2018 European Control Conference (ECC)
June 12-15, 2018. Limassol, Cyprus

A Diffusion-Based Solution Technique for Certain Schrödinger Equation Dynamical Systems

William M. McEneaney[1] and Ruobing Zhao[2]

***Abstract*— A particular class of Schrödinger initial value problems is considered, wherein a particle moves in a scalar field centered at the origin, and more specifically, the distribution associated to the solution of the Schrödinger equation has negligible mass in the neighborhood of the origin. The Schrödinger equation is converted to a stochastic control problem using the Maslov dequantization transform. A non-inertial frame centered along the trajectory of a classical particle is employed. A solution approximation as a series expansion in a small parameter is obtained through the use of complex-valued diffusion-process representations, where under a smoothness assumption, the expansion converges to the true solution. In the case of an expansion up through only the cubic terms in the space variable, there exist approximate solutions that are periodic with the period of a classical particle, but with an additional secular perturbation. The computations required for solution up to a finite order are purely analytical.**

I. Introduction

Diffusion representations have long been a useful tool in solution of second-order Hamilton-Jacobi partial differential equations (HJ PDEs). The bulk of such results apply to real-valued HJ PDEs, that is, to HJ PDEs where the coefficients and solutions are real-valued. The Schrödinger equation is complex-valued, although generally defined over a real-valued space domain, which presents difficulties for the development of stochastic control representations. In [9], [10], a representation for the solution of a Schrödinger-equation initial value problem over a scalar field was obtained as a stationary value for a complex-valued diffusion process control problem. Although there is substantial existing work on the relation of stochastic processes to the Schrödinger equation (cf. [6], [13], [14], [19]), the approach considered in [9], [10] is along a slightly different path, closer to [2], [3], [4], [5], [8]. However, the representation in [9], [10] employs stationarity of the payoff [11] rather than optimization of the payoff, where stationarity can be used to overcome the limited-duration constraints of methods that use optimization.

Here we will consider a specific type of weak field problem, and use diffusion representations as a tool for approximate solution of the Schrödinger equation. Suppose we have a particle in a scalar field centered at the origin, and the particle is sufficiently far from the origin that the distribution associated to the corresponding Schrödinger equation has negligible density near the origin. Let m denote the particle mass, and let $\hbar$ denote Planck's constant. Suppose the potential energy generated by the field interacting with the particle takes the form $\bar{V}(x) = -\bar{c}/|x|$. Let the solution of the Schrödinger equation at time, t, and position, x, be denoted by $\psi(t,x)$, and consider the associated distribution given by $\tilde{P}(t,x) \doteq [\psi^*\psi](t,x)$. We will consider a non-inertial frame where the origin will be centered at $\xi(t)$ for all t. In particular, we consider a case where $\xi(t)$ follows a circular orbit with constant angular velocity. That is, we consider $\xi(t) = \hat{\delta}\big(\cos(\omega t), \sin(\omega t)\big)$ where $\hat{\delta} \in (0,\infty)$. In the case of $\bar{V}(x) = -\bar{c}/|x|$, we have $\omega \doteq [\bar{c}/(m\hat{\delta}^3)]^{1/2}$. We suppose that $\hat{\delta}$ is sufficiently large such that $\tilde{P}(t,x) \ll 1$ for $|x| < \hat{\delta}/2$, and thus that one may approximate $\bar{V}$ in the vicinity of $\xi(t)$ by a finite number of terms in a power series expansion centered at $\xi(t)$. We will use a complex-valued diffusion representations to obtain an approximation to the resulting Schrödinger equation solution. If the solution is holomorphic in x and a small parameter, then the approximate solution converges as the number of terms in the set of diffusion representations approaches infinity.

As our motivation is the case where $\hat{\delta}$ is large relative to the associated position distribution, one expects that the case of a $-\bar{c}/|x|$ potential may be sufficiently well-modeled by a finite number of terms in a power series expansion. An analysis of the errors induced by such an approximation is beyond the scope of this already long paper, and may be addressed in a later effort; the focus here is restricted to the diffusion-representation based method of solution approximation method *given* such an approximation to the potential. We remark that in the case of a quadratic potential, we recover the quantum harmonic oscillator solution. Also, in the case of $\bar{V}(x) = -\bar{c}/|x|$, as $\hat{\delta} \to \infty$, the solution approaches that of the free particle case. *The computations required for solution up to any finite polynomial-in-space order may be performed analytically.* As an example, the solution approximation up to a cubic order in the case of a cubic approximation of the classic $1/r$ type of potential are included.

II. Dequantization

We recall the Schrödinger initial value problem, given as

$$0 = i\hbar\psi_t(s,x) + \tfrac{\hbar^2}{2m}\Delta_x\psi(s,x) - \psi(s,x)\bar{V}(x), \quad (s,x) \in \mathcal{D}, \tag{1}$$

$$\psi(0,x) = \psi_0(x), \quad x \in \mathbb{R}^n, \tag{2}$$

where initial condition ψ_0 takes values in $\mathbb{C}$, and subscript t will denote the derivative with respect to the time variable regardless of the symbol being used for time in the argument

[1]Univ. of California San Diego, La Jolla, CA 92093-0411, USA, e-mail: wmceneaney@ucsd.edu
[2]Univ. of California San Diego, La Jolla, CA 92093-0411, USA, e-mail: ruz015@ucsd.edu

list. We also let $\overline{\mathcal{D}} \doteq [0,t)\times\mathbb{R}^n$. We consider the Maslov dequantization of the solution of the Schrödinger equation (cf. [7]), which is $S:\overline{\mathcal{D}}\to\mathbb{C}$ given by $\psi(s,x)=\exp\{\frac{i}{\hbar}S(s,x)\}$. We find that (1)–(2) become

$$0=-S_t(s,x)+\tfrac{i\hbar}{2m}\Delta_x S(s,x)+H^0(x,S_x(s,x)),\quad (s,x)\in\mathcal{D}, \tag{3}$$

$$S(0,x)=\bar\phi(x),\quad x\in\mathbb{R}^n, \tag{4}$$

where $H:\mathbb{R}^n\times\mathbb{C}^n\to\mathbb{C}$ is the Hamiltonian given by

$$\begin{aligned} H^0(x,p) &= -\left[\tfrac{1}{2m}|p|_c^2+\bar V(x)\right] \\ &= \operatorname*{stat}_{v\in\mathbb{C}^n}\left\{v\cdot p+\tfrac{m}{2}|v|_c^2-\bar V(x)\right\}, \end{aligned} \tag{5}$$

and for $y\in\mathbb{C}^n$, $|y|_c^2\doteq\sum_{j=1}^n y_j^2$. (We remark that notation $|\cdot|_c^2$ is not intended to indicate a squared norm; the range is complex.) stat is defined in Section III-A. We look for solutions in the space $\mathcal{S}\doteq\{S:\overline{\mathcal{D}}\to\mathbb{C}\,|\,S\in C_p^{1,2}(\mathcal{D})\cap C(\overline{\mathcal{D}})\}$, where $C_p^{1,2}$ denotes the space of functions which are continuously differentiable once in time and twice in space, and which satisfy a polynomial-growth bound.

