Discrete Mathematics
Lec 3: Basic Structures: Sets, Functions, Sequences, and Sums

馬誠佑

Sets

A *set* is an unordered collection of objects.

- the students in this class
- the chairs in this room

The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.

The notation $a \in A$ denotes that a is an element of the set A.

If a is not a member of A, write $a \notin A$

Describing a Set: Roster Method

$S = \{a,b,c,d\}$

Order not important

$S = \{a,b,c,d\} = \{b,c,a,d\}$

Each distinct object is either a member or not; listing more than once does not change the set.

$S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$

Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

$S = \{a,b,c,d, \ldots\ldots,z\}$

Roster Method

Set of all vowels in the English alphabet:

$V = \{a,e,i,o,u\}$

Set of all odd positive integers less than 10:

$O = \{1,3,5,7,9\}$

Set of all positive integers less than 100:

$S = \{1,2,3,........,99\}$

Set of all integers less than 0:

$S = \{...., -3,-2,-1\}$

Some Important Sets

N = *natural numbers* = {0,1,2,3….}

Z = *integers* = {…,-3,-2,-1,0,1,2,3,…}

Z⁺ = *positive integers* = {1,2,3,…..}

R = set of *real numbers*

R⁺ = set of *positive real numbers*

C = set of *complex numbers*.

Q = set of rational numbers

Set-Builder Notation

Specify the property or properties that all members must satisfy:

$S = \{x \mid x \text{ is a positive integer less than } 100\}$

$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$

A predicate may be used:

$S = \{x \mid \mathrm{P}(x)\}$

Example: $S = \{x \mid \mathrm{Prime}(x)\}$

Positive rational numbers:

$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$

Set Cardinality

Definition: If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Example

B = {{a,b,c}, {x}, y, z,x}

|B| = ?

{a,b,c}$\in B$, {x} $\in B$, y $\in B$, z $\in B$, x $\in B$

C={{{ø}}}, |C| = ?

Interval Notation

$$[a,b] = \{x \mid a \le x \le b\}$$
$$[a,b) = \{x \mid a \le x < b\}$$
$$(a,b] = \{x \mid a < x \le b\}$$
$$(a,b) = \{x \mid a < x < b\}$$

closed interval [a,b]

open interval (a,b)

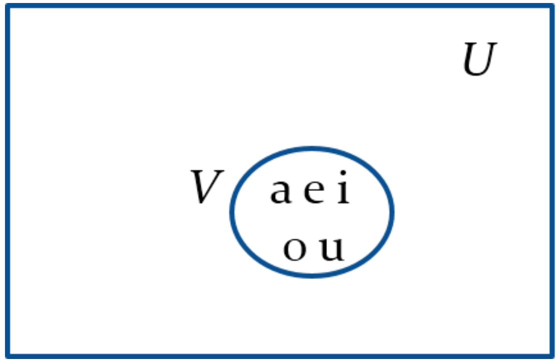
Universal Set and Empty Set

The *universal set U* is the set containing everything currently under consideration.

- Sometimes implicit
- Sometimes explicitly stated.
- Contents depend on the context.

The empty set is the set with no elements. Symbolized ∅, but {} also used.

Venn Diagram



John Venn (1834-1923)
Cambridge, UK

Some things to remember

Sets can be elements of sets.

$\{\{1,2,3\}, a, \{b,c\}\}$

$\{\mathbf{N},\mathbf{Z},\mathbf{Q},\mathbf{R}\}$

The empty set is different from a set containing the empty set.

$\emptyset \neq \{ \emptyset \}$

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x \left(x \in A \leftrightarrow x \in B \right)$
- We write *A* = *B* if *A* and *B* are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \to x \in B)$ is true.
  1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$,for every set S.
  2. Because $a \in S \to a \in S$, $S \subseteq S$, for every set S.

Showing a Set is or is not a Subset of Another Set

Showing that A is a Subset of B: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.

Showing that A is not a Subset of B: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Another look at Equality of Sets

Recall that two sets A and B are *equal*, denoted by $A = B$, iff

$$\forall x\left(x \in A \leftrightarrow x \in B\right)$$

Using logical equivalences we have that $A = B$ iff

$$\forall x\left[\left(x \in A \rightarrow x \in B\right) \wedge \left(x \in B \rightarrow x \in A\right)\right]$$

This is equivalent to

$A \subseteq B$ and $B \subseteq A$

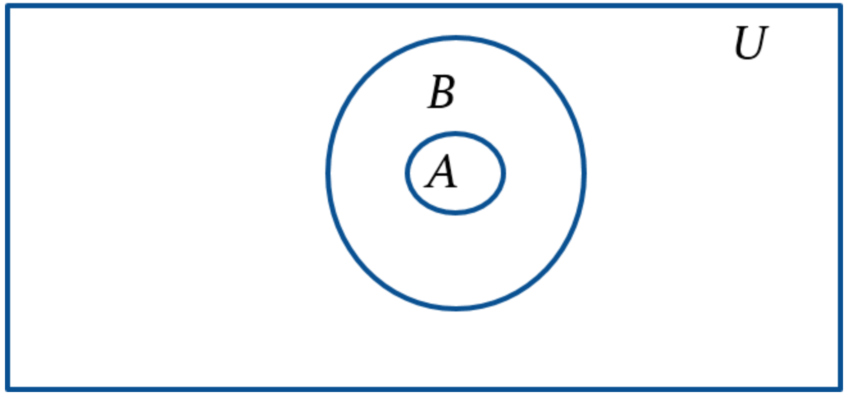
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x \wedge (x \in A \rightarrow x \in B) \wedge \exists x (x \in B \wedge x \notin A)$$

is true.

Venn Diagram



Power Sets

Definition: The set of all subsets of a set *A*, denoted *P*(*A*), is called the *power set* of *A*.

Example: If *A* = {a,b} then

$$P(A) = \{\emptyset, \{a\},\{b\},\{a,b\}\} = 2^A$$

If a set has *n* elements, then the cardinality of the power set is 2^n.

$$|2^A| = 8 = 2^{|A|}$$

Tuples

The *ordered n-tuple* $(a_1, a_2, \ldots\ldots, a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.

Two n-tuples are equal if and only if their corresponding elements are equal.

2-tuples are called *ordered pairs*.

The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product [1]

René Descartes
(1596-1650)

Definition: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) \mid a \in A \land b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

$|A \times B| = |A| X |B|$

Definition: A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B. (Relations will be covered in depth in Chapter 9.)

Cartesian Product $_2$

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_{i} for i = 1, … n.

$$A_1 \times A_2 \times \cdots \times A_n =$$

$$\{(a_1, a_2 \cdots , a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times$ C where A = {0,1}, B = {1,2} and C = {0,1,2}

Solution: $A \times B \times$ C **=** {(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)}

Truth Sets of Quantifiers

Given a predicate *P* and a domain *D*, we define the *truth set* of *P* to be the set of elements in *D* for which *P*(*x*) is true. The truth set of *P*(x) is denoted by

$$\{x \in D \mid P(x)\}$$

Example: The truth set of *P*(*x*) where the domain is the integers and *P*(*x*) is "$|x| = 1$" is the set {-1,1}

Section Summary 2

Set Operations

- Union
- Intersection
- Complementation
- Difference

More on Set Cardinality

Set Identities

Proving Identities

Membership Tables

Boolean Algebra

Propositional calculus and set theory are both instances of an algebraic system called a *Boolean Algebra*. This is discussed in Chapter 12.

