Discrete Mathematics
Lec11: Relation 2

馬誠佑

Relations on a Set

Def:

A (binary) relation from a set A to itself is called a relation on the set A.

- E.g., the “<” relation from earlier was defined as relation on the set N of natural numbers.
- The identity relation $I_A\ on\ a\ set\ A\ is\ the\ set\ \{(a,a)|a \in A)\}$.
- Let A be the set {1,2,3,4}. Which ordered pairs are in the relation R={(a,b)|a divides b}?
- How many relations are there on a set with n elements?

Binary Relations on a Set [1]

Definition: A binary relation *R on a set A* is a subset of $A \times A$ or a relation from A to A.

Example:

- Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b), (a,c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation $R = \{(a,b) \mid a \text{ divides } b\}$ are $(1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3)$, and $(4, 4)$.
- $|A|=n$

$$R: A \leftrightarrow A \;\; a\ relation\ on\ A$$

$$R \subseteq A\times A \Rightarrow 2^{|A\times A|} = 2^{|A|\times|A|}$$

Reflexive Relations（反身性）

Definition: *R* is *reflexive* iff (*a,a*) ∊ *R* for every element *a* ∊ A.
Written symbolically, R is reflexive if and only if

$$\forall x\left[\mathrm{x} \in U \rightarrow (x,x) \in R\right]$$

Example: The following relations on the integers are reflexive:

$$R_1 = \left\{(a,b) \middle| a \le b\right\},$$
$$R_3 = \left\{(a,b) \middle| a = b \text{ or } a = -b\right\},$$
$$R_4 = \left\{(a,b) \middle| a = b\right\}.$$

The following relations are not reflexive:

$$R_2 = \left\{(a,b) \middle| a > b\right\} \quad (\text{note that } 3 \quad 3),$$
$$R_5 = \left\{(a,b) \middle| a = b+1\right\} \quad (\text{note that } 3 \ne 3+1),$$
$$R_6 = \left\{(a,b) \middle| a+b \le 3\right\} \quad (\text{note that } 4+4 \quad 3).$$

If *A* = ∅ then the empty relation is reflexive vacuously. That is the empty relation on an empty set is reflexive!

- A relation is irreflexive iff its complementary relation is reflexive.
- “irreflexive” ≠ ”not relfexive”
- “likes” between people is not reflexive, but not irreflexive either. (Not everyone likes themselves, but not everyone dislike themselves either.)

Example

Consider the following relations on $\{1, 2, 3, 4\}$:

$$
\begin{aligned}
R_1 &= \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},\\
R_2 &= \{(1, 1), (1, 2), (2, 1)\},\\
R_3 &= \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},\\
R_4 &= \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},\\
R_5 &= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},\\
R_6 &= \{(3, 4)\}.
\end{aligned}
$$

Which of these relations are reflexive? Irreflexive ? Not reflexive?

Symmetric Relations

Definition: R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. Written symbolically, R is symmetric if and only if

$$\forall x \forall y \left[\left(x, y \right) \in R \rightarrow \left(y, x \right) \in R \right] \text{ i.e. } R = R^{-1}$$

Example: The following relations on the integers are symmetric:

$$R_3 = \left\{ \left(a,b \right) \middle| a = b \text{ or } a = -b \right\},$$
$$R_4 = \left\{ \left(a,b \right) \middle| a = b \right\},$$
$$R_6 = \left\{ \left(a,b \right) \middle| a + b \leq 3 \right\}.$$

The following are not symmetric:

$$R_1 = \left\{ \left(a,b \right) \middle| a \leq b \right\} \quad \left(\text{note that } 3 \leq 4, \text{ but } 4 \quad 3 \right),$$
$$R_2 = \left\{ \left(a,b \right) \middle| a > b \right\} \quad \left(\text{note that } 4 \leq 3, \text{ but } 3 \quad 4 \right),$$
$$R_5 = \left\{ \left(a,b \right) \middle| a = b + 1 \right\} \quad \left(\text{note that } 4 = 3 + 1, \text{ but } 3 \neq 4 + 1 \right).$$

