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A novel framework for trust network analysis: Connectivity-based intuitionistic fuzzy rough digraph

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ABSTRACT

Network connectivity analysis enables information source tracing and spread regulation in social systems. While existing studies have explored intuitionistic fuzzy rough (IFR) digraphs to address the representation needs of pervasive uncertainties and dual-polarity information in real-world networks, their neglect of connectivity characteristics has limited applicability in information diffusion scenarios. This study breaks through conventional framework and proposes a connectivity-based IFR digraph model, which achieves comprehensive representation of information oppositionality, uncertainty, and propagative characteristic. First, we explore minimum equivalent intuitionistic fuzzy subgraph (MEIFS) and semi-maximum equivalent intuitionistic fuzzy supergraph (SEIFS). MEIFS preserves original strength of connectedness through minimal arc sets, while SEIFS achieves the same objective via redundant arc augmentation. This complementarity provides a mathematical tool for approximating complex networks. Then, a connectivity-based IFR digraph model is established through the synergy of MEIFS and SEIFS. Finally, according to the co-occurrence characteristics of trust and distrust in society, the community detection algorithm and multi-core-node mining method for IFR trust networks are developed. Comparative analysis with three existing methods demonstrates the superiority of the proposed technique in approximate modeling of adversarial information propagation systems.

1. Introduction

Social network analysis is an interdisciplinary field integrating sociology, mathematics, computer science, and other disciplines. It focuses on studying network structures formed by interpersonal relationships, aiming to uncover general patterns of human social behavior and extract critical insights to enhance societal efficiency. Fuzzy interpersonal relationships in social networks can be abstracted as a fuzzy graph model [29], where the membership function quantifies relationship strength between nodes. By incorporating dynamic weight analysis to track community evolution, fuzzy graph theory provides a mathematical framework for modeling uncertain network structures. However, classical fuzzy graphs, relying on single-membership mechanisms, struggle to simultaneously express opposing semantics such as support and opposition, trust and skepticism, etc. To address this, Parvathi and Karunambigai [25] introduced the concept of intuitionistic fuzzy (IF) graph by leveraging IF set theory [1]. IF graph theory enhances semantic descriptions through dual parameters (membership and non-membership degrees) and introduces a hesitation degree to capture decision-making uncertainties. This dynamic game-theoretic mechanism significantly improves analytical precision and decision cred-

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ibility in contradictory information systems. Later, Akram and Davvaz [2] proposed a refined IF graph by revising non-membership constraints, achieving structural alignment with classical graph logic. Moreover, extensions including interval-valued fuzzy graphs [15] and picture fuzzy graphs [8] have been developed, substantially advancing the framework of fuzzy graph theory.

Despite demonstrating significant theoretical and applied potential (see for example [9,19]), IF graph theory is constrained by its underlying logic, leading to certain application limitations. In reality, relationships between social entities not only exhibit fuzziness in intensity but may also involve uncertainty regarding the very existence of such connections. This uncertainty could stem from privacy considerations, information sensitivity, or data incompleteness. To circumvent prior knowledge constraints, rough graph theory [13] emerged. Rough set theory [24] was proposed by Pawlak in 1982 and fundamentally seeks to approximate hard-to-define information using precise data, addressing the expressive limitations of classical set theory in handling incomplete data, redundant attributes, or ambiguous boundaries. To date, this theory has proven valuable across multiple domains, particularly in predictive analytics, decision-making, and granular computing. Rough graphs leverage the approximation concept from rough set theory, using deterministic network structures to approximate uncertain ones. Specifically, it classifies relationships into definitely existing and possibly existing categories, thereby addressing another dimension of uncertainty in real-world networks, i.e., uncertainty about relationship existence. This theoretical method and fuzzy graph theory adapt to the complexity of real networks from two complementary angles.

Research on rough graphs has enabled approximate reasoning for imprecise relational networks. Building on classical rough graph theory, Mathew et al. [22] further considered scenarios with rough vertex sets, developed a vertex rough graph model, and investigated its rough properties. During the same period, Sarwar [33] explored digraphs from a rough set perspective, proposing the concept of vertex rough digraph and applying it to investment analysis in agricultural systems. To further enhance the noise resistance of these models, El Atik et al. [11] introduced neighborhood systems to construct neighborhood-based vertex rough graphs, reconciling local correlations and global uncertainties in complex data. Soft covering rough graphs [26] and neighborhood-based soft covering rough graphs [31] are widely used to represent uncertainty in networks with multi-state structures, demonstrating unique advantages in multi-criteria group decision-making problems. The essence of these studies lies in transcending the limitations of classical rough set theory while extending pivotal results from classical graph theory within the rough graph framework. For instance, Majumder and Kar [21] proposed two methods for identifying multi-criteria shortest paths in rough graphs, namely a modified Dijkstra algorithm and a rough programming approach. Fatima et al. [12] investigated product operations for rough graphs, defined distance metrics, and applied them to a group decision-making problem involving organ trafficking networks.

As a complementary tool to rough set theory, fuzzy set theory [38] enhances the practical value of rough graphs in analyzing imprecise networks by integrating its quantitative principles. This integration underpins the development of fuzzy rough graphs [4,5] and rough fuzzy graphs [3]. Subsequent studies on dominating sets in rough fuzzy digraphs [6] have further enriched the theoretical exploration of rough fuzzy graphs. The introduction of non-membership functions into fuzzy rough graphs leads to intuitionistic fuzzy rough (IFR) graphs [20], significantly improving adaptability for modeling networks with conflicting semantics. Zhan et al. [39] later constructed an IFR digraph model, thereby enhancing expressiveness of rough graph methods in decision-making. However, current IFR graph theory remains confined to a set-theoretic perspective, failing to transcend the limitations of rough sets and overlooking its inherent topological properties as graphs. While it excels in decision optimization, its applicability to complex problems like trust network analysis remains constrained by the underutilization of graph-theoretic knowledge.

Trust constitutes a fundamental bond underpinning human societal functioning. Networks formed by mutually trusting groups serve as critical architectures enabling information dissemination and consensus building. Research on trust network analysis carries profound significance for revealing group decision-making mechanisms, optimizing policy transmission pathways, and identifying pivotal influence nodes. Its application value continues to escalate across domains including social opinion governance, blockchain consensus algorithms, and public health interventions. Current IFR graphs merely depict static binary trust relationships between nodes while overlooking the dynamic propagation properties of trust. They fail to characterize multi-hop propagation effects within networks and cannot quantify trust attenuation or information distortion. Such theoretical deficiencies prevent accurate prediction of information diffusion paths and effective identification of latent key nodes, urgently necessitating novel propagation-aware computational frameworks. Therefore, to enhance the depth and practical applicability of IFR graph theory, it is essential to treat IFR graphs as a specific type of graph and introduce relevant graph-theoretic knowledge to capture the propagation characteristics of information in network structure, while this research direction remains under-explored.

Connectivity, as a foundational element in analyzing complex networks, reflects the degree of interconnection among nodes, directly influencing social behaviors such as information dissemination and resource allocation, while also determining system stability. Consequently, analyzing network connectivity holds practical significance in scenarios involving propagation pathways, such as rumor containment [16,37] and policy advocacy. Additionally, studying connectivity aids in identifying critical nodes [28,40] or communities, which may act as hubs for information diffusion or centers of influence. Understanding the social dynamics of these pivotal nodes offers actionable insights for addressing societal challenges like public opinion guidance. Current connectivity analysis of IF graphs has provided reliable solutions for practical challenges in transport networks [23], disease transmission [7], and water supply management [10]. On this basis, the paper advances a novel framework for connectivity-based IFR digraph theory to achieve three objectives:

(1) Overcoming limitations of IF graphs in handling incomplete data;
(2) addressing the deficiency of rough graphs in describing the uncertainty of continuous gradual change;
(3) enhancing the applicability of classical IFR graphs in information propagation systems.

For achieving the above research goals, the paper mainly makes the following contributions:

(1) The minimum equivalent intuitionistic fuzzy subgraph (MEIFS) and semi-maximum equivalent intuitionistic fuzzy supergraph (SEIFS) are defined, and their construction methods and topological properties are revealed. These structures not only fully preserve the connectivity strength of target systems but also exhibit structural duality (maximal conciseness and maximal complexity), providing feasibility for approaching target systems.
(2) A connectivity-based rough digraph model under IF environment is established, and the approximation and uncertainty quantitative expression of target IF digraphs are realized.
(3) Aiming at the bipolar characteristics (trust and distrust coexist) of social trust network, a community detection algorithm and a rule of discovering multiple key persons based on communities are designed under the framework of IFR theory.

The remaining part of the paper is organized as follows. In Section 2, some notions related to fuzzy digraphs as well as that related of IF digraphs are presented. Two special equivalent IF digraphs, i.e., MEIFS and SEIFS, are introduced in Section 3. Also, their structural attributes are explored in this section. In Section 4, the concept of R-IFR digraphs and its basic properties and rough characteristics are introduced. Section 5 studies an application of R-IFR digraphs in trust networks and points out the limitation of traditional method in this field. Section 6 summarizes the paper and clarifies the future work.

2. Preliminaries

This section reviews some notions that are used throughout the paper, mainly involving fuzzy digraph theory and IF digraph theory. For more details, please refer to [27].

In the following, symbols $\vee$ and $\wedge$ are used to indicate the maximum value and the minimum value, respectively, and symbol $|\cdot|$ represents the cardinality of a set.

A fuzzy digraph [29] with a crisp vertex set V is represented as $D=(V,\tau)$, where $\tau: V\times V\to[0,1]$. The arc set of D is $A=\{(v_i,v_j)\in V\times V : \tau(v_i,v_j)>0\}$. The sequence $P=v_1(v_1,v_2)v_2(v_2,v_3)v_3\ldots(v_{m-1},v_m)v_m$ consisting of different vertices and arcs alternately is termed as a v_1v_m path and its length and strength are $m-1$ and $\bigwedge_{i=1}^{m-1}\tau(v_i,v_{i+1})$, respectively. The strength of connectedness between v_1 and v_m in fuzzy digraph D refers to the maximum strength of all v_1v_m paths and it is denoted by $CONN_D(v_1,v_m)$. If the strength of v_1v_m path P equals to $CONN_D(v_1,v_m)$, then P is called the strongest v_1v_m path. When $(v_1,v_m)\in A$, it is referred to as an α-strong arc if $\tau(v_1,v_m)>CONN_{D-\{(v_1,v_m)\}}(v_1,v_m)$, a β-strong arc if $\tau(v_1,v_m)=CONN_{D-\{(v_1,v_m)\}}(v_1,v_m)$, and a δ-arc if $\tau(v_1,v_m)<CONN_{D-\{(v_1,v_m)\}}(v_1,v_m)$. Fuzzy digraph $D'=(V',\tau')$ is called the fuzzy subgraph of D if $V'\subseteq V$ and $\tau'(v_i,v_j)=\tau(v_i,v_j)$ for any $v_i,\ v_j\in V'$, and D is called the fuzzy supergraph of D'.

Definition 1. [36] Let $D=(V,\tau)$ be a fuzzy digraph and D^M be a fuzzy subgraph of D. Then, D^M is termed as the minimum equivalent fuzzy subgraph of D if D^M is the fuzzy subgraph with the least number of arcs satisfying that $CONN_{D^M}(v_i,v_j)=CONN_D(v_i,v_j)$ for any $v_i,\ v_j\in V$.

Definition 2. [2] An IF digraph with V as its vertex set is denoted by $G=(V,(\sigma',\eta'),(\sigma'',\eta''))$, in which

(1) the functions $\sigma': V\to[0,1]$ and $\eta': V\to[0,1]$ denote the membership value and non-membership value of vertex $v_i\in V$, respectively, and $0\le\sigma'(v_i)+\eta'(v_i)\le 1$;
(2) the functions $\sigma'': V\times V\to[0,1]$ and $\eta'': V\times V\to[0,1]$ denote the membership value and non-membership value of arc $(v_i,v_j)\in V\times V$, respectively, and $0\le\sigma''(v_i,v_j)+\eta''(v_i,v_j)\le 1$, $\sigma''(v_i,v_j)\le\sigma'(v_i)\wedge\sigma'(v_j)$, and $\eta''(v_i,v_j)\ge\eta'(v_i)\vee\eta'(v_j)$.

The arc set of G is $E=\{(v_i,v_j)\in V\times V : \sigma''(v_i,v_j)\vee\eta''(v_i,v_j)>0\}$. In particular, we call $(v_i,v_j)\in E$ a μ-arc if $\sigma''(v_i,v_j)>0$ and an υ-arc if $\eta''(v_i,v_j)>0$. If vertex v is not connected by any arc, then v is called an isolated vertex. Because the discussion of connectivity does not involve the membership values or non-membership values of vertices, the paper focuses on IF digraphs with crisp vertex sets, i.e., $\sigma'(v)=1$ for any $v\in V$. For the sake of simplicity, IF digraph G is expressed as $G=(V,(\sigma,\eta))$ in the following, where σ and η are the membership function and non-membership function on $V\times V$, respectively.

The IF digraph $G_0=(V_0,(\sigma_0,\eta_0))$ with the arc set E_0 is considered as a partial IF subgraph of G if $V_0\subseteq V$ and $\sigma_0(v_i,v_j)\le\sigma(v_i,v_j)$ and $\eta_0(v_i,v_j)\ge\eta(v_i,v_j)$ for any $(v_i,v_j)\in E_0$. Besides, G is called the partial IF supergraph of G_0. We denote it as $G_0\subseteq G$ in this case. In particular, if $V_0\subseteq V$ and $\sigma_0(v_i,v_j)=\sigma(v_i,v_j)$ and $\eta_0(v_i,v_j)=\eta(v_i,v_j)$ for any $(v_i,v_j)\in E_0$, then G_0 is an IF subgraph of G and G is an IF supergraph of G_0. If $G_0\subseteq G$ and $G\subseteq G_0$, then $G_0=G$. If there exists a bijection $f: V_0\to V$ such that $\sigma_0(v_i,v_j)=\sigma(f(v_i),f(v_j))$ and $\eta_0(v_i,v_j)=\eta(f(v_i),f(v_j))$ for any $v_i,\ v_j\in V_0$, then G_0 is isomorphic to G, denoted by $G_0\cong G$.

Given two IF digraphs $G_1=(V_1,(\sigma_1,\eta_1))$ and $G_2=(V_2,(\sigma_2,\eta_2))$ and their arc sets are E_1 and E_2, respectively. The union of G_1 and G_2 is defined as $G_1\cup G_2=(V_1\cup V_2,(\sigma_1\cup\sigma_2,\eta_1\cup\eta_2))$, in which

$$(\sigma_1\cup\sigma_2)(v_i,v_j)=\begin{cases}\sigma_1(v_i,v_j), & \text{if } (v_i,v_j)\in E_1-E_2;\\ \sigma_2(v_i,v_j), & \text{if } (v_i,v_j)\in E_2-E_1;\\ \sigma_1(v_i,v_j)\vee\sigma_2(v_i,v_j), & \text{if } (v_i,v_j)\in E_1\cap E_2;\end{cases}$$

and

$$(\eta_1 \cup \eta_2)(v_i, v_j) = \begin{cases} \eta_1(v_i, v_j), & \text{if } (v_i, v_j) \in E_1 - E_2; \\ \eta_2(v_i, v_j), & \text{if } (v_i, v_j) \in E_2 - E_1; \\ \eta_1(v_i, v_j) \wedge \eta_2(v_i, v_j), & \text{if } (v_i, v_j) \in E_1 \cap E_2. \end{cases}$$

The intersection of G_1 and G_2 is defined as $G_1 \cap G_2 = (V_1 \cup V_2, (\sigma_1 \cap \sigma_2, \eta_1 \cap \eta_2))$, in which

$$(\sigma_1 \cap \sigma_2)(v_i, v_j) = \begin{cases} \sigma_1(v_i, v_j), & \text{if } (v_i, v_j) \in E_1 - E_2; \\ \sigma_2(v_i, v_j), & \text{if } (v_i, v_j) \in E_2 - E_1; \\ \sigma_1(v_i, v_j) \wedge \sigma_2(v_i, v_j), & \text{if } (v_i, v_j) \in E_1 \cap E_2; \end{cases}$$

and

$$(\eta_1 \cap \eta_2)(v_i, v_j) = \begin{cases} \eta_1(v_i, v_j), & \text{if } (v_i, v_j) \in E_1 - E_2; \\ \eta_2(v_i, v_j), & \text{if } (v_i, v_j) \in E_2 - E_1; \\ \eta_1(v_i, v_j) \vee \eta_2(v_i, v_j), & \text{if } (v_i, v_j) \in E_1 \cap E_2. \end{cases}$$

Definition 3. [17] A $v_1 v_m$ path $P = v_1(v_1, v_2)v_2(v_2, v_3)v_3 \dots (v_{m-1}, v_m)v_m$ in an IF digraph $G = (V, (\sigma, \eta))$ is termed as a μ-path if $\sigma(v_i, v_{i+1}) > 0$ and an v-path if $\eta(v_i, v_{i+1}) > 0$, where $1 \le i \le m-1$. For the case of P is a μ-path, its μ-strength is $\bigwedge_{i=1}^{m-1} \sigma(v_i, v_{i+1})$; for the case of P is an v-path, its v-strength is $\bigvee_{i=1}^{m-1} \eta(v_i, v_{i+1})$. The μ-strength of connectedness between v_1 and v_m in IF digraph G, denoted by $CONN_G^{\mu}(v_1, v_m)$, refers to the maximum μ-strength of all $v_1 v_m$ μ-paths; the v-strength of connectedness between v_1 and v_m in IF digraph G, denoted by $CONN_G^{v}(v_1, v_m)$, refers to the minimum v-strength of all $v_1 v_m$ v-paths. The strength of connectedness between v_1 and v_m in IF digraph G is $CONN_G(v_1, v_m) = (CONN_G^{\mu}(v_1, v_m), CONN_G^{v}(v_1, v_m))$.

