Discrete Mathematics
Lec9: Advanced Counting

馬誠佑

Recurrence Relation

- Example : Consider the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2} \; for\; any\; n \geq 2.$$

Which of the following are solutions?

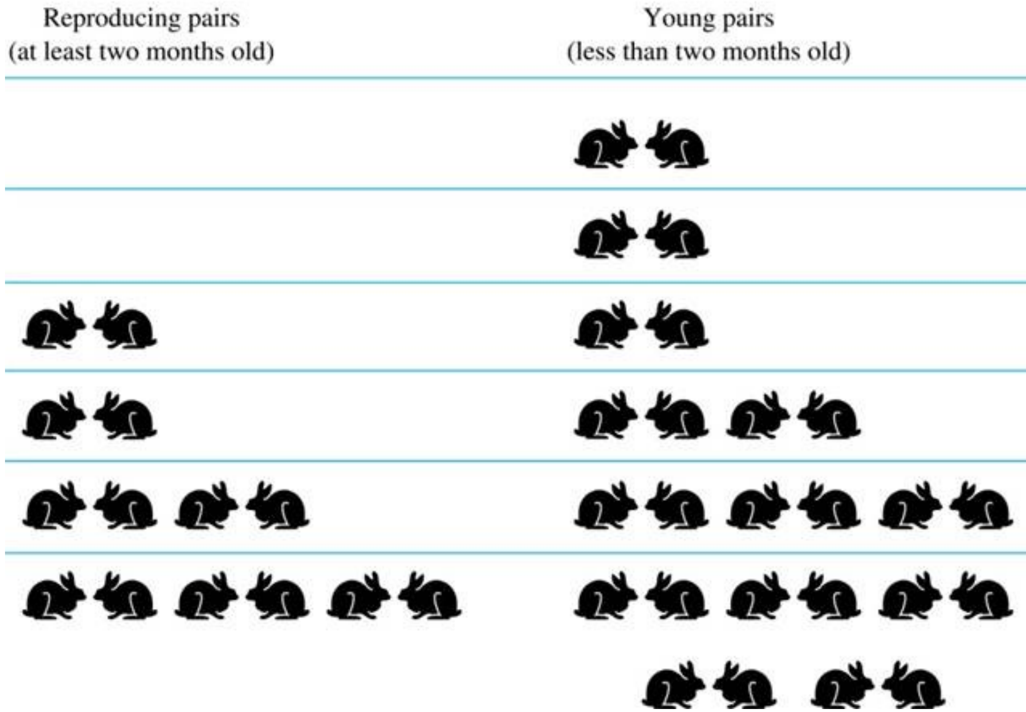
- $a_n = 3n$: $2a_{n-1} - a_{n-2} = 2\big(3(n-1)\big) - \big(3(n-2)\big) = 3n\ (Yes)$
- $a_n = 2^n$: $2a_{n-1} - a_{n-2} = 2(2^{n-1}) - (2^{n-2})\ \neq 2^n (No)$
- $a_n = 5$: $2a_{n-1} - a_{n-2} = 2(5) - (5) = 5\ (Yes)$

Rabbits and the Fibonacci Numbers [1]

Example: A young pair of rabbits (one of each gender) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that rabbits never die.

This is the original problem considered by Leonardo Pisano (Fibonacci) in the thirteenth century.

Rabbits and the Fibonacci Numbers 2



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

Modeling the Population Growth of Rabbits on an Island

[Jump to long description]

Rabbits and the Fibonacci Numbers

Solution: Let f_n be the number of pairs of rabbits after n months.

- There are is $f_1 = 1$ pairs of rabbits on the island at the end of the first month.
- We also have $f_2 = 1$ because the pair does not breed during the first month.
- To find the number of pairs on the island after n months, add the number on the island after the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2}, because each newborn pair comes from a pair at least two months old.

Consequently the sequence $\{f_n\}$ satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ with the initial conditions $f_1 = 1$ and $f_2 = 1$.

The number of pairs of rabbits on the island after n months is given by the nth Fibonacci number.

