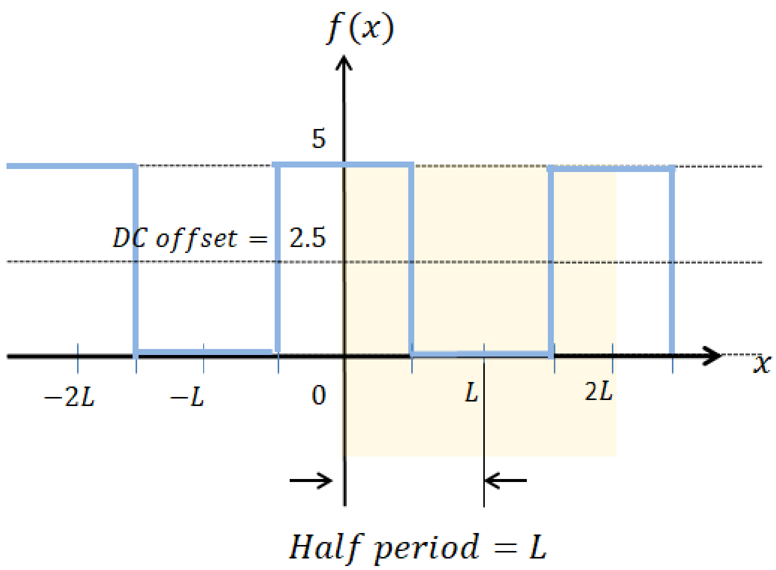
Problem 2 Overview A: Intro to wavelets

1. A quick reminder on Fourier series........ with sines and cosines

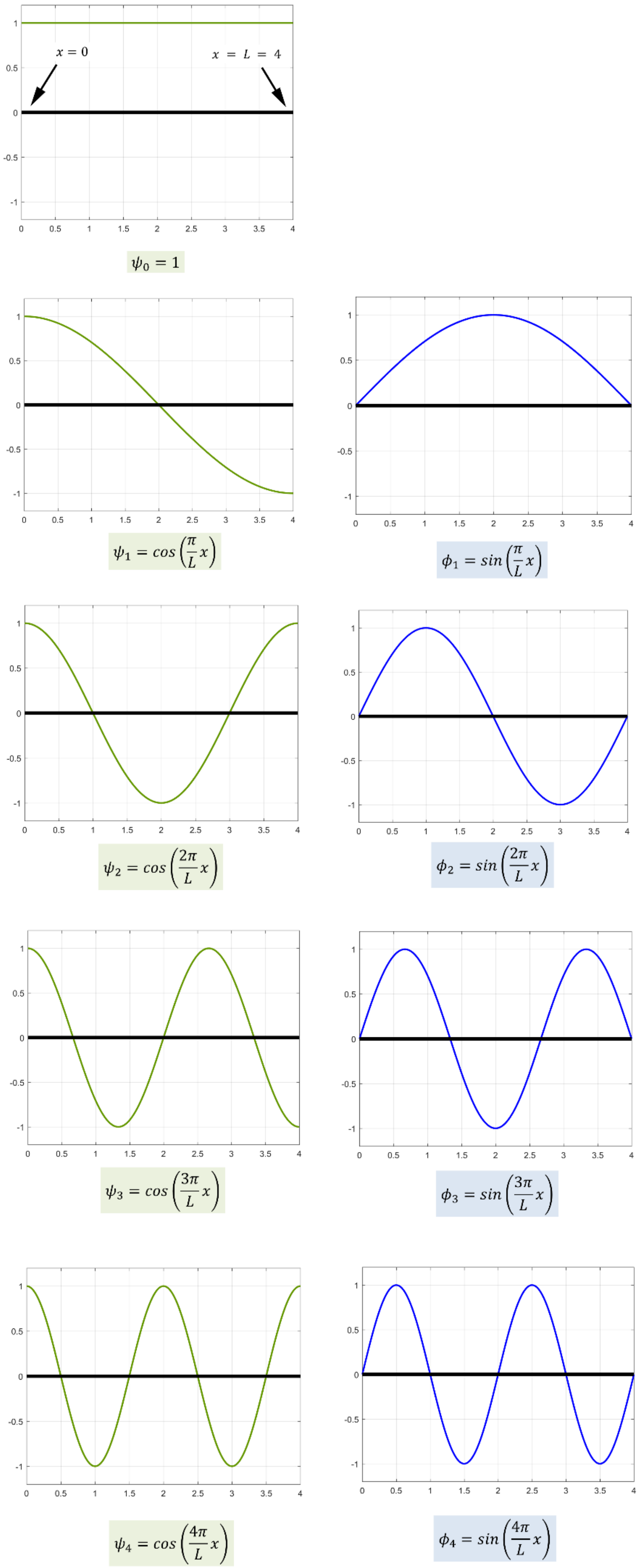
Consider an *semi*-even target function $f(x)$ in Figure 1 (*Note: This function will be* <u>*properly even*</u>*" if you subtract the DC offset value from* $f(x)$ *).*



