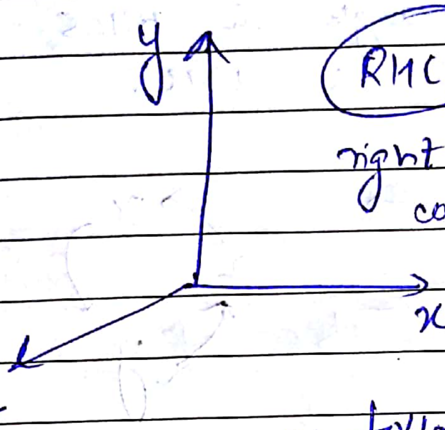
COMPUTER GRAPHICS

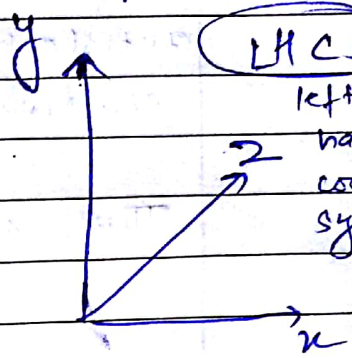
3D TRANSFORMATIONS



RHCS

right handed coord. system

~~x → thumb~~ follows right hand rule, i.e. pt. thumb towards z, then fingers will ~~x → thumb~~ curl ~~towards~~ from x to y (in right hand)



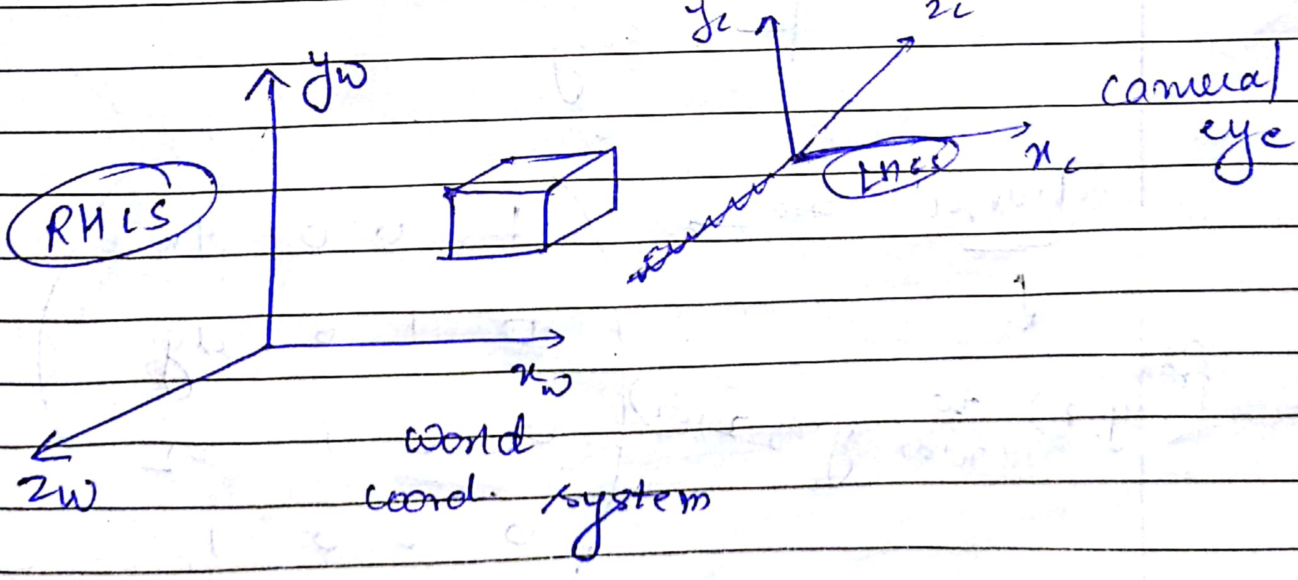
LHCS

left handed coord. system

follows left hand rule, point thumb to z, then fingers curl from x to y (in left hand)

x → thumb
y → index finger
z → middle finger

another way to understand these systems

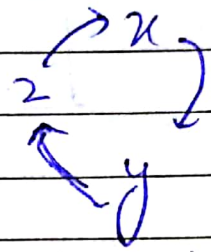


defining clockwise / anticlockwise, positive / negative angle

→ clockwise / anticlockwise defined by the respective hand rule of the system (RHCS or LHCS)

anticlockwise is +ve angle in RHCS
Clockwise is +ve angle in LHCS

Thus, in both systems →



dirn of arrow is +ve angle, opp. dirn is -ve angle.

Transformation
- transl^n
- rotation
- scaling

Transl^N → $\begin{pmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{pmatrix}$

↓

from (x, y, z) to $(x+dx, y+dy, z+dz)$

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Rotation

(1) rotation about x axis

$$R_{x,\theta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$x' = x$

$y' = y\cos\theta - z\sin\theta$

$z' = y\sin\theta + z\cos\theta$

(2) Rotation about y axis

$$R_{y,\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$x' = x\cos\theta + \cancel{z\sin\theta} + z\sin\theta$

$y' = y$

$z' = -x\sin\theta + z\cos\theta$

(3) Rotation about z axis

$$R_{z,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$x' = x\cos\theta - y\sin\theta$

$y' = x\sin\theta + y\cos\theta$

$z' = z$

Scaling

$$S_x = \begin{pmatrix} S_x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

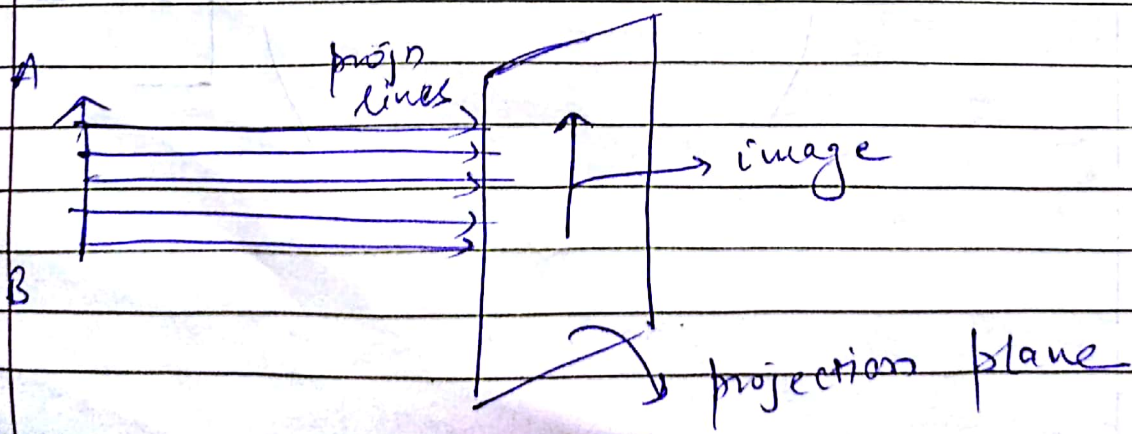
Similarly, others

3D view (Overview)
→ projections

side view	front view	Top view
Isometric view	dimetric view	persp. view

Proj^n (3D → 2D)

