

# CSE400 – Fundamentals of Probability in Computing

## Lecture 4: Joint Probability and Conditional Probability

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### Introduction to Probability Theory

#### Experiments, Sample Space, and Events

**Experiment (E)** An experiment is a procedure that produces an outcome. The result is not known in advance but follows well-defined rules.

**Example:** Tossing a coin five times.

**Outcome ( $\omega$ )** An outcome is a single possible result of an experiment.

**Example:** One possible outcome of tossing a coin five times is HHTHT.

**Event** An event is a collection (set) of outcomes from the same experiment. Events are usually denoted by capital letters.

**Example:** For the coin-tossing experiment, let event  $C$  be:

$$C = \{\text{all outcomes consisting of an even number of heads}\}$$

This means  $C$  contains many outcomes, not just one.

**Sample Space ( $S$ )** The sample space is the set of all possible distinct outcomes of an experiment.

**Key properties:**

- **Mutually Exclusive:** No two outcomes can occur at the same time.
- **Collectively Exhaustive:** One of the outcomes must occur.

**Example:** For a single coin flip:

- Heads and tails are mutually exclusive.
- Heads and tails together exhaust all possibilities.

The sample space  $S$  is the universal set for the experiment.

#### Types of Sample Spaces

A sample space can be:

- Discrete
- Countably infinite

- Continuous

**Examples:**

- Flipping a fair coin once
- Rolling a cubical die
- Rolling two dice
- Flipping a coin until a tail occurs
- Random number generator on interval  $[0, 1)$

## Axioms of Probability

**Probability** Probability is a numerical measure of how likely an event is to occur. It is a function that maps events to numbers between 0 and 1.

### Axioms

1. For any event  $A$ :

$$0 \leq \Pr(A) \leq 1$$

2. If  $S$  is the sample space:

$$\Pr(S) = 1$$

3. If  $A$  and  $B$  are mutually exclusive ( $A \cap B = \emptyset$ ):

$$\Pr(A \cup B) = \Pr(A) + \Pr(B)$$

4. For an infinite collection of mutually exclusive events  $A_1, A_2, \dots$ :

$$\Pr\left(\bigcup_i A_i\right) = \sum_i \Pr(A_i)$$

## Corollary from Probability Axioms

**Corollary 2.1** For a finite number  $M$  of mutually exclusive events  $A_1, A_2, \dots, A_M$ :

$$\Pr\left(\bigcup_i A_i\right) = \sum_i \Pr(A_i)$$

## Propositions from Probability Axioms

**Proposition 2.1 (Complement Rule)**

$$\Pr(A^c) = 1 - \Pr(A)$$

**Proposition 2.2 (Monotonicity)** If  $A \subseteq B$ , then:

$$\Pr(A) \leq \Pr(B)$$

### Proposition 2.3 (Addition Rule)

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

### Proposition 2.4 (Inclusion–Exclusion Principle)

$$\Pr\left(\bigcup_{i=1}^M A_i\right) = \sum_i \Pr(A_i) - \sum_{i < j} \Pr(A_i \cap A_j) + \sum_{i < j < k} \Pr(A_i \cap A_j \cap A_k) - \dots + (-1)^{M+1} \Pr\left(\bigcap_{i=1}^M A_i\right)$$

## Assigning Probabilities

### Classical Approach

Assumes all outcomes in the sample space are equally likely.

#### Examples:

- Coin flip
- Dice roll
- Pair of dice

$$\Pr(\text{Event}) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

### Relative Frequency Approach

$$\Pr(A) \approx \frac{n_A}{n}$$

where  $n$  is the total number of trials and  $n_A$  is the number of times  $A$  occurs.

## Joint Probability

### Definition

$$\Pr(A, B) = \Pr(A \cap B)$$

### Multiple Events

$$\Pr(A_1, A_2, \dots, A_M)$$

### Calculation Approaches

#### Relative Frequency

$$\Pr(A, B) = \lim_{n \rightarrow \infty} \frac{n_{AB}}{n}$$

#### Example: Card Deck

$$\Pr(A) = \frac{26}{52}, \quad \Pr(B) = \frac{40}{52}, \quad \Pr(C) = \frac{13}{52}$$

$$\Pr(A, B) = \frac{20}{52}, \quad \Pr(A, C) = \frac{13}{52}, \quad \Pr(B, C) = \frac{10}{52}$$

## Conditional Probability

### Definition

$$\Pr(A | B) = \frac{\Pr(A \cap B)}{\Pr(B)}, \quad \Pr(B) > 0$$

### Product Rule

$$\Pr(A, B) = \Pr(A | B) \Pr(B)$$

### Chain Rule

$$\Pr(A_1, \dots, A_M) = \prod_{i=1}^M \Pr(A_i | A_1, \dots, A_{i-1})$$

### Example: Cards Without Replacement

$$\Pr(B | A) = \frac{12}{51}$$

### Poker Flush

$$\Pr(\text{Spade Flush}) = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48}$$

$$\Pr(\text{Any Flush}) = 4 \times \Pr(\text{Spade Flush})$$

### Missing Key Problem

$$\Pr(R | L^c) = \frac{0.4}{0.6} = \frac{2}{3}$$