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On the Eigenvalue Placement by Static Output Feedback via Quantifier Elimination

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Abstract— **The static state feedback controller design, which achieves a desired behavior of the closed loop system, was solved in linear control theory decades ago. This is not true for static output feedback controller design, which is still an open problem. Here an approach is presented which handles the existence and design problem using quantifier elimination methods. Our approach allows the formal verification of arbitrary eigenvalue placement and enables the construction of controllers which lead to specified dynamic behavior. The design method is illustrated on examples known from literature.**

I. INTRODUCTION

One of the few still open problems in linear control theory is the so-called static output feedback (SOF) problem. In a nutshell, it consists of finding a static feedback for a given linear time-invariant system such that the closed-loop exhibits a specified dynamic behavior. These dynamics can be described as a set of desired roots of the closed-loop characteristic polynomial or simply by requiring stability of the closed-loop system maybe together with additional constraints.

In contrast to the problem of state feedback (for which solvability conditions are well-known since the work of Kalman in the 1960s [1]), in general, not even the solvability of the SOF can be decided. Although in the 1990s significant progress has been made in this topic, there are still no easily checkable conditions available. Some results shall be mentioned here: Wang [2] showed that in the generic case a solution exists if the product of the number of inputs and the number of outputs is greater than the dynamic order of the system. The term generic refers here to systems whose parameters do not fulfill any algebraic equation. Despite the fact that, from a set-theoretic point of view, this holds true for almost all systems, a mathematical model of a technical system is usually *not* generic.

He also proposed a sufficient condition and an algorithm for feedback calculation for particular systems [3]. This approach was generalized later to a necessary and sufficient condition [4], which is still hard to check. However, this condition also leads to new algorithms for calculating feedback matrices with low gain. The calculation of specific static feedback laws has been continuously investigated by many researchers, see [5], [6] and the references therein, for instance. For a more comprehensive treatise of the literature with respect to the output feedback problem, in particular also regarding the early developments, we refer to the surveys [7], [8], and [9].

The SOF requirements can be formulated using expressions containing existence and/or universal quantifiers, respectively. For an useful application quantifier-free conditions for the control matrix entries are needed. So we are looking for an algorithm which eliminates the quantifiers such that only quantifier-free expressions remain. This leads us to the concept of quantifier elimination (QE). Quantifier elimination is a collective term for a bunch of methods which cancel quantifiers in an algorithmizable process.

To our knowledge, the usage of quantifier elimination to compute a static output feedback was first suggested in [10]. In that paper, the quantifier elimination was carried out by hand. In [9], the authors used a software implementation of QE based on the partial cylindrical algebraic decomposition method to solve the static output stabilization problem. The considered example was low-dimensional and thus for most practical applications not feasible.

The paper is structured as follows. The basics on quantifier elimination are reviewed in Section II, followed by the problems associated with static output feedback that are stated in Section III. The feasibility of our approach will be illustrated on some example systems in Section IV. Finally, we will draw some conclusions in Section V.

II. QUANTIFIER ELIMINATION

All following conditions to static output feedback control can be generalized with the form

$$G(y,z) := (Q_1 y_1)\cdots(Q_l y_l)\, F(y,z), \tag{1}$$

with $Q_i \in \{\exists, \forall\}$ and the quantifier-free formula $F(y,z)$. These formulations are called *prenex formulas*. The variables y are connected to the quantifiers Q_i and therefore called *quantified variables*. The variables z are called *free*, respectively. If the quantifier-free formula $F(y,z)$ is a boolean combination of polynomials with rational coefficients

$$\varphi(y_1 \ldots, y_l, z_1, \ldots, z_k)\, \tau\, 0,$$

with $\tau \in \{=, \neq, <, >, \leq, \geq\}$, several useful strategies exists which eliminate the quantified variables. Thereby a quantifier-free expression results, which is equivalent to former prenex formula. These strategies are summarized under the term *quantifier elimination* methods. The first researcher who showed that always such a quantifier-free equivalent

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exists was A. Tarski [11]. He also proposed the first algorithm to determine an equivalent quantifier-free expression. Certainly, this procedure was not applicable, because its computational effort can not be bounded by any tower of exponentials. The first practical relevant algorithm was stated by Collins [12]. Based on cylindrical algebraic decomposition (CAD) it was possible to eliminate quantifiers with a doubly exponential complexity [13]. In the last decades this algorithm has been significantly improved and some other approaches have been presented, like virtual substitution or real root classification based on Sturm-Habicht sequences. They all have in common, that they are very sensitive with respect to the order of the polynomial and the number of variables, at least the quantified ones. The computational effort of virtual substitution grows exponential in the number of the quantified variables, while the complexity of Sturm-Habicht based quantifier elimination can be reduced to an exponential growing in the polynomial degree (This holds true for so called sign definite conditions: $\forall x : x \geq 0 \implies f(x) > 0$).

