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Linear-Quadratic Optimal Control for Hybrid Systems with State–driven Jumps

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***Abstract*— In this paper, we lay preliminary foundations towards the comprehensive formulation, together with some constructive results, of the Linear-Quadratic (LQ) optimal control problem for hybrid systems in the presence of state-driven jumps. This objective is achieved by introducing the notion of N-jump optimal control law in terms of a policy that minimizes a certain (quadratic) cost functional and, at the same time, is capable of inducing N (state-driven) jumps of the resulting closed-loop hybrid system. Herein, we then focus on the constructive solution to the 1-jump optimal control problem, by stating necessary and sufficient conditions, together with the rather unexpected implications of the comparison with the, somewhat trivial, 0-jump optimal solution. The paper is concluded by an illustrative example that highlights some of the interesting features of the LQ optimal control problem for such a class of hybrid systems, which derive from the fact that linearity of the resulting hybrid arc is not preserved.**

I. INTRODUCTION

The class of systems characterized by a peculiar interplay between continuous-time evolution and discrete-time events, referred to as *hybrid systems* [1], has gained increasing interest in the last decades. This is mainly due to their flexibility in effectively modeling most modern applications that envision the presence of a continuous-time process monitored and/or controlled by means of digital devices. It is then not surprising that a significant effort has been devoted to extending the solutions to classical problems for non-hybrid systems also to the hybrid contexts, encompassing, for instance, stabilization and optimal control [2], [3], output regulation, see [4], [5], [6] for the nominal case and [7] for an approach to robust internal model-based regulation, dynamic games [8] as well as the characterization of structural properties [9]. However, the majority of the results above deal with a specially structured class of hybrid systems, namely systems characterized by linear (flow and jump) dynamics in the presence of time-driven periodic jumps. The latter assumption, in particular, is crucial since it preserves *linearity* of the resulting solutions.

As a consequence, very few attempts have been made in the literature to provide general results for classes of hybrid systems characterized by alternative jump patterns, such as those undergoing discrete-time events induced by current value of the state variables. These include, for instance, [10] in the context of disturbance decoupling, and [11], which discusses the link between point-wise asymptotic stability and optimality in hybrid systems.

The main objective of this paper consists in laying the foundation for the formulation of the Linear-Quadratic optimal control problem in the presence of hybrid systems with state-driven jumps, together with some preliminary constructive results and insights. Note that, despite that the flow and jump dynamics exhibit linear vector fields, such desirable property is potentially *lost* for the resulting hybrid arcs. Some of the unexpected features that may arise due to this specific aspect are pointed out and discussed in the illustrative example that concludes the paper.

The rest of the paper is organized as follows. In Section II few basic results and preliminaries concerning the class of hybrid systems with linear dynamics and state-driven jumps are briefly reviewed and the problem under investigation is properly formulated. In particular, the definition of N-jump optimal control law is provided as the control input that minimizes a certain (quadratic) cost functional and that, at the same time, is capable of inducing N (state-driven) jumps of the closed-loop hybrid system. This definition is then specialized to the cases of $0-$jump and $1-$jump, by stating necessary and sufficient (constructive) conditions for the solutions of the two scenarios above.

II. PRELIMINARIES AND PROBLEM DEFINITION

Let $\mathbb{R}_{\geqslant 0}$ ($\mathbb{R}_{>0}$) and $\mathbb{Z}_{\geqslant 0}$ ($\mathbb{Z}_{>0}$) denote the set of non-negative (positive) real and integer numbers, respectively. $\mathbb{B}$ denotes the unit ball of suitable dimension. Given $M \in \mathbb{R}^{n\times \ell}$, let $\operatorname{img} M = \{x \in \mathbb{R}^n : \exists d \in \mathbb{R}^\ell \text{ such that } x = Md\}$, $\ker M = \{x \in \mathbb{R}^\ell : Mx = 0\}$, and $M^\dagger$ denote the Moore–Penrose pseudoinverse of M. A set $\mathcal{E} \subset \mathbb{R}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ is a *hybrid time domain* if $\mathcal{E} = \bigcup_{j=0}^{j_{\max}} [t_j, t_{j+1}] \times \{j\}$ with $j_{\max} \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ and $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_{j_{\max}+1}$. Given a hybrid time domain, its length is defined as $\operatorname{length} \mathcal{E} = \sup_t \mathcal{E} + \sup_j \mathcal{E}$, where $\sup_t \mathcal{E} = \sup\{t \in \mathbb{R}_{\geqslant 0} : \exists j \in \mathbb{Z}_{\geqslant 0} \text{ such that } (t, j) \in \mathcal{E}\}$ and $\sup_j \mathcal{E} = \sup\{j \in \mathbb{Z}_{\geqslant 0} : \exists t \in \mathbb{R}_{\geqslant 0} \text{ such that } (t, j) \in \mathcal{E}\}$. A *hybrid arc* is a function $x : \mathcal{E} \to \mathbb{R}^s$ where $\mathcal{E}$ is a hybrid time domain and, for each $j \in \mathbb{Z}_{\geqslant 0}$, the mapping $t \mapsto x(t, j)$ is locally absolutely continuous on $I_j := \{t \in \mathbb{R}_{\geqslant 0} : (t, j) \in \mathcal{E}\}$.

