Probability

1. Fundamental Concepts

Random Experiment

An experiment that satisfies two key conditions:

- Has more than one possible outcome
- Cannot predict the outcome in advance

Trial

A single execution of a random experiment. Each trial produces an outcome.

Outcome

A single possible result of a trial.

Sample Space

The set of all possible outcomes that can occur in a random experiment. Generally, one random experiment has one sample space.

Event

A specific set of outcomes from a random experiment - essentially a subset of the sample space. An event can include:

- A single outcome
- Multiple outcomes
- One random experiment can have multiple events

2. Types of Events

Simple Event (Elementary Event)

An event consisting of exactly one outcome.

- **Example**: Rolling a 3 on a six-sided die

Compound Event

An event consisting of two or more simple events.

- **Example**: Rolling an odd number on a die (includes rolling 1, 3, or 5)

Independent Events

Two events where the occurrence of one does not affect the probability of the other.

- **Example**: Flipping a coin and rolling a die - the coin flip doesn't affect the die roll

Dependent Events

Events where the occurrence of one affects the probability of the other.

- **Example**: Drawing two cards from a deck without replacement - the first draw affects the second

Mutually Exclusive Events (Disjoint)

Two events that cannot both occur at the same time.

- **Example**: Rolling a 2 and rolling a 4 on a single die roll

Exhaustive Events

A set of events where at least one event must occur when the experiment is performed.

- **Example**: "Roll an even number" and "roll an odd number" when rolling a die

Impossible Event and Certain Event

- **Impossible Event**: Probability = 0 (will not happen)
- **Certain Event**: Probability = 1 (will certainly happen)

3. What is Probability?

Probability is a measure of the likelihood that a particular event will occur. It's expressed as a number between 0 and 1, inclusive:

- **Probability = 0**: Event will not happen
- **Probability = 1**: Event will certainly happen
- **Probability = 0.5**: Event will happen half the time (equally likely to happen or not)

4. Types of Probability

Empirical Probability (Experimental Probability)

Based on observed data rather than theoretical assumptions.

Formula: Number of times event occurs ÷ Total number of trials

Examples:

- Coin toss: 100 tosses, 55 heads → Empirical probability of heads = 55/100 = 0.55
- Marble draw: 200 draws, 80 red marbles → Empirical probability of red = 80/200 = 0.40

Theoretical Probability (Classical Probability)

Used when each outcome in a sample space is equally likely to occur.

Formula: Number of favorable outcomes ÷ Total number of outcomes in sample space

Examples:

- Tossing a fair coin 3 times: Find probability of exactly 2 heads
- Rolling 2 dice: Find probability of sum = 7

5. Random Variables

A random variable is a function that maps outcomes of a random process to real numbers.

Components:

- **Input**: Outcome from the sample space of a random process
- **Output**: Real number assigned to each possible outcome

Types:

- **Discrete Random Variable**: Countable outcomes
- **Continuous Random Variable**: Uncountable outcomes within a range

6. Probability Distribution

A list of all possible outcomes of a random variable along with their corresponding probability values. It shows how probabilities are distributed across all possible values of the random variable.

7. Statistical Measures

Mean of a Random Variable (Expected Value)

The average outcome of a random process repeated many times. It's a weighted average of possible outcomes, where each outcome is weighted by its probability of occurrence.

Variance of a Random Variable

A statistical measurement describing how much individual observations differ from the mean (expected value). It measures the spread or dispersion of the probability distribution.

Key Applications

Probability is used across multiple disciplines including:

- Science and Engineering
- Medicine
- Economics
- Social Sciences
- Decision making and predictions

Core Probability Concepts

Joint Probability

- **Definition**: The probability that two events occur simultaneously
- **Notation**: P(X = x, Y = y) - probability that X takes value x AND Y takes value y at the same time
- **Example**: If X = passenger class and Y = survival status, joint probability tells us the likelihood of a passenger being in a specific class AND surviving

Marginal Probability

- **Definition**: The probability of an event occurring regardless of other events
- **Also called**: Unconditional probability P(A)
- **Purpose**: Gives the probability of a single variable without considering other variables
- **Example**: Probability of survival regardless of passenger class

Conditional Probability

- **Definition**: The probability of an event occurring given that another event has already occurred
- **Notation**: P(A|B) - probability of A given B has occurred
- **Formula**: $P(A|B) = P(A \cap B) / P(B)$

Sample Problems

1. **Three Coins Problem**: Three unbiased coins are tossed. What is the conditional probability that at least two coins show heads, given that at least one coin shows heads?
2. **Dice Sum Problem**: Two fair six-sided dice are rolled. What is the conditional probability that the sum equals 7, given that the first die shows an odd number?
3. **Dice Constraint Problem**: Two fair dice (D1 and D2) are rolled. What is the conditional probability that D1 equals 2, given that the sum of D1 and D2 is ≤ 5?

Understanding Conditional Probability Formula

Intuition Behind $P(A|B) = P(A \cap B) / P(B)$

The formula works by **reducing our sample space**:

- **Denominator P(B)**: Represents our new "universe" where event B has occurred

- **Numerator P(A ∩ B)**: Represents cases where both A and B occur within this new universe
- **Result**: The proportion of times A occurs when we know B has occurred

Think of it as normalizing the joint probability by the probability of the given condition.

Types of Events

Independent Events

- **Definition**: Events where the occurrence of one does not affect the occurrence of another
- **Examples**:
  - Flipping a coin and rolling a die
  - Drawing a card with replacement

Dependent Events

- **Definition**: Events where the occurrence of one affects the occurrence of another
- **Example**: Drawing cards without replacement

Mutually Exclusive Events

- **Definition**: Events that cannot occur simultaneously
- **Key Point**: If one event occurs, the other cannot
- **Examples**:
  - Getting heads or tails on a coin flip
  - Rolling different numbers on a single die

Advanced Concepts

Bayes' Theorem

- **Purpose**: Updates probabilities of hypotheses when given new evidence
- **Applications**: Machine learning, statistics, game theory
- **Named after**: Thomas Bayes
- **Key Insight**: Provides a systematic way to revise existing predictions when new information becomes available

Visualization Tools

Venn Diagrams

- Useful for visualizing relationships between events
- Help understand intersections, unions, and complements

Contingency Tables

- Organize joint probability data in tabular format
- Helpful for calculating marginal and conditional probabilities
- Particularly useful when dealing with categorical variables

Key Formulas Summary

- **Joint Probability**: $P(X = x, Y = y)$
- **Marginal Probability**: $P(A)$
- **Conditional Probability**: $P(A|B) = P(A \cap B) / P(B)$
- **Bayes' Theorem**: Updates prior probabilities with new evidence