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A new criterion for boundedness of solutions for a class of periodic systems

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***Abstract*—A wide range of practical systems exhibits dynamics, which are periodic with respect to several state variables and which possess multiple invariant solutions. Yet, when analyzing stability of such systems, many classical techniques often fall short in that they only permit to establish *local* stability properties. Motivated by this, we present a new sufficient criterion for *global* stability of such a class of nonlinear systems. The proposed approach is characterized by two main properties. First, it develops the conventional cell structure framework to the case of multiple periodic states. Second, it extends the standard Lyapunov theory by relaxing the usual definiteness requirements of the employed Lyapunov functions to sign-indefinite functions.**

I. INTRODUCTION

Stability of dynamical systems is one of the most fundamental problems studied in control systems theory [7], [9], [19], [21], [22], [23], [27], [32] and related domains, such as mechanics, electric circuits, power systems, systems biology, *etc*. In a general (nonlinear) setting, the main approach employed for stability analysis is based on Lyapunov theory [27]. A key advantage of a Lyapunov-based stability analysis is that boundedness and convergence properties of the system's solutions can be assessed without explicit computation of the latter. Instead, it suffices to verify a set of inequalities for the Lyapunov function and its time derivative, which is derived with respect to the system's equations. More precisely, the existence of a continuously differentiable (or at least Lipschitz continuous) Lyapunov function, which is positive definite with respect to an equilibrium (or an invariant set) and the time derivative of which is non-positive along the solutions of the system under investigation, is equivalent to stability of that equilibrium (or set). Similarly, instability of an equilibrium can be studied using the Chetaev function approach [9], [14]. A Chetaev function may be sign-indefinite[1] with a negative or positive definite derivative.

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[1]A function $V : \mathbb{R}^n \to \mathbb{R}$ is called sign-definite if $V(0) = 0$ and $V(x) > 0$ for all $x \in \mathbb{R} \setminus \{0\}$ or $V(x) < 0$ for all $x \in \mathbb{R} \setminus \{0\}$; and it is called sign-indefinite if $V(x)$ takes both, positive and negative, values.

Classical stability theory is mainly concerned with the analysis of a single equilibrium. However, in numerous applications, such as biological and power systems or distributed optimization, there exist several equilibria or invariant sets (including hidden attractors [12]). Hence, a rigorous analysis of such systems with several disjoint invariant sets represents an important special case of stability theory, which requires suitable methods [4], [28], [19], [31], [6], [35], [17], [13]. For this case the stability notions have to be significantly modified and relaxed as, in particular, it has been done in [13] and further in [2], [3] for the input-to-state stability (ISS) property. See also [1], [5], [8] for other results on robust stability analysis of systems with multiple equilibria.

Unfortunately, the application of these existing results to periodic systems is, in many cases, not straightforward. The main reason for this is the technical difficulty of constructing Lyapunov functions. For example, when some of the states of the system are periodic (*e.g.*, they evolve on the circle), the corresponding Lyapunov function of [3] also has to be periodic with respect to these states, which is a severe requirement. Paramount examples of such systems are the forced nonlinear pendulum [16], [18], power systems [30], [33], [38], [39] and microgrids [34], distributed or centralized optimization [10], [37], phase-locked loops [24], [25], and complex networks of oscillators [35], [11], [36].

Motivated by this wide range of potential applications, we consider a special class of systems, which possess periodic right-hand sides with respect to several state components. The presented analysis builds upon our previous work [15], where an extension of the ISS theory from [3] to periodic systems has been proposed. However, that extension has been derived for systems on manifolds and, thus, does not allow to establish boundedness of trajectories in $\mathbb{R}^n$. The latter is possible by using the cell structure approach proposed in [26] (and later in [29]) and further developed in [19], [40]. Yet, it is only applicable to systems with a scalar periodic variable, which is a severe restriction when considering complex networks. By seeking to overcome the aforementioned limitations of the existing approaches, in the present paper the cell structure framework is extended to autonomous systems, the state of which evolves in $\mathbb{R}^n$ and which possess multiple periodic variables. As in [15], our developments are inspired by the conventional cell structure approach [26]. The presented results are illustrated via application to a complex nonlinear system.

