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Null Space Strong Structural Controllability via Skew Zero Forcing Sets

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Abstract— **In this paper, we examine strong structural controllability of a particular subspace of linear time-invariant networks, namely, the null space of the parameterized family of system matrices. In this direction, we establish a one-to-one correspondence between the set of input nodes for null space controllability and the notion of skew zero forcing sets. Using this class of zero forcing sets, we provide conditions to guarantee that the null space of the parameterized set of state matrices sharing a common network topology is trivial. Moreover, the uncontrollability of the zero mode of directed and undirected networks from a single node is discussed. In addition, methods for growing a network while preserving strong structural controllability of its null space from a set of control nodes is presented. Finally, we provide an application of the developed results for the bipartite consensus dynamics.**

I. Introduction

Key to understanding complex dynamic networks is characterizing methods to effectively interact with them. This objective fits in the general area of controllability of dynamic systems. In order to disseminate the role of the network structure, it has become important for controllability conditions to be provided with respect to the network topology. With this objective in mind, controllability features that have combinatorial and graph-theoretic interpretations have proved to be particularly relevant. One such notion of controllability is known as *strong structural controllability* (ss-controllability) of linear time-invariant (LTI) systems. The ss-controllability serves to show that *all* pairs of state and control matrices, with a given zero/non-zero pattern are controllable. One attraction for such an analysis is its consequences pertaining to *control robustness* for networked systems. In fact, an approach facilitated by (strong) structural framework can ensure that network remains controllable independent of variations in network uncertainty in the edge weights.

The ss-controllability for networks has been pursued by a number of research works, including that of Olesky *et al.* [1] and Chapman and Mesbahi [2] using constrained t-matchings. Other graph-theoretic methods for checking ss-controllability using spanning cycles have been explored in [3], [4]. Moreover, in [5], a one-to-one correspondence between zero forcing sets of the graph and the ss-controllability has been established. In [6], a correspondence between zero forcing sets [5] and constrained matching [2] has been provided. These results have been extended in [7]–[9].

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The focus of this paper is on the ss-controllability with respect to a particular invariant subspace, namely the null space of the parameterized family of system matrices. In this direction, we show that an extension of the notion of zero forcing sets, referred to as the skew zero forcing set, provides a necessary and sufficient condition for ss-controllability of this null space. The notion of skew zero forcing sets and the corresponding skew color-change process has recently been introduced in the literature for investigating the minimum rank problem for patterned skew-symmetric matrices [10].

Invariant subspaces related to a linear transformation represented by the matrix A, have a direct connection to the controllability of the corresponding LTI system. For example, the controllability subspace corresponds to the minimum invariant space under the matrix A, which contains the range space of B [11]. Kalman's canonical decomposition is based on these decompositions; a useful observation in this venue is that the intersection of invariant subspaces is also an invariant subspace [12]. For non-defective A, the controllable subspace is the union of controllable invariant subspaces of A, spanned by the right eigenvectors of A. These eigenvectors correspond to the controllable modes.

This work examines ss-controllability of an invariant subspace characterized by the null space of the parametrized family of system matrices. Our work provides insights into the network controllability with respect to a subspace; however, more to the point, the controllability of the null subspace is of particular interest for analysis of certain classes of networked dynamic systems. For *designed* system dynamics, the available sensors and actuators can lead to invariant subspaces and thus knowledge of the controllability with respect to these subspaces is valuable. Furthermore, in [13], a necessary condition for the stabilizability of sign networks has been presented. Our method, on the other hand, provides a sufficient stabilizability condition for all networks with eigenvalues that have non-positive real-parts. For example, when the system dynamics are naturally dissipative, external control would only be required to control the null space of A to guarantee stabilizability. One such example is networked systems with significant self-damping compared with the coupling strength; this dynamics in turn induces a diagonally dominant state matrix. An example of this class of systems is the bipartite consensus dynamics [14].

The contribution of the paper is using a node coloring process applied to the topology of a networked dynamic system as a method for inferring the ss-controllability of null spaces associated with a structured system matrix, which is discussed in §III. If all nodes in the graph have been colored

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by the termination of the coloring process, null space ss-controllability is then ensured. In this direction, we establish a one-to-one correspondence between ss-controllability of the null space and skew zero forcing sets. It is noted that a minimum cardinality skew zero forcing set is smaller than a minimum zero forcing set [5] that renders a network strongly structurally controllable. Along the way, we derive graph theoretic conditions that characterize when the null space for the parameterized family of system matrices is trivial. The uncontrollability of the "zero" mode of the corresponding LTI systems for directed and undirected networks is also investigated for the single control node case. Furthermore, in §IV, methods for growing controllable graphs in order to preserve ss-controllability of the null space are presented. The paper is concluded with an illustrative example in §V.