Consider linear hybrid systems described by

$$\dot{x} = Ax + Bu, \qquad x \in \mathcal{C}, \tag{1a}$$

$$x^+ = Ex, \qquad x \in \mathcal{D}, \tag{1b}$$

where $x : \mathcal{E} \to \mathbb{R}^n$ and $u : \mathcal{E} \to \mathbb{R}^m$ are hybrid arcs denoting the state and the input of system (1), respectively, whereas

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$\mathcal{C} = \mathbb{R}^n$ is the *flow set* and $\mathcal{D} = \{x \in \mathbb{R}^n : Sx + b = 0\}$, for some $S \in \mathbb{R}^{\ell \times n}$ and $b \in \mathbb{R}^{\ell}$, is the *jump set*. Note that the set $\mathcal{D}$ is a *multi–affine set* according to the definition given in [10] and briefly recalled hereafter.

Definition 1. A subset $\mathcal{Z} \subset \mathbb{R}^n$ is called a *multi-affine* set if it can be decomposed as the finite union of affine subspaces, namely

$$\mathcal{Z} = \bigcup_{k=1}^{N} \mathcal{A}_k, \quad \mathcal{A}_k = v_k + \mathcal{W}_k, \tag{2}$$

where $v_k \in \mathbb{R}^n$ is a fixed constant vector and $\mathcal{W}_k \subset \mathbb{R}^n$ is a linear subspace with $v_k \notin \mathcal{W}_k$. $\diamondsuit$

Note that, since $\mathcal{C}$ and $\mathcal{D}$ are both closed, system (1) satisfies the Hybrid Basic Conditions given in [1, Ass. 6.5], hence it is well–posed.

Before proceeding with the analysis and the description of the optimal control problem to be investigated, few basic concepts from the theory of hybrid systems are briefly recalled. Given $u : \mathcal{E} \to \mathbb{R}^m$, a hybrid arc $x : \mathcal{E} \to \mathbb{R}^n$ is a solution to system (1) starting at x_0 with input u if $x(0,0) = x_0$ and

- for all $j \in \mathbb{Z}_{\geqslant 0}$, $\frac{d}{dt}x(t,j) = Ax(t,j) + Bu(t,j)$ for all $(t,j) \in I_j$ and $x(t,j) \in \mathcal{C}$ for all $(t,j) \in \operatorname{int} I_j$;
- if $(t,j) \in \mathcal{E}$ and $(t,j+1) \in \mathcal{E}$, then $x(t,j+1) = Ex(t,j)$ and $x(t,j) \in \mathcal{D}$.

A solution $x : \mathcal{E} \to \mathbb{R}^n$ to system (1) is *maximal* if it cannot be extended, whereas it is *complete* if $\operatorname{length} \mathcal{E} = \infty$. Let $\mathcal{S}_u(x_0)$ denote the set of all the maximal solutions to system (1) starting at x_0 with input u. By [1, Prop. 2.11], it can be easily derived that, given $u : \mathcal{E} \to \mathbb{R}^m$, if there exists a maximal solution $x : \mathcal{E} \to \mathbb{R}^n$ to system (1) starting at x_0 with input u, then $\mathcal{S}_u(x_0)$ is a singleton. Hence, let

$$\mathcal{U} := \{u : \operatorname{length} \operatorname{dom} u = \infty \wedge \mathcal{S}_u(x_0) \neq 0\}.$$

Given $x_0 \in \mathbb{R}^n$, a hybrid time domain $\mathcal{E} \subset \mathbb{R}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$, and a control $u : \mathcal{E} \to \mathbb{R}^m$, $u \in \mathcal{U}$, let $x \in \mathcal{S}_u(x_0)$ and consider the following quadratic cost functional[1]

$$J(u, x_0) = \sum_{j=0}^{j_{\max}} \int_{t_j}^{t_{j+1}} (x(t,j)^\top Q x(t,j) + u(t,j)^\top R u(t,j)) dt, \tag{3}$$

where $Q = Q^\top \in \mathbb{R}^{n \times n}$, $R = R^\top \in \mathbb{R}^{m \times m}$ satisfy $Q \succeq 0$ and $R \succ 0$. Determining $u^\star : \mathcal{E} \to \mathbb{R}^m$, $u^\star \in \mathcal{U}$, together with the underlying time domain, such that

$$J(u^\star, x_0) \leqslant J(u, x_0), \quad \forall u \in \mathcal{U},$$

means solving the Linear-Quadratic (briefly, LQ) optimal control problem for (1), (3). From a computational perspective, it might be useful to divide the task into a sequence of sub-problems, each of which corresponding to the task of minimizing the cost functional (3) together with the constraint of inducing a given - and fixed - number of jumps to the resulting closed-loop solution. To this end, for any $N \in \mathbb{Z}_{\geqslant 0}$, the cost J can be split into

$$\sum_{j=0}^{N} \int_{t_j}^{t_{j+1}} (x(t,j)^\top Q x(t,j) + u(t,j)^\top R u(t,j)) dt$$
$$+ \sum_{j=N+1}^{j_{\max}} \int_{t_j}^{t_{j+1}} (x(t,j)^\top Q x(t,j) + u(t,j)^\top R u(t,j)) dt, \tag{4a}$$

under the assumption

$$x(t_{j+1}, j) \in \mathcal{D}, \quad \forall j \in \mathbb{Z}_{\geqslant 0}. \tag{4b}$$

