Computer Vision

(Summer Semester 2020)

Lecture 6, Part 2

Camera Calibration

Camera Calibration

- Transforms
- **Least Squares Fitting**
- **Least Squares Extrinsic and Intrinsic Parameter Fitting**

- Note: The core of these slides stems from the class CSCI 1430: “Introduction to Computer Vision” by James Tompkin, Fall 2017, Brown University.

Recap: Two Common Optimization Problems

Problem statement	Solution
$\text{minimize } \lVert \mathbf{Ax} - \mathbf{b} \rVert^2$	$\mathbf{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$
$\text{least squares solution to } \mathbf{Ax} = \mathbf{b}$	$\mathbf{x} = \mathbf{A} \setminus \mathbf{b}$ (matlab)

Problem statement	Solution
$\text{minimize } \mathbf{x}^T\mathbf{A}^T\mathbf{Ax} \quad \text{s.t. } \mathbf{x}^T\mathbf{x} = 1$	$[\mathbf{v}, \lambda] = \text{eig}(\mathbf{A}^T\mathbf{A})$
$\text{minimize} \dfrac{\mathbf{x}^T\mathbf{A}^T\mathbf{Ax}}{\mathbf{x}^T\mathbf{x}}$	$\lambda_1 < \lambda_{2..n} : \mathbf{x} = \mathbf{v}_1$
$\text{non - trivial lsq solution to } \mathbf{Ax} = 0$	

Least squares (global) optimization

Good

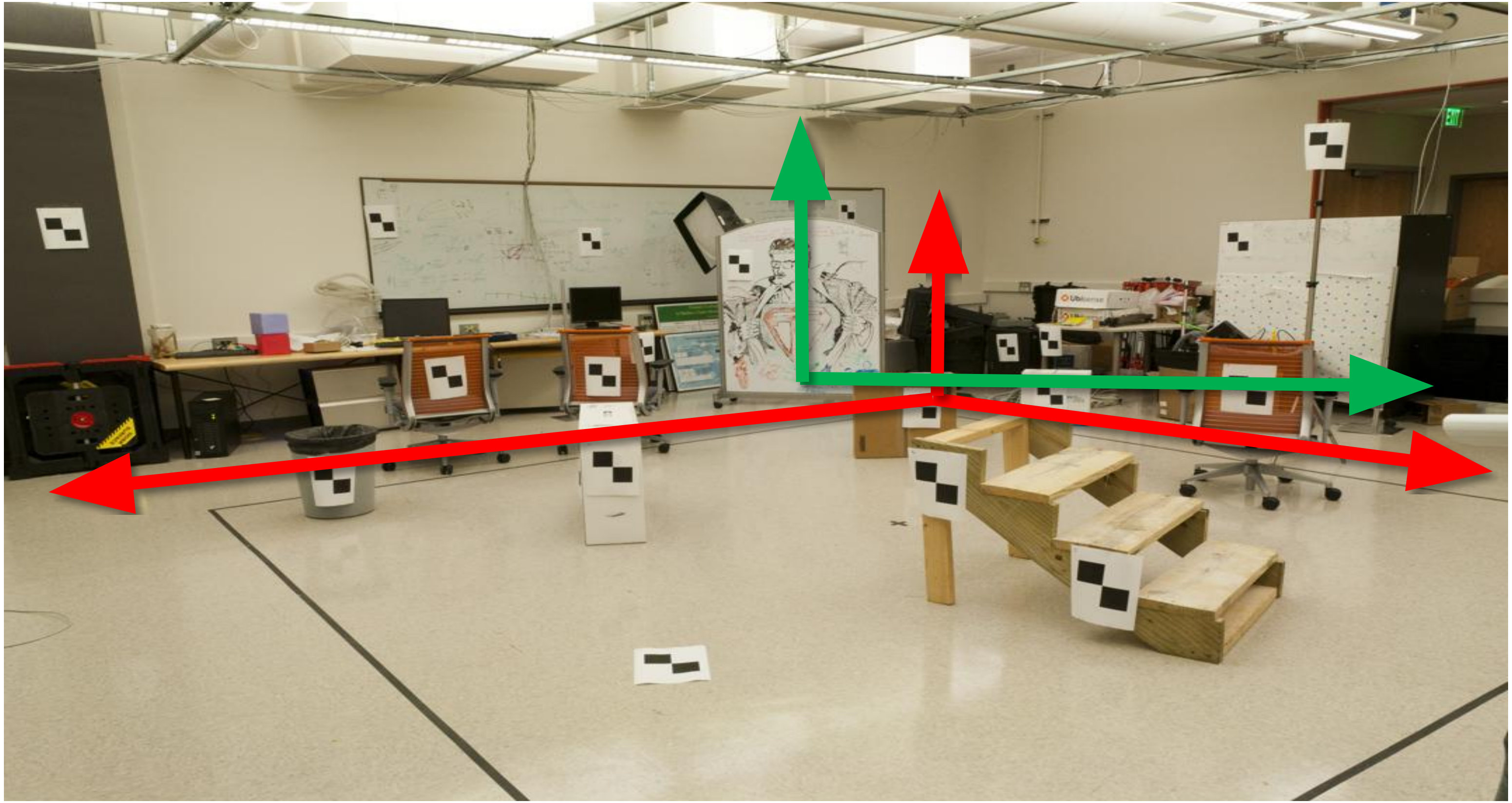
- Clearly specified objective
- Optimization is easy

