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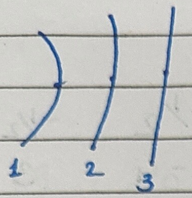
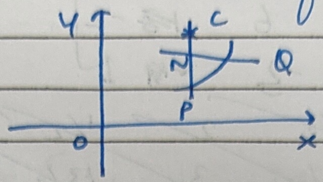
CALCULUS

Curvature :- It is measure of bending of curve.





Let P be a point on a curve and Q be a neighbouring point of P ~~B~~, draw normals through P & Q. These normals may meet at a point N.
Now ~~the~~ when Q tending to P $(Q \to P)$
$N \to$ to a definite C
This C is called the centre of curvature.
CP is radius of curvature and a circle with centre at C & radius as CP is called a circle of curvature. and If a chord is drawn from the point P in the circle of curvature is called the chord of curvature.
The reciprocal of radius of curvature is called curvature. and is denoted by $= \frac{1}{\rho}$

$$\text{curvature} = \frac{1}{\rho} \quad , (\rho = \text{radius of curvature})$$

① Intrinsic for
$S = F(\psi)$
② $r = F(\theta)$
$S = F(x, y)$

} not in syllabus.

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① $y = f(x)$ or $x = f(y)$

Cartesian form when $y=f(x)$ → $\rho = \dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \dfrac{(1+y'^2)^{3/2}}{y''}$

② $x = f(t)$, $y = \phi(t)$ (Parametric form)

$$\rho = \frac{[\dot{x}^2 + \dot{y}^2]^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$\dot{x} = \frac{\partial f}{\partial t}, \quad \ddot{x} = \frac{\partial^2 f}{\partial t^2}, \quad \dot{y} = \frac{\partial \phi}{\partial t}, \quad \ddot{y} = \frac{\partial^2 \phi}{\partial t^2}$$

Q-1 Show that the radius of curvature of the curve whose equation in parametric form is

$x = F(t)$

$y = \phi(t)$

~~is~~ at any point ~~P~~ t is given by $\rho = \dfrac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$

when $\cdot$ denotes differentiation w.r.t t.

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \quad \text{———} (i)$$

$$\frac{dy}{dx} = \frac{dy}{dx} \times \frac{dt}{dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}} \quad \text{———} (2)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\dot{y}}{\dot{x}}\right) = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)\frac{dt}{dx}$$

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$$\frac{d^2y}{dx^2} = \left[\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2}\right]\frac{1}{\dot{x}}$$

$$\rho = \frac{\left[1 + \left(\frac{\dot{y}}{\dot{x}}\right)^2\right]^{3/2}}{\left[\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2}\right]\frac{1}{\dot{x}}} = \frac{\left(\frac{\dot{x}^2 + \dot{y}^2}{\cancel{\dot{x}^2}}\right)^{3/2}}{\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\cancel{\dot{x}^3}}}$$

$$= \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

Q-2 find radius of curvature for $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$2y\frac{dy}{dx} = b^2\left(0 - \frac{2x}{a^2}\right)$$

$$\cancel{2}y\frac{dy}{dx} = \frac{-b^2 x \cancel{2}x}{a^2}$$

$$\frac{dy}{dx} = \frac{-b^2x}{a^2y}$$

$$y'' = \frac{-b^2}{a^2}\left[\frac{y - xy'}{y^2}\right]$$

$$y'' = \frac{-b^2}{a^2y^2}(y - xy')$$

$$y'' = \frac{-b^2}{a^2y^2}\left(y - x\left(\frac{-b^2x}{a^2y}\right)\right)$$

$$y'' = \frac{-b^2}{a^2y^2}\left(\frac{a^2y^2 + x^2b^2}{a^2y}\right)$$

$$= \frac{-b^2}{a^2y^2}\left(\frac{a^2b^2}{a^2y}\right) = \frac{-b^4}{a^2y^3}$$

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$$\rho = \frac{[1+y'^2]^{3/2}}{y''}$$

$$= \frac{\left(1+\left(\frac{-b^2x}{a^2y}\right)^2\right)^{3/2}}{\left(\frac{-b^4}{a^2y^3}\right)} = \frac{\left(\frac{a^4y^2+b^4x^2}{a^4y^2}\right)^{3/2}}{\frac{-b^4}{a^2y^2}}$$

$$= \frac{(a^4y^2+b^4x^2)^{3/2}}{(a^2y)^3 \times -b^4} \times a^2y^3$$

$$= \frac{(a^4y^2+b^4x^2)^{3/2}}{a^{\cancel{4}}y^3 \times b^4} \times \cancel{a^2}\cancel{y^3}$$

$$\rho = \frac{(a^4y^2+b^4x^2)^{3/2}}{a^4b^4}$$ Ans.

