# Partially Optimal Cubic Subspace Clustering

Research Project Machine Learning

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# Problem Statement (1)

Finite sample set $S$, cost function $\mathrm{c} \colon \binom{S}{3} \to \mathbb{R}$.
Instance of the **Cubic Clique Partition Problem**:

$$\min_{\mathrm{y} \colon \binom{S}{2} \to \{0,1\}} \sum_{abc \in \binom{S}{3}} \mathrm{c}_{abc} \, \mathrm{y}_{ab} \, \mathrm{y}_{bc} \, \mathrm{y}_{ac}$$

subject to $\mathrm{y}_{ab} + \mathrm{y}_{bc} - 1 \leq \mathrm{y}_{ac}$ for all distinct $a, b, c \in S$.

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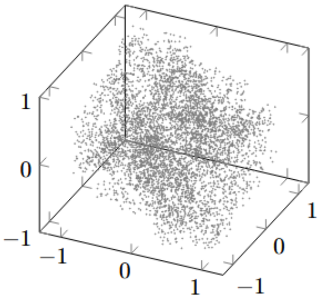
Find a **partially optimal solution**, i.e. fix some labels $\mathrm{y}_{ab}$ for distinct $a, b \in S$

$$\begin{cases} \mathrm{y}_{ab} = 1 & \text{join } a, b \\ \mathrm{y}_{ab} = 0 & \text{cut } a, b \\ \mathrm{y}_{ab} = ? & \text{unknown} \end{cases}$$

in such way that there still exists an optimal solution.

**Subspace Instances** of the Cubic Clique Partition Problem

Samples $S$: points $S \subset \mathbb{R}^3$




(a) Samples $S$

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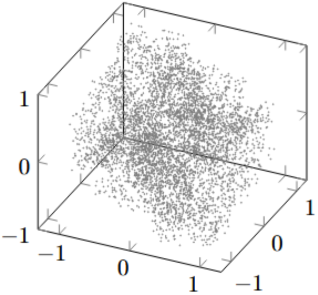
Samples $S$: points $S \subset \mathbb{R}^3$
Point generation: 3 distinct planes containing the origin, noise $\sigma$

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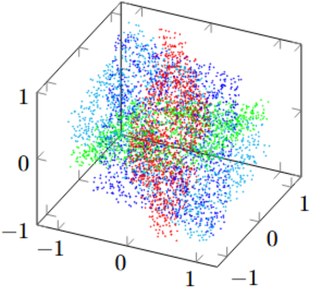


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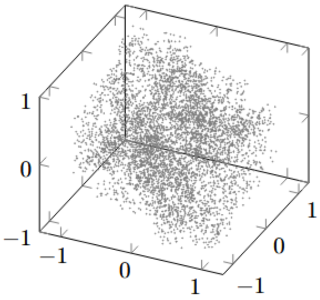


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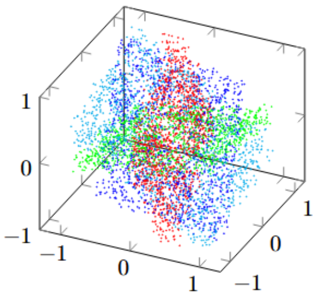


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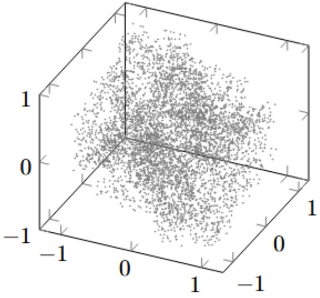
Optimal labeling $y^*$: original planes
Cost function c? (no concrete plane information given)

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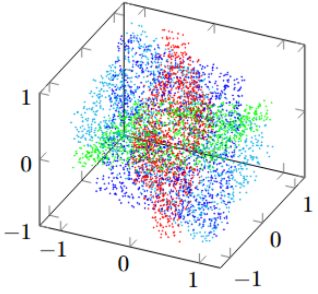


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# Research Goals and Contributions

**Related Work:**

TODO: 1 + 2 (citation at the end!)

