On a Correspondence between Probabilistic and Robust Invariant Sets for Linear Systems

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Abstract— Dynamical systems with stochastic uncertainties are ubiquitous in the field of control, with linear systems under additive Gaussian disturbances a most prominent example. The concept of probabilistic invariance was introduced to extend the widely applied concept of invariance to this class of problems. Computational methods for their synthesis, however, are limited. In this paper we present a relationship between probabilistic and robust invariant sets for linear systems, which enables the use of well-studied robust design methods. Conditions are shown, under which a robust invariant set, designed with a confidence region of the disturbance, results in a probabilistic invariant set. We furthermore show that this condition holds for common box and ellipsoidal confidence regions, generalizing and improving existing results for probabilistic invariant set computation. We finally exemplify the synthesis for an ellipsoidal probabilistic invariant set. Two numerical examples demonstrate the approach and the advantages to be gained from exploiting robust computations for probabilistic invariant sets.

I. INTRODUCTION

Characterizing the effect of unknown disturbances on a dynamical system via invariant sets is an important problem in system analysis [1], and is a key ingredient of many modern robust control methods, including robust model predictive control [2]. The concept generally used to formalize the impact of unknown perturbations is that of robust positively invariant sets (RIS), see [1]. For example, the minimum RIS describes the set of states that can be reached from the origin under a bounded state disturbance. For many systems, however, more information than a bound on the disturbance is available, e.g. in terms of a probability distribution, which can be exploited by a stochastic concept for perturbation analysis. Moreover, if the considered disturbance distribution has infinite support, e.g. the most commonly employed Gaussian distribution, there does not exist a finite upper bound on the disturbance, which is often approximated by truncating the distribution.

In order to extend invariance analysis to a stochastic setting, the concept of probabilistic invariant sets (PIS) has been introduced in [3] and recently extended to continuous-time linear systems [4]. The main concept is that, at all times, the state trajectory has to lie inside a set with at least a certain probability. The notion of PIS has potentially wide-ranging applications, most notably in probabilistic and stochastic model predictive control (MPC).

In stochastic MPC, related concepts are widely used, e.g. for the satisfaction of chance constraints on the states. For linear systems with additive stochastic disturbance the following methods have been proposed. In [5], probabilistic tubes of fixed cross-section and variable scaling enable to tighten constraints to guarantee a certain level of violation probability. In [6], strong feasibility of a stochastic MPC scheme for distributions with finite support is provided by the use of RIS. In [7], a stochastic MPC problem is solved with a combination of randomized and robust optimization techniques. The approach in [8] generalizes some previous efforts based on constraint-tightening in order to provide strong recursive feasibility. For a comprehensive review of stochastic MPC we also point the reader to the summary presented in [9].

In this paper we formalize a correspondence between robust treatment of confidence regions and probabilistic invariance for linear systems. The consequences of this result are twofold. On the one hand it provides justification for the use of RIS for systems with disturbance distributions of infinite support by considering confidence regions of truncated support, as commonly done in practical applications [10]. On the other hand, the link enables the use of various computational methods of RIS for the construction of PIS. For example, the minimum RIS can be calculated arbitrarily accurately for linear time-invariant (LTI) systems [11] and linear difference inclusions [12] in case of polytopic constrained disturbances, or can be approximately obtained as ellipsoids via Lyapunov based arguments, see e.g. [13]. Maximum robust invariant sets for constrained systems can similarly be obtained, see e.g. [1], [13], [14]. Using methods from minimum RIS computation, we are able to recover and generalize the PIS construction of [3], which, to the best of our knowledge, is the only method for the computation of PIS proposed in the literature. Applying similar concepts, we also present an algorithm for the synthesis of an ellipsoidal PIS of given probability level, as well as for deriving a probability level for a given ellipsoidal invariant set.

The paper is organized as follows: In Section II we introduce notation as well as the different concepts of invariant sets and confidence regions. In Section III we state the main result and analyze the condition under which RIS and PIS for a certain probability are equivalent. We then show that for box-shaped and ellipsoidal confidence regions this condition holds. In Section IV we present a method to determine a minimum probability level for which an ellipsoidal set is a

PIS, or, equivalently, how to rescale a RIS such that it is a PIS for a given probability level. In Section V, we use the example in [3] to illustrate the PIS obtained from the introduced link to minimal RIS, compare it against previous results and demonstrate the ellipsoidal algorithm presented in Section IV. We finish with concluding remarks in Section VI.

