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An Output Feedback Control with State Estimation for the Containment of the HIV/AIDS Diffusion*

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Abstract— **An optimal control problem formulated to reduce the infection diffusion of an epidemic disease is solved in a locally linearised context. The approach has been driven by the necessity of using a state observer, since the state variable to be controlled is not directly measurable. The solution is computed making reference to a recently introduced model for HIV/AIDS diffusion and control, in which five classes of individuals are considered: two classes of susceptible subjects, the wise and the incautious individuals, and three classes of infectious subjects, the ones not aware of their condition and the subjects in the pre-aids or in the aids status. The control inputs are represented by information campaigns and medication actions. The initial formulation as an optimal control problem, aiming also at minimising the non measurable number of infected subjects, is enriched with an asymptotic state observer for which a local linear approximation has been chosen for simplicity purpose. The structure of the cost index for the reformulated control problem after the observer introduction, suggests to compute the solution in an approximated way referring to a classical LQR control design. The effectiveness of the control, as well as the effects of the linear approximations, are evidenced by some numerical simulation results.**

I. INTRODUCTION

The optimal control theory is the natural framework to study the control of the epidemic models in which conflicting issues must usually be addressed [1]; in epidemic spread, the control represents the general prevention effort (in particular the vaccine action, if possible), the medication, the quarantine, all under a possible resources limitation.

Optimal control has been successfully applied to the classical SIR model (susceptible-infectious-removed subjects) [2], [3], [4], to the influenza [5], [6], [7], to the Dengue disease [8], to schedule vaccination strategies [9], [10], to study complex networks and identify spread process [11].

Motivated by its diffusion and its highly potential dangerousness, in the last three decades researchers have made great progresses in the attempts of eradicating the Human Immunodeficiency Virus (HIV) responsible of the Acquired Immune Deficiency Syndrome (AIDS), [12], [13], [14]. The virus attacks the cells of the immune system, destroying or impairing their functions; consequently, the immune system is inhibited and the individual is less protect against infections. Virus transmission is facilitated by contacts with infected body fluids such as blood, semen, pre-seminal fluid, rectal fluid, vaginal fluid, and breast milk. The AIDS represents the most advanced stage of the HIV infection; it can be reached in 10-15 years from the initial HIV infection.

Up to now, no vaccine exists and the control actions are the prevention and the medication after a positive diagnosis. Despite the well-known modalities of its transmission, it is still one of the most diffused disease.

The mathematical modelling of the HIV/AIDS diffusion among populations is focused on the dynamic of interactions between individuals [13], [15], [16], [17]. In classical HIV/AIDS spread models, four main classes are introduced: the Susceptible subjects (S), that are the healthy people that may contract the virus, the Infectious one (I) that are individuals not aware of their condition, the pre-AIDS patients (P), the AIDS patients (A). The control actions introduced are mainly focused on the prevention, as for example in [18], where the attention is devoted to risky subjects, drug users and sex workers.

In this paper, the model proposed in [16], [17] is assumed for a control problem formulation and the control scheme design. The susceptible individuals, S, are divided into two categories, the one that adopts wise behaviours and the one that does not take into account the dangerousness of the disease. Such a distinction is meaningful since introduces the concept of safe and unsafe behaviours to characterise the contagion. Therefore, five categories are considered: three classes of infected individuals (I,P,A) and two classes of susceptible subjects.

As far as the input characterisation is considered, the model adopted follows the suggestions of the World Health Organization (WHO), which indicates three lines of intervention: (1), an action aiming at reducing the possibility of new infections; it corresponds to the effort of convincing susceptible subjects to adopt cautious behaviours; (2), a facilitation of a fast diagnosis for unaware infectious patients, thus reducing the percentage of subjects responsible of the virus spread; (3), a medication support to the aware infectious subjects.