<u>Figure 1</u>: The target function $f(x)$ can be reconstructed using a linear combination of cosines !

In the previous homework set, you know that $f(x)$ can be best approximated by taking linear combinations of sines and cosines (with $m = 0$ being the DC offset componenet):

$$\begin{cases} Target: \quad f(x) = a_0\, \psi_0(x) + a_1\, \psi_1(x) + a_2\, \psi_2(x) + \cdots + a_m\, \psi_m(x) \\ \qquad\qquad\qquad\qquad + b_1\, \phi_1(x) + b_2\, \phi_2(x) + \cdots + b_n\, \phi_n(x) \\ Basis\ set: \quad \psi_m(x) = \left\{ \cos\left(\frac{m\pi}{L}x\right) \right\}, \quad where \quad m = 0, 1, 2, \cdots \quad positive\ integers \\ \qquad\qquad \phi_m(x) = \left\{ \sin\left(\frac{n\pi}{L}x\right) \right\}, \quad where \quad n = 1, 2, \cdots \quad positive\ integers \end{cases}$$



m = 0

Lowest frequency basis (global DC average)

m, n = 1

"Really rough" spatial frequency basis

m, n = 2

"Coarse" spatial frequency basis

m, n = 3

"Medium" spatial frequency basis

m, n = 4

"Fine" spatial frequency basis

Figure 2:

The $(m = 0)$ basis can be used to approximate the global average value of our target function $f(x)$, while the higher-frequency basis at $(m, n = 3)$ will be used to capture fine details within $f(x)$

Suppose our *half period for* $f(x)$ *in Figure 1 was* $L = 4.$ If you examine Figure 2, we see that the as the values of m and n increase, our Fourier reconstruction of $f(x)$ will be more and more accurate. This is because at high m, n-values, the spatial frequencies of the sine / cosine basis functions will be able to capture finer and finer details of $f(x)$.

A corollary statement of the above paragraph would be:

$If\ you\ can\ figure\ out\ how\ many\ basis\ terms\ you\ can\ throw\ away\ while\ still\ retaining\ a\ good\ approximation\ to\ f(x),\ \ you\ can\ drastically\ save\ computational\ time\ and\ memory\ in\ your\ reconstruction\ of\ f(x)$!!!

This begs the question: How many c's can we throw out *(In other words, how many of these c's can we set to zero ?)*

$$f(x) = a_0\ \psi_0(x) + a_1\,\psi_1(x) + a_2\,\psi_2(x) + \cancel{a_3}\,\psi_3(x) + \cancel{a_4}\,\psi_4(x) + \cdots + \cancel{a_m}\,\psi_m(x)$$
$$+ \cancel{b_1}\,\phi_1(x) + \cancel{b_2}\,\phi_2(x) + \cdots + \cancel{b_n}\ \phi_n(x)$$

<u>Filter out</u> unnecessary Fourier coefficients

Good enough for $f(x)$ reconstruction

Saves memory + computation time !!

This is the essence of ***<u>data compression</u>*** !... and this is where wavelets are useful in engineering !

