Friedrich-Alexander-Universität
Technische Fakultät

Pattern Recognition Lab

Epipolar Geometry

Projective Two-View Geometry

About the speaker

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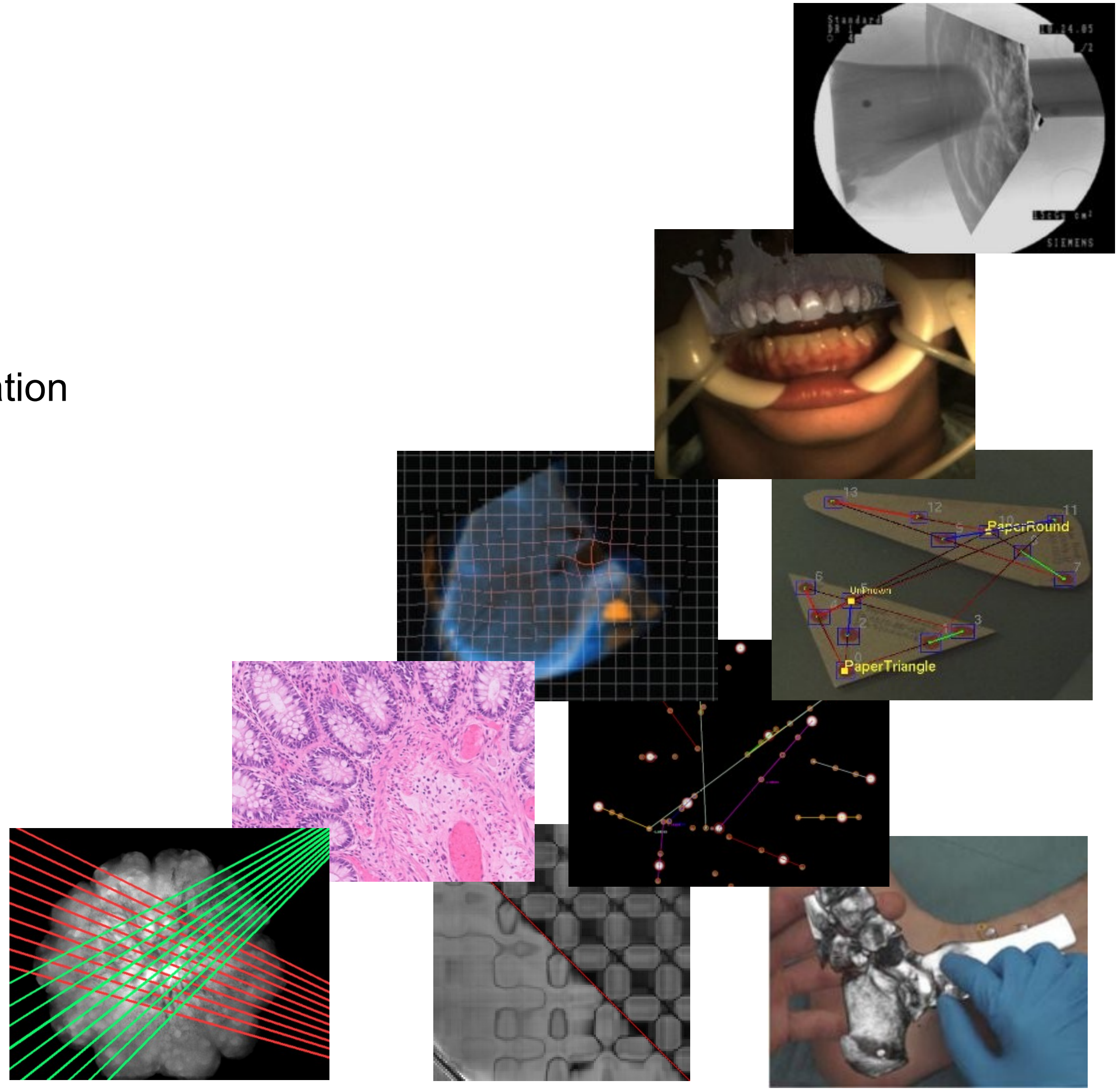
Today's capacity:
Former member of Pattern Recognition Lab



Epipolar Geometry

Projective Two-View Geometry

Julius Plücker
(16 June 1801 – 22 May 1868)

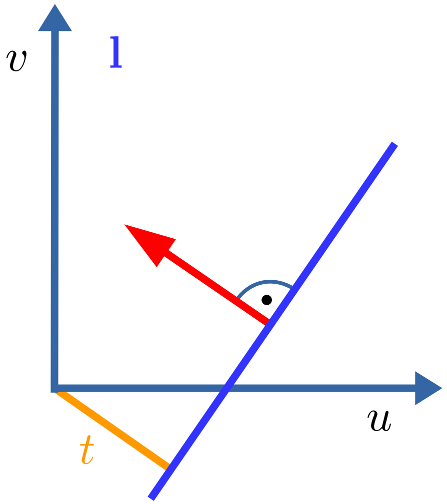
Felix Klein
(25 April 1849 – 22 June 1925)

Recap:
Projective Geometry of Two-Space

Homogeneous Equation of Lines

Points on line $\mathbf{l}$: $l_0 u + l_1 v + l_2 = 0$

Normal $\begin{pmatrix} l_0 \\ l_1 \end{pmatrix}$ Distance $t = \frac{-l_2}{\sqrt{l_0^2 + l_1^2}}$



Recap: Projective Geometry of Two-Space

Homogeneous Coordinates of Lines

Homogeneous Equation of Lines

Points on line $\mathbf{l}$: $l_0 u + l_1 v + l_2 = 0$

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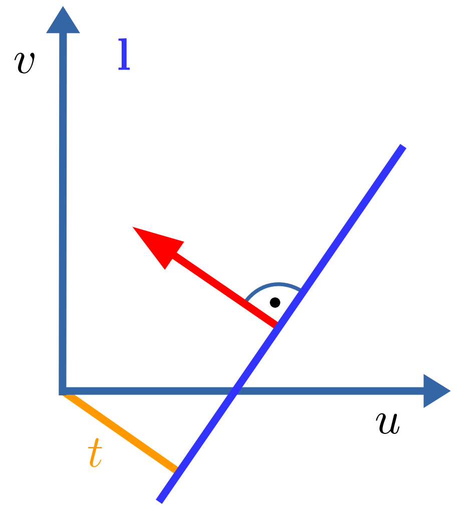

Homogeneous Coordinates of Lines

$$\mathbf{l} \cong \begin{pmatrix} l_0 \\ l_1 \\ l_2 \end{pmatrix} \in \mathbb{P}^2$$

where $\cong$ denotes “equality up scale”

- $\mathbb{R}^3$ without zero-vector
- up to non-zero scalar multiplication

Homogeneous Coordinates of Points

$$\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbf{l} \iff \boxed{(u,\, v,\, 1)} \begin{pmatrix} l_0 \\ l_1 \\ l_2 \end{pmatrix} = 0$$

- Zero-vector does not represent a point.
- Scalar multiples represent same point.

$$\boxed{\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 \leftrightarrow \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \cong \begin{pmatrix} \lambda u \\ \lambda v \\ \lambda \end{pmatrix} \in \mathbb{P}^2}$$

Homogeneous Coordinates of Points

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Duality

→ Looks familiar?

Representation of <u>both</u> points and lines in $\mathbb{P}^2$!

What about zero in last component of points?

$$\boxed{(u,\ v,\ 0)}\begin{pmatrix} l_0 \\ l_1 \\ l_2 \end{pmatrix} = 0$$

- Does not represent a point $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$

What about zero in last component of points?

$$\boxed{(u,\ v,\ 0)} \begin{pmatrix} l_0 \\ l_1 \\ l_2 \end{pmatrix} = 0$$

- Does not represent a point $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$

- l_2 is multiplied by zero!

  $\Rightarrow$ incident with all parallel lines with $\boxed{\text{direction} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} l_1 \\ -l_0 \end{pmatrix}}$

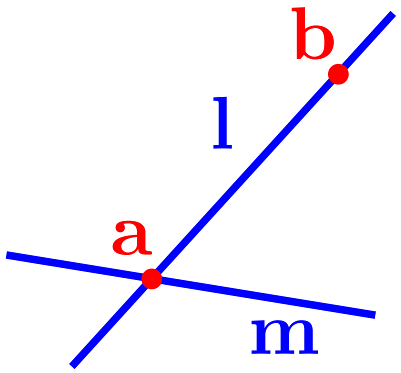
Incedence, Orthogonality and Duality

- Two points define a line ("join")

$$\mathbf{a},\ \mathbf{b} \in \mathbb{P}^2 \ \text{ are joined by } \mathbf{l} \in \mathbb{P}^2$$
$$\iff \boxed{\mathbf{a}^\top \mathbf{l} = 0 \text{ and } \mathbf{b}^\top \mathbf{l} = 0}$$

- Two lines intersect in one point ("meet")

$$\mathbf{l},\ \mathbf{m} \in \mathbb{P}^2 \ \text{ meet in } \mathbf{a} \in \mathbb{P}^2$$
$$\iff \boxed{\mathbf{l}^\top \mathbf{a} = 0 \text{ and } \mathbf{m}^\top \mathbf{a} = 0}$$



Recap: Projective Geometry of Two-Space

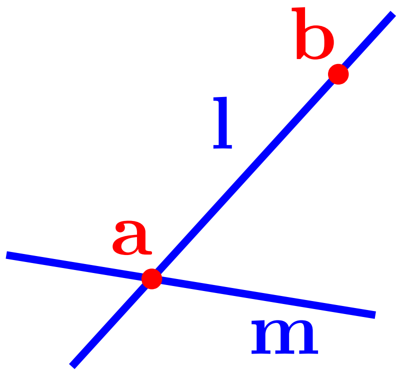
Incedence, Orthogonality and Duality

- Two points define a line ("join")

$$\mathbf{a},\ \mathbf{b} \in \mathbb{P}^2 \text{ are joined by } \mathbf{l} \in \mathbb{P}^2 \iff \boxed{\mathbf{l} \cong \mathbf{a} \times \mathbf{b}}$$

- Two lines intersect in one point ("meet")

$$\mathbf{l},\ \mathbf{m} \in \mathbb{P}^2 \text{ meet in } \mathbf{a} \in \mathbb{P}^2 \iff \boxed{\mathbf{a} \cong \mathbf{l} \times \mathbf{m}}$$



(wake up please, new material :)

Plücker Matrix in 2D

Suppose we have a line $\mathbf{l} \cong \mathrm{join}(\mathbf{a}, \mathbf{b})$ through two points $\mathbf{a}, \mathbf{b} \in \mathbb{P}^2$

$$\mathbf{l} \cong \mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 b_2 - b_1 a_2 \\ b_0 a_2 - a_0 b_2 \\ a_0 b_1 - a_1 b_0 \end{pmatrix} = \begin{pmatrix} l_0 \\ l_1 \\ l_2 \end{pmatrix},$$

(wake up please, new material :)

Plücker Matrix in 2D

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and observe that we can also compute the anti-symmetric matrix

$$[\mathbf{l}]_\times = \mathbf{a}\mathbf{b}^\top - \mathbf{b}\mathbf{a}^\top$$

$$= \begin{pmatrix} a_0 b_0 - b_0 a_0 & a_0 b_1 - b_0 a_1 & a_0 b_2 - b_0 a_2 \\ a_1 b_0 - b_1 a_0 & a_1 b_1 - b_1 a_1 & a_1 b_2 - b_1 a_2 \\ a_2 b_0 - b_2 a_0 & a_2 b_1 - b_2 a_1 & a_2 b_2 - b_2 a_2 \end{pmatrix} = \begin{pmatrix} 0 & l_2 & -l_1 \\ -l_2 & 0 & l_0 \\ l_1 & -l_0 & 0 \end{pmatrix}$$

(wake up please, new material :)

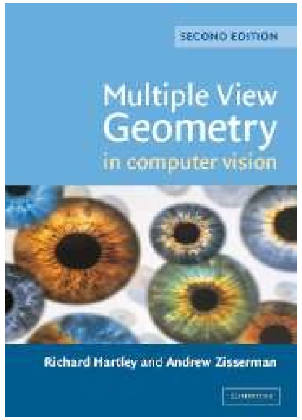
Plücker Matrix in 2D

The operator $[\cdot]_\times$ assembles an anti-symmetric matrix from the components of a three-vector. Multiplication of a vector with the anti-symmetric matrix $[\mathbf{l}]_\times$ is exactly the meet operation with the line $\mathbf{l}$:

$$[\mathbf{l}]_\times \mathbf{m} = \mathbf{l} \times \mathbf{m} = \mathrm{meet}(\mathbf{l}, \mathbf{m}) \tag{19}$$

and by argument of duality

$$\mathbf{l} = [\mathbf{a}]_\times \mathbf{b} = \mathrm{join}(\mathbf{a}, \mathbf{b}) \tag{20}$$



Book Recommendation
Multiple View Geometry in Computer Vision

Richard Hartley and Andrew Zisserman,
Cambridge University Press, March 2004.

https://www.robots.ox.ac.uk/~vgg/hzbook/

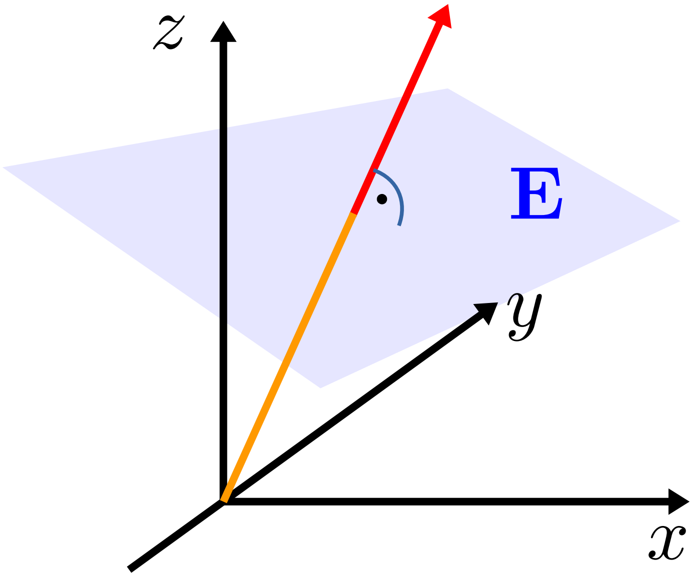
Recap:
Projective Geometry of Three-Space

Points on line $\mathbf{l}$: $l_0 u + l_1 v + l_2 = 0$

Points on plane $\mathbf{E}$: $e_0 x + e_1 y + e_2 z + e_3 = 0$

Plane equation and Hessian Normal Form

$$\mathbf{E} \cong \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \cong \begin{pmatrix} | \\ \mathbf{n} \\ | \\ -t \end{pmatrix} = \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$



Projective Geometry of Lines in Three-Space

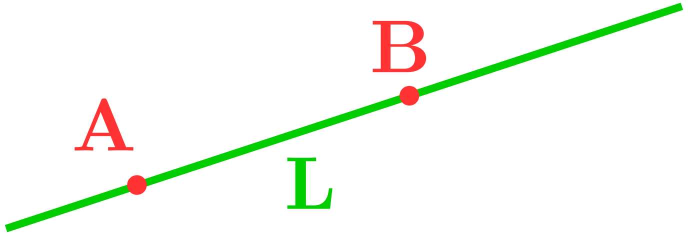
Duality of Points and Planes in Three-Space

BONUS

$$\text{Points } \mathbf{A} \cong \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} \text{ and } \mathbf{B} \text{ accordingly}$$

$$\text{Plane } \mathbf{P} \cong \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} \text{ and } \mathbf{Q} \text{ accordingly}$$

… now what about lines?



Projective Geometry of Lines in Three-Space

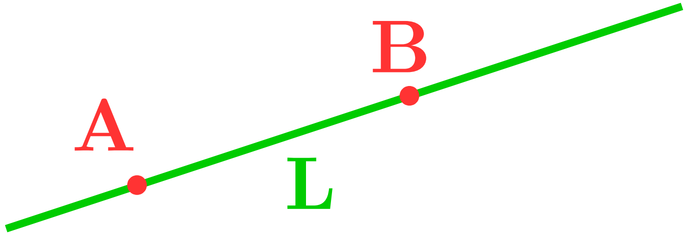
Plücker Matrix

BONUS

Idea: Use Plücker Matrix to Represent 3D Lines

$$[\mathbf{L}]_\times = \begin{pmatrix} 0 & -L_{01} & -L_{02} & -L_{03} \\ L_{01} & 0 & -L_{12} & -L_{13} \\ L_{02} & L_{12} & 0 & -L_{23} \\ L_{03} & L_{13} & L_{23} & 0 \end{pmatrix} = \mathbf{A} \cdot \mathbf{B}^\top - \mathbf{B} \cdot \mathbf{A}^\top$$

six distinct values are called Plücker coordinates.



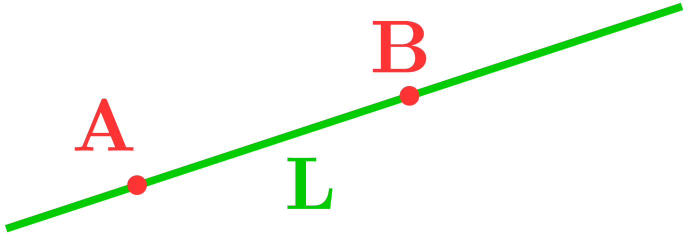
Idea: Use Plücker Matrix to Represent 3D Lines

$$[\mathbf{L}]_\times = \begin{pmatrix} 0 & -L_{01} & -L_{02} & -L_{03} \\ L_{01} & 0 & -L_{12} & -L_{13} \\ L_{02} & L_{12} & 0 & -L_{23} \\ L_{03} & L_{13} & L_{23} & 0 \end{pmatrix} = \begin{pmatrix} [\mathbf{m}]_\times & \mathbf{d} \\ -\mathbf{d}^\top & 0 \end{pmatrix} = \mathbf{A}\cdot\mathbf{B}^\top - \mathbf{B}\cdot\mathbf{A}^\top$$

The **six distinct values** are often written as two vectors

- the line direction $\mathbf{d} = (-L_{03},\, -L_{13},\, -L_{23})^\top$
- the line moment $\mathbf{m} = (L_{12}, -L_{02}, L_{01})^\top$
- which can be shown to be always orthogonal.

- the distance to the origin is $d = \dfrac{\|\mathbf{m}\|}{\|\mathbf{d}\|}$



Projective Geometry of Lines in Three-Space

Example: How to intersect 3D line with plane.

BONUS

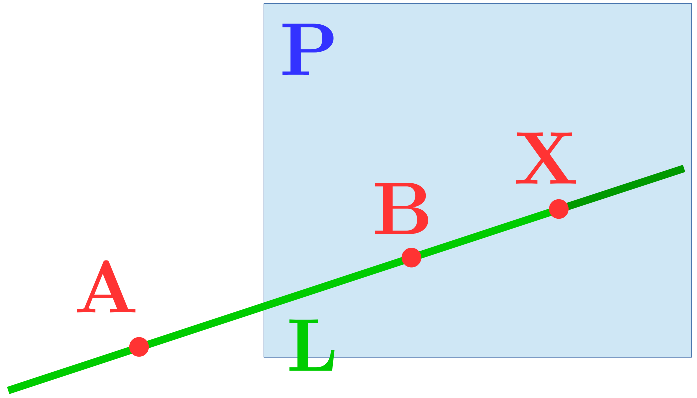
$$\mathbf{X} = [\mathbf{L}]_\times \mathbf{P} = \mathbf{B} \cdot \underbrace{\mathbf{A}^T\mathbf{P}}_{\alpha} - \mathbf{A} \cdot \underbrace{\mathbf{B}^T\mathbf{P}}_{\beta}$$

$\rightarrow$ linear combination!

$\rightarrow$ $\mathbf{X}$ is on the line $\mathbf{L}$

$$\mathbf{P}^T\mathbf{X} = \mathbf{P}^T[\mathbf{L}]_\times \mathbf{P} = \beta\alpha - \alpha\beta = 0$$

$\rightarrow$ $\mathbf{X}$ is on the plane $\mathbf{P}$

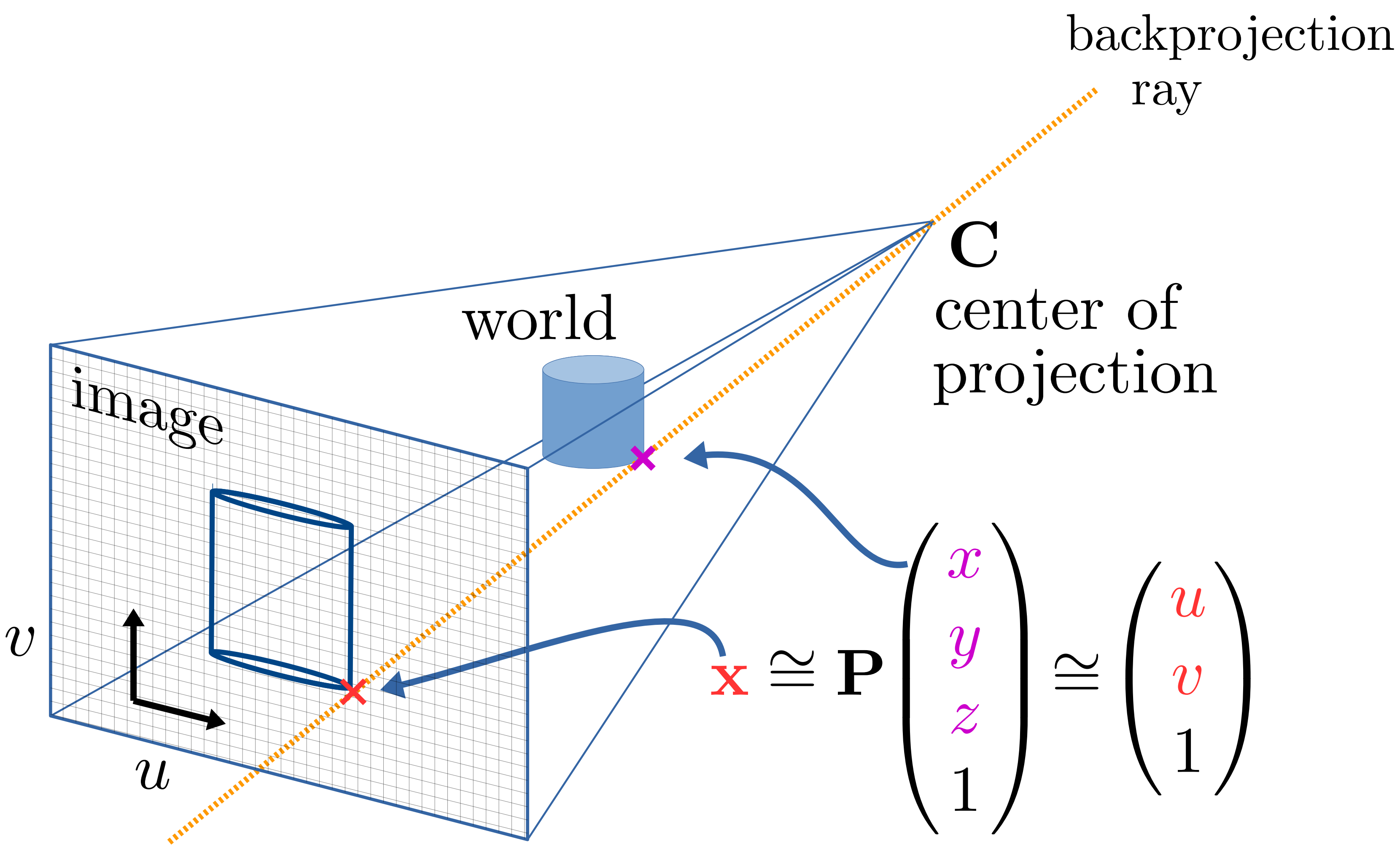


There is much more to be said!

