A Micro → Macro Bridge for Self-Interacting Dark Matter: SU(2) Skyrme Solitons with Profile-Locked Predictions

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https://github.com/Voxtrium/GR-DM-Interaction-Theory

August 13, 2025

Abstract

We present a short, calibration–to–prediction framework for self–interacting dark matter (SIDM) based on an SU(2) Skyrme effective field theory (EFT). A single hedgehog soliton profile is solved once and reused everywhere, fixing three dimensionless "shape constants" that lock (i) the mass–size relation, (ii) the per–mass cross–section normalization at low velocity, and (iii) a finite–size suppression scale that governs the decline of σ_T/m with velocity. Using two measurements at dwarf scales—the particle mass m and the low–v anchor $(\sigma/m)_0$—we algebraically determine the EFT couplings (K_s, e) and then *predict* $\sigma_T/m(v)$ from dwarfs to clusters with no per–halo retuning. The prediction includes the s–wave effective–range expansion and a profile–derived transfer factor $C_T(k)$; an optional single internal mode with $m_\phi \sim R_*^{-1}$ provides a transparent, tunable high–v suppression. On the renormalization side, in the hidden–local–symmetry (HLS) matched scheme and background–field gauge, the one–loop divergences renormalize the $O(p^2)$ and $O(p^4)$ operators equally along the a=1 line, making eK_s marginal at that order and removing a leading RG–drift concern. We also record a thin FRW "bridge" that feeds these micro scales into a conserved continuity system via horizon–entropy source terms. The micro sector is unit–clean and predictive; the macro source is intentionally minimal and testable (and reduces to ΛCDM when gated off).

Keywords: Skyrme EFT, solitons, self–interacting dark matter, effective range, form factors, hidden local symmetry, cosmology.

1 Microphysics: SU(2) Skyrme soliton (locked profile)

Field and Lagrangian. Let $U(x)\in SU(2)$ and $L_\mu \equiv U^\dagger \partial_\mu U$. We use the Skyrme normalization

$$\mathcal{L} \;=\; \frac{F^2}{16}\,\mathrm{Tr}(L_\mu L^\mu) \;+\; \frac{1}{32e^2}\,\mathrm{Tr}\big([L_\mu, L_\nu]^2\big)\,, \qquad K_s \equiv F/2. \tag{1}$$

For bookkeeping one may equivalently write in $O(3)$ unit vector components H with $|H| = 1$: $\mathcal{L} = (K_s^2/2)(\partial H)^2 + (1/4e^2)(\partial H \times \partial H)^2$. Units: $[K_s] = \mathrm{GeV}$, $[e] = 1$, so $[\mathcal{L}] = \mathrm{GeV}^4$.

Hedgehog and dimensionless radius. Use $U(r) = \cos f(r) + i(\tau\cdot\hat{\mathbf{r}})\sin f(r)$ with $f(0) = \pi$, $f(\infty) = 0$ and $x \equiv eK_s r$. Solving the static EOM once by shooting yields a unique regular profile $f(x)$ with initial slope $f'(0) = -2.00754\ldots$.

Locked shape constants (from the solved profile). Define the dimensionless constants

$$c_m = 145.84694, \qquad c_R = 1.44786, \qquad c_\sigma = \frac{\pi c_R^2}{c_m} = 0.04515.$$

They fix the micro observables

$$m=\frac{c_m K_s}{e},\qquad R_*=\frac{c_R}{eK_s},\qquad \left(\frac{\sigma}{m}\right)_{\rm nat}=\frac{c_\sigma\, e}{K_s^3}. \tag{2}$$

Calibration to dwarf scale. Given a target mass m and anchor $(\sigma/m)_0$ at $v\to 0$ (in cgs), convert $(\sigma/m)_0$ to natural units and solve

$$K_s^2=\frac{c_m c_\sigma}{m\,(\sigma/m)_{\rm nat}},\qquad e=\frac{c_m K_s}{m}. \tag{3}$$