For a $v_1 v_m$ μ-path P, if its μ-strength equals to $CONN_G^{\mu}(v_1, v_m)$, then P is called the strongest $v_1 v_m$ μ-path. In the same way, for a $v_1 v_m$ v-path P', if its v-strength equals to $CONN_G^{v}(v_1, v_m)$, then P' is called the strongest $v_1 v_m$ v-path. In addition, the μ-arc with the minimum membership value in a μ-path is called the weakest μ-arc and the v-arc with the maximum non-membership value in an v-path is called the weakest v-arc.

Definition 4. [32] A μ-arc (v_i, v_j) in IF digraph $G = (V, (\sigma, \eta))$ is called an α_μ-strong arc, a β_μ-strong arc, and a δ_μ-arc, if $\sigma(v_i, v_j) > CONN^{\mu}_{G-\{(v_i,v_j)\}}(v_i, v_j)$, $\sigma(v_i, v_j) = CONN^{\mu}_{G-\{(v_i,v_j)\}}(v_i, v_j)$, and $\sigma(v_i, v_j) < CONN^{\mu}_{G-\{(v_i,v_j)\}}(v_i, v_j)$, respectively; v-arc (v_i, v_j) is called an α_v-strong arc, a β_v-strong arc, and a δ_v-arc, if $\eta(v_i, v_j) < CONN^{v}_{G-\{(v_i,v_j)\}}(v_i, v_j)$, $\eta(v_i, v_j) = CONN^{v}_{G-\{(v_i,v_j)\}}(v_i, v_j)$, and $\eta(v_i, v_j) > CONN^{v}_{G-\{(v_i,v_j)\}}(v_i, v_j)$, respectively.

3. Two special equivalent intuitionistic fuzzy digraphs

In this section, MEIFS and SEIFS are defined, which both keep the connectivity of a given IF digraph. Moreover, their structural characteristics and constructions are analyzed.

3.1. Minimum equivalent intuitionistic fuzzy subgraphs

This subsection starts by defining MEIFS. Analyzing the influence of removing various types of arcs on connectivity promotes the acquisition of the construction of MEIFS.

Definition 5. Let $G = (V, (\sigma, \eta))$ be an IF digraph and $\underline{G}$ be an IF subgraph of G. Then, $\underline{G}$ is termed as the minimum equivalent intuitionistic fuzzy subgraph (MEIFS) of G if:

(1) $CONN_{\underline{G}}(u, v) = CONN_G(u, v)$ for any u, $v \in V$, and
(2) $\underline{G}$ is the IF subgraph of G with the least number of arcs of satisfying (1).

When only membership values are considered, an IF digraph degenerates into a fuzzy digraph. Let $G_\mu = (V, \sigma)$ be the fuzzy digraph composed of μ-arcs in IF digraph $G = (V, (\sigma, \eta))$ and B_{G_μ} be the set of β-strong arcs in G_μ. Suppose that e_i, $e_j \in B_{G_\mu}$ and $e_j = (u, v)$. If there exists a strongest uv path P in G_μ such that e_i is an arc in P, then it is expressed as $e_i \bigtriangleup_{G_\mu} e_j$. Let $\beta_{G_\mu}(e_i) = \{e_j \in B_{G_\mu} - \{e_i\} : e_i \bigtriangleup_{G_\mu} e_j\}$, $\beta^*_{G_\mu}(e_i) = \{e_j \in B_{G_\mu} - \{e_i\} : e_j \bigtriangleup_{G_\mu} e_i\}$, and $\Phi_{G_\mu} = \{e_i \in B_{G_\mu} : \beta_{G_\mu}(e_i) \cap \beta^*_{G_\mu}(e_i) = \emptyset\}$. In [36], it has pointed out that if $B_{G_\mu} - \Phi_{G_\mu} \neq \emptyset$, then the minimum equivalent fuzzy subgraph of G_μ may not be unique. Next, let B_G^{μ} be the set of β_μ-strong arcs in IF digraph G. For any e_i, $e_j \in B_G^{\mu}$, where $e_j = (u, v)$, if there is a strongest uv μ-path P in G such that e_i is a μ-arc in P, then denote it as $e_i \bigtriangleup_G^{\mu} e_j$. Let $\beta_G^{\mu}(e_i) = \{e_j \in B_G^{\mu} - \{e_i\} : e_i \bigtriangleup_G^{\mu} e_j\}$, $\beta_G^{\mu*}(e_i) = \{e_j \in B_G^{\mu} - \{e_i\} : e_j \bigtriangleup_G^{\mu} e_i\}$, and $\Phi_G^{\mu} = \{e_i \in B_G^{\mu} : \beta_G^{\mu}(e_i) \cap \beta_G^{\mu*}(e_i) = \emptyset\}$. In the same way, B_G^{v} and Φ_G^{v} can be defined. Clearly, $B_G^{\mu} = B_{G_\mu}$ and $\Phi_G^{\mu} = \Phi_{G_\mu}$, so G_μ has a unique minimum equivalent fuzzy

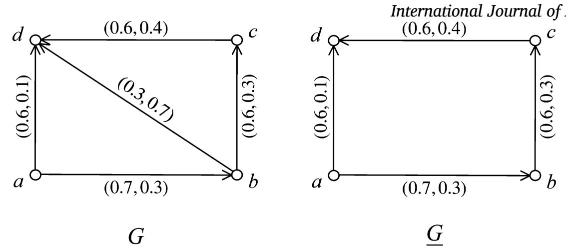

Fig. 1. IF digraph G and its MEIFS $\underline{G}$.

subgraph when $B^{\mu}_{G} = \Phi^{\mu}_{G}$. The introduced MEIFS is a generalization of minimum equivalent fuzzy subgraph in IF context, so to avoid its multiplicity, the IF digraph G considered below always satisfies $B^{\mu}_{G} = \Phi^{\mu}_{G}$ and $B^{\upsilon}_{G} = \Phi^{\upsilon}_{G}$.

Example 1. An IF digraph G and its MEIFS $\underline{G}$ are given in Fig. 1.

According to Definition 5, $\underline{G}$ can be obtained by deleting certain arcs in G. Clearly, $CONN_{\underline{G}}(u,v) = CONN_{G}(u,v)$ means $CONN^{\mu}_{\underline{G}}(u,v) = CONN^{\mu}_{G}(u,v)$ and $CONN^{\upsilon}_{\underline{G}}(u,v) = CONN^{\upsilon}_{G}(u,v)$. We consider only the μ-strength of connectedness first, which is equivalent to considering the strength of connectedness in a fuzzy digraph. It is easy to know that to keep the μ-strength of connectedness unchanged, α_{μ}-strong arcs in G cannot be removed, while β_{μ}-strong arcs and δ_{μ}-arcs in G can be removed. However, this may affect the υ-strength of connectedness between certain vertices. For example, as shown in Fig. 1, (a,d) is a β_{μ}-strong arc in G and removing it will not affect the μ-strength of connectedness between any pair of vertices, but this will increase the υ-strength of connectedness between a and d. In fact, $CONN^{\upsilon}_{G}(a,d) = 0.1$, while $CONN^{\upsilon}_{G-\{(a,d)\}}(a,d) = 0.4$. Hence, to satisfy the condition $CONN_{\underline{G}}(u,v) = CONN_{G}(u,v)$, all α_{μ}-strong arcs in G should be preserved, and β_{μ}-strong arcs and δ_{μ}-arcs in G can be deleted without affecting the υ-strength of connectedness between vertices.

Next, we analyze υ-arcs in G. According to the definition of α_{υ}-strong arc, removing an α_{υ}-strong arc will lead to the increase of υ-strength of connectedness between certain vertices, so α_{υ}-strong arcs in G should be preserved to obtain $\underline{G}$. It can be known that β_{υ}-strong arc is similar to β_{μ}-strong arc in structure. Removing one β_{υ}-strong arc will not change the υ-strength of connectedness between any pair of vertices, but the case of removing multiple β_{υ}-strong arcs is still worth analyzing. Additionally, unlike δ_{μ}-arcs, removing a δ_{υ}-arc may affect the υ-strength of connectedness. For instance, as shown in Fig. 1, (a,b) is a δ_{υ}-arc in G, but $CONN^{\upsilon}_{G-\{(a,b)\}}(a,b) < CONN^{\upsilon}_{G}(a,b)$. In fact, any δ_{υ}-arc (u,v) satisfying $CONN^{\upsilon}_{G-\{(u,v)\}}(u,v) = 0$ should be preserved, because $CONN^{\upsilon}_{G-\{(u,v)\}}(u,v) = 0 < \upsilon(u,v) = CONN^{\upsilon}_{G}(u,v)$.

Lemma 1. *Let G be an IF digraph, A^{υ}_{G}, B^{υ}_{G}, and C^{υ}_{G} be the set of α_{υ}-strong arcs, the set of β_{υ}-strong arcs, and the set of δ_{υ}-arcs in G, respectively, and $\Omega_G = \{e = (u,v) \in C^{\upsilon}_{G} : CONN^{\upsilon}_{G-\{(u,v)\}}(u,v) > 0\} = \{e_1, e_2, \dots, e_k\}$. Then,*

$$A^{\upsilon}_{G-\{e_{i_1},e_{i_2},\dots,e_{i_t}\}} = A^{\upsilon}_{G},$$

$$B^{\upsilon}_{G-\{e_{i_1},e_{i_2},\dots,e_{i_t}\}} = B^{\upsilon}_{G},$$

$$\Omega_{G-\{e_{i_1},e_{i_2},\dots,e_{i_t}\}} = \Omega_G - \{e_{i_1}, e_{i_2}, \dots, e_{i_t}\},$$

where $1 \le t \le k$. In particular, $A^{\upsilon}_{G-\Omega_G} = A^{\upsilon}_{G}$, $B^{\upsilon}_{G-\Omega_G} = B^{\upsilon}_{G}$, and $\Omega_{G-\Omega_G} = \emptyset$.

Proof. First, we prove that $\Omega_{G-\{e_{i_1}\}} = \Omega_G - \{e_{i_1}\}$. Let $G = (V,(\sigma,\eta))$ and $G - \{e_{i_1}\} = (V,(\sigma',\eta'))$. Suppose that $e_{i_1} = (u,v)$ and $e_{i_2} = (a,b) \in \Omega_G - \{e_{i_1}\}$. If there is no strongest ab υ-path in G such that e_{i_1} is an υ-arc in this υ-path, then there is no strongest ab υ-path in $G - \{e_{i_2}\}$ such that e_{i_1} is an υ-arc in this υ-path, and further, removing e_{i_1} from $G - \{e_{i_2}\}$ will not affect the υ-strength of connectedness between a and b, i.e., $CONN^{\upsilon}_{G-\{e_{i_1},e_{i_2}\}}(a,b) = CONN^{\upsilon}_{G-\{e_{i_2}\}}(a,b) > 0$. Additionally, $\eta'(a,b) = \eta(a,b) > CONN^{\upsilon}_{G}(a,b) = CONN^{\upsilon}_{G-\{e_{i_1}\}}(a,b)$. So $e_{i_2} \in \Omega_{G-\{e_{i_1}\}}$. If there exists a strongest ab υ-path $P_1 = P_{au} e_{i_1} P_{vb}$ in G such that e_{i_1} is an υ-arc in P_1, where P_{au} and P_{vb} are an au υ-path and a vb υ-path in G, respectively, then there is another ab υ-path $P_2 = P_{au} P_{uv} P_{vb}$ in G, in which P_{uv} is a strongest uv υ-path in G. As P_1 is the strongest ab υ-path, the υ-strength of P_2 is not less than that of P_1. Since $e_{i_1} \in \Omega_G \subseteq C^{\upsilon}_{G}$, it can be known that the υ-strength of P_2 is not greater than that of P_1. So P_2 is also a strongest ab υ-path in G. Based on this, the removal of e_{i_1} will not affect the υ-strength of connectedness between a and b, and then, $e_{i_2} \in \Omega_{G-\{e_{i_1}\}}$ can be obtained similarly. Consequently, we have $\Omega_{G-\{e_{i_1}\}} \supseteq \Omega_G - \{e_{i_1}\}$. On the contrary, assume that there exists $e = (x,y) \in \Omega_{G-\{e_{i_1}\}} - (\Omega_G - \{e_{i_1}\})$. Then, $CONN^{\upsilon}_{G-\{e_{i_1},e\}}(x,y) > 0$ and this indicates that there is an xy υ-path in G that does not pass through e_{i_1} and e. However, since $\eta(e) = \eta'(e) > CONN^{\upsilon}_{G-\{e_{i_1}\}}(x,y) = CONN^{\upsilon}_{G-\{e_{i_1},e\}}(x,y) \ge CONN^{\upsilon}_{G}(x,y)$ and $e \notin \Omega_G$, we have $e \in C^{\upsilon}_{G} - \Omega_G$, which implies that there is no xy υ-path in G that does not pass through e, a contradiction. Therefore, $\Omega_{G-\{e_{i_1}\}} = \Omega_G - \{e_{i_1}\}$.

Next, we prove that $\Omega_{G-\{e_{i_1},e_{i_2},\dots,e_{i_t}\}} = \Omega_G - \{e_{i_1}, e_{i_2}, \dots, e_{i_t}\}$. For any $e_{i_2} \in \Omega_G - \{e_{i_1}\} = \Omega_{G-\{e_{i_1}\}}$, we have $\Omega_{G-\{e_{i_1},e_{i_2}\}} = \Omega_{(G-\{e_{i_1}\})-\{e_{i_2}\}} = \Omega_{G-\{e_{i_1}\}} - \{e_{i_2}\} = \Omega_G - \{e_{i_1}, e_{i_2}\}$. Similarly, for any $e_{i_3} \in \Omega_G - \{e_{i_1}, e_{i_2}\} = \Omega_{G-\{e_{i_1},e_{i_2}\}}$, we have $\Omega_{G-\{e_{i_1},e_{i_2},e_{i_3}\}} = \Omega_{(G-\{e_{i_1},e_{i_2}\})-\{e_{i_3}\}} = \Omega_{G-\{e_{i_1},e_{i_2}\}} - \{e_{i_3}\} = \Omega_G - \{e_{i_1}, e_{i_2}, e_{i_3}\}$. Through iterative calculation, $\Omega_{G-\{e_{i_1},e_{i_2},\dots,e_{i_t}\}} = \Omega_G - \{e_{i_1}, e_{i_2}, \dots, e_{i_t}\}$ can be obtained, where $1 \le t \le k$. In a similar way, it can be proved that $A^{\upsilon}_{G-\{e_{i_1},e_{i_2},\dots,e_{i_t}\}} = A^{\upsilon}_G$ and $B^{\upsilon}_{G-\{e_{i_1},e_{i_2},\dots,e_{i_t}\}} = B^{\upsilon}_G$. □

Arcs in B^{υ}_G has something in common with arcs in Ω_G in structure. Specifically, for any υ-arc $e = (u, v) \in B^{\upsilon}_G \cup \Omega_G$, it holds that $CONN^{\upsilon}_{G-\{e\}}(u, v) = CONN^{\upsilon}_G(u, v) > 0$ and this means that there is a strongest uv υ-path in G that does not pass through e, so the removal of e will not affect the υ-strength of connectedness between any pair of vertices. Consequently, an argument similar to Lemma 1 can be obtained as follows.