Q-3 For a curve $y = ae^{x/a}$. prove $\rho = a\sec^2\theta\,\mathrm{cosec}\,\theta$ where $\theta = \tan^{-1}\left(\frac{y}{a}\right)$

$$y' = e^{x/a}, \quad y'' = \frac{e^{x/a}}{a}$$

$\left\{ \tan\theta = \frac{y}{a} = e^{x/a} \right\}$ given

$$\rho = \frac{[1+(e^{x/a})^2]^{3/2}}{\frac{e^{x/a}}{a}}$$

$$= \frac{a[1+\tan^2\theta]^{3/2}}{\tan\theta}$$

$$= \frac{a(\sec^2\theta)^{3/2}\cos\theta}{\sin\theta}$$

$$= a\sec^3\theta\cos\theta\,\mathrm{cosec}\,\theta$$

$$= a\sec^2\theta\,\mathrm{cosec}\,\theta$$

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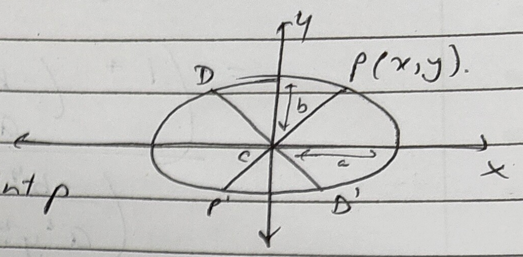
Q4 a) If CP, CD be a pair of conjugate semi diameters of the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



then prove that the radius of curvature at the point P is $\rho = \frac{(CD)^3}{ab}$

where a & b are the length of semi axis of the ellipse.

b) prove that for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\rho = \frac{a^2b^2}{p^3}$ (p is the perpendicular from the centre upon the tangent at any point x, y)

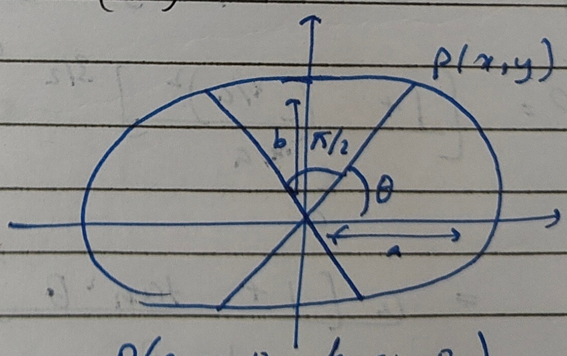
c) If ρ_1 & ρ_2 be the radius of curvature at the end points of two conjugate diameter of the ellipse then prove that

$$\rho_1^{2/3} + \rho_2^{2/3} = \frac{(a^2+b^2)}{(ab)^{2/3}}$$

a) $x = a\cos\theta, \quad y = b\sin\theta$

$\dot{x} = -a\sin\theta, \ \dot{y} = b\cos\theta$

$\ddot{x} = -a\cos\theta ; \ \ddot{y} - b\sin\theta$



$$\rho = \frac{[\dot{x}^2 + \dot{y}^2]^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$= \cancel{(a\cos\theta)^2} \frac{[(a\sin\theta)^2 + (b\cos\theta)^2]^{3/2}}{ab\sin^2\theta + ab\cos^2\theta}$$

$$= \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab} \quad \text{——— (3)}$$

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coordinates of $D(x_1, y_1)$

$x_1 = a\cos\left(\frac{\pi}{2} + \theta\right) = \cancel{a\cos\theta} - a\sin\theta$

$y_1 = b\sin\left(\frac{\pi}{2} + \theta\right) = \cancel{b\sin\theta}\ \ b\cos\theta$

$(CD)^2 = (0 + a\sin\theta)^2 + (0 - b\cos\theta)^2$

$CD^2 = a^2\sin^2\theta + b^2\cos^2\theta$ ——— (4)

using (3) & (4)

