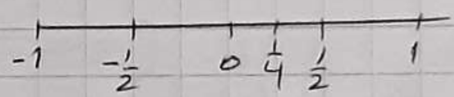
Evaluación Sumativa 2
Cálculo Integral.
Ingeniería Civil en Informática.
Miércoles 14 de julio de 2021.

① Sumas de Riemann y Área entre curvas.

(a) $f(x) = 1 + x^3$, $\Delta = \{-1, -\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{2}, 1\}$

Solución.

Dado que f es creciente en todo $\mathbb{R}$, específicamente en el intervalo $[-1,1]$, calcularemos la Suma Inferior de f con Δ, con x_{i-1} de $[x_{i-1}, x_i] \subseteq [-1,1]$.



$$\underline{S}(f,\Delta) = \left(-\frac{1}{2} - (-1)\right)\cdot f(-1) + \left(0 - \left(-\frac{1}{2}\right)\right)\cdot f\left(-\frac{1}{2}\right)$$

$$+ \left(\frac{1}{4} - 0\right)\cdot f(0) + \left(\frac{1}{2} - \frac{1}{4}\right)\cdot f\left(\frac{1}{4}\right) + \left(1 - \frac{1}{2}\right)\cdot f\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot\frac{7}{8} + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot\frac{65}{64} + \frac{1}{2}\cdot\frac{9}{8}$$

$$= \frac{7}{16} + \frac{1}{4} + \frac{65}{256} + \frac{9}{16}$$

$$= \frac{5\cdot 64}{4\cdot 64} + \frac{65}{256}$$

$$= \frac{320 + 65}{256}$$

$$= \frac{385}{256} \approx 1{,}504\ (u^2)$$

(b) $R: \begin{cases} y = x^3 + 3x^2 + 2 \\ y = x^3 + 6x^2 - 25 \end{cases}$

$x = -3$

$x = \frac{-1-\sqrt{21}}{2} = \alpha$

$x = 1$

solución.

$$A_R = \int_{-3}^{\alpha} (x^3 + 6x^2 - 25)\, dx$$

$$+ \int_{\alpha}^{1} [\underbrace{(\cancel{x^3} + 3x^2 + 2) - (\cancel{x^3} + 6x^2 - 25)}_{(-3x^2 + 27)}]\, dx$$

$$= \left(\frac{1}{64}(-1-\sqrt{21})^4 - \frac{81}{4}\right) + 6\left(9 + \frac{1}{24}(-1-\sqrt{21})^3\right)$$

$$-25\left(\frac{1}{2}(-1-\sqrt{21}+3\right) - 3\left(\frac{1}{3} - \frac{1}{24}(-1-\sqrt{21})^3\right)$$

$$+ 27\left(1 + \frac{1+\sqrt{21}}{2}\right)$$

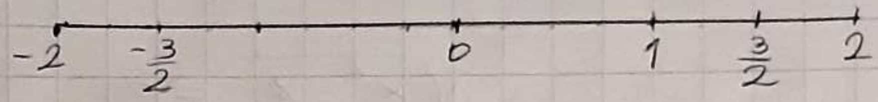
$$\approx 0{,}21 + 79{,}62$$

$$= 79{,}83 \ (u^2)$$

① ⓒ $f(x) = 8 + x^3$, $\Delta = \{-2, -\frac{3}{2}, 0, 1, \frac{3}{2}, 2\}$

Solución.

Dado que $\underline{\underline{f}}$ es creciente en todo $\mathbb{R}$, específicamente en el intervalo $[-2,2]$, calcularemos la Suma Superior de f con Δ, con x_i de $[x_{i-1}, x_i] \subseteq [-2,2]$.



