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Guidance Vector Field Encoding based on Contraction Analysis

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***Abstract*— The guidance vector field encoding problem is revisited via exploiting the recently developed differential Lyapunov framework for contraction analysis. In control applications, a designed guidance vector field is utilized to guide a particle (such as a robot, or an end actuator) to converge to and circulate along a target curve, such as robot navigation, or motion planing. This paper proposes some new methods for guidance vector filed encoding. The methods analyze the guidance vector field encoding problem from a theoretical perspective providing insights into the problem, and it shows that the contraction analysis is a natural theoretical tool for guidance vector encoding. Furthermore, the differences and relations between the proposed result and the previous methods are discussed. Simulation examples are presented to illustrate the effectiveness of the proposed methods.**

I. INTRODUCTION

Vector field based control methods have been widely adopted in many applications, such as robot navigation [1], contour/path following [2]–[4], pattern generation [5], and unmanned aerial vehicle control [6]–[9]. One major advantage of these methods is that an appropriately designed vector field can guide the system to approach the task configuration in a well behaved manner, while guaranteeing provable stability and convergence.

For the applications mentioned above, the key is to appropriately encode such a guidance vector field (GVF). In [10]–[12], a specially defined potential function [13], namely navigation function, is built, and then its negative gradient information is utilized to derive the GVF for obstacle avoidance and target following. However, the task configuration of this method is limited to a fixed point, because a closed orbit attractor is impossible to become a limit set in the gradient system (with integrable vector field) according to Theorem 7.2.1 in [14]. Recently, a control Lyapunov-Barrier function based strategy is proposed to address a stabilization problem with guaranteed safety [15], which is a similar problem solved by the navigation based strategy.

For the case that the task configuration is a closed curve without self-intersections, a method is proposed for $\mathbb{R}^n$ systems where the target curve is defined as the intersection of $n-1$ surfaces with *codimension* [16] of one [1]. This method utilizes the $n-1$ surfaces to construct a potential function, the negative gradient of which is then combined with additional terms to control the object to circulate on the target curve. A similar technique as [1] is used in [9] to derive a GVF for a sine curve in $\mathbb{R}^2$. In [2], a general GVF encoding method is developed for the case that the configuration manifolds are compact Lie groups. This approach relies on the derivations of a group error function and a potential function. The construction of the group error and the potential functions can be vastly different for some specific application scenarios as shown in [2]. Furthermore, [2] adopts a suspension technique to encode GVF for closed curves with self-intersections. To the best of our knowledge, the self-intersection case has only been studied in [2]. There are some other GVF encoding methods without considering the self-intersection case. An electrostatic analogy method is proposed in [5], and an attractor shape variation based method by warping a circular curve is proposed in [8]. In [17], the authors proposed a GVF encoding method for star curves in $\mathbb{R}^3$.

The purpose of this paper is to develop new GVF encoding approaches which can overcome the disadvantages of the existing methods. Specifically, two methods are proposed by revoking the works in [18] and [19], respectively. In [18], the authors propose a horizontal contraction [20] based stabilization law for nonlinear systems. In [19], the authors propose a robust online motion planing method for a class of nonlinear systems with disturbance. We show that the ideas of above two methods can be naturally adopted to encoding a new class of guidance vector fields. Based on [18], a horizontal contraction based GVF encoding (HC-GVF) method is proposed. The implementation of this method requires to solve a set of partial differential inequalities (PDIs). It is shown that the PDIs can be simplified by considering the encoding problem on an extended configuration, which is similar to the suspension technique in [2]. Based on [19], a integral differential GVF encoding method is proposed by taking advantages of the contraction property and online optimization. As a contrast, this method does not require to solve the PDIs, which simplifies its implementation. For the proposed result, the differential Lyapunov framework offers a powerful tool to theoretically analyze the encoded GVF. Furthermore, the differences between the proposed methods and the previous methods on GVF encoding are discussed.

This paper is organized as follows. Section II presents the horizontal contraction based GVF encoding method. Section III proposes the integral differential GVF encoding method.

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Section IV presents simulation results of some examples and further observations. Conclusions are given in Section V.

