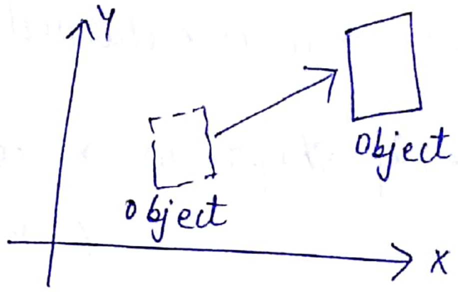
Transformed ob. matrix = [or. ob matrix] × [T.m]. ∴ $[X^*] = [X][T]$

2D Transformations

- Process of changing of sizes, orientation or positions of object by a mathematical operation is called Tr.
- The change is accomplished by 2 methods:-

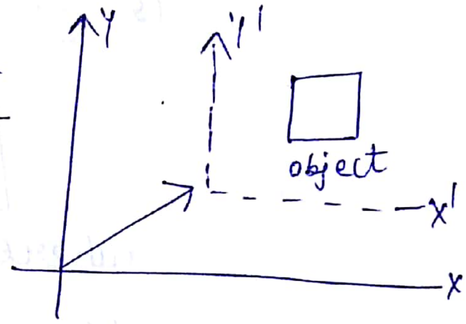
① Geometric Transformation (TRS)

- It is a type of trans. in which an object itself is changed relative to stationary CS.
- Applied to each pt. of object.



② Coordinate Transformation

- Type of trans. in which the object is held stationary while CS is transformed relative to object.



So, (T) is the process by which we can change the shape, position and direction of any object w.r.t any of coordinate system on the background by T, S, R, Refle

Position Size Orientation shapes

→ Transformation Matrix

- Basic tool for Transformation.
- A matrix with $n \times m$ dimensions is multiplied with coordinate of objects.
- Usually 3×3 & 4×4 are used for (T).
- For rotation op. :- $R = \begin{bmatrix} \cos\theta & -\cos\theta \\ \sin\theta & \sin\theta \end{bmatrix}$

2DT :- only planar coordinates are used. For this we use 2x2 (T).

① Translation

→ we shift object parallel to itself in any direction in X-Y plane.

→ this is achieved by a shift in (X) direction + (Y) direc (horizontally) (vertically)

→ If shift in x is t_x
y " t_y

then translation of a pt. $P(x,y)$ into $P(x',y')$ is given as:-

$$\boxed{\begin{aligned} x' &= x + t_x \\ y' &= y + t_y \end{aligned}}$$

where $\vec{V} = t_x\hat{i} + t_y\hat{j}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

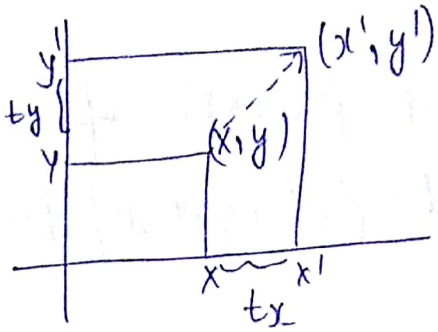
- Translation distance pair (t_x, t_y) is called as (T) vector / shift vector.

- In column-major representation form:-

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\boxed{P' = T \cdot P}$$

Translation :-



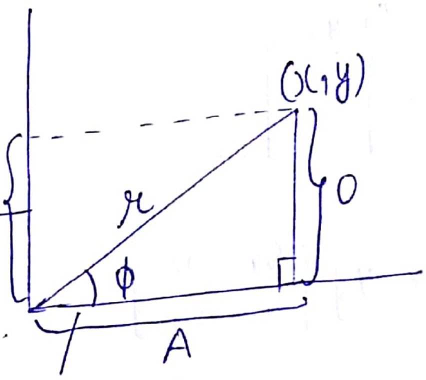
Translation vector.

