Discrete Mathematics
Lec6: Induction and Recursion

馬誠佑

Climbing an Infinite Ladder

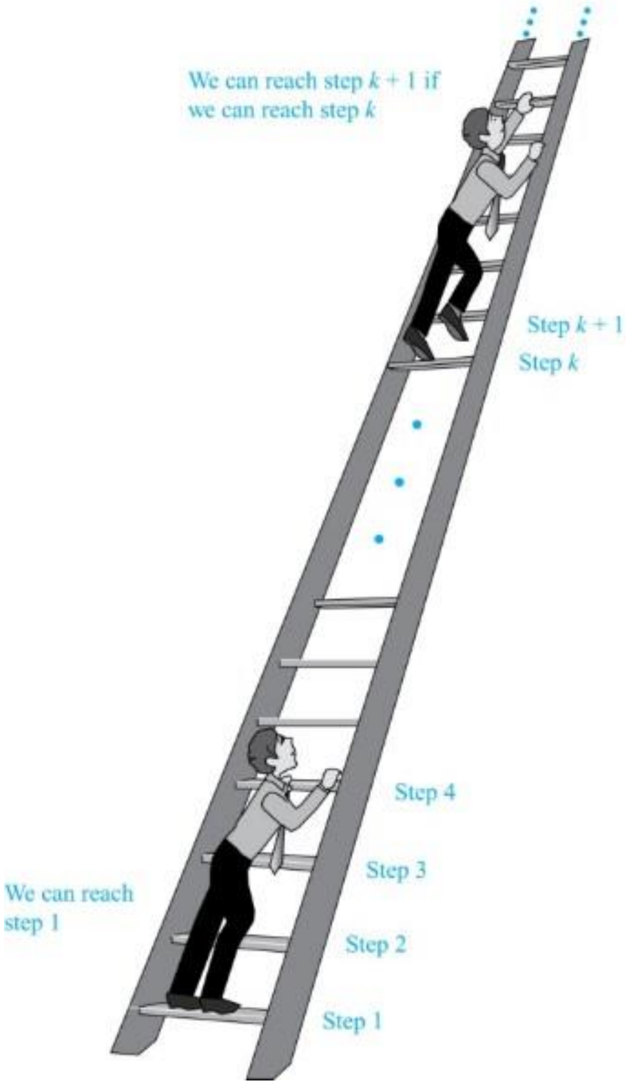
Suppose we have an infinite ladder:

1. We can reach the first rung of the ladder.

2. If we can reach a particular rung of the ladder, then we can reach the next rung.

From (1), we can reach the first rung. Then by applying (2), we can reach the second rung. Applying (2) again, the third rung. And so on. We can apply (2) any number of times to reach any particular rung, no matter how high up.

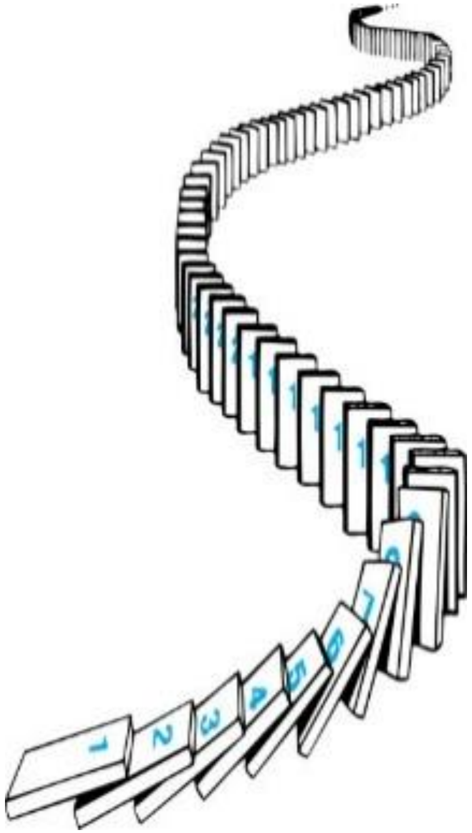
This example motivates proof by mathematical induction.



Remembering How Mathematical Induction Works

Consider an infinite sequence of dominoes, labeled 1,2,3, …, where each domino is standing.

Let $P(n)$ be the proposition that the nth domino is knocked over.



We know that the first domino is knocked down, i.e., $P(1)$ is true .

We also know that if whenever the kth domino is knocked over, it knocks over the $(k + 1)$st domino, i.e, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.

Hence, all dominos are knocked over.