Notation and preliminaries: In this paper, we adopt the notations that are used in [20] and [21]. The *tangent space* of a *n-dimensional manifold* $\mathcal{M}$ at $x \in \mathcal{M}$ is denoted by $T_x\mathcal{M}$, and the *tangent bundle* of $\mathcal{M}$ is denoted by $T\mathcal{M} = \cup_{x\in\mathcal{M}}\{x\} \times T_x\mathcal{M}$. A *curve* γ on a given manifold $\mathcal{M}$, is a mapping $\gamma : \mathcal{I} \subset \mathbb{R} \to \mathcal{M}$, and γ is said to be closed if $\mathcal{I} = [a, b]$ and $\gamma(a) = \gamma(b)$. If a closed curve is without self-intersections, then it is defined as a simple closed curve; otherwise, it is called a singular closed curve.

Consider a manifold $\mathcal{M}$ and a system

$$\dot{x} = f(t, x) \tag{1}$$

where f is a C^1 and time varying *vector field*, which maps each $(t, x) \in \mathbb{R} \times \mathcal{M}$ to a tangent vector $f(t, x) \in T_x\mathcal{M}$. A C^1 curve γ_1 is an *integral curve* of f if $\dot{\gamma}_1 = f(t, \gamma_1)$. The *flow* of f is the collection of maps $\varphi(\cdot) : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ such that $t \to \varphi(t, x_0)$ is the integral curve of f with initial condition x_0. We then give a formal definition of GVF as follows.

Definition 1 (Guidance vector field): if the limit set of the flow of a vector field f on the manifold $\mathcal{M}$ is a curve γ, then f is said to be a *guidance vector field* (GVF) for γ.

Fig.1 depicts a GVF for an ellipse in $\mathbb{R}^2$. The arrows in Fig.1 show the directions of the vector field at local positions. The flow of this vector field ends up with an ellipse as shown by the red line in Fig.1.

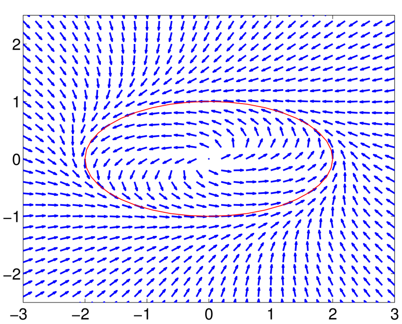

Fig. 1: GVF for an ellipse

II. Horizontal Contraction Based GVF Encoding

This section proposes the horizontal contraction based GVF encoding method, which is concluded by the following theorem.

Theorem 1: Given a differentiable simple closed curve Υ defined by a mapping $\pi(\tau) : \mathcal{I} \to \mathcal{M}$ with $\mathcal{I} \subset \mathbb{R}$ and $\mathcal{M}$ being a manifold of dimension n, if there exists a vector field $f(x) : \mathcal{M} \to T_x\mathcal{M}$, and a mapping $g : \mathbb{R} \to \mathbb{R}_{>0}$(or $\mathbb{R}_{<0}$), such that the following conditions are satisfied:

(A1) For all $\tau \in \mathcal{I}$,

$$f(\pi(\tau)) = \nabla_\tau \pi(\tau) g(\tau). \tag{2}$$

(A2) There exists a mapping $\phi : \mathbb{R}^n \to \mathbb{R}^{n-1}$, such that $\nabla\phi(x)$ is of full rank for all $x \in \mathcal{M}$, and the following set equality holds

$$\{x \in \mathcal{M} | x = \pi(\tau), \tau \in \mathcal{I}\} \triangleq \{x \in \mathcal{M} | \phi(x) = 0\}. \tag{3}$$

Here, with a minor abuse of notation, we also use Υ to denote the target-curve set, i.e.,

$$\Upsilon \triangleq \{x \in \mathcal{M} | x = \pi(\tau), \tau \in \mathcal{I}\}; \tag{4}$$

