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Multivariable Systems Design of Desired Accuracy Based on LQ and H_∞ Optimization Procedures

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***Abstract*— Authors consider continuous linear time invariant multivariable systems affected by external bounded deterministic piecewise continuous disturbances that admit Fourier series decomposition. On top of that, for every component of the disturbance, the sum of the harmonics amplitudes absolute values is supposed to be bounded by a known number. The problem is to design a linear state or output feedback controller that guarantees a given accuracy characterized by the maximum deviation of the controlled variables from zero during the system steady state. To solve the problem, authors use well-known LQ- and H∞-optimization procedures for which the accurate rules for choosing the weighting matrices of the corresponding quadratic cost are determined. The efficiency of the proposed design technique is validated by applying it to the controller design problem for an electromechanical system.**

I. INTRODUCTION

Guaranteeing a given control accuracy for a system affected by external disturbances is one of the main problems of automatic control theory and practice. The controller design approaches based on LQ and H_∞-optimization remain one of the most popular design approaches [1]–[3]. However, their practical application is complicated by the determining problem of the weighting matrices of the optimization criterion. The present work focuses on solution to this problem.

The closest to the described guaranteeing accuracy design problem research areas are l_1-optimization [4] and invariant ellipsoids approach [5]. However, these approaches provide minimum (in certain sense) control errors that can lead to unnecessary increase of controller gain and as a result to small gain and phase margins [6], [7]. The proposed approach just provides a desired accuracy that is more reasonable for practical application.

The first work that solves this kind of problem for step disturbances using LQ optimization was [8]. The works [9]–[11] consider polyharmonic disturbances with bounded power and known finite number of harmonics.

The present work considers the case of piecewise continuous disturbances that admit Fourier series decomposition and the harmonics amplitudes absolute values sum is bounded by a given number. Unlike the work [12], in this work we consider the general case of several controlled outputs and several disturbances. In this case, similarly to [9] and [10], [11], the LQ and H_∞-optimization procedures are used and strict rules for determining the weighting matrices of the corresponding quadratic cost are given.

II. STATEMENT OF THE PROBLEM

Let's consider the linear time invariant plant described by the equation

$$\dot{x} = Ax + B_1 w + B_2 u,\ z = C_1 x,\ y = C_2 x, \qquad (1)$$

where $x(t) \in R^n$ is the state vector of the plant, $w(t) \in R^\mu$ is the unmeasured disturbances vector, $u(t) \in R^m$ is the control input, $z(t) \in R^{m_1}$ is the controlled output, $y(t) \in R^{m_2}$ is the measured output, A, B_1, B_2, C_1, C_2 are known matrices of the plant. The pair (A, B_2) is supposed to be controllable and the pairs (C_1, A) and (C_2, A) are observable.

The components of the piecewise continuous disturbances are bounded polyharmonic functions that admit Fourier series decomposition

$$w_i(t) = \sum_{k=1}^{\infty} w_{ik} \sin(\omega_k t + \phi_{ik}),\ i = \overline{1, \mu}, \qquad (2)$$

where w_{ik} are unknown amplitudes, ϕ_{ik} are unknown phases and ω_k are unknown frequencies.

We assume that the sum of the absolute values of the amplitudes w_{ik} of each component of the disturbance is bounded:

$$\sum_{k=1}^{\infty} |w_{ik}| \le w_i^*,\ i = \overline{1, \mu}, \qquad (3)$$

where w_i^* are given numbers.

The controller is sought in the form of the linear time invariant system described by the equation

$$\dot{x}_c = A_c x_c + B_c y,\ u = C_c x_c + D_c y, \qquad (4)$$

where $x_c(t) \in R^{n_c}$ is the state of the controller, and A_c, B_c, C_c, D_c are unknown controller matrices that are to be found.

Steady-state controlled variables errors $z_{i,st}$ and steady-state control variables values $u_{i,st}$ are defined as

$$\begin{aligned} z_{i,st} &= \lim_{t\to\infty} \sup |z_i(t)|,\ i = \overline{1, m_1}, \\ u_{i,st} &= \lim_{t\to\infty} \sup |u_i(t)|,\ i = \overline{1, m}. \end{aligned} \qquad (5)$$

Problem 1. Find a stabilizing controller (4) such that the inequalities

$$z_{i,st} \le z_i^*,\ i = \overline{1, m_1}, \qquad (6)$$

for the controlled variables errors of the closed-loop system (1), (4) are fulfilled for all disturbances from the class (2), (3), where z_i^* are given numbers.