II. Preliminaries

First, we provide the preliminaries. The set of real numbers is denoted by $\mathbb{R}$. The transpose of the matrix M is denoted by M^T. The ith entry of the vector v is denoted by v_i, and M_{ij} is the entry in the row i and the column j of M. A subvector v_X refers to the set of v_i's, for $i \in X$, lexicographically ordered in a vector form. We denote by $\mathbf{1}_n$ the vector of all ones in $\mathbb{R}^n$. We also denote the $n \times n$ identity matrix by I_n and represent its jth column by e_j. The dimension of a space, the span of a set of vectors, and the rank of a matrix are denoted, respectively, by $\mathbf{dim}(.)$, $\mathrm{span}(.)$, and $\mathrm{rank}(.)$. The cardinality of a set is denoted by $|.|$.

Graphs: A graph G is denoted by $G = (V(G), E(G))$, where $V(G) = \{1, \ldots, n\}$ is the vertex set and $E(G) \subseteq V(G) \times V(G)$ is the edge set of the graph. When $(i, j) \in E(G)$, there is an edge from the node i to the node j in G. The node j (resp., i) is said to be an out-neighbor (resp., in-neighbor) of the node i (resp., j). We denote by $N_{out}(i)$ and $N_{in}(i)$ the set of out-neighbors and in-neighbors of the node i, respectively. An *undirected* graph is a graph such that $(i, j) \in E(G)$ if and only if $(j, i) \in E(G)$, with j referred to as simply the neighbor of i. Note that in this paper, a graph can contain a loop (i, i) for some $i \in V(G)$.

Eigenvalues: Let $\Lambda(A)$ denote the set of eigenvalues of the matrix A. We denote by $\delta_A(\lambda)$ the algebraic multiplicity of $\lambda \in \Lambda(A)$, which is the multiplicity of λ as a root of the characteristic equation. The geometric multiplicity of the eigenvalue $\lambda \in \Lambda(A)$, denoted by $\psi_A(\lambda)$, is the dimension of the subspace $\mathcal{S}_A(\lambda) = \{\nu \in \mathbb{R}^n | \nu^T A = \lambda \nu^T\}$. For $\mathcal{M} \subseteq \Lambda(A)$, the maximum geometric multiplicity of the eigenvalues of A belonging to $\mathcal{M}$ is defined as $\Psi_{\mathcal{M}}(A) = \max\{\psi_A(\lambda) | \lambda \in \mathcal{M}\}$. For symmetric matrices A, $\delta_A(\lambda) = \psi_A(\lambda)$, for all $\lambda \in \Lambda(A)$.

Pattern Matrices: Let $\mathcal{P}(G) = \{A \in \mathbb{R}^{n\times n} : A_{ij} \neq 0 \Leftrightarrow (j, i) \in E(G)\}$ be the set of pattern matrices for the graph G. In a similar way, for an undirected graph G_u, a set of pattern matrices is defined as $\mathcal{P}_u(G_u) = \{A \in \mathbb{R}^{n\times n} : A = A^T,\ A_{ij} \neq 0 \Leftrightarrow\ (i, j) \in E(G_u)\}$.

A. Problem Formulation

Consider the following LTI network

$$\dot{x} = Ax + Bu, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector of the nodes, and $u \in \mathbb{R}^m$ is the input vector. Moreover, the matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are respectively, the system matrix and the input matrix. We also assume that $B = [e_{j_1}, \ldots, e_{j_m}]$, where the nodes j_k, $k = 1, \ldots, m$, are referred to as the *control nodes*, into which control signals are injected. We let $V_C = \{j_1, \ldots, j_m\}$ denote the set of control nodes.

In this paper, for a network with dynamics (1), the controllability of $\lambda = 0$ of all $A \in \mathcal{P}(G)$, or equivalently, the controllability of the null invariant subspace associated with all $A \in \mathcal{P}(G)$ from a given set of control nodes is discussed. More generally, features of invariant subspaces of a dynamic system are important for how it can be controlled.

For $A \in \mathbb{R}^{n\times n}$, the subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is an *A-invariant subspace* if $Ax \in \mathcal{V}$ for all $x \in \mathcal{V}$ [15]. Two significant A-invariant subspaces are the $\mathrm{span}\{v_1, v_2, \ldots, v_m\}$, where v_i's are the right eigenvectors of A, and the controllable subspace of the pair (A, B). In fact, every invariant subspace is spanned by a set of right eigenvectors and generalized eigenvectors of A. The notion of a *controllable invariant subspace* is summarized in the following definition and is not to be confused with a *controlled invariant subspace* which is necessary (but not sufficient) for the former [16].

Definition 1: An A-invariant subspace $\mathcal{V}$ is controllable if for all $x(0) \in \mathcal{V}$ and any final state $x_f \in \mathcal{V}$, there exists an input $u(t)$ such that $x(T) = x_f$ for a finite time $T > 0$.