The operators in set theory are analogous to the corresponding operator in propositional calculus.

As always there must be a universal set U. All sets are assumed to be subsets of U.

Union

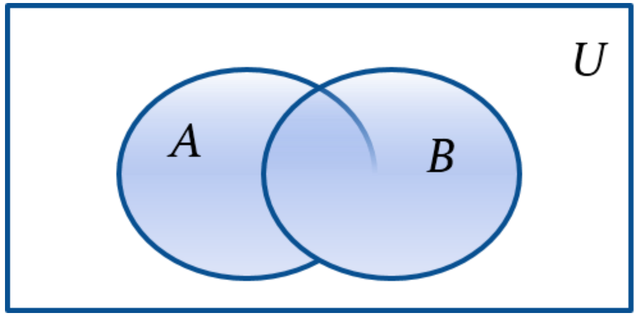
Definition: Let *A* and *B* be sets. The *union* of the sets *A* and *B*, denoted by *A* ∪ *B*, is the set:

$$\{x \mid x \in A \vee x \in B\}$$

Example: What is {1,2,3} ∪ {3, 4, 5}?

Solution: {1,2,3,4,5}

Venn Diagram for *A* ∪ *B*



Intersection

Definition: The *intersection* of sets A and B, denoted by $A \cap B$, is $\{x | x \in A \wedge x \in B\}$

Note if the intersection is empty, then A and B are said to be *disjoint*.

Example: What is? $\{1,2,3\} \cap \{3,4,5\}$?

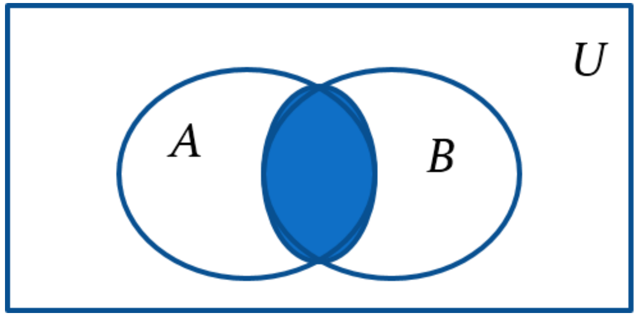
Solution: $\{3\}$

Example: What is?

$\{1,2,3\} \cap \{4,5,6\}$?

Solution: $\emptyset$

Venn Diagram for $A \cap B$



Complement

Definition: If A is a set, then the *complement* of the A (with respect to U), denoted by $\bar{A}$ is the set $U - A$

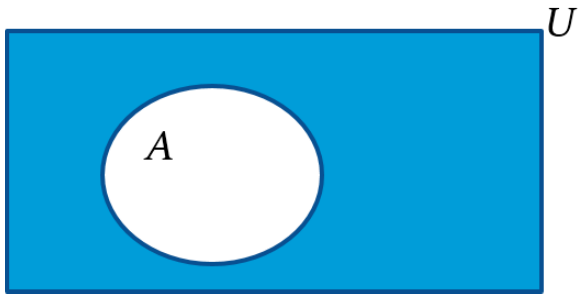
$$\bar{A} = \{x \mid x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

$$\text{Solution}: \{x \mid x \leq 70\}$$

Venn Diagram for Complement

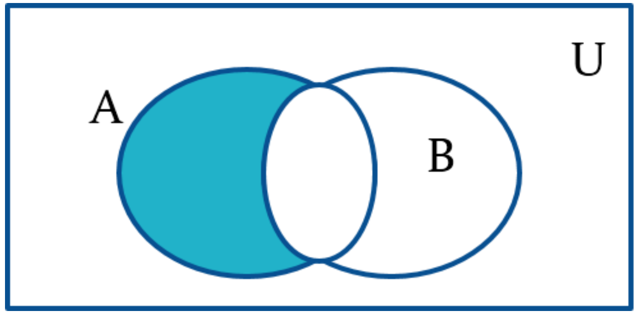


Difference

Definition: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

Venn Diagram for $A - B$



Disjoint: $A \cap B = \emptyset$

Example

$U = \{1,2, \dots , 10\}$
$A = \{1,3,5,7,9\}$
$B = \{2,4,6,8,10\}$
$C = \{3,4,5,6\}$
$A \cap B =?$
$A \cap C =?$
$A \cup B =?$
$\overline{A} =?$
A, B are partitions of U.

The Cardinality of the Union of Two Sets

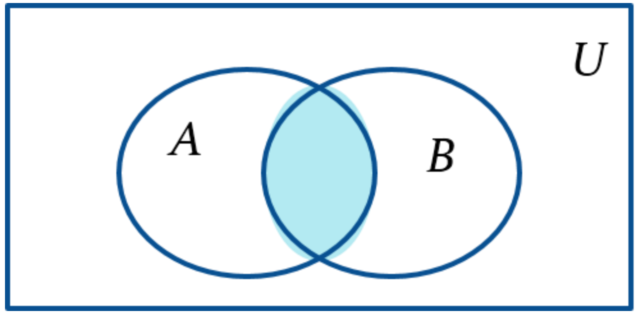
Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$

Example: Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

We will return to this principle in Chapter 6 and Chapter 8 where we will derive a formula for the cardinality of the union of n sets, where n is a positive integer.

Venn Diagram for A, B, A ∩ B, A ∪ B



Example

How many integers between 1 and 100 can be divided by 2 or 3?

Sol:

A= $\{x | 1 \leq x \leq 100 \ \wedge 2 | x\}$

B= $\{x | 1 \leq x \leq 100 \ \wedge 3 | x\}$

$|A \cup B| = |A| + |B| - |A \cap B|$

$= \lfloor 100/2 \rfloor + \lfloor 100/3 \rfloor - \left\lfloor \frac{100}{6} \right\rfloor$ =50 + 33 – 16 = 67

Inclusion-Exclusion

$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$

$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$

$| A_1 \cup A_2 \cup \cdots \cup A_n | = ?$

$| \mathrm{A} \cup B \cup C | = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$$\left| \bigcup_{i=1}^{n} A_i \right| = |A_1| + |A_2| + \cdots + |A_N| - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \cdots - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i \ne j \le n} |A_i \cap A_j| + \sum_{\substack{1 \le i,j,k \le n \\ i \ne j \ne k}} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{n+1} \left| \bigcap_{i=1}^{n} A_i \right|$$

Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: {1,2,3,4,5,6,7,8}

2. $A \cap B$

   Solution: {4,5}

   在這裡鍵入方程式。

3. $\bar{A}$

   Solution: {0,6,7,8,9,10}

4. $\bar{B}$

   Solution: {0,1,2,3,9,10}

5. $A - B$

   Solution: {1,2,3}

6. $B - A$

   Solution: {6,7,8}

Set Identities [1]

Identity laws

$$A \cup \varnothing = A \qquad A \cap \boldsymbol{U} = A$$

Domination laws

$$A \cup \boldsymbol{U} = \boldsymbol{U} \qquad A \cap \varnothing = \varnothing$$

Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

Complementation law

$$\left(\overline{\overline{A}}\right) = A$$

Set Identities 2

Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Set Identities 3

De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \varnothing$$

Membership Table

- **Example**: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

A	B	$A \cap B$	$\overline{A \cap B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Proof of Second De Morgan Law

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

$$1)\ \overline{A \cap B} \subseteq \overline{A} \cup \overline{B} \qquad \text{and}$$

$$2)\ \overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$$

$$\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} = \{x \mid \neg x \in A \cap B\} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} \\
&= \{x \mid \neg x \in A \vee \neg x \in B\} \\
&= \{x \mid x \notin A \vee x \notin B\} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} \\
&= \overline{A} \cup \overline{B}
\end{aligned}$$

Proof of Second De Morgan Law

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$\left(x \in \overline{A}\right) \vee \left(x \in \overline{B}\right)$	by defn. of union
$\left(x \notin A\right) \vee \left(x \in \overline{B}\right)$	defn. of complement
$\neg\left(x \in A\right) \vee \neg\left(x \in B\right)$	defn. of negation
$\neg\left(\left(x \in A\right) \wedge \neg\left(x \in B\right)\right)$	1st De Morgan law for Prop Logic
$\neg\left(x \in A \cap B\right)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

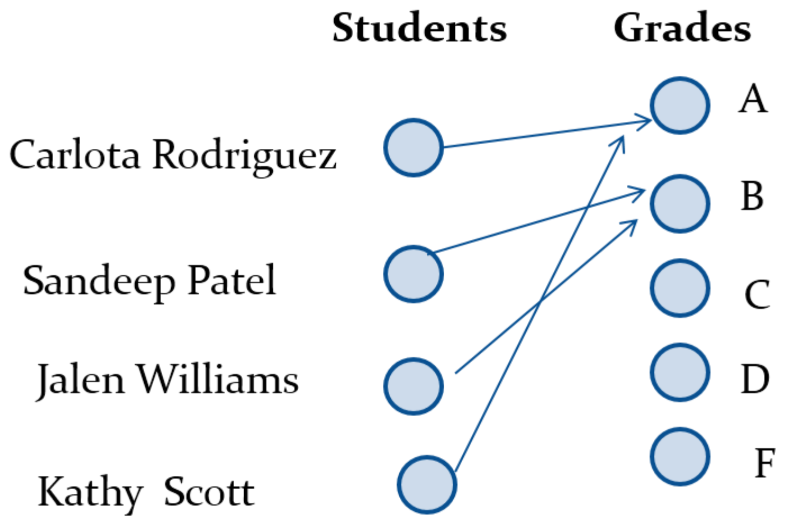
Set-Builder Notation: Second De Morgan Law

$\overline{A \cap B}$	$= x \in \overline{A \cap B}$	by defn. of complement
	$= \left\{ x \mid \neg\left(x \in \left(A \cap B \right) \right) \right\}$	by defn. of does not belong symbol
	$= \left\{ x \mid \neg\left(x \in A \wedge x \in B \right\} \right.$	by defn. of intersection
	$= \left\{ x \mid \neg\left(x \in A \right) \vee \neg\left(x \in B \right) \right\}$	by 1st De Morgan law for Prop Logic
	$= \left\{ x \mid x \notin A \vee x \notin B \right\}$	by defn. of not belong symbol
	$= \left\{ x \mid x \in \overline{A} \vee x \in \overline{B} \right\}$	by defn. of complement
	$= \left\{ x \mid x \in \overline{A} \cup \overline{B} \right\}$	by defn. of union
	$= \overline{A} \cup \overline{B}$	by meaning of notation

Functions[1]

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \to B$ is an assignment of **each element of A** to **exactly one element** of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.



Functions $_2$

A function f: $A \rightarrow B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.

Specifically, a function *f* from *A* to *B* contains **one, and only one ordered pair (*a, b*)** for **every element** $a \in A$.

$$\forall x \left[x \in A \rightarrow \exists y \left[y \in B \wedge (x, y) \in f \right] \right]$$

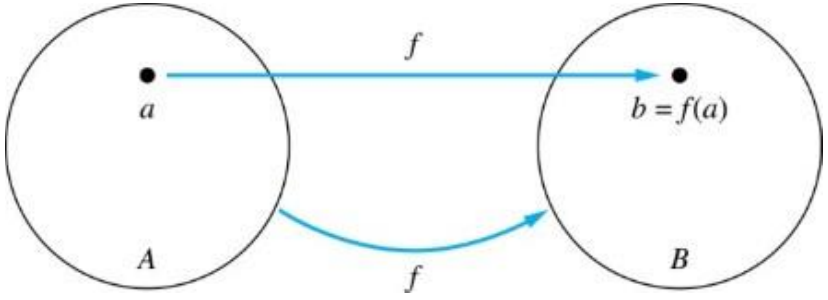
and

$$\forall x, y_1, y_2 \left[\left[(x, y_1) \in f \wedge (x, y_2) \in f \right] \rightarrow y_1 = y_2 \right]$$

Functions [3]

Given a function f: $A \rightarrow B$:

- We say f *maps* A to B *or* f is a *mapping* from A to B.
- A is called the ***domain*** of f.
- B is called the ***codomain*** of f.
- If $f(a) = b$,
  - then b is called the ***image*** of a under f.
  - a is called a *preimage* of b.
- The range of f is the set of all images of points in **A** under f. We denote it by $f(\boldsymbol{A})$.
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.



Jump to long description

Representing Functions

Functions may be specified in different ways:

- An explicit statement of the assignment. Students and grades example.
- A formula.

$$f(x) = x + 1$$

- A computer program.
  - A Java program that when given an integer *n*, produces the *n*th Fibonacci Number (covered in the next section and also in Chapter 5).