The control problem addressed in the paper aims at minimising the number of infectious subjects with as less resources as possible. This is formulated in the framework of optimal control theory, introducing a cost function which weights the number of unaware infectious individuals $I(t)$ and the controls. The dynamical model is non-linear, whereas for the cost index a quadratic lagrangian is chosen. Since

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only the number of the patients with HIV (P) and AIDS (A) are available, a state observer is introduced to estimate the full state and, in particular, the required number of the subjects $I(t)$. Both for the asymptotic state observer design and for the optimal control problem solution, a linearisation of the model in the neighbourhood of one equilibrium point is discussed, also aiming at applying the linear quadratic (LQ) regulator theory, [19], which provides a state feedback control law.

The paper is organized as follows. In Section II the adopted non-linear model is briefly recalled and the optimal control problem is formulated. In Section III the control design is proposed; three subsections are introduced, describing the linear approximation of the model, the state estimator and the regulator in the linear quadratic framework. In Section IV numerical results are presented and discussed.

II. MODEL DESCRIPTION AND OPTIMAL CONTROL PROBLEM FORMULATION

A. A Short Recall of the Mathematical Model

In this paper, the model of the HIV/AIDS diffusion presented in [16], [17] is adopted and is here briefly recalled. It suitably models the two main particularities of the HIV/AIDS spread that significantly distinguish this disease from the others: (1) there is a period, more or less long, in which the symptoms of the infection are not evident; (2) the HIV can be transmitted only by some body fluids or sharing needles or syringes.

The first characteristic is responsible of the dangerousness of HIV/AIDS, since an infectious individual could be unaware of his status for a long time and could infect unwise susceptible subjects. Therefore, it is useful to stress (and to model) the second characteristic described: the infection can be transmitted when unsafe behaviours are adopted. In fact, everyone is susceptible, but one can distinguish between the category of wise people adopting safe behaviours, and the one of unwary subjects that could become infectious. These two particularities of the HIV/AIDS spread are modelled in this paper, where control actions consistent with the three levels of intervention previously recalled are introduced.

The effort to induce the population to participate to test campaign has two main advantages: it reduces the risky time in which an infectious subject, not aware of his status, could infect healthy unwise susceptible ones, and it allows the infected individuals to start a medication program. A schedule of the control action is advisable, since the costs of primary and secondary preventions represent an immediate economic effort, whereas their effects could be appreciated only in the future, as will be discussed later.

Taking into account all these aspects, the variables introduced in the model denote the healthy people $S_1(t)$, not aware of dangerous behaviours and then can be infected, and $S_2(t)$, the ones that, suitably informed, give great attention to the protection, and the three previously described infectious and infected subjects $I(t)$, $P(t)$ and $A(t)$.

As far as the control actions is concerned, they are the information campaign, denoted by $u_1(t)$, the effort to improve a test campaign to the discovery of the infection as soon as possible, denoted by $u_2(t)$, and the therapy which aims at reducing the transition from HIV to AIDS, $u_3(t)$. Therefore, the final model is

$$
\begin{aligned}
\dot{S}_1(t) &= Z - dS_1(t) - \frac{\beta S_1(t)I(t)}{N_c(t)} + \gamma S_2(t) - S_1(t)u_1(t) \\
\dot{S}_2(t) &= -(\gamma + d)S_2(t) + S_1(t)u_1(t) \\
\dot{I}(t) &= \frac{\beta S_1(t)I(t)}{N_c(t)} - (d+\delta)I(t) - \psi\frac{I(t)}{N_c(t)}u_2(t) \\
\dot{P}(t) &= \varepsilon\delta I(t) - (\alpha + d)P(t) + \phi\psi\frac{I(t)}{N_c(t)}u_2(t) + \\
&\quad +P(t)u_3(t) \\
\dot{A}(t) &= (1-\varepsilon)\delta I(t) + \alpha P(t) - (\mu + d)A(t) + \\
&\quad +(1-\phi)\psi\frac{I(t)}{N_c(t)}u_2(t) - P(t)u_3(t)
\end{aligned}
\tag{1}
$$