J. F. Blinn, "A homogeneous formulation for lines in 3 space," SIGGRAPH Comput. Graph., vol. 11, no. 2, pp. 237–241, Jul. 1977.

Recap:
Single View Geometry and the Projection Matrix

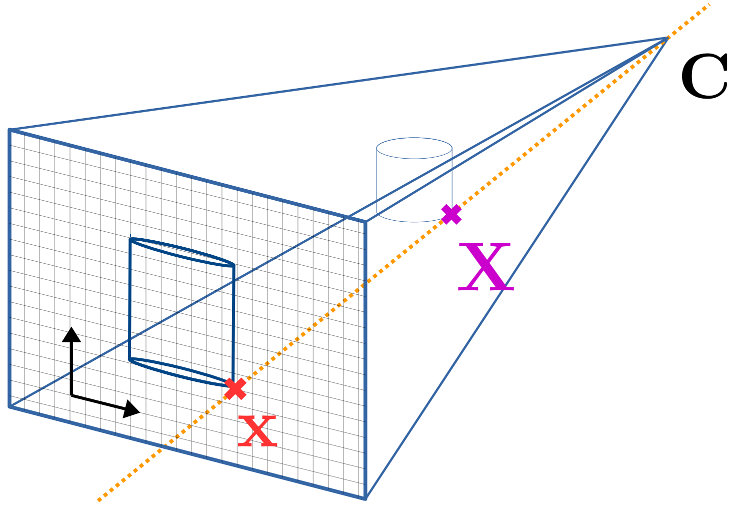
Central Projection: Notaion of 2D and 3D Coordinates



$$\mathbf{x} \cong \mathbf{P}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \cong \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$$

Algebraic Estimation of the Projection Matrix

Central Projection



$$\mathbf{x} \cong \mathbf{P}\mathbf{X}$$

image point in $\mathbb{P}^2$

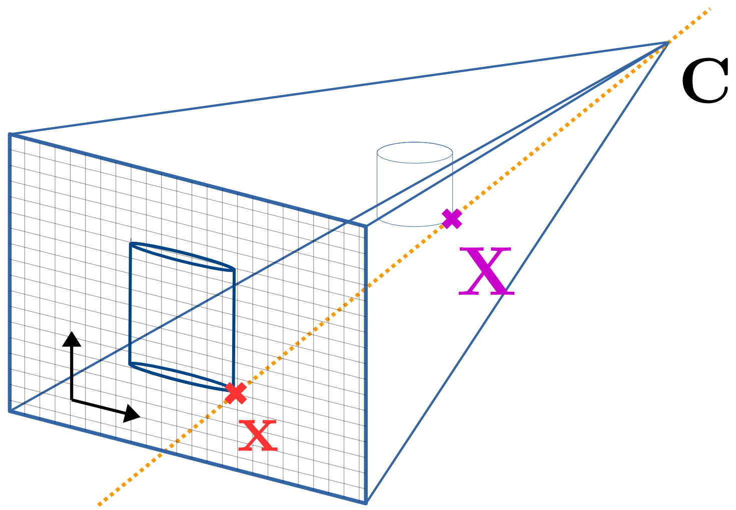
$\mathbb{R}^{3\times 4}$ up to scale

world point in $\mathbb{P}^3$

Algebraic Estimation of the Projection Matrix

Central Projection

- Rays through camera center $\mathbf{C}$ and world point $\mathbf{X}$ intersect image plane at image point $\mathbf{x}$
- Projection described by a single 3x4 matrix up to scale
- Projects both points in “front of” and “behind” camera

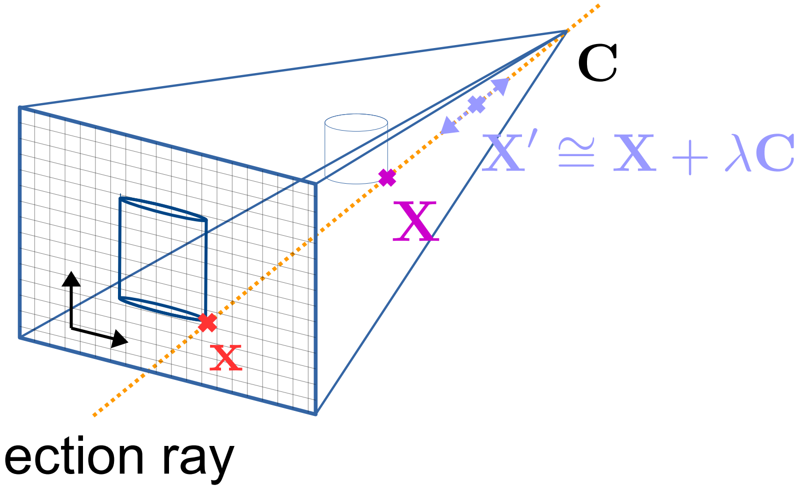


The Anatomy of the Projection Matrix

The center of projection is the right null-space of the projection matrix

- If $\mathbf{X}$ is a point on a projection ray, then so is $\mathbf{C}$
- All points on a ray map to the same image point
- And thus any linear combination of the two does, too:

$$\mathbf{x} \cong \mathbf{P}\left(\mathbf{X} + \lambda\mathbf{C}\right) \cong \mathbf{PX} + \underbrace{\lambda\mathbf{PC}}_{=0}$$



Anatomy of the Projection Matrix

Camera Center

Changing the image leaves the camera center intact

Transform image points with homography

$$\mathbf{Hx} \cong \mathbf{HPX} = (\mathbf{HP})\mathbf{X}$$

transformed
projection matrix

Can simulate rotation about the camera center!

Backprojection

Of points and lines.

The pseudo-inverse allows us to compute a point with identical projection:

$$\mathbf{X}^{+} = \mathbf{P}^{+}\mathbf{x}\,, \qquad \mathbf{P}\mathbf{X}^{+} = \mathbf{P}\mathbf{P}^{+}\mathbf{x} = \mathbf{x}$$

The **backprojection ray** is defined by the camera center and another point.

$$\mathbf{R} = \mathrm{join}(\mathbf{X}^{+}, \mathbf{C})$$

Let $\mathbf{l}$ denote a 2D line through two image points $\mathbf{a}$ and $\mathbf{b}$.
Further, let $\mathbf{A}^{+} \cong \mathbf{P}^{+}\mathbf{a}$ and $\mathbf{B}^{+}$, accordingly denote their backprojection points.
Now, the plane $\mathbf{E} \cong \mathbf{P}^{\top}\mathbf{l}$ is the **backprojection plane** of $\mathbf{l}$ because it contains both these points and the camera center:

$$\mathbf{E}^{\top}\mathbf{A}^{+} = \mathbf{l}^{\top}\underbrace{\mathbf{P}\mathbf{P}^{+}}_{identity}\mathbf{a} = \mathbf{l}^{\top}\mathbf{a}, \qquad \mathbf{E}^{\top}\mathbf{C} = \mathbf{l}^{\top}\underbrace{\mathbf{P}\mathbf{C}}_{=0} = 0.$$

Anatomy of the Projection Matrix

Axis- and Principal Planes

Rows of the matrix can be interpreted as planes

$$\mathbf{P} \cong \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{pmatrix} \qquad \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \cong \mathbf{PX} = \begin{pmatrix} - & \mathbf{p}^{1\top} & - \\ - & \mathbf{p}^{2\top} & - \\ - & \mathbf{p}^{3\top} & - \end{pmatrix} \mathbf{X}$$

Anatomy of the Projection Matrix

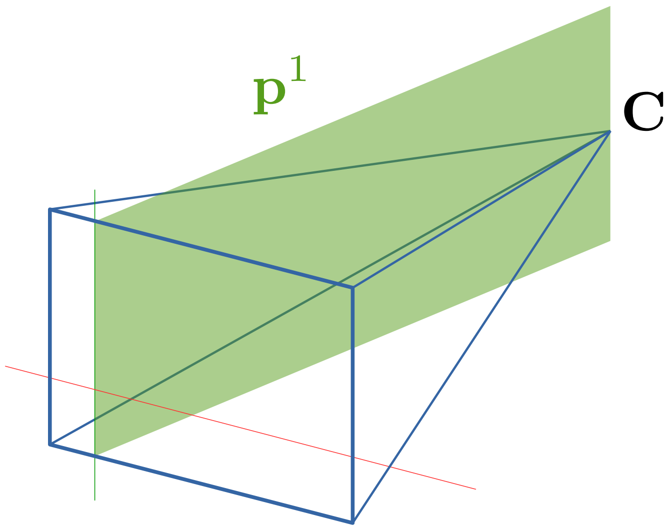
Axis- and Principal Planes

Rows of the matrix can be interpreted as planes

$$\mathbf{P} \cong \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{pmatrix}$$

- First row: $u = \mathbf{p}^{1\top}\mathbf{X}$
- Points on $\mathbf{p}^1$ fulfill $u = 0$

$\Rightarrow \mathbf{p}^1$ contains v-axis!



$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \cong \mathbf{PX} = \begin{pmatrix} - & \mathbf{p}^{1\top} & - \\ - & \mathbf{p}^{2\top} & - \\ - & \mathbf{p}^{3\top} & - \end{pmatrix}\mathbf{X}$$

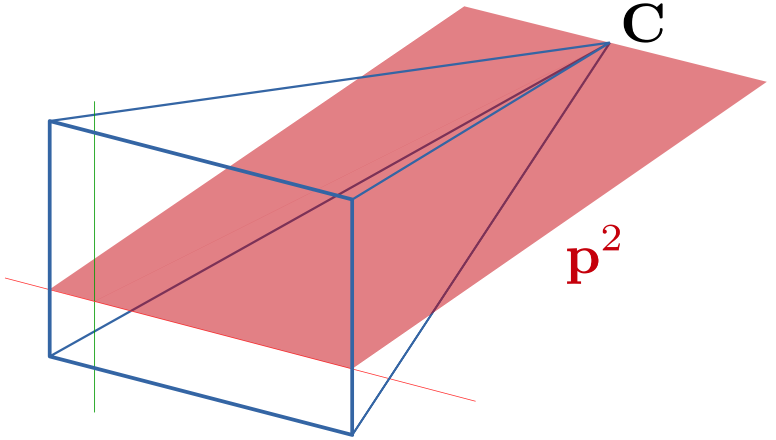
Anatomy of the Projection Matrix

Axis- and Principal Planes

Rows of the matrix can be interpreted as planes

$$\mathbf{P} \cong \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{pmatrix}$$

- Second row: $v = \mathbf{p}^{2\top}\mathbf{X}$
- Points on $\mathbf{p}^2$ fulfill $v = 0$



$\Rightarrow$ $\mathbf{p}^2$ contains u-axis!

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \cong \mathbf{PX} = \begin{pmatrix} - & \mathbf{p}^{1\top} & - \\ - & \mathbf{p}^{2\top} & - \\ - & \mathbf{p}^{3\top} & - \end{pmatrix}\mathbf{X}$$

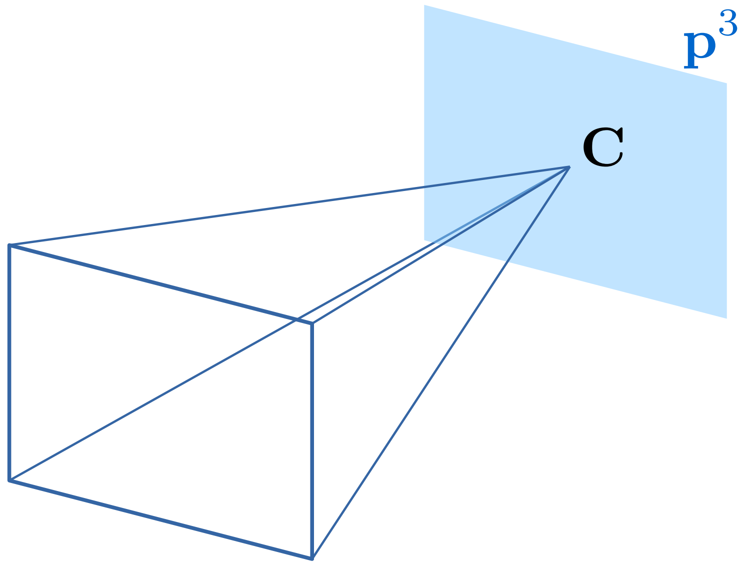
Anatomy of the Projection Matrix

Axis- and Principal Planes

The principal plane

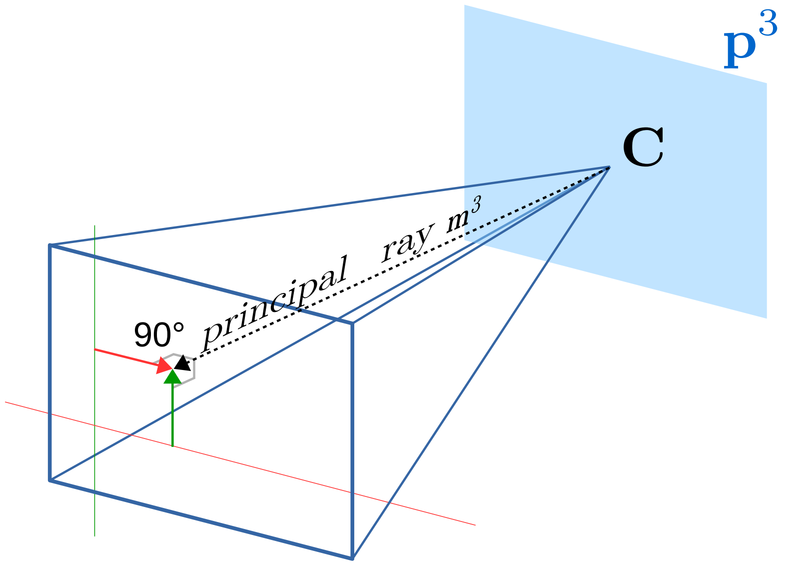
$$\mathbf{P} \cong \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{pmatrix}$$

- Points on $\mathbf{p}^3$ fulfill $0 = \mathbf{p}^{3\top}\mathbf{X}$ and lie on a plane parallel to the image
- They are mapped to infinity!
- $\mathbf{C} = \mathrm{null}(\mathbf{P})$ lies on all planes.



The principal plane

$$\mathbf{P} \cong \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ \boxed{P_{31} \quad P_{32} \quad P_{33}} & & & P_{34} \end{pmatrix}$$



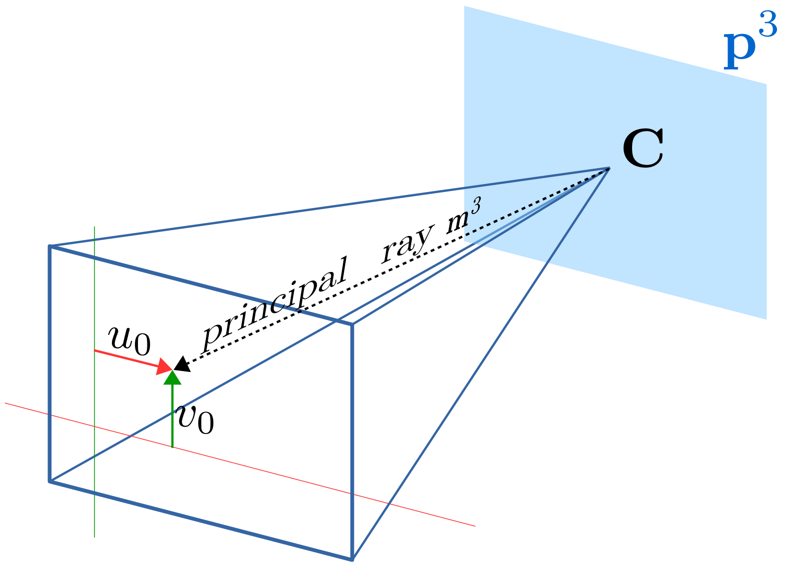
The principal ray is the normal of the principal plane and is orthogonal to the image plane.

The principal plane

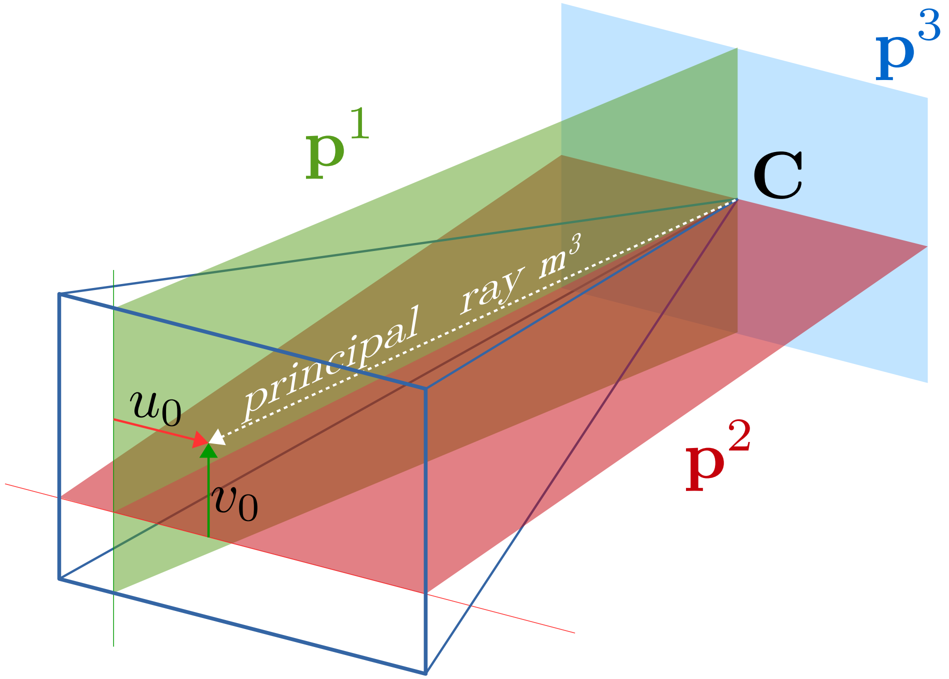
$$\mathbf{P} \cong \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ \underbrace{P_{31} \quad P_{32} \quad P_{33}}_{\mathbf{m}^3} & & & P_{34} \end{pmatrix}$$

The principal point is the orthogonal projection of the camera center to the image.

$$\begin{pmatrix} u_0 \\ v_0 \\ 1 \end{pmatrix} \cong \boxed{\mathbf{M}\mathbf{m}^3}$$



Summary



$$\mathbf{P} \cong \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ \underbrace{P_{31} \quad P_{32} \quad P_{33}}_{\mathbf{m}^3} & & & P_{34} \end{pmatrix}$$

$$\begin{pmatrix} u_0 \\ v_0 \\ 1 \end{pmatrix} \cong \mathbf{M}\mathbf{m}^3$$

Anatomy of the Projection Matrix

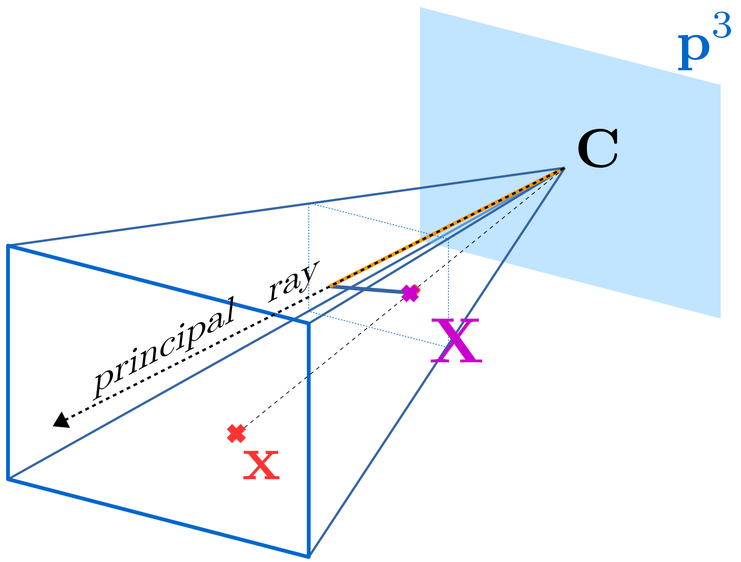
Depth

- Depth is the distance to the principal plane
- Assuming:
  - $\|\mathbf{m}^3\| = 1$ with $\det(\mathbf{M}) > 0$
  - $\mathbf{X} = (x, y, z, 1)^\top$

…the depth of

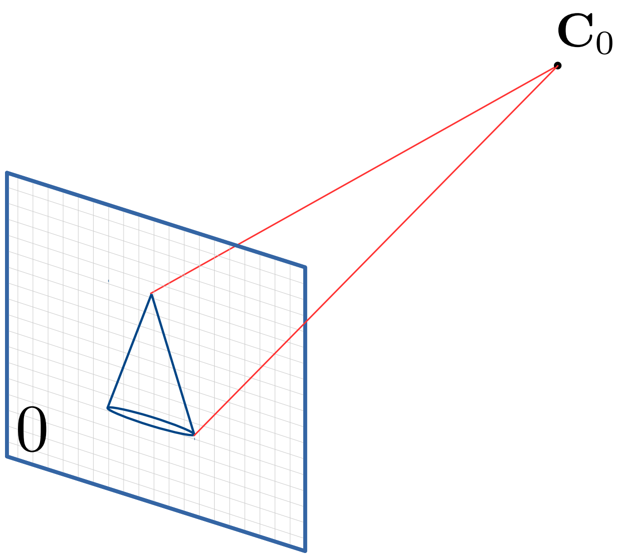
$$\mathbf{x} = \begin{pmatrix} d \cdot u \\ d \cdot v \\ d \end{pmatrix} = \mathbf{P}\mathbf{X}$$

is exactly d.



Two View Geometry: The Fundamental Matrix

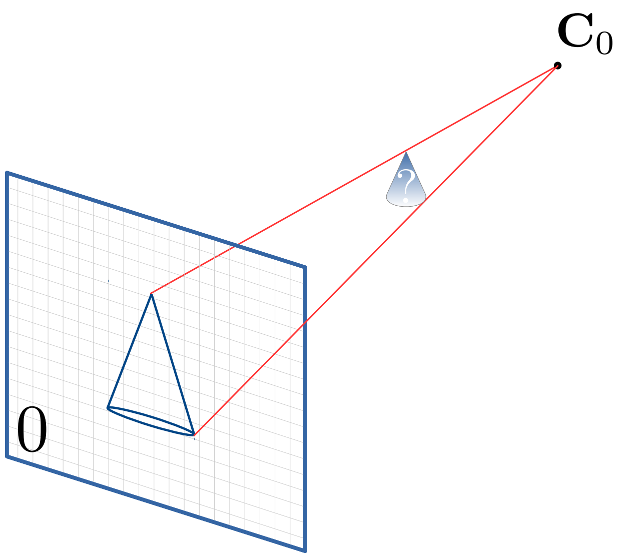
(e.g. photograph, X-ray image ...)



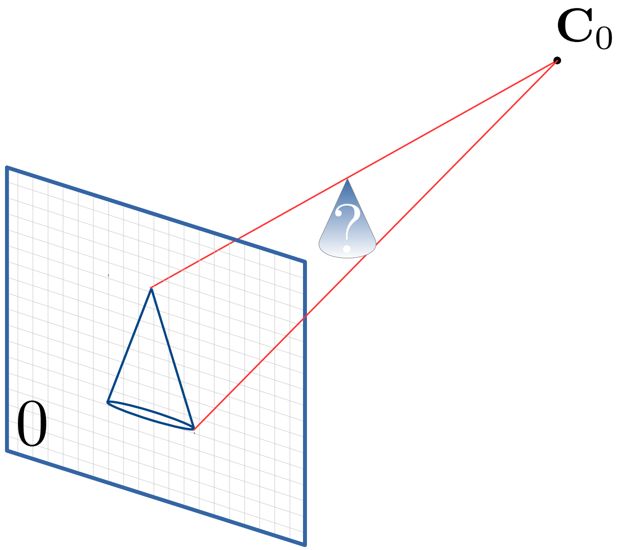
Epipolar Geometry

The geometry of two views

(e.g. photograph, X-ray image ...)



