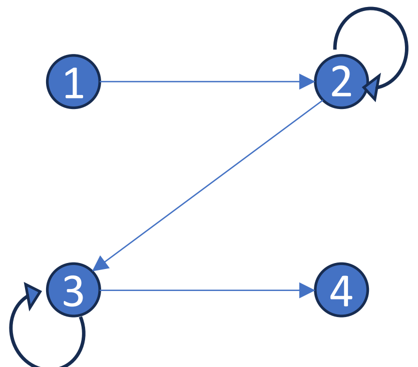
Discrete Mathematics
Lec11: Relation 3

馬誠佑

Closure of Relations

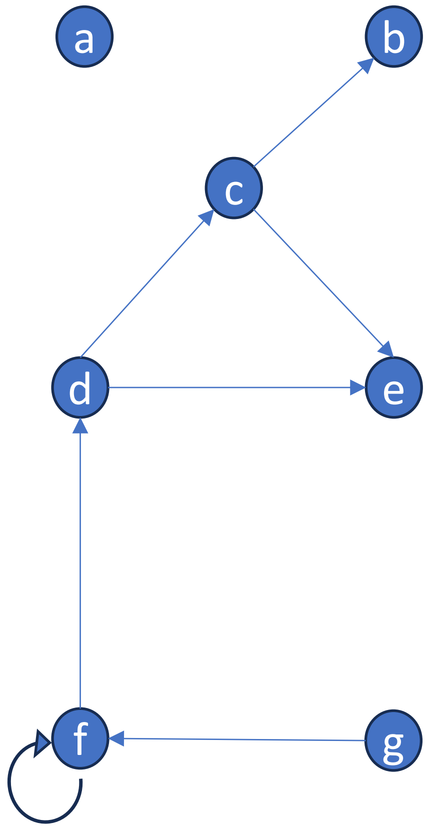
- For any property X, the "X closure" of a set (or relation)R is defined as the "smallest" superset of R that has the given property.
- The reflexive closure of a relation R on A is obtained by adding (a,a) to R for each $a \in A. I.e., it\ is\ R \cup I_A$.
- The symmetric closure of R is obtained by adding (b,a) to R for each (a,b) in R. I.e., it is $R \cup R^{-1}$.
- The transitive closure or connectivity relation of R is obtained by repeatedly adding (a,c) to R for each (a,b) and (b,c) in R. I.e., it is $R^* = \bigcup_{n \in z^+} R^n$

  - A={1,2,3,4}
  - R = {(1,2), (2,3), (2,2), (3,3), (3,4)}



$$A = \{a, b, c, d, e, f, g\}$$

$$R = \{(c, b), (c, e), (d, c), (d, e), (f, d), (f, f), (g, f)\}$$



$$R^{+(ref)} = R \cup I = \{\dots, (a, a), (b, b), (c, c), (d, d), (e, e), (g, g)\}$$

$$R^{+(sym)} = R \cup R^{-1} = \{\dots, (b, c), (e, c), (c, d), (e, d), (d, f), (f, g)\}$$

$R \circ R = R^2$= {路徑長度為2的都會留下來}

$$R^{+(tra)} = R^* = \bigcup_{i=1}^{\infty} R^i$$

Closures of Relations

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Paths in Digraphs/Binary Relations

- Def: A path of length n from node a to b in the directed graph G (or the binary relation R) is a sequence $(a, x_1), (x_1, x_2), \ldots, (x_{n-1}, b)$ $of\ n\ ordered\ pairs\ in\ E_G(or\ R).\ A\ path\ of\ length\ n \geq$ 1 from a to is called a circuit or a cycle.

- **Theorem:**

There exists a path of length n from a to b in R if and only if (a,b)∈ $\boldsymbol{R^n}$.

  - An empty sequence of edges is considered a path of length 0 from a to a.

  - If any path from a to b exists, then we say that a is connected to b. ("You can get there from here.")

Simple Transitive Closure Algorithm

- Lemma

Let A be a set with n element, and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n.

Procedure transClosure(M_R: $rank - n\ 0 \sim 1\ matrix$)

// A procedure computes R^* with 0-1 matrices.