Antisymmetric Relations

Definition: A relation *R* on a set *A* such that for all *a,b* ∈ *A* if (*a*,*b*) ∈ *R* and (*b*,*a*) ∈ *R,* then *a* = *b* is called *antisymmetric*. Written symbolically, *R* is antisymmetric if and only if

$$\forall x \forall y \left[(x,y) \in R \wedge (y,x) \in R \rightarrow x = y \right]$$

Example: The following relations on the integers are antisymmetric:

$$R_1 = \{(a,b) \mid a \le b\},$$
$$R_2 = \{(a,b) \mid a > b\},$$
$$R_4 = \{(a,b) \mid a = b\},$$
$$R_5 = \{(a,b) \mid a = b+1\}.$$

For any integer, if a $a \le b$ and $a \le b$, then $a = b$.

The following relations are notantisymmetric:

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$$
$$\left(\text{note that both } (1,-1) \text{ and } (-1,1) \text{ belongs to } R_3\right),$$
$$R_6 = \{(a,b) \mid a + b \le 3\} \ \left(\text{note that both } (1,2) \text{ and } (2,1) \text{ belongs to } R_6\right).$$

Example

Consider the following relations on $\{1, 2, 3, 4\}$:

$$
\begin{aligned}
R_1 &= \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},\\
R_2 &= \{(1, 1), (1, 2), (2, 1)\},\\
R_3 &= \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},\\
R_4 &= \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},\\
R_5 &= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},\\
R_6 &= \{(3, 4)\}.
\end{aligned}
$$

Which of these relations are Symmetric? Antisymmetric?

Transitive Relations (遞移性)

Definition: A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if

$$\forall x \forall y \forall z \left[(x,y) \in R \wedge (y,z) \in R \rightarrow (x,z) \in R \right]$$

Example: The following relations on the integers are transitive:

$$R_1 = \{(a,b) \mid a \le b\},$$

For every integer, $a \le b$ and $b \le c$, then $a \le c$.

$$R_2 = \{(a,b) \mid a > b\},$$

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a,b) \mid a = b\}.$$

The following are not transitive:

$R_5 = \{(a,b) \mid a = b+1\}$ (note that both $(3,2)$ and $(4,3)$ belongs to R_5, but not $\boxed{(4,2)}$),

$R_6 = \{(a,b) \mid a + b \le 3\}$ (note that both $(2,1)$ and $(1,2)$ belongs to R_6, but not $(2,2)$).

The Power of A Relation

- Def:
- The nth power R^n of a relation R on a set A can be defined recursively by $\begin{cases} R^0 :\equiv I_A; \\ R^{n+1} :\equiv R^n \circ R \; for\ all\ n \geq 0. \end{cases}$

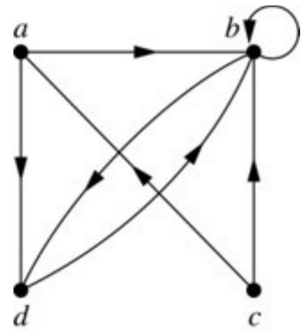
- The negative powers of R can also be defined if desired, by $R^{-n} :\equiv (R^{-1})^n$.

Representing Relations Using Digraphs

Definition: A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a,b), and the vertex b is called the *terminal vertex* of this edge.

- An edge of the form (a,a) is called a *loop*.

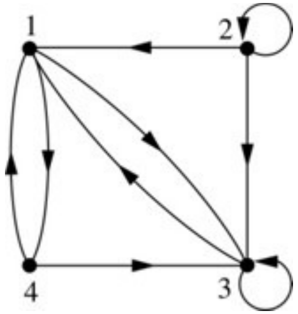
Example 7: A drawing of the directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b) is shown here.



Jump to long description

Examples of Digraphs Representing Relations

Example 8: What are the ordered pairs in the relation represented by this directed graph?



