Problem #4 Overview B: "full-period" inner products in Fourier series

1. Full-period Fourier series = Required when $f(x)$ is ***neither*** odd nor even !!

Consider the function $f(x)$ in Figure 2, where the signal is neither odd nor even:

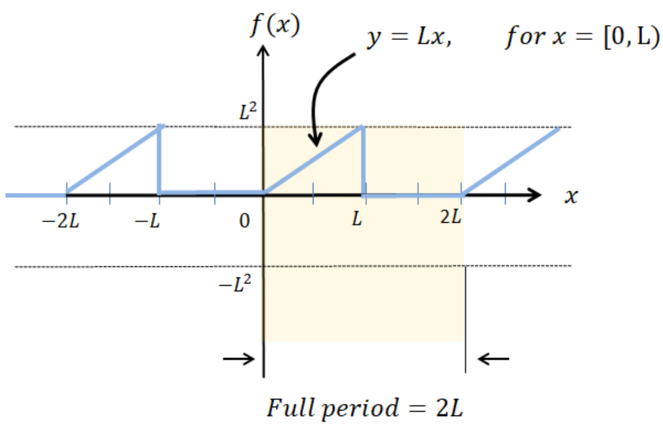

Figure 2: A weird signal $f(x)$ that requires *both* sines, cosines, and *full-period* inner product rules for Fourier series approximations

Since $f(x)$ has *neither odd nor even symmetry*, one might suspect $f(x)$ can be better approximated by using linear combinations of both sines (odd components) and cosines (even components):

$$\psi_m(x) = \left\{ \cos\left(\frac{m\pi}{L}x\right) \right\} \quad and \quad \phi_n(x) = \left\{ \sin\left(\frac{n\pi}{L}x\right) \right\}$$

Where: $$m = 0,\ 1,\ 2\ \cdots, \quad and \quad n = 1,\ 2, \cdots \ positive\ integers$$

Again, notice we have a special case when the basis indices are zero:

$$m\ or\ n = 0 \ \rightarrow \ \begin{cases} \psi_0(x) = exists\ ! = \cos(0 \cdot x) = 1 \ (a\ constant\ basis\ !) \\ \phi_0(x) = zero \end{cases}$$

Facilitates the "DC offset" term in engineering !

Let's write out the reconstruction of our target function (x) :

$$f(x) = a_0\, \psi_0(x) + a_1\,\psi_1(x) + a_2\,\psi_2(x) + \cdots + a_m\,\psi_m(x)$$
$$+ b_1\,\phi_1(x) + b_2\,\phi_2(x) + \cdots + b_n\,\phi_n(x)$$

We can easily rewrite this in a compact form.... of which you might have seen before in undergrad !

$$f(x) = \sum_{m=0}^{\infty} a_m \cos\left(\frac{m\pi}{L}x\right) + \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) = \underbrace{\sum_{m=0}^{\infty} a_m\,\psi_m(x)}_{even\ basis} + \underbrace{\sum_{n=0}^{\infty} b_n\,\phi_n(x)}_{odd\ basis}$$

Using the ***full-period*** inner product rules to find Fourier coefficients:

First, let's check out the individual orthogonality and (norm)^2 rules for sines and cosines on a full period. Notice that the (norm)^2 values have all doubled ! This is because the overlap area now covers the full -period interval $x = [0,\ 2L]$, which is now twice as wide as before.

$$\langle \phi_p,\ \phi_q \rangle = \int_0^{2L} sin\left(\frac{p\pi}{L}x\right) sin\left(\frac{q\pi}{L}x\right) dx = \begin{cases} 0 & if\ p \neq q \quad (orthogonality) \\ L & if\ p = q \quad (norm)^2 \end{cases}$$

A nearly-identical statement can be made with cosines !

$$\langle \psi_p,\ \psi_q \rangle = \int_0^{2L} cos\left(\frac{p\pi}{L}x\right) cos\left(\frac{q\pi}{L}x\right) dx = \begin{cases} 0 & if\ p \neq q \quad (orthogonality) \\ L & if\ p = q \neq 0 \quad (norm)^2 \\ 2L & if\ p = q = 0 \quad (special\ norm)^2 \end{cases}$$

Now, it can be shown that a *__mutual orthogonality__* exists between sines and cosines:

$$\langle \psi_m,\ \phi_n \rangle = \int_0^{2L} cos\left(\frac{m\pi}{L}x\right) sin\left(\frac{n\pi}{L}x\right) dx = 0 \quad \left(\begin{matrix} for\ all\ m,n, and\ it\ also \\ works\ for\ the\ special\ case \\ m = 0 \end{matrix}\right)$$

We can use all 3 orthogonality rules to find the Fourier coefficients a_m and b_n.

