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On the finite determinedness of maximal RPI sets for linear systems with scaled disturbances

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Abstract— **This paper presents new results on robust positively invariant (RPI) sets for linear discrete-time systems with additive disturbances. In particular, we study the finite determinedness of the maximal RPI (MRPI) set for scaled disturbance sets and we show that the number of iterations necessary to exactly compute the MRPI set using standard procedures grows the closer we get to a certain critical scaling factor both from below and above. We characterize this critical scaling factor and illustrate our findings with three examples.**

I. INTRODUCTION

Robust positively invariant (RPI) sets are essential for performance analysis and controller synthesis of disturbed systems (see, e.g., [1] or [2, Sects. 6.4 and 6.5]). In particular, RPI sets are used to design robust model predictive control (MPC) schemes with guaranteed stability (see, e.g., [3], [4], [5], [6]). In this paper, we address RPI sets for linear disturbed systems of the form

$$x(k+1) = A\,x(k) + E\,d(k) \qquad (1)$$

with state and disturbance constraints

$$x(k) \in \mathcal{X} \quad \text{and} \quad d(k) \in \mathcal{D}^\alpha \quad \text{for every} \quad k \in \mathbb{N}. \qquad (2)$$

A special feature of this paper is the consideration of scaled disturbance sets $\mathcal{D}^\alpha := \alpha\,\mathcal{D}^*$, where $\mathcal{D}^*$ denotes some nominal disturbance set and where α is a positive scaling factor. Scaled disturbance sets allows to study the effect of disturbances with different strengths. Moreover, scaled disturbance sets can be of interest for determining allowable state and input constraints and acceptable disturbances when designing actuators, sensors, and controllers [7].

Roughly speaking, an RPI set $\mathcal{P}$ for system (1) with constraints (2) is such that the trajectory of the disturbed system (1) remains in $\mathcal{P}$ at all times $k \in \mathbb{N}$ for every initial condition $x(0) \in \mathcal{P}$ and for all disturbances $d(k) \in \mathcal{D}^\alpha$ (see Def. 1 further below for a formal definition of RPI sets). When studying RPI sets for a given system, special attention is paid to the maximal (i.e., the largest) and the minimal (i.e., the smallest) RPI set [8], [9]. Both sets can be described as the limit sets of certain set-valued sequences. These sequences are not only of theoretical interest but also used for the computation of the minimal and the maximal RPI set (see [1], [8], [10].) In this context, the sequences characterizing the minimal and the maximal RPI set show an important difference. In fact, the sequence associated with the minimal RPI (mRPI) set is usually not finitely determined. In other words, there does usually not exist an element in the sequence that is equivalent to the mRPI set (see [1, Rem. 4.2] for an exception). In contrast, the maximal RPI (MRPI) set is usually finitely determined (see [11, Exmp. 2] for an exception). It is easy to see that finite determinedness is of great interest for the numerical computation of RPI sets. In particular, finite determinedness is important for the computation of parametric RPI sets as studied in [7], [12], and [13].

In this paper, we further investigate the finite determinedness of MRPI sets. Clearly, this property has been intensively studied in the literature especially for linear systems (see, e.g., [1, Sect. 6], [14], or [15] and references therein). However, research on MRPI sets for the considered scaled disturbance sets is rare [7], [11], [12], [13]. This paper provides the following novel contributions to this interesting field. We introduce the finite determinedness index (FDI) as the index of the first sequence element that is equivalent to the MRPI set. We then study how the FDI depends on the scaling factor α. In particular, we show that the FDI is monotone for certain domains of scaling factors. In fact, (an overestimation of) the FDI is nondecreasing for those scaling factors that lead to nonempty MRPI sets. In contrast, the FDI is nonincreasing for scaling factors that result in empty RMPI sets. Both domains are separated by a certain critical scaling factor that marks the transition from nonempty to empty MPRI sets and that was previously studied in [11, Sect. 3].

The paper is organized as follows. We state notation in the remainder of this section and provide background on RPI sets in Section II. The main results of paper, i.e., general properties of the FDI are presented in Section III. The findings are illustrated with three examples in Section IV. Finally, conclusions are stated in Section V.

Notation and assumptions

Real, positive real, and natural numbers are denoted by $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$, respectively. We further introduce $\mathbb{N}_{[i,k]} := \{j \in \mathbb{N} \,|\, i \leq j \leq k\}$. For bounded and convex sets $\mathcal{C}, \mathcal{V} \subset \mathbb{R}^n$, the operations

$$\mathcal{C} \oplus \mathcal{V} := \{x + v \in \mathbb{R}^n \,|\, x \in \mathcal{C}, v \in \mathcal{V}\} \qquad \text{and}$$
$$\mathcal{C} \ominus \mathcal{V} := \{x \in \mathbb{R}^n \,|\, \forall v \in \mathcal{V} : x + v \in \mathcal{C}\},$$

describe the Minkowski addition and the Pontryagin difference. We further define $\beta\,\mathcal{C} := \{\beta x \in \mathbb{R}^n \,|\, x \in \mathcal{C}\}$, $F\,\mathcal{C} := \{Fx \in \mathbb{R}^n \,|\, x \in \mathcal{C}\}$, and $G^{-1}\,\mathcal{C} := \{x \in \mathbb{R}^n \,|\, G\,x \in \mathcal{C}\}$ for $\beta \in \mathbb{R}_+$, $F \in \mathbb{R}^{l \times n}$, and $G \in \mathbb{R}^{n \times n}$, respectively. We stress that $G^{-1}\,\mathcal{C}$ is well defined even if G is not invertible. In this

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case, however, we usually find $G\,G^{-1}\,\mathcal{X} \neq \mathcal{X}$. Nevertheless, the relation

$$G\,G^{-1}\,\mathcal{X} \subseteq \mathcal{X} \subseteq G^{-1}G\,\mathcal{X} \tag{3}$$

holds for every $G \in \mathbb{R}^{n\times n}$. The closure and the boundary of $\mathcal{C}$ are denoted by $\mathrm{cl}(\mathcal{C})$ and $\partial\mathcal{C}$. Furthermore, the support function for any given row vector $\xi \in \mathbb{R}^{1\times n}$ is defined as

$$h_{\mathcal{C}}(\xi) := \sup_{x\in\mathcal{C}} \xi\, x.$$

When used with matrix arguments, $h_{\mathcal{C}}(F)$ is understood as

$$h_{\mathcal{C}}(F) := \begin{pmatrix} h_{\mathcal{C}}(e_1^\top F) & \dots & h_{\mathcal{C}}(e_l^\top F)\end{pmatrix}^\top,$$

where $e_i \in \mathbb{R}^l$ refers to the ith Cartesian unit vector. Finally, a convex and compact set containing the origin as an interior point is called a C-set. Throughout the paper, we assume that the eigenvalues of the system matrix $A \in \mathbb{R}^{n\times n}$ are strictly stable (i.e., $|\lambda| < 1$) and the sets $\mathcal{X}$ and $\mathcal{D}^*$ are C-sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

II. Background on RPI sets

We begin with a formal definition of RPI sets.