Solution: The ordered pairs in the relation are $(1, 3)$, $(1, 4)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(3, 1)$, $(3, 3)$, $(4, 1)$, and $(4, 3)$

Determining which Properties a Relation has from its Digraph

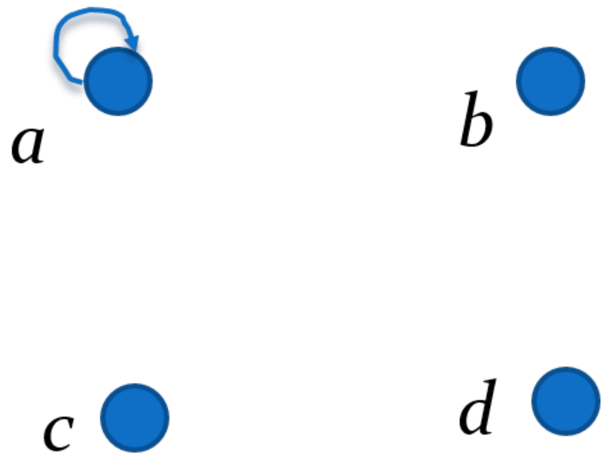
Reflexivity: A loop must be present at all vertices in the graph.

Symmetry: If (x,y) is an edge, then so is (y,x).

Antisymmetry: If (x,y) with $x \neq y$ is an edge, then (y,x) is not an edge.

Transitivity: If (x,y) and (y,z) are edges, then so is (x,z).

Determining which Properties a Relation has from its Digraph – Example 1



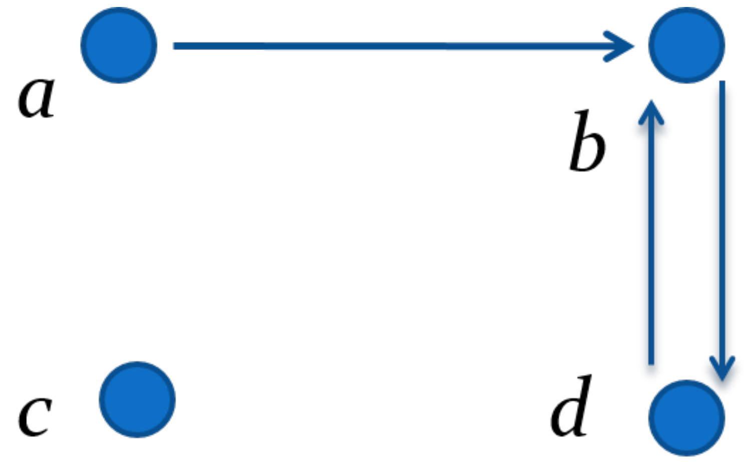
Reflexive? No, not every vertex has a loop

Symmetric? Yes (trivially), there is no edge from one vertex to another

Antisymmetric? Yes (trivially), there is no edge from one vertex to another

Transitive? Yes, (trivially) since there is no edge from one vertex to another

Determining which Properties a Relation has from its Digraph – Example 2



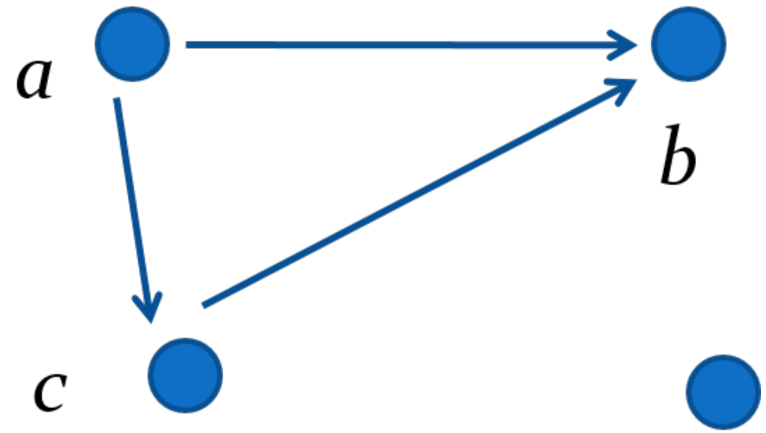
Reflexive? No, there are no loops

Symmetric? No, there is an edge from *a* to *b*, but not from *b* to *a*

Antisymmetric? No, there is an edge from *d* to *b* and *b* to *d*

Transitive? No, there are edges from *a* to *c* and from *c* to *b*, but there is no edge from *a* to *d*

