Discrete Mathematics
Lec10: Advanced Counting Part 2 & relation

馬誠佑

Generating Functions

Definition: The *generating function for the sequence* a_0, $a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Examples:

- The sequence $\{a_k\}$ with $a_k = 3$ has the generating function $\sum_{k=0}^{\infty} 3x^k$.
- The sequence $\{a_k\}$ with $a_k = k + 1$ has the generating function
$$\sum_{k=0}^{\infty} (k+1) x^k.$$
- The sequence $\{a_k\}$ with $a_k = 2^k$ has the generating function $\sum_{k=0}^{\infty} 2^k x^k$.

生成函數的主要功用

1. 解決遞迴關係（recurrence relations）
許多數列（例如費波那契數列）是透過遞迴關係定義的，透過生成函數可以把遞迴轉換為代數方程來求解。

2. 推導封閉式（closed-form）
有時可以藉由生成函數找到數列的封閉公式，或至少是一種精確的表示法。

3. 計數問題（combinatorial enumeration）
生成函數常用來處理計數問題，例如計算某種排列、分割或組合方式的數量。

4. 找出規律與對稱性
一些數列的性質不容易直接看出，但在生成函數表示下可能呈現更清晰的結構。

Useful Generating Functions

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k = 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

Note: The series for the last two generating functions can be found in most calculus books when power series are discussed.

Counting Problems and Generating Functions [1]

Example: Find the number of solutions of

$e_1 + e_2 + e_3 = 17$,

where e_1, e_2, and e_3 are nonnegative integers with $2 \leq e_1 \leq 5$, $3 \leq e_2 \leq 6$, and $4 \leq e_3 \leq 7$.

Solution: The number of solutions is the coefficient of x^{17} in the expansion of

$(x^2 + x^3 + x^4 + x^5)\ (x^3 + x^4 + x^5 + x^6)\ (x^4 + x^5 + x^6 + x^7)$.

This follows because a term equal to is obtained in the product by picking a term in the first sum x^{e1}, a term in the second sum x^{e2}, and a term in the third sum x^{e3}, where $e_1 + e_2 + e_3 = 17$.

There are three solutions since the coefficient of x^{17} in the product is 3.

Example

How many ways to pay r dollars into a vending machine with tokens worth \$1, \$2, and 5\$.

Sol:

If the order in which the tokens are inserted doesn't matter, the answer is given by the coefficient of x^r in the generating function.

$G(x) = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^5 + x^{10} + \cdots)$.

Sol (Cont.)

If the order in which the tokens are inserted matters and exactly n tokens are used, the answer is the coefficient of x^r in the generating function

$$G(x) = (x + x^2 + x^5)^n$$

If the order in which the tokens are inserted matters , the answer is the coefficient of x^r in the generating function

$$G(x) = 1 + (x + x^2 + x^5)^1 + (x + x^2 + x^5)^2 + \cdots$$

$$= \frac{1}{1 - \ (x + x^2 + x^5)}$$

Counting Problems and Generating Functions 2

Example: Use generating functions to find the number of *k*-combinations of a set with *n* elements, i.e., $C(n,k)$.

Solution: Each of the n elements in the set contributes the term $(1 + x)$ to the generating function

$$f(x) = \sum_{k=0}^{n} a^k x^k .$$

Hence $f(x) = (1 + x)^n$ where $f(x)$ is the generating function for $\{a^k\}$, where a^k represents the number of k-combinations of a set with n elements.

By the binomial theorem, we have

$$f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Hence,

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

Solve Recurrence Relation:

Solve the recurrence relation $a_k = 3a_{k-1}\ for\ k = 1,2,\ldots\ and\ initial\ condition\ a_0 = 2.$

Sol:

Let G(x) be the generating function for the sequence $\{a_n\}. Then$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + \left(\sum_{n=1}^{\infty} a_n x^n\right) \Rightarrow a_0 + \left(\sum_{n=1}^{\infty} 3a_{n-1}x^n\right)$$

$$= a_0 + 3x\left(\sum_{n=1}^{\infty} a_{n-1}x^{n-1}\right) \overset{令k = n-1}{=} a_0 + 3x\left(\sum_{k=0}^{\infty} a_k x^k\right) = 3xG(x) + 2$$

$$\Rightarrow (1-3x)G(x) = 2 \therefore G(x) = \frac{2}{1-3x} = 2\sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty} 2 \cdot 3^n \cdot x^n$$