The explicit characterization of a solution to the LQ problem described by (1), (3) appears to be a rather daunting challenge. Therefore, in order to build the basic and preliminary tools towards such a complete solution, consider the following auxiliary problem.

Problem 1. Consider the hybrid system (1), with $x_0 \in \mathbb{R}^n$, together with the cost functional (4), and suppose that $N \in \mathbb{Z}_{\geqslant 0}$ is given. Let $\mathcal{U}_N = \{u \in \mathcal{U} : \sup_j \operatorname{dom} u = N\}$. Find, if any, a hybrid time domain $\mathcal{E}_N^\star \subset \mathbb{R}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$, $\sup_j \mathcal{E}_N^\star = N$, and an input $u_N^\star : \mathcal{E}_N^\star \to \mathbb{R}^m$ such that

$$J(u_N^\star, x_0) \leqslant J(u_N, x_0),$$

for any $u_N \in \mathcal{U}_N$. $\circ$

Roughly speaking, determining a solution to Problem 1 corresponds to design a control input that induces precisely N discrete-time events (jumps) to the trajectories of the closed-loop system and that minimizes the cost functional (4) within the set of the possible control laws that impose N jumps to the plant (1), *i.e.* a solution to Problem 1 solves the *N-jump LQ optimal control problem* (1),(3). Note that, differently from classical results in the linear-quadratic setting, the optimal solution - in addition to the value of the cost - may depend on the initial condition x_0, as it will be illustrated in Section IV by means of an explanatory example. Clearly, by inspecting the structure of (4), it appears evident that solving the LQ optimal control problem hinges upon the ability of determining a solution to Problem 1 for any $N \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$. Therefore, by introducing the notation $J_N^\star(x_0) := J(u_N^\star, x_0)$, the control policy that solves LQ optimal control problem (1),(3) coincides $u^\star = u_{N^\star}$ for

$$N^\star \in \arg \min_{N \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}} J_N^\star(x_0).$$

The main objective of this paper is then twofold: first, to propose a constructive characterization of the solutions to Problem 1 for $N \in \{0, 1\}$, second, to provide some hints on how such a solution can be exploited to solve the original LQ optimal control problem (1),(3).

III. 0-JUMP AND 1-JUMP LQ HYBRID OPTIMAL CONTROL

In this section, a technique to determine a solution to Problem 1 for $N \in \{0, 1\}$ is proposed. Such a solution will be exploited to hint a procedure to tackle the N-jump optimization problem.

[1] Note that, due to the completeness of solutions assumption $\operatorname{length} \operatorname{dom} x = \infty$, one has $j_{\max} = \infty \vee t_{j_{\max}+1} = \infty$.

A. 0–jump optimal control problem

Consider Problem 1 with $N = 0$. Since, by assumption, $\sup_j \operatorname{dom} u_0^\star = 0$, and $\operatorname{dom} x^\star = \operatorname{dom} u_0^\star$, solving Problem 1 with $N = 0$ corresponds essentially to solve the classical continuous-time LQ optimal control problem

$$\min_u J(u_0, x_0) := \int_0^\infty (x^\top(t) Q x(t) + u^\top(t) R u(t)) dt, \\ \text{s.t. } \dot{x} = Ax + Bu_0.$$

By relying on classical results about LQ optimization [12], if the pairs (A, B) and (A, Q) are controllable and detectable, respectively, letting P_+ be the maximal solution[2] to the Algebraic Riccati Equation (briefly, ARE)

$$-Q - PA - A^\top P + PBR^{-1}B^\top P = 0, \tag{5}$$

then the 0–jump LQ optimal control input is given by

$$u_0^\star(t) = -R^{-1}B^\top P_+ x(t), \tag{6}$$

whereas the corresponding cost is given by

$$J_0^\star(x_0) := \min_{u \in \mathcal{U}_0} J(u, x_0) = x_0^\top P_+ x_0\,, \tag{7}$$

where $\mathcal{U}_0$ essentially coincides with the set of continuous functions on $\mathbb{R}^m$. Note that, by resorting to an equivalent Hamiltonian formulation [14], the input $u_0^\star$ given in (6) can be obtained as

$$u_0^\star(t) = -R^{-1}B^\top \lambda(t),$$

where $\lambda(t)$ is the solution to the following linear system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = H \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix},$$

with $x(0) = x_0$, $\lambda(0) = P_+ x_0$, and

$$H = \begin{bmatrix} A & -B^\top R^{-1} B \\ -Q & -A^\top \end{bmatrix} \tag{8}$$

is referred to as the *Hamiltonian matrix.*

Remark 1. It is worth emphasizing that the proposed design does not prevent 0-jump optimal solutions from reaching the jump set $\mathcal{D}$. In fact the flow set $\mathcal{C} = \mathbb{R}^n$, this meaning that 0-jump solutions are allowed to pass through the jump set $\mathcal{D}$ with neither performing the jump nor updating their hybrid time pair (t, j). ▲

