CSE400: Fundamentals of Probability in Computing

Lecture 12 — Transformation of Random Variables

Slide 1 — Lecture Title

Transformation of Random Variables

Given:

- Continuous Random Variable (CRV) (X)
- Known PDF (f_X(x)) and CDF (F_X(x))
- New Random Variable:

$$Y = g(X)$$

Goal:
Find:

$$f_Y(y), \quad F_Y(y)$$

Also transformations:

$$Z = g(X, Y)$$

Examples shown:

$$X + Y, \quad X - Y, \quad X/Y, \quad X \cdot Y$$

Slide 2 — Outline

1. **Transformation of Random Variables**
Learning of transformation techniques for random variables.
2. **Function of Two Random Variables**
Joint transformations and derived distributions.
3. **Illustrative Example**
Detailed derivation for:

$$Z = X + Y$$

Slide 3 — Transformation of RVs (Single Variable)

Assume:

$$Y = g(X)$$

where (g(·))*ismonotonicandinvertible*.

Step-1: CDF Method

$$F_Y(y) = P(Y \le y)$$

Substitute:

$$= P(g(X) \le y)$$

Using inverse:

$$= P(X \le g^{-1}(y))$$

Therefore:

$$F_Y(y) = F_X(g^{-1}(y))$$

Differentiate:

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= \frac{d}{dy} F_X(g^{-1}(y))$$

Apply chain rule:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

Equivalently written:

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|_{x=g^{-1}(y)}$$

Slide 4 — Transformation Cases

Case S1 (Monotone decreasing)

$$F_Y(y) = P(Y \le y)$$

$$= P(X \ge g^{-1}(y))$$

$$= 1 - F_X(g^{-1}(y))$$

Case S2 — Differentiate

$$f_Y(y) = -f_X(g^{-1}(y)) \frac{d}{dy}[g^{-1}(y)]$$

or

$$f_Y(y) = \frac{f_X(x)}{|dy/dx|} \Big|_{x=g^{-1}(y)}$$

(PDF of (X) is assumed known.)

Case S3

Change limits according to transformation.

Slide 5 — Example (Single RV Transformation)

Given:

$$X \sim \text{Uniform}(-1, 1)$$

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Transformation:

$$Y = \sin\left(\frac{\pi X}{2}\right)$$

Step-1: Inverse

$$x = \frac{2}{\pi} \sin^{-1}(y)$$

Step-2: Derivative

$$\frac{dx}{dy} = \frac{2}{\pi} \frac{1}{\sqrt{1 - y^2}}$$

Step-3: PDF

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

Substitute:

$$= \frac{1}{2} \cdot \frac{2}{\pi} \frac{1}{\sqrt{1 - y^2}}$$

$$f_Y(y) = \frac{1}{\pi \sqrt{1 - y^2}}, \quad -1 < y < 1$$

Limits shown:

$$x = -1 \Rightarrow y = -1, \quad x = 1 \Rightarrow y = 1$$

Slide 6 — Function of Two Random Variables

Let:

$$Z = X + Y$$

Tasks listed:
1. Find PDF (f_Z(z))
2. If (X,Y) independent
3. If (X,Y $\sim N(0,1)), show:$

$$Z \sim N(0,2)$$

4. If exponential RVs with parameter $(\lambda), find(f\_Z(z))$

Slide 7 — CDF of Sum

$$F_Z(z) = P(Z \leq z)$$

$$= P(X + Y \leq z)$$

Using joint density:

$$F_Z(z) = \iint_{x+y \leq z} f_{XY}(x,y), dx, dy$$

Integral form shown:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{XY}(x,y), dx, dy$$

Slide 8 — Leibnitz Rule

Let:

$$G(z) = \int_{a(z)}^{b(z)} h(z,y), dy$$

Then:

$$\frac{dG(z)}{dz} = \frac{db(z)}{dz} h(z, b(z))$$

* da(z)$\frac{}{dz h(z,a(z))}$
- $\int_{a(z)}^{b(z)} \frac{\partial}{\partial z} h(z,y), dy$

Differentiate CDF:

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \int_{-\infty}^{\infty} f_{XY}(z-y, y), dy$$

Equivalent expression:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z - x), dx$$

(Result labelled (i) in slide.)

Slide 10 — Convolution (Independent Case)

If (X) and (Y) independent:

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

Therefore:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x), f_Y(z - x), dx$$

Named:
Convolution Integral

Gaussian Example

Given:

$$X \sim N(0, 1), \quad Y \sim N(0, 1)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

Substitute into convolution:

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-(x^2 + (z-x)^2)/2}, dx$$

Slide 11 — Completing Gaussian Result

Integral manipulation shown on slide:

$$= \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-z^2/(2 \cdot 2)}$$

Hence:

$$Z \sim N(0, 2)$$

Slide 12 — Exponential RVs Example

Given:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0$$

$$f_Y(y) = \lambda e^{-\lambda y}, \quad y > 0$$

Use convolution:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x), dx$$

Substitute:

$$= \int_0^z \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda(z-x)} dx$$

$$= \lambda^2 e^{-\lambda z} \int_0^z dx$$

$$= \lambda^2 z e^{-\lambda z}, \quad z > 0$$

Otherwise:

$$0$$

Slide 13 — Difference of RVs

Example:

$$Z = X - Y$$

CDF:

$$F_Z(z) = P(Z \leq z) = P(X - Y \leq z)$$

Integral form shown:

$$\iint f_{XY}(x, y), dx, dy$$

PDF obtained by:

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

Note written:
Same procedure as (Z=X+Y).

Slide 14 — Ratio and Radial Transformation

Ratio

$$Z = \frac{X}{Y}$$

CDF setup:

$$F_Z(z) = P\left(\frac{X}{Y} \leq z\right)$$

Cases indicated:

- (Y¿0)
- (Y¡0)

Radial Example

$$R = \sqrt{X^2 + Y^2}$$

Goal stated:

$$f_R(r) = ?$$

(No further derivation shown on slide.)

End of Lecture 12 Scribe