III. Preliminaries

A. Stationarity definitions

Recall that classical systems obey the stationary action principle, where the path taken by the system is that which is a stationary point of the action functional. Suppose $(\mathcal{Y},|\cdot|)$ is a generic normed vector space over $\mathbb{C}$ with $\mathcal{G}\subseteq\mathcal{Y}$, and suppose $F:\mathcal{G}\to\mathbb{C}$. We say $\bar y\in\operatorname{argstat}\{F(y)\,|\,y\in\mathcal{G}\}$ if $\bar y\in\mathcal{G}$ and either $\limsup_{y\to\bar y,y\in\mathcal{G}\setminus\{\bar y\}}|F(y)-F(\bar y)|/|y-\bar y|=0$, or there exists $\delta>0$ such that $\mathcal{G}\cap B_\delta(\bar y)=\{\bar y\}$ (where $B_\delta(\bar y)$ denotes the ball of radius δ around $\bar y$). If $\operatorname{argstat}\{F(y)\,|\,y\in\mathcal{G}\}\neq\emptyset$, we define the possibly set-valued stat^s operator by

$$\begin{aligned}\operatorname*{stat}^s_{y\in\mathcal{G}}F(y) &\doteq \operatorname{stat}^s\{F(y)\,|\,y\in\mathcal{G}\} \\ &\doteq \left\{F(\bar y)\,\middle|\,\bar y\in\operatorname{argstat}\{F(y)\,|\,y\in\mathcal{G}\}\right\}.\end{aligned}$$

For further discussion, we refer the reader to [11]. The following is immediate from the above definitions.

Lemma 1: Suppose $\mathcal{Y}$ is a Hilbert space, with open set $\mathcal{G}\subseteq\mathcal{Y}$, and that $F:\mathcal{G}\to\mathbb{C}$ is Fréchet differentiable at $\bar y\in\mathcal{G}$ with Riesz representation $F_y(\bar y)\in\mathcal{Y}$ Then, $\bar y\in\operatorname{argstat}\{F(y)\,|\,y\in\mathcal{G}\}$ if and only if $F_y(\bar y)=0$.

B. The non-inertial frame

We suppose a central scalar field such that a particular solution for the motion of a classical particle in the field takes the form $\xi(t)=\hat\delta\big(\cos(\omega t),\sin(\omega t)\big)$ where $\hat\delta,\omega\in(0,\infty)$. In particular, we concentrate on the potential $\bar V(x)=-\bar c/|x|$, in which case $\omega\doteq[\bar c/(m\hat\delta^3)]^{1/2}$. We consider a two-dimensional space model and a non-inertial frame centered at $\xi(t)$ for all $t\in(0,\infty)$, with the first basis axis in the positive radial direction and the second basis vector in the direction of the velocity of the particle. Let positions in the non-inertial frame be denoted by $z\in\mathbb{R}^2$, where the transformation between frames at time $t\in\mathbb{R}$ is given by

$$z=\begin{pmatrix}z_1\\z_2\end{pmatrix}=\begin{pmatrix}\cos(\omega t)&\sin(\omega t)\\-\sin(\omega t)&\cos(\omega t)\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}-\begin{pmatrix}\hat\delta\\0\end{pmatrix}.$$

We will denote this transformation as $z=z^*(x)$, with its inverse denoted similarly as $x=x^*(z)$.

For $z\in\mathbb{R}^2$, define $V(z)\doteq\bar V(x^*(z))$ and $\phi(z)\doteq\bar\phi(x^*(z))$. Then, $\tilde S^f:\overline{\mathcal{D}}\to\mathbb{C}$ defined by $\tilde S^f(s,z)\doteq\hat{\tilde S}^f(s,x^*(z))$ is a solution of the forward-time dequantized HJ PDE problem given by

$$\begin{aligned}0=&-S_t(s,z)+\tfrac{i\hbar}{2m}\Delta_z S(s,z)-(A_0z+b_0)^TS_z(s,z)\\&-\tfrac{1}{2m}|S_z(s,z)|_c^2-V(z),\ \ (s,z)\in\mathcal{D},\end{aligned} \tag{6}$$

$$S(0,z)=\phi(z),\quad z\in\mathbb{R}^2, \tag{7}$$

$$\text{where}\quad A_0\doteq\omega\begin{pmatrix}0&1\\-1&0\end{pmatrix}\ \text{and}\ b_0\doteq-\omega\hat\delta\begin{pmatrix}0\\1\end{pmatrix}, \tag{8}$$

if and only is $\hat{\tilde S}^f$ is a solution of (3)–(4). (We remark that one may see [17] for further discussion of non-inertial frames in the context of the Schrödinger equation.) In order to apply the diffusion representations as an aid in solution, we will find it helpful to reverse the time variable, and hence we look instead, and equivalently, at the Hamilton-Jacobi partial differential equation (HJ PDE) problem given by

$$\begin{aligned}0=&\,S_t(s,z)+\tfrac{i\hbar}{2m}\Delta_z S(s,x)-(A_0z+b_0)^TS_z(s,z)\\&-\tfrac{1}{2m}|S_z(s,z)|_c^2-V(z),\quad (s,z)\in\mathcal{D},\end{aligned} \tag{9}$$

$$S(t,z)=\phi(z),\quad z\in\mathbb{R}^2. \tag{10}$$

In this last form, we will fix $t\in(0,\infty)$, and allow s to vary in $(0,t]$.

C. Extensions to the complex domain

Various details of extensions to the complex domain must be considered prior to the development of the representation. Models (1)–(2), (3)–(4) and (9)–(10) are typically given as HJ PDE problems over real space domains. However, as in Doss et al. [1], [2], we will find it convenient to change the domain to one where the space components lie over the complex field. We also extend the domain of the potential to $\mathbb{C}^2$, i.e., $V:\mathbb{C}^2\to\mathbb{C}$, and we will abuse notation by employing the same symbol for the extended-domain functions. Throughout, for $k\in\mathbb{N}$, and $z\in\mathbb{C}^k$ or $z\in\mathbb{R}^k$, we let $|z|$ denote the Euclidean norm. Let $\mathcal{D}_\mathbb{C}\doteq(0,t)\times\mathbb{C}^2$ and $\overline{\mathcal{D}}_\mathbb{C}=(0,t]\times\mathbb{C}^2$, and define

$$\begin{aligned}\mathcal{S}_\mathbb{C}\doteq\{S:\overline{\mathcal{D}}_\mathbb{C}\to\mathbb{C}|\,&S\text{ is continuous on }\overline{\mathcal{D}}_\mathbb{C}\text{, continuously}\\&\text{differentiable in time on }\mathcal{D}_\mathbb{C}\text{, and}\\&\text{holomorphic on }\mathbb{C}^2\text{ for all }r\in(0,t]\,\},\end{aligned} \tag{11}$$

$$\begin{aligned}\mathcal{S}^p_\mathbb{C}\doteq\{S\in\mathcal{S}_\mathbb{C}\mid\,&S\text{ satisfies a polynomial growth condition}\\&\text{in space, uniformly on }(0,t]\,\}.\end{aligned} \tag{12}$$

