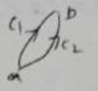
Line integral

$\int_C \vec{F}\cdot d\vec{r}$ is called line integral of function $\vec{F}$ along curve C.

→ line integral depends on path of integration.  $\int_{C_1} F\cdot dr \neq \int_{C_2} F\cdot dr$

if F_1, F_2, F_3 are scalar components of $\vec{F}$

$$\int_C \vec{F}\cdot d\vec{r} = \int (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_C (F_1 dx + F_2 dy + F_3 dz)$$

other forms of line integral

$$\int_C \phi\, d\vec{r} \;\&\; \int_C \vec{F}\times d\vec{r}.$$

here ϕ is scalar function of x, y, z. to evaluate integrals, value of y & z are substituted in terms of x by using eqn of the curve.

$\int_C \vec{F}\cdot d\vec{r}$ gives work done by the force $\vec{F}$ in displacing the particle along the curve C.

conservative field :— If the work done in moving a partic[le] from A to B depends only on points A & B & not on the path of in reaching B from A. the force field is call[ed] conservative field. if $F = \nabla\phi$, F is conservative & ϕ is its scalar potentia[l]

A vector field is conservative if & only if $\nabla\times\vec{F} = \vec{0}$.

if $\vec{F} = \vec{\nabla}\phi$.

$\therefore$ work done $\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{\nabla}\phi\cdot d\vec{r} = \int_C \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}$

$= \int_C d\phi = [\phi(x,y,$

$= \phi_B - \phi_A$

this formula should be applied whenever a line integral is $\therefore$ independent of path

Rate of loss ...

Circulation $\int_C \vec{F}\cdot d\vec{R} = \int_C F_1 dx + F_2 dy + F_3 dz$ ~~tangent line~~ $\vec{F}$ around C is called circulation about C.

where C is closed curve

surface integral if S is a surface, we divide S in finite sub surfaces δs_i. then we have

$$\delta\vec{a}_i = \hat{n}_i \delta s_i$$

where $\delta\vec{a}_i$ is vector area of subsurface δs_i & $\hat{n}_i$ is unit vector (in normal outward dir) $\hat{n}_i$

// Normal surface integral over $\vec{F}$ is denoted by

$$\int_S \vec{F}\cdot d\vec{a} = \int_S \vec{F}\cdot\hat{n}\, ds$$

or $$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

S is projected on the xy plane & normal is $\hat{k}$ (shadow)

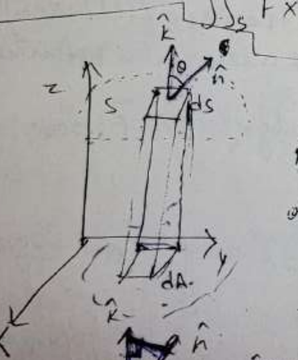
if S_2 & S_3 are projections of S on yz & zx planes we have (normals of these are $\hat{i}$ & $\hat{j}$ respectively).

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_2} \vec{F}\cdot\hat{n}\,\frac{dy\,dz}{\hat{n}\cdot\hat{i}} \quad \& \quad \iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_3} \vec{F}\cdot\hat{n}\,\frac{dz\,dx}{\hat{n}\cdot\hat{j}}$$

we can also define surface integrals

$$\iint_S \vec{F}\times d\vec{s} \quad \& \quad \iint_S \phi\, d\vec{s} \quad (d\vec{s} = \hat{n}\, ds)$$

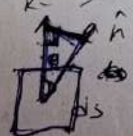
[$\hat{n}$ = unit outward normal for surface s is given by $\hat{n} = \frac{\nabla s}{|\nabla s|}$



$$\hat{n}\cdot\hat{k} = |\hat{n}||\hat{k}|\cos\theta$$

$$\text{or } \cos\theta = \frac{\hat{n}\cdot\hat{k}}{|\hat{n}||\hat{k}|} \quad \& \; |\hat{n}| = 1, \; |\hat{k}| = 1 \Rightarrow |\hat{n}||\hat{k}| = 1$$