Principle of Inclusion-Exclusion

In Section 2.2, we developed the following formula for the number of elements in the union of two finite sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

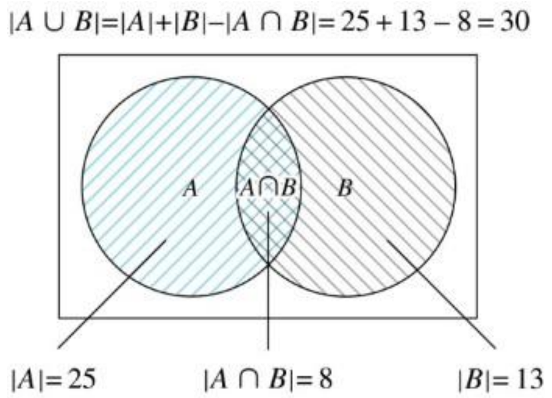
We will generalize this formula to finite sets of any size.

Two Finite Sets

Example: In a discrete mathematics class every student is a major in computer science or mathematics or both. The number of students having computer science as a major (possibly along with mathematics) is 25; the number of students having mathematics as a major (possibly along with computer science) is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in the class?

Solution:

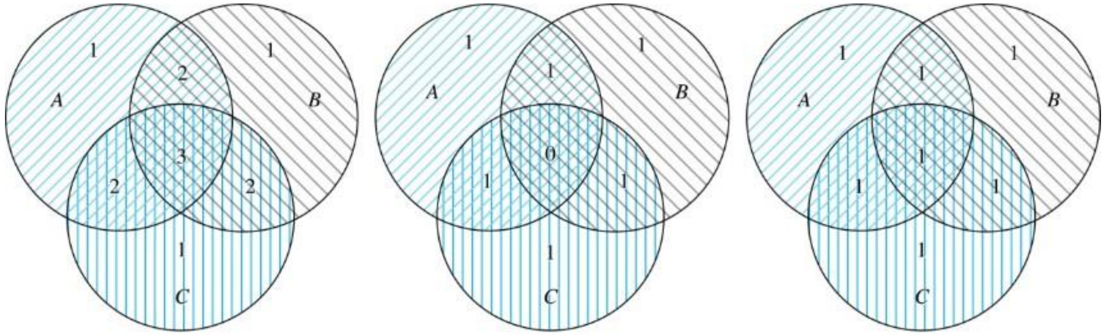
$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 25 + 13 - 8 = 30$$



Jump to long description

Three Finite Sets [1]

$$|A \cup B \cup C| =$$
$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



(a) Count of elements by $|A|+|B|+|C|$

(b) Count of elements by $|A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|$

(c) Count of elements by $|A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|+|A \cap B \cap C|$

Jump to long description

Three Finite Sets [2]

Example: A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken a course in at least one of Spanish French and Russian, how many students have taken a course in all 3 languages.

Solution: Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian. Then, we have

$|S| = 1232$, $|F| = 879$, $|R| = 114$, $|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$, and $|S \cup F \cup R| = 2092$.