The extended-domain form of problem (9)–(10) is

$$\begin{aligned}0=&\,S_t(s,z)+\tfrac{i\hbar}{2m}\Delta_z S(s,z)-(A_0z+b_0)^TS_z(s,z)\\&-\tfrac{1}{2m}|S_z(s,z)|_c^2-V(z),\ \ (s,z)\in\mathcal{D}_\mathbb{C},\end{aligned} \tag{13}$$

$$S(t,z)=\phi(z),\quad z\in\mathbb{C}^2. \tag{14}$$

Remark 2: We remark that a holomorphic function on $\mathbb{C}^2$ is uniquely defined by its values on the real part of its domain. In particular, $\tilde S:\overline{\mathcal{D}}\to\mathbb{C}$ uniquely defines its extension to a time-indexed holomorphic function over complex space, say $\bar{S} : \overline{\mathcal{D}}_{\mathbb{C}} \to \mathbb{C}$, if the latter exists. Consequently, (13)–(14) is an equivalent formulation of the HJ PDE problem (9)–(10), under the assumptions that a holomorphic solution exists and one has uniqueness for both.

Throughout the remainder, we will assume the following.

$$V, \phi : \mathbb{C}^2 \to \mathbb{C} \text{ are holomorphic on } \mathbb{C}^2. \tag{A.1}$$

We will refer to a linear space over the complex [real] field as a complex [real] space. Although (13)–(14) form an HJ PDE problem for a complex-valued solution over real time and complex space, there is an equivalent formulation as a real-valued solution over real time and a double-dimension real space. We begin from the standard mapping of the complex plane into $\mathbb{R}^2$, denoted here by $\mathcal{V}_{00} : \mathbb{C} \to \mathbb{R}^2$, with $\mathcal{V}_{00}(z) \doteq (x,y)^T$, where $x = \mathbf{Re}(z)$ and $y = \mathbf{Im}(z)$. This immediately yields the mapping $\mathcal{V}_0 : \mathbb{C}^2 \to \mathbb{R}^{2n}$ given by $\mathcal{V}_0(x+iy) \doteq (x^T, y^T)^T$, where component-wise, $(x_j, y_j)^T = \mathcal{V}_{00}(x_j)$ for all $j \in]1,n[$, where throughout, for integer $a \le b$, we define $]a,b[= \{a, a+1, \ldots b\}$. Also in the interests of a reduction of cumbersome notation, we will henceforth frequently abuse notation by writing (x,y) in place of $(x^T, y^T)^T$ when the meaning is clear. Lastly, we may decompose any function in $\mathcal{S}_{\mathbb{C}}$, say $F \in \mathcal{S}_{\mathbb{C}}$, as

$$(\bar{R}(r, \mathcal{V}_0(z)), \bar{T}(r, \mathcal{V}_0(z)))^T \doteq \mathcal{V}_{00}(F(r,z)), \tag{15}$$

where $\bar{R}, \bar{T} : \overline{\mathcal{D}}_2 \doteq (0,t] \times \mathbb{R}^{2n} \to \mathbb{R}$, and we also let $\mathcal{D}_2 \doteq (0,t) \times \mathbb{R}^{2n}$.

IV. An expansion

We now reduce our problem class to the two-dimensional space case (i.e., $n = 2$). We will expand the desired solutions of our problems, and use these expansions as a means for approximation of the solution. First, we consider holomorphic V in the form of a power series. The scalar field of most interest takes the form $-\bar{V}(x) = \bar{c}/|x|$, yielding $-V(z) = \bar{c}/|z + (\hat{\delta}, 0)|$. We recall from Section III-B that $\omega \doteq [\bar{c}/(m\hat{\delta}^3)]^{1/2}$, or $\bar{c} = m\omega^2\hat{\delta}^3$. The expansion up to the fourth-order term in z is

$$-\check{V}^2(z) = -\sum_{k=0}^{2} \hat{\epsilon}^k \hat{V}^k(z), \tag{16}$$
$$-\hat{V}^0(z) = m\omega^2\big[\hat{\delta}^2 - \hat{\delta}z_1 + (z_1^2 - z_2^2/2)\big],$$
$$-\hat{V}^1(z) = m\omega^2\big[-z_1^3 + 3z_1z_2^2/2\big],$$
$$-\hat{V}^2(z) = m\omega^2\big[z_1^4 - 3z_1^2z_2^2 + 3z_2^4/8\big].$$

Here, we find it helpful to explicitly consider the dependence of $\tilde{S}$ and $\bar{S}$ (solutions of (9)–(10) and (13)–(14), respectively) on $\hat{\epsilon}$, where for convenience of exposition, we also allow $\hat{\epsilon}$ to take complex values. Abusing notation, we let $\tilde{S} : \overline{\mathcal{D}} \times \mathbb{C} \to \mathbb{C}$ and $\bar{S} : \overline{\mathcal{D}}_{\mathbb{C}} \times \mathbb{C} \to \mathbb{C}$, and denote the dependence on their arguments as $\tilde{S}(s,z,\hat{\epsilon})$ and $\bar{S}(s,z,\hat{\epsilon})$. We let $\check{\mathcal{D}} \doteq \mathcal{D} \times \mathbb{C}$, $\bar{\check{\mathcal{D}}} \doteq \overline{\mathcal{D}} \times \mathbb{C}$, $\check{\mathcal{D}}_{\mathbb{C}} \doteq \mathcal{D}_{\mathbb{C}} \times \mathbb{C}$ and $\bar{\check{\mathcal{D}}}_{\mathbb{C}} \doteq \overline{\mathcal{D}}_{\mathbb{C}} \times \mathbb{C}$, where we recall that the physical-space components are now restricted to the two-dimensional case. We also let

$$\begin{aligned}\check{\mathcal{S}}_{\mathbb{C}} \doteq \{S : \bar{\check{\mathcal{D}}}_{\mathbb{C}} \to \mathbb{C} \,|\, &S \text{ is continuous on } \bar{\check{\mathcal{D}}}_{\mathbb{C}}, \text{ continuously}\\ &\text{differentiable in time on } \check{\mathcal{D}}_{\mathbb{C}}, \text{ and}\\ &S(r,\cdot,\cdot) \text{ is holomorphic on } \mathbb{C}^2 \times \mathbb{C}\\ &\text{for all } r \in (0,t]\,\},\end{aligned} \tag{17}$$

$$\begin{aligned}\check{\mathcal{S}}^p_{\mathbb{C}} \doteq \{S \in \check{\mathcal{S}}_{\mathbb{C}} \,|\, &S \text{ satisfies a polynomial growth condition}\\ &\text{in space, uniformly on } (0,t]\,\}.\end{aligned} \tag{18}$$