Definition 1: A set $\mathcal{P} \subseteq \mathcal{X}$ is called RPI for system (1) with constraints (2), if $A\,x + Ed \in \mathcal{P}$ for every $x \in \mathcal{P}$ and every $d \in \mathcal{D}^\alpha$. The union of all RPI sets for (1) with (2) is called the maximal robust positively invariant (MRPI) set and denoted by $\mathcal{P}_{\max}^\alpha$.

Now, it is well-known that the MRPI set $\mathcal{P}_{\max}^\alpha$ is equivalent to the limit set of the sequence

$$\mathcal{S}_{k+1}^\alpha := A^{-1}(\mathcal{S}_k^\alpha \ominus E\,\mathcal{D}^\alpha) \cap \mathcal{X} \quad \text{with} \quad \mathcal{S}_0^\alpha := \mathcal{X}. \tag{4}$$

In fact, as apparent from [16],

$$\mathcal{P}_{\max}^\alpha = \mathcal{S}_\infty^\alpha := \lim_{k\to\infty} \mathcal{S}_k^\alpha = \bigcap_{k=0}^{\infty} \mathcal{S}_k^\alpha.$$

We say that $\mathcal{P}_{\max}^\alpha$ is finitely determined for a given $\alpha \in \mathbb{R}_+$ if there exists a finite $k \in \mathbb{N}$ such that

$$\mathcal{S}_{k+1}^\alpha = \mathcal{S}_k^\alpha = \mathcal{P}_{\max}^\alpha. \tag{5}$$

In this context, it is easy to show that $\mathcal{P}_{\max}^\alpha$ is finitely determined if

$$0 \in \mathrm{int}(\mathcal{P}_{\max}^\alpha) \tag{6}$$

(cf. [1, Thm. 6.3]). Now, conditions that guarantee (6) were recently studied in [11]. It was shown that (6) holds whenever $\alpha \in (0, \alpha^*)$, where the critical scaling α^* is defined as the largest scaling α such that the limit set

$$\mathcal{R}_\infty^\alpha := \lim_{k\to\infty} \mathcal{R}_k^\alpha = \bigcup_{k=0}^{\infty} \mathcal{R}_k^\alpha$$

of the sequence

$$\mathcal{R}_{k+1}^\alpha := \mathcal{R}_k^\alpha \oplus A^k E\mathcal{D}^\alpha \quad \text{with} \quad \mathcal{R}_0^\alpha := \{0\} \tag{7}$$

is contained in $\mathcal{X}$. More precisely, we find

$$\alpha^* := \sup\{\alpha \in \mathbb{R}_+ \,|\, \mathcal{R}_\infty^\alpha \subseteq \mathcal{X}\}. \tag{8}$$

In [11], it was also proven that $\mathcal{P}_{\max}^\alpha$ is finitely determined but empty whenever $\alpha > \alpha^*$. For the special case $\alpha = \alpha^*$, $\mathcal{P}_{\max}^\alpha$ is guaranteed to be nonempty but it may or may not be finitely determined (see Rem. 12 in Sect. IV).

III. Properties of the finite determinedness index

The paper investigates the finite determinedness of the MRPI set for different scalings of the disturbance set. To this end, we briefly introduced the FDI in the introduction. A more formal definition of the FDI $N : \mathbb{R}_+ \to \mathbb{N}$ is

$$N(\alpha) := \begin{cases} \min \mathcal{K}^\alpha & \text{if} \quad \mathcal{K}^\alpha \neq \emptyset, \\ \infty & \text{otherwise,} \end{cases} \tag{9}$$

where the set

$$\mathcal{K}^\alpha := \{k \in \mathbb{N} \,|\, \mathcal{S}_{k+1}^\alpha = \mathcal{S}_k^\alpha\} \tag{10}$$

collects all k satisfying (5). Obviously, $\mathcal{K}^\alpha$ is nonempty if and only if $\mathcal{P}_{\max}^\alpha$ is finitely determined. In this case, $N(\alpha)$ denotes the smallest $k \in \mathbb{N}$ such that (5) holds, i.e., the index of the first element of the sequence (4) that satisfies $\mathcal{S}_k^\alpha = \mathcal{P}_{\max}^\alpha$. In the following, we study general properties of the FDI N. We start with some basic insights. Afterwards, we investigate N on the domains $(0, \alpha^*)$ and (α^*, ∞).

A. Basic properties of $N(\alpha)$

Lemma 2: Let α^* and $\mathcal{K}^\alpha$ be defined as in (8) and (10). Then, $\mathcal{K}^\alpha$ is nonempty for every $\alpha \in \mathbb{R}_+$ with $\alpha \neq \alpha^*$.

Proof: According to [11, Thm. 6], there exists a finite $k \in \mathbb{N}$ such that (5) holds whenever $\alpha \neq \alpha^*$. Based on the definition of $\mathcal{K}^\alpha$ in (10), this observations immediately proves nonemptiness $\mathcal{K}^\alpha$ for every $\alpha \in \mathbb{R}_+$ with $\alpha \neq \alpha^*$. ■

As a direct consequence of Lemma 2, we find the following corollary.

Corollary 3: Let α^* and N be defined as in (8) and (9). Then, $N(\alpha)$ is finite for every $\alpha \in \mathbb{R}_+$ with $\alpha \neq \alpha^*$.

Note, however, that $\mathcal{K}^\alpha$ may or may not be empty for $\alpha = \alpha^*$. Consequently, $N(\alpha^*)$ may or may not be finite (see Rem. 12 in Sect. IV). We next summarize a condition that is necessary and sufficient for any k being contained in $\mathcal{K}^\alpha$. This condition can be easily derived from the following relation between the sets $\mathcal{S}_k^\alpha$ and $\mathcal{R}_j^\alpha$. In fact, realizing that

$$\mathcal{S}_k^\alpha = \bigcap_{j=0}^{k} (A^j)^{-1}(\mathcal{X} \ominus \mathcal{R}_j^\alpha). \tag{11}$$

holds as a consequence of [1, Eqs. (5.1)–(5.2)], allows one to identify the following condition that result in (5).

Lemma 4: Let $\alpha \in \mathbb{R}_+$, let $k \in \mathbb{N}$, and let $\mathcal{K}^\alpha$ be defined as in (10). Then, $k \in \mathcal{K}^\alpha$ if and only if

$$\mathcal{S}_k^\alpha \subseteq (A^{k+1})^{-1}(\mathcal{X} \ominus \mathcal{R}_{k+1}^\alpha). \tag{12}$$

Proof: According to (10), we have $k \in \mathcal{K}^\alpha$ if and only if $\mathcal{S}_{k+1}^\alpha = \mathcal{S}_k^\alpha$. Clearly, (11) implies

$$\mathcal{S}_{k+1}^\alpha = (A^{k+1})^{-1}(\mathcal{X} \ominus \mathcal{R}_{k+1}^\alpha) \cap \mathcal{S}_k^\alpha \tag{13}$$

for every $k \in \mathbb{N}$. As a consequence, we obtain $\mathcal{S}_{k+1}^\alpha = \mathcal{S}_k^\alpha$ whenever (12) holds and $\mathcal{S}_{k+1}^\alpha \subset \mathcal{S}_k^\alpha$ otherwise. ■

B. $N(\alpha)$ *is nonincreasing for supercritical scalings*

Based on condition (12), it is straightforward to show that $N(\alpha)$ is nonincreasing on (α^*, ∞).