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III. PRELIMINARY RESULTS: FREQUENCY PROPERTIES OF LQ AND H_∞-OPTIMAL SYSTEMS WITH STATE FEEDBACK CONTROLLERS

In this work, the proposed controller design approach is based on the LQ and H_∞ optimization procedures. Frequency matrix inequalities for the transfer matrices (from the disturbance w to the controlled output z and the control input u) play an important role in the result derivation.

We first consider the case when the plant state is available ($C_2 = I$ where I is unit matrix), and the controls and disturbances are applied at the same point ($B_1 = B_2$, $\mu = m$, $m_2 = n$). Then the plant equations (1) can be written as

$$\dot{x} = Ax + B_2(w+u),\ z = C_1x,\ y = x,\ x(0) = 0. \quad (7)$$

It is known [1] that the following control law for plant (7)

$$u = Kx,\ K = -R^{-1}B_2^TP, \quad (8)$$

where P is the symmetric positive definite matrix ($P = P^T > 0$) that satisfies the matrix Riccati equation

$$A^TP + PA - PB_2R^{-1}B_2^TP = -C_1^TQC_1, \quad (9)$$

minimizes the quadratic cost criterion (for $w = 0$ and $x(0) \neq 0$ in (7))

$$J = \min_u \int_0^\infty (z^TQz + u^TRu)dt, \quad (10)$$

where Q and R are given (by user) symmetric positive definite weighting matrices.

The closed-loop LQ optimal system (7), (8) transfer matrices have the form:

$$\begin{array}{c} T_{uw}(s) = K(sI - A_{cl})^{-1}B_2, \\ T_{zw}(s) = C_1(sI - A_{cl})^{-1}B_2, \\ A_{cl} = A + B_2K, \end{array} \quad (11)$$

where $T_{uw}(s)$ connects control input u with disturbances w, $T_{zw}(s)$ connects controlled output z with disturbances w.

The following frequency properties hold true [9].

Theorem 1. The transfer matrices (11) of the optimal system (7)-(10) satisfy the following frequency matrix inequalities

$$T_{zw}^T(-j\omega)QT_{zw}(j\omega) \leq R,\ \omega \in [0,\infty); \quad (12)$$

$$T_{uw}^T(-j\omega)T_{uw}(j\omega) \leq 4I,\ R = rI,\ \omega \in [0,\infty), \quad (13)$$

where r is the user-defined parameter.

If the symmetric positive definite matrix $P = P^T > 0$ of the control law (8) for plant (1) satisfies the matrix Riccati equation [1]

$$\begin{array}{c} A^TP + PA - PB_2R^{-1}B_2P + \gamma^{-2}PB_1B_1^TP \\ = -C_1^TQC_1, \end{array} \quad (14)$$

then the control law (8) is optimal in the sense of the H_∞ minimax quadratic cost criterion

$$J = \min_u \max_w \int_0^\infty (z^TQz + u^TRu - \gamma^2w^Tw)dt, \quad (15)$$

where the minimum is taken over $u \in L_2[0,\infty)$ and the maximum is taken over $w \in L_2[0,\infty)$. In this case, the worst disturbance (in the $L_2[0,\infty)$ sense) has the form $w_b = \gamma^{-2}B_1^TPx$, where γ is given number or a parameter for minimization.

Because the plant (1) in general case has $B_1 \neq B_2$, then one needs to modify the close-loop system (1), (8) transfer matrices

$$\begin{array}{c} T_{uw}(s) = K(sI1 - A_{cl})^{-1}B_1, \\ T_{zw}(s) = C_1(sI - A_{cl})^{-1}B_1, \end{array} \quad (16)$$

that connect disturbances w with control input u and controlled output z accordingly.

The following theorem holds true [10], [11].

Theorem 2. The transfer matrices (16) of the optimal system (1), (8), (14), (15) satisfy the following frequency matrix inequalities at frequencies $\omega \in [0,\infty)$

$$T_{zw}^T(-j\omega)QT_{zw}(j\omega) + T_{uw}^T(-j\omega)RT_{uw}(j\omega) \leq \gamma^2I. \quad (17)$$

IV. BASIC LEMMA

Now we formulate an important statement that is the basis for the work's main results.