We will make the following assumption throughout the sequel.

$$\text{There exists a unique solution, } \bar{S} \in \check{\mathcal{S}}_{\mathbb{C}}, \text{ to (13)–(14).} \tag{A.2}$$

We also let the power series expansion for ϕ be arranged as

$$\phi(z) = \sum_{k=0}^{\infty} \hat{\epsilon}^k \phi^k(z) \doteq \phi^0(z) + \sum_{k=1}^{\infty} \hat{\epsilon}^k \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} b^{\phi}_{k+2,l,j} z_1^j z_2^{l-j}, \tag{19}$$

where $\phi^0(z)$ is quadratic in z. We consider the following terminal value problems. The zeroth-order problem is

$$0 = S^0_t + \tfrac{i\hbar}{2m}\Delta_z S^0 - \big(A_0 z + b_0\big)^T S^0_z - \tfrac{1}{2m}|S^0_z|^2_c - \hat{V}^0, \quad (s,z) \in \mathcal{D}_{\mathbb{C}}, \tag{20}$$

$$S^0(t,z) = \phi^0(z), \qquad z \in \mathbb{C}^2. \tag{21}$$

For $k \ge 1$, the k^{th} terminal value problem is

$$\begin{aligned}0 = S^k_t + \tfrac{i\hbar}{2m}\Delta_z S^k - \big(A_0 z + b_0 + \tfrac{1}{m}S^0_z\big)^T S^k_z\\ - \tfrac{1}{2m}\sum_{\kappa=1}^{k-1}\big(S^\kappa_z\big)^T S^{k-\kappa}_z - \hat{V}^k, \quad (s,z) \in \mathcal{D}_{\mathbb{C}},\end{aligned} \tag{22}$$

$$S^k(t,z) = \phi^k(z), \qquad z \in \mathbb{C}^2. \tag{23}$$

Note that for $k \ge 1$, given the $\hat{S}^\kappa$ for $\kappa < k$, (22) is a linear, parabolic, second-order PDE. (20) is (22) in the case of $k = 0$, but as its form is different, it is worth breaking it out separately. Also, if the S^k are all polynomial in z of order up to k, then the right-hand side of (22) is polynomial in z of order up to k, as is the right-hand side of (23).

Theorem 3: Assume there exists a unique solution, $\hat{S}^0$, in $\check{\mathcal{S}}_{\mathbb{C}}$ to (20)–(21), and that for each $k \ge 1$, there exists a unique solution, $\hat{S}^k$, in $\check{\mathcal{S}}_{\mathbb{C}}$ to (22)–(23). Then, $\bar{S} = \sum_{k=0}^{\infty} \hat{\epsilon}^k \hat{S}^k$.

Proof: The proof is done by inducting on k. ■

V. Periodic $\hat{S}^0$ solutions

To begin computation of the terms in the expansion of Theorem 3, we must obtain a solution of the HJ PDE problem given by (20)–(21). We will choose the initial condition, ϕ^0, such that the resulting solution will be periodic with frequency that is an integer multiple of ω, where we include the case where the multiple is zero (i.e., the steady-state case). Note that we are seeking periodic solutions, $\hat{S}^0$ that are themselves clearly physically meaningful.

Recall that the original, forward-time solution, $\tilde{S}^f$, of (6)-(8) is a solution of the dequantized version of the original Schrödinger equation. Let $\tilde{\psi}^f(s,z) \doteq \exp\{\frac{i}{\hbar}\tilde{S}^f\}$ for all $(s,z) \in \overline{\mathcal{D}}^f \doteq [0,t) \times \mathbb{R}^2$. Recall also that for physically meaningful solutions, at each $s \in [0,t)$, $\tilde{P}^f(s,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ given by $\tilde{P}(s,\cdot) \doteq [\psi^*\psi](s,\cdot)$ represents an unnormalized density associated to the particle at time s. This suggests that we should seek $\tilde{S}^f$ such that $\tilde{P}^f(s,\cdot) \doteq \exp\{\frac{-2}{\hbar}\tilde{T}^f(s,\cdot)\}$ represents an unnormalized probability density for all $s \in [0,t)$, where $\tilde{T}^f(s,z) \doteq \mathbf{Im}[\tilde{S}^f(s,z)]$ for all $(s,z) \in \overline{\mathcal{D}}^f$.

Although the goal in this section is to generate a set of physically meaningful periodic solutions to the zeroth-order term, we do not attempt a full catalog of all possible such solutions. Let $\hat{S}^{0,f}(s,z) \doteq \hat{S}^0(t-s,z)$ for all $(s,z) \in \overline{\mathcal{D}}^f$. Let the resulting time-dependent coefficients be defined by

$$\hat{S}^{0,f}(s,z) = \tfrac{1}{2} z^T Q(s) z + \Lambda^T(s) z + \rho(s). \qquad (24)$$

The condition that $\exp\left\{\frac{-\hbar}{2}\tilde{T}^f(s,\cdot)\right\}$ represent an unnormalized density implies that the imaginary part of $Q(s)$ should be nonnegative definite for all $s \in [0,t)$,

As $\hat{S}^{0,f}(s,\cdot)$ is holomorphic, it is sufficient to solve the problem on the real domain. The forward-time version of (20)–(21), with domain restricted to $\overline{\mathcal{D}}^f$ is

$$\begin{aligned} 0 = -S_t^{0,f} + \tfrac{i\hbar}{2m}\Delta_z S^{0,f} - \big(A_0 z + b_0\big)^T S_z^{0,f} & \\ - \tfrac{1}{2m}|S_z^{0,f}|_c^2 - \hat{V}^0, \quad (s,z) \in (0,t) \times \mathbb{R}^2, & \qquad (25) \\ S^{0,f}(0,z) = \phi^0(z) \quad \forall\, z \in \mathbb{R}^2. & \qquad (26) \end{aligned}$$

Remark 4: It is worth noting that any solution of form (24) to (25)–(26) is the unique solution in $\mathcal{S}_{\mathbb{C}}^p$, and in particular, where this uniqueness is obtained through a controlled-diffusion representation [9], [10].