By PBH test, the invariant subspace $\mathrm{span}\{v_1, v_2, \ldots, v_m\}$ is controllable if and only if for all left eigenvectors w_i of A associated with v_i's, $w_i^T B \neq 0$ [17].

The eigenvalues (or modes), associated with a controllable invariant subspaces can be similarly defined. The following summarizes this property for the controllability of the zero eigenvalue, that corresponds to the null space of A.

Proposition 1 ([17]): For a system with dynamics (1), the zero eigenvalue (mode) of A, corresponding to the invariant subspace $\mathcal{V} = \{x | Ax = 0\}$, is controllable if and only if for all nonzero w for which $w^T A = 0$, $w^T B \neq 0$.

As controllability is invariant under the equivalence transformation, when the LTI system (1) is not controllable, there exists a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ such that

$$T^{-1}AT = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{21} \end{bmatrix}, T^{-1}B = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}, \tag{2}$$

where $\hat{A}_{11} \in \mathbb{R}^{q\times q}$ and $\hat{B}_1 \in \mathbb{R}^{q\times m}$. In this case, $\Lambda(\hat{A}_{11})$ is the set of controllable eigenvalues of the system (1). Given $\lambda \in \Lambda(A)$, if $\lambda \notin \Lambda(\hat{A}_{11})$, we refer to it as a completely uncontrollable eigenvalue of the system (1).

B. Skew Zero Forcing Sets

We now introduce a coloring process on graphs. The process is initiated by first coloring a subset of nodes Z with a target color, say black; the remaining nodes are colored

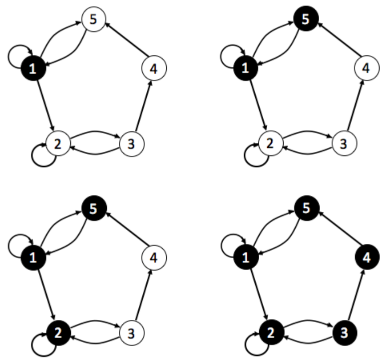

Fig. 1. An example of a skew color-change process.

with another color, say white. Then, an iterative coloring process is defined. The graph is called "forced" if by the termination of the coloring process, all of its nodes are black.

The *skew color-change rule* proceeds as follow: If there is a node $v \in V$ which has exactly one white out-neighbor $u \in V$, then change the color of u to black. When this occurs, we say that v skew forces u; if v is black (resp., white), we denote this process by $v \to u$ (resp., $v \xrightarrow[\text{skew}]{} u$). Apply the skew color-change process for the graph G with the initial set of black nodes Z. The set of final black nodes is referred to as the *skew derived set* of Z and denoted by $\mathcal{D}^{\text{skew}}(Z)$. If for the initial set of black nodes Z, $\mathcal{D}^{\text{skew}}(Z) = V$, Z is called a *skew zero forcing set* (ZFS^{skew}). The *skew zero forcing number* $Z^{\text{skew}}(G)$ is the minimum size of a skew zero forcing set. A skew zero forcing set Z with $|Z| = Z^{\text{skew}}(G)$ is called a *minimal skew zero forcing set* and is denoted by $\text{ZFS}^{\text{skew}}_{\text{min}}$.

For example, in Fig. 1, the steps of the skew color-change process are: (1) $4 \xrightarrow[\text{skew}]{} 5$, (2) $1 \to 2$, (3) $2 \to 3$, and $3 \xrightarrow[\text{skew}]{} 4$.

For more information about skew zero forcing sets, the reader is referred to [10] and the references therein.

III. Controllability of the Zero Mode

In this section, we derive a relation between ss-controllability of the null subspace of (parameterized family of) networks and their corresponding skew zero forcing sets. In this direction, the controllability of the (invariant) null space of the system matrix- or equivalently the controllability of the zero eigenvalue- for a family of networks with dynamics (1) and $A \in \mathcal{P}(G)$ is studied. The next lemma proves useful for this purpose.

Lemma 1: Consider a graph G whose nodes in $Z \subset V(G)$ are colored black with the rest of the nodes colored as white. Let A be a singular matrix in $\mathcal{P}(G)$, and $\nu \in \mathbb{R}^n$ be a left eigenvector of A associated with eigenvalue $\lambda = 0$. If $\nu_i = 0$, for all $i \in Z$, then $\nu_i = 0$, $\forall i \in \mathcal{D}^{\text{skew}}(Z)$.