The complexity of these methods motivated the development of software tools for quantifier elimination. Thus there are some open-source tools like QEPCAD [14], [15] and the REDUCE package REDLOG [16], [17] as well as toolboxes for commercial tools like Mathematica or Maple [18]–[21]. The resulting expressions are generally very complex and redundant, especially using virtual substitution and real root classification. Thus the open-source tool SLFQ [22] enables a subsequent simplification. The problem is carried out in the next section. Afterwards the QE methods are applied to the resulting conditions.

III. Problem Statement

A. Existence Conditions

Considering the state-space system

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{2}$$

with the state vector $x(t) \in \mathbb{R}^n$, the input vector $u(t) \in \mathbb{R}^m$, the output vector $y(t) \in \mathbb{R}^p$, the system matrix $A \in \mathbb{R}^{n\times n}$, the input matrix $B \in \mathbb{R}^{n\times m}$ and the output matrix $C \in \mathbb{R}^{p\times n}$. The system is controlled with a static output feedback controller

$$u = -Ky, \tag{3}$$

in which $K \in \mathbb{R}^{m\times p}$ gives the control matrix. This leads to the closed-loop system

$$\dot{x} = Ax - BKCx = (A - BKC)x, \tag{4}$$

whose system behavior is completely determined by the matrix $A - BKC$. In particular, the eigenvalues of this matrix specify (beside stability) the damping and oscillation behavior of solutions $t \mapsto x(t)$ of (4). For this reason, it is appealing to look for a matrix K which places the eigenvalues of $A - BKC$ as desired.

To this end, we consider the characteristic polynomial of the closed-loop system has form

$$\begin{aligned} \mathrm{CP}(s) &= \det\left(sI - (A - BKC)\right) \\ &= a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + s^n, \end{aligned} \tag{5}$$

where the coefficients $a_0, \ldots, a_{n-1}$ depend on the entries k_{ij} of the matrix $K = (k_{ij})$. If a gain matrix K – if possible – should be computed such that the closed-loop system has a prescribed characteristic polynomial

$$\mathrm{CP}^*(s) = a_0^* + a_1^* s + \cdots + a_{n-1}^* s^{n-1} + s^n, \tag{6}$$

the following problems arise:

Problem 1 (Arbitrary Eigenvalue Placement): Can any characteristic polynomial (6) be assigned to the closed-loop system by (5) with an appropriate matrix K?

Problem 2 (Specific Eigenvalue Placement): Consider a given characteristic polynomial (6). Exists a gain matrix K such that this polynomial can be assigned to the closed-loop system by (5)?

Problem 3 (Stabilizability): Exists a gain matrix K such that all eigenvalues of the closed-loop system (4) have a negative real part?

The polynomials (5) and (6) are identical if and only if they have the same coefficients. There, Problem 1 can be formalized as

$$\forall a_0^* \cdots \forall a_{n-1}^* \, \exists k_{11} \cdots \exists k_{mp} : \; a_0 = a_0^* \wedge \ldots \wedge a_{n-1} = a_{n-1}^*. \tag{7}$$

If the system does not depend on unspecified parameters, the result of QE applied to (7) should be either `true` or `false`. On the other hand, unspecified parameters are free variables in the sense of QE. Then, the elimination process would yield a quantifier-free expression in these parameters.

In case of Problem 2 we have a given desired polynomial (6). The existence of an appropriate gain matrix K can be stated as

$$\exists k_{11} \cdots \exists k_{mr} : \; a_0 = a_0^* \wedge \ldots \wedge a_{n-1} = a_{n-1}^*. \tag{8}$$

The stability requirement associated with Problem 3 can be formulated in terms of Hurwitz oder Routh conditions [23]. In [10], the stability conditions are derived from the Liénard-Chipart criterion. Alternatively, one could rely on Lyapunov equations [24], [25].

