№1

$e_{ijk} a_{i\alpha} a_{j\beta} a_{k\gamma} = a_{1\alpha}a_{2\beta}a_{3\gamma} + a_{2\alpha}a_{3\beta}a_{1\gamma} + a_{3\alpha}a_{1\beta}a_{2\gamma} -$
$a_{3\alpha}a_{2\beta}a_{1\gamma} - a_{1\alpha}a_{3\beta}a_{2\gamma} - a_{2\alpha}a_{1\beta}a_{3\gamma}$

Получаем

$$e_{ijk} a_{i\alpha} a_{j\beta} a_{k\gamma} = \begin{cases} 0, & \alpha=\beta \vee \beta=\gamma \vee \alpha=\gamma \\ \det a, & P(\alpha,\beta,\gamma)=1 \\ -\det a, & P(\alpha,\beta,\gamma)=-1 \end{cases} \Rightarrow e_{ijk} a_{i\alpha} a_{j\beta} a_{k\gamma} = e_{\alpha\beta\gamma} \det a$$

№2

1) в координатной системе x:

$$g_{\alpha\beta} = \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial y^\nu}{\partial x^\beta} \cdot g_{\mu\nu}^{y}$$

Т.к. y_i - евклидово пространство, то

$$g_{ij} = \frac{\partial y^a}{\partial x^i} \cdot \frac{\partial y^a}{\partial x^j}$$

Символ Кристоффеля:

$$\Gamma_{i,jk} = \frac{1}{2}\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right)$$

Найдём составляющие

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial\left(\frac{\partial y_\alpha}{\partial x_i}\cdot\frac{\partial y_\alpha}{\partial x_j}\right)}{\partial x_k} = \sum_\alpha \frac{\partial^2 y_\alpha}{\partial x^i \partial x_k}\frac{\partial y_\alpha}{\partial x_j} + \frac{\partial^2 y_\alpha}{\partial x_j \partial x_k}\cdot\frac{\partial y_\alpha}{\partial x_i}$$

$$\frac{\partial g_{ik}}{\partial x^j} = \sum_\alpha \frac{\partial^2 y_\alpha}{\partial x_i \partial x_j}\cdot\frac{\partial y_\alpha}{\partial x_k} + \frac{\partial^2 y_\alpha}{\partial x_k \partial x_j}\cdot\frac{\partial y_\alpha}{\partial x_i}$$

$$\frac{\partial g_{jk}}{\partial x^i} = \sum_\alpha \frac{\partial^2 y_\alpha}{\partial x_j \partial x_i}\frac{\partial y_\alpha}{\partial x_k} + \frac{\partial^2 y_\alpha}{\partial x_k \partial x_i}\cdot\frac{\partial y_\alpha}{\partial x_j}$$

В итоге

$$\Gamma_{i,jk} = \sum_\alpha \frac{\partial^2 y_\alpha}{\partial x^k \partial x^j}\frac{\partial y_\alpha}{\partial x^i}$$

2) $\nabla_\beta A^\alpha = \frac{\partial A^\alpha}{\partial x^\beta} + A^\alpha\Gamma^\alpha_{\alpha\beta}$

Докажем что это тензор преобразовав координаты. Для удобства преобразуем слагаемые отдельно.

$$\frac{\partial \bar{A}^\alpha}{\partial \bar{x}^\beta} = \frac{\partial\left(A^i\frac{\partial \bar{x}^\alpha}{\partial x^i}\right)}{\partial \bar{x}^\beta} = A^i\frac{\partial^2 \bar{x}^\alpha}{\partial x^i\partial \bar{x}^\beta} + \frac{\partial A^i}{\partial \bar{x}^\beta}\cdot\frac{\partial \bar{x}^\alpha}{\partial x^i}$$

$$\bar{A}^\alpha\bar{\Gamma}^\alpha_{\alpha\beta} = A^i\frac{\partial \bar{x}^\alpha}{\partial x^i}\cdot\frac{\partial x^k}{\partial \bar{x}^\alpha}\cdot\frac{\partial x^j}{\partial \bar{x}^\beta}\cdot\Gamma^l_{jk}\cdot\frac{\partial \bar{x}^\alpha}{\partial x^l} + \frac{\partial \bar{x}^\alpha}{\partial x^l}\cdot\frac{\partial^2 x^l}{\partial \bar{x}^\beta\partial \bar{x}^\alpha}\cdot A^i\frac{\partial \bar{x}^\alpha}{\partial x^i}$$

