- 1 Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).

a) $\neg\forall x\forall y P(x, y)$

b) $\neg\forall y\exists x P(x, y)$

c) $\neg\forall y\forall x(P(x, y) \vee Q(x, y))$

d) $\neg(\exists x\exists y\neg P(x, y) \wedge \forall x\forall y Q(x, y))$

e) $\neg\forall x(\exists y\forall z P(x, y, z) \wedge \exists z\forall y P(x, y, z))$

a) $\exists x\exists y\neg P(x, y)$
b) $\exists y\forall x\neg P(x, y)$
c) $\exists y\exists x(\neg P(x, y) \wedge \neg Q(x, y))$
d) $(\forall x\forall y P(x, y)) \vee (\exists x\exists y\neg Q(x, y))$
e) $\exists x(\forall y\exists z\neg P(x, y, z) \vee \forall z\exists y\neg P(x, y, z))$

- 2 Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.

a) $\exists x\forall y(xy = y)$

b) $\forall x\forall y(((x < 0) \wedge (y < 0)) \rightarrow (xy > 0))$

c) $\exists x\exists y((x^2 > y) \wedge (x < y))$

d) $\forall x\forall y\exists z(x + y = z)$

a)There is a multiplicative identity for the real numbers.
b) The product of two negative real numbers is always a positive real number.
c) There exist real numbers *x* and *y* such that x_2 exceeds *y* but *x* is less than *y*.
d) The real numbers are closed under the operation of addition.

- 3 For each of these collections of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

a) "If I take the day off, it either rains or snows." "I took Tuesday off or I took Thursday off." "It was sunny on Tuesday." "It did not snow on Thursday."

b) "If I eat spicy foods, then I have strange dreams." "I have strange dreams if there is thunder while I sleep." "I did not have strange dreams."

c) "I am either clever or lucky." "I am not lucky." "If I am lucky, then I will win the lottery."

d) "Every computer science major has a personal computer." "Ralph does not have a personal computer." "Ann has a personal computer."

e) "What is good for corporations is good for the United States." "What is good for the United States is good for you." "What is good for corporations is for you to buy lots of stuff."

f) "All rodents gnaw their food." "Mice are rodents." "Rabbits do not gnaw their food." "Bats are not rodents."

a) Valid conclusions are "I did not take Tuesday off," "I took Thursday off," and "It rained on Thursday."

b) "I did not eat spicy foods and it did not thunder" is a valid conclusion.

c) "I am clever" is a valid conclusion.

d) "Ralph is not a CS major" is a valid conclusion.

e) "That you buy lots of stuff is good for the U.S. and is good for you" is a valid conclusion.

f) "Mice gnaw their food" and "Rabbits are not rodents" are valid conclusions.

- $(\forall d)(T(d) \rightarrow (R(d) \vee s(d))$---(1)
- $T(d)$ = "I take take d off"
- $R(d)$ = "It rains on d"
- s(d) = "It snows on d"
- S(d) = "it was sunny on d"

$T(\text{Tue}) \vee T(\text{Thu})$---(2)
$S(\text{Tuesday})$---(3)
$\neg s(\text{Thursday})$---(4)
$(\forall d)(\neg R(d) \vee \neg S(d))$---(5)
$(\forall d)(\neg s(d) \vee \neg S(d))$---(6)
$\neg R(\text{Tue}) \vee \neg S(\text{Tue})$---(7) {(5) Universal instantiation}
$\neg R(\text{Tue})$---(8) {(3) & (7) Disjunctive Syll…}
$\neg s(\text{Tue}) \vee \neg S(\text{Tue})$---(9) {(6) Universal insta…}
$\neg s(\text{Tue})$---(10) {(3) & (9) Disjunctive Syll…}
$\neg R(\text{Tue}) \wedge \neg s(\text{Tue})$ ---(11) {(8) & (10) conjunction}
$\neg(R(\text{Tue}) \vee s(\text{Tue}))$---(12) {(11) De Morgan's law}
$T(\text{Tue}) \rightarrow (R(\text{Tue}) \vee s(\text{Tue}))$---(13) {(1) Universal insta…}
$\neg T(\text{Tue})$---(14) {(13) Modus Tollens}
$T(\text{Thu})$---(15) {(14) & (2) Disjunctive Syll…}

$T(\text{Thu}) \rightarrow (R(\text{Thu}) \vee s(\text{Thu}))$ ---(16) {(1) Univeral instan…}
$R(\text{Thu}) \vee s(\text{Thu})$---(17) {(15) & (16) Modus Ponens}
$R(\text{Thu})$---(18). {(17) & (4) Disjunctive …}

