Computer Systems

Lecture 4

Boolean Algebra and Arithmetic

Dr José Cano, Dr Lito Michala
School of Computing Science
University of Glasgow

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Outline

- Boolean algebra
- Addition and subtraction
  - Half adder
  - Full adder
  - Ripple carry adder
  - Subtraction
- Circuit simulation

Boolean algebra

- Elementary algebra is about properties of operations on numbers: + - x ÷
  - **Example**: $x + y = y + x$
  - Elementary algebra helps you solve many problems involving numbers

- There are many different algebras, for different kinds of object

- Boolean algebra is about the operations on truth values: 0 and 1
  - **Example**: $x \vee y = y \wedge x$
  - Boolean algebra helps you solve many problems involving truth values

Why learn about Boolean algebra?

- We will use it just a little, not very much

- But you will encounter it again and again in computer science, so a brief introduction helps!

- Boolean algebra makes it easier to understand tricky logic
  - In digital circuits
  - In programming with conditionals

- It's also important in the history and philosophy of mathematics

Constants and operations of Boolean algebra

- There are two values: 0, 1
- We can use variables to stand for a value: x, y, z, …
- There are three operators
- Each operator corresponds exactly to a logic gate
- Boolean algebra describes the values of signals in a digital circuit
- There is also another operator, logical implication, but it can be expressed using the others

Algebra	Name	Circuit
$\neg x$	not	inv x
$x \vee y$	or	or2 x y
$x \wedge y$	and	and2 x y

Definition of the operations

x	$\neg x$
0	1
1	0

x	y	$x \wedge y$	$x \vee y$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

- These are the same as the definitions of the logic gates
  - Except we don't have a Boolean operator for the "exclusive or" gate

Laws of Boolean algebra

- The basic rules describing logical operations
- Called laws, axioms, rules (what you call them isn't important)
- You can use these to reason about circuits
  - Simplify expressions
  - Calculate signal values
  - Prove that two circuits calculate the same value
  - Derive circuits to meet a specification
- For this course: don't memorise the laws (shown on the next slides), but do know the truth tables for the logic gates

Operations with constants

- You can verify these using the truth tables for the operations

$$
\begin{aligned}
x \wedge 0 &= 0 \\
x \wedge 1 &= x \\
x \vee 0 &= x \\
x \vee 1 &= 1
\end{aligned}
$$

- These help to work out the values of signals in a circuit

Idempotence

- In general, an operation is idempotent if doing it several times is the same as doing it once
  - On some web sites, registering twice is the same as registering once: here, registration is idempotent
  - But on some other web sites, if you register three times it gives you three separate accounts: here, registration is not idempotent

- Here's the mathematical denition

$$\begin{aligned} x \vee x &= x \\ x \wedge x &= x \end{aligned}$$

- It's straightforward to check these with truth tables
  - Check that they hold for both cases: when x=0 and when x=1

Commutativity

- Very important!

- These say that you can swap around the order of inputs to an and-gate or an or-gate without changing the result

$$
\begin{aligned}
x \vee y &= y \vee x \\
x \wedge y &= y \wedge x
\end{aligned}
$$

Associativity

- These give a way to compute the logical and/or of many inputs, by using several 2-input and/or gates

$$
\begin{aligned}
x \vee (y \vee z) &= (x \vee y) \vee z \\
x \wedge (y \wedge z) &= (x \wedge y) \wedge z
\end{aligned}
$$

- These laws are extremely important in advanced computer science
  - High performance circuit design
  - Programming massively parallel high performance computers

Distributive and absorption laws

- Useful in proving correctness of circuits, but we won't need them in this course

$$\begin{aligned} x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \\ x \vee (y \wedge z) &= (x \vee y) \wedge (x \wedge z) \end{aligned}$$

$$\begin{aligned} x \wedge (x \vee y) &= x \\ x \vee (x \wedge y) &= x \end{aligned}$$

Logical reasoning

- We want to be able to work out the consequences of assumptions
- Work with unambiguous statements that are either true or false
- Typical problem: show that two Boolean expressions always have the same value
- One approach: use truth tables
  - But a truth table with n variables contains 2^n lines
- Often it's easier to carry out a calculation using Boolean algebra

Example: equational reasoning

- Equational reasoning means "substituting equals for equals" using the laws of algebra

