

Partially Optimal Cubic Subspace Clustering

Research Project Machine Learning

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21.07.2025

- 1 Introduction
- 2 Partial Optimality for Cubic Clique Partition Problem
- 3 Cubic Subspace Instance Construction
- 4 Experiments and Evaluation
- 5 Research Results

Problem Statement (1)

Finite sample set S , cost function $c: \binom{S}{3} \rightarrow \mathbb{R}$.

Instance of the **Cubic Clique Partition Problem**:

$$\min_{y: \binom{S}{2} \rightarrow \{0,1\}} \sum_{\{a,b,c\} \in \binom{S}{3}} c_{\{a,b,c\}} y_{\{a,b\}} y_{\{b,c\}} y_{\{a,c\}}$$

subject to $y_{\{a,b\}} + y_{\{b,c\}} - 1 \leq y_{\{a,c\}}$ for all distinct $a, b, c \in S$.

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Find a **partially optimal solution**, i.e. fix some labels $y_{\{a,b\}}$ for distinct $a, b \in S$

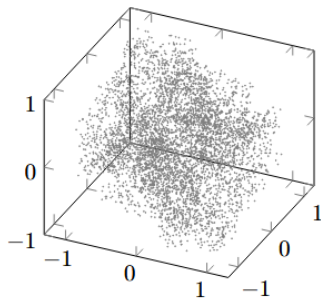
$$\begin{cases} y_{\{a,b\}} = 1 & \text{join } a, b \\ y_{\{a,b\}} = 0 & \text{cut } a, b \\ y_{\{a,b\}} = ? & \text{unknown} \end{cases}$$

in such way that there still exists an optimal solution.

Problem Statement (2)

Subspace Instances of the Cubic Clique Partition Problem

Samples S : points $S \subset \mathbb{R}^3$



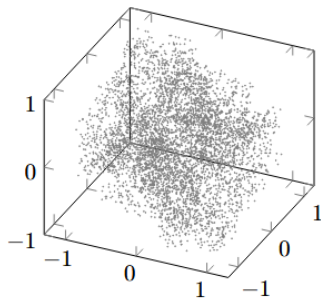
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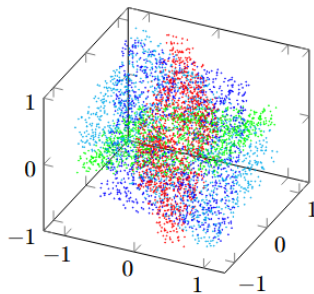
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Point generation: 3 distinct planes containing the origin, noise σ



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(b) Optimal labeling y^*

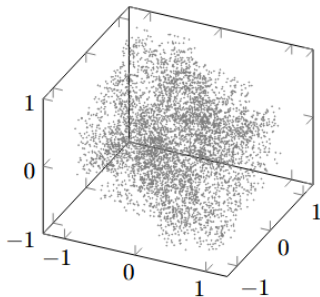
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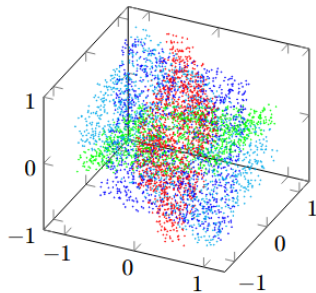
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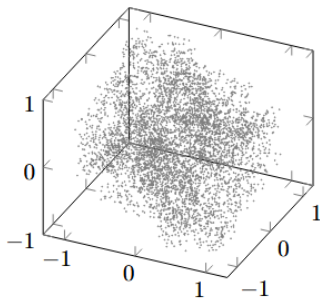
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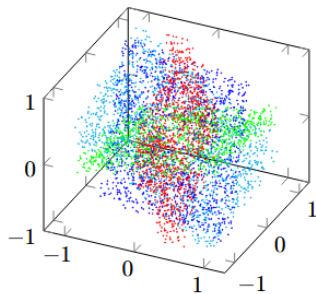
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Optimal labeling y^* : original planes

Cost function c ? (no concrete plane information given)



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Research Goals and Contributions

Task	Solution
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Partial Optimality for Cubic Clique Partition Problem

Extended cost function $c: \binom{S}{3} \cup \binom{S}{2} \cup \emptyset \rightarrow \mathbb{R}$

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Apply partial optimality conditions → solve subproblems

