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On Parametric Model Order Reduction Based on Projections

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***Abstract*— This paper reviews in a geometric way two methods ([1] and [2]) of parametric model order reduction based on projection and matrix interpolation. Then an alternative approach is proposed combining the concept of generalised coordinate system used in the first method with the oblique projection used in the second one. Then the different methods are compared by applying them on an academic models.**

I. Introduction

Numerical simulations of dynamical systems are increasingly used in numerous applications. To enhance this process by reducing the simulation time and/or the needed processing power, model order reduction aims to approximate a complex model with a simpler one, easier (faster and/or requiring less computations) to simulate. Model order reduction for Linear Time-Invariant (LTI) systems has become a quite mature field, and the focus is now on larger classes of models, such as nonlinear models, Time-Varying models or parameterised models. This work studies the so-called parametric Model Order Reduction (pMOR), where the reduction of models depends on some parameters (constant in time). This parameter dependency can be used to model a parametric uncertainty, a different configuration of the system, etc.

This work is organised as follows: after having introduced the problem, two main methods are reviewed in Section III with a focus on the geometric interpretation. Then, a combination of these two approaches is proposed in Section IV and some results on academic examples are studied in Section V.

II. Problem Statement

This work addresses the problem of parametric model order reduction (pMOR). That is the approximation of a parametric model of order n by a model of order $\hat{n} \ll n$ with the parameter dependency of the original model (the parameters are constant and their range is known).

The approach considered here is to first sample over the parameter space a set of ℓ LTI models, then to reduce and adapt them (the *off-line computations*) for finally being able to compute via a matrix interpolation, possibly in real-time, a reduced order model (ROM) at any desired parameter value (the *on-line computations*).

Each of these steps can be tackled in many ways, what [3] pointed out as the *degrees of freedom*. The choices to be made depends on the considered system, the memory storage and processing power available, as well as specifications like real-time capability, maximum error allowed, etc.

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In this work, a few techniques are investigated and compared, mostly those concerning the reduction and adaptation of the local models and their interpolation.

A. Local Models Sampling

The parametric model $\mathbf{\Sigma}(\rho)$ is discretised over the parameter space into a set of local LTI models $\{\mathbf{\Sigma}_i\}_{i=1,\dots,\ell}$. As detailed in [4], a few different methods can be used to sample the parameter space. The most intuitive way is to construct a grid (or a sparse grid) over the parameter space and sample a model at each node. Another possibility is to randomly sample many models over the parameter space. Although these two methods can be effective for models with few parameters and limited parameter range, they are unsuitable for models with a large number of parameters (more than a dozen for instance), so other techniques are used, which better represent the parameter variations while sampling fewer models.
An iterative method that can be used is parameter sampling via greedy search, it is an algorithm that searches for the point on the parameter space where the error between the original model and the reduced one is maximal. Then it samples a model at this point, uses it to recompute the reduced model, and finds the new point where the error is now maximal, samples it, etc. until the error is acceptable.
Another interesting method is to use local sensitivity analysis to determine where to sample, consequently more samples are placed where the system behaviour depends greatly on the parameter variations and fewer elsewhere.
Although parameter sampling via local sensitivity analysis seems a very interesting technique, only grid-based sampling is used in this work, for reasons of simplicity.

B. Local Models Order Reduction

All local models $\{\mathbf{\Sigma}_i\}_{i=1,\dots,\ell}$ are reduced to the same order $\hat{n} \ll n$ by means of a projection-based model order reduction technique. In this work, balanced truncation or a closely related method are considered. Model order reduction by projection of a model $\mathbf{\Sigma}$ restricts the original state $x \in \mathbb{R}^n$ to a lower dimensional subspace $\mathcal{V}$ of dimension $\hat{n}$.

Therefore, if the local models $\mathbf{\Sigma}_i$ are reduced separately, their reduced states $\hat{\mathbf{x}}_i$ lie in a priori different subspaces $\mathcal{V}_i$ preventing any interpolation. To tackle this drawback, the two main solutions known in the literature introduce a *mean* subspace $\mathcal{R}$ of dimension $\hat{n}$ obtained by concatenating the matrices V_i whose columns span the subspaces $\mathcal{V}_i$ into a matrix V_{all} and then computing its singular value decomposition

$$V_{\text{all}} = [\, V_1 \; V_2 \; \cdots \; V_\ell \,] = U \Sigma N^{\mathsf{T}}.$$

Then, the *mean* subspace,$\mathcal{R}$ is spanned by the first $\hat{n}$ columns of U, that is $\mathcal{R} = \operatorname{span} R$ with $R = [\,U_1\ U_2\ \cdots\ U_{\hat{n}}\,]$.

