Задачи

(1) [illegible]

$\xi \sim R(0, \theta)$

вер. модель (параметрич. модель), $\theta > 0$

$\theta \in \Theta = (0, +\infty)$

Выборка $\vec{X_n}$

$\tilde{\theta}_1 = 2\bar{x} = 2 \cdot \frac{1}{n}\sum_{i=1}^{n} x_i$

$\tilde{\theta}_2 = X_{min}$

$\tilde{\theta}_3 = X_{max}$

$\tilde{\theta}_4 = x_1 + \frac{1}{(n-1)}\sum_{i=2}^{n} x_i$

$\underline{M\xi} = \int_{-\infty}^{\infty} x\, dF(x, \theta) = \int_0^{\theta} x \frac{1}{\theta} dx = \underline{\frac{\theta}{2}}$

$p(x, \theta) = \frac{1}{\theta} 1\{(0, \theta)\}$

$\xi \sim R(a, b)$: $p(x) = \frac{1}{b-a}\{[a, b]\}$

$M\xi^2 = \int_0^{\theta} x^2 \cdot \frac{1}{\theta} dx = \frac{\theta^2}{3}$

$\underline{D\xi} = M\xi^2 - (M\xi)^2 = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \underline{\frac{\theta^2}{12}}$

Парам. модель

несм. $\forall \theta \in \Theta \quad M\tilde{\theta} = \theta$.

сост. $\forall \theta \in \Theta \quad \forall \varepsilon > 0 \quad P(|\tilde{\theta} - \theta| \geq \varepsilon) \xrightarrow[n \to \infty]{} 0$

1) $\tilde{\theta}_1$: $\forall \theta \in \Theta, \theta > 0 \quad M\tilde{\theta}_1 = M(2 \frac{1}{n} \sum_{i=1}^{n} x_i) =$

x_i – незав. сл. велич, $x_i \sim R(0, \theta)$

$= \frac{2}{n} \sum M x_i = \frac{2}{n} n M\xi = \theta$. (перв. оценка <u>несмещ.</u>)

$D\tilde{\theta}_1 = D(2 \frac{1}{n} \sum x_i) = \frac{4}{n^2} \sum D x_i = \frac{4}{n^2} n \cdot \underset{\text{(одина. распр)}}{D\xi} = \frac{\theta^2}{3n} \xrightarrow[n \to \infty]{} 0 \quad \forall \theta > 0$

<u>состоят</u> по дост. услов.