[2]By [13], if the pairs (A, B) and (A, Q) are controllable and detectable, respectively, then the ARE (5) admits two solutions P_+ and P_- (denoted maximal and minimal solution, respectively) such that the matrices $A_+ = A - BR^{-1}B^\top P_+$ and $A_- = A - BR^{-1}B^\top P_-$ are stable and anti–stable, respectively.

B. 1–jump optimal control problem

Before investigating optimality conditions for the hybrid plant (1) together with the cost functional (3), hence determining a solution to Problem 1 for $N = 1$, it is useful to distinguish between two different reasonable concepts of optimality in the setting of hybrid systems with state-driven jumps, by firstly introducing the notion of *weak optimality*.

Definition 2. *(Weak Optimality).* Consider the hybrid system (1), with $x_0 \in \mathbb{R}^n$, together with the cost functional (4), and suppose that $N = 1$. A control input $\hat{u}_1^\star(t) : \mathcal{E} \to \mathbb{R}^m$, $\sup_j \mathcal{E} = 1$, is said to be 1*-jump weakly optimal* if $\hat{u}_1^\star \in \mathcal{U}_1$ and, letting $t_1^\star$ be such that $(t_1^\star, 0), (t_1^\star, 1) \in \mathcal{E}$ and $x \in \mathcal{S}_{\hat{u}_1^\star}(x_0)$, one has that $x(t_1^\star, 0) \in \mathcal{D}$ and

(a) letting
$$\tilde{J}_0(u, t_1, x_0) = \int_0^{t_1} (x^\top(t,0) Q x(t,0) + u^\top(t,0) R u(t,0)) dt, \tag{9}$$
the control input $\tilde{u}_1 : [0, t_1] \to \mathbb{R}^m$, $\tilde{u}_1(t) = u(t, 0)$ for all $t \in \{t \in \mathbb{R}_{\geqslant 0} : (t, j) \in \operatorname{dom} \hat{u}_1^\star\}$ and $t_1^\star \in \mathbb{R}$ are such that
$$\tilde{J}_0(\tilde{u}_1, t_1^\star, x_0) \leqslant \tilde{J}_0(u, t_1, x_0),$$
for any absolutely continuous $u \in \mathcal{U}_1$ and any $t_1 \in \mathbb{R}_{\geqslant 0}$;

(b) letting
$$\tilde{J}_\infty(u, x_0) = \int_{t_1^\star}^{\infty} (x^\top(t,1) Q x(t,1) + u^\top(t,1) R u(t,1)) dt,$$
the control input $\tilde{u}_\infty : [t_1^\star, \infty) \to \mathbb{R}^m$, $\tilde{u}_\infty(t) = u(t, 1)$ for all $t \in \{t \in \mathbb{R}_{\geqslant 0} : (t, j) \in \operatorname{dom} u_1^\star\}$, is such that
$$\tilde{J}_\infty(\tilde{u}_\infty, x_0) \leqslant \tilde{J}_\infty(u, x_0),$$
for any absolutely continuous $u : [t_1^\star, \infty) \to \mathbb{R}^m$. ⋄

The above definition of (weak) optimality entails that the candidate control input $\tilde{u}_1$ and jumping time should minimize the cost functional with respect to any other discrete-time instants in $\mathbb{R}_{\geqslant 0}$ and any absolutely continuous input that induces a jump at the same continuous time as $\tilde{u}_1$, namely such that $\operatorname{dom} u = \operatorname{dom} \tilde{u}_1$. Clearly, if $u_1^\star$ is a solution to Problem 1 for $N = 1$ it is also a 1–jump weakly optimal control, while the converse may not hold. As it will be shown in the following discussions, such a weaker notion of optimality is instrumental for the design of computationally viable strategies to address the, otherwise daunting, problem in this context. The following statement provides sufficient conditions for existence of the above weak solution.

