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Compensation of Simultaneous Input/Output Delay and Unknown Sinusoidal Disturbances for Known LTI Systems

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***Abstract*— This paper focuses on estimation and cancellation of unknown sinusoidal disturbances in a known LTI system with the presence of simultaneous known input-output delay. Parametrizing the disturbance and representing the delays as a transport PDE, the problem is converted to an adaptive control problem for ODE-PDE cascade. An existing state observer is used to estimate the ODE system states. The exponential stability of the equilibrium of the closed-loop and error system is proved. The perfect estimation of the disturbance and state is shown. Moreover, the convergence of the state to zero as $t \to \infty$ is achieved in the closed loop system. The effectiveness of the controller is demonstrated in a numerical simulation.**

I. INTRODUCTION

Cancellation of sinusoidal disturbances has been among difficult challenges faced by control engineers. A common method to address this problem is to model the disturbance as the output of a linear dynamic system which is called an exosystem. Including the exosystem in the feedback loop, disturbance effect can be compensated in the plant response. This method is known as internal model principle [1] and achieves robust output regulation.

Since time delay is a common phenomenon observed in most real-world applications, the studies have focused on developing control methods in which delays arise. The idea of representing time delay as dynamic of PDE is introduced in [8]. Inspiring by [8], an adaptive observer for PDEs is developed in [7] with a backstepping like design technique to compensate a delay.

The cancellation of sinusoidal disturbance for known and unknown LTI systems with input delay is studied in [2] and [3], respectively. The output regulation problem is addressed in [4] for output-delayed known linear systems. An observer design for output delayed systems with model parameter uncertainty is given in [10]. Moreover, for known linear systems with simultaneous state, input and output delay, disturbance cancellation algorithms are proposed in [5] and [6]. However, the studies [4]–[6] assume that exosystem is known and can be used in the controller.

The problem that we consider in this paper is the combination of disturbance cancellation by output feedback and delay in the measurement and controller. Contrary to [4]–[6], the unknown disturbance in our system is generated by an uncertain exosystem, i.e., the frequencies of the sinusoidal disturbances are unknown. Our main contribution is to solve this type of a problem by combining two methods. We first use the technique given in [9] to express the disturbance in a parametrized form and then, employ an adaptive observer proposed in [10]. In addition to this, by employing the perfect estimation of the disturbance and the state, we design an adaptive controller that rejects the disturbance and makes the equilibrium of the closed-loop system exponentially stable.

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The paper is organized as follows. In Section II, the problem definition is stated. The disturbance representation and disturbance parametrization are given in Sections III and IV, respectively. In Section V, the controller design and stability theorem are presented. In Section VI, the proof of stability theorem is given. Finally, an example simulation is illustrated in Section VII.

Notation. Throughout the paper, we use the following notations; B_i is a column vector whose i^{th} element is 1 and the rest is 0, state/parameter estimation and estimation errors are denoted with the symbols " ˆ " and " ˜ ", respectively. As an example, estimation error of X state is $\tilde{\boldsymbol{X}} = \hat{\boldsymbol{X}} - \boldsymbol{X}$ where $\hat{X}$ is the estimation of X. We use subscript i for i^{th} scalar element in general, however I_i and 0_i denote $i \times i$ identity matrix and $i \times 1$ column zero vector, respectively. The Euclidean norm is denoted by $|\cdot|$. We use ∂_t and ∂_x to denote time and spatial derivatives of a function respectively.

II. PROBLEM STATEMENT

We consider the single-input single-output LTI system

$$\dot{X}(t) = AX(t) + B\big(U(t - D_u) + \nu(t)\big), \tag{1}$$

$$Y(t) = CX(t - D_y), \tag{2}$$

where $D_u \in \mathbb{R}$ and $D_y \in \mathbb{R}$ are the known input and output delay, respectively. $X = [X_1, \ldots, X_n]^T \in \mathbb{R}^n$ is the system state, $U(t) \in \mathbb{R}$ is the input and

$$A = \begin{bmatrix} -a_{n-1} & \\ \vdots & I_{n-1} \\ \vdots & \\ -a_0 & 0_{n-1}^T \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T \tag{3}$$

with $0_{n-1} = [0, \ldots, 0]^T \in \mathbb{R}^{n-1}$. The unknown sinusoidal disturbance $\nu(t) \in \mathbb{R}$ is given by

$$\nu(t) = d + \sum_{i=1}^{q} g_i \sin(w_i t + \phi_i) \tag{4}$$

where $d, g_i, w_i, \phi_i \in \mathbb{R}$ are unknown. The output delay can be modelled by the following first-order hyperbolic PDE

$$\partial_t y(x,t) = \partial_x y(x,t), \quad x \in \big[0, D_y\big] \tag{5}$$

$$y(D,t) = CX(t). \tag{6}$$

The solution to these equations is $y(x,t) = CX(t+x-D)$.

The sinusoidal disturbance $\nu(t)$ can be represented as the output of a linear exosystem,

$$\dot{W}(t) = SW(t), \tag{7}$$
$$\nu(t) = h_\nu^T W(t), \tag{8}$$

where the state $W(t) \in \mathbb{R}^{2q+1}$. The matrix S comprises the unknown frequency of the sinusoidal disturbance $\nu(t)$. Constant bias term d, amplitude g_i and phase ϕ_i are determined by initial condition of (7), are thus unknown. Without loss of generality, one can choose output vector h_ν^T such that (h_ν^T, S) becomes observable pair.