$$\overline{S}(f,\Delta) = \left(-\tfrac{3}{2}-(-2)\right)\cdot f\left(-\tfrac{3}{2}\right) + \left(0-\left(-\tfrac{3}{2}\right)\right)\cdot f(0)$$

$$+ (1-0)\cdot f(1) + \left(\tfrac{3}{2}-1\right)\cdot f\left(\tfrac{3}{2}\right) + \left(2-\tfrac{3}{2}\right)\cdot f(2)$$

$$= \frac{1}{2}\cdot\frac{37}{8} + \frac{3}{2}\cdot 8 + 1\cdot 9 + \frac{1}{2}\cdot\frac{91}{8} + \frac{1}{2}\cdot 16$$

$$= \cancel{\frac{37}{16}} + \cancel{\frac{24}{2}} + \cancel{9} + \cancel{\frac{91}{16}} + \cancel{\frac{16}{2}}$$

$$= \frac{128}{16} + \frac{29\cdot 16}{1\cdot 16}$$

$$= \frac{128+464}{16}$$

$$= \frac{592}{16}$$

$$= 37\ (u^2)$$

① (a) $R: \quad y = \sqrt{x^2-3}$ $\quad y = |x-1|$

$x = -2$
$x = -\sqrt{3}$
$x = 1$
$x = \sqrt{3}$
$x = 2$

Solución.

$$A_R = \int_{-2}^{-\sqrt{3}} \left[(-x+1) - \sqrt{x^2-3}\right] dx + \int_{-\sqrt{3}}^{1} (-x+1)\, dx$$

$$+ \int_{1}^{\sqrt{3}} (x-1)\, dx + \int_{\sqrt{3}}^{2} \left[(x-1) - \sqrt{x^2-3}\right] dx$$

$$A_R = -\frac{1}{2}\cdot x^2\Big|_{-2}^{-\sqrt{3}} + x\Big|_{-2}^{-\sqrt{3}} - \left(\frac{1}{2}x\sqrt{x^2-3} - \frac{3}{2}\ln\left|x+\sqrt{x^2-3}\right|\right)\Big|_{-2}^{-\sqrt{3}}$$

$$+ -\frac{1}{2}\cdot x^2\Big|_{-\sqrt{3}}^{1} + x\Big|_{-\sqrt{3}}^{1} + \frac{1}{2}x^2\Big|_{1}^{\sqrt{3}} - x\Big|_{1}^{\sqrt{3}}$$

$$+ \frac{1}{2}x^2\Big|_{\sqrt{3}}^{2} - x\Big|_{\sqrt{3}}^{2} - \left(\frac{1}{2}x\cdot\sqrt{x^2-3} - \frac{3}{2}\ln\left|x+\sqrt{x^2-3}\right|\right)\Big|_{\sqrt{3}}^{2}$$

$$A_R = -\frac{1}{2}(3-4) + (-\sqrt{3}+2) - \left(\frac{-\sqrt{3}}{2}\overset{0}{\cancel{\sqrt{3-3}}} - \frac{3}{2}\ln(\sqrt{3}) + 1 + \frac{3}{2}\ln(1)\right)$$

$$-\frac{1}{2}(1-3) + (1+\sqrt{3}) + \frac{1}{2}(3-1) - (\sqrt{3}-1)$$

$$+\frac{1}{2}(4-3) - (2-\sqrt{3}) - \left(1 - \frac{3}{2}\cdot\ln(3) - \frac{\sqrt{3}}{2}\overset{0}{\cancel{\sqrt{3-3}}} + \frac{3}{2}\ln(\sqrt{3})\right)$$

$$= \frac{1}{2} - \cancel{\sqrt{3}} + \cancel{2} + \cancel{\frac{3}{2}\ln(\sqrt{3})} - \cancel{1} - \overset{0}{\cancel{\frac{3}{2}\ln(1)}} + 1 + 1 + \cancel{\sqrt{3}}$$

$$+ \cancel{1} - \cancel{\sqrt{3}} + \cancel{1} + \frac{1}{2} - \cancel{2} + \cancel{\sqrt{3}} - \cancel{1} + \frac{3}{2}\ln 3 - \cancel{\frac{3}{2}\ln(\sqrt{3})}$$

$$= 3 + \frac{3}{2}\ln(3) - \frac{3}{2}\ln(1)_{\to 0}$$

$$\approx 4{,}65\ (u^2)$$

① (a) Dato: $h(\frac{\pi}{2}) = h(-\frac{\pi}{2}) = h'(\frac{\pi}{2}) = \frac{\pi}{2}$

$$F(x) = \operatorname{sen}^3(x^2-12x+36) + \int_{-h(x)}^{h(x)} h(t)\,dt$$

y

$$f(x) = \int_{-\pi}^{x} F(u)\,du$$

Calcular $f''(\frac{\pi}{2})$.