The outline of this paper is as follows. Preliminaries and the theory from [19], [40] are given in Section II. The problem statement is given in Section III with the main results in Section IV.

II. Preliminaries

Denote the real and integer numbers by $\mathbb{R}$ and $\mathbb{Z}$, respectively, and $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$.

Let the map $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ be of class $\mathcal{C}^1$, $f(0) = 0$, and consider a nonlinear system of the following form:

$$\dot{x}(t) = f(x(t)) \quad \forall t \geq 0 \tag{1}$$

with the state $x(t) \in \mathbb{R}^n$. We denote by $X(t,x)$ the uniquely defined solution of (1) at time t fulfilling $X(0,x) = x$. A set $S \subset M$ is invariant for the system (1) if $X(t,x) \in S$ for all $t \in \mathbb{R}$ and for all $x \in S$; for $x \in \mathbb{R}^n$ the point $y \in \mathbb{R}^n$ belongs to its ω-limit (α-limit) set if there is a sequence t_i, $\lim_{i\to+\infty} t_i = +\infty$, such that $\lim_{i\to+\infty} X(t_i,x) = y$ ($\lim_{i\to+\infty} X(-t_i,x;0) = y$); for any $x \in \mathbb{R}^n$ its α- and ω-limit sets are invariant [20]. Define the distance from a point $x \in \mathbb{R}^n$ to the set $S \subset \mathbb{R}^n$ as $|x|_S = \inf_{a\in S} |x - a|$, where $|x| = |x|_{\{0\}}$ for $x \in \mathbb{R}^n$ is a usual Euclidean norm of a vector $x \in \mathbb{R}^n$, and denote

$$|x|_\infty = \max_{1\leq i\leq n} |x_i|,\ |x|_1 = \sum_{i=1}^{n} |x_i|,$$

then

$$|x|_\infty \leq |x| \leq |x|_1 \leq \sqrt{n}|x| \leq n|x|_\infty.$$

A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot,t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_+$ and $\lim_{t\to+\infty} \beta(s,t) = 0$ for each fixed $s \in \mathbb{R}_+$.

The notation $DV(x)f(x)$ stands for the directional (or Dini) derivative of a continuously differentiable (or locally Lipschitz continuous) function $V : \mathbb{R}^n \to \mathbb{R}_+$ with respect to the vector field f evaluated at the point x.

A. Boundedness of solutions of periodic systems

As outlined in Section I, the present paper is dedicated to the stability analysis of periodic systems [19], [40]. More precisely, we assume in the following that for the system (1) there exists $\xi \in \mathbb{R}^n$, $\xi \neq 0$, such that for all $x \in \mathbb{R}^n$

$$f(x) = f(x + \xi).$$

Next, we recall a sufficient criterion derived in [26], [19], [40], which allows to establish *boundedness* of solutions of periodic systems. To this end consider a special case of the system (1) given by

$$f(x) = Px + b\varphi(c^\top x),$$

where $P \in \mathbb{R}^{n\times n}$ is a singular matrix, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^n$, $\varphi : \mathbb{R} \rightrightarrows \mathbb{R}$ is a Δ-periodical set-valued function, which is upper semicontinuous, with a nonempty, convex and closed set of values for any value of its argument. We note that a time-varying version of φ has been considered in [19], [40], but we restrict ourselves to the autonomous version of φ. Then under these restrictions and for any initial condition $x_0 \in \mathbb{R}^n$ the system (1) has a solution $X(t,x_0)$. Assume also that for all $\sigma \in \mathbb{R} \setminus \{0\}$ and all $\phi \in \varphi(\sigma)$

$$\mu_1 \leq \frac{\phi}{\sigma} \leq \mu_2;\ \mu_1^{-1}\mu_2^{-1}\varphi(0) = 0$$

for some $\mu_1 \in \mathbb{R} \cup \{-\infty\}$ and $\mu_2 \in \mathbb{R} \cup \{+\infty\}$. The periodicity of φ implies that either $\mu_1 < 0$, $\mu_2 > 0$ or $\mu_1 = \mu_2 = 0$, and the latter case is excluded from consideration due to its triviality.