(A3) There exists a candidate Finsler-Lyapunov function $V : T\mathcal{M} \to \mathbb{R}_{\geq 0}$ given by

$$V(x, \delta x) = (\delta x)^T \nabla^T \phi(x) P(x) \nabla\phi(x) \delta x \tag{5}$$

where $P : \mathbb{R}^n \to \mathbb{R}^{(n-1)\times(n-1)}, P = P^T > 0$, such that for a class $\mathcal{K}$ function $\alpha(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, the following inequality holds

$$\dot{V}(x, \delta x) \leq -\alpha(V(x, \delta x)) \tag{6}$$

along the trajectories of the prolonged system

$$\dot{x} = f(x) \tag{7}$$

$$\dot{\delta x} = \nabla f(x) \delta x; \tag{8}$$

(A4) All the trajectories of the system $\dot{x} = f(x)$ are bounded.

Then, $f(x)$ is a guidance vector field for the closed curve Υ.

The proof of Theorem is omitted here for compactness, and it is inspired by [18, Prop. 2].

Remark 1: In Theorem 1, the vertical distribution and the horizontal distribution of $T_x\mathcal{M}$ is $\mathcal{V}_x = \text{Span}\{f(\pi(\tau))\}$ and $\mathcal{H}_x = \text{Span}\{\nabla^T\phi_1, \nabla^T\phi_2, \cdots, \nabla^T\phi_{n-1}\}$ respectively according to the horizontal contraction theory. Here, ϕ_i denotes the ith element of $\phi(x)$. By (2), $x = \pi(\tau)$ with $\dot{\tau} = g(\tau)$ is a solution of the differential equation $\dot{x} = f(x)$. Additionally, condition (3), together with (6), renders the solution $x = \pi(\tau)$ attractive.

Remark 2: According to [1], the way of developing $\phi(x)$ is to express the target curve as the intersection of $n-1$ surfaces of codimension one. That is, the surface functions $\alpha_i(x)$ in [1], for $i = 1, 2, \cdots, n-1$, is a nature selection of $\phi_i(x)$. The difference between the method in [1] and Theorem 1 is that the circulation vector field in [1] is obtained by calculating the wedge product [22] of $\nabla\phi_i$'s, and in Theorem 1, it is defined by $\nabla_\tau\pi(\tau)g(\tau)$.

Example: Consider the GVF encoding problem for a unit circle in $\mathbb{R}^2$, which can be defined by the following mapping

$$\pi(\tau) = [\cos(\tau), \sin(\tau)]^T \tag{9}$$

where $\tau \in [0, 2\pi)$. Then, condition (3) is verified with

$$\phi(x) = x_1^2 + x_2^2 - 1. \tag{10}$$

Based on (5) in Theorem 1, we selected the following Finsler-Lyapunov function

$$V(x, \delta x) = \frac{\|\nabla\phi(x)\delta x\|^2}{(x_1^2 + x_2^2)^2}. \tag{11}$$

Then, by solving (2) and (6)-(8), the GVF for the unit circle can be obtained as follows

$$f(x)=\begin{bmatrix} 1-x_1^2-x_2^2 & -1 \\ 1 & 1-x_1^2-x_2^2 \end{bmatrix} x \tag{12}$$

which satisfies

$$\dot{V}=-4(x_1^2+x_2^2)V. \tag{13}$$

By (13), the contraction region is $\mathcal{C} \triangleq \mathbb{R}^2\backslash\{0\}$. Note that the origin is not in the contraction region since the origin is actually an unstable equilibrium point.

It is worth pointing out that the HC-GVF, in general, is difficult to implement in practical applications, because we need to solve the partial differential inequalities (PDIs) (6)-(8) subject to the contour condition (2) to determine $f(x)$.

III. Integral Differential GVF

In this section, the integral differential GVF method is presented. This method has freedom to be updated online via optimization while guarantee the attractivity of the target curve. As a result, it can achieves an almost shortest-distance tracking result and can encode GVF for target curves with self-intersections.

