# Computer Vision

## (Summer Semester 2020)

Lecture 5, Part 3

Singular Value Decomposition (SVD)

# Singular Value Decomposition (SVD)

- Requirements: Matrix Algebra and Vector Calculus
- Core Idea - Theoretical Background, Interpretation
- Applications

*Detailed explanations: 1. https://blog.statsbot.co/singular-value-decomposition-tutorial-52c695315254*
*2. https://towardsdatascience.com/svd-8c2f72e264f*

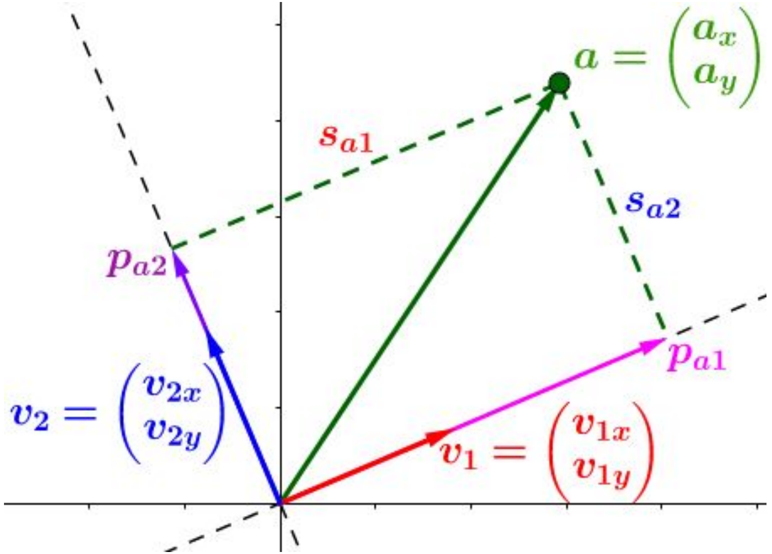
# Core Idea

$$A = U \, \Sigma \, V^T$$

- $A$ is an $m \times n$ matrix
- $U$ is an $m \times n$ *orthogonal* matrix
- $\sum$ is an $n \times n$ *diagonal matrix*
- $V$ is an $n \times n$ orthogonal matrix

$$U^T U = V V^T = I$$

# Core Idea -- Vector Decomposition




$$a^T \cdot v_1 = \begin{pmatrix} a_x & a_y \end{pmatrix} \cdot \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = s_{a1}$$

$$a^T \cdot v_2 = \begin{pmatrix} a_x & a_y \end{pmatrix} \cdot \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} = s_{a2}$$

**Matrix Form:**

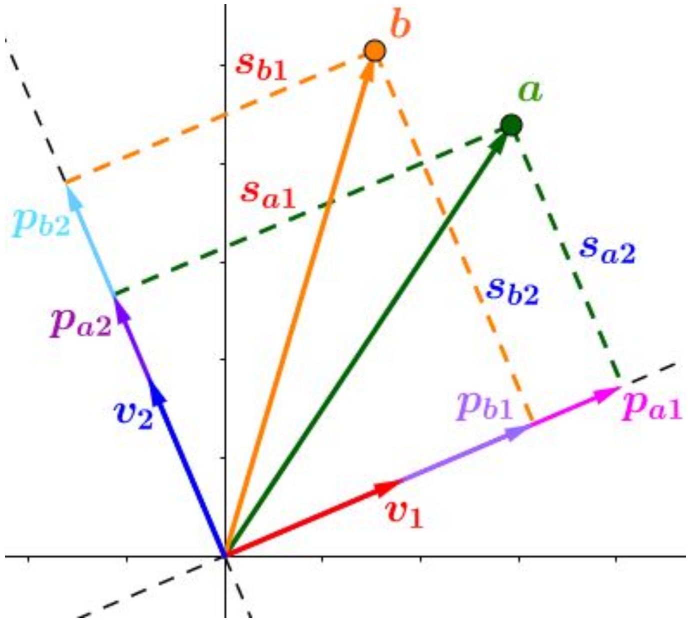
$$a^T \cdot V = \begin{pmatrix} a_x & a_y \end{pmatrix} \cdot \begin{pmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{pmatrix} = \begin{pmatrix} s_{a1} & s_{a2} \end{pmatrix}$$

