Partially Optimal Cubic Subspace Clustering

Research Project Machine Learning

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1 Introduction

2 Partial Optimality for Cubic Clique Partition Problem

3 Cubic Subspace Instance Construction

4 Experiments and Evaluation

5 Research Results

Problem Statement (1)

Finite sample set S, cost function $\mathrm{c}\colon \binom{S}{3} \to \mathbb{R}$.
Instance of the **Cubic Clique Partition Problem**:

$$\min_{\mathrm{y}\colon \binom{S}{2} \to \{0,1\}} \sum_{abc \in \binom{S}{3}} \mathrm{c}_{abc}\, \mathrm{y}_{ab}\, \mathrm{y}_{bc}\, \mathrm{y}_{ac}$$

subject to $\mathrm{y}_{ab} + \mathrm{y}_{bc} - 1 \leq \mathrm{y}_{ac}$ for all distinct $a, b, c \in S$.

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Find a **partially optimal solution**, i.e. fix some labels y_{ab} for distinct $a, b \in S$

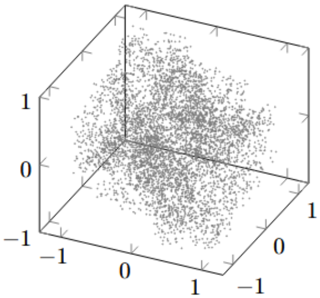
$$\begin{cases} \mathrm{y}_{ab} = 1 & \text{join } a, b \\ \mathrm{y}_{ab} = 0 & \text{cut } a, b \\ \mathrm{y}_{ab} = ? & \text{unknown} \end{cases}$$

in such way that there still exists an optimal solution.

Problem Statement (2)

Subspace Instances of the Cubic Clique Partition Problem

Samples S: points $S \subset \mathbb{R}^3$



(a) Samples S

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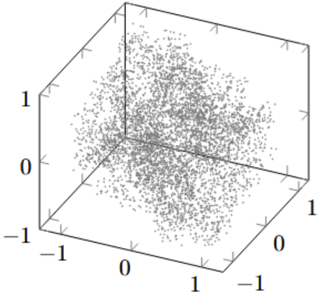

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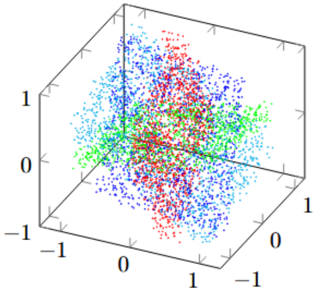

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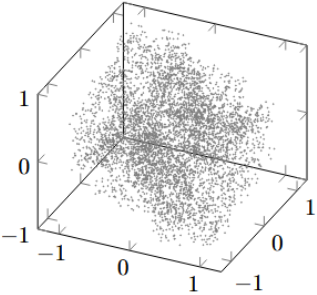

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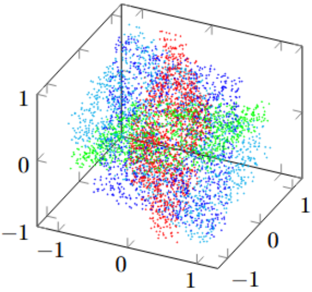

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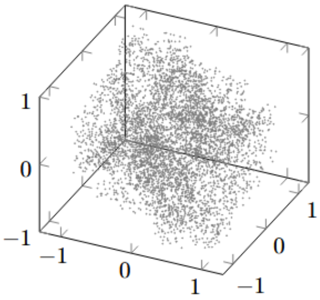
Optimal labeling y^*: original planes
Cost function c? (no concrete plane information given)



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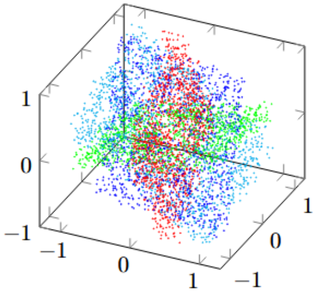

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Related Work:
TODO: 2 + 1

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   $\rightarrow$ implementation in C++ (with some adjustments)
2. Construct subspace instances of increasing difficulty
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3. Apply algorithm to the subspace instances, assess partial optimality, accuracy and computation time
   $\rightarrow$ experiments and evaluation (prove the quality of c)