A. Design of GVF

For the target curve Υ, we introduce its tangential vector field as $K_\Upsilon : \Upsilon \to T_x\mathcal{M}$. Therefore, moving each point along K_Υ simply rotates the target curve, i.e., the target curve is a forward invariant with respect to the K_Υ. With the parameterized mapping $\pi(\tau)$ for the target curve Υ in Theorem 1, K_Υ can be calculated as follows

$$K_\Upsilon(x)=\nabla\pi(\tau)g(\tau),\ x=\pi(\tau)\in\Upsilon. \tag{14}$$

Inspired by [19], the GVF for Υ can be designed to as following structure

$$\dot{x}=K_\Upsilon(x_d(t))+g(x,x_d(t)),\ x_d(t)\in\Upsilon \tag{15}$$

where $x_d(t)$ is a time-vary signal which need to be specified when $x(t)\notin\Upsilon$. When $x(t)\in\Upsilon$, $x_d(t)=x(t)$. By borrowing the name form [19], the GVF (15) is coined as a integral differential GVF.

With the above design, note that if $g(x^d,x^d)=0$, then the target curve Υ is a integral curve of (15) because (15) reduces to the K_Υ for Υ in this case. Therefore, if $x(t)=x_d(t)$, $x_d(t)$ is naturally updated via $\dot{x}_d=K_\Upsilon(x_d(t))$ and the target curve is forward invariant. Additionally, as an another required for the GVF, $g(x,x^d)$ should be designed to guarantee the attractivity of Υ.

To facilitate the design of $g(x(t),x_d(t)$, we define the Finsler-Lyapunov function as follows

$$V(x,\delta x)=\delta x^T M(x)\delta x \tag{16}$$

where $M(x):\mathbb{R}^n\to\mathbb{R}^{n\times n}$ is a positive definite symmetry matrix. Furthermore, let $l(t,s)\in\Gamma(x(t),x_d(t))$ be the minimizing geodesic with respect to the metric tensor $M(x)$. Therefore, we have $l(t,0)=x_d(t)$, and $l(t,1)=x(t)$. Then, based on the contraction theory, $g(x,x^d)$ is designed as the following integral-differential form

$$g(x,x_d)=-\int_0^1 R(l(s,t))\frac{\partial l(s,t)}{\partial s}\mathrm{d}s \tag{17}$$

where the matrix $R(\cdot):\mathbb{R}^n\to\mathbb{R}^{n\times n}$ is positive definite.

Based on the above analysis, the following theorem is readily concluded.

Theorem 2: Given the Finsler-Lyapunov function $V(x,\delta)$ defined in (16), the minimizing geodesic $l(t,s)$, if the following condition holds

$$\dot{M}(x)-2M(x)R(x)\leq-\lambda M(x) \tag{18}$$

where λ is a positive constant, then (15) with (17) is a GVF of the target curve Υ.

Remark 3: The proof of Theorem 2 is similar to [19, Th. III.2]. Therefore, the proof procedure is omitted here for compactness. A not rigorous but simple way to understand Theorem 2, is that the design in (15) and (17) is intended to let the variational system for (15) has the following form

$$\dot{\delta x}=-R(x)\delta x. \tag{19}$$

That is, (15) is a contracting system. Since the target curve Υ is a integral curve of (15), then it can be concluded that all the trajectories of (15) converge to Υ.

B. Online optimization of $x_d(t)$

The subsequent analysis proposes a update law for $x_d(t)$ when $x(t)\notin\Upsilon$ based on an online optimization method. According to the definition for $V(x,\delta x)$ in (16), the length of the minimizing geodesic connecting $x(t)$ and $x_d(t)$, i.e., $l(s,t)$, is defined as follows

$$d(x(t),x_d(t))\triangleq\int_0^1\sqrt{V(l(s,t),\delta_l(s,t))}\mathrm{d}s \tag{20}$$

Then, $x_d(t)$ is updated online by solving the following optimization problem.

Online Optimization Problem: At time $t\geq0$, given a minimizing geodesic $l(\cdot,t)$, solve

$$x_d(t)=\min_{x^*\in\Upsilon} d(x(t),x^*). \tag{21}$$

Note that if $x(T)\in\Upsilon$, then the solution for (21) is $x_d(T)=x(T)$ and $x(t)\in\Upsilon$ for $t\geq T$.

Remark 4: Considering the computation time required for the above optimization problem, it can be concluded that the resulting $x_d(t)$ is constant piecewise continuous when $x(t)\notin\Upsilon$.