$x' = x + t_x$

$y' = y + t_y$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

(2) Rotation



$\sin\phi = \frac{O}{r}$

$\boxed{O = r\sin\phi}$

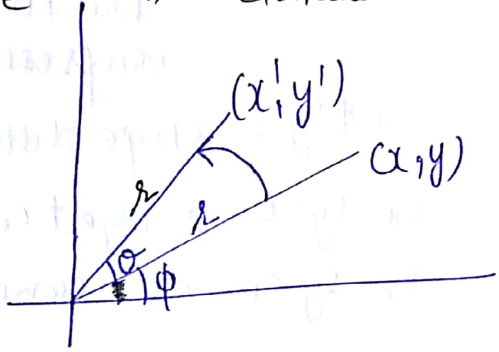
$\cos\phi = \frac{A}{r}$

$\boxed{r\cos\phi = A}$

- A 2D rotation is applied to an object by repositioning it along a circular path in xy-plane. $y = r\sin\phi$

$x = r\cos\phi$

- ∠sign determines direction of rotation. +ve value for rotation ∠ defines counterclockwise rotation
  -ve " clockwise rotation of object.



Clockwise

$x' = r\cos(\phi - \theta)$

$y' = r\sin(\phi - \theta)$

$$[x' \; y'] = [x \; y]\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$x = r\cos\phi$

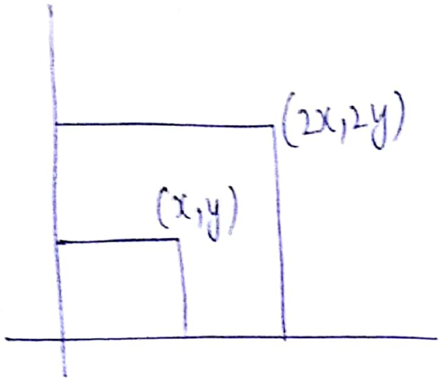
$y = r\sin\phi$

$x' = r\cos(\theta+\phi)$	$y' = (\theta+\phi)\, r\sin\phi$
$x' = r\cos\theta\cos\phi - r\sin\theta\sin\phi$	$y' = r\sin\theta\cos\phi + r\cos\theta\sin\phi$
$\boxed{x' = x\cos\theta - y\sin\theta}$	$\boxed{y' = x\sin\theta + y\cos\theta}$

Row major :- $[x' \; y'] = [x \; y]\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

3) Scaling :-

Defined as process of changing the size/sh[ape] of an object (elongation/shrinking) in x & y directi[on]



$$\left.\begin{aligned} x' &= x \cdot S_x \\ y' &= y \cdot S_y \end{aligned}\right\} \text{scaling factor}$$

$$[x' \ y'] = [x \ y] \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

$$= [x S_x \quad y S_y]$$

$$[X'] = [T][X]$$

$S_x = S_y = 1 \Rightarrow$ no change in object

$S_x = S_y > 1 \rightarrow$ uniform scaling / elongation / extention
object large

$S_x = S_y < 1 \rightarrow$ reduction / shrinking / compression.

$S_x \neq S_y$ = shape distortion

$S_x = S_y = 0 \rightarrow$ object collapses

$S_x = S_y < 0 \rightarrow$ no meaning.

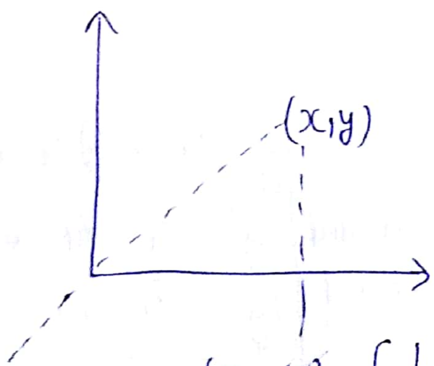
Non-uniform = $S_x = 2$
$S_y = 0.5$

Uniform = $S_x = S_y = 2$.

4) Reflection

(Specific case of scaling)

$(x', y') = (-x, y)$



$$[x' \ y'] = [x \ y] \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

y-axis

$(-x, -y)$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

origin

$(x, -y)$ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

x-axis

② Scaling

- Non uniform expansion/ compression also occur.
- Depends on whether S_x and S_y are individually >1/<1 but unequal. Such a scaling is called Differentials. (both shape/size changes)

→ In uniform scaling basic object shape remain unaltered.