The disturbance $\nu(t)$ is not measured. The output $Y(t)$ is the only available measurement. Regarding the plant (1)–(2) and the exosystem (7)–(8), we make the following assumptions:

Assumption 1 The frequencies of the disturbance are distinct, $\omega_i \neq \omega_j$ for $i \neq j$, and the number of the distinct frequencies q is known.

Assumption 2 The bias $d \neq 0$ and amplitude $g_i \neq 0$ for all $i \in \{1, \dots, q\}$.

Our ultimate goal is to design an observer achieving accurate online estimation of state $X(t)$ as well as the disturbance $\nu(t)$. Using the observer states, we design a controller stabilizing the equilibrium of the closed loop system. Moreover, we aim the state $X(t)$ to converge to zero as $t \to \infty$ in the presence of simultaneous input-output delay and unmeasured sinusoidal disturbance.

III. Disturbance Representation

Our main interest here is a preparation for disturbance observer design which is presented in the next section. Firstly, we employ a filter introduced in [7] for systems under no disturbance effect. However, because of the unknown disturbance in our system, we show that the error between the system states and the filter states is driven by unknown sinusoidal terms. Main motivation of obtaining this error is to use it in disturbance representation and then, disturbance parametrization.

Inspiring [7], we propose the following filter

$$\dot{\hat{X}}_d(t) = A\hat{X}_d(t) + BU(t-D_u) + e^{AD_y}L(Y(t) - \hat{y}_d(0,t)), \tag{9}$$
$$\partial_t \hat{y}_d(x,t) = \partial_x \hat{y}_d(x,t) + Ce^{Ax}L\big(Y(t) - \hat{y}_d(0,t)\big), \tag{10}$$
$$\hat{y}_d(D_y,t) = C\hat{X}_d(t), \tag{11}$$

where L is chosen such that $A - LC$ is Hurwitz. Since the pair (A, C) is observable, there exists an L such that this condition is satisfied. The error is given as follows,

$$\tilde{X}_d(t) = \hat{X}_d(t) - X(t), \tag{12}$$
$$\dot{\tilde{X}}_d(t) = A\tilde{X}_d - e^{AD_y}L\tilde{y}_d(0,t) - B\nu(t), \tag{13}$$
$$\partial_t \tilde{y}_d(x,t) = \partial_x \tilde{y}_d(x,t) - Ce^{Ax}L\tilde{y}_d(0,t), \tag{14}$$
$$\partial_t \tilde{y}_d(D_y,t) = C\tilde{X}_d(t). \tag{15}$$

The following transformation

$$\tilde{w}(x,t) = \tilde{y}_d(x,t) - Ce^{A(x-D_y)}\tilde{X}_d(t) \tag{16}$$

transforms (12), (13) into the form of

$$\dot{\tilde{X}}_d(t) = A_e\tilde{X}_d(t) - e^{AD_y}L\tilde{w}(0,t) - B\nu(t), \tag{17}$$
$$\partial_t \tilde{w}(x,t) = \partial_x \tilde{w}(x,t) + Ce^{A(x-D_y)}B\nu(t), \tag{18}$$
$$\tilde{w}(D_y,t) = 0, \tag{19}$$

where $A_e = A - e^{AD_y}LCe^{-AD_y}$. By using similarity transformation e^{AD_y} and noting that $A - LC$ is Hurwitz, it can be proved that A_e is Hurwitz.

If there is no disturbance in the system as it is shown in [7], the error $\tilde{X}_d(t)$ converges to 0 as $t \to \infty$. However, $\tilde{X}_d(t)$ is driven by $\nu(t)$ and $\tilde{w}(0,t)$ as seen in (17). In Lemma 1, we show that $\tilde{\omega}(0,t)$ can be expressed as a sum of sinusoidal signals whose frequencies are same as $\nu(t)$.

Lemma 1: The signal $\tilde{w}(0,t)$ can be expressed in the following form

$$\tilde{w}(0,t) = \bar{d} + \sum_{i=1}^{q} \bar{g}_i \sin(w_i t + \bar{\phi}_i) \tag{20}$$

where $\bar{g}_i = \sqrt{(\bar{g}_i^s)^2 + (\bar{g}_i^c)^2}, \bar{\phi}_i = \phi_i + \arctan(\frac{\bar{g}_i^c}{\bar{g}_i^s}), \bar{d} = \Big(C\int_0^{D_y} e^{A(\xi-D_y)}Bd\xi\Big)d$ with

$$\bar{g}_i^s = \Big(C\int_0^{D_y} e^{A(\xi-D_y)}B\cos(w_i\xi)d\xi\Big)g_i, \tag{21}$$
$$\bar{g}_i^c = -\Big(C\int_0^{D_y} e^{A(\xi-D_y)}B\sin(w_i\xi)d\xi\Big)g_i. \tag{22}$$

Proof Solution of $\tilde{w}(x,t)$ with Laplace Transformation method gives us

$$\tilde{w}(x,t) = \int_x^{D_y} Ce^{A(\xi-D_y)}B\nu(t+x-\xi)d\xi. \tag{23}$$

Using necessary trigonometric formulas, $\nu(t-\xi)$ can be written as

$$\begin{aligned}\nu(t-\xi) =& d + \sum_{i=1}^{q} g_i \sin(w_i t + \phi_i)\cos(w_i\xi) \\ & - g_i\cos(w_i t + \phi_i)\sin(w_i\xi).\end{aligned} \tag{24}$$

Substituting (24) into (23), writing for $x = 0$ and using trigonometric identities, we get (20). ■

From Lemma 1, we show that the signal $\tilde{w}(0,t)$ excites at the same frequencies with the disturbance $\nu(t)$. From (23), we prove the boundedness of $\tilde{\omega}(0,t)$ with the boundedness of $\nu(t)$. Considering the boundedness of $\tilde{w}(0,t)$ and $\nu(t)$ and noting that A_e is Hurwitz, from (17), we can conclude that $\tilde{X}_d(t)$ is bounded and driven by the unknown sinusoidal terms.