Let $T_{\bar{z}\bar{w}}(s)$ be arbitrary $(r \times l)$ stable transfer matrix that connects the input vector $\bar{w}$ from the class (2), (3) with the certain output variables vector $\bar{z}$ (with the replacement of the dimension μ by l)

$$\bar{z}(s) = T_{\bar{z}\bar{w}}(s)\bar{w}(s). \quad (18)$$

And let this matrix satisfy the frequency matrix inequality

$$T_{\bar{z}\bar{w}}^T(-j\omega)\bar{Q}T_{\bar{z}\bar{w}}(j\omega) \leq \bar{R},\ \omega \in [0,\infty), \quad (19)$$

where $\bar{Q} = \mathrm{diag}[\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_r]$ and $\bar{R} = \mathrm{diag}[\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_l]$ are some positive definite diagonal matrices of appropriate dimensions.

Lemma 1. If the frequency inequality (19) is valid, then the steady-state values of the stable system (18) output variables fed by the input signal from the class (2), (3) satisfy the inequality

$$\bar{q}_i\bar{z}_{i,st}^2 \leq \left(\sum_{j=1}^{l} \sqrt{\bar{r}_j}\bar{w}_j^*\right)^2,\ i = \overline{1,r}, \quad (20)$$

where $\bar{w}_j^*$ are bounds of the disturbances components from the right part of the inequality (3) analogs just for disturbance $\bar{w}$.

The proof of the lemma is given in the Appendix.

V. STATE FEEDBACK

The purpose of this section is to solve the *Problem 1*, when the plant state is available to measure. The controller design method uses LQ and H_∞ optimization procedures. All statements of this section are direct consequences of the section III frequency inequalities.

A. State Feedback Controllers via LQ optimization

Theorem 3. The steady-state values of closed-loop system (7)-(10) controlled variables $z_{i,st}$ (controlled errors) affected by the disturbances (2), (3) in the case of the diagonal weighting matrices Q and R of the criterion (10) satisfy the following inequalities

$$\sqrt{q_i} z_{i,st} \leq \sum_{j=1}^{\mu} \sqrt{r_j} w_j^*,\ i = \overline{1, m_1}, \tag{21}$$

and the steady-state values control variables $u_{i,st}$ for fixed $R = rI$, $r > 0$ and $Q > 0$ (not necessarily diagonal) are bounded by the equations

$$u_{i,st} \leq 2 \sum_{j=1}^{\mu} w_j^*,\ i = \overline{1, m}, \tag{22}$$

where q_i and r_j are elements of the diagonal weighting matrices Q and R, w_j^* are the bounds of the disturbance components from the right parts of the inequalities (3).

This result is obvious in consequence of the *lemma 1* and inequalities (12), (13).

Corollary 1. The control law (8), (9) resolves the *Problem 1* for the plant (7) if the elements of the diagonal weighting matrices Q and R of the criterion (10) satisfy the conditions

$$q_i = \frac{(\sum_{j=1}^{\mu} w_j^*)^2}{(z_i^*)^2},\ i = \overline{1, m_1},\ r_j = 1,\ j = \overline{1, m}. \tag{23}$$

In fact, the inequality $z_{i,st}/z_i^* \leq 1$ follows from (21) because of (23) and hence the inequality (6) holds true.

Note, that the control law (8), (9), (23) allows to maintain the high control accuracy (z_i^* are any given numbers) independently from the disturbance (2) frequencies ω_k. Moreover, it is obvious that any plant state variable (in this case $z = x$ or $C_1 = I$, $m_1 = n$) can de reduced in the steady-state mode as small as needed, but only if disturbances and controls are applied at the same point $B_1 = B_2$.

B. State Feedback Controllers via H_∞ optimization

Inequality (17) implies the following obvious inequalities ($\omega \in [0, \infty)$)

$$\begin{array}{l} T_{zw}^T(-j\omega) Q T_{zw}(j\omega) \leq \gamma^2 I, \\ T_{uw}^T(-j\omega) R T_{uw}(j\omega) \leq \gamma^2 I. \end{array} \tag{24}$$

From these inequalities and taking into account *Lemma 1* we obtain the following result.