With $m=6.283\,\mathrm{GeV}$ and $(\sigma/m)_0=0.10\,\mathrm{cm}^2\,\mathrm{g}^{-1}$ we obtain

$$K_s\simeq 0.04785\,\mathrm{GeV},\quad e\simeq 1.11063,\quad R_*\simeq 27.25\,\mathrm{GeV}^{-1}\simeq 5.38\times 10^{-13}\,\mathrm{cm}.$$

Define $X\equiv eK_s$ and the reduced mass $\mu=m/2$. A natural velocity scale is $v_0\equiv(\mu R_*)^{-1}\simeq 0.0117\,c$ ($\sim$3500 km/s).

2 Predictive scattering: $\sigma_T/m(v)$ from one curve

Effective range (s–wave) and transfer factor. Let $k=\mu v$ (with v in units of c), and write $k\cot\delta_0(k)=-1/a+(r_e/2)k^2$. We set $r_e=\xi\,R_*$ with $\xi=\mathcal{O}(1)$. The per–mass transfer cross–section is

$$\frac{\sigma_T}{m}(v)=\frac{(\sigma/m)_0}{\left(1-\frac{1}{2}ar_ek^2\right)^2+(ak)^2}\ \times\ C_T(k), \tag{4}$$

where

$$C_T(k)=\frac{1}{4\pi}\int d\Omega\,(1-\cos\theta)\,|F_{\rm prof}(q)|^2,\qquad q=2k\sin(\theta/2), \tag{5}$$

and the *profile–derived* form factor is

$$F_{\rm prof}(q)=\frac{\int_0^\infty dx\,\varepsilon(x)\,j_0\big((q/X)\,x\big)}{\int_0^\infty dx\,\varepsilon(x)}. \tag{6}$$

Here $\varepsilon(x)$ is the (dimensionless) hedgehog energy density and j_0 a spherical Bessel function. This C_T falls as v^2 at low v and produces the finite–size suppression at cluster speeds.

Optional threshold (one internal mode). Eliminating a single nonpropagating internal mode of mass $m_\phi\sim R_*^{-1}$ multiplies the amplitude by $S_\phi(q)=1/(1+q^2/m_\phi^2)$, so $C_T\to C_T^\phi$ with $|F_{\rm prof}(q)|^2\ \to\ |F_{\rm prof}(q)|^2S_\phi(q)^2$. Choosing $m_\phi\approx R_*^{-1}$ gives a gentle suppression around $v\sim v_0$; smaller m_ϕ pushes the turnover to lower v.

What is fixed vs. what is chosen. Once $(m,(\sigma/m)_0)$ are set at dwarfs, (K_s,e,R_*) are fixed and the *shape* of $\sigma_T/m(v)$ follows from the universal profile. The only residual choice is ξ (order one). If stronger high–v suppression is demanded by data, m_ϕ is a single, transparent knob.

3 Renormalization hygiene (one loop, matched scheme)

In the HLS–matched EFT (parameters F,a,g_H) and background–field gauge, the divergent one–loop effective action below the vector mass is a gauge–invariant functional of the Maurer–Cartan forms and renormalizes the two independent invariants equally along the $a=1$ line:

$$\delta\mathcal{L}_{\rm div}\propto\Big[\,\mathcal{I}_2+\frac{1}{2e^2}\,\mathcal{I}_4\Big],\qquad\Rightarrow\qquad \mu\frac{d}{d\mu}\ln(eK_s)=0\quad\text{(one loop, }\ a=1). \tag{7}$$

Thus eK_s is marginal at that order; two–loop *sub*divergences preserve equality. This removes the leading RG–drift worry for R_*, m, and the anchor.