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   → point generation, appropriate cost function c
   (high accuracy, significant noise tolerance)

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**Tasks and Solutions:**

1. Read [**?**], implement the partial optimality algorithm
   $\rightarrow$ implementation in C++ (with some adjustments)
2. Construct subspace instances of increasing difficulty
   $\rightarrow$ point generation, appropriate cost function c
   (high accuracy, significant noise tolerance)
3. Apply algorithm to the subspace instances, assess partial optimality, accuracy and computation time
   $\rightarrow$ experiments and evaluation (prove the quality of c)

Extended cost function $c\colon \binom{S}{3} \cup \binom{S}{2} \cup \emptyset \to \mathbb{R}$

# Partial Optimality for Cubic Clique Partition Problem

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Instance of the extended cubic clique partition problem:

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**Input:** labeling y without fixed labels

**while** condition applied **do**

apply subproblem-CUT-condition exhaustively

apply one of JOIN-conditions (in effective order)

**end while**

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2. JOIN-Conditions: fix JOIN labels for elements of the sample subset; add the join-cost to $c_\emptyset$; solve the problem where the subset is considered as one sample;

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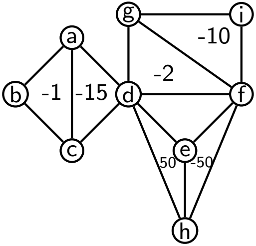


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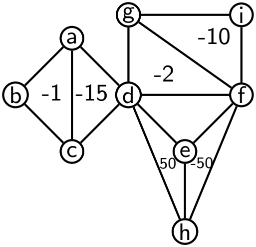


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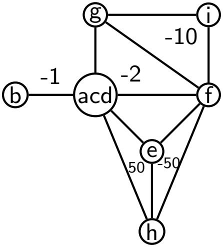


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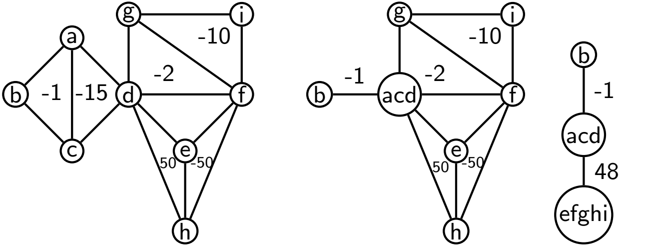


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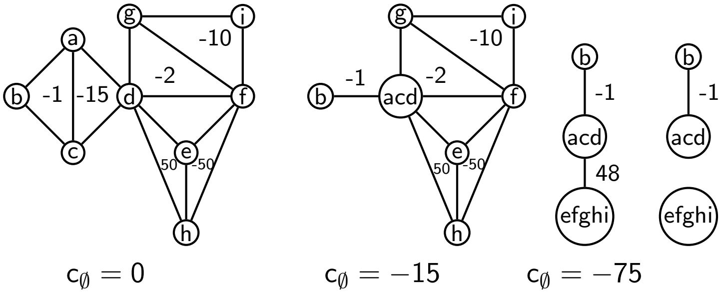

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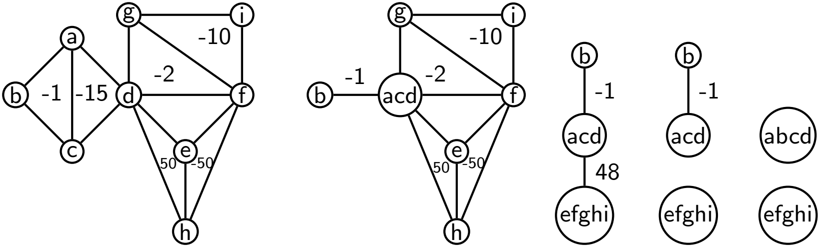


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# Other JOIN-conditions

**Pair-JOIN-1:** join samples $i, j$ if their overall joining reward $\geq$ the sum of rewards and penalties for joining some subset $R$ with $i \in R$ and $\overline{R}$ with $j \in \overline{R}$ ($\approx$ i-j min-cut)

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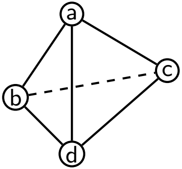
**Triple-JOIN:** join samples $i, j, k$ if the condition holds (similar to Pair-JOIN-1) ($\approx$ i-jk min-cut)

# Pyramid Instance and CUT-conditions

$c_{bcd} = 10$

$c_{abc} = c_{abd} = c_{acd} = -50$



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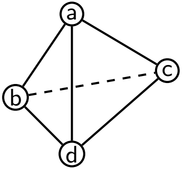


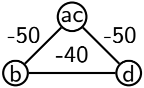


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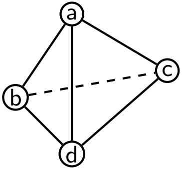


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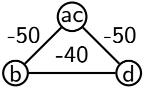
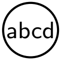



