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Robust Output Regulation for a Class of Nonlinear Systems not Detectable by Regulated Output

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Abstract— **This paper considers the robust output regulation problem for a class of nonlinear systems, not detectable by the regulated output. Differently from the previous results on the subject, it is supposed that the considered class of systems is in the normal form not defined on the conventional *regulated output*. We propose a novel internal model structure that, joined to a high-gain stabilizing action, solves the output regulation problem robustly with respect to possible uncertainties affecting the systems.**

I. Introduction

The output regulation problem aims to achieve asymptotic tracking or rejection of exogenous inputs, that are thought of as generated by an autonomous system typically referred to as *exosystem*. This problem has attracted the research interest in the last decades, with both feedforward methods (in which the knowledge of the exosystem is typically used to design feedforward steady state actions) and internal model-based methods (in which the exosystem structure is appropriately embedded in the regulator). Due to its capability of handling uncertainties, the latter one has received the major attention, particularly since the celebrated contributions [1] for linear systems and [2] for nonlinear systems.

In general the internal model is designed so that the desired output regulation problem is transformed into a stabilization problem for an appropriately augmented system with a "residual" control input to be designed (see [3], [4], [5], [6], [7], [8]). The design of the regulator is in particular divided in two steps. The first one is to establish an appropriate internal model so as to reproduce the steady-state input. So far, several systematic design methods have been reported to construct the internal model such as [3], [9], [4], in which different kinds of polynomial assumptions on the solution of the regulator equations are utilized. These polynomial type assumptions are removed in [7] by proposing a high-order nonlinear internal model. For a comprehensive introduction of internal model design, see [3], [14] and references therein. After augmenting the regulated system with the internal model and operating appropriate change of variables (meant to transform the output regulation problem into a problem of stabilization to the origin), the regulator design proceeds with the second step, that is to design a (dynamical) stabilizer through a residual input that is typically left free in the internal model design. Small-gain technique [16], [17] and nonlinear separation principles [11], [12], [13] are typical methods that are adopted to design the residual input to stabilize the origin of the augmented systems, thus solving the problem of output regulation.

In order to apply the above conventional two-step method, the resulting augmented system is required to be stabilizable by a dynamical output feedback, uniformly with respect to exogenous variables. As shown in [9] and the forthcoming motivating example in section II-A, the desired stabilizability is guaranteed if the controlled system is detectable by the regulated output when the exogenous variables are set to zero[1]. For general multivariable linear systems (see [1]), it is well known that such a detectability assumption is fulfilled if a postprocessing solution (see [5]) is adopted in the phase of internal model design, and if certain non-resonance conditions are fulfilled. However, as far as nonlinear systems are concerned, all existing solutions are still strongly dependent on the assumption that the regulated plant is detectable by the regulated output and it is not clear how to design the regulator if the detectability is indeed guaranteed by other (i.e. different from the regulated ones) outputs.

In this paper we consider the robust output regulation for a class of nonlinear systems in the normal form, not detectable by the regulated output. Different from the existing results, our normal form is not defined on the conventional *regulated output*. We propose a new internal model structure that, properly complemented with an high-gain stabilization unit, solves the problem at hand robustly with respect to uncertainties affecting the regulated systems under certain assumptions.

Notations: For convenience, we use matrix A_r to denote the shift matrix of dimension $r \times r$ and B_r to denote a $r \times 1$ vector whose elements are all zeros except the last one being equal to one.

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[1]In what follows, when we say that a system is or is not detectable by the regulated output, we always mean the situation when exogenous variables (in the following denoted by w) are zero.

II. PRELIMINARIES

A. A Motivating Example

Consider the output regulation problem for the nonlinear system

$$\begin{array}{rcl} \dot{z} & = & -z + x_1 + w_1^2 - w_2^2 \\ \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \cos(-2z + x_1) - x_1 + bu \end{array} \tag{1}$$

where $e = -2z + x_1$ is the regulated output, $b > 0$ is an unknown positive constant, and the exogenous inputs w_1, w_2 are generated by an harmonic oscillator

$$\dot{w} = \begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = S_0 w \triangleq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} . \tag{2}$$