$P(n)$ is true for all positive integers n.

Principle of Mathematical Induction

- Let P(n) be a propositional logic. To prove P(n) is true for all $n \in Z$, we can
  - Basis step: Show. P(1) is true.
  - Induction step: Prove $P(k) \rightarrow P(k+1)\ is\ true\ \forall k \in Z^{+}$. (if P(k) then P(k+1))
  - Conclude $\left(\forall n \in Z^{+}, P(n)\right)\ is\ true$.

Important Points About Using Mathematical Induction

Mathematical induction can be expressed as the rule of inference

$$\left(P(1) \wedge \forall k\left(P(k) \rightarrow P(k+1)\right)\right) \rightarrow \forall n\, P(n),$$

where the domain is the set of positive integers.

In a proof by mathematical induction, we don't assume that $P(k)$ is true for all positive integers! We show that if we assume that $P(k)$ is true, then $P(k + 1)$ must also be true.

Proofs by mathematical induction **do not always start at the integer 1**. In such a case, the basis step begins at a starting point b where b is an integer. We will see examples of this soon.

Proving a Summation Formula by Mathematical Induction

Example: Show that: $\forall n \in Z^+, 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
Solution:

> Note: Once we have this conjecture, mathematical induction can be used to prove it correct.

- BASIS STEP: $P(1)$ is true since $1(1 + 1)/2 = 1$.
- INDUCTIVE STEP: Assume true for $P(k)$.

The inductive hypothesis is $\sum_{i=1}^{k} = \frac{k(k+1)}{2}$

Under this assumption,

$$1+2+\ldots+k+(k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2} = \frac{(k+1)[(k+1)+1]}{2} \quad So, P(k+1) is\ true.$$

- Therefore, $\mathrm{P(n)}$ is true $\forall n \in Z^+$

Proving Inequalities[1]

Example: Use mathematical induction to prove that $n < 2^n$ for all positive integers n.

Solution: Let $P(n)$ be the proposition that $n < 2^n$.

- BASIS STEP: $P(1)$ is true since $1 < 2^1 = 2$.
- INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $k < 2^k$, for an arbitrary positive integer k.
- Must show that $P(k + 1)$ holds. Since by the inductive hypothesis, $k < 2^k$, it follows that:

$$k + 1 < 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

Therefore $n < 2^n$ holds for all positive integers n.

Proving Inequalities[2]

Example: Use mathematical induction to prove that $2^n < n!$, for every integer $n \geq 4$.

Solution: Let $P(n)$ be the proposition that $2^n < n!$.

- BASIS STEP: $P(4)$ is true since $2^4 = 16 < 4! = 24$.
- INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $2^k < k!$ for an arbitrary integer $k \geq 4$. To show that $P(k + 1)$ holds:

$$
\begin{aligned}
2^{k+1} &= 2 \cdot 2^k \\
&< 2 \cdot k! \qquad \textit{(by the inductive hypothesis)} \\
&< (k+1)k! \\
&= (k+1)!
\end{aligned}
$$

Therefore, $2^n < n!$ holds, for every integer $n \geq 4$.

Note that here the basis step is $P(4)$, since $P(0)$, $P(1)$, $P(2)$, and $P(3)$ are all false.

The Cardinality of Power Sets

- Prove $|2^s| = 2^{|s|}\ for\ all\ finite\ set\ S$.
  - **Basis step:** if |S|=0, then S=ϕ. $\because\ 2^s = \{\phi\}\ \therefore\ |2^s| = 2^{|s|}\ is\ true$.
  - **Inductive step:** Consider |S|=k is true $\forall k \in N,\ i.e., |2^S| = 2^k$.

    If |S|=k+1 and WLOG, assume $a \in S. Let\ T = S - \{a\}$.

    $\because\ |T| = k\ \therefore\ |2^T| = 2^k$

    $\because\ 2^s = 2^{T\cup\{a\}}\ \therefore\ |2^S| = |2^{T\cup\{a\}}| = 2 \cdot 2^k = 2^{k+1}$

    So, $|2^S| = 2^{|S|}\ is\ also\ true$.
  - Therefore, $|2^s| = 2^{|s|}$ for all finite set S.