A:=B:=M_R;

for i:=2 to n begin

A:=$A \odot M_R$; $B := B \vee A$;

end

Return B

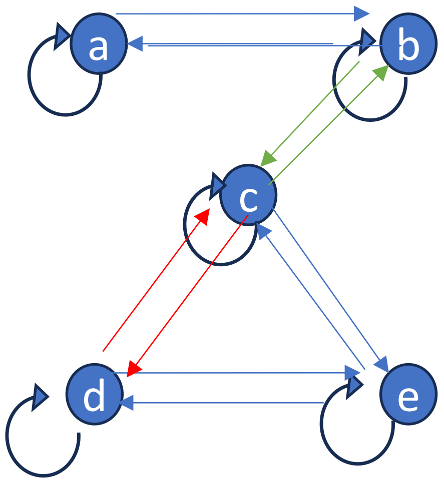
Equivalence Relations

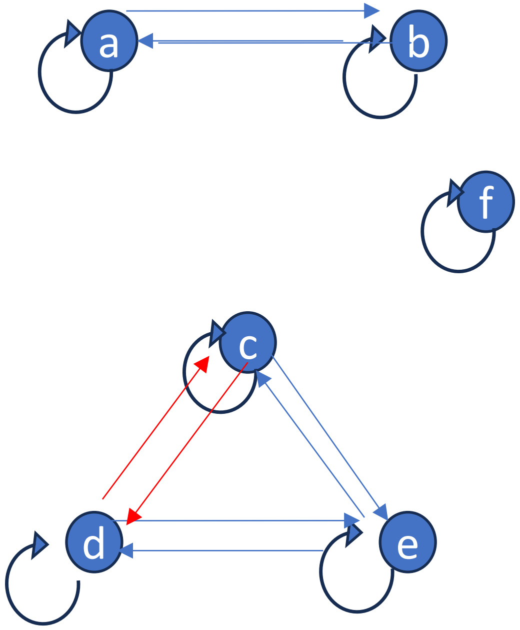
- Def:

An equivalence relation (e.r.) on a set A is simply any binary relation on A that is reflexive, symmetric, and transitive.

  - E.g., “=” itself is an equivalence relation.
  - For any function f: $A \rightarrow B, the\ relation$ “$have\ the\ same\ f\ value$”$, or = f :\equiv \{(a_1, a_2) | f(a_1) = f(a_2)\}$ is an equivalence relation.
    - E.g., let m = “mother of ”, then = m:≡ “have the same mother” is an e.r..

Example:





- Pf: R is a equivalence relation (E.R.)

1. Reflexive: $\forall a \in A, aRa$
2. Symmetric: $\forall a, b \in A, aRb \rightarrow bRa$
3. Transitive: $\forall a, b, c \in A,\ aRb \wedge bRc \rightarrow aRc$

Strings

Example: Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold.

- *Reflexivity*: Because $l(a) = l(a)$, it follows that aRa for all strings a.
- *Symmetry*: Suppose that aRb. Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and bRa.
- *Transitivity*: Suppose that aRb and bRc. Since $l(a) = l(b)$,and $l(b) = l(c)$, $l(a) = l(a)$ also holds and aRc.

Congruence Modulo m

Example: Let m be an integer with $m > 1$. Show that the relation
$R = \{(a,b) \mid a \equiv b \pmod{m}\}$
is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- *Reflexivity*: $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- *Symmetry*: Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)\, m$, so $b \equiv a \pmod{m}$.
- *Transitivity*: Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations: $a - c = (a - b) + (b - c) = km + lm = (k + l)\, m$.

$$a - c = (a - b) + (b - c) = km + lm = (k + l)m.$$

Therefore, $a \equiv c \pmod{m}$.

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$.

When only one relation is under consideration, we can write $[a]$, without the subscript R, for this equivalence class.

Note that $[a]_R = \{s \mid (a,s) \in R\}$.

If $b \in [a]_R$, then b is called a representative of this equivalence class. Any element of a class can be used as a representative of the class.

The equivalence classes of the relation congruence modulo m are called the *congruence classes modulo m*. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\ldots,\ a-2m,\ a-m,\ a+2m,\ a+2m,\ \ldots\}$.

