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Distributed Secondary Frequency Control Design for Microgrids: Trading Off L_2-Gain Performance and Communication Efforts under Time-Varying Delays

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Abstract— **Consensus algorithms are promising control schemes for secondary control tasks in microgrids. Since consensus algorithms are distributed protocols, communication efforts and time delays are significant factors for the control design and performance. Moreover, both the electrical and the communication layer in a microgrid are continuously exposed to exogenous disturbances. Motivated by this, we derive a synthesis for a consensus-based secondary frequency controller that guarantees robustness with respect to time-varying delays and in addition provides the option to trade off L_2 disturbance attenuation against the number of required communication links. The efficacy of the proposed approach is illustrated via simulations based on the CIGRE benchmark medium voltage distribution network.**

I. Introduction

A. Motivation and Related Work

The rapid deployment of distributed renewable energy sources poses tremendous challenges for power system control and operation [1]. In particular, the replacement of a few bulk conventional power plants with a large number of small-scale renewable generators significantly increases the complexity of coordinating demand and generation in real-time. Clearly, in such a setting centralized solutions are inappropriate and instead distributed architectures need to be developed. In that spirit, the microgrid (MG) concept has been identified as a core element of future power systems [1], [2]. A MG is a small-scale power system, which is composed of a combination of distributed generation units, energy storage devices and loads at the distribution level, with the ability to operate either in grid-connected mode or islanded mode [1], [3]. Thus, future power systems could be operated as a cell-structure of interconnected MGs.

For this type of networks many new control challenges arise [4]. Amongst these, frequency regulation is a fundamental operational objective [2], [4]. As in bulk power systems, in MGs this objective is typically realized via a hierarchical control layer consisting of primary, secondary and tertiary control [4], [5]. While the primary controllers are usually implemented in a completely decentralized manner [4], the secondary control layer requires (distributed) integral action, in order to restore the frequency to its nominal value following a change in the systems' power balance [4], [5].

In recent years, distributed consensus-based algorithms have gained increasing popularity for secondary frequency control in MGs [6]–[10]. Consensus protocols are distributed protocols and peer-to-peer communication between participating units is essential for their implementation. Thus, the design of the communication network and the robustness of the closed-loop system with respect to communication uncertainties, such as time delays and exogenous disturbances, is of paramount importance [2]. Likewise, the electrical layer of the MG is continuously exposed to perturbations, e.g., in the power demand. Robustness of consensus-based secondary controllers with respect to delays has been investigated in [11]–[13], but the analysis is either limited to a linearized (small-signal) model or does not consider the electrical dynamics and is partially restricted to constant delays. Bounded input-output performance of linearized models of secondary controlled MGs has been considered using the $\mathcal{H}_2$-norm in [10] and the $\mathcal{H}_\infty$-norm in [14]. A very similar setup for bulk power systems with distributed frequency control is employed in [9], where in addition to minimizing the $\mathcal{H}_2$-norm also sparsity of the communication network is promoted. Yet, the simultaneous consideration of these three objectives has not been reported in the literature.

B. Contributions

As a consequence of the above discussion and extending our previous work on delay-robust stability analysis [15], the main contribution of the present paper is a design procedure for consensus-based secondary frequency controllers in MGs, which jointly considers the objectives of delay robustness, bounded L_2-gain performance for disturbance attenuation (i.e., the maximum energy amplification ratio of the system) and sparsity of the communication network.

Inspired by related work on sparsity-promoting control for power systems and MGs [9], [16], [17], we use the (re)weighted ℓ_1-norm as a proxy for the sparsity of the communication network, see also [18]. Then we formulate the proposed robust and sparsity-promoting control synthesis as a convex optimization problem, which is derived for a nonlinear MG model via the Lyapunov-Krasovskii and the descriptor methods [19]. The employed cost function provides the option to trade off L_2-gain performance against the number of communication links.

Compared to an analysis based on linearization, our design criterion is equilibrium-independent (besides the usual requirement that the stationary angle differences don't exceed

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$|\frac{\pi}{2}|$). Thus, if it is feasible, the desired performance specifications hold true in a wide range of operating conditions. This is illustrated via numerical experiments using the CIGRE benchmark medium voltage (MV) distribution network [20].

Notation. We define the sets $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} | x \geq 0\}$, $\mathbb{R}_{>0} := \{x \in \mathbb{R} | x > 0\}$ and $\mathbb{R}_{<0} := \{x \in \mathbb{R} | x < 0\}$. For a set $\mathcal{V}$, $|\mathcal{V}|$ denotes its cardinality and $[\mathcal{V}]^k$ denotes the set of all subsets of $\mathcal{V}$ that contain k elements. Let $x := \mathrm{col}(x_i) \in \mathbb{R}^n$ denote a vector with entries x_i for $i = 1, \ldots, n$, $\mathbb{1}_n$ the vector with all entries equal to one, I_n the $n \times n$ identity matrix, $\mathbb{0}$ a zero matrix of appropriate dimensions and $\mathrm{diag}(a_i), i = 1, \ldots, n$ an $n \times n$ diagonal matrix with diagonal entries $a_i \in \mathbb{R}$. For $A \in \mathbb{R}^{n \times n}$, $A > 0$ ($A < 0$) means that A is symmetric positive (negative) definite. The lower-diagonal elements of a symmetric matrix are denoted by $*$. We denote by $W[-h, 0]$, $h \in \mathbb{R}_{>0}$, the Banach space of absolutely continuous functions $\phi : [-h, 0] \to \mathbb{R}^n$, $h \in \mathbb{R}_{>0}$, with $\dot{\phi} \in L_2(-h, 0)^n$ and with the norm $\|\phi\|_W = \max_{\theta \in [a,b]} |\phi(\theta)| + \left(\int_{-h}^{0} \dot{\phi}^2 d\theta\right)^{0.5}$. Also, ∇f denotes the gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$.

II. Preliminaries

A. L_2-Gain of Dissipative Systems

We briefly recall some standard results on dissipative systems based on [21], [22]. Consider the state space system

$$\begin{aligned} \dot{x} &= f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \\ y &= h(x, u), \quad y \in \mathbb{R}^p. \end{aligned} \tag{II.1}$$

Definition 2.1: [21] The state space system (II.1) is dissipative with respect to the supply rate $s : \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}$ if there exists a function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, called the storage function, such that for all $x_0 \in \mathbb{R}^n$, all $t_1 \geq t_0$ and all input functions u,

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t)) dt.$$

Definition 2.2: [21] The system (II.1) has a L_2-gain less than or equal to γ if it is dissipative with respect to the supply rate $s(u, y) = \frac{1}{2}(\gamma^2 \|u\|^2 - \|y\|^2)$.
Based on [22, Definition 6.2], we employ the following notion of a *small-signal* L_2-gain.