Proposition 1. *Consider the hybrid system* (1)*, with $x_0 \in \mathbb{R}^n$, together with the cost functional* (4)*, and suppose that $N = 1$ and that the pairs (A, B) and (A, Q) are controllable and detectable, respectively. Let P_+ and P_- be the maximal and minimal solution to the ARE* (5)*, respectively, and let $\Delta = P_+ - P_-$. Let V_f be a basis matrix of* $\ker S$ *and define the following matrices*

$$K_+ = R^{-1}B^\top P_+, \qquad K_- = R^{-1}B^\top P_-, \\ A_+ = A - BK_+, \qquad A_- = A - BK_-.$$

Thus, letting

$$\Omega(t) = \Delta^{-1} - e^{A_+ t}\Delta^{-1}e^{A_+^\top t},$$
$$L(t) = \begin{bmatrix} I & e^{-A_- t} \\ e^{A_+ t} & I \end{bmatrix},$$
$$X(t) = \begin{bmatrix} P_+ e^{A_+^\top t}\Omega^{-1}(t)e^{A_+ t} & -e^{A_+^\top t}\Omega^{-1}(t) \\ -\Omega^{-1}(t)e^{A_+ t} & \Omega^{-1}(t) - P_+ \end{bmatrix},$$
$$N(t) = \begin{bmatrix} I & e^{-A_- t} \\ Se^{A_+ t} & S \\ 0 & 0 \\ -V_f^\top P_+ e^{A_+ t} & -V_f^\top P_- \end{bmatrix},$$
$$h = [\ x_0^\top \quad -b^\top \quad 0 \quad 0\ ]^\top,$$

define

$$\pi(t) = N^\dagger(t)h + K(t)\nu(t)$$

where $K(t)$ is a basis matrix for $\ker N(t)$ *and $\nu(t)$ is an arbitrary vector. Then, then there exists a 1-jump weakly optimal control law if the function*

$$\Gamma(t) := \pi^\top(t)L^\top(t)X(t)L(t)\pi(t) \tag{10}$$

attains a global minimum with respect to $t \in [0, \infty)$. ∘

Remark 2. The condition requiring the function $\Gamma(t)$ in (10) to attain a minimum for a finite value of continuous time t specializes the conditions in Assumption 5.1 of [11] to the context of Linear-Quadratic optimal control problems for hybrid systems with state-driven jumps. Moreover, the proof of the above statement also suggests a constructive procedure, explored in more details in the following theorem, for the computation of a weakly optimal control law: an expression for the candidate policy is firstly computed as a function of the jumping time t_1, then the corresponding optimal value for the cost is minimized with respect to the time of the discrete-time event. ▲

The following theorem shows that, if the assumptions of Proposition 1 hold, a 1-jump weakly optimal control can be determined by solving a constrained minimization problem.

Theorem 1. *Consider the hybrid system* (1)*, with $x_0 \in \mathbb{R}^n$, together with the cost functional* (4)*, and suppose that the assumptions of Proposition 1 hold. Let $t_1^\star, \lambda_0^\star$ be a solution to the following minimization problem*

$$\left| \begin{array}{rl} \min_{t_1,\lambda_0,d} & \xi_0^\top \Psi(t_1)\xi_0, \\ \text{with} & S[\ I \quad 0\ ]e^{H(t_1)}\xi_0 + b = 0, \\ & [\ 0 \quad I\ ]e^{H(t_1)}\xi_0 = S^\top d, \end{array} \right. \tag{11}$$

where $\xi_0 = [\ x_0^\top \quad \lambda_0^\top\ ]^\top$, H is as in (8)*, and*

$$\Psi(t_1) = \int_0^{t_1} e^{H^\top \tau} \begin{bmatrix} Q & 0 \\ 0 & BR^{-1}B^\top \end{bmatrix} e^{H\tau} d\tau. \tag{12}$$

Then, letting $\mathcal{E} = ([0, t_1] \times \{0\}) \cup ([t_1, \infty) \times \{1\})$ and $\xi_0^\star = [\ x_0^\top \quad (\lambda_0^\star)^\top\ ]^\top$, the control input $\hat{u}_1^\star : \mathcal{E} \to \mathbb{R}^m$,

$$\hat{u}_1^\star(t, j) = \begin{cases} -[\ 0 \quad R^{-1}B^\top\ ]e^{Ht}\xi_0^\star, & \textit{if } (t, 0) \in \mathcal{E}, \\ -R^{-1}B^\top P_+ x(t), & \textit{if } (t, 1) \in \mathcal{E}, \end{cases}$$

where P_+ is the maximal solution to the ARE (5)*, is a* 1*-jump weakly optimal control.* ∘

In the remainder of this section, by using the tools just developed to solve the 1-jump weak LQ optimal control problem, we propose a technique to determine a solution to the original Problem 1 with $N = 1$. Using the results concerning infinite horizon LQ design, briefly recalled in Section III-A, it can be easily derived that the cost functional to be minimized to solve Problem 1 for $N = 1$ can be rewritten as

$$\min_{u,t_1} \left\{ \int_0^{t_1} (x(t,0)^\top Qx(t,0) + u(t,0)^\top Ru(t,0))dt + x(t_1,0)^\top E^\top P_+ Ex(t_1,0) \right\} \tag{13}$$

subject to the constraint $x(t_1, 0) \in \mathcal{D}$. Note that, in the cost functional (13), the jump time t_1 and the control input u are both minimization variables, while the *tail* of the cost due to the infinite horizon sub-problem in $[t_1, \infty]$ is incorporated in the second quadratic term of (13). The following statement, which is based on tools similar to those employed in Proposition 1, guarantees the existence of a solution to Problem 1 for $N = 1$.