Then consider the solution $(\Pi_0(w), Y(w), \Psi(w))$ of the regulator equations

$$\begin{array}{rcl} L_{S_0 w}\Pi_0(w) & = & -\Pi_0(w) + Y(w) + w_1^2 - w_2^2 \\ L^2_{S_0 w} Y(w) & = & \cos(-2\Pi_0(w) + Y(w)) - Y(w) + b\Psi(w) \\ 0 & = & -2\Pi_0(w) + Y(w) \end{array} \tag{3}$$

that, as a simple calculation shows, are given by

$$\begin{array}{rcl} \Pi_0(w) & = & -0.8 w_1 w_2 - 0.2 w_1^2 + 0.2 w_2^2 \,, \\ Y(w) & = & -1.6 w_1 w_2 - 0.4 w_1^2 + 0.4 w_2^2 \,, \\ \Psi(w) & = & \frac{1}{b}(4.8 w_1 w_2 + 1.2 w_1^2 - 1.2 w_2^2) \,. \end{array}$$

Considering the change of variables

$$\tilde{z} = z - \Pi_0(w) \,, \quad \tilde{x}_1 = x_1 - Y(w) \,, \quad \tilde{x}_2 = x_2 - L_{Sw} Y(w) \,,$$

it is immediately seen that the *error system* takes the form

$$\begin{array}{rcl} \dot{\tilde{z}} & = & -\tilde{z} + \tilde{x}_1 \\ \dot{\tilde{x}}_1 & = & \tilde{x}_2 \\ \dot{\tilde{x}}_2 & = & \cos(-2\tilde{z} + \tilde{x}_1) - \tilde{x}_1 + b(u - \Psi(w)) \\ e & = & -2\tilde{z} + \tilde{x}_1 \end{array} \tag{4}$$

By observing that the function Ψ and the exosystem dynamics (2) fulfill the assumption of "immersion into a linear systems", the "standard" design paradigm (see [9] and [18]) would suggest to design the internal model as a linear system of the form

$$\begin{array}{rcl} \dot{\eta} & = & \Phi\eta + G(u - \bar{\Gamma}\eta) \\ u & = & \bar{\Gamma}\eta + v \end{array} \tag{5}$$

where [2]

$$\Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 0 \end{pmatrix} , \quad G = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} , \quad \bar{\Gamma}^\top = \begin{pmatrix} 6 \\ 7 \\ 6 \end{pmatrix} \tag{6}$$

and v denotes the "residual" control to be designed. We observe that there exists a unique $\Sigma(w)$ such that

$$\begin{array}{rcl} \frac{\partial \Sigma(w)}{\partial w} S_0 w & = & \Phi \Sigma(w) \\ \Psi(w) & = & \bar{\Gamma}\Sigma(w) \end{array}$$

namely, (5) has the internal model property. Thus, setting $\tilde{\eta} = \eta - \Sigma(w)$ yields

$$\dot{\tilde{\eta}} = \Phi\tilde{\eta} + Gv \,. \tag{7}$$

[2]The indicated $\bar{\Gamma}$ is a possible choice to make $\Phi - G\bar{\Gamma}$ Hurwitz.

Towards this end, by augmenting (7) with (4), the original output regulation problem is transformed to the stabilization problem to the origin of the resulting augmented system with "residual" control v. What make the stabilization problem hard to be solved is that $\tilde{z}, \tilde{x}_1, \tilde{x}_2, \tilde{\eta}$ are not known since since w is not known. In addition, by letting $w_1 = w_2 = 0$ and linearizing the system (1) or its "tilde" form (4), it can be seen that the resulting linear system is *not detectable* by the regulated output e, which brings obstacles to design an observer to estimate $\tilde{z}, \tilde{x}_1, \tilde{x}_2$. In view of the above analysis, the remaining stabilization problem of the augmented system becomes very difficult (and possibly unsolvable). It is noted that, following the method in [9], one can construct additional compensators to reproduce the steady state for the state x_2 but due to the absence of detectable regulated output, stabilizing the augmented system turns out difficult and possibly unsolvable once again. With this being the case, in this paper we are going to explore a new design paradigm such that the output regulation problem of system (1) can be solved.

B. Problem statement

Consider a class of nonlinear systems having the normal form

$$\begin{array}{rcl} \dot{z} & = & f_0(w, z, \xi) \\ \dot{\xi} & = & A_r\xi + B_r[q(w, z, \xi) + b(w, z, \xi)u] \\ e & = & h_e(w, z, \xi) \end{array} \tag{8}$$

where state $x = \mathrm{col}(z, \xi)$ with $z \in \mathbb{R}^{n-r}$ and $\xi = \mathrm{col}(\xi_1, \ldots, \xi_r) \in \mathbb{R}^r$, control input $u \in \mathbb{R}$, regulated output $e \in \mathbb{R}$, and the variable w denotes exogenous inputs, that might represent reference/commands (to be tracked/rejected) or uncertain parameters. It is assumed that $w \in \mathbb{R}^{n_w}$ is generated by the dynamics of $\dot{w} = Sw$ with the matrix S being *neutrally stable*. The mappings $f_0(w, z, \xi)$, $q(w, z, \xi)$, $b(w, z, \xi)$ and $h_e(w, z, \xi)$ are smooth and the high-frequency gain $b(w, z, \xi) > b_0$ with a constant $b_0 > 0$ uniformly in (w, z, ξ).

