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Lecture 6

Quantum Fourier Transform and Phase Estimation

- Quantum Fourier transform
- Phase Estimation

1 Quantum Fourier transform

Hadamard transform: $|x\rangle \xmapsto{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$ where x is an integer modulo 2^n.
This is a Fourier transform over $\underbrace{\mathbb{Z}_2 \otimes \cdots \otimes \mathbb{Z}_2}_{n}$.

How about Fourier transform over $\mathbb{Z}_{2^n}$? That has the form:

$$|x\rangle \longmapsto \frac{1}{\sqrt{2^n}} \sum_{y \in \mathbb{Z}_{2^n}} e^{\frac{2\pi i x y}{2^n}} \; |y\rangle := |\tilde{x}\rangle$$

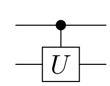
where $x \in \mathbb{Z}_{2^n}$ represents an integer modulo 2^n.
For $n = 1$, the transform is

$$|0\rangle \longmapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |1\rangle \longmapsto \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$
$$|x\rangle \xmapsto{H} \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}} e^{\frac{2\pi i x y}{2^n}}$$

In general, the $|\tilde{x}\rangle$ states form an orthonormal basis, the Fourier basis: $\langle \tilde{x} \mid \tilde{x}' \rangle = \delta_{x,x'}$.
When do we need the quantum Fourier transform?

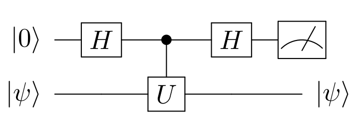
1.1 Phase estimation

- Given: Ability to implement a controlled unitary operator U and a quartum state $|\psi\rangle$ with $U|\psi\rangle = e^{i\theta}|\psi\rangle$.



- Problem: Learn θ.

1.2 Hadamard test



$$|0\rangle|\psi\rangle \xmapsto{H\otimes I} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\psi\rangle \xmapsto{\text{controlled-U}} \frac{1}{\sqrt{2}}(|0\rangle|\psi\rangle + |1\rangle U|\psi\rangle)$$
$$= \frac{1}{\sqrt{2}} \left(|0\rangle + e^{i\theta}|1\rangle\right)|\psi\rangle$$
$$\xmapsto{H\otimes I} \frac{1}{2}\left[(|0\rangle + |1\rangle) + e^{i\theta}(|0\rangle - |1\rangle)\right]|\psi\rangle = \left(\frac{1+e^{i\theta}}{2}|0\rangle + \frac{1-e^{i\theta}}{2}|1\rangle\right)|\psi\rangle$$

$$\Pr(0) = \left|\frac{1+e^{i\theta}}{2}\right|^2 = \frac{1}{4}\left[(1+\cos\theta)^2 + \sin^2\theta\right] \quad \Pr(1) = \left|\frac{1-e^{i\theta}}{2}\right|^2 = \sin^2\frac{\theta}{2}.$$
$$= \frac{1}{4}[2+2\cos\theta] = \cos^2\frac{\theta}{2}$$

If $\theta = 0$ or $\theta = \pi$, we learn θ perfectly: After we measure the state in the computational basis, we get $|0\rangle$ with probability 1 when $\theta = 0$ and $|1\rangle$ with probability 1 when $\theta = 1$.

If $0 < \theta < \pi$. learn the probability distribution by samples to get information of θ.

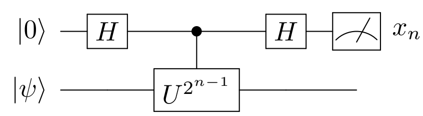
Now, consider $\theta = 2\pi \cdot \sum_{j=1}^{n} \frac{x_j}{2^j}$, $x_j \in \{0,1\}$. i.e., $\theta = 2\pi \cdot 0.\overbrace{x_1 \dots x_n}^{n \text{ bits}}$

Consider what happens if we apply U^{2^k}.