Lemma 2. *Let G be an IF digraph, A^{υ}_G, B^{υ}_G, and C^{υ}_G be the set of α_{υ}-strong arcs, the set of β_{υ}-strong arcs, and the set of δ_{υ}-arcs in G, respectively, and $\Omega_G = \{e = (u, v) \in C^{\upsilon}_G : CONN^{\upsilon}_{G-\{(u,v)\}}(u, v) > 0\}$. Suppose that $B^{\upsilon}_G = \{e_1, e_2, \dots, e_s\}$. Then,*

$$A^{\upsilon}_{G-\{e_{i_1},e_{i_2},\dots,e_{i_j}\}} = A^{\upsilon}_G,$$
$$B^{\upsilon}_{G-\{e_{i_1},e_{i_2},\dots,e_{i_j}\}} = B^{\upsilon}_G - \{e_{i_1}, e_{i_2}, \dots, e_{i_j}\},$$
$$\Omega_{G-\{e_{i_1},e_{i_2},\dots,e_{i_j}\}} = \Omega_G,$$

where $1 \le j \le s$. In particular, $A^{\upsilon}_{G-B^{\upsilon}_G} = A^{\upsilon}_G$, $B^{\upsilon}_{G-B^{\upsilon}_G} = \emptyset$, and $\Omega_{G-B^{\upsilon}_G} = \Omega_G$.

Proposition 1. *Let G be an IF digraph, B^{υ}_G and C^{υ}_G be the set of β_{υ}-strong arcs and the set of δ_{υ}-arcs in G, respectively, and $\Omega_G = \{e = (u, v) \in C^{\upsilon}_G : CONN^{\upsilon}_{G-\{(u,v)\}}(u, v) > 0\}$. Then, $CONN^{\upsilon}_{G-(B^{\upsilon}_G\cup\Omega_G)}(u, v) = CONN^{\upsilon}_G(u, v)$ for any vertices u and v.*

Proof. Suppose that $B^{\upsilon}_G = \{e_1, e_2, \dots, e_p\}$ and $\Omega_G = \{e_{p+1}, e_{p+2}, \dots, e_{p+q}\}$. For any s υ-arcs $e_{i_1}, e_{i_2}, \dots, e_{i_s} \in B^{\upsilon}_G$ and any t υ-arcs $e_{i_{s+1}}, e_{i_{s+2}}, \dots, e_{i_{s+t}} \in \Omega_G$, since $B^{\upsilon}_G = B^{\upsilon}_{G-\{e_{i_{s+1}},e_{i_{s+2}},\dots,e_{i_{s+t}}\}}$ and $\Omega_G = \Omega_{G-\{e_{i_1},e_{i_2},\dots,e_{i_s}\}}$, according to Lemmas 1 and 2, we have

$$\begin{aligned}
B^{\upsilon}_{G-\{e_{i_1},e_{i_2},\dots,e_{i_s},e_{i_{s+1}},e_{i_{s+2}},\dots,e_{i_{s+t}}\}} &= B^{\upsilon}_{(G-\{e_{i_1},e_{i_2},\dots,e_{i_s}\})-\{e_{i_{s+1}},e_{i_{s+2}},\dots,e_{i_{s+t}}\}} \\
&= B^{\upsilon}_{G-\{e_{i_1},e_{i_2},\dots,e_{i_s}\}} \\
&= B^{\upsilon}_G - \{e_{i_1}, e_{i_2}, \dots, e_{i_s}\}, \\
\Omega_{G-\{e_{i_1},e_{i_2},\dots,e_{i_s},e_{i_{s+1}},e_{i_{s+2}},\dots,e_{i_{s+t}}\}} &= \Omega_{(G-\{e_{i_{s+1}},e_{i_{s+2}},\dots,e_{i_{s+t}}\})-\{e_{i_1},e_{i_2},\dots,e_{i_s}\}} \\
&= \Omega_{G-\{e_{i_{s+1}},e_{i_{s+2}},\dots,e_{i_{s+t}}\}} \\
&= \Omega_G - \{e_{i_{s+1}}, e_{i_{s+2}}, \dots, e_{i_{s+t}}\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&B^{\upsilon}_{G-\{e_{i_1},e_{i_2},\dots,e_{i_s},e_{i_{s+1}},e_{i_{s+2}},\dots,e_{i_{s+t}}\}} \cup \Omega_{G-\{e_{i_1},e_{i_2},\dots,e_{i_s},e_{i_{s+1}},e_{i_{s+2}},\dots,e_{i_{s+t}}\}} \\
=&B^{\upsilon}_G \cup \Omega_G - \{e_{i_1}, e_{i_2}, \dots, e_{i_s}, e_{i_{s+1}}, e_{i_{s+2}}, \dots, e_{i_{s+t}}\}.
\end{aligned}$$

Based on this and the fact that $CONN^{\upsilon}_{G-\{e_{k_1}\}}(u, v) = CONN^{\upsilon}_G(u, v)$ for any $e_{k_1} \in B^{\upsilon}_G \cup \Omega_G$ and any vertices u and v, we have $CONN^{\upsilon}_{G-\{e_{k_1},e_{k_2}\}}(u, v) = CONN^{\upsilon}_{G-\{e_{k_1}\}}(u, v) = CONN^{\upsilon}_G(u, v)$ for any $e_{k_2} \in B^{\upsilon}_G \cup \Omega_G - \{e_{k_1}\} = B^{\upsilon}_{G-\{e_{k_1}\}} \cup \Omega_{G-\{e_{k_1}\}}$. By repeating this behavior, we can finally get that

$$\begin{aligned}
CONN^{\upsilon}_{G-\{e_{k_1},e_{k_2}\dots,e_{k_{p+q}}\}}(u, v) &= CONN^{\upsilon}_{G-\{e_{k_1},e_{k_2}\dots,e_{k_{p+q-1}}\}}(u, v) \\
&= CONN^{\upsilon}_{G-\{e_{k_1},e_{k_2}\dots,e_{k_{p+q-2}}\}}(u, v) \\
&= \cdots = CONN^{\upsilon}_G(u, v)
\end{aligned}$$

for any $e_{k_{p+q}} \in B^{\upsilon}_G \cup \Omega_G - \{e_{k_1}, e_{k_2}, \dots, e_{k_{p+q-1}}\} = B^{\upsilon}_{G-\{e_{k_1},e_{k_2},\dots,e_{k_{p+q-1}}\}} \cup \Omega_{G-\{e_{k_1},e_{k_2}\dots,e_{k_{p+q-1}}\}}$. Therefore, $CONN^{\upsilon}_{G-(B^{\upsilon}_G\cup\Omega_G)}(u, v) = CONN^{\upsilon}_G(u, v)$ for any vertices u and v. □

According to Proposition 1, removing all υ-arcs in set $B^{\upsilon}_G \cup \Omega_G$ from IF digraph G will not change the υ-strength of connectedness between any pair of vertices. However, the removal of certain υ-arcs in set $B^{\upsilon}_G \cup \Omega_G$ may decrease the μ-strength of connectedness. Accordingly, in combination with the previous analysis, all α_{υ}-strong arcs in G and δ_{υ}-arcs in set $C^{\upsilon}_G - \Omega_G$ should be kept, and β_{υ}-strong arcs in G and δ_{υ}-arcs in set Ω_G should be deleted without affecting the μ-strength of connectedness. Overall, $\underline{G}$ can be obtained by deleting all arcs that do not belong to the set $A^{\mu}_G \cup [A^{\upsilon}_G \cup (C^{\upsilon}_G - \Omega_G)]$, where A^{μ}_G represents the set of α_{μ}-strong arcs in G.

Proposition 2. *Let $G = (V, (\sigma, \eta))$ be an IF digraph and $O = V \times V - [A^{\mu}_{G} \cup A^{\upsilon}_{G} \cup (C^{\upsilon}_{G} - \Omega_G)]$, in which A^{μ}_{G}, A^{υ}_{G}, and C^{υ}_{G} are the set of α_μ-strong arcs, the set of α_υ-strong arcs, and the set of δ_υ-arcs in G, respectively, and $\Omega_G = \{e = (u,\upsilon) \in C^{\upsilon}_{G} : CONN^{\upsilon}_{G-\{(u,\upsilon)\}}(u,\upsilon) > 0\}$. Then, $G - O$ is the MEIFS of G.*

According to the above proposition, it can be seen that each IF digraph has a unique MEIFS.

3.2. Semi-maximum equivalent intuitionistic fuzzy supergraphs

We try to construct a maximum equivalent IF supergraph $\overline{G}$ for any IF digraph G, namely, $G \subseteq \overline{G}$, $CONN_{\overline{G}}(u,\upsilon) = CONN_G(u,\upsilon)$ for any vertices u and υ, and if there is an IF digraph G' such that $G \subseteq G'$ and $CONN_{G'}(u,\upsilon) = CONN_G(u,\upsilon)$ for any vertices u and υ, then $G' \subseteq \overline{G}$. To obtain $\overline{G}$, a natural idea is to increase the membership value of any arc (u,υ) in G to the maximum value $CONN^{\mu}_{G}(u,\upsilon)$ (too large membership value will lead to $CONN^{\mu}_{\overline{G}}(u,\upsilon) > CONN^{\mu}_{G}(u,\upsilon)$) and reduce the non-membership value of (u,υ) to the minimum value $CONN^{\upsilon}_{G}(u,\upsilon)$ (too small non-membership value will lead to $CONN^{\upsilon}_{\overline{G}}(u,\upsilon) < CONN^{\upsilon}_{G}(u,\upsilon)$). However, $CONN^{\mu}_{G}(u,\upsilon) + CONN^{\upsilon}_{G}(u,\upsilon) \le 1$ is not always true, so the sum of the membership value and the non-membership value of arc (u,υ) may be greater than 1, which does not meet the binding condition of IF digraph. Consequently, we need to sacrifice the membership value or non-membership value of arc (u,υ) to solve this problem, namely, (i) the non-membership value of arc (u,υ) is reduced to the minimum and its membership value is as close as possible to the maximum, or (ii) the membership value of arc (u,υ) is increased to the maximum and its non-membership value is set to 0. In the following, we use method (i) to construct the concept of SEIFS.

Proposition 3. *For any IF digraph $G = (V, (\sigma, \eta))$, let $\overline{G} = (V, (\overline{\sigma}, \overline{\eta}))$, and for any $u, \upsilon \in V$ it holds that*

$$\overline{\sigma}(u,\upsilon) = CONN^{\mu}_{G}(u,\upsilon) \wedge (1 - \overline{\eta}(u,\upsilon)),$$

$$\overline{\eta}(u,\upsilon) = CONN^{\upsilon}_{G}(u,\upsilon) \wedge \eta(u,\upsilon).$$

Then, $\overline{G}$ is an IF digraph and $G \subseteq \overline{G}$.

Proof. Clearly, $\overline{\sigma}(u,\upsilon) \le 1 - \overline{\eta}(u,\upsilon)$, and thus, $\overline{\sigma}(u,\upsilon) + \overline{\eta}(u,\upsilon) \le 1$ for any u, $\upsilon \in V$, so $\overline{G}$ is an IF digraph. Next, we prove that $G \subseteq \overline{G}$. Suppose that (x,y) is an arc in G. Clearly, $\eta(x,y) \ge \overline{\eta}(x,y)$. Since $\sigma(x,y) \le 1 - \eta(x,y) \le 1 - \overline{\eta}(x,y)$ and $\sigma(x,y) \le CONN^{\mu}_{G}(x,y)$, we have $\sigma(x,y) \le CONN^{\mu}_{G}(x,y) \wedge (1 - \overline{\eta}(x,y)) = \overline{\sigma}(x,y)$. Hence, $G \subseteq \overline{G}$. □

Let $\Gamma_G = \{(u,\upsilon) \in V \times V : \sigma(u,\upsilon) = 0, CONN^{\mu}_{G}(u,\upsilon) > 0\}$ and $\Omega_G = \{(u,\upsilon) \in C^{\upsilon}_{G} : CONN^{\upsilon}_{G-\{(u,\upsilon)\}}(u,\upsilon) > 0\}$. It is easy to see that only when $(u,\upsilon) \in C^{\mu}_{G} \cup \Gamma_G$, there may be $\sigma(u,\upsilon) < \overline{\sigma}(u,\upsilon)$; only when $(u,\upsilon) \in \Omega_G$, it holds that $\overline{\eta}(u,\upsilon) = CONN^{\upsilon}_{G}(u,\upsilon) < \eta(u,\upsilon)$. Thus, $\overline{G}$ can be obtained by increasing the membership value of each $(u,\upsilon) \in C^{\mu}_{G} \cup \Gamma_G$ to $CONN^{\mu}_{G}(u,\upsilon) \wedge (1 - \overline{\eta}(u,\upsilon))$ and decreasing the non-membership values of all δ_υ-arcs in set Ω_G so that they become β_υ-strong.

Let $G_\mu = (V, \sigma)$ and $(\overline{G})_\mu = (V, \overline{\sigma})$ be the fuzzy digraph composed of μ-arcs in G and the fuzzy digraph composed of μ-arcs in $\overline{G}$, respectively, and G^{T}_{μ} be the fuzzy transitive closure of G_μ. As introduced in [36], G^{T}_{μ} can be got by increasing the membership value of each (u,υ) satisfying $\sigma(u,\upsilon) < CONN_{G_\mu}(u,\upsilon)$ (i.e., $(u,\upsilon) \in C^{\mu}_{G} \cup \Gamma_G$) so that it becomes β-strong, then we have $G_\mu \subseteq (\overline{G})_\mu \subseteq G^{T}_{\mu}$, and based on this, it can be inferred that the fuzzy transitive closure of $(\overline{G})_\mu$ is same as that of G_μ. As two fuzzy digraphs with the same fuzzy transitive closure have the same connectivity (see [36], Theorem 1), it can be obtained that $CONN^{\mu}_{\overline{G}}(u,\upsilon) = CONN_{(\overline{G})_\mu}(u,\upsilon) = CONN_{G_\mu}(u,\upsilon) = CONN^{\mu}_{G}(u,\upsilon)$ for any vertices u and υ.

Proposition 4. *Let $G = (V, (\sigma, \eta))$ be an IF digraph and $\overline{G} = (V, (\overline{\sigma}, \overline{\eta}))$, where $\overline{\sigma}(u,\upsilon) = CONN^{\mu}_{G}(u,\upsilon) \wedge (1 - \overline{\eta}(u,\upsilon))$ and $\overline{\eta}(u,\upsilon) = CONN^{\upsilon}_{G}(u,\upsilon) \wedge \eta(u,\upsilon)$ for any u, $\upsilon \in V$. Then, $CONN_{\overline{G}}(u,\upsilon) = CONN_G(u,\upsilon)$ for any vertices u and υ.*

Proof. We just need to prove that $CONN^{\upsilon}_{\overline{G}}(u,\upsilon) = CONN^{\upsilon}_{G}(u,\upsilon)$.

(1) Suppose that $CONN^{\upsilon}_{G}(u,\upsilon) = 0$. Then, $CONN^{\upsilon}_{\overline{G}}(u,\upsilon) = 0$. Otherwise, assume that $CONN^{\upsilon}_{\overline{G}}(u,\upsilon) > 0$. This suggests that there is a strongest $u\upsilon$ υ-path $P = u(u,x_1)x_1(x_1,x_2)x_2 \dots x_k(x_k,\upsilon)\upsilon$ in $\overline{G}$, and then, $\eta(u,x_1) \ge \overline{\eta}(u,x_1) > 0$, $\eta(x_k,\upsilon) \ge \overline{\eta}(x_k,\upsilon) > 0$, and $\eta(x_i,x_{i+1}) \ge \overline{\eta}(x_i,x_{i+1}) > 0$, namely, P is also a $u\upsilon$ υ-path in G, which contradicts $CONN^{\upsilon}_{G}(u,\upsilon) = 0$.