In this paper, we are mainly interested in the partial state feedback output regulation, which means that in addition to the regulated output e, the partial state ξ is also measurable. Thus the objective of the paper, given arbitrary compact sets $\mathcal{W} \in \mathbb{R}^{n_w}$, $\mathcal{X} \in \mathbb{R}^n$ and $\mathcal{X}_c \in \mathbb{R}^{n_c}$, is to design an appropriate partial state feedback controller of the form

$$\begin{array}{rcl} \dot{x}_c & = & f_c(x_c, e, \xi) \\ u & = & h_c(x_c, e, \xi) \,, \end{array} \tag{9}$$

with state $x_c \in \mathbb{R}^{n_c}$, and for any initial condition $(w(0), x(0), x_c(0)) \in \mathcal{W} \times \mathcal{X} \times \mathcal{X}_c$ such that the resulting closed-loop system satisfies

- the resulting trajectory of (8)-(9) is bounded, and
- $\lim\limits_{t\to\infty} e(t) = 0$.

Remark 1: We observe that the normal form (8) is not related to the input-error pair (u, e), namely the output ξ_1 with respect to which the normal form (8) is defined is not necessarily coincident with the regulated error.

C. Reduction of Relative Degree and Assumptions

It is well known that, by redesigning the output, the relative degree of system (8) can be lowered from r to one. This is achieved by picking a new output variable as

$$\vartheta := \xi_r + \sum_{j=1}^{r-1} c_j \xi_j \,, \tag{10}$$

in which the coefficients c_j's are such that the matrix

$$F = \begin{pmatrix} 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \ddots & \cdot \\ 0 & 0 & \cdot & 1 \\ -c_1 & -c_2 & \cdot & -c_{r-1} \end{pmatrix}$$

is Hurwitz. Thus, using ζ to denote $\text{col}\{\xi_1, \ldots, \xi_{r-1}\}$, the dynamics of (8) with ξ replaced by the new coordinates (ζ, ϑ) can be seen as a system in normal form having relative degree one between input u and redesigned variable ϑ, i.e.

$$\begin{aligned} \dot{w} &= Sw \\ \dot{z} &= f_0(w, z, \zeta, \vartheta) \\ \dot{\zeta} &= F\xi + B_{r-1}\vartheta \\ \dot{\vartheta} &= \bar{q}(w, z, \zeta, \vartheta) + b(w, z, \zeta, \vartheta)u \\ e &= h_e(w, z, \zeta, \vartheta) \end{aligned} \tag{11}$$

where $\bar{q}(w, z, \zeta, \vartheta) = C\zeta + c_{r-1}\vartheta + q(w, z, \zeta, \vartheta)$ with $C^\top = \text{col}\{-c_{r-1}, c_1 - c_{r-1}c_2, c_2 - c_{r-1}c_3, \ldots, c_{r-2} - c_{r-1}^2\}$, and with a bit abuse of notations, we insist on using the same symbols to denote functions such as f_0, q, b, h_e, after their argument ξ is changed into (ζ, ϑ).

In what follows, we mainly focus on the design of regulator driven by regulated output e and redesigned variable ϑ, which is a linear function of partial state ξ. In fact, if only ξ_1, instead of the whole partial states ξ is measurable, in order to further obtain the regulator via measurement feedback, we can use the technique of rough high-gain observer driven by ξ_1 to (practically) estimate the whole partial states ξ (see the forthcoming Remark 3).

Towards this end, following the conventional internal model approach, we assume that the *regulator equations* for system (8) admits a solution as formalized in the following.

Assumption 1: There exist smooth maps $\Pi_0(w)$, $Y(w)$, $i = 1, 2, \ldots, m$ and $\Psi(w)$ such that the regulator equations

$$\begin{aligned} &\frac{\partial \Pi_0}{\partial w} Sw = f_0(w, \Pi_0(w), \Pi_\xi(w)) \\ &0 = h_e(w, \Pi_0(w), \Pi_\xi(w)) \\ &L_{Sw}^r Y(w) = q(w, \Pi_0(w), \Pi_\xi(w)) + b(w, \Pi_0(w), \Pi_\xi(w))\Psi(w) \end{aligned} \tag{12}$$

with $\Pi_\xi(w) = \text{col}\{Y(w), L_s Y(w), \ldots, L_s^{r-1} Y(w)\}$, is satisfied.