II. PRELIMINARIES

In this section we review some definitions and results that will be used throughout the paper.

A. Notation

We denote real numbers, strictly positive real numbers, real vectors of dimension n and real matrices of dimension $n \times m$ with $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$, respectively. Given a vector $x \in \mathbb{R}^n$, the i'th component of x is indicated with $[x]_i$. Given two sets $\mathcal{A}$ and $\mathcal{B}$, the Minkowski sum is $\mathcal{A} \oplus \mathcal{B} = \{x \mid x = x_1 + x_2, x_1 \in \mathcal{A}, x_2 \in \mathcal{B}\}$. Given a set $\mathcal{S}$, a matrix $A \in \mathbb{R}^{n \times m}$ and a vector $z \in \mathbb{R}^n$, we define an affine set transformation as $A\mathcal{S} + z = \{x \mid x = As + z, s \in \mathcal{S}\}$. Positive definite and semidefinite matrices are indicated with $A \succ 0$ and $A \succeq 0$, respectively. The function $\operatorname{diag}(A_1, \dots, A_M)$ defines the block-diagonal matrix composed of M blocks $A_i, i \in \{1, \dots, M\}$. We indicate with $x \sim \mathcal{Q}$ a random variable x of distribution $\mathcal{Q}$, and with $\mathcal{N}(\mu, \Sigma)$ a multivariate Gaussian distribution with mean $\mu \in \mathbb{R}^n$ and covariance $\Sigma \in \mathbb{R}^{n \times n}$, $\Sigma \succeq 0$. We use $\Pr(E)$ for the probability of an event E. For a random variable x, $\mathbb{E}(x)$ and $\operatorname{var}(x)$ denote expected value and variance, respectively.

B. Invariant Sets

Consider the following linear time-invariant autonomous system affected by an additive disturbance $w_k \in \mathbb{R}^n$, together with a nominal system state $z_k \in \mathbb{R}^n$ and error state $e_k \in \mathbb{R}^n$

$$z_{k+1} = Az_k, \tag{1a}$$
$$x_{k+1} = Ax_k + w_k, \tag{1b}$$
$$e_k = x_k - z_k, \tag{1c}$$

where $A \in \mathbb{R}^{n \times n}$. In the following, we recall some standard definitions of positively invariant sets.

Definition 1 (Positively Invariant Set (IS) [1]).
A set $\mathcal{P}$ is said to be a positively invariant set (IS) for system (1a) if

$$z_0 \in \mathcal{P} \Rightarrow z_k \in \mathcal{P} \quad \forall k \geq 0.$$

Definition 2 (Robust Positively Invariant Set (RIS) [1]).
A set $\mathcal{P}$ is said to be a robust positively invariant set (RIS) for system (1b) with $w_k \in \mathcal{W} \subseteq \mathbb{R}^n$ if

$$x_0 \in \mathcal{P} \Rightarrow x_k \in \mathcal{P} \quad \forall k \geq 0.$$

Definition 3 (Probabilistic Positively Invariant Set (PIS) [3]).
A set $\mathcal{P}$ is said to be a probabilistic positively invariant set (PIS) of probability level p for system (1b) with disturbances $w_k \sim \mathcal{Q}^w$ if

$$x_0 \in \mathcal{P} \Rightarrow \Pr(x_k \in \mathcal{P}) \geq p \quad \forall k \geq 0.$$

C. Confidence Region

In order to connect probabilistic invariance to robust invariance we make use of the concept of conservative confidence regions [15]. To this end, we first provide a definition of a conservative confidence region and then present two specific types of regions commonly used. Since throughout this paper we are only interested in bounding the relevant probabilities, we simply use the term confidence region.

