

CSE 400: Fundamentals of Probability in Computing

Lecture 8 – Continuous Random Variables and Gaussian Random Variable

1. Continuous Random Variable (CRV)

Definition

A random variable (X) is a **continuous random variable** if it is characterized by a **probability density function (PDF)** ($f_X(x)$).

Properties of PDF

- $(f_X(x) \geq 0)$ for all (x)
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$$\int_{-\infty}^{\infty} f_X(x), dx = 1$$

Probability from PDF

For any interval $([x_1, x_2])$:

$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f_X(x), dx$$

2. Cumulative Distribution Function (CDF)

Definition

The **CDF** of a continuous random variable (X) is defined as:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(t), dt$$

Interval Probability Using CDF

$$P(x_1 \leq X \leq x_2) = F_X(x_2) - F_X(x_1)$$

3. Expectation of a Continuous Random Variable

Expectation of a Function of X

For a function ($g(X)$):

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x), dx$$

Mean (Expected Value)

$$\mu_X = E[X] = \int_{-\infty}^{\infty} xf_X(x), dx$$

4. Moments of a Random Variable

n-th Order Moment (About the Origin)

$$E[X^n]$$

Special cases:

- ($n = 0$): ($E[X^0] = E[1] = 1$)
 - ($n = 1$): ($E[X] = \mu_X$)
 - ($n = 2$): ($E[X^2]$)
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5. Central Moments

Definition

The **n-th order central moment** is defined as:

$$E[(X - \mu_X)^n]$$

Special Cases

- ($n = 0$):
$$E[(X - \mu_X)^0] = E[1] = 1$$
 - ($n = 1$):
$$E[(X - \mu_X)] = E[X] - \mu_X = 0$$
 - ($n = 2$): Variance
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6. Variance

Definition

$$\sigma_X^2 = E[(X - \mu_X)^2]$$

Derivation

$$E[(X - \mu_X)^2] = E[X^2 - 2\mu_X X + \mu_X^2]$$

Using linearity of expectation:

$$= E[X^2] - 2\mu_X E[X] + \mu_X^2$$

Since ($E[X] = \mu_X$):

$$\sigma_X^2 = E[X^2] - \mu_X^2$$

7. Moments vs Central Moments

- Moments: ($E[X^n]$)
- Central Moments: ($E[(X - \mu_X)^n]$)

Key distinction:

$$E[X]^2 \neq E[X^2]$$

8. Skewness

Definition (3rd Central Moment)

$$C_s = \frac{E[(X - \mu_X)^3]}{\sigma_X^3}$$

Interpretation

- ($C_s > 0$): Right-skewed PDF
 - ($C_s < 0$): Left-skewed PDF
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9. Kurtosis

Definition (4th Central Moment)

$$C_k = E[(X - \mu_X)^4]$$

Observation

- Large kurtosis indicates a large peak near the mean
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10. Linearity of Expectation

Theorem

For constants (a) and (b):

$$E[aX + b] = aE[X] + b$$

Extension

If ($g(x) = g_1(x) + g_2(x) + \dots + g_n(x)$), then:

$$E[g(X)] = E[g_1(X)] + E[g_2(X)] + \dots + E[g_n(X)]$$

11. Gaussian (Normal) Random Variable

Definition

A random variable (X) is Gaussian if:

$$X \sim \mathcal{N}(\mu_X, \sigma_X^2)$$

Probability Density Function

$$f_X(x) = \frac{1}{\sqrt{2\pi}, \sigma_X} \exp\left(-\frac{(x - \mu_X)^2}{2\sigma_X^2}\right)$$

12. Standard Normal Distribution

Definition

If:

$$\mu_X = 0, \quad \sigma_X^2 = 1$$

Then (X) follows the **standard normal distribution**.

13. Symmetry of Gaussian Distribution

- PDF is symmetric about ($x = \mu_X$)
 - Left and right areas around the mean are equal
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14. Worked Probability Example (from Lecture)

Given:

$$|X + 3| < 2$$

Step-by-step:

$$\begin{aligned} -2 &< X + 3 < 2 \\ -5 &< X < -1 \end{aligned}$$

Probability:

$$P(-5 < X < -1) = F_X(-1) - F_X(-5)$$

End of Lecture 8 Scribe