Subset-JOIN

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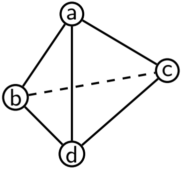


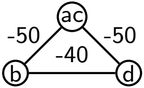


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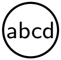


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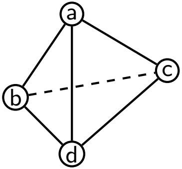


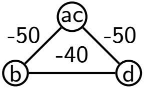


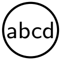


Pair-JOIN-2 Subset-JOIN $c_\emptyset = -140$

**Pair-CUT:** cut samples $i, j$ if the direct joing penalty $\geq$ the sum of rewards for joining some subset $R$ with $i \in R$ and $\overline{R}$ with $j \in \overline{R}$ ($\approx$ i-j min-cut)

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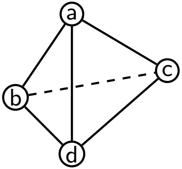
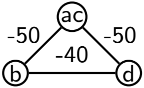
# Pyramid Instance and CUT-conditions

$c_{bcd} = 10$

$c_{abc} = c_{abd} = c_{acd} = -50$

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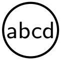


Subset-JOIN $\quad c_{\emptyset} = -140$

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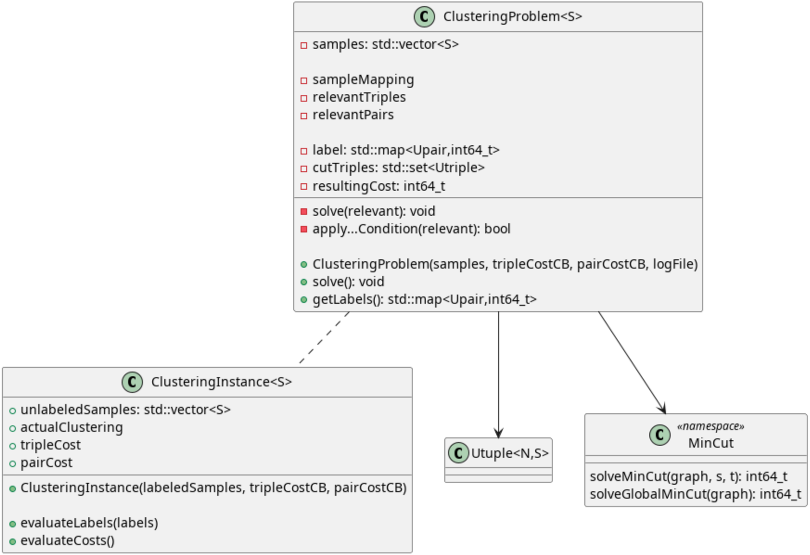
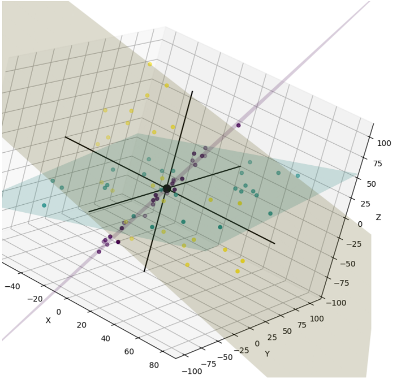
Samples in the pyramid with $c_{bcd} = 100$ are unjoinable!
Triple-CUT is applied to the triple *bcd*

# Program Structure



**Plane Generation:**

- generate 3 planes
  as distinct normal vectors
  $\vec{n}_1, \vec{n}_2, \vec{n}_3$ (normalized)



# Plane and Point Generation

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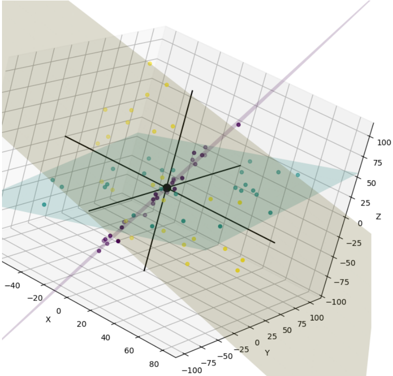
- generate 3 planes as distinct normal vectors $\vec{n}_1, \vec{n}_2, \vec{n}_3$ (normalized)
- compute a $\vec{r}_{i,1}$ (normalized) orthogonal to $\vec{n}_i$ $(i \in \{1, 2, 3\})$