B. Optimization

Assume Problem 2 is solvable for a given system with specified closed-loop eigenvalues. In the case of $mp > n$, when the number mp of entries of the matrix K is strictly larger than the dimension n of the state-space we could expect some degrees of freedom in the choice of the gain matrix. Then, one could use these degrees of freedom to select a low gain matrix. Formally, this leads to the optimization problem

$$\min_K f(K) \tag{9}$$

with the cost functional f. A typical choice of the cost functional would be the square $f(K) = \|K\|_F^2$ of the Frobenius norm

$$\|K\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{p} k_{ij}^2}. \tag{10}$$

There are several approaches to solve this optimization problem. First, we could determine a parametrization of the gain matrix K with a minimum number of free parameters. In this case, we would solve (9) as an unrestricted optimization problem w.r.t. the free parameters, which could be calculated using the first derivative test. Alternatively, we could formulate this task as a restricted optimization problem

$$\min_K f(K) \quad \text{s.t.} \quad h(K) = 0, \tag{11}$$

where the equation-based constraints resulting from the prescribed closed-loop characteristic polynomial are formulated with the function h:

$$\begin{array}{rcl} h_1(K) & = & a_0(K) - a_0^*, \\ & \vdots & \\ h_n(K) & = & a_{n-1}(K) - a_{n-1}^*. \end{array}$$

The Lagrangian associated with (11) is

$$L(K,\lambda) = f(K) + \sum_{i=1}^{n} \lambda_i h_i(K),$$

with the Lagrangian multipliers $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. The existence conditions of the first derivative test associated with (11) can be written as the following prenex formula:

$$\begin{array}{l} \exists k_{11} \cdots \exists k_{mp}\, \exists \lambda_1 \cdots \exists \lambda_n : \\ \frac{\partial L(K,\lambda)}{\partial k_{11}} = 0 \wedge \ldots \wedge \frac{\partial L(K,\lambda)}{\partial k_{mp}} = 0 \wedge \\ h_1(K) = 0 \wedge \ldots \wedge h_n(K) = 0. \end{array} \tag{12}$$

In the following the conditions gathered in this section are applied on three examples.

IV. Case Studies

A. Example 1

We consider a system (2) taken from [26, Example 2] with the following matrices:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The controller gain is a 3×2-matrix

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \\ k_{31} & k_{32} \end{pmatrix}. \tag{13}$$

Note that in this example the number of unknowns mp equals the number of eigenvalues to place n ($m = 3$, $p = 2$, $n = 6$). Hence, only a finite number of solutions assigning a particular characteristic polynomial can be expected. In the generic case, this number of complex-valued solutions is determined by the so-called Schubert number

$$d(m,p) = \frac{1!2!\cdots(p-1)!1!2!\cdots(m-1)!(mp)!}{1!2!\cdots(m+p-1)!},$$

see [27], for instance. Since in this example $d(2,3) = 5$ is odd and the solutions are complex conjugates, at least one solution K must be real. However, this holds true only in the generic case.

The systems characteristic polynomial (5) has the coefficients

$$\begin{array}{rcl} a_0 & = & k_{11}k_{32} - k_{12}k_{31} + k_{31} \\ a_1 & = & k_{11}k_{32} + k_{32} - k_{12}k_{31} + k_{31} \\ a_2 & = & k_{32} + k_{11}k_{22} - k_{12}k_{21} + k_{21} \\ a_3 & = & k_{11}k_{22} + k_{22} - k_{12}k_{21} + k_{21} \\ a_4 & = & k_{22} + k_{11} \\ a_5 & = & k_{12} + k_{11}. \end{array} \tag{14}$$

The condition (7) for arbitrary eigenvalue placement can be formulated as follows:

$$\forall a_0^* \cdots \forall a_5^*\, \exists k_{11} \cdots \exists k_{23} :\; a_0 = a_0^* \wedge \ldots \wedge a_5 = a_5^*. \tag{15}$$

We employed the computer algebra system REDUCE with the package REDLOG. The computations were carried out on a PC with Intel® Core™ i3-4130 CPU at 3.4 GHz and 32 GiB RAM under the Linux system Fedora 25 (64 bit). For QE we used the virtual substitution method from [17] (i.e., function `rlqe` with the switch `on ofsfvs`). The source code of a prototype implementation is given in the appendix. In some cases we will compare the computational effort with the CAD method (REDLOG function `rlcad`).

This prenex formula (15) contains no free variables. The computation yields the result `false`, where quantifier elimination with the virtual substitution method was carried out in less than 100 ms. Therefore, arbitrary eigenvalue placement by static output feedback is not possible for this system. We also tried to obtain this results with the CAD method, but aborted the computation after 24 h computation time.