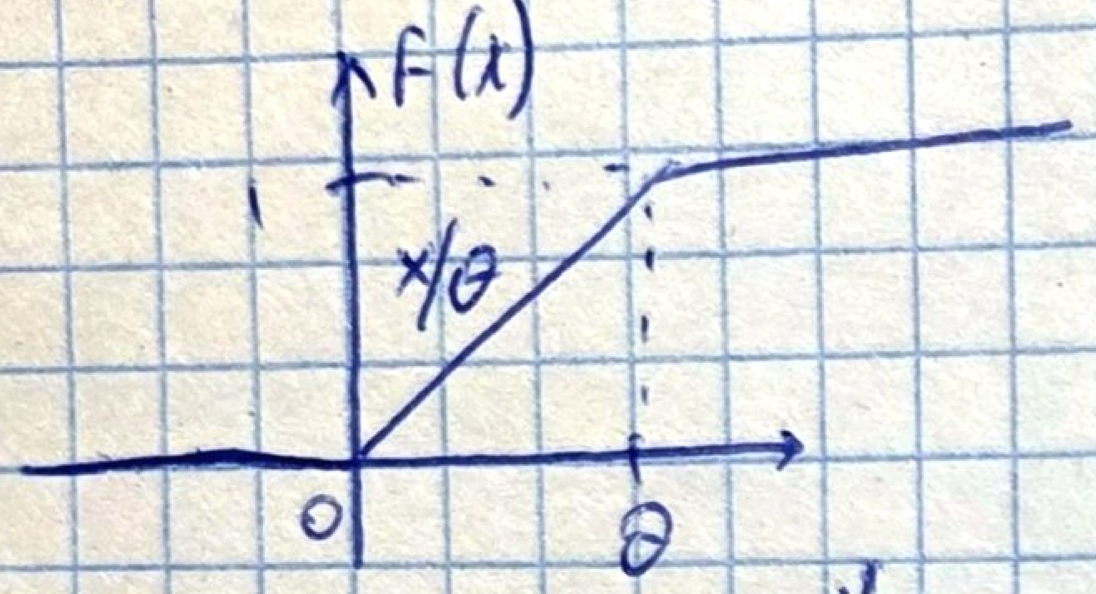
② $\hat{\theta}_2 = x_{min}$

$\forall \theta > 0 \quad M\hat{\theta}_2 = M x_{min}$

$\xi \sim F(x)$

$\xi_{min} \sim \underbrace{1-(1-F(x))^n}_{\varphi(x)}$

$\psi(x) = \varphi'(x) = n(1-F(x))^{n-1}\cdot F'(x) = n\left(1-\frac{x}{\theta}\right)^{n-1}\frac{1}{\theta}\, I\{(0,\theta)\}$



$M x_{min} = \int_0^\theta x\, n \Big(\underbrace{1-\frac{x}{\theta}}_{t}\Big)^{n-1}\frac{1}{\theta}\,dx = \quad x = \theta(1-t)$

$= \int_1^0 n(t-1)\, t^{n-1}\, \theta\, dt =$

$= n\theta\int_0^1 \left(t^{n-1}-t^n\right)dt = n\theta\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{\theta}{n+1}$ – смещенн

$\hat{\theta}_2' = (n+1)x_{min}$

$M\hat{\theta}_2' = (n+1)\,M x_{min} = \theta$ – несмещ

$D\hat{\theta}_2' = D((n+1)x_{min}) = (n+1)^2 D x_{min} \ominus$

$M x^2_{min} = \int_0^\theta x^2 n \Big(\underbrace{1-\frac{x}{\theta}}_{t}\Big)^{n-1}\frac{1}{\theta}\,dx = \int_0^1 \theta^2(1-t)^2\, n\, t^{n-1}\,dt =$

$n\theta^2\int_0^1 \left(t^{n-1}-2t^n+t^{n+1}\right)dt = n\theta^2\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) = \frac{2\theta^2}{(n+1)(n+2)}$

$D x_{min} = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} = \theta^2\left(\frac{2(n+1)-(n+2)}{(n+1)^2(n+2)}\right) =$

$= \frac{n\theta^2}{(n+1)^2(n+2)}$

$\ominus \quad \frac{n\theta^2}{n+2} \underset{n\to\infty}{\not\to} 0$ (ничего не след, дост. усл не раб)

<u>По опр:</u>

$\forall \theta > 0 \quad \forall \varepsilon > 0 \quad P\left(|\hat{\theta}_2' - \theta| \geq \varepsilon\right) \underset{n\to\infty}{\to} 0$

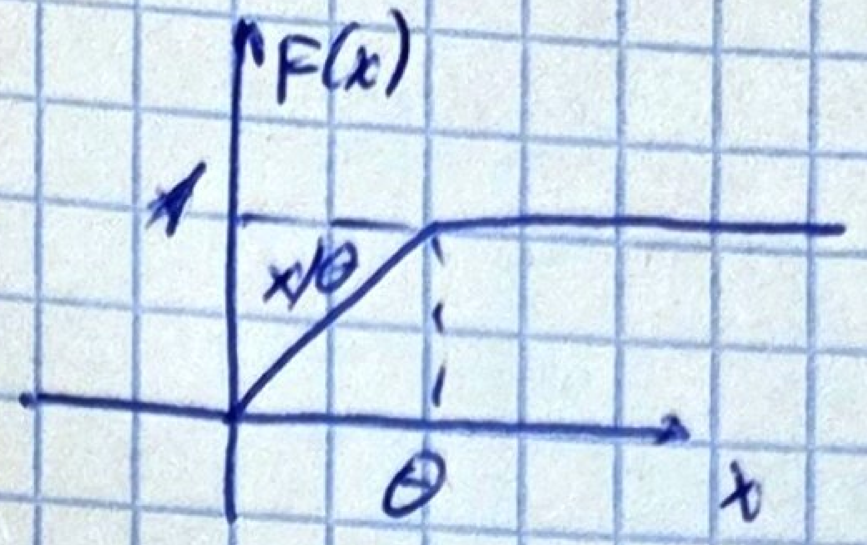
$P\left(|\hat{\theta}_2' - \theta| \geq \varepsilon\right) \geq P\left(\hat{\theta}_2' \geq \theta + \varepsilon\right) =$