- parallel
  - proj^n lines are parallel
  - → orthographic
  - → oblique
- non parallel



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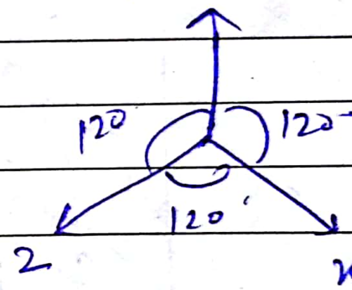
if projn lines are ⊥ to projn plane → orthographic ($\alpha = 90°$)

if at an angle → oblique ($\alpha \neq 90°$)

Orthographic

① projn lines are //
② They make $\alpha = 90°$ with projn plane

- isometric → projn lines make equal ∠'s with all 3 coord. axes



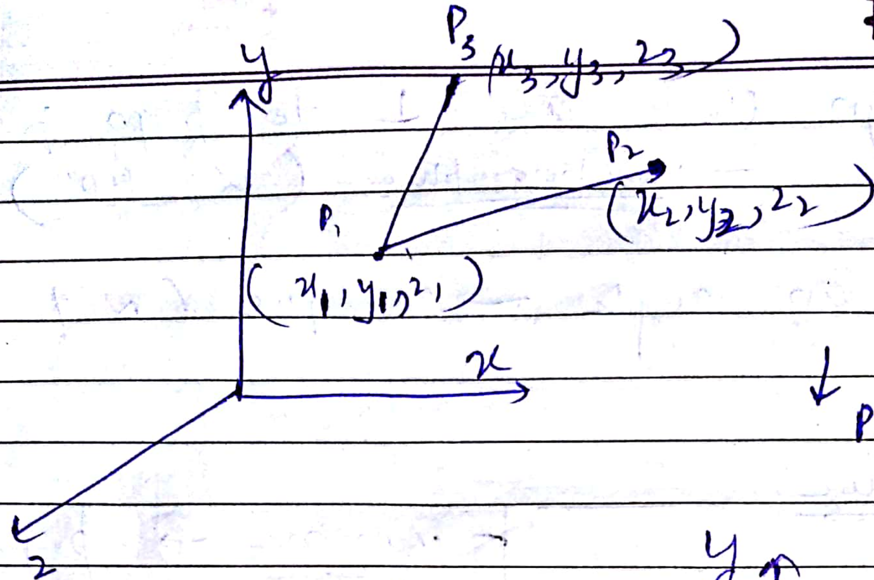
- dimetric → projn lines make equal ∠'s with any 2 axes
- Trimetric → all 3 ∠'s are unequal

Oblique projN

- cabinet
- Cavalier

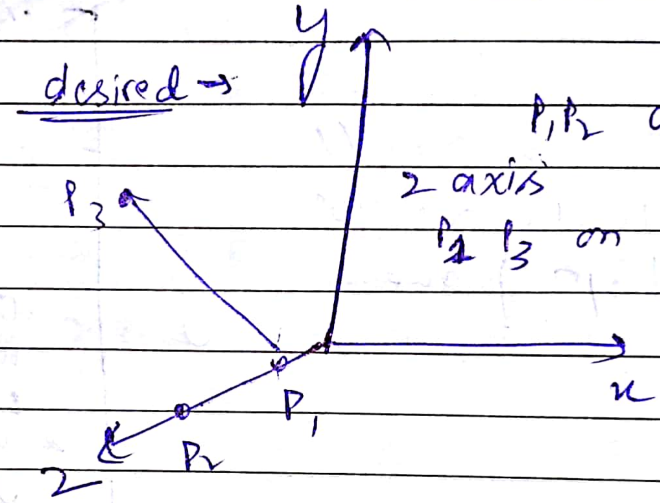
non parallel projn (perspective)
(for real time view)

Ex 1



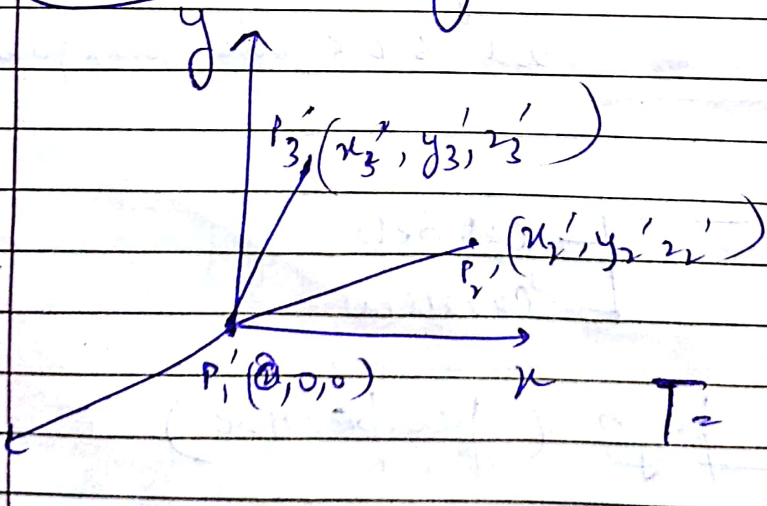
given
↓
P_1, P_2, P_3 in 3D space

desired →



$P_1 P_2$ along z axis
$P_1 P_3$ on yz plane

Step 1 bring P_1 to origin



$$T = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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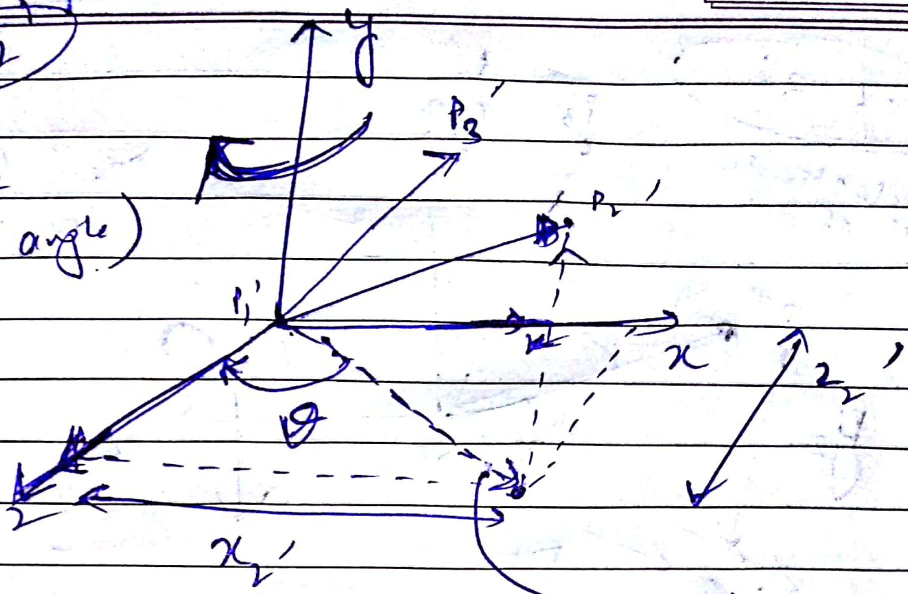
Step 2