Theorem 5: Let α^* and N be defined as in (8) and (9), respectively. Then, $N(\alpha_1) \geq N(\alpha_2)$ for every $\alpha_1, \alpha_2 \in \mathbb{R}_+$ with $\alpha^* < \alpha_1 \leq \alpha_2$.

Proof: By definition of $N(\alpha)$, it is sufficient to show that $\mathcal{K}^{\alpha_1} \subseteq \mathcal{K}^{\alpha_2}$ or, equivalently, that

$$k \in \mathcal{K}^{\alpha_1} \quad \Longrightarrow \quad k \in \mathcal{K}^{\alpha_2} \tag{14}$$

for some $k \in \mathbb{N}$. Now, $\alpha_1 > \alpha^*$ implies $\mathcal{R}_\infty^{\alpha_1} \nsubseteq \mathcal{X}$ by definition of α^* in (8). As a consequence, we find $\mathcal{R}_k^{\alpha_1} \nsubseteq \mathcal{X}$ and thus

$$\mathcal{X} \ominus \mathcal{R}_k^{\alpha_1} = \emptyset \tag{15}$$

for a finite $k \in \mathbb{N}$. Clearly, (15) implies

$$\mathcal{S}_{k+1}^{\alpha_1} = \mathcal{S}_k^{\alpha_1} = \emptyset \tag{16}$$

according to (13) and thus $k \in \mathcal{K}^{\alpha_1}$. It is easy to see that (15) is not only sufficient but also necessary for $k \in \mathcal{K}^{\alpha_1}$. To this end, we assume that (15) does not hold for some $k \in \mathcal{K}^{\alpha_1}$ and show that a contradiction results. Clearly, having $\mathcal{X} \ominus \mathcal{R}_k^{\alpha_1} \neq \emptyset$ implies $\mathcal{X} \ominus \mathcal{R}_j^{\alpha_1} \neq \emptyset$ for every $j \in \mathbb{N}_{[0,k-1]}$ (due to $\mathcal{R}_j^{\alpha_1} \subseteq \mathcal{R}_k^{\alpha_1}$) and thus $\mathcal{S}_k^{\alpha_1} \neq \emptyset$ according to (11) (since $0 \in \mathcal{S}_k^{\alpha_1}$ is guaranteed). Now, $k \in \mathcal{K}^{\alpha_1}$ implies $\mathcal{S}_\infty^{\alpha_1} = \mathcal{S}_{k+1}^{\alpha_1} = \mathcal{S}_k^{\alpha_1}$. However, $\mathcal{R}_\infty^{\alpha_1} \nsubseteq \mathcal{X}$ implies $\mathcal{S}_\infty^{\alpha_1} = \emptyset$, which contradicts $\mathcal{S}_\infty^{\alpha_1} = \mathcal{S}_k^{\alpha_1} \neq \emptyset$.

In summary, for $\alpha_1 > \alpha^*$, $k \in \mathcal{K}^{\alpha_1}$ is equivalent to (15). As a consequence, (14) is equivalent to

$$\mathcal{X} \ominus \mathcal{R}_k^{\alpha_1} = \emptyset \quad \Longrightarrow \quad \mathcal{X} \ominus \mathcal{R}_k^{\alpha_2} = \emptyset. \tag{17}$$

Clearly, a sufficient condition for (17) would be

$$\mathcal{R}_k^{\alpha_1} \subseteq \mathcal{R}_k^{\alpha_2} \tag{18}$$

for every $k \in \mathbb{N}$. To see that (18) holds for $\alpha_2 \geq \alpha_1$, note that $\mathcal{R}_k^{\alpha} = \alpha \mathcal{R}_k^1$ for every $\alpha \in \mathbb{R}_+$ as apparent from (7). We thus have $\mathcal{R}_k^{\alpha_2} = \frac{\alpha_2}{\alpha_1} \mathcal{R}_k^{\alpha_1}$, which implies (18) since $\mathcal{R}_k^{\alpha}$ is known to be convex with $0 \in \mathcal{R}_k^{\alpha}$ for every $\alpha \in \mathbb{R}_+$. ∎

Remark 6: The proof of Theorem 5 shows that $\mathcal{S}_\infty^{\alpha} = \emptyset$ for every $\alpha \in (\alpha^*, \infty)$. Analogously, it is easy to show that $\mathcal{S}_\infty^{\alpha} \neq \emptyset$ for every $\alpha \in (0, \alpha^*]$. Both observations are in line with [11, Thm. 6].

C. *Overestimation is nondecreasing for subcritical scalings*

We next address the behavior of $N(\alpha)$ for scalings $\alpha < \alpha^*$. It will turn out that proving monotonicity of N on $(0, \alpha^*)$ is more complicated than on (α^*, ∞) as in Section III-B. We thus first analyze a nondecreasing overestimation for N on $(0, \alpha^*)$. As a preparation, we introduce and characterize the following set $\underline{\mathcal{K}}^{\alpha}$.

Lemma 7: Let α^* and $\mathcal{K}^{\alpha}$ be defined as in (8) and (10). Then, $\underline{\mathcal{K}}^{\alpha} \subseteq \mathcal{K}^{\alpha}$ for every $\alpha \in \mathbb{R}_+$ with $\alpha < \alpha^*$, where

$$\underline{\mathcal{K}}^{\alpha} := \left\{ k \in \mathbb{N} \,|\, A^{k+1} \mathcal{X} \subseteq \left(1 - \frac{\alpha}{\alpha^*}\right) \mathcal{X} \right\} \tag{19}$$

is a nonempty set.