Т.к. $\frac{\partial \bar{x}^\alpha}{\partial x^l}\cdot\frac{\partial^2 x^l}{\partial \bar{x}^\beta\partial x^i} = -\frac{\partial^2 \bar{x}^\alpha}{\partial x^i\partial \bar{x}^\beta}$

$$\nabla_\beta A^{\bar{\alpha}} = \frac{\partial A^i}{\partial x^j} \cdot \frac{\partial x^j}{\partial \bar{x}^\alpha} \cdot \frac{\partial \bar{x}^\alpha}{\partial x^i} + A^k \Gamma^i_{kj} \cdot \frac{\partial x^j}{\partial \bar{x}^\beta} \cdot \frac{\partial \bar{x}^\alpha}{\partial x^i} \Rightarrow$$ это смешанный тензор (1,1)

3) $$\nabla_\alpha A^\mu B_\mu + A^\mu \nabla_\alpha B_\mu = A^\mu \cdot \left(\frac{\partial B_\mu}{\partial x^\alpha} - B_\lambda \Gamma^\lambda_{\mu\alpha} \right) +$$

$$+ B_\mu \left(\frac{\partial A^\mu}{\partial x^\alpha} + A^\lambda \cdot \Gamma^\mu_{\lambda\alpha} \right) = A^\mu \cdot \frac{\partial B_\mu}{\partial x^\alpha} + B_\mu \cdot \frac{\partial A^\mu}{\partial x^\alpha}$$

№3

1) Т.к. матрица тензора g^{ij} является обратной к g_{ij}, то

$$g^{ji} = \frac{G_{ij}}{g}$$, где G_{ij} - алгебраическое дополнение g_{ij}

Воспользовавшись формулой для производной детерминанта

$$\frac{\partial g}{\partial x^\alpha} = \frac{\partial g_{ik}}{\partial x^\alpha} G_{ik} = g \cdot g^{ki} \cdot \frac{\partial g_{ik}}{\partial x^\alpha}$$

2) $$\Gamma^i_{i\alpha} = \frac{1}{2} g^{i\beta} \left(\frac{\partial g_{\beta i}}{\partial x^\alpha} + \frac{\partial g_{\alpha\beta}}{\partial x^i} - \frac{\partial g_{i\alpha}}{\partial x^\beta} \right) = \frac{1}{\sqrt{g}} \cdot \left(\frac{1}{2} \cdot \frac{1}{\sqrt{g}} \right) \frac{\partial g}{\partial x^\alpha} =$$

$$= \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^\alpha}$$

3) $$\nabla_i A^i = \frac{\partial A^i}{\partial x^i} + A^\alpha \Gamma^i_{\alpha i} = \frac{\partial A^i}{\partial x^i} + \frac{A^\alpha}{\sqrt{g}} \cdot \frac{\partial \sqrt{g}}{\partial x^\alpha}$$

С другой стороны $$\frac{1}{\sqrt{g}} \cdot \frac{\partial \sqrt{g} A^i}{\partial x^i} = \frac{\partial A^i}{\partial x^i} + \frac{A^i}{\sqrt{g}} \cdot \frac{\partial \sqrt{g}}{\partial x^i}$$

Следовательно

$$\nabla_i A^i = \frac{1}{\sqrt{g}} \cdot \frac{\partial(\sqrt{g} A^i)}{\partial x^i}$$

№4

1) $E_{ijk} = \sqrt{g} \cdot e_{ijk} \Rightarrow \sqrt{g} = \frac{E_{ijk}}{e_{ijk}}$