Definition 2.3: The system (II.1) has a small-signal L_2-gain less than or equal to γ if it is dissipative with respect to the supply rate $s(u, y) = \frac{1}{2}(\gamma^2 \|u\|^2 - \|y\|^2)$ for all $u \in L_2$ with $\sup_{0 \leq t \leq \tau} \|u\| \leq r$ for some positive real constant r.

B. Algebraic Graph Theory

An undirected weighted graph of order n is a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, z)$, with set of nodes $\mathcal{V} = \{1, \ldots, n\}$, set of undirected edges $\mathcal{E} \subseteq [\mathcal{V}]^2$, $\mathcal{E} = \{e_1, \ldots, e_m\}$, $m = |\mathcal{E}|$ and weight function $z : \mathcal{E} \to \mathbb{R}_{\geq 0}$. By associating an arbitrary ordering to the edges, the node-edge incidence matrix $\mathcal{B} \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ of an undirected graph is defined element-wise as $b_{il} = 1$, if node i is the source of the l-th edge e_l, $b_{il} = -1$, if i is the sink of e_l and $b_{il} = 0$ otherwise. The Laplacian matrix of an undirected weighted graph is given by [23], [24]

$$\mathcal{L} = \mathcal{B} \mathcal{Z} \mathcal{B}^\top, \quad \mathcal{Z} = \mathrm{diag}\left(z_l\right), \tag{II.2}$$

where $z_l \geq 0$ is the weight of the l-th edge, $l = 1, \ldots, m$. An ordered sequence of nodes such that any pair of consecutive nodes in the sequence is connected by an edge is called a path. A graph $\mathcal{G}$ is called connected if for all pairs $\{i, k\} \in [\mathcal{V}]^2$ there exists a path from i to k. The Laplacian matrix $\mathcal{L}$ of an undirected graph is positive semidefinite with a simple zero eigenvalue if and only if the graph is connected. The corresponding right eigenvector to this simple zero eigenvalue is $\mathbb{1}_n$, i.e., $\mathcal{L}\mathbb{1}_n = \mathbb{0}_n$ [24]. We refer the reader to [23], [24] for further information on graph theory.

III. Microgrid Model with Distributed Secondary Frequency Control and Time Delay

A. Microgrid Model

We consider a Kron-reduced representation of a MG with mixed generation pool consisting of rotational and electronic-interfaced units [5], [25]. The set of nodes is denoted by $\mathcal{N} = \{1, \cdots, n\}$, $n \geq 1$. Following standard practice [5], [6], [15], we assume that the line admittances are purely inductive and that the voltage amplitudes $V_i \in \mathbb{R}_{>0}$ at all nodes are constant. This assumption is admissible in MG analysis, since the inverter output impedance is typically highly inductive [6], [26]. Then, two nodes i and k are connected via a non-zero susceptance $B_{ik} \in \mathbb{R}_{<0}$. If there is no line between i and k, then $B_{ik} = 0$. We denote the set of neighboring nodes of node i by $\mathcal{N}_i = \{k \in \mathcal{N} | B_{ik} = 0\}$. We assume that for all $\{i, k\} \in [\mathcal{N}]^2$ there exists an ordered sequence of nodes from i to k such that any pair of consecutive nodes in the sequence is connected by a power line represented by an admittance, i.e., the electrical network is connected. We assign to each node a phase angle $\theta_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and a frequency $\omega_i = \dot{\theta}_i$ and define $\theta = \mathrm{col}(\theta_i)$ and $\omega = \mathrm{col}(\omega_i)$. With the potential function $U : \mathbb{R}^n \to \mathbb{R}$,

$$U(\theta) = - \sum_{\{i,k\} \in [\mathcal{N}]^2} |B_{ik}| V_i V_k \cos(\theta_{ik}),$$

the active power flows $P : \mathbb{R}^n \to \mathbb{R}^n$ can be written as

$$P(\theta) = \nabla U(\theta).$$

Furthermore, we assume that all units are equipped with the standard frequency droop controller [4], [5], [25]. Then, the MG dynamics can be compactly written as [25], [26]

$$\begin{aligned} \dot{\theta} &= \omega, \\ M\dot{\omega} &= -D(\omega - \mathbb{1}_n \omega^d) - \nabla U(\theta) + P^{\text{net}} + u, \end{aligned} \tag{III.1}$$

where $D = \mathrm{diag}(D_i) \in \mathbb{R}^n_{>0}$ is the matrix of (inverse) droop coefficients, $\omega^d \in \mathbb{R}_{>0}$ is the reference frequency and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is the secondary frequency control input. Moreover, the matrix of (virtual) inertia coefficients is given by $M = \mathrm{diag}(M_i) \in \mathbb{R}^n_{>0}$, where for any inverter-interfaced unit $M_i = \tau_{p_i} D_i$ with $\tau_{p_i} \in \mathbb{R}_{>0}$ being the time constant of the power measurement filter. In addition, P^{net} is given by $P^{\text{net}} = \mathrm{col}(P_i^d - G_{ii} V_i^2)$, where $P_i^d \in \mathbb{R}$ denotes the active power set point and $G_{ii} V_i^2$, $G_{ii} \in \mathbb{R}_{\geq 0}$, represents the active power demand at the i-th node. See [3] for further details on the modeling of the system components.

B. Secondary Frequency Control: Objectives and Distributed Control Scheme

Suppose that the solutions of the system (III.1) evolve along a motion with constant frequency $\omega^s = \mathbb{1}_n \omega^*$, $\omega^* \in \mathbb{R}$.

Then, summing over all frequency dynamics yields

$$\mathbb{1}_n^\top M\dot{\omega}^s = 0 \quad \Rightarrow \quad \omega^* = \omega^d + \frac{\mathbb{1}_n^\top P^{\text{net}} + \mathbb{1}_n^\top u^*}{\mathbb{1}_n^\top D\mathbb{1}_n}, \quad \text{(III.2)}$$

where we have used the fact that $\mathbb{1}_n^\top \nabla U(\theta) = 0$. A standard requirement in power system operation is that $\omega^* = \omega^d$, i.e., the network synchronizes to the nominal frequency [4], [5]. However, in practice, the load demands $G_{ii}V_i^2$ contained in P^{net} are unknown and thus, typically, $\mathbb{1}_n^\top P^{\text{net}} \neq 0$. Therefore, the control inputs u^* have the task to compensate this power imbalance such that indeed $\omega^* = \omega^d$, see (III.2). This task is termed secondary frequency control [4], [5].