$\rho = \frac{((CD^2))^{3/2}}{ab} = \frac{CD^3}{ab}$

b) eq^n of tangent at $p(x, y)$ for ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

from (x_1, y_1)

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

$$\frac{x(a\cos\theta)}{a^2} + \frac{y(b\sin\theta)}{b^2} = 1$$

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \text{ ——— (i)}$$

$ax + by + c = 0$

from (x_1, y_1)

$$p = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

drawing $\perp$ from $C(0,0)$

$$p = \frac{+1}{\sqrt{\left(\frac{\cos\theta}{a}\right)^2 + \left(\frac{\sin\theta}{b}\right)^2}}$$

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$$p = \frac{1 \times ab}{\sqrt{(b^2\cos^2\theta + a^2\sin^2\theta)}}$$

$$\frac{1}{p^3} = \frac{(b^2\cos^2\theta + a^2\sin^2\theta)^{3/2}}{(ab)^3} \quad \text{———} (6)$$

using (6) in (3)

$$\rho = \frac{(ab)^2}{p^3}$$

c) $$\rho_1 = \frac{[a^2\sin^2\theta + b^2\cos^2\theta]^{3/2}}{ab}$$

$$\rho_2 = \frac{\left[a^2\sin^2\left(\frac{\pi}{2}+\theta\right) + b^2\cos^2\left(\frac{\pi}{2}+\theta\right)\right]^{3/2}}{ab}$$

$$= \frac{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}{ab}$$

LHS $\rho_1^{2/3} + \rho_2^{2/3}$

$$= \frac{a^2\sin^2\theta + b^2\cos^2\theta}{(ab)^{2/3}} + \frac{a^2\cos^2\theta + b^2\sin^2\theta}{(ab)^{2/3}}$$

$$\rho_1^{2/3} + \rho_2^{2/3} = \frac{a^2+b^2}{(ab)^{2/3}}$$

Q-5. Show that the parabola $y^2 = 4ax$ the radius of curvature at any point P is

$$\rho = \frac{2(SP)^{3/2}}{\sqrt{a}}$$

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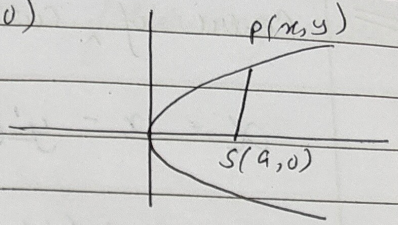
whereas the focus of parabola $(a,0)$



$x = at^2, \quad y = 2at$

$\dot{x} = 2at, \quad \dot{y} = 2a$

$\ddot{x} = 2a, \quad \ddot{y} = 0$

$$\rho = \frac{\left((2at)^2 + (2a)^2\right)^{3/2}}{0 \mp 4a^2}$$

$$\rho = \frac{[(2a)^2]^{3/2}[t^2+1]^{3/2}}{+4a^2} \quad \text{——(1)}$$

$$(SP)^2 = (a - at^2)^2 + (0 - 2at)^2$$

$$(SP)^2 = a^2 + a^2t^4 - 2a^2t^2 + 4a^2t^2$$

$$= \cancel{a^2(1+t^2-2t+4t^2)}$$

$$= \cancel{a^2(1+5t^2-2t)}$$

$$= (a+at^2)^2.$$

$$= a^2(1+t^2)^2 \quad \text{——(2)}$$

$$SP = a(1+t^2)$$

$$\rho = \frac{\cancel{8}^{2} a^{\cancel{3}} [t^2+1]^{3/2}}{\cancel{4a^2}} = 2a(t^2+1)^{3/2}$$

$$\rho = 2a \frac{(SP)^{3/2}}{(a)^{3/2}} = \frac{2(SP)^{3/2}}{\sqrt{a}} \quad \text{Ans.}$$

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Centre of curvature (circle of curvature) :-

$$\alpha = x - \frac{y'(1+y'^2)}{y''}$$

$$\beta = y + \frac{(1+y'^2)}{y''}$$

Q-1 Find the centre of circle of curvature for the curve $y^2 = 4ax$.