where $N_c(t) = S_1(t) + S_2(t) + I(t)$. Dynamics (1) can be expressed in the compact form

$$
\dot{X} = f(X) + g_1(X)u_1 + g_2(X)u_2 + g_3(X)u_3 = F(X,U) \tag{2}
$$

once $X = \{S_1\ S_2\ I\ P\ A\}^T$, $U = \{u_1\ u_2\ u_3\}^T$ and

$$
f(\cdot) = \begin{pmatrix} Z - dS_1 - \frac{\beta S_1 I}{N_c} + \gamma S_2 \\ -(\gamma+d)S_2 \\ \frac{\beta S_1 I}{N_c} - (d+\delta)I \\ \varepsilon\delta I - (\alpha+d)P \\ (1-\varepsilon)\delta I + \alpha P - (\mu+d)A \end{pmatrix} \qquad g_1(\cdot) = \begin{pmatrix} -S_1 \\ S_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3}
$$

$$
g_2(\cdot) = \begin{pmatrix} 0 \\ 0 \\ -\psi\frac{I}{N_c} \\ \phi\psi\frac{I}{N_c} \\ (1-\phi)\psi\frac{I}{N_c} \end{pmatrix} \qquad g_3(\cdot) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ P \\ -P \end{pmatrix} \tag{4}
$$

are defined.

In (1), d denotes the rate of natural death; Z denotes the flux of new subjects in the class S_1; β is related to the dangerous interactions between S_1 and I categories; γ is the rate of wise subjects that could change, incidentally, their status, increasing $S_1(t)$; ψ is related to the control action aiming at helping the individuals in class I to discover their infectious condition, and therefore to flow to the P or the A class; ϕ is the percentage of test positive subjects with HIV ($(1-\phi)$ the percentage with AIDS); δ is the rate of transition from I to P (percentage ε) or A (percentage $(1-\varepsilon)$) without any external action; α is the rate of the natural transition from P to A; μ is the rate of death in class A caused by the infection.

As far as the output is concerned, the physically and realistically available measurements of the dynamics are represented by the number of the subjects with a positive diagnosis of HIV and /or AIDS, i.e. $P(t)$ and $A(t)$. Assuming to measure the total number of diagnosed individuals, the output is $P(t)+A(t)$ giving

$$y(t) = CX(t), \qquad C = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \end{pmatrix} \tag{5}$$

B. The Control Problem Formulation

For the model considered, the critical class is represented by I, that is the individuals infectious but not conscious of their status and then, with their behaviour, can make the number of sufferers increase.

Since the control $u_3(t)$ does not affect the $I(t)$ dynamic at all, the control inputs considered hereinafter are the first two only and the vector $\hat{U}(t) = (u_1(t)\, u_2(t))^T$ is introduced and the vector field $g_3(\cdot)$ in (4) is neglected.

On these basis, the goal of the proposed control action is the minimization of the number of infectious subjects $I(t)$ making use of as less resources as possible. Then, in the framework of optimal control problem formulation, the following quadratic cost index is assumed

$$\begin{aligned} J(X,\hat{U}) = & \; \tfrac{1}{2}\int_{t_0}^{\infty} \left(qI^2(t) + r_1u_1^2(t) + r_2u_2^2(t)\right) dt = \\ = & \; \tfrac{1}{2}\int_{t_0}^{\infty} \left(X^T(t)QX(t) + \hat{U}^T(t)R\hat{U}(t)\right) dt \end{aligned} \tag{6}$$

with Q the square matrix of order five with all zero entries except $Q(3,3) = q$, and $R = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$, $r_1 > 0$, $r_2 > 0$; moreover, a closed form solution is required.