Let $A \in \mathbb{R}^{n\times n}_{>0}$ be a diagonal positive definite weighting matrix, $K \in \mathbb{R}^{n\times n}_{>0}$ be a diagonal feedback gain matrix and $\mathcal{L} \in \mathbb{R}^{n\times n}$ be the Laplacian matrix of an undirected and connected graph with incidence matrix $\mathcal{B}$ and diagonal matrix of nonnegative edge weights $\mathcal{Z}$, see (II.2). Consider the distributed secondary frequency control [6], [15]

$$u = -p, \quad \dot{p} = K(\omega - \mathbb{1}_n\omega^d) - KA\mathcal{L}Ap(t-\tau), \quad \text{(III.3)}$$

where $\tau : \mathbb{R}_{\geq 0} \to [0, h]$, $h \in \mathbb{R}_{\geq 0}$, denotes a fast-varying delay [19], [27]. Physically, this delay represents communication delays between different nodes in the network. As a consequence of digital control [19] and the communication network conditions [28] this delay may be time-varying.

It has been shown in [29], [30], that the control (III.3) restores the frequency to its nominal value, while ensuring economic optimality in a synchronized state, i.e.,

$$A_{ii}u_i^s = A_{kk}u_k^s \quad \forall i \in \mathcal{N}, \quad \forall k \in \mathcal{N}.$$

Thus, usually the matrix A is fixed by economic considerations. Hence, given (III.3), the distributed secondary control design problem consists in suitably determining the matrices K and $\mathcal{L}$. This problem is addressed in the present paper.

C. Closed-Loop System

Combining (III.1) with (III.3) yields

$$\begin{aligned}\dot{\theta} &= \omega,\\ M\dot{\omega} &= -D(\omega - \mathbb{1}_n\omega^d) + P^{\text{net}} - p - \nabla U(\theta), \qquad \text{(III.4)}\\ \dot{p} &= K(\omega - \mathbb{1}_n\omega^d) - KA\mathcal{L}Ap(t-\tau).\end{aligned}$$

For the subsequent controller synthesis, the following notion is useful, see also [15], [26].

Definition 3.1: The system (III.4) admits a synchronized motion if it has a solution for all $t \geq 0$ of the form

$$\theta^s(t) = \theta_0^s + \omega^s t, \quad \omega^s = \omega^*\mathbb{1}_n, \quad p^s \in \mathbb{R}^n,$$

where $\omega^* \in \mathbb{R}$ and $\theta_0^s \in \mathbb{R}^n$ are such that

$$|\theta_{0,i}^s - \theta_{0,k}^s| < \frac{\pi}{2} \quad \forall i \in \mathcal{N}, \ \forall k \in \mathcal{N}_i.$$

It has been shown in [29], [30] that the system (III.4) possesses at most one synchronized motion and that this motion satisfies

$$u^s = -p^s, \quad p^s = \lambda A^{-1}\mathbb{1}_n, \quad \lambda = \frac{\mathbb{1}_n^\top P^{\text{net}}}{\mathbb{1}_n^\top A^{-1}\mathbb{1}_n}.$$

IV. Controller Synthesis

A. Coordinate Transformation and Error System

Following the approach in [15], we perform both a coordinate transformation and reduction that are instrumental to our synthesis. Let $K = \kappa\mathcal{K}$, where $\mathcal{K} \in \mathbb{R}^{n\times n}$ is a diagonal matrix with positive diagonal entries and $\kappa > 0$ is a parameter. Note that the fact that $\mathcal{L}\mathbb{1}_n = \mathbb{0}_n$ yields to an invariant subsystem in the p-variables. Consider the change of coordinates

$$\begin{bmatrix}\bar{p}\\ \zeta\end{bmatrix} = \mathcal{W}^\top(\kappa\mathcal{K})^{-\frac{1}{2}}p, \ \ \mathcal{W} = \begin{bmatrix} W & \frac{1}{\sqrt{\mu}}\mathcal{K}^{-\frac{1}{2}}A^{-1}\mathbb{1}_n\end{bmatrix}, \quad \text{(IV.1)}$$

where $W \in \mathbb{R}^{n\times(n-1)}$ is chosen such that $W^\top\mathcal{K}^{-\frac{1}{2}}A^{-1}\mathbb{1}_n = \mathbb{0}_{n-1}$ and $\mu = \|\mathcal{K}^{-\frac{1}{2}}A^{-1}\mathbb{1}_n\|_2^2$. Hence, the column vectors of W form an orthonormal basis that is orthogonal to $\mathcal{K}^{-\frac{1}{2}}A^{-1}\mathbb{1}_n$. Thus, the transformation matrix $\mathcal{W} \in \mathbb{R}^{n\times n}$ is orthogonal. By using (IV.1) and following the procedure in [15, Section 3.2], we can represent the closed-loop system (III.4) in new reduced order coordinates by

$$\begin{aligned}\dot{\theta} &= \omega,\\ M\dot{\omega} &= -D(\omega - \mathbb{1}_n\omega^d) + P^{\text{net}} - \nabla U(\theta) - (\kappa\mathcal{K})^{\frac{1}{2}}W\bar{p}\\ &\quad -\frac{\kappa}{\mu}A^{-1}\mathbb{1}_n(\mathbb{1}_n^\top A^{-1}(\theta - \theta_0 - \mathbb{1}_n\omega^d t + \kappa^{-1}\mathcal{K}^{-1}p_0),\\ \dot{\bar{p}} &= \kappa^{\frac{1}{2}}W^\top\mathcal{K}^{\frac{1}{2}}(\omega - \mathbb{1}_n\omega^d) - \kappa W^\top\mathcal{K}^{\frac{1}{2}}A\mathcal{L}A\mathcal{K}^{\frac{1}{2}}W\bar{p}(t-\tau),\end{aligned} \quad \text{(IV.2)}$$

where we have expressed the variable ζ in (IV.1) in terms of θ, ω^d, θ_0 and p_0, see [15] for details. We make the following standard assumption [15], [26].