4 Macro bridge (thin, testable, optional)

For FRW with baryons ρ_b, DM $\rho_{\rm DM}$, standard radiation $\rho_r^{\rm std}$, GW $\rho_{\rm GW}$, and vacuum ρ_Λ,

$$H^2 = \frac{8\pi G}{3}\,(\rho_b + \rho_{\rm DM} + \rho_r^{\rm std} + \rho_{\rm GW} + \rho_\Lambda). \tag{8}$$

We add a conserved source system

$$\dot\rho_\Lambda = (\alpha_h/V_c)\,\dot S_{\rm hor}, \qquad \dot\rho_{\rm DM} + 3H\rho_{\rm DM} = p_{\rm DM}\,(\varepsilon_h/V_c)\,\dot S_{\rm hor}, \tag{9}$$

$$\dot\rho_{\rm GW} + 4H\rho_{\rm GW} = p_{\rm GW}\,(\varepsilon_h/V_c)\,\dot S_{\rm hor}, \qquad p_\Lambda + p_{\rm DM} + p_{\rm GW} = 1, \tag{10}$$

with micro–informed coefficients

$$\alpha_h = C_\alpha\Big(\frac{\kappa}{K_s}\Big)(|\Omega| R_*), \qquad \varepsilon_h = C_\varepsilon\Big(\frac{\kappa}{K_s}\Big)(|\Omega| R_*), \tag{11}$$

where $C_\alpha, C_\varepsilon = \mathcal{O}(1)$, $\kappa[\mathrm{GeV}^2]$, $|\Omega|[\mathrm{GeV}]$ (a rate), $R_*[\mathrm{GeV}^{-1}]$. If $\dot S_{\rm hor} \to 0$ the system reduces to ΛCDM. We track safety via

$$\epsilon_{\rm DE} \equiv \frac{(\alpha_h/V_c)\dot S_{\rm hor}}{3H\rho_\Lambda} \ll 1, \qquad f_{\rm inj} \equiv \frac{p_{\rm DM}(\varepsilon_h/V_c)\dot S_{\rm hor}}{3H\rho_{\rm DM}} \ll 1.$$

This section is deliberately thin; it is a bridge, not a claim of new gravity. (See the project README for unit checks and usage notes.)

5 What is already done; what is next

Done: (i) locked hedgehog profile and constants (c_m, c_R, c_σ); (ii) algebraic calibration (K_s, e) to $(m, (\sigma/m)_0)$; (iii) $\sigma_T/m(v)$ with effective range + *profile* transfer $C_T(k)$ and optional m_ϕ; (iv) one–loop $Z_2 = Z_4$ equality (matched/BFG, $a = 1$).
Next: (1) full phase–shift computation (beyond ER) to fix $(a, r_e, \ldots)$ directly from the soliton potential; (2) irreducible two–loop check (for completeness); (3) astrophysical comparison: dwarfs→clusters with no per–halo knobs; (4) empirical calibration of the thin FRW bridge if used (else set it to zero and the micro result stands on its own).

6 Predictions and falsifiability

- **Velocity trend:** one curve anchored at dwarfs predicts a *finite–size* decline toward clusters; scale $v_0 = (\mu R_*)^{-1}$ is fixed by the dwarf calibration.
- **Optional threshold:** $m_\phi \sim R_*^{-1}$ is a *single* physical knob to steepen the cluster suppression if demanded by data.
- **No per–halo retuning:** after dwarf anchoring, the same parameters apply to all systems (predictivity).

- **RG stability (one loop):** eK_s marginal $\Rightarrow$ the calibrated mass/size do not undergo a leading runaway.
- **FRW bridge (if used):** any nonzero drift in ρ_Λ or injection to $\rho_{\rm DM}$ scales with $(|\Omega| R_*)$ and can be gated to satisfy $|w_{\rm eff}+1| \lesssim$ few% and $f_{\rm inj} \ll 1$.

Data/code. Tables and constants are in the GitHub repository: `https://github.com/Voxtrium/GR-DM_Interaction_Theory`