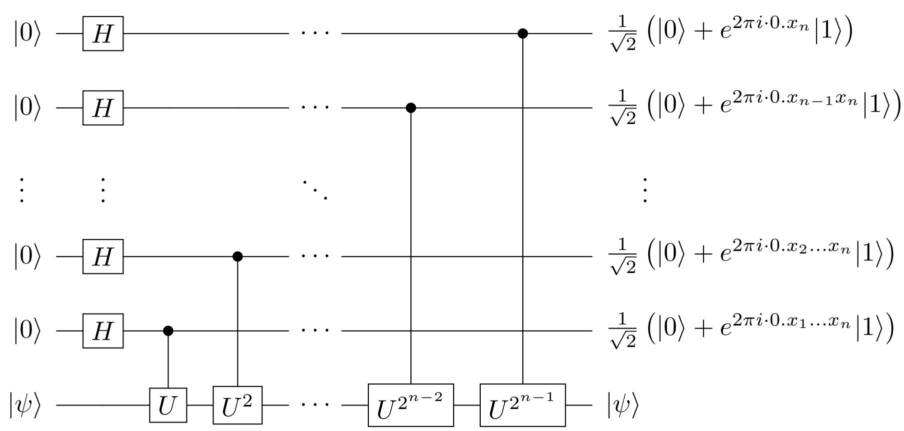
$$U^{2^k}|\psi\rangle = e^{i2^k\theta}|\psi\rangle = e^{2\pi i \cdot x_1 \dots x_k . x_{k+1} \dots x_n}|\psi\rangle = e^{2\pi i \cdot 0.x_{k+1}\dots x_n}|\psi\rangle$$

In particular, $U^{2^{n-1}}|\psi\rangle = e^{2\pi i \cdot 0.x_n}|\psi\rangle$.

In other words, $U^{2^{n-1}}|\psi\rangle = \begin{cases} |\psi\rangle & x_n = 0 \\ -|\psi\rangle & x_n = 1 \end{cases}$



Idea: Combine n such experiments for different exponents of U:



The output state is:

$$\bigotimes_{i=1}^{n} \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \cdot 0.x_{n+1-i}\dots x_n}|1\rangle\right)$$
$$= \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{\frac{2\pi i \cdot 2^{n-1} x y_{n-1}}{2^n}} |y_{n-1}\rangle\, e^{\frac{2\pi i \cdot 2^{n-2} x y_{n-2}}{2^n}} |y_{n-2}\rangle \cdots e^{\frac{2\pi i x y_0}{2^n}} |y_0\rangle$$
$$= \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^{n-1}} e^{\frac{2\pi i x y}{2^n}} |y\rangle = |\tilde{x}\rangle$$

Conclusion: Phase estimation can be solved by inverse of quantum Fourier transform.

Note: $QFT : |x\rangle \longrightarrow |\tilde{x}\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{\frac{2\pi i x y}{2^n}} |y\rangle \quad x \in \{0, 1, \cdots, 2^n - 1\}$

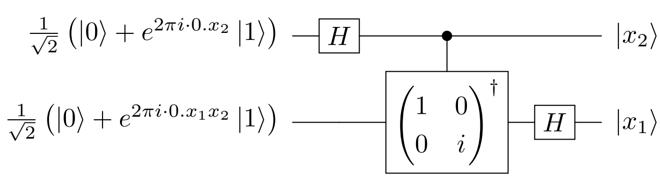
For phase estimation: Ideally, we want to have $x_1, \cdots, x_n$ directly as outputs. How do we implement this?

The first qubit is $\frac{|0\rangle + e^{2\pi i \cdot 0.x_n}|1\rangle}{\sqrt{2}} = H|x_n\rangle$. Therefore, applying H reveals x_n.
The second qubit is $\frac{|0\rangle + e^{2\pi i \cdot 0.x_{n-1}x_n}|1\rangle}{\sqrt{2}}$:

$$e^{2\pi i \cdot 0.x_{n-1}x_n} = e^{2\pi i \cdot 0.x_{n-1}} \cdot e^{2\pi i \cdot 0.0x_n} = e^{x_{n-1}\cdot \pi i} \cdot e^{x_n \cdot \frac{\pi i}{2}} = (-1)^{x_{n-1}} \cdot i^{x_n}.$$

We can remove the dependence on x_n :
If $x_n = 0$, do I; if $x_n = 1$, do $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}^\dagger$.
Then the state becomes $\frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x_{n-1}}|1\rangle)$. Then Hadamard gate H reveals x_{n-1}.