Bad

- Sensitive to outliers
  - Bad matches, extra points
- Doesn’t allow you to get multiple good fits
  - Detecting multiple objects, lines, etc.

Iterative solutions are better

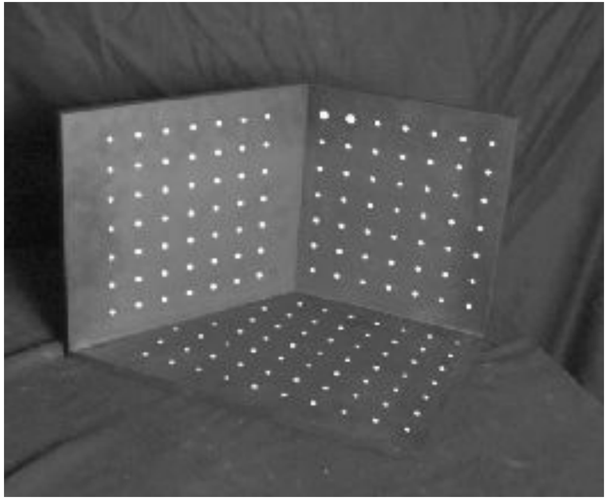
World vs Camera coordinates



Calibrating the Camera

Use an scene with known geometry

- Correspond image points to 3d points
- Get least squares solution (or non-linear solution)



Known 2d image coords

Known 3d world locations

$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \overset{\mathbf{M}}{\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix}} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Unknown Camera Parameters

How do we calibrate a camera?

Known 2d image coords

880	214
43	203
270	197
886	347
745	302
943	128
476	590
419	214
317	335
783	521
235	427
665	429
655	362
427	333
412	415
746	351
434	415