TABLE 2 **Rules of Inference for Quantified Statements.**

Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

Rules	Tautology	Name
$p \rightarrow q$ p $\therefore q$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus Ponens
$p \rightarrow q$ $\neg q$ $\therefore \neg p$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus Tollens
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical Syllogism
$p \vee q$ $\neg p$ $\therefore q$	$(\neg p \wedge (p \vee q)) \rightarrow q$	Disjunctive Syllogism
p $\therefore p \vee q$	$p \rightarrow (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \rightarrow p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$\neg p \vee r$ $p \vee q$ $\therefore q \vee r$	$((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$	Resolution

- 4 Identify the error or errors in this argument that supposedly shows that if $\exists x P(x) \wedge \exists x Q(x)$ is true then $\exists x(P(x) \wedge Q(x))$ is true.

1.	$\exists x P(x) \vee \exists x Q(x)$	Premise
2.	$\exists x P(x)$	Simplification from (1)
3.	$P(c)$	Existential instantiation from (2)
4.	$\exists x Q(x)$	Simplification from (1)
5.	$Q(c)$	Existential instantiation from (4)
6.	$P(c) \wedge Q(c)$	Conjunction from (3) and (5)
7.	$\exists x(P(x) \wedge Q(x))$	Existential generalization

The error occurs in step (5), because we cannot assume, as is being done here, that the *c* that makes *P* true is the same as the *c* that makes *Q* true.

- 5 Use resolution to show that the compound proposition $(p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$ is not satisfiable.

Assume that this proposition is satisfiable. Using resolution on the first two clauses enables us to conclude $q \vee q$; in other words, we know that q has to be true. Using resolution on the last two clauses enables us to conclude $\neg q \vee \neg q$; in other words, we know that $\neg q$ has to be true. This is a contradiction. So this proposition is not satisfiable.

- 6 Use a direct proof to show that the sum of two odd integers is even.

Let $n=2k+1$ and $m = 2l + 1$ be odd integers. Then $n+m=2(k+l+1)$ is even.

- 7 Show that if n is an integer and $n^3 + 5$ is odd, then n is even using

a) a proof by contraposition.
b) a proof by contradiction.

a) Assume that n is odd, so $n=2k+1$ for some integer k. Then $n^3+5 = 2(4k^3+6k^2+3k+3)$. Because n^3+5 is two times some integer, it is even.

b) Suppose that n^3+5 is odd and n is odd. Because n is odd and the product of two odd numbers is odd, it follows that n^2 is odd and then that n^3 is odd. But then $5 = (n^3+5)-n^3$ would have to be even because it is the difference of two odd numbers. Therefore, the supposition that n^3+5 and n were both odd is wrong.

- 8 Show that the propositions p_1, p_2, p_3, p_4, and p_5 can be shown to be equivalent by proving that the conditional statements $p_1 \to p_4$, $p_3 \to p_1$, $p_4 \to p_2$, $p_2 \to p_5$, and $p_5 \to p_3$ are true.

Suppose that $p_1 \to p_4 \to p_2 \to p_5 \to p_3 \to p_1$.
To prove that one of these propositions implies any of the others, just use hypothetical syllogism repeatedly.

- 9 Prove that $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$.

$1_2+1=2 \geq 2=2_1$
$2_2+1 = 5 \geq 4=2_2$
$3_2+1 = 10\geq8=2_3$
$4_2+1=17\geq16=2_4$

- 10 Let $S = x_1y_1 + x_2y_2 + \cdots + x_ny_n$, where $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are orderings of two different sequences of positive real numbers, each containing n elements.

  a) Show that S takes its maximum value over all orderings of the two sequences when both sequences are sorted (so that the elements in each sequence are in nondecreasing order).

  b) Show that S takes its minimum value over all orderings of the two sequences when one sequence is sorted into nondecreasing order and the other is sorted into nonincreasing order.

a) Without loss of generality, we can assume that the *x* sequence is already sorted into nondecreasing order, because we can relabel the indices. There are only a finite number of possible orderings for the *y* sequence, so if we can show that we can increase the sum (or at least keep it the same) whenever we find *yi* and *yj* that are out of order (i.e., *i* < *j* but *yi* > *yj*) by switching them, then we will have shown that the sum is largest when the *y* sequence is in nondecreasing order. Indeed, if we perform the swap, then we have added *xiyj* + *xjyi* to the sum and subtracted *xiyi* + *xjyj*. The net effect is to have added *xiyj* +*xjyi* −*xiyi* −*xjyj* = (*xj* − *xi*)(*yi* − *yj*), which is nonnegative by our ordering assumption

b) Similar to a)