(e.g. photograph, X-ray image ...)



Epipolar Geometry

The geometry of two views

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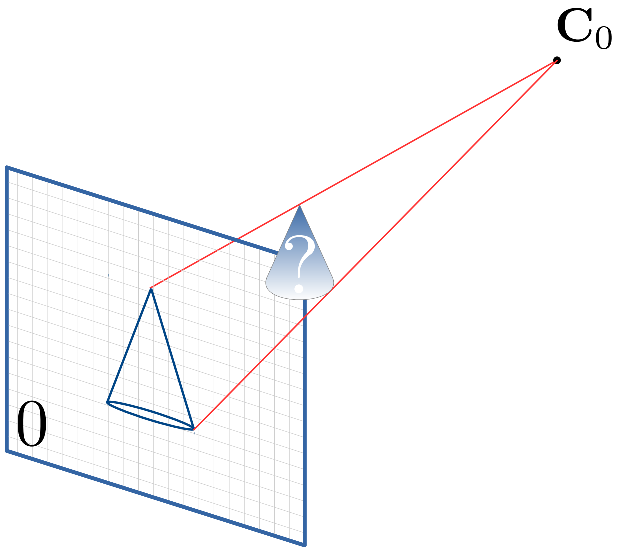
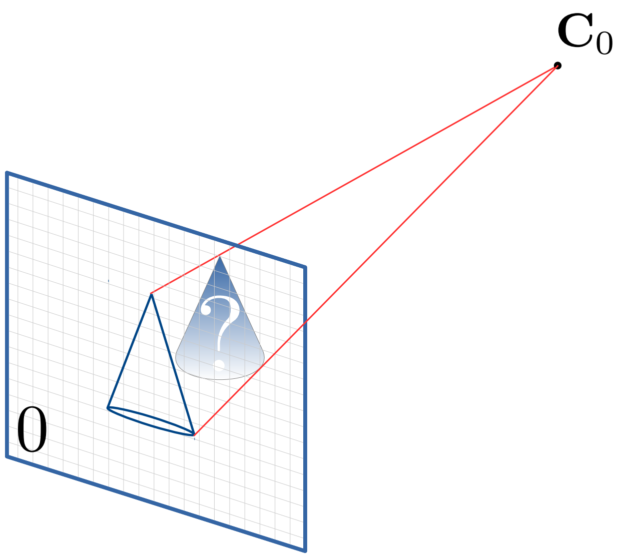


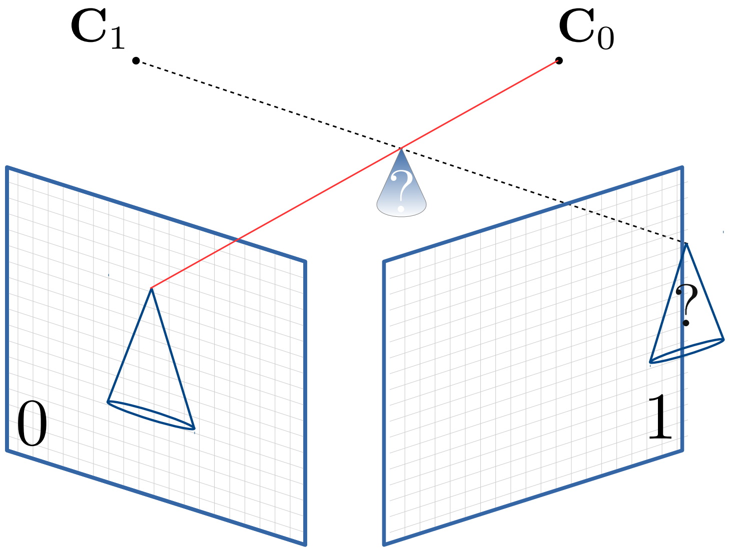
Epipolar Geometry

The geometry of two views

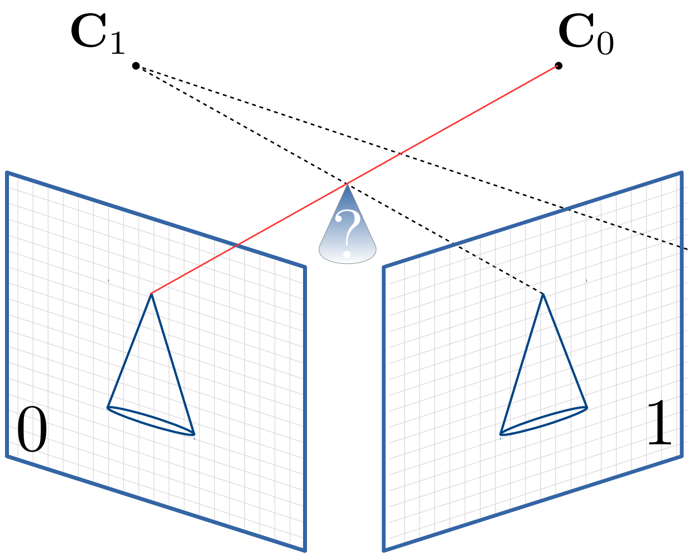
(e.g. photograph, X-ray image ...)



Constraints on points in two central projections?



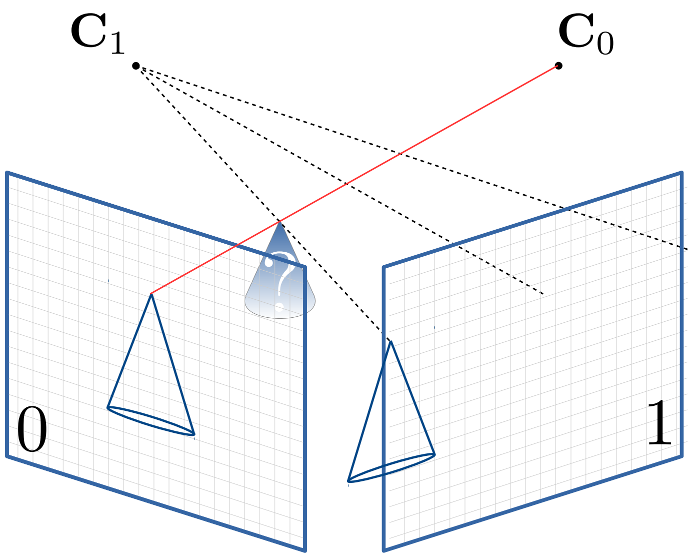
Constraints on points in two central projections?



Epipolar Geometry

The geometry of two views

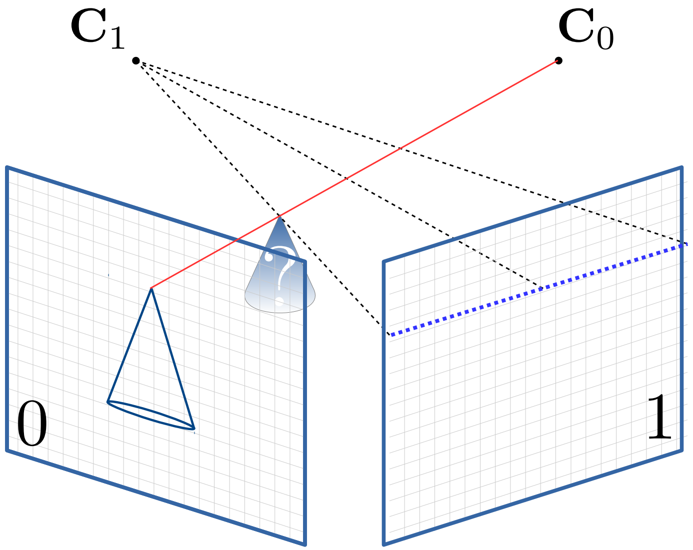
Constraints on points in two central projections?



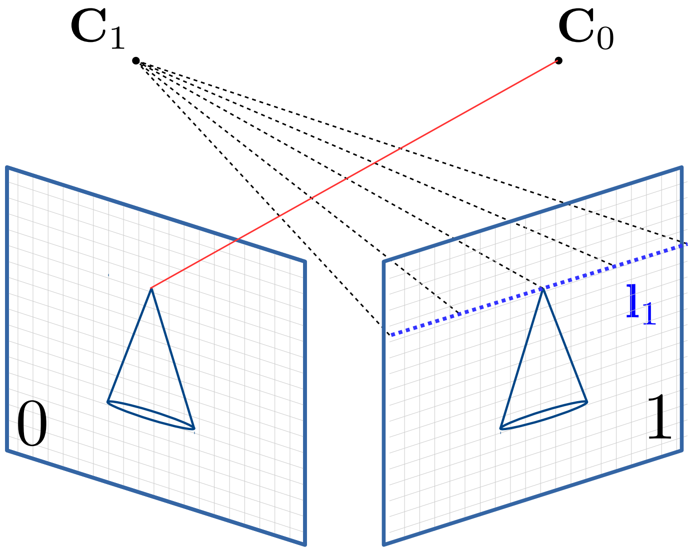
Epipolar Geometry

The geometry of two views

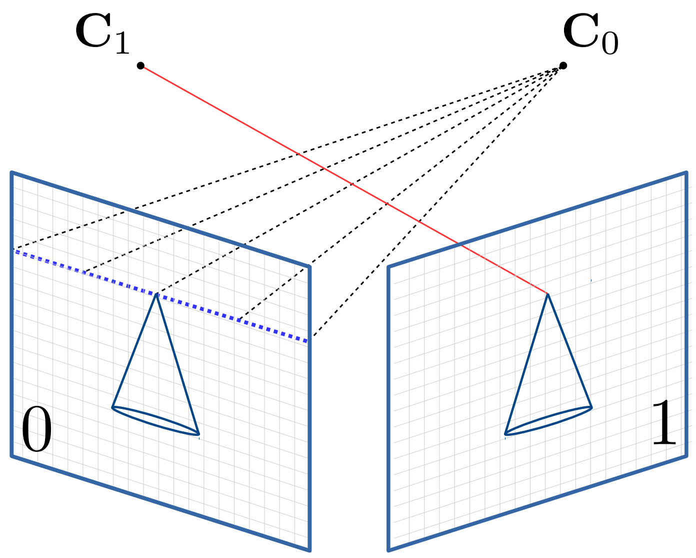
Constraints on points in two central projections?



Construction is symmetric.



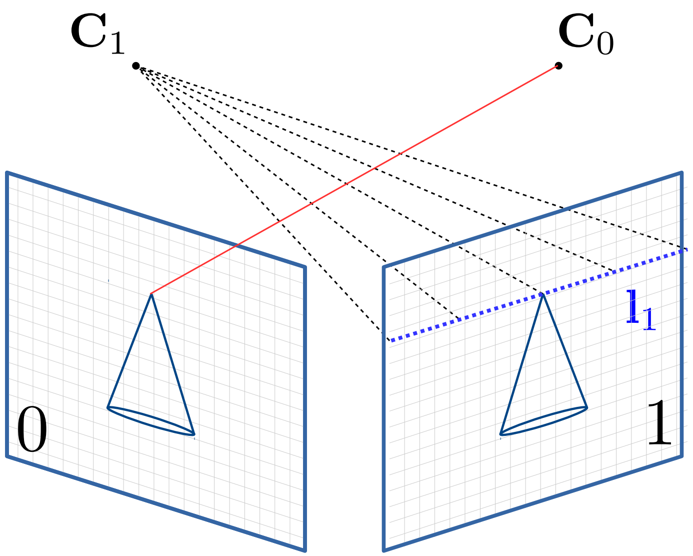
Construction is symmetric.



Epipolar Geometry

The geometry of two views

Construction is symmetric.

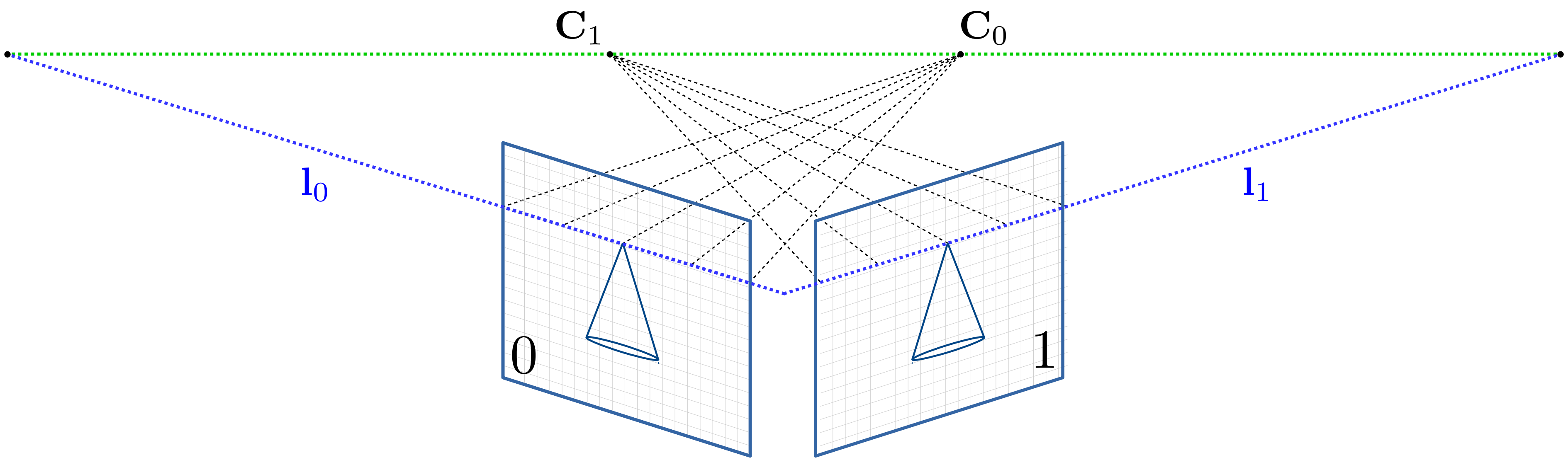


Epipolar Geometry

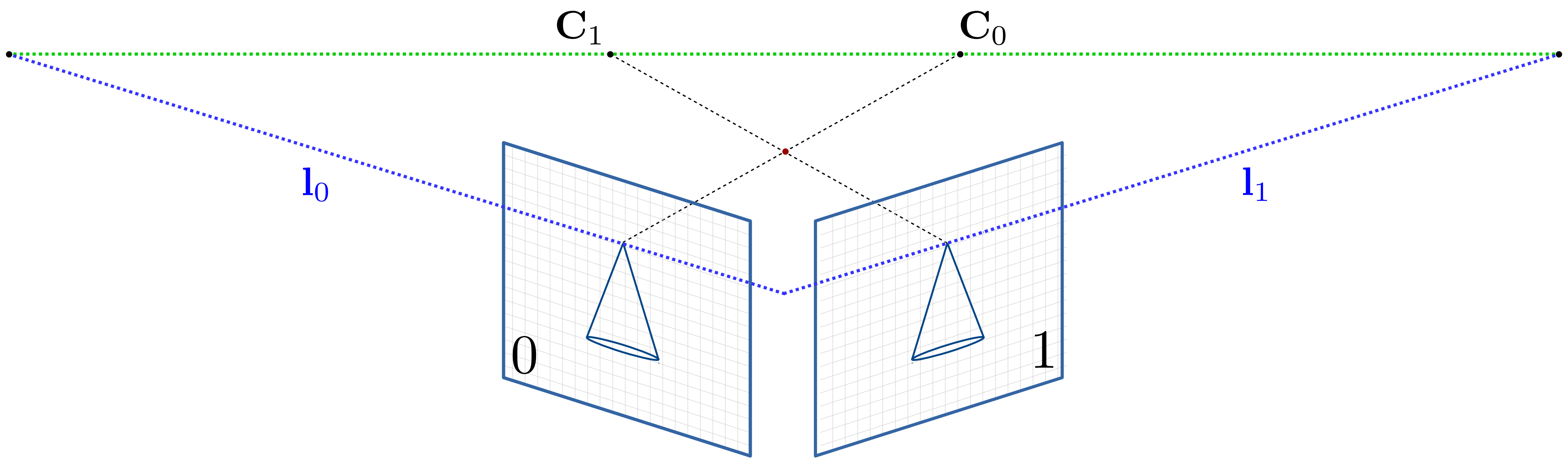
The geometry of two views

Plane through both source positions

and its intersection line with both detectors



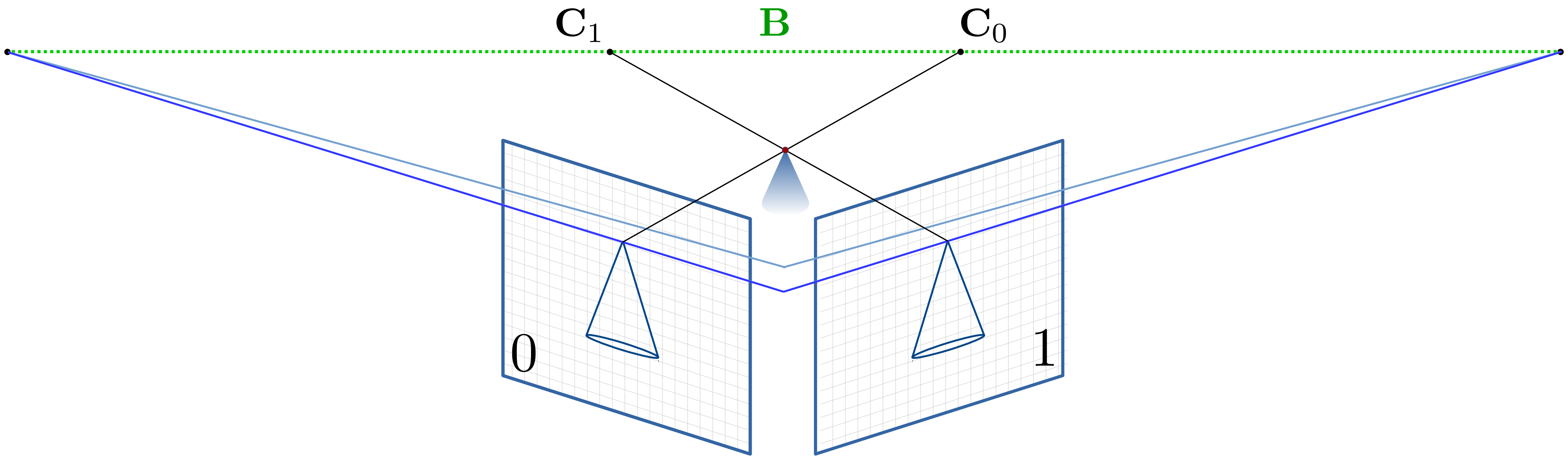
Observe: $\mathbf{l}_0$ and $\mathbf{l}_1$ are **corresponding epipolar lines**.



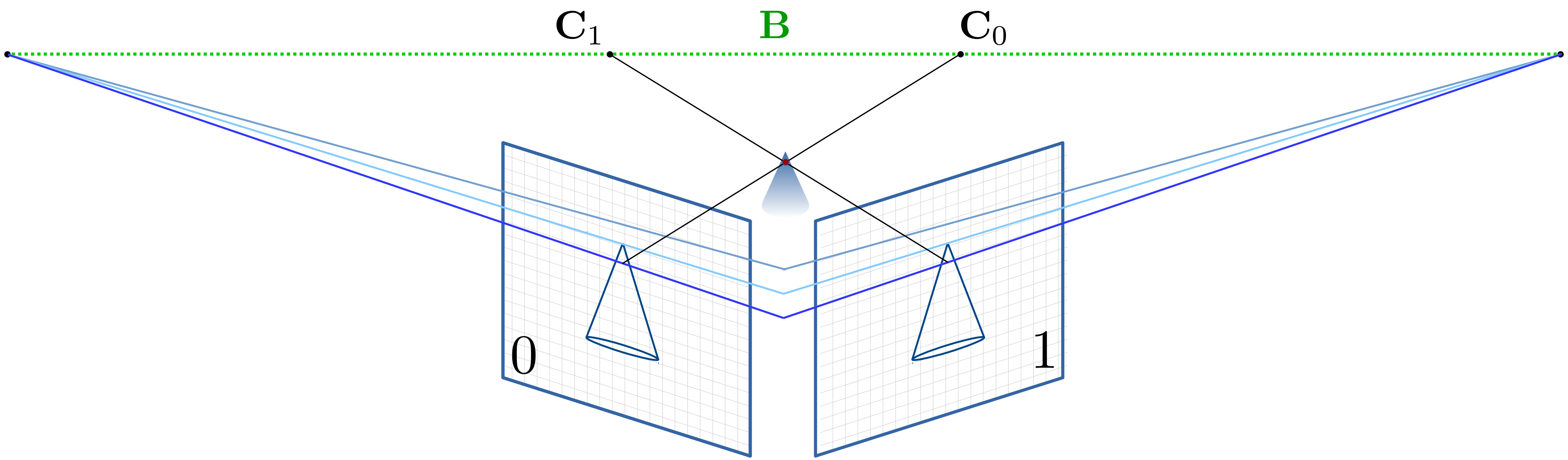
Epipolar Geometry

The geometry of two views

There exists a pencil of epipolar lines!



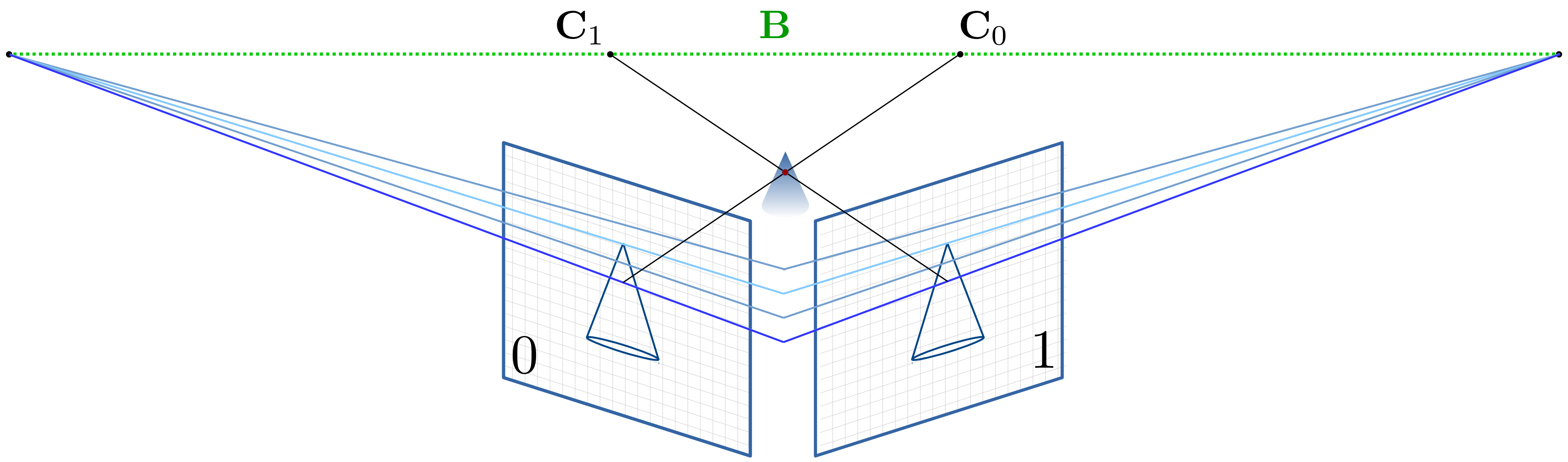
There exists a pencil of epipolar lines!



