Topological Ramsey Theory: Ellentuck's Theorem

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Introduction

Recall the classical formulation to Ramsey's Theorem:

Theorem (Ramsey)

For any $k, n, r \in \mathbb{Z}^+$, there exists an N such that $N \to (k)^n_r$.

This can be easily generalized to the case when our set is infinite. In fact, this is the original formulation that Ramsey came up with[1]:

Theorem (Ramsey)

Let χ be a finite coloring on $[\omega]^k$ for some $k \in \mathbb{Z}^+$. Then, there exists an infinite $W \subseteq \omega$ such that $\chi|_{[W]^k}$ is constant.

- Can this result be generalized further?

[1] A little caveat: Ramsey actually proved something a little bit stronger, as he showed the same result for arbitrary infinite sets.

Motivation

What if we consider arbitrary collections of subsets on ω?

- Unfortunately, no. As an example, $[\omega]^{<\omega}$ doesn't works:

Counterexample

For any $a \in [\omega]^{<\omega}$, let $\chi(a) = |a| \mod 2$. In other words, color a red if $|a|$ is even, and color it blue otherwise.

- Clearly, no subset of $[\omega]^{<\omega}$ cannot be homogeneous in χ.

Ellentuck's work gave a topological characterization for which subsets of $[\omega]^{\omega}$ can have "Ramsey like" properties.

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Notational Conventions

The following notation will be used throughout the presentation:

- $a, b, x \ldots$ are finite subsets of ω, i.e. $a \in [\omega]^{<\omega}$
- $A, B, X \ldots$ are infinite subsets of ω, i.e. $A \in [\omega]^{\omega}$
- $[a, A]^{\omega} = \{B \in [\omega]^{\omega} \mid a \subseteq B \subseteq a \cup A_{>a}\}$, where[2]:
  - $A_{>a} = \{n \in A \mid \forall m \in a, n > m\}$
- Symmetric difference: $A \triangle B = (A - B) \cup (B - A)$

[2] To amuse you, sets of this form are called donuts. The motivation behind this name is quite clear.

The Ramsey Property

The following properties on collections of infinite subsets in ω determine whether they behave like "colorings" or not.

Definition (Ramsey property)

Given a set $W \subseteq [\omega]^\omega$, we say that W is Ramsey if there is some $a \in [\omega]^\omega$, such that either $[a]^\omega \subseteq W$ or $[a]^\omega \cap W = \emptyset$.

A stronger (and more useful) condition is:

Definition (complete Ramsey property)

Given a set $W \subseteq [\omega]^\omega$, we say that W is completely Ramsey if for each a and A, either $[a, A]^\omega \subseteq W$ or $[a, A]^\omega \cap W = \emptyset$.

The Baire Property

We need a few basic topology concepts for our characterization.

Definition (topology terms)

Let X be a topological space, then:

- $A \subseteq X$ is dense if $X \cap O$ is nonempty for any open set O.
- $B \subseteq X$ is nowhere dense if $\overline{X}$ contains a dense open set.
- A countable union of nowhere dense sets is meager.

Now, we can define the Baire property:

Definition (Baire property)

A is Baire if there is some open set O such that $A \triangle O$ is meager.

The Ellentuck Topology

Ellentuck defined[3] the following topology on $[\omega]^\omega$:

Definition (Ellentuck topology)

Denote $\mathcal{O} = \{\emptyset\} \cup \{[a, A]^\omega\}$. Then, $([\omega]^\omega, \mathcal{O})$ is a topological space and $\mathcal{O}$ is called the Ellentuck topology.

We will not prove them here, but $\mathcal{O}$ intuitively satisfies the three topology axioms. Notably, $\mathcal{O}$ also has the following property:

- $[a, A]^\omega \cap [b, B]^\omega$ is either empty or $[a \cup b, A \cap B]^\omega$.
- Singleton sets are nowhere dense, countable sets are meager.
- Open sets are completely Ramsey.

[3] Ellentuck himself, in fact, did not name it after himself. He simply referred to this construction as the "classical" topology.

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Ellentuck's Theorem

Ellentuck, in his 1974 paper, showed the following fact about collections of infinite subsets in ω:

Theorem (Ellentuck, 1974)

Let $W \subseteq [\omega]^\omega$. Then, W is completely Ramsey if and only if W is Baire in the Ellentuck Topology.

We now give an outline to the proof of Ellentuck's theorem.