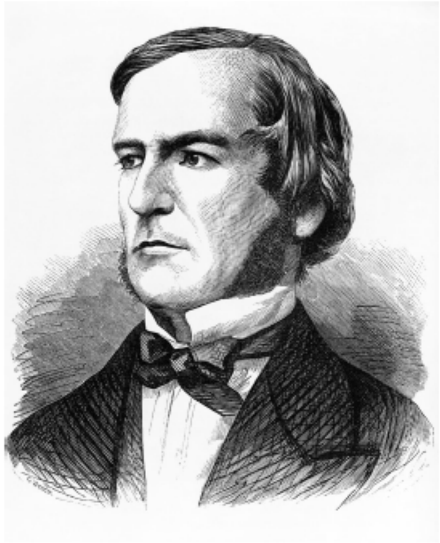
- **Problem**: Use Boolean algebra to simplify $(0 \wedge P) \vee Q$

- Each of these steps uses one of the laws

$$
\begin{array}{ll}
(0 \wedge P) \vee Q & \\
= (P \wedge 0) \vee Q & \wedge \text{ is commutative} \\
= 0 \vee Q & \text{constant operation on } \wedge \\
= Q \vee 0 & \vee \text{ is commutative} \\
= Q & \text{constant operation on } \vee
\end{array}
$$

- For large and complicated calculations, it is easier to use the laws carefully
  - It helps to prevent mistakes

George Boole, 1815-1864



- English mathematician and logician
- Applied algebra to logic
- **Symbolic logic**: formal reasoning instead of arguments in natural language

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- Addition and subtraction
  - Half adder
  - Full adder
  - Ripple carry adder
  - Subtraction
- Circuit simulation

Addition and subtraction

- Addition, subtraction, and negation are all done by one circuit: rippleAdd, the "ripple carry adder"

- We will proceed in stages
  - Adding two bits: halfAdd
  - Adding three bits: fullAdd
  - Adding two integers: rippleAdd
  - Subtracting an integer from another

- Multiplication and division are more complicated
  - Not too complicated, but we won't do them in this course

Review: binary addition

- You can add two binary numbers x and y the same way as decimal numbers
- Write one number above the other
- Work through each column, from right to left
- In each column, add the bit from x, the bit from y, and the carry from the column to the right
- This gives the sum bit s for the column, and the carry output which goes to the left
- In each column we add three bits: a carry input, a bit from x, and a bit from y

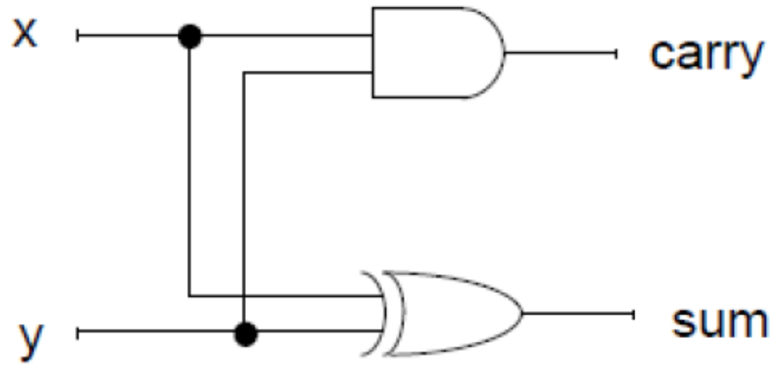
The half adder: adding two bits

- A half adder adds two bits
  - Since the result could be 0, 1, or 2, we need a two-bit representation of the sum, called (carry, sum)

- Specify the circuit abstractly by writing the complete addition table
  - Then we recognise that the carry function is just and2, and the sum function is just xor2

x	y	result	carry	sum
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	2	1	0

The half adder



x	y	result	carry	sum
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	2	1	0

The full adder: adding three bits

- To add two binary numbers, we must add three bits for each column
  - The two data bits (one from each word)
  - The carry input from the column to the left
- **Solution**: the full adder
- A full adder adds three bits x, y, z and outputs a carry c and sum s

