Partially Optimal Cubic Subspace Clustering

Research Project Machine Learning

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Technische Universität Dresden

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Partial Optimality for Cubic Clique Partition Problem

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Apply partial optimality connditions → solve subproblems

Partial Optimality Algorithm

Partial Optimality Algorithm:

Input: clustering y without fixed labels

while condition applied **do**

    apply subproblem-CUT-condition exhaustively

    apply one of JOIN-conditions (in effective order)

end while

apply CUT-conditions exhaustively

Output: partially optimal clustering y with some fixed labels

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Reduction to subproblems:

1. Subproblem-CUT-condition: fix CUT labels for element pairs from different sample subsets; solve each subset as an independent problem and accumulate the results in $c_\emptyset$;
2. JOIN-Conditions: fix JOIN labels for elements of the sample subset; add the join-cost to $c_\emptyset$; solve the problem where the subset is considered as one sample;

Subproblem-CUT and Subset-JOIN

Subproblem-CUT: cut sample subsets $R_1, R_2, \ldots, R_k$ that are only connected via non-negative costs (applied if $k > 1$)

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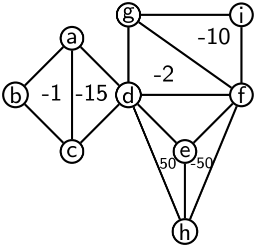

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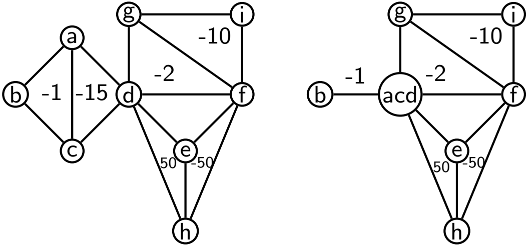

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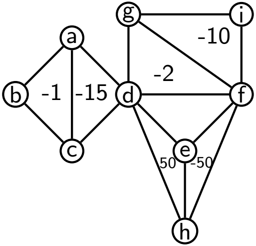

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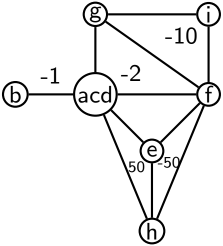

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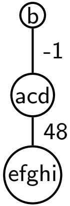

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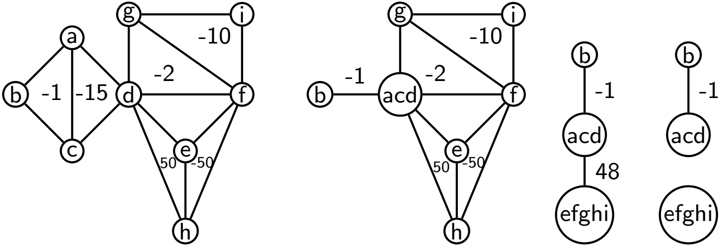

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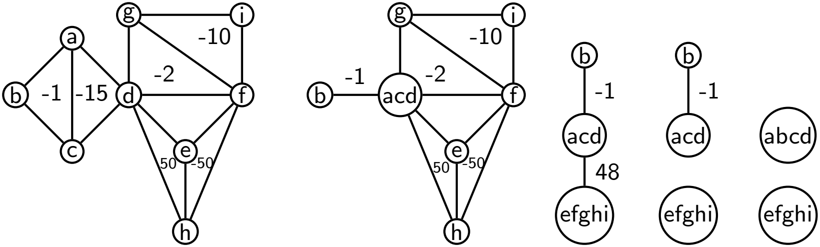

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Other JOIN-conditions and CUT-conditions

Pair-JOIN-1: join samples i, j if their overall joining reward $\geq$ the sum of rewards and penalties for joining some subset R with $i \in R$ and $\overline{R}$ with $j \in \overline{R}$ (rhs $\approx$ min-cut)

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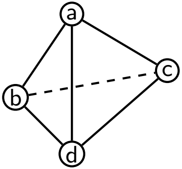

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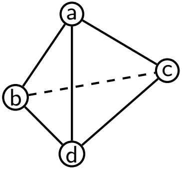

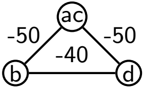

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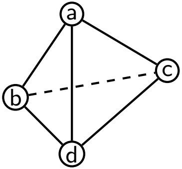

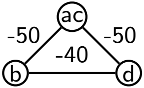

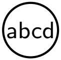

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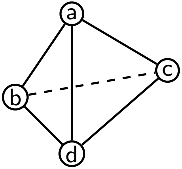

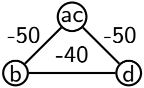

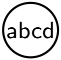

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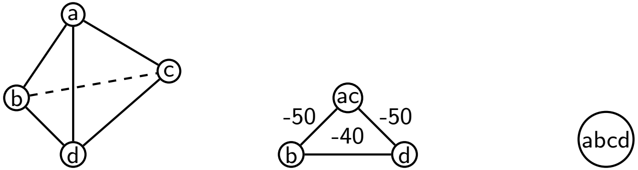

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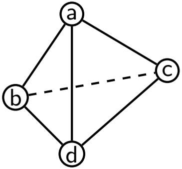

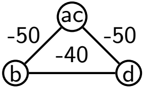

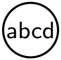

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Samples in pyramid with $c_{\{b,c,d\}} = 100$ are unjoinable!
Triple-CUT is applied to the triple *bcd*

Program Structure

TODO: class Diagram TODO: features Algorithm implementation in ClusteringProblem Features: ClusteringProblem is generally defined for all types of Cubic Clique Partition Problem (not necessarily points), cost function + sparse costs!, label computation, cut triples, logs joins and cuts! (add screenshots)