The purpose of pMOR is to obtain a low order representation of the initial state-space $\mathbb{S}(\rho) = \left[\begin{array}{c|c} A(\rho) & B(\rho) \\ \hline C(\rho) & D(\rho) \end{array}\right]$ by interpolating the state-space representations $\left\{\hat{\mathbb{S}}_i = \left[\begin{array}{c|c} \hat{A}_i & \hat{B}_i \\ \hline \hat{C}_i & \hat{D}_i \end{array}\right]\right\}_{i=1,\dots,\ell}$ of the reduced local models (obtained for ℓ fixed values $(\rho_i)_{i=1,\dots,\ell}$ of the parameter ρ):

$$\hat{\mathbb{S}}(\rho) = \left[\begin{array}{c|c} \hat{A}(\rho) & \hat{B}(\rho) \\ \hline \hat{C}(\rho) & \hat{D}(\rho) \end{array}\right] = \left[\begin{array}{c|c} \sum_i \alpha_i(\rho)\hat{A}_i & \sum_i \alpha_i(\rho)\hat{B}_i \\ \hline \sum_i \alpha_i(\rho)\hat{C}_i & \sum_i \alpha_i(\rho)\hat{D}_i \end{array}\right]$$

where α_i are weighting coefficients.

A common case is the polytopic representation when the parameter ρ is a vector of dimension p lying in a convex polytope of the parameter vector space. Such a system has $\ell = 2^p$ vertices $(A_i, B_i, C_i, D_i) = (A(\omega_i), B(\omega_i), C(\omega_i), D(\omega_i))$ corresponding to all possible combinations of minimum $\underline{\rho}_i$ and maximum $\overline{\rho}_i$ values for each component ρ_i of parameter vector ρ. In this case, the weights are defined such that : $\sum_{i=1}^{\ell} \alpha_i(\rho) = 1$, $\alpha_i(\rho) \geq 0$ and $\alpha_i(\omega_j) = \delta_{ij}\ \forall i,j \in [\![1;\ell]\!]$. The explicit expression is $\alpha_i(\rho) = \prod_{j=1}^{p} \frac{|\rho_j - \omega_i^{\complement}(j)|}{\overline{\rho}_j - \underline{\rho}_j}$ where $\omega_i^{\complement}$ denotes the complement of $\omega_i : \omega_i^{\complement}(j) = \overline{\rho}_j$ if $\omega_i(j) = \underline{\rho}_j$ or $= \underline{\rho}_j$ if $\omega_i(j) = \overline{\rho}_j$.

Similar to the polytopic representation, grid representation consists of sampling a set of models over the parameter space equally distant from one another to build a set of local LTI models. The model $\mathbf{\Sigma}(\rho)$ is then computed as an interpolation between the set of models.

For this, the ℓ state-space representations $\hat{\mathbb{S}}_i$ need to be compatible in some sense, that is, the states $\hat{x}_i$ have to be expressed in the same basis.

One approach to this problem is to separately reduce the local models by balanced truncation, then to adapt them with a common basis. Another approach is to reduce the local models by projection onto a single subspace, so the adaptation process is no more needed.

These order reductions can be viewed by mean of oblique projections. The projection onto a subspace $\mathcal{V} = \operatorname{span}(V)$ orthogonally to a subspace $\mathcal{W} = \operatorname{span}(W)$ is obtained by

$$\Pi = V(W^{\mathsf{T}}V)^{-1}W^{\mathsf{T}}. \tag{1}$$

If $\mathcal{W} = \mathcal{V}$ the projection is said to be orthogonal. If the matrix V and W are biorthonormal (that is $W^{\mathsf{T}}V = I$) the expression of the projector is $\Pi = VW^{\mathsf{T}}$.

III. Review of existing Methods

A. Local Balanced Truncation and Adaptation [1]

1) Reduction via balanced truncation: Each local model $\mathbf{\Sigma}_i$ is separately reduced via balanced truncation. This projection is expressed in accordance with the algorithm of Laub [5] as

$$\Pi_i = \underbrace{L_{\mathrm{r},i}N_i\Lambda_i^{-1/2}}_{V_i}\underbrace{\Lambda_i^{-1/2}U_i^{\mathsf{T}}L_{\mathrm{o},i}^{\mathsf{T}}}_{W_i^{\mathsf{T}}}$$

where $L_{\mathrm{r},i}$ and $L_{\mathrm{o},i}$ are the lower Cholesky factors of respectively the reachability and observability Gramians of the realisation $\mathbb{S}_i$ and the matrices $N_i, U_i \in \mathbb{R}^{n\times\hat{n}}$ and $\Lambda_i \in \mathbb{R}^{\hat{n}\times\hat{n}}$ are obtained from the SVD

$$L_{\mathrm{o},i}^{\mathsf{T}}L_{\mathrm{r},i} = \left[\begin{array}{cc} U_i & U \end{array}\right]\left[\begin{array}{cc} \Lambda_i & \\ & \Lambda \end{array}\right]\left[\begin{array}{cc} N_i & N \end{array}\right]^{\mathsf{T}}.$$

