

CSE400 – Fundamentals of Probability in Computing

Lecture 4: Joint Probability and Conditional Probability

Instructor: Dhaval Patel, PhD

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Introduction to Probability Theory

Experiments, Sample Space, and Events

Experiment (E) An experiment is a procedure that produces an outcome. The result is not known in advance but follows well-defined rules.

Example: Tossing a coin five times.

Outcome (ω) An outcome is a single possible result of an experiment.

Example: One possible outcome of tossing a coin five times is HHTHT.

Event An event is a collection (set) of outcomes from the same experiment. Events are usually denoted by capital letters.

Example: For the coin-tossing experiment, let event C be:

$$C = \{\text{all outcomes consisting of an even number of heads}\}$$

This means C contains many outcomes, not just one.

Sample Space (S) The sample space is the set of all possible distinct outcomes of an experiment.

Key properties:

- **Mutually Exclusive:** No two outcomes can occur at the same time.
- **Collectively Exhaustive:** One of the outcomes must occur.

Example: For a single coin flip:

- Heads and tails are mutually exclusive.
- Heads and tails together exhaust all possibilities.

The sample space S is the universal set for the experiment.

Types of Sample Spaces

A sample space can be:

- Discrete
- Countably infinite

- Continuous

Examples:

- Flipping a fair coin once
- Rolling a cubical die
- Rolling two dice
- Flipping a coin until a tail occurs
- Random number generator on interval $[0, 1)$

Axioms of Probability

Probability Probability is a numerical measure of how likely an event is to occur. It is a function that maps events to numbers between 0 and 1.

Axioms

1. For any event A :

$$0 \leq \Pr(A) \leq 1$$

2. If S is the sample space:

$$\Pr(S) = 1$$

3. If A and B are mutually exclusive ($A \cap B = \emptyset$):

$$\Pr(A \cup B) = \Pr(A) + \Pr(B)$$

4. For an infinite collection of mutually exclusive events A_1, A_2, \dots :

$$\Pr\left(\bigcup_i A_i\right) = \sum_i \Pr(A_i)$$

Corollary from Probability Axioms

Corollary 2.1 For a finite number M of mutually exclusive events A_1, A_2, \dots, A_M :

$$\Pr\left(\bigcup_i A_i\right) = \sum_i \Pr(A_i)$$

Propositions from Probability Axioms

Proposition 2.1 (Complement Rule)

$$\Pr(A^c) = 1 - \Pr(A)$$

Proposition 2.2 (Monotonicity) If $A \subseteq B$, then:

$$\Pr(A) \leq \Pr(B)$$

Proposition 2.3 (Addition Rule)

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

Proposition 2.4 (Inclusion–Exclusion Principle)

$$\Pr\left(\bigcup_{i=1}^M A_i\right) = \sum_i \Pr(A_i) - \sum_{i < j} \Pr(A_i \cap A_j) + \sum_{i < j < k} \Pr(A_i \cap A_j \cap A_k) - \cdots + (-1)^{M+1} \Pr\left(\bigcap_{i=1}^M A_i\right)$$

Assigning Probabilities**Classical Approach**

Assumes all outcomes in the sample space are equally likely.

Examples:

- Coin flip
- Dice roll
- Pair of dice

$$\Pr(\text{Event}) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

Relative Frequency Approach

$$\Pr(A) \approx \frac{n_A}{n}$$

where n is the total number of trials and n_A is the number of times A occurs.

Joint Probability**Definition**

$$\Pr(A, B) = \Pr(A \cap B)$$

Multiple Events

$$\Pr(A_1, A_2, \dots, A_M)$$

Calculation Approaches**Relative Frequency**

$$\Pr(A, B) = \lim_{n \rightarrow \infty} \frac{n_{AB}}{n}$$

Example: Card Deck

$$\Pr(A) = \frac{26}{52}, \quad \Pr(B) = \frac{40}{52}, \quad \Pr(C) = \frac{13}{52}$$

$$\Pr(A, B) = \frac{20}{52}, \quad \Pr(A, C) = \frac{13}{52}, \quad \Pr(B, C) = \frac{10}{52}$$

Conditional Probability

Definition

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}, \quad \Pr(B) > 0$$

Product Rule

$$\Pr(A, B) = \Pr(A \mid B) \Pr(B)$$

Chain Rule

$$\Pr(A_1, \dots, A_M) = \prod_{i=1}^M \Pr(A_i \mid A_1, \dots, A_{i-1})$$

Example: Cards Without Replacement

$$\Pr(B \mid A) = \frac{12}{51}$$

Poker Flush

$$\Pr(\text{Spade Flush}) = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48}$$

$$\Pr(\text{Any Flush}) = 4 \times \Pr(\text{Spade Flush})$$

Missing Key Problem

$$\Pr(R \mid L^c) = \frac{0.4}{0.6} = \frac{2}{3}$$