$$2yy' = 4a$$

$$y' = \frac{4a}{2y} = \frac{2a}{y}$$

$$y'' = 2a\left[\frac{-1}{y^2}\right]$$

$$\alpha = x - \frac{\frac{2a}{y}\left(1+\left(\frac{2a}{y}\right)^2\right)}{\frac{-2a}{y^2}} = x - \frac{\frac{2a}{y}\left(\frac{y^2+4a^2}{y^2}\right)}{\left(\frac{-2a}{y^2}\right)}$$

$$= x + \frac{2a}{y}\times\frac{y^2}{2a}\left[1+\frac{4a^2}{y^2}\right]$$

$$= x - \frac{\left(\frac{2ay^2+8a^3}{y^3}\right)}{\frac{2a}{y^2}}$$

$$= \frac{x}{2}\left(\frac{2y^2+8a^2}{2y}\right)$$

$$= x + y\left[1+\frac{4a^2}{y^2}\right]$$

$$\beta = y + \frac{\left(1+\frac{4a^2}{y^2}\right)}{\frac{-2a}{y^2}} = y - \frac{y^2}{2a}\left(1+\frac{4a^2}{y^2}\right)$$

$(x-\bar{x})^2+(y-\bar{y})=\rho^2$

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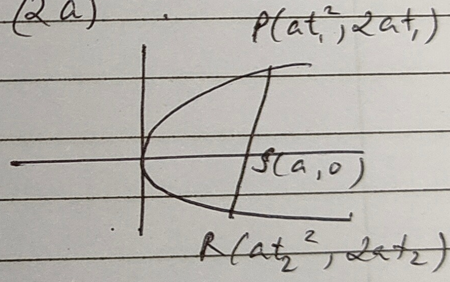
Q-2 find the radius of curvature for following curves

1) $ay^2=x^3$
2) $x^2=4ay$
3) $xy=a^2$
4) $x=a(t+\sin t)$
   $y=a(1-\cos t)$
5) $x=a\cos^3\theta$
   $y=a\sin^3\theta$
6) $x=6t^2-3t^4$
   $y=8t^3$

Q-3 If ρ_1 & ρ_2 are the radius of curvature at the end points of a focal chord of the parabola $y^2=4ax$ then show that $(\rho_1)^{-2/3}+(\rho_2)^{-2/3}=(2a)^{-2/3}$.

$t_1 t_2=-1$



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Q. ~~A straight line~~

Asymptote :-

A straight line at a finite distance from the origin to which a tangent tends, as the distance from the origin of the point of contact tends to infinity is called Asymptote of the curve.

procedure to find eq^n of Asymptote :-

Q $3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0$

$\phi_n(m)$ $\quad x = 1,\ y = m$

$\phi_3(m) = 3 + 2m - 7m^2 + 2m^3$

$\phi_2(m) = -14m + 17m^2$

$\downarrow$

$\phi_{n-1}(m)$

$\phi_1(m) = \phi_{n-2}(m) = 4 + 5m$

It is easily seen that $\phi_n(m)$ is obtained by putting $x = 1,\ y = m$ in the n^{th} degree terms of the eq^n of the curve. Similarly we can find $\phi_{n-1}(m)$.
Solve $\phi_n(m) = 0$ for the values of m.

Let its roots be $m_1, m_2, \dots, m_n$, the value of c i.e $c_1, c_2, \dots c_n$ corresponding to each m are given by

$$c = \frac{-\phi_{n-1}(m)}{\phi_n'(m)}$$

provided $\phi_n'(m) \neq 0$

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$\boxed{y = mx + c}$

Oblique Asymptotes :-

Asymptotes parallel to each other :

If for some values of m both,

$\phi_{n-1}(m)$ & $\phi_n'(m) = 0$

So that c cannot be find ~~for~~ by the previous formula $c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}$

then we will find c by following formulae

for 2 equal roots of m: $\frac{c^2}{\lfloor 2}\phi_n''(m) + \frac{c}{\lfloor 1}\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$

for 3 equal values of m:

$\frac{c^3}{\lfloor 3}\phi_n'''(m) + \frac{c^2}{\lfloor 2}\phi_{n-1}''(m) + \frac{c}{\lfloor 1}\phi_{n-2}'(m) + \phi_{n-3}(m) = 0$

Asymptotes parallel to x & y axis :-

The Asymptotes parallel to x is obtained by equating to 0 the coefficient of highest power of x in the eq^n of curve, provided it is not a constant

Similarly we can find the Asymptote parallel to y axis.