Solución:

Usando el T.F.C.-1 para $f(x)$ tenemos que:

$$f'(x) = \frac{d}{dx}\left[\int_{-\pi}^{x} F(u)\,du\right] = F(x)$$

Ahora, dado que $\underline{h}$ es diferenciable en todo $\mathbb{R}$ (por hipótesis), calculamos $f''(x)$. Esto es:

$$f''(x) = F'(x) = \left[3\operatorname{sen}^2(x^2-12x+36)\right]\cdot(2x-12) + \frac{d}{dx}\left[\int_{-h(x)}^{h(x)} h(t)\,dt\right]$$

$$f''(x) = (6x-36)\operatorname{sen}^2(x^2-12x+36) + h(h(x))\cdot h'(x) - h(-h(x))\cdot(-h(x))'$$

Evaluando en $x=\frac{\pi}{2}$, obtenemos:

$$f''(\tfrac{\pi}{2}) = (3\pi-36)\operatorname{sen}^2\left(\frac{\pi^2}{4}-6\pi+36\right) + h(h(\tfrac{\pi}{2}))\cdot h'(\tfrac{\pi}{2}) + h(-h(\tfrac{\pi}{2}))\cdot h'(\tfrac{\pi}{2})$$

$$= (3\pi-36)\operatorname{sen}^2\left(\frac{\pi^2}{4}-6\pi+36\right) + h(\tfrac{\pi}{2})\cdot\frac{\pi}{2} + h(\tfrac{\pi}{2})\cdot\frac{\pi}{2}$$

$$\therefore f''(\tfrac{\pi}{2}) = (3\pi-36)\operatorname{sen}^2\left(\frac{\pi^2}{4}-6\pi+36\right) + \frac{\pi^2}{2}$$

2(b) Calcular $\widetilde{B} = \int_0^4 x^5 e^{x^2} dx$

Solución.

$$B = \int x^5 \cdot e^{x^2} dx = \int x^4 \cdot (x \cdot e^{x^2}) dx$$

Usando Integración por partes, tenemos lo siguiente:

$$u = x^4 \quad du = 4x^3 dx$$

$$dv = x e^{x^2} dx \quad v = \frac{1}{2}\int 2x e^{x^2} dx = \frac{1}{2} e^{x^2}$$

$$B = \frac{x^4}{2} \cdot e^{x^2} - 2\underbrace{\int x^3 e^{x^2} dx}_{I}$$

$$I = \int x^3 e^{x^2} dx = \int x^2 \cdot (x \cdot e^{x^2}) dx$$

$$\tilde{u} = x^2, \quad d\tilde{u} = 2x\, dx$$

$$d\tilde{v} = x \cdot e^{x^2} dx \quad \tilde{v} = \frac{1}{2}\int 2x e^{x^2} dx = \frac{1}{2} e^{x^2}$$

$$I = \frac{x^2}{2} \cdot e^{x^2} - \frac{1}{2}\int 2x e^{x^2} dx$$

$$I = \frac{x^2}{2} \cdot e^{x^2} - \frac{1}{2} \cdot e^{x^2}$$

$$\therefore B = \frac{x^4}{2} \cdot e^{x^2} - 2\left[\frac{x^2}{2} e^{x^2} - \frac{1}{2} e^{x^2}\right]$$

$$B = \frac{x^4}{2} \cdot e^{x^2} - x^2 e^{x^2} + e^{x^2} + C$$

Volviendo al Problema, calculamos:

$$\widetilde{B} = \left(\frac{x^4}{2} e^{x^2} - x^2 e^{x^2} + e^{x^2}\right)\Big|_0^4$$