Since $x_d(t)$ is not a trajectory timed by τ, the above design can avoid the radial reduction phenomenon. Additionally, $x_d(t)$ is obtained by minimizing $d(x(t),x_d(t))$, therefore, an almost shortest-distance tracking result can be achieved via the following GVF

$$\dot{x}=\frac{bK_\Upsilon(x_d)}{\|d(x,x_d)\|+\varepsilon}+g(x,x_d(t)) \tag{22}$$

where b, ε are designable positive constants. Note from (22) that when $x(t)$ is far away the target curve, $g(x, x_d)$, i.e., the convergence term, plays a dominant role in the designed GVF, otherwise, the tangential vector field K_Υ is the principal part.

The subsequent analysis shows that (22) is also a GVF for the target curve Υ. Since $x_d(t)$ is obtained from (21), it can be claimed that $x_d(t)$ depends on $x(t)$. Therefore, we denote $x_d(t)$ as $x_d(x(t))$ with a minor abuse of notation. Let $x_a(t)$ is a solution for (22), i.e.,

$$\dot{x}_a = \frac{bK_\Upsilon(x_d(x_a))}{\|d(x_a, x_d(x_a))\| + \varepsilon} + g(x_a, x_d(x_a)). \tag{23}$$

Then, we consider the following virtual system

$$\dot{x} = \frac{bK_\Upsilon(x_d(x_a))}{\|d(x_a, x_d(x_a))\| + \varepsilon} + g(x, x_d(x_a)). \tag{24}$$

Notice that $x_a(t)$ and the target curve are two particular integral curves of (24), additionally, the variational system for (24) is same as the one of (15). Therefore, $x_a(t)$ will converge to the target curve eventually. Due to that $x_a(t)$ is selected arbitrarily, it can be concluded that every trajectory of (22) will converge to and circulate along the target curve Υ, thus, (22) defines a GVF for Υ.

C. Normalization of the GVF

In practical implementations, to simplify the GVF based control, the designed GVF usually need to be normalized like in [9], [17]. As a result, the controlled object such as a wheeled robot can move at a steady speed. For an encoded GVF $f(x)$, its normalized version is as follows

$$\dot{x} = \frac{f(x)}{\|f(x)\|}. \tag{25}$$

Since we consider a GVF for a closed curve, it is guaranteed that $f(x) \neq 0$. This normalization technique is curve parameterization in essence, which can be shown by the following analysis. Consider the following two systems

$$\dot{\xi} = f(\xi), \tag{26}$$

$$\dot{\zeta} = \eta(\zeta) f(\zeta) \tag{27}$$

where $\xi, \zeta \in \mathbb{R}^n$, $\eta : \mathbb{R}^n \to \mathbb{R}_{>0}$. We utilize $\psi(t, \xi_0)$ to denote the solution of (26) with initial condition ξ_0, i.e., $\psi(t_0, \xi_0) = \xi_0$, and $\varphi(t, \zeta_0)$ as the solution of (27) with initial condition ζ_0. Here, $\eta(\cdot)$ is the normalization term.

We give the following theorem to analyze transient states explicitly.

Theorem 3: For the systems in (26) and (27), if $\xi_0 = \zeta_0$, then their solutions are of the same shapes in the sense that $\forall \bar{t}' \in (t_0, \infty)$, there exists $\bar{t}''$, such that

$$\{\psi(t', \xi_0) \in \mathbb{R}^n | t' \in [t_0, \bar{t}']\} = \{\varphi(t'', \zeta_0) \in \mathbb{R}^n | t'' \in [t_0, \bar{t}'']\} \tag{28}$$

i.e., the trajectory of system (26) on $t \in [t_0, \bar{t}']$ matches the trajectory of system (27) on $t \in [t_0, \bar{t}'']$.