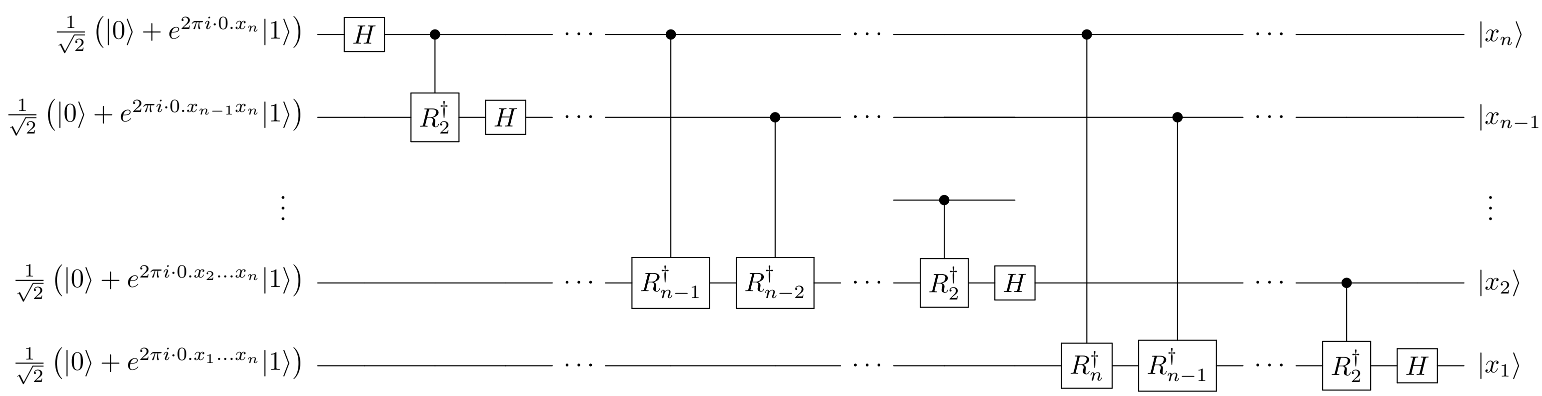
Inverse QFT with $n = 2$:



For convenience, denote $R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^k}} \end{pmatrix}$.
Move generally, since $e^{2\pi i \cdot 0.x_{n-k+1}\dots x_n} = e^{2\pi i\left(\frac{x_{n-k+1}}{2^1} + \frac{x_{n+12}}{2^2} + \dots + \frac{x_n}{2^k}\right)}$, the k^{th} qubit is

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i \cdot 0.x_{n-k+1}\dots x_n}|1\rangle\right) = R_k^{x_n} \cdots R_3^{x_{n-k+3}} R_2^{x_{n-k+2}} H \left|x_{n-k+1}\right\rangle.$$

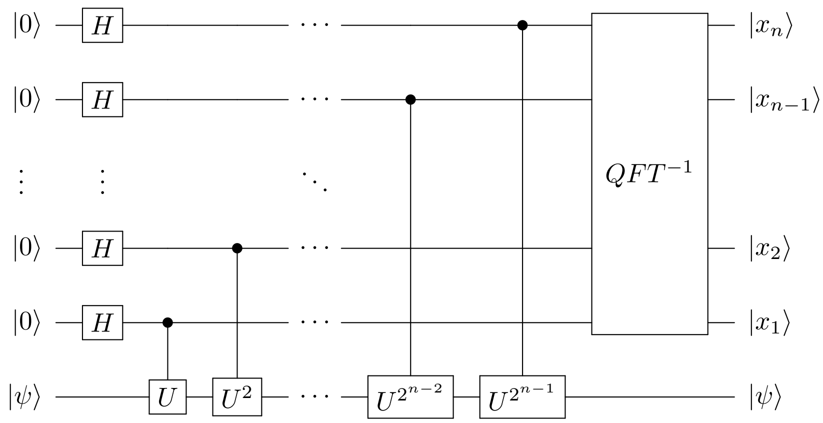
The circuit of inverse QFT:



Gate complexity of QFT^{-1}: $O(n^2)$

2 Phase Estimation

Big picture of phase estimation:



If $U|\psi\rangle = e^{2\pi i \cdot 0.x_1 \dots x_n}|\psi\rangle$, this works. How about $e^{i\varphi}$ for general $\varphi \in [0, 2\pi)$?

$$|0\rangle^{\otimes n} \xmapsto{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \qquad x = \sum_{i=1}^{n} x_i \cdot 2^{n-i} \qquad |x_i\rangle \xmapsto{c-U^{2^{i-1}}} \begin{cases} |x_i\rangle & \text{if } x_i = 0 \\ e^{i\varphi 2^{i-1}} |x_i\rangle & \text{if } x_i = 1 \end{cases}$$

$$\xmapsto{c-U_s} \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} e^{i\varphi x}|x\rangle \qquad |x_1 \dots x_n\rangle \longmapsto \prod_{i=1}^{n} e^{i\varphi_{x_i} 2^{i-1}} |x_i\rangle$$

The QFT^{-1} is $\sum_{x=0}^{2^n-1} |x\rangle \langle \tilde{x}| = \frac{1}{\sqrt{2^n}} \sum_{x,y=0}^{2^n-1} e^{-\frac{2\pi i x y}{2^n}} |x\rangle \langle y|$.