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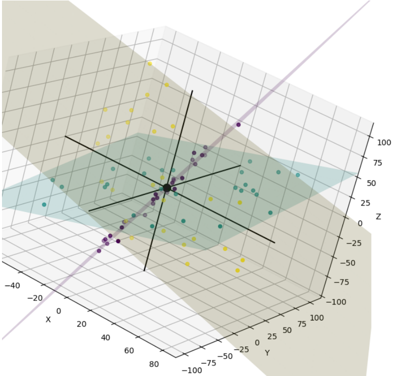

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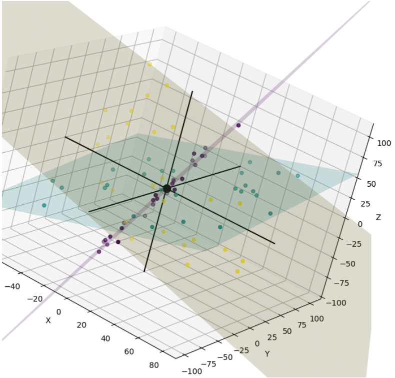


**Point Generation** on the plane $(\vec{n}, \vec{r}_1, \vec{r}_2)$, parameters $(D, \sigma)$:

- random variables $k_1, k_2 \in [-D, D]$ (uniform distribution)
- random variable $k_n$ (normal distribution based on $\sigma$)
- generate point $p = k_1\vec{r}_1 + k_2\vec{r}_2 + k_n\vec{n}$

# Cost Function (1)

**Triangle** $abc \in \binom{S}{3}$

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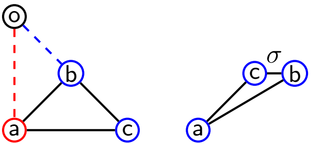

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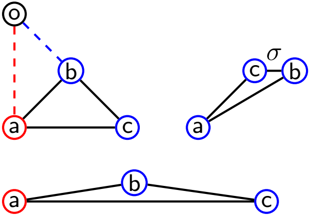

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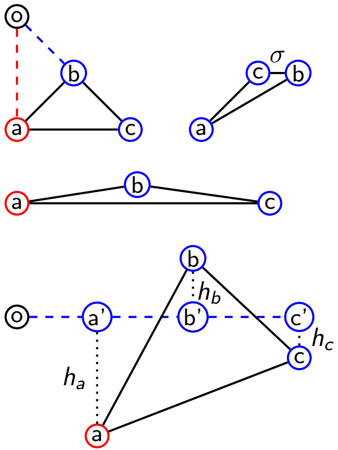
$h_a + h_b + h_c > 3\sigma + 10^{-6}$
$\rightarrow c_{abc} = \frac{(h_a+h_b+h_c)-(3\sigma+10^{-6})}{3D}$



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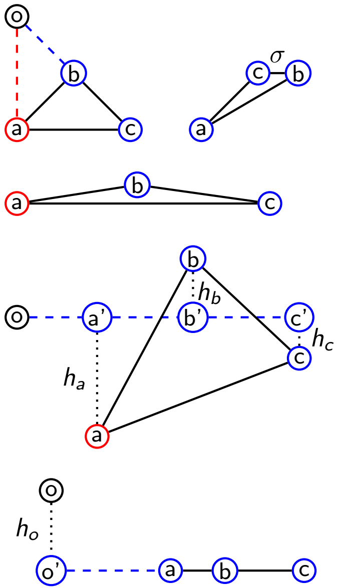

# Cost Function (2)

(5) for all points $p$:
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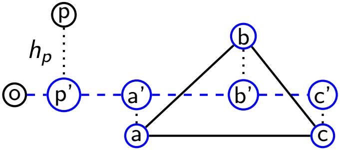

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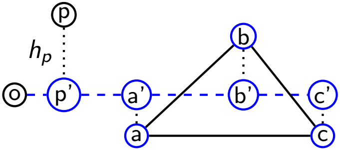

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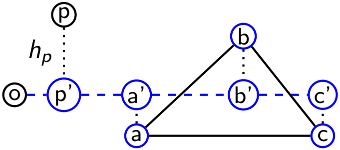


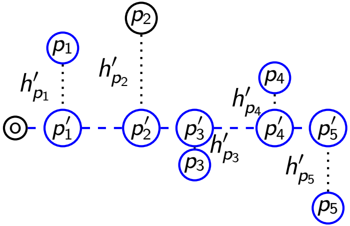

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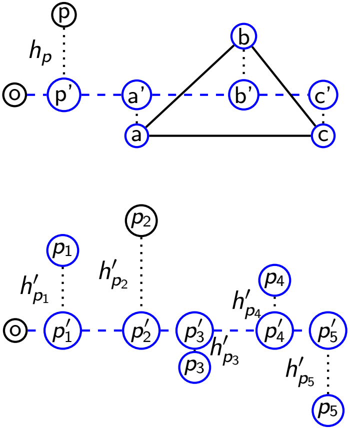
$M := \{p \mid p \text{ chosen } \wedge \delta_p < 0\}$



