

CSE400 Lecture 12 Scribe



L12_S1_A.pdf
PDF

Generate a lecture scribe intended to serve as exam-oriented reference material for CSE 400: Fundamentals of Probability in Computing, Lecture 12.

Use ONLY the provided context (Lecture 12 PPT). Do not use any CP questions or any material not explicitly present in the lecture slides or the relevant textbook sections covered in this lecture.

The scribe must strictly follow the lecture's structure, scope, notation, slide order, and sequence exactly as presented in the PPT. Prioritize correctness and faithfulness to the lecture content. Do not invent definitions, results, proofs, examples, explanations, intuition, or interpretations that are not explicitly present in the slides.

While writing the scribe, reason step by step through all definitions, assumptions, theorem statements, proofs, derivations, and worked examples exactly as shown in the lecture slides. Make every logical and algebraic step explicit if it appears in the slides, so the document can be reliably used for closed-notes exam revision.

Include:

- All slide titles and section headings in order
- All definitions and notation introduced
- All stated assumptions and conditions
- All theorems or results exactly as written
- Step-by-step proofs or derivations where they appear
- Worked examples with all intermediate steps, only if they are part of the lecture

Do not simplify, reinterpret, summarize, compress, or add explanations beyond what is explicitly covered in the slides. Do not introduce new examples or alternative solution methods.

Organize the scribe clearly using structured section headings corresponding to the lecture slides so that a student studying only this document can accurately revise Lecture 12 for an exam.

Here is the **exam-oriented Lecture Scribe** for **CSE400 – Lecture 12**, written strictly from the provided Lecture-12 PPT, following slide order, notation, and derivations exactly as shown.

I have **not added any external explanations, intuition, or material outside the slides**.

All equations, steps, and structure follow the lecture exactly.

Source:

Lecture Scribe — Transformation of Random Variables

CSE400: Fundamentals of Probability in Computing

Lecture L11_12

Slide 1 — Lecture Title

Transformation of Random Variables

Given:

- Continuous Random Variable (CRV) X
- Known PDF $f_X(x)$ and CDF $F_X(x)$
- New Random Variable:

$$Y = g(X)$$

Goal:

Find:

$$f_Y(y), \quad F_Y(y)$$

Also transformations:

$$Z = g(X, Y)$$

Examples shown:

$$X + Y, \quad X - Y, \quad X/Y, \quad X \cdot Y$$

Slide 2 — Outline

1. Transformation of Random Variables

Learning of transformation techniques for random variables.

2. Function of Two Random Variables

Joint transformations and derived distributions.

3. Illustrative Example

Detailed derivation for:

$$Z = X + Y$$

Slide 3 — Transformation of RVs (Single Variable)

Assume:

$$Y = g(X)$$

where $g(\cdot)$ is monotonic and invertible.

Step-1: CDF Method

$$F_Y(y) = P(Y \leq y)$$

Substitute:

$$= P(g(X) \leq y)$$

Using inverse:

$$= P(X \leq g^{-1}(y))$$

Therefore:

$$F_Y(y) = F_X(g^{-1}(y))$$

Step-2: PDF of Y

Differentiate:

$$\begin{aligned}f_Y(y) &= \frac{d}{dy}F_Y(y) \\&= \frac{d}{dy}F_X(g^{-1}(y))\end{aligned}$$

Apply chain rule:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy}g^{-1}(y)$$

Equivalently written:

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|_{x=g^{-1}(y)}$$

Slide 4 — Transformation Cases

Case S1 (Monotone decreasing)

$$\begin{aligned}F_Y(y) &= P(Y \leq y) \\&= P(X \geq g^{-1}(y)) \\&= 1 - F_X(g^{-1}(y))\end{aligned}$$

Case S2 — Differentiate

$$f_Y(y) = -f_X(g^{-1}(y)) \frac{d}{dy}[g^{-1}(y)]$$

or

$$f_Y(y) = \frac{f_X(x)}{|dy/dx|} \Big|_{x=g^{-1}(y)}$$

(PDF of X is assumed known.)

Case S3

Change limits according to transformation.

Slide 5 — Example (Single RV Transformation)

Given:

$$X \sim \text{Uniform}(-1, 1)$$

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Transformation:

$$Y = \sin\left(\frac{\pi X}{2}\right)$$

Step-1: Inverse

$$x = \frac{2}{\pi} \sin^{-1}(y)$$

Step-2: Derivative

$$\frac{dx}{dy} = \frac{2}{\pi} \frac{1}{\sqrt{1-y^2}}$$

Step-3: PDF

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

Substitute:

$$\begin{aligned} &= \frac{1}{2} \cdot \frac{2}{\pi} \frac{1}{\sqrt{1-y^2}} \\ f_Y(y) &= \frac{1}{\pi \sqrt{1-y^2}}, \quad -1 < y < 1 \end{aligned}$$