Proof: First note that $\underline{\mathcal{K}}^{\alpha}$ is nonempty for every $\alpha \in (0, \alpha^*)$ since Schur stability of A implies the existence of a finite k such that $A^{k+1}\mathcal{X}$ is contained in the C-set $\left(1 - \frac{\alpha}{\alpha^*}\right) \mathcal{X}$. We will next show that

$$k \in \underline{\mathcal{K}}^{\alpha} \quad \Longrightarrow \quad k \in \mathcal{K}^{\alpha} \tag{20}$$

by proving that fulfillment of the condition

$$A^{k+1} \mathcal{X} \subseteq \left(1 - \frac{\alpha}{\alpha^*}\right) \mathcal{X} \tag{21}$$

implies (12). To this end, first note that we have

$$\mathcal{R}_{k+1}^{\alpha} \subseteq \mathcal{R}_\infty^{\alpha} \subseteq \frac{\alpha}{\alpha^*} \mathcal{R}_\infty^{\alpha^*} \subseteq \frac{\alpha}{\alpha^*} \mathcal{X}$$

by construction and by definition of α^*. We thus obtain

$$\left(1 - \frac{\alpha}{\alpha^*}\right) \mathcal{X} \subseteq \mathcal{X} \ominus \mathcal{R}_{k+1}^{\alpha}. \tag{22}$$

Combining (3), (21), and (22) yields

$$\mathcal{X} \subseteq (A^{k+1})^{-1} A^{k+1} \mathcal{X} \subseteq (A^{k+1})^{-1} \left(\mathcal{X} \ominus \mathcal{R}_{k+1}^{\alpha}\right). \tag{23}$$

This completes the proof since the r.h.s. in (12) and (23) are equivalent and since $\mathcal{S}_k^{\alpha} \subseteq \mathcal{X}$ holds by construction. ∎

Based on the findings in Lemma 7, we easily identify the following overestimation for N on $(0, \alpha^*)$.

Corollary 8: Let α^* and N be defined as in (8) and (9). Then, $N(\alpha) \leq \overline{N}(\alpha)$ for every $\alpha \in \mathbb{R}_+$ with $\alpha < \alpha^*$, where

$$\overline{N}(\alpha) := \min \underline{\mathcal{K}}^{\alpha}. \tag{24}$$

As summarized in the following lemma, the overestimation $\overline{N}$ is indeed nondecreasing on $(0, \alpha^*)$.

Lemma 9: Let α^* and $\overline{N}$ be defined as in (8) and (24). Then, $\overline{N}(\alpha_1) \leq \overline{N}(\alpha_2)$ for every $\alpha_1, \alpha_2 \in \mathbb{R}_+$ with $\alpha_1 \leq \alpha_2 < \alpha^*$.

Proof: By definition of $\overline{N}(\alpha)$, it is sufficient to show that $\underline{\mathcal{K}}^{\alpha_2} \subseteq \underline{\mathcal{K}}^{\alpha_1}$ or, equivalently, that

$$k \in \underline{\mathcal{K}}^{\alpha_2} \quad \Longrightarrow \quad k \in \underline{\mathcal{K}}^{\alpha_1} \tag{25}$$

for any $k \in \mathbb{N}$. Since we obviously have

$$\left(1 - \frac{\alpha_2}{\alpha^*}\right) \mathcal{X} \subseteq \left(1 - \frac{\alpha_1}{\alpha^*}\right) \mathcal{X},$$

relation (25) is trivially fulfilled by definition of $\underline{\mathcal{K}}^{\alpha}$. ∎

D. *Nondecrease of* $N(\alpha)$ *for subcritical scalings*

Roughly speaking, the construction and the monotonicity of the overestimation $\overline{N}$ build on the observation that (21) implies (12). Obviously, the conditions in (12) and (21) both depend on α. However, the dependence on α in (21) is much "simpler" than in (12). In fact, while only the r.h.s. in (21) is changing with α, the sets on both sides of the condition in (12) depend on α. This characteristic makes it difficult to prove monotonicity of N on $(0, \alpha^*)$. To see this, note that proving nondecrease of N on $(0, \alpha^*)$ requires to show that

$$k \in \mathcal{K}^{\alpha_2} \quad \Longrightarrow \quad k \in \mathcal{K}^{\alpha_1} \tag{26}$$

for every $\alpha_1, \alpha_2 \in \mathbb{R}_+$ with $\alpha_1 \leq \alpha_2 < \alpha^*$. Taking Lemma 4 into account and introducing the sets

$$\mathcal{T}_k^{\alpha} := (A^k)^{-1} (\mathcal{X} \ominus \mathcal{R}_k^{\alpha}) \tag{27}$$

for every $k \in \mathbb{N}$, we find that (26) is equivalent to

$$\mathcal{S}_k^{\alpha_2} \subseteq \mathcal{T}_{k+1}^{\alpha_2} \quad \Longrightarrow \quad \mathcal{S}_k^{\alpha_1} \subseteq \mathcal{T}_{k+1}^{\alpha_1}. \tag{28}$$

Now, proving (28) is difficult since both $\mathcal{S}_k^\alpha$ and $\mathcal{T}_{k+1}^\alpha$ are shrinking with increasing α. In other words, it is not straightforward to show that the second condition in (28) is satisfied whenever the first condition holds. For some special situations, however, (28) can be proven.

Proposition 10: Let α^* and N be defined as in (8) and (9). Assume there exists an $\eta \in [0,1)$ such that $A^2 = \eta\, A$. Then, $N(\alpha_1) \leq N(\alpha_2)$ for every $\alpha_1, \alpha_2 \in \mathbb{R}_+$ with $\alpha_1 \leq \alpha_2 \leq \alpha^*$.

Proof: As discussed above, $N(\alpha_1) \leq N(\alpha_2)$ holds if and only if (28) is satisfied. Now, assume the first condition in (28) holds for $k = 0$, i.e., $\mathcal{S}_0^{\alpha_2} = \mathcal{X} \subseteq \mathcal{T}_1^{\alpha_2}$. Then, we obviously have $\mathcal{S}_0^{\alpha_1} = \mathcal{X} \subseteq \mathcal{T}_1^{\alpha_1}$ due to $\mathcal{T}_1^{\alpha_2} \subseteq \mathcal{T}_1^{\alpha_1}$ (as apparent from (27)). In contrast, if $\mathcal{S}_0^{\alpha_2} \nsubseteq \mathcal{T}_1^{\alpha_2}$, we will show that $A^2 = \eta\, A$ implies that both conditions in (28) hold for $k = 1$ and thus for every $k \in \mathbb{N}$ with $k > 1$. As a consequence, (28) holds for every $k \in \mathbb{N}$. It remains to show that

$$\mathcal{S}_1^\alpha \subseteq \mathcal{T}_2^\alpha \tag{29}$$

for every $\alpha \in (0, \alpha^*]$. To see this, first note that

$$\mathcal{S}_1^\alpha = \mathcal{S}_0^\alpha \cap \mathcal{T}_1^\alpha = \mathcal{X} \cap \mathcal{T}_1^\alpha$$

according to (13). A sufficient condition for (29) to hold would thus be

$$\mathcal{T}_1^\alpha \subseteq \mathcal{T}_2^\alpha. \tag{30}$$

Taking $A^2 = \eta\, A$ and (27) into account, (30) can be rewritten as

$$(A)^{-1}(\mathcal{X} \ominus \mathcal{R}_1^\alpha) \subseteq (\eta\, A)^{-1}(\mathcal{X} \ominus \mathcal{R}_2^\alpha). \tag{31}$$

To prove (31), we will show that

$$\eta\, (\mathcal{X} \ominus \mathcal{R}_1^\alpha) \subseteq \mathcal{X} \ominus \mathcal{R}_2^\alpha. \tag{32}$$