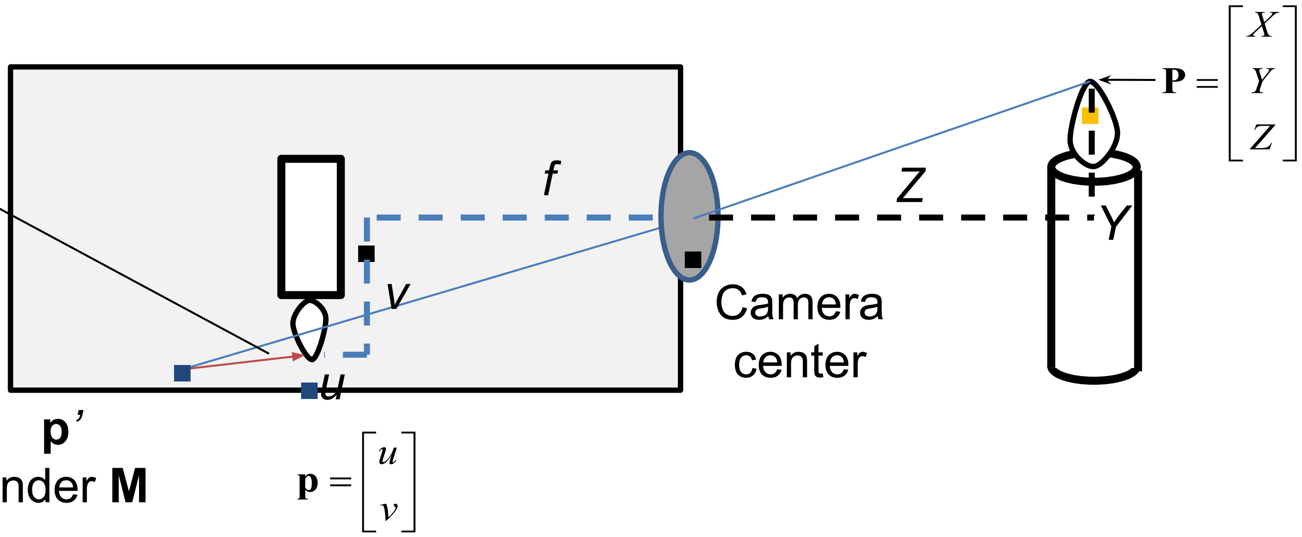
Known 3d world locations

312.747	309.140	30.086
305.796	311.649	30.356
307.694	312.358	30.418
310.149	307.186	29.298
311.937	310.105	29.216
311.202	307.572	30.682
307.106	306.876	28.660
309.317	312.490	30.230
307.435	310.151	29.318
308.253	306.300	28.881
306.650	309.301	28.905
308.069	306.831	29.189
309.671	308.834	29.029
308.255	309.955	29.267
307.546	308.613	28.963
311.036	309.206	28.913
307.518	308.175	29.069

What is least squares doing?

- Given 3D point evidence, find best **M** which minimizes error between estimate (**p**’) and known corresponding 2D points (**p**).