The obtained set of reduced local models $\{\hat{\mathbf{\Sigma}}_i\}_{i=1,\dots,\ell}$ have the following state-space representations $\{\hat{\mathbb{S}}_i^{\star}\}_{i=1,\dots,\ell}$

$$\left\{\hat{\mathbb{S}}_i^{\star} = \left[\begin{array}{c|c} \hat{A}_i^{\star} & \hat{B}_i^{\star} \\ \hline \hat{C}_i^{\star} & \hat{D}_i^{\star} \end{array}\right] \overset{\text{def.}}{=} \left[\begin{array}{c|c} W_i^{\mathsf{T}}AV_i & W_i^{\mathsf{T}}B \\ \hline CV_i & D_i \end{array}\right]\right\}_{i=1,\dots,\ell}.$$

Their states evolve in different subspaces $\{\mathcal{V}_i\}_{i=1,\dots,\ell}$.

2) Adaptation of the reduced local models: Each reduced local model $\hat{\mathbf{\Sigma}}_i$ has its state $\hat{\mathbf{x}}_i$ lying in a subspace $\mathcal{V}_i$ (spanned by matrices V_i) and expressed in the basis built by the columns of the matrix V_i. The concept of generalised coordinates gives a meaning to state-space representation interpolation having their states lying in different subspaces. It has been introduced in [1] and further used in [6] and [3]. Although in these articles the concept of generalised coordinates has been described by means of matrix transformations, here another slightly more geometric approach is adopted.

Thus, it is important to distinguish a geometric vector (preexisting to any basis) from its algebraic expression in a basis. The considered geometric vectors $\hat{\mathbf{x}}_i$ are the reduced state-vectors. Three different algebraic expressions of these geometric vectors $\hat{\mathbf{x}}_i \in \mathcal{V}_i$ represented in different bases are used:

- $\hat{x}_i^{\star} = [\hat{\mathbf{x}}_i]_{\mathcal{B}_{\mathcal{V}_i}^{\star}} \in \mathbb{R}^{\hat{n}}$, coordinates in the basis $\mathcal{B}_{\mathcal{V}_i}^{\star}$ formed by the columns of V_i.
- $V_i[\hat{\mathbf{x}}_i]_{\mathcal{B}_{\mathcal{V}_i}} \in \mathbb{R}^n$, expression in the canonical basis of the ambient space.
- $\hat{x}_i = [\hat{\mathbf{x}}_i]_{\mathcal{B}_{\mathcal{V}_i}} \in \mathbb{R}^{\hat{n}}$: coordinates in the new basis $\mathcal{B}_{\mathcal{V}_i}$.

Definition *Generalised coordinate system*: Given a set of subspaces $\{\mathcal{V}_i\}_{i=1,\dots,\ell}$ and $\mathcal{R}$ of dimension $\hat{n}$ with bases $\mathcal{B}_{\mathcal{V}_i} = \{v_{ij}\}_{j=1,\dots,\hat{n}}$ and $\mathcal{B}_{\mathcal{R}} = \{r_j\}_{j=1,\dots,\hat{n}}$. The bases $\mathcal{B}_{\mathcal{V}_i}$ describe a generalised coordinate system w.r.t. the basis $\mathcal{B}_{\mathcal{R}}$ if $\Pi_{\mathcal{R}}v_{i_j} = r_j$ (for all i and j) where $\Pi_{\mathcal{R}}$ is a given projection onto $\mathcal{R}$. The bases $\mathcal{B}_{\mathcal{V}_i}$ are said to be compatible w.r.t. the basis $\mathcal{B}_{\mathcal{R}}$.

In other words, the bases $\mathcal{B}_{\mathcal{V}_i}$ define a set of generalised coordinates w.r.t. the basis $\mathcal{B}_{\mathcal{R}}$ if the projection of the basis vectors v_{i_j} onto $\mathcal{R}$ is equal to the basis vectors r_j.