Now, since $\mathcal{R}_2^\alpha = \mathcal{R}_1^\alpha \oplus AE\mathcal{D}^\alpha$, (32) can be rewritten as

$$(\mathcal{X} \ominus \mathcal{R}_1^\alpha) \ominus (1-\eta)\, (\mathcal{X} \ominus \mathcal{R}_1^\alpha) \subseteq (\mathcal{X} \ominus \mathcal{R}_1^\alpha) \ominus AE\mathcal{D}^\alpha. \tag{33}$$

As a consequence, (33) holds if and only if

$$(1-\eta)\, (\mathcal{X} \ominus \mathcal{R}_1^\alpha) \supseteq AE\mathcal{D}^\alpha. \tag{34}$$

To show (34), first note that $\mathrm{cl}(\mathcal{R}_\infty^\alpha) \subseteq \mathcal{X}$ for every $\alpha \in (0, \alpha^*]$ by definition of α^*. Moreover, $A^2 = \eta\, A$ implies

$$\mathcal{R}_\infty^\alpha = \bigoplus_{k=0}^{\infty} A^k E\, \mathcal{D}^\alpha = E\, \mathcal{D}^\alpha \oplus \left(\bigoplus_{k=1}^{\infty} \eta^{k-1} A\, E\, \mathcal{D}^\alpha \right)$$

according to (7) and consequently

$$\mathrm{cl}(\mathcal{R}_\infty^\alpha) = E\, \mathcal{D}^\alpha \oplus \frac{1}{1-\eta}\, A\, E\, \mathcal{D}^\alpha. \tag{35}$$

From (35), we infer

$$\mathrm{cl}(\mathcal{R}_\infty^\alpha) \ominus \mathcal{R}_1^\alpha = \frac{1}{1-\eta}\, A\, E\, \mathcal{D}^\alpha.$$

The set on the l.h.s. in (34) thus satisfies

$$(1-\eta)\, (\mathcal{X} \ominus \mathcal{R}_1^\alpha) \supseteq (1-\eta)\, (\mathrm{cl}(\mathcal{R}_\infty^\alpha) \ominus \mathcal{R}_1^\alpha) = A\, E\, \mathcal{D}^\alpha,$$

which completes the proof. ∎

Note that the condition $A^2 = \eta A$ in Proposition 10 is similar to but even stricter than the condition in [1, Rem. 4.2]. Nevertheless, there exist systems in the literature that satisfy this special condition and we will discuss two examples in Section IV-A. We currently have no proof for N being nondecreasing on $(0, \alpha^*)$ for the more general case $A^2 \neq \eta A$. We show, however, that N is nondecreasing on $(0, \alpha^*)$ for an example with $A^2 \neq \eta A$ in Section IV-B. Moreover, we subsequently present some observations that may allow to prove monotonicity of N on $(0, \alpha^*)$ in general. To this end, note that (28) is equivalent to

$$\mathcal{S}_k^{\alpha_1} \nsubseteq \mathcal{T}_{k+1}^{\alpha_1} \quad \Longrightarrow \quad \mathcal{S}_k^{\alpha_2} \nsubseteq \mathcal{T}_{k+1}^{\alpha_2}. \tag{36}$$

We could thus prove $N(\alpha_1) \leq N(\alpha_2)$ by showing that (36) applies. To study this argument in more detail, let us assume the constraints $\mathcal{X}$ are polytopic and can be written as

$$\mathcal{X} = \{x \in \mathbb{R}^n \,|\, H\, x \leq c\} \tag{37}$$

for some $H \in \mathbb{R}^{l \times n}$, $c \in \mathbb{R}^l$, and $l \in \mathbb{N}$ with $l \geq n+1$. As apparent from [17, p. 200], we then find

$$\mathcal{T}_k^\alpha = \{x \in \mathbb{R}^n \,|\, HA^k x \leq c - h_{\mathcal{R}_k^\alpha}(H)\},$$

where

$$h_{\mathcal{R}_k^\alpha}(H) = \alpha\, h_{\mathcal{R}_k^1}(H) = \alpha \sum_{j=0}^{k-1} h_{\mathcal{D}^*}(HA^j E).$$

Based on this observation, we can specify (36) as follows. Having $\mathcal{S}_k^{\alpha_1} \nsubseteq \mathcal{T}_{k+1}^{\alpha_1}$ implies the existence of an $i \in \mathbb{N}_{[1,l]}$ such that

$$h_{\mathcal{S}_k^{\alpha_1}}(e_i^\top HA^{k+1}) > h_{\mathcal{T}_{k+1}^{\alpha_1}}(e_i^\top HA^{k+1}) \tag{38}$$

and

$$h_{\mathcal{T}_{k+1}^{\alpha_1}}(e_i^\top HA^{k+1}) = e_i^\top c - \alpha_1\, h_{\mathcal{R}_{k+1}^1}(e_i^\top H), \tag{39}$$

where (39) is a technical condition that guarantees that the hyperplane $\mathcal{H}_{k+1,i}^{\alpha_1}$ with

$$\mathcal{H}_{k,i}^{\alpha_1} := \{x \in \mathbb{R}^n \,|\, e_i^\top HA^k x = e_i^\top c - \alpha_1\, h_{\mathcal{R}_k^1}(e_i^\top H)\}$$

has a nonempty intersection with $\partial\mathcal{T}_{k+1}^{\alpha_1}$ and is thus not redundant in the description of $\mathcal{T}_{k+1}^{\alpha_1}$. To prove (36), it is now sufficient to show that (38) and (39) imply

$$h_{\mathcal{S}_k^{\alpha_2}}(e_i^\top HA^{k+1}) > h_{\mathcal{T}_{k+1}^{\alpha_2}}(e_i^\top HA^{k+1}). \tag{40}$$

At this point, note that (40) is not necessary for (36) to be satisfied. In fact, the violated constraint $i \in \mathbb{N}_{[1,l]}$ in (38) and (40) could, in principle, be different. However, having the same i in (38), (39) and (40), promotes the following conclusions. First, the technical condition (39) implies $\mathcal{H}_{k+1,i}^{\alpha_2} \cap \partial\mathcal{T}_{k+1}^{\alpha_2} \neq \emptyset$ and thus

$$h_{\mathcal{T}_{k+1}^{\alpha_1}}(e_i^\top HA^{k+1}) - h_{\mathcal{T}_{k+1}^{\alpha_2}}(e_i^\top HA^{k+1}) = \Delta\alpha\, h_{\mathcal{R}_{k+1}^1}(e_i^\top H), \tag{41}$$

where $\Delta\alpha := \alpha_2 - \alpha_1$. Second, to prove (40), it remains to show that (38) and (39) imply

$$h_{\mathcal{S}_k^{\alpha_1}}(e_i^\top H A^{k+1}) - h_{\mathcal{S}_k^{\alpha_2}}(e_i^\top H A^{k+1}) \leq \Delta\alpha \, h_{\mathcal{R}_{k+1}^1}(e_i^\top H). \tag{42}$$

In fact, (38) in combination with (41) and (42) obviously leads to (40). Now, while it is non-trivial to prove (42) in general, it can be verified for the examples in Section IV-B.