Error between **M** estimate and known projection point

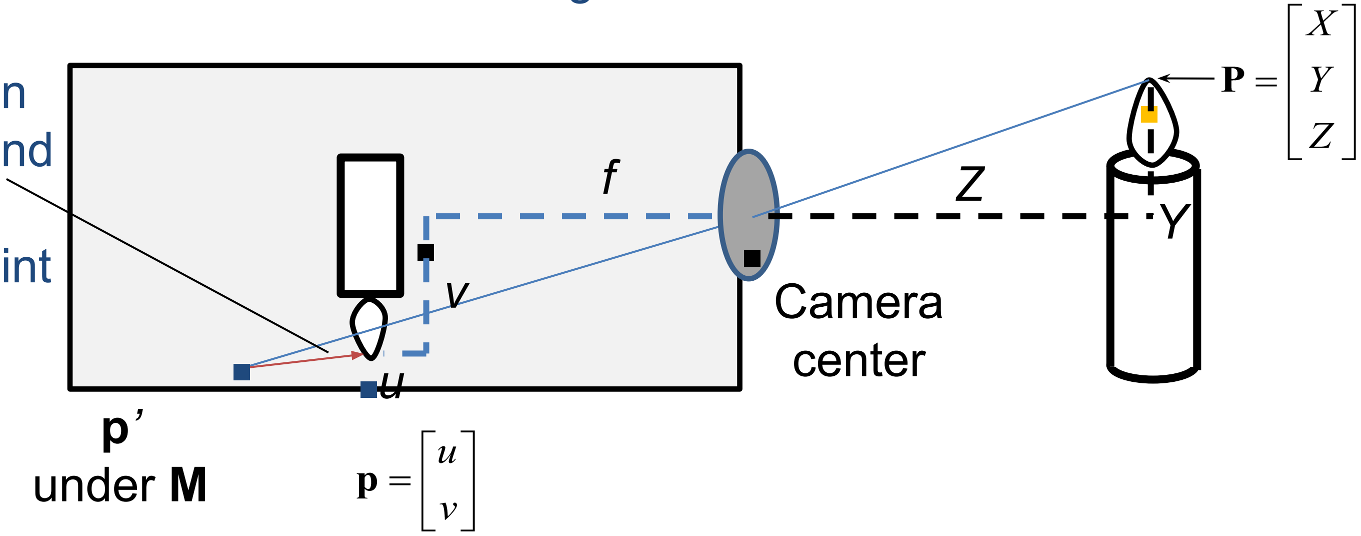


p = distance from image center

What is least squares doing?

- Best **M** occurs when **p'** = **p**, or when **p'** – **p** = 0
- Form these equations from all point evidence
- Solve for model via closed-form regression

Error between **M** estimate and known projection point



p = distance from image center

Unknown Camera Parameters

Known 2d image coords

$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Known 3d locations

First, work out where X,Y,Z projects to under candidate **M.**

$$su = m_{11}X + m_{12}Y + m_{13}Z + m_{14}$$

$$sv = m_{21}X + m_{22}Y + m_{23}Z + m_{24}$$

$$s = m_{31}X + m_{32}Y + m_{33}Z + m_{34}$$

Two equations per 3D point correspondence

$$u = \frac{m_{11}X + m_{12}Y + m_{13}Z + m_{14}}{m_{31}X + m_{32}Y + m_{33}Z + m_{34}}$$

$$v = \frac{m_{21}X + m_{22}Y + m_{23}Z + m_{24}}{m_{31}X + m_{32}Y + m_{33}Z + m_{34}}$$

Unknown Camera Parameters

Known 2d image coords

$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Known 3d locations

Next, rearrange into form where all **M** coefficients are individually stated in terms of X,Y,Z,u,v.
-> Allows us to form lsq matrix.

$$u = \frac{m_{11}X + m_{12}Y + m_{13}Z + m_{14}}{m_{31}X + m_{32}Y + m_{33}Z + m_{34}}$$

$$v = \frac{m_{21}X + m_{22}Y + m_{23}Z + m_{24}}{m_{31}X + m_{32}Y + m_{33}Z + m_{34}}$$

$$(m_{31}X + m_{32}Y + m_{33}Z + m_{34})u = m_{11}X + m_{12}Y + m_{13}Z + m_{14}$$
$$(m_{31}X + m_{32}Y + m_{33}Z + m_{34})v = m_{21}X + m_{22}Y + m_{23}Z + m_{24}$$

$$m_{31}uX + m_{32}uY + m_{33}uZ + m_{34}u = m_{11}X + m_{12}Y + m_{13}Z + m_{14}$$
$$m_{31}vX + m_{32}vY + m_{33}vZ + m_{34}v = m_{21}X + m_{22}Y + m_{23}Z + m_{24}$$

Unknown Camera Parameters

Known 2d image coords

$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Known 3d locations

Next, rearrange into form where all **M** coefficients are individually stated in terms of X,Y,Z,u,v.
-> Allows us to form lsq matrix.

$$m_{31}uX + m_{32}uY + m_{33}uZ + m_{34}u = m_{11}X + m_{12}Y + m_{13}Z + m_{14}$$
$$m_{31}vX + m_{32}vY + m_{33}vZ + m_{34}v = m_{21}X + m_{22}Y + m_{23}Z + m_{24}$$

$$0 = m_{11}X + m_{12}Y + m_{13}Z + m_{14} - m_{31}uX - m_{32}uY - m_{33}uZ - m_{34}u$$
$$0 = m_{21}X + m_{22}Y + m_{23}Z + m_{24} - m_{31}vX - m_{32}vY - m_{33}vZ - m_{34}v$$