Substituting form (24) into (25), and collecting terms, yields the system of ordinary differential equations (ODEs) given as

$$\begin{aligned} \dot{Q}(s) &= -\big(A_0^T Q(s) + Q(s) A_0\big) - \tfrac{1}{m} Q^2(s) + m\omega^2 T^V, \quad (27) \\ \dot{\Lambda}(s) &= -\big(A_0^T + \tfrac{1}{m} Q(s)\big)\Lambda + \omega\hat{\delta} Q(s) u^2 - m\omega^2 \hat{\delta} u^1, \quad (28) \\ \dot{\rho}(s) &= \tfrac{i\hbar}{2m}\,\mathrm{tr}[Q(s)] + \omega\hat{\delta}(u^2)^T \Lambda(s) - \tfrac{1}{2m}\Lambda^T(s)\Lambda(s) \\ &\quad + m\omega^2 \hat{\delta}^2, \quad (29) \\ T^V &= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \qquad u^1 \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad u^2 \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (30) \end{aligned}$$

where $Q : [0,t) \to \mathbb{C}^{2\times 2}$, $\Lambda : [0,t) \to \mathbb{C}^2$ and $\rho : [0,t) \to \mathbb{C}$. Throughout, we assume that $Q(s)$ is symmetric for all $s \in [0,t)$. Note that if $Q(s)$ is nonsingular for all $s \in [0,t)$, then (24) may also be written as

$$\begin{aligned} \hat{S}^{0,f}(s,z) =& \tfrac{1}{2}\big(z + Q^{-1}(s)\Lambda(s)\big)^T Q(s)\big(z + Q^{-1}(s)\Lambda(s)\big) \\ &+ \rho(s) - \Lambda^T(s) Q^{-1}(s)\Lambda(s), \end{aligned}$$

where we see that $-Q^{-1}(s)\Lambda(s)$ may be interpreted as a mean of the associated distribution at each time s. Consequently, we look for solutions with $-Q^{-1}(s)\Lambda(s) \in \mathbb{R}^2$ for all s.

As this paper is already of substantial length, we will restrict ourselves to the steady-state case (modulo the real part of ρ^0). One easily finds that the unique steady state solution for Q is

$$Q(s) = \bar{Q}^0 \doteq \begin{bmatrix} im\omega & -m\omega \\ -m\omega & 0 \end{bmatrix} \quad \forall\, s \in [0,t), \qquad (31)$$

$$\begin{aligned} &\Lambda(s) = \bar{\Lambda}^0 \doteq \big(id, m\omega\hat{\delta} - d/2\big)^T, d \in \mathbb{R} \\ &\rho^0(s) = \rho^0(0) + \bar{c}_1(d) s \quad \forall\, s \in [0,t). \end{aligned} \qquad (32)$$

It should be noted here that although we restrict ourselves to the steady-state case given by (31),(32), for our actual computations of succeeding terms in the expansion, the theory will be sufficiently general to encompass the periodic case as well.

VI. Diffusion representations

We will use diffusion representations to obtain the solutions to the HJ PDEs (22)–(23) that define the succeeding terms in the expansion, i.e., to obtain the $\hat{S}^k$ for $k \in \mathbb{N}$. We need to define the complex-valued diffusion dynamics and the expected payoffs that will yield the $\hat{S}^k$. An extention of the Itô rule to the complex domain is obtained, and the proof of the representation is straightforward.

A. The underlying stochastic dynamics

We let $(\Omega, \mathcal{F}, P)$ be a probability triple, where Ω denotes a sample space, $\mathcal{F}$ denotes a σ-algebra on Ω, and P denotes a probability measure on $(\Omega, \mathcal{F})$. Let $\{\mathcal{F}_s \,|\, s \in [0,t]\}$ denote a filtration on $(\Omega, \mathcal{F}, P)$, and let $B_\cdot$ denote an $\mathcal{F}_\cdot$-adapted Brownian motion taking values in $\mathbb{R}^n$. We will be interested in diffusion processes given by the linear stochastic differential equation (SDE) in integral form

$$\begin{aligned} \zeta_r = \zeta_r^{(s,z)} &= z + \int_s^r -\big(A_0\zeta_\rho + b_0 + \tfrac{1}{m}\hat{S}_z^0(\rho,\zeta_\rho)\big)\,d\rho \\ &\quad + \sqrt{\tfrac{\hbar}{m}}\tfrac{1+i}{\sqrt{2}}\int_s^r dB_\rho \qquad (33) \\ &\doteq z + \int_s^r \lambda(\rho,\zeta_\rho)\,d\rho + \sqrt{\tfrac{\hbar}{m}}\tfrac{1+i}{\sqrt{2}}\,B_r^\Delta, \end{aligned}$$

where $z \in \mathbb{C}^2$, $s \in [0,t)$, $B_r^\Delta \doteq B_r - B_s$ for $r \in [s,t)$, and

$$\begin{aligned} \lambda(\rho,z) &\doteq -[A_0 z + b_0 + \tfrac{1}{m} S_z^0(\rho,z)] \\ &\doteq -A_{>0}(\rho) z - b_{>0}(\rho). \end{aligned} \qquad (34)$$

Let $\bar{f} : [0,t] \times \mathbb{C}^2 \to \mathbb{C}^2$, and suppose there exists some $K_{\bar{f}} < \infty$ such that $|\bar{f}(s,z^1) - \bar{f}(s,z^2)| \le K_{\bar{f}}|z^1 - z^2|$ for all $(s,z^1),(s,z^2) \in \overline{\mathcal{D}}_{\mathbb{C}}$. For $(s,z) \in \overline{\mathcal{D}}_{\mathbb{C}}$, consider the complex-valued diffusion, $\zeta_\cdot \in \mathcal{X}_s$, given by

$$\zeta_r = \zeta_r^{(s,z)} = z + \int_s^r \bar{f}(\rho,\zeta_\rho)\,d\rho + \int_s^r \tfrac{1+i}{\sqrt{2}}\sigma\,dB_\rho, \qquad (35)$$

where $\sigma \in \mathbb{R}^{n\times n}$, and note that this is a slight generalization of (33). For $s \in (0,t]$, let

$$\begin{aligned} \mathcal{X}_s \doteq \{\zeta : [s,t] \times \Omega \to \mathbb{C}^2 \,|\, \zeta \text{ is } \mathcal{F}_\cdot\text{-adapted, right-cts} \\ \text{and such that } \mathbb{E}\sup_{r\in[s,t]}|\zeta_r|^m < \infty\ \forall m \in \mathbb{N}\}. \end{aligned} \qquad (36)$$

It is important to note here that complex-valued diffusions have been discussed elsewhere in the literature; see for example, [18] and the references therein.

We also define the isometric isomorphism, $\mathcal{V} : \mathcal{X}_s \to \mathcal{X}_s^v$ by $[\mathcal{V}(\zeta)]_r \doteq [\mathcal{V}(\xi + i\nu)]_r \doteq (\xi_r^T, \nu_r^T)^T$ for all $r \in [s,t]$ and $\omega \in \Omega$, where

$$\begin{aligned} \mathcal{X}_s^v \doteq \{(\xi,\nu) : [s,t] \times \Omega \to \mathbb{R}^{2n} \,|\, (\xi,\nu) \text{ is } \mathcal{F}_\cdot\text{-adapted,} \\ \text{right-cts and}\, \mathbb{E}\sup_{r\in[s,t]}[|\xi_r|^m + |\nu_r|^m] < \infty\ \forall m \in \mathbb{N}\}, \end{aligned} \qquad (37)$$

Under transformation by $\mathcal{V}$, (35) becomes

$$\begin{pmatrix} \xi_r \\ \nu_r \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \int_s^r \hat{f}(\rho,\xi_\rho,\nu_\rho)\,d\rho + \int_s^r \tfrac{1}{\sqrt{2}}\hat{\sigma}\,dB_\rho, \qquad (38)$$

where $\hat{f}(\rho,\xi_\rho,\nu_\rho) \doteq \big((\mathbf{Re}[\bar{f}(\rho,\xi_\rho+i\nu_\rho)])^T,(\mathbf{Im}[\bar{f}(\rho,\xi_\rho+i\nu_\rho)])^T\big)^T$ and $\hat{\sigma}\doteq(1,1)^T$. Throughout, concerning stochastic differential equations, *solution* refers to a strong solution, unless otherwise noted. The following are easily obtained from existing results; see [10], [15].