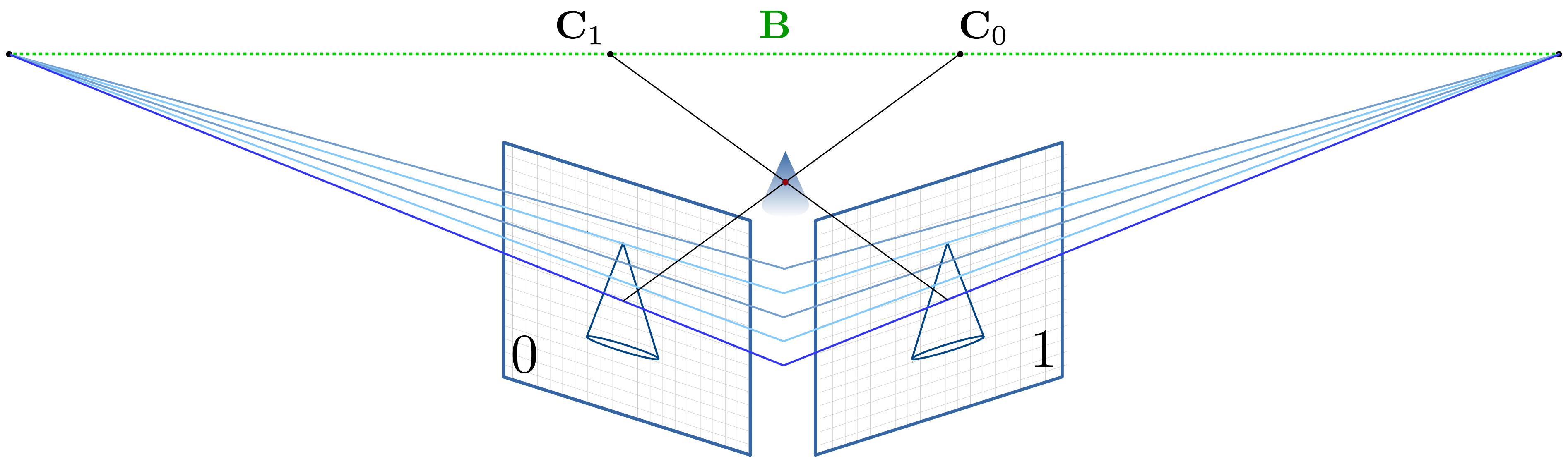
Epipolar Geometry

The geometry of two views

There exists a pencil of epipolar lines!



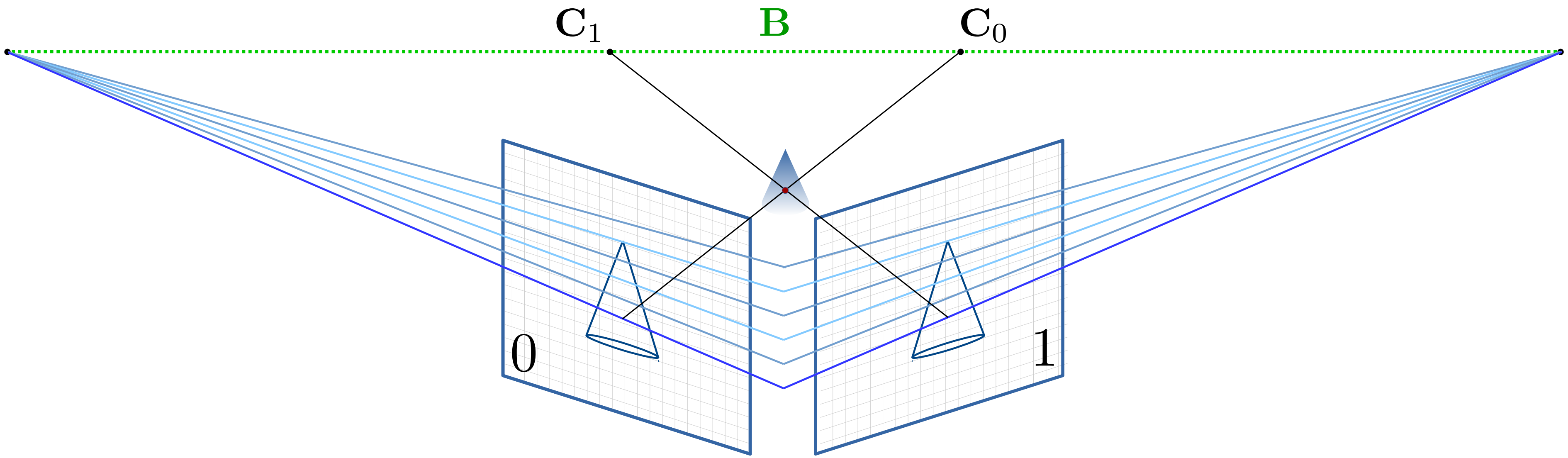
There exists a pencil of epipolar lines!



Epipolar Geometry

The geometry of two views

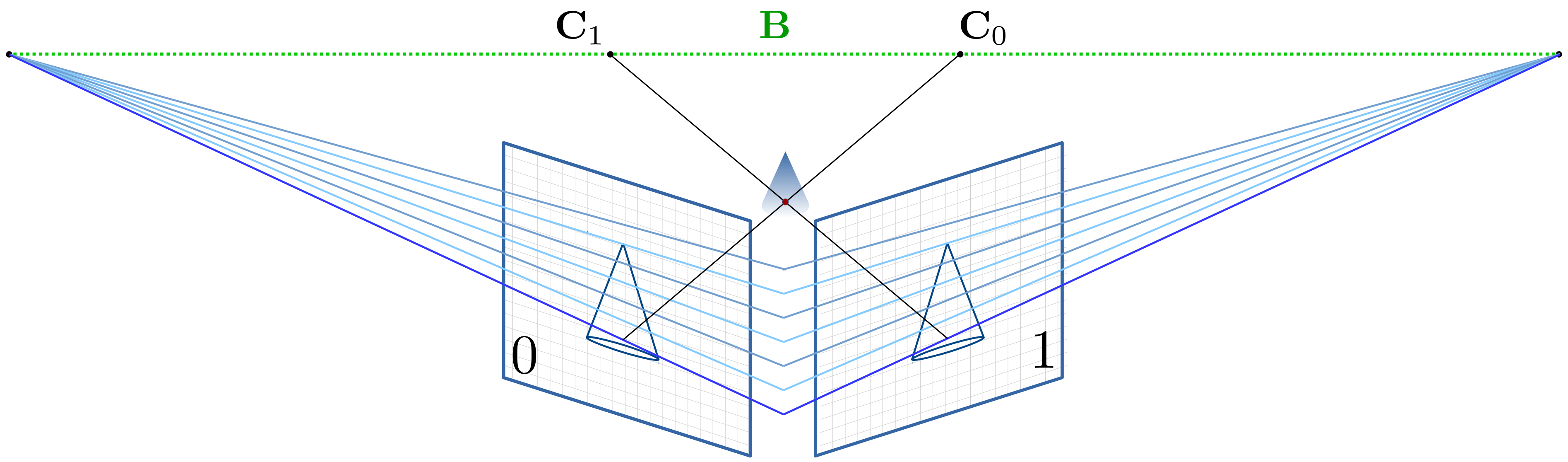
There exists a pencil of epipolar lines!



Epipolar Geometry

The geometry of two views

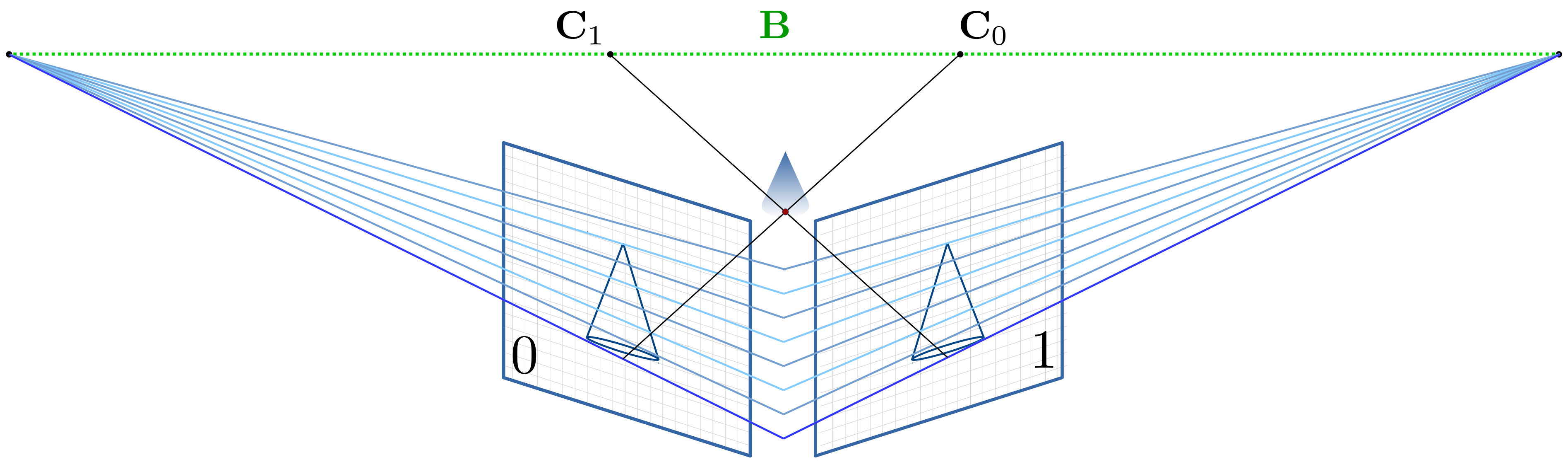
There exists a pencil of epipolar lines!



Epipolar Geometry

The geometry of two views

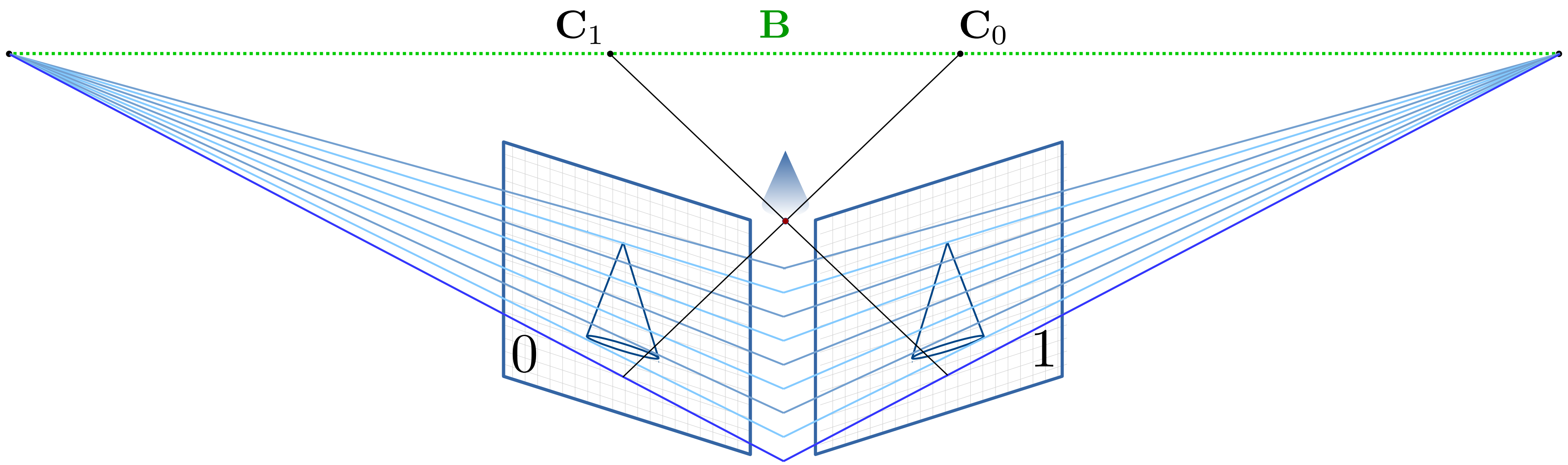
There exists a pencil of epipolar lines!



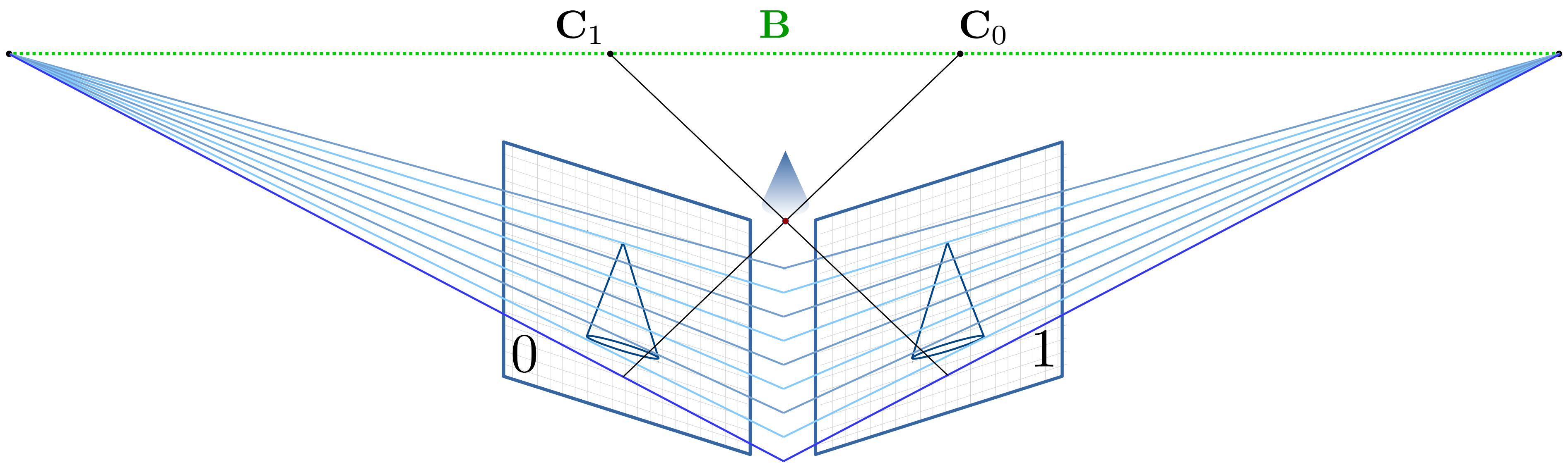
Epipolar Geometry

The geometry of two views

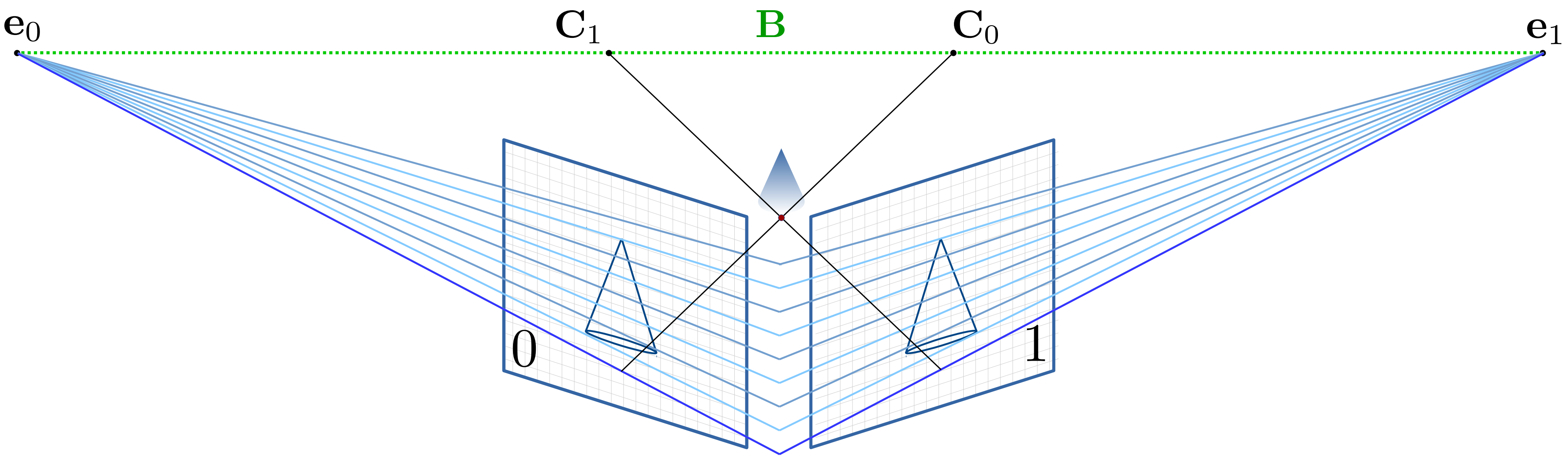
There exists a pencil of epipolar lines!



Observe: All epipolar planes contain the **stereo baseline** $\mathbf{B}$.



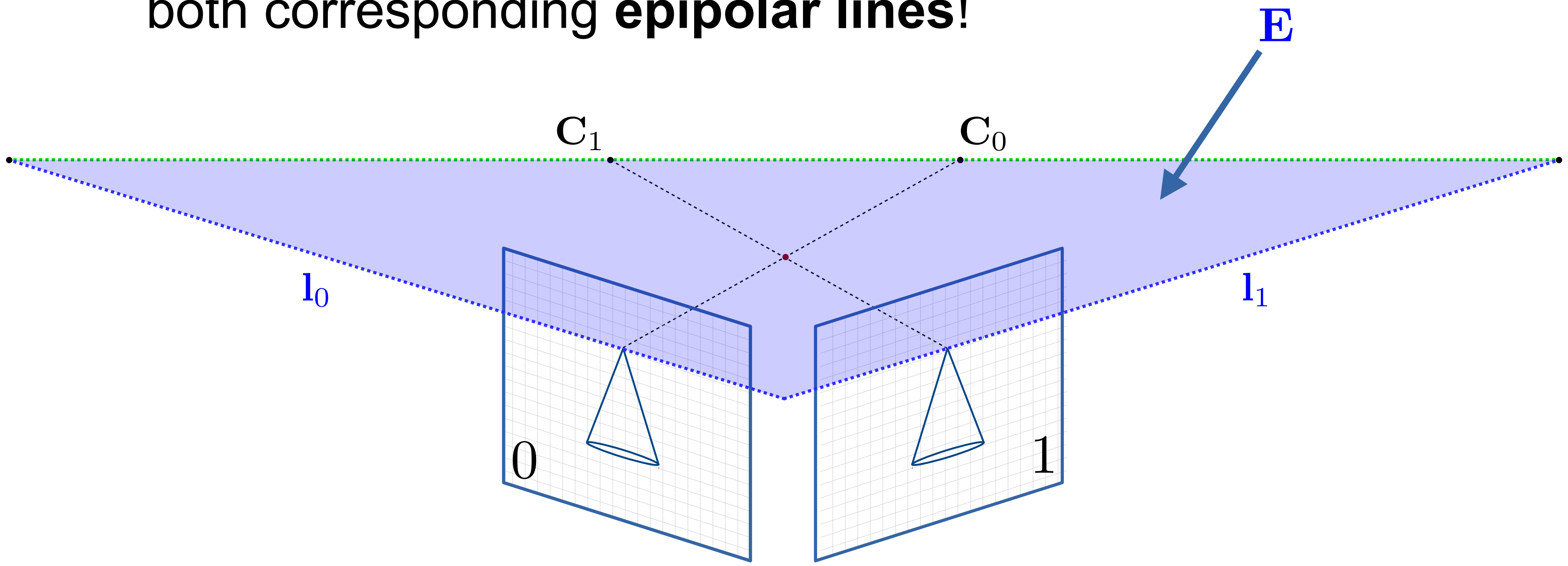
Observe: The image planes intersect the baseline $\mathbf{B}$ in the so-called epipoles.



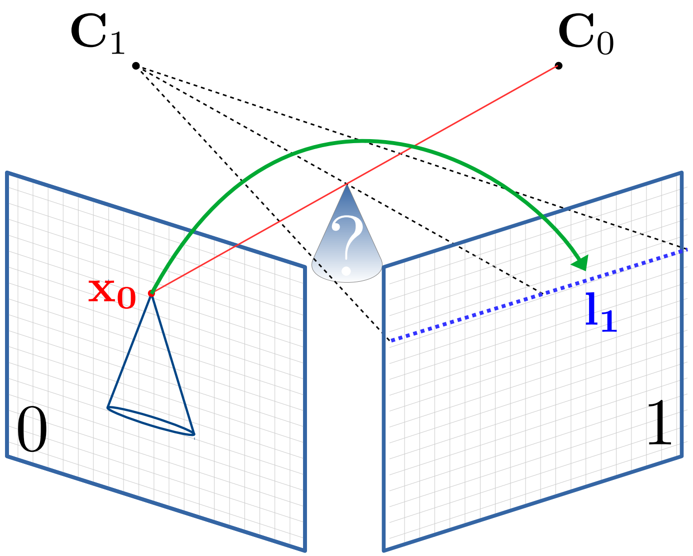
Epipolar Geometry

The geometry of two views

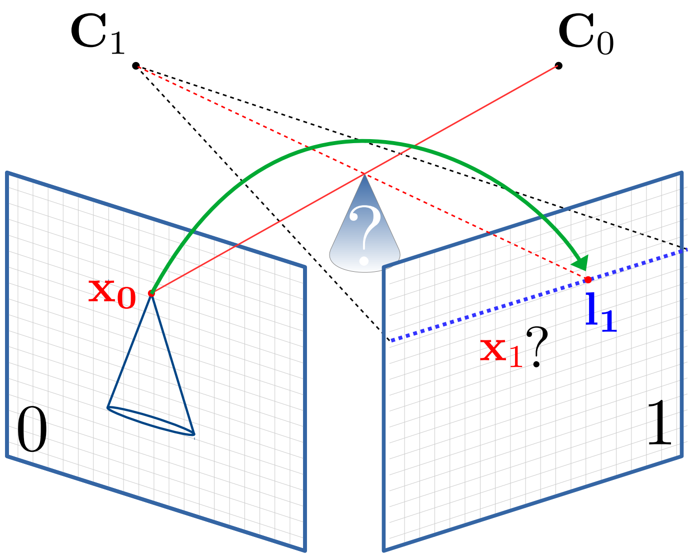
The **epipolar plane** contains both camera centers, the **epipoles** and both corresponding **epipolar lines**!



The epipolar line contains the corresponding image point: $\mathbf{x}_1^\top \mathbf{l}_1 = 0$



Idea: estimate depth from stereo disparity!



