Exp. C-1 Mathematical Models of Chaotic Physical Systems

(Simulation with software EXCEL)

References:
1. EXCEL Help Function or Manual
2. Classical Dynamics by J. B. Marion and S. T. Thornton
   (Sections 4.6, 4.7, and 4.8 of Chapter 4)
3. Melissinos, “Experiments in Modern Physics”, 2^{nd} ed., Sec. 3.7

I. Objectives: *To study “chaos” using a mathematical model and a simple “dripping faucet” experiment.*

II. Simulation Experiment
(You will view a video on Chaos at the beginning of the experiment.)

Part 1. Using EXCEL (Skip if you are familiar with Excel)

EXCEL software enables one to do various numerous computations in scientific investigations without writing computer programs. The graphic capability of the spreadsheet is useful in plotting graphs. Use the on-line help on EXCEL to learn the following basic operations:

a. To enter a value or formula into a cell.
b. How to edit the content of a cell.
c. To copy a formula from one cell to a range of cells.
d. To know the available internal mathematical functions and how to use them.
e. To insert or delete a column or row.
f. To graph an x-y plot.
g. To print either a spreadsheet or a graph.
h. To save a spreadsheet or a graph onto a file.

Part 2. To study the logistic equation using EXCEL

Most phenomena we observe in nature are related to non-linear problems such as the reversals of the earth's magnetic field, dripping of a faucet, and weather prediction. A system that is deterministic and yet unpredictable can be traced to the system's sensitivity to the initial conditions. A chaotic motion may exhibit certain characteristics such the period doubling and strange attractor. In this lab simulation, one is to study the logistic equation or population growth model by the use of a spreadsheet.

The population growth can be represented by the logistic equation: $x_{n+1} = r\, x_n\, (1-x_n)$ where $0< x_n <1$ and $r<4$. The non-linearity term is rx_n^2. It was found that the period doubling regime is below $r=3.56994$. A chaotic region may be found for $r>3.56994$.
The following steps may be used to study the logistic equation:

a. Study of the period doubling for different r and x.
This can be done rather easily by the use of EXCEL and its graphic capability. Tabulate the values and generate plots of x_n vs. n for $0 < r < 3.56994$.

b. Exploring the chaotic region
Tabulate and plot x_n versus n for $r>3.56994$.
Investigate the behavior of X_n by varying x_o by a small amount such as from 0.8 to 0.801. Compare the change of x_n graphically and find the n where x_n's are changed by about 1%.

c. Plot x_{n+1} vs x_n for the cases studied in steps a and b, and observe the order in chaos.

The period doubling process continues until the value r=3.56994. The sequence of period-doubling parameter values approaches a limiting value called the Feigenbaum number: $(r_i-r_{i-1})/(r_{i+1}-r_i) = 4.66920$, where r_i refers to the value of r when the i^{th} period-doubling initiates.

Do computations of x_n by varying r with fixed x_o and determine the Feigenbaum number. This bifurcation parameter scaling has been a universal property for certain class of chaotic dynamical process.

(d) Lyapunov exponents
A small initial difference, ε_o, in x_o will grow exponentially for chaos in a deterministic system, i.e. $\varepsilon_n = \varepsilon_o e^{\lambda n}$ where $\varepsilon_n = x_n(x_o+\varepsilon_o) - x_n(x_o)$ and λ is Lyapunov exponent. A practical method of determining λ is through the use of the following formula:

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \ln\left|\frac{df(x_k)}{dx}\right|$$

$$\textit{where } f(x_n) = x_{n+1} = rx_n(1-x_n) \textit{ in this case.}$$

Compute λ and verify that $\lambda>0$ in the chaotic region and $\lambda<0$ in the regular region.

(e) (optional) Plot x_n's versus r.

<u>Part 3. Dripping faucet</u>

Measure the time between formation of drops in a dripping faucet (or burette) and observe the onset of chaotic behavior as a function of water flow. Repeat the analysis of Part 2 on this data.

(Original 1/14/04 by R.S. Jin, revised 1/13/07 M.Hohlmann)