Equivalently, considering any vector instead of the basis vector,

$$\forall \mathbf{x}_i \in \mathcal{V}_i, \quad [\mathbf{x}_i]_{\mathcal{B}_{\mathcal{V}_i}} = [\Pi_{\mathcal{R}}\mathbf{x}_i]_{\mathcal{B}_{\mathcal{R}}}$$

where $x_i = [\mathbf{x}_i]_{\mathcal{B}_{\mathcal{V}_i}}$ is the expression of the geometric vector $\mathbf{x}_i$ in the basis $\mathcal{B}_{\mathcal{V}_i}$, and $[\Pi_{\mathcal{R}}\mathbf{x}_i]_{\mathcal{B}_{\mathcal{R}}}$ is the expression of the projection of $\mathbf{x}_i$ onto $\mathcal{R}$ in the basis $\mathcal{B}_{\mathcal{R}}$.

a) Generalised coordinate system by orthogonal projection: As it is used in [1], new bases $\mathcal{B}_{\mathcal{V}_i}$ are constructed to obtain a generalised coordinate system defined via an orthogonal projection $\Pi_{\mathcal{R}}$. Since the subspaces $\mathcal{V}_i$ are spanned by matrices V_i and the subspace $\mathcal{R}$ is spanned by R, the vector $\hat{\mathbf{x}}_i$ projected onto $\mathcal{R}$ and expressed in $\mathcal{B}_{\mathcal{R}}$ (the basis described by the columns of R) is

$$[\Pi_{\mathcal{R}}\hat{\mathbf{x}}_i]_{\mathcal{B}_{\mathcal{R}}} = R^{\mathsf{T}} V_i [\hat{\mathbf{x}}_i]_{\mathcal{B}^\star_{\mathcal{V}_i}} \tag{2}$$

Note that $V_i[\mathbf{x}_i]_{\mathcal{B}^\star_{\mathcal{V}_i}}$ is the vector $\mathbf{x}_i$ expressed in the original state-space $\mathbb{R}^n$ and then multiplying by R^{T} we get its projection onto $\mathcal{R}$ expressed in $\mathcal{B}_{\mathcal{R}}$.
The expression of the $\hat{\mathbf{x}}_i$ in the new bases $\mathcal{B}_{\mathcal{V}_i}$ are obtained by suitable changes of bases

$$[\hat{\mathbf{x}}_i]_{\mathcal{B}^\star_{\mathcal{V}_i}} = T_i [\hat{\mathbf{x}}_i]_{\mathcal{B}_{\mathcal{V}_i}} \iff [\hat{\mathbf{x}}_i]_{\mathcal{B}_{\mathcal{V}_i}} = T_i^{-1} [\hat{\mathbf{x}}_i]_{\mathcal{B}^\star_{\mathcal{V}_i}} \tag{3}$$

The new bases must define a set of generalised coordinates, hence must satisfy

$$[\hat{\mathbf{x}}_i]_{\mathcal{B}_{\mathcal{V}_i}} = [\Pi_{\mathcal{R}}\hat{\mathbf{x}}_i]_{\mathcal{B}_{\mathcal{R}}}$$

introducing equations (2) and (3) into this last equation, the appropriate change of bases T_i are obtained

$$T_i = (R^{\mathsf{T}} V_i)^{-1}$$

After the state-space transformations T_i, the state-space representations of the reduced local models

$$\left\{ \hat{\mathbb{S}}_i \overset{\text{def.}}{=} \left[\begin{array}{c|c} T_i^{-1}\hat{A}_i^\star T_i & T_i^{-1}\hat{B}_i^\star \\ \hline \hat{C}_i^\star T_i & \hat{D}_i^\star \end{array} \right] \right\}_{i=1,\dots,\ell}$$

are now expressed in generalised coordinates w.r.t R, hence the interpolation between these new state-space representation of the reduced local models is possible.

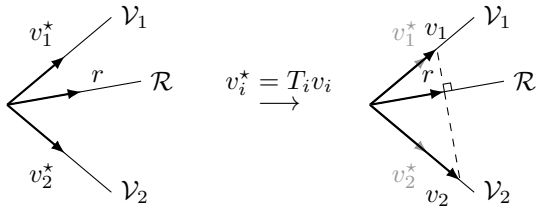

Fig. 1. v_1 and v_2 generalised coordinates w.r.t. r

B. Reduction via adapted balanced truncation [2]

In [2], the authors propose a modified balanced truncation to tackle the problem of LPV model order reduction. For LPV models, the Petrov-Galerkin projection gives a reduced model of the form

$$\left\{ \begin{array}{rcl} \dot{\hat{x}}(t) & = & W_\rho^{\mathsf{T}} \left(AV_\rho - \sum_{j=1}^{d} \frac{\partial V_\rho}{\partial \rho_j}\dot{\rho}_j \right) \hat{x}(t) + W_\rho^{\mathsf{T}} B u(t) \\ y(t) & = & CV_\rho \hat{x}(t) + Du(t) \end{array} \right.$$

