PHY 4210-01 Senior Lab
Lab C1: Mathematical Models of Chaotic Physical Systems

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Abstract

Chaotic behavior was studied by creating mathematical models and altering key parameters. A simple chaotic model was then observed by measuring time intervals of droplet formation from a leaking faucet. A small change in initial conditions can equate to much larger changes in the behavior of a system.

Contents

1 The Logistic Equation

A population model can be represented with equation 1, where $0 < x_n < 1$ and $r < 4$. These parameters can be modified to observe the onset of chaotic behavior. The period doubling and chaotic regions will be investigated.

$$x_{n+1} = rx_n(1 - x_n) \tag{1}$$

1.1 Period Doubling Region of the Logistic Equation

A period doubling region exists for $r < 3.56994$. Plots of x_n versus x are produced for $0 < r < 3.56994$. Figure 1 shows this behavior for a range of r values. For $r < 3$, the set of x_n converge to a particular value, although they may fluctuate for the first few iterations. Around $r = 3$ the values of x_n oscillate between two values, as seen in figure 1. Around $r = 3.5$, the values of x_n begin oscillating between 4 values. This exhibits the period doubling region.

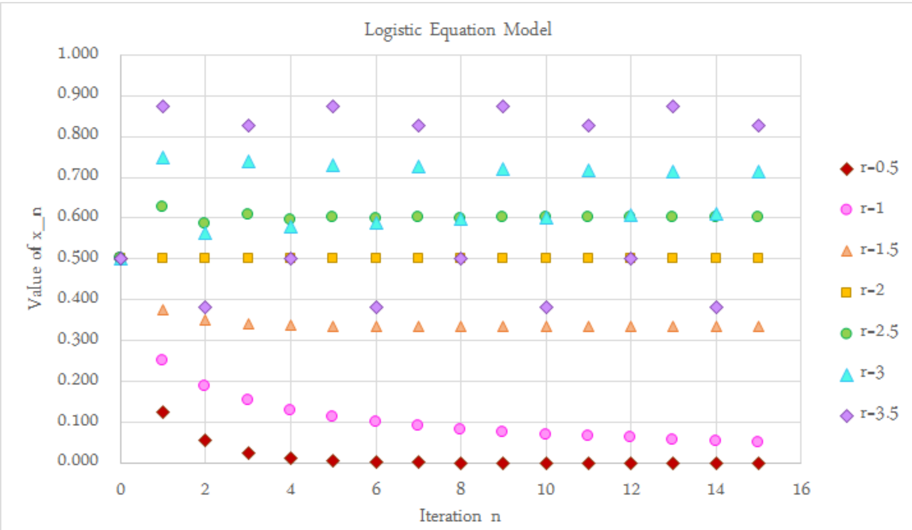


Figure 1: Series of x_n vs x for varying parameters $r < 3.56994$. Plots of x_n versus x are produced for initial condition $x_0 = 0.5$. This is the period doubling region.

In the period doubling region, the period doubling parameters allows for a limit on the quantity $\frac{r_i - r_{i-1}}{r_{i+1} - r_i}$ equal to the Feigenbaum number, 4.66920. Here, r_i is the value of r during the i^{th} period doubling. As mentioned previously, the data set will oscillate between two points for some r, then oscillate between four points for a greater r, and so on. Figure 2 shows the r values that correspond to oscillation between 2, 4, and 8 values. Their associated parameters will constitute r_{i-1}, r_i, and r_{i+1}, respectively.

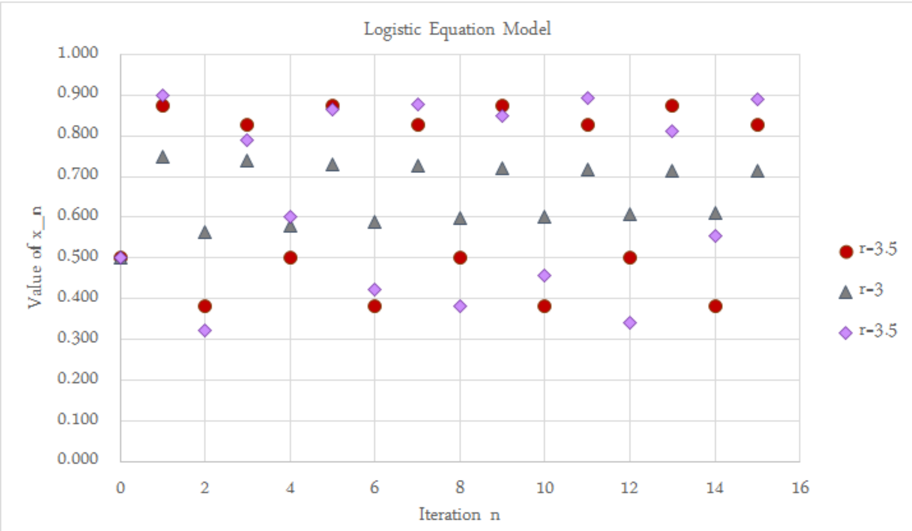


Figure 2: Series of x_n vs x for three r values, used to compute the Feigenbaum number.

