Lecture 1 Notes

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1 Basic Notations

The following notations will be used in this class:

- $\emptyset = \{\}$ is the empty set
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ are the integers
- $\mathbb{C} = \{a + bi \mid i = \sqrt{-1}, a, b \in \mathbb{Z}\}$ are the complex numbers
- $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$ are the rational numbers
- $\mathbb{Z}^+ = \{1, 2, \dots\}$ are the positive integers

2 The Well-Ordering Principle

Definition. *Suppose $S \subseteq \mathbb{R}$. Then, $t \in \mathbb{R}$ is a least element of S if:*

(i) $t \in S$;

(ii) *if* $t_1 \in S$, *then* $t \leq t_1$

Example. *Let $S = \{t_1 \in \mathbb{R} \mid 0 < t_1\}$. Then, S has no least element.*

Proof. If t is a least element of S, then $t > 0$. But $0 < \frac{t}{2} < t$, so there is $\frac{t}{2} < t$ such that $\frac{t}{2} \in S$. This is a contradiction, so S must not have a least element. $\square$

Theorem. *(Well Ordering Property of $\mathbb{Z}^+$) if $S \subseteq \mathbb{Z}^+$ and $S \neq \emptyset$, then S has a least element.*

Corollary. *There does not exist a $t_1 \in \mathbb{Z}$, such that $0 < t_1 < 1$.*

Proof. Let $S = \{t_1 \in \mathbb{Z}^+ \mid 0 < t_1 < 1$, and suppose that $S \neq \emptyset$. Then, let t be the least element of S. We have $0 < t^2 < t < 1$, and it is clear that $t^2 \in S$. This is a contradiction, so S has no least element. And since $S \subseteq \mathbb{Z}^+$, $S = \emptyset$ by the Well-Ordering Principle. $\square$

3 The Principle of Mathematical Induction

The Principle of Induction is implied by the Well-Ordering Principle.

Definition. *$P(n)$ is a statement about $n \in \mathbb{Z}^+$ which is either true or false.*

Example. *Let $P(n)$ be the statement "$n^3 + 1$ is divisible by 3". Then, $P(1)$ is false but $P(2)$ is true.*

Theorem. *(Principle of Induction) Suppose that we know two things about P:*

(i) *$P(1)$ is true;*

(ii) *(Inductive Hypothesis) For any $n \in \mathbb{Z}^+$, $P(m)$ is true $\forall m < n \implies P(n)$ is true.*

Then, we can conclude that $P(n)$ is true $\forall n \in \mathbb{Z}^+$.

Proof. Let $S = \{n \mid n \in \mathbb{Z}^+, P(n) \text{ is false}\}$. From given, $P(1)$ is true so $1 \notin S$. Now suppose that $P(k)$ is false for some k. Then, $S \neq \emptyset$ and S has a least element $n_1 \in S$ by the Well-Ordering Principle. Therefore, $P(n_1 - 1), P(n_1 - 2), \ldots, P(1)$ are all true. Then by the Inductive Hypothesis given, $P(n_1)$ must be true. Thus, $P(n_1) \notin S$. This is a contradiction, so $S = \emptyset$ i.e. $P(n)$ is true $\forall n \geq 1$. $\square$