2. The wavelet basis $\approx$ Cosines + sines on steroids !

Once you're comfortable with the concepts of orthogonal basis, Fourier coefficients, and change of basis (the Lecture #8 material involving $Qc = Ib$, $Q^{-1}AQc = b$, and $Ac = Sb$), you will have no problem understanding how digital wavelets work. Why ? This is because:

$$\begin{aligned} Digital\ wavelet\ basis\ vectors\ &= \quad orthogonal\ basis\ sets\ (can\ be\ normalized\ if necessary) \\ &\approx \quad Essentially\ a\ digital\ cousin\ of\ cosines\ and\ sines\,! \end{aligned}$$

Ok – let's start ! Suppose we know the values within a ***4-point segment*** of some ***digitized*** function $f(x)$ depicted in Figure 4:

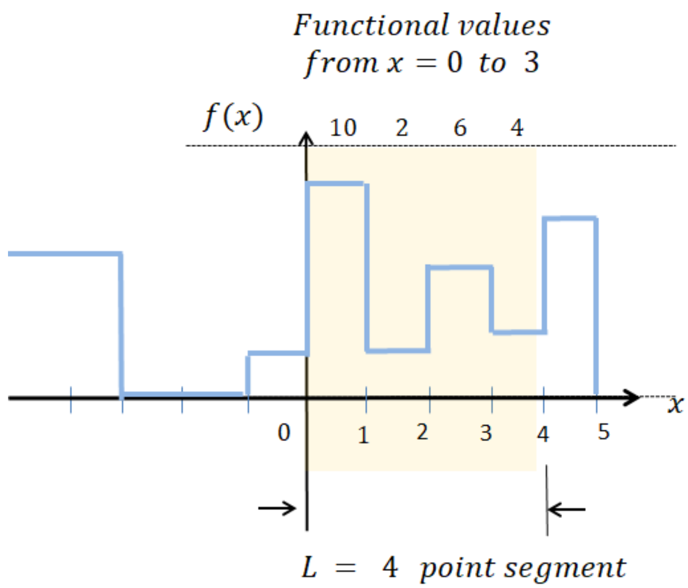


Figure 4: A 4-point segment of a *non-even, non-odd, non-periodic* digital function $f(x)$

The first couple things to notice are:

- The function $f(x)$ does not have to be periodic
- $f(x)$ exhibits neither odd nor even symmetry
- The functional values 10, 2, 6, 3 were sampled from the *left edge* of each histogram bar. In other words, our 4-point segment can be written as a target vector b :

$$Target\ vector\!: \quad b = \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 6 \\ 4 \end{bmatrix}$$

2a) The "4-point Haar wavelet" basis set: The Fourier series perspective

Just like Fourier series, the quintessential question here is: Can we reconstruct our target digital sample (vector b) as a *finite* linear combination of some digital basis vectors $\overrightarrow{w_n}$?

$$\begin{cases} Target: & b \;=\; c_0\,\overrightarrow{w_0} \;+\; c_1\,\overrightarrow{w_1} \;+\; c_2\,\overrightarrow{w_2} \;+\; c_3\,\overrightarrow{w_3} \\ \\ Basis\ set: & \overrightarrow{w_n}\ , \qquad where \quad n = 0, 1, 2, 3 \end{cases}$$

In this homework problem, we will explore using the "4-point Haar basis" set denoted by matrix W:

$$W \;=\; \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} \\ \begin{array}{cccc} \overrightarrow{w_0} & \overrightarrow{w_1} & \overrightarrow{w_2} & \overrightarrow{w_3} \end{array} \end{array} \;\rightarrow\; \begin{array}{c} Orthogonal\ column\ vectors \\ (but\ not\ normal\,!) \end{array}$$

Like all good orthogonal basis sets, they must obey a certain set of *orthogonality* and (norm)[2] rules !

$$\langle \overrightarrow{w_p},\ \overrightarrow{w_q} \rangle \;=\; \overrightarrow{w_p} \cdot \overrightarrow{w_q} \;=\; \begin{cases} 0 & & if\ \ p \neq q \quad (orthogonality) \\ \|\overrightarrow{w_0}\|^2 = 4 \qquad \|\overrightarrow{w_2}\|^2 = 2 & & \\ \|\overrightarrow{w_1}\|^2 = 4 \qquad \|\overrightarrow{w_3}\|^2 = 2 & & if\ \ p = q \qquad (norm)^2 \end{cases}$$

Hmmm….. this suggests the Haar wavelet coefficients c_n are also *generalized Fourier coefficients !!* *=)* Hence, we can easily find these coefficients by applying inner products ! Given:

$$\begin{array}{l} Target\ vector \\ reconstruction \end{array} : \qquad \vec{b} \;=\; c_0\,\overrightarrow{w_0} \;+\; \cdots \;+\; c_n\,\overrightarrow{w_n} \ , \qquad where\ n = 0, 1, 2, 3$$