A quick proof of the (norm)^2 for a full-period inner product for cosines

The (norm)^2 for a cosine basis over a full-period interval can be written as:

$$\langle \psi_p,\ \psi_p \rangle \ = \ \int_0^{2L} cos^2\left(\frac{p\pi}{L}x\right)\, dx \ = \ \int_0^{2L} \frac{1}{2}\left[1 + cos\left(\frac{2p\pi}{L}x\right)\right]\, dx$$

$$= \frac{1}{2}\left[x + \left(\frac{L}{2p\pi}\right) sin\left(\frac{2p\pi}{L}x\right)\right]_0^{2L}$$

$$= \frac{1}{2}\left[2L + \left(\frac{L}{2p\pi}\right) sin\left(\frac{2p\pi}{L}\cdot 2L\right) - \ 0 \ - \ 0\right]$$

$$= \frac{1}{2}\left[2L + \cancel{\left(\frac{L}{2p\pi}\right) sin(4p\pi)}\ \right] \qquad ** \ Note:\ sin(Intgers\ of\ \pi) \ = \ 0$$

$$\boxed{\langle \psi_p,\ \psi_p \rangle \ = \ L \quad \rightarrow \quad (norm)^2\ \ if \ \ p \neq 0}$$

For the special "DC offset" basis member ($p = 0$), the (norm)^2 relationship is:

$$\langle \psi_0,\ \psi_0 \rangle \ = \ \int_0^{2L} cos^2(0\cdot x)\, dx \ = \ \int_0^{2L} 1\ dx \ = \ \boxed{2L} \quad \rightarrow \quad (norm)^2\ \ if \ \ p = 0$$

A quick proof of the (norm)^2 for a full-period inner product for sines

The (norm)^2 for a sine basis over a full-period interval can be written as:

$$\langle \phi_p,\ \phi_p \rangle \ = \ \int_0^{2L} sin^2\left(\frac{p\pi}{L}x\right)\, dx \ = \ \int_0^{2L} \frac{1}{2}\left[1 - cos\left(\frac{2p\pi}{L}x\right)\right]\, dx$$

$$= \frac{1}{2}\left[x \ - \ \left(\frac{L}{2p\pi}\right) sin\left(\frac{2p\pi}{L}x\right)\right]_0^{2L}$$

$$= \frac{1}{2}\left[2L \ - \ \left(\frac{L}{2p\pi}\right) sin\left(\frac{2p\pi}{L}\cdot 2L\right) + \ 0 \ + \ 0\right]$$

$$= \frac{1}{2}\left[2L \ - \ \cancel{\left(\frac{L}{2p\pi}\right) sin(4p\pi)}\ \right] \qquad ** \ Note:\ sin(Intgers\ of\ \pi) \ = \ 0$$

$$\boxed{\langle \phi_p,\ \phi_p \rangle \ = \ L \quad \rightarrow \quad (norm)^2\ \ if \ \ p \neq 0}$$

a) Sine coefficients for a mixed basis approximation (on full-period interval):

Take the inner product with ϕ_n on both sides of our $f(x)$ approximation equation !

Mutual orthogonality !

$$\langle f,\ \phi_n \rangle = a_0\, \cancelto{0}{\langle \psi_0,\ \phi_n \rangle} + a_1\, \cancelto{0}{\langle \psi_1,\ \phi_n \rangle} + a_2\, \cancelto{0}{\langle \psi_2,\ \phi_n \rangle} + \cdots + a_m\, \cancelto{0}{\langle \psi_m,\ \phi_n \rangle}$$

$$+\ b_1\, \cancelto{0}{\langle \phi_1,\ \phi_n \rangle} + b_2\, \cancelto{0}{\langle \phi_2,\ \phi_n \rangle} + \cdots + b_n\, \underbrace{\langle \phi_n,\ \phi_n \rangle}_{(norm)^2}$$

We can now solve for b_n :

$$b_n = \frac{\langle f,\ \phi_n \rangle}{\langle \phi_n,\ \phi_n \rangle} = \frac{\langle f,\ \phi_n \rangle}{(full\ period\ norm)^2} = \frac{1}{L} \int_0^{2L} f(x)\, sin\left(\frac{n\pi}{L} x\right)\ dx$$

b) Cosine coefficients for a mixed basis approximation (on full-period interval):

Take the inner product with ψ_m on both sides of our $f(x)$ approximation equation !

$$\langle f,\ \psi_m \rangle = a_0\, \cancelto{0}{\langle \psi_0,\ \psi_m \rangle} + a_1\, \cancelto{0}{\langle \psi_1,\ \psi_m \rangle} + a_2\, \cancelto{0}{\langle \psi_2,\ \psi_m \rangle} + \cdots + a_m\, \overbrace{\langle \psi_m,\ \psi_m \rangle}^{(norm)^2}$$

$$+\ b_1\, \cancelto{0}{\langle \phi_1,\ \psi_m \rangle} + b_2\, \cancelto{0}{\langle \phi_2,\ \psi_m \rangle} + \cdots + b_n\, \cancelto{0}{\langle \phi_n,\ \psi_m \rangle}$$

Mutual orthogonality !

We can immediately solve for a_m , if $m \neq 0$:

$$a_{m\neq 0} = \frac{\langle f,\ \psi_m \rangle}{\langle \psi_m,\ \psi_m \rangle} = \frac{\langle f,\ \psi_m \rangle}{(full\ period\ norm)^2} = \frac{1}{L} \int_0^{2L} f(x)\ cos\left(\frac{m\pi}{L} x\right)\ dx$$

For the special case of $m = 0$, the "DC offset" Fourier coefficient is uber-easy to find !! =)

$$a_{m=0} = \frac{\langle f,\ \psi_m \rangle}{(full\ period\ norm)^2} = \frac{1}{2L} \int_0^{2L} f(x)\ dx \qquad [\ The\ \ DC\ offset\ ]$$

$$= \frac{Total\ area\ under\ the\ curve}{Total\ period\ width} = \begin{array}{c} The\ average\ value\ of \\ signal\ f(x)\ over\ period\ 2L \end{array}$$