IV. Numerical Examples

We subsequently analyze three illustrative examples from the literature. The two first examples satisfy the special condition $A^2 = \eta A$. For those examples, N being nondecreasing on $(0, \alpha^*)$ and nonincreasing on (α^*, ∞) is guaranteed by Proposition 10 and Theorem 5. The third example offers $A^2 \neq \eta\, A$. While this case in not covered by Proposition 10, we still find that N is nondecreasing on $(0, \alpha^*)$.

A. Two examples satisfying $A^2 = \eta\, A$

Example 1: Consider system (1) with $A = 0.5$ and $E = 1$ and the constraints $\mathcal{X} = [-2, 2]$ and $\mathcal{D}^* = [-1, 1]$ as in [11, Exmp. 1]. Note that the same system, without state constraints, was also analyzed in [17, Exmp. 6.10]. As shown in [11], we find

$$\mathcal{R}_k^\alpha = \{x \in \mathbb{R} \,|\, |x| \leq \rho_k^\alpha\} \quad \text{with} \quad \rho_k^\alpha := 2\,\alpha\left(1 - 0.5^k\right) \tag{43}$$

for every $k \in \mathbb{N}$. As a consequence, we obtain $\mathcal{R}_\infty^1 = (-2, 2)$ and thus $\alpha^* = 1$. For supercritical scalings $\alpha > \alpha^*$, the smallest k such that $\mathcal{X} \ominus \mathcal{R}_k^\alpha = \emptyset$ determines $N(\alpha)$ as apparent from the proof of Theorem 5. In this context, we first find

$$\mathcal{X} \ominus \mathcal{R}_k^\alpha = \{x \in \mathbb{R} \,|\, |x| \leq 2 - \rho_k^\alpha\}$$

for every $k \in \mathbb{N}$ (and every $\alpha \in \mathbb{R}_+$). We thus have

$$\mathcal{X} \ominus \mathcal{R}_k^\alpha = \emptyset \iff 2 - \rho_k^\alpha < 0 \iff 0.5^k < \frac{\alpha - 1}{\alpha} \tag{44}$$

and consequently

$$N(\alpha) = \min\left\{ k \in \mathbb{N} \,\middle|\, k > \log_2\left(\frac{\alpha}{\alpha - 1}\right)\right\} \tag{45}$$

for every $\alpha \in (1, \infty)$. Evaluating $\mathcal{R}_k^\alpha$ for $k = 1$ also allows to compute

$$\mathcal{T}_1^\alpha = \{x \in \mathbb{R} \,|\, |0.5\, x| \leq 2 - \alpha\}$$

according to (27). Clearly, $\mathcal{T}_1^\alpha \supseteq \mathcal{X}$ for every $\alpha \in (0, 1]$. As a consequence, we find

$$\mathcal{S}_1^\alpha = \mathcal{T}_1^\alpha \cap \mathcal{S}_0^\alpha = \mathcal{T}_1^\alpha \cap \mathcal{X} = \mathcal{X}$$

and thus $N(\alpha) = 0$ for every $\alpha \in (0, 1]$. The FDI for this example is illustrated in Figure 1.(a). Clearly, N is nondecreasing on $(0, \alpha^*)$ and nonincreasing on (α^*, ∞) as predicted by Proposition 10 and Theorem 5. Note that this example is covered by Proposition 10, since we obviously have $A^2 = 0.5\, A$. Figure 1.(a) also shows the nondecreasing overestimation $\overline{N}$ from (24) for this example. In this context, we find

$$A^{k+1}\mathcal{X} \subseteq \left(1 - \frac{\alpha}{\alpha^*}\right)\mathcal{X} \iff 0.5^{k+1} \leq 1 - \alpha$$

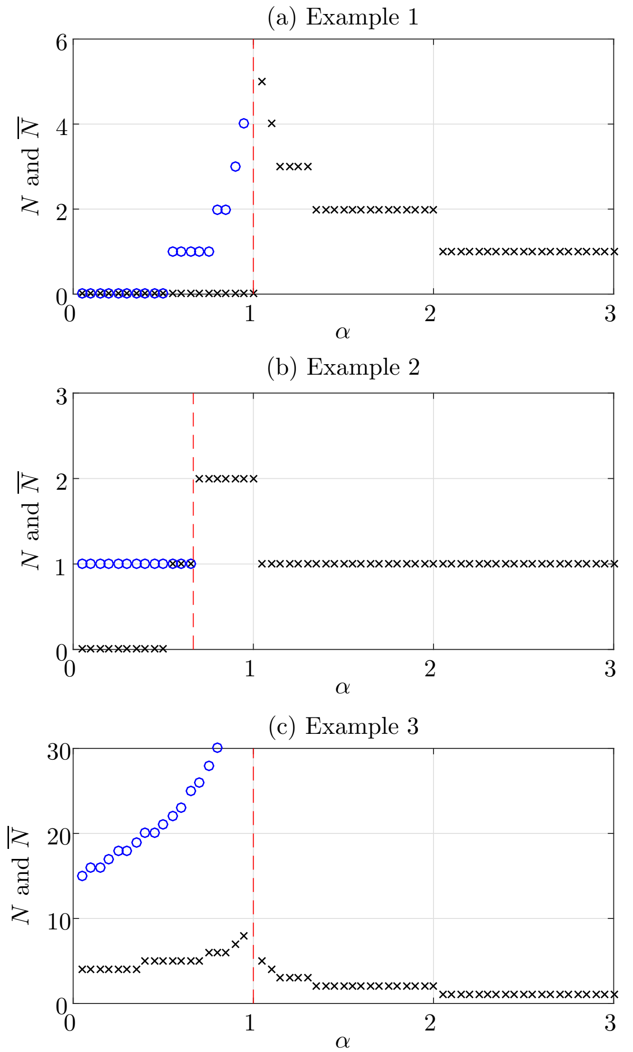

Fig. 1. Illustration of the FDI N for (a) Example 1, (b) Example 2, and (c) Example 3 for some scalings $\alpha \in \mathbb{R}_+$ (balck crosses). In addition, the overestimation $\overline{N}$ is shown for some subcritical scalings $\alpha < \alpha^*$ (blue circles). For every example, the critical scaling α^* is marked by the red dashed line.

and thus

$$\overline{N}(\alpha) = \min\left\{k \in \mathbb{N} \,|\, k \geq -\log_2(1 - \alpha) - 1\right\}$$

for every $\alpha \in (0, 1)$.