The experimental value for the Feigenbaum number is calculated as follows:

$$\begin{aligned} F &= \frac{r_i - r_{i-1}}{r_{i+1} - r_i} \\ &= \frac{3.5 - 3}{3.6 - 3.5} \\ &= 5 \end{aligned}$$

This is roughly equal to the expected value of $F = 4.6692$, with a percent error of $\delta F = 7.08\%$. This error is a result of the approximation of the r values that correspond to the start of each successive period doubling region, as there is no clear demarcation between these regions. Thus, there exists a random error in the determination of r_{i-1}, r_i, and r_{i+1}.

The period doubling region is further investigated by constructing a plot of x_{n+1} versus x_n. For larger values of the parameter r, where periodicity begins emerging, ordered pairs (x_n, x_{n+1}) began overlapping, as they repeat the same values in a given period. For smaller values of r, nearby x_n values will also have nearby values for x_{n+1}. This trend is shown in figure 3.

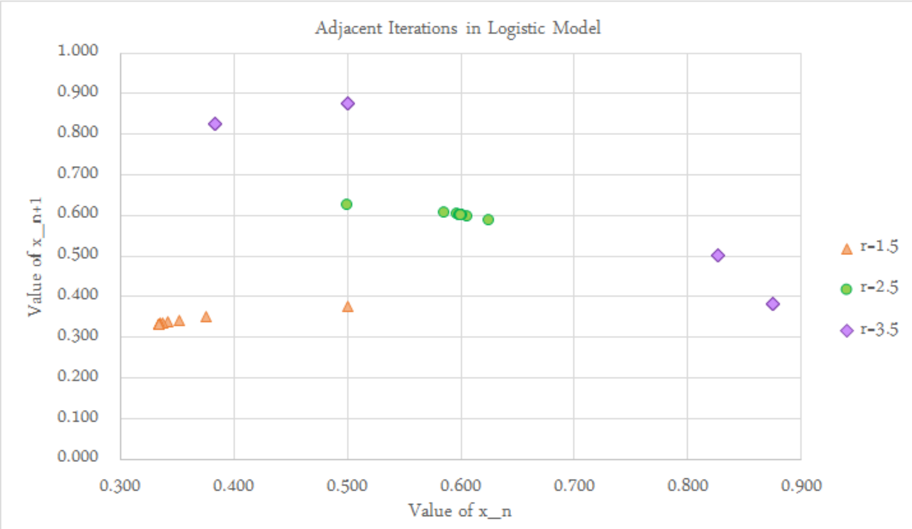


Figure 3: The data from figure 1 was re-processed for three r values to show the relationship between x_n values for adjacent iterations

1.2 Chaotic Region of the Logistic Equation

A chaotic region occurs for $r > 3.56994$. Plots of x_n versus x are produced for this region and graphed in figure 4. The distributions for each r value appear seemingly random because, although the system is deterministic, it is very sensitive to small changes.

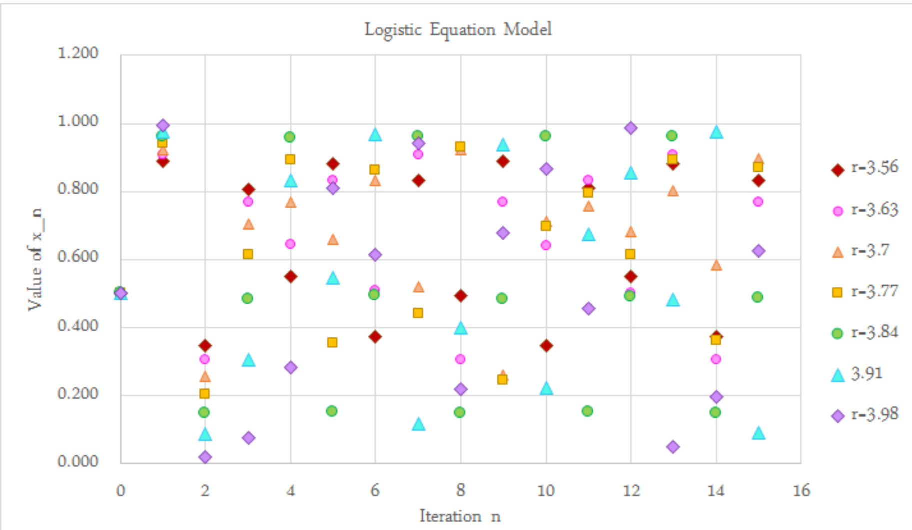


Figure 4: Series of x_n vs x for varying parameters $r > 3.56994$. Plots of x_n versus x are produced for initial condition $x_0 = 0.5$. This is the chaotic region.

