.

[6] S. Gugercin, A. C. Antoulas, and C. Beattie, "H_2 model reduction for large-scale linear dynamical systems," *SIAM journal on matrix analysis and applications*, vol. 30, no. 2, pp. 609–638, 2008.

[7] J. R. Phillips, "Projection frameworks for model reduction of weakly nonlinear systems," in *Proceedings of the 37th Annual Design Automation Conference*. ACM, 2000, pp. 184–189.

[8] T. Breiten and T. Damm, "Krylov subspace methods for model order reduction of bilinear control systems," *Systems & Control Letters*, vol. 59, no. 8, pp. 443–450, 2010.

[9] M. Condon and R. Ivanov, "Krylov subspaces from bilinear representations of nonlinear systems," *COMPEL: The International Journal for Computation and Mathematics in Electrical and Electronic Engineering*, vol. 26, no. 2, pp. 399–406, 2007.

[10] P. Benner and T. Breiten, "Interpolation-based H_2-model reduction of bilinear control systems," *SIAM Journal on Matrix Analysis and Applications*, vol. 33, no. 3, pp. 859–885, 2012.

[11] I. J. Couchman, E. C. Kerrigan, and C. Böhm, "Model reduction of homogeneous-in-the-state bilinear systems with input constraints," *Automatica*, vol. 47, no. 4, pp. 761–768, 2011.

[12] M. Espana and I. Landau, "Reduced order bilinear models for distillation columns," *Automatica*, vol. 14, no. 4, pp. 345–355, 1978.

[13] R. Mohler and C. Barton, "Compartmental control model of the immune process," in *Optimization Techniques Part 1*. Springer, 1978, pp. 421–430.

[14] R. R. Mohler, *Bilinear control processes: with applications to engineering, ecology and medicine*. Academic Press, Inc., 1973.

[15] J. Mula, D. Peidro, M. Díaz-Madroñero, and E. Vicens, "Mathematical programming models for supply chain production and transport planning," *European Journal of Operational Research*, vol. 204, no. 3, pp. 377–390, 2010.

[16] L. Feng and P. Benner, "A note on projection techniques for model order reduction of bilinear systems," in *AIP Conference Proceedings, Numerical Analysis and Applied Mathematics*, vol. 936, 2007, pp. 208–211.

[17] R. Choudhary and K. Ahuja, "Stability analysis of bilinear iterative rational krylov algorithm," *Linear Algebra and its Applications*, vol. 538, pp. 56–88, 2018.

[18] I. Zajıc, "A Hammerstein-bilinear approach with application to heating ventilation and air conditioning systems," Ph.D. dissertation, Coventry University, 2013.

[19] W. J. Rugh, *Nonlinear system theory*. Johns Hopkins University Press Baltimore, 1981.

[20] O. Agbaje, "Krylov subspace model order reduction for bilinear and nonlinear control systems," Ph.D. dissertation, Coventry University, 2017.