Example 2: Consider system (1) with

$$A = \begin{pmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the constraints $\mathcal{X} = \{x \in \mathbb{R}^2 \,|\, \|x\|_\infty \leq 1\}$ and $\mathcal{D}^* = \{d \in \mathbb{R}^2 \,|\, \|d\|_1 \leq 1\}$ as in [6, p. 114]. Clearly, A is nilpotent since $A^2 = 0$. We thus have

$$\mathcal{R}_\infty^\alpha = \mathcal{R}_2^\alpha = A\,\mathcal{D}^\alpha \oplus \mathcal{D}^\alpha = \alpha\,\mathrm{conv}\left\{\pm\begin{pmatrix}1.5\\0.5\end{pmatrix}, \pm\begin{pmatrix}0.5\\1.5\end{pmatrix}\right\}$$

and consequently $\alpha^* = 2/3$. In addition, we obtain

$$\mathcal{X} \ominus \mathcal{R}_1^\alpha = \mathcal{X} \ominus \mathcal{D}^\alpha = \{x \in \mathbb{R}^2 \,|\, \|x\|_\infty \leq 1 - \alpha\} \tag{46}$$

and

$$\mathcal{X} \ominus \mathcal{R}_k^\alpha = \mathcal{X} \ominus \mathcal{R}_2^\alpha = \{x \in \mathbb{R}^2 \,|\, \|x\|_\infty \leq 1 - 1.5\,\alpha\}$$

for every $k \in \mathbb{N}$ with $k \geq 2$. We thus find $N(\alpha) = 2$ for every $\alpha \in (2/3, 1]$ and $N(\alpha) = 1$ for every $\alpha \in (1, \infty)$. From (46), we also infer

$$\begin{aligned}\mathcal{T}_1^\alpha &= \{x \in \mathbb{R}^2 \,|\, \|A\,x\|_\infty \leq 1 - \alpha\} \\ &= \{x \in \mathbb{R}^2 \,|\, |x_1 - x_2| \leq 2 - 2\,\alpha\}\end{aligned}$$

and

$$\mathcal{T}_2^\alpha = \{x \in \mathbb{R}^2 \,|\, \|A^2 x\|_\infty \leq 1 - 1.5\,\alpha\} = \{x \in \mathbb{R}^2 \,|\, 1.5\,\alpha \leq 1\},$$

where the description of $\mathcal{T}_2^\alpha$ results from $A^2 = 0$. It is easy to see that $\mathcal{T}_1^\alpha \supseteq \mathcal{X}$ for every $\alpha \in (0, 0.5]$. We thus have $\mathcal{S}_1^\alpha = \mathcal{S}_0^\alpha$ and $N(\alpha) = 0$ for every $\alpha \in (0, 0.5]$. Moreover, we obtain $\mathcal{T}_2^\alpha = \mathbb{R}^2$ for every $\alpha \in (0, 2/3]$ (and $\mathcal{T}_2^\alpha = \emptyset$ for $\alpha > 2/3$). We thus have $\mathcal{S}_2^\alpha = \mathcal{S}_1^\alpha \subset \mathcal{S}_0^\alpha = \mathcal{X}$ and consequently $N(\alpha) = 1$ for every $\alpha \in (0.5, 2/3]$. Finally, regarding the overestimation $\overline{N}$, we find that condition (21) is violated for $k = 0$ and any $\alpha \in (0, \alpha^*)$ due to

$$A\,\mathcal{X} = \{x \in \mathbb{R}^2 \,|\, -1 \leq x_1 = x_2 \leq 1\} \nsubseteq \left(1 - \frac{\alpha}{\alpha^*}\right)\mathcal{X}.$$

However, (21) is satisfied for $k = 1$ and every $\alpha \in (0, \alpha^*)$ since we have

$$A^2 \mathcal{X} = \{0\} \subseteq \left(1 - \frac{\alpha}{\alpha^*}\right)\mathcal{X}.$$

We consequently obtain $\overline{N}(\alpha) = 1$ for every $\alpha \in (0, \alpha^*)$. The FDI and the overestimation $\overline{N}$ are illustrated in Figure 1.(b). Clearly, the monotonicities of both functions are as expected.

Remark 11: We have $N(\alpha) \in \{0, 1\}$ for every subcritical scaling in Examples 1 and 2. This observations is in line with the proof of Proposition 10. In fact, from that proof, we easily infer that $N(\alpha) \in \{0, 1\}$ is guaranteed for every $\alpha \in (0, \alpha^*)$ whenever the condition $A^2 = \eta\, A$ is satisfied.

B. An example with $A^2 \neq \eta\, A$

Example 3: Consider system (1) with

$$A = \begin{pmatrix} 0.5 & 2.0 \\ 0.0 & 0.9 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the constraints $\mathcal{X} = \{x \in \mathbb{R}^2 \,|\, \|x\|_\infty \leq 2\}$ and $\mathcal{D}^* = [-1, 1]$ as in [11, Exmp. 2]. Note that the system can be seen as an extension of Example 1. In fact, for $x_2 = 0$, the system state x_1 has the same dynamics and constraints as the system in Example 1. Taking this into account, it is not surprisingly that $\mathcal{R}_k^\alpha$ and $\mathcal{X} \ominus \mathcal{R}_k^\alpha$ evaluate to

$$\begin{aligned}\mathcal{R}_k^\alpha &= \{x \in \mathbb{R}^2 \,|\, |x_1| \leq \rho_k^\alpha,\, x_2 = 0\} \quad \text{and} \\ \mathcal{X} \ominus \mathcal{R}_k^\alpha &= \{x \in \mathbb{R}^2 \,|\, |x_1| \leq 2 - \rho_k^\alpha,\, |x_2| \leq 2\}\end{aligned}$$

with ρ_k^α as in (43). We thus find the same critical scaling $\alpha^* = 1$ as in Example 1. Moreover, the relations in (44) also hold for the present example, which implies that $N(\alpha)$ is as in (45) for every $\alpha \in (\alpha^*, \infty)$. To study the FDI for subcritical scalings, first note that