Clockwise rotation about y (Thus -ve angle)



proj^n of P_2' on x-z plane. This proj^n is aligned with z axis

$$\cos(-\theta) = \cos\theta = \frac{z_2'}{D_1}$$

$$D_1 = \sqrt{z_2'^2 + x_2'^2}$$

$$\sin(-\theta) = -\sin\theta = \frac{-x_2'}{D_1}$$

$$R_{y,\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ -0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{z_2'}{D_1} & 0 & \frac{-x_2'}{D_1} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{x_2'}{D_1} & 0 & \frac{z_2'}{D_1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now, P_1'' P_2'' are in yz plane

~~4 possib~~

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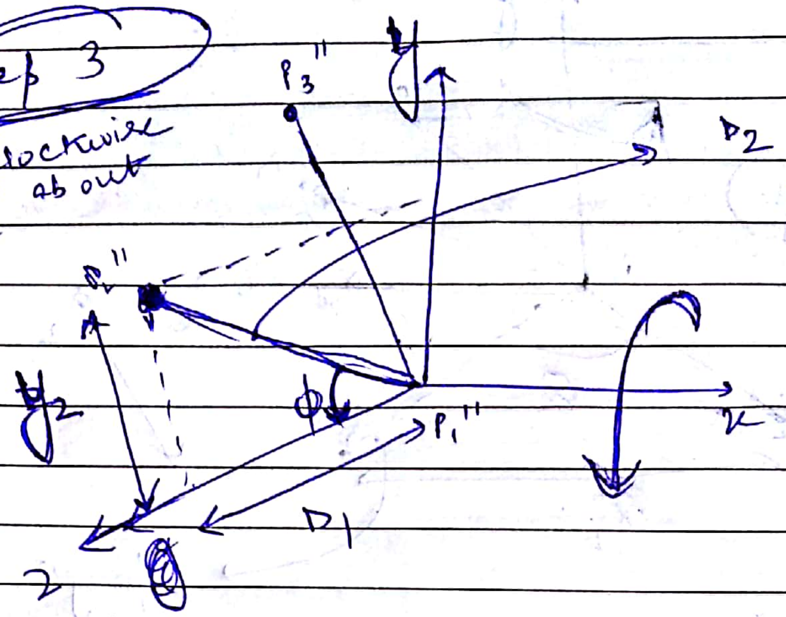
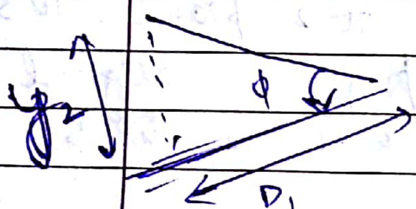
Step 3

anticlockwise rotn about x



$$D_2 = \sqrt{D_1^2 + y_2^2}$$

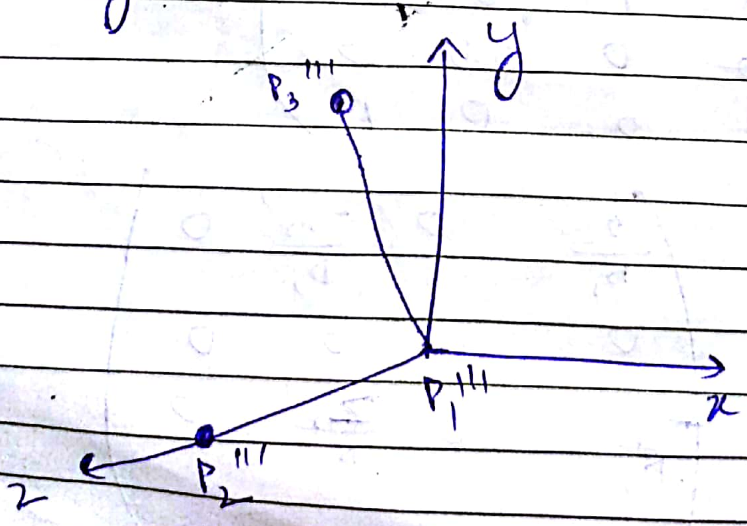
$$= \sqrt{x_2^2 + y_2^2 + z_2^2}$$



$$\cos\phi = \frac{D_1}{D_2}, \quad \sin\phi = \frac{y_2}{D_2}$$

$$R_{x,\phi} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & D_1/D_2 & -\frac{y_2}{D_2} & 0 \\ 0 & \frac{y_2}{D_2} & \frac{D_1}{D_2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Finally, we have:



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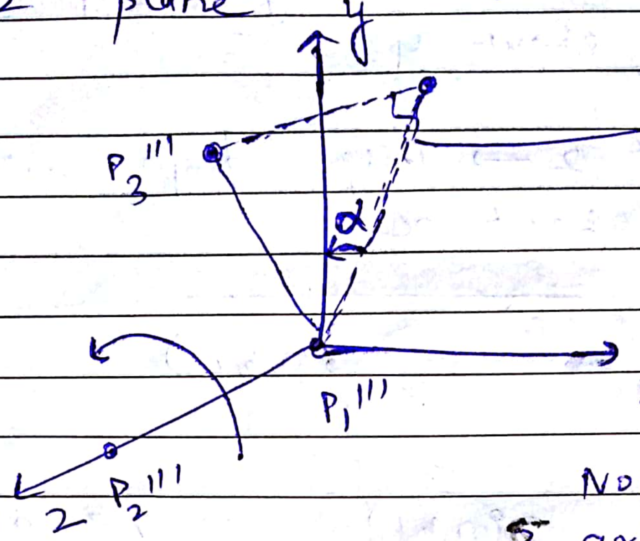
$\overrightarrow{P_1''' P_2'''}$ is finally along $\vec{z}$.

Now, we need to ~~only~~ bring P_1''' P_3''' in ~~xy~~ yz plane

Step 4



Now rotate about $\vec{z}$ axis to align projn of P_3''' with y axis

Thus, finally P_3''' will come in yz plane

Why we rotated about z axis, and not y axis as in Step 2

because rotation about any other axis will displace $\overrightarrow{P_1''' P_2'''}$ which ~~is~~ ~~a~~has already been aligned with $\vec{z}$.