Lemma 5: Let $s\in[0,t)$, $z\in\mathbb{C}^2$, $(x,y)=\mathcal{V}_0(z)$. There exists a unique solution, $\zeta\in\mathcal{X}_s$, to (35). In addition, $\zeta\in\mathcal{X}_s$ is a solution of (35) if and only if $\mathcal{V}(\zeta)\in\mathcal{X}_s^v$ is a solution of (38).

We remark that one may apply Lemmas 5 to the specific case of (33) in order to establish existence and uniqueness. In particular, for the dynamics of (33), the corresponding process $(\xi,\nu)=\mathcal{V}(\zeta)$ satisfies

$$\begin{aligned}\begin{pmatrix}\xi_r\\ \nu_r\end{pmatrix}&=\begin{pmatrix}x\\ y\end{pmatrix}\\&\quad-\int_s^r\left[\begin{pmatrix}A^r_{>0}(\rho)&-A^i_{>0}(\rho)\\A^i_{>0}(\rho)&A^r_{>0}(\rho)\end{pmatrix}\begin{pmatrix}\xi_r\\ \nu_r\end{pmatrix}+\begin{pmatrix}b^r_{>0}(\rho)\\ b^i_{>0}(\rho)\end{pmatrix}\right]d\rho\\&\quad+\sqrt{\tfrac{\hbar}{2m}}\begin{pmatrix}I_{n\times n}\\ I_{n\times n}\end{pmatrix}B^\Delta_r \qquad (39)\\&\doteq\begin{pmatrix}x\\ y\end{pmatrix}+\int_s^r-\bar{A}_{>0}(\rho)\begin{pmatrix}\xi_r\\ \nu_r\end{pmatrix}-\bar{b}_{>0}(\rho)\,d\rho+\sqrt{\tfrac{\hbar}{2m}}\,\bar{\mathcal{I}}\,B^\Delta_r\end{aligned}$$

where $A^r_{>0}(\rho)\doteq\mathbf{Re}(A_{>0}(\rho))$, $A^i_{>0}(\rho)\doteq\mathbf{Im}(A_{>0}(\rho))$, $\big((b^r_{>0}(\rho))^T,(b^i_{>0}(\rho))^T\big)^T\doteq\mathcal{V}_0(b_{>0}(\rho))$ for all $\rho\in[0,t)$.

B. Itô's rule

The representation results will rely on a minor generalization of Itô's rule to the specific complex-diffusion dynamics of interest here. The following complex-case Itô rule is similar to existing results (cf., [18]).

Lemma 6: Let $\bar{g}\in\mathcal{S}_\mathbb{C}$ and $(s,z)\in\overline{\mathcal{D}}_\mathbb{C}$, and suppose diffusion process $\zeta.$ is given by (35). Then, for all $r\in[s,t]$,

$$\bar{g}(r,\zeta_r)=\bar{g}(s,z)+\int_s^r\bar{g}_t(\rho,\zeta_\rho)+\bar{g}_z^T(\rho,\zeta_\rho)\bar{f}(\rho,\zeta_\rho)\,d\rho \qquad (40)$$
$$+\int_s^r\tfrac{1+i}{\sqrt{2}}\bar{g}_z^T(\rho,\zeta_\rho)\sigma\,dB_\rho+\tfrac{1}{2}\int_s^r\operatorname{tr}\left[\bar{g}_{zz}(\rho,\zeta_\rho)(\sigma\sigma^T)\right]d\rho.$$

Theorem 7: Let $k\in\mathbb{N}$. Let $\hat{S}^\kappa\in\mathcal{S}^p_\mathbb{C}$ satisfy (22)–(23) for all $\kappa\in]1,k[$. Let $(s,z)\in\overline{\mathcal{D}}_\mathbb{C}$, and let $\zeta\in\mathcal{X}_s$ satisfy (33). Then,

$$\hat{S}^k(s,z)=\mathbb{E}\bigg\{\int_s^t-\tfrac{1}{2m}\sum_{\kappa=1}^{k-1}\left[S_z^\kappa(r,\zeta_r)\right]^TS_z^{k-\kappa}(r,\zeta_r)\\ -\hat{V}^k(\zeta_r)\,dr+\phi^k(\zeta_t)\bigg\}.$$

C. Moments and Iteration

Note that Theorem 7 yields an expression for the k^{th} term in our expansion for $\bar{S}$, $\hat{S}^k$, from the previous terms, $\hat{S}^\kappa$ for $\kappa<k$. We now examine how this generates a computationally tractable scheme. It is heuristically helpful to examine the first two iterates. For $(s,z)\in\overline{\mathcal{D}}_\mathbb{C}$, we have

$$\begin{aligned}\hat{S}^1(s,z)&=\mathbb{E}\bigg\{\int_s^tm\omega^2\Big(-[\zeta_r]_1^3+(3/2)[\zeta_r]_1[\zeta_r]_2^2\Big)\,dr\\&\qquad+\sum_{l=0}^3\sum_{j=0}^lb^\phi_{3,l,j}[\zeta_t]_1^j[\zeta_t]_2^{l-j}\bigg\}, \qquad (41)\\ \hat{S}^2(s,z)&=\mathbb{E}\bigg\{\int_s^t-\tfrac{1}{2m}\big|\hat{S}^1_z(r,\zeta_r)\big|_c^2\\&\qquad+m\omega^2\Big([\zeta_r]_1^4-3[\zeta_r]_1^2[\zeta_r]_2^2+(3/8)[\zeta_r]_2^4\Big)\,dr\\&\qquad+\sum_{l=0}^4\sum_{j=0}^lb^\phi_{4,l,j}[\zeta_t]_1^j[\zeta_t]_2^{l-j}\bigg\}. \qquad (42)\end{aligned}$$

Note that the right-hand side of (41) consists of an expectation of a polynomial in ζ_t and an integral of a polynomial in $\zeta.$, and further, that the dynamics of ζ are linear in the state variable. Thus, we may anticipate that $\hat{S}^1(s,\cdot)$ may also be polynomial. Applying this anticipated form on the right-hand side of (42) suggests that the polynomial form will be inherited in each $\hat{S}^k$. This will form the basis of our computational scheme.