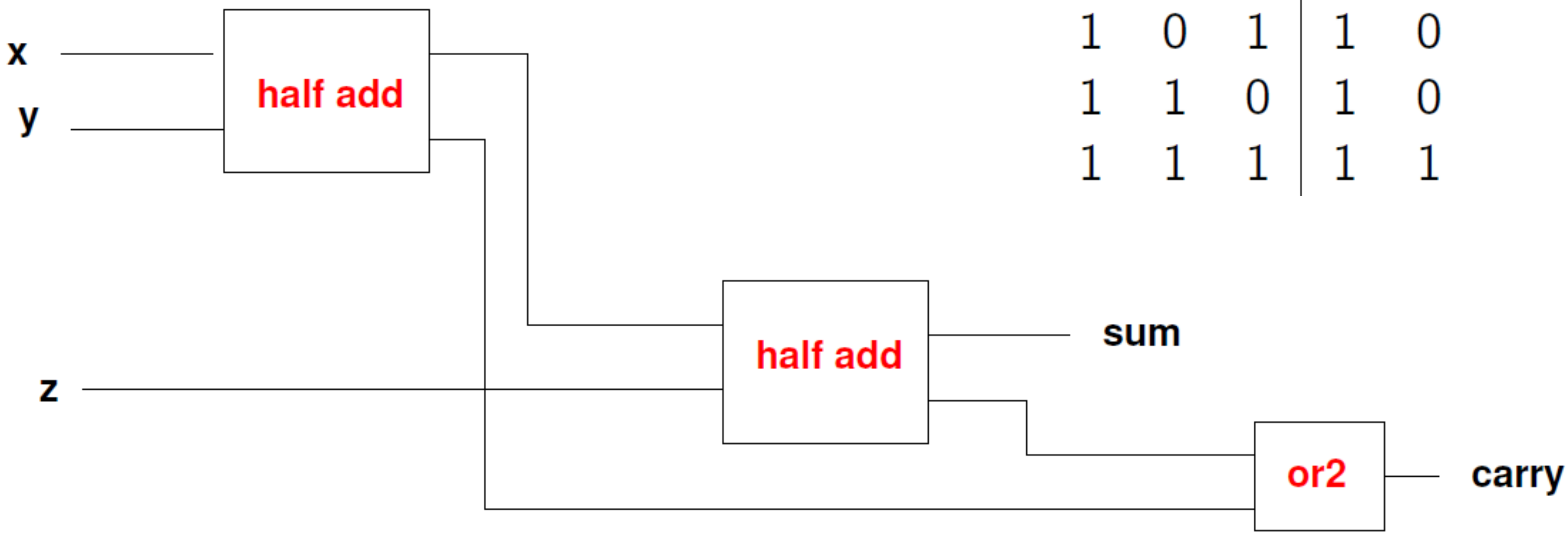
Truth table of full adder

x	y	z	c	s
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

- The sum is 1 if an odd number of inputs are 1
- The carry is 1 if two or more inputs are 1
- You can view *c* and *s* as a 2-bit binary number giving the result

The full adder circuit

x	y	z	c	s
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1



Binary word addition

- Use a full adder circuit for each bit position
- To add two 16-bit words x and y, we have 16 separate full adders
- Each full adder receives
  - A bit from x and a bit from y
  - The carry output from the full adder to the right

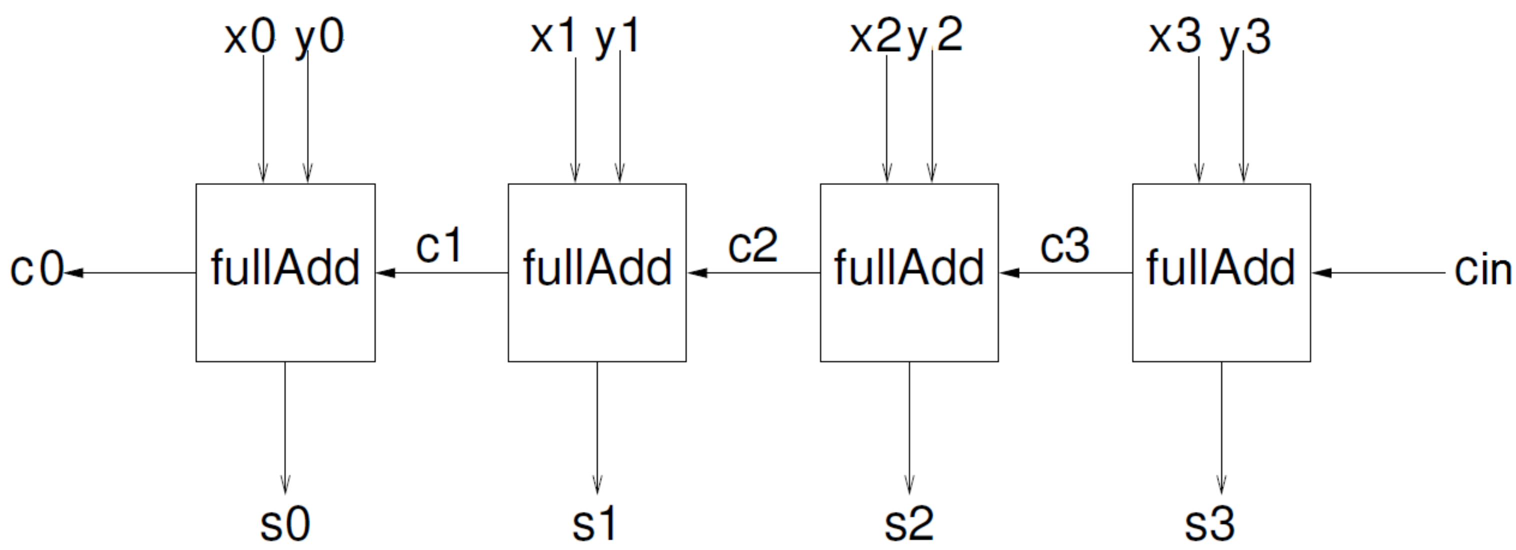
Add the weight 4 column (from Lecture 2)