(2) Suppose that $CONN^{\upsilon}_{G}(u,\upsilon) > 0$ and $P_{u\upsilon}$ is a strongest $u\upsilon$ υ-path in G. Let $S^{\upsilon}_{G}(P_{u\upsilon})$ be the υ-strength of $P_{u\upsilon}$ in G. At least one of all the weakest υ-arcs in $P_{u\upsilon}$ does not belong to the set $\Omega_G = \{(a,b) \in C^{\upsilon}_{G} : CONN^{\upsilon}_{G-\{(a,b)\}}(a,b) > 0\}$, where C^{υ}_{G} represents the set of δ_υ-arcs in G. Otherwise, assume that $P_{u\upsilon}$ contains t weakest υ-arcs (x_1,y_1), (x_2,y_2), …, (x_t,y_t), and $(x_i,y_i) \in \Omega_G$ for $1 \le i \le t$. Then, $P_{u\upsilon}$ can be expressed as $P_{u\upsilon} = P_{ux_1}(x_1,y_1)P_{y_1x_2}(x_2,y_2)P_{y_2x_3} \dots (x_t,y_t)P_{y_t\upsilon}$, where P_{ux_1}, $P_{y_ix_{i+1}}$, and $P_{y_t\upsilon}$ represent a ux_1 υ-path, a y_ix_{i+1} υ-path, and a $y_t\upsilon$ υ-path in G, respectively. Then, there is another $u\upsilon$ υ-path $P'_{u\upsilon} = P_{ux_1}P_{x_1y_1}P_{y_1x_2}P_{x_2y_2}P_{y_2x_3} \dots P_{x_ty_t}P_{y_t\upsilon}$, where $P_{x_iy_i}$ is a strongest x_iy_i υ-path in G. Since $(x_i,y_i) \in \Omega_G$, we have $S^{\upsilon}_{G}(P_{x_iy_i}) < \eta(x_i,y_i)$ and further $S^{\upsilon}_{G}(P'_{u\upsilon}) < S^{\upsilon}_{G}(P_{u\upsilon})$, which contradicts that $P_{u\upsilon}$ is the strongest $u\upsilon$ υ-path in G. Hence, at least one weakest υ-arc in $P_{u\upsilon}$ does not belong to the set Ω_G and then we have $S^{\upsilon}_{\overline{G}}(P_{u\upsilon}) = S^{\upsilon}_{G}(P_{u\upsilon})$ according to the construction method of $\overline{G}$.

Next, we analyze any other uv υ-path P''_{uv}, which is not the strongest. If all υ-arcs in P''_{uv} do not belong to the set Ω_G, then $S^{\upsilon}_{\overline{G}}(P''_{uv}) = S^{\upsilon}_{G}(P''_{uv})$ according to the construction method of $\overline{G}$. In this case, we have $CONN^{\upsilon}_{\overline{G}}(u,\upsilon) = S^{\upsilon}_{\overline{G}}(P_{uv}) \wedge S^{\upsilon}_{\overline{G}}(P''_{uv}) = S^{\upsilon}_{G}(P_{uv}) \wedge S^{\upsilon}_{G}(P''_{uv}) = S^{\upsilon}_{G}(P_{uv}) = CONN^{\upsilon}_{G}(u,\upsilon)$. For the case of there are s υ-arcs (x''_1, y''_1), (x''_2, y''_2), $\ldots$, (x''_s, y''_s) in P''_{uv} belong to the set Ω_G, let $P''_{uv} = P_{ux''_1}\ (x''_1, y''_1)\ P_{y''_1x''_2}\ (x''_2, y''_2)\ P_{y''_2x''_3}\ \ldots (x''_s, y''_s)\ P_{y''_s\upsilon}$, where $P_{ux''_1}$, $P_{y''_ix''_{i+1}}$, and $P_{y''_s\upsilon}$ represent a ux''_1 υ-path, a $y''_ix''_{i+1}$ υ-path, and a $y''_s\upsilon$ υ-path in G, respectively. Then, there is another uv υ-path $P'''_{uv} = P_{ux''_1}P_{x''_1y''_1}P_{y''_1x''_2}P_{x''_2y''_2}P_{y''_2x''_3}\ldots P_{x''_sy''_s}P_{y''_s\upsilon}$, where $P_{x''_iy''_i}$ is a strongest $x''_iy''_i$ υ-path in G and there is no υ-arc in $P_{x''_iy''_i}$ belongs to the set Ω_G (as IF digraph G is finite), and further, there is no υ-arc in P'''_{uv} belongs to the set Ω_G. Consequently, $S^{\upsilon}_{\overline{G}}(P''_{uv}) \geq S^{\upsilon}_{\overline{G}}(P'''_{uv}) = S^{\upsilon}_{G}(P'''_{uv})$. Therefore, $CONN^{\upsilon}_{\overline{G}}(u,\upsilon) = S^{\upsilon}_{\overline{G}}(P_{uv}) \wedge S^{\upsilon}_{\overline{G}}(P''_{uv}) \wedge S^{\upsilon}_{\overline{G}}(P'''_{uv}) = S^{\upsilon}_{G}(P_{uv}) \wedge S^{\upsilon}_{G}(P'''_{uv}) = S^{\upsilon}_{G}(P_{uv}) = CONN^{\upsilon}_{G}(u,\upsilon)$ in this case. □

From the above proposition and the foregoing analysis of the construction of $\overline{G}$, it can be inferred that

$$A^{\mu}_{\overline{G}} = A^{\mu}_{G}, \quad B^{\mu}_{\overline{G}} \subseteq B^{\mu}_{G} \cup C^{\mu}_{G} \cup \Gamma_G, \quad C^{\mu}_{\overline{G}} \subseteq C^{\mu}_{G} \cup \Gamma_G, \quad \Gamma_{\overline{G}} \subseteq \Gamma_G;$$
$$A^{\upsilon}_{\overline{G}} = A^{\upsilon}_{G}, \quad B^{\upsilon}_{\overline{G}} = B^{\upsilon}_{G} \cup \Omega_G, \quad C^{\upsilon}_{\overline{G}} = C^{\upsilon}_{G} - \Omega_G, \quad \Omega_{\overline{G}} = \emptyset.$$

According to Propositions 3 and 4, the IF digraph $\overline{G}$ we construct satisfies the description of SEIFS at the beginning of this subsection.

Definition 6. Let $G = (V, (\sigma, \eta))$ be an IF digraph and $\overline{G} = (V, (\overline{\sigma}, \overline{\eta}))$, where $\overline{\sigma}(u,\upsilon) = CONN^{\mu}_{G}(u,\upsilon) \wedge (1 - \overline{\eta}(u,\upsilon))$ and $\overline{\eta}(u,\upsilon) = CONN^{\upsilon}_{G}(u,\upsilon) \wedge \eta(u,\upsilon)$ for any u, $\upsilon \in V$. Then, $\overline{G}$ is called the semi-maximum equivalent intuitionistic fuzzy supergraph (SEIFS) of G.

The following proposition explains the semi-maximum characteristic of $\overline{G}$, i.e., under certain conditions, $\overline{G}$ is the maximum partial IF supergraph that preserves the connectivity of G.

Proposition 5. *Let $G = (V, (\sigma, \eta))$ be an IF digraph and $\overline{G}$ be the SEIFS of G. For any partial IF supergraph $G' = (V, (\sigma', \eta'))$ of G that satisfies $CONN_{G'}(u,\upsilon) = CONN_{G}(u,\upsilon)$ for any vertices u and υ, if $\eta'(u,\upsilon) > 0$ for any $(u,\upsilon) \in B^{\upsilon}_{G} \cup \Omega_G$, where $\Omega_G = \{e = (u,\upsilon) \in C^{\upsilon}_{G} : CONN^{\upsilon}_{G-\{(u,\upsilon)\}}(u,\upsilon) > 0\}$ and B^{υ}_{G} and C^{υ}_{G} represent the set of β_{υ}-strong arcs and the set of δ_{υ}-arcs in G, respectively, then $G' \subseteq \overline{G}$.*

Proof. Since $G \subseteq G'$, it holds that $0 \leq \eta'(u,\upsilon) \leq \eta(u,\upsilon)$ for any υ-arc (u,υ) in G. However, according to $CONN^{\upsilon}_{G'}(u,\upsilon) = CONN^{\upsilon}_{G}(u,\upsilon)$, it can be known that $\eta'(u,\upsilon) > 0$ for the case of $(u,\upsilon) \in A^{\upsilon}_{G} \cup (C^{\upsilon}_{G} - \Omega_G)$; otherwise, $CONN^{\upsilon}_{G'}(u,\upsilon) = 0 < CONN^{\upsilon}_{G}(u,\upsilon)$, a contradiction. Thus, only when $(u,\upsilon) \in B^{\upsilon}_{G} \cup \Omega_G$, it may hold that $\eta'(u,\upsilon) = 0$. Next, suppose that $G' \nsubseteq \overline{G}$. Let $\overline{G} = (V, (\overline{\sigma}, \overline{\eta}))$. Then, there exists an arc (u', υ') in G' such that $\sigma'(u', \upsilon') > \overline{\sigma}(u', \upsilon')$ or $\eta'(u', \upsilon') < \overline{\eta}(u', \upsilon')$. If $\sigma'(u', \upsilon') > \overline{\sigma}(u', \upsilon')$, according to the fact that $\sigma'(u', \upsilon') \leq CONN^{\mu}_{G'}(u', \upsilon') = CONN^{\mu}_{G}(u', \upsilon')$, we have $\overline{\sigma}(u', \upsilon') = CONN^{\mu}_{G}(u', \upsilon') \wedge (1 - \overline{\eta}(u', \upsilon')) = 1 - \overline{\eta}(u', \upsilon') < CONN^{\mu}_{G}(u', \upsilon')$. Then, $\overline{\eta}(u', \upsilon') > 1 - CONN^{\mu}_{G}(u', \upsilon') \geq 0$, and thus, $\eta(u', \upsilon') > 0$, i.e., (u', υ') is an υ-arc in G, and $\overline{\eta}(u', \upsilon') = CONN^{\upsilon}_{G}(u', \upsilon') \wedge \eta(u', \upsilon') = CONN^{\upsilon}_{G}(u', \upsilon')$. Since $\sigma'(u', \upsilon') > \overline{\sigma}(u', \upsilon')$, $\sigma'(u', \upsilon') + \eta'(u', \upsilon') \leq 1$, and $\overline{\sigma}(u', \upsilon') + \overline{\eta}(u', \upsilon') = 1$, we have $\eta'(u', \upsilon') < \overline{\eta}(u', \upsilon') = CONN^{\upsilon}_{G}(u', \upsilon') = CONN^{\upsilon}_{G'}(u', \upsilon')$. Hence, $\eta'(u', \upsilon') = 0$, and thus, $(u', \upsilon') \in B^{\upsilon}_{G} \cup \Omega_G$. However, this contradicts $\eta'(u,\upsilon) > 0$ for any $(u,\upsilon) \in B^{\upsilon}_{G} \cup \Omega_G$. Next, assume that $\eta'(u', \upsilon') < \overline{\eta}(u', \upsilon')$. This indicates that $\overline{\eta}(u', \upsilon') > 0$, so $\eta(u', \upsilon') > 0$, i.e., (u', υ') is an υ-arc in G, and thus, $\overline{\eta}(u', \upsilon') = CONN^{\upsilon}_{G}(u', \upsilon') \wedge \eta(u', \upsilon') = CONN^{\upsilon}_{G}(u', \upsilon')$. Therefore, $\eta'(u', \upsilon') < CONN^{\upsilon}_{G}(u', \upsilon') = CONN^{\upsilon}_{G'}(u', \upsilon')$ and this means that $\eta'(u', \upsilon') = 0$, and further, $(u', \upsilon') \in B^{\upsilon}_{G} \cup \Omega_G$. This is still in contradiction with $\eta'(u,\upsilon) > 0$ for any $(u,\upsilon) \in B^{\upsilon}_{G} \cup \Omega_G$. In conclusion, $G' \subseteq \overline{G}$. □

As introduced at the beginning of this subsection, we consider the former of two methods to construct the SEIFS. However, it is worth explaining that the SEIFS constructed by method (ii) does not always exist. For instance, we consider an IF digraph $G = (V, (\sigma, \eta))$, in which $V = \{a, b, c\}$, $\sigma(a,b) = 0.4$, $\sigma(a,c) = 0.1$, and $\sigma(b,c) = 0.8$; $\eta(a,b) = 0.2$, $\eta(a,c) = 0.7$, and $\eta(b,c) = 0$. Then, according to method (ii), we can get the SEIFS $\overline{G}' = (V, (\overline{\sigma}', \overline{\eta}'))$, where $\overline{\sigma}'(a,b) = \sigma(a,b)$, $\overline{\sigma}'(b,c) = \sigma(b,c)$, and $\overline{\sigma}'(a,c) = 0.4$; $\overline{\eta}'(a,b) = \eta(a,b)$, $\overline{\eta}'(b,c) = \eta(b,c)$, and $\overline{\eta}'(a,c) = 0$. It can be seen that $CONN^{\upsilon}_{\overline{G}'}(a,c) = 0 < 0.7 = CONN^{\upsilon}_{G}(a,c)$, which does not meet the requirement of maintaining the connectivity of the original IF digraph.

4. *R*-intuitionistic fuzzy rough digraphs

This section puts forward the notion of R-IFR digraph by approximating an IF digraph with its MEIFS and SEIFS and studies related properties and rough characteristics.

4.1. The construction of *R*-intuitionistic fuzzy rough digraphs

In this subsection, we use MEIFS and SEIFS to approximate an IF digraph, so as to obtain an R-IFR digraph. A number of important properties of R-IFR digraphs are explored.

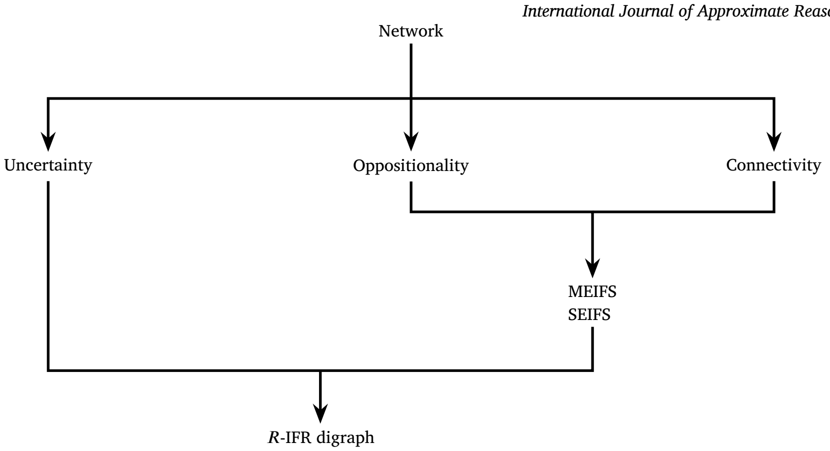

Fig. 2. Theoretical architecture of R-IFR digraph.

Table 1
Membership values and non-membership values of arcs in IF digraph G.

(v_1,v_2)	(v_1,v_4)	(v_2,v_1)	(v_2,v_4)	(v_3,v_4)	(v_3,v_5)	(v_5,v_4)
$(0.2,0.6)$	$(0, 0.3)$	$(0.4,0.6)$	$(0.2,0.7)$	$(0.2,0.7)$	$(0.2,0.5)$	$(0.2,0.4)$

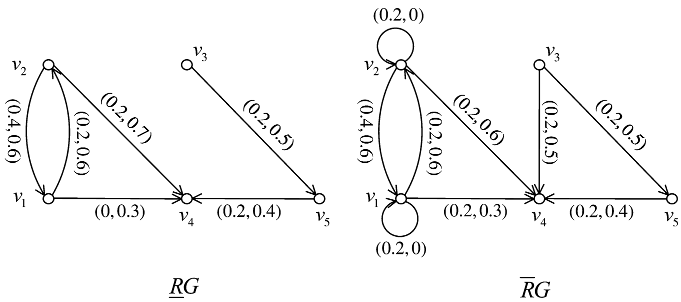

Fig. 3. R-IFR digraph $G=(\underline{R}G,\overline{R}G)$.

Definition 7. Let $G'=(V',(\sigma',\eta'))$ be an IF digraph with E' as its arc set and $G'=\overline{G'}$. Suppose that R is a relation on E' and for any $G\subseteq G'$, its lower and upper approximate IF digraphs with respect to R are $\underline{R}G=\underline{G}$ and $\overline{R}G=\overline{G}$, respectively. Then, the pair (G',R) is referred to as an approximate IF digraph space based on strength of connectedness, and the pair $(\underline{R}G,\overline{R}G)$ is called an IFR digraph based on strength of connectedness of G or an R-IFR digraph of G, denoted by $G=(\underline{R}G,\overline{R}G)$.

The above approximate IF digraph space generates approximation operators by conditioning on connectivity information, and then divides any substructure of IF digraph G' into two parts, in which the lower approximate IF digraph preserves definitely existing connections, while the upper approximate IF digraph encompasses all potentially existing connections. The discrepancy between them reveals cognitive blind spots regarding network structures when connectivity information is known, establishing a topological approximation framework for handling path-dependent systems with partial observations. The theoretical architecture of the proposed model is illustrated in Fig. 2 and it achieves comprehensive representation of information oppositionality, uncertainty, and propagative characteristic.