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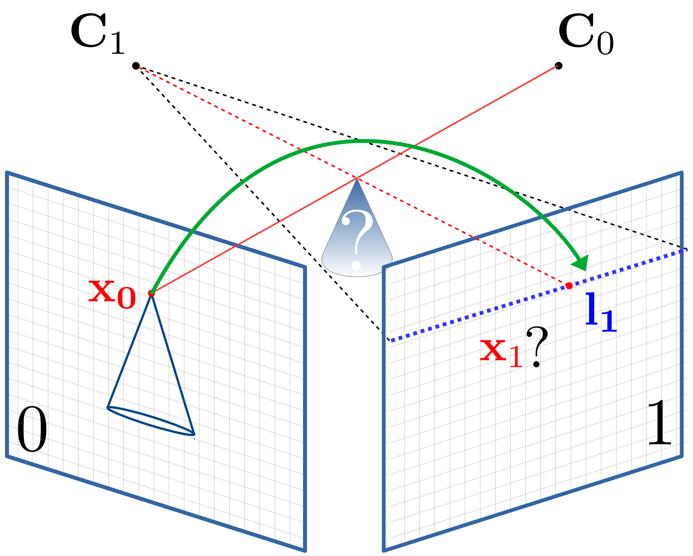

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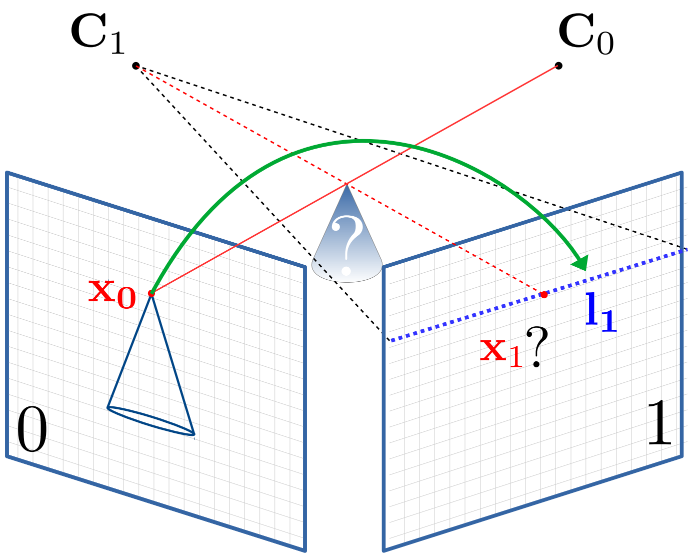
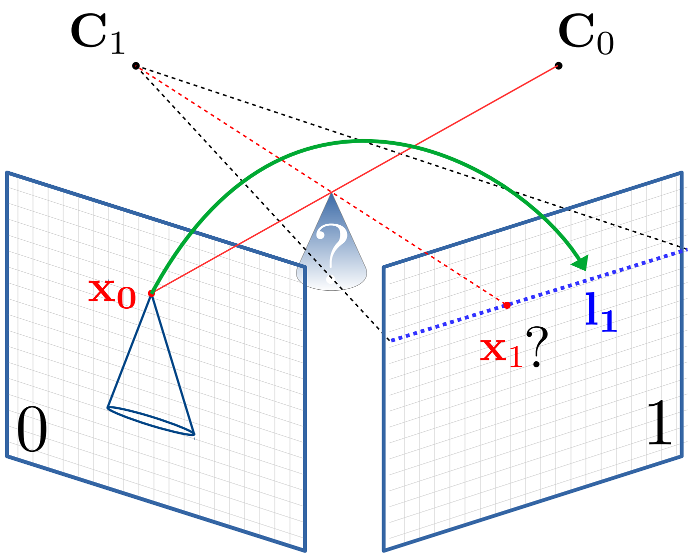
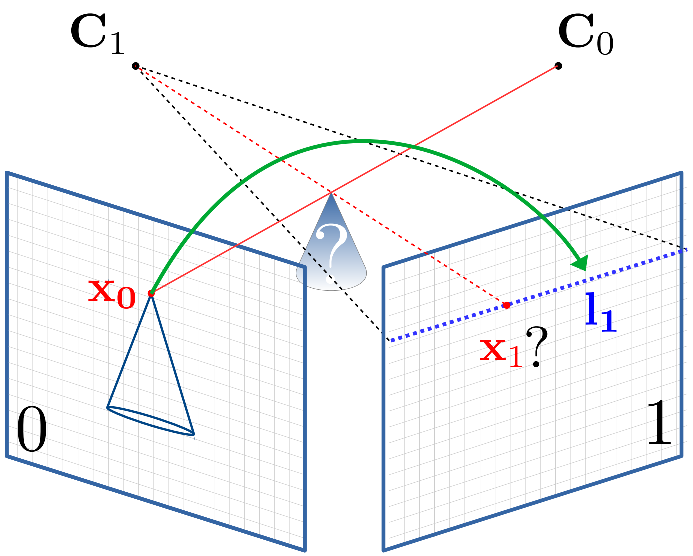


Idea: estimate depth from stereo disparity!



Idea: estimate depth from stereo disparity!

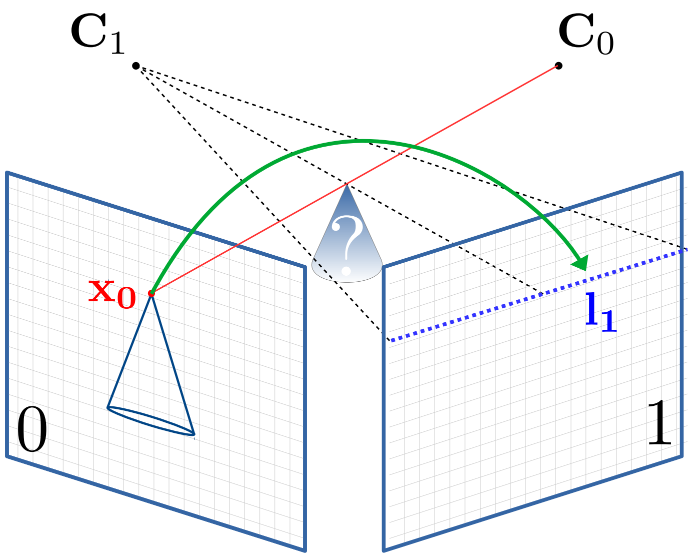


Algebraic Estimation of the Fundamental Matrix

Epipolar Geometry

Fundamental matrix

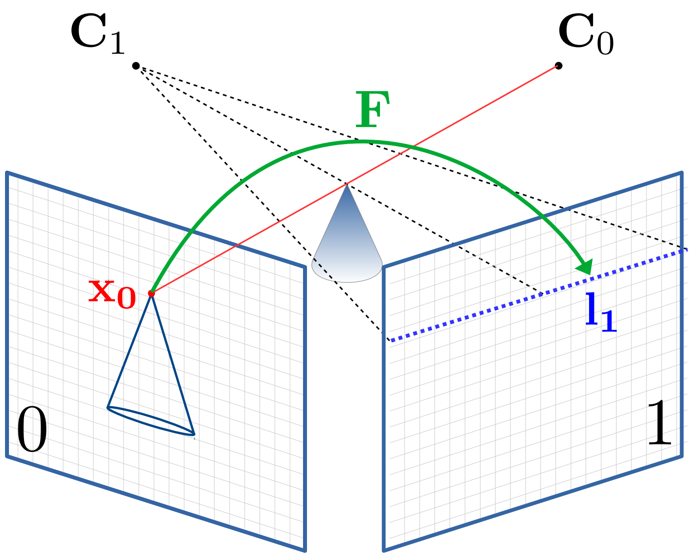
For any point in image 0, we get a corresponding line in image 1.



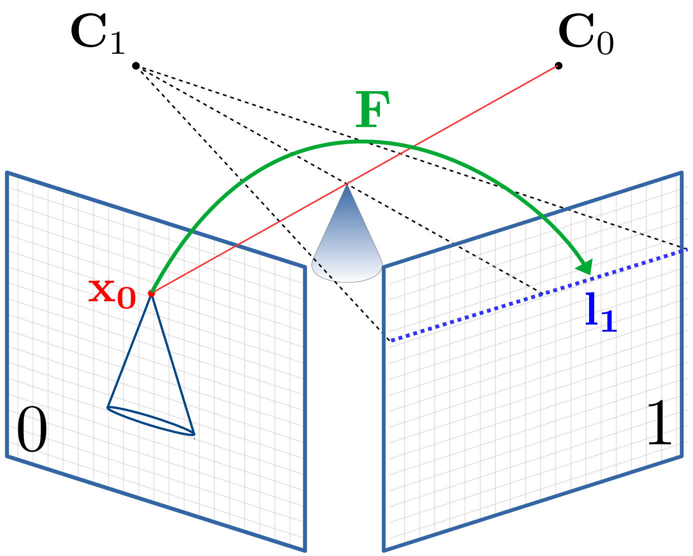
Epipolar Geometry

Fundamental matrix

There is a linear mapping $\mathbf{F}$ from points to lines in the other image.

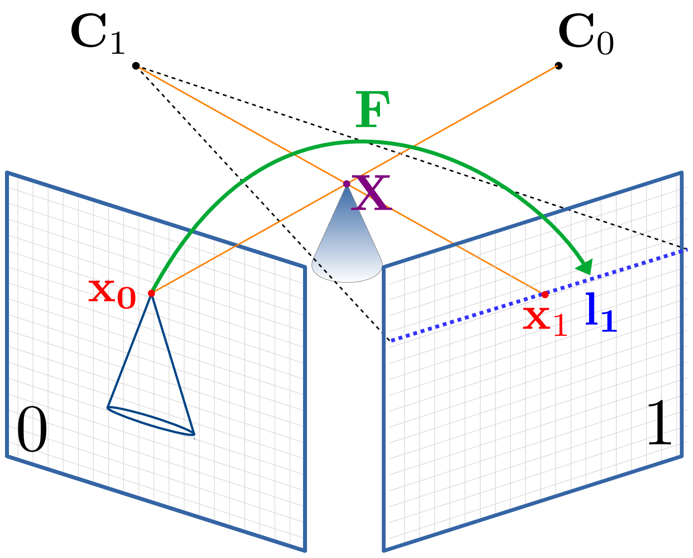


There is a linear mapping $\mathbf{F}$ from points to lines in the other image. Let's find it!

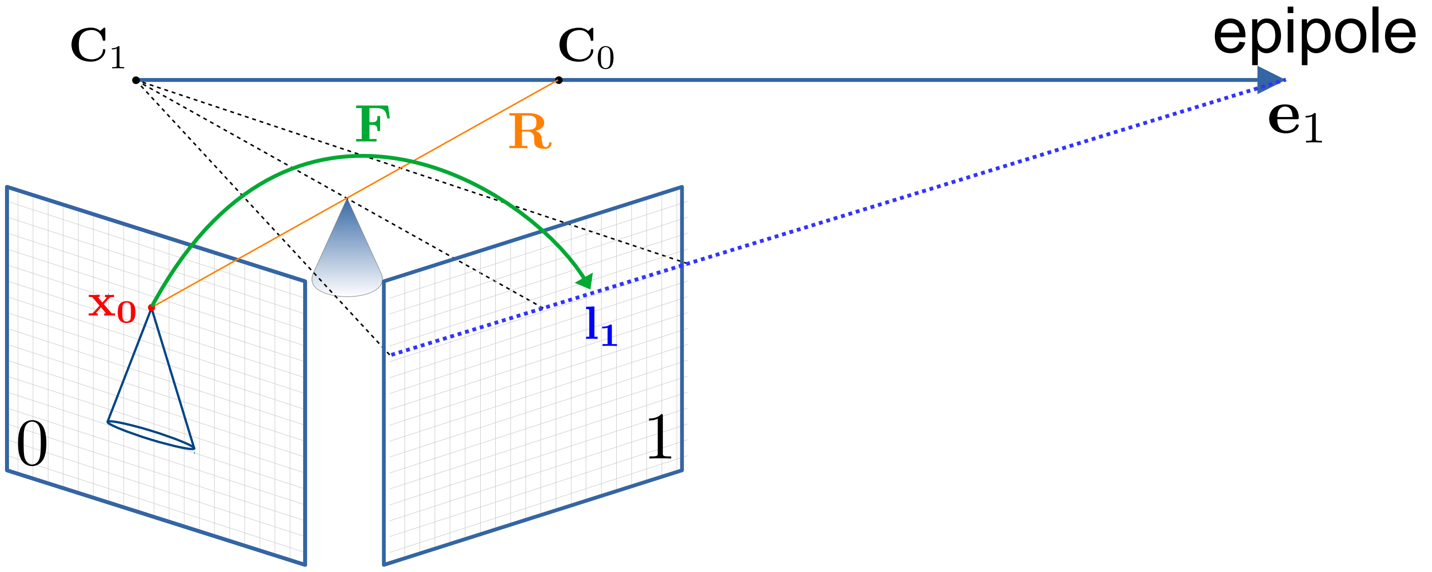


Suppose that the same world point is seen by two cameras

$$\mathbf{x}_0 = \mathbf{P}_0\mathbf{X}; \quad \mathbf{x}_1 = \mathbf{P}_1\mathbf{X}$$

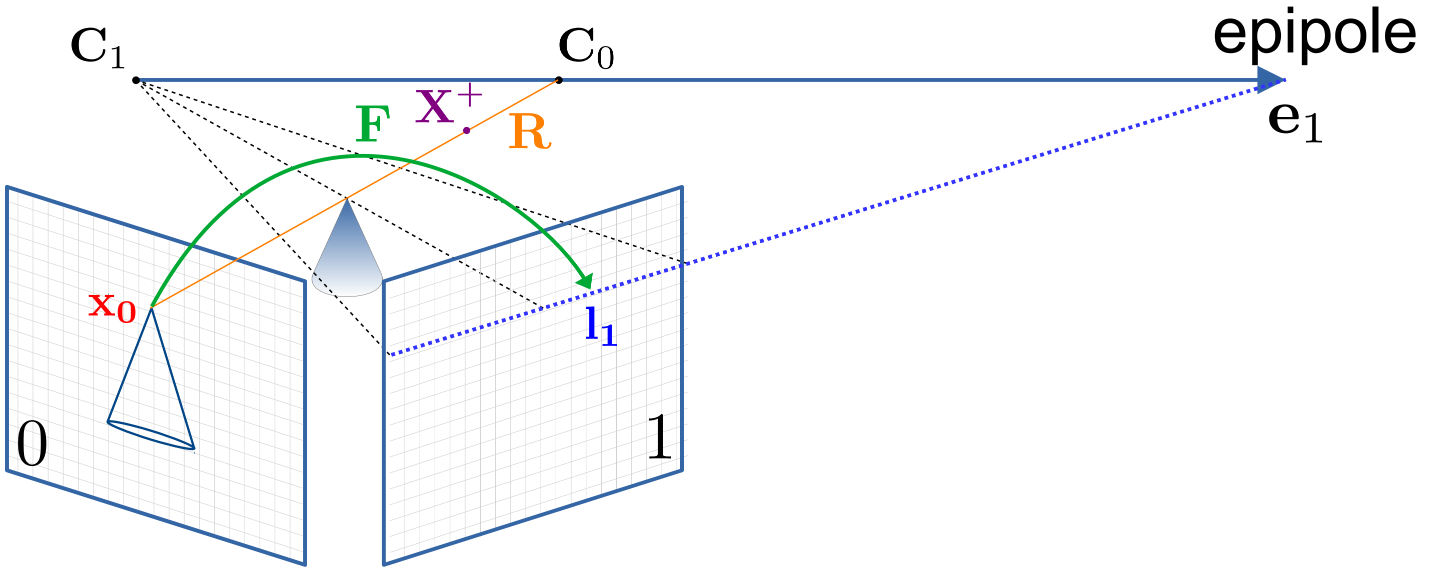


A line is defined by two distinct points. The backprojection ray passes through the center of projection, so $\mathbf{e}_1 \cong \mathbf{P}_1\mathbf{C}_0$ is on the ray and on the epipolar line.



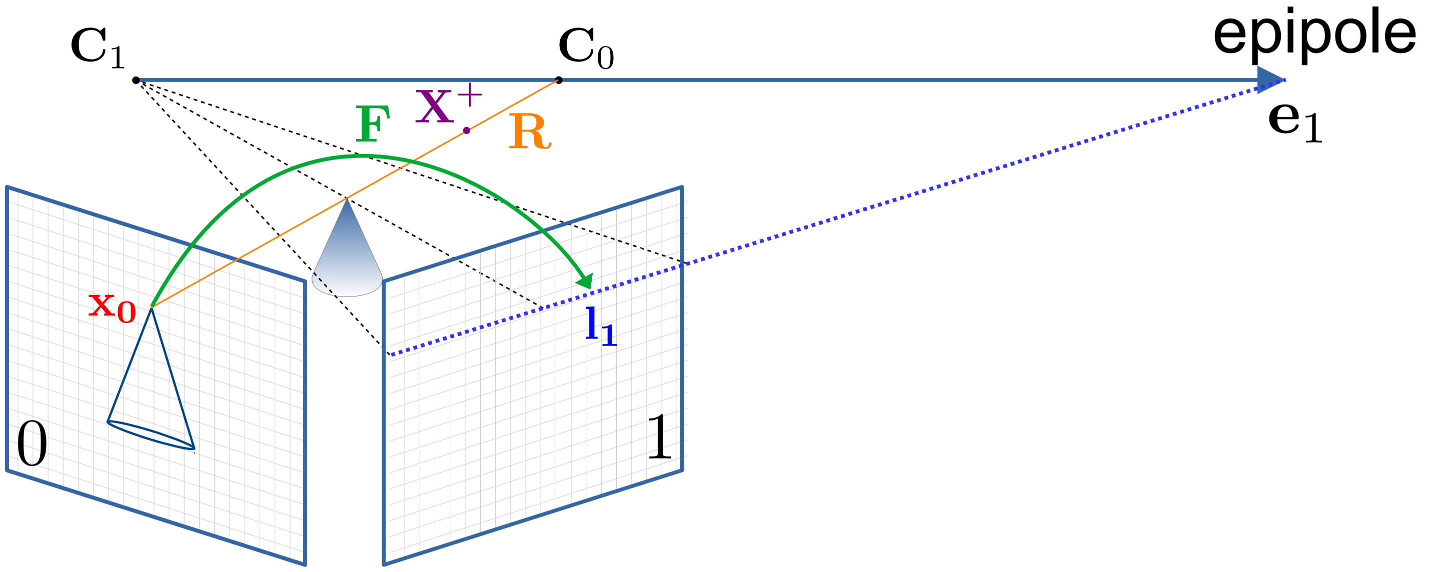
A second point is obtained by backprojection of $\mathbf{x}_0$:

$\mathbf{X}^+ \cong \mathbf{P}^+\mathbf{x}_0$ (it is somewhere on the ray!)



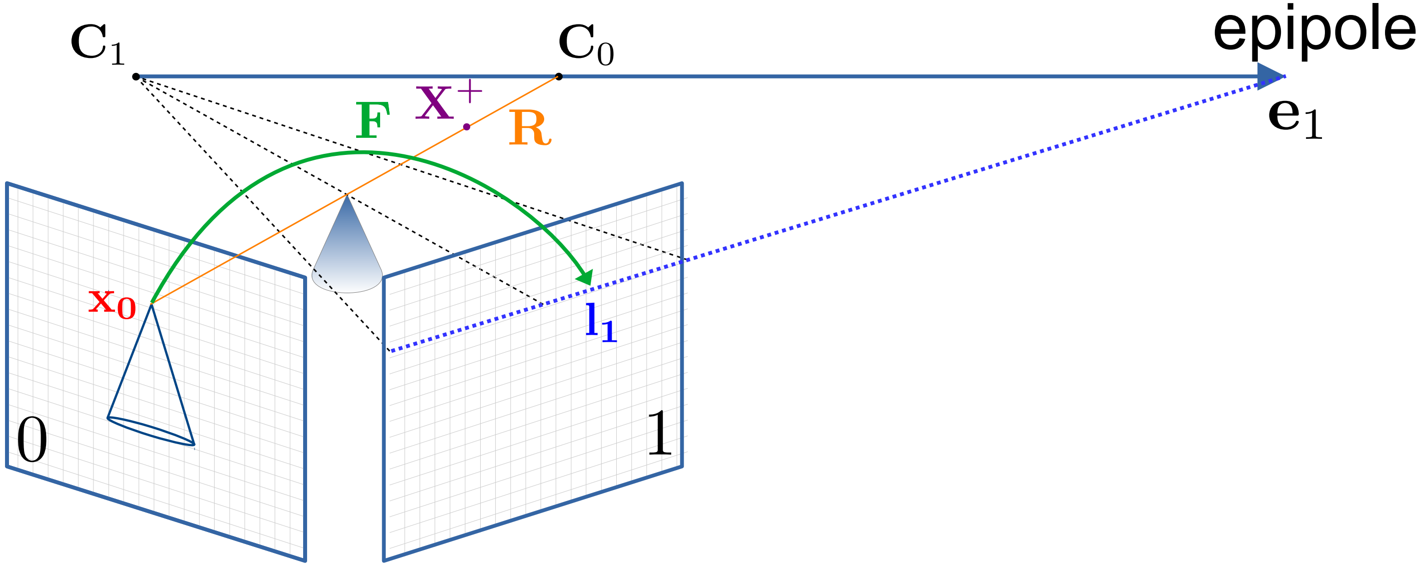
From these two points, we can compute the epipolar line:

$$\mathbf{l}_1 \cong \mathbf{P}_1\mathbf{C}_0 \times \mathbf{P}_1\mathbf{P}_0^+\mathbf{x}_0$$



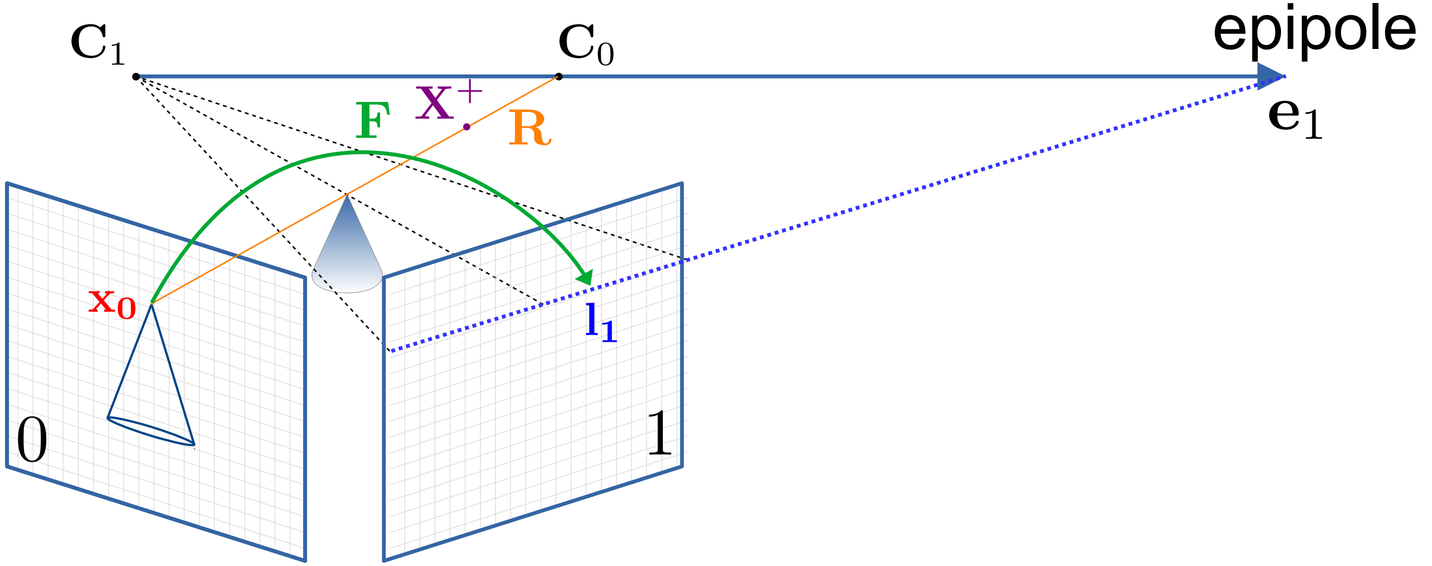
From these two points, we can compute the epipolar line:

$$\mathbf{l}_1 \cong \mathbf{P}_1\mathbf{C}_0 \times \mathbf{P}_1\mathbf{P}_0^{+}\mathbf{x}_0$$
$$= \left([\mathbf{P}_1\mathbf{C}_0]_\times \mathbf{P}_1\mathbf{P}_0^{+}\right)\mathbf{x}_0$$



From these two points, we can compute the epipolar line:

$$\mathbf{l}_1 \cong \mathbf{P}_1\mathbf{C}_0 \times \mathbf{P}_1\mathbf{P}_0^{+}\mathbf{x}_0$$
$$= \left([\mathbf{P}_1\mathbf{C}_0]_\times \mathbf{P}_1\mathbf{P}_0^{+}\right)\mathbf{x}_0$$