For example, $[0]_4 = \{\ldots,\ -8,\ -4, 0, 4, 8,\ \ldots\}$ $\quad [1]_4 = \{\ldots,\ -7,\ -3, 1, 5, 9,\ \ldots\}$

$[2]_4 = \{\ldots,\ -6,\ -2, 2, 6, 10,\ \ldots\}$ $\quad [3]_4 = \{\ldots,\ -5,\ -1, 3, 7, 11,\ \ldots\}$

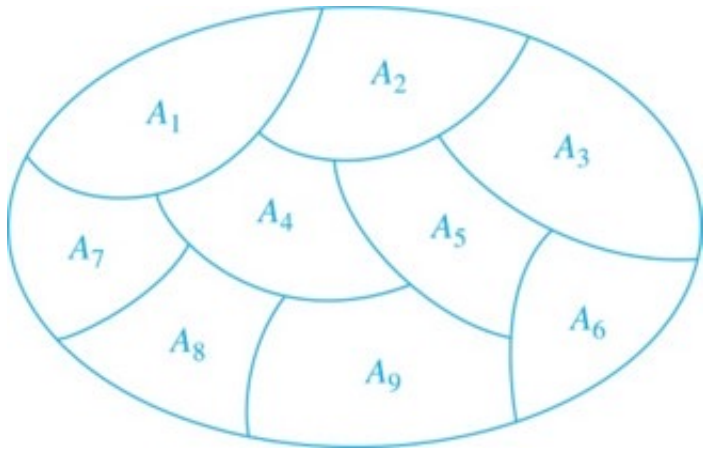
Examples of E.R.'s

- "Strings a and b are the same length."
  - [a] = the set of all strings of the same length as a.
- "Integers a and b have the same absolute value."
  - [a] = the set {a, -a}
- "Real numbers a and b have the same fractional part (i.e., a-b$\in Z$)."
  - $[a] = the\ set\ \{\dots, a-2, a-1, a, a+1, a+2, \dots\}$.
- "Integers a and b have the same residue modulo m." (for a given m > 1) $[a] = the\ set\ \{\dots, a-2m, a-m, a, a+m, a+2m, \dots\}$.

Partition of a Set

Definition: A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and $$\bigcup_{i \in I} A_i = S.$$



A Partition of a Set

Partial Orderings 1

Definition 1: A relation R on a set S is called a *partial ordering,* or *partial order,* if it is reflexive, antisymmetric, and transitive. A set together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

- The "greater than or equal" relation $\geq$ is a partial ordering on the set of integers.
- The divisibility relation $|$ is a partial ordering on the set of positive integers.
- The inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Partial Orderings 2

Example 1: Show that the "greater than or equal" relation (≥) is a partial ordering on the set of integers.

- *Reflexivity*: $a \geq a$ for every integer a.
- *Antisymmetry*: If $a \geq b$ and $b \geq a$, then $a = b$.
- *Transitivity*: If $a \geq b$ and $b \geq c$, then $a \geq c$.

These properties all follow from the order axioms for the integers. (*See Appendix* 1).

Partial Orderings 3

Example 2: Show that the divisibility relation (**|**) is a partial ordering on the set of integers.

- *Reflexivity*: $a \mid a$ for all integers a. (*see Example* 9 *in Section* 9.1)
- *Antisymmetry*: If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$. (*see Example* 12 *in Section* 9.1)
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

$(Z^+, \mid)$ is a poset.

Partial Orderings 4

Example 3: Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S.

- *Reflexivity*: $A \subseteq A$ whenever A is a subset of S.
- *Antisymmetry*: If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- *Transitivity*: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The properties all follow from the definition of set inclusion.

Comparability

Definition 2: The elements a and b of a poset $(S, \preccurlyeq)$ are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called i*ncomparable*.

The symbol $\preccurlyeq$ is used to denote the relation in any poset.

Definition 3: If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order.* A totally ordered set is also called a *chain.*

Definition 4: $(S, \preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Lexicographic Order

Definition: Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), that is,

$(a_1, a_2) \prec (b_1, b_2)$,

either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$.

This definition can be easily extended to a lexicographic ordering on strings (*see text*).