$$= \frac{256}{2} \cdot e^{16} - 16 \cdot e^{16} + e^{16} - 1$$

$$\approx 113 e^{16} - 1 \approx 1.004\,130.488$$

② (c) Dato: $h(\frac{\pi}{2}) = h(-\frac{\pi}{2}) = h'(\frac{\pi}{2}) = \frac{\pi}{2}$

$$F(x) = \operatorname{Sen}^3(4x^2 - 36x + 81) + \int_{-h(x)}^{h(x)} h(t)\,dt$$

y

$$f(x) = \int_{-\pi}^{x} F(u)\,du \qquad \text{Calcular } f''(\tfrac{\pi}{2}).$$

Solución.

Idem a ②(a), procedemos a calcular $f''(x)$.

$$f'(x) = \frac{d}{dx}\left[\int_{-\pi}^{x} F(u)\,du\right] = F(x)$$

$$f''(x) = F'(x) = \left[3\operatorname{Sen}^2(4x^2 - 36x + 81)\right]\cdot(8x - 36)$$
$$+ h(h(x))\cdot h'(x) - h(-h(x))\cdot(-h(x))'$$

Luego:

$$f''(\tfrac{\pi}{2}) = (12\pi - 36)\cdot \operatorname{Sen}^2((\pi - 9)^2) + \frac{\pi^2}{2}.$$

② (d) Calcular $\tilde{D} = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos^3(x)}{\sqrt{\operatorname{sen}^3(x)}} dx$

Solución:

Calculamos:

$$D = \int \frac{\cos^3(x)}{\sqrt{\operatorname{sen}^3(x)}} dx = \int \frac{\cos^2(x) \cdot \cos(x)}{\sqrt{\operatorname{sen}^3(x)}} dx$$

$$\therefore D = = \int \frac{(1 - \operatorname{sen}^2(x)) \cdot \cos(x)}{\sqrt{\operatorname{sen}^3(x)}} dx$$

Usando el Método de Sustitución, tendremos:

$$u = \operatorname{sen}(x), \qquad du = \cos(x) dx$$

$$D = \int \frac{(1 - u^2)}{u^{\frac{3}{2}}} du = \int u^{-\frac{3}{2}} du - \int u^{\frac{1}{2}} du$$

$$= \frac{u^{-\frac{3}{2} + \frac{2}{2}}}{-\frac{3}{2} + \frac{2}{2}} - \frac{u^{\frac{1}{2} + \frac{2}{2}}}{\frac{1}{2} + \frac{2}{2}} + C$$

$$= -2 \cdot u^{-\frac{1}{2}} - \frac{2}{3} \cdot u^{\frac{3}{2}} + C$$

$$D = \frac{-2}{\sqrt{\operatorname{sen}(x)}} - \frac{2}{3} \sqrt{\operatorname{sen}^3(x)} + C$$

Volviendo al problema, calculamos:

$$\tilde{D} = -\left(\frac{2}{\sqrt{\operatorname{sen}(x)}} + \frac{2}{3} \sqrt{\operatorname{sen}^3(x)} \right) \Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= -\left(\frac{2}{\sqrt{1}} + \frac{2}{3}\sqrt{1} - \frac{2}{\sqrt{\frac{1}{2}}} - \frac{2}{3}\sqrt{\left(\frac{1}{2}\right)^3} \right)$$

$$= -\left(2 + \frac{2}{3} - 2\sqrt{2} - \frac{2}{3} \cdot \frac{\sqrt{2}}{4} \right) \simeq 5{,}73$$

③ (a) $R \begin{cases} y^2 = x^3 \\ y = 8 \\ x = 0 \text{ (Eje Y)} \end{cases}$

(Nos reducimos al primer cuadrante).

Girar R encanada alrededor de $x=4$.

Solución.