Theorem 1. *[26], [19], [40] Assume that there exists $\lambda > 0$ such that:*

1) the matrix $P + \lambda I_n$, where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix, has $n - 1$ eigenvalues with negative real parts;

2) for all $\omega \in \mathbb{R}$

$$\mu_1^{-1}\mu_2^{-1} + (\mu_1^{-1} + \mu_2^{-1})Re\chi(i\omega - \lambda) + |\chi(i\omega - \lambda)|^2 \leq 0,$$

where $\chi(s) = c^T(P - sI_n)^{-1}b$.

Then, for any initial condition $x_0 \in \mathbb{R}^n$ the solution $X(t,x_0)$ of the periodic system (1) is bounded for $t \in [0,+\infty)$.

Remark 1. Under the conditions of this theorem the system (1) admits only one periodic coordinate, *i.e.* the vector ξ has only one non-zero element and the remaining $n - 1$ state variables are not periodic.

The proof of Theorem 1 (see Theorem 4.3.1 in [19], or Theorem 4.7 in [40]) is based on the fact that under the introduced conditions there is $H = H^\top \in \mathbb{R}^{n\times n}$ (which has one negative and $n - 1$ positive eigenvalues) such that for $V_0(x) = x^\top Hx$ we have that $dV_0(x(t))/dt \leq -2\lambda V_0(x(t))$ for all $t \in [0,+\infty)$, which implies that the set $\Omega_0 = \{x \in \mathbb{R}^n : V_0(x) \leq 0\}$ is forward invariant for (1), *i.e.* $X(t,x_0) \in \Omega_0$ for all $t \in [0,+\infty)$ provided that $x_0 \in \Omega_0$. Next, introducing the functions $V_j(x) = V_0(x - j\delta)$ and sets $\Omega_j = \{x \in \mathbb{R}^n : V_j(x) \leq 0\}$, where j is any integer and the vector $\delta \in \mathbb{R}^n$ satisfies the conditions $\delta \neq 0$, $P\delta = 0$ and $c^\top\delta = \Delta$, by periodicity of f in (1) we obtain that $dV_j(x(t))/dt \leq -2\lambda V_j(x(t))$ for all $t \in [0,+\infty)$, then the sets Ω_j are forward invariant for (1). Finally, it is shown in [19], [40] that for any $x_0 \in \mathbb{R}^n$ there is an index j_0 such that $x_0 \in \Gamma_{j_0}$, where $\Gamma_j = \Omega_j \cap \Omega_{-j} \cap \{x \in \mathbb{R}^n : |h^\top x| \leq j|h^\top\delta|\}$ with $h \in \mathbb{R}^n$ being the eigenvector of the matrix H corresponding to the negative eigenvalue. As it has been shown above $X(t,x_0) \in \Gamma_{j_0}$ for all $t \in [0,+\infty)$ (since it is true for $\Omega_{j_0} \cap \Omega_{-j_0}$). In addition the set Γ_{j_0} is bounded, which was necessary to prove. In other words, an important observation of [26], [19], [40] is that any intersection of the sets Ω_j for all integers j forms a kind of cell cover of $\mathbb{R}^n$, where each cell is bounded and forward invariant. Therefore, this framework is commonly known as cell structure approach.