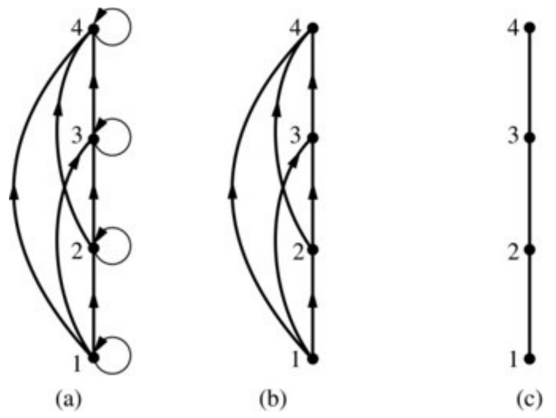
Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- *discreet* $\prec$ *discrete*, because these strings differ in the seventh position and $e \prec t$.
- *discreet* $\prec$ *discreetness*, because the first eight letters agree, but the second string is longer.

$$A = \{2,3,4,5,6,9,10\}$$
$$B = \{1,2,3,4,5\}$$
$$(2,1) \leq (4,3)$$
$$(3,1) \nleq (2,4)$$
$$(3,1) \leq (3,4)$$

Hasse Diagrams (哈斯圖)

Definition: A *Hasse diagram* is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.



A partial ordering is shown in (a) of the figure above. The loops due to the reflexive property are deleted in (b). The edges that must be present due to the transitive property are deleted in (c). The Hasse diagram for the partial ordering (a), is depicted in (c).

Procedure for Constructing a Hasse Diagram

To represent a finite poset $(S, \preccurlyeq)$ using a Hasse diagram, start with the directed graph of the relation:

- Remove the loops (a, a) present at every vertex due to the reflexive property.
- Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
- Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

Example

$A = \{2,3,4,5,6,9,10\}$ (A,|)

| = {(2,2),(2,4),(2,6),(2,10),

(3,3),(3,6),(3,9),

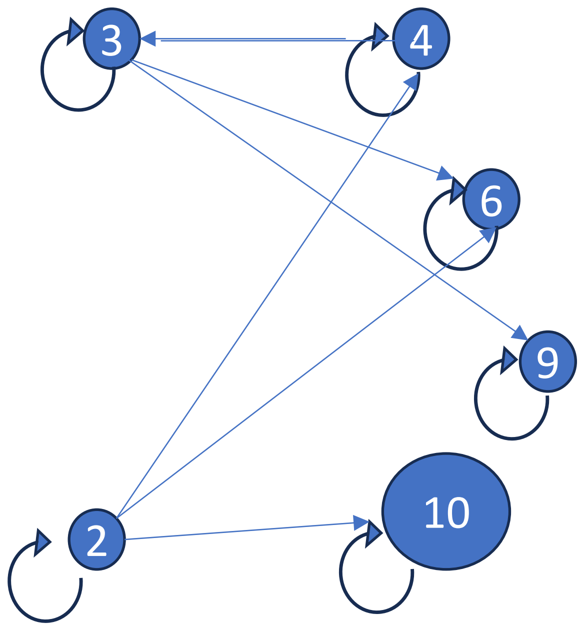
(4,4),

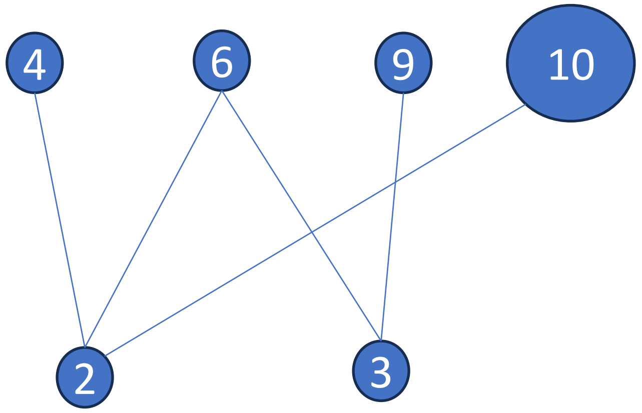
(6,6),

(9,9),

(10,10)