With this being the case, define

$$\begin{aligned} \tilde{z} &= z - \Pi_0(w) \,, \\ \tilde{\zeta}_1 &= \zeta_1 - Y(w) \end{aligned}$$

and recursively for $2 \le j \le r-1$,

$$\tilde{\zeta}_j = \zeta_j - L_{Sw}^j Y(w) \,,$$

and

$$\tilde{\vartheta} = \vartheta - L_{Sw}^{r-1} Y(w) - \sum_{j=1}^{r-1} c_j L_{Sw}^{j-1} Y(w) \,.$$

Therefore, combining Assumption 1 with (11) we can obtain

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{f}_0(w, \tilde{z}, \tilde{\zeta}, \tilde{\vartheta}) \\ \dot{\tilde{\zeta}} &= F\tilde{\zeta} + B_{r-1}\tilde{\vartheta} \\ \dot{\tilde{\vartheta}} &= \tilde{\bar{q}}(w, \tilde{z}, \tilde{\zeta}, \tilde{\vartheta}) + b(w, z, \zeta, \vartheta)[u - \Psi(w)] \\ e &= \tilde{h}_e(w, \tilde{z}, \tilde{\zeta}, \tilde{\vartheta}) \end{aligned} \tag{13}$$

in which, for the sake of simplicity, we have set

$$\begin{aligned} \tilde{f}_0(w, \tilde{z}, \tilde{\zeta}, \tilde{\vartheta}) &= f_0(w, z, \zeta, \vartheta) - f_0(w, \Pi_0(w), \Pi_\xi(w)) \\ \tilde{\bar{q}}(w, \tilde{z}, \tilde{\zeta}, \tilde{\vartheta}) &= \bar{q}(w, z, \zeta, \vartheta) - \bar{q}(w, \Pi_0(w), \Pi_\xi(w)) \\ &\quad + b(w, z, \zeta, \vartheta) - b(w, \Pi_0(w), \Pi_\xi(w)) \\ \tilde{h}_e(w, \tilde{z}, \tilde{\zeta}, \tilde{\vartheta}) &= h_e(w, z, \zeta, \vartheta) - h_e(w, \Pi_0(w), \Pi_\xi(w)) \,. \end{aligned}$$

Note that $\tilde{f}_0(w, 0, 0, 0) = 0$, $\tilde{\bar{q}}(w, 0, 0, 0) = 0$ and $\tilde{h}_e(w, 0, 0, 0) = 0$.

III. Main Results

A. Regulator Design

Before we present the dynamical regulator, as in [3] we make the following assumption on the solution of regulator equations (1).

Assumption 2: Both $Y(w)$ and $\Psi(w)$ are polynomials in w.

With this assumption, now we turn to the design of internal model, which is going to be embedded into the controller. Making full use of the polynomial property indicated in Assumption 2, we know that it is always possible to define a new variable $\bar{w} \in \mathbb{R}^\nu$ for some positive integer ν (dependent on the maximal degree of the polynomials of $Y(w)$ and $\Psi(w)$), such that there exists matrix $\bar{S} \in \mathbb{R}^{\nu \times \nu}$, yielding

$$\frac{\partial \bar{w}}{\partial w}(Sw) = \bar{S}\bar{w} \,,$$

and

$$Y(w) = \bar{Y}\bar{w} \,, \qquad \Psi(w) = \bar{\Psi}\bar{w}$$

for some appropriate constant matrices $\bar{Y}$ and $\bar{\Psi}$, i.e. $Y(w)$ and $\Psi(w)$ can be rewritten as linear functions of $\bar{w}$. Furthermore, let

$$\mathcal{P}_{\bar{S}}(\lambda) = \lambda^d + \bar{s}_{d-1}\lambda^{d-1} + \ldots + \bar{s}_1(\lambda) + \bar{s}_0$$

denote the minimal polynomial of $\bar{S}$.