Proof: If $\mathcal{D}^{\text{skew}}(Z) = Z$, the proof is complete. Let us thus assume that $\mathcal{D}^{\text{skew}}(Z) \neq Z$. Hence, there exists at least one node $i \in V$ which has exactly one white out-neighbor $j \in V \setminus Z$, and j is skew forced by i. Now, consider the ith column of the matrix equation $\nu^T A = 0$, namely,

$$\sum_{k \in N_{out}(i)} \nu_k a_{ki} = \sum_{k \in N_{out}(i), k \neq j} \nu_k a_{ki} + \nu_j a_{ji} = 0. \quad (3)$$

Since all out-neighbors of i except for j are black, they are in Z, and $\nu_k = 0$, for all $k \in N_{out}(i), k \neq j$. Consequently, the equation (3) reduces to $\nu_j a_{ji} = 0$, which results in $\nu_j = 0$. As such, before the skew color-change process terminates, there exist nodes which are the only white out-neighbor of some nodes in the graph, and are colored black by the application of the skew color-change rule; thereby, the corresponding entries in ν are zero. Hence, after repeated application of the skew color-change rule, we have $\nu_x = 0$, for all $x \in \mathcal{D}^{\text{skew}}(Z)$. ■

Theorem 1: Consider a network with dynamics (1) and the graph G. For all $A \in \mathcal{P}(G)$, $\lambda = 0$ is controllable if and only if V_C is a skew zero forcing set of G.

Proof: Assume that V_C is a skew zero forcing set, but there is some $A \in \mathcal{P}(G)$ for which $\lambda = 0$ is not controllable from V_C. Then, there exists a left eigenvector $\nu \neq 0$ of A associated with $\lambda = 0$ that $\nu^T B = 0$. Thus, $\nu_i = 0$, for all $i \in V_C$. Since $\mathcal{D}^{\text{skew}}(V_C) = V$, Lemma 1 implies that $\nu = 0$, which is a contradiction.

For the converse assertion, assume that $\lambda = 0$ of all singular $A \in \mathcal{P}(G)$ is controllable, but V_C is not a skew zero forcing set, that is, $\mathcal{D}^{\text{skew}}(V_C) \neq V$. Let $|\mathcal{D}^{\text{skew}}(V_C)| = n_1$. Without loss of generality, let the nodes in $\mathcal{D}^{\text{skew}}(V_C)$ be indexed first. Now, consider a partitioning of A as:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad (4)$$

where $A_{11} \in \mathbb{R}^{n_1 \times n_1}$, $A_{12} \in \mathbb{R}^{n_1 \times (n-n_1)}$, $A_{21} \in \mathbb{R}^{(n-n_1) \times n_1}$, and $A_{22} \in \mathbb{R}^{(n-n_1) \times (n-n_1)}$. Let $\nu = \begin{bmatrix} 0^T_{n_1} & \mathbf{1}^T_{n-n_1} \end{bmatrix}$. We claim that no column of A_{21} and A_{22} has exactly one nonzero entry. Otherwise, there exists a node which has exactly one white out-neighbor. Then, every column of A_{21} and A_{22} has either no nonzero entry or has more than one nonzero entry. Hence, the nonzero entries in each column of A_{21} and A_{22} can be chosen such that $\mathbf{1}^T_{n-n_1} A_{21} = \mathbf{1}^T_{n-n_1} A_{22} = 0$. Thereby, we have $\nu^T A = 0$ for some singular $A \in \mathcal{P}(G)$. Moreover, since $V_C \subseteq \mathcal{D}^{\text{skew}}(V_C)$, we have $\nu^T B = 0$. Then, for some $A \in \mathcal{P}(G)$, the zero eigenvalue is not controllable, a contradiction. ■

Note that there may be some graph G, for which no $A \in \mathcal{P}(G)$ has any zero eigenvalue. In what follows, we discuss conditions on the graph G that ensure no singular $A \in \mathcal{P}(G)$; conditions for singularity of all $A \in \mathcal{P}(G)$ are also dervied.

Given a graph G, the next proposition provides an upper bound for the maximum geometric multiplicity of the zero eigenvalue for all $A \in \mathcal{P}(G)$.

Proposition 2: Let G be a graph with skew zero forcing number $Z^{\text{skew}}(G)$. For all $A \in \mathcal{P}(G)$, $\psi_A(0) \leq Z^{\text{skew}}(G)$.

Proof: Suppose that $\dim(\mathcal{S}_A(0)) = k$. Then, for every $X \subset V$ with $|X| = k-1$, there is a nonzero $\nu \in \mathcal{S}_A(0)$ such that $\nu_X = 0$ (see the proof of Proposition 2.2. of [18]). Now, assume that there exists some $A \in \mathcal{P}(G)$ for which $\psi_A(0) > Z^{\text{skew}}(G)$. Then, for a skew zero forcing set Z with $|Z| =$

$Z^{\text{skew}}(G)$, there is a nonzero $\nu \in \mathcal{S}_A(0)$ such that $\nu_i = 0$, for all $i \in Z$. By Lemma 1, for every $X \subset V$, if $\nu_X = 0$, then $\nu_i = 0$, for all $i \in \mathcal{D}^{\text{skew}}(X)$. Consequently, since $\mathcal{D}^{\text{skew}}(Z) = V$, we have $\nu = 0$, which is a contradiction. ■

We now describe graphs G for which all $A \in \mathcal{P}(G)$ are nonsingular.