The behavior of x_n is investigated by varying the initial condition x_0 by small amounts. A small change in this initial condition will produce even greater variations as n gets large. For the range of n values shown in figure 5, the percent change in x_n values was computed between the series corresponding to initial conditions $x_0 = 0.8$ and $x_0 = 0.801$. The percent change was minimized for the zeroth iteration, at 0.125%. The difference was at a maximum of 437% for the fourteenth iteration, and the change was closest to 1% for the eighth iteration.

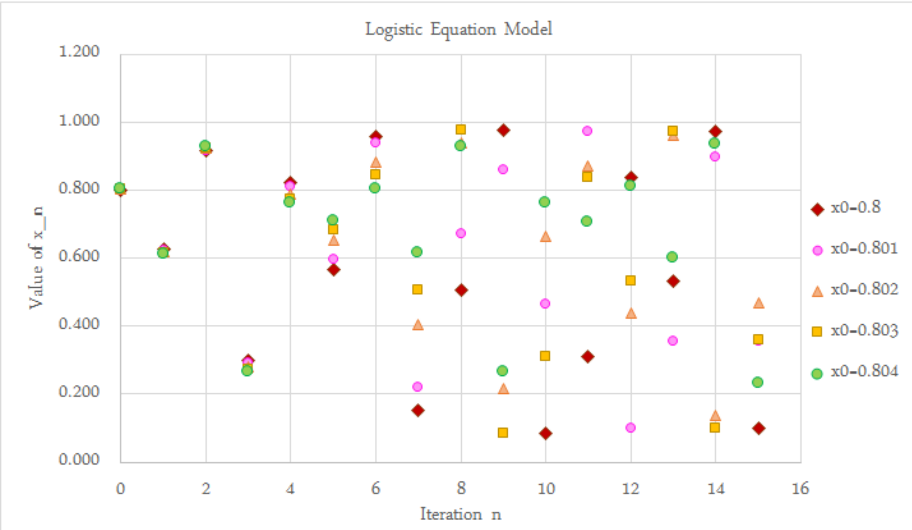


Figure 5: Series of x_n vs x for varying initial conditions x_0. Plots of x_n versus x are produced for $r = 3.91$, which falls in the chaotic region.

The chaotic region is further investigated by constructing a plot of x_{n+1} versus x_n. An interesting pattern emerges, as shown in figure 6, that depicts how a chaotic process is still deterministic. For all three r values, the relationship between x_{n+1} and x_n follows the same roughly quadratic distribution. Thus, there is a relatively predictable relationship between x_n and x_{n+1}. For a small neighborhood of variation around x_n, we can also define a small neighborhood of variation around the corresponding values of x_{n+1}.

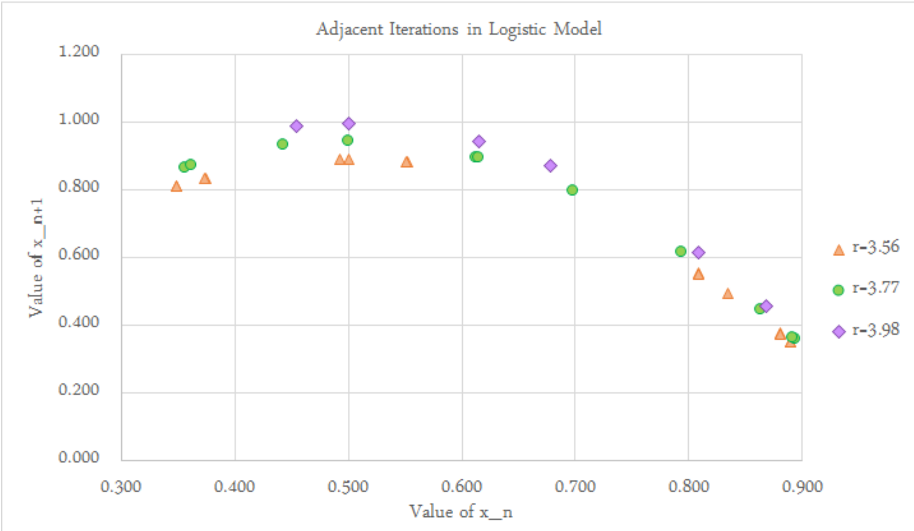
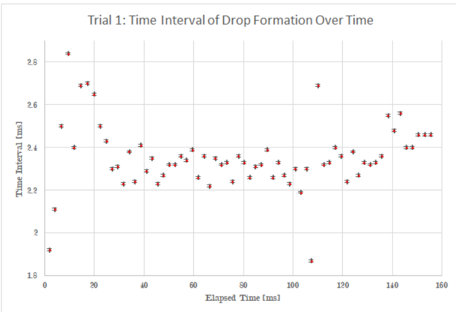