The solution of such a problem cannot be easily obtained if the variables involved ($I(t)$ in the present case) are not directly measurable and, then, a state estimation is needed. Clearly, the use of the estimated state values instead of their actual ones introduces an error that affects the cost index value, producing a solution which looses its optimality.

Then, before the control law design in Subsection III-C, a preliminary state estimator is computed in Subsection III-B.

The simple solution of the estimation problem by means of a linear observer, designed from a linear approximation of the dynamics in a neighbourhood of an equilibrium point, is adopted, with the assumption that the approximation error is well compensated by the simplicity of the design procedure and implementation.

According to the previous considerations, all the design phases are preceded by the computation and the analysis of a suitable linear approximation of the dynamics, performed in the next Subsection III-A.

III. THE CONTROL DESIGN

In this Section, the proposed control design procedure is described. As previously discussed, a preliminary linearisation is performed and analysed (Subsection III-A).

A. The Linear Approximated Dynamics

According to the control design proposed, a linear approximation in the neighbourhood of an equilibrium point is required. Following a classical approach, the computation of the state space points which verify the equation $F(X^e, 0) = 0$ is performed, yielding the solutions ([16], [17])

$$X_1^e = \begin{pmatrix} 1/d \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad X_2^e = \begin{pmatrix} 1/H \\ 0 \\ \frac{H-d}{H(d+\delta)} \\ \frac{\varepsilon\delta(H-d)}{H(\alpha+d)(d+\delta)} \\ \frac{\delta(H-d)[(1-\varepsilon)d+\alpha]}{H(\alpha+d)(d+\delta)(\mu+d)} \end{pmatrix} \tag{7}$$

where $H = \beta - \delta$. The non negativeness of the elements in the vector state X_2^e implies the conditions $H > 0$ and $H \geq d$; therefore the equilibrium point X_2^e is a feasible one if and only if $H \geq d$, being $X_1^e = X_2^e$ if $H = d$. Bifurcation analysis has been performed in [17] to better investigate equilibria and stability properties.

In this work, it is assumed that condition $H > d$ holds, so that the both equilibrium points exist and a full analysis can be carried on. In addition, in the considered case, for the parameter values chosen, the condition is verified.

So, the two linearised dynamics are computed in the neighbourhood of the two equilibrium points

$$\dot{\tilde{X}}(t) = A_i\tilde{X}(t) + \hat{B}_i\hat{U}(t) \tag{8}$$

with

$$A_i = \left.\frac{\partial F}{\partial X}\right|_{X=X_i^e,\hat{U}=0} \qquad \hat{B}_i = \left.\frac{\partial F}{\partial \hat{U}}\right|_{X=X_i^e,\hat{U}=0} \tag{9}$$

$i = 1, 2$, both with the output function (5) rewritten as

$$\tilde{y}(t) = C\tilde{X} \tag{10}$$

according to the state translation

$$\tilde{X}(t) = X(t) - X_i^e \tag{11}$$

once $\tilde{y}(t) = y(t) - CX_i^e$ is posed.

All computations for stability analysis of the two equilibrium points can be found in [16], [17]. It is shown that if condition $H > d$ holds, as it is supposed, X_1^e is an unstable equilibrium point while X_2^e exists and is locally asymptotically stable; otherwise, X_1^e is the only equilibrium point and it is asymptotically stable.

Despite the procedure can be adopted making reference to the both equilibrium points, easy computations show that the linear dynamics which approximates the non-linear one in the neighbourhood of X_1^e is neither detectable nor controllable. The lack in detectability makes impossible the linear observer design and then the choice of X_2^e is necessarily performed.

Then, in the sequel, the matrices of the linear approximation are given by (9) for index $i = 2$, and (10)

$$\begin{aligned} \dot{\tilde{X}}(t) &= A_2\tilde{X}(t) + \hat{B}_2\hat{U}(t) \\ \tilde{y}(t) &= C\tilde{X}(t) \end{aligned} \tag{12}$$

B. The State Observer Design

A linear observer is chosen to estimate the state variables, assuming that the additional approximation error introduced by linearisation is compensated by the simplicity in the design and in the consequent implementation.