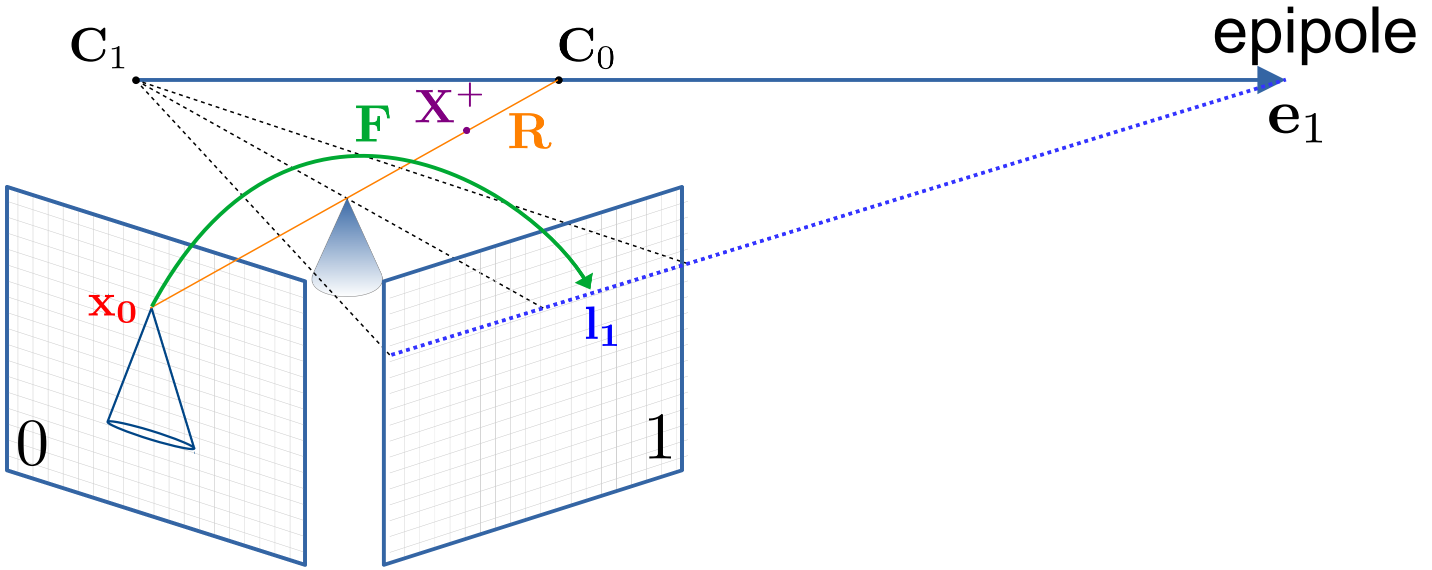


Epipolar Geometry

Fundamental matrix

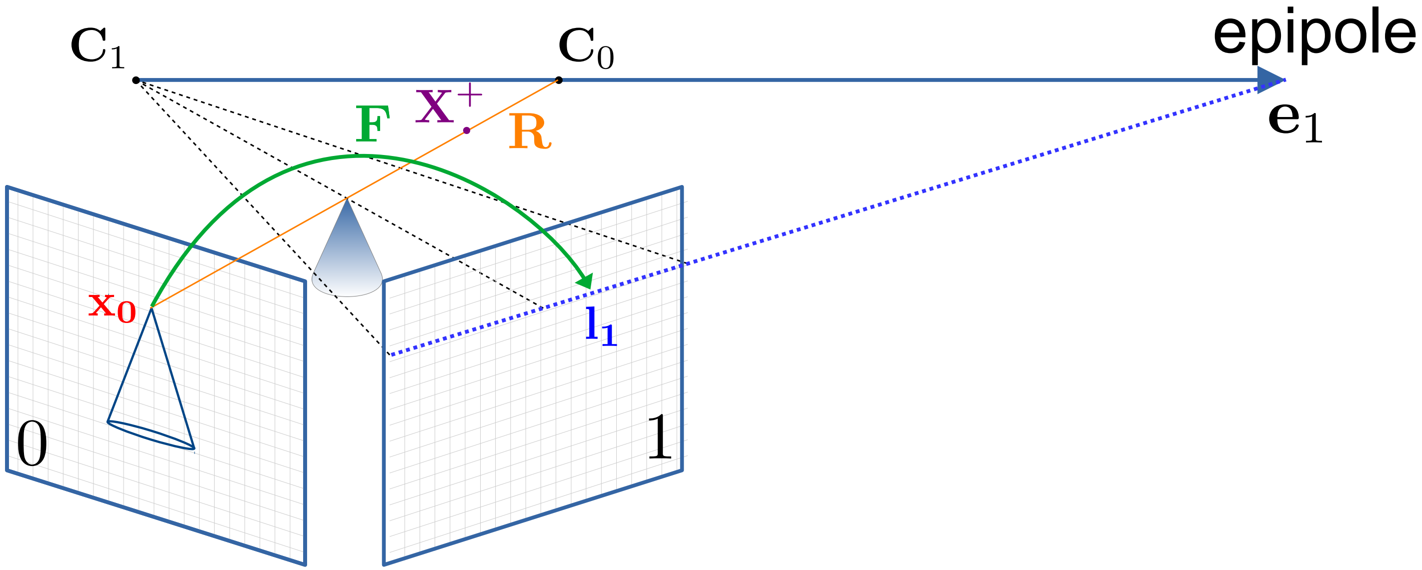
From these two points, we can compute the epipolar line:

$$\mathbf{l}_1 \cong \mathbf{P}_1\mathbf{C}_0 \times \mathbf{P}_1\mathbf{P}_0^{+}\mathbf{x}_0$$
$$= \left([\mathbf{e}_1]_\times \mathbf{P}_1\mathbf{P}_0^{+}\right) \mathbf{x}_0$$



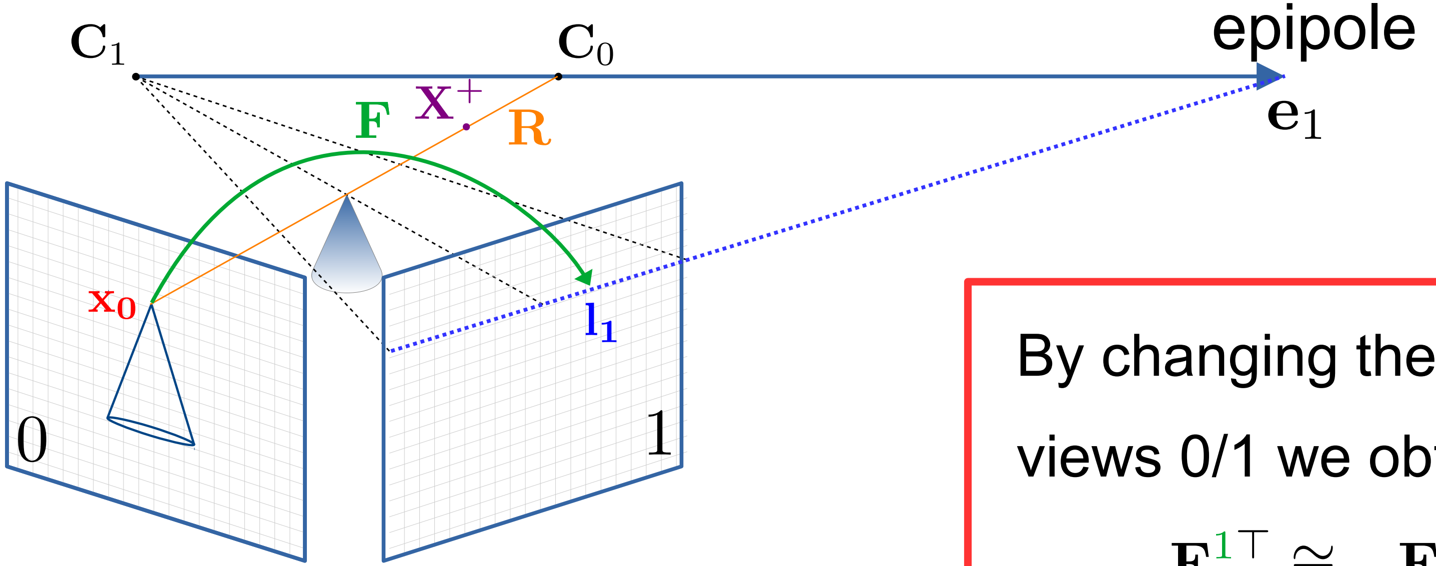
From these two points, we can compute the epipolar line:

$$\mathbf{l}_1 \cong \left([\mathbf{e}_1]_\times \mathbf{P}_1 \mathbf{P}_0^+ \right) \mathbf{x}_0$$
$$= \mathbf{F}_0^1 \mathbf{x}_0$$



From these two points, we can compute the epipolar line:

$$\mathbf{l}_1 \cong \left([\mathbf{e}_1]_\times \mathbf{P}_1 \mathbf{P}_0^+\right) \mathbf{x}_0$$
$$= \mathbf{F}_0^1 \mathbf{x}_0$$



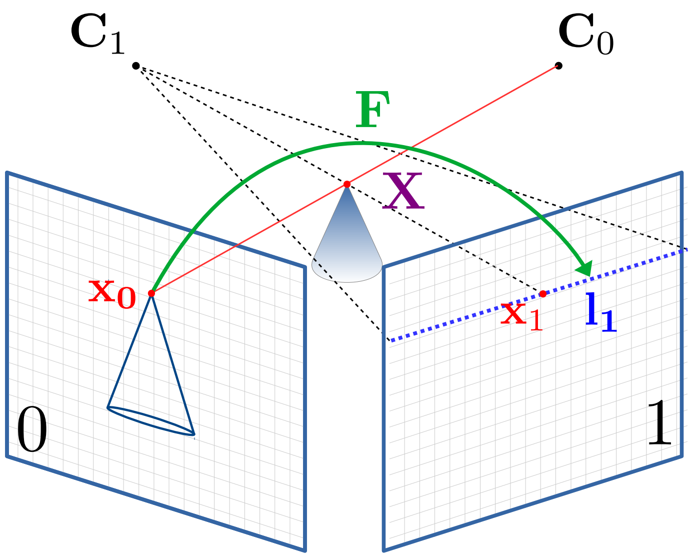
By changing the order of views 0/1 we obtain:

$$\mathbf{F}_0^{1\top} \cong -\mathbf{F}_1^{0\top}$$

Epipolar Geometry

Fundamental matrix

The epipolar line contains the corresponding image point: $\mathbf{x}_1^\top \mathbf{l}_1 = 0$



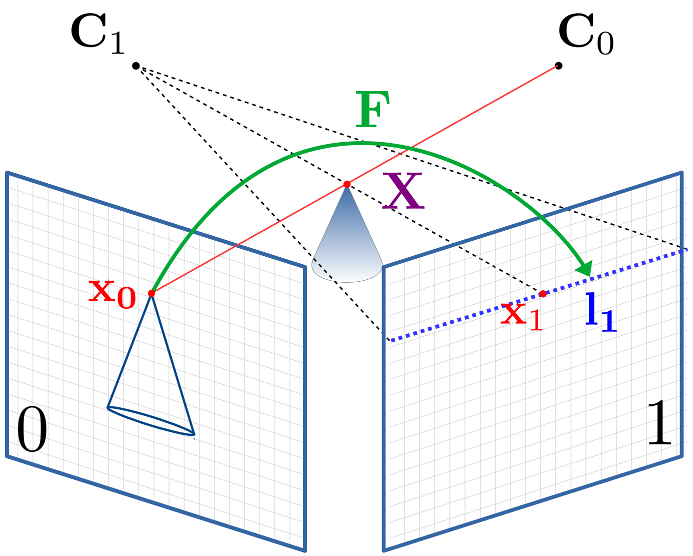
$\mathbf{F}$ is called the **fundamental matrix** and has shape 3x3:

$$\mathbf{l}_1 = \mathbf{F}\mathbf{x}_0$$

Epipolar Geometry

Epipolar constraint

The **epipolar constraint** has to hold for corresponding image points.



$$\mathbf{l}_1 = \mathbf{F}\mathbf{x}_0$$

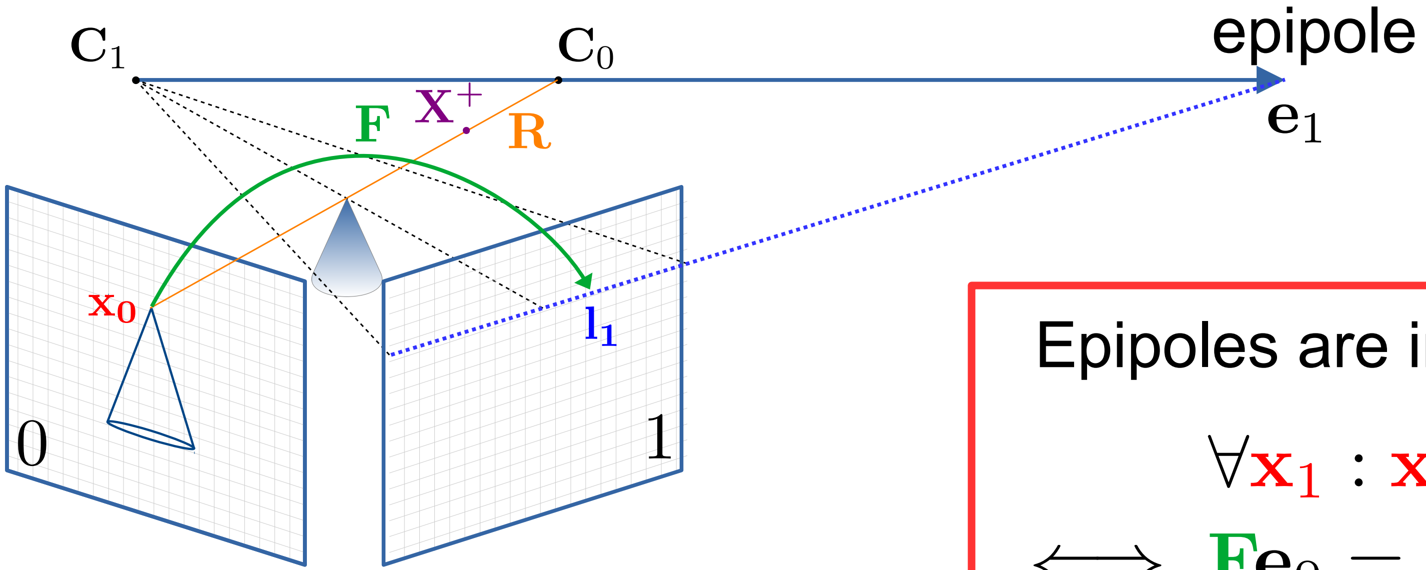
$$\mathbf{x}_1^\top \mathbf{l}_1 = 0$$

Together we have:

$$\mathbf{x}_1^\top \mathbf{F}\mathbf{x}_0 = 0$$

The epipoles are in the right and left null-space of $\mathbf{F}$.

$\mathbf{F}$ is rank two.



Epipoles are in null-space

$$\forall \mathbf{x}_1 : \mathbf{x}_1^\top \mathbf{F}\mathbf{e}_0 = 0$$

$$\iff \mathbf{F}\mathbf{e}_0 = 0$$

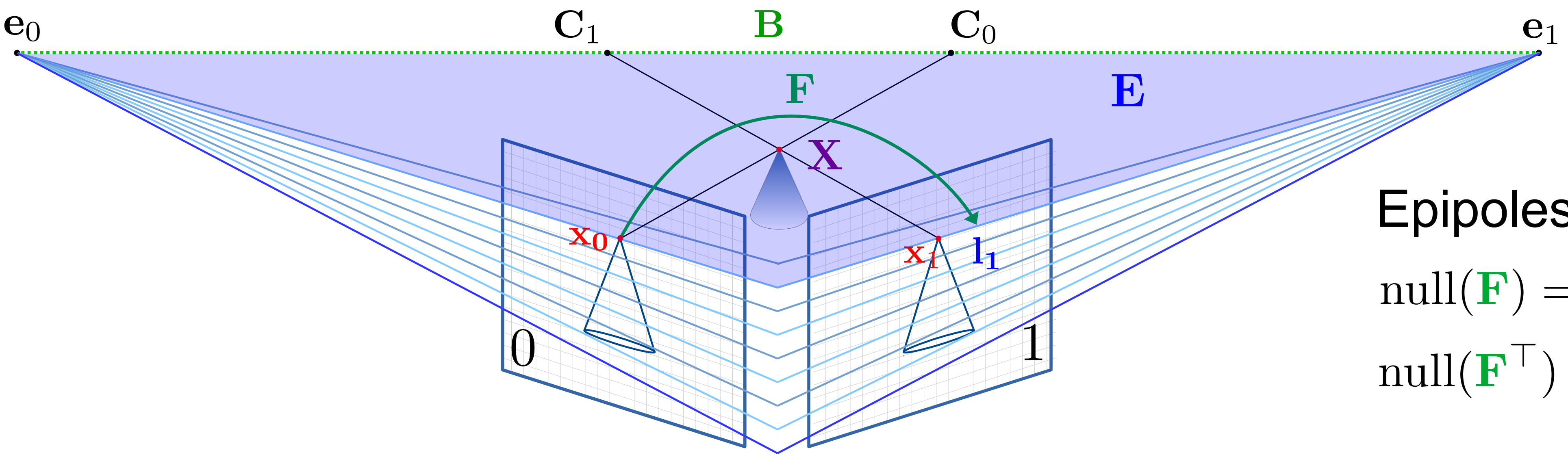
Epipolar Geometry

Summary

Corresponding image points $\mathbf{x}_0 = \mathbf{P}_0\mathbf{X}; \quad \mathbf{x}_1 = \mathbf{P}_1\mathbf{X}$ fulfill $\mathbf{x}_1^\top \mathbf{F}\mathbf{x}_0 = 0$.

An epipolar plane $\mathbf{E}$ is defined by two camera centers and an additional point $\mathbf{X}$.

The epipolar line contains the corresponding image point $\mathbf{l}_1 = \mathbf{F}\mathbf{x}_0$ with $\mathbf{x}_1^\top \mathbf{l}_1 = 0$.



Epipoles are in null-space

$$\mathrm{null}(\mathbf{F}) = \mathbf{e}_0$$

$$\mathrm{null}(\mathbf{F}^\top) = \mathbf{e}_1$$

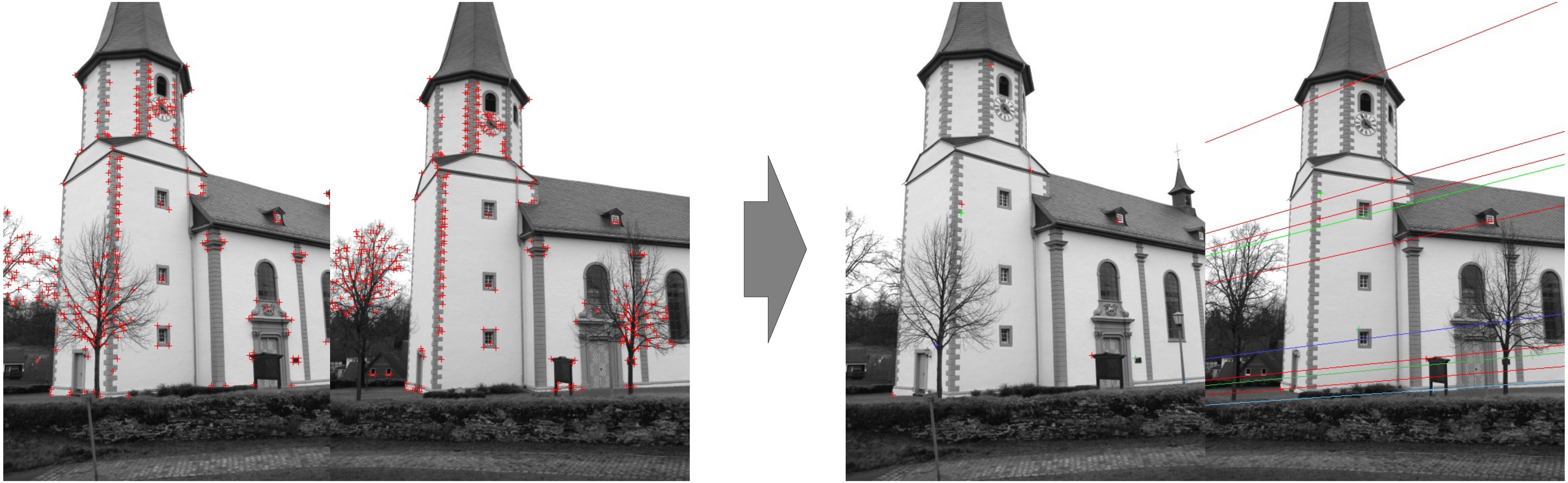
Epipolar planes contain the stereo baseline $\mathbf{B}$. Epipolar lines meet in the epipoles.

Algebraic Estimation of the Fundamental Matrix

Algebraic Estimation

Eight-point algorithm

Goal: From point correspondences in two images, estimate fundamental matrix.



For each point, the epipolar constraint gives a constraints of the form:

$$\mathbf{x_1}^\top \mathbf{F} \mathbf{x_0} = 0$$

$$\text{with} \quad \mathbf{F} = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}$$

expands to

$$\boxed{u_1 u_0 f_{11} + u_1 v_0 f_{12} + u_1 f_{13} + v_1 u_0 f_{21} + v_1 v_0 f_{22} + v_1 f_{23} + u_0 f_{31} + v_0 f_{32} + f_{33} = 0}$$

Algebraic Estimation

Eight-point algorithm

Stack constrains for at least 8 linearly independent pionts

$$\begin{pmatrix} u_1^0 u_0^0 & u_1^0 v_0^0 & u_1^0 & v_1^0 u_0^0 & v_1^0 v_0^0 & v_1^0 & u_0^0 & v_0^0 & 1 \\ & & & & & & & & 1 \\ & \dots & & \dots & & \dots & & \dots & 1 \\ & & & & & & & & 1 \\ u_1^n u_0^n & u_1^n v_0^n & u_1^n & v_1^n u_0^n & v_1^n v_0^n & v_1^n & u_0^n & v_0^n & 1 \end{pmatrix} \begin{pmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{pmatrix} = 0$$

Solve with SVD or similar

Algebraic Estimation

Eight-point algorithm

Note on instability due to algebraic error
being optimized in vastly differently scaled variables.

The below order of magnitudes assume pixel coordinates in the hundreds.

$$\boxed{u_1 u_0 f_{11} + u_1 v_0 f_{12} + u_1 f_{13} + v_1 u_0 f_{21} + v_1 v_0 f_{22} + v_1 f_{23} + u_0 f_{31} + v_0 f_{32} + f_{33} = 0}$$

~10000 ~10000 ~100 ~10000 ~10000 ~100 ~100 ~100 ~1

Mitigation: change coordinate system to be centered in image and coordinates in the order of one.
(ideally, scaled centroid with square root of two diameter)

Problem: Fundamental matrix is a singular matrix

(Why? Consider that skew symmetric matrices $[\ .\ ]_\times$ are rank deficient!)