The computation of the expectations that generate the $\hat{S}^k$ for $k\geq1$ will be obtained through the moments of the underlying diffusion process. Thus, the first step is solution of (33). We let the state transition matrices for deterministic linear systems $\dot{y}_r=-A_{>0}(r)y_r$ and $\dot{y}_r^{(2)}=-\bar{A}_{>0}(r)y_r^{(2)}$ be denoted by $\Phi(r,s)$ and $\Phi^{(2)}(r,s)$, respectively. More specifically, with initial (or terminal) conditions, $y_s=\bar{y}$ and $y_s^{(2)}=\bar{y}^{(2)}$, the solutions at time r are given by $y_r=\Phi(r,s)\bar{y}$ and $y_r^{(2)}=\Phi^{(2)}(r,s)\bar{y}^{(2)}$, respectively. The solutions of our SDEs are given by the following.

Lemma 8: Linear SDE (33) has solution given by $\zeta_r=\mu_r+\Delta_r$, where

$$\mu_r=\Phi(r,s)z+\int_s^r\Phi(r,\rho)\big(-b_{>0}(\rho)\big)\,d\rho,$$
$$\Delta_r=\sqrt{\tfrac{\hbar}{m}}\,\tfrac{1+i}{\sqrt{2}}\int_s^r\Phi(r,\rho)\,dB_\rho$$

for all $r\in[s,t]$. Linear SDE (39) has solution given by $X_r^{(2)}=\mu_r^{(2)}+\Delta_r^{(2)}$, where

$$\mu_r^{(2)}=\Phi^{(2)}(r,s)x^{(2)}+\int_s^r\Phi^{(2)}(r,\rho)\big(-\bar{b}_{>0}(\rho)\big)\,d\rho,$$
$$\Delta_r^{(2)}=\sqrt{\tfrac{\hbar}{2m}}\int_s^r\Phi^{(2)}(r,\rho)\bar{\mathcal{I}}\,dB_\rho$$

for all $r\in[s,t]$, where $x^{(2)}\doteq(x^T,y^T)^T$.

Lemma 9: For all $r\in[s,t]$, $X_r^{(2)}$ and ζ_r have normal distributions, and μ_r is the mean of ζ_r, and Δ_r is a zero-mean normal random variable with covariance given by $\mathbb{E}\big[\Delta_r\Delta_r^T\big]=\frac{i\hbar}{m}\int_s^r\Phi(r,\rho)\Phi^T(r,\rho)\,d\rho$, where further, $\mathbb{E}\big[(\zeta_r-\mu_r)(\zeta_r-\mu_r)^T\big]=\mathbb{E}\big[\Delta_r\Delta_r^T\big]$.

As noted above, we will perform the computations mainly in the simpler, steady-state case. In this case, we have

$$-A_{>0}=\omega\begin{pmatrix}-i&0\\2&0\end{pmatrix},\quad\text{and}\quad-b_{>0}=\tfrac{d}{2m}\begin{pmatrix}-2i\\1\end{pmatrix}. \qquad (43)$$

In the case $d=0$, we have $-b_{>0}=0$, while in the case $d=-2m\omega\hat{\delta}$, we have $-b_{>0}=\omega\hat{\delta}(2i,-1)^T$.

Theorem 10: In the steady state case, for all $r\in[s,t]$, ζ_r is a normal random variable with mean and covariance given by, with $\hat{d}\doteq d/(m\omega)$,

$$\mu_r=\begin{pmatrix}\mu_r^1\\ \mu_r^2\end{pmatrix}\quad\text{and}\quad\tilde{\Sigma}_r\doteq\begin{pmatrix}\tilde{\Sigma}_r^{1,1}&\tilde{\Sigma}_r^{1,2}\\ \tilde{\Sigma}_r^{2,1}&\tilde{\Sigma}_r^{2,2}\end{pmatrix},\quad\text{where}$$
$$\mu_r^1=e^{-i\omega(r-s)}z_1+\hat{d}(e^{-i\omega(r-s)}-1),$$

$$\mu_r^2 = 2i[e^{-i\omega(r-s)} - 1]z_1 + z_2 + \hat{d}\big[2i((e^{-i\omega(r-s)} - 1) - 3\omega(r-s)/2\big],$$
$$\tilde{\Sigma}_r^{1,1} = \tfrac{\hbar}{m\omega}\tfrac{1}{2}\big(1 - e^{-2i\omega(r-s)}\big),$$
$$\tilde{\Sigma}_r^{1,2} = \tilde{\Sigma}_r^{2,1} = \tfrac{\hbar}{m\omega} i\big[2\big(e^{-i\omega(r-s)} - 1\big) - \big(e^{-2i\omega(r-s)} - 1\big)\big],$$
$$\tilde{\Sigma}_r^{2,2} = \tfrac{\hbar}{m\omega}\big[2\big(e^{-2i\omega(r-s)} - 1\big) - 8\big(e^{-i\omega(r-s)} - 1\big) - 3i\omega(r-s)\big].$$

Theorem 11: For $k \geq 1$ and $(s,z) \in \overline{\mathcal{D}}_{\mathbb{C}}$, $\hat{S}^k(s,z) = \sum_{l=0}^{k+2}\sum_{j=0}^{l} \hat{c}_{l,j}^k(s) z_1^j z_2^{l-j}$. Given the coefficient functions $\hat{c}_{l,j}^\kappa(s)$ for $\kappa < k$, the time-indexed coefficients $\hat{c}_{l,j}^k(s)$ are obtained by the evaluation of linear combinations of moments of up to $(k+2)^{th}$-order of the normal random variables ζ_r and closed-form time-integrals.

VII. The $\hat{S}^1$ term

Here, we proceed an additional step, computing $\breve{S}^1 \doteq \hat{S}^0 + \frac{1}{\delta}\hat{S}^1$. We perform the actual computations for $\hat{S}^1$ only in the steady-state case. For $(s,z) \in \overline{\mathcal{D}}_{\mathbb{C}}$, we may obtain $\hat{S}^1(s,z)$ from (41), using the expressions for the mean and variance of normal ζ_r given in Theorem 10. There are well-known expressions for all moments of normal random variables. We note that, as our interest is in the solution of the original forward-time problem, it is sufficient to take $s = 0$. Further, as our interest will be in solutions that exhibit periodic behavior, we take $t = \tau \doteq 2\pi/\omega$. We find

$$\mathbb{E}\Big\{ \int_0^\tau -\hat{V}^1(\zeta_r)\,dr \Big\} = m\omega^2 \Big\{ \tfrac{3\pi\hat{d}}{\omega}[z_1^2 + iz_1z_2 - z_2^2] + c_1(\tau)(1, 2i)z + c_2(\tau) \Big\}, \tag{44}$$

where

$$c_1(\tau) = (3\pi/\omega)\big[\hat{d}^2(1 - 3i\pi)/2 - \hbar/(m\omega)\big],$$
$$c_2(\tau) = \tfrac{\pi\hat{d}\hbar}{m\omega^2}(18i\pi - 9/2) + \tfrac{3\pi\hat{d}^3}{2\omega}\big((1/3) - 3i\pi - 6\pi^2\big).$$