Then, once the detectability property is satisfied, the state estimation $\xi(t)$ of the state $\tilde{X}$ of (12), verifying the condition $\lim_{t\to+\infty} \|\tilde{X}(t) - \xi(t)\| = 0$, can be obtained as the state evolution of the linear observer dynamics

$$\dot{\xi}(t) = (A_2 - GC)\,\xi(t) + \hat{B}_2\hat{U}(t) + G\tilde{y}(t) \tag{13}$$

with matrix G chosen in order to have all the eigenvalues of the dynamic matrix $(A_2 - GC)$ with negative real part.

This computation, as well as the preliminary check of the detectability property are performed in Section IV for the particular choice of parameter values there adopted.

C. The Optimal Control Problem Solution

The problem initially formulated has to be reconsidered after the introduction of the state observer. In particular, the cost index (6) is rewritten putting in evidence the coordinates change (11) performed:

$$\begin{aligned} J(X,\hat{U}) &= J(\tilde{X} + X_2^e, \hat{U}) = \tilde{J}(\tilde{X},\hat{U}) = \\ &= \tfrac{1}{2}\int_{t_0}^{\infty} \left(\left(\tilde{X}(t) + X_2^e\right)^T Q \left(\tilde{X}(t) + X_2^e\right) + \right. \\ &\qquad \left. + \hat{U}^T(t) R \hat{U}(t) \right) dt = \\ &= \tfrac{1}{2}\int_{t_0}^{\infty} \left(q\left(\tilde{I}(t) + \tfrac{H-d}{H(d+\delta)}\right)^2 + r_1 u_1^2(t) + r_2 u_2^2(t) \right) dt \end{aligned} \tag{14}$$

With this cost function, if the dynamics to be controlled were linear, the optimal control problem would have the classical structure of a Linear Quadratic Regulator problem with a tracking term. A consequence would be the possibility of computing a state feedback control law solving an algebraic Riccati Equation, so having a simple solution with a (robust) state feedback implementation.

This possibility should also take into account that in the case under investigation, a true state feedback cannot be obtained due to the unavailability of a measure of $I(t)$, problem that can be overcome by the use of the observer introduced in previous Subsection III-B.

Due to the presence of a linear approximation in the control scheme referring to the state estimation, the idea is to use the linear approximation of the original dynamics also for the control design, so getting a true LQR with tracking problem to be solved, yielding a state feedback control law. Clearly, the result obtained is not optimal for the given system, but the approximation error so introduced is of the same order of magnitude of the one present in the observer dynamics.

Then, the solution proposed is the design of both the controller and the observer on the basis of the linearised dynamics, which assures also the possibility to invoke the separation principle for justifying the procedure for the linear dynamics.

The linear dynamics considered is (12), the same used for the observer design. The aim is to compute the optimal control $\hat{U}(t)$ which minimises the cost index (14). The minimization of the cost index (14) for the linear dynamical system is equivalent to solve an LQ tracking problem where the reference $\bar{r}_{\tilde{I}}$ for $\tilde{I}(t)$ is the constant value

$$\bar{r}_{\tilde{I}} = -\frac{H-d}{H(d+\delta)} \tag{15}$$

The existence and uniqueness of the solution of the stationary tracking problem over an infinite time interval is guaranteed only if the matrix Q is positive definite. For a positive semi-definite matrix Q, the cost index is not influenced by all the states components; therefore, it could be possible that an optimal feedback law could not be stabilizing, if the states not weighted are not stable by themselves, [20]. Nevertheless, in this case the linearised system (12) results to be stable, as already pointed out in [16], [17]. The state feedback optimal control law obtained is given by ([1])