}



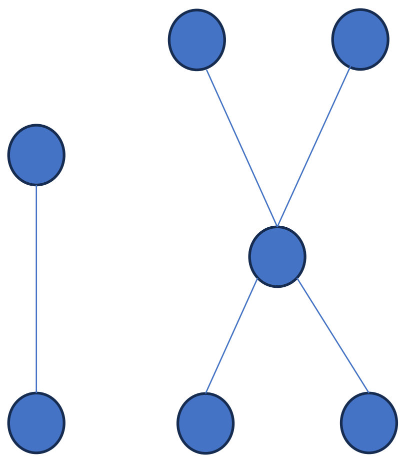


Maximal and Minimal Elements

- Def: a is a maximal (resp., minimal) element in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$ $(resp., b \prec a)$. (極大)（沒有人比a大）（有可能是不能比）
- Def: a is the greatest (resp., least) element of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ (resp., $a \preccurlyeq b$) for all $b \in S$ （最大）（有可能不存在）

- Lemma

Every finite nonempty poset $(S, \preccurlyeq)$ has a minimal element.

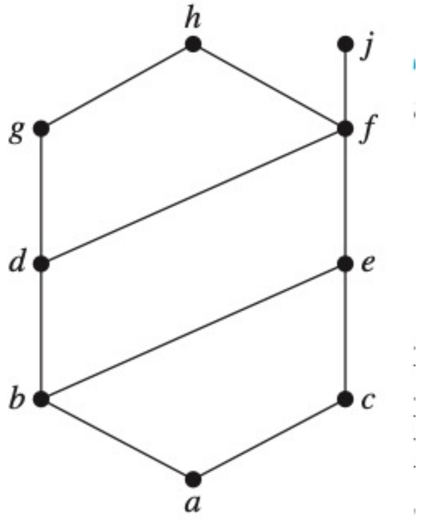


Maximal and Minimal Elements-2

- Def: A is a subset of a poset (S,≼).
  - $u \in S$ is called an upper bound (resp., lower bound) of A if $a \preccurlyeq u$ $(resp., u \preccurlyeq a)$ for all $a \in A$.
  - $x \in S$ is called the least upper bound (resp., greatest lower bound) of A if x is an upper bound (resp., lower bound) that is less than every other upper bound (resp., lower bound) of A.
- Def: (S,≼) is a well-ordered set if it is a poset such that ≼ is a total ordering and every nonempty subset of S has a least element.
  - E.g., $(Z^+, \leq)$ is well-ordered but (R,≤) is not.
  - There is "well-ordered induction".

Lattices

- Def: A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a lattice.

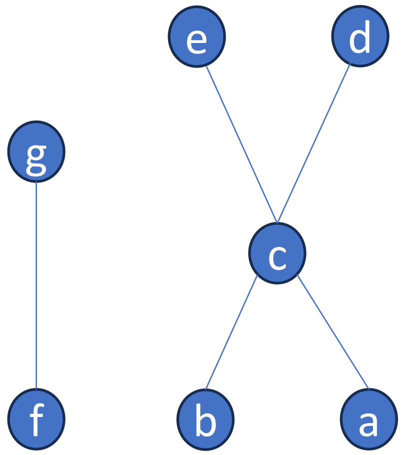


- Example: Determine whether the posets $(\{1,2,3,4,5\},|)$ and $(\{1,2,4,8,16\},|)$ are lattices.

Topological Sorting

- Motivation: A project is made up of 20 different tasks. Some tasks can be completed only after others have been finished. How can an order be found for these tasks?

- Topological sorting: Given a partial ordering R, find a total ordering $\preccurlyeq$ such that a $\preccurlyeq$ b whenever aRb. $\preccurlyeq$ is said compatible with R.



Topological Sorting for Finite Posets

- Procedure topological_sort(S: finite poset)

  k:=1

  while S$\neq \emptyset$

  begin

  $a_k \coloneqq a\ minimal\ element\ of\ S$

  $S \coloneqq S - \{a_k\}$

  $k \coloneqq k + 1$

  end $\{a_1, a_2, a_3, \dots, a_n\ is\ a\ compatible\ total\ ordering\ of\ S\}$