Questions

f(a) = ? z

The image of d is ? z

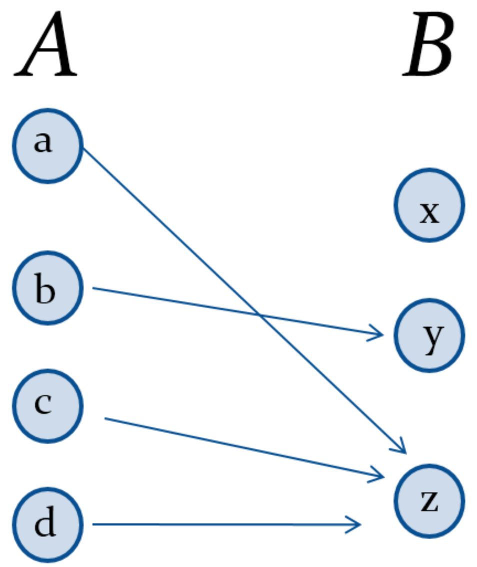
The domain of f is ? A

The codomain of f is ? B

The preimage of y is ? b

$f(A)$ = ? {y,z}

The preimage(s) of z is (are) ? {a,c,d}



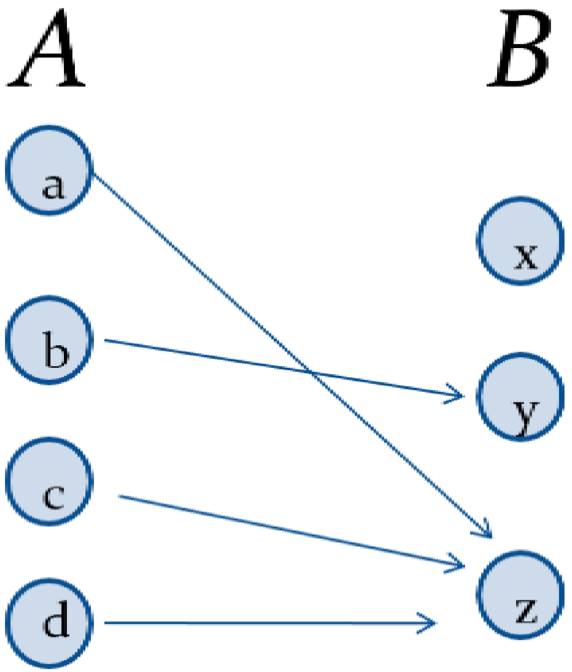
Question on Functions and Sets

If $f : A \rightarrow B$ and S is a subset of A, then

$$f(S) = \{ f(s) \mid s \in S \}$$

f {a,b,c,} is ? {y,z}

f {c,d} is ? {z}



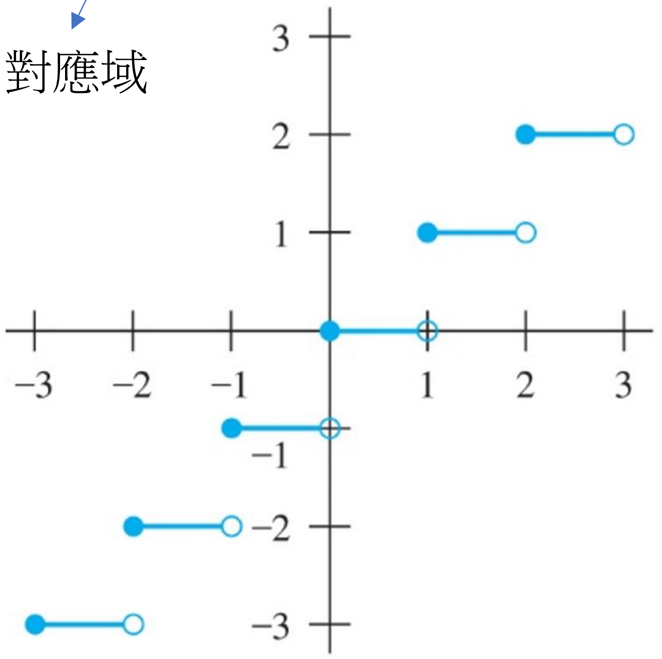
Example

$f(x) = \lfloor x \rfloor$ the floor function

$$f(1.5) = 1$$
$$f(0) = 0$$
$$f(-1.5) = -2$$

$f: R \to R$ 值域：Z

定義域 對應域



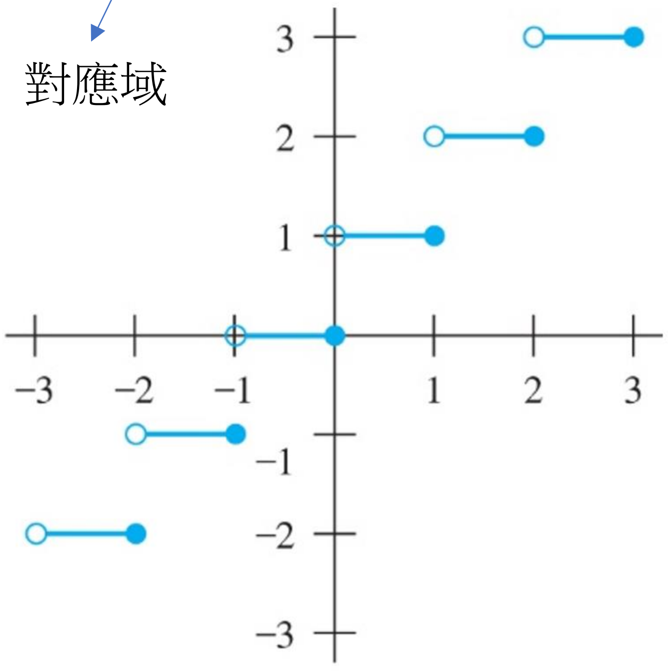
(a) $y = \lfloor x \rfloor$

Example

$f(x) = \lceil x \rceil$ the ceiling function $f: R \to R$ 值域：Z

定義域 對應域

$$f(1.5) = 2$$
$$f(0) = 0$$
$$f(-1.5) = -1$$



(b) $y = \lceil x \rceil$

$$x - 1 < \lfloor x \rfloor \le x \equiv \lfloor x \rfloor \le x < \lfloor x \rfloor + 1$$

$$\frac{x \le \lceil x \rceil < x + 1 \equiv \lceil x \rceil - 1 \le x < \lceil x \rceil}{x = n + \varepsilon \; for \; some \; n \in Z, 0 \le \varepsilon < 1}$$

$$Prove: \lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$$

Pf: Assume $x = n + \varepsilon, where\ n \in Z, 0 \le \varepsilon < 1$

Case1: $0 \le \varepsilon \le 0.5$

$$LHS = \lfloor 2(n+\varepsilon) \rfloor = \lfloor 2n + 2\varepsilon \rfloor = 2n$$

$$RHS = \lfloor (n+\varepsilon) \rfloor + \left\lfloor n + \varepsilon + \frac{1}{2} \right\rfloor = 2n$$

Case2: $0.5 \le \varepsilon < 1$

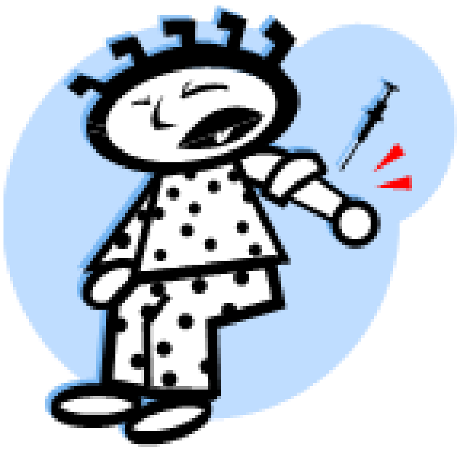
$$LHS = \lfloor 2(n+\varepsilon) \rfloor = \lfloor (2n+1) + (2\varepsilon - 1) \rfloor = 2n + 1$$

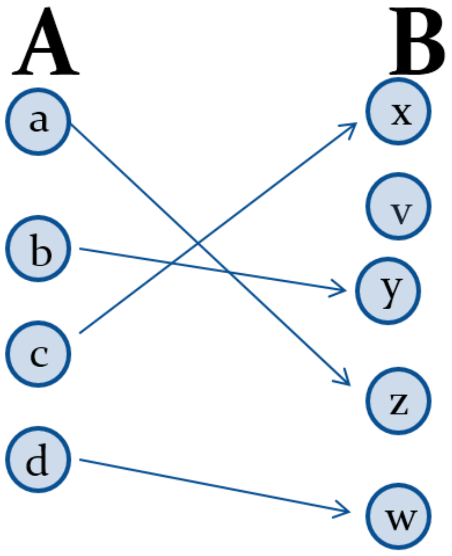
$$RHS = n + n + 1 = 2n + 1$$

$$\begin{cases} 0.5 \le \varepsilon < 1 \\ 1 \le 2\varepsilon < 2 \\ 0 \le 2\varepsilon - 1 < 1 \end{cases}$$

Injections （嵌射）

Definition: A function f is said to be *one-to-one* , or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.