Theorem 2: For a graph G, there is no singular $A \in \mathcal{P}(G)$ if and only if $Z^{\text{skew}}(G) = 0$.

Proof: Assume $Z^{\text{skew}}(G) = 0$, and there exists some $A \in \mathcal{P}(G)$ that is singular. Thus, $\delta_A(0) > 0$, and accordingly, $\psi_A(0) > 0$. On the other hand, by Proposition 2, $\psi_A(0) \leq 0$, implying that $\psi_A(0) = 0$; this establishes a contradiction.

To prove the converse, assume all $A \in \mathcal{P}(G)$ are nonsingular, but $Z^{\text{skew}}(G) > 0$. Then, for every subset $\mathcal{X} \subset V$ that $|\mathcal{X}| = Z^{\text{skew}}(G) - 1$, $\mathcal{D}^{\text{skew}}(\mathcal{X}) \neq V$. Let $\mathcal{Z}_{\mathcal{X}} = \mathcal{D}^{\text{skew}}(\mathcal{X})$ with $n_{\mathcal{X}} = |\mathcal{Z}_{\mathcal{X}}|$, and $\mathcal{N}_{\mathcal{X}} = V \setminus \mathcal{Z}_s$. Then, there is no vertex in V with exactly one out-neighbor in $\mathcal{N}_{\mathcal{X}}$. Without loss of generality, we index nodes of $\mathcal{Z}_{\mathcal{X}}$ first. Let $\nu = \begin{bmatrix} 0_{n_{\mathcal{X}}}^T & \mathbf{1}_{n-n_{\mathcal{X}}}^T \end{bmatrix}^T$. Since there is no nodes of V with exactly one out-neighbor in $\mathcal{N}_{\mathcal{X}}$, for every $i \in V$, either $A_{ki} = 0$, for all $k \in \mathcal{N}_{\mathcal{X}}$, or $A_{ki} \neq 0$, for at least two $k \in \mathcal{N}_{\mathcal{X}}$. Hence, one can choose the nonzero entries of A such that for every $i \in V$, $\sum_{k \in \mathcal{N}_{\mathcal{X}}} A_{ki} = 0$, implying that $\nu^T A = 0$. Thus, A is singular, establishing a contradiction. ■

We can thus conclude from Theorem 2 that if $Z^{\text{skew}}(G) = 0$, there is no $A \in \mathcal{P}(G)$ with zero eigenvalues. Conversely, if $Z^{\text{skew}}(G) > 0$, there exist a singular matrix in $\mathcal{P}(\mathcal{G})$. In the following, we provide a condition for checking whether all $A \in \mathcal{P}(G)$ are singular. First, we provide a brief introduction to bipartite matchings and the notion of cycle covers.

Consider an undirected bipartite graph $H = (V^+(H), V^-(H), E(H))$ associated with a matrix $M = [M_{ij}] \in \mathbb{R}^{p \times q}$, where $V^+ = \{1, \ldots, q\}$ and $V^- = \{1, \ldots, p\}$; $\{i, j\} \in E(H)$ if and only if $M_{ji} \neq 0$. A pattern matrix carrying the structure of a graph G can be represented by a bipartite graph H. Recall that a perfect matching in a bipartite graph H is a set of edges which share no end points and cover all nodes of H.

Lemma 2 ([19]): A bipartite graph H has a perfect matching if and only if $\det(M)$ is not identically zero.

Definition 2: Given a graph, a *cycle cover* is a set of vertex-disjoint directed cycles, covering all of its nodes.

The technical ingredients are now in place for the following result on the singularity of matrices $A \in \mathcal{P}(G)$.

Theorem 3: Given a graph G, all matrices $A \in \mathcal{P}(G)$ are singular if and only if G has no cycle cover.

Proof: Consider the bipartite graph H associated with matrices $A \in \mathcal{P}(G)$. The existence of a perfect matching in H is equivalent to the existence of a cycle cover in G. Hence, from Lemma 2, there is no $A \in \mathcal{P}(G)$ with $\det(A) \neq 0$ if and only if G has no cycle cover. ■

The problem of finding a cycle cover in a graph can be transformed into the problem of finding a perfect matching in a larger graph, and it can be solved in a polynomial time [20]. From Theorems 2 and 3, one can establish a necessary condition for a graph G to have $Z^{\text{skew}}(G) = 0$.

Fig. 2. An example of a strongly structurally controllable network.

Corollary 1: For a graph G, if $Z^{\text{skew}}(G) = 0$, then G has a cycle cover.

The next proposition provides some sufficient conditions for uncontrollability (unobservability) of the zero eigenvalue of $A \in \mathcal{P}(G)$.