Q $3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0$

① Asymptotes ∥ to the axis

∥ to x axis

$3 \neq 0$ [coeff of highest power of x = 3 which is not equal to 0]

$2 \neq 0$ [coeff of y]

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$\phi_3(m) = 3 + 2m - 7m^2 + 2m^3 = 0$

$2m^3 - 7m^2 + 2m + 3 = 0$

$m = -\frac{1}{2}, 1, 3$

$\phi_3'(m) = 2 - 14m + 6m^2$

$6m^2 - 14m + 2 = 0$

$\cancel{3m^2 - 7m + 1 = 0}$

$n = 3 \quad C = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{-(7m^2 - 14m)}{6m^2 - 14m + 2}$

$\phi_2(m) = -14m + 7m^2$

$m_1 = -\frac{1}{2}$

$C_1 = \frac{-\left(7\left(\frac{1}{4}\right) + 7\right)}{36\left(\frac{1}{4}\right) + 7 + 2} = \frac{-\left(\frac{7+28}{4}\right)}{\frac{3+14+4}{2}} = -\frac{5}{6}$

$m_2 = 1 \longrightarrow C_2 = -\frac{7}{6}$

$m_3 = 3, \quad C_3 = -\frac{3}{2}$

$y = -\frac{1}{2}x - \frac{5}{6}$

$y = x - \frac{7}{6}$

$y = 3x - \frac{3}{2}$

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8 $x^3+3x^2y-4y^3-x+y+3=0$

$\phi_n(m) = \phi_3(m) = 1+3m-4m^3$

$\phi_3(m) = 0 = 4m^3-3m-1$

$m = \frac{-1}{2}, \frac{-1}{2}, 1$

for $m=1$, $c = \frac{-\phi_{n-1}(m)}{\phi_n'(m)} = \frac{-\phi_2(m)}{\phi_3'(m)}$

$\phi_3'(m) = 3-12m^2 \Rightarrow \phi_3''(m) = -24m$

$\phi_2(m) = 0$

$\cancel{\phi}$ $\phi_1(m) = -1+m$

$c = \frac{0}{3-12m^2}$ for $m=1 \Rightarrow c=0$

$y=x$

$\frac{c^2}{\lfloor 2}\phi_3''(m) + \frac{c}{\lfloor 1}\phi_2'(m) + \phi_1(m)$

for $m = \frac{-1}{2}, \frac{-1}{2}$

$\frac{c^2}{2}(-24m) + c(0) + m-1 = 0$

putting $m = \frac{-1}{2}$

$\frac{c^2}{2}(12) + m-1 = 0$

$\frac{c^2}{\cancel{2}}(\cancel{12}^{6}) + \left(\frac{-1}{2}\right) - 1 = 0$

$\cancel{6}^{2}c^2 = \frac{\cancel{3}}{2}$ | $c^2 = \frac{1}{4}$ | $c = \pm\frac{1}{2}$

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$$y = \frac{-1}{2}x - \frac{1}{2}$$

$$y = \frac{-1}{2}x + \frac{1}{2}$$

Q-2 $y^3 - 6xy^2 + 11x^2y - 6x^3 + x + y = 0$

The coeff of highest power of $x \neq 0$ so no parallel assymptote to x-axis

In the same way no assymptote is $\parallel^{el}$ to y-axis.

$n = 3$ $\phi_3(m)$ $x = 1, y = m$

$$\phi_3(m) = m^3 - 6m^2 + 11m - 6 = 0$$

$$m = 1, 2, 3$$

$$c = \frac{-\phi_2(m)}{\phi_3'(m)}$$

$\phi_2 = 0$

$\phi_3'(m) = 3m^2 - 12m + 11 = 0$

$$c = \frac{0}{\phi_3'(m)} = 0 \text{ for all values of } m.$$

$y = x$

$y = 2x$

$y = 3x$

Q-3 ~~4x~~ $4x^3 - 3xy^2 - y^3 + 2x^2 - xy - y^2 - 1 = 0$

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Q-4 find asymptote parallel to following curve :-

1. $x(y-3)^3 = 4y(x-1)^3$ $y=0, \quad x=0$

2. $a^2x^2 = (x-y)^2(y^2+a^2)$ $\left(-(y^2+a^2)-a^2\right)y=0$, $y^2=0$

3. $(x+a)(y^2) - x^2y = bx^2$

4. $x^2y^2 - a^2(x^2+y^2) - a^3(x+y) + a^4 = 0$

5. $x^3y + y^3x = a^4$

6. $a^2y^2 + b^2y^2 = x^2y^2$

7. $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$

1. $x(y-3)^3 - 4y(x-1)^3 = 0$

coeff of highest power of $x^3 = -4y$

$-4y = 0$

$y=0 \longrightarrow$ x-axis itself is asymptote.