The term $\sum_{j=1}^{d} \frac{\partial V_\rho}{\partial \rho_j}\dot{\rho}_j$ can be avoided by projecting on a constant subspace $\hat{\mathcal{V}}$ spanned by a matrix V.
The idea is to project all the local models $\mathbf{\Sigma}_i$ onto a single subspace $\mathcal{R}$ and orthogonally to different subspaces $\mathcal{W}_i$. Since the reduced states all lie in the same subspace and are expressed in the same basis, the interpolation can then be immediately applied.
For pMOR this method allows us to avoid the adaptation step, at the cost of a larger error in the Petrov-Galerkin approximation. Indeed, each projection is done along the orthogonal complement of $\mathcal{W}_i$, same as balanced truncation. However, all projections are done onto a unique subspace $\mathcal{R}$, which induces larger errors than if it were done onto the subspaces $\mathcal{V}_i$ as in balanced projection.
First, the projection achieving balanced truncation has to be rewritten, then the new formulation is used to project the local models onto the *mean* subspace $\mathcal{R}$ orthogonally to different subspaces $\mathcal{W}_i$.

1) Rewriting the projection of balanced truncation: The projection achieving balanced truncation is an oblique projection (1) and can be rewritten in the form

$$\Pi_i = V_i (V_i^{\mathsf{T}} Q_i V_i)^{-1} V_i^{\mathsf{T}} Q_i$$

It can be seen that choosing V_i as an orthonormal basis for $\operatorname{span}(L_{\mathrm{r},i} N_i)$ (using for example the QR decomposition of $L_{\mathrm{r},i} N_i$) and $Q_i = W_{\mathrm{o},i}$, the projection achieves balanced truncation.

2) Projection onto a same subspace along different directions: To be close to balanced truncation, the (constant) subspace $\mathcal{V}$ is chosen to best approximates the set of subspaces V_i that would have been obtained with the balanced truncation for the set of sampled models $\{\mathbf{\Sigma}_i\}_{i=1,\dots,\ell}$. This can be done through the principal component analysis of V_{all}.

The subspaces $\mathcal{W}_i$, orthogonal to which the projections are done, are exactly the subspaces obtained by balanced truncation for each $\mathbf{\Sigma}_i$.
Then the state-space representations $\{\hat{\mathbb{S}}_i\}_{i=1,\dots,\ell}$ of the obtained reduced models can be directly interpolated since their states lie in the same subspace $\mathcal{V}$.

C. Interpolation

To interpolate the adapted reduced local state-space representations $\hat{\mathbb{S}}_i$, choices have to be made on the manifold in which to interpolate and the method for computing the interpolation coefficients.

1) Vector space where to interpolate: The interpolation of the matrices of $\hat{\mathbb{S}}_i$ can be done in different vector spaces, in the vector space of matrices (of which they are an element) or in the tangent space to an element of the matrix manifold. Note that the set of Hurwitz matrices is not a convex set, hence interpolating matrices corresponding to stable systems does not necessarily results in a stable interpolated model.

a) Matrix space: The matrices $\hat{A}_i, \hat{B}_i, \hat{C}_i$ can be interpolated in their respective vector spaces, specifically the spaces $\mathbb{R}^{\hat{n}\times\hat{n}}$ for the matrices $\hat{A}_i$, $\mathbb{R}^{\hat{n}\times m}$ for matrices $\hat{B}_i$ and $\mathbb{R}^{p\times\hat{n}}$ for matrices $\hat{C}_i$.

b) Tangent space [7], [8], [9], [10]: This technique uses the tangent space to the matrix manifold at a particular point. For interpolating the matrices $\hat{A}_i$, one of the matrices to be interpolated is selected as a reference in the matrix

manifold $\mathbb{R}^{\hat{n}\times\hat{n}}$. Its tangent space is constructed and the other matrices to be interpolated are mapped into this tangent space via the logarithmic map. The matrices are then interpolated in the tangent space and the obtained interpolated matrix is mapped back to the matrix manifold via the exponential map.

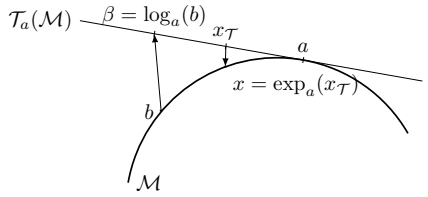

Fig. 2. Interpolation in the tangent space of $\mathcal{M}$ at a

As shown in the illustration of Fig. 2, a manifold $\mathcal{M}$ is considered. To interpolate two of its elements a and b in the tangent space of a, b is mapped to this tangent space $\mathcal{T}_a(\mathcal{M})$, obtaining $\beta = \log_a(b)$. Then a and β are interpolated using any interpolation method (e.g. linear, splines) and the result $x_\mathcal{T}$ is mapped back to $\mathcal{M}$.