Using the singular value decomposition of our initial estimate

$$\tilde{\mathbf{F}} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top = \mathbf{U}\begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix}\mathbf{V}^\top$$

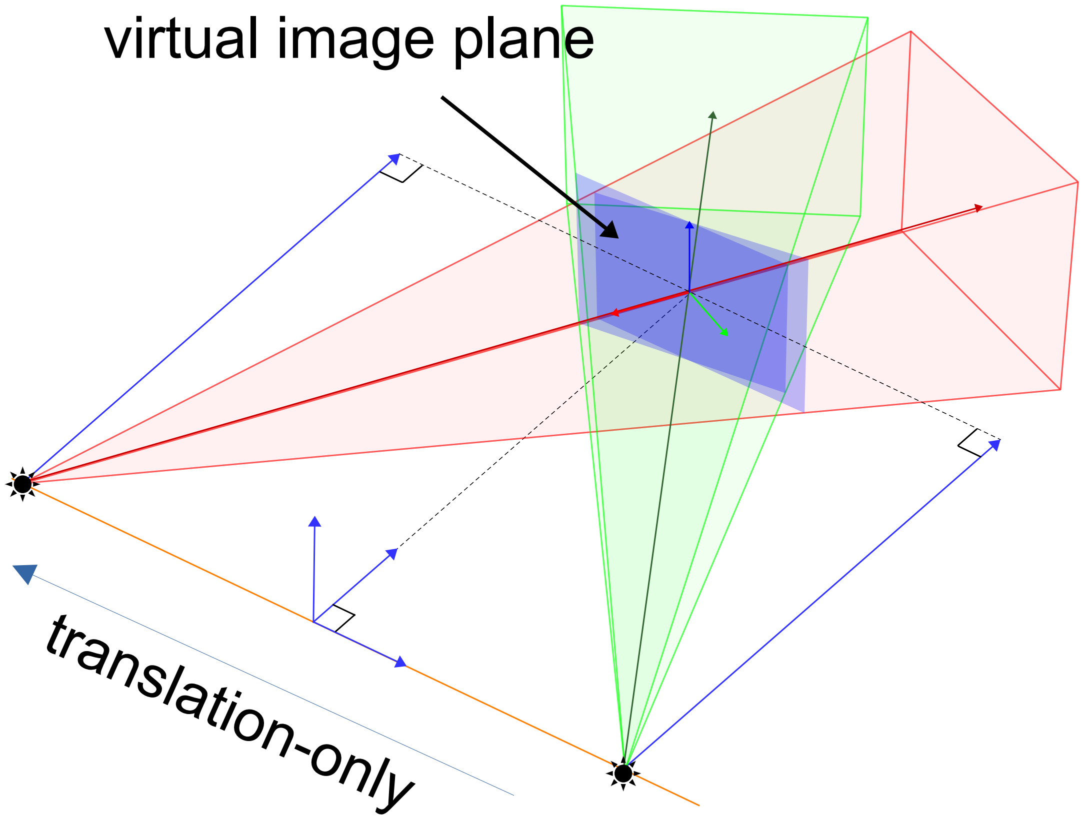
We can determine the closest rank-deficient matrix according to Mahalanobis distance:

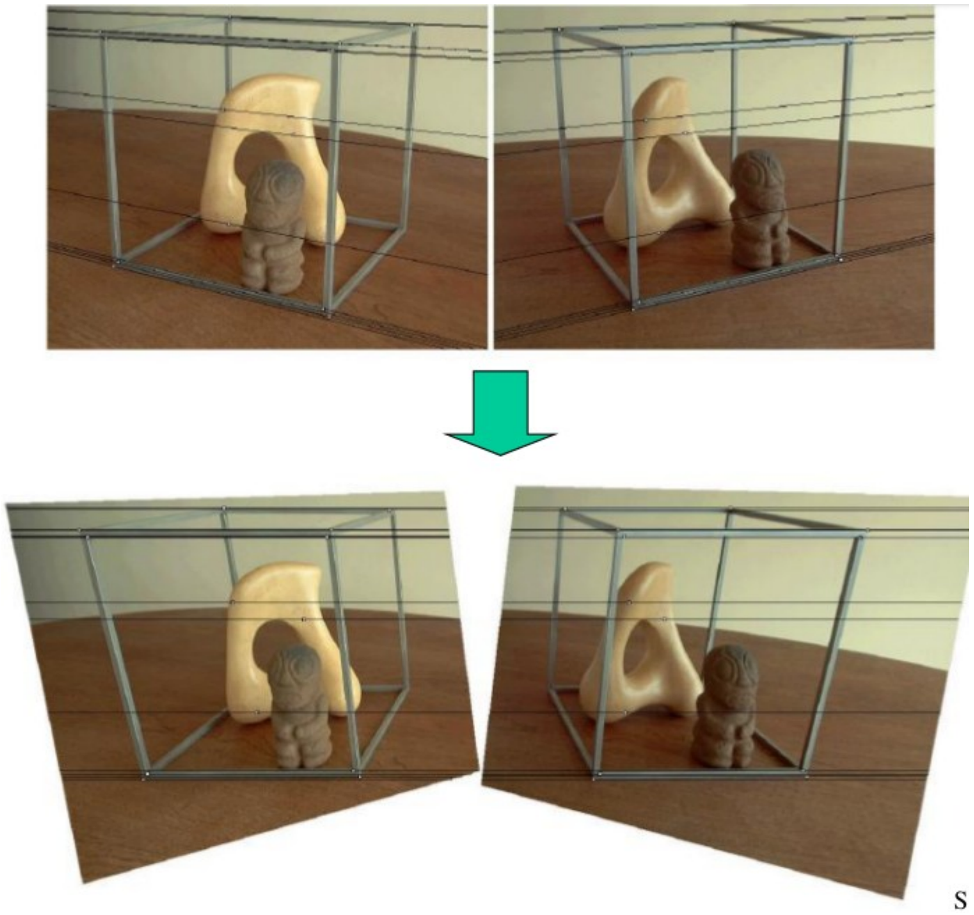
$$\mathbf{F} = \mathbf{U}\begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & 0 \end{pmatrix}\mathbf{V}^\top$$

Rectification

Speeding up disparity search

Idea: Given fundamental matrix, transform both images to make all epipolar lines parallel to image axis.

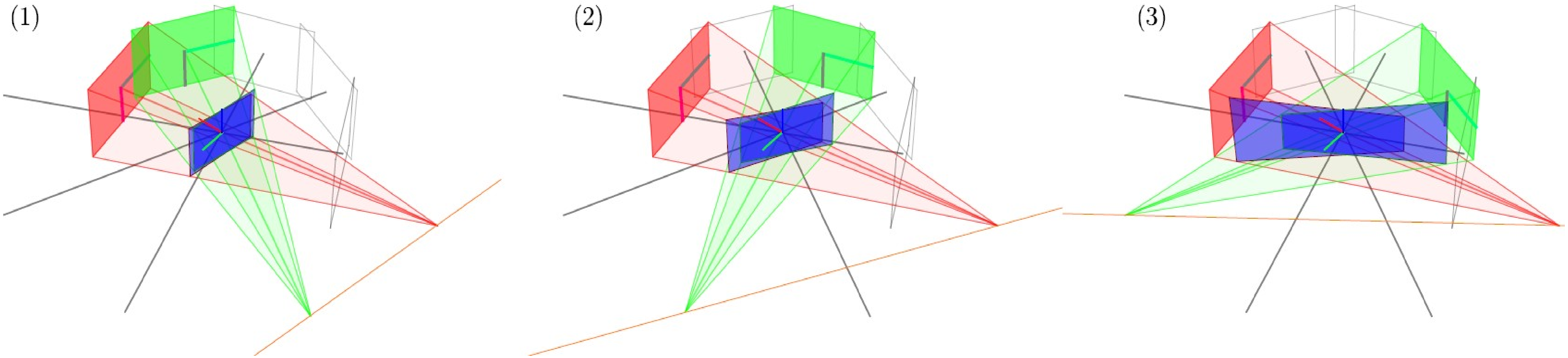




Rectification

Rectification by virtual rotation

The larger the angle between views, the more image distortion is caused.



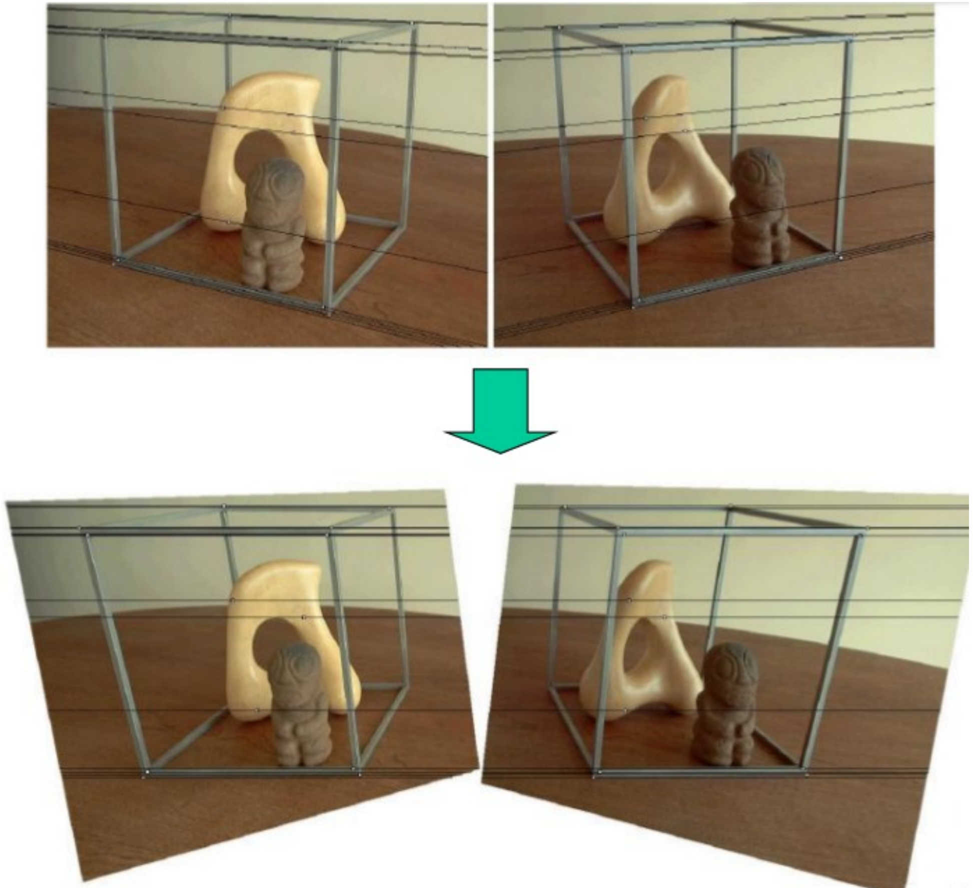
Accomplishes with two image transformations

Desired properties after transformation:

- Principal rays are parallel (epipoles are at infinity)
- Rotation aligned with pixels for fast sliding window correspondence search.
- Minimize distortion

- Several methods exist (even non-linear ones)

(e.g. Pollefeys M, Koch R, Van Gool L. A simple and efficient rectification method for general motion. In Proceedings of the Seventh IEEE International Conference on Computer Vision 1999)



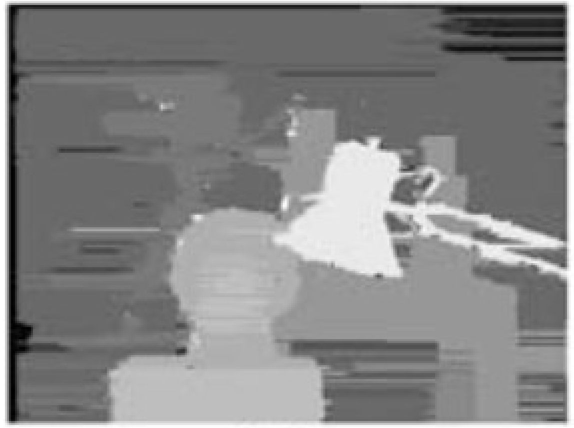
S

Rectification

Example of disparity map

Input Stereo Pair

Block Matching



Ground Truth

Friedrich-Alexander-Universität
Technische Fakultät

Pattern Recognition Lab

Thank You.

Literature.

H. Coxeter, Projective Geometry. Springer New York, 2003.

J. Richter-Gebert, Perspectives on Projective Geometry: A Guided Tour Through Real and Complex Geometry. Springer, 2011.

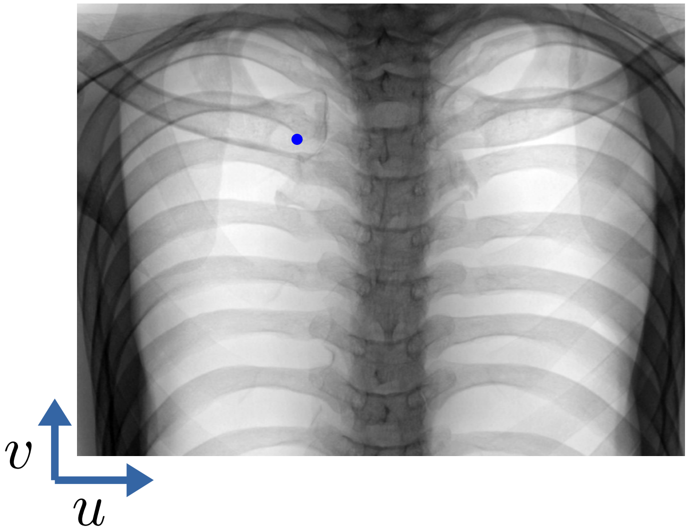
R. I. Hartley and A. Zisserman, Multiple View Geometry in Computer Vision. Cambridge University Press, ISBN: 0521623049, 2000. Pages 72/73

Bonus material:
Application to X-ray imaging

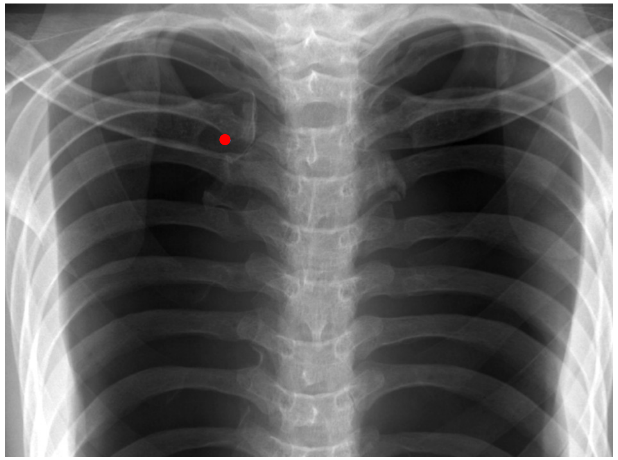
FDCT C-Arm Calibration

X-Ray Attenuation

$$I(u,v) = I_0 e^{-\int \mu(x)dx}$$



$$-log\left(\frac{I(u,v)}{I_0}\right)$$



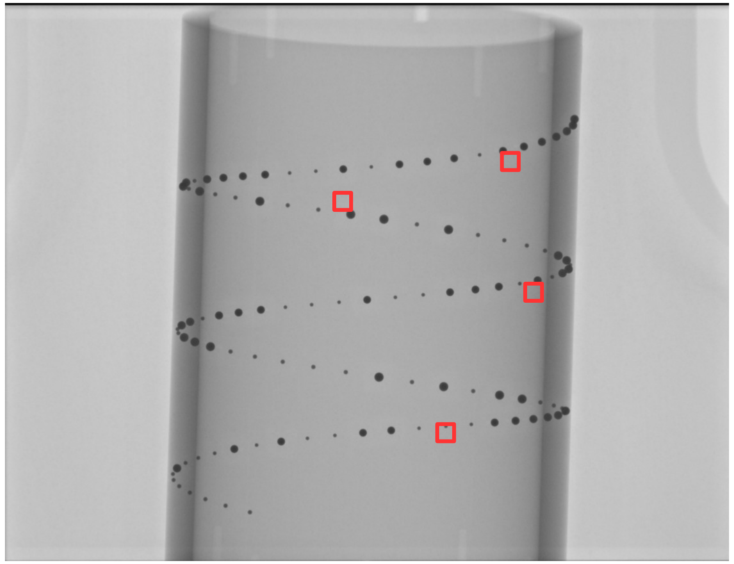
Detected Intensity

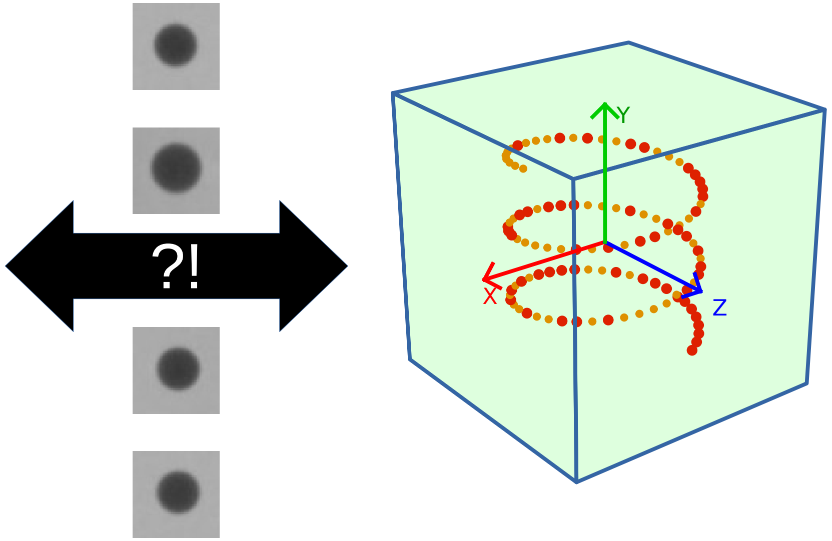
Line Integrals of
Absorption Coefficients

FDCT C-Arm Calibration

Estimation of the projection matrix

CV answer: find analytically special image points
→ **Locally** defined for a certain scale
→ **Invariant** w.r.t. scale, lighting etc.
→ **Salient** compared to other points





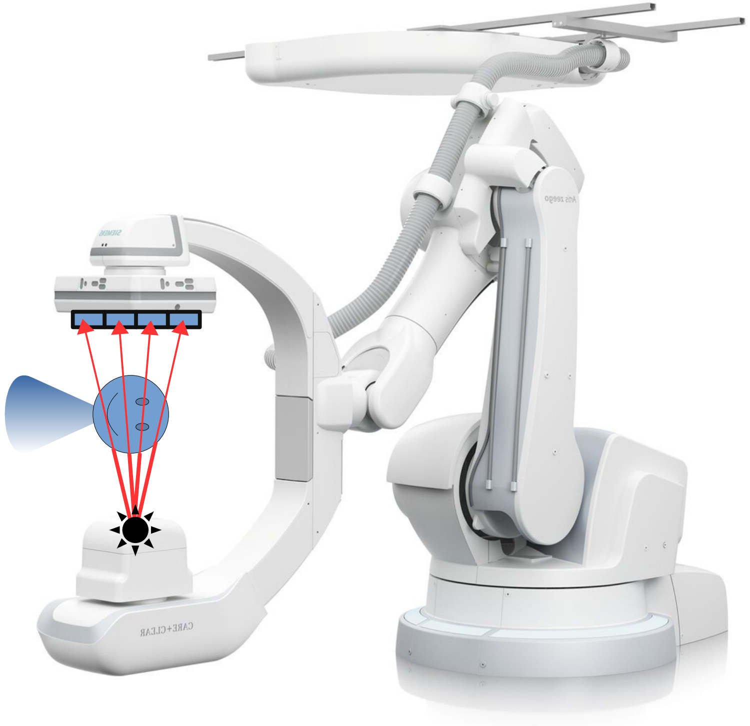
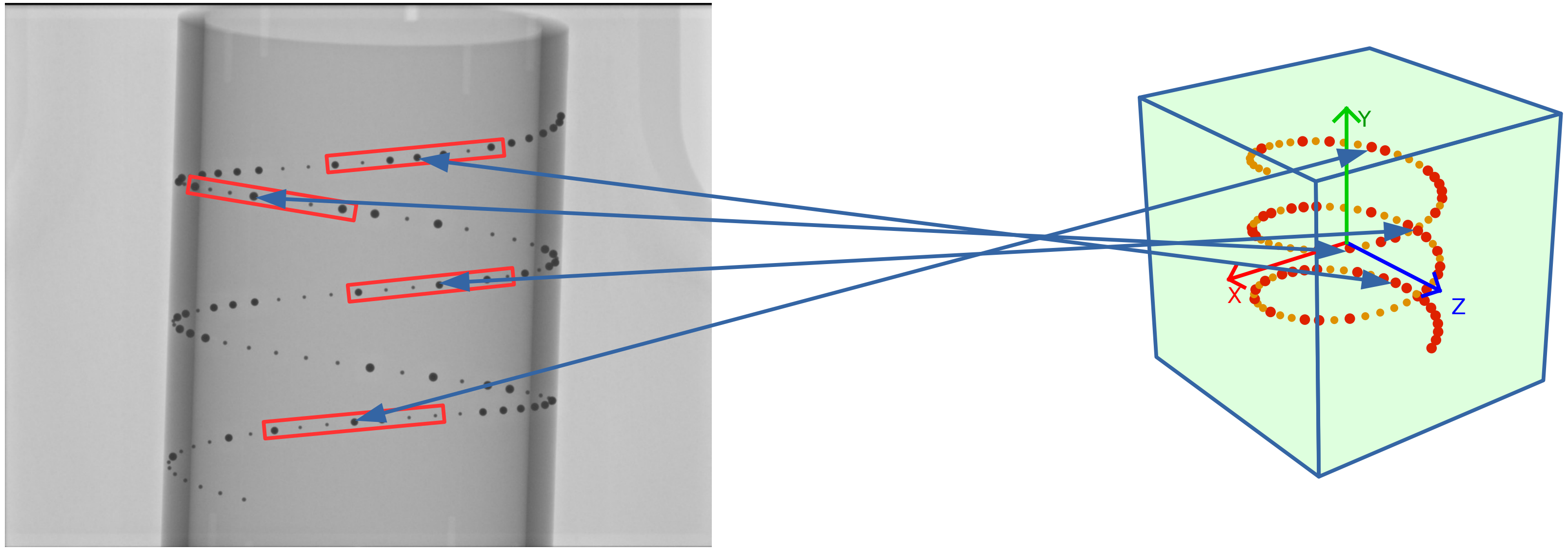
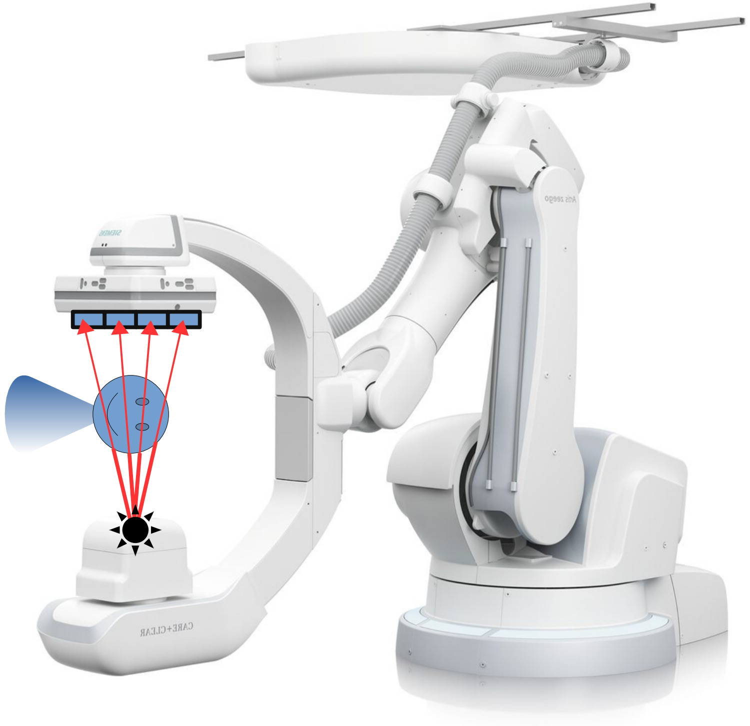


Image Source: Siemens Healthineers.

PDS2 solution

→ Every sequence of 7 large/small beads is unambiguous.
→ Once point matches are established: Estimate projection matrix!