Note that as shown in Section II-A, since we have no knowledge that e is detectable, the conventional internal model-based approach cannot be applied. In order to solve the output regulation problem at hand, we propose a novel internal model having the form

$$\begin{aligned} \dot{x}_s &= A_s x_s + B_s \Gamma \eta \\ \dot{\eta} &= \Phi\eta + Ge \\ \mu &= C_s x_s + D_s \Gamma \eta \end{aligned} \tag{14}$$

where $x_s \in \mathbb{R}^{n_s}$ with positive integer n_s to be fixed and $\eta = \text{col}\{\eta_1, \eta_2, \ldots, \eta_d\}$, matrices (A_s, B_s, C_s, D_s) are free design parameters to be determined later, and $\Phi \in \mathbb{R}^{d\times d}$, $G \in \mathbb{R}^{d}$ and $\Gamma \in \mathbb{R}^{1\times d}$ have the canonical form

$$\begin{array}{rcl}\Phi & = & \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 1 \\ -\bar{s}_0 & -\bar{s}_1 & -\bar{s}_2 & \cdot & -\bar{s}_{d-1} \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{pmatrix}, \\ \Gamma & = & \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \end{array}$$

Clearly, the matrix pair (Φ, G) is *controllable*, and (Φ, Γ) is *observable*.

With the dynamical compensator (14), the desired (high-gain) control is chosen as

$$u = -\kappa(\vartheta - \mu)\,. \tag{15}$$

Remark 2: Compared to the p-copy internal model [3], (14) has the additional x_s-dynamics, which, as it will be shown in the sequel, plays a crucial row in guaranteeing that the regulator possesses the internal model property and that the stabilization task can be accomplished. On the other hand, compared to the internal model in [2], [4], which is driven by control u, (14) is driven by the regulated output e.

B. Stability Analysis

As it is well known, to show that the regulator (14)-(15) solves the output regulation at hand, there are two issues to be verified. The first one (internal model property) is to show that there exist matrices $\chi_s \in \mathbb{R}^{n_s\times\nu}$ and $\Sigma \in \mathbb{R}^{d\times\nu}$ such that

$$\begin{array}{rcl}\chi_s\bar{S} & = & A_s\chi_s + B_s\Gamma\Sigma \\ \Sigma\bar{S} & = & \Phi\Sigma \\ \bar{\Psi} & = & -\kappa(\Theta - C_s\chi_s - D_s\Gamma\Sigma)\end{array} \tag{16}$$

where $\Theta = \bar{Y}\bar{S}^{r-1} + \sum_{j=1}^{r-1} c_j\bar{Y}\bar{S}^{j-1}$. With this in mind, set

$$\Sigma = \begin{pmatrix} \Sigma_1 \\ \Sigma_1\bar{S} \\ \cdot \\ \Sigma_1\bar{S}^{d-1} \end{pmatrix} \tag{17}$$

where $\Sigma_1 \in \mathbb{R}^{1\times\nu}$ is a free matrix. This yields

$$\Sigma\bar{S} = \Phi\Sigma\,. \tag{18}$$

Thus the first issue to be verified can be simplified to verify that there exists matrix pair (χ_s, Σ_1) such that

$$\begin{array}{rcl}\chi_s\bar{S} & = & A_s\chi_s + B_s\Sigma_1 \\ \frac{1}{\kappa}\bar{\Psi} + \Theta & = & C_s\chi_s + D_s\Sigma_1\end{array} \tag{19}$$

With this being the case, we formulate the following lemma, whose proof can be easily obtained using the PBH test.

Lemma 1: Given any Hurwitz matrix A_s and any matrix Ψ_κ, the linear matrix equation

$$\begin{pmatrix} A_s & B_s \\ C_s & D_s \end{pmatrix}\begin{pmatrix} \chi_s \\ \Sigma_1 \end{pmatrix} + \begin{pmatrix} -I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \chi_s \\ \Sigma_1 \end{pmatrix}\bar{S} = \begin{pmatrix} 0 \\ \Psi_\kappa \end{pmatrix} \tag{20}$$

has a unique solution (χ_s, Σ_1) if and only if the *nonresonance condition*

$$\det\left(C_s(\lambda I - A_s)^{-1}B_s + D_s\right) \neq 0\,, \qquad \forall\lambda \in \sigma(\bar{S}) \tag{21}$$

is fulfilled.

Thus by choosing A_s to be Hurwitz and (A_s, B_s, C_s, D_s) such that (21) is fulfilled, Lemma 1 implies that (19) has the unique solution (χ_s, Σ), which indicates that the first issue is verified. Then let us proceed to verify the second issue (stability requirement), i.e., the manifold

$$\begin{array}{r}\mathcal{M} = \{(w, z, \xi, x_s, \eta) | z = \Pi_0(w), \xi = \Pi_\xi(w), \\ x_s = \chi_s\bar{w}, \eta = \Sigma\bar{w}\}\end{array} \tag{22}$$

is attractive.