For details on the interpretation of such an interpolation, see [11]. For matrices A and B in the manifold of non-singular matrices, B is mapped on the tangent space to the manifold at A by $\beta = \log(BA^{-1})$ and the exponential map from the tangent space to the manifold is $B = \exp(\beta)A$ using the matrix exponential and logarithmic exponential.

2) Interpolation method: There exists many interpolation method such as piecewise constant, linear, polynomial, spline, etc. It is not clear wether a method is better than another one and as [3] pointed out, it may depend on the considered model.

IV. Proposed method

The proposed technique combines the ideas of the projection orthogonally to a varying subspace from [2] and of the generalised coordinates [1]. The idea is to reduce the local models by balanced truncation, then to construct a set of generalised coordinates w.r.t. a *mean* subspace $\mathcal{R}$, except it is not an orthogonal projection onto $\mathcal{R}$ that is used to construct the new bases but a projection onto $\mathcal{R}$ along the same direction as the balanced truncation.

Figures 3, 4 and 5 illustrates the differences between all the considered methods. The illustrations depict a reduction from $n = 2$ to $\hat{n} = 1$ of the i-th local model having state $\mathbf{x}_i$. The subspaces $\mathcal{V}_i$ and $\mathcal{W}_i$ are the ones obtained by balanced truncation, and the subspace $\mathcal{R}$ is computed via an SVD to approximate at best the subspaces $\mathcal{V}_i$.

a) Oblique projection: In this method, the states are projected onto the subspace $\mathcal{R}$ orthogonally to the subspaces $\mathcal{W}_i$, hence the reduction of the local models is not exactly a balanced truncation (due to the projection being onto $\mathcal{R}$ and not onto $\mathcal{V}_i$). The advantage is that the interpolation can be directly applied and this method is also valid for LPV models.

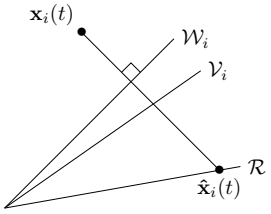

Fig. 3. Oblique projection method

b) Generalised coordinates via orthogonal projection: This method first achieves an exact balanced truncation of the reduced models and then build a set of generalised coordinates w.r.t. $\mathcal{R}$ so the interpolation is meaningful. Here the set of generalised coordinates is define via an orthogonal balanced projection.

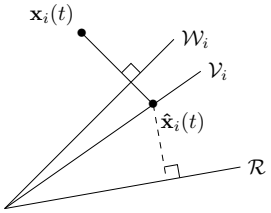

Fig. 4. Generalised coordinates via orthogonal projection

c) Generalised coordinates via oblique projection: Here, an exact balanced truncation of the local models is achieved, then a set of generalised coordinates w.r.t. $\mathcal{R}$ is constructed by means of an oblique projection. The projection is done orthogonally to the subspaces $\mathcal{W}_i$ so the obtained models are more in accordance to the balanced truncation.

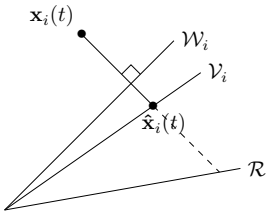

Fig. 5. Generalised coordinates via oblique projection

The subspaces $\mathcal{R}$, $\mathcal{V}_i$, $\mathcal{W}_i$ are spanned respectively by the matrices R, V_i, W_i. The expression of the projection (Fig. 5) is

$$[\Pi_{\mathcal{R}\mathcal{W}_i}\hat{\mathbf{x}}_i]_{\mathcal{B}_\mathcal{R}} = (W_i^\mathsf{T}R)^{-1}W_i^\mathsf{T}V_i[\hat{\mathbf{x}}_i]_{\mathcal{B}^\star_{\mathcal{V}_i}}$$

The change of bases is $[\hat{\mathbf{x}}_i]_{\mathcal{B}^\star_{\mathcal{V}_i}} = T_i[\hat{\mathbf{x}}_i]_{\mathcal{B}_{\mathcal{V}_i}}$, thus the generalised coordinates definition (Fig. 6) $[\hat{\mathbf{x}}_i]_{\mathcal{B}_{\mathcal{V}_i}} = [\Pi_\mathcal{R}\hat{\mathbf{x}}_i]_{\mathcal{B}_\mathcal{R}}$ gives the state transformations

$$T_i = (W_i^\mathsf{T}V_i)^{-1}W_i^\mathsf{T}R.$$

After the state-space transformations T_i, the state-space representations of the reduced local models

$$\left\{\hat{\mathbb{S}}_i = \left[\begin{array}{c|c} \hat{A}_i & \hat{B}_i \\ \hline \hat{C}_i & \hat{D}_i \end{array}\right] = \left[\begin{array}{c|c} T_i^{-1}\hat{A}_i^\star T_i & T_i^{-1}\hat{B}_i^\star \\ \hline \hat{C}_i^\star T_i & \hat{D}_i^\star \end{array}\right]\right\}_{i=1,\ldots,\ell}$$

are now expressed in generalised coordinates w.r.t R, hence the interpolation between these new state-space representation of the reduced local models is possible.