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Apply partial optimality connditions → solve subproblems

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Input: clustering y without fixed labels

while condition applied **do**

apply subproblem-CUT-condition exhaustively

apply one of JOIN-conditions (in effective order)

end while

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Reduction to subproblems:

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2. JOIN-Conditions: fix JOIN labels for elements of the sample subset; add the join-cost to $c_\emptyset$; solve the problem where the subset is considered as one sample;

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Subproblem-CUT: cut sample subsets $R_1, R_2, \ldots, R_k$ that are only connected via non-negative costs (applied if $k > 1$)

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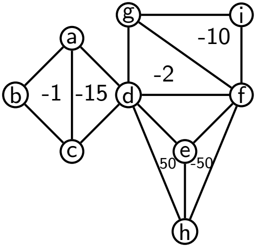

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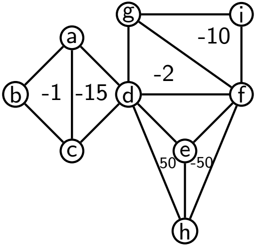

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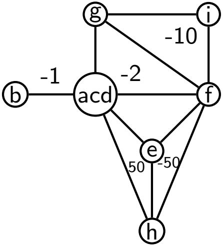

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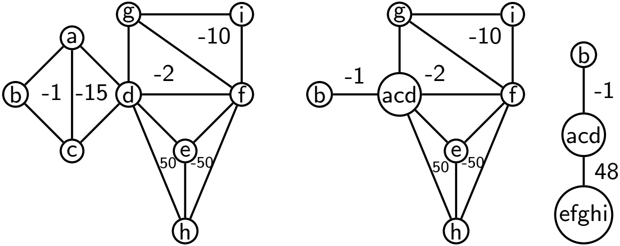

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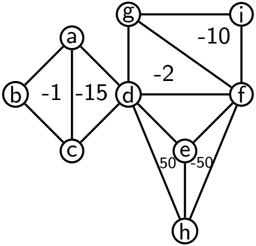

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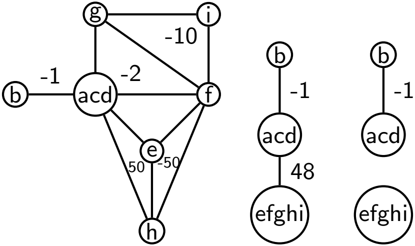

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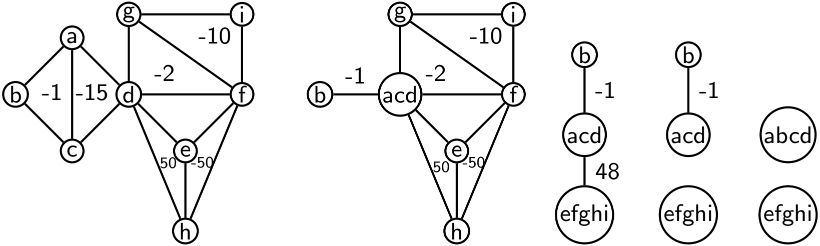

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Other JOIN-conditions

Pair-JOIN-1: join samples i, j if their overall joining reward $\geq$ the sum of rewards and penalties for joining some subset R with $i \in R$ and $\overline{R}$ with $j \in \overline{R}$ ($\approx$ i-j min-cut)

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Triple-JOIN: join samples i, j, k if the condition holds (similar to Pair-JOIN-1) ($\approx$ i-jk min-cut)

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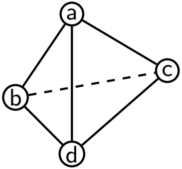

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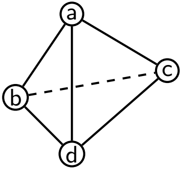

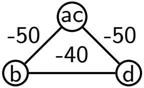

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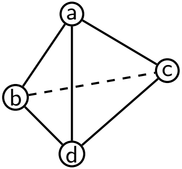

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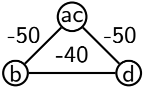
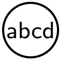