Determining which Properties a Relation has from its Digraph – Example 3



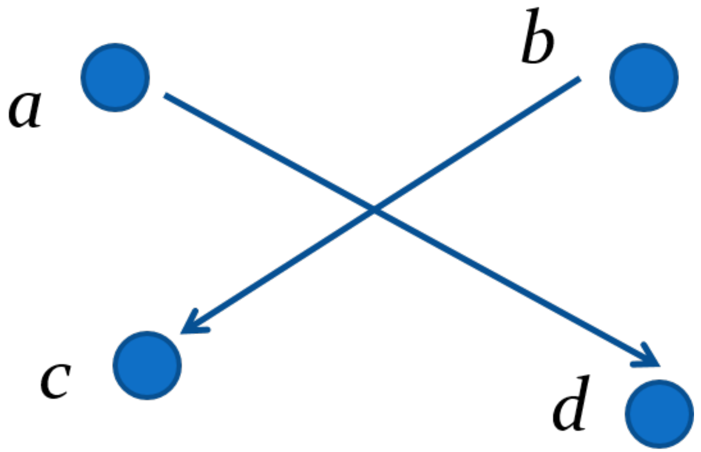
Reflexive? No, there are no loops

Symmetric? No, for example, there is no edge from *c* to *a*

Antisymmetric? Yes, whenever there is an edge from one vertex to another, there is not one going back

Transitive? Yes.

Determining which Properties a Relation has from its Digraph – Example 4

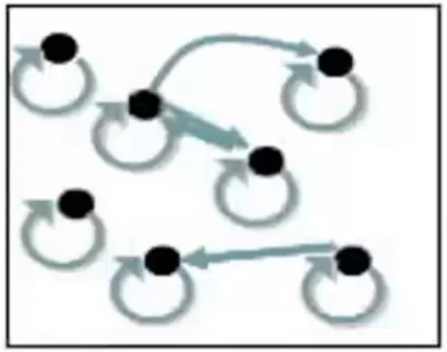


Reflexive? No, there are no loops

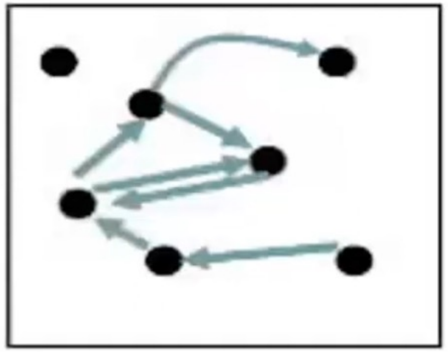
Symmetric? No, for example, there is no edge from *d* to *a*

Antisymmetric? Yes, whenever there is an edge from one vertex to another, there is not one going back

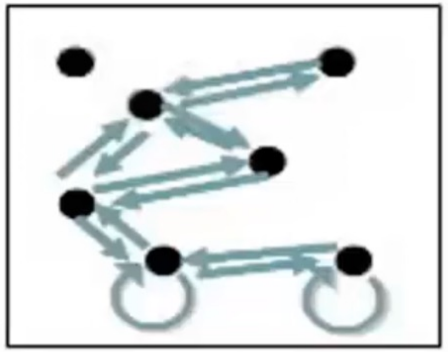
Transitive? Yes (trivially), there are no two edges where the first edge ends at the vertex where the second edge begins