Proposition 2. *Consider the hybrid system* (1)*, with $x_0 \in \mathbb{R}^n$, together with the cost functional* (4)*, and suppose that $N = 1$ and that the pairs (A, B) and (A, Q) are controllable and detectable, respectively. Let the matrices P_+, P_-, A_+, A_-, Δ, $\Omega(t)$, V_f, h, $L(t)$, and $X(t)$ be defined as in Proposition 1. Thus, letting*

$$M(t) = \begin{bmatrix} I & e^{-A_- t} \\ Se^{A_+ t} & S \\ 0 & 0 \\ V_f^\top (E^\top P_+ E - P_+)e^{A_+ t} & V_f^\top (E^\top P_+ E - P_-) \end{bmatrix},$$
$$P = \mathrm{diag}(0, E^\top P_+ E)$$

define

$$\varpi(t) = M^\dagger(t)h + H(t)\mu(t)$$

where $H(t)$ is a basis matrix for $\ker M(t)$ *and $\mu(t)$ is an arbitrary vector. Then, there exists a 1-jump* strong *optimal control law, namely a solution to Problem 1 for $N = 1$, if the function $\Gamma(t) := \varpi^\top(t)L^\top(t)(P + X(t))L(t)\varpi(t)$ attains a global minimum with respect to $t \in [0, \infty)$.* ∘

The following theorem shows that, under the assumptions of Proposition 2, a solution to Problem 1 for $N = 1$ can be determined by solving a constrained minimization problem.

Theorem 2. *Consider the hybrid system* (1)*, with $x_0 \in \mathbb{R}^n$, together with the cost functional* (4)*, and suppose that the assumptions of Proposition 2 hold. Let $t_1^\star, \lambda_0^\star$ be a solution to the following minimization problem*

$$\left| \begin{array}{rl} \min_{t_1,\lambda_0,d} & \xi_0^\top \Psi(t_1)\xi_0 + x_1^\top E^\top P_+ Ex_1, \\ \text{with} & x_1 = [\ I \quad 0\ ]e^{H(t_1)}\xi_0 \\ & Sx_1 + b = 0, \\ & [\ 0 \quad I\ ]e^{H(t_1)}\xi_0 = S^\top d, \end{array} \right. \tag{14}$$

where $\xi_0 = [\ x_0^\top \quad \lambda_0^\top\ ]^\top$, H and $\Psi(t_1)$ are defined as in (12)*, and P_+ is the maximal solution to the ARE* (5)*. Then, letting $\mathcal{E} = ([0, t_1] \times \{0\}) \cup ([t_1, \infty) \times \{1\})$ and $\xi_0^\star = [\ x_0^\top \quad (\lambda_0^\star)^\top\ ]^\top$, the control input $u_1^\star : \mathcal{E} \to \mathbb{R}^m$,*

$$u_1^\star(t, j) = \begin{cases} -[\ 0 \quad R^{-1}B^\top\ ]e^{Ht}\xi_0^\star, & \text{if } (t, 0) \in \mathcal{E}, \\ -R^{-1}B^\top P_+ x(t), & \text{if } (t, 1) \in \mathcal{E}, \end{cases}$$

is a solution to Problem 1 for $N = 1$.

Remark 3. The definition of N-jump weak optimality can be easily given by adapting Definition 2. More precisely, the control input $\hat{u}_N^\star$ and the jumping times $t_{k+1} \in (t_k, \infty)$ are N-jump weak optimal if $\tilde{u}_k(t) = \hat{u}_N(t, k)$ for all t such that $(t, k) \in \operatorname{dom} \hat{u}_N$ minimizes the cost functional

$$\tilde{J}_k(u, t_{k+1}, x_0) = \int_{t_k}^{t_{k+1}} (x^\top(t, k)Qx(t, k) + u^\top(t, k)Ru(t, k))dt$$

for all $k \in \mathbb{Z}_{\geqslant 0}$, $k < N$, and item (b) of Definition 2 holds with t_1 substituted by t_{N+1}. Such a N-jump weakly optimal control input $\hat{u}_N^\star$ can be determined by solving iteratively multiple problems wholly similar to the problem given in (11) and existence of such optimal controls can be argued with assumptions entirely similar to the ones made in Proposition 1.

On the other hand, in order to determine a solution to Problem 1 for $N \in \mathbb{Z}_{\geqslant 0}$, $N > 1$, one has to solve a nonlinear programming problem similar to (14), but with nested conditions about $x_1, x_2, \ldots, x_N$ in order to relate pre– and post–jump conditions and with a cost functional to be minimized that depends on such variables. Conditions on the existence of such solutions can be argued with recursive assumptions similar to the ones made in Proposition 2. ▲