→ In pure " " with factors <1 object moves closer
" >1 " farther from origin.

* Scaling w.r.t to arbitary point

① Point A(h,K) is arbitary that we wish to scale. Then we firstly translate A(h,K) to origin O(0,0).

② scaling performed w.r.t O(S_x,S_y)

③ Finally Translate A(h,K) back to (h,K).

$$T = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & K \\ 0 & 0 & 1 \end{bmatrix} \quad S = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \quad T' = \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -K \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Final Scaling matrix is :-

$$P = T \cdot S \cdot T' = \begin{bmatrix} S_x & 0 & -hS_x + h \\ 0 & S_y & -KS_y + K \\ 0 & 0 & 1 \end{bmatrix}$$

Col. matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & " & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P' = S \cdot P$$

$$\left.\begin{aligned} x' &= xS_x - hS_x + h \\ y' &= yS_y - KS_y + K \end{aligned}\right]$$

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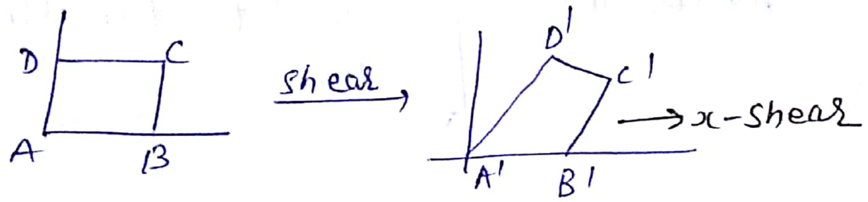
④ Shearing Transformation

Process of applying Tangential force to any object which distorts the shape of object such that transformed shape appears as if object were composed of internal layers that has been caused to slide over each other is called shear. & Ⓣ is called S.T / twist / torque / tilt.

① x- direction :- Y coordinate remain unchanged but x is changed.



$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

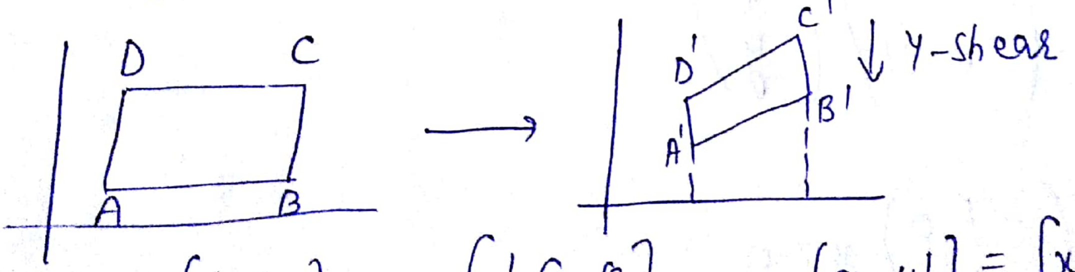
$$[x' \; y] = [x \; y] \cdot \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

$$= [x + yb, \; y]$$

$$x' = x + yb$$

$$y' = y$$

② y -direction



$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(x \; y') = [x \; y] \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

$$= [x, \; xa + y]$$

$$x' = x$$

$$y' = xa + y$$

(5) Reflection

→ Produces mirror image of an object. In 2D reflection we consider any line in 2D plane as the mirror.

→ Reflection is also described as rotation by 180°.