Theorem 4. The steady-state values of closed-loop system (1), (8), (14) controlled variables (controlled errors) affected by the piecewise continuous disturbances (2), (3) in the case of the diagonal weighting matrix Q from the criterion (15) satisfy the following frequency inequalities

$$\sqrt{q_i} z_{i,st} \leq \gamma \sum_{j=1}^{\mu} w_j^*,\ i = \overline{1, m_1},$$

and the steady-state control variables values with the diagonal weighting matrix R satisfy the following inequalities

$$\sqrt{r_i} u_{i,st} \leq \gamma \sum_{j=1}^{\mu} w_j^*,\ i = \overline{1, m},$$

where q_i and r_j are elements of the diagonal weighting matrices Q and R, w_j^* are the bounds of the disturbances components from the right parts of inequalities (3).

Proof of theorem 4. These inequalities are the direct consequence of (24) and *Lemma 1*.

Corollary 2. If the diagonal weighting matrices Q and R elements from the criterion (15) satisfy the conditions

$$q_i = \left(\sum_{j=1}^{\mu} w_j^*\right)^2 / (z_i^*)^2,\ i = \overline{1, m_1},$$
$$r_i = \left(\sum_{j=1}^{\mu} w_j^*\right)^2 / (u_i^*)^2,\ i = \overline{1, m},$$

then the control law (8), (14) guarantees the fulfillment of the following inequalities

$$z_{i,st} \leq \gamma z_i^*,\ i = \overline{1, m_1}, \quad u_{i,st} \leq \gamma u_i^*,\ i = \overline{1, m}.$$

Proof of the corollary 2. The corollary inequalities follow from those presented in *Theorem 4*, where the diagonal matrices Q and R weighting coefficients are chosen according to *Corollary 2*.

Obviously, if $\gamma \leq 1$ then *Problem 1* is solved but it's clear that this condition is only sufficient.

VI. OUTPUT FEEDBACK

The purpose of this section is to design output feedback controller (4) for plant (1) to solve the *Problem 1*. In this case the controls and disturbances can be applied in different points ($B_1 \neq B_2$).

A. Output Feedback Controller via LQ optimization

When one uses design technique based on the LQ optimization the controller is based on the full order observer, so that it additionally guarantees the phase and gain stability margins at the plant input as it was in the case of the state feedback controller (8) (see [1], [2]).

The plant should be the minimum-phase type (all transfer matrix $C_2(sI - A)^{-1} B_2$ zeros strictly lie in the left half-plane) with the same number of controlled variables and controls $m_1 = m$, and the vector $y = C_2 x$ that coincides with the vector of controlled variables $z = C_1 x$ ($C_1 = C_2$) is available to direct measurement.

We introduce vector f of the equivalent plant (1) disturbances that is shifted to the plant input by the following formula [9]

$$\begin{array}{c} f(s) = [C_2(sI - A)^{-1} B_2]^{-1} C_2 (sI - A)^{-1} B_1 w(s) = \\ = T_{fw}(s) w(s), \end{array}$$

and find such a parameter ρ that the frequency inequalities

$$T_{fw}^T(-j\omega) T_{fw}(j\omega) \leq \rho^2 I,\ \omega \in [0, \omega^*]$$

are valid [9], where ω^* is chosen number such that frequency range $[0, \omega^*]$ contains all main harmonics of the unmeasured disturbances.

Let's determine control law in the form $u = K x_c$, where K is the matrix of the controller (8), where $x_c \in R^n$ is the state vector of the full-order observer (filter)

$$\dot{x}_c = A x_c + B_2 u + K_f (y - C_2 x_c),\ K_f = Y C_2^T, \tag{25}$$

where $Y = Y^T > 0$ is the solution to the matrix Riccati equation for the observer (see [1], [9])

$$A^T Y + YA - YC_2^T C_2 Y = -\alpha B_2 B_2^T, \qquad (26)$$

where α is large enough positive weighting coefficient that determines frequency range in which properties (in frequency sense) of the state feedback and the output feedback systems are close.

In this case the controller matrices have the form $A_c = A + B_2K - K_f C_2$, $B_c = K_f$, $C_c = K$, $D_c = 0$. Then, in the case of the diagonal weighting matrices Q and $R = rI$ the following Theorem 5 takes place.

Theorem 5. If the plant (1) is minimum-phase type and the number of controls and controlled variables (coinciding with measured variables) is the same ($m_1 = m_2 = m$), then the statements of the *Theorem 3* for sufficiently large α and *Corollary 1* hold true for the system with the observer with accuracy up to replacement from w_j^* to ρw_j^* in (21)–(23).

The proof of the *Theorem 5* using the *Lemma 1* is analogous [9].