III. Problem statement

The functions proposed in [26] for the analysis of boundedness of trajectories of the system (1) with a scalar periodic state are sign-indefinite with a sign-indefinite derivative. Clearly, such a relaxation of the definiteness of a Lyapunov function might simplify significantly its construction. Usually sign-indefinite functions with a sign-definite derivative are used to establish instability of an equilibrium of (1), *e.g.* Chetaev functions [9], [14]. An important observation of [26] is that the combination of "instability" and periodicity leads to boundedness of trajectories: under periodicity the existence of invariant solutions separating the domain of periodic variables, with probably repulsing trajectories around those invariants, implies the existence of a certain cell structure created by periodically repeated invariant solutions, which bounds the admissible behavior of the trajectories. A major restriction of the cell structure approach in [26], [19], [40] is that it can only be applied in the case of a scalar periodic component in $x(t)$. As a consequence, the main objective of this work is to extend that approach to a generic multidimensional case.

Inspired by [26], [3], an equivalent characterization of the ISS property for a periodic system on *manifolds* has been proposed in [15] in terms of ISS-Leonov functions, which are sign-indefinite. In the present work the concept of Leonov functions is further developed for (1) with $x \in \mathbb{R}^n$. More precisely, we assume the following:

Assumption 1. *Let $x = (z, \theta) \in \mathbb{R}^n$, where $z \in \mathbb{R}^k$ and $\theta \in \mathbb{R}^q$ are two subsets of the state vector, $n = k + q$, $k > 0$ and $q > 0$. The vector field f in (1) is 2π-periodic with respect to θ.*

In other words, we suppose that the system (1) can be embedded into a manifold $M = \mathbb{R}^k \times \mathbb{S}^q$ by a simple projection of the variables $\theta(t)$ on the sphere $\mathbb{S}^q$. For our subsequent derivations, it is convenient to introduce an auxiliary set

$$\mathcal{W} \quad = \quad \{x = (z, \theta) \in \mathbb{R}^n : z = 0,\ |\theta|_\infty = \pi\}. \qquad (2)$$

Definition 1. We say that a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ is a Leonov function for (1) if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\sigma_1, \sigma_2 \in \mathcal{K}$, and a continuous function $\lambda : \mathbb{R} \to \mathbb{R}$, $\lambda(0) = 0$ and $\lambda(s)s > 0$ for all $s \neq 0$, and scalars $0 < g_1 \leq g_2 \leq \sigma_2(\pi)$, such that for all $x = (z, \theta) \in \mathbb{R}^n$

$$\begin{aligned} \alpha_1(|z|) - \sigma_1(||\theta|_\infty - \pi|) + g_1 \leq V(x) \\ \leq \alpha_2(|z|) - \sigma_2(||\theta|_\infty - \pi|) + g_2, \end{aligned} \qquad (3)$$

and the following dissipation holds:

$$DV(x)f(x) \leq -\lambda\left(V(x)\right). \qquad (4)$$

Roughly speaking, the function V is positive definite with respect to the non-periodic variable z and negative definite with respect to the distance to the set $\{x = (z, \theta) \in \mathbb{R}^n : |\theta|_\infty = \pi\}$. The restriction $g_1 > 0$ ensures that on the set $\mathcal{W}$ (and in some of its vicinity) the function V takes positive values, while $g_2 \leq \sigma_2(\pi)$ implies that $V(0) \leq 0$, and that there exists $c \in (\pi, 2\pi)$ such that $V(x) \leq 0$ for all $z = 0$ and $|\theta|_\infty \geq c$.

Remark 2. Chetaev's theorem about instability of a set can be formulated as follows [14]: for a $\mathcal{C}^1$ function $U : \mathbb{R}^n \to \mathbb{R}$, with $U(x) = 0$ for all $x \in \mathcal{A}$, where $\mathcal{A} \subset \mathbb{R}^n$ is a compact invariant set of (1), if there exists $\epsilon_0 > 0$ such that $\mathcal{U}^+ \cap \mathcal{B}_\mathcal{A}(\epsilon) \neq \emptyset$ for any $\epsilon \in (0, \epsilon_0]$, where $\mathcal{U}^+ = \{x \in \mathcal{B}_\mathcal{A}(\epsilon_0) : U(x) > 0\}$ and $\mathcal{B}_\mathcal{A}(\varepsilon) = \{x \in \mathbb{R}^n : |x|_\mathcal{A} < \varepsilon\}$, and if