Assumption 4.1: The system (IV.2) possesses a synchronized motion.

With Assumption 4.1, we define the error states

$$\begin{aligned}\tilde{\omega} &= \omega - \omega^s, \quad \tilde{\theta} = \theta_0 - \theta_0^s + \int_0^t \tilde{\omega}(\tau)d\tau,\\ \tilde{p} &= \bar{p} - \bar{p}^s, \quad x = \text{col}(\tilde{\theta}, \tilde{\omega}, \tilde{p}).\end{aligned}$$

Then, the error system corresponding to (IV.2) is given by

$$\begin{aligned}\dot{\tilde{\theta}} &= \tilde{\omega},\\ M\dot{\tilde{\omega}} &= -D\tilde{\omega} - \nabla U(\tilde{\theta} + \theta^s) + \nabla U(\theta^s) - (\kappa\mathcal{K})^{\frac{1}{2}}W\tilde{p}\\ &\quad -\frac{1}{\mu}\kappa A^{-1}\mathbb{1}_n\mathbb{1}_n^\top A^{-1}\tilde{\theta} + d_\omega,\\ \dot{\tilde{p}} &= \kappa^{\frac{1}{2}}W^\top\mathcal{K}^{\frac{1}{2}}\tilde{\omega} - \kappa W^\top\mathcal{K}^{\frac{1}{2}}A\mathcal{L}A\mathcal{K}^{\frac{1}{2}}W\tilde{p}(t-\tau) + d_p,\\ y &= \text{col}\left(W_1^{\frac{1}{2}}\tilde{\omega}, W_2^{\frac{1}{2}}\tilde{p}\right), \quad d = \text{col}(d_\omega, d_p),\end{aligned} \quad \text{(IV.3)}$$

where we have added the disturbance inputs d_ω and d_p, as well as—inspired by [9]—defined the performance output y with weighting matrices

$$\begin{aligned}W_1 &= M > 0, \quad W_2 = W^\top\mathcal{K}^{\frac{1}{2}}\bar{W}_2\mathcal{K}^{\frac{1}{2}}W,\\ \bar{W}_2 &= I_n - \frac{1}{\mathbb{1}_n^\top A^{-1}\mathbb{1}_n}A^{-\frac{1}{2}}\mathbb{1}_n\mathbb{1}_n^\top A^{-\frac{1}{2}}.\end{aligned} \quad \text{(IV.4)}$$

Note that W_2 quantifies the deviation of the controller states from their average (scaled by $\kappa^{-1}A^{\frac{1}{2}}$) and W_1 accounts for the system's kinetic energy. Moreover, with Assumption 4.1, the system (IV.3) has an equilibrium point $x^s = \text{col}(\tilde{\theta}^s, \tilde{\omega}^s, \tilde{p}^s)$ at the origin.

B. Problem Statement

Given the secondary control law (III.3), the key problem addressed in the present paper is how to select the controller matrices K and $\mathcal{L}$, such that the closed-loop system possesses desired properties. Compared to existing work, e.g., [6], [8]–[10], our proposed design takes robustness with respect to time-varying delays and external perturbations into account, while minimizing the required communication efforts, i.e., the number of network links.

With regard to the number of communication links, an obvious approach is to, in addition to the L_2-gain, minimize the 0-norm of the vector $\mathcal{Z}\mathbb{1}_m$, i.e., $\|\mathcal{Z}\mathbb{1}_m\|_0 = \{\text{number of } z_i |\, z_i \neq 0\}$ (recall from (II.2) that $\mathcal{Z} \geq 0$ is a diagonal matrix). Yet, the difficulty in using this approach is that the problem is non-convex. To overcome the non-convexity, the ℓ_1-norm $\|\mathcal{Z}\mathbb{1}_m\|_1 = \sum_{i=1}^m |z_i|$ is often used as a convex relaxation of the 0-norm [9], [17], [18]. To improve this relaxation, the reweighted ℓ_1-norm $\|W_{\mathcal{Z}}\mathcal{Z}\mathbb{1}_m\|_1$ can be used [18], where the diagonal entries of the diagonal matrix $W_{\mathcal{Z}}$ are chosen as $w_{\mathcal{Z},i} = (z_i + \upsilon)^{-1}$, $i = 1, \dots, m$, with υ being a small positive number. This, however, implies that an iteration scheme is needed, since the assigned values of the weighting matrix $W_{\mathcal{Z}}$ depend on the solution of the optimization problem. Alternatively, in the MG case power system engineering insights could be used to determine the weighting matrix $W_{\mathcal{Z}}$, see also [17]. The above discussion leads to the following problem statement.

Problem 4.2: Consider the system (IV.3) with Assumption 4.1. Determine κ and $\mathcal{Z}$, such that given $h \in \mathbb{R}_{\geq 0}$ with $\tau \leq h$, $x^s = \mathbb{0}_{3n-1}$ is a uniformly asymptotically stable equilibrium point of the system (IV.3), the system (IV.3) is dissipative with respect to the supply rate $s(d,y) = \frac{1}{2}(\gamma^2\|d\|_2^2 - \|y\|_2^2)$, where d and y are given in (IV.3), and the number of communication links is minimized, i.e., $\min_{\mathcal{Z}\geq 0} \operatorname{trace}(\mathcal{Z})$.

C. Main Result

In this section, we provide a solution to Problem 4.2. Recall the definition of $\mathcal{L}$ in (II.2). To present our main result, it is convenient to introduce the scaled matrix of edge weights and the corresponding scaled Laplacian matrix, i.e.,

$$\bar{\mathcal{Z}} = \kappa\mathcal{Z}, \qquad \bar{\mathcal{L}} = \mathcal{K}^{\frac{1}{2}} A\mathcal{B}\bar{\mathcal{Z}}\,\mathcal{B}^\top A\mathcal{K}^{\frac{1}{2}}. \tag{IV.5}$$