Proposition 3: In a graph G, if there is a node $i \in V$ with exactly one out-neighbor j, then with $V_C = \{j\}$, for any singular $A \in \mathcal{P}(G)$, the zero mode is not controllable.

Proof: Assume there is some singular $A \in \mathcal{P}(G)$ whose zero mode is controllable. Let $B = \{e_j\}$. Then, for any $\nu \neq 0$ that $\nu^T A = 0$, we have $\nu^T B \neq 0$; this yields to $\nu_j \neq 0$. Since the node i has exactly one out-neighbor j, for all $k \neq j$, we have $A_{ki} = 0$. Now, consider the ith column of the equation $\nu^T A = 0$. This results in $\nu_j A_{ji} = 0$. Since $A_{ji} \neq 0$, one can find that $\nu_j = 0$, contradicting the assumption. ■

The next proposition is an extension of Proposition 3 for undirected network.

Proposition 4: Let G be an undirected graph. If for $i \in V(G)$, $V_C = \{i\}$ has exactly one neighbor j, the zero mode of every singular $A \in \mathcal{P}_u(G)$ is uncontrollable.

Proof: Similar to the proof of Proposition 3, for every eigenvector ν of A associated with eigenvalue $\lambda = 0$, $\nu^T B = 0$. Then, $\lambda = 0$ is not controllable. In other words, considering (2), we have $0 \in \Lambda(\hat{A}_{22})$. Let $\Lambda_{nz}(\hat{A}_{22})$ be set of the nonzero uncontrollable eigenvalues, and let $p = |\Lambda_{nz}(\hat{A}_{22})|$. Assume $0 \in \Lambda(\hat{A}_{11})$. Then, it holds that $n - q < p + \delta_A(0)$. On the other hand, for a symmetric A, $n - q = \dim\{\nu \in \mathbb{R}^n : \text{for some } \lambda \in \mathbb{R}, A\nu = \lambda\nu, \nu^T B = 0\}$, and thus, $n - q = p + \delta_A(0)$; this completes the proof. ■

Note that before applying Propositions 3 and 4, one should ensure that $Z^{\text{skew}}(G) > 0$. Otherwise, there is no singular $A \in \mathcal{P}(G)$, and the results cannot be applied. For example, node 1 of the graph G in Fig. 2 has one neighbor, i.e., node 2. However, an LTI network with graph G and $A \in \mathcal{P}(G)$ is controllable from node 2. Indeed, $Z^{\text{skew}}(G) = 0$, and by Theorem 2, there is no $A \in \mathcal{P}(G)$ with a zero mode.

IV. Growing Networks and the Zero Mode

We now discuss a growth process for a network such that ss-controllability of its zero mode is preserved. Similarly, the network can be reduced (through removing some nodes and edges) such that the controllability of the zero mode remains intact. In this direction, we introduce additional notation.

For a graph G and a node $v \notin V(G)$, we say that $G \dashrightarrow v$ (resp., $v \dashrightarrow G$) if there is no edges directed from v to any nodes of G (resp., from any nodes of G to v), i.e., all potential edges between v and G are from v to some nodes of G. Moreover, we use $G \dashleftarrow\dashrightarrow v$ when there is some directed edges from v to some nodes of G and from some nodes G to v. Lastly, when there is no edge directed from v to any nodes of G (resp., from any nodes of G to v), we use the notation $v \not\dashrightarrow G$ (resp., $G \not\dashrightarrow v$).

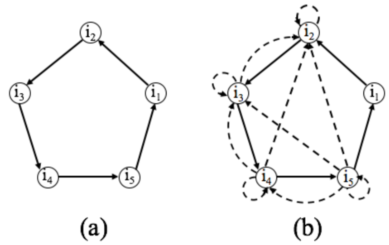

Fig. 3. a) A simple directed cycle and b) an extended cycle.

Consider a directed simple cycle C of size r, $r \geq 2$, as shown in Fig. 3 (a), with $V(C) = \{i_1, \ldots, i_r\}$ and $E(C) = \{(i_1, i_2), \ldots, (i_{r-1}, i_r), (i_r, i_1)\}$. Let $E'(C) = \{(i_k, i_j) : 1 < j \leq k \leq r\}$. The directed edges in $E'(C)$ are shown in Fig. 3(b) with dashed lines. An extended cycle C^e is a cycle with some (potential) extra edges from $E'(C)$.

For a graph $\widetilde{G}$ and an extended cycle C^e with the node set $V(C^e) = \{i_1, \ldots, i_r\}$, we write $G = \widetilde{G} \oplus C^e$ if for a unique k, $1 < k \leq r$, we have $\widetilde{G} \dashleftarrow\dashrightarrow i_k$, and for $1 < j < k$, $\widetilde{G} \dashrightarrow i_j$.