In Step 2 & Step 3 → we have 4 ways to bring $P_1' P_2'$ to ~~y plane~~ z axis

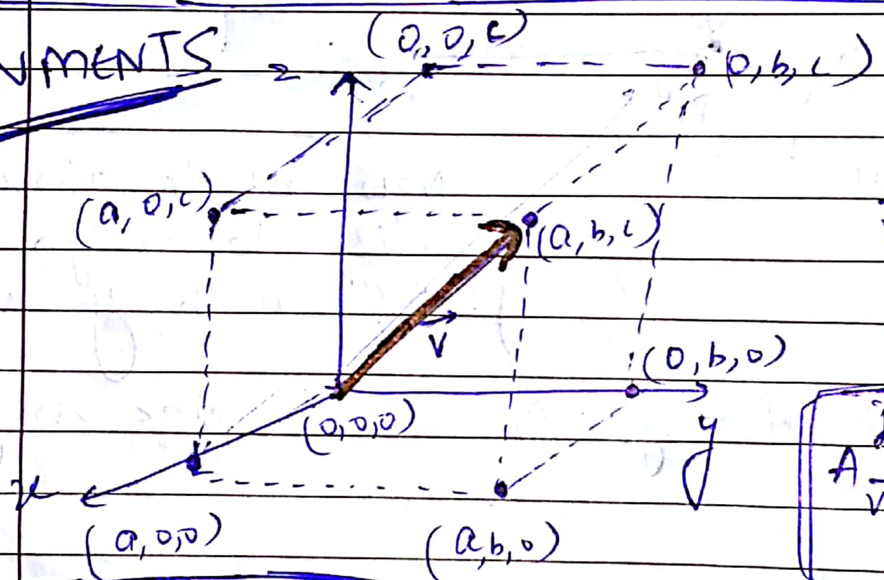
(1) Rotate about y axis → align projn with z ~~→ rotate about x axis~~
→ rotate about x axis

(2) Rotate about x → align projn with z axis
→ rotate about y

(3) Rotate about z → align projn with x axis
→ rotate about y

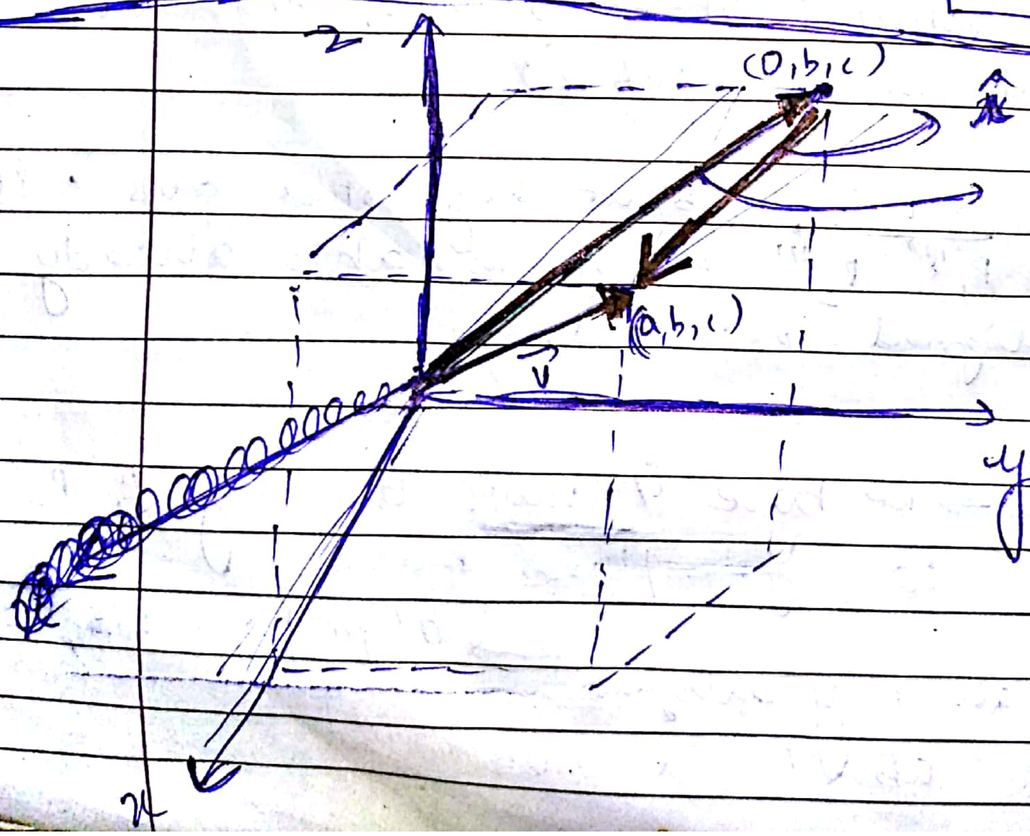
(4) Rotate about z → align projn with y axis
→ rotate about x

ALIGNMENTS



$\vec{V} = a\hat{i} + b\hat{j} + c\hat{k}$

$\boxed{A_{\vec{v}}^{\hat{j}}}$ Align $\vec{v}$ to $\hat{j}$ i.e y axis



$\hat{k}$ component of $\vec{v}$

projn of $\vec{v}$ in yz plane ($\vec{P}$)