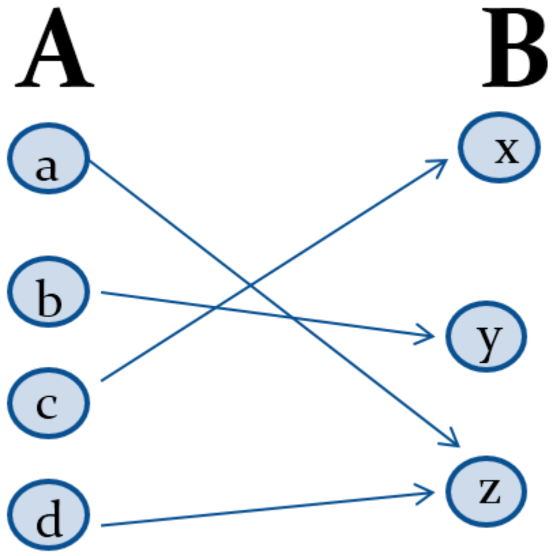




Surjections(蓋射)

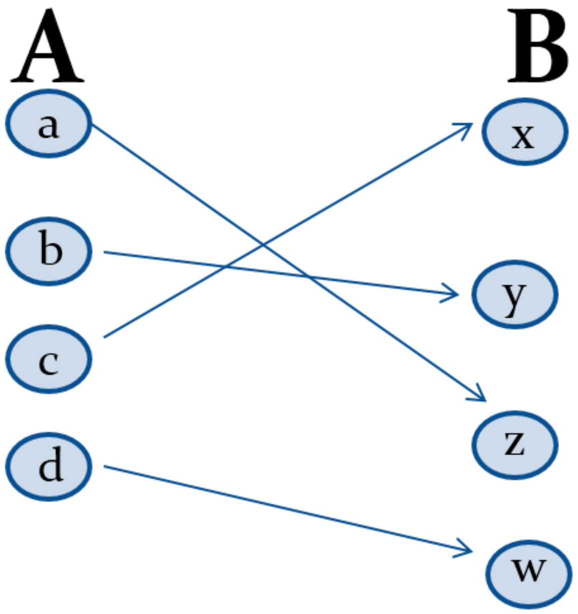
Definition: A function *f* from *A* to *B* is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

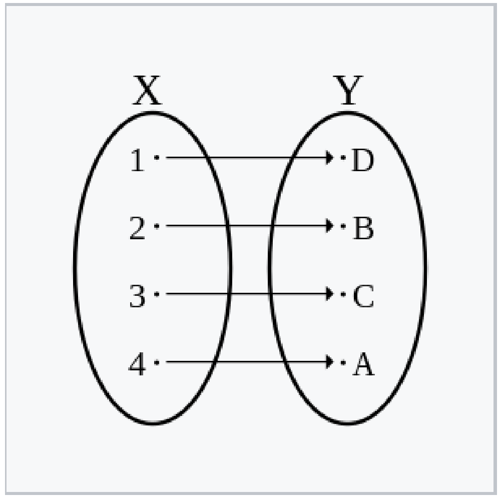
A function *f* is called a *surjection* if it is *onto*.



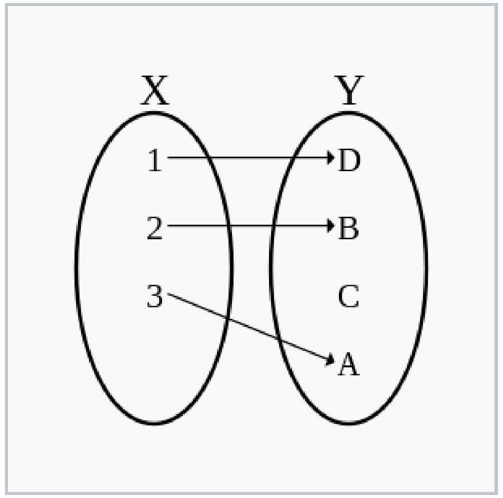
Bijections （對射）

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).

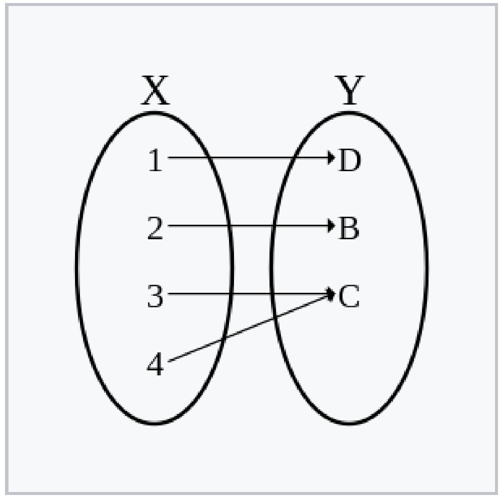




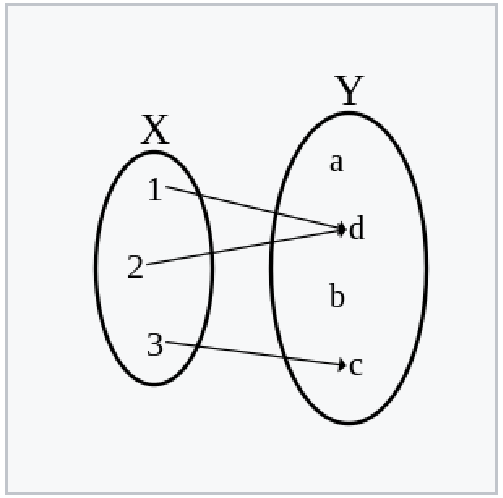
對射（嵌射與蓋射）



嵌射但非蓋射



蓋射但非嵌射



非蓋射非嵌射

Showing that f is one-to-one or onto [1]

Suppose that $f : A \rightarrow B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Showing that f is one-to-one or onto 2

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

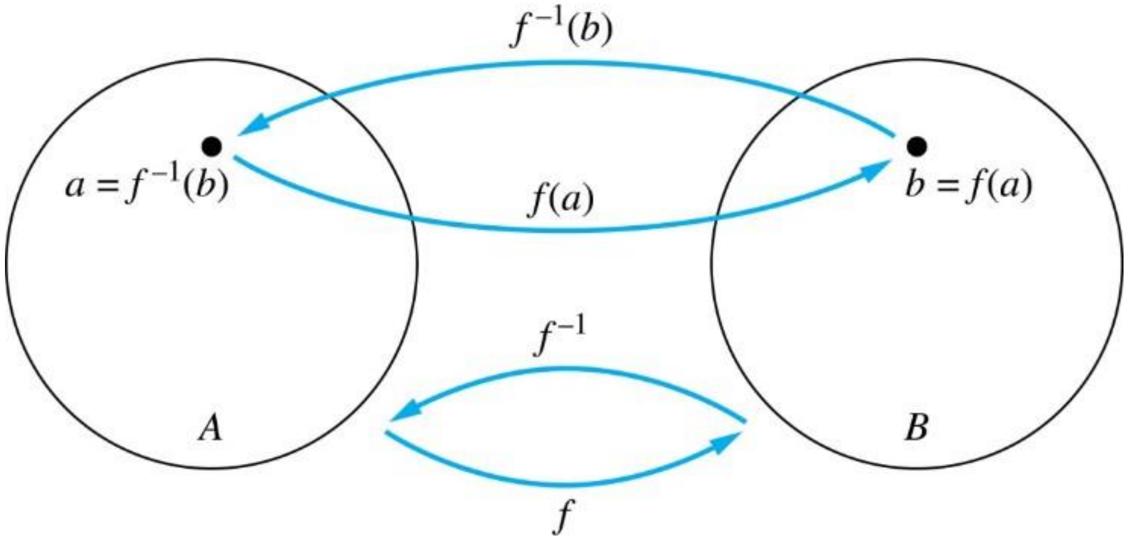
Example 2: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