Como nos reducimos al primer cuadrante consideraremos:

$$R: \begin{cases} y = \sqrt{x^3} \\ y = 8 \\ x = 0 \\ y = 0 \end{cases}$$

y hacemos girar esta región alrededor del eje $x=4$.
De esta manera usamos la definición de volúmenes referido al método de capas cilíndricas concéntricas, parte (a).

usamos la fórmula:

$$V(S) = 2\pi \int_a^b |x-c| \cdot (f(x) - g(x))\, dx$$

donde: $a=0$, $b=4$, $f(x)=8$, $g(x)=\sqrt{x^3}$

Además, $0 \leq x \leq 4 \leq 4 \Rightarrow |x-4| = 4-x$.

Con lo anterior, calculamos:

$$V(S) = 2\pi \int_0^4 (4-x)(8-\sqrt{x^3})\, dx$$

$$= 2\pi \int_0^4 (32 - 4\sqrt{x^3} - 8x + x\sqrt{x^3})\, dx$$

$$\therefore V(S) = 2\pi \left[32\int_0^4 dx - 4\int_0^4 x^{\frac{3}{2}} dx - 8\int_0^4 x\, dx + \int x^{\frac{5}{2}} dx \right]$$

$$V(S) = 2\pi \left[32x \Big|_0^4 - 4\frac{x^{\frac{5}{2}}}{\frac{5}{2}} \Big|_0^4 - \frac{8}{2}x^2 \Big|_0^4 + \frac{x^{\frac{7}{2}}}{\frac{7}{2}} \Big|_0^4 \right]$$

$$= 2\pi \left[32(4-0) - \frac{8}{5}\left(4^{\frac{5}{2}} - 0\right) - 4(16-0) + \frac{2}{7}\left(4^{\frac{7}{2}} - 0\right) \right]$$

$$= 2\pi \left[128 - \frac{8}{5} \cdot 32 - 64 + \frac{2}{7} \cdot 128 \right]$$

$$= 2\pi \left(128 - \frac{256}{5} - 64 + \frac{256}{7} \right)$$

$$= 2\pi \left(64 + \frac{-1.792 + 1.280}{35} \right)$$

$$= 2\pi \left(64 - \frac{512}{35} \right)$$

$$= 2\pi \left(\frac{2.240 - 512}{35} \right)$$

$$= 2\pi \cdot \frac{1728}{35}$$

$$= \frac{3.456}{35} \pi \quad (u^3)$$

$$\approx 98{,}74 \quad (u^3)$$

③ (b) $R \begin{cases} x = \sqrt{y} \\ x = \dfrac{y^3}{32} \end{cases}$

Girar R alrededor del eje X

Solución

Dado que debemos girar R alrededor del eje X, nos reducimos al primer cuadrante, por lo que consideraremos:

$$R: \begin{array}{l} y = x^2 \\ y = \sqrt[3]{32x} \end{array}$$

Además, notamos que:

$$x^2 = \sqrt[3]{32x} \quad (\;)^3$$

$$\Rightarrow x^6 = 32x$$

$$\Rightarrow x^6 - 32x = 0$$

$$\Rightarrow x(x^5 - 32) = 0$$

$$\Rightarrow x = 0 \lor x^5 = 32 \; \sqrt[5]{}$$

$$\Rightarrow \underline{x = 0} \lor \underline{x = 2}$$

Las gráficas se intersectan en $x=0$ y en $x=2$.

Dadas las características del problema, considerando 2 funciones (curvas) y un eje de giro con respecto al eje X, usaremos la definición de volúmenes referida al método de arandelas (o anillos)

Usaremos la fórmula:

$$V(S) = \pi \int_a^b [(f(x))^2 - (g(x))^2]\, dx$$

donde $a=0$, $b=2$, $f(x)=\sqrt[3]{32x}$
y $g(x)=x^2$.