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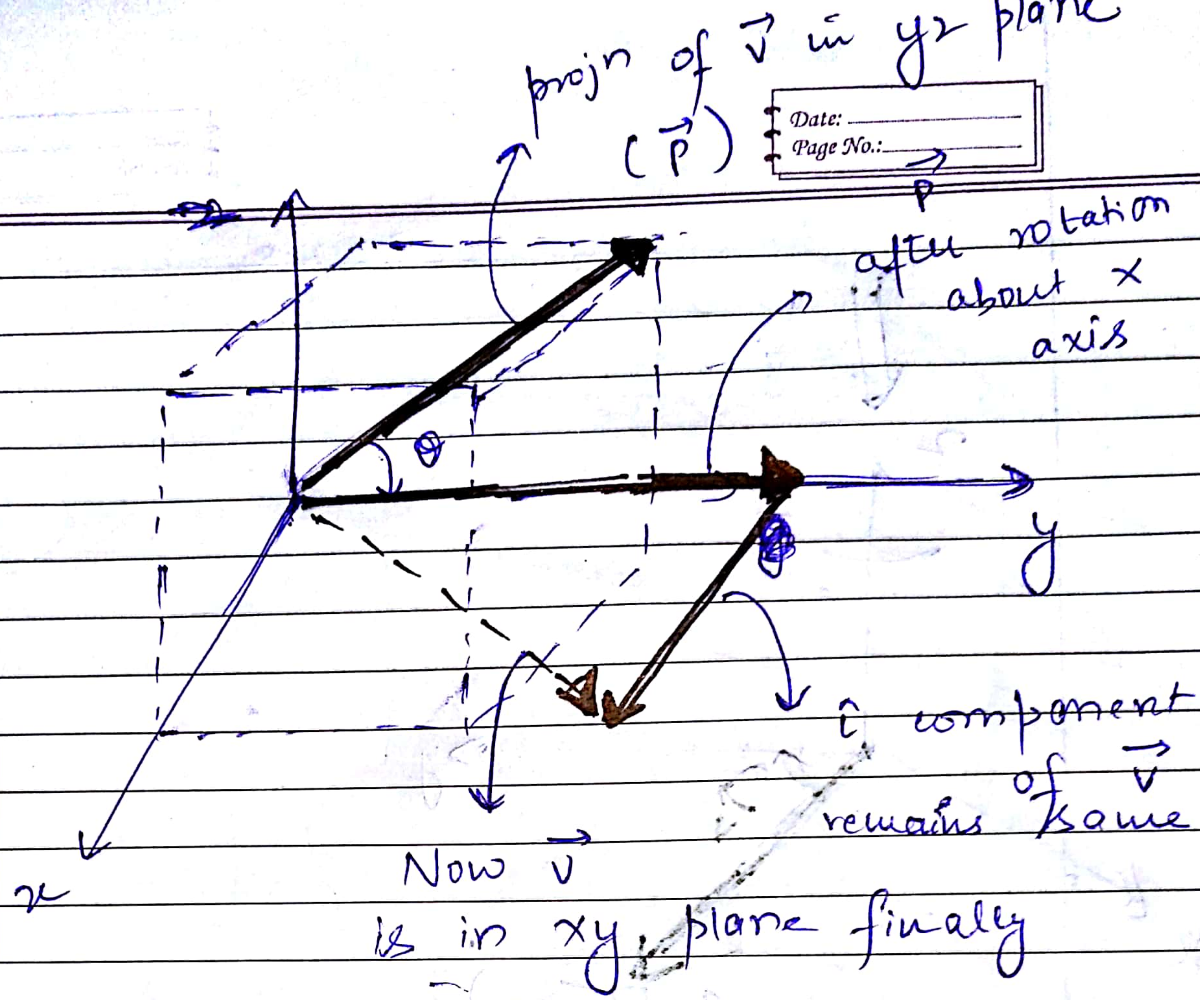
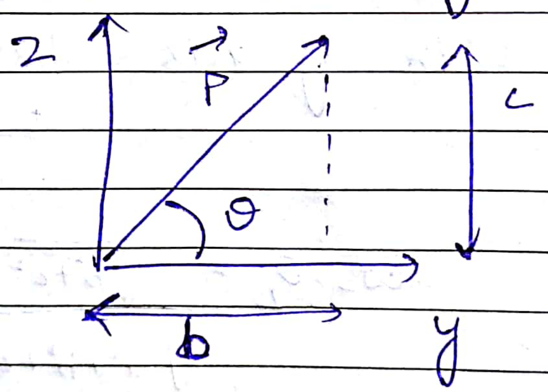


Step 1 Thus, clockwise rotation about x axis to align $\vec{P}$ with y axis

$R_{x, -\theta}$



$$\cos(-\theta) = \frac{b}{\sqrt{b^2+c^2}}$$

$$\sin(-\theta) = \frac{-c}{\sqrt{b^2+c^2}}$$

$$R_{x,-\theta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{b}{\sqrt{b^2+c^2}} & \frac{c}{\sqrt{b^2+c^2}} & 0 \\ 0 & \frac{-c}{\sqrt{b^2+c^2}} & \frac{b}{\sqrt{b^2+c^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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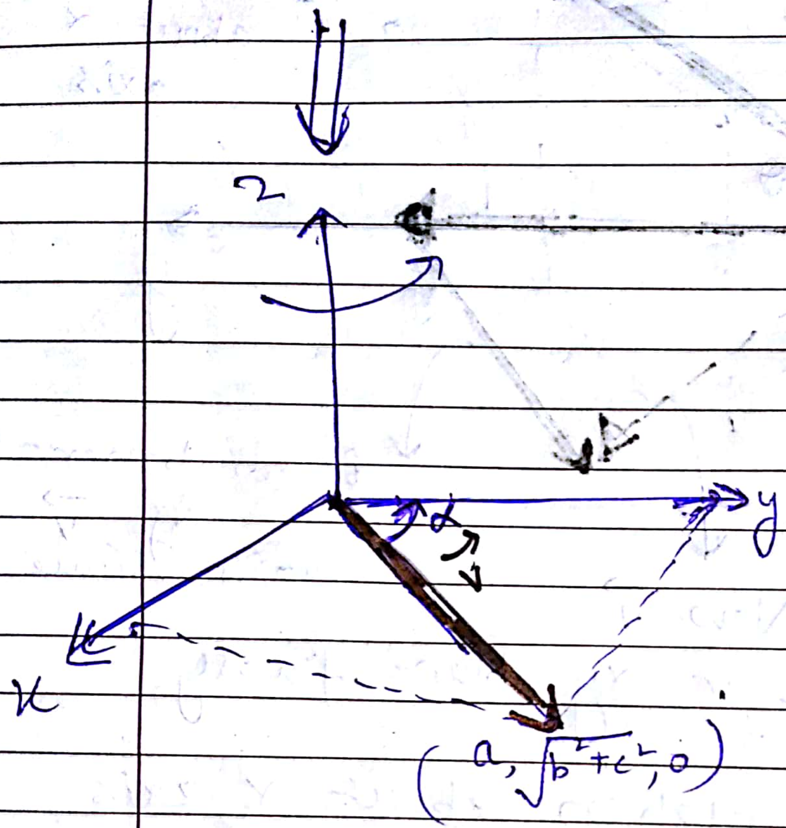
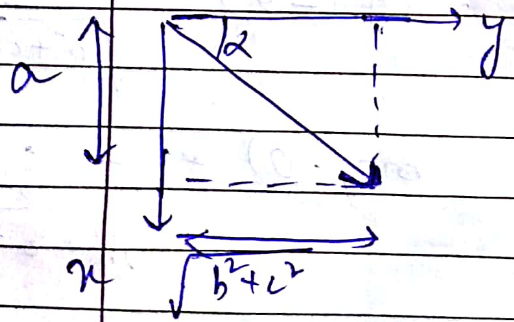


Step 2 anticlockwise rotation about z to align $\vec{V}$ with y axis finally $\hookrightarrow R_{z}, \theta_2$



$$\cos\theta_2 = \frac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}}$$

$$\sin\theta_2 = \frac{a}{\sqrt{a^2+b^2+c^2}}$$

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$$R_{2,\theta_2} = \begin{pmatrix} \frac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}} & \frac{-a}{\sqrt{a^2+b^2+c^2}} & 0 & 0 \\ \frac{a}{\sqrt{a^2+b^2+c^2}} & \frac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let, $\lambda = \sqrt{b^2+c^2}$, $|\vec{V}| = \sqrt{a^2+b^2+c^2}$