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos$$



$\hat{k}$ is normal to dA i.e. (xy plane)

$\hat{n}$ is normal to ds

BAP HHB

$\hat{k}\cdot = \cos\theta \Rightarrow \hat{k} = \hat{n}\cos\theta$ ∴ $dA = ds\cos\theta$ $\quad ds = \frac{dA}{\cos\theta} = \frac{dx\,dy}{\hat{n}\cdot\hat{k}}$ (1.1) the normal surface integral given by $\int_S \vec{F}\cdot d\vec{a} = \int_S \vec{F}\cdot\hat{n}\, ds$

of ~~continuous~~ a vector point function $\vec{F}$ over a closed surface S

is defined as flux of $\vec{F}$ across S.

the vector point f^n $\vec{F}$ is said to the solenoidal in a region if

flux of $\vec{F}$ across every closed curve in the region is zero.

Along

Q. 1. If a force $\vec{F} = 2x^2y\,\hat{i} + 3xy\,\hat{j}$ displaces a particle in the xy- from $(0,0)$ to $(1,4)$ along a curve $y = 4x^2$. Find the work do

Sol^n:- work done $= \int_c \vec{F}\cdot d\vec{r}$

$$= \int_c (2x^2y\,\hat{i} + 3xy\,\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_c 2x^2y\,dx + 3xy\,dy)$$

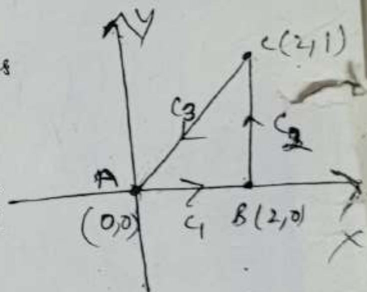
$$= \int_0^1 2x^2 \cdot 4x^2 dx + 3x \cdot 4x^2 \cdot 8x\,dx \qquad [\because y = 4x^2,\ dy = 8x\,dx$$

$$= \int_0^1 (8x^4 + 96x^4)\,dx$$

$$= (104)\left(\frac{x^5}{5}\right)_0^1 = \frac{104}{5}.$$

Q2. If $F = (2x+y^2)\,\hat{i} + (3y - 4x)\,\hat{j}$. Evaluate $\oint_c F \cdot dr$ around a triangle ABC in the xy-plane with $A(0,0)$, $B(2,0)$; $C(2,1)$ in counter clockwise direction. What is its value in clockwise direction.

Sol^n:- The curve C is union of three curves C_1, C_2 & C_3.



$$\therefore \oint_c \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r}$$

$$= I_1 + I_2 + I_3 \text{ (say)}$$

Along C_1:- straight line AB, $y=0$, $z=0$ & x varies from 0 to

$$\therefore \vec{r} = x\,\hat{i} \Rightarrow d\vec{r} = dx\,\hat{i}$$

$$I_1 = \int_c \vec{F}\cdot d\vec{r} = \int_{C_1} [(2x+y^2)\,\hat{i} + (3y-4x)\,\hat{j}] \cdot (dx\,\hat{i})$$

$$= \int_{C_1} (2x + y^2)\,dx = \int_0^2 2x\,dx \qquad [\because y = 0]$$

$$= [x^2]_0^2 = 4$$

Along C_2 The straight line BC $\therefore$ $x=2$; $z=0$, y varies from 0 to 1 (8)

$\Rightarrow dx=0$

$\therefore \vec{r} = 2\hat{i} + y\hat{j} \Rightarrow d\vec{r} = dy\,\hat{j}$

$$I_2 = \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_2} (3y-4x)\,\hat{j}\cdot dy\,\hat{j}$$

$$= \int_0^1 (3y-4\times 2)\,dy \quad \left[\because x=2,\ dx=0\right]$$

$$= -\frac{13}{2}$$

Along C_3 straight line CA.