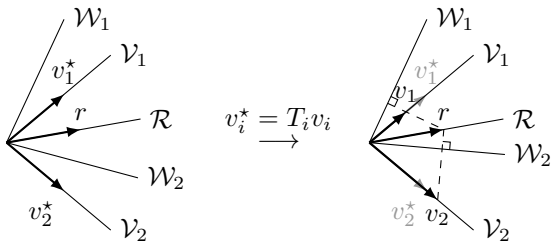

Fig. 6. v_1 and v_2 generalised coordinates w.r.t. r

V. Results on academic examples

The investigated methods were implemented and applied to two academic mass-spring-damper systems. Each of them emphasize different aspects of the methods.

1) 1D example: This first example has been used to assess pMOR methods in [6] and then in [9]. The model has $n = 8$ states with 1 uncertain parameter $\mu \in [\,0\,,\,1\,]$ which impacts multiple elements of the system described in Table I and Fig. 7.

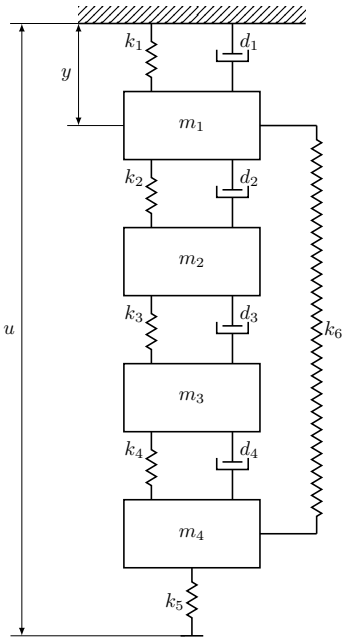

Fig. 7. Mass-spring-damper system (example 1)

TABLE I
Parameters values

masse [kg]	Stiffness [N/m]	Damping [Ns/m]
$m_1 = 125$	$k_1 = 2 + 2\mu$	$d_1 = \mu$
$m_2 = 25$	$k_2 = 1$, $k_3 = 3$	$d_2 = 1.6$
$m_3 = 5$	$k_4 = 9$, $k_5 = 27$	$d_3 = 0.4$
$m_4 = 1$	$k_6 = 1 + 2\mu$	$d_4 = 0.1$

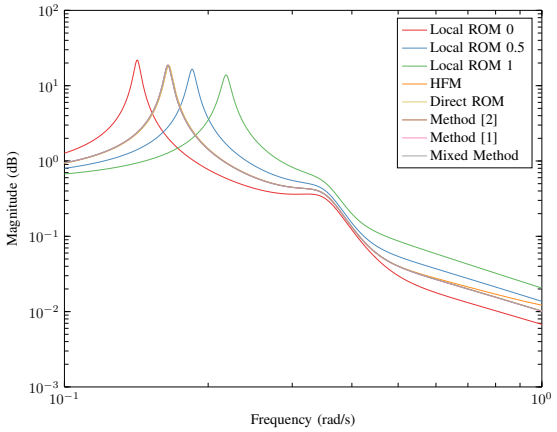

Fig. 8. Bode plot (example 1)

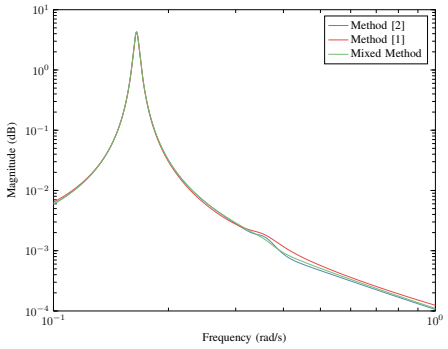

Fig. 9. Bode plot of the error between the pMOR methods and the direct ROM (example 1)

In the parameter space $[\,0\,,\,1\,]$, three models are sampled at $\mu \in \{\,0\,,\,0.5\,,\,1\,\}$ and reduced to the order $\hat{n} = 4$.