Subset-JOIN



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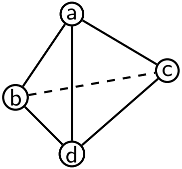

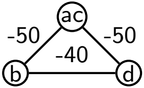

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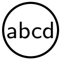

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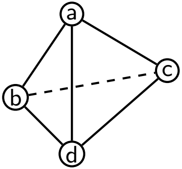
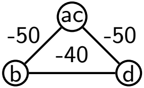
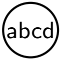






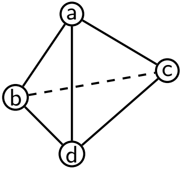
Pair-JOIN-2 Subset-JOIN $c_{\emptyset} = -140$

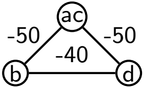
Pair-CUT: cut samples i, j if the direct joing penalty $\geq$ the sum of rewards for joining some subset R with $i \in R$ and $\overline{R}$ with $j \in \overline{R}$ ($\approx$ i-j min-cut)

Triple-CUT: cut samples i, j, k if the condition holds (similar to Pair-CUT) ($\approx$ i-jk min-cut)

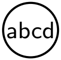
Pyramid Instance and CUT-conditions

$c_{\{b,c,d\}} = 10$
$c_{\{a,b,c\}} = c_{\{a,b,d\}} = c_{\{a,c,d\}} = -50$





Pair-JOIN-2



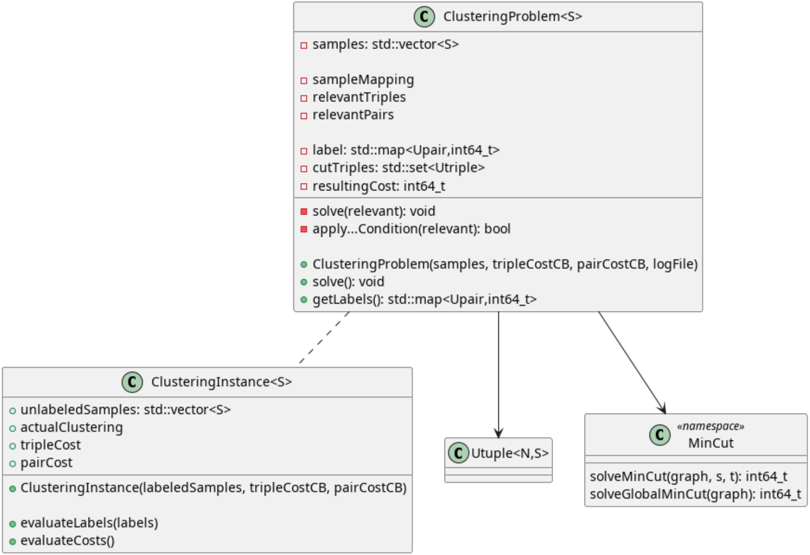
Subset-JOIN $c_{\emptyset} = -140$

Pair-CUT: cut samples i, j if the direct joing penalty $\geq$ the sum of rewards for joining some subset R with $i \in R$ and $\overline{R}$ with $j \in \overline{R}$ ($\approx$ i-j min-cut)

Triple-CUT: cut samples i, j, k if the condition holds (similar to Pair-CUT) ($\approx$ i-jk min-cut)

Samples in the pyramid with $c_{\{b,c,d\}} = 100$ are unjoinable!
Triple-CUT is applied to the triple *bcd*

Program Structure



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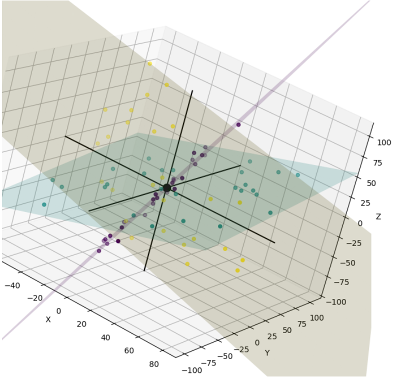
Plane and Point Generation

Plane Generation:

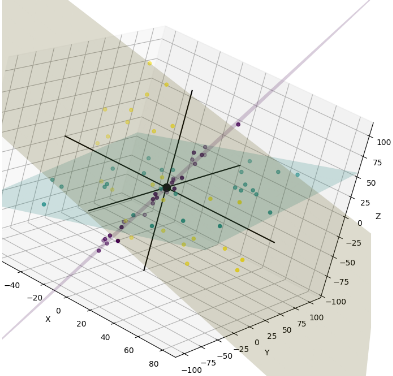
- generate 3 planes
  as distinct normal vectors
  $\vec{n}_1, \vec{n}_2, \vec{n}_3$ (normalized)



Plane and Point Generation

Plane Generation:

- generate 3 planes as distinct normal vectors $\vec{n}_1, \vec{n}_2, \vec{n}_3$ (normalized)
- compute a $\vec{r}_{i,1}$ (normalized) orthogonal to $\vec{n}_i$ $(i \in \{1, 2, 3\})$



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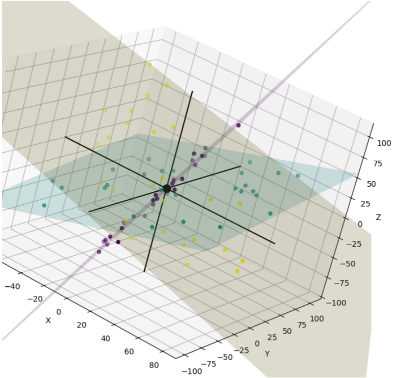

Plane and Point Generation

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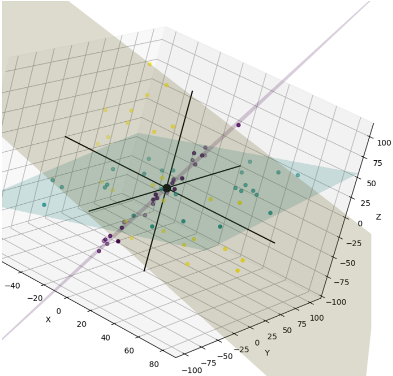

Point Generation on the plane $(\vec{n}, \vec{r}_1, \vec{r}_2)$, parameters (D, σ):

- random variables $k_1, k_2 \in [-D, D]$ (uniform distribution)
- random variable k_n (normal distribution based on σ)
- generate point $p = k_1\vec{r}_1 + k_2\vec{r}_2 + k_n\vec{n}$

Cost Function

Triangle $abc \in \binom{S}{3}$

1. Smallest side $s < D/2 \to c_{abc} = 0$
2. Largest angle $\alpha > 150^\circ \to c_{abc} = 0$
3. ha, hb, hc: distances to the best fitting plane
   $ha + hb + hc > 3\sigma + 10^{-6}$
   $\to c_{abc} = \frac{(ha+hb+hc)-(3\sigma+10^{-6})}{3D}$
4. ho: distance from the origin to the triangle plane
   $ho > \frac{10\sigma}{\#points} + 10^{-6} \to c_{abc} = 0$
5. for all points p: hp: distance to the best fitting plane
   choose p if $hp < \sigma + 10^{-6}$ and $|\vec{p}| > 0.3D$
   hp': distance to the best fitting plane of all chosen points
   $\delta_p = \frac{hp'-(\sigma+10^{-6})}{D}$, $SAME = \{p \colon \delta_p < 0\}$, $rew = \sum_{p \in SAME} \delta_p$,
   $|SAME| \leq 3 \to c_{abc} = 0$
   else $\to c_{abc} = 2^{|SAME|-4} rew$

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Experiments

My laptop characteristics, Random cubic subspace instances with different seeds max point component size 100 (noise are the percents then) no noise 0, small noise 1, significant noise 3, large noise 5, (Table: instance size + noise + instance count)

Cost Function Evaluation

blue and red dots, conflicts and and their effect (picture of the typical cost function evaluation)

Experiment Results for 3x7 Points

3x7 points: all results + time-optimality-accuracy (min-max-average)

Experiment Results

3x(7x12x17x22) time-optimality-accuracy (min-max-average) DIAGRAM!!! mention the coefficients to prove the efficiency, partial optimality and accuracy!!!

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Conclusion

TODO: summarize the research results Partial optimality reimplementation, Dedicated Cost Function, Efficiency and Accuracy (shown by the experiments)
Future Work :