Thus, $R_{2,\theta_2} = \begin{pmatrix} \frac{\lambda}{|\vec{V}|} & \frac{-a}{|V|} & 0 & 0 \\ \frac{a}{|\vec{V}|} & \frac{\lambda}{|\vec{V}|} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2. $R_{x,-\theta_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{b}{\lambda} & \frac{c}{\lambda} & 0 \\ 0 & \frac{-c}{\lambda} & \frac{b}{\lambda} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(from step 1)

composite matrix

$$C = R_{z,\theta_2} \cdot R_{x,-\theta_1}$$

$$= \cancel{\begin{pmatrix} \frac{\lambda}{|\vec{V}|} & \frac{-a}{|\vec{V}|} & 0 & 0 \\ \frac{a}{|\vec{V}|} & \frac{\lambda}{|\vec{V}|} & 0 & 0 \\ 0 & 0 & & \end{pmatrix}}$$

$$= \begin{pmatrix} \frac{\lambda}{|V|} & \frac{-a}{|V|} & 0 & 0 \\ \frac{a}{|V|} & \frac{\lambda}{|V|} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{b}{\lambda} & \frac{c}{\lambda} & 0 \\ 0 & \frac{-c}{\lambda} & \frac{b}{\lambda} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow A_{\vec{V}, \hat{J}} = \boxed{\begin{pmatrix} \frac{\lambda}{|\vec{V}|} & \frac{-ab}{|\vec{V}|\lambda} & \frac{-ac}{|\vec{V}|\lambda} & 0 \\ \frac{a}{|\vec{V}|} & \frac{b}{|\vec{V}|} & \frac{c}{|\vec{V}|} & 0 \\ 0 & \frac{-c}{\lambda} & \frac{b}{\lambda} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

imp

$$A_{\hat{j}}^{\vec{v}} = \left(A_{\vec{v}}^{\hat{j}}\right)^{-1}$$

$$= \left(R_{z,\theta_2} * R_{x,-\theta_1}\right)^{-1}$$

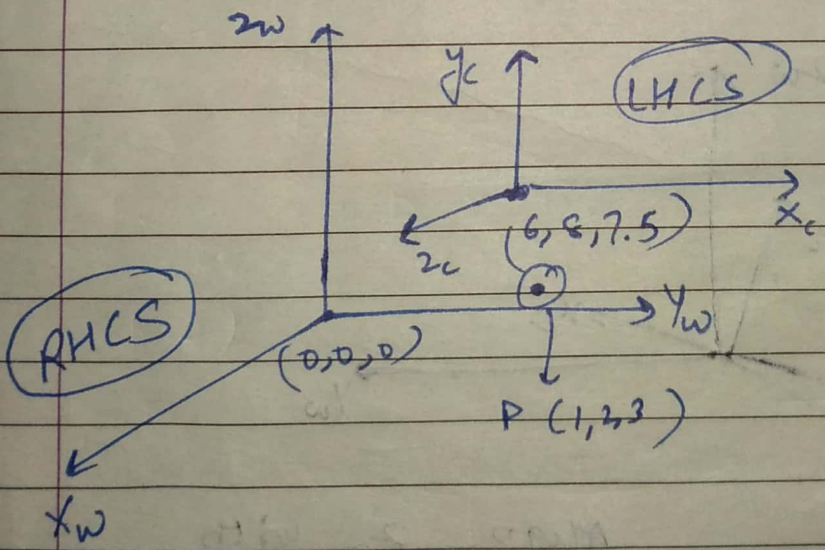
$$= \left(R_{x,-\theta_1}\right)^{-1} \cdot \left(R_{z,\theta_2}\right)^{-1}$$

$$\Rightarrow \boxed{R_{x,\theta_1} \cdot R_{z,-\theta_2}}$$

$\#\; A_{\vec{v}}^{\hat{k}} = \boxed{R_{x,90}\; A_{\vec{v}}^{\hat{j}}}$

$\#\; A_{\vec{v}}^{\hat{i}} = \boxed{R_{z,-90} \cdot A_{\vec{v}}^{\hat{j}}}$

View through a camera



- z_c is pointing to origin
- y_c is up (doesn't mean $\parallel$ to z_w)
- x_c is on $z_w = 7.5$ plane

$P(1,2,3)$

$x_w \quad y_w \quad z_w$

Q. find coordinates of P in camera coord system.

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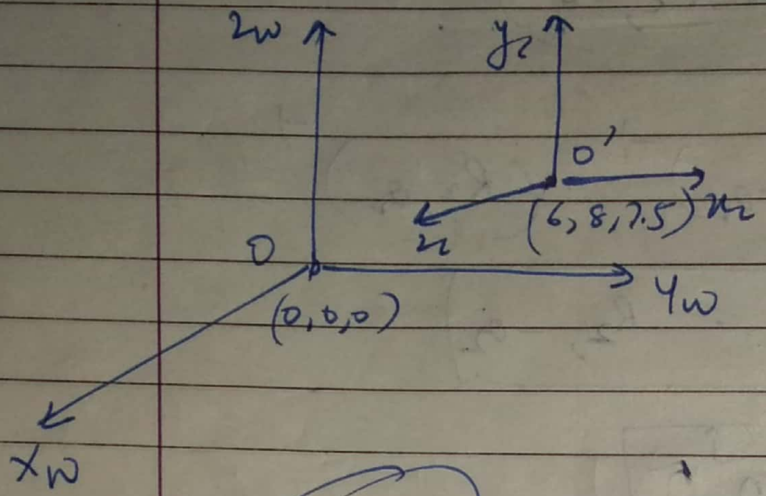
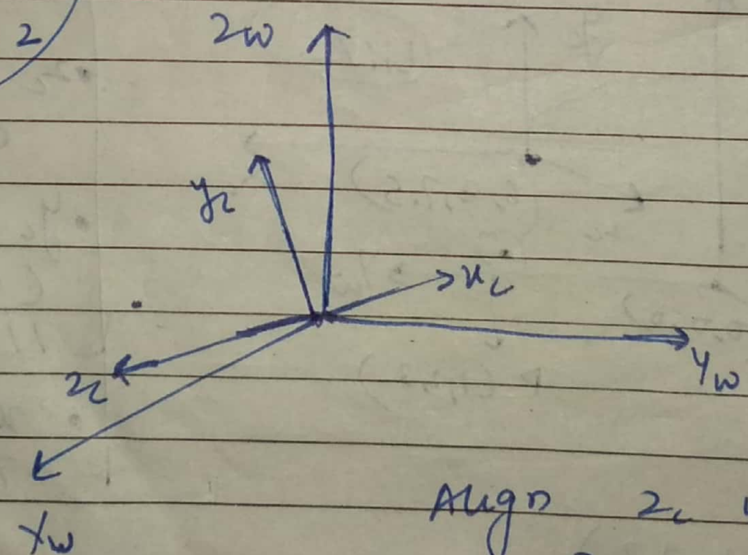
Sol^N Bring camera coord. system to origin of WCS and align the camera axes with world axes.