To show this, let us consider the change of coordinates

$$\tilde{\eta} = \eta - \Sigma\bar{w}\,, \qquad \tilde{x}_s = x_s - \chi_s\bar{w}\,,$$

and

$$\tilde{y} = \tilde{\vartheta} - \tilde{\mu}\,, \qquad \tilde{\mu} = (C_s\tilde{x}_s + D_s\tilde{\eta}_1)\,.$$

By combining (14) with (16), this yields

$$\begin{array}{rcl}\dot{\tilde{x}}_s & = & A_s\tilde{x}_s + B_s\tilde{\eta}_1 \\ \dot{\tilde{\eta}} & = & \Phi\tilde{\eta} + G\tilde{h}_e(w, \tilde{z}, \tilde{\zeta}, C_s\tilde{x}_s + D_s\tilde{\eta}_1 + \tilde{y}) \\ u & = & \Psi(w) - k\tilde{y}\,.\end{array} \tag{23}$$

In summary, augmenting (23) with (13), we can obtain an augmented system having the form

$$\begin{array}{rcl}\dot{\tilde{z}} & = & \tilde{f}_0(w, \tilde{z}, \tilde{\zeta}, C_s\tilde{x}_s + D_s\tilde{\eta}_1 + \tilde{y}) \\ \dot{\tilde{\zeta}} & = & F\tilde{\zeta} + B_{r-1}(C_s\tilde{x}_s + D_s\tilde{\eta}_1 + \tilde{y}) \\ \dot{\tilde{x}}_s & = & A_s\tilde{x}_s + B_s\tilde{\eta}_1 \\ \dot{\tilde{\eta}} & = & \Phi\tilde{\eta} + G\tilde{h}_e(w, \tilde{z}, \tilde{\zeta}, C_s\tilde{x}_s + D_s\tilde{\eta}_1 + \tilde{y}) \\ \dot{\tilde{y}} & = & \Delta_1(w, \tilde{z}, \tilde{\zeta}, \tilde{x}_s, \tilde{\eta}, \tilde{y}) - b(w, z, \xi)k\tilde{y}\,.\end{array} \tag{24}$$

in which for simplicity, we have defined

$$\begin{array}{c}\Delta_1(w, \tilde{z}, \tilde{\zeta}, \tilde{x}_s, \tilde{\eta}, \tilde{y}) = \tilde{\bar{q}}(w, \tilde{z}, \tilde{\zeta}, \tilde{\vartheta}) - C_s(A_s\tilde{x}_s + B_s\tilde{\eta}_1) \\ -D_s(\Phi\tilde{\eta} + G\tilde{h}_e(w, \tilde{z}, \tilde{\zeta}, C_s\tilde{x}_s + D_s\tilde{\eta}_1 + \tilde{y}))\end{array}$$

with $\Delta_1(w, 0, 0, 0, 0, 0) = 0$. Note that the manifold $\mathcal{M}$ is attractive, if system (24) is shown to be asymptotically stable at the origin.

With this in mind, for convenience set $Z = (\tilde{z}, \tilde{\zeta}, \tilde{x}_s, \tilde{\eta})$. The interconnected system (24) can be rewritten as

$$\begin{array}{rcl}\dot{Z} & = & F_0(w(t), Z, \tilde{y}) \\ \dot{\tilde{y}} & = & \Delta_1(w(t), Z, \tilde{y}) - b(w(t), z, x)\kappa\tilde{y}\,.\end{array} \tag{25}$$

Clearly, system (25) can be viewed as an interconnected time-varying system, which consists of two time-varying subsystems: the Z subsystem and the $\tilde{y}$ subsystem. It follows that the desired objective is achieved, if given some compact sets $\mathcal{W}, \mathcal{Z}, \mathcal{A}$, for any $(w(0), Z(0), \tilde{y}(0)) \in \mathcal{W}\times\mathcal{Z}\times\mathcal{A}$ system (25) is asymptotically stable at the equilibrium $(Z, \tilde{y}) = (0, 0)$. In view of this, we will show that if (A_s, B_s, C_s, D_s) are chosen appropriately, then asymptotic stability of system (25) can be achieved by choosing large enough κ. To show this, a fundamental assumption is made as follows, whose feasibility will be discussed in next section.

Assumption 3: There exists (A_s, B_s, C_s, D_s) such that the system

$$\dot{Z} = F_0(w(t), Z, 0) \tag{26}$$

is exponentially stable at $Z = 0$ with some domain of attraction $\mathcal{K} \supseteq \mathcal{Z}$, uniformly in $w(t) \in \mathcal{W}$ for a compact set $\mathcal{W}$. That is, for any initial condition $Z(0) \in \mathcal{K}$, there exist a candidate Lyapunov function $V_0(Z)$, and a positive constant $\alpha_0 > 0$, independent of κ such that

$$\frac{\partial V_0}{\partial Z} F_0(w(t), Z, 0) \leq -\alpha_0 V_0(Z)$$

for all $w(t) \in \mathcal{W}$.

