CSE400 — Fundamentals of Probability in Computing
Lecture L11/L12: Transformation of Random Variables

February 10–12, 2026

Outline (as given)

Transformation of Random Variables
Learning transformation techniques for random variables.
Function of Two Random Variables
Joint transformations and derived distributions.
Illustrative Example
Detailed derivation for the case:

$$Z = X + Y$$

L12_S1_A (2)

1. Transformation of a Single Random Variable

Let X be a known random variable with known PDF.
Let a new random variable be defined as

$$Y = g(X)$$

The goal is to find:
CDF of Y:

$$F_Y(y)$$

PDF of Y:

$$f_Y(y)$$

Step-by-Step Method (as presented)

Step 1 — Start from the CDF definition

$$F_Y(y) = \Pr(Y \leq y)$$

Step 2 — Substitute $Y = g(X)$

$$F_Y(y) = \Pr(g(X) \leq y)$$

Step 3 — Convert inequality into an equivalent condition on X
This produces an interval (or union of intervals) in terms of X:

$$= \Pr(X \in \text{corresponding region})$$

Step 4 — Express using the CDF of X

$$F_Y(y) = F_X(\text{boundary})$$

or equivalently

$$F_Y(y) = 1 - F_X(\text{boundary})$$

depending on monotonicity.
Step 5 — Differentiate to obtain the PDF

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

This yields the required PDF of Y.
These steps are explicitly shown in the slides as:
Write CDF of Y
Replace Y by function of X
Convert probability to X-domain
Use known CDF/PDF of X
Differentiate to obtain $f_Y(y)$
L12_S1_A (2)

Worked Example 1 (Single RV)

Given
X is uniformly distributed on $(-1, 1)$
Hence,

$$f_X(x) = \frac{1}{2}, \quad -1 < x < 1$$

Define

$$Y = \sin(\pi X)$$

Required
Find the PDF of Y.

Solution (exact lecture flow)

Step 1 — Start with CDF of Y

$$F_Y(y) = \Pr(Y \le y)$$

Step 2 — Substitute transformation

$$= \Pr(\sin(\pi X) \le y)$$

Step 3 — Convert inequality to bounds on X
This produces inverse–sine type limits on X (shown graphically/analytically in slides).

Step 4 — Use PDF of uniform X
Since
$$f_X(x) = \frac{1}{2}$$
the probability becomes an integral over the corresponding X-interval.
Step 5 — Differentiate CDF to obtain PDF
Final result (as written in slides):
$$f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}}, \quad |y| < 1$$
and
$$f_Y(y) = 0 \text{ otherwise.}$$
L12_S1_A (2)

2. Function of Two Random Variables

Let two random variables X, Y be given.
Define a new RV:
$$Z = g(X, Y)$$
Goal: find CDF/PDF of Z.

General Definition

$$F_Z(z) = \Pr(Z \le z) = \Pr(g(X, Y) \le z)$$
This is evaluated by integrating the joint PDF over the appropriate region:
$$F_Z(z) = \iint_{g(x,y)\le z} f_{X,Y}(x, y)\, dx\, dy$$
L12_S1_A (2)

Illustrative Example: $Z = X + Y$

Step 1 — Write CDF
$$F_Z(z) = \Pr(X + Y \le z)$$
Step 2 — Convert to double integral
$$F_Z(z) = \iint_{x+y\le z} f_X(x) f_Y(y)\, dx\, dy$$
(Independence is assumed in the lecture examples.)
Step 3 — Express limits explicitly
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x) f_Y(y)\, dy\, dx$$
Step 4 — Differentiate to obtain PDF

Using Leibniz rule for differentiation under the integral sign (explicitly shown):
If

$$G(z) = \int h(x, z)\, dx$$

then

$$\frac{d}{dz}G(z) = \int \frac{\partial}{\partial z}h(x, z)\, dx$$

Applying this,

$$f_Z(z) = \frac{d}{dz}F_Z(z)$$

leads to the convolution form:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z - x)\, dx$$

L12_S1_A (2)

Example: Exponential Random Variables

Given:
X, Y are exponential RVs
PDFs shown in slides as:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$f_Y(y) = \lambda e^{-\lambda y}, \quad y \geq 0$$

Define:

$$Z = X + Y$$

Using convolution:

$$f_Z(z) = \int_0^z \lambda e^{-\lambda x}\lambda e^{-\lambda(z-x)}\, dx$$

Step-by-step (as written)
Substitute PDFs
Combine exponentials
Take constants outside
Integrate over x from 0 to z
Final expression in slides:

$$f_Z(z) = \lambda^2 z e^{-\lambda z}, \quad z \geq 0$$

and

$$f_Z(z) = 0 \text{ otherwise.}$$

L12_S1_A (2)

Additional Slide Examples (listed)

Linear combinations such as:

$$Z = X - 2Y$$

CDF written as:

$$F_Z(z) = \Pr(X - 2Y \leq z)$$

and converted into joint-probability regions in the x-y plane, following the same steps:
Write CDF
Express inequality
Set integration region
Integrate joint PDF
L12_S1_A (2)

Key Exam Takeaways (from slides)

Always start with CDF definition.
Substitute transformation explicitly.
Convert probability into integration region.
Use known PDF(s).
Apply Leibniz rule when differentiating integrals.
For $Z = X + Y$, final PDF is obtained via convolution.
For single-RV transformations, use CDF $\rightarrow$ differentiate.