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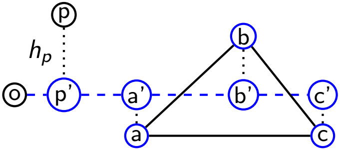


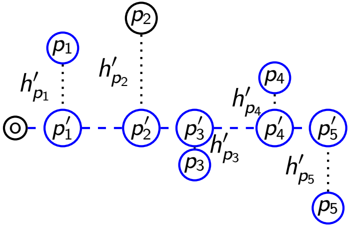

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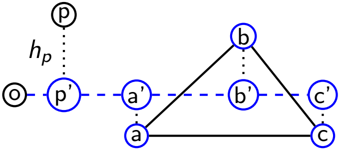
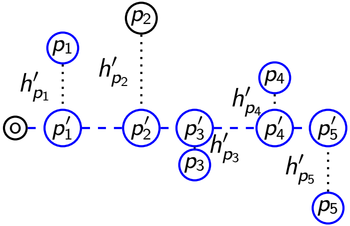
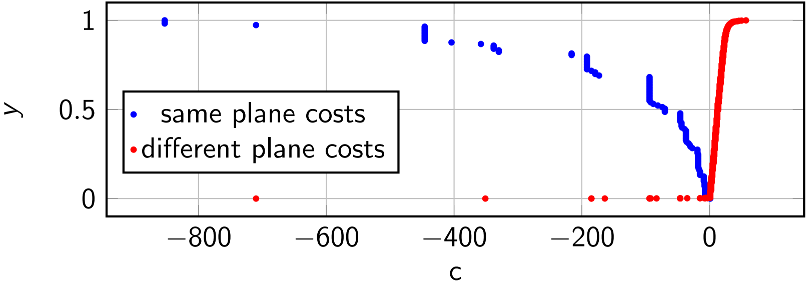


# Cost Function Evaluation (3x15 points, $\sigma = 1$)



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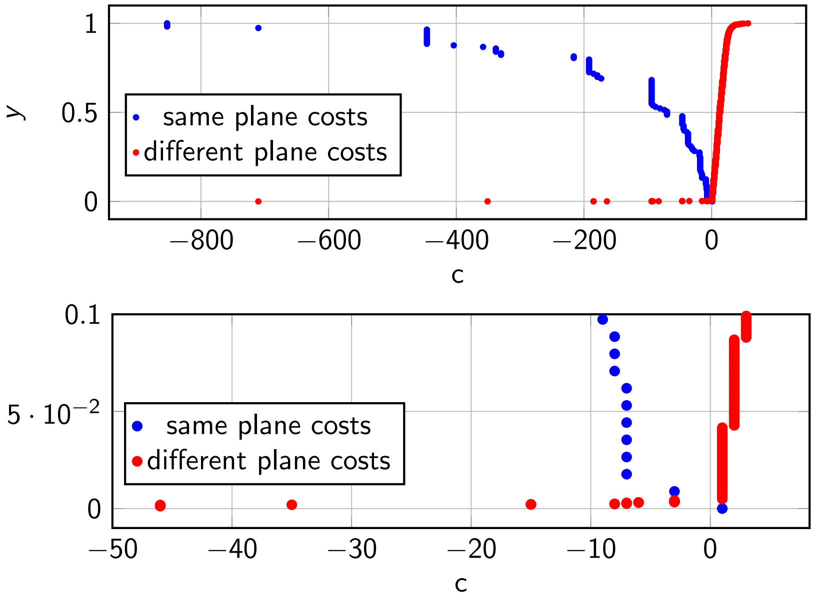