$= P((n+1)X_{min} \geq \theta + \varepsilon) = P\left(X_{min} \geq \frac{\theta+\varepsilon}{n+1}\right) =$

$= 1 - \Phi\left(\frac{\theta+\varepsilon}{n+1}\right) \ominus$

$\Phi(x) = 1 - (1 - F(x))^n$

$n \geq N$

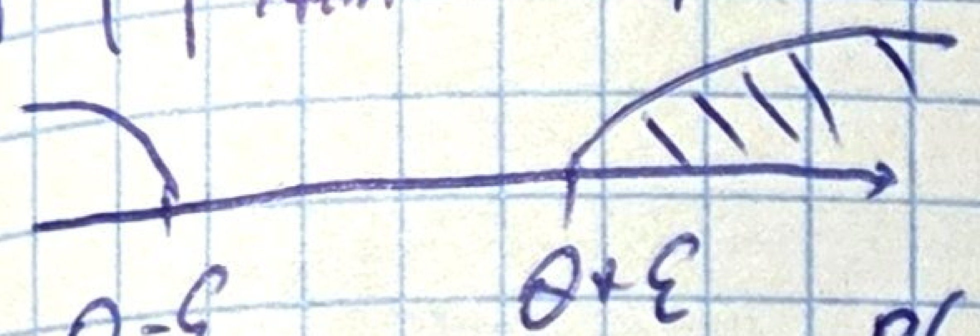


$\ominus\ 1 - \left(1 - \left(1 - \frac{\theta+\varepsilon}{\theta(n+1)}\right)^n\right) =$

$= \left(1 - \frac{\theta+\varepsilon}{\theta(n+1)}\right)^n \xrightarrow[n\to\infty]{} e^{-\frac{\theta+\varepsilon}{\theta}} > 0.$

не явл сост

$P(|X_{min} - \theta| \geq \varepsilon) = P(X_{min} \leq \theta - \varepsilon) = \Phi(\theta - \varepsilon) =$



$P(X_{min} \geq \theta + \varepsilon) = 0$

$= 1 - (1 - F(\theta - \varepsilon))^n = 1 - \left(1 - \frac{\theta-\varepsilon}{\theta}\right)^n = 1 - \left(\frac{\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 1$

$\frac{\varepsilon}{\theta} < 1$

$\exists \varepsilon > 0,\ \exists \theta > 0 : \frac{\varepsilon}{\theta} < 1$ (отриц. опред $\Rightarrow X_{min}$ не явл состоят)

③ $\tilde{\theta}_3 = X_{max}$

$M\tilde{\theta}_3 = M X_{max}$

$X_{max} \sim \underbrace{(F(x))^n}_{\Psi(x)}$

$\psi(x) = \Psi'(x) = n\left(\frac{x}{\theta}\right)^{n-1} \frac{1}{\theta} \{(0,\theta)\}$

$MX_{max} = \int_0^\theta x\, n\left(\frac{x}{\theta}\right)^{n-1} \frac{1}{\theta}\, dx = \frac{n}{\theta^n} \frac{\theta^{n+1}}{n+1} = \frac{n}{n+1}\theta$

$\tilde{\theta}_3' = \frac{n+1}{n} X_{max}$

$M\tilde{\theta}_3' = \frac{n+1}{n} M X_{max} = \theta$ — несмещ

$D\tilde{\theta}_3' = \left(\frac{n+1}{n}\right)^2 D X_{max} \ominus$

$M X_{max}^2 = \int_0^\theta x^2 n \left(\frac{x}{\theta}\right)^{n-1} \frac{1}{\theta}\, dx = \frac{n}{\theta^n} \frac{\theta^{n+2}}{n+2} = \frac{\theta^2 n}{n+2}$

$$D X_{max} = \theta^2\left(\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right) = \theta^2\left(\frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2}\right) = \frac{n\theta^2}{(n+2)(n+1)^2}$$