B. Output Feedback Controller via H_∞ optimization

Let the plant (1) measured output be described by the equation $y = C_2 x + \eta$, where η is measurement noise. The noise η components are the piecewise continuous functions that admit Fourier series decomposition:

$$\eta_j(t) = \sum_{k=1}^{\infty} \eta_{jk} \sin(\omega_k t + \psi_{jk}),\ j = \overline{1, m_2}, \qquad (27)$$

where η_{jk} are unknown amplitudes, ψ_{jk} are unknown phases, ω_k are unknown frequencies.

We assume that the amplitudes absolute values sum for each noise component is bounded as follows:

$$\sum_{k=1}^{\infty} |\eta_{jk}| \le \eta_j^*,\ j = \overline{1, m_2}, \qquad (28)$$

where η_j^* are given numbers.

We introduce extended disturbances vector as $\bar{w} = [w^T,\ \ \eta^T]^T$ and extended controlled variables vector as $\bar{z} = [(Q^{1/2}z)^T,\ \ (R^{1/2}u)^T]^T = [(Q^{1/2}C_1x)^T,\ \ (R^{1/2}u)^T]^T$ (where $Q = Q^T > 0$ and $R = R^T > 0$ are weighting matrices) and $T_{\bar{z}\bar{w}}(s)$ is the closed loop system transfer matrix that connects these vectors.

Then the solution to the H_∞-optimization problem

$$T_{\bar{z}\bar{w}}^T(-j\omega) T_{\bar{z}\bar{w}}(j\omega) \le \gamma^2 I,\ \omega \in [0, \infty) \qquad (29)$$

is reduced to solving the two Riccati equations: the first is the equation (14) and the second is the following

$$\begin{aligned} YA^T + AY + \gamma^{-2} YC_1^T Q C_1 Y - YC_2^T C_2 Y = \\ = -B_1 B_1^T, \end{aligned} \qquad (30)$$

relative to positive definite matrices P and Y and to checking the condition

$$\lambda_{\max}(PY) \le \gamma^2, \qquad (31)$$

where $\lambda_{\max}$ is the matrix PY maximum eigenvalue.

In this case the controller (4) matrices are written as $A_c = A + B_2K - K_f C_2 + B_1 K_w$, $B_c = K_f$, $C_c = K$, $D_c = 0$, where

$$\begin{aligned} K = -R^{-1} B_2^T P, \quad K_f = (I - \gamma^{-2} YP)^{-1} YC_2^T, \\ K_w = \gamma^{-2} B_1^T P. \end{aligned} \qquad (32)$$

We obtain the inequalities similar to (24) from the cost inequality (29):

$$\begin{aligned} T_{z\bar{w}}^T(-j\omega) Q T_{z\bar{w}}(j\omega) \le \gamma^2 I, \\ T_{u\bar{w}}^T(-j\omega) R T_{u\bar{w}}(j\omega) \le \gamma^2 I,\ \omega \in [0, \infty). \end{aligned}$$

Using *Lemma 1* and these frequency inequalities we obtain the following result.

Theorem 6. The controlled variables (controlled errors) steady-state values of the closed-loop system (1), (4), (14), (30), (31), (32) affected by the piecewise continuous disturbances (2), (3) and noises (27), (28) in the case of the criterion (15) diagonal weighting matrix Q satisfy the following frequency inequalities

$$\sqrt{q_i} z_{i,st} \le \gamma \left(\sum_{l=1}^{\mu} w_l^* + \sum_{j=1}^{m_2} \eta_j^* \right), \quad i = \overline{1, m_1},$$

and the control variables steady-state values with the diagonal weighting matrix R satisfy the inequalities

$$\sqrt{r_i} u_{i,st} \le \gamma \left(\sum_{l=1}^{\mu} w_l^* + \sum_{j=1}^{m_2} \eta_j^* \right), \quad i = \overline{1, m}.$$

Corollary 3. If the elements of the criterion (15) diagonal weighting matrices Q and R satisfy the conditions

$$\begin{aligned} q_i = \left(\textstyle\sum_{i=1}^{\mu} w_i^* + \sum_{j=1}^{m_2} \eta_j^* \right)^2 / (z_i^*)^2,\ i = \overline{1, m_1}, \\ r_i = \left(\textstyle\sum_{i=1}^{\mu} w_i^* + \sum_{j=1}^{m_2} \eta_j^* \right)^2 / (u_i^*)^2,\ i = \overline{1, m}, \end{aligned}$$

then the control law (4), (14), (30), (31) (32) leads to the inequalities

$$z_{i,st} \le \gamma z_i^*,\ i = \overline{1, m_1}, \quad u_{i,st} \le \gamma u_i^*,\ i = \overline{1, m}.$$

The proof of *Corollary 3* is analogous to that of *Corollary 2*. As in the state feedback case it is clear that the condition $\gamma \le 1$ is only sufficient in which *Problem 1* is solved.