Nevertheless, we want to assign the eigenvalues $-0.5, -2, -2.5, -3, -3.5$ and -4 to the closed-loop system. This corresponds to the desired characteristic polynomial (6) with the coefficients

$$\begin{array}{rclrcl} a_0^* & = & 105, & a_3^* & = & 303.125, \\ a_1^* & = & 395.75 & a_4^* & = & 96.25, \\ a_2^* & = & 500.875, & a_5^* & = & 15.5. \end{array} \tag{16}$$

The test for this specific eigenvalue placement can be formulated as

$$\exists k_{11} \cdots \exists k_{32} :\; a_0 = a_0^* \wedge \ldots \wedge a_5 = a_5^* \tag{17}$$

with the coefficients given in (14) and (16). Again, this formula contains no free variables. In less than 100 ms computation time, REDUCE with the virtual substitution method yields the result `true`, i.e, this problem has a (real) solution. Solving the same problem with CAD took slightly more than $11\,\mathrm{min}$ computation time.

Now, we discuss the calculation of the gain matrix. In [26, Example 2], a numerical approach converged after approximately 30000 iterations to the gain matrix

$$K = \begin{pmatrix} 3.27253 & 12.2223 \\ 8.38933 & 92.9134 \\ 75.3207 & 290.325 \end{pmatrix}. \tag{18}$$

After 7 iteration steps, the algorithm described in [4], [28] results in

$$K = \begin{pmatrix} 3.25 & 12.25 \\ 8.18889 & 93 \\ 74.6611 & 290.75 \end{pmatrix}. \tag{19}$$

To compute the gains using QE we omit the existence quantifier for one gain. The corresponding gain becomes a free variable. QE results in a quantifier-free expression regarding to this variable. Then, we select an admissible value for this variable and proceed with the next gain. Starting with k_{11}, we successive obtain the following values:

$$\begin{array}{rcl} \texttt{4*k11 - 13 = 0} & \Rightarrow & k_{11} = 13/4 \\ \texttt{4*k12 - 49 = 0} & \Rightarrow & k_{12} = 49/4 \\ \texttt{90*k21 - 737 = 0} & \Rightarrow & k_{21} = 737/90 \\ \texttt{k22 - 93 = 0} & \Rightarrow & k_{22} = 93 \\ \texttt{180*k31 - 13439 = 0} & \Rightarrow & k_{31} = 13439/180 \\ \texttt{4*k32 - 1163 = 0} & \Rightarrow & k_{32} = 1163/4 \end{array}$$

This finally results in the gain matrix

$$K = \begin{pmatrix} 3.25 & 12.25 \\ 8.1\bar{8} & 93 \\ 74.66\bar{1} & 290.75 \end{pmatrix}, \tag{20}$$

which is the exact (namely rational) solution of the above mentioned eigenvalue placement problem. Clearly, the numerical solutions (18) and (19) are reasonable good approximations of the exact solution (20).

B. Example 2

The system matrices of the next example are given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

see [26, Example 3]. The gain matrix has the form (13) as in the previous example. For the closed-loop system we obtain the characteristic polynomial

$$\begin{array}{rcl} a_0 & = & k_{11}k_{32} - k_{12}k_{31} + k_{31} \\ a_1 & = & k_{32} + k_{11}k_{22} - k_{12}k_{21} + k_{21} \\ a_2 & = & k_{11}k_{22} + k_{22} - k_{12}k_{21} + k_{21} \\ a_3 & = & k_{22} + k_{11} \\ a_4 & = & k_{12} + k_{11}. \end{array} \tag{21}$$

First, we test the system regarding to arbitrary eigenvalue placement. Applying QE to the associated prenex formula (7) yields `true`.

The gain matrix K should be calculated such that the closed-loop system has the eigenvalues $-3, -4, -5, -2 \pm 2j$. The desired characteristic polynomial is

$$\mathrm{CP}^*(s) = s^5 + 16s^4 + 103s^3 + 344s^2 + 616s + 480.$$

In [26], the authors obtained the gain matrix

$$K = \begin{pmatrix} 5 & 11 \\ 24.3999 & 98.0002 \\ 137.001 & 370 \end{pmatrix} \tag{22}$$

numerically after several iterations.

The associated condition (8) is fulfilled due to the already verified arbitrary eigenvalue placement. Omitting the existence quantifier for k_{11}, REDUCE yields the equivalent quantifier-free formula

$$k_{11} \neq 15. \tag{23}$$

We set $k_{11} := 5$. In the next step, we omit the existence quantifier for k_{12} and obtain an constraint regarding to k_{12}. Continuing this process we obtain the exact rational gain matrix

$$K = \begin{pmatrix} 5 & 11 \\ 24.4 & 98 \\ 137 & 370 \end{pmatrix}, \tag{24}$$

of which (22) is a good approximation.