Invoking inner products and orthogonality with respect to $\overrightarrow{w_n}$:

$$\langle \vec{b},\ \overrightarrow{w_n} \rangle = c_0 \underbrace{\cancel{\langle \overrightarrow{w_0},\ \overrightarrow{w_n} \rangle}}_{0} \;+\; c_1 \underbrace{\cancel{\langle \overrightarrow{w_1},\ \overrightarrow{w_n} \rangle}}_{0} \;+\; \cdots \;+\; c_n \underbrace{\langle \overrightarrow{w_n},\ \overrightarrow{w_n} \rangle}_{(norm)^2}$$

We can easily solve for c_n:

$$c_n \;=\; \frac{\langle \vec{b},\ \overrightarrow{w_n} \rangle}{\langle \overrightarrow{w_n},\ \overrightarrow{w_n} \rangle} \;=\; \frac{\vec{b} \cdot \overrightarrow{w_n}}{(norm)^2} \;=\; \frac{b^T\, w_n}{(norm)^2}$$

2b) The Haar wavelet basis set: A *multi-scale mixture* of cosines and sines!

Similar to what we've done in Figure 2 for the cosine basis, let's plot the Haar wavelet basis set against the backdrop of our data axis x with a data *word length* of $L = 4$ in Figure 5:

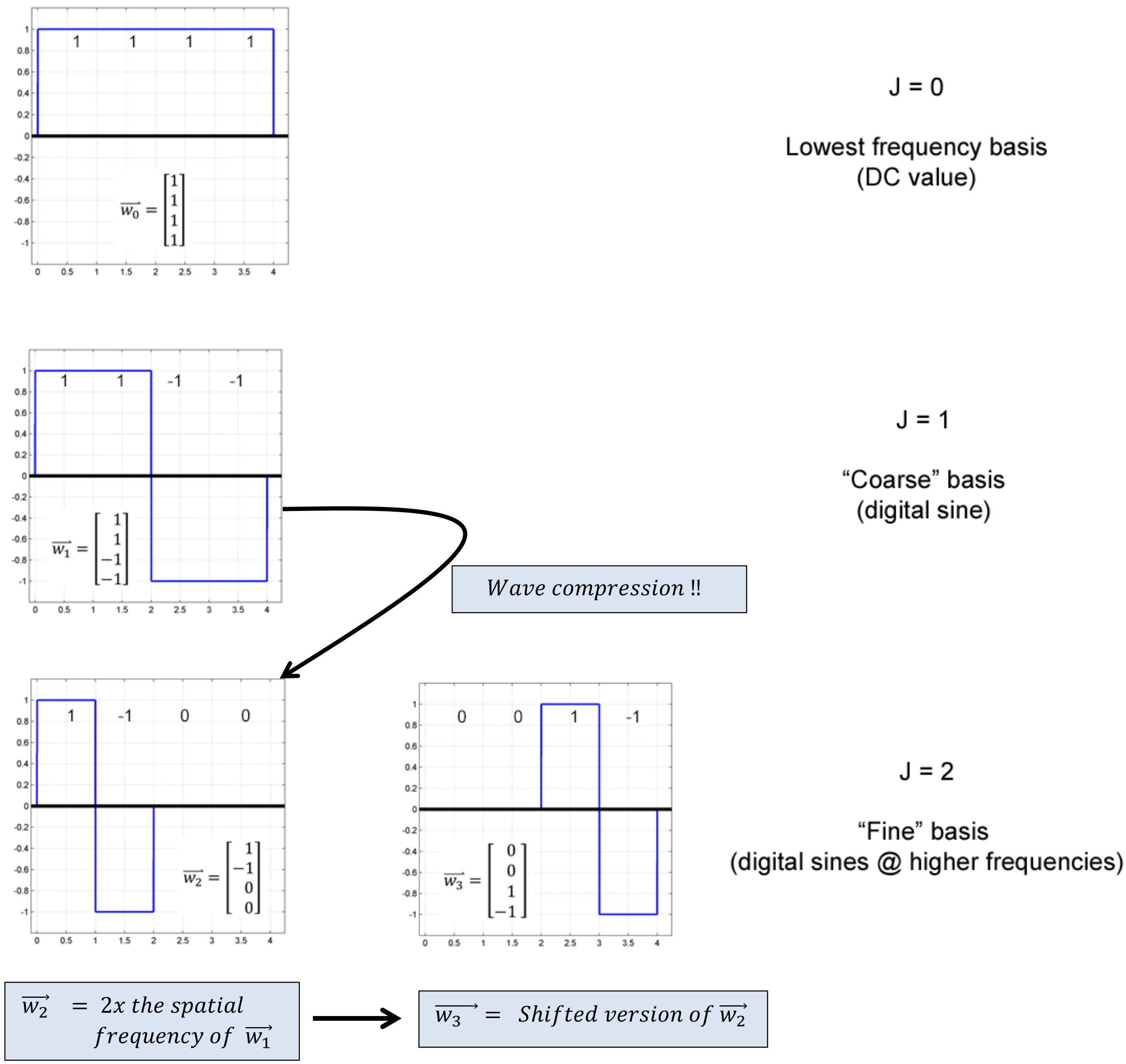


Figure 5: The 4-point Haar wavelet bases $\overrightarrow{w_n}$ look just like *digital* sines and cosines at different roughness scales.

After a quick peek at Figure 4, you might be thinking: *Wait a minute…. these digital square pulses kind of look like:*

$$i) \quad The \quad \overrightarrow{w_0} \ wavelet \ \leftrightarrow \quad Similar\ to\ the\ cosine\ DC\ basis \quad : \ cos\left(\frac{2^J\ \pi}{L}x\right), \qquad with \ \ J\ = 0$$

$$ii) \quad The \quad \overrightarrow{w_1} \ wavelet \ \leftrightarrow \quad Analogous\ to\ a\ coarse\ sine\ basis \quad : \ sin\left(\frac{2^J\ \pi}{L}x\right), \qquad with \ \ J\ = 1$$