Figure 6: The data from figure 4 was re-processed for three r values to show the relationship between x_n values for adjacent iterations

2 Lyapunov Experiments

2.1 Water Drop Experiment



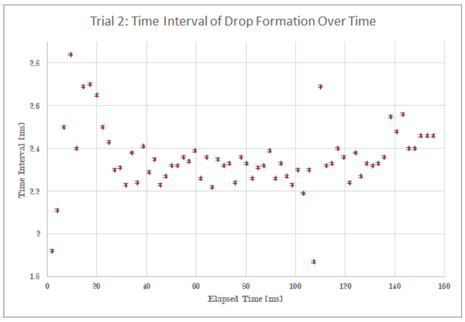


Figure 7: sink1

Figure 8: sink2

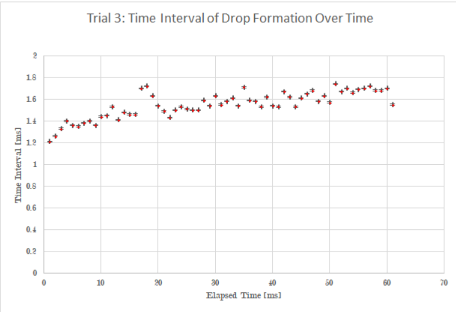


Figure 9: sink3

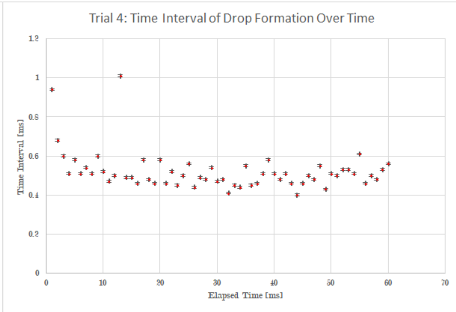


Figure 10: sink4

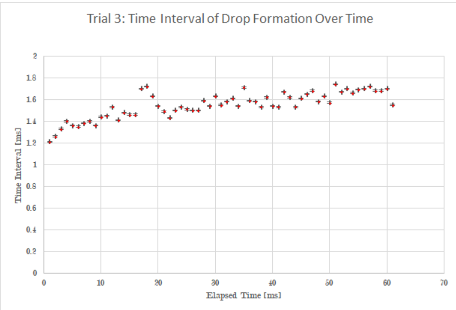


Figure 11: sink3

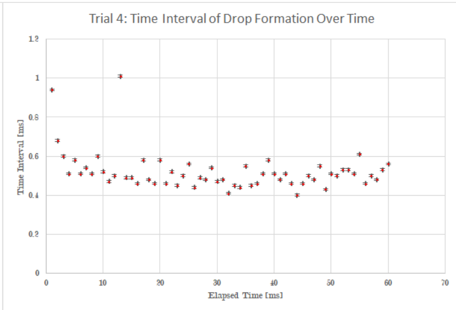


Figure 12: sink4

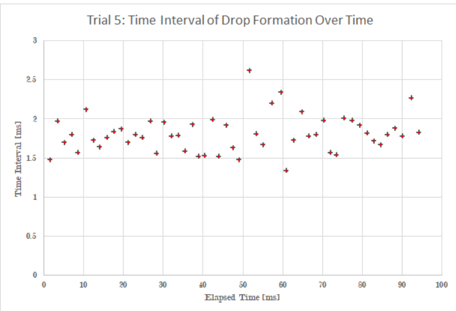


Figure 13: sink5

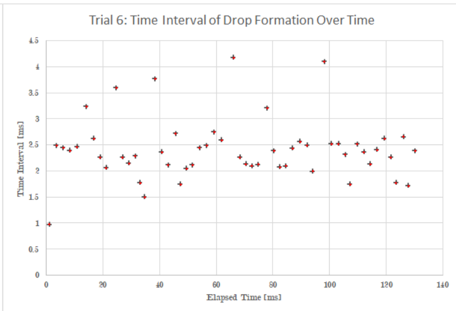


Figure 14: sink6

3 Sources of Error

4 Conclusion

5 Appendices

5.1 Appendix A: Data

5.2 Appendix B: Source Code

5.2.1 Error Propagation and Data Processing