$$DU(x)f(x) > 0 \quad \forall x \in \mathcal{U}^+, \qquad (5)$$

then (1) is unstable with respect to $\mathcal{A}$ with the region of repulsion $\mathcal{B}_\mathcal{A}(\epsilon_0)$. Therefore, take $U(x) = -V(x)$, where $V : \mathbb{R}^n \to \mathbb{R}$ is a Leonov function for (1), then due to (4) the property (5) may be verified, and if the set $\mathcal{A} = \{x \in \mathbb{R}^n : V(x) = 0\}$ (since V is continuous and sign indefinite according to (3) then this set is not empty) is compact and invariant, or it contains an equilibrium of (1), then existence of a Leonov function V implies instability of (1), with the region of repulsion belonging to $\{x \in \mathbb{R}^n : V(x) \leq 0\}$.

Appearance of the norm $|\cdot|_\infty$ is originated by the topology induced by periodicity in θ of f: replicating the equilibrium at the origin using periodicity in $\theta \in \mathbb{R}^q$ creates a set of equilibria located on intersections of the levels $|\theta|_\infty = j\pi$ for $j \in \mathbb{Z}$.

IV. Main result

In this section, at first, it is shown that the existence of a Leonov function implies boundedness of trajectories for a periodic system (1). Second, a particular scenario is analyzed with conditions to check applying the proposed concept.

Theorem 2. *If for the system (1) under Assumption 1 there exists a Leonov function, then all its trajectories are globally bounded.*

Proof. Denote the set of negative values of V as

$$\Omega = \{x \in \mathbb{R}^n : V(x) \leq 0\}.$$

Then from the definition of a Leonov function

$$\underline{\Omega} \subseteq \Omega \subseteq \overline{\Omega},$$

where

$$\begin{aligned} \overline{\Omega} = \{x \in \mathbb{R}^n : \alpha_1(|z|) + g_1 \leq \sigma_1(||\theta|_\infty - \pi|)\}, \\ \underline{\Omega} = \{x \in \mathbb{R}^n : \alpha_2(|z|) + g_2 \leq \sigma_2(||\theta|_\infty - \pi|)\}. \end{aligned}$$

Recall that by assumption $g_1 \leq g_2 \leq \sigma_2(\pi)$ and note that at the origin the relations (3) take the form:

$$0 \geq g_2 - \sigma_2(\pi) \geq V(0) \geq g_1 - \sigma_1(\pi).$$

Thus, $g_1 \leq \sigma_1(\pi)$ and both these sets are non-empty and can be decomposed into two subsets:

$$\underline{\Omega} = \underline{\Omega}' \cup \underline{\Omega}'',\ \overline{\Omega} = \overline{\Omega}' \cup \overline{\Omega}''$$

with $\underline{\Omega}' \subseteq \overline{\Omega}' \subset \{x \in \mathbb{R}^n : |z| \leq \alpha_1^{-1}(\sigma_1(\pi) - g_1),\ |\theta|_\infty < \pi\}$ being compact sets, and $\underline{\Omega}'' \subseteq \overline{\Omega}'' \subset \{x \in \mathbb{R}^n : |\theta|_\infty \geq$

$c\}$ for $c \in (\pi, 2\pi)$ being not necessarily bounded sets. The compact set $\mathcal{W}$ defined in (2) is not in $\underline{\Omega}$ or $\overline{\Omega}$, and it separates $\underline{\Omega}'$ and $\overline{\Omega}'$ with $\underline{\Omega}''$ and $\overline{\Omega}''$ for $z = 0$, respectively. By inclusion property, the set Ω can also be decomposed into two parts:

$$\Omega = \Omega' \cup \Omega'',$$

where $\Omega' \subset \{x \in \mathbb{R}^n : |z| \leq \alpha_1^{-1}(\sigma_1(\pi) - g_1),\ |\theta|_\infty < \pi\}$ is a compact set and $\Omega'' \subset \{x \in \mathbb{R}^n : |\theta|_\infty \geq c\}$ is unbounded.