$$\hat{U}^0(t) = -R^{-1}\hat{B}_2^T K \tilde{X}^0(t) + R^{-1}\hat{B}_2^T g_{\bar{r}} \tag{16}$$

where K is the solution of the Algebraic Riccati Equation

$$0 = K\hat{B}_2 R^{-1}\hat{B}_2^T K - KA_2 - A_2^T K - Q \tag{17}$$

and $\tilde{X}^0(t)$ is the optimal evolution for the linear dynamics with control (16)

$$\dot{\tilde{X}}^0(t) = \left(A_2 - \hat{B}_2 R^{-1}\hat{B}^T K\right)\tilde{X}^0(t) + \hat{B}_2 R^{-1}\hat{B}_2^T g_{\bar{r}} \tag{18}$$

In (18), $g_{\bar{r}} = \left(K\hat{B}_2 R^{-1} B_2^T - A_2^T\right)^{-1} Q\,\bar{r}$, where $\bar{r}$ denotes the tracking term; in this case, it is given by $\bar{r} = (**\bar{r}_{\tilde{I}}**)^T$, from which $Q\,\bar{r} = (0\,0\,q_{\bar{r}_{\tilde{I}}}\,0\,0)^T$.

The complete solution is then obtained using the observer output, i.e. the state estimation, instead of the actual state requested by the so obtained control law given by (16).

The separation principle assures the stability of the whole control scheme for the linear system and then also the local stability of the given non-linear one.

The introduction in the control scheme of the estimation of the unmeasurable state variable I considered in the cost function is a novelty in the optimal approaches to HIV/AIDS control. The effectiveness and the performance of the obtained control law, in the form of a dynamical output control scheme, are then evaluated by simulation results in Section IV, once the designed controller is applied to the original non linear dynamics (1).

IV. NUMERICAL RESULTS AND DISCUSSION

In this Section, a numerical analysis is performed to show the feasibility of the proposed control scheme and to evaluate, for the specific given dynamics, the effects of the approximations and the errors introduced by the linear control design and by the observer.

The values for the parameters in the dynamics (1) adopted for the numerical computations have been taken, for comparative purposes, from [12] and [16]; they have been firstly used in [21] on the basis of epidemiological research conducted at the San Francisco City Clinic:

$$d = 0.02, \quad \beta = 1.5, \quad \delta = 0.4, \quad \varepsilon = 0.6, \quad \phi = 0.95$$
$$\gamma = 0.2 \quad \psi = 10^5, \quad \alpha = 0.5, \quad \mu = 1, \quad Z = 1000$$

Consequently, $H = \beta - \delta = 1.1 > 0$ and, then, the equilibrium point X_2^e exists and is locally asymptotically stable. Numerically, $X_2^e = (0.91\ 0\ 2.34\ 1.08\ 0.9)^T \cdot 10^3$.

The linear approximation in the neighbourhood of this equilibrium point yields the following matrices:

$$A_2 = \begin{pmatrix} -0.80 & 0.20 & -0.12 & 0 & 0 \\ 0 & -0.22 & 0 & 0 & 0 \\ 0.78 & 0 & 0.30 & 0 & 0 \\ 0 & 0 & 0.24 & -0.52 & 0 \\ 0 & 0 & 0.16 & 0.5 & -1.02 \end{pmatrix} \tag{19}$$

$$\hat{B}_2 = \begin{pmatrix} -0.91 & 0 \\ 0.91 & 0 \\ 0 & -72 \\ 0 & 68.40 \\ 0 & 3.60 \end{pmatrix} 10^3 \tag{20}$$

with C in (5).

For this numerically defined system, the property of controllability and detectability are easily verified. Then, the design procedures for both the state observer and the feedback control law can be performed.

Following the steps described in Section III, a preliminary asymptotic state observed is required. Since the solution here adopted, as described in Subsection III-B, is based on the use of a linear state observer, the design procedure requires the computation of matrix G in (13) such that the matrix $(A_2 - GC)$ is asymptotically stable.