Inverse Functions [1]

Definition: Let *f* be a bijection from *A* to *B*. Then the *inverse* of *f*, denoted f^{-1}, is the function from *B* to *A* defined as $f^{-1}(y) = x$ iff $f(x) = y$

No inverse exists unless *f* is a bijection. Why?



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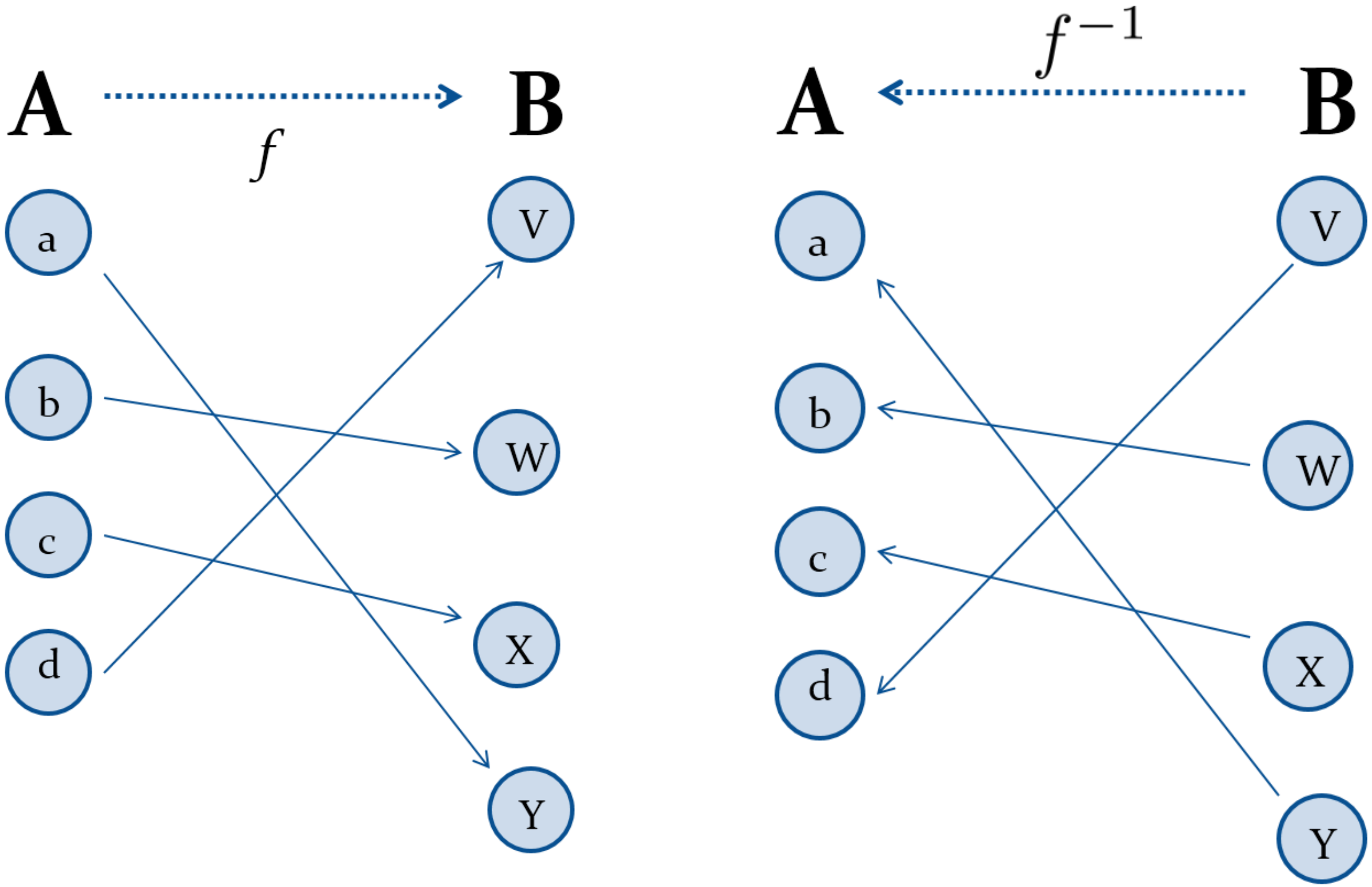
Injection, Surjection, and Bijection

Definition

Let f: $A \rightarrow B$ be a function.

- $f\ is\ called\ one-to-one, or\ an\ injection, iff$
$\forall a, b \in A(f(a) = f(b) \rightarrow a = b)$ is true.
- $f\ is\ called\ onto, or\ surjection, iff$
$\forall b \in B(\exists a \in A(f(a) = b))$ is true.
- $f\ is\ called\ one-to-one\ correspondence, or\ a\ bijection, if\ it\ is\ both\ one-to-one\ and\ onto.$

Inverse Functions



Questions

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions

Example 2: Let f*:* **Z** $\rightarrow$ **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

Questions

Example 3: Let f: R $\rightarrow$ R be such that $f(x) = x^2$

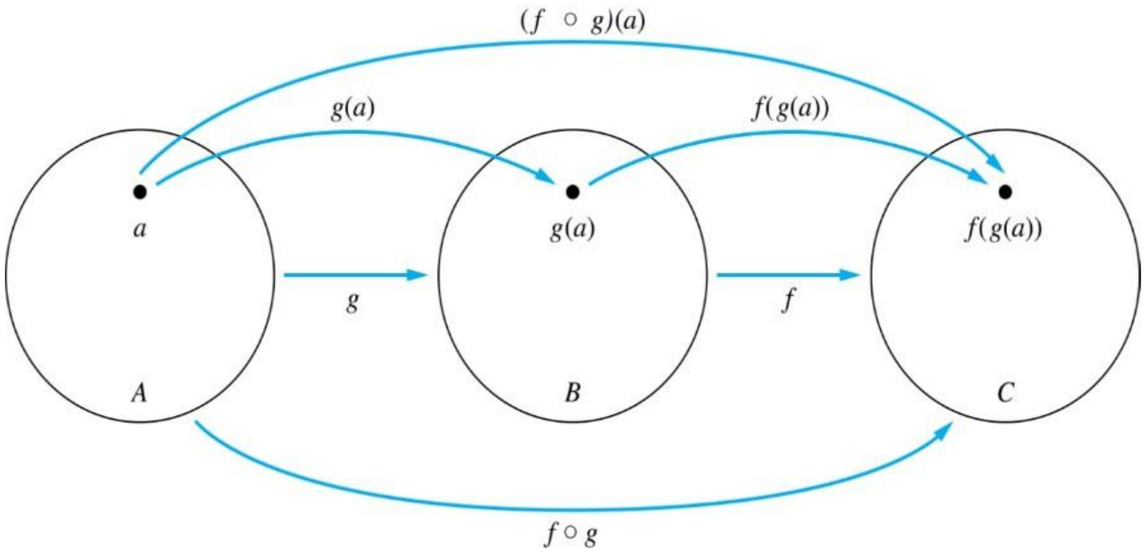
Is f invertible, and if so, what is its inverse?