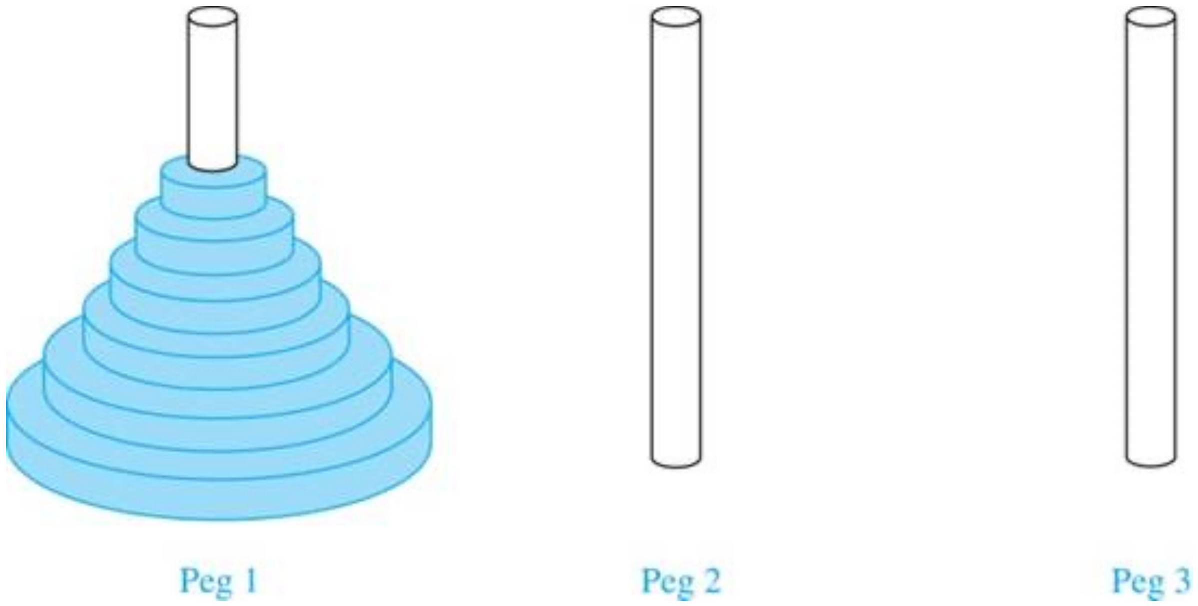
Example

- Recurrence relation for growth of a bank account with P% interest per given period:
- $M_n = M_{n-1} + \left(\frac{P}{100}\right) M_{n-1} = (1 + P/100)\, M_{n-1} = rM_{n-1}$ (let r = 1 + P/100)

$$= r(rM_{n-2}) = r \cdot r \cdot (rM_{n-3}) \ldots and\ so\ on\ to \ldots = r^n M_0$$

The Tower of Hanoi 2

The Initial Position in the Tower of Hanoi Puzzle



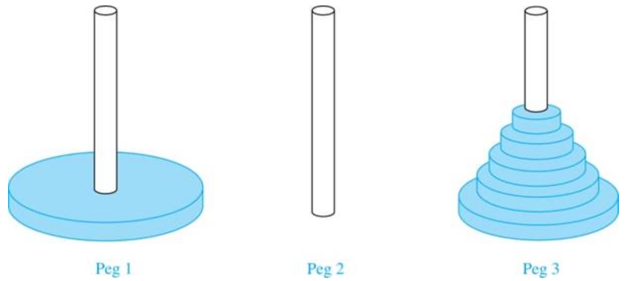
Problem:

Get all disks form peg 1 to peg 2.

- Only move 1 disk at a time.
- Never set a larger disk on a smaller one.

The Tower of Hanoi 3

Solution: Let $\{H_n\}$ denote the number of moves needed to solve the Tower of Hanoi Puzzle with n disks. Set up a recurrence relation for the sequence $\{H_n\}$. Begin with n disks on peg 1. We can transfer the top $n-1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves.