VII. NUMERICAL EXAMPLE

Let us consider the electromechanical system with its mathematical model and parameters described in [13]. It is a minimum-phase plant, but the number of controls is less than that of the controlled variables. The similar plant is discussed by K. J. Astrom and T. Hagglund in section *Parallel systems* in [14].

The components of the state vector $x \in R^5$ have the following meaning: x_1 and x_2 are the deviations of the output voltages of thyristor converters from the rated values, x_3 and x_4 are the deviations of the engine rotor currents from the rated values, x_5 is the deviation of the angular speed from the rated one (the motors are loaded by the common load and connected rigidly, i.e. they have the same angular speed). The control vector $u \in R^2$ contains components u_1 and u_2 that are the deviations of the control voltages from the rated value (inputs of the thyristor converters). The disturbance w is the resistance moment (load) deviation from the rated value. The controlled variables are represented by engine currents and angular speed.

The plant matrices have the following view:

$$A = \begin{pmatrix} -100 & 0 & 0 & 0 & 0 \\ 0 & -83.33 & 0 & 0 & 0 \\ 137.81 & 0 & -11.29 & 0 & -1123.16 \\ 0 & 132.46 & 0 & -11.07 & -1101.13 \\ 0 & 0 & 0.25 & 0.25 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -0.03 \end{pmatrix},\ B_2 = \begin{pmatrix} 16120 & 0 \\ 0 & 13702 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$C_1 = C_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We notice that the plant is characterized by the essential property $B_1 \neq B_2$. We use the design approach proposed in *Section VI* and *Subsection B* to find the output feedback controller.

The weight matrices Q and R were selected from *Corollary 3* equalities where signals bounds were taken as the following $z_1^* = z_2^* = 375,\ z_3^* = 1,\ u_1^* = u_2^* = 5,\ w^* = 600,\ \eta_1^* = \eta_2^* = \eta_3^* = 0$.

Then, $Q^{1/2} = \mathrm{diag}[1.6,\ 1.6,\ 600]$, $R^{1/2} = \mathrm{diag}[120,\ 120]$. The plant measured output is $y = C_2 x + \beta\eta$, where β is the user-defined parameter, we use $\beta = 0.11$.

The output feedback controller was designed and studied in MATLAB where function `hinfric` was used to solve the corresponding H_∞-optimization problem and for this controller the corresponding γ value is 0.9865.

The closed loop system amplitude Bode diagram for the main controlled variable z_3 is shown in the upper part of the Fig. 1. This characteristic decreases monotonously and therefore the worst external disturbance is step function with amplitude $w = w^* = 600$. The angular speed deviation step response in closed loop from this worst load torque (disturbance) is also shown in the Fig. 1 below the Bode diagram.

The Fig. 2 pictures the equally laden motors currents step response. The controls step response that attenuate the load torque change are shown on the Fig. 3.

All the represented plots and characteristics satisfy the target inequalities (6) that determine engineering requirements to the closed loop system performance:

$$z_{1,st} < z_1^* = 375,\ z_{2,st} < z_2^* = 375,\ z_{3,st} < z_3^* = 1.$$

In this example $\gamma < 1$ and sufficient condition for *Problem 1* solving holds true. However, even for $\gamma > 1$ the accuracy requirements (6) are achieved for the majority of the minimum-phase plants that were numerically examined.