Partial Optimality Algorithm

Partial Optimality Algorithm:

Input: clustering y without fixed labels

while condition applied **do**

 apply subproblem-CUT-condition exhaustively

 apply one of JOIN-conditions (in effective order)

end while

 apply CUT-conditions exhaustively

Output: partially optimal clustering y with some fixed labels

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Reduction to subproblems:

- 1 Subproblem-CUT-condition: fix CUT labels for element pairs from different sample subsets; solve each subset as an independent problem and accumulate the results in c_\emptyset ;
- 2 JOIN-Conditions: fix JOIN labels for elements of the sample subset; add the join-cost to c_\emptyset ; solve the problem where the subset is considered as one sample;

Subproblem-CUT and Subset-JOIN

Subproblem-CUT: cut sample subsets R_1, R_2, \dots, R_k that are only connected via non-negative costs (applied if $k > 1$)

Subproblem-CUT and Subset-JOIN

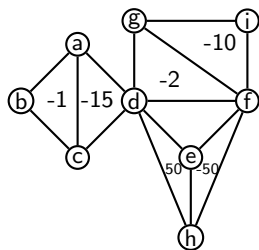
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(The worst bipartition joining cost \approx min-cut)

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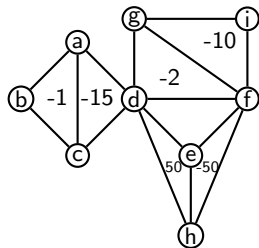


$$c_{\emptyset} = 0$$

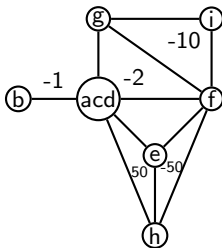
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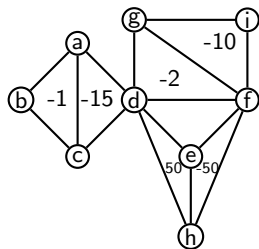


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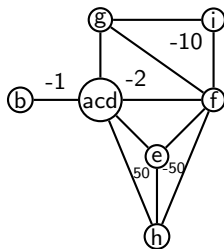
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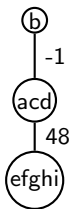
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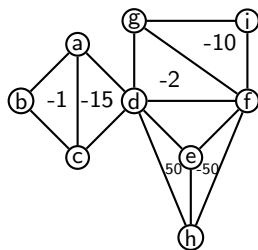


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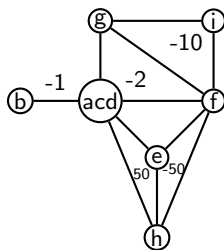
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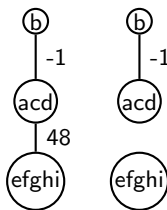
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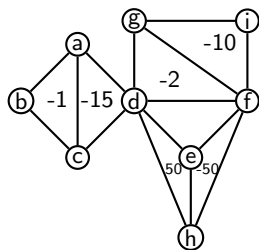


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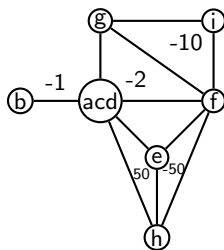
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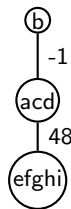
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Other JOIN-conditions

Pair-JOIN-1: join samples i, j if their overall joining reward \geq the sum of rewards and penalties for joining some subset R with $i \in R$ and \bar{R} with $j \in \bar{R}$ (rhs \approx s-t min-cut)

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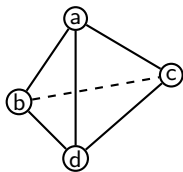
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Pyramid Instance and CUT-conditions

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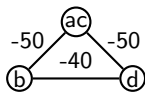
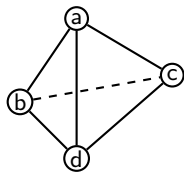
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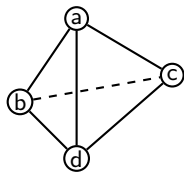


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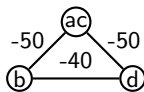
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Subset-JOIN