- His original proof used the Galvin-Prikry theorem, a result that formulates a similar characterization for Borel sets.
- It generalizes a theorem of Silver about analytic (Σ^1_1) sets.
- However, we will follow the modified approach published by Matet in 2001 that simplifies the argument.

Proving Ellentuck's Theorem

We first show the forward direction of Ellentuck's theorem.

- Let $W \subseteq [\omega]^\omega$ be completely Ramsey. Then, define:

$$O_W = \bigcup \{[a, A]^\omega \mid [a, A]^\omega \subseteq W\}$$

- Since any open set in the Ellentuck topology has the form $S = [a, A]^\omega$, any such S must either be in O_W or $O_{\overline{W}}$. Thus, $O_W \cup O_{\overline{W}}$ is dense.
- Clearly, $O_W \subseteq W$ and $O_{\overline{W}} \subseteq \overline{W}$. Thus, O_W and $O_{\overline{W}}$ both have trivial intersections with $W - O_W$; that is, $(W - O_W) \cap (O_W \cup O_{\overline{W}}) = \emptyset$. So $W - O_W$ is nowhere dense.
- Now $W \triangle O_W = W - O_W$ is meager, i.e. W is Baire. $\square$

Lemmas

The backwards direction of Ellentuck's theorem is more involved. We need a few lemmas which will not be proved here.

Lemma 1

Define the set $\mathcal{N}$ to be the collection of all W such that:

- For all a, A, there exists b, B with $[b, B]^\omega \subseteq [a, A]^\omega - W$.

Then, the following fact holds about $\mathcal{N}$.

Lemma

$\mathcal{N}$ is closed under countable unions.

This is a consequence of a lemma due to Matet based on an argument of Mathias.

Lemma (Matet)

Let $W_i \subseteq [\omega]^\omega$ such that $[a, B]^\omega \notin \bigcap W_i$ for each $B \in [A]^\omega$. Then, there exists c, C such that for some W_i, $[d, D]^\omega \not\subseteq W_i$ for every $[d, D]^\omega \subseteq [c, C]^\omega$.

Lemma 2

Similarly, we define the set $\mathcal{C}$ to be all W such that:

- there exists b, B with $[b, B]^\omega \subseteq [a, A]^\omega$, and either $[b, B]^\omega \subseteq W$ or $[b, B]^\omega \cap W = \emptyset$.

Then, the following fact holds about $\mathcal{C}$.

Lemma

Every member of $\mathcal{C}$ is completely Ramsey.

This can also be derived from Matet's lemma from above.[4]

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[4] In fact, this lemma is the key concept that allowed Matet to achieve a short proof. The conventional proof of the theorem follows the same "track", but relied on a different (more stringent) construction to $\mathcal{N}$ and $\mathcal{C}$: replacing $[a, B]^\omega$ instead of $[b, B]^\omega$ in each construction.

Proving Ellentuck's Theorem

Now, we proceed with the backwards direction of the theorem:

- Let W have the Baire property. Then, $W \triangle O$ is meager for some open set O.

We first show that $W \triangle O \in \mathcal{N}$.

- As $W \triangle O$ is meager, $W \triangle O = \bigcup S_i$ with S_i nowhere dense. There exists $\tilde{S}_i$ disjoint from S_i that is dense; i.e. $\tilde{S}_i \subseteq \overline{S_i}$.
- Then since each $[a, A]^\omega$ is open, we have:

$$[a, A]^\omega - S_i = [a, A]^\omega \cap \overline{S_i} \supseteq [a, A]^\omega \cap \tilde{S}_i = [\tilde{a}, \tilde{A}]^\omega$$

  Since $\tilde{S}_i$ is dense and $[a, A]^\omega \cap \tilde{S}_i$ is open.
- Thus $S_i \in \mathcal{N}$, and by Lemma 1 $W \triangle O \in \mathcal{N}$ holds as well.

Proving Ellentuck's Theorem

We then show that $W \in \mathcal{C}$.

- Let a, A be arbitrary. Then by above, there are b, B such that:

$$[b, B]^\omega \subseteq [a, A]^\omega - (W \triangle O) \subseteq (W \cap O) \cup (\overline{W \cup O})$$

- $[b, B]^\omega \subseteq [a, A]^\omega$ is clear. By above, if it falls into the first half of the union, then clearly $[b, B]^\omega \subseteq W$; otherwise, it is also evident that $[b, B]^\omega \cap W = \emptyset$.
- By construction, we have $W \in \mathcal{C}$.

Now by Lemma 2, we have that W is completely Ramsey. $\square$

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