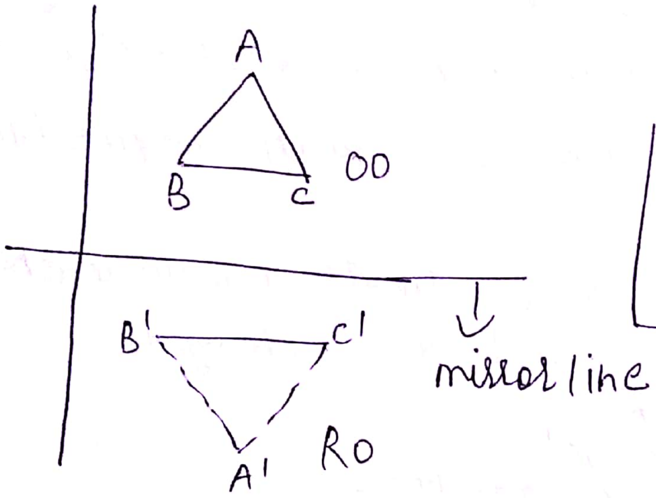
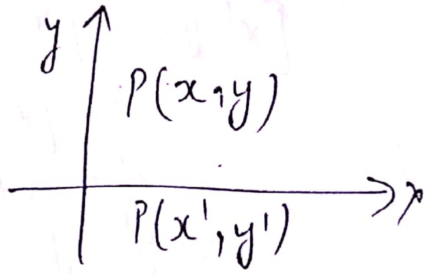


image of an object is formed on opp. side to where object is lying w.r.t to mirror line.

→ Reflection about <u>x-axis</u> changes y-coordinate but x-coordinate remain unchanged.

$$x' = x$$
$$y' = -y$$

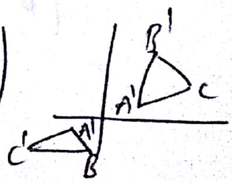


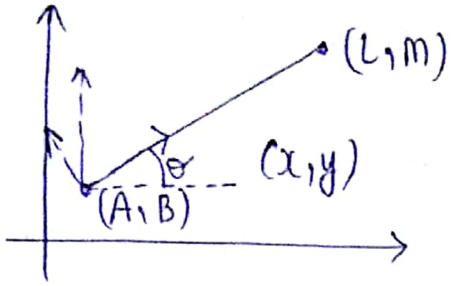
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad y = 0$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

→ y-axis $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad x = 0$

→ origin $x' = -x$, $y' = -y$ $\quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$





→ Reflect around an inclined axis.

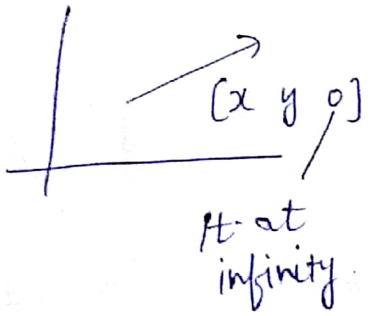
1. Translate origin :- $[-A, -B]$ — T_1
2. Rotate — T_2
3. Reflect about x axis T_3
4. Reverse

$$\boxed{T = T_1 T_2 T_3 T_2^{-1} T_1^{-1}}$$

→ Uniform scaling can be captured by single parameter.

→ Transformations can be combined.

→ Pt. at infinity can be easily captured.



$[x \quad y \quad h]$

$[x/h \quad y/h \quad 1]$

SC

$$\begin{bmatrix} a & Ro & b & c \\ d & & e & f \\ g & & i & h \end{bmatrix}$$

perspective view

T

US

Homogenous Coordinates

$[x \;\; y]$ $\quad [x_h \;\; y_h \;\; h]$ $\quad [x \;\; y \;\; 1]$

$[2 \;\; 3]$ $\quad [\frac{2}{1} \;\; \frac{3}{1} \;\; 1]$ $\quad [\frac{4}{2} \;\; \frac{6}{2} \;\; 2]$ $\quad [1 \;\; 1.5 \;\; 0.5]$

Advantages :-

① Translation

$$[x' \; y'] = [x \;\; y] + [T_x \;\; T_y]$$

$$[x' \; y' \; 1] = [x \;\; y \;\; 1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ T_x & T_y & 1 \end{bmatrix}$$

② Rotation

$$[x' \; y' \; h] = [x \;\; y \;\; 1] \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

③ Scaling

$$[x' \; y' \; h] = [x \;\; y \;\; 1] \begin{bmatrix} S_x^{①} & 0 & 0 \\ 0 & S_y^{①} & 0 \\ 0 & 0 & 1 \end{bmatrix}_{1/s}$$ (uniform scaling)

- Transformation should be captured by matrix mul.