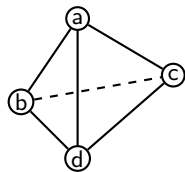


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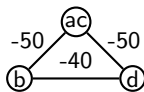
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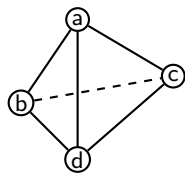
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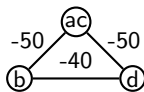
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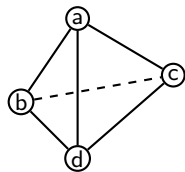
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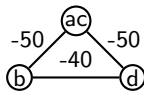
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$$c_{\{b,c,d\}} = 10$$

$$c_{\{a,b,c\}} = c_{\{a,b,d\}} = c_{\{a,c,d\}} = -50$$



Pair-JOIN-2



Subset-JOIN



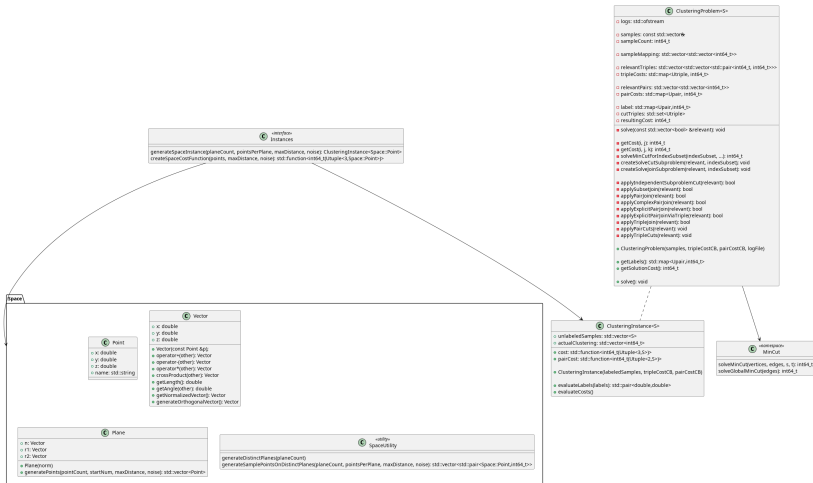
$$c_{\emptyset} = -140$$

Pair-CUT: cut samples i, j if the direct joining penalty \geq the sum of rewards for joining some subset R with $i \in R$ and \bar{R} with $j \in \bar{R}$ (rhs \approx s-t min-cut)

Triple-CUT: cut samples i, j, k if the condition holds (similar to Pair-CUT) (rhs \approx s-t min-cut)

Samples in the pyramid with $c_{\{b,c,d\}} = 100$ are unjoinable!
Triple-CUT is applied to the triple bcd

Program Structure



Pyramid Instance: Program Logs

Clustering problem has been initied with 4 relevant triples,
(negative: 3, positive: 1)

Trying independent subproblem cut (3.1)...

Trying subset join (3.11)...

Trying pair join (3.4)...

Trying complex pair join (3.6)...

* Applying the complex pair join (3.6)

Join: a c

Trying independent subproblem cut (3.1)...

Trying subset join (3.11)...

* Applying the subset join (3.11)

Join: ac b d

(0: cut; 1: joint; x: unknown)

Labeling:

a b c d

a - 1 1 1

b 1 - 1 1

c 1 1 - 1

d 1 1 1 -

Unjoinable cluster triples:

Clustering:

Cluster 0: abcd

Problem solved: completely

Cost: -140

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Plane and Point Generation

plane Generation point Generation save as the labeled data, use the unlabeled data + cost function to init the ClusteringProblem

Cost Function

My contribution!!! Cost function does not depend on the point distribution bounds!!! Steps to assign the cost !!!!!

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Experiments

My laptop characteristics, Random cubic subspace instances with different seeds max point component size 100 (noise are the percents then) no noise 0, small noise 1, significant noise 3, large noise 5, (Table: instance size + noise + instance count)

Cost Function Evaluation

blue and red dots, conflicts and their effect (picture of the typical cost function evaluation)

Experiment Results for 3x7 Points

3x7 points: all results + time-optimality-accuracy
(min-max-average)

Experiment Results

3x(7x12x17x22) time-optimality-accuracy (min-max-average)
DIAGRAM!!! mention the coefficients to prove the efficiency,
partial optimality and accuracy!!!

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Research Results

Partial optimality reimplementation, Dedicated Cost Function,
Efficiency and Accuracy (shown by the experiments)

Future Work

Optimize the partial optimality algorithm Determine better parameters for the cost function Extend the cost function with more advanced conditions