Denote $V(t) = V(X(t, x_0))$ for any $x_0 \in \mathbb{R}^n$, then under the conditions of the theorem we have:

$$\dot{V}(t) + \lambda(V(t)) \leq 0 \quad \forall t \geq 0,$$

and, clearly, $V(t)$ is strictly decreasing while $X(t, x_0) \in \mathbb{R}^n \setminus \Omega$. Therefore, if $x_0 \in \Omega$ then $X(t, x_0) \in \Omega$ for all $t \geq 0$ and the set Ω is forward invariant for (1). Conversely, if $x_0 \in \mathbb{R}^n \setminus \Omega$ then there exists $0 \leq T_{x_0} < +\infty$ such that $X(T_{x_0}, x_0) \in \Omega$ and, by invariance, the trajectory remains in this set for all $t \geq T_{x_0}$. Thus, the set Ω is globally attractive and forward invariant. Note, that this property does not imply any kind of stability since Ω may be unbounded (it can also be interpreted as *instability* of a set close to $\mathcal{W}$ with the domain of repulsion $\mathbb{R}^n \setminus \Omega$).

To establish stability, we exploit the *periodicity* of (1). Denote by $j = [j_1, \ldots, j_q]$ a multi-index vector, where $j_i \in \mathbb{Z}$ for all $i = 1, \ldots, q$. Introduce the new variable $x_j = x - \begin{bmatrix} 0_k \\ 2\pi j \end{bmatrix}$ and the function $V_j(x) = V\left(x - \begin{bmatrix} 0_k \\ 2\pi j \end{bmatrix}\right)$ for any such multi-index vector j (*i.e.* $V_0(x) = V(x)$), where 0_k is the zero vector of dimension k. Then, by 2π-periodicity of f in θ,

$$V_j(x) = V_j\left(x_j + \begin{bmatrix} 0_k \\ 2\pi j \end{bmatrix}\right) = V(x_j)$$

and

$$\dot{x}_j(t) = f(x(t)) = f(x_j(t)) \quad \forall t \geq 0.$$

Therefore, taking into account the properties substantiated for (1) and V, we obtain that the set $\{x_j \in \mathbb{R}^n : V(x_j) \leq 0\}$ is globally attractive and forward invariant, which in the original coordinates x implies these properties for the set

$$\Omega_j = \{x \in \mathbb{R}^n : V_j(x) \leq 0\} = \left\{x \in \mathbb{R}^n : V\left(x - \begin{bmatrix} 0_k \\ 2\pi j \end{bmatrix}\right) \leq 0)\right\}.$$

Using similar arguments as for Ω it is possible to show that $\Omega_j = \Omega_j' \cup \Omega_j''$, where

$$\Omega_j' \subset \{x \in \mathbb{R}^n : |z| \leq \alpha_1^{-1}(\sigma_1(\pi) - g_1),\ |\theta - 2\pi j|_\infty < \pi\},$$
$$\Omega_j'' \subset \{x \in \mathbb{R}^n : |\theta - 2\pi j|_\infty \geq c\}.$$