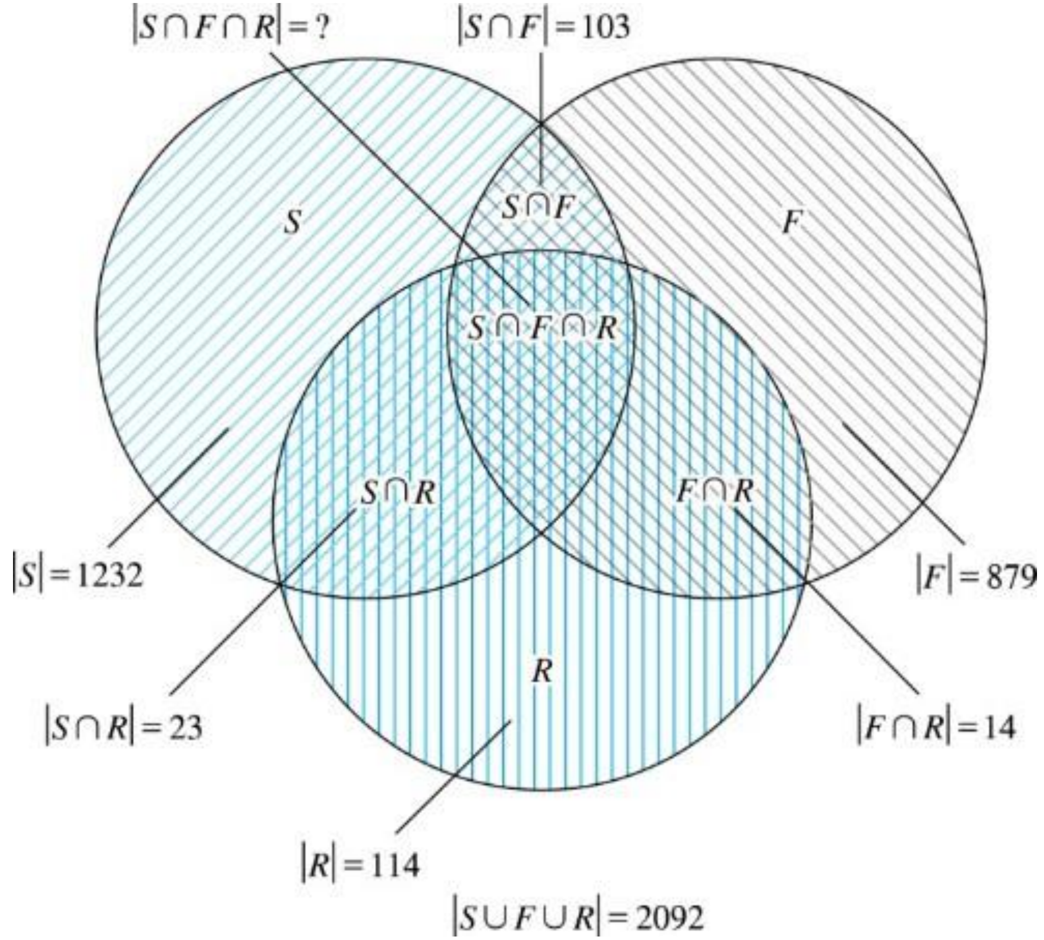
Using the equation

$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$,

we obtain $2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$.

Solving for $|S \cap F \cap R|$ yields 7.

Illustration of Three Finite Set Example



The Number of Onto Functions [1]

Example: How many onto functions are there from a set with six elements to a set with three elements?

Solution: Suppose that the elements in the codomain are b_1, b_2, and b_3. Let P_1, P_2, and P_3 be the properties that b_1, b_2, and b_3 are not in the range of the function, respectively. Let U be the set of all possible functions. The function is onto if none of the properties P_1, P_2, and P_3 hold.

By the inclusion-exclusion principle the number of onto functions from a set with six elements to a set with three elements is

$$|\overline{P_1 \cup P_2 \cup P_3}| = |U - (P_1 \cup P_2 \cup P_3)|$$
$$= |U|$$
$$- [|P_1| + |P_2| + |P_3| - (|P_1 \cap P_2| + |P_1 \cap P_3| + |P_2 \cap P_3|) + |P_1 \cap P_2 \cap P_3|]$$
$$= 3^6 - [\binom{3}{1} 2^6 - \binom{3}{2} 1^6 + \binom{3}{3} 0^6]$$

Hence, the number of onto functions from a set with six elements to a set with three elements is:

$3^6 - 3 \cdot 2^6 + 3 = 729 - 192 + 3 = 540$

Derangements(重排)[1]

Definition: A *derangement* is a permutation of objects that leaves no object in the original position.

Example: The permutation of 21453 is a derangement of 12345 because no number is left in its original position. But 21543 is not a derangement of 12345, because 4 is in its original position.

Derangements 2

Theorem 2: The number of derangements of a set with *n* elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \left(-1\right)^n \frac{1}{n!}\right].$$

Proof follows from the principle of inclusion-exclusion (*see text*).

Proof:

U: all arrangement (permutation)

$$P_1: 1\ is\ at\ the\ 1st\ place.$$

$$P_2: 2\ is\ at\ the\ 2nd\ palce$$

$$P_i: i\ is\ at\ the\ i-th\ palce$$

$$|\overline{P_1 \cup P_2 \cup \cdots \cup P_n}| = |U| - |P_1 \cup P_2 \cup \cdots \cup P_n| = |U| - [\ldots]$$

$$|P_i| = (n-1)! \Rightarrow \binom{n}{1}(n-1)! \Rightarrow \frac{n}{1!}(n-1)!$$

$$\left|P_i \cap P_j\right| = (n-2)! \Rightarrow \binom{n}{2}(n-2)! \Rightarrow \frac{n(n-1)}{2!}(n-2)!$$