Second step of this section deals with unknown sinusoidal terms in the output dynamics. For this aim, we write the derivative of the output using (3) as the following

$$\begin{aligned}\dot{Y}(t) =& -a_{n-1}X_1(t-D_y) + X_2(t-D_y) + b_1\nu(t-D_y) \\ & + b_1U(t-D_u-D_y).\end{aligned} \tag{25}$$

Using (12), we get

$$X_2(t) = \hat{X}_{d2}(t) - \tilde{X}_{d2}(t) \tag{26}$$

where $\hat{X}_{d2}(t) = B_2^T\hat{X}_d(t) \in \mathbb{R}$ and $\tilde{X}_{d2}(t) = B_2^T\tilde{X}_d(t) \in$

$\mathbb{R}$. Substituting (26) into (25), we get

$$\begin{aligned}\dot{Y}(t) =& - a_{n-1}X_1(t-D_y) + \hat{X}_{d2}(t-D_y) - \tilde{X}_{d2}(t-D_y) \\ &+ b_1U(t-D_u-D_y) + b_1\nu(t-D_y). \end{aligned} \tag{27}$$

In (27), the output dynamics consists of output signal, known filter state, input and two unknown terms $\tilde{X}_{d2}(t-D_y)$ and $\nu(t-D_y)$. Therefore, we need to obtain the response of $\tilde{X}_{d2}(t)$ from (17), so we represent signal $\tilde{X}_{d2}(t)$ as summation of the steady state $\tilde{X}^{ss}_{d2}(t)$ and transient responses $\tilde{X}^{in}_{d2}(t)$,

$$\tilde{X}_{d2}(t) = \tilde{X}^{ss}_{d2}(t) + \tilde{X}^{in}_{d2}(t). \tag{28}$$

The states $\tilde{X}^{ss}_{d2}(t), \tilde{X}^{in}_{d2}(t)$ are given in the proof of next lemma. Substituting (28) with the delay D_y into (27), output dynamics is rewritten as

$$\begin{aligned}\dot{Y}(t) =& - a_{n-1}X_1(t-D_y) + \hat{X}_{d2}(t-D_y) + \epsilon_x(t-D_y) \\ &- \tilde{X}^{in}_{d2}(t-D_y) + b_1U(t-D_u-D_y) \end{aligned} \tag{29}$$

where

$$\epsilon_x(t) = b_1\nu(t) - \tilde{X}^{ss}_{d2}(t). \tag{30}$$

The representation of $\epsilon_x(t)$ is given in the following lemma.

Lemma 2: The signal $\epsilon_x(t)$ can be represented in the form

$$\epsilon_x(t) = d_\epsilon + \sum_{i=1}^{q} g^s_{\epsilon_i}\sin\left(w_it+\phi_i\right) + g^c_{\epsilon_i}\cos\left(w_it+\phi_i\right) \tag{31}$$

where $d_\epsilon = b_1d - d_{\tilde{x}}, g^s_{\epsilon_i} = b_1g_i - g^s_{\tilde{x}_i}, g^c_{\epsilon_i} = -g^c_{\tilde{x}_i}$ with

$$d_{\tilde{x}} =|G_{d_1}(0)|d + |G_{d_2}(0)|\bar{d}, \tag{32}$$

$$\begin{aligned}g^s_{\tilde{x}_i} =&|G_{d_2}(jw_i)|\Big(\bar{g}^s_i\cos(\underline{/G_{d_2}(jw_i)}) - \bar{g}^c_i\sin(\underline{/G_{d_2}(jw_i)})\Big) \\ &+ |G_{d_1}(jw_i)|g_i\cos(\underline{/G_{d_1}(jw_i)}), \end{aligned} \tag{33}$$

$$\begin{aligned}g^c_{\tilde{x}_i} =&|G_{d_2}(jw_i)|\Big(\bar{g}^s_i\sin(\underline{/G_{d_2}(jw_i)}) + \bar{g}^c_i\cos(\underline{/G_{d_2}(jw_i)})\Big) \\ &+ |G_{d_1}(jw_i)|g_i\sin(\underline{/G_{d_1}(jw_i)}), \end{aligned} \tag{34}$$

and

$$G_{d_1}(s) = - B_2^T(sI-A_e)^{-1}B, \tag{35}$$

$$G_{d_2}(s) = - B_2^T(sI-A_e)^{-1}(e^{AD_y}L). \tag{36}$$

Proof The states $\tilde{X}^{ss}_{d2}(t)$ and $\tilde{X}^{in}_{d2}(t)$ in (28) are given by