First, we use 1 move to transfer the largest disk to the second peg. Then we transfer the $n-1$ disks from peg 3 to peg 2 using H_{n-1} additional moves. This can not be done in fewer steps. Hence,

$$H_n = 2H_{n-1} + 1.$$

The initial condition is $H_1 = 1$ since a single disk can be transferred from peg 1 to peg 2 in one move.

The Tower of Hanoi 4

We can use an iterative approach to solve this recurrence relation by repeatedly expressing H_n in terms of the previous terms of the sequence.

$$
\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\
&= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&\vdots \\
&= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \ \ldots\ + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ \ldots\ + 2 + 1 \qquad \textit{because } H_1 = 1 \\
&= 2^n - 1 \qquad \textit{using the formula for the sum of the terms of a geometric series}
\end{aligned}
$$

- There was a myth created with the puzzle. Monks in a tower in Hanoi are transferring 64 gold disks from one peg to another following the rules of the puzzle. They move one disk each day. When the puzzle is finished, the world will end.
- Using this formula for the 64 gold disks of the myth,

  $2^{64} - 1 = 18,446, 744,073, 709,551,615$

  days are needed to solve the puzzle, which is more than 500 billion years.

Linear Homogeneous Recurrence Relations (LiHoReCoCos)

Definition: A *linear homogeneous recurrence relation of* ***degree*** k *with* ***constant coefficients*** is a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k\, a_{n-k}$, where $c_1, c_2, \ldots., c_k$ are real numbers, and $c_k \neq 0$

- it is *linear* because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of n.
- it is *homogeneous* because no terms occur that are not multiples of the a_js. Each coefficient is a constant.
- the *degree* is k because a_n is expressed in terms of the previous k terms of the sequence.

By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions $a_0 = C_1$, $a_0 = C_1$,… , $a_{k-1} = C_{k-1}$.

Examples of Linear Homogeneous Recurrence Relations

$$P_n = (1.11)P_{n-1}$$

linear homogeneous recurrence relation of degree one

$$f_n = f_{n-1} + f_{n-2}$$

linear homogeneous recurrence relation of degree two

$$a_n = a_{n-1} + a^2{}_{n-2}$$

not linear

$$H_n = 2H_{n-1} + 1$$

not homogeneous

$$B_n = nB_{n-1}$$

coefficients are not constants

Solving Linear Homogeneous Recurrence Relations

The basic approach is to look for solutions of the form $a_n = r^n$, where r is a constant.

Note that $a_n = r^n$ is a solution to the recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r_{n-k}.$$

Algebraic manipulation yields the *characteristic equation*:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution to the characteristic equation.

The solutions to the characteristic equation are called the *characteristic roots* of the recurrence relation. The roots are used to give an explicit formula for all the solutions of the recurrence relation.

Solving Linear Homogeneous Recurrence Relations of Degree Two

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

if and only if

$$a_n = \alpha r_1^n + \alpha_2 r_2^n$$

for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

Solving LiHoReCoCos

- Basic idea: Look for solutions of the form $a_n = r^n, where\ r\ is\ a\ constant$.
- Bring $a_n = r^n$ back to the recursive equation.

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$
$$i.e., r^{n-k}\left(r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k\right) = 0$$

- The characteristic equation:

$$\left(r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k\right) = 0$$

- The solutions (characteristic roots) can yield an explicit formula for the sequence

Example

- $a_n = 3a_{n-1} + 4a_{n-2}$

Let $a_n = r^n$ bring back to the recursive equation.

$$r^n = 3r^{n-1} + 4r^{n-2}$$

$$r^n - 3r^{n-1} - 4r^{n-2} = 0$$

$r^{n-2}(r^2 - 3r - 4) = 0$ <-Characteristic equation (特徵多項式/方程式)

$$r^{n-2}(r-4)(r+1) = 0$$

$$\therefore r = 4 \; or \; -1$$

- $a_n = \alpha 4^n + \beta(-1)^n$ 若此時再給你$a_0 = 5, a_1 = 10$

=> α+ β=5, $4\alpha - \beta$=10 => $\alpha = 3, \beta = 2$ => $a_n = 3 \cdot 4^n + 2(-1)^n$

Using Theorem 1

Example: What is the solution to the recurrence relation

$a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution: The characteristic equation is $r^2 - r - 2 = 0$. => $(r-2)(r+1)$

Its roots are $r = 2$ and $r = -1$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and

only if $a_n = \alpha_1 2^n + \alpha_2(-1)^n$, for some constants α_1 and α_2.