Relation

A={1,2,3}

B={a,b,c,d}

A X B =
{(1,a),(1,b),(1,c),(1,d),(2,a),(2,b),(2,c),(2,d),(3,a),(3,b),(3,c),(3,d)}

A X B 的子集合

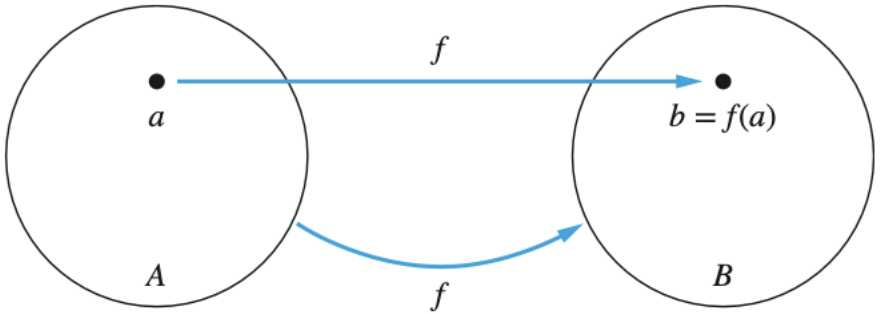
$$R_1 = \{(1, a), (2, b), (3, c)\}$$
$$R_2 = \{(1, a), (1, b), (1, c)\}$$
$$R_3 = \{(2, a), (2, c), (3, d)\}$$

以上都是A->B的關係（relation）

Function 是 Relation 的一種

R = $\{(1, a), (2, b), (3, c)\}$

$(1, a) \in R, 1Ra$



R

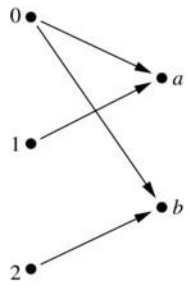
Binary Relations

Definition: A *binary relation R* from a set A to a set B is a subset

$$R: A \leftrightarrow B \quad R \subseteq A \times B.$$

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a) , (2, b)\}$ is a relation from A to B.
- The notation **0Ra** means $(0, a) \in R$, we may say "a is related to b (by relation R)", or "a relates to b (under relation R)".
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where exactly one element of B is related to each element of A.

Example

- $<: N \leftrightarrow N :\equiv \{(n, m) | n < m\}. a < b \; means \; (a, b) \in <.$
- A binary relation R corresponds to a predicate function

$$P_R: A \times B \to \{T, F\} defined\ over\ the\ 2\ sets\ A\ and\ B$$

Binary Relations on a Set 3

Example: Consider these relations on the set of integers:

$$R_1 = \{(a,b) | a \le b\},$$

$$R_2 = \{(a,b) | a > b\},$$

$$R_3 = \{(a,b) | a = b \text{ or } a = -b\},$$

$$R_4 = \{(a,b) | a = b\},$$

$$R_5 = \{(a,b) | a = b + 1\},$$

$$R_6 = \{(a,b) | a + b \le 3\}.$$

Note that these relations are on an infinite set and each of these relations is an infinite set.

Which of these relations contain each of the pairs

(1,1), (1, 2), (2, 1), (1, −1), and (2, 2)?

Solution: Checking the conditions that define each relation, we see that the pair (1,1) is in R_1, R_3, R_4 , and R_6: (1,2) is in R_1 and R_6: (2,1) is in R_2, R_5, and R_6: (1, −1) is in R_2, R_3, and R_6 : (2,2) is in R_1, R_3, and R_4.

Examples of Binary Relations

- Let A = {0,1,2} and B = {a,b}. Then R={(0,a),(0,b),(1,a),(2,b)} is a relation from A to B. For instance, we have 0Ra, 0Rb, etc..
- Let A be the set of all cities, and let B be the set of the 50 states in the US. Define the relation R by specifying that (a,b) belongs to R if city a is in state b. For instance, (Boulder, Colorado), (Bangor, Maine), (Ann Arbor, Michigan), (Middletown, New Jersey), (Middletown, New York), (Cupertino, California), and (Red Bank, New Jersey) are in R.
- ”eats”:≡ $\{(a,b) | organism\ a\ eats\ food\ b\}$.