$\parallel$ to y-axis

$x=0 \longrightarrow$ y-axis itself is asymptote.

2. $a^2x^2 - (x-y)^2(y^2+a^2) = 0$

$a^2 - (y^2+a^2) = 0$

$y=0 \longrightarrow$ x-axis itself is asymptote.

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Partial differentiation :-

$z = F(x,y)$

or $u = F(x,y,z)$

$y = F(x)$

$\frac{dy}{dx} =$

$\frac{\partial z}{dx}, \frac{\partial z}{\partial y}, \frac{\partial U}{dx}, \frac{\partial U}{dy}, \frac{\partial U}{dz}$

$\frac{\partial^2 z}{dx^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y \partial x}$

$\frac{\partial^3 U}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial U}{dz}\right)\right]$

⇒ If funct^n is cont^n.

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

Q Find the first order partial differential coefficient of $x^3 + x^2 y^2 + y^4 + 3z^3 - xyz$

$f(x,y,z) = x^3 + x^2 y^2 + y^4 + 3z^3 - xyz$

$\frac{\partial F}{\partial x} = 3x^2 + 2xy^2 + 0 + 0 - yz$

$\frac{\partial F}{\partial y} = 2x^2 y + 4y^3 - xz$

$\frac{\partial F}{\partial z} = +9z^2 - xy$

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Q.2. if $u = e^{xyz}$ then show that

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2) e^{xyz}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z$$

$$\frac{\partial u}{\partial z} = e^{xyz}(xy)$$

$$\frac{\partial^2 u}{\partial y \partial z} = x[e^{xyz} + xze^{xyz}\, y]$$

$$= xe^{xyz} + x^2yz\, e^{xyz}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz} + yze^{xyz}x + 2xyze^{xyz} + yze^{xyz}x^2yz$$

$$= e^{xyz}(1 + xyz + 2xyz + x^2y^2z^2)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2) e^{xyz}$$

Hence proved.

Q-3 $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$

prove that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = x^2\left[\frac{1}{1 + \frac{y^2}{x^2}}\right]\left(\frac{1}{x}\right) - 2y \tan^{-1}\frac{x}{y} - y^2\left(\frac{1}{1 + \frac{x^2}{y^2}}\right)\left(\frac{-x}{y^2}\right)$$

$$= \frac{x^3}{x^2 + y^2} - 2y \tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2 + y^2}$$

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$$\frac{\partial^2 U}{\partial x \partial y} = \left[-2y\left(\frac{1}{1+\frac{x^2}{y^2}}\right)\left(\frac{1}{y}\right)\right]$$

$$= \frac{x^3 + xy^2}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y}$$

$$= \cancel{2}x - 2y \tan^{-1}\frac{x}{y}$$

$$\frac{\partial^2 U}{\partial x \partial y} = 1 - 2y \left(\frac{1}{1+\frac{x^2}{y^2}}\right)\left(\frac{1}{y}\right)$$

$$= 1 - \frac{2y^2}{x^2+y^2}$$

$$= \frac{x^2+y^2-2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$ Hence proved.

Q-4. $U = \tan^{-1}\frac{xy}{\sqrt{1+x^2+y^2}}$

show that $\frac{\partial^2 U}{\partial x \partial y} = \frac{1}{[1+x^2+y^2]^{3/2}}$

$$\frac{\partial U}{\partial y} = \frac{1}{1+\frac{x^2y^2}{1+x^2+y^2}}\left(x\left(\sqrt{1+x^2+y^2}\right) - \frac{\frac{x}{2}(1+x^2+y^2)^{-1/2}\, xy^2}{x}\right.$$