- In the middle of an addition

		128	64	32	16	8	4	2	1
c						1	0	0	0
x		0	0	1	0	1	1	0	1
y		0	1	0	0	1	1	1	0
s							0	1	1

- A full adder will calculate the carry and sum in every column

4-Bit Ripple Carry Adder



- **Inputs**: word x, word y, carry input bit c
- **Outputs**: sum s, carry output
- Each bit position receives a bit from x, a bit from y, and a carry input from the position to the right
- A full adder circuit adds these three bits

Pascal's adder

- Carry propagation using gears!
- Designed by the French philosopher and mathematician (and early computer scientist!) Blaise Pascal (1623-1662)

Subtraction

- Work with two's complement numbers
- Note that $x - y = x + (-y)$
- Recall that to negate a number, you invert the bits and add 1
- To invert the bits of y, just put an inverter on each bit of y
- To add 1, just set the carry input to the entire ripple carry adder to 1
- **So**: use a ripple carry adder, with each bit of y inverted, and with carry input = 1, and the output will be $x - y$
- Modern computers use the adder circuit to perform subtraction and addition

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Circuit simulation

- After designing a circuit, how do we find out whether it works?
- **One way**: build the circuit out of transistors, and test it
- To fabricate a design on a chip takes a long time and costs a lot of money
- **Better way**: use a circuit simulator