Complementary Relations

- Let R: $A \leftrightarrow B$ $be\ any\ binary\ relation. Then, \bar{R}: A \leftrightarrow B, the\ complement\ of\ R, is\ the\ binary\ relation\ defined\ by\ \bar{R} :\equiv \{(a, b) | (a, b) \notin R\} = (A \times B) - R$.

- Note this is just $\bar{R}$ if the universe of discourse is $U = A \times B$; thus the name complement.
- The complement of $\bar{R}$ is R.

Inverse Relations

- Def:

Any binary relation $R: A \leftrightarrow B\ has\ an\ inverse\ relation\ R^{-1}: B \leftrightarrow A, defined\ by\ R^{-1} :\equiv \{(b, a) | (a, b) \in R\}$.

Examples:

1. $<^{-1} = \{(b, a) | a < b\} = \{(b, a) | b > a\} = >$.
2. If R: People$\rightarrow Foods\ is\ defined\ by$ "$aRb \Leftrightarrow a\ eats\ b$"$, then\ bR^{-1}a \Leftrightarrow$b is eaten by a.

Example

- Let $A = \{1,2,3,4,5\}$ $and\ R: A \leftrightarrow A: \equiv \{(a,b): a|b\}. What\ are\ \bar{R}\ and\ R^{-1}$?
- Sol:
- R={(1,1),(1,2),(1,3),(1,4),(1,5),(2,2),(2,4),(3,3),(4,4),(5,5)}
- $\bar{R} = \begin{Bmatrix} (2,1),(2,3),(2,5),(3,1),(3,2),(3,4),(3,5), \\ (4,1),(4,2),(4,3),(4,5),(5,1),(5,2),(5,3),(5,4) \end{Bmatrix}$
- $R^{-1} = \{(1,1),(2,1),(3,1),(4,1),(5,1),(2,2),(4,2),(3,3),(4,4),(5,5)\}$
- $|R|, |R^{-1}|, |\bar{R}|$

$|R|$ = $|R^{-1}|$

$|R|$+ $|\bar{R}| = |\mathrm{A} \times A|$

Combining Relations

Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.

Example: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

Quiz: What is $R_1 \oplus R_2$?

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2)(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\} \qquad R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

Composite Relations

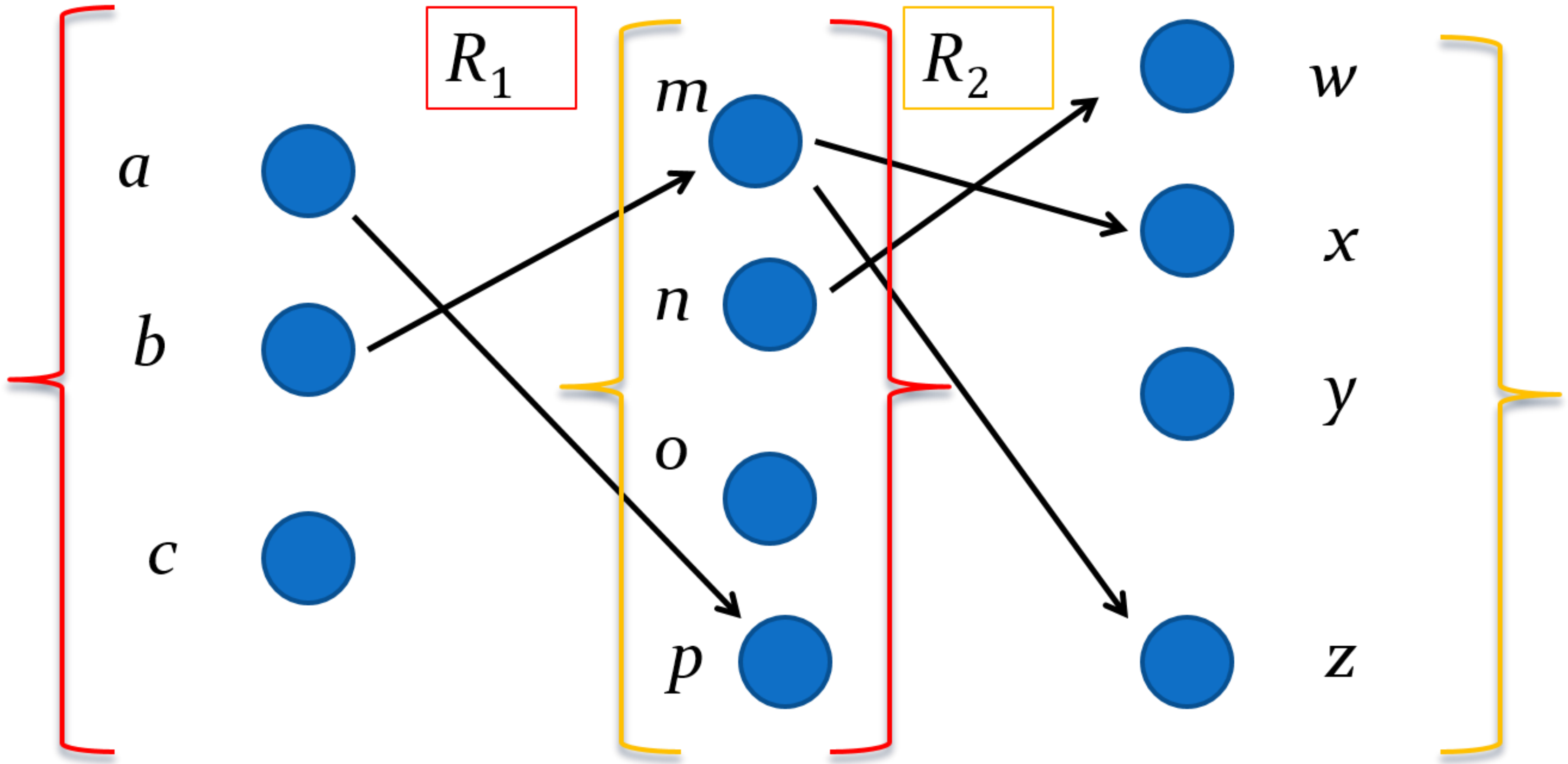
Let $R: A \leftrightarrow B, and\ S: B \leftrightarrow C. Then\ the\ composite\ S \circ R\ of\ R\ and\ S\ is\ defined\ as:\ S \circ R = \{(a, c) | aRb \wedge bSc\}$.