$$[x \;\; y \;\; 1/s] = [xs \;\; ys \;\; 1]$$

Combining different Transformation

$$P_1 = PT_1$$

$$P_2 = PT_1T_2$$

$\vdots$

$$P_n = PT_1 T_2 \cdots T_n$$

$$= P(T_1T_2 \cdots T_n)$$

$$= PT$$

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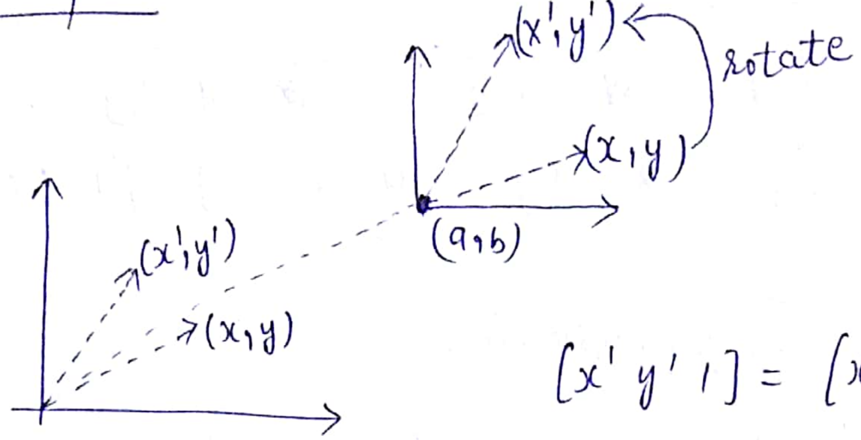
Example:-

$a+(-a)$
$b+(-b)$



$$[x' \; y' \; 1] = [x \; y \; 1] T_x \text{ (origin)}$$

→ Translate axis such that (a,b) becomes origin. → (arbitary pt.)

Step1 :- Translate origin to (a,b)

S2 :- Rotate by θ about origin

S3 :- origin back.

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & -b & 1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{bmatrix}$$

$$T = T_1 T_2 T_3 \, / \, T_1 T_2 T_1^{-1}$$

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & -b & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{bmatrix}$$

$$\boxed{P' = PT}$$

Translate a sq. with foll. coordinates:- $A(0,0)$ $B(3,0)$ $C(3,3)$ $D(0,3)$

Translate it by 2 units in both direction.

$t_x = 2$, $t_y = 2$

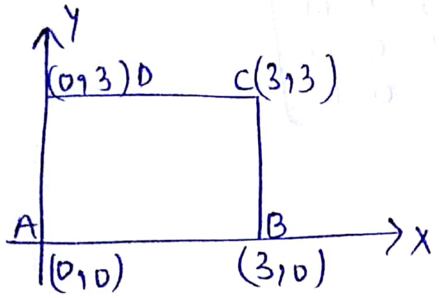
$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

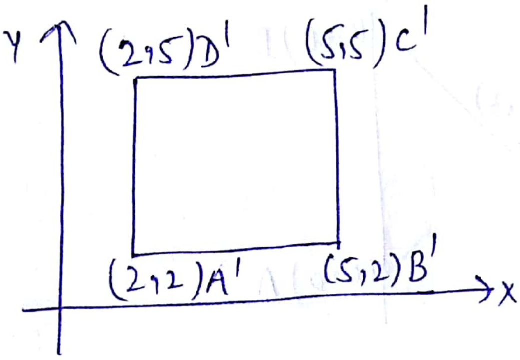
$$X = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 0 & 1 \\ 3 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

$$X^* = X \cdot T$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 3 & 0 & 1 \\ 3 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 5 & 2 & 1 \\ 5 & 5 & 1 \\ 2 & 5 & 1 \end{bmatrix}$$





② Rotate a Δ at $A(0,0)$
$B(6,0)$
$C(3,3)$ by 90° about origin in anticlockwise direction.