By definition $\bigcap_j \Omega_j' = \emptyset$ and $\bigcap_j \Omega_j'' = \emptyset$ (the former property is true since the sets Ω_j' are isolated, and the latter one due to the fact that $\Omega_j'' \subset \mathbb{R}^n \setminus \Upsilon_j$ with $\Upsilon_j = \{x \in \mathbb{R}^n : |\theta - 2\pi j|_\infty < c\}$, and thus $\bigcap_j \Omega_j'' \subset \bigcap_j \mathbb{R}^n \setminus \Upsilon_j = \mathbb{R}^n \setminus \bigcup_j \Upsilon_j = \mathbb{R}^n \setminus \mathbb{R}^n = \emptyset$). Since $c \in (\pi, 2\pi)$, then $(0_k, 2\pi j_1) \in \Omega_{j_1}'$ and $(0_k, 2\pi j_1) \in \Omega_{j_2}''$ for any $j_1 \neq j_2$. Consequently, $\Omega_{j_1}' \cap \Omega_{j_2}'' \neq \emptyset$ for any multi-index vectors $j_1 \neq j_2$. Therefore, $\bigcap_j \Omega_j = \bigcup_j \Omega_j'$ and the compact sets Ω_j' form a kind of cell cover of $\{x \in \mathbb{R}^n : |z| \leq \alpha_1^{-1}(\sigma_1(\pi) - g_1)\}$. Recall that for all multi-index vectors j the corresponding sets Ω_j are globally attracting and forward invariant. Clearly then $\bigcup_j \Omega_j'$ possesses the same properties (and each Ω_j' is isolated). Take any $x_0 \in \mathbb{R}^n$, then $X(t, x_0)$ asymptotically enters and stays in a cell Ω_j' for some j. Hence, for any $x_0 \in \mathbb{R}^n$ the corresponding solution $X(t, x_0)$ is bounded. □

Theorem 2 provides a very general result on the application of the theory of Leonov functions. In the sequel, we develop equivalent conditions for a special case.

Corollary 1. *Suppose that there exist $\mathcal{C}^1$ functions $W : \mathbb{R}^n \to \mathbb{R}$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, constants $\lambda > 0$ and $\overline{a}_i \geq \underline{a}_i > 0$, $\overline{c}_i \geq \underline{c}_i \geq 0$, $\underline{b}_i \geq \overline{b}_i > 0$, $i = 1, \ldots, q$, such that for all $x \in \mathbb{R}^n$*

$$\alpha_1(|z|) \leq W(x) \leq \alpha_2(|z|),$$
$$\sum_{i=1}^q \underline{a}_i\theta_i^2 - \underline{b}_i|\theta_i| + \underline{c}_i \leq \varphi(x) \leq \sum_{i=1}^q \overline{a}_i\theta_i^2 - \overline{b}_i|\theta_i| + \overline{c}_i,$$
$$DV(x)f(x) + \lambda V(x) \leq 0,$$

where $V(x) = W(x) - \varphi(x)$ and $x = (z, \theta)$. Then $X(t, x_0)$ of (1) is bounded for any $x_0 \in \mathbb{R}^n$ and $t \geq 0$ provided that

$$\underline{a}_{\min}\pi + \sum_{i=1}^q \underline{c}_i\pi^{-1} < q\underline{b}_{\max} < 2\underline{a}_{\min}\pi + 0.5\sum_{i=1}^q \underline{c}_i\pi^{-1},$$
$$q^2\overline{a}_{\max}\pi + \sum_{i=1}^q \overline{c}_i\pi^{-1} < \overline{b}_{\min} < 2q^2\overline{a}_{\max}\pi + 0.5\sum_{i=1}^q \overline{c}_i\pi^{-1},$$

where $\underline{a}_{\min} = \min_{1\leq i\leq q}\underline{a}_i$, $\underline{b}_{\max} = \max_{1\leq i\leq q}\underline{b}_i$, $\overline{a}_{\max} = \max_{1\leq i\leq q}\overline{a}_i$ and $\overline{b}_{\min} = \min_{1\leq i\leq q}\overline{b}_i$.

Proof. The claim is established by applying Theorem 2. To this end, it is only necessary to show that V is a Leonov function for (1), *i.e.*

$$\sigma_2(||\theta|_\infty - \pi|) - g_2 \leq \varphi(x) \leq \sigma_1(||\theta|_\infty - \pi|) - g_1$$

for some $\sigma_1, \sigma_2 \in \mathcal{K}$ and some $0 < g_1 \leq g_2 \leq \sigma_2(\pi)$, since all the remaining conditions are already satisfied under the standing assumptions.