Definition 4 (Confidence Region).
A set $\mathcal{E}_p(\mathcal{Q}^x)$ is called a confidence region of level p for a random variable $x \sim \mathcal{Q}^x$ if

$$\Pr(x \in \mathcal{E}_p(\mathcal{Q}^x)) \geq p.$$

Using this definition, we can construct different types of confidence regions. We first introduce a box-shaped confidence region as follows:

Lemma 1 (Box-shaped Confidence Region).
Let $x \sim \mathcal{Q}^x$ be a random variable taking values in $\mathbb{R}^n$, with $\mathbb{E}(x) = 0$, $var(x) = \Sigma$ and

$$\mathcal{E}_p^b(\mathcal{Q}^x) := \left\{ \bar{x} \,\middle|\, [\bar{x}]_i^2 \leq \frac{[\Sigma]_{ii}}{\tilde{p}_i}, i = \{1, \dots, n\} \right\}, \tag{2}$$

where $\tilde{p}_i \in \mathbb{R}, i = \{1, \dots, n\}$, are parameters that determine the violation probability for each component of the random vector. Then $\mathcal{E}_p^b(\mathcal{Q}^x)$ is a confidence region of probability level $p = 1 - \sum_{i=1}^{n} \tilde{p}_i$.

Proof. Using the Chebyshev inequality for each dimension i we have $\Pr\left([x]_i^2 \geq \frac{[\Sigma]_{ii}}{\tilde{p}_i}\right) \leq \tilde{p}_i$. By taking the union bound and forming the complementary probability, the claim follows. □

Under the assumption that the disturbance distribution is Gaussian, we can state a tighter confidence region.

Corollary 1. *For $x \sim \mathcal{Q}^x = \mathcal{N}(0, \Sigma)$, Lemma 1 holds with*

$$\mathcal{E}_p^b(\mathcal{Q}^x) := \left\{ \bar{x} \,\middle|\, [\bar{x}]_i^2 \leq [\Sigma]_{ii} \left(\Phi^{-1}\left(\frac{\tilde{p}_i}{2}\right)\right)^2, i = \{1, \dots, n\} \right\},$$

where $p = 1 - \sum_{i=1}^{n} \tilde{p}_i$, and Φ^{-1} is the quantile function of the standard normal distribution.

The result follows directly using the same arguments and confidence intervals of the marginal Gaussian distribution in each dimension.

As a second type of confidence region we consider commonly used ellipsoidal sets of the following form.

Lemma 2 (Ellipsoidal Confidence Region).
Let $x \sim \mathcal{Q}^x$ be a random variable taking values in $\mathbb{R}^n$, with $\mathbb{E}(x) = 0$, $var(x) = \Sigma$ and

$$\mathcal{E}_p^e(\mathcal{Q}^x) := \left\{ \bar{x} \,\middle|\, \bar{x}^T \Sigma^{-1} \bar{x} \leq \frac{n}{p} \right\}. \tag{3}$$

Then $\mathcal{E}_p^e(\mathcal{Q}^x)$ is a confidence region of probability level p.

Proof. This follows directly from the multivariate Chebyshev inequality [16]. □

For a Gaussian distribution, the confidence region in Lemma 2 can similarly be tightened using the following well-known result:

Corollary 2. *For $x \sim \mathcal{Q}^x = \mathcal{N}(0, \Sigma)$, Lemma 2 holds for*

$$\mathcal{E}_p^e(\mathcal{Q}^x) := \left\{ \bar{x} \mid \bar{x}^T \Sigma^{-1} \bar{x} \leq \chi_n^2(p) \right\},$$

where χ_n^2 is the chi-squared distribution of degree n.

In the next section we will use the definitions of confidence regions and invariant sets to link the concept of PIS to that of RIS, where robustness is with respect to a disturbance contained in a given confidence region.

III. PROBABILISTIC INVARIANT SETS FROM ROBUST INVARIANT SETS

We present a theorem which, under a central condition, shows a direct relationship between probabilistic and robust invariant sets for linear dynamical systems. We then show that this condition holds for the box-shaped and ellipsoidal confidence intervals introduced in II-C.