To find the constants α_1 and α_2, note that

$a_0 = 2 = \alpha_1 + \alpha_2$ and $a_1 = 7 = \alpha_1 2 + \alpha_2(-1)$.

Solving these equations, we find that $\alpha_1 = 3$ and $\alpha_2 = -1$.

Hence, the solution is the sequence $\{a_n\}$ with $a_n = 3 \cdot 2^n - (-1)^n$.

Example

$$a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$$

令$a_n = r^n$代入

$$r^n = 2r^{n-1} + 5r^{n-2} - 6r^{n-3}$$

$$r^n - 2r^{n-1} - 5r^{n-2} + 6r^{n-3} = 0$$

$$r^{n-3}(r^3 - 2r^2 - 5r + 6) = 0$$

$$(r^3 - 2r^2 - 5r + 6) = 0$$

$$(r-3)(r+2)(r-1) = 0$$

$$\therefore r = -2, 1, 3$$

$$a_n = \alpha(-2)^n + \beta + \gamma 3^n$$

EXAMPLE 6 Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Solution: The characteristic polynomial of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6.$$

The characteristic roots are $r = 1$, $r = 2$, and $r = 3$, because $r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3)$. Hence, the solutions to this recurrence relation are of the form

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n.$$

To find the constants α_1, α_2, and α_3, use the initial conditions. This gives

$$\begin{aligned} a_0 &= 2 = \alpha_1 + \alpha_2 + \alpha_3, \\ a_1 &= 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3, \\ a_2 &= 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9. \end{aligned}$$

When these three simultaneous equations are solved for α_1, α_2, and α_3, we find that $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$ ◀

The Solution when there is a Repeated Root

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has one repeated root r_0. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha r_0^n + \alpha_2 n r_0^n$$

for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
has repeat root x_0 => $f'(x_0) = 0$

Using Theorem 2

Example: What is the solution to the recurrence relation
$a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution: The characteristic equation is $r^2 - 6r + 9 = 0$.

The only root is $r = 3$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 3^n + \alpha_2 n(3)^n$$

where α_1 and α_2 are constants.

To find the constants α_1 and α_2, note that

$$a_0 = 1 = \alpha_1 \quad \text{and} \quad a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3.$$

Solving, we find that $\alpha_1 = 1$ and $\alpha_2 = 1$.

Hence,

$$a_n = 3^n + n3^n .$$

Example

$$a_n = 2a_{n-1} - a_{n-2}$$

令$a_n = r^n$代入

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$\therefore r = 1$$

$$a_n = \alpha(1)^n + \beta n(1)^n = \alpha + \mathrm{n}\beta$$

Example

$$a_n = 6a_{n-1} - 12a_{n-2} - 8a_{n-3}$$

C.E. $r^3 - 6r^2 + 12r - 8 = 0$

$$(r-2)^3 = 0$$

$$\therefore r = 2(3\, repeated\; roots)$$

$a_n = \alpha(2)^n + \beta n(2)^n + \gamma n^2\, (2)^n$

Solving Linear Homogeneous Recurrence Relations of Arbitrary Degree

This theorem can be used to solve linear homogeneous recurrence relations with constant coefficients of any degree when the characteristic equation has distinct roots.

Theorem 3: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$r^k - c_1 r^{k-1} - \cdots - c_k = 0$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$

if and only if $$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

The General Case with Repeated Roots Allowed

Theorem 4: Let c_1, c_2 ,…, c_k be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots r_1, r_2, …, r_t with multiplicities m_1, m_2, …, m_t, respectively so that $m_i \geq 1$ for i = 0, 1, 2, …,t and $m_1 + m_2 + \ldots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = \ & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for n = 0, 1, 2, …, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_{i-1}$.