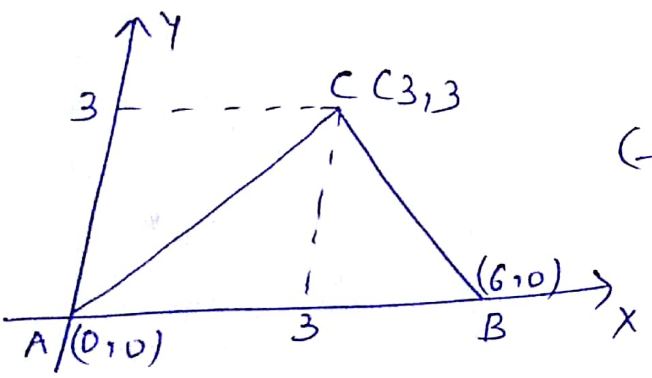
$$T = \begin{bmatrix} 0 & 0 & 1 \\ 6 & 0 & 1 \\ 3 & 3 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

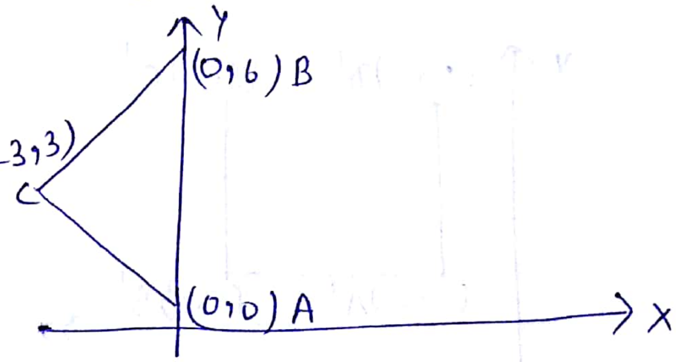
$\theta = 90^\circ$

$$R = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T' = T \cdot R = \begin{bmatrix} 0 & 0 & 1 \\ 6 & 0 & 1 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 6 & 1 \\ -3 & 3 & 1 \end{bmatrix}$$





Find Transformation matrix that transforms square ABCD whose center is at (2,2) is reduced to half of its size, with center still at (2,2). The coordinates of ABCD are A(0,0), B(0,4), C(4,4), D(4,0). Find coordinates of new square.

Ans.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 4 & 1 \\ 4 & 4 & 1 \\ 4 & 0 & 1 \end{bmatrix}$$

→ For reducing sq. ABCD to half of its size we scale it by $S_x = \frac{1}{2}$, $S_y = \frac{1}{2}$.

$$\begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 4 & 1 \\ 4 & 4 & 1 \\ 4 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

After scaling coordinates of sq. ABCD are :-

A(0,0)
B(0,2)
C(2,2)
D(2,0)

& now centre is (1,1) but we want center at (2,2) ∴ we translate square ABCD by $\left.\begin{matrix} T_x = 1 \\ T_y = 1 \end{matrix}\right]$

$$Tm = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ T_x & T_y & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

<u>After translation</u> :-

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 3 & 3 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

new coordinates are :- A(1,1)

B(1,3)

(3,3)

(3,1)

representing all transformations as matrix (x).

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

$$P' = P \cdot T(t_x \; t_y)$$

- capturing composite transformations conviently.
- Representing pt. at infinity.

$$\begin{bmatrix} \frac{x}{0} & \frac{y}{0} \end{bmatrix} = \begin{bmatrix} x & y & 0 \end{bmatrix}$$

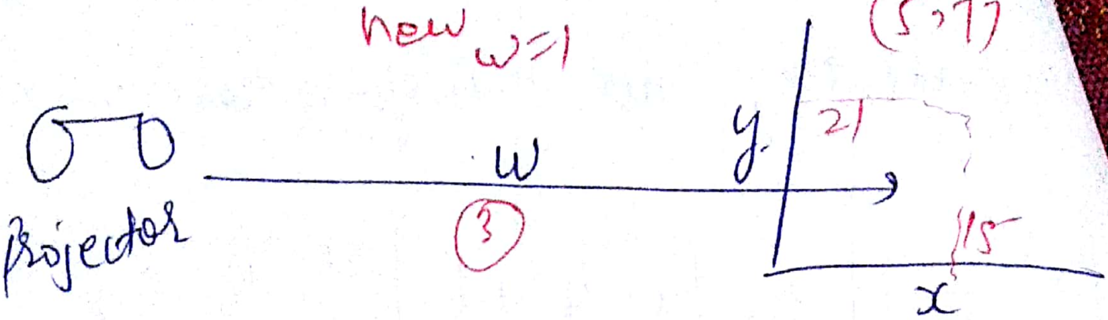
when we want to represent a pt. at infinity in a certain direction.