As a consequence, the following result can be formulated.

Theorem 1: Consider the system (8) with the regulator (14)-(15). Suppose Assumptions 1–3 hold, A_s is Hurwitz, and (A_s, B_s, C_s, D_s) is such that the nonresonance condition (21) is satisfied. Then there exists $\kappa^* > 0$ such that for all $\kappa \geq \kappa^*$, all the resulting trajectories of closed-loop system are bounded and $\lim_{t\to\infty} \|e(t)\| = 0$ with the attraction of domain containing a compact set $\mathcal{W} \times \mathcal{X} \times \mathcal{X}_c$.

Moreover, if the system (26) is also globally asymptotically stable, then it is possible to claim that the system (25) can be semiglobally asymptotically stabilized at the equilibrium $(Z, \tilde{y}) = (0, 0)$ (see [12], [10]). More explicitly, the following result can be concluded.

Corollary 1: Consider the system (8) with the regulator (14)-(15). Suppose Assumptions 1–2 hold, and choose A_s to be Hurwitz and (A_s, B_s, C_s, D_s) such that the nonresonance condition (21) is satisfied. Suppose the equilibrium point of the system (26) is uniformly globally asymptotically stable and also locally exponentially stable. Then for every choice of compact set $\mathcal{W} \times \mathcal{X} \times \mathcal{X}_c$, there exists a $\kappa^* > 0$ such that for all $\kappa \geq \kappa^*$, all the resulting trajectories of closed-loop system are bounded and $\lim_{t\to\infty} \|e(t)\| = 0$ with a domain of attraction that contains $\mathcal{W} \times \mathcal{X} \times \mathcal{X}_c$.

Remark 3: Note that the regulator (14)-(15) utilizes the information of regulated output e and ϑ, which is a linear function of partial state $\xi = \mathrm{col}(\xi_1, \ldots, \xi_r)$. If only ξ_1, instead of the whole vector ξ is measurable, to achieve the desired measurement feedback output regulation, the variable ϑ can be replaced by

$$\hat{\vartheta} = \hat{x}_r + \sum_{j=1}^{r-1} c_j \hat{x}_j \tag{27}$$

in which $\hat{x}_j$s are estimates provided by the rough high-gain observer of the form

$$\begin{array}{rcl} \dot{\hat{x}}_j &=& \hat{x}_{j+1} + \ell^j a_j (x_1 - \hat{x}_1)\,, \quad 1 \leq j \leq r-1 \\ \dot{\hat{x}}_r &=& \ell^r a_r (x_1 - \hat{x}_1) \end{array} \tag{28}$$

in which a_j's are such that the polynomial $\mathcal{P}(\lambda) = \lambda^r + a_r \lambda^{r-1} + \ldots + a_2 \lambda + a_1$ is Hurwitz and ℓ is a design parameter.

Thus according to [12], [10], by choosing sufficiently large ℓ and appropriately saturating the control (15) to avoid the occurrence of finite escape times, one is able to obtain that in the resulting closed-loop system, the regulated output $e(t)$ practically decays to zero.

C. Feasibility Analysis of Assumption 3

In this section, we will investigate the feasibility of Assumption 3. Let us have a look at the dynamics

$$\dot{Z} = F_0(0, Z, 0) \tag{29}$$

Thus by linearizing (26) at $Z = 0$, we can obtain

$$\begin{array}{rcl} \dot{\tilde{z}} &=& F_0 \tilde{z} + F_0' \tilde{\zeta} + G_1 (C_s \tilde{x}_s + D_s \tilde{\eta}_1) \\ \dot{\tilde{\zeta}} &=& F \tilde{\zeta} + B (C_s \tilde{x}_s + D_s \tilde{\eta}_1) \\ \dot{\tilde{\eta}} &=& \Phi \tilde{\eta} + G[H_0 \tilde{z} + H_1 \tilde{\zeta} + K (C_s \tilde{x}_s + D_s \tilde{\eta}_1)] \\ \dot{\tilde{x}}_s &=& A_s \tilde{x}_s + B_s \tilde{\eta}_1 \end{array} \tag{30}$$