Unknown Camera Parameters

Known 2d image coords

$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Known 3d locations

- Finally, solve for m's entries using linear least squares
- Method 1 –

$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -u_1X_1 & -u_1Y_1 & -u_1Z_1 \\ 0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -v_1X_1 & -v_1Y_1 & -v_1Z_1 \\ & & & & & & \vdots & & & & \\ X_n & Y_n & Z_n & 1 & 0 & 0 & 0 & 0 & -u_nX_n & -u_nY_n & -u_nZ_n \\ 0 & 0 & 0 & 0 & X_n & Y_n & Z_n & 1 & -v_nX_n & -v_nY_n & -v_nZ_n \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{24} \\ m_{31} \\ m_{32} \\ m_{33} \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ u_n \\ v_n \end{bmatrix}$$

```
M = A\Y;
M = [M;1];
M = reshape(M,[],3)';
```

Unknown Camera Parameters

Known 2d image coords

$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Known 3d locations

- Or, solve for m’s entries using total linear least-squares.
- Method 2 –

$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -u_1X_1 & -u_1Y_1 & -u_1Z_1 & -u_1 \\ 0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -v_1X_1 & -v_1Y_1 & -v_1Z_1 & -v_1 \\ & & & & & & \vdots & & & & & \\ X_n & Y_n & Z_n & 1 & 0 & 0 & 0 & 0 & -u_nX_n & -u_nY_n & -u_nZ_n & -u_n \\ 0 & 0 & 0 & 0 & X_n & Y_n & Z_n & 1 & -v_nX_n & -v_nY_n & -v_nZ_n & -v_n \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{24} \\ m_{31} \\ m_{32} \\ m_{33} \\ m_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

```
[U, S, V] = svd(A);
M = V(:,end);
M = reshape(M,[],3)';
```

How do we calibrate a camera?

Known 2d image coords		Known 3d world locations		
880	214	312.747	309.140	30.086
43	203	305.796	311.649	30.356
270	197	307.694	312.358	30.418
886	347	310.149	307.186	29.298
745	302	311.937	310.105	29.216
943	128	311.202	307.572	30.682
476	590	307.106	306.876	28.660
419	214	309.317	312.490	30.230
317	335	307.435	310.151	29.318
783	521	308.253	306.300	28.881
235	427	306.650	309.301	28.905
665	429	308.069	306.831	29.189
655	362	309.671	308.834	29.029
427	333	308.255	309.955	29.267
412	415	307.546	308.613	28.963
746	351	311.036	309.206	28.913
434	415	307.518	308.175	29.069

How do we calibrate a camera?

Known 2d image coords

Known 3d world locations

1st point

(u_1, v_1)

880	214
43	203
270	197
886	347
745	302
943	128
476	590
419	214
317	335
783	521
235	427
665	429
655	362
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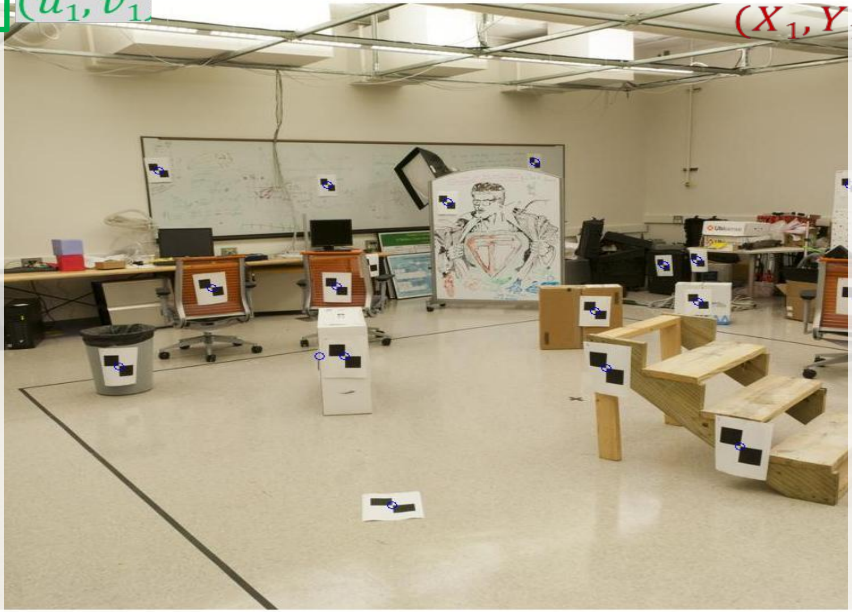