$$= \frac{1}{1+\frac{x^2y^2}{1+x^2+y^2}}\left(x \sqrt{\right.$$

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Q. if $x^x y^y z^z = c$, show that at $x=y=z$
$$\frac{\partial^2 z}{\partial x \partial y} = -[x(1+\log x)]^{-1}$$

$$x^x y^y z^z = c$$

$$x\log x + y\log y + z\log z = \log c$$

diff. this w.r.t y

$$\log y + \cancel{y}\frac{1}{\cancel{y}} + \frac{\partial z}{\partial y}\log z + \cancel{z}\frac{1}{\cancel{z}}\frac{\partial z}{\partial y} = 0$$

$$1+\log y + \log z\frac{\partial z}{\partial y} + \frac{\partial z}{\partial y} = 0$$

$$1+\log y + \frac{\partial z}{\partial y}(1+\log z) = 0$$

$$\frac{\partial z}{\partial y} = \frac{-(1+\log y)}{1+\log z}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -(1+\log y)\left[\frac{-\frac{1}{z}\frac{\partial z}{\partial x}}{(1+\log z)^2}\right]$$

Since this is a symmetric eq^n.

$$\frac{\partial z}{\partial x} = \frac{-(1+\log x)}{1+\log z}$$

$$\frac{\partial^2 z}{\partial x \partial y} = +(1+\log y)\frac{\frac{1}{z}\left(\frac{-(\log x+1)}{1+\log z}\right)}{(1+\log z)^2}$$

$$= -\frac{1}{z}(1+\log y)\frac{(\log x+1)}{(1+\log z)^3}$$

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at $x=y=z$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-\cancel{(1+\log x)^2}}{x(1+\log x)^{\cancel{3}}}$$

$$= \frac{-1}{x(1+\log x)}$$

$$= -[(x(1+\log x))]^{-1}$$

Hence proved.

Q-2. if $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$ —— (i)

Show that.

$$u_x^2 + u_y^2 + u_z^2 = 2(x u_x + y u_y + z u_z)$$

diff (i) w.r.t x

$$\frac{2x(a^2+u) - x^2(u_x)}{(a^2+u)^2} + \frac{[-y^2 u_x]}{[b^2+u]^2} + \frac{[-z^2 u_x]}{(c^2+u)^2} = 0$$

$$u_x\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] = \frac{2x}{a^2+u} \text{ —— (ii)}$$

diff (i) w.r.t y

$$\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]u_y = \frac{2y}{b^2+u} \text{ —— (iii)}$$

diff (i) w.r.t z

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$$u_z\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]=\frac{2z}{c^2+u} \quad \text{--- (iv)}$$

Squaring (2), (3), (4) & adding

$$\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]^2\left[u_x^2+u_y^2+u_z^2\right]$$
$$=4\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

$$4x^2+4y^2+4z^2=\frac{4}{\left(\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right)} \quad \text{--- (5)}$$

Multiply eq (ii) by x, eq(iii) by y & eq (iv) by z & adding them.

$$\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right](xu_x+yu_y+zu_z)$$
$$=2\left[\frac{x^2}{a^2+u}+\cancel{\frac{y^2}{b^2+u}}+\frac{z^2}{c^2+u}\right] \quad \text{--- (6)}$$

(the crossed term has "1" written above it)

5 ÷ 6

$$\frac{u_x^2+u_y^2+u_z^2}{xu_x+yu_y+zu_z}=2$$

Q if $z=f(x+ct)+\phi(x-ct)$ prove $\frac{\partial^2 z}{\partial t^2}=c^2\frac{\partial^2 z}{\partial x^2}$ --- (1)

$$\frac{\partial z}{\partial t}=cF'(x+ct)-c\phi'(x-ct)$$

$$\frac{\partial^2 z}{\partial t^2}=c^2F''(x+ct)+c^2F''(x-ct) \quad \text{--- (2)}$$

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$$\frac{\partial z}{\partial x} = F'(x+ct) + \phi'(x-ct)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct) \quad — (3)$$

using (2) & (3)

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

Q if $\theta = t^n e^{-\frac{r^2}{4t}}$ what values of n will make

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$$

Q $u = x^y$ show that

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$$

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Q. If $u = \log(x^3+y^3+z^3-3xyz)$
Show that

① $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -9(x+y+z)^{-2}$

② $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = -3(x+y+z)^{-2}$

$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$ — (i)

$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$ — (ii)