$$iii) \quad The \quad \overrightarrow{w_2} \ wavelet \ \leftrightarrow \quad 2x - compressed\ \ version\ of\ \ J = 1 : \ sin\left(\frac{2^J\ \pi}{L}x\right), \qquad with \ \ J\ = 2$$

$$iv) \quad The \quad \overrightarrow{w_3} \ wavelet \ \leftrightarrow \quad A\ shifted\ version\ of\ of\ the\ above \quad : \ sin\left(\frac{2^J\ \pi}{L}\ x\right), \qquad with \ \ J\ = 2$$

where:

$$J \ = \ 'The\ \ granularity' \quad (or\ the\ level\ of\ roughness)\ of\ the\ spatial\ frequency\ of\ each\ set\ of\ basis\ vectors$$

$$= \ The\ spatial\ frequency\ will\ double \quad for\ an\ unit\ increase\ in\ \ J - level$$

Hence, the 4-point Haar wavelets basis are essentially distant digital cousins of the sines and cosines bases used in Fourier series !! And again, the best part about the basis set $\overrightarrow{w_n}$ is that they obey a set of <u>*orthogonality and (norm)*</u>^{2} relationships !

2c) You can "build" larger Haar wavelet basis sets for longer data word length !

Suppose we extend our data word length such that $L = 8$. If were going to reconstruct the yellow-shaded target data in Figure 6, we must expand our wavelet basis set in a way in which:

1) Our new Haar matrix W is an 8×8 matrix
2) Our new basis set $\overrightarrow{w_n}$ will be able to reconstruct "super-fine" details within our target data
3) Our new basis set $\overrightarrow{w_n}$ must contain a larger selection of "coarse" and "fine" spatial frequency basis vectors for us to choose from.

