CSE400 – Fundamentals of Probability in Computing

Lecture 8: Gaussian, Uniform, Exponential, and Gamma Random Variables

Instructor: Dhaval Patel, PhD

January 29, 2026

---

1 Gaussian Random Variable: Definition and Properties

Definition

A Gaussian (Normal) random variable is a continuous random variable whose probability density function (PDF) is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

where:

- μ is the mean (location parameter),
- σ^2 is the variance (spread parameter),
- σ is the standard deviation.

Notation:

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

This PDF is symmetric about $x = \mu$, which implies equal probability mass on both sides of the mean.

Properties

- Mean: $E[X] = \mu$
- Variance: $\mathrm{Var}(X) = \sigma^2$
- Symmetry: The distribution is symmetric around the mean.

- Skewness: Zero (due to symmetry).
- Kurtosis: Fixed value characteristic of Gaussian distributions.

The cumulative distribution function (CDF) does not have a closed-form expression in elementary functions and is evaluated using special functions.

2 Gaussian Random Variable: Standard Forms

2.1 Error Function (erf)

The error function is defined as:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

It arises naturally when integrating the Gaussian PDF.

2.2 Complementary Error Function (erfc)

The complementary error function is defined as:

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

This function is useful for tail probability calculations.

2.3 Φ-function (Standard Normal CDF)

Let $Z \sim \mathcal{N}(0, 1)$. The CDF of a standard normal random variable is:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2}\, dt$$

2.4 Q-function (Gaussian Tail Function)

The Q-function is defined as:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt$$

It represents the right-tail probability of a standard Gaussian random variable.

Key Identity

$$Q(x) = 1 - \Phi(x)$$

3 Relation Between Φ-function and Q-function

For a general Gaussian random variable $X \sim \mathcal{N}(\mu, \sigma^2)$, the CDF is evaluated by standardization:

$$F_X(x) = \Pr(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

This step converts X into a standard normal variable.

Evaluating Tail Probabilities

$$\Pr(X > x) = Q\left(\frac{x-\mu}{\sigma}\right)$$

Key Identities

$$F_X(x) = 1 - Q\left(\frac{x-\mu}{\sigma}\right)$$

Interpretation

- $\Phi(x)$: Area under the left tail of the Gaussian curve.
- $Q(x)$: Area under the right tail of the Gaussian curve.

4 Gaussian Random Variable: Example

Given a random variable X with PDF:

$$f_X(x) = \frac{1}{\sqrt{8\pi}} e^{-x^2/8}$$

By comparison with the standard Gaussian form, the parameters μ and σ^2 are identified.

Required Probabilities

- $\Pr(X < 0)$ expressed using the Φ-function.
- $\Pr(X > 4)$ expressed using the Q-function.
- $\Pr(|X + 3| < 2)$ converted into an interval probability.
- $\Pr(|X - 2| > 1)$ converted into tail probabilities.

Hint from lecture

- Use $Q(x)$ for right-tail probabilities.
- Use $\Phi(x)$ for left-tail probabilities.

5 Applications of Gaussian Random Variables

Gaussian models arise naturally in many engineering and computing contexts:

- Thermal noise in electronic circuits.
- Measurement errors in sensors:

$$\text{Measured Value} = \text{True Value} + \text{Gaussian Noise}$$

- Packet delay variation (jitter) in communication networks.

6 Problem Solving: CDF Analysis Exercise

Given CDF:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x > 1 \end{cases}$$

Step 1: Find the PDF

$$f_X(x) = \frac{d}{dx} F_X(x) = 2x, \quad 0 < x < 1$$

Step 2: Mean

$$\mu_X = \int_0^1 x(2x)\, dx = \frac{2}{3}$$

Step 3: Variance

$$\sigma_X^2 = E[X^2] - (E[X])^2$$

Step 4: Skewness and Kurtosis

Derived from central moments.

Conclusion

The distribution is left-skewed and platykurtic.

7 Gaussian Modeling: From Noise to Math

Noise is modeled using a Gaussian random variable:

$$X = \sigma Z + \mu, \quad Z \sim \mathcal{N}(0, 1)$$

This transformation maps standard randomness to a physical model.

Key Idea

Probability models the shape of uncertainty, not exact values.

8 Density Estimation: Learning from Data

In practice, the true μ and σ are unknown.

Observed data is represented as a histogram.

The estimated PDF smooths noise into a predictive model.

Key Concept

We estimate probability distributions, not deterministic functions.

9 Applications: Network Systems

Case Study: Packet Delay Modeling

500 ICMP ping measurements show delay variation due to queuing and congestion.

Engineering Goals

- Estimate typical delay (mean).
- Measure jitter (variance).

High jitter is more damaging to real-time video than constant delay.

10 Applications in Broader Computing Contexts

- Image Processing: Gaussian priors for denoising.
- Cloud Systems: Tail latency modeling.
- IoT & Robotics: Sensor fusion using probabilistic filters.

Final Insight

Uncertainty is unavoidable and probability is the key tool to manage it.