$$\begin{aligned}\tilde{X}^{ss}_{d2}(t) =&|G_{d_1}(0)|d + \sum_{i=1}^{q}|G_{d_1}(jw_i)|g_i\Big(\cos(\underline{/G_{d_1}(jw_i)}) \\ &\times\sin(w_it+\phi_i) + \sin(\underline{/G_{d_1}(jw_i)})\cos(w_it+\phi_i)\Big) \\ &+ |G_{d_2}(0)|\bar{d} + \sum_{i=1}^{q}|G_{d_2}(jw_i)|\bar{g}^s_i\Big(\cos(\underline{/G_{d_2}(jw_i)}) \\ &\times\sin(w_it+\phi_i) + \sin(\underline{/G_{d_2}(jw_i)})\cos(w_it+\phi_i)\Big) \\ &+ \sum_{i=1}^{q}|G_{d_2}(jw_i)|\bar{g}^c_i\Big(\cos(\underline{/G_{d_2}(jw_i)})\times \\ &\cos(w_it+\phi_i) - \sin(\underline{/G_{d_2}(jw_i)})\sin(w_it+\phi_i)\Big), \end{aligned} \tag{37}$$

$$\tilde{X}^{in}_{d2}(t) =B_2^T\tilde{X}^{in}_d(t), \tag{38}$$

with (35), (36) and

$$\dot{\tilde{X}}^{in}_d(t) = A_e\tilde{X}^{in}_d(t), \quad \tilde{X}^{in}_d(0) = \tilde{X}_d(0). \tag{39}$$

We can rewrite the expression (37) in a more compact form

$$\tilde{X}^{ss}_{d2}(t) = d_{\tilde{x}} + \sum_{i=1}^{q} g^s_{\tilde{x}_i}\sin(w_it+\phi_i) + \sum_{i=1}^{q} g^c_{\tilde{x}_i}\cos(w_it+\phi_i) \tag{40}$$

where $d_{\tilde{x}}, g^s_{\tilde{x}_i}$ and $g^c_{\tilde{x}_i}$ are given in (32)–(34). Substituting (40) and (4) into (30), we get (31). ■

From (31), it is shown that $\epsilon_x(t)$ is a sinusoidal signal with a bias term, which excites at the same frequency as the disturbance $\nu(t)$.

In the final step of this section, we write $\epsilon_x(t)$ signal as the input of a linear stable system whose system state is unknown and estimated in the next section. Considering the exosystem (7)–(8), the signal $\epsilon_x(t)$ can be represented as $\epsilon_x(t) = h_\epsilon^TW(t)$ where (h_ϵ, S) is an observable pair. Let $G \in \mathbb{R}^{(2q+1)\times(2q+1)}$ be a Hurwitz matrix and let (G, l) be a controllable pair. Since the spectra of S and G are disjoint, this guarantees that the unique solution $M \in \mathbb{R}^{(2q+1)\times(2q+1)}$ of the Sylvester equation

$$MS - GM = lh_\epsilon^T \tag{41}$$

is invertible [12]. The change of coordinates

$$Z(t) = MW(t) \tag{42}$$

transforms the exosystem (7), (8) into the form

$$\dot{Z}(t) = GZ(t) + l\epsilon_x(t) \tag{43}$$

where $\epsilon_x(t) = h_\epsilon^TM^{-1}Z(t)$. Using (42), we rewrite (8) as

$$\nu(t) = h_\nu^TM^{-1}Z(t). \tag{44}$$

Using (44), we write the disturbance in terms of the unknown constant vectors and the unknown $Z(t)$ signal. In the next section, we design a disturbance observer so that we cancel the unknown sinusoidal effect with an observer based adaptive controller.

IV. Parametrization of Disturbance

In this section, since we cannot estimate $Z(t)$ due to the delay in the output, we design filters to estimate $Z(t-D_y)$ signal. Then, we represent the unknown disturbance by using the filter states. The following lemma presents the estimation of $Z(t-D_y)$ signal and establishes the properties of the filters.

Lemma 3: The unmeasured signal $Z(t-D_y)$ can be represented as

$$Z(t-D_y) = \Xi(t) + \epsilon_\nu(t) \tag{45}$$

with

$$\Xi(t) =\eta(t) + lY(t), \tag{46}$$

$$\begin{aligned}\dot{\eta}(t) =&G\Xi(t) - l\Big(\hat{X}_{d2}(t-D_y) - a_{n-1}X_1(t-D_y) \\ &+ b_1U(t-D_u-D_y)\Big). \end{aligned} \tag{47}$$

The estimation error defined by $\epsilon_\nu = Z(t-D_y) - \Xi(t)$ obeys the equation

$$\dot{\epsilon}_\nu(t) = G\epsilon_\nu(t) + lB_2^T e^{-A_e D_y}\tilde{X}_d^{in}(t). \tag{48}$$

Proof Since (43) is linear and G is a Hurwitz matrix, we can write

$$\dot{Z}(t-D_y) = GZ(t-D_y) + l\epsilon_x(t-D_y). \tag{49}$$

Taking derivative of (45) in view of (29), (46), (47), (49) and recalling (38), we get

$$\dot{\epsilon}_\nu(t) = G\epsilon_\nu(t) + lB_2^T\tilde{X}_d^{in}(t-D_y). \tag{50}$$

Solution of (39) gives $\tilde{X}_d^{in}(t) = e^{A_e t}\tilde{X}_d^{in}(0)$, the delayed signal is written by

$$\begin{aligned}\tilde{X}_d^{in}(t-D_y) =& e^{A_e(t-D_y)}\tilde{X}_d^{in}(0) = e^{-A_eD_y}e^{A_et}\tilde{X}_d^{in}(0)\\ =& e^{-A_eD_y}\tilde{X}_d^{in}(t).\end{aligned} \tag{51}$$

Substitution of (51) into (50) yields (48). ■

In the following lemma, we write the disturbance signal $\nu(t)$ by using (45).