A crucial aspect is the transient characteristics of the error dynamics, which are related to the eigenvalues of matrix $(A_2 - GC)$. For this purpose, the set of eigenvalues chosen is $\Lambda = \{-1.0, -1.1, -1.2, -1.3, -1.4\}$. They are taken real and a little bit greater, in module, than the ones of the given linear dynamics.

Then, the observer dynamics (13), with inputs $\hat{U}(t)$ and $\tilde{y}(t)$, is fully defined.

Now, according to the procedure described in Subsection III-C, the control law has to be computed as the solution of a LQR problem with offset (tracking) term. Numerical computation of (15) gives $\bar{r}_{\tilde{I}} = -2.34 \cdot 10^3$.

The weights in the cost index are chosen as $q = 10^{-4}$, $r_1 = 1$, $r_2 = 1000$. The reason is to compute a control law that, for the linear approximated dynamics, gives a great attention to the error term (low cost) and a lower one to the control cost, expecially to the second one (higher cost).

The solution K of the Algebraic Riccati Equation (17) gives

$$\begin{pmatrix} 0.07 & -0.01 & 0.14 & 0 & 0 \\ -0.01 & 0.02 & -0.05 & 0 & 0 \\ 0.14 & -0.05 & 4.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{21}$$

and then $g_{\bar{r}} = \begin{pmatrix} 0.05 & 0 & -1.03 & 0 & 0 \end{pmatrix}^T \cdot 10^{-2}$

The optimal control (16) so obtained, which should drive the state variable $\tilde{I}$ of the linearised system to the reference value $\bar{r}_{\tilde{I}}$, is

$$\hat{U}^0(t) = 10^{-4} \begin{pmatrix} -0.71 & 0.25 & -1.78 & 0 & 0 \\ -0.10 & 0.04 & -3.12 & 0 & 0 \end{pmatrix} \tilde{X}(t) + \\ + \begin{pmatrix} 0.41 \\ 0.74 \end{pmatrix} \tag{22}$$

Due to the state transformation (11), the same control law (22), once applied to the non-linear dynamics, should drive the state variable $I(t)$ to a low value.

The output feedback control law just computed is applied to the non linear dynamics (1) and some simulations are performed to illustrate the behaviour of the non linear control scheme. The values of the parameters adopted imply that the time scale is expressed in years.

Figures 1-3 show the time evolution of the five state components; the susceptible subjects $S_1(t)$ and $S_2(t)$ are shown in Fig. 1; the infected subjects are depicted separating the infectious individuals not aware of their status $I(t)$, plotted in Fig. 2, and the patients with a positive diagnosis $P(t)$ and $A(t)$, shown in Fig. 3. The corresponding control law is depicted in Fig. 4

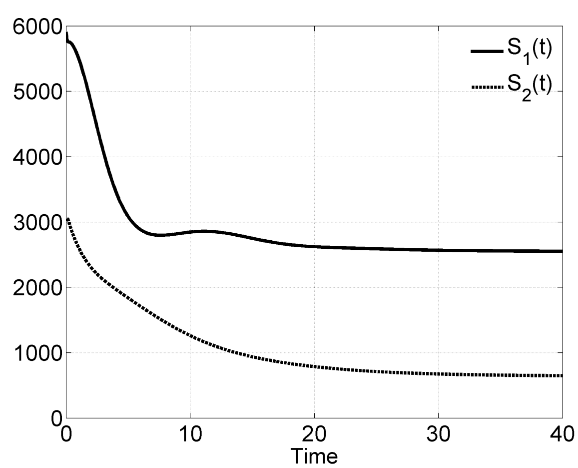

Fig. 1. Time history of the number of unwise $S_1(t)$ and wise $S_2(t)$ population members

It can be noted that the state evolution of the non-linear controlled system asymptotically reaches a steady state value X_c^e, corresponding to the equilibrium point for the closed loop system. Such a value can be computed and, for the considered case, it is $X_c^e = 10^3 \cdot (2.55\ 0.64\ 1.03\ 1.39\ 0.87)^T$.