Ex1 Function composition $f \circ g\ is\ an\ example$.

Ex2 $A = \{1,2,3\}, B = \{a, b, c, d\}, C = \{x, y, z\}$.

- $R: A \leftrightarrow B, R = \{(1, a), (1, b), (2, b), (2, c)\}$.
- $S: B \leftrightarrow C, S = \{(a, x), (a, y), (b, y), (d, z)\}$.
- $S \circ R = \{(1, \mathrm{x}), (1, \mathrm{y}), (2, \mathrm{y})\}$

Representing the Composition of Relations



$$R_1 \circ R_2 = \{(b, x), (b, z)\}$$

Relation on a Set

Def:

A (binary) relation from a set A to itself is called a relation on the set A.

- E.g., the "<" relation from earlier was defined as relation on the set N of natural numbers.
- The identity relation $I_A\ on\ a\ set\ A\ is\ the\ set\ \{(a,a)|a \in A)\}$.
- Let A be the set {1,2,3,4}. Which ordered pairs are in the relation R={(a,b)|a divides b}?
- How many relations are there on a set with n elements?

Binary Relations on a Set [1]

Definition: A binary relation *R on a set A* is a subset of $A \times A$ or a relation from A to A.

Example:

- Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b),(a,c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation R $= \{(a,b) \mid a$ divides $b\}$ are
  (1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3), and (4, 4).
- |A|=N

$$R: A \leftrightarrow A \;\; a\, relation\, on\, A$$

$$R \subseteq A \times A \Rightarrow 2^{|A \times A|} = 2^{|A| \times |A|}$$

Reflexive Relations

Definition: *R* is *reflexive* iff (*a,a*) ∊ *R* for every element *a* ∊ A. Written symbolically, R is reflexive if and only if

$$\forall x\left[\mathrm{x} \in U \rightarrow (x,x) \in R\right]$$

Example: The following relations on the integers are reflexive:

$$R_1 = \{(a,b) \mid a \leq b\},$$
$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a,b) \mid a = b\}.$$

The following relations are not reflexive:

$$R_2 = \{(a,b) \mid a > b\} \quad (\text{note that } 3 \not> 3),$$
$$R_5 = \{(a,b) \mid a = b+1\} \quad (\text{note that } 3 \neq 3+1),$$
$$R_6 = \{(a,b) \mid a+b \leq 3\} \quad (\text{note that } 4+4 \not\leq 3).$$

If *A* = ∅ then the empty relation is reflexive vacuously. That is the empty relation on an empty set is reflexive!