Solution: The function f is not invertible because it is not one-to-one.

Composition

Definition: Let *f*: *B*→*C*, *g*: *A*→*B*. The *composition of f with g*, denoted $f \circ g$ is the function from *A* to *C* defined by

$$f \circ g(x) = f(g(x))$$



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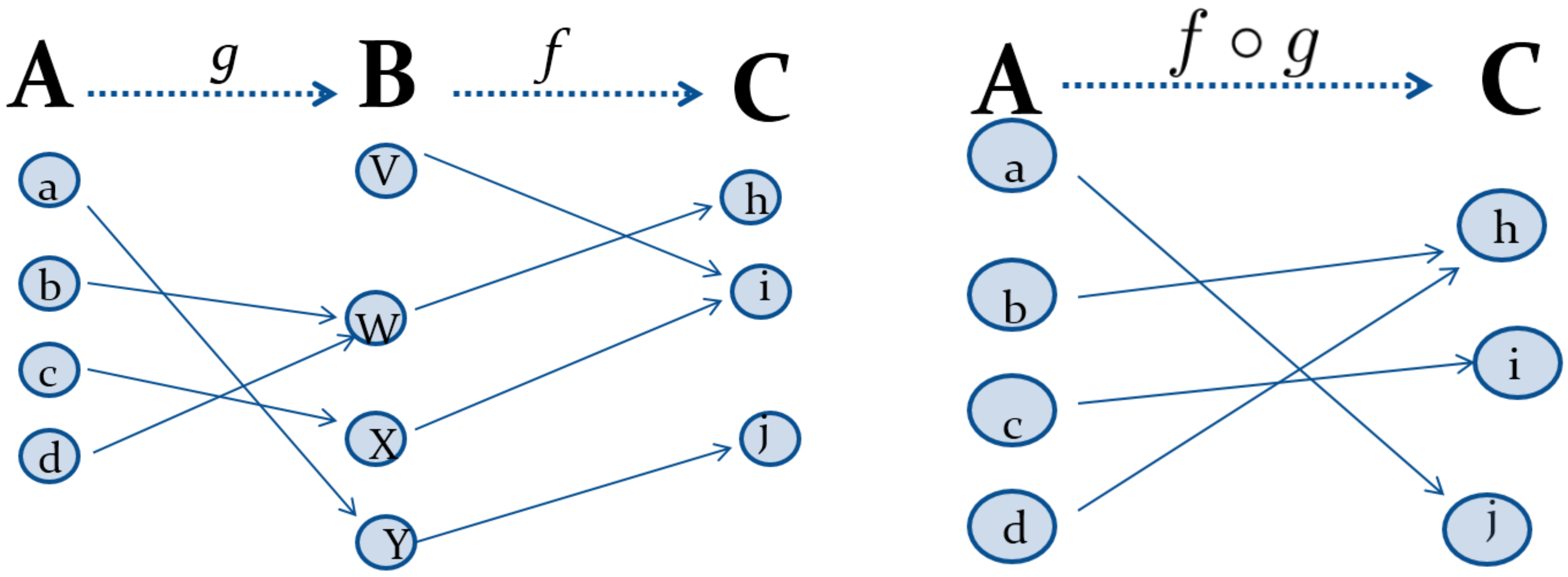
Composition



Composition

Example 1: If

$$f(x) = x^2 \text{ and } g(x) = 2x + 1,$$

then

$$f(g(x)) = (2x+1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition Questions

Example 2: Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f.

Solution: The composition $f \circ g$ is defined by

$$f \circ g(a) = f(g(a)) = f(b) = 2.$$

$$f \circ g(b) = f(g(b)) = f(c) = 1.$$

$$f \circ g(c) = f(g(c)) = f(a) = 3.$$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition Questions

Example 2: Let f and g be functions from the set of integers to the set of integers defined by

$$f(x) = 2x + 3 \quad \text{and} \quad g(x) = 3x + 2.$$

What is the composition of *f* and *g*, and also the composition of *g* and *f* ?

Solution:

$$f \circ g(x) = f(g(x)) = f(3x+2) = 2(3x+2) + 3 = 6x + 7$$

$$g \circ f(x) = g(f(x)) = g(3x+2) = 3(3x+2) + 2 = 6x + 11$$

Sequences

Definition: A *sequence* is a function from a subset of the integers (usually either the set {0, 1, 2, 3, 4, ...} or {1, 2, 3, 4, ...}) to a set *S*.

The notation a_n is used to denote the image of the integer *n*. We can think of a_n as the equivalent of *f(n)* where *f* is a function from {0,1,2,.....} to *S*. We call a_n a *term* of the sequence.

Example: Consider the sequence $\{a_n\}$ where

$$a_n = \frac{1}{n} \qquad \{a_n\} = \{a_1,\ a_2,\ a_3 ...\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$$

Summations

Sum of the terms $a_m, a_m + 1, \ldots, a_n$

from the sequence $\{a_n\}$

The notation:

$$\sum_{j=m}^{n} a_j \qquad \sum\nolimits_{j=m}^{n} a_j \qquad \sum\nolimits_{m \le j \le n} a_j$$

represents

$$a_m + a_{m+1} + \cdots + a_n$$

The variable *j* is called the *index of summation*. It runs through all the integers starting with its *lower limit* *m* and ending with its *upper limit n*.

Summations

More generally for a set S:

$$\sum_{j \in s} a_j$$

Examples:

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_0^n r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_1^\infty \frac{1}{i}$$

If $S = \{2, 5, 7, 10\}$ then $\sum_{j \in s} a_j = a_2 + a_5 + a_7 + a_{10}$

Arithmetic Progression

Definition: A arithmetic progression is a sequence of the form: $a,\ a+d,\ a+2d, \ldots,\ a+nd, \ldots$

where the *initial term a* and the *common difference d* are real numbers.

$$S_n = \sum_{i=0}^{n} (a + id)$$

$$= \frac{1}{2}(a + (a + nd))(n + 1)$$

Examples :

1. Let $a = -1$ and $d = 4$:

$$\{s_n\} = \{s_0,\ s_1,\ s_2,\ s_3,\ s_4, \ldots\} = \{\text{-1, 3, 7, 11, 15} \quad \ldots\}$$

2. Let $a = 7$ and $d = -3$:

$$\{t_n\} = \{t_0,\ t_1,\ t_2,\ t_3,\ t_4, \ldots\} = \{7,\ 4,\ 1, -2,\ -5, \ldots\}$$