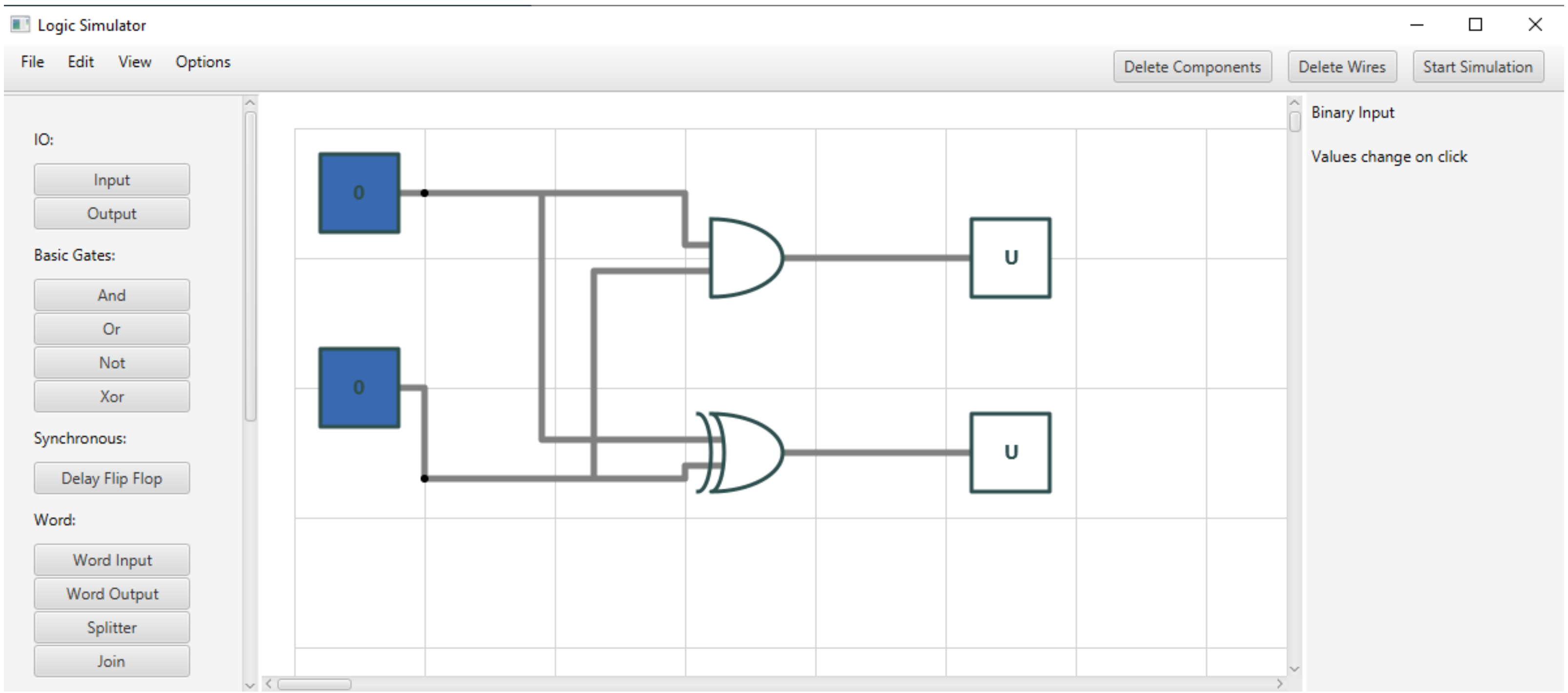
Software tools for circuit design

- "Real world" hardware is designed using special languages called computer hardware description languages

- An easier approach (though less powerful) is to use a schematic capture application
  - You draw the circuit interactively
  - A diagram of a circuit is called a schematic diagram
  - The software then simulates it