Theorem 1. *Let $\mathcal{P}$ be a RIS for system (1b) with $w_k \in \mathcal{E}_p(\mathcal{Q}^w) \forall k \geq 0$, where $\mathcal{E}_p(\mathcal{Q}^w)$ is a confidence region of probability level p. Let $\mathcal{Q}_k^e$ denote the distribution of e_k in (1) at each time step k. If*

$$\mathcal{E}_p(\mathcal{Q}_{k+1}^e) \subseteq A \mathcal{E}_p(\mathcal{Q}_k^e) \oplus \mathcal{E}_p(\mathcal{Q}^w) \quad \forall k \geq 0, \qquad (4)$$

then the set $\mathcal{P}$ is also a PIS of probability level p for system (1b) with $w_k \sim \mathcal{Q}^w$.

Proof. Let $\mathcal{R}_k$ be the k-step reachable set of e_k in (1) starting at $e_0 = 0$ under $w_k \in \mathcal{E}_p(\mathcal{Q}^w)$, which is recursively defined through $\mathcal{R}_{k+1} = A\mathcal{R}_k \oplus \mathcal{E}_p(\mathcal{Q}^w)$ and $\mathcal{R}_0 = \{0\}$. For $x_0 \in \mathcal{P}$ we thus have $z_k + \mathcal{R}_k \in \mathcal{P} \, \forall k$, since $\mathcal{P}$ is a RIS. Showing $\mathcal{E}_p(\mathcal{Q}_k^e) \subseteq \mathcal{R}_k$ for all $k \geq 0$ proves the claim, since then $z_k + \mathcal{E}_p(\mathcal{Q}_k^e) \in \mathcal{P} \, \forall k \geq 0$, which implies $\Pr(x_k \in \mathcal{P}) \geq p \, \forall k \geq 0$. Assuming $\mathcal{E}_p(\mathcal{Q}_k^e) \subseteq \mathcal{R}_k$ for some k we have with condition (4)

$$\begin{aligned} \mathcal{R}_{k+1} &= A\mathcal{R}_k \oplus \mathcal{E}_p(\mathcal{Q}_k^w) \\ &\supseteq A\mathcal{E}_p(\mathcal{Q}_k^e) \oplus \mathcal{E}_p(\mathcal{Q}^w) \\ &\supseteq \mathcal{E}_p(\mathcal{Q}_{k+1}^e). \end{aligned} \qquad (5)$$

Since $\mathcal{E}_p(Q_0^e) \subseteq \mathcal{R}_0$, the proof follows by induction. $\square$

The necessary condition (4) of the theorem requires that at each time step the evolution of the confidence region of the error is contained in the Minkowski sum of the confidence region of the disturbance and previous error. As we have shown in (5), condition (4) implies that the confidence region of the error will always be contained in the respective reachable set considering a disturbance bounded by the confidence region $\mathcal{E}_p(\mathcal{Q}^w)$.

Remark 1. Using Theorem 1, the correspondence between a RIS and a PIS thus comes down to verifying condition (4) for given system dynamics and confidence region of the disturbance.

Remark 2. Note that if, according to Theorem 1, set $\mathcal{P}$ is guaranteed to be a PIS of probability level p, there might still exist a $\tilde{p} > p$ at which level $\mathcal{P}$ is a PIS. Theorem 1 therefore only provides a lower bound on the maximum probability level at which a set is a PIS.

In the following, we show that condition (4) holds for the confidence regions presented in Section II-C and for certain classes of LTI system dynamics.

Corollary 3. *Let $\mathcal{P}$ be a RIS for system (1b) with $w_k \in \mathcal{E}_p^b(\mathcal{Q}^w) \forall k \geq 0$ and $A \in \mathbb{R}^{n \times n}$ diagonal, where $\mathcal{E}_p^b(\mathcal{Q}^w)$ is a box-shaped confidence region of probability level p. Then $\mathcal{P}$ is also a PIS of probability level p for system (1b) with $w_k \sim \mathcal{Q}^w$.*

Proof. Let $\text{var}(e_k) = \Sigma_k$ and $\text{var}(w_k) = Q$, then $\Sigma_{k+1} = A\Sigma_k A^T + Q$ and we have

$$\begin{aligned} \mathcal{E}_p^b(\mathcal{Q}_{k+1}^e) &= \left\{ e \,\middle|\, [e]_i^2 \leq \frac{[A\Sigma_k A^T + Q]_{ii}}{\tilde{p}_i} \right\} \\ &= \left\{ e \,\middle|\, [e]_i^2 \leq \frac{[A]_{ii}^2 [\Sigma_k]_{ii}}{\tilde{p}_i} + \frac{[Q]_{ii}}{\tilde{p}_i} \right\} \\ &\subseteq A\mathcal{E}_p^b(\mathcal{Q}_k^e) \oplus \mathcal{E}_p^b(\mathcal{Q}^w). \end{aligned}$$

The claim follows from Theorem 1. $\square$

Remark 3. Corollary 3 also holds for complex system dynamics and can be applied to any real diagonalizable system by using an eigenvalue decomposition [3].