Luego, calculamos:

$$V(S)=\pi\int_0^2\left[\left(\sqrt[3]{32x}\right)^2-\left(x^2\right)^2\right]dx$$

$$=\pi\left[\int_0^2(32x)^{\frac{2}{3}}dx-\int_0^2 x^4dx\right]$$

$$=\pi\left[32^{\frac{2}{3}}\int_0^2 x^{\frac{2}{3}}dx-\int_0^2 x^4dx\right]$$

$$=\pi\left[32^{\frac{2}{3}}\cdot\frac{x^{\frac{5}{3}}}{\frac{5}{3}}\Big|_0^2-\frac{x^5}{5}\Big|_0^2\right]$$

$$=\pi\left[8\sqrt[3]{2}\cdot\frac{3}{5}\left(2\sqrt[3]{4}-0\right)-\left(\frac{32}{5}-0\right)\right]$$

$$=\pi\left[\frac{24}{5}\sqrt[3]{2}\cdot 2\sqrt[3]{4}-\frac{32}{5}\right]$$

$$=\pi\left(\frac{48}{5}\sqrt[3]{8}-\frac{32}{5}\right)$$

$$=\pi\left(\frac{48}{5}\cdot 2-\frac{32}{5}\right)$$

$$=\pi\left(\frac{96-32}{5}\right)$$

$$=\pi\cdot\frac{64}{5}$$

$$=12{,}8\,\pi\ (u^3)$$

④ (a) Teorema del Valor Medio para integrales. (TVM).

Sea $f \in C[a,b]$. Entonces, existe $c \in [a,b]$ tal que:

$$\int_a^b f(x)\,dx = f(c)\cdot(b-a)$$

(a) $f(x) = \dfrac{1}{(x+1)^2}$, con $[a,b] = [0,2]$

Solución.

Dado que $f \in C[0,2]$, procederemos con el T.V.M. para integrales.

$$\int_0^2 \frac{1}{(x+1)^2}\,dx = f(c)\cdot(2-0)$$

Calculamos el lado izquierdo.

$$\int_0^2 \frac{1}{(x+1)^2}\,dx = \int_0^2 (x+1)^{-2}\,dx$$

$$= \left.\frac{(x+1)^{-1}}{-1}\right|_0^2$$

$$= -\left.\left[\frac{1}{(x+1)}\right]\right|_0^2$$

$$= -\left(\frac{1}{3} - 1\right)$$

$$= -\left(-\frac{2}{3}\right)$$

$$\therefore \int_0^2 \frac{1}{(x+1)^2}\,dx = \frac{2}{3}$$

Luego:

$$\frac{2}{3} = f(c) \cdot 2$$

donde $f(c) = \frac{1}{(c+1)^2}$

Con esto, tenemos lo siguiente:

$$\frac{2}{3} = \frac{2}{(c+1)^2}$$

$$\Leftrightarrow (c+1)^2 = 3 \quad / \sqrt{\ }$$

$$|c+1| = \sqrt{3}$$

$$c+1 = \sqrt{3} \quad \vee \quad c+1 = -\sqrt{3}$$

$$c = -1+\sqrt{3} \quad \vee \quad c = -1-\sqrt{3}$$

Pero, $-1-\sqrt{3} \notin [0,2]$.

Luego, el valor de $\underline{c}$ que satisface el T.V.M. es $c = -1+\sqrt{3}$.

④ (b) $f(x) = \sqrt{x+1}$ con $[a,b] = [0,3]$.

Solución.

Dado que $f \in C[0,3]$, procederemos con el uso del T.V.M.
Decimos que

$$\int_0^3 \sqrt{x+1}\, dx = f(c)\cdot(3-0)$$

Calculamos el lado izquierdo.

$$\int_0^3 (x+1)^{\frac{1}{2}} dx = \left.\frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^3$$

$$= \frac{2}{3}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$$

$$= \frac{2}{3}(8-1)$$

$$= \frac{2}{3}\cdot 7$$

$$\therefore \int_0^3 (x+1)^{\frac{1}{2}} dx = \frac{14}{3}$$

De esta manera:

$$\frac{14}{3} = f(c)\cdot 3, \text{ donde } f(c) = \sqrt{c+1}$$

Así, nos queda:

$$\frac{14}{3} = \sqrt{c+1}\cdot 3$$

$$\Longleftrightarrow \sqrt{c+1} = \frac{14}{9} \quad (\;)^2$$

$$c+1 = \frac{196}{81}$$

$$c = \frac{196}{81} - \frac{81}{81} = \frac{115}{81} \approx 1{,}42 \in [0,3]$$