This linear closed-loop system (30) actually can be viewed as an interconnected system, composing of the driven system

$$\begin{array}{rcl} \dot{\tilde{z}} &=& F_0 \tilde{z} + F_0' \tilde{\zeta} + G_1 u_a \\ \dot{\tilde{\zeta}} &=& F \tilde{\zeta} + B u_a \\ \dot{\tilde{\eta}} &=& \Phi \tilde{\eta} + G[H_0 \tilde{z} + H_1 \tilde{\zeta} + K u_a] \\ \tilde{\mathrm{y}}_a &=& \tilde{\eta}_1 \end{array} \tag{31}$$

where $\tilde{\mathrm{y}}_a$ is the output, u_a is the feedback control, and the driving system (an output feedback stabilizer)

$$\begin{array}{rcl} \dot{\tilde{x}}_s &=& A_s \tilde{x}_s + B_s \tilde{\mathrm{y}}_a \\ u_a &=& C_s \tilde{x}_s + D_s \tilde{\mathrm{y}}_a\,. \end{array} \tag{32}$$

In view of this, the problem in question can be transformed into the problem of checking whether there exists such an output feedback stabilizer (32) such that system (31) is stabilized at origin. As a matter of fact, system (31) is able to be stabilized by the output feedback controller (32), if system (30) is stabilizable by state feedback and detectable. In this way, the following lemma is made, whose proof can be derived by applying the PBH test and it is omitted for reasons of space.

Lemma 2: Suppose F_0 is a Hurwitz matrix, then the system (30) is stabilizable by state feedback and detectable if and only if the non-resonance condition

$$\det \begin{pmatrix} H_0 & H_1 & K \\ F_0 - \lambda I & F_0' & G_1 \\ 0 & F - \lambda I & B \end{pmatrix} \neq 0\,, \qquad \forall \lambda \in \sigma(\bar{S}) \tag{33}$$

holds.

With this lemma in mind, if F_0 is a Hurwitz matrix and the non-resonance condition (33) holds, then there exists (A_s, B_s, C_s, D_s) such that system (26) is locally exponentially stable.

IV. Continuation of the Motivating Example

In this section, we are going to continue solving the output regulation problem for the motivating example.

For system (1), operate the reduction to vector relative 1 by redesigning the new variable

$$\tilde{\vartheta} = \tilde{x}_1 + \tilde{x}_2\,.$$

Then, design the internal model having the form (14)-(15) with

$$A_s = \begin{pmatrix} -3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 11 & -4 & 0 & -2 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 4.4 & -5.6 & 3.6 & -4 \end{pmatrix}, \quad B_s = \begin{pmatrix} 3 \\ -1 \\ -11 \\ 0 \\ 0 \end{pmatrix},$$
$$C_s = \begin{pmatrix} 1 & 4.4 & -5.6 & 3.6 & -3 \end{pmatrix}, \qquad D_s = 0$$

By simple calculations, one can find that A_s is Hurwitz and the condition (21) is satisfied. Moreover, the resulting dynamics (29) is globally exponentially stable, which further implies that the desired output regulation is globally achieved.

The simulation is done with unknown parameter $b = 1$, high gain $\kappa = 30$, initial conditions $(w_1(0), w_2(0)) = (1, 0)$, $(x_1(0), x_2(0), x_3(0)) = (2, -1, 3)$, and all the other initial conditions being zero. The simulation results are given in Figure 1.

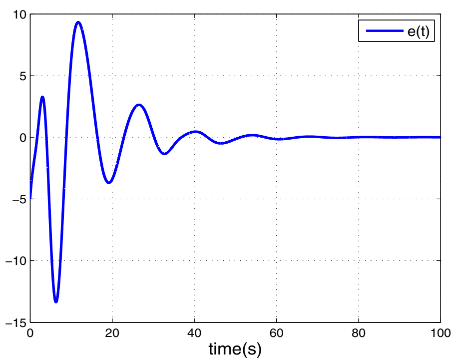

Fig. 1. Trajectory of regulated output e

V. Conclusion

This paper deals with output regulation problems for a class of nonlinear systems. The normal form utilized in this paper is not defined on the conventional *regulated output*, which is quite different from the existing results. With the augment of a novel internal model, a high-gain feedback law is established such that the resulting augmented system can be (locally) asymptotically stabilized to zero under certain assumptions, i.e., the desired robust output regulation is achieved. Note that the proposed regulator is linear, which to some extent restricts its application to deal with more general class of nonlinear systems and to obtain global results. The future research related to this paper might be developing a more general nonlinear dynamical controller so as to deal with a broad class of nonlinear systems such as lateral-vertical plane and mobile robots.

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