PAUTA Certamen 3 de Cálculo Integral(220146)

RESULTADOS DE APRENDIZAJES

Calcula área de coordenadas polares,determina Convergencia o divergencia de integrales impropias sucesiones y series, utilizando criterios y luego aplica a problemas contextualizado.

Ejercicio 1. **20 puntos**

Determine si la integral impropia converge o diverge, justifique.

(a) $\displaystyle\int_{-\infty}^{2} \frac{dx}{(4-x)^2}$

Solución

$$\begin{aligned}
\int_{-\infty}^{2} \frac{dx}{(4-x)^2} &= \lim_{t\to-\infty} \int_{t}^{2} \frac{dx}{(4-x)^2} \\
&= \lim_{t\to-\infty} \left(\frac{1}{4-x}\right)\Big|_t^2 \\
&= \lim_{t\to-\infty} \left(\frac{1}{2} - \frac{1}{4-t}\right) \\
\int_{-\infty}^{2} \frac{dx}{(4-x)^2} &= \frac{1}{2}.
\end{aligned}$$

Por tanto, la integral impropia converge.

(b) $\displaystyle\int_{-\infty}^{3} \frac{dx}{(5-x)^2}$

Solución

$$\begin{aligned}
\int_{-\infty}^{3} \frac{dx}{(5-x)^2} &= \lim_{t\to-\infty} \int_{t}^{3} \frac{dx}{(5-x)^2} \\
&= \lim_{t\to-\infty} \left(\frac{1}{5-x}\right)\Big|_t^3 \\
&= \lim_{t\to-\infty} \left(\frac{1}{2} - \frac{1}{5-t}\right) \\
\int_{-\infty}^{3} \frac{dx}{(5-x)^2} &= \frac{1}{2}.
\end{aligned}$$

Por tanto, la integral impropia converge.

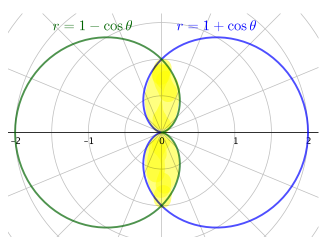
Ejercicio 2. **20 puntos**

(a) Calcule el área interior común a las curvas polares $r = 1+\cos(\theta)$, $\quad r = 1-\cos(\theta)$

Solución

Intersección entre curvas

$$\begin{aligned}
1+\cos(\theta) &= 1-\cos(\theta) \\
2\cos(\theta) &= 0 \\
\theta &= \left\{\tfrac{\pi}{2},\ \tfrac{3\pi}{2}\right\}.
\end{aligned}$$



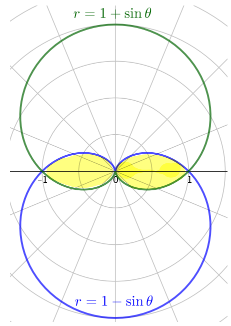
Área

$$\begin{aligned}
A &= 4\left(\tfrac{1}{2}\right) \int_0^{\frac{\pi}{2}} (1-\cos\theta)^2 d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \left[1 - 2\cos\theta + \frac{1+\cos(2\theta)}{2}\right] d\theta \\
&= 2\left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right]\Big|_0^{\frac{\pi}{2}} \\
&= 2\left[\tfrac{\pi}{2} - 2(1) + \tfrac{\pi}{4}\right] \\
A &= \frac{3\pi-8}{2} u^2.
\end{aligned}$$

Profesora Yrina Vera-Damián

(b) Calcule el área interior común a las curvas polares $r = 1 + \sin(\theta)$, $\quad r = 1 - \sin(\theta)$

SOLUCIÓN



Intersección entre curvas

$$\begin{aligned} 1 + \sin(\theta) &= 1 - \sin(\theta) \\ 2\sin(\theta) &= 0 \\ \theta &= \{0,\ \pi,\ 2\pi\}\,. \end{aligned}$$

Área

$$\begin{aligned} A &= 4\left(\tfrac{1}{2}\right)\int_0^{\frac{\pi}{2}} (1 - \sin\theta)^2 d\theta \\ &= 2\int_0^{\frac{\pi}{2}} \left[1 - 2\sin\theta + \frac{1 - \cos(2\theta)}{2}\right] d\theta \\ &= 2\left[\theta + 2\cos\theta + \frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]\Big|_0^{\frac{\pi}{2}} \\ &= 2\left[\tfrac{\pi}{2} + 2(0) + \tfrac{\pi}{4} - 2(1)\right] \\ A &= \frac{3\pi - 8}{2} u^2. \end{aligned}$$