Remark 4. Computation of the minimum RIS for a diagonalized system with confidence region $\mathcal{E}_p^b$ recovers the PIS computations of [3], Corollary 3 therefore provides a different interpretation and generalization of the available computation method for PIS.

Next, we show the correspondence for the case of ellipsoidal confidence intervals, defined in (3), for general linear systems, namely, also for systems with non-diagonalizable dynamics matrix A.

Corollary 4. *Let $\mathcal{P}$ be a RIS for system (1b) with $w_k \in \mathcal{E}_p^e(\mathcal{Q}^w) \forall k \geq 0$, where $\mathcal{E}_p^e(\mathcal{Q}^w)$ is an ellipsoidal confidence region of probability level p. Then $\mathcal{P}$ is also a PIS of probability level p for system (1b) with $w_k \sim \mathcal{Q}^w$.*

Proof. For the sake of brevity, the proof is given for A invertible. Using a coordinate transformation T, such that $T^{-1}AT = \text{diag}(\bar{A}, 0)$ with $\bar{A}$ non singular, the results can then be extended to a general dynamics matrix A.

Consider the error state $e_k \sim \mathcal{Q}_k^e$ and disturbance $w_k \sim \mathcal{Q}^w$ with $\mathbb{E}(e_k) = 0$, $\text{var}(e_k) = \Sigma$ and $\mathbb{E}(w_k) = 0$, $\text{var}(w_k) = Q$. Then $\Sigma_{k+1} = A\Sigma_k A^T + Q$ and we have [17], [18],

$$e^T \Sigma_{k+1}^{-1} e = \inf_{\tilde{e} + w = e} \tilde{e}^T (A^{-1})^T \Sigma_k^{-1} A^{-1} \tilde{e} + w^T Q^{-1} w.$$

We can see that for each e and arbitrary $\bar{p} > 0$ with $e^T \Sigma_{k+1}^{-1} e = \bar{p}$ there exist $\tilde{e}$ and w, such that $\tilde{e} + w = e$ and $\tilde{e}(A^{-1})^T \Sigma_k^{-1} A^{-1} \tilde{e} \leq \bar{p}$ and $w^T Q^{-1} w \leq \bar{p}$. By noting

that

$$A\mathcal{E}_p^e(\mathcal{Q}_k^e) = \left\{ e \,\middle|\, e^T(A^{-1})^T\Sigma_k^{-1}A^{-1}e \leq \frac{n}{p} \right\},$$

this implies that for all $e \in \mathcal{E}_p^e(\mathcal{Q}_{k+1}^e)$ there exist $\tilde{e} + w = e$, with $\tilde{e} \in A\mathcal{E}_p^e(\mathcal{Q}_k^e)$ and $w \in \mathcal{E}_p^e(\mathcal{Q}^w)$, and therefore

$$\mathcal{E}_p^e(\mathcal{Q}_{k+1}^e) \subseteq A\mathcal{E}_p^e(\mathcal{Q}_k^e) \oplus \mathcal{E}_p^e(\mathcal{Q}^w).$$

The claim follows from Theorem 1. □

Corollary 4 extends PIS computation to non-diagonalizable systems using ellipsoidal confidence regions.

Corollary 3 and Corollary 4 thus establish a correspondence between RIS and PIS for two important classes of confidence regions for the disturbance. Using these results, the task of designing a large PIS subject to constraints can be related to the design of a maximum RIS. They furthermore allow for using the PIS concept and its probabilistic certificates in optimization-based analysis and controller design procedures, such as those commonly used in robust control, see e.g. [13].