$(X_1, Y_1$

312.747	309.140	30.086
305.796	311.649	30.356
307.694	312.358	30.418
310.149	307.186	29.298
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307.546	308.613	28.963
311.036	309.206	28.913
307.518	308.175	29.069

Projection error defined by two equations – one for *u* and one for *v*

$$
\begin{bmatrix}
312.747 & 309.140 & 30.086 & 1 & 0 & 0 & 0 & 0 & -880\times312.747 & -880\times309.140 & -880\times30.086 & -880 \\
0 & 0 & 0 & 0 & 312.747 & 309.140 & 30.086 & 1 & -214\times312.747 & -214\times309.140 & -214\times30.086 & -214 \\
 & & & & & & \vdots & & & & & \\
X_n & Y_n & Z_n & 1 & 0 & 0 & 0 & 0 & -u_nX_n & -u_nY_n & -u_nZ_n & -u_n \\
0 & 0 & 0 & 0 & X_n & Y_n & Z_n & 1 & -v_nX_n & -v_nY_n & -v_nZ_n & -v_n
\end{bmatrix}
\begin{bmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{24} \\ m_{31} \\ m_{32} \\ m_{33} \\ m_{34} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}
$$

How many points do I need to fit the model?

$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

Degrees of freedom?

$$w\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \overset{5}{\begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix}} \underbrace{\overset{6}{\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix}}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Think 3:
Rotation around x
Rotation around y
Rotation around z

How many points do I need to fit the model?

$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

Degrees of freedom?

$$w\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \overset{5}{\begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{6}{\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- **M** is 3x4, so 12 unknowns, but projective scale ambiguity – 11 deg. freedom. One equation per unknown -> 5 1/2 point correspondences determines a solution (e.g., either u or v).
- More than 5 1/2 point correspondences -> overdetermined, many solutions to **M**.

How many points do I need to fit the model?

$$\mathbf{x} = \mathbf{K}[\mathbf{R} \quad \mathbf{t}]\mathbf{X}$$

Degrees of freedom?

$$w\begin{bmatrix}u\\v\\1\end{bmatrix} = \begin{array}{c}\begin{array}{ccc} & 5 & \end{array}\\ \begin{bmatrix}\alpha & s & u_0\\0 & \beta & v_0\\0 & 0 & 1\end{bmatrix}\end{array}\begin{array}{c}\begin{array}{cccc} \phantom{r_{11}} & 6 & \phantom{r_{13}} & \end{array}\\ \begin{bmatrix}r_{11} & r_{12} & r_{13} & t_x\\r_{21} & r_{22} & r_{23} & t_y\\r_{31} & r_{32} & r_{33} & t_z\end{bmatrix}\end{array}\begin{bmatrix}x\\y\\z\\1\end{bmatrix}$$

- Least squares is finding the solution that best satisfies the overdetermined system.
- Why use more than 6? Robustness to error in feature points.