$E^{ijk} = \frac{1}{\sqrt{g}} e_{ijk} = e_{ijk} \cdot \frac{1}{E_{ijk}}$

$E^{ijk} = \frac{c}{T_{ijk}}$, где c - константа

T_{ijk} - коваріантний тензор 3 ранга

очевидно контрваріантний тензор 3го ранга

2) $\{\nabla \times A\}^i = E^{ijk} \frac{\partial A_j}{\partial x^k}$

Очевидно $\{\nabla \times A\}^i$ раскладывается на

~~$\{\nabla \times A\}^i = \partial$~~ сумму из компонентов вида $\frac{1}{\sqrt{g}}\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right)$,

каждый из которых тензор $\Rightarrow \{\nabla \times A\}^i$ тоже тензор

№5

$$ds^2 = dt^2 - a^2(t)\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

Перейдём к η. Т.к. t зависит от η, то

$a(t(\eta)) = a(\eta)$

$$ds^2 = a(\eta)^2 \cdot [d\eta^2 - d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)]$$

найдём метрический тензор

$g_{\eta\eta} = a(\eta)^2$

$g_{\chi\chi} = -a(\eta)^2$

$g_{\theta\theta} = a(\eta)^2 \cdot \sin^2\chi$

$g_{\varphi\varphi} = a(\eta)^2 \cdot \sin^2\chi \cdot \sin^2\theta$

$\Gamma_{\eta,\eta\eta} = \frac{\partial a(\eta)}{\partial \eta}$ $\qquad \Gamma_{i,jk} = 0, \text{ if } i \neq j \cap j \neq k \cap k \neq i$

$\Gamma_{\eta,\eta\chi} = \Gamma_{\eta,\eta\theta} = \Gamma_{\eta,\eta\varphi} = 0$

$\Gamma_{\eta,\chi\chi} = \frac{\partial a(\eta)}{\partial \eta}$

$\Gamma_{\eta,\theta\theta} = -\sin^2\chi \cdot \frac{\partial a(\eta)}{\partial \eta} \cdot a(\eta)$

$\Gamma_{\eta,\varphi\varphi} = -a(\eta) \cdot \sin^2\chi \cdot \sin^2\theta \cdot \frac{\partial a(\eta)}{\partial \eta}$

$\Gamma_{\chi,\chi\eta} = -a(\eta) \cdot \frac{\partial a(\eta)}{\partial \eta}$

$\Gamma_{\chi,\chi\chi} = \Gamma_{\chi,\chi\varphi} = \Gamma_{\chi,\chi\theta} = \Gamma_{\chi,\eta\eta} = 0$

$\Gamma_{\chi,\theta\theta} = -a(\eta)^2 \cdot \sin\chi \cdot \cos\chi$

$\Gamma_{\chi,\varphi\varphi} = -a(\eta)^2 \cdot \sin^2\theta \cdot \sin\chi \cdot \cos\chi$

$\Gamma_{\theta,\theta\theta} = \Gamma_{\theta,\theta\varphi} = 0$

$\Gamma_{\theta,\theta\eta} = a(\eta) \cdot \sin^2\chi \cdot \frac{\partial a(\eta)}{\partial \eta}$

$\Gamma_{\theta,\theta x} = a^2(\eta) \cdot \sin x \cos x$

$\Gamma_{\theta,\eta\eta} = \Gamma_{\theta,xx} = 0$

$\Gamma_{\theta,\varphi\varphi} = -a(\eta)^2 \cdot \sin^2 x \cdot \sin\theta \cdot \cos\theta$

$\Gamma_{\varphi,\varphi\varphi} = \Gamma_{\varphi,xx} = \Gamma_{\varphi,\eta\eta} = \Gamma_{\varphi,\theta\theta} = 0$

$\Gamma_{\varphi,\varphi\eta} = a(\eta) \cdot \sin^2 x \cdot \sin^2\theta \cdot \dfrac{\partial a(\eta)}{\partial \eta}$

$\Gamma_{\varphi,\varphi x} = a(\eta)^2 \cdot \sin^2\theta \cdot \sin x \cdot \cos x$

$\Gamma_{\varphi,\varphi\theta} = a(\eta)^2 \cdot \sin^2 x \cdot \sin\theta \cdot \cos\theta$