Lemma 4: The unknown disturbance $\nu(t)$ can be represented in the form

$$\nu(t) = \theta_1^T\Xi(t-D_u) + \theta_2^T\epsilon_\nu(t) + \theta_3^T\tilde{X}_d^{in}(t) \tag{52}$$

where $\theta_1^T = h_\nu^T e^{S(D_u+D_y)}M^{-1}$, $\theta_2^T = \theta_1^T e^{-GD_u}$, $\theta_3^T = -\theta_2^T\int_0^{D_u} e^{G\tau}lB_2^T e^{-A_e(\tau+D_u)}d\tau$ are unknown.

Proof Solving (7), we get $W(t) = e^{St}W(0)$. The delayed signal is given by

$$\begin{aligned}W(t-D_u-D_y) &= e^{S(t-D_u-D_y)}W(0)\\ &= e^{-S(D_u+D_y)}e^{St}W(0)\\ &= e^{-S(D_u+D_y)}W(t).\end{aligned} \tag{53}$$

Using (42) and (53), $Z(t-D)$ is expressed as follows

$$\begin{aligned}Z(t-D_u-D_y) &= MW(t-D_u-D_y)\\ &= Me^{-S(D_u+D_y)}W(t).\end{aligned} \tag{54}$$

Substitution of $W(t) = M^{-1}Z(t)$ into (54) and writing for $Z(t)$ gives

$$Z(t) = Me^{S(D_u+D_y)}M^{-1}Z(t-D_u-D_y). \tag{55}$$

Substituting (55) into (44), we obtain

$$\nu(t) = h_\nu^T e^{S(D_u+D_y)}M^{-1}Z(t-D_u-D_y). \tag{56}$$

Representing $Z(t-D_y-D_u)$ by using (45) and substituting it into (56), we get (52). ■

Lemma 4 gives us the representation of the unknown disturbance as the multiplication of unknown constant vector with a known regressor and two exponentially vanishing terms. This method converts the disturbance cancellation problem to an adaptive control problem. In the next section, we propose an adaptive controller together with a disturbance and state observer.

Remark 1: Our state observer design given in the next section is based on the idea introduced in [10]. Contrary to [10] where a parametric uncertainty is considered, here we consider an unknown sinusoidal disturbance as given in (4). Using Lemmas 1–4, we formulate the problem that is similar to one given in [10].

V. Observer Based Adaptive Controller Design

Substituting the representation of the disturbance given in (52) into (1), we get

$$\begin{aligned}\dot{X}(t) =& AX(t) + B\Big(u(0,t) + \theta_1^T\xi(0,t) + \theta_2^T\epsilon_\nu(t),\\ &+ \theta_3^T\tilde{X}_d^{in}(t)\Big)\end{aligned} \tag{57}$$

$$\partial_t u(x,t) = \partial_x u(x,t), \quad x\in[0,D_u] \tag{58}$$

$$u(D_u,t) = U(t) \tag{59}$$

$$\partial_t \xi(x,t) = \partial_x \xi(x,t), \quad x\in[0,D_u] \tag{60}$$

$$\xi(D_u,t) = \Xi(t) \tag{61}$$

The solutions of the transport PDEs are $u(x,t) = U(x+t-D_u), \xi(x,t) = \Xi(x+t-D_u)$. Considering (57)–(61), (5)–(6) and following the idea given in [10], we design the following observer based adaptive controller

$$\begin{aligned}U(t) =& Ke^{AD_u}\hat{X}_s(t) - \hat{\theta}_1^T(t)\Xi(t) + K\int_0^{D_u} e^{A(D_u-y)}B\\ &\times\Big(u(y,t) + \hat{\theta}_1^T(t)\xi(y,t)\Big)dy\end{aligned} \tag{62}$$

where the state observer is

$$\begin{aligned}\dot{\hat{X}}_s(t) =& A\hat{X}_s(t) + BU(t-D_u) + B\Xi^T(t-D_u)\hat{\theta}_1(t)\\ &+ e^{AD_y}L(Y(t)-\hat{y}_s(0,t)) + \lambda_0(t)\dot{\hat{\theta}}_1(t),\end{aligned} \tag{63}$$

$$\begin{aligned}\partial_t\hat{y}_s(x,t) =& \partial_x\hat{y}_s(x,t) + Ce^{Ax}L(Y(t)-\hat{y}_s(0,t))\\ &+ \Big(\lambda_1(x,t) + Ce^{(x-D_y)A}\lambda_0(t)\Big)\dot{\hat{\theta}}_1(t),\end{aligned} \tag{64}$$

$$\hat{y}_s(D_y,t) = C\hat{X}_s(t), \tag{65}$$

with the auxiliary states

$$\dot{\lambda}_0(t) = A_e\lambda_0(t) + B\Xi^T(t-D_u) - e^{AD_y}L\lambda_1(0,t), \tag{66}$$

$$\partial_t\lambda_1(x,t) = \partial_x\lambda_1(x,t) - Ce^{(x-D_y)A}B\Xi^T(t-D_u) \tag{67}$$

$$\lambda_1(D_y,t) = 0 \in \mathbb{R}^{1\times(2q+1)},\ \lambda_1(x,0) = 0. \tag{68}$$

The control gain $K\in\mathbb{R}^{1\times n}$ is chosen such that $(A+BK)$ becomes Hurwitz. The least-squares parameter update law is given by

$$\dot{\hat{\theta}}_1(t) = -\rho R(t)\Lambda(t)\tilde{y}_s(0,t), \tag{69}$$

$$\dot{R}(t) = R(t) - R(t)\Lambda(t)\Lambda(t)^T R(t), \tag{70}$$

$$\Lambda(t) = (Ce^{-AD_y}\lambda_0(t) + \lambda_1(0,t))^T, \tag{71}$$

where $\rho > 0$. We now state the main theorem and then prove it in the next section.