$z=0$; $2y=x$ & x varies from 2 to 0

$\therefore \vec{r} = x\hat{i} + \frac{x}{2}\hat{j} + 0\hat{k} \Rightarrow$

$\Rightarrow d\vec{r} = dx\,\hat{i} + \frac{dx}{2}\hat{j} = \left(\hat{i} + \frac{\hat{j}}{2}\right)dx$

$$I_3 = \int_C \vec{F}\cdot d\vec{r} = \int_C \left[(2x+y^2)\hat{i} + (3y-4x)\hat{j}\right]\cdot\left(\hat{i}+\frac{\hat{j}}{2}\right)dx$$

$$= \int_C \left\{(2x+y^2) + \frac{1}{2}(3y-4x)\right\}dx$$

$$= \int_2^0 \left(2x + \frac{x^2}{4} + \frac{3}{4}x - \frac{4x}{2}\right)dx$$

$$= \left(\frac{x^3}{12} + \frac{3x^2}{8}\right)_2^0$$

$$= -\frac{8}{12} - \frac{12}{8} = \frac{-13}{6} \quad \text{—— (3)}$$

the required integral in counter clockwise direction is

$$\oint_C \vec{F}\cdot d\vec{r} = I_1 + I_2 + I_3 = 4 - \frac{13}{2} - \frac{13}{6} = \frac{-14}{3}.$$

the value of the integral in clockwise direction.

$$\oint_C \vec{F}\cdot d\vec{r} = -\oint_C \vec{F}\cdot d\vec{r} = \frac{14}{3}.$$

Compute $\int_c F\cdot ds$ when $F = \dfrac{y\hat{i} - x\hat{j}}{x^2+y^2}$ & c is circle $x^2+y^2=1$

sol^n. $r = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

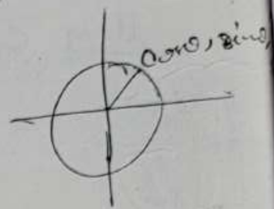
$x = \cos\theta$; $y = \sin\theta$; $z = 0$

radius 1 for this circle

$dx = -\sin\theta d\theta$; $dy = \cos\theta d\theta$; $dz = 0$

$$\int_c \vec{F}\cdot d\vec{r} = \int_c \left(\frac{y\hat{i} - x\hat{j}}{x^2+y^2}\right)\cdot\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \int_c \left(\frac{y\,dx}{x^2+y^2} - \frac{x\,dy}{x^2+y^2}\right)$$



$$= \int_0^{2\pi} \frac{\sin\theta(-\sin\theta)d\theta}{\sin^2\theta + \cos^2\theta} - \int_0^{2\pi} \frac{\cos\theta\cos\theta d\theta}{\sin^2\theta+\cos^2\theta}$$

$$= -\int_0^{2\pi} \frac{(\sin^2\theta + \cos^2\theta)d\theta}{\sin^2\theta+\cos^2\theta} = -\int_0^{2\pi} d\theta = -(\theta) = -2\pi$$

-2π

Q. Find work done by force $F = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2+2)k$ in moving a particle from $P(0,1,-1)$ to $Q\left(\frac{\pi}{2}, -1, 2\right)$

Sol^n :- We are not given curve C but, we are given here end point P & Q so probably force field $\vec{F}$ is conservative.

ie $\nabla\times F = 0$

find $\nabla\times F = 0$ thus F is conservative, it means ϕ is s.t $\vec{F} = \nabla\phi \Rightarrow d\phi = \vec{F}\cdot d\vec{r} = \vec{\nabla}\phi\cdot d\vec{r}$

find $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = (y^2\cos x + z^3)dx + (2y\sin x - 4)dy + (3xz^2+2)dz$

$= (y^2\cos x\,dx + 2y\sin x\,dy) + (z^3dx + 3xz^2dz) + (-4dy + 2dz)$

$= d(y^2\sin x) + d(xz^3) + d(-4y + 2z)$

$\phi = y^2\sin x + xz^3 - 4y + 2z + c$.

$\therefore$ work done $= \int_P^Q \vec{F}\cdot d\vec{r} = \int_P^Q d\phi = \phi(Q) - \phi(P)$.

$= 15 + 4\pi$

surface.