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  - 1.quartile (Q1)
  - median (Q2)
  - 3.quartile(Q3)
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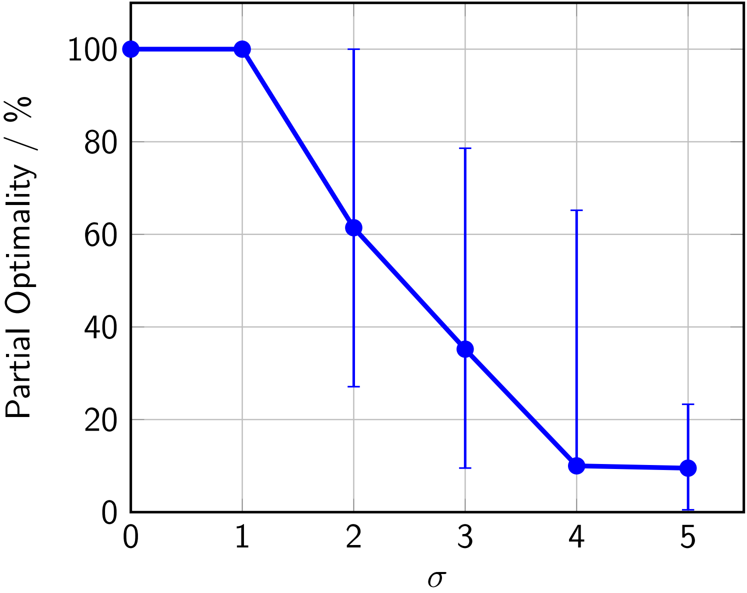
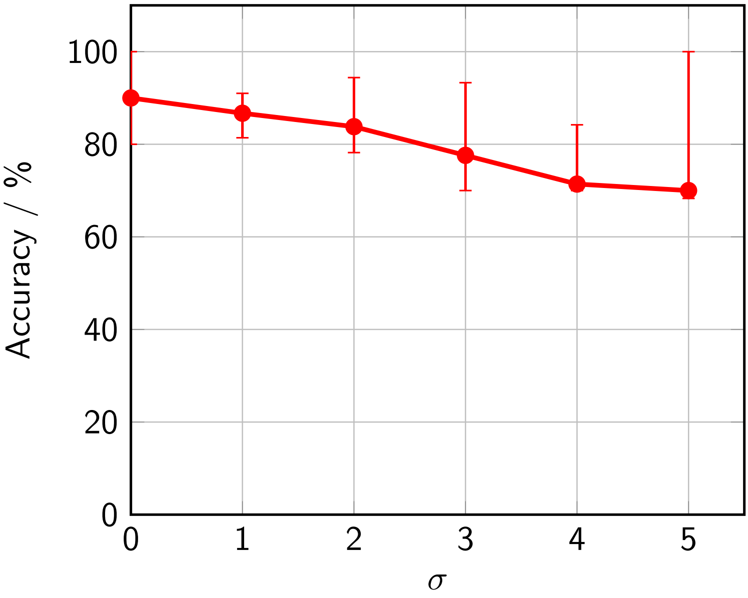
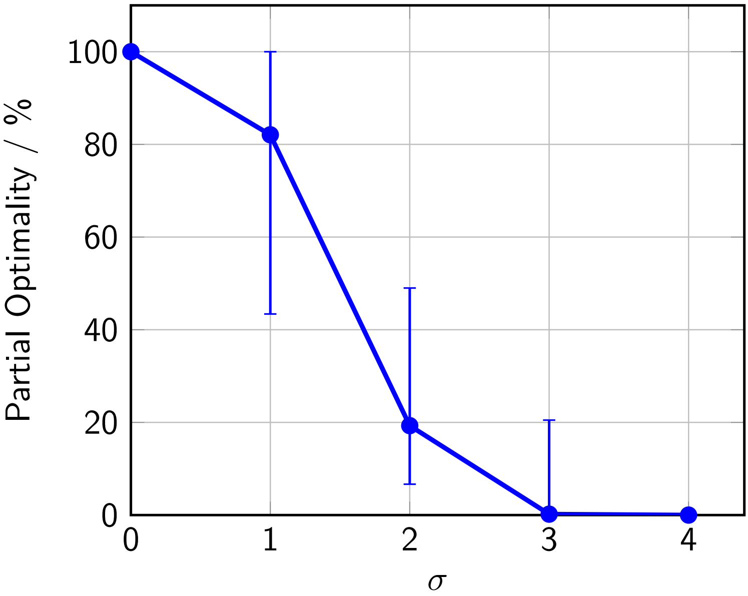
# Partial Optimality (3x7 points)