By leveraging these insights, the next section presents a synthesis and analysis method for ellipsoidal PIS.

IV. Ellipsoidal PIS for LTI Systems

The following shows how to synthesize an ellipsoidal PIS based on the correspondence to RIS and how to derive a lower bound on the probability level associated with a given invariant set if the system is subject to a random disturbance. We consider system (1), which is assumed to be asymptotically stable, i.e., the eigenvalues of A are strictly inside the unit circle, and $w_k \sim \mathcal{Q}^w = \mathcal{N}(0, Q)$, $Q \succ 0$. Given an invariant set for the nominal system (1a) of the form

$$\mathcal{P} = \left\{ x \,\middle|\, x^T P x \leq 1 \right\}, \tag{6}$$

where $P \in \mathbb{R}^{n \times n}$, $P \succ 0$, and defining a scaling of the invariant set as $\mathcal{P}^\alpha = \left\{ x \,\middle|\, x^T P x \leq \alpha \right\}$, we address the following problem:

Problem 1.
  a) What is the smallest $\alpha \in \mathbb{R}_+$ such that the set $\mathcal{P}^\alpha$ is a PIS of given probability level p?
  b) What is the largest probability level p at which set $\mathcal{P}$ is a PIS?

Remark 5. Note that assuming A to be asymptotically stable and $\mathcal{P}$ an invariant set for the nominal system is necessary for the existence of a RIS $\mathcal{P}^\alpha$.

We make use of Theorem 1, offering a means to assess a lower bound on the maximum probability level, to find an approximate solution to Problem 1 by defining ellipsoidal confidence regions of the disturbance of probability level p, i.e. $\mathcal{E}_p^e(\mathcal{Q}^w)$. We use the fact that according to Corollary 4, if $\mathcal{P}^\alpha$ is a RIS for $w_k \in \mathcal{E}_p^e(\mathcal{Q}^w)$, it is also a PIS for the corresponding stochastic system. The problem therefore reduces to the synthesis of a RIS, which has been well-studied in the literature [1]. We briefly summarize the procedure,

which will also form the basis for addressing Problem 1b. The robust invariance condition can be expressed as

$$\begin{aligned} x_k^T P x_k &\leq \alpha, \\ w_k^T Q^{-1} w_k &\leq \chi_n^2(p) \end{aligned} \Rightarrow (Ax_k + w_k)^T P(Ax_k + w_k) \leq \alpha. \tag{7}$$

By applying the S-procedure, this implication is true if and only if the following LMI admits a feasible solution [19], [20]:

$$\begin{bmatrix} A^T P A - \tau_0 P & A^T P \\ PA & P - \tau_1 Q^{-1} \end{bmatrix} \preceq 0, \tag{8a}$$

$$\alpha - \tau_0 \alpha - \tau_1 \chi_n^2(p) \geq 0, \tag{8b}$$

$$\tau_0, \tau_1 > 0. \tag{8c}$$

In this case equivalence follows from the fact that the S-procedure is lossless for positive definite P and Q ([21] Theorem 4.2).

Since $\alpha > 0$, equation (9b) is equivalent to

$$1 - \tau_0 - \tau_1\bar{p} \geq 0$$

with $\bar{p} = \frac{\chi^2(p)}{\alpha}$. In order to compute the minimum scaling factor α, we can thus maximize the scalar $\bar{p}$ for a given probability level p.

Given $\alpha = 1$, the same formulation can be used to compute a maximum probability p^* via $\bar{p}^*$ providing the best lower bound on the probability level according to Theorem 1. The following optimization problem thus solves both Problem 1a and 1b

$$\max_{\bar{p}, \tau_0, \tau_1} \quad \bar{p} \tag{9a}$$

$$\text{s.t.} \quad 1 - \tau_0 - \tau_1\bar{p} \geq 0, \tag{9b}$$

$$(8a), (8c). \tag{9c}$$

Problem (9) has the following property:

Proposition 1. Let $p_1, p_2 \in \mathbb{R}_+$ be two scalars such that $p_2 \geq p_1$. The following two facts hold:

  1) If there exists a feasible solution for (9) with $\bar{p} = p_2$, there also exists a feasible solution for $\bar{p} = p_1$.
  2) If there does not exist a feasible solution for (9) with $\bar{p} = p_1$, there also does not exist a feasible solution for $\bar{p} = p_2$.

