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Lecture 3

More on Basic Definitions, Protocols and Quantum Circuits

- Quantum measurements
- Superdense coding
- Teleportation
- Circuit representation
- 1-qubit gate

Last time we've introduced: quantum states, dynamics, and compositions.
How to determine what state we have?

1 Orthonormal basis

A group of vectors $\{|\varphi_i\rangle\}$ such that:

$$\langle\varphi_i|\varphi_j\rangle = \delta_{ij} \quad \text{and} \quad \sum_i |\varphi_i\rangle\langle\varphi_i| = \mathbb{I}$$

For example: For 1-qubit system, $\{|0\rangle, |1\rangle\}$ is orthonormal.

Denote : $\begin{cases} |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{cases}$ $\quad |+\rangle\langle+| + |-\rangle\langle-| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Thus, $\{|+\rangle, |-\rangle\}$ is orthonormal.
For 2-qubit system, $\{|+0\rangle, |+1\rangle, |-0\rangle, |-1\rangle\}$ is orthonormal.

- Let $B = \{|\varphi_i\rangle\}$ be an orthonormal basis for the state space of a quantum system. Then if the system is in a state $|\psi\rangle = \sum_i a_i |\varphi_i\rangle$, a quantum measurement with respect to B outputs "i" with probability $|a_i|^2$, leaving the system in state $|\varphi_i\rangle$.

  Notice that the probability $|a_i|^2 = |\langle\varphi_j|\psi\rangle|^2$.

- If the system is in $\sum_i a_i |\varphi_i\rangle \otimes |\omega_i\rangle$ where the states $|\omega_i\rangle$ is normalized. Then a measurement of the first subsystem w.r.t. B gives "i" with probability $|a_i|^2$, leaving the system in state $|\varphi_i\rangle \otimes |\omega_i\rangle$.

- Global phase does not matter.

For example: Measure $|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$ in the basis $\{|+\rangle, |-\rangle\}$.

$$\langle +|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \frac{1+\sqrt{3}}{2\sqrt{2}} \Rightarrow \Pr(+) = |\langle +|\psi\rangle|^2 = \frac{2+\sqrt{3}}{4} \quad \Pr(-) = 1 - \Pr(+) = \frac{2-\sqrt{3}}{4}$$

$$\text{Or}: |\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle = \frac{1}{2}\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right) + \frac{\sqrt{3}}{2}\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}}|+\rangle + \frac{1-\sqrt{3}}{2\sqrt{2}}|-\rangle$$

We could also realize this by a unitary followed by a computational basis measurement.

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \begin{matrix} H|+\rangle \longmapsto |0\rangle \\ H|-\rangle \longmapsto |1\rangle \end{matrix} \quad H\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{3}}{2\sqrt{2}} \\ \frac{1-\sqrt{3}}{2\sqrt{2}} \end{pmatrix}$$

2 Partial measurement

For example: $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, measure 1^{st} qubit in computational basis:

$$\Pr(0) = \frac{1}{2}, \ |\psi\rangle \longmapsto |00\rangle \quad \Pr(1) = \frac{1}{2}, \ |\psi\rangle \longmapsto |11\rangle$$

For example: $|\psi\rangle = \sqrt{\frac{1}{10}}|00\rangle + \sqrt{\frac{2}{10}}|01\rangle + \sqrt{\frac{3}{10}}|10\rangle + \sqrt{\frac{4}{10}}|11\rangle$, measure 1^{st} qubit in computational basis:

$$|\psi\rangle = \sqrt{\frac{3}{10}}|0\rangle \otimes (\sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle) + \sqrt{\frac{7}{10}}|1\rangle \otimes (\sqrt{\frac{3}{7}}|0\rangle + \sqrt{\frac{4}{7}}|1\rangle)$$

$$\Pr(0) = \frac{3}{10}, \ \Pr(1) = \frac{7}{10}$$

If we measure both qubits in the computational basis, we get:

$$\Pr(00) = \frac{1}{10}, \ \Pr(01) = \frac{2}{10}, \ \Pr(10) = \frac{3}{10}, \ \Pr(11) = \frac{4}{10}.$$

Notice that

$$\Pr(0) = \Pr(00) + \Pr(01), \Pr(1) = \Pr(10) + \Pr(11).$$

3 Superdense coding

Suppose Alice and Bob share an entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$(an ebit), where Alice has the first qubit and Bob has the second qubit.

Suppose Alice applies one of $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ ZX = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = iY$

$$(I \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\beta_{00}\rangle$$

$$(X \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) = |\beta_{01}\rangle$$

$$(Z \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\beta_{10}\rangle$$

$$(ZX \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(-|10\rangle + |01\rangle) = |\beta_{11}\rangle$$

$\{|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle\}$ forms an orthonormal basis, the Bell basis.

A procedure that Alice can send two classical bits to Bob:

1. If Alice wants to send $zx \in \{0,1\}^2$ to Bob, she performs $Z^z X^x$ on the first qubit of $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and obtains $|\beta_{zx}\rangle$.
2. Alice sends her qubit to B.
3. B measures in the Bell basis $\longrightarrow$ learn x and z

The procedure requires an ebit, sends a qubit, and Bob can decode two classical bits.

4 Teleportation

Now, Alice wants to send Bob a qubit, but she only has a classical channel. Suppose that they also share a $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, a Bell pair.

Mathematically, Alice has $|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle$. They start with $|\psi\rangle \otimes |\beta_{00}\rangle$, where Alice has qubits 1 & 2, Bob has qubit 3.