Example: Harmonic Numbers

The harmonic numbers H_j $for\ j = 1,2,\ldots are\ given\ by\ H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}. Prove\ \forall n \in N, H_{2^n} \geq 1 + \frac{n}{2}.$

Pf:

1. Basis step: Consider n=0.

$$LHS = H_{2^0} = H_1 = 1; RHS = 1 + \frac{0}{2} = 1 \therefore H_{2^0} \geq 1 + \frac{0}{2}\ is\ true.$$

2. Inductive step: Assume $H_{2^k} \geq 1 + \frac{k}{2}\ is\ true\ any\ given\ k \in N.$

$H_{2^{k+1}} = \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^k}\right) + \left(\frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}}\right) \geq \left(1 + \frac{k}{2}\right) + 2^k \cdot \frac{1}{2^{k+1}} = 1 + \frac{k+1}{2}. So, H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$ is also true.

Therefore, ($\forall n \in N, H_{2n} \geq 1 + \frac{n}{2}$) is true.

Example: Generalized De Morgan's law for set

- $\overline{A_1 \cap A_2 \ldots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}\ \forall n \in Z^+$
- Pf:
  - 1. Basis step: n =2. $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$
  - 2. Inductive step: assume n=k is true $\overline{A_1 \cap A_2 \ldots \cap A_k} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_k}$
  - if n = k+1 prove: $\overline{A_1 \cap A_2 \ldots \cap A_k \cap A_{k+1}} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_k} \cup \overline{A_{k+1}}$

$$\overline{(A_1 \cap A_2 \ldots \cap A_k) \cap A_{k+1}} = \overline{(A_1 \cap A_2 \ldots \cap A_k)} \cup \overline{A_{k+1}}$$

$$= (\overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_k}) \cup \overline{A_{k+1}}$$

3. Therefore $\overline{A_1 \cap A_2 \ldots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}\ \forall n \in Z^+, n \geq 2\ is\ true$

Example: Proving Divisibility Results

Use mathematical induction to prove that $n^3 - n$ is divisible by 3, for every positive integer n.

Solution: Let $P(n)$ be the proposition that $n^3 - n$ is divisible by 3.

- BASIS STEP: $P(1)$ is true since $1^3 - 1 = 0$, which is divisible by 3.
- INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $k^3 - k$ is divisible by 3, for an arbitrary positive integer k. To show that $P(k + 1)$ follows:

$$\begin{aligned}(k+1)^3 - (k+1) &= \left(k^3 + 3k^2 + 3k + 1\right) - (k+1) \\ &= \left(k^3 - k\right) + 3\left(k^2 + k\right)\end{aligned}$$

By the inductive hypothesis, the first term $(k^3 - k)$ is divisible by 3 and the second term is divisible by 3 since it is an integer multiplied by 3. So by part (i) of Theorem 1 in Section 4.1 , $(k + 1)^3 - (k + 1)$ is divisible by 3.

Therefore, $n^3 - n$ is divisible by 3, for every integer positive integer n.

Guidelines:
Mathematical Induction Proofs

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form “for all $n \geq b$, $P(n)$” for a fixed integer b.
2. Write out the words “Basis Step.” Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words “Inductive Step”.
4. State, and clearly identify, the inductive hypothesis, in the form “assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$.”
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k + 1)$ says.
6. Prove the statement $P(k + 1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying “this completes the inductive step.”
8. After completing the basis step and the inductive step, state the conclusion, namely, by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.

Strong Induction

Strong Induction: To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, complete two steps:

- *Basis Step*: Verify that the proposition $P(1)$ is true.
- *Inductive Step*: Show the conditional statement

$$\left[P(1) \wedge P(2) \wedge \cdots \wedge P(k)\right] \rightarrow P(k+1)$$

holds for all positive integers k.

Strong Induction is sometimes called the *second principle of mathematical induction* or *complete induction*.

Completion of the proof of the Fundamental Theorem of Arithmetic

Example: *n* >1: integer

that *n* can be written as the product of primes.

Solution: Let *P*(*n*) be the proposition that *n* can be written as a product of primes.

- BASIS STEP: *P*(2) is true since 2 itself is prime.
- INDUCTIVE STEP: The inductive hypothesis is *P*(*j*) is true for all integers *j* with 2 ≤ *j* ≤ *k*. To show that *P*(*k* + 1) must be true under this assumption, two cases need to be considered:
  - If *k* + 1 is prime, then *P*(*k* + 1) is true.
  - Otherwise, *k* + 1 is composite and can be written as the product of two positive integers *a* and *b* with 2 ≤ *a* ≤ *b* < *k* + 1.
  - $k+1 = a \cdot b \; for \; some \; 2 \leq a, b < k+1.$

    $\because a = p_1 p_2 \dots p_s \; and \; b = q_1 q_2 \dots q_t$ where $p_1, p_2, \dots, p_s, q_1, q_2, \dots, q_t \; are \; primes.$

    $\therefore k+1 = (p_1 p_2 \dots p_s)(q_1 q_2 \dots q_t)$

    The statement is also true for n=k+1

  - Hence, it has been shown that every integer greater than 1 can be written as the product of primes.