Θ $\frac{\theta^2}{n(n+2)} \xrightarrow[n\to\infty]{} 0$ состоят

~~$\tilde\theta_3' = \frac{n+1}{n} X_{max}$ состоят? по опр~~

~~$\tilde\theta_3$ состоят (по опр) 9-го~~

(4) $\tilde\theta_4 = X_1 + \frac{1}{n-1}\sum_{i=2}^{n} X_i$

$M\tilde\theta_4 = MX_1 + \frac{1}{n-1}\sum_{i=2}^{n} MX_i = M\xi + \frac{1}{n-1}\sum_{i=2}^{n} M\xi =$

$= \frac{\theta}{2} + \frac{\theta}{2} = \theta$ — несмещ.

$D\tilde\theta_4 = \cancel{DX_1 + \sum_{i=2}^{n} DX_i \frac{1}{n-1}}\ DX_1 + \frac{1}{(n-1)^2}\sum_{i=2}^{n} DX_i = \frac{\theta^2}{12} + \frac{1}{n-1}\frac{\theta^2}{12} \not\to 0$

не вып. дост. усл

Сост по опр:

$\tilde\theta_4 \xrightarrow{P} \theta$

$\boxed{X_1 + \underbrace{\frac{1}{n-1}\sum_{i=2}^{n} X_i}_{\to M\xi = \frac{\theta}{2}} \xrightarrow{P} \xi + \frac{\theta}{2}}$

$X_1 \xrightarrow{P} \xi$

$\begin{cases}\xi_n \xrightarrow{P} \xi \\ \eta_n \xrightarrow{P} \eta\end{cases} \to \xi_n + \eta_n \xrightarrow{P} \xi + \eta$, $\xi_n \eta_n \xrightarrow{P} \xi\eta$

не сост.