Theorem 4: For a graph G, if $G = \widetilde{G} \oplus C^e$, then $Z^{\text{skew}}(G) = Z^{\text{skew}}(\widetilde{G})$.

Proof: Assume that $\widetilde{\text{ZFS}}^{\text{skew}}_{\text{min}}$, a minimal skew zero forcing of $\widetilde{G}$, is the set of initially black nodes of G. Start the skew color-change process with $i_1 \xrightarrow{\text{skew}} i_2$. Then, for $j = 2, \ldots, k-1$, in step j of the skew color-change process, since i_j is a black node with exactly one out-neighbor i_{j+1}, we have $i_j \rightarrow i_{j+1}$. Then, i_k is a black node which has one white out-neighbor in C^e. If $i_k \not\dashrightarrow \widetilde{G}$, i_k forces its white out-neighbor in C^e, and by the skew color-change process in C^e, all of its nodes are forced to be black. Then, one can apply the skew color-change rule in $\widetilde{G}$, and since the set of initial black nodes is its skew zero forcing set, $\widetilde{G}$ will be eventually colored black. On the other hand, if $i_k \dashrightarrow \widetilde{G}$, the skew color-change process terminates in C^e. Now, let the skew color-change process be initiated in $\widetilde{G}$. Note that for every $v \in V(\widetilde{G})$, we have $N_{out}(v) \cap C^e \subseteq \{i_j : 1 < j \leq k\}$. In other words, the out-neighbors of nodes of $\widetilde{G}$ in C^e are all black, and they cannot affect the ability of their in-neigbors in $\widetilde{G}$ to force a color change. Thus, by the skew color-change process, $\widetilde{G}$ will be black, and afterwards, i_k can force its one white out-neighbor in C^e. The process can be continued in C^e until all of its nodes are black. Then, $\widetilde{\text{ZFS}}^{\text{skew}}_{\text{min}}$ is a skew zero forcing set of G, and accordingly $Z^{\text{skew}}(G) \leq Z^{\text{skew}}(\widetilde{G})$.

Now, suppose that nodes of $Z = \text{ZFS}^{\text{skew}}_{\text{min}}$, a minimal skew zero forcing of G, are initially colored black. First, assume that all nodes of C^e are white. Then, $Z \subset V(\widetilde{G})$. Let $\widetilde{\mathcal{D}}(Z)$ be the skew derived set of Z in $\widetilde{G}$, and assume that $\widetilde{\mathcal{D}}(Z) \neq V(\widetilde{G})$. Then, either there should be some nodes $v, w \in V(\widetilde{G})$ and some node $u \in V(C^e)$ such that $v \rightarrow u$ and $u \rightarrow w$, or there should be some node $v \in V(\widetilde{G})$ and some node $u \in V(C^e)$ such that $u \xrightarrow{\text{skew}} v$. Since i_k is the only node in C^e which may have some out-neighbors in $\widetilde{G}$, we have $u = i_k$. On the other hand, i_k has at least one white out-neighbor in C^e, say x, which cannot be forced by any nodes of $\widetilde{G}$ as it is not the out-neighbor of any node in $\widetilde{G}$. Then, i_k cannot force any nodes in $\widetilde{G}$ before x is black, and x cannot be colored black before other out-neighbors of i_k are black. This, on the other hand, is a contradiction, and $\widetilde{\mathcal{D}}(Z) = V(\widetilde{G})$. In other words, $Z^{\text{skew}}(\widetilde{G}) \leq Z^{\text{skew}}(G)$.

Now, assume that $Z \cap V(C^e) \neq \emptyset$. Let $v = i_1$ if $k = r$ and $v = i_{k+1}$ if otherwise. Then, we should have $Z \cap V(C^e) = v$. Otherwise, $Z \setminus (Z \cap V(C^e))$ is a skew zero forcing set of G as well. Thereby, Z is not a minimum skew zero forcing set of G. Now. let all nodes of C^e become black by the application of the skew color-change rule. Then, i_k is a black node which has no white out-neighbor in C^e. If i_k does not force any nodes in $\widetilde{G}$, $Z \setminus v$ can also be a skew zero forcing set of G, which has fewer nodes. Then, in some step of the skew color-change process, we should have $i_k \rightarrow u$, where $u \in V(\widetilde{G})$. Before this step, since i_k has a white out-neighbor u, it could not force other nodes of $\widetilde{G}$. Hence, one can consider $\widetilde{Z} = Z \setminus \{v\} \cup \{u\}$ as a minimal skew zero forcing set of $\widetilde{G}$. Consequently, $Z^{\text{skew}}(\widetilde{G}) \leq Z^{\text{skew}}(G)$. ■

The next corollary follows from Theorems 1 and 4; its consequence is a method for reducing or growing a network while preserving ss-controllability of its zero modes.