Three different methods are used to reduce and adapt the models, namely the oblique projection (method [2]), and the balanced truncation in conjunction with the generalised coordinates via orthogonal projection (method [1]) and via oblique projection (mixed method). The interpolation is done directly in the matrix space via a cubic spline interpolation using *interp1* Matlab function.

Fig. 8 describes the reduced models of order $\hat{n} = 4$ at parameter value $\mu = 0.25$, and also shows the three local model from which the interpolation is done. Since the reduced models obtained by interpolation are very close to the reference (the model reduced by balanced truncation at the parameter value $\mu = 0.25$), Fig. 9 shows the error between the considered models and the original one for a better comparison.

To assess the validity of the pMOR methods over the parameter space, it is possible to compute the H_∞ (Fig. 10) or the H_2 norm of the error between the reduced order models and the original one. In this case the objective is to be as close as possible to the error obtained by direct balanced truncation at each parameter value.

2) 2D example: Another example is needed to understand how valuable the generalised coordinates property is. This example is made of 40 masses, linked one to another by

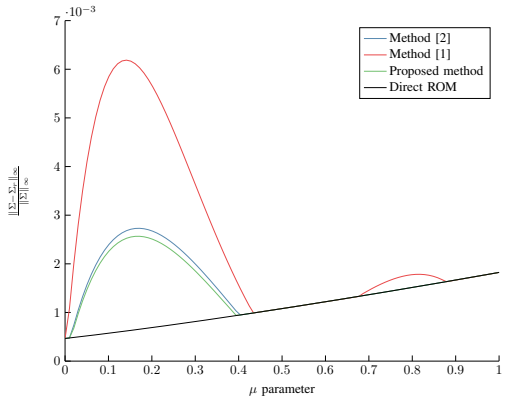

Fig. 10. Relative H_∞ error between the reduced models and the original one (1D example)

springs and dampers, and attached at both extremities by springs and dampers (Fig. 11). The $n = 80$ states are the position and velocity of each mass. The input is the force exerced on the first mass, and the output is the position of the first mass. The masses, springs and dampers have all the same values $m = 10$ kg, $d = 3$ Ns/m, $k = 5$ N/m. The first spring and damper have uncertain value, ranging in their nominal value $\pm 20\%$, normalised with $\delta_c, \delta_k \in [\,-1\,,\,1\,]$.

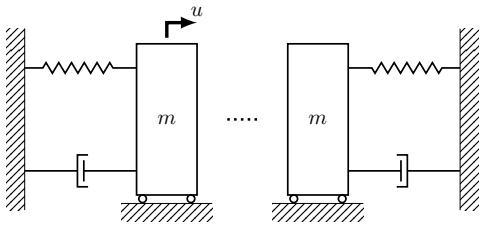

Fig. 11. Mass-spring-damper system (example 2)

An interesting advantage of the method [1] and the proposed method is that the obtain models at the sampling points are exactly the one obtained by balanced truncation. In this example the polytopic approach is used. The 2 parameters are sampled at the 4 vertices of the parameter space (describing the variation of the first spring and first damper coefficients). After reduction of the local models to the order $\hat{n} = 8$, the interpolation is performed linearly in the matrix space. Due to the generalised coordinate system, in Fig. 12 the reduced order models obtained by method [1] and the proposed method match exactly the objective function (obtained by balanced truncation at each parameter point) at the sampling points (the 4 vertices).

VI. Conclusion

After having studied two main pMOR methods and having rephrased them more geometrically, a combination have been proposed to take the best advantage of both approaches. The table II summarizes the difference between the two existing methods and the proposed one.

Furture studies will focus on the tools developped in [12] to perform alternative local or global order reduction.

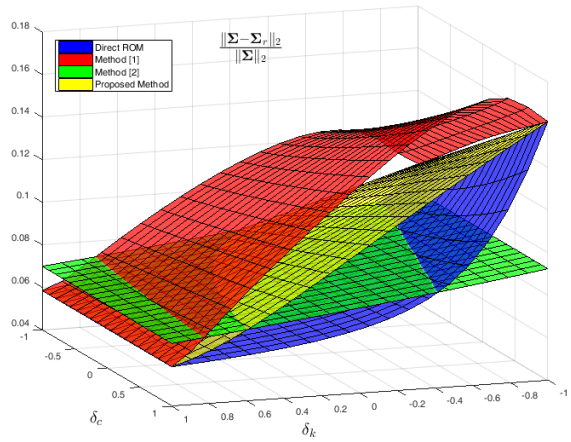

Fig. 12. Relative error of the H_2 norm (2D example)

TABLE II
Methods Comparison

Method	[1]	[2]	Proposed
Type	Local	Semi-local	Local
Reduction	Balanced	Pseudo-balanced via oblique	Balanced
Adaptation	Orthogonal	-	Oblique

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