3. Let $a = 1$ and $d = 2$:

$$\{u_n\} = \{u_0,\ u_1,\ u_2,\ u_3,\ u_4, \ldots\} = \{1,\ 3,\ 5,\ 7,\ 9, \ldots\}$$

Geometric Progression

Definition: A *geometric progression* is a sequence of the form: $a, ar^2, \ldots, ar^n, \ldots$

where the *initial term a* and the *common ratio r* are real numbers.

$$S_n = \sum_{i=0}^{n} ar^i = \frac{a(r^{n+1} - 1)}{r - 1}$$

Examples :

1. Let $a = 1$ and $r = -1$. Then :

$$\{b_n\} = \{b_0,\ b_1,\ b_2,\ b_3,\ b_4, \ldots\} = \{1,\ -1,\ 1,\ -1,\ 1, \ldots\}$$

2. Let $a = 2$ and $r = 5$. Then :

$$\{c_n\} = \{c_0,\ c_1,\ c_2,\ c_3,\ c_4, \ldots\} = \{2,\ 10,\ 50,\ 250,\ 1250, \ldots\}$$

3. Let $a = 6$ and $r = 1/3$. Then :

$$\{d_n\} = \{d_0,\ d_1,\ d_2,\ d_3,\ d_4, \ldots\} = \left\{6,\ 2,\ \frac{2}{3},\ \frac{2}{9},\ \frac{2}{27}, \ldots\right\}$$

More Summation Notations

- An infinite series

$$\sum_{i=j}^{\infty} a_i = a_j + a_{j+1} + \cdots$$

- Summation over all members of a set $X = \{x_1, x_2, \dots\}$

$$\sum_{x \in X} f(x) = f(x_1) + f(x_2) + \cdots$$

- By the help of a predicate logic, e.g. $X = \{x | P(x)\}$

$$\sum_{P(x)} f(x) = f(x_1) + f(x_2) + \cdots$$

Summation Examples

1. An infinite series

$$\sum_{i=0}^{\infty} 2^{-i} = 2^0 + 2^{-1} + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$$

2. Using a predicate logic

$$\sum_{(x\ is\ prime) \wedge (x<10)} x^2 = 2^2 + 3^2 + 5^2 + 7^2 =$$

4+9+25+49 = 87

Summation Manipulations

$$\sum_{n=1}^{10} 2n = (2*1+2*2+2*3+\cdots+2*10) = 2(1+2+3+\cdots+10)$$

$$= 2\sum_{n=1}^{10} n$$

$$a_n = 2n, b_n = 2^n$$

$$\sum_{n=1}^{10}(2n+2^n) = [(2*1+2^1)+(2*2+2^2)+\cdots+(2*10+2^{10})]$$

$$= (2*1+2*2+2*3+\cdots 2*10)+(2^1+2^2+2^3+\cdots+2^{10})$$

$$= \sum_{n=1}^{10} 2n + \sum_{n=1}^{10} 2^n$$

Summation Manipulations

$\sum_{i=1}^{10} a_n = a_1 + a_2 + a_3 + \cdots + a_{10} = a_{1+2-2} + a_{2+2-2} + \cdots + a_{10+2-2} = a_{3-2} + a_{4-2} + \cdots + a_{12-2} = \sum_{i=3}^{12} a_{i-2}$

Summation Manipulations1

- Distributive law

$$\sum_n c f_n = c \sum_n f_n$$

- Application of commutativity

$$\sum_n f_n + g_n = \left(\sum_n f_n\right) + (\sum_n g_n)$$

- Index shifting

$$\sum_{n=i}^{j} f_n = \sum_{n=i+k}^{j+k} f_{n-k}$$

Summation Manipulations2

- Series splitting

$$\sum_{n=i}^{k} f_n = \left(\sum_{n=i}^{j} f_n\right) + \left(\sum_{n=j+1}^{k} f_n\right), \text{ if } i < j \leq k.$$

- Order reversal

$$\sum_{n=0}^{k} f_n = \sum_{n=0}^{k} f_{k-n}$$

- Grouping

$$\sum_{n=0}^{2k} f_n = \sum_{n=0}^{k} f_{2n} + \sum_{n=0}^{k-1} f_{2n+1} = \sum_{n=0}^{k} f_n + \sum_{n=k+1}^{2k} f_n$$

Example

Double summations arise in many contexts (as in the analysis of nested loops in computer programs). An example of a double summation is

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij.$$

To evaluate the double sum, first expand the inner summation and then continue by computing the outer summation:

$$\begin{aligned}\sum_{i=1}^{4}\sum_{j=1}^{3} ij &= \sum_{i=1}^{4}(i + 2i + 3i)\\ &= \sum_{i=1}^{4} 6i\\ &= 6 + 12 + 18 + 24 = 60.\end{aligned}$$

We can also use summation notation to add all values of a function, or terms of an indexed set, where the index of summation runs over all values in a set. That is, we write

$$\sum_{s \in S} f(s)$$

to represent the sum of the values $f(s)$, for all members s of S.

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

EXAMPLE 23 Find $\sum_{k=50}^{100} k^2$.

Solution: First note that because $\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2$, we have

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2.$$

Using the formula $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$ from Table 2 (and proved in Exercise 38), we see that

$$\sum_{k=50}^{100} k^2 = \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6} = 338{,}350 - 40{,}425 = 297{,}925.$$

◀

Cardinality [1]

Definition: The *cardinality* of a set A is equal to the cardinality of a set B, denoted

$$|A| = |\mathrm{B}|,$$

if and only if there is a one-to-one correspondence (*i.e.*, a bijection) from A to B.

If there is a one-to-one function (*i.e.*, an injection) from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.

When $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and write $|A| < |B|$.

Cardinality $_2$

Definition: A set that is either finite or has the same cardinality as the set of positive integers (**Z**$^+$) is called *countable*. A set that is not countable is *uncountable*.

The set of real numbers **R** is an uncountable set.

When an infinite set is countable (*countably infinite*) its cardinality is $\aleph_0$ (where $\aleph$ is aleph, the 1st letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

Showing that a Set is Countable

An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).

The reason for this is that a one-to-one correspondence *f* from the set of positive integers to a set *S* can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$ where $a_1 = f(1)$, $a_2 = f(2), \ldots, a_n = f(n), \ldots$

Showing that a Set is Countable 1

Example 1: Show that the set of positive even integers E is countable set.

Solution: Let $f(x) = 2x.$

1	2	3	4	5	6
b	b	b	b	b	b
2	4	6	8	10	12

Then f is a bijection from **N** to E since f is both one-to-one and onto. To show that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$. To see that it is onto, suppose that t is an even positive integer. Then $t = 2k$ for some positive integer k and $f(k) = t$.

Showing that a Set is Countable 2

Example 2: Show that the set of integers **Z** is countable.

Solution: Can list in a sequence:

0, 1, − 1, 2, − 2, 3, − 3 ,..........

Or can define a bijection from **N** to **Z**:

- When n is even: $f(n) = n/2$
- When n is odd: $f(n) = -(n-1)/2$

n (natural number)	$f(n)$ (integer)
1	0
2	1
3	-1
4	2
5	-2
6	3
7	-3
8	4
9	-4
...	...