$$\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} e^{i\varphi x}|x\rangle \xmapsto{QFT^{-1}} \frac{1}{2^n} \sum_{x,y=0}^{2^n-1} e^{i\varphi x} e^{-\frac{2\pi i x y}{2^n}} |y\rangle$$

$$= \sum_{y=0}^{2^n-1} \alpha_y |y\rangle \quad \alpha_y := \frac{1}{2^n} \sum_{x=0}^{2^n-1} e^{i-\frac{2\pi y}{2^n}x} e^{i\varphi x} = \frac{1}{2^n} \sum_{x=0}^{2^n-1} e^{i\left(\varphi - \frac{2\pi y}{2^n}\right)x}.$$

where α_y is a geometric series. Denote $\tilde{\varphi} = \varphi - \frac{2\pi y}{2^n}$:

$$\sum_{x=0}^{2^n-1} e^{i\tilde{\varphi}x} = \frac{1 - e^{i\tilde{\varphi}2^n}}{1 - e^{i\tilde{\varphi}}} = \frac{e^{i\tilde{\varphi}2^{n-1}} \left(e^{-i\tilde{\varphi}2^{n-1}} - e^{i\tilde{\varphi}2^{n-1}}\right)}{e^{i\tilde{\varphi}/2} \left(e^{-i\tilde{\varphi}/2} - e^{i\tilde{\varphi}/2}\right)} \Rightarrow \left| \sum_{x=0}^{2^{n-1}} e^{i\tilde{\varphi}x} \right|^2 = \frac{\sin^2\left(\tilde{\varphi}2^{n-1}\right)}{\sin^2(\tilde{\varphi}/2)}$$

Therefore: $\Pr(y) = |\alpha_y|^2 = \frac{1}{2^{2n}} \frac{\sin^2\left(\left(\varphi - \frac{2\pi y}{2n}\right) \cdot 2^{n-1}\right)}{\sin^2\left(\varphi - \frac{2\pi y}{2}\right) \cdot \frac{1}{2}\right)}$. $\quad \frac{\sin mx}{\sin x} \to m$ when $x \to 0$.

This distribution is tightly peaked around those y for which $\frac{2\pi y}{2^n} \approx \varphi$.

If $\varphi = \frac{2\pi y}{2^n}$, $\quad \Pr(y) = \frac{1}{2^{2n}} \cdot (2^{n-1} \cdot 2)^2 = 1$.

Claim: Let $\frac{2\pi k}{2^n} \leqslant \varphi \leqslant \frac{2\pi(k+1)}{2^n}$. Then the probebility of outputting either k or $k+1$ is at least $\frac{8}{\pi^2}$.

Proof. The probability of success is $\Pr(k) + \Pr(k+1)$.

$$\Pr(k) + \Pr(k+1) = \frac{1}{2^{2n}} \left(\frac{\sin^2\left(2^{n-1}\varphi - \pi k\right)}{\sin^2\left(\frac{\varphi}{2} - \frac{\pi k}{2^n}\right)} + \frac{\sin^2\left(2^{n-1}\varphi - \pi(k+1)\right)}{\sin^2\left(\frac{\varphi}{2} - \frac{\pi(k+1)}{2^n}\right)} \right)$$

$$\left(\text{min when } \varphi = \frac{2\pi\left(k+\frac{1}{2}\right)}{2^n}\right) \geqslant \frac{2}{2^{2n}} \frac{\sin^2\left(\pi\left(k+\frac{1}{2}\right) - \pi k\right)}{\sin^2\left(\frac{\pi\left(k+\frac{1}{2}\right)}{2^n} - \frac{\pi k}{2^n}\right)}$$

$$= \frac{1}{2^{2n-1}} \cdot \frac{1}{\sin^2 \frac{\pi}{2^{n+1}}}$$

$$(\sin x \leqslant x) \geqslant \frac{1}{2^{2n-1}} \cdot \frac{1}{\left(\frac{\pi}{2^{n+1}}\right)^2} = \frac{8}{\pi^2}.$$

□

Summary: Given $|\psi\rangle$ with $U|\psi\rangle = e^{i\varphi}|\psi\rangle$. we can produce an estimate of φ thet differs from the true value by at most $\varepsilon\left(\frac{2\pi}{2^n}\right)$ with probability $\geqslant \frac{8}{\pi^2}$. This use QFT^{-1} with gate complexity $O\left((\log\frac{1}{\epsilon})^2\right)(n^2)$ and $O(1/\varepsilon)$ controlled $- U_s\left(2^n\right)$.