Lecture 17

Linear Combination of Unitaries

- Going Beyond Unitary

Suppose we have N unitaries $U_1, \ldots U_N$, and coefficients $\alpha_1 \ldots \alpha_N \geq 0$, Denote $\alpha = \sum_{i=1}^N \alpha_i$.
Goal: Implement a linear function $M = \sum_{i=1}^N \alpha_i U_i$
This is not a unitary in general – we have to pay some cost.
Theorem(*LCU*). Let V be a unitary such that $V|0^n\rangle = \frac{1}{\sqrt{\alpha}} \sum_{i=1}^N \sqrt{\alpha_i}\,|i\rangle$
U be a the unitary $U = \sum_{i=1}^N |i\rangle\langle i| \otimes U_i$
Then $W := (V^\dagger \otimes I)U(V \otimes I)$ satisfies:

$$W|0^n\rangle|\psi\rangle = \frac{1}{\alpha}|0^n\rangle M|\psi\rangle + |\Phi^\perp\rangle$$

for all states $|\psi\rangle$, where $M = \sum_{i=1}^N \alpha_i U_i$ and $\Pi|\Phi^\perp\rangle = 0$, where $\Pi = |0^n\rangle\langle 0^n| \otimes I$.
Proof.

$$\begin{aligned}
W|0^n\rangle|\psi\rangle &= (V^\dagger \otimes I)U(V \otimes I)|0^n\rangle|\psi\rangle \\
&= (V^\dagger \otimes I)U\Big(\frac{1}{\sqrt{\alpha}}\sum_{i=1}^N \sqrt{\alpha_i}\,|i\rangle|\psi\rangle\Big) \\
&= (V^\dagger \otimes I)\Big(\frac{1}{\sqrt{\alpha}}\sum_{i=1}^N \sqrt{\alpha_i}\,|i\rangle \otimes U_i|\psi\rangle\Big) \\
&= \underbrace{\Pi(V^\dagger \otimes I)}_{\langle 0^n|V^\dagger = (V|0^n\rangle)^\dagger}\Big(\frac{1}{\sqrt{\alpha}}\sum_{i=1}^N \sqrt{\alpha_i}\,|i\rangle \otimes U_i|\psi\rangle\Big) + \underbrace{(I-\Pi)(V^\dagger \otimes I)\Big(\frac{1}{\sqrt{\alpha}}\sum_{i=1}^N \sqrt{\alpha_i}\,|i\rangle \otimes U_i|\psi\rangle\Big)}_{:=|\Phi^\perp\rangle,\ \Pi|\Phi^\perp\rangle = 0 \text{ because } \Pi(I-\Pi)=0} \\
&= \Big(|0^n\rangle\Big(\frac{1}{\sqrt{\alpha}}\sum_{j=1}^N \sqrt{\alpha_j}\,\langle j|\Big) \otimes I\Big)\Big(\frac{1}{\sqrt{\alpha}}\sum_{i=1}^N \sqrt{\alpha_i}\,|i\rangle \otimes U_i|\psi\rangle\Big) + |\Phi^\perp\rangle \\
&= |0^n\rangle \frac{1}{\alpha}\sum_{i=1}^N \alpha_i U_i|\psi\rangle + |\Phi^\perp\rangle \\
&= \frac{1}{\alpha}|0^n\rangle M|\psi\rangle + |\Phi^\perp\rangle
\end{aligned}$$

□

In this theorem, the unitary operator W can be regarded as a probabilistic implementation of M:

If we measure the first n qubits of $W\left|0^n\right\rangle|\psi\rangle$ and observe $\underbrace{0 \ldots 0}_{n}$, the state of the second register is proportional to $M|\psi\rangle$.

Success probability: $(\|M|\psi\rangle\| / \alpha)^2$

Quantum speedup: amplitude amplification. Repeat $O(\alpha / \|M|\psi\rangle\|)$ rounds of alternative reflections. This requires reflection about $|\psi\rangle$, which can be implemented by two uses of a unitary preparing $|\psi\rangle$: If $U_0\left|0^n\right\rangle=|\psi\rangle$, apply $U_0^{\dagger}(I-2|0\rangle\langle 0|) U_0$.

Furthermore the theorem can be generalize to $M=\sum_{i=1}^N \alpha_i T_i$ where T_i is a block of unitary.

Theorem(Non-unitary LCU). Let $M=\sum_{i=1}^N \alpha_i T_i$ with $\alpha_i>0$ for some linear operator T_i s.t. $U_i\left|0^t\right\rangle|\phi\rangle= \left|0^t\right\rangle T_i|\phi\rangle+\left|\Phi_i^{\perp}\right\rangle$ for all states $|\phi\rangle$, where each U_i is a unitary, t is a non-negative integer, and $\left(\left|0^t\right\rangle\left\langle 0^t\right| \otimes I\right)\left|\Phi_i^{\perp}\right\rangle=0$.

Given an algorihthm U_B for creating a state $|b\rangle$, there is a quantum algorithm that exactly prepares the state $M|b\rangle / \|M|b\rangle\|$ with constant probability, using $O(\alpha / \|M|b\rangle\|)$ times of U_B, U and V, where

$$U=\sum_i|i\rangle\langle i| \otimes U_i, \quad V\left|0^n\right\rangle=\frac{1}{\alpha} \sum_i \sqrt{\alpha_i}|i\rangle, \quad \alpha=\sum_i \alpha_i .$$

and an output bit indicating whether it was successful or not.

Proof. Denote $M^{\prime}=\sum_i \alpha_i V_i$. By the LCU theorem, denoting $W=\left(V^{\dagger} \otimes I\right) U(V \otimes I)$, we have $W\left|0^n\right\rangle|\psi\rangle= \frac{1}{\alpha}\left|0^n\right\rangle M^{\prime}|\psi\rangle+\left|\Phi^{\perp}\right\rangle$.