From (44), we see that the expected value, $\mathbb{E}\int_0^\tau -\hat{V}^1(\zeta_r)\,dr$ has at most quadratic terms in z. (In contrast, for typical $t \neq \tau$, this integral is cubic in z.) Consequently, it may be of interest to take terminal cost, ϕ^1 to be quadratic rather than the more general hypothesized cubic form. Suppose we specifically take $\phi^1(z) \doteq \frac{1}{2}z^T Q^1 z$, where Q^1 has components $Q_{j,k}^1$. Noting that we are seeking a solution of form $\breve{S}^1 = \hat{S}^0 + \frac{1}{\delta}\hat{S}^1$, we find it helpful to now allow general $d \in \mathbb{C}$ with corresponding $\bar{\Lambda}^0$ given by (32). Combining (41) and (44), we find

$$\hat{S}^1(\tau, z) = \tfrac{1}{2}z^T(Q^1 + Q^\Delta)z + b^T z + \rho^1(\tau), \tag{45}$$

where

$$Q^\Delta = 6\pi d \begin{pmatrix} 1 & i/2 \\ i/2 & -1 \end{pmatrix}, \quad b = \Big[\tilde{k}_1\big(Q^1 + Q^\Delta\big) + \tilde{k}_2 Q^\Delta\Big]\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{46}$$

$$\rho^1(\tau) = \tilde{k}_1\big[\tfrac{\tilde{k}_1}{2} + \tfrac{i\hbar}{d}\big]Q_{2,2}^1 + \tfrac{\pi d\hbar}{m\omega}(18i\pi - 9/2) + \tfrac{3\pi d^3}{2m^2\omega^2}\big((1/3) - 3i\pi - 6\pi^2\big), \tag{47}$$

$$\tilde{k}_1 = \tfrac{-3\pi d}{m\omega}, \qquad \tilde{k}_2 = \tfrac{i\hbar}{d} + \tfrac{d}{2m\omega}(3\pi - i), \tag{48}$$

and Q^1, d are free. Recalling that $\hat{S}^0(\tau,z) = \frac{1}{2}z^T\bar{Q}^0 z + (\bar{\Lambda}^0)^T z + \rho^0(\tau)$, we find

$$\breve{S}^1(\tau,z) = \hat{S}^0(\tau,z) + \tfrac{1}{\delta}\hat{S}^1(\tau,z) = \tfrac{1}{2}z^T\big[\bar{Q}^0 + \tfrac{1}{\delta}\big(Q^1 + Q^\Delta\big)\big]z + \big[\bar{\Lambda}^0 + \tfrac{1}{\delta}b\big]^T z + \rho^0(\tau) + \tfrac{1}{\delta}\rho^1(\tau) \tag{49}$$

A. Periodicity

We investigate whether there exist approximate solutions, $\breve{S}^1$, and corresponding forward-time approximate solution, $\breve{S}^{f,1}$, such that the distribution, $\breve{P}^1(t,x) \doteq [\psi^*\psi](t,x) = \exp\big\{\frac{-2}{\hbar}\mathbf{Im}\big[\breve{S}^{f,1}(\tau,z)\big]\big\}$ is periodic, i.e. $\breve{P}^1(0,\cdot) = \breve{P}^1(\tau,\cdot)$. We find that there do not exist values of such coefficients. However, there exist solutions $\breve{S}^1$ of periodic-plus-small-angular-drift form. Namely, after one nominal circular rotation (i.e. at $t = \tau$), the distribution of the particle with $\hbar > 0$ travels an angular distance of $2\pi + \delta_\theta$, while the classical particle travels 2π. If one takes $\hbar \downarrow 0$, then this excess angular travel, δ_θ, goes to zero as well. Future work will address the question of whether the effect of angular drift will diminish rapidly with the inclusion of higher order terms in the approximate solution $\breve{S}^k$.

Acknowledgment

Research partially supported by AFOSR Grant FA9550-15-1-0131 and NSF Grant DMS-1312569.

References

[1] R. Azencott and H. Doss, “L’équation de Schrödinger quand $\hbar$ tend vers zéro: une approche probabiliste”, Stochastic Aspects of Classical and Quantum Systems, Lecture Notes in Math., 1109 (1985), 1–17.
[2] H. Doss, “Sur une resolution stochastique de l’equation de Schrödinger à coefficients analytiques”, Comm. Math, Phys., 73 (1980), 247–264.
[3] R.P. Feynman, “Space-time approach to non-relativistic quantum mechanics”, Rev. of Mod. Phys., 20 (1948), 367–387.
[4] W.H. Fleming, “Stochastic calculus of variations and mechanics”, J. Optim. Theory and Applics., 41 (1983), 55–74.
[5] V.N. Kolokoltsov, *Semiclassical Analysis for Diffusions and Stochastic Processes,* Springer Lec. Notes in Math., 1724, Springer, Berlin, 2000.
[6] A.J. Krener, “Reciprocal diffusions in flat space”, Prob. Theory and Related Fields, 107 (1997), 243–281.
[7] G.L. Litvinov, “The Maslov dequantization, idempotent and tropical mathematics: A brief introduction”, J. Math. Sciences, 140 (2007), 426–444.
[8] V.P. Maslov. “A new superposition principle for optimization problem”, Russian Math. Surveys [translation of Uspekhi Mat. Nauk], 42 (1987), 39–48.
[9] W.M. McEneaney, “A stationary-action control representation for the dequantized Schrödinger Equation”, Proc. 22nd Intl. Symp. Math. Theory Networks and Systems (2016).
[10] W.M. McEneaney, “A stochastic control verification theorem for the dequantized Schrödinger equation not requiring a duration restriction”, Applied Math. Optim. (to appear).
[11] W.M. McEneaney and P.M. Dower, “Staticization, its dynamic program and solution propagation”, Automatica (to appear).
[12] W.M. McEneaney and P.M. Dower, “The principle of least action and fundamental solutions of mass-spring and n-body two-point boundary value problems”, SIAM J. Control and Optim., 53 (2015), 2898–2933.
[13] M. Nagasawa, *Schrödinger Equations and Diffusion Theory,* Birkhäuser, Basel, 1993.
[14] E. Nelson, “Derivation of the Schrödinger equation from Newtonian mechanics”, Phys. Rev., 150 (1966), 1079–1085.
[15] C. Perret, “The Stability of Numerical Simulations of Complex Stochastic Differential Equations”, ETH, Zurich, 2010.
[16] R.M. Range, *Holomorphic Functions and Integral Representations in Several Complex Variables,* Springer, New York, 1986.
[17] S. Takagi, “Quantum Dynamics and Non-Inertial Frames of Reference. I. Generality”, Prog. Theoretical Physics, 85 (1991), 463–479.
[18] J. Uboe, “Complex valued multiparameter stochastic integrals”, J. Theoretical Prob., 8 (1995), 601–624.
[19] J.C. Zambrini, “Probability in quantum mechanics according to E. Schrödinger”, Physica B+C, 151 (1988), 327–331.