Idea: Alice measures qubit 1 & 2 in Bell basis:

$$\begin{aligned}
|\psi\rangle |\beta_{00}\rangle &= (a_0 |0\rangle + a_1 |1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
&= \frac{1}{\sqrt{2}}(a_0 |000\rangle + a_0 |011\rangle + a_1 |100\rangle + a_1 |111\rangle) \\
&= \frac{1}{2}(a_0(|\beta_{00}\rangle + |\beta_{10}\rangle) |0\rangle + a_0(|\beta_{01}\rangle + |\beta_{11}\rangle) |1\rangle + a_1(|\beta_{01}\rangle - |\beta_{11}\rangle) |0\rangle + a_1(|\beta_{00}\rangle - |\beta_{10}\rangle) |1\rangle) \\
&= \frac{1}{2}[|\beta_{00}\rangle (a_0 |0\rangle + a_1 |1\rangle) + |\beta_{01}\rangle (a_0 |1\rangle + a_1 |0\rangle) + |\beta_{10}\rangle (a_0 |0\rangle - a_1 |1\rangle) + |\beta_{11}\rangle (a_0 |1\rangle - a_1 |0\rangle)] \\
&= \frac{1}{2}[|\beta_{00}\rangle |\psi\rangle + |\beta_{01}\rangle X |\psi\rangle + |\beta_{10}\rangle Z |\psi\rangle + |\beta_{11}\rangle XZ |\psi\rangle]
\end{aligned}$$

Alice gets each outcome with probability $\frac{1}{4}$.

Procedure:

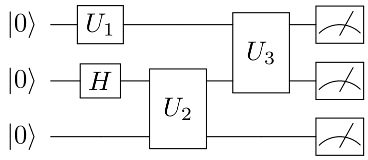
1. Alice measures qubit 1 & 2 in Bell basis.
2. If the outcome is $\begin{cases} |\beta_{00}\rangle \\ |\beta_{01}\rangle \\ |\beta_{10}\rangle \\ |\beta_{11}\rangle \end{cases}$, Alice sends $xz = \begin{cases} 00 \\ 10 \\ 01 \\ 11 \end{cases}$ to Bob.
3. Bob applies $Z^z X^x$ to qubit 3.

- Superdense coding: The procedure requires an ebit, sends a qubit, and Bob can decode two classical bits.
- Teleportation: The procedure requires an ebit, sends two classical bits, and Bob can decode a qubit.

In the following sections, we discuss about chapter of Quantum Circuits, which should have been scheduled in the next lecture, but was put here corresponding with the lecture's pace.

How do we draw quantum computing procedures, i.e., quantum cricuits?

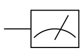
For example:



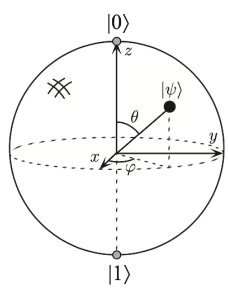
Quantum circuits compute from left to right, the above example illustrates such a calculation process:

$$(U_3 \otimes I)(I \otimes U_2)(U_1 \otimes H \otimes I) |000\rangle .$$

Common rules:

- Quantum circuits run from left to right.
- Qubits start in $|0\rangle$(unless specifically mentioned).
-  for quantum measurement.
- Measure normally in the computational basis.
- — for quantum wire, ═ for classical wire.

5 Single-qubit gates



$$|\psi\rangle = \cos\tfrac{\theta}{2}|0\rangle + e^{i\phi}\sin\tfrac{\theta}{2}|1\rangle$$

$$2 \times 2 \text{ unitary matrix} \longleftrightarrow \text{ rotation on Bloch sphere}$$

$$\text{Recall } X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X^2 = Y^2 = Z^2 = I \;\; \text{Consider :}$$

$$\begin{aligned} R_X(\theta) = e^{-i\frac{\theta}{2}X} &= I - i\frac{\theta}{2}X + \frac{(i\frac{\theta}{2})^2}{2!}X^2 - \frac{(i\frac{\theta}{2})^3}{3!}X^3 + \frac{(i\frac{\theta}{2})^4}{4!}X^4 + \cdots \\ &= I\left(1 - \frac{(\frac{\theta}{2})^2}{2!} + \frac{(\frac{\theta}{2})^4}{4!} + \cdots\right) - iX\left(\frac{\theta}{2} - \frac{(\frac{\theta}{2})^3}{3!} + \frac{(\frac{\theta}{2})^5}{5!} + \cdots\right) \\ &= \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}X = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \end{aligned}$$

Similarly, we can define $R_Y(\theta) = e^{-i\frac{\theta}{2}Y}$, $R_Z(\theta) = e^{-i\frac{\theta}{2}Z}$:

$$R_Y(\theta) = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_Z(\theta) = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Z = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$$

Consider acting $R_Z(\theta)$ on $|\psi\rangle = \cos\frac{\xi}{2}|0\rangle + e^{i\phi}\sin\frac{\xi}{2}|1\rangle$:

$$R_Z(\theta)|\psi\rangle = e^{-i\frac{\theta}{2}}\cos\frac{\xi}{2}|0\rangle + e^{i\frac{\theta}{2}}e^{i\phi}\sin\frac{\xi}{2}|1\rangle = e^{-i\frac{\theta}{2}}\left(\cos\frac{\xi}{2}|0\rangle + e^{i(\theta+\phi)}\sin\frac{\xi}{2}|1\rangle\right)$$

$$\Longrightarrow R_Z(\theta) \text{ is a rotation about z} - \text{axis by } \theta$$

More generally, $\hat{n} = (n_x, n_y, n_z) \in \mathbb{R}^3$, $R_{\hat{n}}(\theta) = e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)}$.
Fact: Any 1-qubit gate, apply $R_{\hat{n}}(\theta)$ is equivalent to rotating angle θ about $\hat{n}$.