Proposition 4.3: Consider the system (IV.3) with Assumption 4.1. Recall the weighting matrices W_1 and W_2 given in (IV.4). Fix $h \geq 0$, $\mathcal{K} > 0$ and $\varepsilon > 0$ as well as weighting parameters $\alpha > 0$, $\beta > 0$ and a diagonal weighting matrix $W_{\mathcal{Z}} > 0$. Suppose that there exist parameters $\bar{\kappa} > 0$ and matrices $\bar{\mathcal{Z}} \geq 0$, $R > 0$, $S > 0$ and S_{12}, such that the following optimization problem is feasible:

$$\min_{\bar{\gamma},\bar{\kappa},\bar{\mathcal{Z}}} \; \alpha\,\bar{\gamma} - \beta\,\bar{\kappa} + \operatorname{trace}\left(W_{\mathcal{Z}}\bar{\mathcal{Z}}\right)$$

subject to

$$Q = \begin{bmatrix} Q_{11} & \mathbb{0} & Q_{13} & \mathbb{0} & \mathbb{0} & \frac{1}{2}I_n & \mathbb{0} \\ * & Q_{22} & -\frac{1}{4}I_{n-1} & S_{12} & Q_{25} & \mathbb{0} & \frac{1}{2}I_{n-1} \\ * & * & Q_{33} & \mathbb{0} & Q_{35} & \mathbb{0} & \frac{1}{4}\varepsilon I_{n-1} \\ * & * & * & Q_{44} & Q_{45} & \mathbb{0} & \mathbb{0} \\ * & * & * & * & Q_{55} & \mathbb{0} & \mathbb{0} \\ * & * & * & * & * & Q_{66} & \mathbb{0} \\ * & * & * & * & * & * & Q_{77} \end{bmatrix} < 0, \tag{IV.6}$$

$$\begin{bmatrix} R & S_{12} \\ * & R \end{bmatrix} \geq 0,$$

where

$$Q_{11} = -D + 0.5W_1,\ Q_{13} = 0.25\varepsilon\bar{\kappa}\mathcal{K}^{\frac{1}{2}}W,\ Q_{22} = S - R + 0.5W_2,$$
$$Q_{25} = R - S_{12} - 0.5W^\top\bar{\mathcal{L}}W,\ Q_{33} = -0.5\varepsilon I_{n-1} + h^2 R,$$
$$Q_{35} = -0.25\varepsilon W^\top\bar{\mathcal{L}}W,\ Q_{44} = -S - R,\ Q_{45} = R - S_{12}^\top,$$
$$Q_{55} = -2R + S_{12} + S_{12}^\top,\ Q_{66} = -0.5\bar{\gamma}I_n,\ Q_{77} = -0.5\bar{\gamma}I_{n-1}.$$

Choose the controller parameters as

$$\kappa = \bar{\kappa}^2, \quad \mathcal{L} = \frac{1}{\kappa}\mathcal{B}\bar{\mathcal{Z}}\mathcal{B}^\top. \tag{IV.7}$$

Then, for all $\tau(t) \in [0,h]$, the origin is a locally uniformly asymptotically stable equilibrium point of the system (IV.3) and the system has a small-signal L_2-gain less than or equal to $\gamma = \sqrt{\bar{\gamma}}$ with respect to the supply rate $s(d,y) = \frac{1}{2}\left(\gamma^2\|d\|_2^2 - \|y\|_2^2\right)$, where d and y are given in (IV.3).

Proof: The proof is established by combining ideas of the related stability analysis conducted in [15] with the control design approach using the descriptor method, which has been applied previously to linear time-delay systems, see, e.g., [19]. By noting that the delay appears only in $\tilde{p}$, consider the Lyapunov-Krasovskii functional (LKF)

$$\begin{aligned} V(x,\dot{x},t) = {}& \frac{1}{2}\tilde{\omega}^\top(t)M\tilde{\omega}(t) + U(\tilde{\theta}(t)+\theta^s) - \nabla U(\theta^s)^\top\tilde{\theta}(t) \\ &+ \frac{1}{4}\tilde{p}^\top(t)\tilde{p}(t) + \epsilon\tilde{\omega}^\top(t)M\mathbb{1}_n\mathbb{1}_n^\top A^{-1}\tilde{\theta}(t) \\ &+ \epsilon\tilde{\omega}^\top(t)AM\left(\nabla U(\tilde{\theta}(t)+\theta^s) - \nabla U(\theta^s)\right) \\ &+ \frac{\kappa}{2\mu}(\mathbb{1}_n^\top A^{-1}\tilde{\theta}(t))^2 + \int_{t-h}^{t}\tilde{p}^\top(s)S\tilde{p}(s)ds \\ &+ h\int_{-h}^{0}\int_{t+\phi}^{t}\dot{\tilde{p}}^\top(s)R\dot{\tilde{p}}(s)dsd\phi, \end{aligned} \tag{IV.8}$$

where $\epsilon > 0$, $S > 0$, $R > 0$ and $\phi \in [-h, 0]$. It follows in a straightforward manner from [15, Proposition 7] that with Assumption 4.1 there always exists an ϵ, such that V is positive definite in a neighborhood of the origin. We seek to design controller gains, such that the L_2-gain of the system (IV.3) is minimized while also ensuring delay robustness. Following [19], we at first set $\epsilon = 0$ in (IV.8). Then differentiating V yields

$$\begin{aligned} \dot{V} = {}& -\tilde{\omega}^\top(t)D\tilde{\omega}(t) - \frac{1}{2}\kappa^{\frac{1}{2}}\tilde{\omega}^\top(t)\mathcal{K}^{\frac{1}{2}}W\tilde{p}(t) + \tilde{\omega}^\top(t)d_\omega(t) \\ &+ \frac{1}{2}\tilde{p}^\top(t)d_p(t) + h^2\dot{\tilde{p}}^\top(t)R\dot{\tilde{p}}(t) + \tilde{p}^\top(t)S\tilde{p}(t) \\ &- \frac{\kappa}{2}\tilde{p}^\top(t)W^\top\mathcal{K}^{\frac{1}{2}}A\mathcal{L}A\mathcal{K}^{\frac{1}{2}}W\tilde{p}(t-\tau) \\ &- h\int_{t-h}^{t}\dot{\tilde{p}}^\top(s)R\dot{\tilde{p}}(s)ds - \tilde{p}^\top(t-h)S\tilde{p}(t-h). \end{aligned} \tag{IV.9}$$