Step 1 O' to O

$$T = \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -7.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2



Align z_c with z_w

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$\vec{V}_{z_c} = -6\hat{i} - 8\hat{j} - 7.5\hat{k}$

$$\boxed{A_{\vec{v}}^{\hat{k}} = R_{x,90}\; A_{\vec{v}}^{\hat{j}}}$$

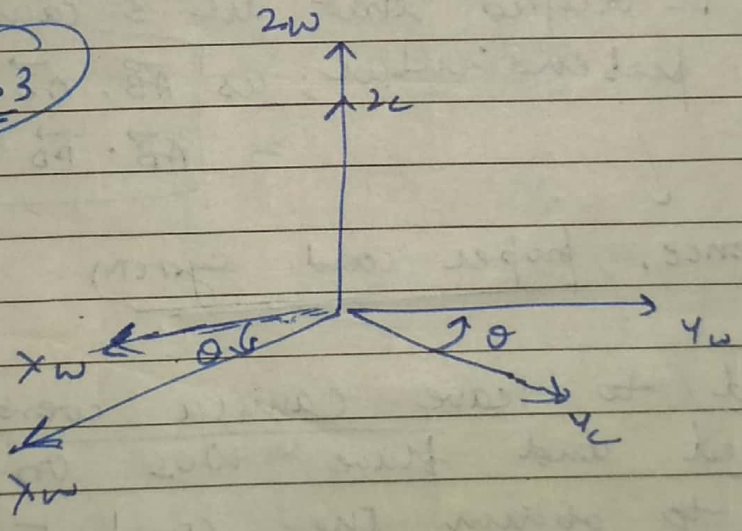
use predefined matrix

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\lambda}{|\vec{V}|} & -\frac{ab}{|\vec{V}|\lambda} & -\frac{ac}{|V|\lambda} & 0 \\ \frac{a}{|V|} & \frac{b}{|V|} & \frac{c}{|V|} & 0 \\ 0 & -\frac{c}{\lambda} & \frac{b}{\lambda} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\lambda = \sqrt{b^2 + c^2}$
$= 10.965$

$|\vec{V}| = 12.5$

Step 3



Trace $\vec{V}_{y_c}$ and align it with y_w

$\Rightarrow R_{z,\theta}$

not possible in this quesn, as dirns of x_c & y_c have not been specified, only dirn of z_c is known

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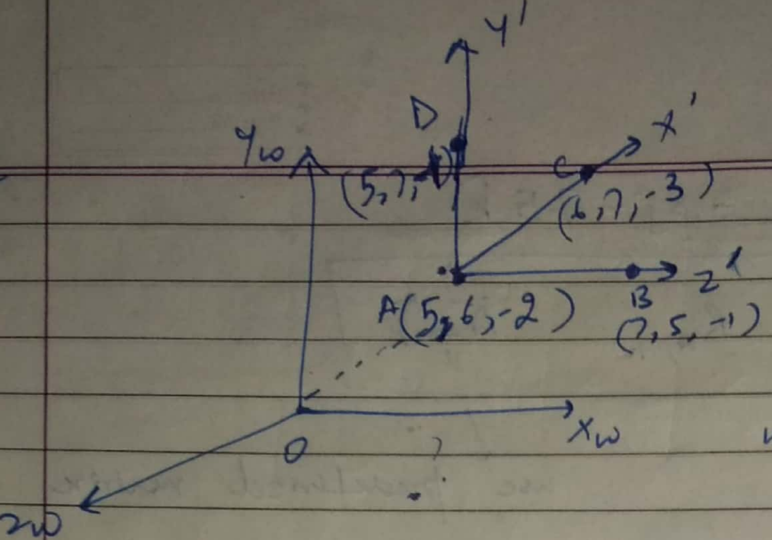

$P(x', y', z') = (1,2,3)$

Find P coord.s wrt WCS

In this quesn, we know the directions of all 3 axes in camera coord. system

$$\vec{V}_{z_c} = \vec{AB} = 2\hat{i} - \hat{j} + \hat{k}$$

$$\vec{V}_{x_c} = \vec{AC} \Rightarrow \hat{i} + \hat{j} - \hat{k}$$

$$\vec{V}_{y_c} = \vec{AD} = \hat{j} + \hat{k}$$

It can be verified that all 3 camera axes are mutually perpendicular; as $\boxed{\vec{AB} \cdot \vec{AC} = \vec{AC} \cdot \vec{AD} = \vec{AB} \cdot \vec{AD} = 0}$

hence, proper coord. system

$\Rightarrow$ We need to leave camera coord. system untouched and place WCS on it (properly aligned) to obtain the composite matrix that will be multiplied with $P(x', y', z')$ to obtain $P(x, y, z)$.

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$\Rightarrow$ However, this can be done by ~~rest~~ using LCS on WCS (properly aligned) and then taking the inverse of the final comp. matrix obtained. Doing this is easier as all rotations will be about origin.

Step 1 Bring A to origin

$$T = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

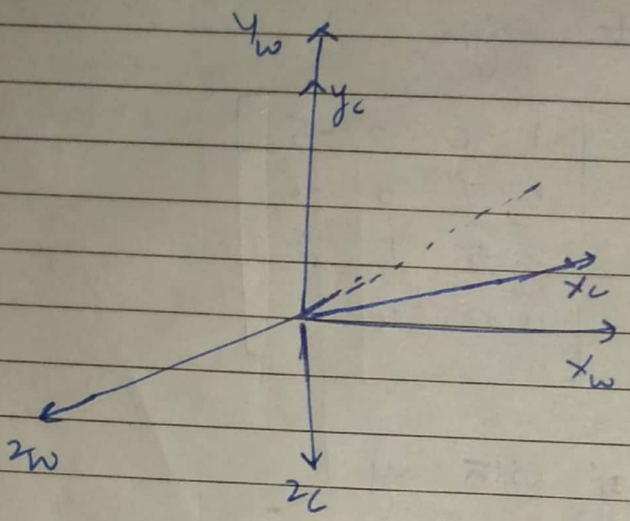
Step 2 Align y', or y_L with y

$$A_{\vec{v}}^{\hat{j}} = \begin{bmatrix} \frac{\lambda}{|V|} & \frac{-ab}{|V|\lambda} & \frac{-ac}{|V|\lambda} & 0 \\ \frac{a}{|V|} & \frac{b}{|V|} & \frac{c}{|V|} & 0 \\ 0 & -\frac{c}{|V|} & \frac{b}{|V|} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\vec{v}_{y_L} = \hat{j}+\hat{k}$, hence, $a=0$, $b=1$, $c=1$ & $\lambda=\sqrt{2}$, $|V|=\sqrt{2}$

$$\hat{J}_{A_{\vec{V}}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3



Trace x_c and align it with x_w by rotation along y_w

x_c (or x') is specified by pt. C

Thus, we need to find new coord's of C

$$C' = \hat{J}_{A_{\vec{V}}} \times T \times C$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ -3 \\ 1 \end{bmatrix}$$