Proof: We proceed by contradiction. Assume that (28) does not hold, then for a bounded positive scalar β, there exist t_1, t_2, and a infinitesimal $\Delta t > 0$, such that

$$\psi(t_1, \xi_0) = \varphi(t_2, \zeta_0), \tag{29}$$

$$\{\psi(t', \xi_0) | t' \in [t_0, t_1]\} = \{\varphi(t'', \zeta_0) | t'' \in [t_0, t_2]\}, \tag{30}$$

$$\psi(t_1 + \Delta t, \xi_0) \neq \varphi(t_2 + \beta \Delta t, \zeta_0). \tag{31}$$

Thus,

$$\begin{aligned}\dot{\psi}(t_1, \xi_0) = &\lim_{\Delta t \to 0} \frac{\psi(t_1 + \Delta t, \xi_0) - \psi(t_1, \xi_0)}{\Delta t} \\ =&\beta \lim_{\Delta t \to 0} \frac{\psi(t_1 + \Delta t, \xi_0) - \psi(t_1, \xi_0)}{\beta \Delta t} \\ \neq&\beta \lim_{\Delta t \to 0} \frac{\varphi(t_2 + \beta \Delta t, \zeta_0) - \varphi(t_2, \zeta_0)}{\beta \Delta t} \\ =&\beta \dot{\varphi}(t_2, \zeta_0).\end{aligned} \tag{32}$$

However, from (26) and (27), we can get

$$\dot{\psi}(t_1, \xi_0) = \frac{1}{\eta(\varphi(t_2, \zeta_0))} \dot{\varphi}(t_2, \zeta_0), \tag{33}$$

which contradicts with (32) when β is selected as $1/\eta(\varphi(t_2, \zeta_0))$. Thus, the proof is completed. ■

According to Theorem 3, the introduction of the normalization term $\eta(\cdot)$ in (27) preserves the shape of the solutions of (26), and naturally, preserves the steady state of (26).

Based on the above analysis, the overall integral differential GVF encoding method is specified in the pseudocode form.

Algorithm 1: Integral Differential GVF Encoding

Input: $x(t)$ (current position), Υ (target curve)
At time t: compute $x_d(t)$ based on (21) .
Apply the updated $x^d(t)$ to (22) and normalize the resulting GVF.

Compared to the surface-intersection method in [1], the proposed online GVF encoding method has the following advantages.

1) The proposed method can encode a GVF for singular curves;
2) The proposed GVF can achieve an almost shortest-distance tracking result;
3) Compared to find a group of surface function α_i such that the intersection of $\alpha_i = 0$ is the target curve, the K_Υ is easier to be defined via parameterizing the target curve. Because the K_Υ is only required to specified on the target curve, as a contrast, the surface functions need to be defined globally. For example, in the practical scenario that the target curve should be constructed based on a set of sample points, the existing interpolation method is always more effective to construct a closed curve than to construct $n-1$ surfaces.

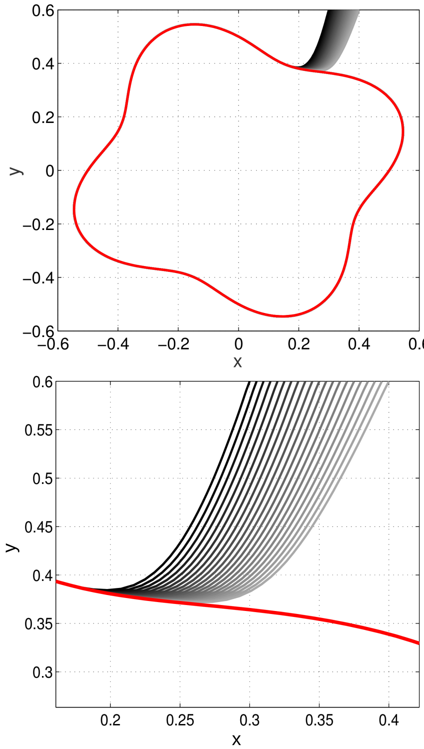

Fig. 2: Integral curves of the GVF for a simple closed curve in $\mathbb{R}^2$.

IV. Simulation Results

The proposed integral differential GVF encoding method is validated on several kinds of target curves. In the simulation, the minimizing geodesic between two points is chosen as the straight line connecting those two points for simplicity. It's worth to point out that the simulation result only shows the integral curve of the encoded GVF, and it is not the tracking result with respect to a physical system. For the latter, a closed-loop controller should be designed based on the encoded GVF, which is a topic of our future research.