Ejercicio 3. **20 puntos**

Determine los siguientes límites de las suceciones. Justifique su respuesta

(a) $\lim\limits_{n\to\infty} \dfrac{2\ln(n)}{n^2}$

SOLUCIÓN

Note que

$$\lim_{n\to\infty} \frac{2\ln(n)}{n^2} = \frac{+\infty}{+\infty},$$

aplicamos L'Hospital.

$$\begin{aligned} \lim_{n\to\infty} \frac{2\ln(n)}{n^2} &= \lim_{n\to\infty} \frac{\frac{d}{dn}[2\ln(n)]}{\frac{d}{dn}(n^2)} \\ &= \lim_{n\to\infty} \left(\frac{\frac{2}{n}}{2n}\right) \\ &= \lim_{n\to\infty} \frac{1}{n^2} \\ \lim_{n\to\infty} \frac{2\ln(n)}{n^2} &= 0. \end{aligned}$$

Así, la sucesión $\left\{\dfrac{2\ln(n)}{n^2}\right\}_{n\in\mathbb{N}}$ converge.

(b) $\lim\limits_{n\to\infty} \dfrac{3\ln(n)}{n^3}$

SOLUCIÓN

Note que

$$\lim_{n\to\infty} \frac{3\ln(n)}{n^3} = \frac{+\infty}{+\infty},$$

aplicamos L'Hospital.

$$\begin{aligned} \lim_{n\to\infty} \frac{3\ln(n)}{n^3} &= \lim_{n\to\infty} \frac{\frac{d}{dn}[3\ln(n)]}{\frac{d}{dn}(n^3)} \\ &= \lim_{n\to\infty} \left(\frac{\frac{3}{n}}{3n^2}\right) \\ &= \lim_{n\to\infty} \frac{1}{n^3} \\ \lim_{n\to\infty} \frac{3\ln(n)}{n^3} &= 0. \end{aligned}$$

Así, la sucesión $\left\{\dfrac{3\ln(n)}{n^3}\right\}_{n\in\mathbb{N}}$ converge.

4a) $\sum_{n=1}^{\infty} \frac{n^3}{4^n}$ **Converge, pues**

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\left|\frac{\frac{(n+1)^3}{4^{n+1}}}{\frac{n^3}{4^n}}\right| = \lim_{n\to+\infty}\left|\frac{4^n (n+1)^3}{4^{n+1}\, n^3}\right| = \lim\left|\frac{(n+1)^3}{4n^3}\right| = \frac{1}{4}$$

4b) $\sum_{n=1}^{+\infty} \frac{n^3}{5^n}$ **Converge, pues**

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\left|\frac{\frac{(n+1)^3}{5^{n+1}}}{\frac{n^3}{5^n}}\right| = \lim_{n\to+\infty}\left|\frac{5^n (n+1)^3}{n^3\, 5^{n+1}}\right| = \lim_{n\to+\infty}\left|\frac{(n+1)^3}{5n^3}\right| = \frac{1}{5}$$

5a) $$\left(20 + \frac{2}{5}(20)\right) + \left(\frac{2}{5}(20) + \left(\frac{2}{5}\right)(20)\right) + \dots$$

$$28 + \frac{2}{5}(28) + \left(\frac{2}{5}\right)^2(28) + \dots = \sum_{n=1}^{+\infty} 28\left(\frac{2}{5}\right)^{n-1}$$

$$S = \frac{28}{1-\frac{2}{5}} = \frac{28\times 5}{3} \approx 46.67$$

5b) $$\left(18 + \frac{3}{5}(18)\right) + \left(\frac{3}{5}(18) + \left(\frac{3}{5}\right)^2(18)\right) + \dots$$

$$\frac{144}{5} + \frac{3}{5}\left(\frac{144}{5}\right) + \left(\frac{3}{5}\right)^2\left(\frac{144}{5}\right) + \dots = \sum_{n=0}^{+\infty} \frac{144}{5}\left(\frac{3}{5}\right)^n$$

$$S = \frac{\frac{144}{5}}{1-\frac{3}{5}} = 72$$