Q. Evaluate $\int_S \frac{\vec{r}}{r^3} \, d\vec{a}$. where s denote the sphere of radius a with centre at the origin. (S1)

Sol^n :- eq^n of sphere of radius a with centre origin

$$x^2+y^2+z^2=a^2$$

Normal vector to above surface is

$$\nabla(x^2+y^2+z^2-a^2)$$

$\therefore$ Unit normal vector is $= \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a} = \hat{n}$$

given $\vec{F} = \frac{\vec{r}}{r^3} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{3/2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a^3}$

$\therefore \int_S \vec{F}\cdot d\vec{a} = \int_S \vec{F}\cdot\hat{n}\,ds$

$$= \int_S \left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a^3}\right)\cdot \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}\,ds$$

$$= \int \frac{x^2+y^2+z^2}{a^4}\,ds = \int \frac{a^2}{a^4}\,ds = \frac{1}{a^2}\int ds$$

$$= \frac{1}{a^2}\,4\pi a^2 = 4\pi$$

Surface area of sphere $= 4\pi a^2$

Q2. $\vec{F} = y\hat{i} + (x-2xz)\hat{j} - xy\hat{k}$ evaluate

$\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,ds$.

where s is the surface of the sphere.

$x^2+y^2+z^2=a^2$ above the xy plane.

Sol^n: $\vec{f} = \vec{\nabla}\times\vec{F} = x\hat{i}+y\hat{j}-2z\hat{k}$

now, the vector normal to the surface s is given by

$\nabla(a) \qquad \nabla(x^2+y^2+z^2) = 2x\hat{i}+2y\hat{j}+2z\hat{k}$

$\therefore \hat{n} =$ unit normal vector

$$= \frac{\vec{\nabla}(a)}{|\vec{\nabla}(a)|} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a} \qquad [\because x^2+y^2+z^2 \cdots$$

$$f.\hat{n} = \nabla\times\vec{F}.\hat{n}$$

$$= (x\hat{i}+y\hat{j}-2z\hat{k}) . \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

$$= \frac{x^2+y^2-2z^2}{a} = \frac{3(x^2+y^2)-2a^2}{a}$$

$$\int_S F.\hat{n}\,ds = \iint_{S_3} \vec{F}.\hat{n}\frac{dx\,dy}{\hat{n}.\hat{k}} \qquad \left[ds = \frac{dx\,dy}{\hat{n}.\hat{k}}\right.$$

$$\hat{n}.\hat{k} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}.\hat{k} = \frac{z}{a} = \frac{\sqrt{a^2-x^2-y^2}}{a}$$

$$\therefore \iint_S (\nabla\times F).\hat{n}\,ds = \iint_{S_3} \frac{3(x^2+y^2)-2a^2}{a} \frac{.\,dx\,dy\,.\,a}{\sqrt{a^2-x^2-y^2}}$$

S_3 is shadow (Projection) of $x^2+y^2+z^2=a^2$ in xy plane given by

$x^2+y^2=a^2$

$x = r\cos\theta$; $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^a \frac{3r^2-2a^2}{a} \frac{r\,dr\,d\theta\,a}{\sqrt{a^2-r^2}}$$

$$= \int_0^a\int_0^{2\pi} \frac{r(3r^2-2a^2)}{\sqrt{a^2-r^2}}\,dr\,d\theta.$$

$$= 2\pi\int_0^a \frac{3r^2-2a^2}{\sqrt{a^2-r^2}}\,r\,dr.$$

put $a^2-r^2=t^2$

$\Rightarrow -r\,dr = t\,dt$

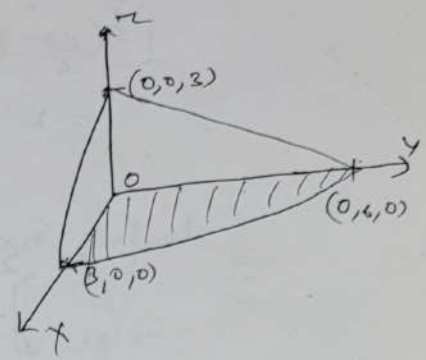
$$2\pi\int_a^0 \frac{3(a^2-t^2)-2a^2}{t}(-t\,dt)$$