As a matter of fact, the gain K is not unique. For example, if we start with $k_{11} = 10$, we would gradually obtain the gain matrix

$$K = \begin{pmatrix} 10 & 6 \\ 679/5 & 93 \\ 634 & 365 \end{pmatrix},$$

which is a different solution of the above given eigenvalue assignment problem.

In this example, we have a state-space of dimension $n = 5$ and $m \times p = 6$ entries in the gain matrix. This suggests that the eigenvalue placement problem is overdetermined. This hypothesis is supported by the condition (23), where k_{11} can be chosen almost freely. If we do not specify k_{11}, we can derive polynomial constraints for the other entires of the gain matrix (13). Finally, we obtain the matrix

$$K(k_{11}) = \begin{pmatrix} k_{11} & 16 - k_{11} \\ \frac{k_{11}^2 - 102k_{11} + 241}{k_{11} - 15} & 103 - k_{11} \\ \frac{k_{11}^2 - 375k_{11} + 480}{k_{11} - 15} & 375 - k_{11} \end{pmatrix} \tag{25}$$

parametrized in k_{11}. This family of gain matrices characterizes together with condition (23) all solutions of the above mentioned eigenvalue placement problem.

For robustness purposes one usually wants to compute a low gain feedback matrix. From the symbolic expression (25) we can easily compute the Forbenius norm (10). We want to determine the minima of this norm. Using the first derivate test

$$\frac{\partial \|K(k_{11})\|_F^2}{\partial k_{11}} \stackrel{!}{=} 0,$$

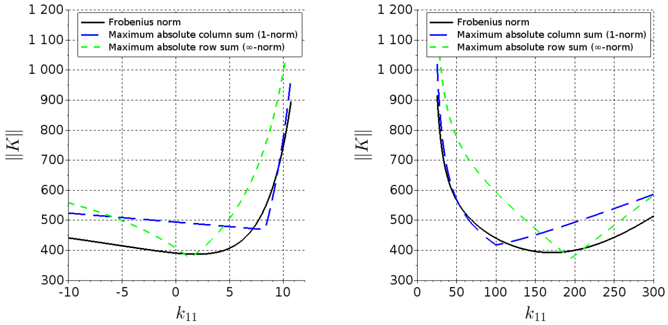

Fig. 1. Different norms of the gain matrix (25) depending on k_{11}

we obtain the two solutions

$$\begin{aligned} k_{11,1} &= \tfrac{63617711}{33554432} \approx 1.89595553278923, \\ k_{11,2} &= \tfrac{5712044743}{33554432} \approx 170.2321989238262, \end{aligned} \tag{26}$$

which are both local minima. The first solution yields $\|K\|_F \approx 387.23$, whereas the second one results in $\|K\|_F \approx 392.55$. Therefore, the first solution is a global minimum. For $k_{11} \to 15$ we have $\|K\|_F \to \infty$. i.e., there is a pole at $k_{11} = 15$. Fig. 1 shows the Frobenius norm depending on the parameter k_{11}. Additionally, the figure shows the norms $\|K\|_1$ and $\|K\|_\infty$.

Alternatively to the minimization of the Frobenius norm using a the representation (25) with a minimum number of free parameters we use the formulation (12) of the restricted optimization problem (11). Omitting the existence quantifier on k_{11} results in the quantifier-free polynomial equation

$$6k_{11}^4 - 1211k_{11}^3 + 46395k_{11}^2 - 2429433k_{11} + 4447499 = 0.$$

This polynomial has two real roots, which are already stated in Eq. (26). Again, we obtain the full gain matrix K via a step-by-step reduction of the existence quantifiers. All other entries are subject to linear constraints, e.g. we obtain a rational gain matrix. The optimal solution is approximately given by

$$K \approx \begin{pmatrix} 1.8959555 & 14.104044 \\ -3.9077388 & 101.10404 \\ 17.352557 & 373.10404 \end{pmatrix}.$$

C. Example 3

The SOF problem for the linearized model of a unicycle was investigated in [4], [29]. The system matrices derived from [30] are given by

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{15}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 15 & -5 & 0 & 0 & 0 \\ 0 & -15 & 13 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{27}{40} & -\frac{3}{10} \\ -\frac{3}{4} & 1 \\ \frac{11}{20} & -\frac{9}{5} \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