Example 2. Let $G=(V,(\sigma,\eta))$ be an IF digraph, where $V=\{v_1,v_2,v_3,v_4,\ v_5\}$ and σ and η are given in Table 1. Then, it can be known that $A^{\mu}_G=\{(v_1,v_2),(v_2,v_1),(v_2,v_4),(v_3,v_5),(v_5,v_4)\}$, $B^{\mu}_G=\{(v_3,v_4)\}$, $C^{\mu}_G=\emptyset$, and $\Gamma_G=\{(v_1,v_4)\}$; $A^{\upsilon}_G=\{(v_1,v_4)\}$, $B^{\upsilon}_G=\emptyset$, $\Omega_G=\{(v_2,v_4),(v_3,v_4)\}$, and $C^{\upsilon}_G-\Omega_G=\{(v_1,v_2),(v_2,v_1),(v_3,v_5),(v_5,v_4)\}$. According to Proposition 2 and Definition 6, an R-IFR digraph of G can be obtained, as shown in Fig. 3.

According to Definition 5 and Proposition 4, we can know that the upper and lower approximate IF digraphs with respect to R contain the connectivity information in G, while the upper and lower approximate IF digraphs of a classical IFR digraph [39] cannot guarantee this.

If $\underline{R}G = \overline{R}G$, then we call the IF digraph G R-exact; otherwise, IF digraph G is considered as R-IFR. The following proposition provides a way to judge whether an IF digraph is R-exact.

Proposition 6. *For any IF digraph G, it is R-exact if it satisfies the following conditions:*

(1) $A^{\mu}_{G} \supseteq B^{\upsilon}_{G}$,
(2) $B^{\mu}_{G} \subseteq A^{\upsilon}_{G} \cup C^{\upsilon}_{G}$,
(3) $C^{\mu}_{G} = \Gamma_G = \Omega_G = \emptyset$.

Proof. Let $\overline{R}G$ and $\underline{R}G$ be the upper and lower approximate IF digraphs of G, respectively. Since $C^{\mu}_{G} = \Gamma_G = \Omega_G = \emptyset$, it can be got that $G = \overline{R}G$ according to the construction method of SEIFS. In addition, since $B^{\mu}_{G} \cup C^{\mu}_{G} \cup \Gamma_G \subseteq A^{\upsilon}_{G} \cup C^{\upsilon}_{G} = A^{\upsilon}_{G} \cup (C^{\upsilon}_{G} - \Omega_G)$ and $B^{\upsilon}_{G} \cup \Omega_G \subseteq A^{\mu}_{G}$, we have $G = \underline{R}G$ according to Proposition 2. Hence, $\underline{R}G = G = \overline{R}G$, i.e., G is R-exact. $\square$

Next, we discuss some properties related to the upper and lower approximate IF digraphs of R-IFR digraphs.

Proposition 7. *Let $G = (\underline{R}G, \overline{R}G)$ be an R-IFR digraph. Then,*

(1) $\underline{R}G \subseteq G \subseteq \overline{R}G$,
(2) $\underline{R}(\underline{R}G) = \underline{R}G$, $\overline{R}(\overline{R}G) = \overline{R}G$,
(3) $\underline{R}G \subseteq \underline{R}(\overline{R}G)$,
(4) $\overline{R}G \subseteq \overline{R}(\underline{R}G)$.

Proof. (1) This argument is obviously valid.

(2) First, we prove that $\underline{R}(\underline{R}G) = \underline{R}G$. It can be known that $\underline{R}(\underline{R}G)$ is an IF subgraph of G and $CONN_{\underline{R}(\underline{R}G)}(u,\upsilon) = CONN_{\underline{R}G}(u,\upsilon) = CONN_G(u,\upsilon)$ for any vertices u and υ, thus the number of arcs in $\underline{R}(\underline{R}G)$ is not less than that in $\underline{R}G$ according to Definition 5. Based on this and the fact that $\underline{R}(\underline{R}G)$ is an IF subgraph of $\underline{R}G$, we have $\underline{R}(\underline{R}G) = \underline{R}G$. Next, we prove that $\overline{R}(\overline{R}G) = \overline{R}G$. Let $\overline{R}G = (V, (\overline{\sigma}, \overline{\eta}))$ and $\overline{R}(\overline{R}G) = (V, (\overline{\overline{\sigma}}, \overline{\overline{\eta}}))$. According to (1), we have $\overline{R}G \subseteq \overline{R}(\overline{R}G)$ and thus we just need to prove that $\overline{R}(\overline{R}G) \subseteq \overline{R}G$, i.e., $\overline{\overline{\sigma}}(x,y) \leq \overline{\sigma}(x,y)$ and $\overline{\overline{\eta}}(x,y) \geq \overline{\eta}(x,y)$ for any arc (x,y) in $\overline{R}(\overline{R}G)$. Since $\overline{\overline{\eta}}(x,y) = CONN^{\upsilon}_{\overline{R}G}(x,y) \wedge \overline{\eta}(x,y)$ and $\Omega_{\overline{R}G} = \emptyset$, which implies that $(x,y) \notin \Omega_{\overline{R}G}$ and thus $CONN^{\upsilon}_{\overline{R}G}(x,y) = \overline{\eta}(x,y)$, further, $\overline{\overline{\eta}}(x,y) = \overline{\eta}(x,y)$. Based on this and Proposition 4, we have $\overline{\overline{\sigma}}(x,y) = CONN^{\mu}_{\overline{R}G}(x,y) \wedge (1 - \overline{\overline{\eta}}(x,y)) = CONN^{\mu}_{G}(x,y) \wedge (1 - \overline{\eta}(x,y)) = \overline{\sigma}(x,y)$. Thus, $\overline{R}(\overline{R}G) \subseteq \overline{R}G$. To sum up, $\overline{R}(\overline{R}G) = \overline{R}G$.

(3) Let $G = (V, (\sigma, \eta))$, $\underline{R}G = (V, (\underline{\sigma}, \underline{\eta}))$, $\overline{R}G = (V, (\overline{\sigma}, \overline{\eta}))$, and $\underline{R}(\overline{R}G) = (V, (\underline{(\overline{\sigma})}, \underline{(\overline{\eta})}))$. Suppose that (u,υ) is an arc in $\underline{R}G$. It is required to prove that $\underline{\sigma}(u,\upsilon) \leq \underline{(\overline{\sigma})}(u,\upsilon)$ and $\underline{\eta}(u,\upsilon) \geq \underline{(\overline{\eta})}(u,\upsilon)$. It can be known that $\underline{(\overline{\sigma})}(u,\upsilon) = 0$ or $\underline{(\overline{\sigma})}(u,\upsilon) = \overline{\sigma}(u,\upsilon)$. For the former case, we have $(u,\upsilon) \notin A^{\mu}_{\overline{R}G} = A^{\mu}_{G}$ and $(u,\upsilon) \notin A^{\upsilon}_{\overline{R}G} \cup (C^{\upsilon}_{\overline{R}G} - \Omega_{\overline{R}G}) = A^{\upsilon}_{G} \cup (C^{\upsilon}_{G} - \Omega_G)$, then $\underline{\sigma}(u,\upsilon) = 0 = \underline{(\overline{\sigma})}(u,\upsilon)$. For the latter case, according to (1), we have $\underline{(\overline{\sigma})}(u,\upsilon) = \overline{\sigma}(u,\upsilon) \geq \underline{\sigma}(u,\upsilon)$. In addition, we have $\underline{(\overline{\eta})}(u,\upsilon) = 0 \leq \underline{\eta}(u,\upsilon)$ or $\underline{(\overline{\eta})}(u,\upsilon) = \overline{\eta}(u,\upsilon) \leq \underline{\eta}(u,\upsilon)$. In conclusion, $\underline{R}G \subseteq \underline{R}(\overline{R}G)$.

(4) Let $\overline{R}(\underline{R}G) = (V, (\overline{(\underline{\sigma})}, \overline{(\underline{\eta})}))$. For any arc (u,υ) in $\overline{R}G$, if $\overline{\eta}(u,\upsilon) < \overline{(\underline{\eta})}(u,\upsilon) = CONN^{\upsilon}_{\underline{R}G}(u,\upsilon) \wedge \underline{\eta}(u,\upsilon) \leq CONN^{\upsilon}_{\underline{R}G}(u,\upsilon) = CONN^{\upsilon}_{G}(u,\upsilon)$, then $\overline{\eta}(u,\upsilon) = CONN^{\upsilon}_{G}(u,\upsilon) \wedge \eta(u,\upsilon) = \eta(u,\upsilon) = 0$, and thus, $\overline{(\underline{\eta})}(u,\upsilon) \leq \underline{\eta}(u,\upsilon) \leq \eta(u,\upsilon) = 0$, which contradicts that $\overline{(\underline{\eta})}(u,\upsilon) > \overline{\eta}(u,\upsilon) \geq 0$. Thus, $\overline{\eta}(u,\upsilon) \geq \overline{(\underline{\eta})}(u,\upsilon)$. Based on this and $CONN^{\mu}_{\underline{R}G}(u,\upsilon) = CONN^{\mu}_{G}(u,\upsilon)$, it can be obtained that $\overline{(\underline{\sigma})}(u,\upsilon) = CONN^{\mu}_{\underline{R}G}(u,\upsilon) \wedge (1 - \overline{(\underline{\eta})}(u,\upsilon)) \geq CONN^{\mu}_{G}(u,\upsilon) \wedge (1 - \overline{\eta}(u,\upsilon)) = \overline{\sigma}(u,\upsilon)$. Therefore, $\overline{R}G \subseteq \overline{R}(\underline{R}G)$. $\square$

It is easy to get the following corollary from the above proposition, which supports a new way to judge whether an IF digraph is R-exact or not.

Corollary 1. *Let $G = (\underline{R}G, \overline{R}G)$ be an R-IFR digraph. Then, the following arguments hold:*

(1) *If $\overline{R}G$ is R-IFR, then G is R-IFR.*
(2) *If $\underline{R}G$ is R-exact, then G is R-exact.*

Proof. (1) According to Proposition 7, we have $\underline{R}G \subseteq \underline{R}(\overline{R}G) \subseteq \overline{R}(\overline{R}G) = \overline{R}G$. Hence, when $\overline{R}G$ is R-IFR, i.e., $\underline{R}(\overline{R}G) \subset \overline{R}(\overline{R}G)$, it holds that $\underline{R}G \subset \overline{R}G$, i.e., G is R-IFR.

(2) It can be obtained that $\underline{R}(\underline{R}G) = \underline{R}G \subseteq \overline{R}G \subseteq \overline{R}(\underline{R}G)$. If $\underline{R}G$ is R-exact, i.e., $\underline{R}(\underline{R}G) = \overline{R}(\underline{R}G)$, then we have $\underline{R}G = \overline{R}G$, i.e., G is R-exact. $\square$

According to the analysis of the topological structure of SEIFS and MEIFS in Section 3, the following conclusions can be drawn.

Proposition 8. *Let $G = (\underline{R}G, \overline{R}G)$ be an R-IFR digraph. If $B^{\upsilon}_G \cup \Omega_G \subseteq A^{\mu}_G$, then $\overline{R}(\underline{R}G) = \overline{R}G$.*

Proof. Let $G = (V, (\sigma, \eta))$, $\underline{R}G = (V, (\underline{\sigma}, \underline{\eta}))$, $\overline{R}G = (V, (\overline{\sigma}, \overline{\eta}))$, and $\overline{R}(\underline{R}G) = (V, (\overline{(\underline{\sigma})}, \overline{(\underline{\eta})}))$. First, we discuss the υ-arcs in G. For any υ-arc (x, y) in G, i.e., $\sigma(x, y) \geq 0$ and $\eta(x, y) > 0$, there are two possible situations, namely, $(x, y) \in A^{\upsilon}_G \cup (C^{\upsilon}_G - \Omega_G)$ or $(x, y) \in B^{\upsilon}_G \cup \Omega_G$. For the former case, it holds that $\underline{\eta}(x, y) = \eta(x, y)$ according to Proposition 2, then we have $\overline{(\underline{\eta})}(x, y) = CONN^{\upsilon}_{\underline{R}G}(x, y) \wedge \underline{\eta}(x, y) = CONN^{\upsilon}_G(x, y) \wedge \eta(x, y) = \overline{\eta}(x, y)$, and then, $\overline{(\underline{\sigma})}(x, y) = CONN^{\mu}_{\underline{R}G}(x, y) \wedge (1 - \overline{(\underline{\eta})}(x, y)) = CONN^{\mu}_G(x, y) \wedge (1 - \overline{\eta}(x, y)) = \overline{\sigma}(x, y)$. For the latter case, since $B^{\upsilon}_G \cup \Omega_G \subseteq A^{\mu}_G$, it can be obtained that $\underline{\eta}(x, y) = \eta(x, y)$ according to Proposition 2, then $\overline{(\underline{\eta})}(x, y) = \overline{\eta}(x, y)$ and $\overline{(\underline{\sigma})}(x, y) = \overline{\sigma}(x, y)$ can be obtained similarly. Second, for any μ-arc (a, b) that is not an υ-arc, i.e., $\sigma(a, b) > 0$ and $\eta(a, b) = 0$, then $\underline{\eta}(a, b) = 0$, and further, $\overline{(\underline{\eta})}(a, b) = CONN^{\upsilon}_{\underline{R}G}(a, b) \wedge \underline{\eta}(a, b) = 0$ and $\overline{\eta}(a, b) = CONN^{\upsilon}_G(a, b) \wedge \eta(a, b) = 0 = \overline{(\underline{\eta})}(a, b)$. Based on this, $\overline{(\underline{\sigma})}(a, b) = \overline{\sigma}(a, b)$ can be obtained similarly. Finally, for any $(u, v) \in V \times V$ that satisfies $\sigma(u, v) = \eta(u, v) = 0$, it is easy to get that $\overline{(\underline{\sigma})}(u, v) = \overline{\sigma}(u, v) = CONN^{\mu}_G(u, v)$ and $\overline{(\underline{\eta})}(u, v) = \overline{\eta}(u, v) = 0$ according to Definition 6. Overall, this argument is valid. □

Proposition 9. *Let $G = (\underline{R}G, \overline{R}G)$ be an R-IFR digraph. If $A^{\mu}_G \cap \Omega_G = \emptyset$ and $(C^{\mu}_G \cup \Gamma_G) \cap [A^{\upsilon}_G \cup (C^{\upsilon}_G - \Omega_G)] = \emptyset$, then $\underline{R}(\overline{R}G) = \underline{R}G$.*

Proof. Let $G = (V, (\sigma, \eta))$, $\underline{R}G = (V, (\underline{\sigma}, \underline{\eta}))$, $\overline{R}G = (V, (\overline{\sigma}, \overline{\eta}))$, and $\underline{R}(\overline{R}G) = (V, (\underline{(\overline{\sigma})}, \underline{(\overline{\eta})}))$. It is required to verify that $\underline{(\overline{\sigma})}(u, v) = \underline{\sigma}(u, v)$ and $\underline{(\overline{\eta})}(u, v) = \underline{\eta}(u, v)$ for any $(u, v) \in V \times V$. First, assume that $(u, v) \in A^{\mu}_G$. Then, $\underline{\sigma}(u, v) = \sigma(u, v) = \overline{\sigma}(u, v)$ and $\underline{\eta}(u, v) = \eta(u, v)$. Since $(u, v) \notin \Omega_G$, we have $\overline{\eta}(u, v) = \eta(u, v) = \underline{\eta}(u, v)$. Since $A^{\mu}_G = A^{\mu}_{\overline{R}G}$, it can be got that $\underline{(\overline{\sigma})}(u, v) = \overline{\sigma}(u, v) = \underline{\sigma}(u, v)$ and $\underline{(\overline{\eta})}(u, v) = \overline{\eta}(u, v) = \underline{\eta}(u, v)$. Second, for any $(u, v) \in B^{\mu}_G$, there are two possible situations, namely, $(u, v) \in A^{\upsilon}_G \cup (C^{\upsilon}_G - \Omega_G)$ or $(u, v) \notin A^{\upsilon}_G \cup (C^{\upsilon}_G - \Omega_G)$. For the former case, we have $\overline{\sigma}(u, v) = \sigma(u, v) = \underline{\sigma}(u, v)$ and $\overline{\eta}(u, v) = \eta(u, v) = \underline{\eta}(u, v)$, and further, $\underline{(\overline{\sigma})}(u, v) = \overline{\sigma}(u, v) = \underline{\sigma}(u, v)$ and $\underline{(\overline{\eta})}(u, v) = \overline{\eta}(u, v) = \underline{\eta}(u, v)$ since $B^{\mu}_G \subseteq B^{\mu}_{\overline{R}G}$ and $A^{\upsilon}_G \cup (C^{\upsilon}_G - \Omega_G) = A^{\upsilon}_{\overline{R}G} \cup (C^{\upsilon}_{\overline{R}G} - \Omega_{\overline{R}G})$. For the latter case, we have $\underline{\sigma}(u, v) = \underline{\eta}(u, v) = 0$, and $\underline{(\overline{\sigma})}(u, v) = \underline{(\overline{\eta})}(u, v) = 0$ since $(u, v) \in B^{\mu}_{\overline{R}G}$ but $(u, v) \notin A^{\upsilon}_{\overline{R}G} \cup (C^{\upsilon}_{\overline{R}G} - \Omega_{\overline{R}G})$. Third, suppose that $(u, v) \in C^{\mu}_G \cup \Gamma_G$. Since $(u, v) \in C^{\mu}_G \cup \Gamma_G \subseteq B^{\mu}_{\overline{R}G} \cup C^{\mu}_{\overline{R}G} \cup \Gamma_{\overline{R}G}$ and $(u, v) \notin A^{\upsilon}_G \cup (C^{\upsilon}_G - \Omega_G) = A^{\upsilon}_{\overline{R}G} \cup (C^{\upsilon}_{\overline{R}G} - \Omega_{\overline{R}G})$, then $\underline{\sigma}(u, v) = \underline{\eta}(u, v) = 0$ and $\underline{(\overline{\sigma})}(u, v) = \underline{(\overline{\eta})}(u, v) = 0$. Finally, suppose that $\sigma(u, v) = CONN^{\mu}_G(u, v) = 0$ and $\eta(u, v) \geq 0$. It is easy to get that $\underline{\sigma}(u, v) = \underline{(\overline{\sigma})}(u, v) = 0$. Since $A^{\upsilon}_G \cup (C^{\upsilon}_G - \Omega_G) = A^{\upsilon}_{\overline{R}G} \cup (C^{\upsilon}_{\overline{R}G} - \Omega_{\overline{R}G})$, it holds that $\underline{\eta}(u, v) = \eta(u, v) = \overline{\eta}(u, v) = \underline{(\overline{\eta})}(u, v)$. To sum up, $\underline{R}(\overline{R}G) = \underline{R}G$. □