Theorem 1: Consider the closed-loop system consisting of the plant (1), (2), the unknown disturbance (4), the filters (9)–(11), (46), (47), the control law (62)–(68) and the update law (69)–(71). Under Assumptions 1-2, the following holds

(a) For any $\alpha > 0$, $\Upsilon(t) \le e^{-\alpha t}\Upsilon(0) \quad \forall t\ge 0$ where

$$\begin{aligned}\Upsilon(t) =& \Big|\tilde{X}_s(t) - \lambda_0(t)\tilde{\theta}_1(t)\Big|^2 + \int_0^{D_y}(\tilde{y}_s(x,t)\\ &- Ce^{(x-D_y)A}\tilde{X}_s(t) - \lambda_1(x,t)\tilde{\theta}_1(t))^2dx\\ &+\Big|\tilde{y}_s(0,t) - Ce^{-AD_y}\tilde{X}_s(t) - \lambda_1(0,t)\tilde{\theta}_1(t)\Big|^2\\ &+\Big|\Lambda^T(t)\tilde{\theta}_1(t)\Big|^2 + \Big|\epsilon_\nu(t)\Big|^2 + \Big|\tilde{X}_d^{in}(t)\Big|^2,\end{aligned}$$

(b) $\left|\hat{X}_s(t) - X(t)\right|^2$, $\left|X(t)\right|^2$ and $\left|\hat{y}_s(x,t) - y(x,t)\right|^2 \to 0$ as $t \to \infty$. Moreover, we achieve perfect estimation of the unknown disturbance, i.e. $\hat{\theta}_1^T(t - D_u)\Xi(t - D_u) - \nu(t) \to 0$ as $t \to \infty$.

VI. Stability Proof

The state observer error system is given by

$$\begin{aligned}\dot{\tilde{X}}_s(t) =& A\tilde{X}_s(t) + B\Xi^T(t - D_u)\tilde{\theta}_1(t) + e^{AD_y}L(Y(t) \\ & - \hat{y}_s(0,t)) + \lambda_0(t)\dot{\hat{\theta}}_1(t) - \epsilon_\theta(t), \end{aligned} \tag{72}$$

$$\begin{aligned}\partial_t\tilde{y}_s(x,t) =& \partial_x\tilde{y}_s(x,t) + Ce^{Ax}L(Y(t) - \hat{y}_s(0,t)) \\ & + \left(\lambda_1(x,t) + Ce^{(x-D_y)A}\lambda_0(t)\right)\dot{\hat{\theta}}_1(t), \end{aligned} \tag{73}$$

$$\tilde{y}_s(D_y,t) = C\tilde{X}_s(t). \tag{74}$$

where $\epsilon_\theta(t) = B\theta_2^T\epsilon_\nu(t) + B\theta_3^T\tilde{X}_d^{in}(t)$. Consider the following backstepping transformation proposed in [10]

$$\Phi(t) = \tilde{X}_s(t) - \lambda_0(t)\tilde{\theta}_1(t), \tag{75}$$

$$\varepsilon(x,t) = \tilde{y}_s(x,t) - Ce^{(x-D_y)A}\tilde{X}_s(t) - \lambda_1(x,t)\tilde{\theta}_1(t) \tag{76}$$

This transformation yields a closed-loop system in the following form

$$\dot{\Phi}(t) = A_e\Phi(t) - e^{AD_y}L\varepsilon(0,t) - \epsilon_\theta(t), \tag{77}$$

$$\partial_t\varepsilon(x,t) = \partial_x\varepsilon(x,t) + Ce^{(x-D_y)A}\epsilon_\theta(t), \tag{78}$$

$$\varepsilon(D,t) = 0. \tag{79}$$

In order to ensure the exponential convergence of the state observer, we need to establish persistent excitation of signal $\Lambda(t)$. The following lemma states this property.

Lemma 5: The signal vector $\Lambda(t)$ is persistently exciting (PE) and there exist $\delta_0, \delta_1, T_0 > 0$ such that the following holds

$$\delta_1 I \geq \int_t^{t+T_0} \Lambda(\tau)\Lambda^T(\tau)d\tau \geq \delta_0 I \quad \forall t \geq 0. \tag{80}$$

Proof Considering Assumptions 1-2, (17), (20) and noting that A_e is Hurwitz, we can state that the signal $\tilde{X}_d(t)$ is sufficiently rich in the order of $2q+1$. From (30), it follows that $\epsilon_x(t)$ is also sufficiently rich in the order of $2q+1$. Considering this fact and controllable pair (G,l) and noting that $Z(t) \in \mathbb{R}^{2q+1}$, we can show that $Z(t)$ is PE signal from (43). This implies that $\Xi(t)$ satisfies the condition of persistent excitation from (45) in view of (50). Since $\lambda_1(0,t)$ and $\lambda_0(t)$ are dependent on $\Xi(t)$ signal as seen in (66)–(67), we ensure that $\lambda_1(0,t)$ and $\lambda_0(t)$ are bounded and sufficiently rich in the order of $2q+1$. As seen in (71), the signal $\Lambda(t)$ consists of two bounded PE signals $C^{-AD_y}\lambda_0(t), \lambda_1(0,t)$. Therefore, the signal $\Lambda(t)$ only depends on the signal $\Xi(t)$ and satisfies (80). ■

As stated in [11], if persistent excitation condition is satisfied, then covariance matrix inverse $R^{-1}(t)$ is a bounded positive definite matrix. Its dynamics is given by

$$\frac{dR^{-1}}{dt} = -R^{-1} + \Lambda(t)\Lambda(t)^T. \tag{81}$$

This guarantees that $\rho_l \leq R^{-1}(t) \leq \rho_u$, $\forall t \geq 0$ for lower and upper bounds $\rho_l, \rho_u > 0$.