The positive effects of the control actions can be observed comparing the original equilibrium point X_2^e with the steady state values. The number of not infectious subjects increases, more sensibly for the wise group $S_1(t)$ (from 910 to 2550) than for the unwise one $S_2(t)$ (from 0 to 640)). The number of unaware infectious individuals considered in the cost function, $I(t)$, strongly decreases (from 2340 to 1030); since such a reduction is not due to a change in the illness conditions but only in a change of awareness of the individual illness status, one expected effect, verified, is the increment of the number of the diagnosed infectious $P(t)+A(t)$ (from 1170 to 2602). The relative distribution among them depends on the choice of the model parameter ϕ.

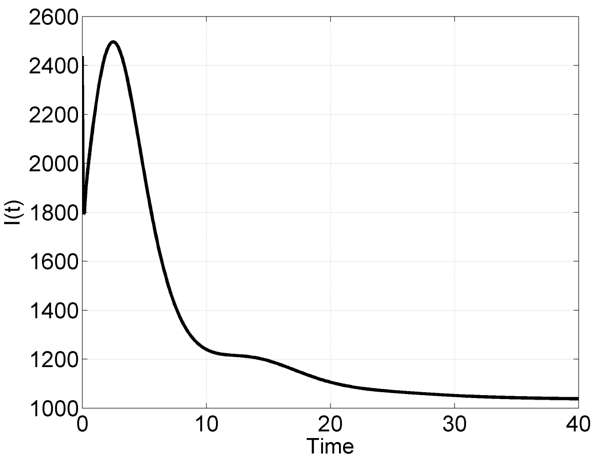

Fig. 2. Time history of the number of the unaware infectious subjects $I(t)$

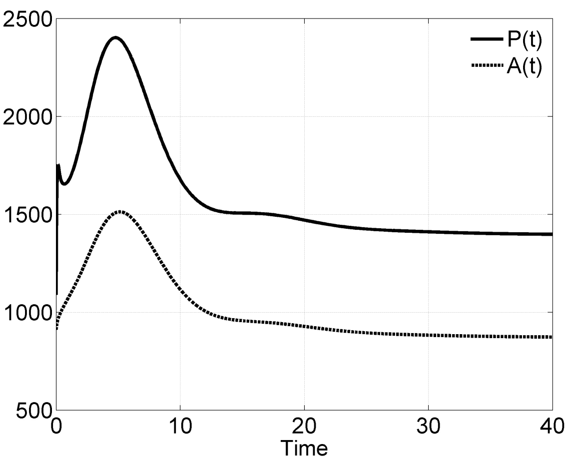

Fig. 3. Time history of the number of the diagnosed infectious patients $P(t)$ and $A(t)$

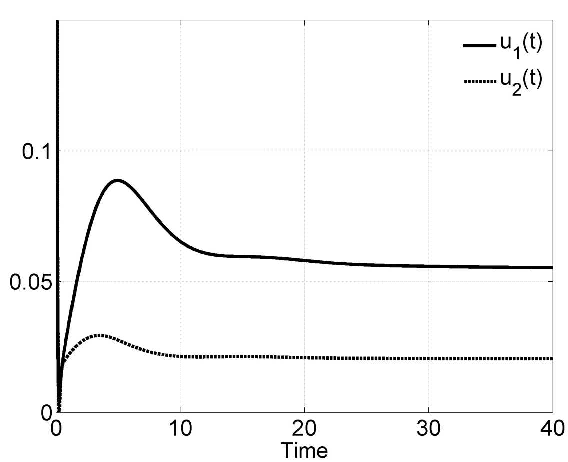

Fig. 4. Time history of the state feedback control actions

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