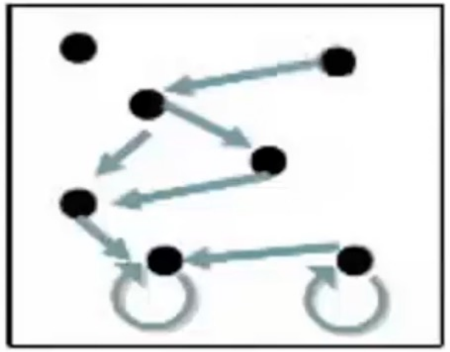
Reflexive:
Every node
has a self-loop



Irreflexive:
No node
links to itself



Symmetric:
Every link is
bidirectional

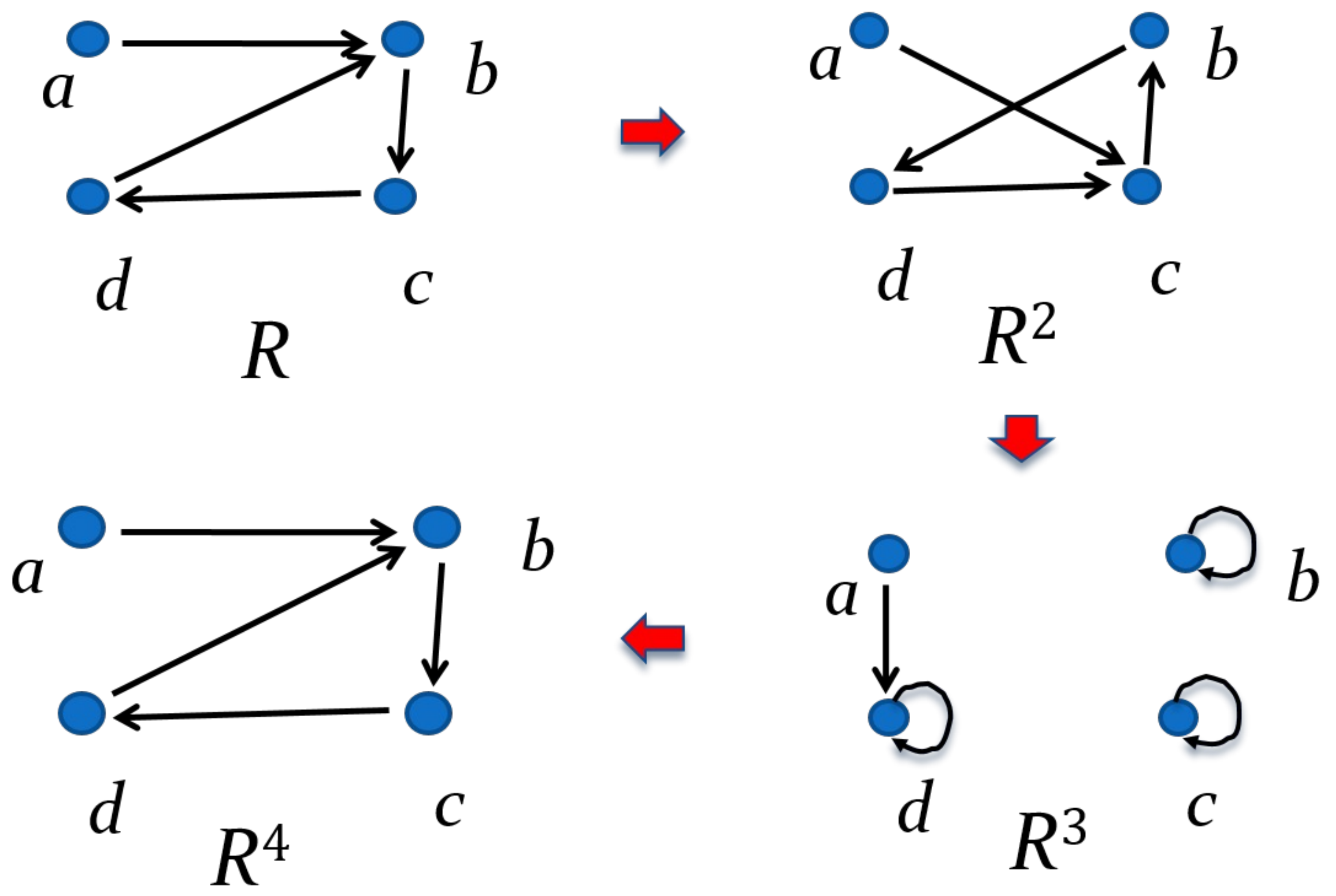


Antisymmetric:
No link is
bidirectional

Asymmetric, non-antisymmetric

Non-reflexive, non-irreflexive

Example of the Powers of a Relation



The pair (x,y) is in R^n if there is a path of length n from x to y in R (following the direction of the arrows).

N-ary Relations

Definition 1 Let $A_1, A_2, \dots, A_n$ be sets. An *n-ary relation* on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The sets $A_1, A_2, \dots, A_n$ are called the *domains* of the relation, and n is called its *degree*.