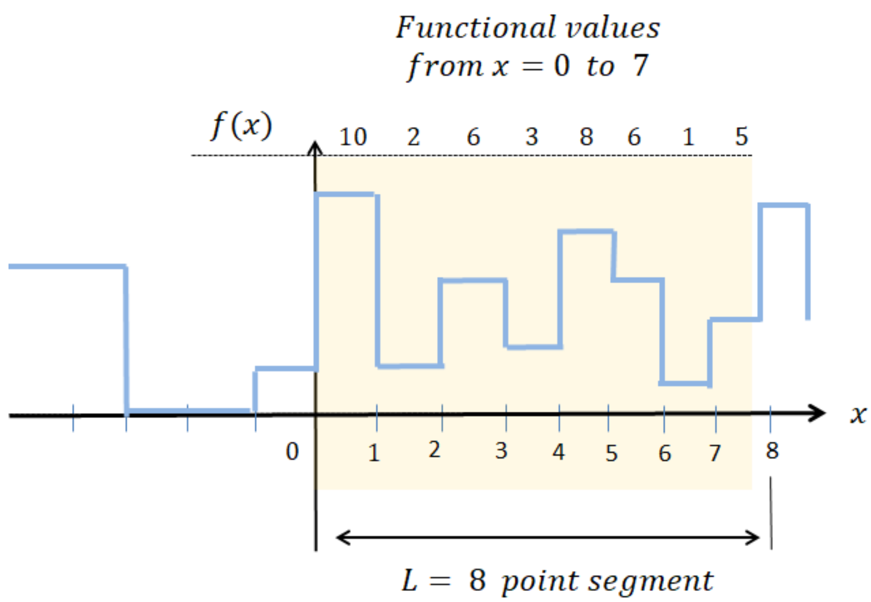
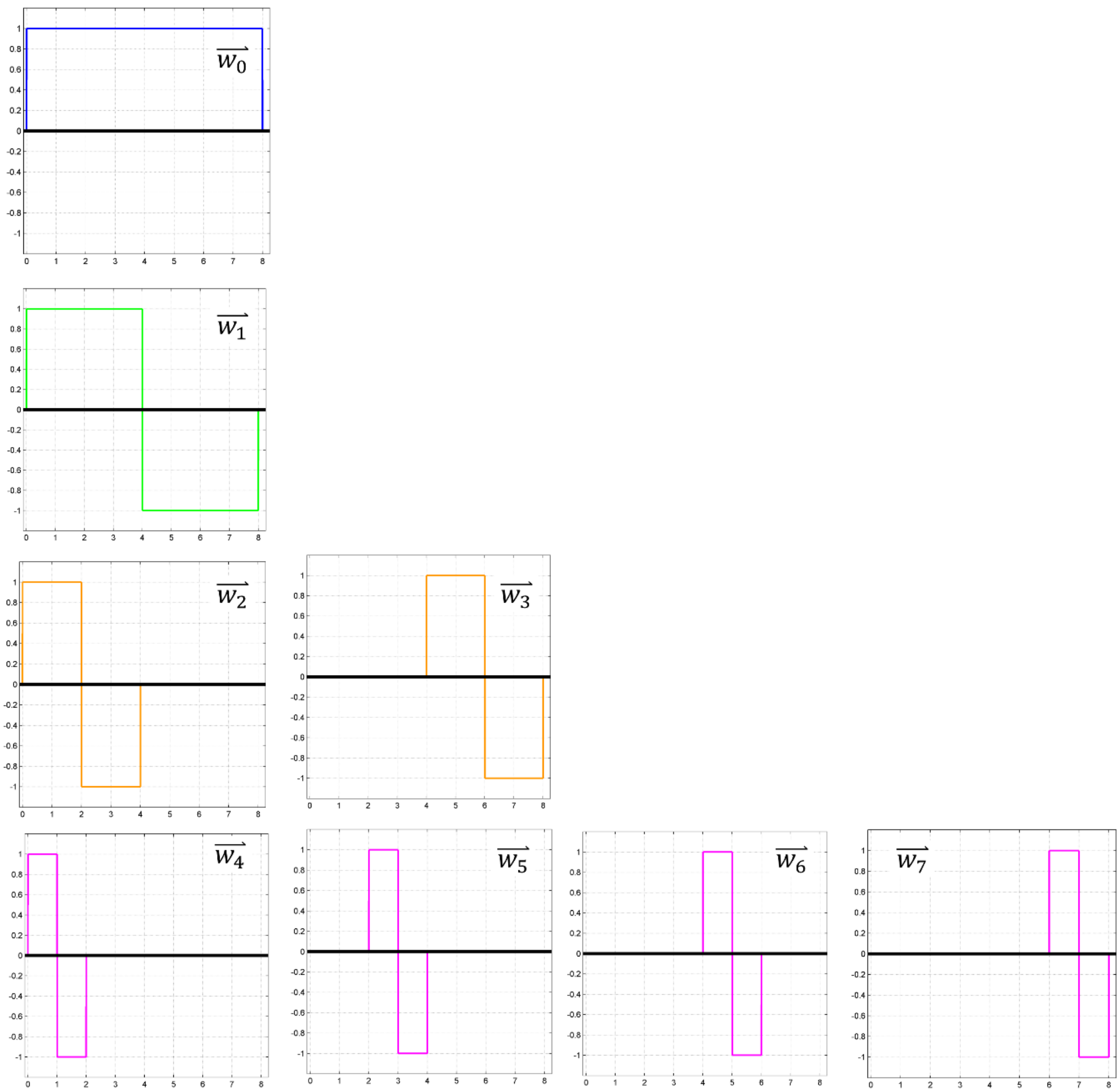


Figure 6: If we extend our "data word length" to 8 data entries, we also need to expand our Haar basis set to accommodate for the longer target vector.