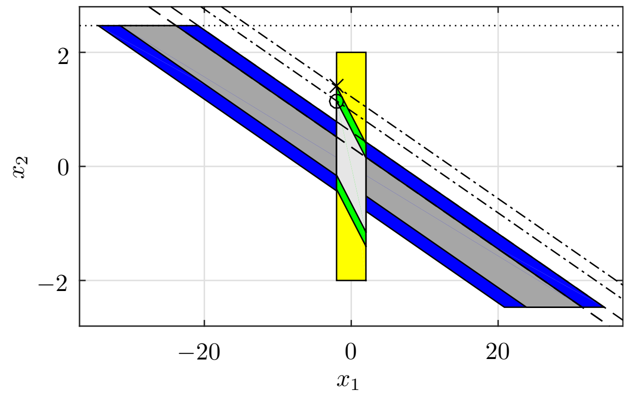

Fig. 2. Illustration of the sets $\mathcal{S}_1^\alpha$, $\mathcal{T}_2^\alpha$, and $\mathcal{X}$ for Example 3 and two different $\alpha \in (0, \alpha^*)$. We see (from the background to the foreground) $\mathcal{T}_2^{\alpha_1}$ in blue, $\mathcal{T}_2^{\alpha_2}$ in dark gray, $\mathcal{X}$ in yellow, $\mathcal{S}_1^{\alpha_1}$ in green, and $\mathcal{S}_1^{\alpha_2}$ in light gray for $\alpha_1 = 0.2$ and $\alpha_2 = 0.7$. The dashed lines show the parallel hyperplanes $\mathcal{H}_{2,1}^{\alpha_1}$ and $\mathcal{H}_{2,1}^{\alpha_2}$. The dotted line depicts the hyperplane $\mathcal{H}_{2,2}^{\alpha_1} = \mathcal{H}_{2,2}^{\alpha_2}$. The cross and the circle refer to the optimizer x^* for (49) with $\alpha = \alpha_1$ and $\alpha = \alpha_2$, respectively. The dash-dotted lines run parallel to $\mathcal{H}_{2,1}^{\alpha_1}$ (and $\mathcal{H}_{2,1}^{\alpha_2}$) through these optimizers.

$$A^k = \begin{pmatrix} 0.5^k & 5\,(0.9^k - 0.5^k) \\ 0.0 & 0.9^k \end{pmatrix}.$$

We thus obtain

$$\mathcal{T}_k^\alpha = \{x \in \mathbb{R}^2 \,|\, |e_1^\top A^k x| \leq 2 - \rho_k^\alpha,\, |0.9^k x_2| \leq 2\} \tag{47}$$

for every $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}_+$, where

$$e_1^\top A^k x = 0.5^k x_1 + 5\,(0.9^k - 0.5^k)\,x_2.$$

Having (47) and taking the description for $\mathcal{S}_k^\alpha$ from (11) into account, allows to verify the condition $\mathcal{S}_k^\alpha \subseteq \mathcal{T}_{k+1}^\alpha$ for given $k \in \mathbb{N}$ and $\alpha \in (0, \alpha^*)$. In fact, it can be shown that

$$\mathcal{S}_k^\alpha \subseteq \mathcal{T}_{k+1}^\alpha \iff 1 + \alpha \leq (1 - \alpha)\left(\frac{0.25}{0.5^k} - \frac{1.25}{0.9^k}\right). \tag{48}$$

Some resulting function values of N are illustrated in Figure 1.(c). Clearly, N in nondecreasing on $(0, \alpha^*)$ as expected. The same observation holds for the overestimation $\overline{N}$, which can be derived from the relation

$$A^{k+1}\mathcal{X} \subseteq \left(1 - \frac{\alpha}{\alpha^*}\right)\mathcal{X} \iff 5 \cdot 0.9^{k+1} - 4 \cdot 0.5^{k+1} \leq 1 - \alpha.$$

For this example, we obviously have $A^2 \neq \eta A$. Thus, N being nondecreasing on $(0, \alpha^*)$ cannot be predicted by Prop. 10. Nevertheless, the argumentation in the last paragraph of Section III-D can be applied to motivate the observed monotonicity of N from a theoretical point of view. To this end, first note that $\mathcal{X}$ can be written as in (37) with

$$H = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}^\top \quad \text{and} \quad c = \begin{pmatrix} 2 & 2 & 2 & 2 \end{pmatrix}^\top.$$

We next study relation (36) for $k = 1$ and $\alpha_1 = 0.2$ and $\alpha_2 = 0.7$. In this context, it is apparent from Figure 2 that (38) holds for $i = 1$. It can also be seen that the technical condition (39) is satisfied in this case. As a consequence, we find

$$h_{\mathcal{T}_2^{\alpha_1}}(e_1^\top HA^2) - h_{\mathcal{T}_2^{\alpha_2}}(e_1^\top HA^2) = \Delta\alpha\, h_{\mathcal{R}_2^1}(e_1^\top H) = 0.75$$

as predicted by (41). Now, to verify (42), first note that

$$h_{\mathcal{S}_1^\alpha}(e_1^\top HA^2) = \sup_{x\in\mathcal{S}_1^\alpha} e_1^\top HA^2 x = 3.7 - 1.4\,\alpha \qquad (49)$$

for every $\alpha \in (0, \alpha^*)$. We thus have

$$h_{\mathcal{S}_1^{\alpha_1}}(e_1^\top HA^2) - h_{\mathcal{S}_k^{\alpha_2}}(e_1^\top HA^2) = 1.4\,\Delta\alpha = 0.7 < 0.75$$

and consequently (40) for $k = i = 1$. Using a similar argumentation, (36) can be verified for every $k \in \mathbb{N}$. In summary, this proves $N(\alpha_1) \leq N(\alpha_2)$ as already observed in Figure 1.(c). At this point, it is interesting to note that (42) does not hold for $i = 2$ (or $i = 4$). In fact, we have $h_{\mathcal{R}_2^1}(e_2^\top H) = 0$ but

$$h_{\mathcal{S}_1^{\alpha_1}}(e_2^\top HA^2) - h_{\mathcal{S}_k^{\alpha_2}}(e_2^\top HA^2) > 0$$

as apparent from Figure 2. While this observation is not critical for (36), it shows that the "direction" i will be important for proving (42).

Remark 12: It is interesting to note that $N(\alpha^*)$ is finite for Examples 1 and 2, which implies that the MRPI set $\mathcal{P}_{\max}^{\alpha^*}$ is finitely determined for these examples. In contrast, as shown in [11, Exmp. 2], $\mathcal{P}_{\max}^{\alpha^*}$ is not finitely determined for Example 3. As a consequence, we obtain $N(\alpha^*) = \infty$ for this example. Clearly, this can also be seen from (48). In fact, for $\alpha = 1 = \alpha^*$, the second condition in (48) is violated for every $k \in \mathbb{N}$ (due to $2 \nleq 0$).

V. Conclusions

The paper investigated the finite determinedness of maximal robust positively invariant (MRPI) sets as a function of the scaling factor α, which determines the size of the disturbance set. We introduced the finite determinedness index (FDI) $N(\alpha)$ and proved that $N(\alpha)$ is nonincreasing on (α^*, ∞) (see Thm. 5), where α^* is the critical scaling factor as defined in (8). We further showed that $N(\alpha)$ is nondecreasing on $(0, \alpha^*)$ under the assumption that the system matrix satisfies the restrictive condition $A^2 = \eta A$ for some $\eta \in [0, 1)$ (see Prop. 10). A proof for N being nondecreasing on $(0, \alpha^*)$ for the more general case $A^2 \neq \eta A$ is currently missing. However, supported by the example in Section IV-B, we expect that N is nondecreasing on $(0, \alpha^*)$ for a large group of linear systems. Clearly, future research has to address a formal proof for this conjecture. Furthermore, future research will investigate scaled state constraints of the form $\mathcal{X}^\beta := \beta\mathcal{X}^*$ instead of scaled disturbance sets $\mathcal{D}^\alpha$.

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