Derivations and Background

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1 Clausius-Clapeyron

The Clausius-Clapeyron equation is defined as the change in pressure over the change in temperature. e_s is the Saturation Vapour pressure. L_v is the latent heat of Vaporization, which is assumed as a constant of temperature for this equation, but varies as defined by Goff-Gratch or another meteorological model depending on temperature range. ρ_v and ρ_l are the vapour and liquid densities for water.

$$\frac{de_s}{dT} = \frac{L_v}{T}[\frac{1}{\rho_v} - \frac{1}{\rho_L}]^{-1} \tag{1}$$

Due to the density of liquid water being much greater than vapour, Equation (1) can be approximated as (2).

$$\frac{de_s}{dT} = \frac{L_v \rho_v}{T} \tag{2}$$

Considering Water vapour as an Ideal gas, the saturation vapour pressure is:

$$e_s = \rho_v R_v T \tag{3}$$

.Substituting this value in to equation (2) for vapour density, Equation (4) is formed. By solving this as an ODE:

$$\frac{de_s}{dT} = \frac{L_v e_s}{R_v T^2} \tag{4}$$

$$\int_{e_0}^{e_s} \frac{de_s}{e_s} = \frac{L_v}{R_v} \int_{T_0}^{T} \frac{dT}{T^2} \tag{5}$$

$$ln(\frac{de_s}{e_s}) = -\frac{L_v}{R_v}[\frac{1}{T} - \frac{1}{T_0}] \tag{6}$$

$$\frac{e_s}{e_0} = exp(\frac{p}{R_v T p_w}) \tag{7}$$

The final solution for th CC equation is produced.

Referenced: Stull, R. B. (2017). Practical meteorology: An algebra-based survey of Atmospheric Science. University of British Columbia.

2 Kelvin Effect

The Laplace formulation can be found in [2]. p_w and p_v are the pressure of water and vapour. σ is the inter-facial tension. a is the radius of the droplet. Equation (8) is the Laplace formula.

$$p_w - p_v = \frac{2\sigma}{a} \tag{8}$$

P_h is the total pressure at a distance h, as seen on slide 6.

$$P_h = P_o + \rho_l g h \tag{9}$$

$$P_h' = P_o + (P_v - P_0) + P_c \tag{10}$$

Equation 10 is a representative equation of the system, and by equating (9) and (10), you produce this form for h.

$$h = \frac{(P_v - P_0) + P_c}{\rho_l g} \tag{11}$$

Utilizing the ideal gas law:

$$\rho_v = \frac{MP}{RT} \tag{12}$$

$$\frac{dP}{dz} = \rho_v g \tag{13}$$

$$\frac{dP}{dz} = \frac{MPg}{RT} \tag{14}$$

$$P_v = \frac{MPg}{RT} \tag{15}$$

$$P_v = P_0 e^{\frac{Mgh}{RT}} \tag{16}$$

The Claudius function is produced, where M is the molecular mass. By inputting this relation for h, we see a from where the laplace formula can be used.

$$P_v = P_0 e^{\frac{M(P_v - P_0 + P_c)}{\rho_l RT}} \tag{17}$$

Which produces this final Equation.

$$P_v = P_0 e^{\frac{M(2\sigma)}{\rho_l RTa}} \tag{18}$$

Referenced: [1] Galvin, K. (2005). A conceptually simple derivation of the Kelvin equation. Chemical Engineering Science, 60(16), 4659-4660. doi:10.1016/j.ces.2005.03.030
[2] Huang, J. (2018). A simple accurate formula for calculating saturation vapor pressure of water and Ice. Journal of Applied Meteorology and Climatology, 57(6), 1265–1272. https://doi.org/10.1175/jamc-d-17-0334.1