Lecture Scribe Probability

CSE400 – Fundamentals of Probability in Computing

Lecture L11–L12: Transformation of Random Variables
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Outline

1. Transformation of Random Variables
   Learning transformation techniques for random variables.

2. Function of Two Random Variables
   Joint transformations and derived distributions.

3. Illustrative Example
   Detailed derivation for the case.

1 Transformation of Random Variables

We consider a transformation:

$$Y = g(X)$$

Our goal is to find the distribution (CDF and PDF) of Y, given that the distribution of X is known.

Step–V: CDF of Y

From the slide:

$$F_Y(y) = P(Y \leq y)$$

Since $Y = g(X)$,

$$F_Y(y) = P(g(X) \leq y)$$

The probability is rewritten in terms of X, because the PDF of X is known.
The slide shows transformation using:

$$F_Y(y) = 1 - F_X(\cdot)$$

depending on the monotonicity of the function.
Thus the method used in the slides:

1. Express event $Y \leq y$
2. Rewrite in terms of X
3. Use known CDF of X
4. Differentiate to get PDF of Y

Finally:

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

2 Example: Uniform Random Variable

Given:

$$X \sim \text{Uniform}(-1, 1)$$

$$f_X(x) = \frac{1}{2}, \quad -1 < x < 1$$

Find:

$$Y = g(X) = \sin\left(\frac{\pi}{2}X\right)$$

Step 1: Define transformation

$$F_Y(y) = P(Y \leq y)$$

$$= P\left(\sin\left(\frac{\pi}{2}X\right) \leq y\right)$$

Rewrite inequality in terms of X, then use uniform PDF.
Final PDF (as shown in slide):

$$f_Y(y) = \frac{1}{\pi\sqrt{1 - y^2}}, \quad -1 < y < 1$$

This result comes from differentiating the transformed CDF and using the Jacobian term that appears after inversion.

3 Function of Two Random Variables

Now consider:

$$Z = g(X, Y)$$

Case:

$$Z = X + Y$$

We start with CDF:

$$F_Z(z) = P(Z \leq z)$$

$$= P(X + Y \leq z)$$

This defines a region in the (x, y)-plane.
Thus:

$$F_Z(z) = \iint_{x+y\leq z} f_{X,Y}(x, y)\, dx\, dy$$

If X and Y are independent:

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

To obtain PDF from CDF:

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

Leibniz Rule (Used in Slides)

If

$$G(z) = \int_{a(z)}^{b(z)} h(x, z)\, dx$$

then

$$\frac{dG(z)}{dz} = h(b(z), z)\frac{db}{dz} - h(a(z), z)\frac{da}{dz} + \int_{a(z)}^{b(z)} \frac{\partial h(x, z)}{\partial z}\, dx$$

This rule is applied to differentiate the CDF integral.

Convolution Result

After applying Leibniz rule and simplification:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx$$

This is the convolution formula (as derived in slides).

Example: Exponential Random Variables

Given independent RVs:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$f_Y(y) = \lambda e^{-\lambda y}, \quad y \geq 0$$

Find:

$$Z = X + Y$$

Using convolution:

$$f_Z(z) = \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx$$

$$= \lambda^2 e^{-\lambda z} \int_0^z dx$$

$$= \lambda^2 z e^{-\lambda z}$$

This matches the derived expression in the slides.

Additional Example

$$Z = X - Y$$

Rewrite:

$$F_Z(z) = P(X - Y \leq z)$$

$$= P(X \leq z + Y)$$

And integrate over joint PDF accordingly.

Summary of Lecture Content

1. Transformation of single RV using CDF method
2. Deriving PDF via differentiation
3. Handling monotonic functions
4. Transformation of two RVs
5. Derivation of convolution formula
6. Application to:
   - Uniform transformation
   - Sum of independent RVs
   - Exponential case
   - Difference case