IV. ILLUSTRATIVE EXAMPLE

As an explanatory - although particularly insightful - example, consider the linear hybrid system

$$\begin{array}{rcll} \dot{x} & = & x + u, & x \in \mathcal{C}, \\ x^+ & = & 0, & x \in \mathcal{D}, \end{array} \tag{15}$$

where $x(t, j) \in \mathbb{R}$, $u(t, j) \in \mathbb{R}$ for all admissible $(t, j) \in \mathbb{R}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$, $\mathcal{C} = \mathbb{R}$ and $\mathcal{D} = \{1\}$. Define the cost functional J as in (3) with $Q = \varepsilon$, $\varepsilon \in \mathbb{R}_{>0}$, and $R = 1$. It is worth noting that, since $E = 0$ and hence $x(t_j, j+1) = 0$ for all $(t_j, j+1) \in \operatorname{dom} x$, determining a solution to such LQ optimization problem is equivalent to solve Problem 1 for $N = 0$ and $N = 1$ and, letting $u_0^\star$ and $u_1^\star$ be the respective solutions, select $u^\star = u_{i^\star}$, where $i^\star \in \arg\min_{i \in \{0,1\}} J(u_i^\star, x_0)$. Furthermore, by (11) and (14), since $E = 0$, in this case 1-jump weakly optimal solutions are also 1-jump strong solutions, hence solving Problem 1 for $N = 1$ (actually, for $N \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$).

To begin with, consider Problem 1 with $N = 0$. The scalar $P_+ = \sqrt{\varepsilon + 1} + 1$ is the maximal solution to the ARE (5) and thus the input $u_0^\star = -(\sqrt{\varepsilon + 1} + 1)x$ is a solution to Problem 1 with $N = 0$, leading to the following minimal value of the cost functional

$$J(u_0^\star, x_0) = (\sqrt{\varepsilon + 1} + 1)x_0^2. \tag{16}$$

Consider now Problem 1 with $N = 1$. The pair (A, B) is reachable (namely, $\operatorname{img} P = \mathbb{R}$) and the pair (A, Q) is detectable, thus guaranteeing that the basic assumptions of Proposition 1 are satisfied. In addition to P_+, let us consider also the minimal solution to the ARE, namely $P_- = -\sqrt{\varepsilon + 1} + 1$, and set $A_+ := A - BRB^T P_+ = -\sqrt{\varepsilon + 1}$, $\quad A_- := A - BRB^T P_- = \sqrt{\varepsilon + 1}$. As the jump set is a singleton, for any fixed $\tau > 0$ the optimal cost with the constraint $x(\tau, 0) = 1$ can be easily computed using [15, Lemma 4] and is given by

$$\Gamma(\tau) = \varepsilon_1((1 + x_0^2)\cosh(\varepsilon_1\tau) - 2x_0)\operatorname{csch}(\varepsilon_1\tau) + x_0^2 - 1$$

where $\varepsilon_1 := \sqrt{\varepsilon + 1}$. As we are interested in minimizing such cost with respect to τ, we must look for a zero of the first derivative $\Gamma'(\tau)$. Computing the latter yields

$$\Gamma'(\tau) = \varepsilon_1(2x_0\cosh(\varepsilon_1\tau) - 1 - x_0^2)\operatorname{csch}^2(\varepsilon_1\tau),$$

thus revealing two distinct regimes: in fact, a (unique) solution to $\Gamma'(\tau) = 0$ exists if and only if $x_0 > 0$, i.e. if and only if the initial state is "on the same side of the jump set with respect to the origin", while there is no solution otherwise. In particular, if $x_0 > 0$ the optimal time $t^\star$ such that $\Gamma'(t^\star) = 0$ is given by the following closed-form expression

$$t^\star = \frac{1}{\varepsilon_1}\log\left(\frac{1 + x_0^2}{2x_0} + \sqrt{\left(\frac{1 + x_0^2}{2x_0}\right)^2 - 1}\right) =: h(x_0, \varepsilon_1)$$

Accordingly, the value of the optimal cost is

$$\Gamma(t^\star) = |x_0^2 - 1|(\varepsilon_1 + \operatorname{sign}(x_0 - 1)) =: g(x_0, \varepsilon_1), \tag{17}$$

with associated optimal input

$$u_1^\star(t, j) = \begin{cases} \operatorname{csch}(\varepsilon_1 h(x_0, \varepsilon_1)) \cdot \\ \quad \cdot[\varepsilon_1\cosh(\varepsilon_1 t) - \sinh(\varepsilon_1 t) \\ \quad + \varepsilon_1\sinh(\varepsilon_1(t - h(x_0, \varepsilon_1)))x_0 & \text{if } (t, 0) \in \mathcal{E} \\ \quad - \cosh(\varepsilon_1(t - h(x_0, \varepsilon_1)))x_0] \\ \qquad 0 & \text{if } (t, 1) \in \mathcal{E} \end{cases}$$

and $J(u_1^\star, x_0) = \Gamma(t^\star)$. In order to select the most convenient control strategy among $\{u_0^\star, u_1^\star\}$, it might be helpful to define the switching function $\sigma(\cdot) : \varepsilon \mapsto x_0$ that associates to each ε the unique positive solution x_0 of $J(u_0^\star, x_0) = J(u_1^\star, x_0)$. Thanks to (16) and (17), for $\zeta \in (0, +\infty)$ the switching function reads as $\sigma(\zeta) = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{1}{\sqrt{1+\zeta}}}$, and the following rule can be stated for the synthesis of the overall optimal control $u^\star(t, j)$:

$$u^\star(t, j) = \begin{cases} u_0^\star(t, j) & \text{if } x_0 \leq \sigma(\varepsilon) \\ u_1^\star(t, j) & \text{if } x_0 \geq \sigma(\varepsilon) \end{cases}$$