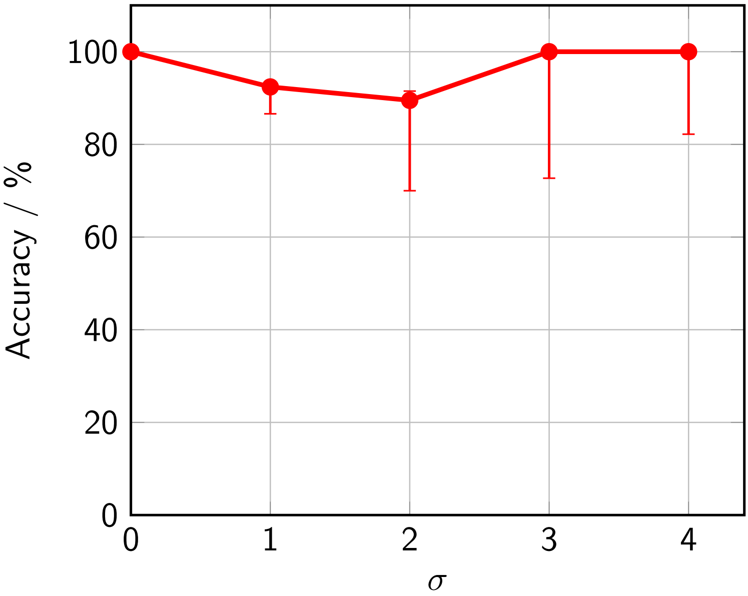
# Accuracy (3x7 points)



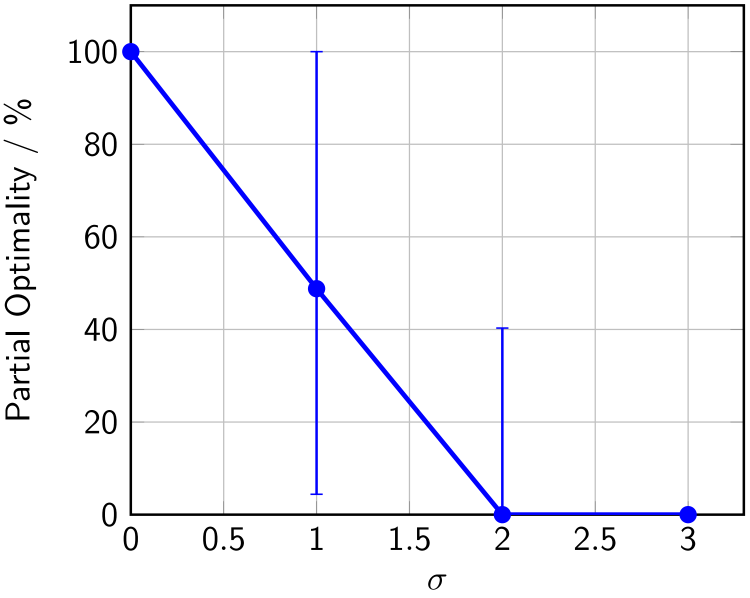
# Partial Optimality (3x10 points)



# Accuracy (3x10 points)



# Partial Optimality (3x15 points)



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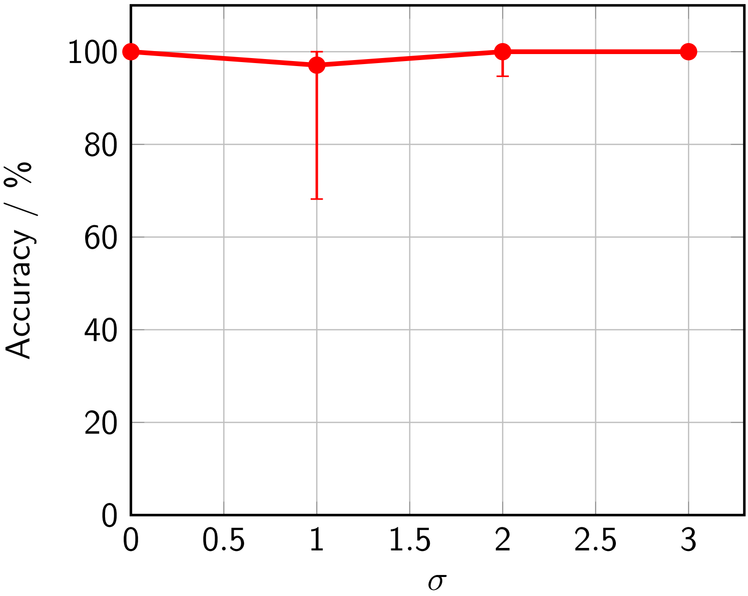