$$= 2\pi\left[a^2t-t^3\right]_0^a = 2\pi\left[a^3-a^3\right] = 0.$$

Q. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ (10)

and S is the surface of the plane $2x+y+2z=6$ in the first octant

Soln :-



vector normal to s is

$\nabla(2x+y+2z) = 2\hat{i} + \hat{j} + 2\hat{k}$

$\therefore \hat{n} = \dfrac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\hat{i}+\hat{j}+2\hat{k})$.

The region of integration is the projection of s on the xy-plane ie bounded by x-axis, y-axis & the line $2x+y=6$; $z=0$.

Given integral $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S \dfrac{\vec{F}\cdot\hat{n}\,dx\,dy}{|\hat{n}\cdot\hat{k}|}$

Now $\vec{F}\cdot\vec{n} = \frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{4}{3}yz$

$= \frac{2}{3}y^2 + \frac{4}{3}yz$

& $\hat{n}\cdot\hat{k} = \frac{2}{3}$

$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S \left(\frac{2}{3}y^2 + \frac{4}{3}yz\right)\frac{3}{2}\,dx\,dy$

$= \iint_S \left\{y^2 + 2y\left(\frac{6-2x-y}{2}\right)\right\}.dx\,dy$

$$= 2\int_0^6 \int_0^{\frac{6-y}{2}} y(3-x)\,dx\,dy$$

$$= 2\int_0^6 y\left[3x - \frac{x^2}{2}\right]_0^{\frac{6-y}{2}} dy$$

$$= 2\int_0^6 y\left[3\left(\frac{6-y}{2}\right) - \frac{1}{2}\left(\frac{6-y}{2}\right)^2\right] dy$$

$$= 2\int_0^6 \left[9y - \frac{3}{2}y^2 - \frac{9}{2}y + \frac{3}{2}y^2 - \frac{y^3}{8}\right] dy.$$

$$= \int_0^6 \left[9y - \frac{y^3}{4}\right] dy = \left[\frac{9}{2}y^2 - \frac{y^4}{16}\right]_0^6 = 81.$$

Q2. If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, evaluate $\iiint_V \vec{F}\,dv$ where V is the region bounded by the surfaces $x=0,\ y=0,\ x=2,\ y=4$ $z = x^2$ & $z = 2$

sol^n $$\iiint_V \vec{F}\,dv = \int_0^2 \int_0^4 \int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\,dz\,dy\,dx$$

$$= \int_0^2 \int_0^4 \left(z^2\hat{i} - xz\hat{j} + yz\hat{k}\right)_{x^2}^2 dy\,dx$$

$$= \int_0^2 \int_0^4 \left[(4 - x^4)\hat{i} - (2x - x^3)\hat{j} + (2y - yx^2)\hat{k}\right] dy\,dx$$

$$= \int_0^2 \left[(4-x^4)y\,\hat{i} - (2x - x^3)y\,\hat{j} + \left(y^2 - \frac{x^2y^2}{2}\right)\hat{k}\right]_{y=0}^4 dx$$

(ii) (2)

$$= \int_0^2 \left[4(4-x^4)\,\hat{i} - 4(2x - x^3)\,\hat{j} + (16 - 8x^2)\,\hat{k}\right] dx$$

$$= \left[4\left(4x - \frac{x^5}{5}\right)\hat{i} - 4\left(x^2 - \frac{x^4}{4}\right)\hat{j} + \left(16x - \frac{8x^3}{3}\right)\hat{k}\right]_0^2$$

$$= \left[\left(32 - \frac{128}{5}\right)\hat{i} - (16 - 16)\hat{j} + \left(32 - \frac{64}{3}\right)\hat{k}\right]$$

$$= \frac{32}{5}\,\hat{i} - \frac{32}{3}\,\hat{k}$$

$\hat{i} - 2xy\,\hat{j} - 4x\,\hat{k}$ Evaluate