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients[1]

Definition: A *linear nonhomogeneous recurrence relation with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k} + F(n),$$

where c_1, c_2, ….,c_k are real numbers, and $F(n)$ is a function not identically zero depending only on n.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k},$$

is called the associated homogeneous recurrence relation.

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients 2

Example: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$.
What is the solution with $a_1 = 3$?
Solution: The associated linear homogeneous equation is $a_n = 3a_{n-1}$.
Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.
Because $F(n) = 2n$ is a polynomial in n of degree one, to find a particular solution we might try a linear function in n, say $= cn + d$, where c and d are constants. Suppose that $p_n = cn + d$ is such a solution.
Then $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n-1) + d) + 2n$.
Simplifying yields $(2 + 2c)n + (2d - 3c) = 0$. It follows that $cn + d$ is a solution if and only if
$2 + 2c = 0$ and $2d - 3c = 0$. Therefore, $cn + d$ is a solution if and only if $c = -1$ and $d = -3/2$.
Consequently, $a_n^{(p)} = -n - 3/2$ is a particular solution.
By Theorem 5, all solutions are of the form $a_n = a_n^{(p)} + a_n^{(h)} = -n - 3/2 + \alpha 3^n$, where α is a constant.
To find the solution with $a_1 = 3$, let $n = 1$ in the above formula for the general solution.
Then $3 = -1 - 3/2 + 3\alpha$, and $\alpha = 11/6$. Hence, the solution is $a_n = -n - 3/2 + (11/6)3^n$.

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients 2

Example: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$.
What is the solution with $a_1 = 3$?
Solution: The associated linear homogeneous equation is $a_n = 3a_{n-1}$.
Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.
Because $F(n)= 2n$ is a polynomial in n of degree one, to find a particular solution we might try a linear function in n, say $a_n^{(p)} = cn + d$, where c and d are constants. Suppose that $a_n^{(p)} = cn + d$ is such a solution.
Then $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n-1) + d)+ 2n$.
Simplifying yields $(2 + 2c)n + (2d - 3c) = 0$. It follows that $cn + d$ is a solution if and only if
$2 + 2c = 0$ and $2d - 3c = 0$. Therefore, $cn + d$ is a solution if and only if $c = - 1$ and $d = - 3/2$.
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To find the solution with $a_1 = 3$, let $n = 1$ in the above formula for the general solution.
Then $3 = -1 - 3/2 + 3\alpha$, and $\alpha = 11/6$. Hence, the solution is $a_n = -n - 3/2 + (11/6)3^n$.

Example

Solve the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2} + 16n + 5\ for\ n \geq 2\ with\ a_0 = 4$ by characteristic equations.

Sol:

Let $a_n = h_n + p_n. For\ the\ homogeneous\ part$

$(h_n = 7h_{n-1} - 10h_{n-2}), we\ have\ h_n = \alpha_1 2^n + \alpha_2 5^n. Now\ assume\ p_n = an + b\ and\ p_n = 7p_{n-1} - 10p_{n-2} + 16n + 5$

$(an + b) = 7(a(n-1) + b) - 10(a(n-2) + b) + 16n + 5.$

Continue

We can get $a = 4, b = {}^{57}/_{4}\,.\,Thus,$

$$a_n = h_n + p_n = \alpha_1 2^n + \alpha_2 5^n + 4n + \frac{57}{4}$$

From $a_0 = 0, we\ have\ \alpha_1 + \alpha_2 = -\frac{57}{4}$. From $a_1 = 1, we\ have\ 2\alpha_1 + 5\alpha_2 = -\frac{69}{4}.\mathrm{Thus, we\ can\ get}\ \alpha_1$=-18 and $\alpha_2 = \frac{15}{4}$. So, $a_n = (-18)2^n + \frac{15}{4}5^n + 4n + \frac{57}{4}$