Now, consider the action of W on $|\psi\rangle=\left|0^t\right\rangle|\phi\rangle$:

$$\begin{aligned} W\left(0^{n+t}\right\rangle|\phi\rangle & =\frac{1}{\alpha}\left|0^n\right\rangle|(\sum_i \alpha_i V_i)\left|0^t\right\rangle|\phi\rangle+\left|\Psi^{\perp}\right\rangle . \\ & =\frac{1}{\alpha}\left|0^n\right\rangle\left|0^t\right\rangle(\sum_i \alpha_i T_i)|\phi\rangle+\frac{1}{\alpha}\left|0^n\right\rangle(\sum_i \alpha_i\left|\Phi_i^{\perp}\right\rangle)+\left|\Psi^{\perp}\right\rangle . \\ & =\frac{1}{\alpha}\left|0^{n+t}\right\rangle M(\phi\rangle+\left|\Theta^{\perp}\right\rangle .\end{aligned}$$

where $\left|\Theta^{\perp}\right\rangle$ satisfies $\left(\left|0^{n+t}\right\rangle\left\langle 0^{n+t}\right| \otimes I\right)\left|\Theta^{\perp}\right\rangle=0$.

Since we want to prepare a state proptional to $M|b\rangle$, we play in $|\phi\rangle=|b\rangle$. This gives

$$W\left|0^{n+t}\right\rangle|b\rangle=\frac{1}{\alpha}\left|0^{n+t}\right\rangle M|b\rangle+\left|\Theta^{\perp}\right\rangle=(\frac{1}{\alpha}\|M|b\rangle\|)\left|0^{n+t}\right\rangle \frac{M|b\rangle}{\|M|b\rangle\|}+\left|\Theta^{\perp}\right\rangle .$$

By amplitude amplification, we can prepare $\frac{M|b\rangle}{\|M|b\rangle\|}$ using $O(\alpha /\|M|b\rangle\|)$ the of $W=\left(V^{\dagger} \otimes I\right) U(V \otimes I)$ and U_B. □

We also prove a lemma for approximation:

Lemma. Let C be a Hermitian with $\left\|C^{-1}\right\| \leq 1$ (ie., all eigenvalues of C in absolute value $\geqslant 1$). Let D be an operator such that $\|C-D\| \leqslant \varepsilon<\frac{1}{2}$. Then the states $|x\rangle:=C(\psi\rangle /\|C|\psi\rangle\|$ and $|\tilde{x}\rangle=D|\psi\rangle /\|D|\psi\rangle\|$ satisfy

$$\||x\rangle-|\tilde{x}\rangle\| \leq 4 \varepsilon .$$

Proof. WLOG, assume $|\psi\rangle$ is normalized, ie., $\||\psi\rangle\|=1$. By the triangle inequality:

$$\||x\rangle-|\tilde{x}\rangle\|=\left\|\frac{C|\psi\rangle}{\|C|\psi\rangle\|}-\frac{D|\phi\rangle}{\|D|\psi\rangle\|}\right\| \leqslant\left\|\frac{C|\psi\rangle}{\|C|\psi\rangle\|}-\frac{C|\psi\rangle}{\|D|\psi\rangle}\right\|+\left\|\frac{C|\psi\rangle}{\|D|\psi\rangle)}-\frac{D|\psi\rangle}{\|D|\psi\rangle\|\|}\right\| .$$

Using the Triangle inequality again, we have: $\|C(\psi\rangle\| \leqslant \|D(\psi)\| + \|(C-D)|\psi\rangle\| \leq \|D|\psi\rangle\| + \varepsilon$, which yields

$$| \; \|D|\psi\rangle\| - \|C|\psi\rangle\| \; | \leqslant \varepsilon \quad \text{ and } \quad \|D|\psi\rangle\| \geqslant \|C|\psi\rangle\| - \varepsilon \geqslant 1-\varepsilon.$$

As a result:

$$\left\| \frac{C|\psi\rangle}{\|C(\psi\rangle\|} - \frac{C|\psi\rangle}{\|D(\psi\rangle)\|} \right\| = \|C|\psi\rangle\| \cdot \left| \frac{1}{\|C|\psi\rangle\|} - \frac{1}{\|D(\psi\rangle)\|} \right| = \frac{|\|D|\psi\rangle\| - \|C(\psi\rangle\| \; |}{\|D|\psi\rangle\|} \leqslant \frac{\varepsilon}{\|D|\psi\rangle)\|} \leqslant \frac{\varepsilon}{1-\varepsilon} \leqslant 2\varepsilon.$$

Similarly:

$$\left\| \frac{C|\psi\rangle}{\|D|\psi\rangle\|} - \frac{D|\psi\rangle}{\|D|\psi\rangle\|} \right\| \leqslant \frac{|\|C(\psi\rangle\| - \|D|\psi\rangle\| \; |}{\|D|\psi\rangle\|} \leqslant \frac{\varepsilon}{\|D|\psi\rangle\|} \leqslant \frac{\varepsilon}{1-\varepsilon} \leqslant 2\varepsilon.$$

In all, we have $\|(x) - |\tilde{x}\rangle\| \leq \| \frac{C|\psi\rangle}{\|c|\psi\rangle\|} - \frac{C|\psi\rangle}{\|D|\psi\rangle\|} \| + \| \frac{C|\psi\rangle}{\|D|\psi\rangle\|} - \frac{D|\psi\rangle}{\|D|\psi\rangle\|} \| \leqslant 4\varepsilon.$ $\square$