EXAMPLE 1 Let R be the relation on $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$ consisting of triples (a, b, c), where a, b, and c are integers with $a < b < c$. Then $(1, 2, 3) \in R$, but $(2, 4, 3) \notin R$. The degree of this relation is 3. Its domains are all equal to the set of natural numbers. ◀

EXAMPLE 2 Let R be the relation on $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$ consisting of all triples of integers (a, b, c) in which a, b, and c form an arithmetic progression. That is, $(a, b, c) \in R$ if and only if there is an integer k such that $b = a + k$ and $c = a + 2k$, or equivalently, such that $b - a = k$ and $c - b = k$. Note that $(1, 3, 5) \in R$ because $3 = 1 + 2$ and $5 = 1 + 2 \cdot 2$, but $(2, 5, 9) \notin R$ because $5 - 2 = 3$ while $9 - 5 = 4$. This relation has degree 3 and its domains are all equal to the set of integers. ◀

EXAMPLE 3 Let R be the relation on $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^+$ consisting of triples (a, b, m), where a, b, and m are integers with $m \geq 1$ and $a \equiv b \pmod{m}$. Then $(8, 2, 3)$, $(-1, 9, 5)$, and $(14, 0, 7)$ all belong to R, but $(7, 2, 3)$, $(-2, -8, 5)$, and $(11, 0, 6)$ do not belong to R because $8 \equiv 2 \pmod{3}$, $-1 \equiv 9 \pmod{5}$, and $14 \equiv 0 \pmod{7}$, but $7 \not\equiv 2 \pmod{3}$, $-2 \not\equiv -8 \pmod{5}$, and $11 \not\equiv 0 \pmod{6}$. This relation has degree 3 and its first two domains are the set of all integers and its third domain is the set of positive integers. ◀

N-ary Relations

EXAMPLE 4 Let R be the relation consisting of 5-tuples (A, N, S, D, T) representing airplane flights, where A is the airline, N is the flight number, S is the starting point, D is the destination, and T is the departure time. For instance, if Nadir Express Airlines has flight 963 from Newark to Bangor

at 15:00, then (Nadir, 963, Newark, Bangor, 15:00) belongs to R. The degree of this relation is 5, and its domains are the set of all airlines, the set of flight numbers, the set of cities, the set of cities (again), and the set of times. ◀

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero-one matrix.

Suppose R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.

- The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.

The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i,\ b_j) \in R, \\ 0 \text{ if } (a_i,\ b_j) \notin R. \end{cases}$$

The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j.

Using Zero-One Matrices

- To represent a relation R by a matrix $M_R = [m_{ij}], let\ m_{ij} = 1\ if\ (a_i, b_j) \in R$, else 0.
- E.g., Joe likes Susan and Mary, Fred likes Mary, and Mark likes Sally. The 0-1 matrix representation of that "Likes" relation:

$$\begin{array}{c c} & \begin{array}{ccc} \text{Susan} & \text{Mary} & \text{Sally} \end{array} \\ \begin{array}{c} \text{Joe} \\ \text{Fred} \\ \text{Mark} \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} = M_R$$

$$R: A \leftrightarrow B$$

$$A = \{Joe, Fred, Mark\}$$

$$B = \{Susan, Mary, Sally\}$$

$$R = \{(Joe, Susan), (Joe, Mary), (Fred, Mary), (Mark, Sally)\}$$

Examples of Representing Relations Using Matrices

Example 2: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

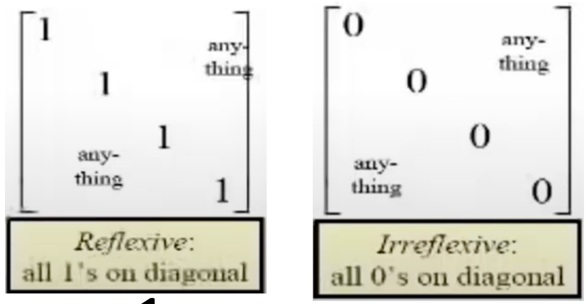
$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that:

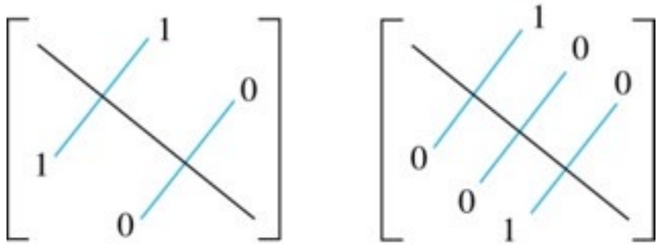
$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$

Matrices of Relations on Sets

If R is a reflexive relation, all the elements on the main diagonal of M_R are equal to 1.