Proof of Theorem 1: Consider the following Lyapunov function

$$\begin{aligned}V =& \mu_1\Phi^T P\Phi + \mu_2\int_0^{D_y}(1+x)\varepsilon(x,t)^2dx \\ & + \mu_3\tilde{\theta}_1^T R^{-1}\tilde{\theta}_1 + q_1\epsilon_v^T P_G\epsilon_v + q_2(\tilde{X}_d^{in})^T P\tilde{X}_d^{in}, \end{aligned} \tag{82}$$

where

$$\mu_3 = \frac{\beta_3}{\rho(2-\zeta_1) - 1} \tag{83}$$

$$\mu_1 = (\beta_1 + \frac{\mu_3\rho}{2\zeta_1}(Ce^{-AD_y}e^{-(AD_y)^T}C^T)(\mu_P - 3)^{-1} \tag{84}$$

$$\mu_2 = \mu_1\lambda_{max}(Pe^{AD_y}LL^T(e^{AD_y})^T P) + \frac{\mu_3\rho}{2\zeta_1} + \beta_2 \tag{85}$$

$$\begin{aligned}q_1 =& (\beta_4 + \mu_1\lambda_{max}(PB\theta_2^T\theta_2 B^T P) + \frac{\mu_2}{2\zeta_2}\lambda_{max}(\theta_2\theta_2^T)) \\ & \times (\mu_G - 1)^{-1} \end{aligned} \tag{86}$$

$$\begin{aligned}q_2 =& (q_1\lambda_{max}\left(P_G l B_2^T e^{-A_e D_y}e^{-(A_e D_y)^T}B_2 l^T P_G\right) \\ & + \mu_1\lambda_{max}(PB\theta_3^T\theta_3 B^T P) + \frac{\mu_2}{2\zeta_2}\lambda_{max}(\theta_3\theta_3^T) \\ & + \beta_5)(\mu_P)^{-1}, \end{aligned} \tag{87}$$

with $PA_e + A_e^T P = -\mu_P I$, $G^T P_G + P_G G = -\mu_G I$ and $\zeta_2 = (1-\beta_6)/(\int_0^{D_y}(1+x)Ce^{(x-D_y)A}Bdx)^2$ for design parameters β_1, β_2, β_3, β_4, β_5 and $\zeta_1 > 0$. Taking the time derivative of (82) by virtue of (39), (50), (77)–(79), (81), (83)–(87) and using the Young's inequality for the cross terms, we obtain

$$\begin{aligned}\dot{V}(t) \leq & -\beta_1\Phi^T\Phi - \mu_3\tilde{\theta}_1^T R^{-1}\tilde{\theta}_1 - \beta_2\varepsilon^2(0,t) - \beta_3\left|\Lambda^T\tilde{\theta}_1\right|^2 \\ & - \beta_4\epsilon_\nu^T\epsilon_\nu - \beta_5(\tilde{X}_d^{in})^T\tilde{X}_d^{in} \\ & - \frac{\beta_6}{1+D_y}\int_0^{D_y}(1+x)\varepsilon^2(x,t)dx. \end{aligned} \tag{88}$$

Considering (82) and noting that $|\Lambda|$ is a bounded and PE signal from Lemma 5, (88) can be written as

$$\dot{V}(t) \leq -\alpha V(t) \tag{89}$$

for $\alpha > 0$.

Recalling (75), (76), from (89), we prove part (a) of Theorem 1. Recalling the boundedness of $\nu(t)$ and $\epsilon_\nu(t)$, from (52), $\Xi(t)$ is bounded. Recalling that $\lambda_0(t)$ is bounded and $\tilde{\theta}_1(t), \Phi(t) \to 0$ as $t \to \infty$, from (75), we can show that $\tilde{X}_s(t)$ is bounded and goes to zero as $t \to \infty$. In view of boundedness of $\lambda_1(0,t)$, this fact implies that, from (76), $\tilde{y}_s(x,t)$ is bounded and goes to zero as $t \to \infty$. We obtain the boundedness of $U(t)$ and $\dot{\hat{\theta}}_1(t)$ from (62) and (69)–(71). Note that input (62) transforms system (57) to

$$\begin{aligned}\dot{X}(t) =& (A + BK)X(t) + B\left(K\tilde{X}_s(t) - \tilde{\theta}_1^T(t)\xi(0,t)\right. \\ & \left. + w_u(0,t) + \theta_2^T\epsilon_\nu(t) + \theta_3^T\tilde{X}_d^{in}(t)\right) \end{aligned} \tag{90}$$

where

$$\begin{aligned}w_u(x,t) =& u(x,t) + \hat{\theta}_1(t)\xi(x,t) - Ke^{Ax}\hat{X}_s(t) - K\int_0^x \\ & \times e^{A(x-y)}B\left(u(y,t) + \hat{\theta}_1^T(t)\xi(y,t)\right)dy \end{aligned} \tag{91}$$

