Problem #4 Overview A: ½-period inner products in Fourier series

1. Half-period Fourier series = Great for even ***or*** odd target functions $f(x)$

Suppose we have two periodic functions $f(x)$ and $g(x)$:

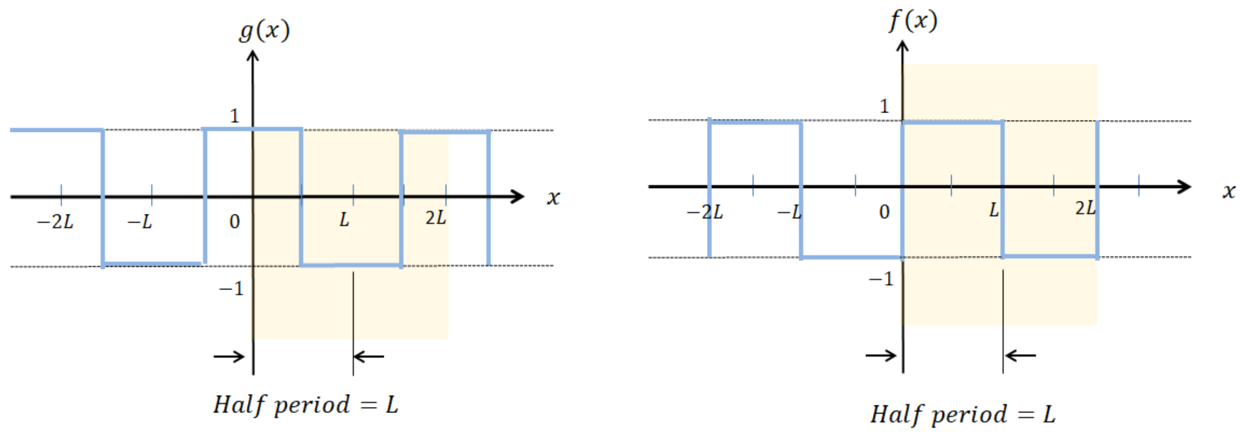

Figure 1: $f(x)$ is an odd function, while $g(x)$ is an even function

a) Odd function = Use sines

For the odd function $f(x)$ in Figure 1, we expect that a linear combination of sines ($sin(x)$ is odd) would be able to approximate (x) :

$$\begin{cases} f(x) = c_1\,\phi_1(x) + c_2\,\phi_2(x) + \cdots + c_n\,\phi_n(x) \\ \\ Basis\ set\!: \quad \phi_n(x) = \left\{ \ sin\left(\frac{n\pi}{L}x\right) \right\}, \quad where \quad n = 1, 2, \cdots \quad positive\ integers \end{cases}$$

And the sine basis set follows these orthogonality and (norm)^2 rules:

$$\langle \phi_p,\ \phi_q \rangle = \int_0^L sin\left(\frac{p\pi}{L}x\right)\, sin\left(\frac{q\pi}{L}x\right)\, dx = \begin{cases} 0 & if\ \ p \neq q \quad (orthogonality) \\ L/_2 & if\ p = q \quad (norm)^2 \end{cases}$$

Inner product can be defined over a ½ -period if $f(x)$ is *purely* odd

A quick proof of the (norm)^2 for a ½-period inner product for sines

The (norm)^2 for a sine basis over a ½-period interval can be written as:

$$\langle \phi_p,\ \phi_p \rangle = \int_0^L sin^2\left(\frac{p\pi}{L}x\right)\, dx \quad = \quad \int_0^L \frac{1}{2}\left[1 - cos\left(\frac{2p\pi}{L}x\right)\right]\, dx$$

$$= \frac{1}{2}\left[x - \left(\frac{L}{2p\pi}\right) sin\left(\frac{2p\pi}{L}x\right)\right]_0^L$$

$$= \frac{1}{2}\left[L - \left(\frac{L}{2p\pi}\right) sin\left(\frac{2p\pi}{L}\cdot L\right) + 0 + 0\right]$$

$$= \frac{1}{2}\left[L - \cancel{\left(\frac{L}{2p\pi}\right) sin(2p\pi)}\right] \qquad ** \; Note:\ sin(Intgers\ of\ \pi) = 0$$

$$= \frac{L}{2}$$

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b) Even functions = Use cosines

For the even function $g(x)$ in Figure 1, we expect that a linear combination of cosines ($cos(x)$ is even) would be able to approximate (x) :

$$\begin{cases} g(x) = c_0\ \psi_0(x) + c_1\,\psi_1(x) + c_2\,\psi_2(x) + \cdots + c_m\,\psi_m(x) \\ \\ Basis\ set\!:\quad \psi_m(x) = \left\{ cos\left(\frac{m\pi}{L}x\right) \right\},\ \ where \quad m = 0, 1, 2, \cdots\ \ positive\ integers \end{cases}$$

Notice the cosine basis set has a special member ! If $m = 0$, we'll have:

$$m = 0 \qquad \rightarrow \qquad \psi_0(x) = exists\,! \quad = \quad cos(0 \cdot x) = 1 \ (a\ constant\ basis\,!)$$

Facilitates the "DC offset" term in engineering !

Now, the cosine basis set follows these orthogonality and (norm)^2 rules. Notice there's an extra (norm)^2 rule for the special case of $p = 0$

$$\langle \phi_p,\ \phi_q \rangle = \int_0^L cos\left(\frac{p\pi}{L}x\right) cos\left(\frac{q\pi}{L}x\right) dx = \begin{cases} 0 & if\ \ p \neq q & (orthogonality) \\ L/2 & if\ p = q \neq 0 & (norm)^2 \\ L & if\ p = q = 0 & (special\ norm)^2 \end{cases}$$

Inner product can be defined over a ½ -period if $g(x)$ is *purely* even

A quick proof of the (norm)^2 for a ½-period inner product for cosines

The (norm)^2 for a cosine basis over a ½-period interval can be written as:

$$\langle \phi_p,\ \phi_p \rangle \ = \ \int_0^L cos^2\left(\frac{p\pi}{L}x\right)\, dx \quad = \quad \int_0^L \ \frac{1}{2}\left[1 + cos\left(\frac{2p\pi}{L}x\right)\right]\ dx$$

$$= \ \frac{1}{2}\left[x + \ \left(\frac{L}{2p\pi}\right)\ sin\left(\frac{2p\pi}{L}x\right)\right]_0^L$$

$$= \ \frac{1}{2}\ \left[L + \ \left(\frac{L}{2p\pi}\right)\ sin\left(\frac{2p\pi}{L}\cdot L\right) - \ 0 \ - \ 0\right]$$

$$= \ \frac{1}{2}\ \left[L + \ \cancel{\left(\frac{L}{2p\pi}\right)\ sin(2p\pi)}\ \right] \qquad\qquad ** \ Note\colon\ sin(Intgers\ of\ \pi) \ = \ 0$$

$$\boxed{\langle \phi_p,\ \phi_p \rangle \ = \ \frac{L}{2} \quad \rightarrow \quad (norm)^2\ \ if \ \ \ p \ \neq \ 0}$$

For the special “DC offset” basis member ($p = 0$), the (norm)^2 relationship is:

$$\langle \phi_p,\ \phi_p \rangle \ = \ \int_0^L cos^2(0\ \cdot x)\, dx \quad = \quad \int_0^L \ 1\ \ dx \quad = \boxed{L} \quad \rightarrow \quad (norm)^2\ \ if \ \ \ p = 0$$