Since under the conditions of the proposition, the second LMI in (IV.6) is feasible, applying Jensen's inequality together with Lemma 3.3 in [19], see also [31], yields

$$\begin{aligned} -h\int_{t-h}^{t}\dot{\tilde{p}}^\top(s)R\dot{\tilde{p}}(s)ds = {}& -h\int_{t-h}^{t-\tau(t)}\dot{\tilde{p}}^\top(s)R\dot{\tilde{p}}(s)ds \\ &- h\int_{t-\tau(t)}^{t}\dot{\tilde{p}}^\top(s)R\dot{\tilde{p}}(s)ds \leq -\eta^\top\begin{bmatrix} R & S_{12} \\ * & R\end{bmatrix}\eta, \end{aligned}$$

where $\eta = \mathrm{col}(\tilde{p}(t) - \tilde{p}(t-\tau(t)), \tilde{p}(t-\tau(t)) - \tilde{p}(t-h))$. Next, we apply the descriptor method, see [19, Chapter 3]. Let $\varepsilon > 0$ and add the expression

$$0 = 0.5\left[\tilde{p}(t)^\top + \varepsilon\dot{\tilde{p}}^\top(t)\right]\Big[\kappa^{\frac{1}{2}}W^\top\mathcal{K}^{\frac{1}{2}}\tilde{\omega}(t)$$
$$-\kappa W^\top\mathcal{K}^{\frac{1}{2}}A\mathcal{L}A\mathcal{K}^{\frac{1}{2}}W\tilde{p}(t-\tau(t)) + d_p(t) - \dot{\tilde{p}}(t)\Big]$$

to (IV.9). This gives

$$\dot{V}(x,\dot{x},t) - \frac{1}{2}\left(\gamma^2\|d(t)\|_2^2 - \|y(t)\|_2^2\right) \le \zeta^\top(t)\mathcal{Q}\zeta(t),$$

where

$$\zeta(t) = \mathrm{col}\left(\tilde{\omega}(t), \tilde{p}(t), \dot{\tilde{p}}(t), \tilde{p}(t-h), \tilde{p}(t-\tau(t)), d_\omega(t), d_p(t)\right)$$

and

$$\mathcal{Q} = \begin{bmatrix} \mathcal{Q}_{11} & \mathbb{0} & \mathcal{Q}_{13} & \mathbb{0} & \mathbb{0} & \frac{1}{2}I_n & \mathbb{0} \\ * & \mathcal{Q}_{22} & -\frac{1}{4}I_{n-1} & S_{12} & \mathcal{Q}_{25} & \mathbb{0} & \frac{1}{2}I_{n-1} \\ * & * & \mathcal{Q}_{33} & \mathbb{0} & \mathcal{Q}_{35} & \mathbb{0} & \frac{1}{4}\varepsilon I_{n-1} \\ * & * & * & \mathcal{Q}_{44} & \mathcal{Q}_{45} & \mathbb{0} & \mathbb{0} \\ * & * & * & * & \mathcal{Q}_{55} & \mathbb{0} & \mathbb{0} \\ * & * & * & * & * & -\frac{1}{2}\gamma^2 I_n & \mathbb{0} \\ * & * & * & * & * & * & -\frac{1}{2}\gamma^2 I_{n-1} \end{bmatrix}, \tag{IV.10}$$

with $\mathcal{Q}_{11} = -D + 0.5W_1$, $\mathcal{Q}_{13} = 0.25\varepsilon(\kappa\mathcal{K})^{\frac{1}{2}}W$, $\mathcal{Q}_{22} = S - R + 0.5W_2$, $\mathcal{Q}_{25} = R - S_{12} - 0.5\kappa W^\top\mathcal{K}^{\frac{1}{2}}A\mathcal{L}A\mathcal{K}^{\frac{1}{2}}W$, $\mathcal{Q}_{33} = -0.5\varepsilon I_{n-1} + h^2R$, $\mathcal{Q}_{35} = -0.25\varepsilon\kappa W^\top\mathcal{K}^{\frac{1}{2}}A\mathcal{L}A\mathcal{K}^{\frac{1}{2}}W$, $\mathcal{Q}_{44} = -S - R$, $\mathcal{Q}_{45} = R - S_{12}^\top$, and $\mathcal{Q}_{55} = -2R + S_{12} + S_{12}^\top$. Furthermore, by recalling $\bar{\mathcal{L}}$ in (IV.5) and defining $\bar{\kappa} = \kappa^{\frac{1}{2}}$, $\bar{\gamma} = \gamma^2$, the matrix $\mathcal{Q}$ in (IV.10) is equivalent to the matrix Q in (IV.6). Note that for $\epsilon = 0$ the time derivative of the LKF is not strict. Yet, under the standing assumptions, $\mathcal{Q} < 0$. Hence, for $\epsilon \neq 0$, $\dot{V}$ can be strictified in a straightforward manner following [15, Proposition 7]. Thus,

$$\dot{V}(x,\dot{x},t) - \frac{1}{2}\left(\gamma^2\|d(t)\|^2 - \|y(t)\|^2\right) \le -\varrho\left(\|x\|_2^2 + \|d(t)\|_2^2\right)$$

for some $\epsilon \in \mathbb{R}_{>0}$ and $\varrho \in \mathbb{R}_{>0}$. By invoking [19, Lemma 4.3] we conclude that the origin of the system (IV.3) is locally uniformly asymptotically stable and that the system has a small-signal L_2-gain less than or equal to $\gamma = \sqrt{\bar{\gamma}}$.

To conclude the proof, we note that the matrix Q in (IV.6) is a LMI in the controller variables $\bar{\kappa}$ and $\bar{\mathcal{L}}$ as well as in the auxiliary variables $\bar{\gamma}$, R, S_{12} and S with additional (fixed) tuning parameter ε. Therefore, sparsity of the communication network can be included in the control design by augmenting the cost function in the optimization problem (IV.6) with the term trace $\left(\bar{\mathcal{Z}}\right)$. This yields the convex optimization problem (IV.6), where we have included additional weighting factors to trade off L_2-gain performance (α) against frequency error convergence[1] (β) and communication efforts ($W_{\mathcal{Z}}$). ■

V. Numerical Example

The performance of the proposed controller synthesis and the inherent design trade-off between the maximum guaranteed L_2-gain and the sparsity of the communication network are illustrated via numerical experiments on the three-phase islanded Subnetwork1 of the CIGRE benchmark MV network [20] shown in Fig. 1.