- We will use a schematic capture and simulation tool called LogicWorks

A schematic diagram in LogicSim

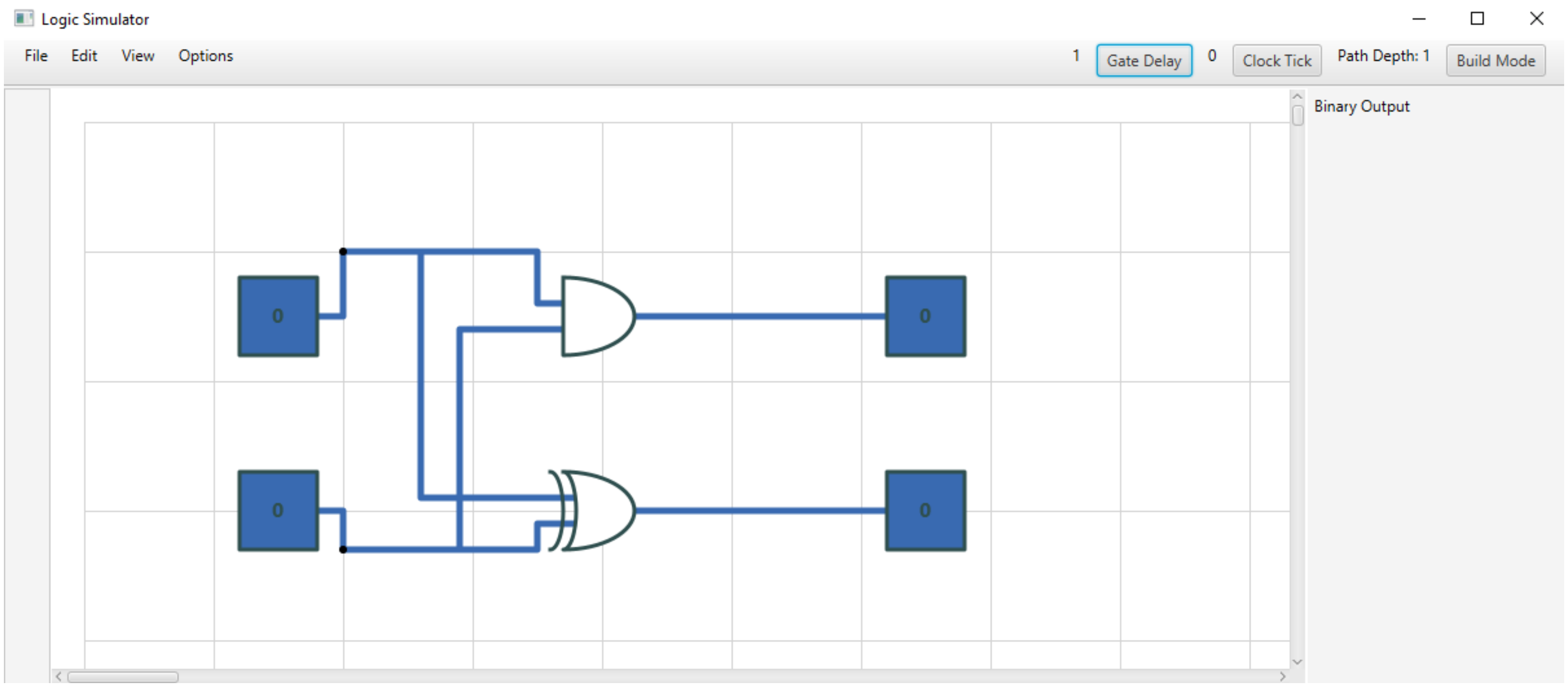


Drawing a circuit

- Some basic points
  - Select a component from combo box in the lower right
  - You'll see the component in a little window on upper right
  - Click it, and drag to where you want to place the component
  - To draw a wire, click the + icon on toolbar
  - For input, use a binary switch
  - For output, use a binary probe

- See document on Moodle for more about how to use LogicWorks

Half adder circuit: inputs x=0, y=0



Aim: find out whether the circuit works

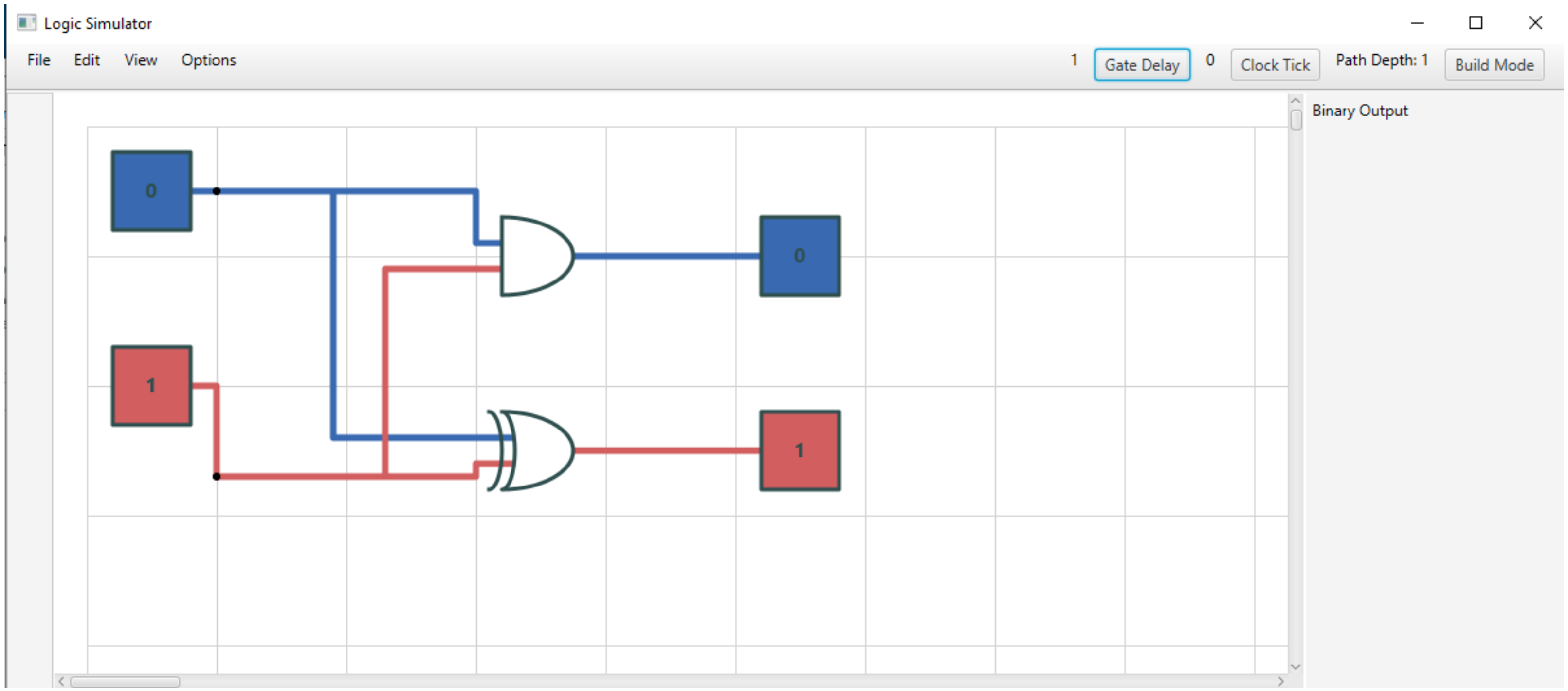
- The circuit has to give the correct outputs for every possible values of the inputs