which satisfies

$$\begin{aligned}\partial_t w_u(x,t) =& \partial_x w_u(x,t) + \dot{\hat{\theta}}_1^T(t)\Big(\xi(x,t) - \int_0^x \xi(y,t) \\ & \times Ke^{A(x-y)}Bdy\Big) + Ke^{A(x-D_y)}L\tilde{y}_s(0,t) \\ & - Ke^{Ax}\lambda_0(t)\dot{\hat{\theta}}_1(t), \end{aligned} \tag{92}$$

$$w_u(D_u,t) = 0. \tag{93}$$

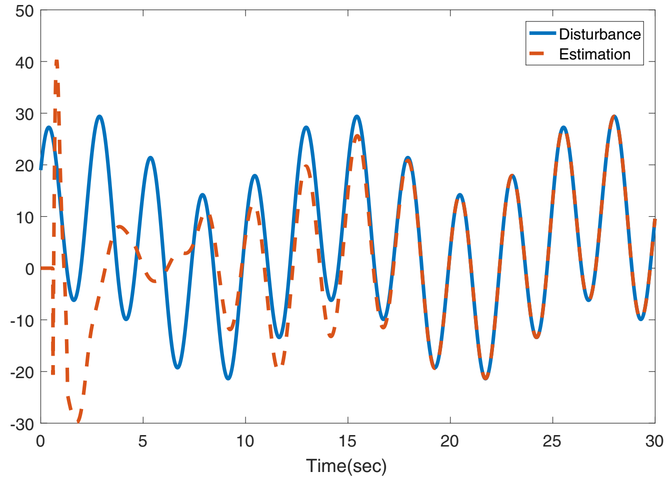
The solution of (92)–(93) is given by

$$\begin{aligned} w_u(x,t) = \int_x^{D_u} & \Big[\dot{\hat{\theta}}_1(t+x-v)\Big(\Xi(t+x-D_u) \\ & - \int_0^v \Xi(y+t-D_u)K^{A(v-y)}Bdy\Big) \\ & + Ke^{A(v-D_u)}L\tilde{y}_s(0,t+x-v) \\ & - Ke^{Av}\lambda_0(t+x-v)\dot{\hat{\theta}}_1(t+x-v)\Big]dv. \end{aligned} \tag{94}$$

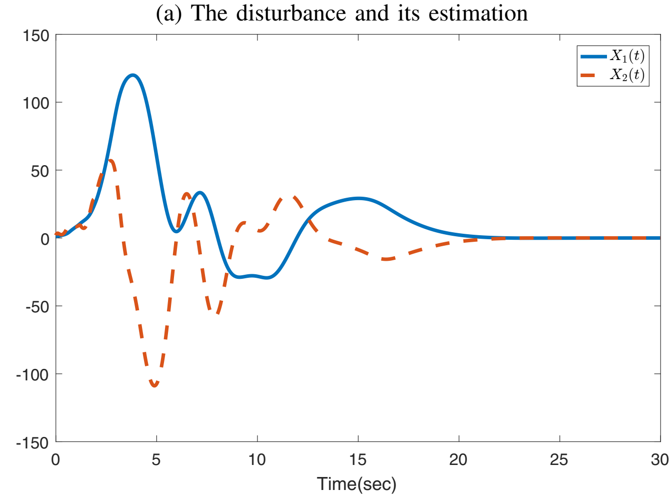
From (94), we prove that $w_u(x,t)$ is bounded and converges to zero as $t \to \infty$ since $\dot{\hat{\theta}}_1^T(t), \Xi(t), \tilde{y}_s(0,t), \lambda_0(t)$ are bounded and $\dot{\hat{\theta}}_1^T(t), \tilde{y}_s(0,t) \to 0$ as $t \to \infty$. Considering this fact and noting that $(A+BK)$ is Hurwitz, from (90), we obtain that $X(t)$ is bounded and goes to zero as $t \to \infty$. Moreover, noting that $\tilde{\theta}_1(t) \to 0$ as $t \to \infty$, from Lemma 4, we obtain that $\hat{\theta}_1^T(t-D_u)\Xi(t-D_u) - \nu(t) \to 0$ exponentially, i.e. we achieve perfect estimation of the unknown disturbance. This proves part (b) of Theorem 1. ■

VII. Numerical Simulation

We test the performance of the controller with an unstable second-order system with $A = \begin{bmatrix} 0.3 & 1 \\ 0.2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, the unknown disturbance $\nu(t) = 4 + 8\sin(0.5t + \pi/6) + 18\sin(2.5t + \pi/5)$, the known input delay $D_u = 0.6$, the known output delay $D_y = 0.7$ and the initial conditions $X(0) = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$. We choose ρ as 1.5 and eigenvalues of $A - LC$ as -2 and -2.5. We choose the controllable pair (G,l) as $l = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$, $G = \begin{bmatrix} 0_4 & I_4 \\ 0_5^T & \end{bmatrix} + l\begin{bmatrix} -19.96 & -54.85 & -60.28 & -33.12 & -9.10 \end{bmatrix}$. The control gain K is chosen such that the eigenvalues of $A+BK$ are -1.5 and -2. Figure 1 shows that $X(t)$ and $\hat{\theta}_1(t-D_u)\Xi(t-D_u) - \nu(t)$ converge to zero as stated in Theorem 1.



(a) The disturbance and its estimation



(b) The system states

Fig. 1: Performance of the controller and the disturbance observer for an unstable system with 0.6 second input delay and 0.7 second output delay.

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