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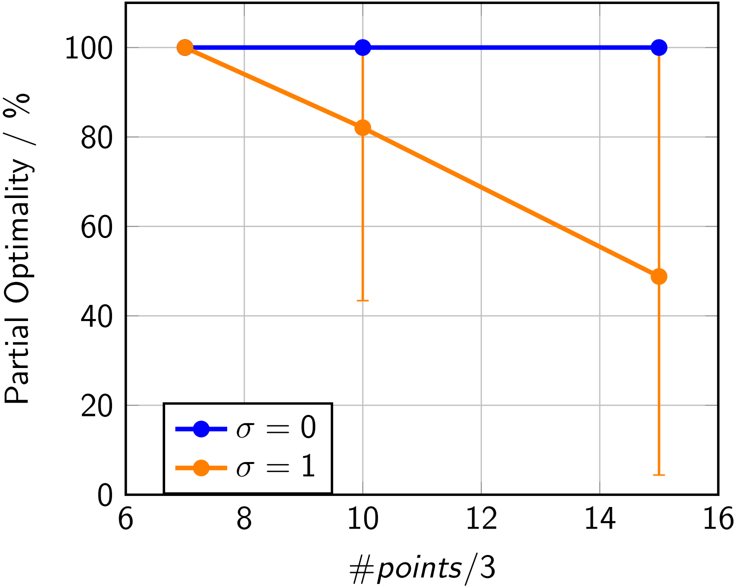

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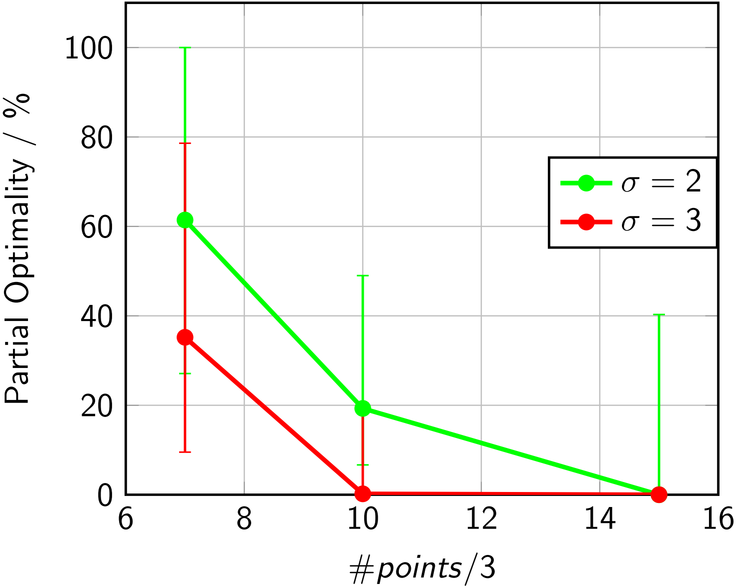
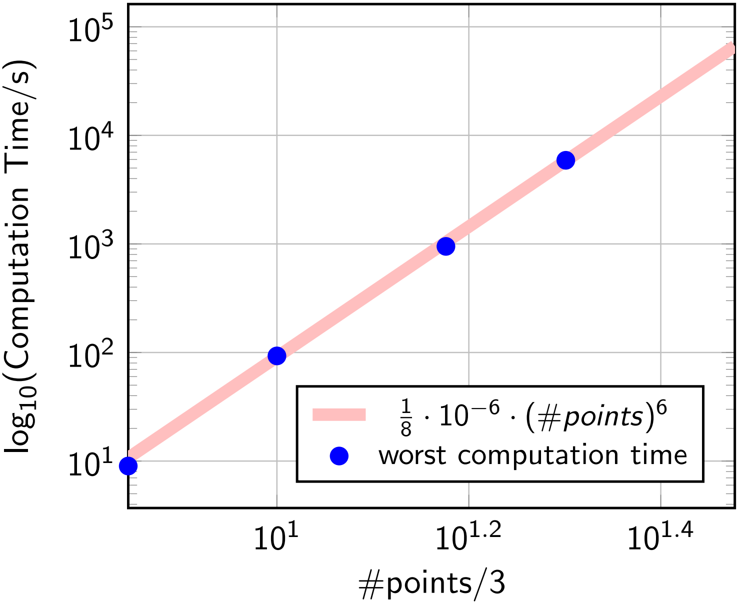


# Computation Time (worst case)



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**Future Work**:

- optimize and parallelize the algorithm
- overcome the partial optimality loss for c
- determine better parameters for c
- update c with advanced criteria

# References

**My program, scripts and presentation:**

`https://github.com/Vovsanka/ResearchProjectML`

TODO: citation