For any two R-IFR digraphs $G_1 = (\underline{R}G_1, \overline{R}G_1)$ and $G_2 = (\underline{R}G_2, \overline{R}G_2)$, and $B^{\upsilon}_{G_1} \cup \Omega_{G_1} \subseteq A^{\mu}_{G_1}$ and $B^{\upsilon}_{G_2} \cup \Omega_{G_2} \subseteq A^{\mu}_{G_2}$, if $\underline{R}G_1 = \underline{R}G_2$, then according to Proposition 8, we have $\overline{R}G_1 = \overline{R}(\underline{R}G_1) = \overline{R}(\underline{R}G_2) = \overline{R}G_2$. Similarly, if $A^{\mu}_{G_i} \cap \Omega_{G_i} = \emptyset$ and $(C^{\mu}_{G_i} \cup \Gamma_{G_i}) \cap [A^{\upsilon}_{G_i} \cup (C^{\upsilon}_{G_i} - \Omega_{G_i})] = \emptyset$ for $i = 1, 2$ and $\overline{R}G_1 = \overline{R}G_2$, we can know from Proposition 9 that $\underline{R}G_1 = \underline{R}(\overline{R}G_1) = \underline{R}(\overline{R}G_2) = \underline{R}G_2$.

Corollary 2. *Let $G_1 = (\underline{R}G_1, \overline{R}G_1)$ and $G_2 = (\underline{R}G_2, \overline{R}G_2)$ be two R-IFR digraphs. Then, the following arguments are valid:*

(1) *If $\underline{R}G_1 = \underline{R}G_2$ and $B^{\upsilon}_{G_i} \cup \Omega_{G_i} \subseteq A^{\mu}_{G_i}$ for $i = 1, 2$, then $\overline{R}G_1 = \overline{R}G_2$.*
(2) *If $\overline{R}G_1 = \overline{R}G_2$ and $A^{\mu}_{G_i} \cap \Omega_{G_i} = (C^{\mu}_{G_i} \cup \Gamma_{G_i}) \cap [A^{\upsilon}_{G_i} \cup (C^{\upsilon}_{G_i} - \Omega_{G_i})] = \emptyset$ for $i = 1, 2$, then $\underline{R}G_1 = \underline{R}G_2$.*

4.2. Rough characteristics of R-intuitionistic fuzzy rough digraphs

This subsection defines some rough characteristics of R-IFR digraphs, such as precision, similarity degree, extended similarity degree, R-IFR equality, and R-IFR isomorphism. A number of conclusions about these concepts are also drawn.

Definition 8. Let $G = (\underline{R}G, \overline{R}G)$ be an R-IFR digraph, where $\underline{R}G = (V, (\underline{\sigma}, \underline{\eta}))$ and $\overline{R}G = (V, (\overline{\sigma}, \overline{\eta}))$. The μ-precision of G with respect to R is defined as

$$\alpha^{\mu}_R(G) = \frac{\sum_{u,v \in V} \underline{\sigma}(u, v)}{\sum_{u,v \in V} \overline{\sigma}(u, v)}, \tag{1}$$

and the υ-precision of G with respect to R is defined as

$$\alpha^{\upsilon}_R(G) = \frac{\sum_{u,v \in V} \underline{\eta}(u, v) \wedge \sum_{u,v \in V} \overline{\eta}(u, v)}{\sum_{u,v \in V} \underline{\eta}(u, v) \vee \sum_{u,v \in V} \overline{\eta}(u, v)}. \tag{2}$$

The precision of G with respect to R is

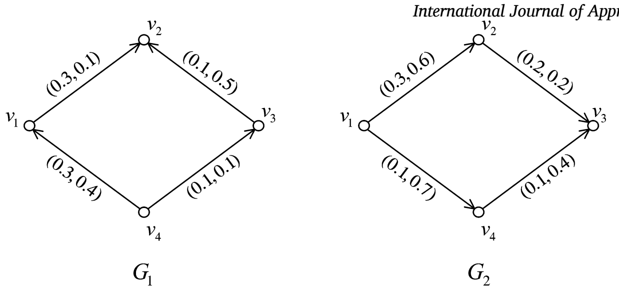

Fig. 4. IF digraphs G_1 and G_2.

$$\alpha_R(G) = \frac{\alpha_R^\mu(G) + \alpha_R^\upsilon(G)}{2}. \tag{3}$$

It can be seen that $0 \le \alpha_R^\mu(G) \le 1$ and $0 \le \alpha_R^\upsilon(G) \le 1$, so $0 \le \alpha_R(G) \le 1$. The smaller $\alpha_R(G)$ is, the rougher G is. In particular, G is R-exact if and only if $\alpha_R(G) = 1$. In addition, it is worth explaining that Equation (2) cannot be presented in a form similar to Equation (1). There is generally $\underline{\eta}(u,v) \ge \overline{\eta}(u,v)$ since $\underline{R}G \subseteq \overline{R}G$, but $\underline{\eta}(u,v) = 0 < \overline{\eta}(u,v)$ when $(u,v) \in \Omega_G - A_G^\mu$.

Definition 9. Let $G_1 = (\underline{R}G_1, \overline{R}G_1)$ and $G_2 = (\underline{R}G_2, \overline{R}G_2)$ be two R-IFR digraphs. If $\underline{R}G_1 = \underline{R}G_2$, then G_1 and G_2 are considered to be lower R-IFR equal, denoted by $G_1 \approx_R^l G_2$; if $\overline{R}G_1 = \overline{R}G_2$, then G_1 and G_2 are considered to be upper R-IFR equal, denoted by $G_1 \approx_R^u G_2$. If $G_1 \approx_R^l G_2$ and $G_1 \approx_R^u G_2$, then G_1 and G_2 are considered to be R-IFR equal, denoted by $G_1 \approx_R G_2$.

According to Corollary 2 (1), for any two R-IFR digraphs G_1 and G_2, and $B_{G_1}^\upsilon \cup \Omega_{G_1} \subseteq A_{G_1}^\mu$ and $B_{G_2}^\upsilon \cup \Omega_{G_2} \subseteq A_{G_2}^\mu$, we can conclude that if $G_1 \approx_R^l G_2$, then $G_1 \approx_R^u G_2$. Similarly, according to Corollary 2 (2), for any two R-IFR digraphs G_1 and G_2, when $A_{G_i}^\mu \cap \Omega_{G_i} = \emptyset$ and $(C_{G_i}^\mu \cup \Gamma_{G_i}) \cap [A_{G_i}^\upsilon \cup (C_{G_i}^\upsilon - \Omega_{G_i})] = \emptyset$ for $i = 1, 2$, it can be concluded that if $G_1 \approx_R^u G_2$, then $G_1 \approx_R^l G_2$.

It can be seen that the connectivity of two R-IFR equal IF digraphs is the same. Next, we consider the concept of similarity degree, which depicts R-IFR equality from the numerical point of view.

Definition 10. Let $G_1 = (\underline{R}G_1, \overline{R}G_1)$ and $G_2 = (\underline{R}G_2, \overline{R}G_2)$ be two R-IFR digraphs, where $\underline{R}G_i = (V_i, (\underline{\sigma_i}, \underline{\eta_i}))$ and $\overline{R}G_i = (V_i, (\overline{\sigma_i}, \overline{\eta_i}))$, $i = 1, 2$. Suppose that $\underline{E_i}$ and $\overline{E_i}$ are the arc sets of $\underline{R}G_i$ and $\overline{R}G_i$, respectively. The μ-similarity degree between G_1 and G_2 with respect to R is defined as

$$sim_R^\mu(G_1, G_2) = \frac{\sum_{(u,v)\in \underline{E_1}\cap \underline{E_2}} (\underline{\sigma_1} \cap \underline{\sigma_2})(u,v)}{\sum_{(u,v)\in \underline{E_1}\cup \underline{E_2}} (\underline{\sigma_1} \cup \underline{\sigma_2})(u,v)} \wedge \frac{\sum_{(u,v)\in \overline{E_1}\cap \overline{E_2}} (\overline{\sigma_1} \cap \overline{\sigma_2})(u,v)}{\sum_{(u,v)\in \overline{E_1}\cup \overline{E_2}} (\overline{\sigma_1} \cup \overline{\sigma_2})(u,v)}. \tag{4}$$

The υ-similarity degree between G_1 and G_2 with respect to R is defined as

$$sim_R^\upsilon(G_1, G_2) = \frac{\sum_{(u,v)\in \underline{E_1}\cap \underline{E_2}} (\underline{\eta_1} \cup \underline{\eta_2})(u,v)}{\sum_{(u,v)\in \underline{E_1}\cup \underline{E_2}} (\underline{\eta_1} \cap \underline{\eta_2})(u,v)} \wedge \frac{\sum_{(u,v)\in \overline{E_1}\cap \overline{E_2}} (\overline{\eta_1} \cup \overline{\eta_2})(u,v)}{\sum_{(u,v)\in \overline{E_1}\cup \overline{E_2}} (\overline{\eta_1} \cap \overline{\eta_2})(u,v)}. \tag{5}$$

The similarity degree between G_1 and G_2 with respect to R is

$$sim_R(G_1, G_2) = \frac{sim_R^\mu(G_1, G_2) + sim_R^\upsilon(G_1, G_2)}{2}. \tag{6}$$

Clearly, $0 \le sim_R^\mu(G_1, G_2) \le 1$ and $0 \le sim_R^\upsilon(G_1, G_2) \le 1$, so $0 \le sim_R(G_1, G_2) \le 1$. Besides, the larger $sim_R(G_1, G_2)$ is, the more the connectivity information of G_1 is the same as that of G_2.

Example 3. Consider two IF digraphs G_1 and G_2, as shown in Fig. 4. It is easy to see that $\underline{R}G_1 = G_1$ and $\underline{R}G_2 = G_2$. Additionally, $\overline{R}G_1 = G_1 \cup \{(v_4, v_2)\}$, and the membership value and non-membership value of arc (v_4, v_2) in IF digraph $\overline{R}G_1$ are 0.3 and 0, respectively; $\overline{R}G_2 = G_2 \cup \{(v_1, v_3)\}$, and the membership value and non-membership value of arc (v_1, v_3) in IF digraph $\overline{R}G_2$ are 0.2 and 0, respectively. Then, according to Definition 10, we have $sim_R^\mu(G_1, G_2) = \frac{0.4}{1.1} \wedge \frac{0.4}{1.6} = 0.25$, $sim_R^\upsilon(G_1, G_2) = \frac{0.2}{2.8} \wedge \frac{0.2}{2.8} = 0.071$, and $sim_R(G_1, G_2) = 0.161$.

Proposition 10. *Let G_1 and G_2 be two R-IFR digraphs with the same vertex set. Then, $G_1 \approx_R G_2$ if and only if $sim_R(G_1, G_2) = 1$.*

Proof. Let $G_i = (\underline{R}G_i, \overline{R}G_i)$, $\underline{R}G_i = (V, (\underline{\sigma_i}, \underline{\eta_i}))$, and $\overline{R}G_i = (V, (\overline{\sigma_i}, \overline{\eta_i}))$, where $i = 1, 2$. Assume that $G_1 \approx_R G_2$, i.e., $\underline{R}G_1 = \underline{R}G_2$ and $\overline{R}G_1 = \overline{R}G_2$. Then, it can be got that $sim^{\mu}_R(G_1, G_2) = sim^{\upsilon}_R(G_1, G_2) = 1$, and thus, $sim_R(G_1, G_2) = 1$. On the contrary, suppose that $sim_R(G_1, G_2) = 1$. Then, $sim^{\mu}_R(G_1, G_2) = sim^{\upsilon}_R(G_1, G_2) = 1$, and thus, we have $\underline{\sigma_1} = \underline{\sigma_2}$, $\underline{\eta_1} = \underline{\eta_2}$, $\overline{\sigma_1} = \overline{\sigma_2}$, and $\overline{\eta_1} = \overline{\eta_2}$. Besides, it is known that $\underline{R}G_1$ and $\underline{R}G_2$ have the same vertex set, and $\overline{R}G_1$ and $\overline{R}G_2$ have the same vertex set. Hence, $\underline{R}G_1 = \underline{R}G_2$ and $\overline{R}G_1 = \overline{R}G_2$, i.e., $G_1 \approx_R G_2$. $\square$

When it is not stated that G_1 and G_2 have the same vertex set, the conclusion that $G_1 \approx_R G_2$ cannot be drawn from $sim_R(G_1, G_2) = 1$. In fact, when there is an isolated vertex in one of these two IF digraphs, it is still possible to hold $sim_R(G_1, G_2) = 1$, but in this case there is obviously $G_1 \not\approx_R G_2$.

Definition 11. Let $G_1 = (\underline{R}G_1, \overline{R}G_1)$ and $G_2 = (\underline{R}G_2, \overline{R}G_2)$ be two R-IFR digraphs. If $\underline{R}G_1 \cong \underline{R}G_2$, then G_1 and G_2 are called lower R-IFR isomorphic, denoted by $G_1 \cong^l_R G_2$; if $\overline{R}G_1 \cong \overline{R}G_2$, then G_1 and G_2 are called upper R-IFR isomorphic, denoted by $G_1 \cong^u_R G_2$. If $G_1 \cong^l_R G_2$ and $G_1 \cong^u_R G_2$, then G_1 and G_2 are regarded as R-IFR isomorphic, denoted by $G_1 \cong_R G_2$.

Given two IF digraphs $G_1 = (V_1, (\sigma_1, \eta_1))$ and $G_2 = (V_2, (\sigma_2, \eta_2))$. Suppose that $G_1 \cong G_2$, i.e., there exists a bijection $f : V_1 \to V_2$ such that $\sigma_2(f(u), f(v)) = \sigma_1(u, v)$ and $\eta_2(f(u), f(v)) = \eta_1(u, v)$ for any $u, v \in V_1$. Then, according to the constructions of MEIFS and SEIFS, it is easy to know that $\underline{R}G_1 \cong \underline{R}G_2$ and $\overline{R}G_1 \cong \overline{R}G_2$, namely, $G_1 \cong_R G_2$.

Proposition 11. *Let $G_1 = (\underline{R}G_1, \overline{R}G_1)$ and $G_2 = (\underline{R}G_2, \overline{R}G_2)$ be two R-IFR digraphs. Then:*

(1) *If $G_1 \cong^l_R G_2$ and $B^{\upsilon}_{G_i} \cup \Omega_{G_i} \subseteq A^{\mu}_{G_i}$, where $i = 1, 2$, then $G_1 \cong^u_R G_2$.*
(2) *If $G_1 \cong^u_R G_2$ and $A^{\mu}_{G_i} \cap \Omega_{G_i} = (C^{\mu}_{G_i} \cup \Gamma_{G_i}) \cap [A^{\upsilon}_{G_i} \cup (C^{\upsilon}_{G_i} - \Omega_{G_i})] = \emptyset$, where $i = 1, 2$, then $G_1 \cong^l_R G_2$.*

Proof. (1) Since $\underline{R}G_1 \cong \underline{R}G_2$, it holds that $\overline{R}(\underline{R}G_1) \cong \overline{R}(\underline{R}G_2)$. Then, according to Proposition 8, we have $\overline{R}G_1 = \overline{R}(\underline{R}G_1) \cong \overline{R}(\underline{R}G_2) = \overline{R}G_2$, i.e., $G_1 \cong^u_R G_2$.