→ Imagine projector was pushed closer to screen so that distance ~~to~~ was (1 m).

→ closer projector goes to screen, image become smaller

→ Projector moves 3 times, so image becomes 3 times smaller.

2D



w = Dis b/w projector & screen.

→ what would happen in 2D if we inc/dec (w) i.e

→ If projector is closer to screen whole 2D image become small. (away.)

→ value of (w)/h affects size of image.

↓

Apply to 3D.

h=1/w=1,

→ Reason is when you scale a coordinate to 1, it doesn't shrink/grow, it just stay at same size.

→ (h=1) no effect on x, y, z values.

∴ in 3D coordinates are said to be "correct" only when h=1

If h>1 everything would look too small.

h<1 " " " " big.

h=0 crash.

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Composite Transformation

Process of calculating the product of matrix of a no. of different transformations in a sequence is known as Concatenation / composite Trans.

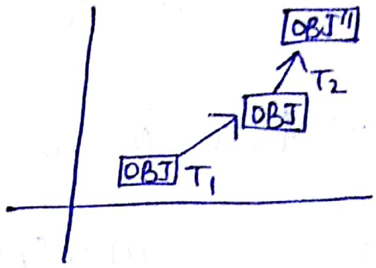
- Doing this will eliminate the calculation of intermediate coordinate values after each sucessive transformation.

- The real problem occurs when there is a translation or rotation or scaling about an arbitary point rather than origin.

→ Rules for concatenation

1) 2 succesive translations (T_1 & T_2) are additive

$$T_1 = \begin{bmatrix} 1 & 0 & tx_1 \\ 0 & 1 & ty_1 \\ 0 & 0 & 1 \end{bmatrix} \quad T_2 = \begin{bmatrix} 1 & 0 & ty_1 \\ 0 & 1 & ty_2 \\ 0 & 0 & 1 \end{bmatrix}$$



$$T_1 \cdot T_2 = \begin{bmatrix} 1 & 0 & tx_1 \\ 0 & 1 & ty_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & ty_1 \\ 0 & 1 & ty_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & tx_1 + ty_1/x_2 \\ 0 & 1 & \cancel{ty_1} + ty_2 \\ 0 & 0 & 1 \end{bmatrix}$$

2) Scaling :-

$$S_1 \begin{bmatrix} Sx_1 & 0 & 0 \\ 0 & Sy_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad S_2 \begin{bmatrix} Sx_2 & 0 & 0 \\ 0 & Sy_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_1 \cdot S_2 = \begin{bmatrix} Sx_1 \cdot Sx_2 & 0 & 0 \\ 0 & Sy_1 \cdot Sy_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation about an arbitary pt.

If (a, b, c, d) are multiplicative factors of T_1 and (m, n) are respective (x, y) Translation factors of T_2.

then we have

$$\begin{bmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{matrix} m \\ n \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix}_{3\times 3} \quad (1)$$

→ But in eq. (1) matrix isn't comfortable for multiplication with (2×1), 2D position vector matrice.

→ Hence we need, to include dummy coordinate to make (2×1) position vector matrix to a (3×1).

→ Now if we (x) $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ with non zero scalar 'h' then matrix will be :- $\begin{bmatrix} xh \\ yh \\ h \end{bmatrix}$ or $\begin{bmatrix} x_h \\ y_h \\ h \end{bmatrix}$ called as homogenous coordinate/ " position vector of same pt. $\begin{bmatrix} x \\ y \end{bmatrix}$ in 2D i.e

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} xh \\ yh \\ h \end{bmatrix} = h \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

weight which is applied to cartesian components.

→ thus, a general homogenous coordinate rep. of any pt. $P(x, y) \longrightarrow (x_h\ y_h\ h)$