$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$ — (iii)

adding (i), (ii) & (iii)

$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3x^2+3y^2+3z^2-\cancel{9xyz}\ 3xy-3yz-3xz}{x^3+y^3+z^3-3xyz}$

$= \frac{3(x^2+y^2+z^2-\cancel{3xyz}\ xy-xz-yz)}{(x+y+z)(x^2+y^2+z^2-xy-yz-xz)}$

$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u = \frac{3}{x+y+z}$ — (v)

$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u = \frac{3}{(x+y+z)^{2}}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)$

$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$

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$$= 3\left(\frac{-1}{(x+y+z)^2} - \frac{1}{(x+y+z)^2} - \frac{1}{(x+y+z)^2}\right)$$

$$= -9\,(x+y+z)^{-2} \quad \underline{\text{Ans}}.$$

(ii) $\dfrac{\partial u}{\partial x} = \dfrac{3x^2 - 3yz}{x^3+y^3+z^3-3xyz}$

$\dfrac{\partial u}{\partial y} = \dfrac{3y^2 - 3xz}{x^3+y^3+z^3-3xyz}$

$\dfrac{\partial u}{\partial z} = \dfrac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{6x(x^3+y^3+z^3-3xyz) - (3x^2-3yz)(3x^2-3yz)}{(x^3+y^3+z^3-3xyz)^2}$$

$$= \cancel{\frac{6x^4 + 6xy^3 + 6xz^3 - 18x^2yz - 9x^4 + 3x^2yz}{-18yz \quad + 9x^2yz - 3y^2z^2}}$$

$$= \frac{6x(A) - (3x^2-3yz)^2}{A^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{6y(A) - (3y^2-3xz)^2}{A^2}$$

$$\frac{\partial^2 z}{\partial z^2} = \frac{6z(A) - (3z^2-3xy)^2}{A^2}$$

$$= \frac{(A)(6x+6y+6z) - (9x^4 + 9y^4 + 9z^4 + 9y^2z^2 + 9x^2z^2 + 9x^2y^2 - 18x^2yz - 18y^2xz - 18xyz^2)}{(x^3+y^3+z^3-3xyz)^2}$$

$$= \frac{(x^3+y^3+z^3-3xyz)(6x+6y+6z) - 9(x^4+y^4+z^4+x^2z^2+y^2z^2 + x^2y^2) + 18xyz(x+y+z)}{(x^3+y^3+z^3-3xyz)^2}$$

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$\cancel{6x^4} + 6xy^3 + 6xz^3 - 18x^2yz + 6x^3y + \cancel{6y^4} +$
$6yz^3 - 18xy^2z + 6zx^3 + 6zy^3 +$
$\cancel{6z^4} - 18xyz^3$

~~Numerator~~ →

$-9(\cancel{x^4} + \cancel{y^4} + \cancel{z^4} + x^2z^2 + y^2z^2 + x^2y^2)$
$+18xyz(x+y+z)$

$\Rightarrow -3x^4 - 3y^4 - 3z^4 + 6xy^3 + 6xz^3 + 6yz^3 +$
$6zx^3 + 6zy^3 + 6x^3y - 18xyz\cancel{(x+y+z)}$
$-9(x^2z^2 + y^2z^2 + x^2y^2)$
$+18xyz\cancel{(x+y+z)}$

$\Rightarrow \frac{1}{A^2} \times -3(x^4 + y^4 + z^2 - 2xy^3 - 2xz^3 - 2yz^3 -$
$2zx^3 - 2zy^3 - 6x^3y + 3x^2z^2$
$+ 3y^2z^2 + 3x^2y^2)$

$$= \frac{-3(x^4 + y^2 + z^2 - 2xy^3 - 2xz^3 - 2yz^3 - 2zx^3 - 2zy^3 - 6x^3y + 3x^2z^2 + 3y^2z^2 + 3x^2y^2)}{(x+y+z)^2(x^2+y^2+z^2-xy-xz-yz)^2}$$

$= -3(x+y+z)^{-2}$

Q2 $u = x^y$ Show that.

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}.$$

Q- $u = f(r)$ & $x = r\cos\theta$, $y = r\sin\theta$

prove: $\frac{\partial^2 w}{\partial x^2} + \frac{\partial u}{\partial y^2} = F''(r) + \frac{1}{r} f'(r)$

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