Unknown Camera Parameters

Known 2d image coords

$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Known 3d locations

- Method 1 – **Ax=b** form - p' = p
- To set scale, we artificially set m_{34} to 1

$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -u_1X_1 & -u_1Y_1 & -u_1Z_1 \\ 0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -v_1X_1 & -v_1Y_1 & -v_1Z_1 \\ & & & & & & \vdots & & & & \\ X_n & Y_n & Z_n & 1 & 0 & 0 & 0 & 0 & -u_nX_n & -u_nY_n & -u_nZ_n \\ 0 & 0 & 0 & 0 & X_n & Y_n & Z_n & 1 & -v_nX_n & -v_nY_n & -v_nZ_n \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{24} \\ m_{31} \\ m_{32} \\ m_{33} \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ u_n \\ v_n \end{bmatrix}$$

- Note – you will see this form called 'inhomogeneous' linear system -> nothing to do with homogeneous coordinates

```
M = A\Y;
M = [M;1];
M = reshape(M,[],3)';
```

Unknown Camera Parameters

Known 2d image coords

$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Known 3d locations

- Method 2 – **Ax=0** form - p' – p = 0
- SVD – singular value decomposition
- Computes pseudo-inverse

$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -u_1X_1 & -u_1Y_1 & -u_1Z_1 & -u_1 \\ 0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -v_1X_1 & -v_1Y_1 & -v_1Z_1 & -v_1 \\ & & & & & & \vdots & & & & & \\ X_n & Y_n & Z_n & 1 & 0 & 0 & 0 & 0 & -u_nX_n & -u_nY_n & -u_nZ_n & -u_n \\ 0 & 0 & 0 & 0 & X_n & Y_n & Z_n & 1 & -v_nX_n & -v_nY_n & -v_nZ_n & -v_n \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{24} \\ m_{31} \\ m_{32} \\ m_{33} \\ m_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

Note – you will see this form called 'homogeneous' linear system -> nothing to do with homogeneous coordinates

```
[U, S, V] = svd(A);
M = V(:,end);
M = reshape(M,[],3)';
```

Calibration with linear method

Advantages

- Easy to formulate and solve
- Provides initialization for non-linear methods

Disadvantages

- Doesn’t directly give you camera parameters
- Doesn’t model radial distortion
- Can’t impose constraints, such as known focal length

Non-linear methods are preferred

- Define error as difference between projected points and measured points
- Minimize error using Newton’s method or other non-linear optimization

Can we factorize M back to K [R | T]?

- Yes!
- We can directly solve for the individual entries of K [R | T].

Can we factorize M back to K [R | T]?

- Yes: there is a direct solution
- We can also use *RQ* factorization (not QR)
  - R in RQ is not rotation matrix R; crossed names!
- *R* (right diagonal) is K
- *Q* (orthogonal basis) is R.
- T, the last column of [R | T], is inv(K) * last column of M.
  - Need post-processing for valid matrices, see [Dissecting the Camera Matrix, Part 1: Extrinsic/Intrinsic Decomposition ←]

Recovering the camera center

$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

$$w\begin{bmatrix}u\\v\\1\end{bmatrix}=\begin{bmatrix}\alpha & s & u_0\\0 & \beta & v_0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}r_{11} & r_{12} & r_{13} & \boxed{t_x}\\r_{21} & r_{22} & r_{23} & \boxed{t_y}\\r_{31} & r_{32} & r_{33} & \boxed{t_z}\end{bmatrix}\begin{bmatrix}x\\y\\z\\1\end{bmatrix}$$

$$\begin{bmatrix}su\\sv\\s\end{bmatrix}=\begin{bmatrix}\underbrace{\begin{matrix}* & * & *\\ * & * & *\\ * & * & *\end{matrix}}_{Q} & \overbrace{\begin{matrix}*\\ *\\ *\end{matrix}}^{m_4}\end{bmatrix}\begin{bmatrix}X\\Y\\Z\\1\end{bmatrix}$$

This is not the camera center C.
It is –RC, as the point is rotated before t_x, t_y, and t_z are added

So we need $-R^{-1} K^{-1} m_4$ to get C.

This is **t** × K
So $K^{-1} m_4$ is **t**

Q is K × R.
So we just need $-Q^{-1}$ m4