The arbitrary pole placement property by static output feedback of the system was already verified in [4]. Now, we want to assign the eigenvalues $-1, -2, -3, -4, -5, -6$ to the closed-loop system as in [4], [29], which corresponds to the desired characteristic polynomial

$$\begin{aligned} \mathrm{CP}^*(s) &= (s+1)(s+2)(s+3)(s+4)(s+5)(s+6) \\ &= s^6 + 21s^5 + 175s^4 + 735s^3 + 1624s^2 + 1764s + 720. \end{aligned}$$

Coefficient matching of the absolute terms of CP and CP* directly yields $k_{11} = 24$. The 2×4 gain matrix K has 8 entries, whereas the state-space has dimension $n = 6$. This suggests that the controller design problem has $8 - 6 = 2$ degrees of freedom. To simplify the design procedure, we set $k_{12} = k_{13} = 0$. Taking k_{14} as a free variable results in the quantifier-free expression

$$160k_{14}^4 - 70770k_{14}^3 - 142110k_{14}^2 - 30051k_{14} - 36774 = 0.$$

This polynomial equation has two real solutions $k_{14,1} \approx -1.919$ and $k_{14,2} \approx 444.312$. Unfortunately, the exact solutions are not rational numbers. Hence, we cannot directly use the solution in the prenex formula. To select the first solution $k_{14,1}$ we add the condition $k_{14} < 0$ and proceed with the variables $k_{21}, k_{22}, k_{23}, k_{24}$. For each of these variables we obtain a $4th$ order polynomial equation, where either the smallest or the larges real root is used, respectively. Solving these polynomial equations numerically results in the controller gains

$$\begin{aligned} k_{14} &= -1.919535487890244, \\ k_{21} &= -83.64122745394707, \\ k_{22} &= 785.2861048281193, \\ k_{23} &= 548.7705494463444, \\ k_{24} &= 162.2573166787624. \end{aligned} \tag{27}$$

To select the second solution $k_{14,2}$ we add the condition $k_{14} > 0$ and proceed as above. This results in the gains

$$\begin{aligned} k_{14} &= 444.3124695122242, \\ k_{21} &= 9.736928671598434, \\ k_{22} &= -3.471433848142624, \\ k_{23} &= -2.449630409479141, \\ k_{24} &= 198.0225431621075, \end{aligned} \tag{28}$$

that constitute an alternative solution of the same eigenvalue assignment problem. We verified our results by a numerical calculation of the eigenvalues of the closed-loop system with the gain matrices (27) and (28).

V. Conclusion

In this paper we present an approach to determine static output feedback controllers with an a-priori chosen dynamic behavior. Utilizing quantifier elimination methods, it can be verified if the closed-loop systems eigenvalues can achieve arbitrary values or specific ones. Based on the calculated conditions, a suitable controller matrix can be computed. In contrast to the procedures known from literature the results are analytical and thus exact. This process is illustrated on three examples. Furthermore it is shown that our approach is suitable for low gain design. Using the virtual substitution method for quantifier elimination we were able to treat higher dimensional problems compared to [9], [10].

```
% Preparation
load_package "redlog";
rlset r;
on ofsfvs;
% Coefficient matching
g0:=a0 = k11*k32-k12*k31+k31;
g1:=a1 = k11*k32+k32-k12*k31+k31;
g2:=a2 = k32+k11*k22-k12*k21+k21;
g3:=a3 = k11*k22+k22-k12*k21+k21;
g4:=a4 = k22+k11;
g5:=a5 = k12+k11;
% Prenex formula
phi:=all({a0,a1,a2,a3,a4,a5},
 ex({k11,k12,k21,k22,k31,k32},
 g0 and g1 and g2 and g3 and g4 and g5));
% Quantifier elimination
psi:=rlqe(phi);
```

Fig. 2. REDUCE code to test the example system from Section IV-A regarding to arbitrary eigenvalue placement

APPENDIX

Fig. 2 shows a simple REDUCE implementation to test the example system from Section IV-A w.r.t. arbitrary eigenvalue placement. First, the package REDLOG is loaded. Then, the algebraic context is defined, namely the real numbers $\mathbb{R}$. The switch `on ofsfvs` activates the advanced algorithms for virtual substitution [17]. After that, the coefficients of the characteristic polynomial are defined as in (14). In the prenex formula, the universal and the existence quantifiers are denoted by `all` and `ex`, respectively. The quantifier elimination is carried out with the command `rlqe`.

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