Problem (9) can be solved up to a predefined accuracy using a bisection on $\bar{p}$.

V. Numerical Results

To exemplify the implications of the presented results and design methods we use two examples: in Subsection V-A we generate two PIS using both box-shaped and ellipsoidal confidence regions making use of minimum RIS computations. In Subsection V-B we show an application of the algorithm presented in Section IV for evaluation of the probability level of an ellipsoidal invariant set.
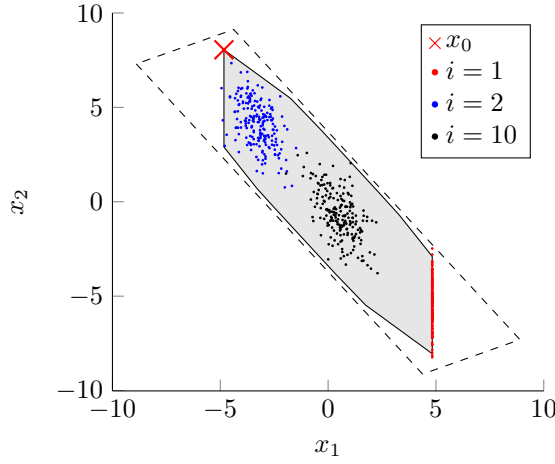
Fig. 1. PIS with $p = 0.99$ for system (10) based on a box-shaped confidence region for the diagonalized system (dashed) and based on an ellipsoidal confidence region for the original system (solid, shaded gray). Samples from the state distributions with initial condition x_0 (red cross) are shown at time instants $i = 1, 2, 10$ indicated with red, blue and black dots, respectively.

A. PIS Synthesis via RIS

We consider the example from [3] to generate PIS using box-shaped and ellipsoidal confidence regions as defined in (2) and (3). The system is of the form (1) with Gaussian disturbance $w_k \sim Q^w = \mathcal{N}(0, Q)$ and

$$A = \begin{bmatrix} 0 & 0.6 \\ 0.3 & -0.5 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (10)$$

To apply Lemma 1 with Corollary 3, we diagonalize the system using an eigenvalue decomposition $A = VDV^{-1}$ and generate the box-shaped confidence region using the coordinate transformed noise characteristic

$$\tilde{Q} = V^{-1}QV^{-*} = \begin{bmatrix} 0.43 & 0.64 \\ 0.64 & 0.94 \end{bmatrix},$$

where V^{-*} is the conjugate transpose of the inverse of V.

Using $\tilde{p}_1 = \tilde{p}_2 = 0.005$, such that $p = 0.99$ the box-shaped confidence region for the coordinate transformed disturbance $\tilde{w}_k = V^{-1}w_k \sim Q^{\tilde{w}}$ is given by

$$\mathcal{E}_p^b(Q^{\tilde{w}}) = \left\{ w \left| \begin{bmatrix} [w]_1^2 \\ [w]_2^2 \end{bmatrix} \leq \begin{bmatrix} 3.40 \\ 7.40 \end{bmatrix} \right. \right\}.$$

By computing the minimum RIS $\tilde{\mathcal{P}}_p^b$ for this confidence region following [11] and transforming the set back to the original coordinate system, we obtain $\mathcal{P}_p^b = V\tilde{\mathcal{P}}_p^b$, shown in Figure 1, and recover the PIS computed for the same system in [3].

Since the synthesis based on the correspondence to RIS according to Theorem 1 only provides a lower bound on the maximum probability level at which a set is a PIS (see also Remark 2), we make use of simulations to estimate the probability of violating the set membership. To this end, we simulate the system for 20 time steps starting from 10000 initial conditions in $\mathcal{P}_p^b$, taken on a regular grid with additional samples on the boundary of the set, and

using 20000 samples of the disturbance at each time step. The probability level at which $\mathcal{P}$ is a PIS is estimated by considering the ratio of states lying within the set $\mathcal{P}$ at each time step to the overall number of samples and taking the maximum ratio over all time steps and initial conditions, which results in the probability $p_{est} = 0.995$.

