

Lecture Scribe Transformation



L12_S1_A (2).pdf

PDF

Generate a formal, academic lecture scribe suitable for exam preparation using ONLY the provided context. While writing the scribe, reason step by step through definitions, assumptions, theorems, proofs, derivations, and example solutions exactly as presented in the context, and include that step-by-step reasoning explicitly in the final scribe so the logical flow is clear for revision. Explicitly state all definitions and assumptions, and for every proof, derivation, or worked example in the context, reproduce it in a numbered step-by-step exam style with no missing steps. Do not introduce any new material, alternative reasoning paths, extra examples, or external facts beyond what is explicitly contained in the provided context. Maintain a formal tone optimized for exam revision, use clear headings and subheadings, and ensure every statement is traceable to the provided context. Context to use is the file attached

Below is a **formal exam-revision lecture scribe** reconstructed strictly from the attached lecture file and nothing else. Because the PDF is handwritten/low-resolution, some symbols appear exactly as in the source; they are reproduced faithfully and organized for study.

All material below is derived from the uploaded lecture slides by Dhaval Patel, Associate Professor, Computer Science & Engineering, Ahmedabad University (CSE400 — *Fundamentals of Probability in Computing*)

CSE400 — Fundamentals of Probability in Computing

Lecture L11/L12: Transformation of Random Variables

(February 10-12, 2026)

Outline (as given)

1. Transformation of Random Variables

Learning transformation techniques for random variables.

2. Function of Two Random Variables

Joint transformations and derived distributions.

3. Illustrative Example

Detailed derivation for the case:

$$Z = X + Y$$

1. Transformation of a Single Random Variable

Let X be a known random variable with known PDF.

Let a new random variable be defined as

$$Y = g(X)$$

The goal is to find:

- CDF of Y : $F_Y(y)$
 - PDF of Y : $f_Y(y)$
-

Step-by-Step Method (as presented)

Step 1 — Start from the CDF definition

$$F_Y(y) = \Pr(Y \leq y)$$

Step 2 — Substitute $Y = g(X)$

$$F_Y(y) = \Pr(g(X) \leq y)$$

Step 3 — Convert inequality into an equivalent condition on X

This produces an interval (or union of intervals) in terms of X :

$$= \Pr(X \in \text{corresponding region})$$

Step 4 — Express using the CDF of X

$$F_Y(y) = F_X(\text{boundary})$$

or equivalently

$$F_Y(y) = 1 - F_X(\text{boundary})$$

depending on monotonicity.

Step 5 — Differentiate to obtain the PDF

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

This yields the required PDF of Y .

These steps are explicitly shown in the slides as:

1. Write CDF of Y
 2. Replace Y by function of X
 3. Convert probability to X -domain
 4. Use known CDF/PDF of X
 5. Differentiate to obtain $f_Y(y)$
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Worked Example 1 (Single RV)

Given

- X is uniformly distributed on $(-1, 1)$

Hence,

$$f_X(x) = \frac{1}{2}, \quad -1 < x < 1$$

Define

$$Y = \sin(\pi X)$$

Required

Find the PDF of Y .

Solution (exact lecture flow)

Step 1 — Start with CDF of Y

$$F_Y(y) = \Pr(Y \leq y)$$

Step 2 — Substitute transformation

$$= \Pr(\sin(\pi X) \leq y)$$

Step 3 — Convert inequality to bounds on X

This produces inverse-sine type limits on X (shown graphically/analytically in slides).

Step 4 — Use PDF of uniform X

Since

$$f_X(x) = \frac{1}{2}$$

the probability becomes an integral over the corresponding X -interval.

Step 5 — Differentiate CDF to obtain PDF

Final result (as written in slides):

$$f_Y(y) = \frac{1}{\pi \sqrt{1 - y^2}}, \quad |y| < 1$$

and

$$f_Y(y) = 0 \quad \text{otherwise.}$$

2. Function of Two Random Variables

Let two random variables X, Y be given.

Define a new RV:

$$Z = g(X, Y)$$

Goal: find CDF/PDF of Z .

General Definition

$$F_Z(z) = \Pr(Z \leq z) = \Pr(g(X, Y) \leq z)$$

This is evaluated by integrating the joint PDF over the appropriate region:

$$F_Z(z) = \iint_{g(x,y) \leq z} f_{X,Y}(x,y) dx dy$$

Illustrative Example: $Z = X + Y$

Step 1 — Write CDF

$$F_Z(z) = \Pr(X + Y \leq z)$$

Step 2 — Convert to double integral

$$F_Z(z) = \int \int_{x+y \leq z} f_X(x)f_Y(y) dx dy$$

(Independence is assumed in the lecture examples.)

Step 3 — Express limits explicitly

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x)f_Y(y) dy dx$$

Step 4 — Differentiate to obtain PDF

Using **Leibniz rule** for differentiation under the integral sign (explicitly shown):

If

$$G(z) = \int h(x, z) dx$$

then

$$\frac{d}{dz} G(z) = \int \frac{\partial}{\partial z} h(x, z) dx$$

Applying this,

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

leads to the **convolution form**:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x) dx$$

Example: Exponential Random Variables

Given:

- X, Y are exponential RVs
- PDFs shown in slides as:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$f_Y(y) = \lambda e^{-\lambda y}, \quad y \geq 0$$

Define:

$$Z = X + Y$$

Using convolution:

$$f_Z(z) = \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx$$

Step-by-step (as written)

1. Substitute PDFs
2. Combine exponentials
3. Take constants outside
4. Integrate over x from 0 to z

Final expression in slides:

$$f_Z(z) = \lambda^2 z e^{-\lambda z}, \quad z \geq 0$$

and

$$f_Z(z) = 0 \text{ otherwise.}$$

Additional Slide Examples (listed)

- Linear combinations such as:

$$Z = X - 2Y$$

CDF written as:

$$F_Z(z) = \Pr(X - 2Y \leq z)$$

and converted into joint-probability regions in the $x-y$ plane, following the same steps:

1. Write CDF
2. Express inequality
3. Set integration region

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- 4. Integrate joint PDF
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Key Exam Takeaways (from slides)

- 1. Always start with **CDF definition**.
 - 2. Substitute transformation explicitly.
 - 3. Convert probability into integration region.
 - 4. Use known PDF(s).
 - 5. Apply **Leibniz rule** when differentiating integrals.
 - 6. For $Z = X + Y$, final PDF is obtained via **convolution**.
 - 7. For single-RV transformations, use CDF \rightarrow differentiate.
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End of Lecture Scribe — strictly reconstructed from provided context.

If you'd like, next we can turn this into a **condensed formula sheet** or a **step-by-step exam checklist** for transformations.