1. Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for the positive integer n.

a) What is the statement $P(1)$? $a)\ 1^2 = 1 \cdot 2 \cdot 3/6$

b) Show that $P(1)$ is true, completing the basis step of a proof that $P(n)$ is true for all positive integers n.

b) $Basis\ step$: $P(1) = 1^2 = 1 \cdot 2 \cdot 3/6$

c) What is the inductive hypothesis of a proof that $P(n)$ is true for all positive integers n?

c) $Inductive\ step$: $P(k) = 1^2 + 2^2 + \ldots + k^2 = k(k+1)(2k+1)/6$

d) What do you need to prove in the inductive step of a proof that $P(n)$ is true for all positive integers n?

d) For each $k \geq 1$ that $P(k)$ implies $P(k+1)$; in other words that assuming the inductive hypothesis [see part (c)] we can show $P(k+1) = 1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = (k+1)(k+2)(2k+3)/6$

e) Complete the inductive step of a proof that $P(n)$ is true for all positive integers n, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

e) $(1^2 + 2^2 + \cdots + k^2) + (k+1)^2 = [k(k+1)(2k+1)/6] + (k+1)^2 = [(\mathrm{k}+1)/6][\mathrm{k}(2\mathrm{k}+1) + 6(\mathrm{k}+1)] = \left[\frac{\mathrm{k}+1}{6}\right](2k^2 + 7k + 6) = \left[\frac{\mathrm{k}+1}{6}\right](\mathrm{k}+2)(2\mathrm{k}+3) = (\mathrm{k}+1)(\mathrm{k}+2)(2\mathrm{k}+3)/6$

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

2. **a)** Find a formula for

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}$$

by examining the values of this expression for small values of n.

b) Prove the formula you conjectured in part (a).

a) $\sum_{j=1}^{n}\frac{1}{2^j}=(2^n-1)/2^n$

b) Basis step:

P(1) is true $\because \frac{1}{2}=\frac{2^1-1}{2^1}$

Inductive step:

Assume that $\sum_{j=1}^{k}\frac{1}{2^j}=(2^k-1)/2^k$

Then) $\sum_{j=1}^{k+1}\frac{1}{2^j}=\sum_{j=1}^{k}\frac{1}{2^j}+1/2^{k+1}=(2^k-1)/2^k+1/2^{k+1}=\frac{2^{k+1}-2+1}{2^{k+1}}=\frac{(2^{k+1}-1)}{2^{k+1}}$

3. What is wrong with this "proof"?

"Theorem" For every positive integer n, if x and y are positive integers with $\max(x, y) = n$, then $x = y$.

Basis Step: Suppose that $n = 1$. If $\max(x, y) = 1$ and x and y are positive integers, we have $x = 1$ and $y = 1$.

Inductive Step: Let k be a positive integer. Assume that whenever $\max(x, y) = k$ and x and y are positive integers, then $x = y$. Now let $\max(x, y) = k + 1$, where x and y are positive integers. Then $\max(x - 1, y - 1) = k$, so by the inductive hypothesis, $x - 1 = y - 1$. It follows that $x = y$, completing the inductive step.

The mistake is in applying the inductive hypothesis to look at $\max(x - 1, y - 1)$, because even though x and y are positive integers, $x - 1$ and $y - 1$ need not be (one or both could be 0)

4. **a)** Determine which amounts of postage can be formed using just 4-cent and 11-cent stamps.

b) Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

c) Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

a) 4, 8, 11, 12, 15, 16, 19, 20, 22, 23, 24, 26, 27, 28, and all values greater than or equal to 30

b) Let $P(n)$ be the statement that we can form n cents of postage using just 4-cent and 11-cent stamps. We want to prove that $P(n)$ is true for all $n \geq 30$.

For the **basis step**, 30 = 11+11+4+4.

Inductive step: Assume that we can form k cents of postage we will show how to form $k + 1$ cents of postage.

If the k cents included an 11-cent stamp, then replace it by three 4-cent stamps. Otherwise, k cents was formed from just 4-cent stamps.

Because $k \geq 30$, there must be at least eight 4-cent stamps involved. Replace eight 4-cent stamps by three 11-cent stamps, and we have formed $k + 1$ cents in postage.

c) To prove that $P(n)$ is true for all $n \geq 30$, we check for the **basis step** that 30 = 11+11+4+4, 31 = 11+4+4+4+4+4, 32 = 4+4+4+4+4+4+4+4, and 33 = 11+11+11. For **the inductive step**, assume the inductive hypothesis, **that $P(j)$ is true for all j with $30 \leq j \leq k$, where k is an arbitrary integer greater than or equal to 33**. **We want to show that $P(k + 1)$ is true**. Because $k-3 \geq 30$, we know that $P(k-3)$ is true, that is, that we can form $k - 3$ cents of postage. **Put one more 4-cent stamp on the envelope, and we have formed $k + 1$ cents of postage**. In this proof, our inductive hypothesis was that $P(j)$ was true for all values of j between 30 and k inclusive, rather than just that $P(30)$ was true.

5. Consider this variation of the game of Nim. The game begins with n matches. Two players take turns removing matches, one, two, or three at a time. The player removing the last match loses. Use strong induction to show that if each player plays the best strategy possible, the first player wins if $n = 4j$, $4j + 2$, or $4j + 3$ for some nonnegative integer j and the second player wins in the remaining case when $n = 4j + 1$ for some nonnegative integer j.

Basis step: There are four base cases. If $n = 1 = 4 \cdot 0+1$,then clearly the second player wins. If there are two, three, or four matches ($n = 4 \cdot 0+2$, $n = 4 \cdot 0+3$, or $n = 4 \cdot 1$), then the first player can win by removing all but one match.

Inductive step: Assume the strong inductive hypothesis, that in games with k or fewer matches, the first player can win if $k \equiv 0$, 2, or 3 (mod 4) and the second player can win if $k \equiv 1$ (mod 4). Suppose we have a game with $k+1$ matches, with $k \geq 4$. If $k+1 \equiv 0$ (mod4), then the first player can remove three matches, leaving $k - 2$ matches for the other player. Because $k - 2 \equiv 1$ (mod 4), by the inductive hypothesis, this is a game that the second player at that point (who is the first player in our game) can win. Similarly, if $k + 1 \equiv 2$ (mod 4), then the first player can remove one match; and if $k + 1 \equiv 3$ (mod 4), then the first player can remove two matches. Finally, if $k + 1 \equiv 1$ (mod 4), then the first player must leave k, $k - 1$, or $k - 2$ matches for the other player. Because $k \equiv 0$ (mod 4), $k-1 \equiv 3$ (mod 4), and $k - 2 \equiv 2$ (mod 4), by the inductive hypothesis, this is a game that the first player at that point (who is the second player in our game) can win.