(2) In the same way, we have $\underline{R}(\overline{R}G_1) \cong \underline{R}(\overline{R}G_2)$ since $\overline{R}G_1 \cong \overline{R}G_2$. Further, according to Proposition 9, it holds that $\underline{R}G_1 = \underline{R}(\overline{R}G_1) \cong \underline{R}(\overline{R}G_2) = \underline{R}G_2$, i.e., $G_1 \cong^l_R G_2$. $\square$

Two R-IFR isomorphic IF digraphs have similar connectivity information. Next, we give the definition of extended similarity degree to depict R-IFR isomorphism from the numerical point of view.

Definition 12. Consider two IF digraphs $G_1 = (V_1, (\sigma_1, \eta_1))$ and $G_2 = (V_2, (\sigma_2, \eta_2))$, where $|V_1| \le |V_2|$. Let F be the set of injections from V_1 to V_2 and

$$K = \left\{ f : f = \arg \bigvee_{f \in F} \sum_{u, v \in V_1} [\sigma_1(u, v) \wedge \sigma_2(f(u), f(v)) + \eta_1(u, v) \wedge \eta_2(f(u), f(v))] \right\}.$$

Let $f(G_1) = (f(V_1), (\sigma_f, \eta_f))$, and for any u, $v \in V_1$ it holds that $\sigma_f(f(u), f(v)) = \sigma_1(u, v)$ and $\eta_f(f(u), f(v)) = \eta_1(u, v)$. The extended similarity degree between G_1 and G_2 with respect to R is defined as

$$esim_R(G_1, G_2) = \bigvee_{f \in K} sim_R(f(G_1), G_2). \tag{7}$$

Example 4. Consider Example 3. Through calculation, it can be known that $K = \{f\}$, where $f(v_1) = v_2$, $f(v_2) = v_3$, $f(v_3) = v_4$, and $f(v_4) = v_1$. According to Definition 10, it can be obtained that $sim^{\mu}_R(f(G_1), G_2) = \frac{0.7}{0.8} \wedge \frac{0.9}{1.1} = 0.818$, $sim^{\upsilon}_R(f(G_1), G_2) = \frac{1}{2} \wedge \frac{1}{2} = 0.5$, and $sim_R(f(G_1), G_2) = 0.659$. Thus, $esim_R(G_1, G_2) = sim_R(f(G_1), G_2) = 0.659$.

Proposition 12. *Let G_1 and G_2 be two R-IFR digraphs with the same number of vertices. Then, the following arguments hold:*

(1) *If $esim_R(G_1, G_2) = 1$, then $G_1 \cong_R G_2$.*
(2) *If $G_1 \cong^l_R G_2$ and $B^{\upsilon}_{G_i} \cup \Omega_{G_i} \subseteq A^{\mu}_{G_i}$ for $i = 1, 2$, then $esim_R(G_1, G_2) = 1$.*
(3) *If $G_1 \cong^u_R G_2$ and $A^{\mu}_{G_i} \cap \Omega_{G_i} = (C^{\mu}_{G_i} \cup \Gamma_{G_i}) \cap [A^{\upsilon}_{G_i} \cup (C^{\upsilon}_{G_i} - \Omega_{G_i})] = \emptyset$ for $i = 1, 2$, then $esim_R(G_1, G_2) = 1$.*

Proof. (1) Let $G_1 = (V_1, (\sigma_1, \eta_1))$, $G_2 = (V_2, (\sigma_2, \eta_2))$, and F be the set of injections from V_1 to V_2. According to Definition 12, there exists $f \in F$ such that $sim_R(f(G_1), G_2) = esim_R(G_1, G_2) = 1$, where $f(G_1) = (f(V_1), (\sigma_f, \eta_f))$ and for any u, $v \in V_1$, it holds that

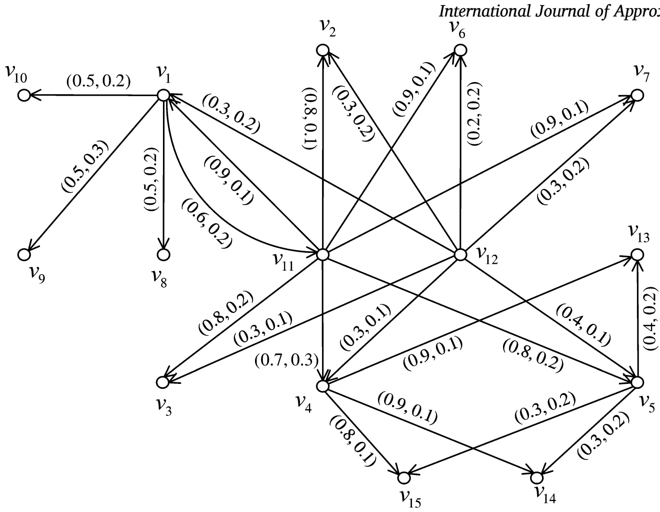

Fig. 5. An IF trust network $G = (V, (\sigma, \eta))$.

$\sigma_f(f(u), f(v)) = \sigma_1(u, v)$ and $\eta_f(f(u), f(v)) = \eta_1(u, v)$. Clearly, $G_1 \cong f(G_1)$, and thus, $\overline{R}G_1 \cong \overline{R}f(G_1)$ and $\underline{R}G_1 \cong \underline{R}f(G_1)$. Additionally, since $f(G_1)$ and G_2 have the same vertex set, according to Proposition 10 we have $f(G_1) \approx_R G_2$, i.e., $\overline{R}f(G_1) = \overline{R}G_2$ and $\underline{R}f(G_1) = \underline{R}G_2$. Therefore, $\overline{R}G_1 \cong \overline{R}G_2$ and $\underline{R}G_1 \cong \underline{R}G_2$, namely, $G_1 \cong_R G_2$.

(2) For any $f \in F$, let $f(G_1) = (f(V_1), (\sigma_f, \eta_f)) = (V_2, (\sigma_f, \eta_f))$, in which $\sigma_f(f(u), f(v)) = \sigma_1(u, v)$ and $\eta_f(f(u), f(v)) = \eta_1(u, v)$ for any $u, v \in V_1$. Then, it holds that $G_1 \cong f(G_1)$, and thus, $\underline{R}G_1 \cong \underline{R}f(G_1)$. Further, we have $\underline{R}f(G_1) \cong \underline{R}G_2$ since $G_1 \cong^l_R G_2$. It can be known that $\underline{R}f(G_1)$ and $\underline{R}G_2$ have the same vertex set, so there is always $f_0 \in F$ such that $\underline{R}f_0(G_1) = \underline{R}G_2$. Since $G_1 \cong f_0(G_1)$ and $B^{\upsilon}_{G_1} \cup \Omega_{G_1} \subseteq A^{\mu}_{G_1}$, we have $B^{\upsilon}_{f_0(G_1)} \cup \Omega_{f_0(G_1)} \subseteq A^{\mu}_{f_0(G_1)}$ and then $\overline{R}(\underline{R}f_0(G_1)) = \overline{R}f_0(G_1)$ according to Proposition 8. Based on this, we have $\overline{R}f_0(G_1) = \overline{R}(\underline{R}f_0(G_1)) = \overline{R}(\underline{R}G_2) = \overline{R}G_2$. So $f_0(G_1) \approx_R G_2$. Hence, we have $esim_R(G_1, G_2) = sim_R(f_0(G_1), G_2) = 1$.

(3) The proof is similar to that for (2). □

5. Application to trust networks

Trust constitutes a fundamental social relationship. A cohesive group with mutual trust forms a social trust network, which enables the dissemination of opinions and information. Identifying influential persons through trust network analysis can therefore enhance information propagation and policy communication. In large-scale networks, however, relying on a single key person may yield suboptimal results, necessitating the identification of multiple pivotal nodes. Existing studies [14,18,30,35] on community-based multi-node selection offer methodological insights for this purpose.

In reality, one person does not always have absolute trust in another person, and there may still be a certain degree of distrust or dubious attitude. Hence, it is necessary to simulate the trust network with an IF digraph. We consider an IF trust network $G = (V, (\sigma, \eta))$ composed of 15 people, as shown in Fig. 5, in which the vertices represent people, and the values of membership, non-membership, and hesitation of an arc represent the degree of trust, distrust, and doubt of one person to another, respectively. Moreover, trust and distrust will lead to positive and negative influence, respectively. As demonstrated in Fig. 5, the trust degree of v_{11} to v_1 is 0.9 and the distrust degree is 0.1, which suggests that v_{11} can accept 90% of the information (such as opinions and behaviors) conveyed by v_1, but will refuse to accept the remaining 10%. For the rejected part, v_{11} will make other possible decisions according to the environment. In this case, it can be considered that v_1 positively affects v_{11} to a greater extent and negatively affects v_{11} to a lesser extent. Additionally, the μ-strength and υ-strength of connectedness indicate the degree of indirect trust and indirect distrust of one person to another person through others, respectively. Overall, our key person selection criteria require: The positive influence on others is large enough, whether it is caused by direct trust or indirect trust; the negative influence on others is small enough, whether it is caused by direct distrust or indirect distrust.

According to the constructions of MEIFS and SEIFS, it can be known that $\underline{R}G$ implies positive influence caused by direct trust and negative influence caused by direct distrust, while $\overline{R}G - \underline{R}G$ implies positive influence caused by indirect trust and negative influence caused by indirect distrust. So it is more intuitive to deal with the trust network with the proposed IFR digraph method than with a single IF digraph method. Since $B^{\mu}_G = \{(v_{12}, v_2), (v_{12}, v_3), (v_{12}, v_4), (v_{12}, v_7)\}$, $C^{\mu}_G = \{(v_{12}, v_6)\}$, and $B^{\upsilon}_G = \{(v_{12}, v_2), (v_{12}, v_6), (v_{12}, v_7)\}$, we have $\underline{R}G = G - \{(v_{12}, v_2), (v_{12}, v_6), (v_{12}, v_7)\}$. According to Definition 6, $\overline{R}G$ can be obtained by completing two steps: (i) Increasing the membership value of arc (v_{12}, v_6) to 0.3, and (ii) adding some μ-arcs to IF digraph G and their membership values are shown in Table 2 and non-membership values are all zero.

The selected multiple core persons should be distributed in various locations of the network, otherwise, their influence may overlap. Community-based technique can effectively avoid overlapping influence. Therefore, we first carry out community detection

Table 2
Membership values of added μ-arcs.

(v_1,v_1)	(v_1,v_2)	(v_1,v_3)	(v_1,v_4)	(v_1,v_5)	(v_1,v_6)	(v_1,v_7)	(v_1,v_{13})
0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
(v_1,v_{14})	(v_1,v_{15})	(v_{11},v_8)	(v_{11},v_9)	(v_{11},v_{10})	(v_{11},v_{11})	(v_{11},v_{13})	(v_{11},v_{14})
0.6	0.6	0.5	0.5	0.5	0.6	0.7	0.7
(v_{11},v_{15})	(v_{12},v_8)	(v_{12},v_9)	(v_{12},v_{10})	(v_{12},v_{11})	(v_{12},v_{13})	(v_{12},v_{14})	(v_{12},v_{15})
0.7	0.3	0.3	0.3	0.3	0.4	0.3	0.3

on the R-IFR trust network in Section 5.1 and then select core persons from each community in Section 5.2. Finally, our method is compared with the other two methods in Section 5.3.

5.1. Community detection of R-IFR networks

Effective community detection techniques for crisp networks have been put forward in [30,34]. Based on these, we explore community detection in the R-IFR context, which is divided into two steps, i.e., the discovery of small communities and the merger of small communities.

In view of the discovery of small communities, we consider dividing the vertices with high enough similarity into the same small community. In an R-IFR network $(\underline{RG}, \overline{RG})$, where $\underline{RG} = (V, (\underline{\sigma}, \underline{\eta}))$ and $\overline{RG} = (V, (\overline{\sigma}, \overline{\eta}))$, we consider the following equation to depict the similarity between vertices:

$$s(v_i, v_j) = \frac{s^{\mu}(v_i, v_j) + s^{\upsilon}(v_i, v_j)}{2}, \tag{8}$$

where

$$s^{\mu}(v_i, v_j) = \frac{\sum_{v_k \in V} \underline{\sigma}(v_k, v_i) \wedge \underline{\sigma}(v_k, v_j)}{\sum_{v_k \in V} \underline{\sigma}(v_k, v_i) \vee \underline{\sigma}(v_k, v_j)} \wedge \frac{\sum_{v_k \in V} \overline{\sigma}(v_k, v_i) \wedge \overline{\sigma}(v_k, v_j)}{\sum_{v_k \in V} \overline{\sigma}(v_k, v_i) \vee \overline{\sigma}(v_k, v_j)}$$

and

$$s^{\upsilon}(v_i, v_j) = \frac{\sum_{v_k \in V} \underline{\eta}(v_k, v_i) \wedge \underline{\eta}(v_k, v_j)}{\sum_{v_k \in V} \underline{\eta}(v_k, v_i) \vee \underline{\eta}(v_k, v_j)} \wedge \frac{\sum_{v_k \in V} \overline{\eta}(v_k, v_i) \wedge \overline{\eta}(v_k, v_j)}{\sum_{v_k \in V} \overline{\eta}(v_k, v_i) \vee \overline{\eta}(v_k, v_j)}.$$

First, vertices are sorted in descending order according to the difference between μ-strength and υ-strength of connectedness. Then, the top vertex v is selected, and another vertex v' which is most similar to v and the similarity between v and v' is not less than the given parameter ε_1 is found. If v' has not been divided into any small communities, v and v' will be divided together to form a small community; otherwise, divide v into the small community where v' is located. However, if the similarity between v and v' is less than ε_1, then v will form a small community alone. Vertices that have been divided into small communities will be marked, and the above operations will be repeated for vertices that have not been marked. Until all vertices are marked, the discovery of small communities is completed. Specific operations are shown in Algorithm 1.

For the merger of small communities, we consider a merging index, as shown in the following:

$$M(SC_i, SC_j) = \frac{\sum_{u \in SC_i} \sum_{v \in SC_j} s(u, v)}{\varepsilon_1 * |SC_i| * |SC_j|}, \tag{9}$$

in which SC_i and SC_j are two small communities. First, the small community SC_k with the smallest size is found. Then, the merging index between SC_k and other small communities is computed and then the small community SC_t with the largest merging index with SC_k is determined. If the merging index between SC_k and SC_t is less than the given parameter ε_2, then no operation is performed on SC_k except marking it. However, if that is not less than ε_2, then SC_k and SC_t are merged to form a new small community and then they are deleted from the original set of small communities. For the update set of small communities, continue to repeat the above operations until the merging behavior can no longer be carried out. Details are shown in Algorithm 2.

Here we set $\varepsilon_1 = \varepsilon_2 = 0.7$. According to Algorithm 1, we can obtain the set of small communities, i.e., $SC = \{SC_1, SC_2, SC_3, SC_4, SC_5, SC_6\}$, where

$$SC_1 = \{v_{13}, v_{14}, v_{15}\}, \quad SC_2 = \{v_2, v_6, v_7\}, \quad SC_3 = \{v_3, v_4, v_5\},$$
$$SC_4 = \{v_1\}, \quad SC_5 = \{v_8, v_9, v_{10}, v_{11}\}, \quad SC_6 = \{v_{12}\}.$$

Then, according to Algorithm 2, the set of communities can be got by merging SC_3 and SC_4, i.e., $C = \{C_1, C_2, C_3, C_4, C_5\}$, where

Algorithm 1 A small community detection algorithm of R-IFR digraph.

