Singular Value Decomposition

Singular Value Decomposition (SVD)

Non-Square Matrix (Rectangular)

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \longrightarrow \underset{\text{output}}{n} \times \underset{\text{input}}{m}$$

2×3: output (2D), input (3D)

This tells us that you are initially in 3D state and you will go towards 2D space.

Rectangular Diagonal Matrix

A Matrix that would be diagonal matrix, if it was a squared matrix, but instead is a rectangular due to extra rows or columns of zeros.

Imp:	If you have a rectangular matrix, it's transformation will be to change the dimensions from $n \longrightarrow m$ $(\underset{\text{output}}{m} \times \underset{\text{input}}{n})$

Intuition of transformations.

$$\underset{\text{result}}{\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \end{bmatrix}} \xrightarrow[\text{Composed of}]{\text{linear transformations}} \underset{2\times 2}{\overset{A}{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}} \underset{2\times 3}{\overset{B}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}}$$

We always go from right to left

So, matrix B is rectangular, and it will be applied first

Since, its rectangular matrix, the transformation will be "changing its dimension"

After that, Matrix A will be applied, and since matrix A is diagonal matrix, the transformation will be scaling transformation.

Another example

$$\begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix} \longrightarrow \underset{2\times 2}{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}$$

Since, the multiplication of this matrix is not possible, we will change the order

$$\underset{3\times 2}{\overset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}} \underset{2\times 2}{\overset{B}{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}}$$

This is possible

Therefore, Matrix B will be applied first resolting in Scailing transformation followed by changing dimension transformation by matrix A (due to rectangolar)

SVD Applications:

- Data science and Machine learning
- Natural Processing Language (NLP).
- Computer Vision
- Signal Processing
- Numerical Linear Algebra
- Psychometrics
- Bioinformatics
- Quantum Computing

etc..

SVD

SVD is a matrix Decomposition / factorization method that decomposes a matrix into three other matrices.

Given a matrix A, the singular value decomposition of A is usually written as:

$$A = U \Sigma V^T$$

Where:

- U and V are orthogonal matrices. U is the left singular vectors and V is the right singular vectors.
- Σ is a diagonal matrix containing what we call the singular values.

If we solve and decode the formula, we can conclude that

$$A = \underbrace{U}_{\text{left Singular}} \; \underbrace{\Sigma}_{\text{Singular Values}} \; \underbrace{V^T}_{\text{Right Singular}}$$

left Singular — Singular vector (AA^T) w.r.t. A

Singular Values (a, b) w.r.t A $\rightarrow (\sqrt{a^2}, \sqrt{b^2})$

Right Singular — Singular vector (A^TA) w.r.t A — A is LHS in the eq of SVD.

$$\left.\begin{aligned} AA^T &\rightarrow a^2 b^2 \\ A^TA &\rightarrow a^2 b^2 \end{aligned}\right\} \text{eigen values}$$

$$\Sigma \rightarrow \sqrt{a^2}\sqrt{b^2}$$

singular values

Geometric Intuition

$A = U \Sigma V^T$

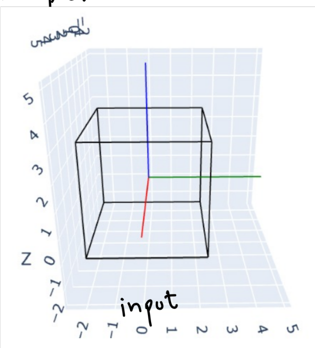
Starting from V^T, $A^TA \longrightarrow$ Anti clockwise Rotation

$\Sigma - \begin{cases} \Sigma_1 & \text{Dimensionality Change (3d to 2d or opp.)} \\ \Sigma_2 & \text{Streching} \end{cases}$

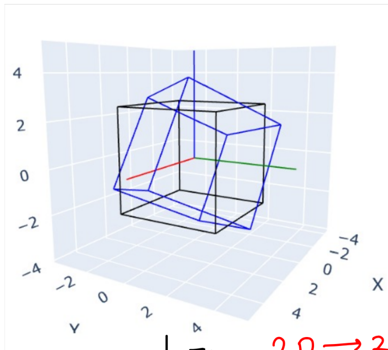
V, $AA^T \longrightarrow$ Clockwise Rotation

By applying these steps, we get the final output.

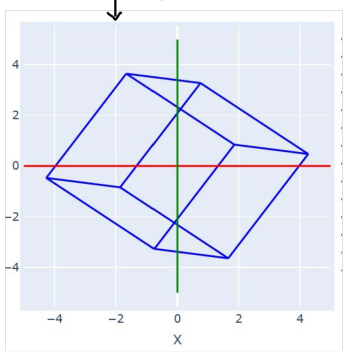
Example:



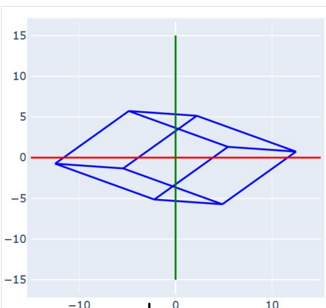
$\xrightarrow{V^T}$ Anti clockwise rotation



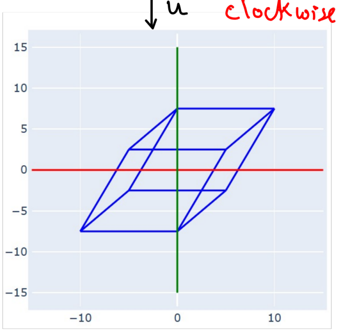
$\downarrow \Sigma 1$ 3D → 2D (getting snapshot from top)



$\xleftarrow{\Sigma 2}$ Streching



$\downarrow u$ clockwise rotation



output

To calculate SVD for A,
np.linalg.svd(A)