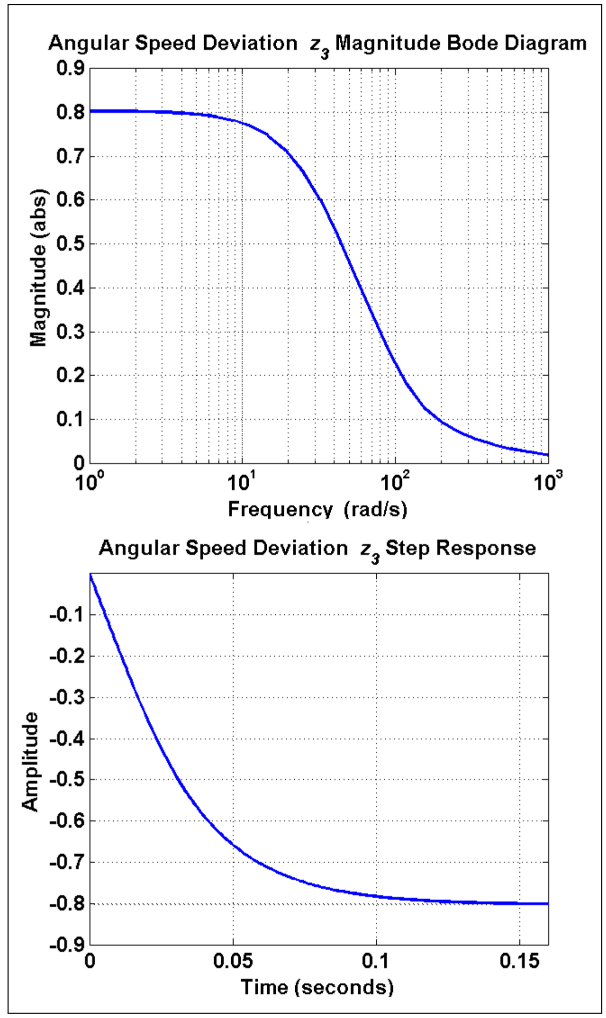

Fig. 1. The closed loop system Bode diagram from w to z_3 and the plant's output z_3 step response from step disturbance $w = 600$

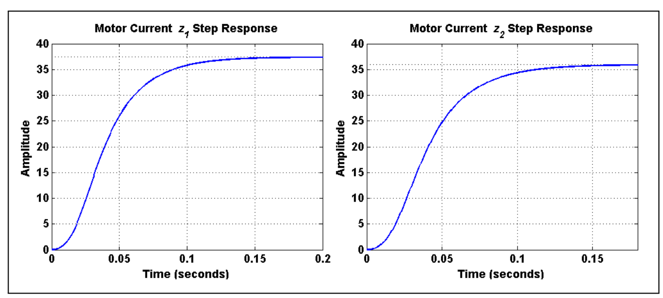

Fig. 2. The plant's outputs z_1 and z_2 step response from step disturbance $w = 600$

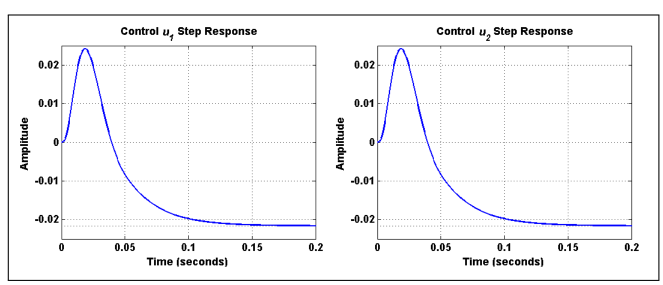

Fig. 3. The plant's controls u_1 and u_2 step response from step disturbance $w = 600$

VIII. CONCLUSION

In this work the controller design procedures for the linear multivariable systems that provide the given system accuracy in the presence of the bounded deterministic piecewise continuous disturbances are proposed. The disturbances admit Fourier series decomposition and the harmonics amplitudes absolute values sum is bounded by the given number. The accuracy is evaluated by maximum of each plant controlled variables deviation from zero during steady state.

Known LQ and H_∞ optimization procedures are used to solve this problem. The main role in these procedures is played by the frequency matrix inequalities for the disturbance-output transfer matrix. Analytic expressions for choosing the weighting matrices of the corresponding quadratic cost criterion are presented. The proposed design approach efficiency is illustrated by numerical example where the output feedback controller that satisfies desired engineering requirements for multivariable electromechanical system is designed.

APPENDIX

Proof of Lemma 1. The system (18) output forced oscillations (when $t \to \infty$) are described by the formula:

$$\bar{z}_i(t) = \sum_{k=1}^{\infty} a_i(\omega_k) \sin(\omega_k t + \phi_i(\omega_k)),\ i = \overline{1, l}, \qquad (33)$$

where $a_i(\omega_k) \geq 0$ and $\phi_i(\omega_k)$ are output oscillations amplitudes and phases, respectively, resulting from the k-th harmonic of the form (2) input signal $\bar{w}$ where the dimension μ is replaced by l.