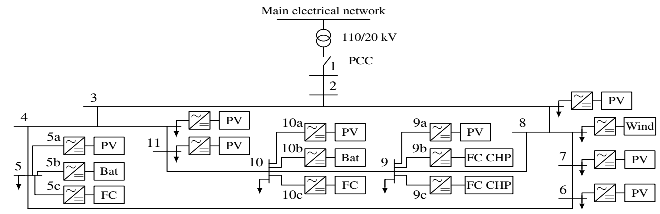

Fig. 1. 20kV MV model with 11 main buses and inverter-interfaced units of type: photovoltaic (PV), fuel cell (FC), battery, combined heat and power (CHP) FC, and wind turbine. The controlled units are located at buses 4, 5b, 5c, 6, 8, 9b, 9c, 10b, 10c and 11. PCC denotes the point of common coupling to the main grid.

The system contains 11 main buses with 15 generation units. The values of the network parameters are mainly taken from [20]. Similarly to [26], the following modifications are made compared to the original system in [20]. At bus 9b, an inverter-interfaced combined heat and power (CHP) fuel cell (FC) is used instead of the CHP diesel generator. Moreover, the power ratings of the controllable generation units (CHPs, batteries, FC, PVs) are scaled by a factor 4 to be able to meet the load demand of the system in islanded mode. In order to integrate the PV units at buses 4, 6, 11 and 7 in the frequency control, we assume that they are operated at 70% of their actual maximum power point and, thus, can increase or decrease their generation. We assume that all controllable units are equipped with frequency droop control.

Non-controlled generation units are connected at buses 3 and 8. The loads in the network represent industrial and household loads. Their data is specified in [20, Table 1]. Moreover, the largest R/X ratio in the reduced admittance matrix is less than 0.3. Thus, the assumption of dominantly inductive admittances is satisfied. The matrix A is chosen as $A = \mathrm{diag}(a_i)$ where $a = \mathrm{col}(0.21, 0.28, 0.56, 0.18, 0.18, 0.26, 0.4, 0.19, 0.3, 0.24)$ (per unit values) and the (inverse) droop gains as $D = 5A$. Also, we set $K = \kappa D$, $\tau_{p_i} = 0.2$s and $P^d = 0.3\,a$. To carry out the secondary control design, i.e., to solve the optimization problem (IV.6), we assume a maximum time delay of $h = 100$ms and set ε=0.3. Then, at first we compute a nominal controller without enforcing any restrictions on the communication network topology. Thus, we set the weighting factors of the objective function in (IV.6) to $\alpha = \beta = 1$ and $W_{\mathcal{Z}} = \mathbb{0}$. The numerical implementation is conducted in Matlab by using Yalmip [32]. This yields a nominal feedback gain of $\kappa = 2.6792$ and a nominal bound for the L_2-gain of $\gamma^* = 0.9637$. The presented results in Fig. 2 illustrate the convergence of the system trajectories to a synchronized motion after being subjected to external perturbations. The fast-varying delay is implemented as a piecewise continuous signal with a sampling time of 2ms.

By taking these values as a benchmark, we redesign the controller with the aim of minimizing the number of communication links, while preserving delay-robustness. To determine $W_{\mathcal{Z}}$ we employ the reweighted ℓ_1-norm, see Section IV-B. In addition, we evaluate the achievable sparsity for different upper bounds $\gamma > \gamma^*$, while keeping κ fixed. Thus, we set $\alpha = \beta = 0$ in (IV.6). As expected the obtained results show a trade-off between the value of γ and the number of non-zero off-diagonal entries of the matrix $\mathcal{Z}$, see Fig. 3.

[1] In our experience, with $\beta = 0$ the numerical value of $\bar{\kappa}$ resulting from the optimization problem is typically very small. This is explained by the fact that $\bar{\kappa}$ only appears in a positive off-diagonal term in Q in (IV.6). Yet, when tested in simulations it turns out that a minimum value of $\bar{\kappa}$ is required to drive the frequency error to zero, thus justifying the choice $\beta > 0$.

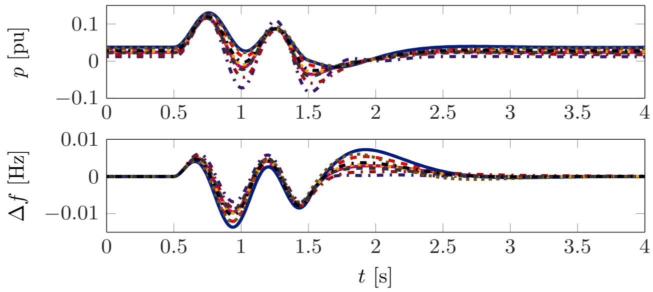

Fig. 2. Simulation results with $\kappa = 2.6792$, $\gamma = 0.9637$, and $h = 100$ms

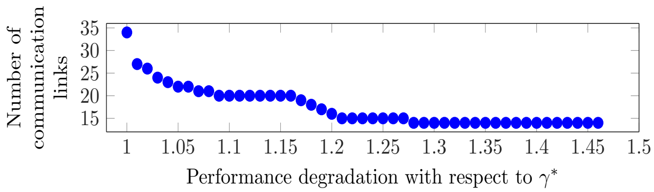

Fig. 3. Number of non-zero elements of $\mathcal{Z}$ for different values of γ. The number of required communication links in the case of $\gamma^* = 0.9637$ is 34.

Note that in all cases, robustness with respect to fast-varying delays $\tau(t) \leq h$ is guaranteed.

Recall that the design approach leading to (IV.6) is based on the descriptor method with fixed tuning parameter ε. The latter could potentially introduce some conservativeness. Thus to improve our estimate of the L_2-gain, we solve (IV.6) and implement the obtained values for κ and $\mathcal{L}$ in a modified version of the conditions for stability *analysis* derived in [15] that incorporates the L_2-gain performance. The resulting performance index γ with the analysis conditions in [15] is only 9.5% lower than the γ^* obtained via (IV.6). Hence, in the present case the descriptor method does not introduce significantly more conservativeness, while providing the advantage that κ and $\mathcal{L}$ are free design variables (in the analysis in [15] they are treated as constant parameters).

VI. Conclusions

Time delays and exogenous disturbances represent significant challenges in distributed control of MGs. In this paper, we have proposed a synthesis for a consensus-based secondary frequency controller in MGs that guarantees delay-robustness and simultaneously permits to trade off finite L_2-gain performance against the sparsity of the required communication network. The design criterion is derived based on a LKF together with the descriptor method and cast as a constraint convex optimization problem. To enforce controller sparsity, we employ the usual reweighted ℓ_1-norm. The presented case study on the CIGRE benchmark MV distribution network illustrates the design trade-off between the number of communication links and the guaranteeable L_2-gain. In future work, we plan to validate our design criterion experimentally and incorporate voltage and reactive power dynamics and control in the analysis. Moreover, we will explore further applications of time-delay stability analysis and control design in MGs and bulk power systems.