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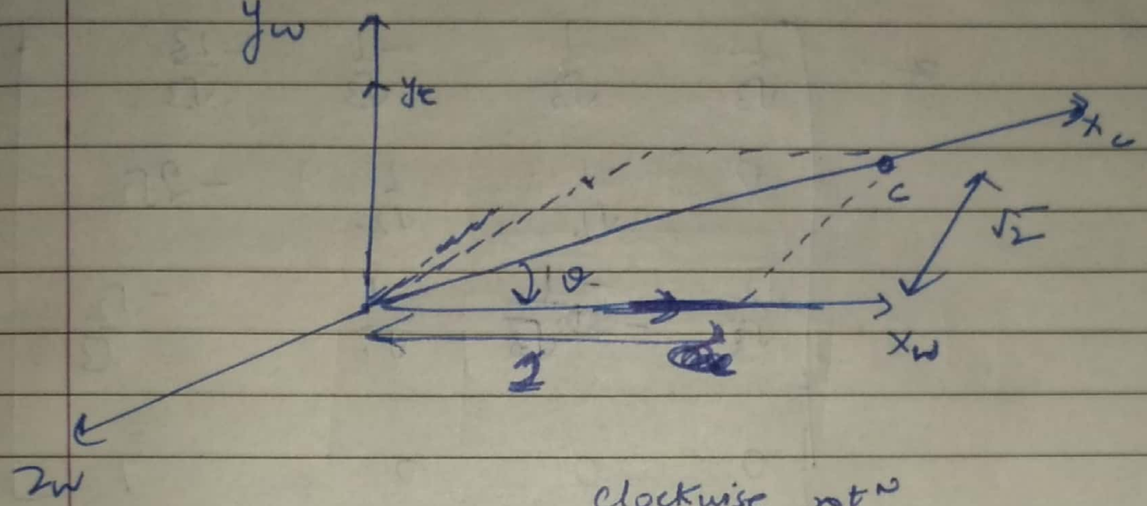
$$= \begin{bmatrix} 1 \\ 0 \\ -\sqrt{2} \\ 1 \end{bmatrix} = (1, 0, -\sqrt{2})$$



clockwise rot^n

$\Rightarrow R_{y,-\theta}$

$$\cos(-\theta) = \cos\theta = \frac{1}{\sqrt{3}}$$

$$\sin(-\theta) = -\sin\theta = \frac{-\sqrt{2}}{\sqrt{3}}$$

Hence, $R_{y,-\theta} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{\sqrt{2}}{\sqrt{3}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Thus, composite matrix C_m obtained is

$$C_m = R_{y,-\theta} \times A_{\vec{V}, \hat{J}} \times T_1$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-\sqrt{2}}{\sqrt{3}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{13}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -2\sqrt{2} \\ \frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

final composite matrix C_m reqd.

$$\Rightarrow \boxed{(C_m)^{-1}}$$

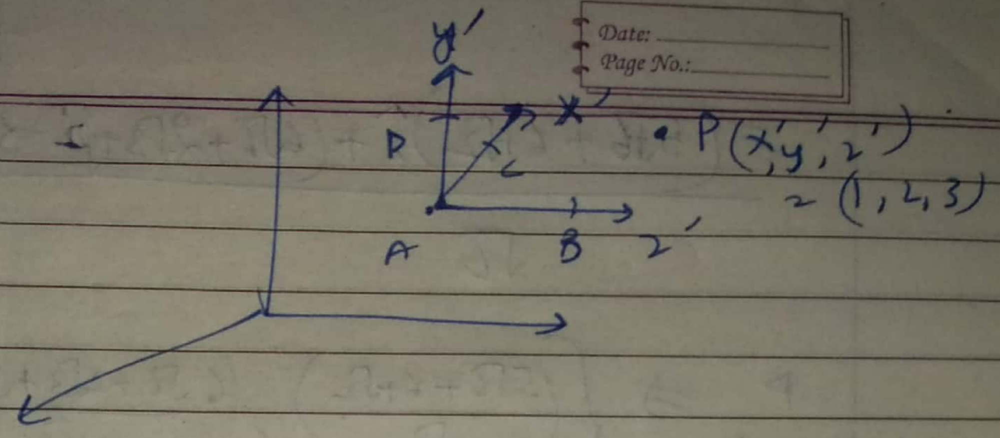
Thus, final coordinates of $P(x, y, z)$

$$= (C_m)^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8.026 \\ 6.766 \\ 0.616 \\ 1 \end{bmatrix} \Rightarrow \underline{(8.026, 6.766, 0.616)}$$

$\boxed{\text{in WCS}}$

Alter:- (for verificⁿ)



$$\vec{v}_{z_c} = 2\hat{i} - \hat{j} + \hat{k}$$

unit vector along $\vec{v}_{z_c} = \dfrac{\vec{v}_{z_c}}{|\vec{v}_{z_c}|} = \dfrac{2i - j + \hat{k}}{\sqrt{6}}$

$\vec{v}_{x_c} = i + j - k$

unit vector $= \dfrac{i + j - k}{\sqrt{3}}$

$\vec{v}_{y_c} = j + k$, unit vector $= \dfrac{j + k}{\sqrt{2}}$

Now, $\vec{AP}$ in WCS $= \dfrac{\vec{v}_{x_c}}{|v_{x_c}|} \times 1 + \dfrac{\vec{v}_{y_c}}{|\vec{v}_{y_c}|} \times 2 + \dfrac{\vec{v}_{z_c}}{|v_{z_c}|} \times 3$

(if A is brought to origin)

coord's (x', y', z') in CCS

$$\Rightarrow \frac{i + j - k}{\sqrt{3}} + 2\frac{(j + k)}{\sqrt{2}} + 3\left(\frac{2i - j + k}{\sqrt{6}}\right)$$

Finally $\vec{OP}$ in WCS $\Rightarrow \vec{OA} + \vec{AP}$

$5i + 6j - 2k$

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$$= \frac{(5\sqrt{6}+6+\sqrt{2})\hat{i} + (6\sqrt{6}+2\sqrt{3}+\sqrt{2}-3)\hat{j} + (3+2\sqrt{3}-\sqrt{2}-2\sqrt{6})\hat{k}}{\sqrt{6}}$$

$$\underset{(x,y,z)}{P} \Rightarrow \left\{ \left(\frac{5\sqrt{6}+6+\sqrt{2}}{\sqrt{6}}\right), \left(\frac{6\sqrt{6}+2\sqrt{3}+\sqrt{2}-3}{\sqrt{6}}\right), \left(\frac{3+2\sqrt{3}-\sqrt{2}-2\sqrt{6}}{\sqrt{6}}\right) \right.$$

$$= (8.026,\ 6.766,\ 0.616)$$

$$\boxed{\text{PROJECTIONS}}$$