(*uniqueness proved in Section* 4.3)

Proof using Strong Induction 2

Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: Let $P(n)$ be the proposition that postage of n cents can be formed using 4-cent and 5-cent stamps.

- BASIS STEP: $P(12)$, $P(13)$, $P(14)$, and $P(15)$ hold.
  - $P(12)$ uses three 4-cent stamps.
  - $P(13)$ uses two 4-cent stamps and one 5-cent stamp.
  - $P(14)$ uses one 4-cent stamp and two 5-cent stamps.
  - $P(15)$ uses three 5-cent stamps.
- INDUCTIVE STEP: The inductive hypothesis states that $P(j)$ holds for $12 \leq j \leq k$, where $k \geq 15$. Assuming the inductive hypothesis, it can be shown that $P(k + 1)$ holds.
- Using the inductive hypothesis, $P(k - 3)$ holds since $k - 3 \geq 12$. To form postage of $k + 1$ cents, add a 4-cent stamp to the postage for $k - 3$ cents. Hence, $P(n)$ holds for all $n \geq 12$.
- $(k+1 = (k-3) + 4 \; and \; 12 \leq k-3 \leq k. Since \; P(k-3) \; add \; a \; 4-cent \; stamp \; to \; get \; postage \; for \; k+1. Therefore, P(k+1) \; is \; true)$

Proof of Same Example using Mathematical Induction

Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: Let $P(n)$ be the proposition that postage of n cents can be formed using 4-cent and 5-cent stamps.

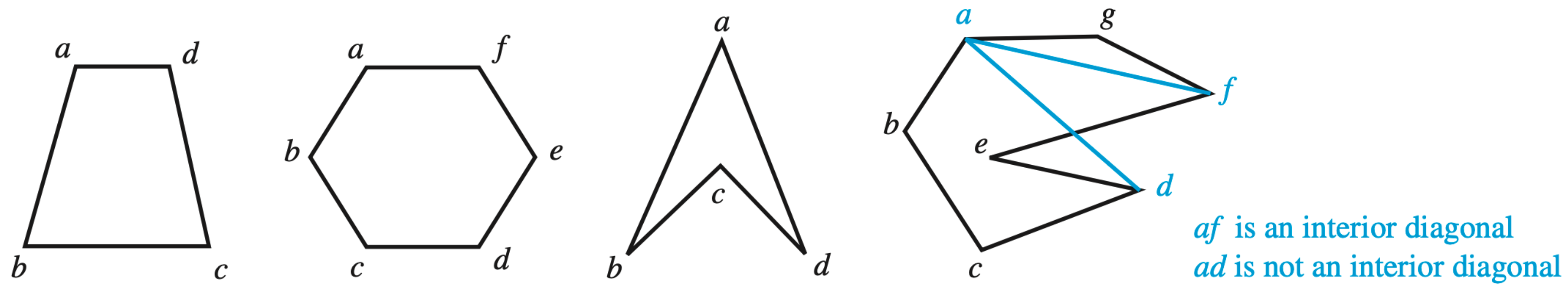
- BASIS STEP: Postage of 12 cents can be formed using three 4-cent stamps.
- INDUCTIVE STEP: The inductive hypothesis $P(k)$ for any positive integer k is that postage of k cents can be formed using 4-cent and 5-cent stamps. To show $P(k + 1)$ where $k \geq 12$, we consider two cases:
  - If at least one 4-cent stamp has been used, then a 4-cent stamp can be replaced with a 5-cent stamp to yield a total of $k + 1$ cents.
  - Otherwise, no 4-cent stamp have been used and at least three 5-cent stamps were used. Three 5-cent stamps can be replaced by four 4-cent stamps to yield a total of $k + 1$ cents.

Hence, $P(n)$ holds for all $n \geq 12$.

5.2.4 Using Strong Induction in Computational Geometry

THEOREM 1 A simple polygon with n sides, where n is an integer with $n \geq 3$, can be triangulated into $n - 2$ triangles.

LEMMA 1 Every simple polygon with at least four sides has an interior diagonal.



(a) (b) (c) (d)