Corollary 2: In a network with dynamics (1) and the graph $\widetilde{G}$, let V_C be a skew zero forcing set. If $G = \widetilde{G} \oplus C^e$ for some extended cycle C^e, then the zero eigenvalue of all singular $A \in \mathcal{P}(G)$ is controllable.

Now, consider a simple cycle C in a graph G. Let $\widetilde{G} = G - C$ which is an induced subgraph of G on the vertex set $V(G) \setminus V(C)$. The cycle C is called a *pendant cycle* of G if there is only one node $v \in V(C)$ such that $\widetilde{G} \dashrightarrow v$ or $v \dashrightarrow \widetilde{G}$. By Corollary 2, any extended cycle C^e (or any pendant cycle C) in a graph G can be removed to find a reduced graph with the same skew zero forcing set. This algorithm can be continued until there is no C^e (or a pendent cycle C) in the reduced graph. For example, in an undirected graph, any *pendant edge*, including a leaf u along with its neighbor v, is a pendant cycle of size two.

Proposition 5 ([21]): Let G be an undirected graph. $\widetilde{G}$ is a graph obtained from G by the repeated removal of the pendant edges until there is no leaves. Then, $Z^{\text{skew}}(G) = Z^{\text{skew}}(\widetilde{G})$. Moreover, $Z^{\text{skew}}(G) = 0$ if and only if $\widetilde{G} = \emptyset$.

Furthermore, one can grow a network with structure $\widetilde{G}$ and a skew zero forcing set V_C and add extended (or pendant) cycles to the graph in each iteration to obtain G. Following this procedure, the controllability of the zero eigenvalues of all $A \in \mathcal{P}(G)$ is preserved. For example, in Fig. 4, the skew zero forcing set of a network (the black node) is preserved under its growth by adding pendant cycles at every iteration.

In Fig. 5, preservation of the skew zero forcing set of an undirected network under adding pendant edges is shown.

V. Example: Stabilizability of Bipartite Consensus

A bipartite consensus [14] is a distributed algorithm on a signed network (edges can admit negative weights); the algorithm converges to a value which is the same for all

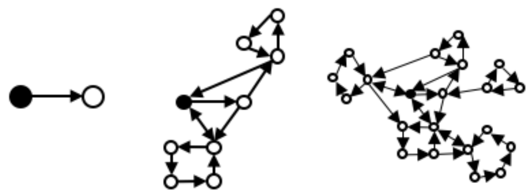

Fig. 4. Growing a network by adding pendant cycles.

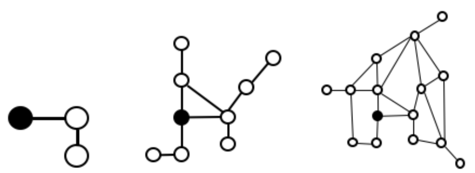

Fig. 5. Growing an undirected network by adding some pendant edges.

nodes in modulus but not in sign. This type of dynamics can be interpreted as allowing antagonistic interactions in a network with application in social networks such as bimodal coalitions. In this section, we show one application of our results in examining the stabilizability of a network with the bipartite consensus dynamics.

The bipartite consensus dynamics is defined over a graph G with the topology that if there exists an edge $(j,i) \in E(G)$ with some strictly positive or negative weight w_{ij}, then there also exists a self-loop $(i,i) \in E(G)$ with edge weight $w_{ii} = \sum_{(j,i)\in E(G)} |w_{ij}|$. The dynamics are,

$$\dot{x}_i = -w_{ii}x_i + \sum_{(j,i)\in E(G)} w_{ij}x_j + e_i^T Bu. \tag{5}$$

More compactly, $\dot{x} = -L_s x + Bu$, where

$$[L_s]_{ij} = \begin{cases} \sum_{(j,i)\in E(G)} |w_{ij}| & i = j \\ -w_{ij} & (j,i) \in E(G) \\ 0 & \text{otherwise} \end{cases}. \tag{6}$$

As L_s is diagonally dominant with positive diagonal elements, due to the Gershgorin's disk theorem, all the eigenvalues of $-L_s$ have negative real part or are identical zero. Consequently, if the null space of L_s is controllable, then the dynamics are stabilizable. Controlling the null space of L_s has the added benefit that the equilibrium subspace can be controlled, and any steady state value can be reached with control applied only over a finite period of time.

VI. Conclusion

Strong structural controllability of the zero mode of networks was discussed in this paper. We examined the problem through the skew zero forcing sets of the graphs and provided a necessary and sufficient condition for the ss-controllability of the zero eigenvalue of a network. Moreover, we provided conditions under which a graph admits a zero eigenvalue over all its pattern matrices as well as discussing scenarios where the pattern matrices must be nonsingular. The uncontrollability of the zero modes and methods for the growth or reduction of a network so that its null space remains strongly structurally controllable from a given set of control nodes were also presented. Finally, we applied the developed theory to the stabilization of bipartite consensus.

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