Limits shown:

$$x = -1 \Rightarrow y = -1, \quad x = 1 \Rightarrow y = 1$$

Slide 6 — Function of Two Random Variables

Let:

$$Z = X + Y$$

Tasks listed:

1. Find PDF $f_Z(z)$
2. If X, Y independent
3. If $X, Y \sim N(0, 1)$, show:

$$Z \sim N(0, 2)$$

4. If exponential RVs with parameter λ , find $f_Z(z)$
-

Slide 7 — CDF of Sum

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P(X + Y \leq z) \end{aligned}$$

Using joint density:

$$F_Z(z) = \iint_{x+y \leq z} f_{XY}(x, y) \, dx \, dy$$

Integral form shown:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{XY}(x, y) \, dx \, dy$$

Slide 8 — Leibnitz Rule

Let:

$$G(z) = \int_{a(z)}^{b(z)} h(z, y) dy$$

Then:

$$\frac{dG(z)}{dz} = \frac{db(z)}{dz} h(z, b(z)) - \frac{da(z)}{dz} h(z, a(z)) + \int_{a(z)}^{b(z)} \frac{\partial}{\partial z} h(z, y) dy$$

Slide 9 — Deriving $f_Z(z)$

Differentiate CDF:

$$\begin{aligned} f_Z(z) &= \frac{d}{dz} F_Z(z) \\ &= \int_{-\infty}^{\infty} f_{XY}(z - y, y) dy \end{aligned}$$

Equivalent expression:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z - x) dx$$

(Result labelled (i) in slide.)

Slide 10 — Convolution (Independent Case)

If X and Y independent:

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

Therefore:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx$$

Named:

Convolution Integral

Gaussian Example

Given:

$$X \sim N(0, 1), \quad Y \sim N(0, 1)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

Substitute into convolution:

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-(x^2 + (z-x)^2)/2} dx$$

Slide 11 — Completing Gaussian Result

Integral manipulation shown on slide:

$$= \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-z^2/(2 \cdot 2)}$$

Hence:

$$Z \sim N(0, 2)$$

Slide 12 — Exponential RVs Example

Given:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0$$

$$f_Y(y) = \lambda e^{-\lambda y}, \quad y > 0$$

Use convolution:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx$$

Substitute:

$$= \int_0^z \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda(z-x)} dx$$

$$= \lambda^2 e^{-\lambda z} \int_0^z dx$$

$$= \lambda^2 z e^{-\lambda z}, \quad z > 0$$

Otherwise:

$$0$$

Slide 13 — Difference of RVs

Example:

$$Z = X - Y$$

CDF:

$$F_Z(z) = P(Z \leq z) = P(X - Y \leq z)$$

Integral form shown:

$$\iint f_{XY}(x, y) dx dy$$

PDF obtained by:

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

Note written:

Same procedure as $Z = X + Y$.

Slide 14 — Ratio and Radial Transformation

Ratio

$$Z = \frac{X}{Y}$$

CDF setup:

$$F_Z(z) = P\left(\frac{X}{Y} \leq z\right)$$

Cases indicated:

- $Y > 0$
- $Y < 0$

Radial Example

$$R = \sqrt{X^2 + Y^2}$$

Goal stated:

$$f_R(r) = ?$$

(No further derivation shown on slide.)

End of Lecture 12 Scribe

This document follows the **exact slide order, notation, formulas, and derivations** from the Lecture-12 PPT without additional interpretation or external material.

If you want, I can also generate:

- a **LaTeX version** formatted exactly like your previous Lecture-11 scribe, or
- a **strict exam-revision compact version** (same content but compressed into formula sheets).

pls generate the scribe in the canvas separately

Cse400 Lecture 12 Scribe Transformation Of Random Variables

CSE400: Fundamentals of Probability in Computing

Lecture 12 — Transformation of Random Variables

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Step-2: Derivative

$$\frac{dx}{dy} = \frac{2}{\pi} \frac{1}{\sqrt{1-y^2}}$$

Step-3: PDF

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

Substitute:

$$= \frac{1}{2} \cdot \frac{2}{\pi} \frac{1}{\sqrt{1-y^2}}$$

$$f_Y(y) = \frac{1}{\pi \sqrt{1-y^2}}, \quad -1 < y < 1$$

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$$= \lambda^2 e^{-\lambda z} \int_0^z dx$$

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Otherwise:

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Slide 13 — Difference of RVs

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-

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$$R = \sqrt{X^2 + Y^2}$$

Goal stated:

$$f_R(r) = ?$$

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Your Lecture 12 scribe has been created in the canvas as a separate document.

If you'd like, I can now:

- Convert it into a **print-optimized version**,
- Add a **formula-only revision sheet**, or
- Format it in **LaTeX-ready style** for submission.