Now, to accommodate an $b = \mathbb{R}^8$ target vector, we shall extend our 4-point Haar basis set by increasing the spatial frequency by 2 (such that we'll get 8 total basis vectors). The new basis set are depicted in Figure 7.

$$\text{Longer target vector:} \quad b = \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \\ f(6) \\ f(7) \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 6 \\ 3 \\ 8 \\ 6 \\ 1 \\ 5 \end{bmatrix}$$

Figure 7: The 8-point Haar basis set



J = 0

Lowest spatial freq basis (global DC average)

J = 1

Coarse basis (digital sine)

J = 2

Fine basis (2x freq of J = 1)

J = 3

Super-fine basis (4x freq of J = 1)

$$W = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix} \\ \begin{matrix} \overrightarrow{w_0} & \overrightarrow{w_1} & \overrightarrow{w_2} & \overrightarrow{w_3} & \overrightarrow{w_4} & \overrightarrow{w_5} & \overrightarrow{w_6} & \overrightarrow{w_7} \end{matrix} \end{array}$$

Just like the 4-point Haar basis set, the 8-point basis set also exhibits the special property: An unit increase in the J-level causes a 2x increase in spatial frequency in the basis vectors.

$i) \quad \overrightarrow{w_0} \quad \leftrightarrow \quad \textit{Similar to the cosine DC basis} \quad : \quad cos\left(\frac{2^J\ \pi}{L}x\right), \qquad with\ \ J\ = 0$

$ii) \quad \overrightarrow{w_1} \quad \leftrightarrow \quad \textit{Analogous to a coarse sine basis} \quad : \quad sin\left(\frac{2^J\ \pi}{L}x\right), \qquad with\ \ J\ = 1$

$iii) \quad \overrightarrow{w_2} \quad \leftrightarrow \quad 2x - compressed\ \ version\ of\ \ J = 1 : \quad sin\left(\frac{2^J\ \pi}{L}x\right), \qquad with\ \ J\ \ = 2$

$iv) \quad \overrightarrow{w_3} \quad \leftrightarrow \quad A\ shifted\ version\ of\ of\ the\ above \quad : \quad sin\left(\frac{2^J\ \pi}{L}\ x\right)$

$v) \quad \overrightarrow{w_4} \quad \leftrightarrow \quad 2x - compressed\ \ version\ of\ \ J = 2 : \quad sin\left(\frac{2^J\ \pi}{L}x\right), \qquad with\ \ J\ \ = 3$

$vi) \quad \overrightarrow{w_5} \quad \leftrightarrow \quad A\ shifted\ version\ of\ of\ the\ above \quad : \quad sin\left(\frac{2^J\ \pi}{L}\ x\right)$

$vii) \quad \overrightarrow{w_6} \quad \leftrightarrow \quad A\ shifted\ version\ of\ of\ the\ above \quad : \quad sin\left(\frac{2^J\ \pi}{L}\ x\right)$

$viii) \quad \overrightarrow{w_7} \quad \leftrightarrow \quad A\ shifted\ version\ of\ of\ the\ above \quad : \quad sin\left(\frac{2^J\ \pi}{L}\ x\right)$

This immediately suggests we can build a Haar matrix that can accommodate any target data sets with a word length of 2^J !! =)

$$\begin{array}{|c|}\hline To\ reconstruct\ a\ target\ vector \\ with\ a\ word\ length\ \ of\ \ \ L\ =\ 2^J \\ \hline\end{array} \rightarrow \left\{ \begin{array}{l} 1.\quad Extend\ the\ Haar\ basis\ set\ to\ \ J - levels \\ 2.\quad Within\ each\ level,\ \ you\ will\ have\ 2^{J-1}\ total\ basis \\ \qquad members\ containing\ equally - spaced\ \ shifts \\ \qquad within\ the\ interval\ [0,\ \ L] \end{array} \right.$$