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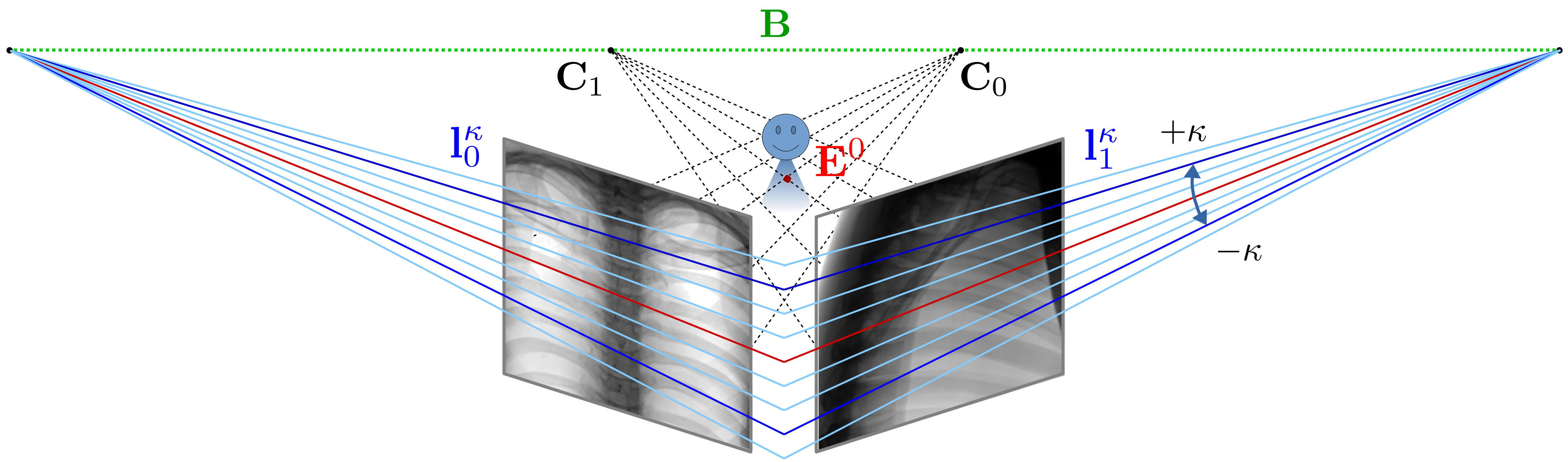
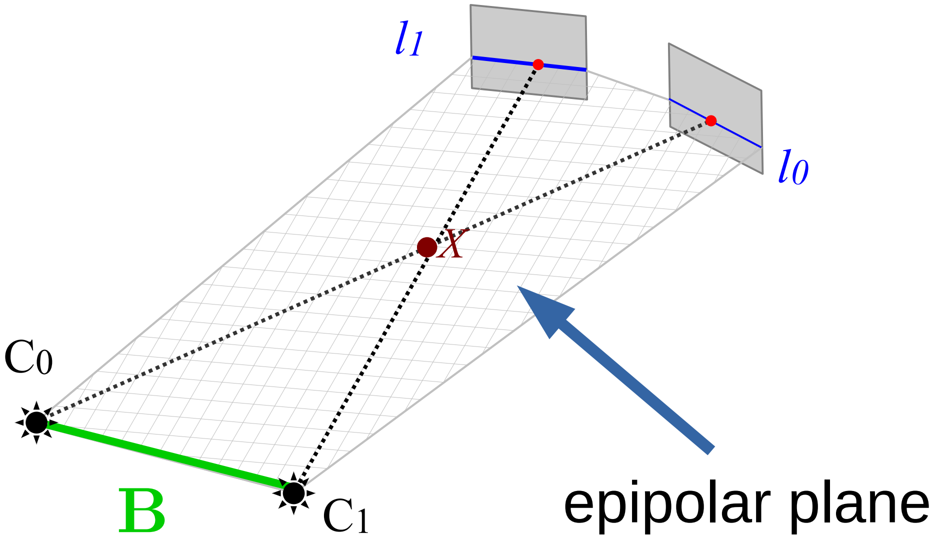
Let’s combine with epipolar geometry:



Consider a single epipolar plane:



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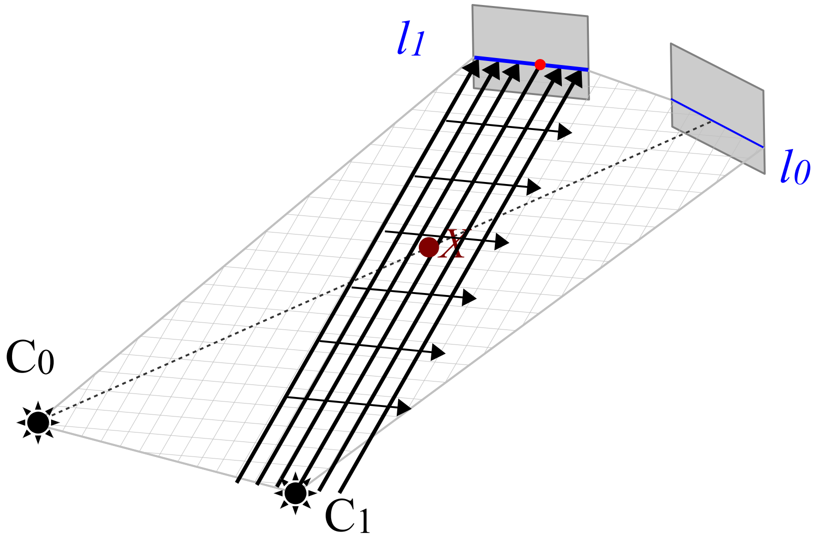

- Assume rays are approximately parallel
- Then: **plane integral = line integral!**

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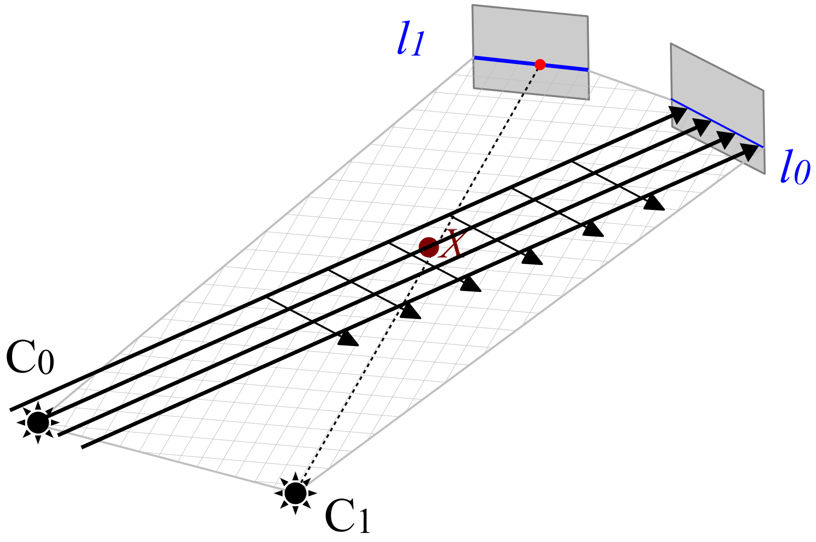

- Assume rays are approximately parallel
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- Symmetry: two ways of computing the same plane integral via lines 0 and 1

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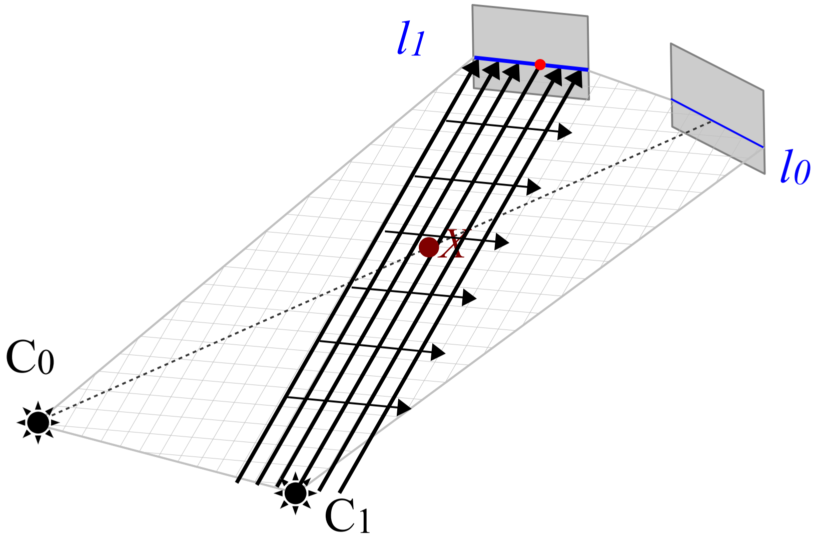

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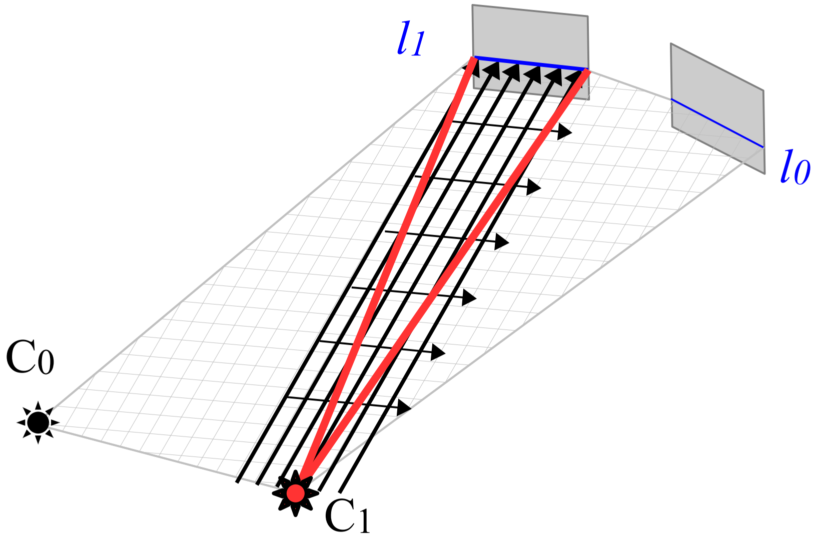

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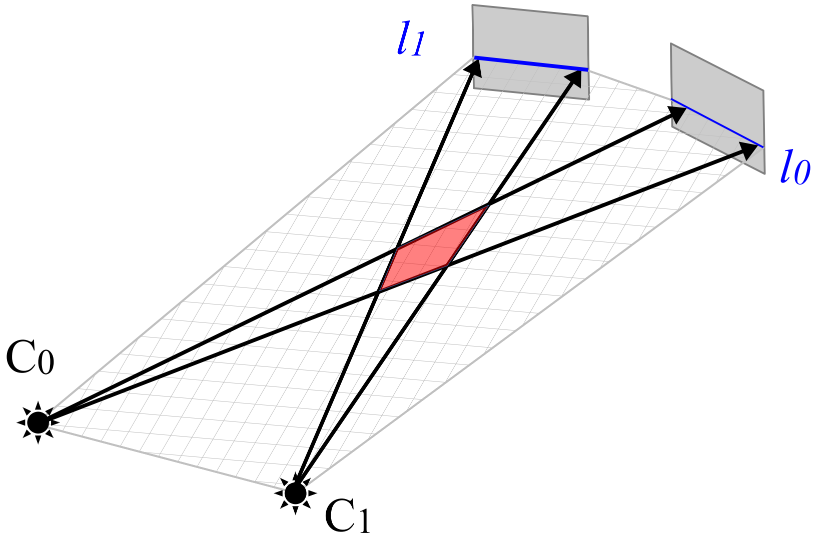

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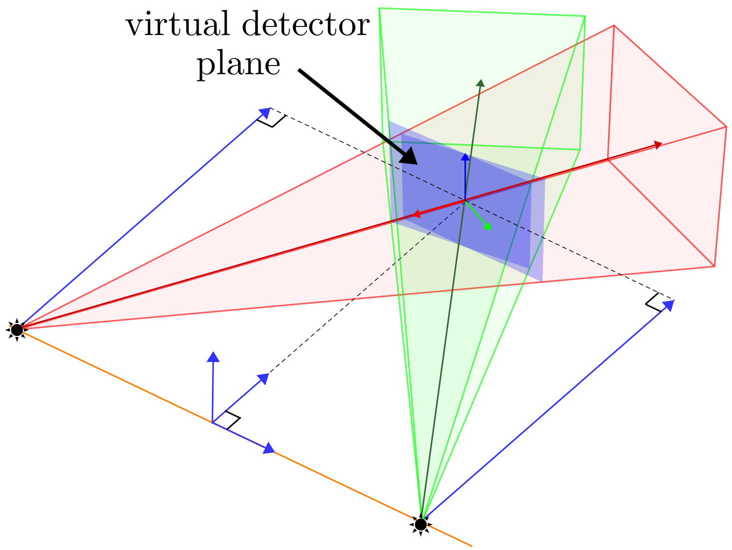
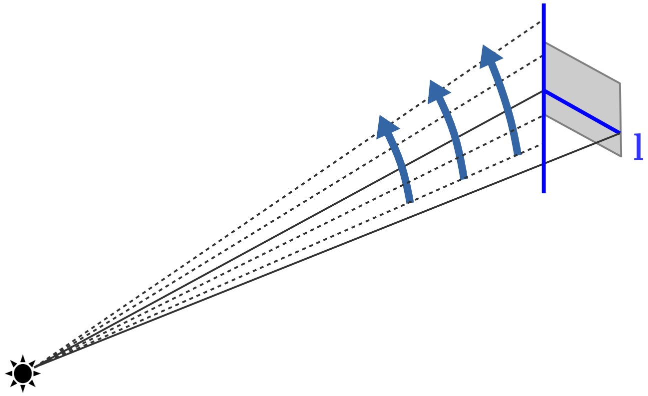
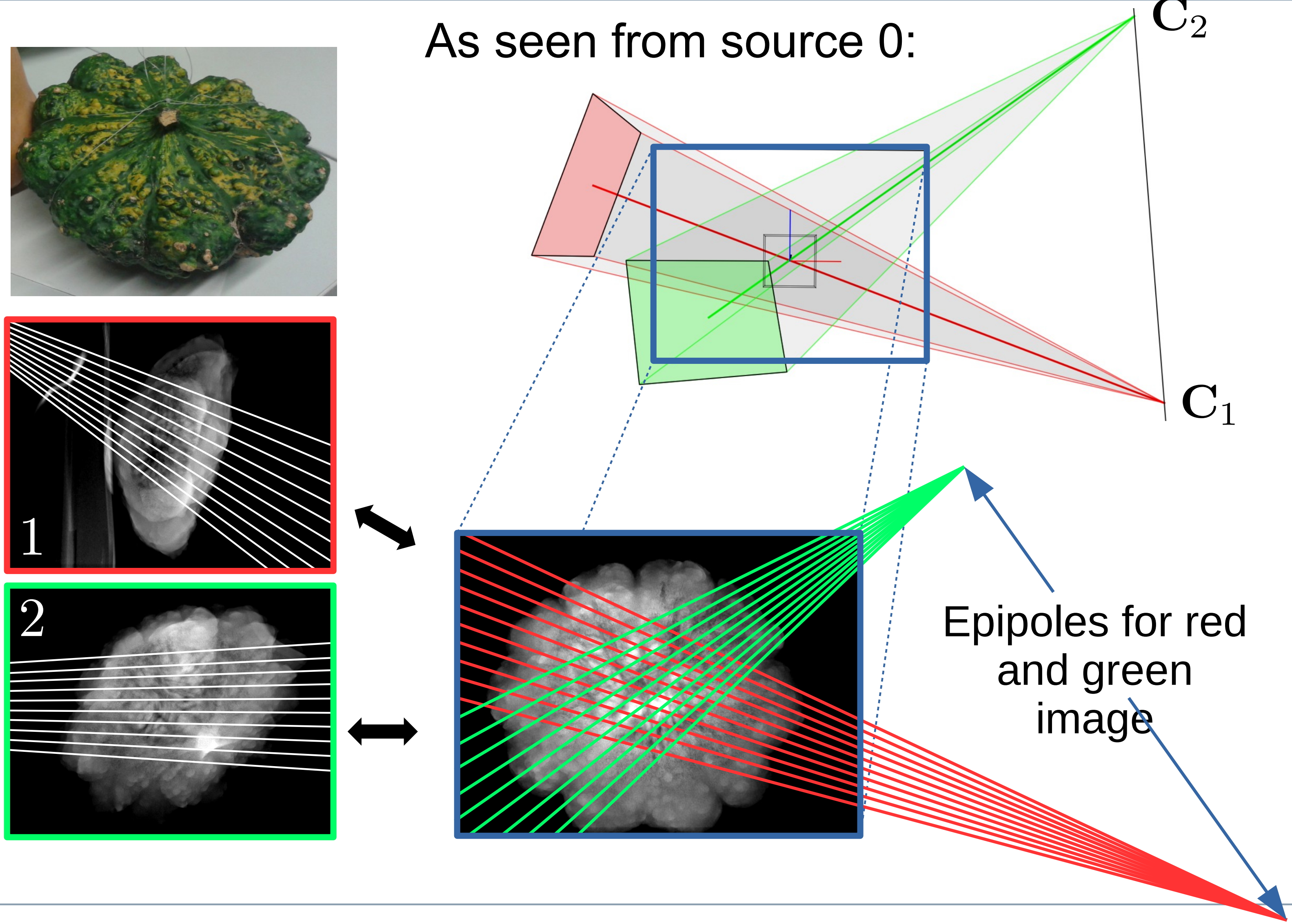
Two ways to handle general cone-beam projections:

Rectification



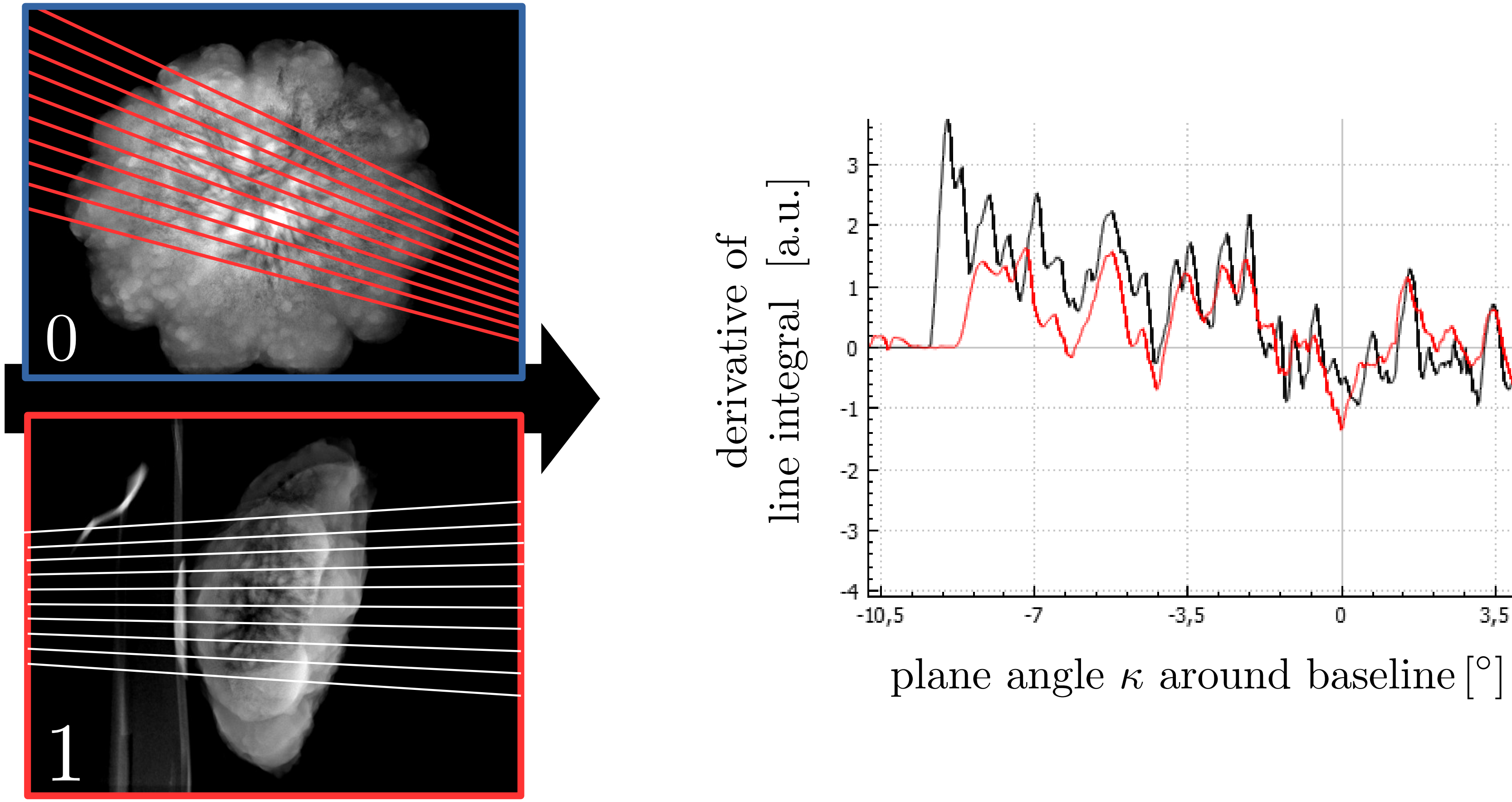
Orthogonal Derivative




As seen from source 0:
$\mathbf{C}_2$
$\mathbf{C}_1$
1
2
Epipoles for red and green image

FDCT C-Arm Motion Correction

Epipolar Consistency



FDCT C-Arm Motion Correction

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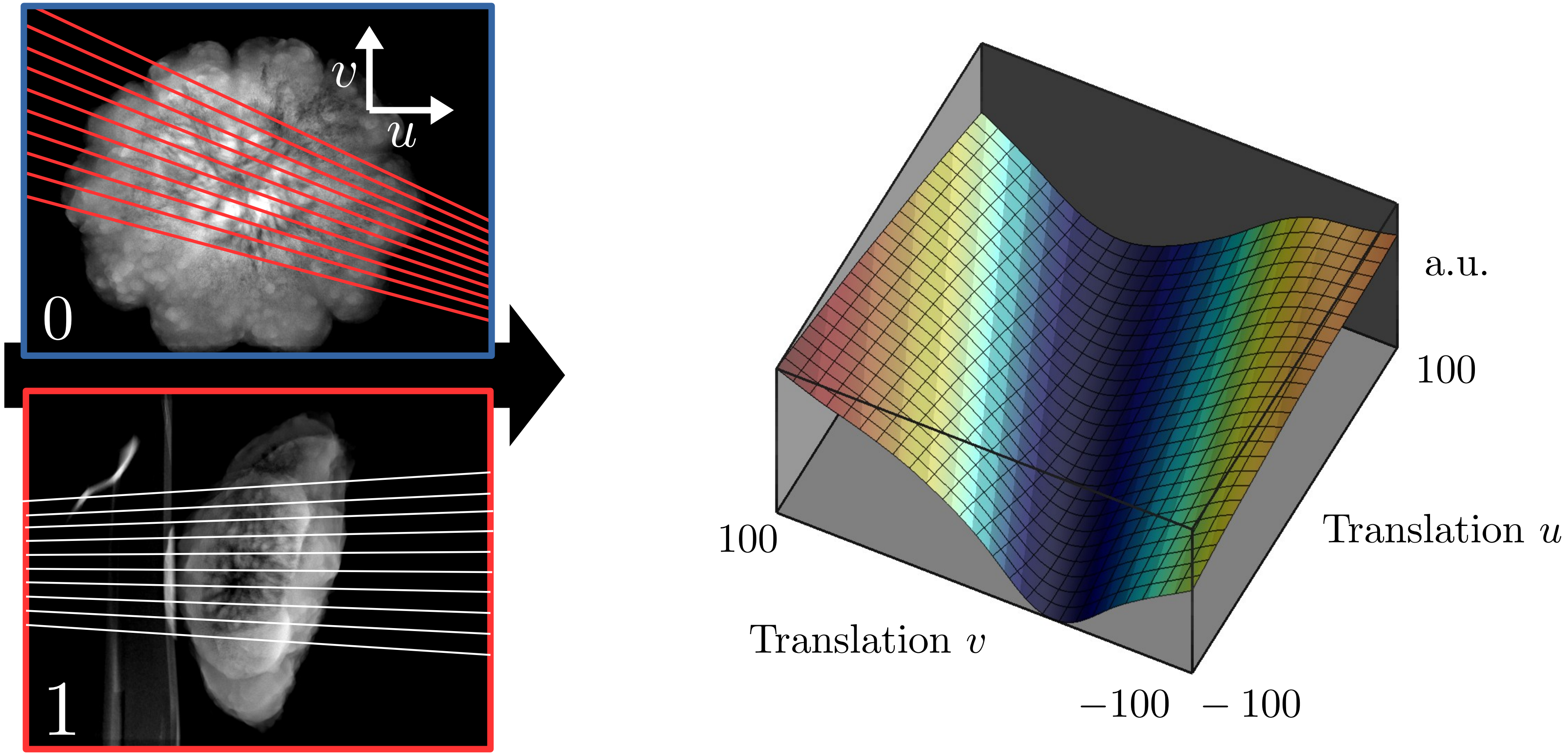

FDCT C-Arm Motion Correction

Epipolar Consistency (Movie Clip)