- So there are four separate simulation problems

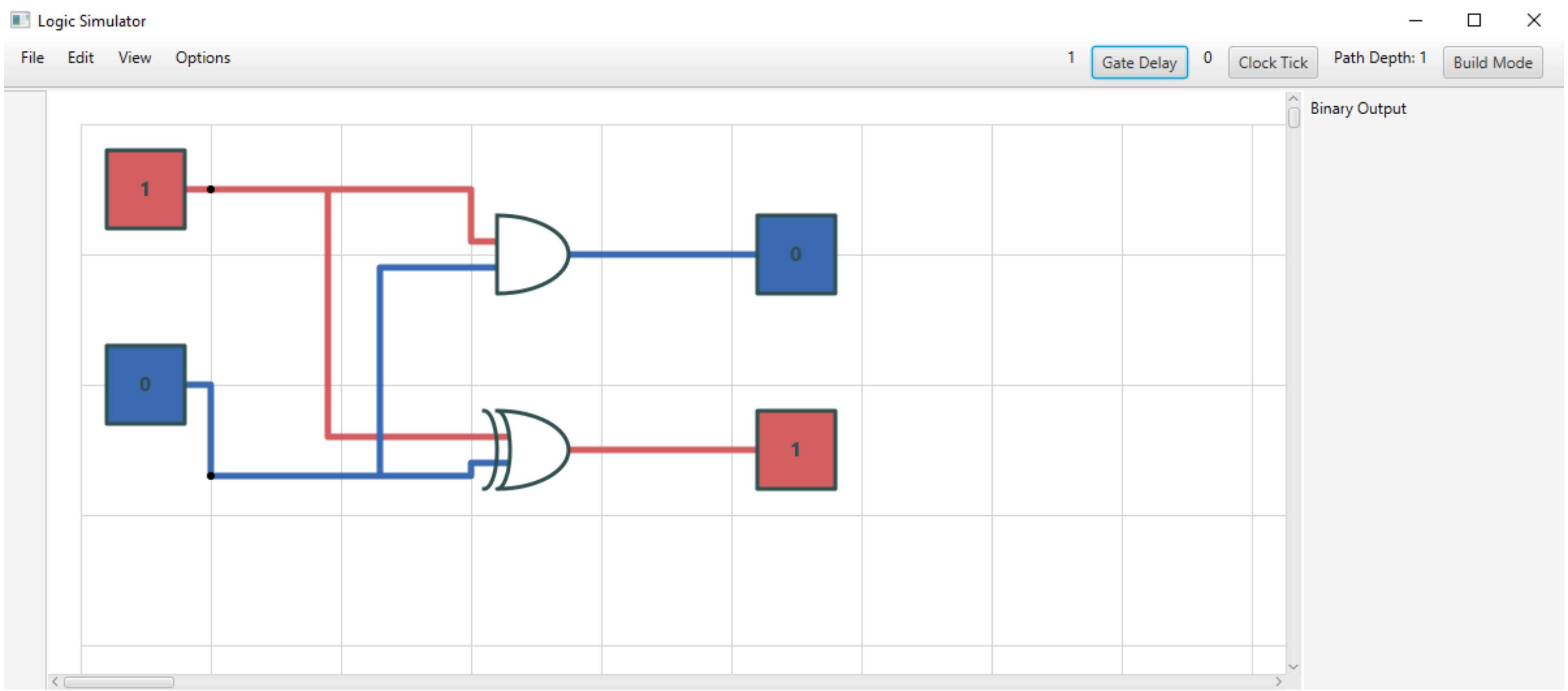
- For each line, set x and y using the switches, and record the outputs c and s in the table

x	y	c	s
0	0	?	?
0	1	?	?
1	0	?	?
1	1	?	?