As an alternative synthesis method, we use an ellipsoidal confidence region on w_k

$$\mathcal{E}_p^e(Q^w) = \left\{ w \left| [w]_1 = 0, [w]_2^2 \leq \chi_1^2(p) \right. \right\},$$

with $p = 0.99$ and construct a minimum RIS according to [11] without prior diagonalization[1]. This directly yields the PIS $\mathcal{P}_p^e$ of probability level p as shown in Figure 1 by the solid line and shaded gray area. Using simulations as before we empirically estimate the maximum probability at which $\mathcal{P}_p^e$ is a PIS to be $p_{est} = 0.99$, meaning that in this example the probability bound provided by the robust treatment is tight. An initial condition leading to this maximum violation probability is on one of the vertices of $\mathcal{P}_p^e$ and is shown in Figure 1. In this example the construction using ellipsoidal confidence regions therefore provides tighter bounds as the established method in [3]. It is, however, based on a potentially expensive RIS computation, whereas the approach with box-shaped confidence intervals and diagonalization is based on a Jordan decomposition and computationally cheap.

B. Ellipsoidal PIS

In the second example we consider the non-diagonalizable LTI system

$$x_{k+1} = \begin{bmatrix} 0.4 & 0.4 \\ 0 & 0.4 \end{bmatrix} x_k + w_k, \quad (11)$$

with $w_k \sim \mathcal{N}(0, Q)$, and $Q = I$. We consider the following ellipsoidal invariant set for the nominal system

$$\mathcal{P} = \left\{ x \left| x^T \begin{bmatrix} 0.98 & 0.15 \\ 0.15 & 0.78 \end{bmatrix} x \leq 100 \right. \right\},$$

for which we use the method presented in Section IV to provide a lower bound on the probability level at which $\mathcal{P}$ is a PIS. We start the bisection with $\bar{p}_{min}^* = 0$ and $\bar{p}_{max}^* = \chi_2^2(0.999)$, corresponding to probability levels of 0 and 0.999, respectively. We set the termination condition of the bisection method at an accuracy of 0.001 probability. After 18 bisection steps we find a probability level at which $\mathcal{P}$ is guaranteed to be a PIS to be $p^* = 0.99$. Figure 2 shows the set and samples of the state evolution starting from an initial condition on the boundary of the set.

Similarly to the previous experiments we perform simulations to estimate the maximum probability level at which $\mathcal{P}$ is a PIS, which results in $p_{est} = 0.998$.

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[1]Strictly speaking, condition (4) was only shown for non-degenerate ellipsoids, i.e. it holds for $\bar{Q} = Q + \varepsilon I$ and arbitrarily small $\varepsilon > 0$.
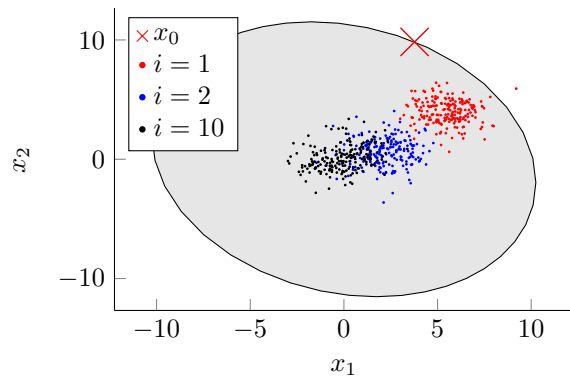
Fig. 2. Ellipsoidal PIS with $p = 0.99$ for system (11). Samples from the state distributions with initial condition x_0 (red cross) are shown at time instants $i = 1, 2, 10$ indicated with red, blue and black dots, respectively

VI. CONCLUSIONS

In this paper we presented a correspondence between robust and probabilistic invariant sets. Under suitable assumptions, this enables an interpretation of robust invariance in terms of probabilistic guarantees, and allows for the construction of sets with probabilistic guarantees from robust methods. In particular, we demonstrated the correspondence for LTI systems with box-shaped and ellipsoidal confidence regions, which allows for generalizing the current state of the art algorithm for computing probabilistic invariant sets. Moreover, we presented an algorithm that constructs probabilistic guarantees for ellipsoidal invariant sets.

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