# Core Idea -- Add another point

$$A \cdot V = \begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix} \cdot \begin{pmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{pmatrix} = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix} = S$$

**Add n-points**

$$A \cdot V = \underbrace{\begin{pmatrix} a_x & a_y & \cdots \\ b_x & b_y & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}}_{n \times d} \cdot \underbrace{\begin{pmatrix} v_{1x} & v_{2x} & \cdots \\ v_{1y} & v_{2y} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}}_{d \times d} = \underbrace{\begin{pmatrix} s_{a1} & s_{a2} & \cdots \\ s_{b1} & s_{b2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}}_{n \times d} = S$$



# Core Idea -- Summary

$$A \cdot V = S$$

Matrix of points

The dot product performs the projection

Matrix of decomposition axes

Matrix of the lengths of projections

# Decomposing *S*

$$A = S\,V^{-1} = S\,V^{T}$$

$$A = U\,\Sigma\,V^{T}$$

$$S = U\,\Sigma$$

$$S = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix}$$

A column vector containing the lengths of projections of each point on the 1st axis *v1*

A column vector containing the lengths of projections of each point on the 2nd axis *v2*

# Decomposing *S*

$$S = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix}$$

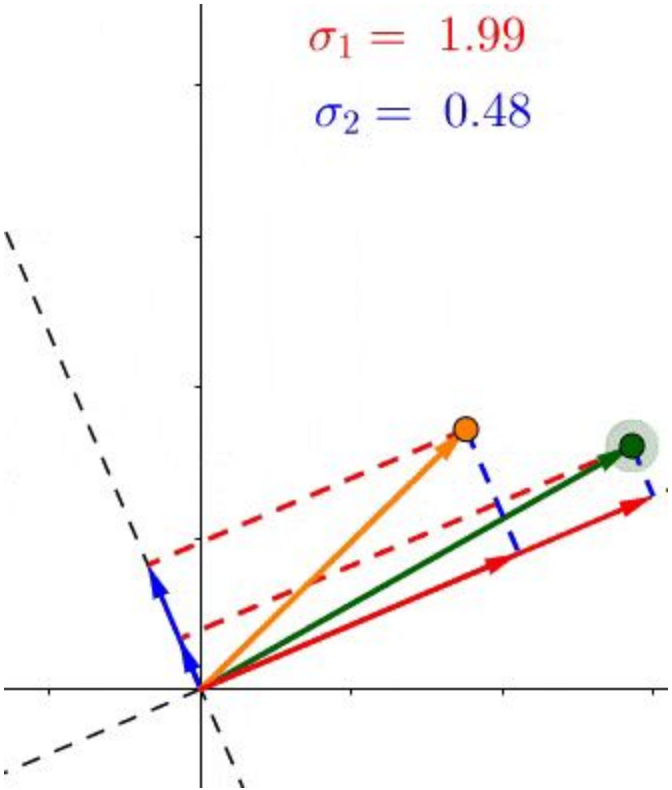
$$\text{Magnitude of 1st column} = \sigma_1 = \sqrt{(s_{a1})^2 + (s_{b1})^2}$$

$$\text{Magnitude of 2nd column} = \sigma_2 = \sqrt{(s_{a2})^2 + (s_{b2})^2}$$

$$S = \begin{pmatrix} \frac{s_{a1}}{\sigma_1} & \frac{s_{a2}}{\sigma_2} \\ \\ \frac{s_{b1}}{\sigma_1} & \frac{s_{b2}}{\sigma_2} \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \underbrace{\begin{pmatrix} u_{a1} & u_{a2} \\ u_{b1} & u_{b2} \end{pmatrix}}_{\downarrow\; U} \underbrace{\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}}_{\downarrow\; \Sigma}$$

# Interpreting sigmas (σ)

...also called singular values



# Applications - Dimensionality reduction

$$a_{ij} \approx \sum_{k=1}^{p} u_{ik} s_k v_{jk}$$

Select the first p (n > p) singular values

# Applications - Computing Homography

$$A = U \, \Sigma \, V^T$$

...the axes are decomposed into the columns of $V^T$ (homography coordinates)

# Applications - Solving A $\boldsymbol{x}$ = b

$$\begin{aligned} A\boldsymbol{x} &= \boldsymbol{b} \\ \boldsymbol{x} &= VS^{-1}U^T\boldsymbol{b} \end{aligned}$$