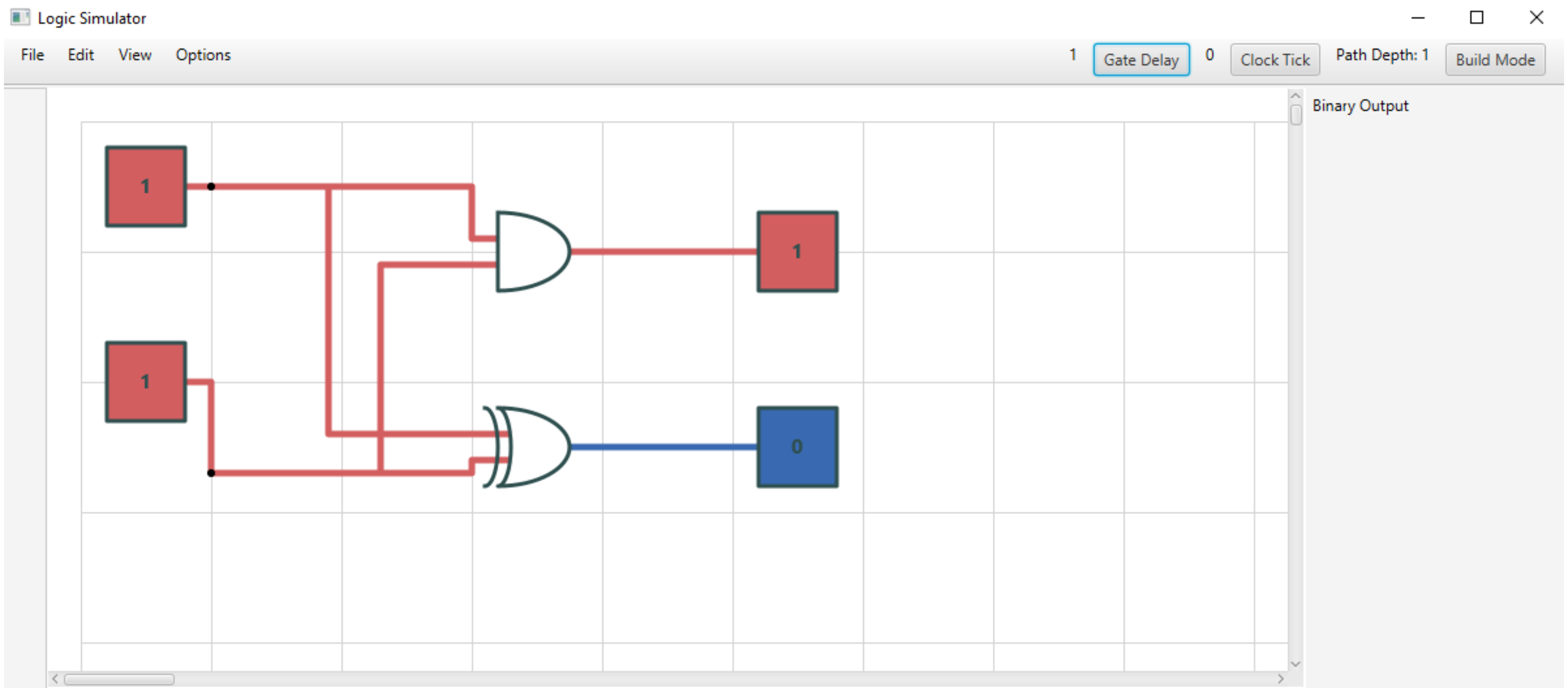
Half adder circuit: inputs x=0, y=1



Half adder circuit: inputs x=1 , y=0



Half adder circuit: inputs x=1 , y=1



Record simulation results in a truth table

- For each row, set the input switches for x, y, observe the outputs c, s and fill in the truth table

x	y	c	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

- The table gives the correct carry and sum outputs, so the circuit works correctly

To do

- Revise the lecture slides
- Finish Quiz 1 by Friday night
- Quiz 2 available
- Check the solutions to Week 2 Lab (on Moodle, at the weekend)
- Always read the solutions, even if you know the answer
  - Sometimes the model solution will give additional information
- Study the ripple carry adder circuit
  - Understand how it's doing the same thing you do when you add numbers by hand, with carry where needed

https://xkcd.com/302/