Input: An R-IFR digraph $G = (\underline{R}G, \overline{R}G)$ with $V = \{v_1, v_2, \ldots, v_{|V|}\}$ as its vertex set and a parameter $\varepsilon_1 \in [0, 1]$
Output: The set SC of small communities

$SC = \emptyset$ and $Unmarked = V$
for $i = 1$ to $|V|$ **do**
  $h(v_i) = \sum_{v_s \in V-\{v_i\}} (CONN_G^{\mu}(v_s, v_i) - CONN_G^{\nu}(v_s, v_i))$
end for
while $Unmarked \neq \emptyset$ **do**
  $v = \arg \bigvee_{v_i \in Unmarked} h(v_i)$
  for $j = 1$ to $|V|$ **do**
    Calculate the similarity $s(v, v_j)$ between v and v_j according to Equation (8)
  end for
  $v' = \arg \bigvee_{v_j \neq v} s(v, v_j)$
  if $s(v, v') \geq \varepsilon_1$ **then**
    if $v' \in Unmarked$ **then**
      $SC = SC \cup \{\{v, v'\}\}$ and $Unmarked = Unmarked - \{v, v'\}$
    else
      Identify the small community to which v' belongs and record it as SC_k
      $SC = SC - \{SC_k\} \cup \{SC_k \cup \{v\}\}$
      $SC_k = SC_k \cup \{v\}$ and $Unmarked = Unmarked - \{v\}$
    end if
  else
    $SC = SC \cup \{\{v\}\}$ and $Unmarked = Unmarked - \{v\}$
  end if
end while
return SC

Algorithm 2 A merging small community algorithm.

Input: A set $SC = \{SC_1, SC_2, \ldots, SC_{|SC|}\}$ of small communities and a parameter $\varepsilon_2 \in [0, 1]$
Output: The set C of communities

$C = SC$ and $Unmerged = SC$
while $|Unmerged| > 1$ **do**
  $SC_k = \arg \bigwedge_{SC_i \in Unmerged} |SC_i|$
  for $SC_i \in Unmerged - \{SC_k\}$ **do**
    Compute the merging index $M(SC_k, SC_i)$ according to Equation (9)
  end for
  $SC_t = \arg \bigvee_{SC_i \in Unmerged - \{SC_k\}} M(SC_k, SC_i)$
  if $M(SC_k, SC_t) \geq \varepsilon_2$ **then**
    $C = C - \{SC_k, SC_t\} \cup \{SC_k \cup SC_t\}$
    $Unmerged = C$
  else
    $Unmerged = Unmerged - \{SC_k\}$
  end if
end while
return C

$$C_1 = \{v_{13}, v_{14}, v_{15}\}, \qquad C_2 = \{v_2, v_6, v_7\}, \qquad C_3 = \{v_1, v_3, v_4, v_5\},$$
$$C_4 = \{v_8, v_9, v_{10}, v_{11}\}, \qquad C_5 = \{v_{12}\}.$$

5.2. *The determination of multiple key persons*

This subsection discusses the method of finding k key persons. Specifically, communities are arranged in descending order according to their size and vertices in a community are ranked by considering both positive and negative influences, and then the top-$\frac{k*|C_i|}{|V|}$ core people are selected from each community in turn until k core people are selected. What needs to be explained here is that when $\frac{k*|C_i|}{|V|}$ is a decimal, we take its rounded value.

The ranking of vertices in a community is designed as follows. Let $\underline{N}_{\mu}^{r}(v_i)$ be the set of r-step μ-in-neighbors of v_i in $\underline{R}G$, i.e., $v_j \in \underline{N}_{\mu}^{r}(v_i)$ if and only if the length of the shortest μ-path from v_j to v_i in $\underline{R}G$ is r. Then, the positive influence of v_i can be characterized by the following equation:

$$I^{\mu}(v_i) = \sum_{v^1 \in \underline{N}_{\mu}^{1}(v_i)} \underline{\sigma}(v^1, v_i) + \sum_{r=1}^{l_\mu - 1} \sum_{v^r \in \underline{N}_{\mu}^{r}(v_i)} [CONN_G^{\mu}(v^r, v_i)]^r * \sum_{v^{r+1} \in \underline{N}_{\mu}^{r+1}(v_i)} \underline{\sigma}(v^{r+1}, v^r),$$

where l_μ represents the maximum length of shortest μ-paths from all other vertices to v_i in $\underline{R}G$. Let $\overline{N}^1_\mu(v_i)$ be the set of 1-step μ-in-neighbors of v_i in $\overline{R}G$. For any vertex v_j, it is obvious that $CONN^\mu_G(v_j, v_i) > 0$ if and only if $v_j \in \overline{N}^1_\mu(v_i)$. So $\bigcup_{r=2}^{l_\mu} \underline{N}^r_\mu(v_i) = \overline{N}^1_\mu(v_i) - \underline{N}^1_\mu(v_i)$. This indicates that $I^\mu(v_i)$ integrates the positive influences caused by direct trust and indirect trust. Similarly, the negative influence of v_i can be described as follows:

$$I^\upsilon(v_i) =$$
$$\sum_{v^1 \in \underline{N}^1_\upsilon(v_i)} \underline{\eta}(v^1, v_i) + \sum_{r=1}^{l_\upsilon - 1} \sum_{v^r \in \underline{N}^r_\upsilon(v_i)} [CONN^\upsilon_G(v^r, v_i)]^r * \sum_{v^{r+1} \in \underline{N}^{r+1}_\upsilon(v_i)} \underline{\eta}(v^{r+1}, v^r),$$

in which $\underline{N}^r_\upsilon(v_i)$ represents the set of r-step υ-in-neighbors of v_i in $\underline{R}G$, i.e., $v_j \in \underline{N}^r_\upsilon(v_i)$ if and only if the length of the shortest υ-path from v_j to v_i in $\underline{R}G$ is r, and l_υ represents the maximum length of shortest υ-paths from all other vertices to v_i in $\underline{R}G$. Let $\overline{N}^{r'}_\upsilon(v_i)$ be the set of r'-step υ-in-neighbors of v_i in $\overline{R}G$ and l'_υ be the maximum length of shortest υ-paths from all other vertices to v_i in $\overline{R}G$. Then, we have $\underline{N}^1_\upsilon(v_i) \subseteq \overline{N}^1_\upsilon(v_i)$, $\underline{N}^r_\upsilon(v_i) \supseteq \overline{N}^{r'}_\upsilon(v_i)$ for the case of r, $r' \geq 2$, and $\bigcup_{r=1}^{l_\upsilon} \underline{N}^r_\upsilon(v_i) = \bigcup_{r=1}^{l'_\upsilon} \overline{N}^{r'}_\upsilon(v_i)$. Therefore, $I^\upsilon(v_i)$ considers both the negative influence caused by direct distrust and the negative influence caused by indirect distrust.

As stated at the beginning of this section, the selected core persons should positively influence others as much as possible and negatively influence others as little as possible, so we consider the following measure to describe the comprehensive influence of v_i:

$$I(v_i) = I^\mu(v_i) + \frac{1}{\mathrm{e}^{I^\upsilon(v_i)}}, \tag{10}$$

where e is the natural constant. Obviously, the larger $I(v_i)$ is, the more likely v_i is to meet our requirements for key person.

In this case, three key persons will be determined. Since $|C_3| = |C_4| > |C_1| = |C_2| > |C_5|$, we first determine one core person in C_3. Through calculation, it can be got that $I(v_1) = 1.941$, $I(v_3) = 2.292$, $I(v_4) = 2.051$, and $I(v_5) = 2.392$. Hence, v_5 is the first core person. In the same way, core persons in C_4 and C_1 can be determined. By calculation, it can be obtained that

$$I(v_8) = 1.871, \quad I(v_9) = 1.778, \quad I(v_{10}) = 1.871, \quad I(v_{11}) = 2.091;$$
$$I(v_{13}) = 3.639, \quad I(v_{14}) = 3.419, \quad I(v_{15}) = 3.219.$$

Thus, v_{11} and v_{13} are the second and third key persons, respectively.

5.3. Comparisons and discussions

The presented method is compared with three other methods, namely, a method that does not pay attention to community detection, the classical IFR digraph method [39], and CBIM method [30].

First, we consider a method that ignores community detection. Through calculation, the comprehensive influence of vertices in communities C_2 and C_5 can be obtained, i.e., $I(v_2) = 2.167$, $I(v_6) = 2.327$, $I(v_7) = 2.327$, and $I(v_{12}) = 0$. Then, it holds that $I(v_{13}) > I(v_{14}) > I(v_{15}) > I(v_5) > I(v_6) = I(v_7) > I(v_3) > I(v_2) > I(v_{11}) > I(v_4) > I(v_1) > I(v_{10}) = I(v_8) > I(v_9) > I(v_{12})$. If the dispersion of key persons is not considered, the persons with great comprehensive influence will be identified as the key persons, and thus, v_{13}, v_{14}, and v_{15} will be selected. The comparison of the direct influence of core persons is considered here. It can be known from $\underline{R}G$ and $\overline{R}G$ that these three persons directly influence v_4 and v_5 and indirectly influence v_1, v_{11}, and v_{12}. However, the three core persons determined by the proposed approach, namely v_5, v_{11}, and v_{13}, can directly influence v_1, v_4, v_5, v_{11}, and v_{12}. In particular, as can be seen from Fig. 5, these five people, i.e., v_1, v_4, v_5, v_{11}, and v_{12}, are all people who can be influenced, which means that the three key persons determined by the presented method directly influence the people in the trust network to the greater extent, whether from the perspective of number or influence.

Next, we consider the classical IFR digraph method. It is constructed by giving an IF binary relation T' on the vertex set of G and an IF binary relation T'' on the arc set of G in advance. We set T' to

$$T'(v_i, v_j) = \begin{cases} (1, 0), & \text{if } i = j; \\ (0.5, 0.1), & \text{if } i \neq j, \end{cases}$$

and T'' to

$$T''((v_i, v_j), (v_m, v_n)) = \begin{cases} (1, 0), & \text{if } (v_i, v_j) = (v_m, v_n); \\ (0.2, 0.1), & \text{if } (v_i, v_j) \neq (v_m, v_n) \text{ and } \sigma(v_i, v_j) > 0.3; \\ (0, 1), & \text{if } (v_i, v_j) \neq (v_m, v_n) \text{ and } \sigma(v_i, v_j) \leq 0.3. \end{cases}$$

Then, the lower approximate IF digraph $\underline{T''}G$ can be obtained by decreasing the membership values of arcs

$$(v_1, v_8),\ (v_1, v_9),\ (v_1, v_{10}),\ (v_1, v_{11}),\ (v_4, v_{13}),\ (v_4, v_{14}),\ (v_4, v_{15}),\ (v_5, v_{13}),$$
$$(v_{11}, v_1), (v_{11}, v_2), (v_{11}, v_3),\ (v_{11}, v_4),\ (v_{11}, v_5),\ (v_{11}, v_6),\ (v_{11}, v_7),\ (v_{12}, v_5)$$

to 0.2 and increasing the non-membership values of arcs

$$(v_4, v_{13}), (v_4, v_{14}), (v_4, v_{15}), (v_{11}, v_1), (v_{11}, v_2), (v_{11}, v_6), (v_{11}, v_7), (v_{12}, v_5)$$

to 0.2; the upper approximate IF digraph $\overline{T''}G$ can be obtained by decreasing the non-membership values of arcs

$$(v_1, v_8), (v_1, v_9), (v_1, v_{10}), (v_1, v_{11}), (v_5, v_{13}), (v_{11}, v_3), (v_{11}, v_4), (v_{11}, v_5)$$

to 0.1. According to Algorithm 1, we have five small communities

$$SC_1 = \{v_{13}, v_{14}, v_{15}\}, \quad SC_2 = \{v_1, v_2, v_6, v_7\}, \quad SC_3 = \{v_3, v_4, v_5\},$$
$$SC_4 = \{v_8, v_9, v_{10}, v_{11}\}, \quad SC_5 = \{v_{12}\}.$$

By calculation, $M(SC_2, SC_3) = 1.063 > 0.7$, so according to Algorithm 2 we have four communities

$$C_1 = \{v_{13}, v_{14}, v_{15}\}, \quad C_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\},$$
$$C_3 = \{v_8, v_9, v_{10}, v_{11}\}, \quad C_4 = \{v_{12}\}.$$

Therefore, we need to identify a core person in C_1, C_2, and C_3, respectively. Calculations show that

$$I(v_1) = 1.170, \quad I(v_2) = 1.317, \quad I(v_3) = 1.372, \quad I(v_4) = 1.271,$$
$$I(v_5) = 1.204, \quad I(v_6) = 1.237, \quad I(v_7) = 1.337, \quad I(v_8) = 1.206,$$
$$I(v_9) = 1.107, \quad I(v_{10}) = 1.206, \quad I(v_{11}) = 1.167, \quad I(v_{12}) = 0,$$
$$I(v_{13}) = 1.698, \quad I(v_{14}) = 1.758, \quad I(v_{15}) = 1.708.$$

Thus, v_3, v_8, and v_{14} will be selected as the key persons.

The result obtained by the classical method is different from that obtained by the proposed method, i.e., v_5, v_{11}, and v_{13}. The comparison of the direct influence of key persons is still considered here. Obviously, v_5 is more trusted by v_{12} than v_3, while v_5 and v_3 are distrusted by v_{12} to the same extent; v_{11} is more trusted by v_1 than v_8, while v_{11} and v_8 are distrusted by v_1 to the same extent; v_{13} is more trusted by v_5 than v_{14}, while v_{13} and v_{14} are distrusted by v_5 to the same extent. Consequently, in the considered scenario, the proposed method performs better.

Finally, we consider CBIM method, a community-based influence maximization approach designed for L-layer ($L \geq 1$) crisp undirected networks. To adapt this method to our application scenario, we set $L = 1$ and examine the underlying undirected graph G^* of G. The small communities calculated by CBIM method are

$$SC_1 = \{v_8, v_9, v_{10}, v_{11}, v_{12}\}, \quad SC_2 = \{v_1, v_2, v_3, v_6, v_7\},$$
$$SC_3 = \{v_4, v_5\}, \quad SC_4 = \{v_{13}, v_{14}, v_{15}\}.$$

Further calculations reveal that merging SC_2 and SC_3 yields the final communities $C = \{C_1, C_2, C_3\}$, where $C_1 = SC_1$, $C_2 = SC_2 \cup SC_3$, and $C_3 = SC_4$.

CBIM method characterizes the influence of v_i through the edge weight sum EWS:

$$EWS(v_i) = \sum_{q=1}^{l} \sum_{t=1}^{|C_j|} \alpha^q d_t H_{it}^q, \tag{11}$$

where C_j denotes the community to which v_i belongs, d_t represents the degree of v_t ($v_t \in C_j$ and $v_t \neq v_i$), H_{it}^q indicates the number of paths of length q between v_i and v_t in graph G^*, l signifies the maximum length among all paths having v_i as one endpoint, and parameter α is set to the default value of 0.1 according to [30]. Next, one core person must be selected from each of these three communities. Calculations show that

$$EWS(v_1) = 0.384, \quad EWS(v_2) = 0.468, \quad EWS(v_3) = 0.468,$$
$$EWS(v_4) = 0.549, \quad EWS(v_5) = 0.549, \quad EWS(v_6) = 0.468,$$
$$EWS(v_7) = 0.468, \quad EWS(v_8) = 0.179, \quad EWS(v_9) = 0.179,$$
$$EWS(v_{10}) = 0.179, \quad EWS(v_{11}) = 0.522, \quad EWS(v_{12}) = 0.592,$$
$$EWS(v_{13}) = 0.082, \quad EWS(v_{14}) = 0.082, \quad EWS(v_{15}) = 0.082.$$

The three core persons selected based on the above results are v_4, v_{12}, and v_{13}. As shown in Fig. 5, v_{12} is not trusted by anyone, making their inclusion as a core person clearly unreasonable. Furthermore, the computational results indicate many persons exhibit identical importance scores, resulting in insufficient discriminative resolution. This prevents effective user ranking and consequently compromises the conclusions. Therefore, the CBIM method is not applicable for this case.

6. Conclusion and future work

The primary contributions are threefold. First, we defined MEIFS and SEIFS to maintain equivalent connectivity with a given IF digraph, and derived their construction methods by analyzing how changes in arc membership and non-membership degrees affect connectivity. Second, regarding MEIFS and SEIFS as the lower and upper approximations, respectively, IFR digraph based on strength of connectedness were proposed. A complete rough characteristic system was established as well. Lastly, our method was successfully applied to an IFR trust network. We developed an approach to community detection for the IFR trust network and then studied how to find multiple key persons based on these communities. Analysis of the results suggests that the presented theory is more effective than the classical theory.

The theoretical research of the paper will contribute to the effective analysis of data transmission, policy propaganda, etc. However, it remains limited by its homogeneous node assumption which fails to capture individual differences in social networks. In the future, we will embed node attribute information and develop a new IFR digraph theoretical framework under joint attribute-connectivity constraints by integrating a connectivity-driven IFR digraph with the classical IFR digraph. Besides, it is worth thinking about using IFR digraph method based on connectivity to identify influential groups in social networks.

CRediT authorship contribution statement

Danyang Wang: Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Ping Zhu:** Writing – review & editing, Supervision, Resources, Funding acquisition.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

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