ЗБЧ Хинчина

ξ_n независ., одинак. распред., $\exists M\xi_n < a$

$\to \frac{1}{n}\sum_{i=1}^{n} \xi_n \xrightarrow{P} M\xi_1$

$\tilde\theta_3' = \frac{n+1}{n} X_{max}$

$\tilde\theta_1 = 2\bar{x}$

$D\tilde\theta_1 = \frac{\theta^2}{3n}$ | должны выбр. наиб. эфф

$D\tilde\theta_3 = \frac{\theta^2}{(n+2)n}$

$\frac{1}{3n}$ $\quad$ $\frac{1}{n(n+2)}$

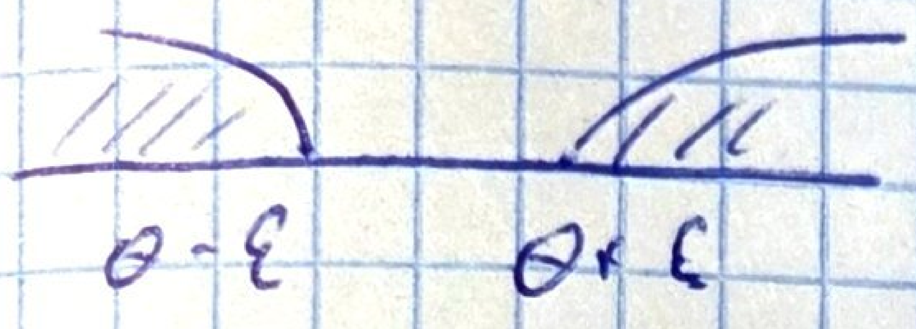
$n^2 + 2n > 3n$, $n > 1$

$\overline{\theta}_3$ более эф. $\overline{\theta}_1$

$\overline{\theta}_3'$

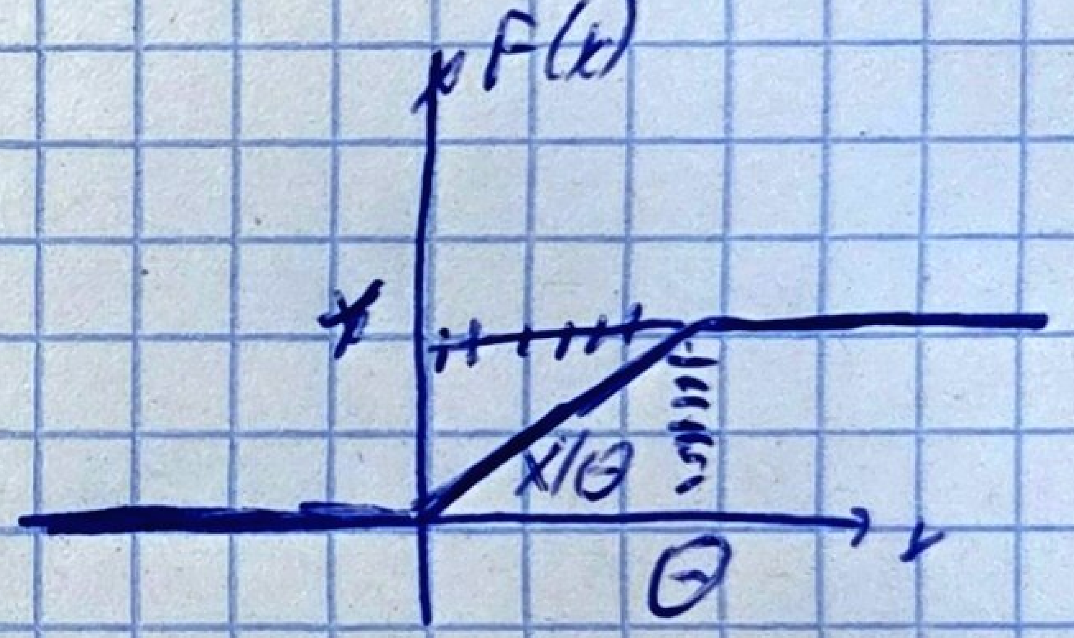
по опр

$\forall\theta>0 \quad \forall\varepsilon>0 \quad P(|\overline{\theta}_3' - \theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$



$$\begin{cases} P\left(\frac{n+1}{n} X_{max} \leq \theta-\varepsilon\right) = P\left(X_{max} \leq \frac{(\theta-\varepsilon)n}{n+1}\right) = \Psi\left(\frac{(\theta-\varepsilon)n}{n+1}\right) = F\left(\frac{(\theta-\varepsilon)n}{(n+1)}\right)^n = \\ = \left(\frac{(\theta-\varepsilon)n}{(n+1)\theta}\right)^n = \left(\frac{n}{n+1}\cdot\left(1-\frac{\varepsilon}{\theta}\right)\right)^n \xrightarrow[n\to\infty]{} 0, \text{ т.к. } \frac{\varepsilon}{\theta}<1 \\ P\left(\frac{n+1}{n} X_{max} \geq \theta+\varepsilon\right) = P\left(X_{max} \geq \frac{(\theta+\varepsilon)n}{n+1}\right) = 1-\Psi\left(\frac{(\theta+\varepsilon)n}{n+1}\right) = \\ = 1 - F\left(\frac{(\theta+\varepsilon)n}{n+1}\right)^n = 0, \text{ т.к.} \end{cases}$$

при равн. распред.



$\Downarrow$

$\overline{\theta}_3'$ - состоят.

$\overline{\theta}_3$

по опр

$$\begin{cases} P(\overline{\theta}_3 \leq \theta-\varepsilon) = P(X_{max} \leq \theta-\varepsilon) = \Psi(\theta-\varepsilon) = (F(\theta-\varepsilon))^n = \\ = \left(\frac{\theta-\varepsilon}{\theta}\right)^n = \left(1-\frac{\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 0, \text{ т.к. } \frac{\varepsilon}{\theta}<1 \\ P(\overline{\theta}_3 \geq \theta+\varepsilon) = 0 \end{cases}$$

(т.к. по больше прав. границы, не может быть больше θ)

$\Downarrow$

состоятельна