R is a symmetric relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$. R is an antisymmetric relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.



(a) Symmetric (b) Antisymmetric

Example of a Relation on a Set

Example 3: Suppose that the relation *R* on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is *R* reflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements are equal to 1, *R* is reflexive. Because M_R is symmetric, *R* is symmetric and not antisymmetric because both $m_{1,2}$ and $m_{2,1}$ are 1.

Composite Relations

Let $R: A \leftrightarrow B, and\ S: B \leftrightarrow C. Then\ the\ composite\ S \circ R\ of\ R\ and\ S\ is\ defined\ as\colon\ S \circ R = \{(a, c) | aRb \wedge bSc\}$.

Ex1 Function composition $f \circ g\ is\ an\ example$.

Ex2 $A = \{1,2,3\}, B = \{a, b, c, d\}, C = \{x, y, z\}$.

- $R: A \leftrightarrow B, R = \{(1, a), (1, b), (2, b), (2, c)\}$.
- $S: B \leftrightarrow C, S = \{(a, x), (a, y), (b, y), (d, z)\}$.
- $S \circ R = \{(1, \mathrm{x}), (1, \mathrm{y}), (2, \mathrm{y})\}$

Composite Relations

- $A = \{1,2,3\}, B = \{a, b, c, d\}, C = \{x, y, z\}$.
  - $R: A \leftrightarrow B, R = \{(1, a), (1, b), (2, b), (2, c)\}$.
  - $S: B \leftrightarrow C, S = \{(a, x), (a, y), (b, y), (d, z)\}$.
  - $S \circ R = \{(1, \mathrm{x}), (1, \mathrm{y}), (2, \mathrm{y})\}$

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.\ M_S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.\ M_{S \circ R} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{S \circ R} = M_R \odot M_S$$

A relation R on A

- $M_{R\circ R} = M_R \odot M_R = (M_R)^2 = M_{R^2}$
- $M_{R\circ R\circ R} = M_R \odot M_R \odot M_R = (M_R)^3 = M_{R^3}$

Transitive

- $\forall a, b, c: (a, b) \in R \land (b, c) \in R \rightarrow (a, c) \in R$
- $(aRb \land bRc \rightarrow aRc)$
- Theroem: The R on a set A is transitive iff $R^n \subseteq R\ for\ all\ n = 1,2,3, \ldots$
  - Think about what $(a, b) \in R^k\ means$?
  - How to prove an “iff” statement?
  - Let R = {(1,1),(2,1),(3,2),(4,3)}. Find the powers $R^n\ for\ n = 2,3, \ldots$
  - Let R = {(1,2),(1,3),(2,2),(2,3),(4,3)}. Find the powers $R^n\ for\ n = 2,3, \ldots$

Transitive

If $(a, c) \in R^2$ $imply$ $\exists b$ $s.t.$ $aRb \wedge bRc$ $\therefore (a, c) \in R$

證：$\forall \mathrm{a}, \mathrm{b}, \mathrm{c}$ $\mathrm{aRb} \wedge bRc \rightarrow aRc$

If $(M_{R^n})_{ij} = 1, then\ (M_R)_{ij} = 1$

Closure of Relations

- For any property X, the "X closure" of a set (or relation)R is defined as the "smallest" superset of R that has the given property.
- The reflexive closure of a relation R on A is obtained by adding (a,a) to R for each $a \in A. I.e., it\ is\ R \cup I_A$.
- The symmetric closure of R is obtained by adding (b,a) to R for each (a,b) in R. I.e., it is $R \cup R^{-1}$.
- The transitive closure or connectivity relation of R is obtained by repeatedly adding (a,c) to R for each (a,b) and (b,c) in R. I.e., it is $R^* = \bigcup_{n \in z^+} R^n$

  - A={1,2,3,4}
  - R = {(1,2), (2,3), (2,2), (3,3), (3,4)}