Note that

$$\sum_{i=1}^q \overline{a}_i\theta_i^2 - \overline{b}_i|\theta_i| + \overline{c}_i \leq \sum_{i=1}^q \overline{a}_{\max}\theta_i^2 - \overline{b}_{\min}|\theta_i| + \overline{c}_i$$
$$= \overline{a}_{\max}|\theta|^2 - \overline{b}_{\min}|\theta|_1 + \sum_{i=1}^q \overline{c}_i$$
$$\leq q^2\overline{a}_{\max}|\theta|_\infty^2 - \overline{b}_{\min}|\theta|_\infty + \sum_{i=1}^q \overline{c}_i = \overline{f}(|\theta|_\infty)$$

and

$$\sum_{i=1}^{q}\underline{a}_i\theta_i^2-\underline{b}_i|\theta_i|+\underline{c}_i\geq\sum_{i=1}^{q}\underline{a}_{\min}\theta_i^2-\underline{b}_{\max}|\theta_i|+\underline{c}_i$$
$$=\underline{a}_{\min}|\theta|^2-\underline{b}_{\max}|\theta|_1+\sum_{i=1}^{q}\underline{c}_i$$
$$\geq\underline{a}_{\min}|\theta|_\infty^2-q\underline{b}_{\max}|\theta|_\infty+\sum_{i=1}^{q}\underline{c}_i=\underline{f}(|\theta|_\infty).$$

Under the introduced restrictions the following properties are satisfied for $\underline{f}$:

$$\sum_{i=1}^{q}\underline{c}_i=\underline{f}(0)\geq 0,$$
$$\underline{a}_{\min}\pi+\sum_{i=1}^{q}\underline{c}_i\pi^{-1}<q\underline{b}_{\max}\Rightarrow 0>\underline{f}(\pi),$$
$$2\underline{a}_{\min}\pi+0.5\sum_{i=1}^{q}\underline{c}_i\pi^{-1}>q\underline{b}_{\max}\Rightarrow\underline{f}(2\pi)>0$$

and $2\pi^2\underline{a}_{\min}>\sum_{i=1}^{q}\underline{c}_i$ is a sufficient condition for existence of $\underline{b}_{\max}$. Similarly for $\overline{f}$:

$$\sum_{i=1}^{q}\overline{c}_i=\overline{f}(0)\geq 0,$$
$$q^2\overline{a}_{\max}\pi+\sum_{i=1}^{q}\overline{c}_i\pi^{-1}<\overline{b}_{\min}\Rightarrow 0>\overline{f}(\pi),$$
$$2q^2\overline{a}_{\max}\pi+0.5\sum_{i=1}^{q}\overline{c}_i\pi^{-1}>\overline{b}_{\min}\Rightarrow\overline{f}(2\pi)>0$$

and the inequality $2\pi^2q^2\overline{a}_{\max}>\sum_{i=1}^{q}\overline{c}_i$ has to be satisfied. From these properties of $\underline{f}$ and $\overline{f}$ we can conclude that the required functions $\sigma_1,\sigma_2\in\mathcal{K}$ and constants $0<g_1\leq g_2\leq\sigma_2(\pi)$ exist, completing the proof (for example, these functions can be selected in the form $\sigma_1(s)=\ell_1\max\{\ell_2 s,s^2\}$ and $\sigma_2(s)=\ell_3\min\{\ell_4 s,s^2\}$ for some $\ell_i>0$ for $i=1,2,3,4$). □

V. Conclusions

By extending ideas of [26], we have introduced the concept of a Leonov function, and have shown that for a class of periodical dynamical systems the existence of a Leonov function implies boundedness of all trajectories. Such a function is in general sign-indefinite and not continuously differentiable on the manifold, on which the system dynamics can be projected. These achievements represent significant relaxations compared to the usual requirements on a standard Lyapunov function [3]. Application of the proposed theory to power systems and microgrids (in particular, to the swing equation) is currently under investigation.

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