References

[1] H. Farhangi, "The path of the smart grid," *IEEE Power Energy Mag.*, vol. 8, no. 1, pp. 18 –28, january-february 2010.

[2] G. Strbac, N. Hatziargyriou, J. P. Lopes, C. Moreira, A. Dimeas, and D. Papadaskalopoulos, "Microgrids: Enhancing the resilience of the European megagrid," *IEEE Power Energy Mag.*, vol. 13, no. 3, pp. 35–43, 2015.

[3] J. Schiffer, D. Zonetti, R. Ortega, A. Stanković, T. Sezi, and J. Raisch, "A survey on modeling of microgrids—from fundamental physics to phasors and voltage sources," *Automatica*, vol. 74, pp. 135–150, 2016.

[4] J. Guerrero, P. Loh, M. Chandorkar, and T. Lee, "Advanced control architectures for intelligent microgrids – part I: Decentralized and hierarchical control," *IEEE Trans. Ind. Electron.*, vol. 60, no. 4, pp. 1254–1262, 2013.

[5] P. Kundur, *Power system stability and control*. McGraw-Hill, 1994.

[6] J. W. Simpson-Porco, F. Dörfler, and F. Bullo, "Synchronization and power sharing for droop-controlled inverters in islanded microgrids," *Automatica*, vol. 49, no. 9, pp. 2603 – 2611, 2013.

[7] A. Bidram, F. Lewis, and A. Davoudi, "Distributed control systems for small-scale power networks: Using multiagent cooperative control theory," *IEEE Control Systems*, vol. 34, no. 6, pp. 56–77, 2014.

[8] C. De Persis and N. Monshizadeh, "Bregman storage functions for microgrid control," *IEEE Trans. on Automatic Control*, 2017.

[9] X. Wu, F. Dörfler, and M. Jovanović, "Topology identification and design of distributed integral action in power networks," in *ACC*, 2016, pp. 5921–5926.

[10] E. Tegling, M. Andreasson, J. W. Simpson-Porco, and H. Sandberg, "Improving performance of droop-controlled microgrids through distributed PI-control," in *ACC*, 2016, pp. 2321–2327.

[11] S. Lee, C. Lee, M. Park, and O. Kwon, "Delay effects on secondary frequency control of micro-grids based on networked multi-agent," in *ICCAS*, 2016, pp. 655–659.

[12] J. Lai, H. Zhou, X. Lu, X. Yu, and W. Hu, "Droop-based distributed cooperative control for microgrids with time-varying delays," *IEEE Trans. on Smart Grid*, vol. 7, no. 4, pp. 1775–1789, 2016.

[13] E. A. A. Coelho, D. Wu, J. M. Guerrero, J. C. Vasquez, T. Dragičević, Č. Stefanović, and P. Popovski, "Small-signal analysis of the microgrid secondary control considering a communication time delay," *IEEE Trans. Ind. Electron.*, vol. 63, no. 10, pp. 6257–6269, 2016.

[14] C. Kammer and A. Karimi, "Robust distributed averaging frequency control of inverter-based microgrids," in *CDC*, 2016, pp. 4973–4978.

[15] J. Schiffer, F. Dörfler, and E. Fridman, "Robustness of distributed averaging control in power systems: Time delays & dynamic communication topology," *Automatica*, vol. 80, pp. 261–271, 2017.

[16] X. Wu, F. Dörfler, and M. Jovanović, "Input-output analysis and decentralized optimal control of inter-area oscillations in power systems," *IEEE Trans. on Pow. Sys.*, vol. 31, no. 3, pp. 2434–2444, 2016.

[17] S. Schuler, U. Münz, and F. Allgöwer, "Decentralized state feedback control for interconnected systems with application to power systems," *Journal of Process Control*, vol. 24, no. 2, pp. 379–388, 2014.

[18] E. Candes, M. Wakin, and S. Boyd, "Enhancing sparsity by reweighted ℓ_1 minimization," *Journal of Fourier Analysis and Applications*, vol. 14, no. 5, pp. 877–905, 2008.

[19] E. Fridman, *Introduction to time-delay systems: analysis and control*. Birkhäuser, 2014.

[20] K. Rudion, A. Orths, Z. Styczynski, and K. Strunz, "Design of benchmark of medium voltage distribution network for investigation of DG integration," in *IEEE PESGM*, 2006.

[21] A. van der Schaft, *L2-gain and passivity techniques in nonlinear control*. Springer, 2000.

[22] H. K. Khalil, *Nonlinear systems*. Prentice Hall, 2002, vol. 3.

[23] R. Diestel, *Graduate texts in mathematics: Graph theory*. Springer, 2000.

[24] C. Godsil and G. Royle, *Algebraic Graph Theory*. Springer, 2001.

[25] J. Schiffer, D. Goldin, J. Raisch, and T. Sezi, "Synchronization of droop-controlled microgrids with distributed rotational and electronic generation," in *52nd CDC*, Florence, Italy, 2013, pp. 2334–2339.

[26] J. Schiffer, R. Ortega, A. Astolfi, J. Raisch, and T. Sezi, "Conditions for stability of droop-controlled inverter-based microgrids," *Automatica*, vol. 50, no. 10, pp. 2457–2469, 2014.

[27] E. Fridman, "Tutorial on Lyapunov-based methods for time-delay systems," *Europ. Journal of Control*, vol. 20, no. 6, pp. 271–283, 2014.

[28] J. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138–162, 2007.

[29] F. Dörfler, J. W. Simpson-Porco, and F. Bullo, "Breaking the hierarchy: Distributed control and economic optimality in microgrids," *IEEE Trans. on Control of Network Sys.*, vol. 3, no. 3, pp. 241–253, 2016.

[30] J. Schiffer and F. Dörfler, "On stability of a distributed averaging PI frequency and active power controlled differential-algebraic power system model," in *ECC*, 2016, pp. 1487–1492.

[31] P. Park, J. Ko, and C. Jeong, "Reciprocally convex approach to stability of systems with time-varying delays," *Automatica*, vol. 47, no. 1, pp. 235–238, 2011.

[32] J. Löfberg, "YALMIP : a toolbox for modeling and optimization in MATLAB," in *IEEE Int. Symposium on Computer Aided Control Systems Design*, Sept. 2004, pp. 284 –289.