Example a: Simple closed curve in R^2

Consider the following target curve

$$\begin{aligned} x^d(\tau) &= 0.25(0.3\sin(4\tau)+2)\cos(\tau) \\ y^d(\tau) &= 0.25(0.3\sin(4\tau)+2)\sin(\tau) \end{aligned} \tag{34}$$

where $\tau = [0, 2\pi)$. In the simulation, the GVF is encoded based on Algorithm 1, and a group of initial points on the line $y = 0.6$ is chosen. Fig. 2 presents the respective integral curves (gray lines) of the encoded GVF. It is shown that a smooth and almost shortest-distance tracking performance can be achieved. Fig. 3 presents the contour following error, i.e., $\|[x(t), y(t)] - [x_d(t), y_d(t)]\|$, where $x_d(t), y_d(t)$ is obtained via solving the online optimization problem specified in (21).

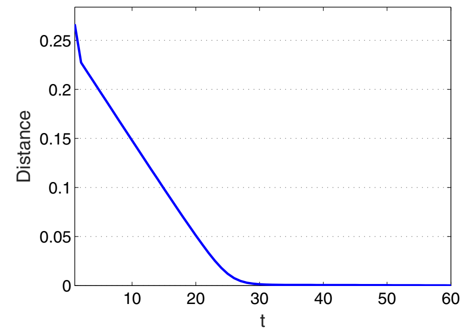

Fig. 3: The tracking error $\|[x(t), y(t)] - [x_d(t), y_d(t)]\|$.

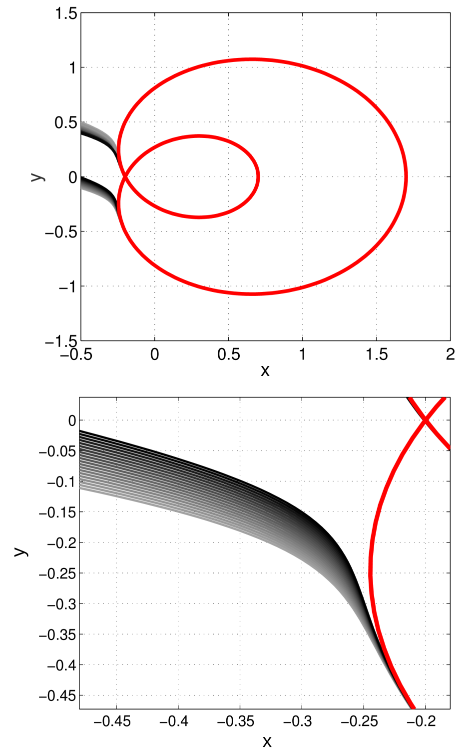

Fig. 4: Integral curves of the GVF for a singular closed curve in $\mathbb{R}^2$.

Example b: Singular closed curve in $\mathbb{R}^2$

Consider the following target curve

$$\begin{aligned} x^d(\tau) =& 0.7\cos(\tau) + 1\cos((1/4)\tau)\cos((1/4)\tau) \\ y^d(\tau) =& 0.7\sin(\tau) + 1\cos((1/4)\tau)\sin((1/4)\tau) \end{aligned} \tag{35}$$

where $\tau \in [0, 4\pi)$. Fig. 4 depicts a group of integral curves of the GVF for this singular closed-curve. It is shown the proposed GVF can also guide a partical to asymptotically converge to the target curve. As a contrast, the method proposed in [1] cannot be applied in this case. Since the circulation term [1] at the intersection section can not be defined from the developed surface functions.

V. CONCLUSION

In this paper, the guidance vector field encoding problem is revisited. First, a guidance vector field encoding method based on horizontal contraction is proposed. The disadvantage of this method is that it requires to solve a set of partial differential inequalities, which is challenging in implementations. Then, an alternative integrable differential encoding method is proposed. It is show that the resulting vector field ensures the attractivity of the target curve. Additionally, it allows to encode a vector field for the closed curve with self-intersections and can achieve an almost shortest-distance tracking result. Finally, the proposed method is validated via numerical simulation.

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