The oscillations amplitudes with the frequencies ω_k along each vector $\bar{z}$ coordinate in (33) are the adjoined vectors $T_{\bar{z}\bar{w}}(j\omega)w_+^{(k)}$ and $T_{\bar{z}\bar{w}}(-j\omega)w_-^{(k)}$ are corresponding components absolute values, where

$$w_+^{(k)} = [w_{1k}e^{j\phi_{1k}},\ w_{2k}e^{j\phi_{2k}},\ \ldots,\ w_{rk}e^{j\phi_{rk}}]^T,$$
$$w_-^{(k)} = [w_{1k}e^{-j\phi_{1k}},\ w_{2k}e^{-j\phi_{2k}},\ \ldots,\ w_{rk}e^{-j\phi_{rk}}]^T.$$

Indeed, it is easy to see that the k-th harmonic of the input vector $\bar{w}$ with the components of (2) can be written as $\left(w_+^{(k)}e^{j\omega_k t} - w_-^{(k)}e^{-j\omega_k t}\right)/(2j)$.

Then, after determining the equation (18) particular solution as $\bar{w}_+ = w_+^{(k)}e^{j\omega_k t}$ and $\bar{w}_- = w_-^{(k)}e^{-j\omega_k t}$ and denoting the corresponding output vectors as z_+ and z_- by virtue of the superposition principle for the k-th harmonic of the output vector $\bar{z}$ with components from (33) we can write

$$(z_+ - z_-)/(2j) =$$
$$= (T_{\bar{z}\bar{w}}(j\omega_k)w_+^{(k)}e^{j\omega_k t} - T_{\bar{z}\bar{w}}(-j\omega_k)w_-^{(k)}e^{j\omega_k t})/(2j).$$

It is obvious that $a_i^2(\omega_k) = z_{-i}z_{+i}$ where z_{-i} and z_{+i} respectively are the i-th components of the vectors z_- and z_+.

Then, taking into account the matrix $\bar{Q}$ diagonal structure we find that

$$\sum_{i=1}^{r} \bar{q}_i a_i^2(\omega_k) = z_-^T\bar{Q}z_+ = {w_-^{(k)}}^T T_{\bar{z}\bar{w}}^T(-j\omega_k)\bar{Q}T_{\bar{z}\bar{w}}(j\omega_k)w_+^{(k)}.$$

and considering the matrix $\bar{R}$ diagonal structure and the inequality (19) we obtain

$$\textstyle\sum_{i=1}^{r} \bar{q}_i a_i^2(\omega_k) \leq {w_-^{(k)}}^T \bar{R} w_+^{(k)} = \sum_{j=1}^{l} \bar{r}_j w_{jk}^2.$$
$$\textstyle\bar{q}_i a_i^2(\omega_k) \leq \sum_{j=1}^{l} \bar{r}_j w_{jk}^2,\ i = \overline{1, r}.$$

Extracting the square root from both parts of the last inequality we find that

$$\sqrt{\bar{q}_i}a_i(\omega_k) \leq \sqrt{\sum_{j=1}^{l} \bar{r}_j w_{jk}^2} \leq \sqrt{\left(\sum_{j=1}^{l} \sqrt{\bar{r}_j}|w_{jk}|\right)^2},\ i = \overline{1, r},$$

and therefore $\sqrt{\bar{q}_i}a_i(\omega_k) \leq \sum_{j=1}^{l} \sqrt{\bar{r}_j}w_{jk},\ i = \overline{1, r}$.

Let's sum over all frequencies

$$\sqrt{\bar{q}_i}\sum_{k=1}^{\infty} a_i(\omega_k) \leq \sum_{j=1}^{l} \sqrt{\bar{r}_j} \sum_{k=1}^{\infty} |w_{jk}|,\ i = \overline{1, r},$$

and considering that $z_{i,st} \leq \sum_{k=1}^{\infty} a_i(\omega_k),\ i = \overline{1, r}$.

Taking into account (3) we obtain

$$\sqrt{\bar{q}_i}z_{i,st} \leq \sum_{j=1}^{l} \sqrt{\bar{r}_j} \sum_{k=1}^{\infty} |w_{jk}| \leq \sum_{j=1}^{l} \sqrt{r_j}w_j^*,\ i = \overline{1, r},$$

which implies (20).

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