Theorem 1. *Let $I \subset \mathbb{R}$ be a nonempty open interval. Then it holds for $f, g \in C^\infty(I)$ and $n \in \mathbb{N}$*

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

Proof. Proof by induction:

For the base case is correct because

$$(fg)' = f'g + fg' = \binom{1}{1} f'g + \binom{1}{0} fg' = \sum_{k=0}^{1} \binom{1}{k} f^{(k)} g^{(1-k)}.$$

For induction step assume that (1) holds for some $n \in \mathbb{N}$. That implies

$$\begin{aligned}
(fg)^{(n+1)} &= \left((fg)^{(n)}\right)' \overset{\text{I.H.}}{=} \left(\sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}\right)' \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k)} g^{(n-k)}\right)' \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k+1)} g^{(n-k)} + f^{(k)} g^{(n-k+1)}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)} g^{(n-k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k+1)} \\
&= \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(k)} g^{(n-k+1)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k+1)} \\
&= \sum_{k=1}^{n} \binom{n}{k-1} f^{(k)} g^{(n-k+1)} + \binom{n}{n} f^{(n+1)} g + \binom{n}{0} f g^{(n+1)} + \sum_{k=1}^{n} \binom{n}{k} f^{(k)} g^{(n-k+1)} \\
&= \left(\sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k}\right) f^{(k)} g^{(n-k+1)}\right) + \binom{n}{n} f^{(n+1)} g + \binom{n}{0} f g^{(n+1)} \\
&= \left(\sum_{k=1}^{n} \binom{n+1}{k} f^{(k)} g^{(n-k+1)}\right) + \binom{n+1}{n+1} f^{(n+1)} g + \binom{n+1}{0} f g^{(n+1)} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n+1-k)}
\end{aligned}$$

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