It is worth noticing that, since $\sigma(\zeta) < 1/\sqrt{2}$ for any $\zeta \in (0, +\infty)$, the optimal control regime always involves a jump when the initial condition satisfies $x_0 \geq 1/\sqrt{2}$, irrespectively

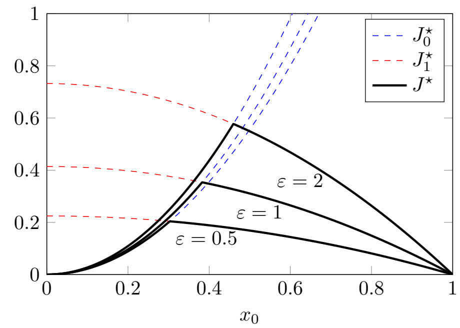

Fig. 1. Optimal costs for different values of ε.

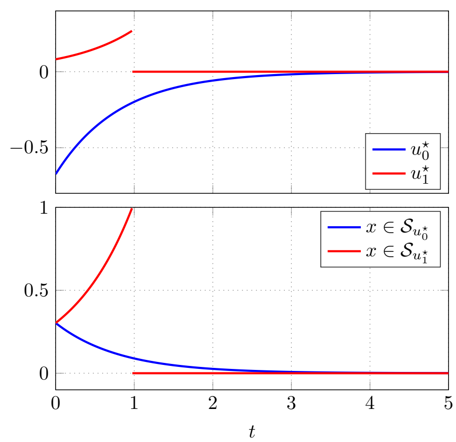

Fig. 2. Optimal controls and state trajectory for $x_0 = \sigma(\varepsilon)$ and $\varepsilon = 0.5$.

of the value of ε. Conversely, whenever $x_0 < 0$, the optimal trajectory will be characterized by flow only.

Figure 1 depicts the optimal cost $J^\star(x_0) := J(u^\star, x_0)$ for all $x_0 \in (0, 1)$ and for three different values of ε. Note that if $x_0 \leqslant \sigma(\varepsilon)$, then $J^\star(x_0) = J_0^\star(x_0)$, whereas $J^\star(x_0) = J_1^\star(x_0)$, if $x_0 \geqslant \sigma(\varepsilon)$.

It is worth noticing that, if $x_0 = \sigma(\varepsilon)$, then both $u_0^\star$ and $u_1^\star$ solve the considered LQ optimal control problem (indeed, $J^\star(\sigma(\varepsilon)) = J_0^\star(\sigma(\varepsilon)) = J_1^\star(\sigma(\varepsilon))$). Figure 2 depicts the optimal controls $u_0^\star$ and $u_1^\star$ and the corresponding state trajectories for $\varepsilon = 0.5$ and $x_0 = \sigma(0.5)$. It is worth stressing that, even tough the two control inputs depicted in Figure 2 lead to substantially different state trajectories (namely, one makes the state jump, whereas the other makes the system just flow) the two corresponding costs are actually equal. This phenomenon is unprecedented in Linear-Quadratic optimal control theory under the assumptions stated herein (in particular with a positive definite matrix R).

Some other interesting aspects arise in the limit case $\varepsilon \to 0^+$. In fact, assuming the initial condition $x_0 \in (0, 1)$, the 0-jump optimal input becomes $u_0^\star = -2x$, while the 1-jump optimal input reduces to $u_1^\star = 0$. Evaluating and comparing the costs $J_0^\star$ and $J_1^\star$ for such initial condition, the 0-jump control $u_0^\star$ turns out to be never optimal (in the *strong* sense), whereas $u_1^\star$ always yields zero cost. Moreover, despite the lack of observability (via Q), the 0-jump control u_0^* guarantees asymptotic stability of the closed-loop system, while the system driven by the 1-jump optimal input $u_1^\star$ is unstable but has an attractive equilibrium in the origin, which is allowed by the nonlinear nature of the closed-loop trajectories.

V. Conclusions

In this paper, we have laid preliminary foundations towards the comprehensive formulation, together with some constructive results, to the Linear-Quadratic (LQ) optimal control problem for hybrid systems in the presence of state-driven jumps. This objective is achieved by introducing the notion of N-jump optimal control law in terms of a policy that minimizes a certain (quadratic) cost functional and, at the same time, is capable of inducing N (state-driven) jumps of the resulting closed-loop hybrid system, and by distinguishing between *weak* and *strong* optimality. In particular, the former has been provided mainly in order to suggest a computationally viable strategy towards the latter. Herein, we have then focused on the constructive solution to the 1-jump optimal control problem, by stating necessary and sufficient conditions, together with the rather unexpected implications of the comparison with the, somewhat trivial, 0-jump optimal solution.

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