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A two-player newsvendor game with competition on demand under ambiguity

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ABSTRACT

We deal with a single period two-player newsvendor game where both newsvendors are assumed to be rational and risk-neutral, and to operate under ambiguity. Each newsvendor needs to choose his/her order quantity of the same perishable product, whose global market demand is modeled by a discrete random variable, endowed with a reference probability measure. Furthermore, the global market demand is distributed to newsvendors according to a proportional allocation rule. We model the uncertainty faced by each newsvendor with an individual ϵ-contamination of the reference probability measure, computed with respect to a suitable class of probability measures. The resulting ϵ-contamination model preserves the expected demand under the reference probability and is used to compute the individual lower expected profit as a Choquet expectation. Therefore, the optimization problem of each player reduces to settle the order quantity that maximizes his/her lower expected profit, given the opponent choice, which is a maximin problem. In the resulting game, we prove that a Nash equilibrium always exists, though it may not be unique. Finally, we provide a characterization of Nash equilibria in terms of best response functions.

1. Introduction

The classical newsvendor model faces a decision problem where a market agent has to settle the quantity of a perishable product to order in a way to satisfy a random demand. Assuming that there is no initial inventory, the aim of the market agent is to maximize his/her expected profit by choosing a volume of inventory, that is also the total amount available for sale. The expected profit is negatively influenced by both underestimation and overestimation of the demand. The optimal order quantity of the classical one-period model can be represented as a fractile of the cumulative distribution function of the random demand [25]. Such optimal solution turns out to be unique for a continuous distribution on the demand, while uniqueness is not guaranteed for discrete distributions.

Competitive versions of the newsvendor model are largely studied in the literature by relying on the game theory framework, under the name of *newsvendor games*, where $N \geq 2$ players have to maximize their own expected profit function.

Newsvendor games can be classified depending on the type of competition players face. The seminal paper by Parlar [24] studies competition on inventory assuming that each of $N = 2$ players has to satisfy an independent random demand and each unsatisfied demand is reallocated to the other player. In a non-cooperative game of this kind, the author proves that there exists a unique Nash equilibrium.

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Differently, in [19] the authors consider a global random market demand that is allocated among $N \geq 2$ newsvendors based on a certain (random or mixed) splitting rule and each newsvendor's excess demand is reallocated to other newsvendors. It follows that competition is on the reallocation process since an increase in the order of a player aims to capture the excess demand and reduces the quantity available to order for others.

On the other hand, [3,29,32] propose a newsvendor game where the global random market demand is distributed among each player proportionally to their order quantities (namely, *proportional demand allocation rule*), therefore, setting a larger order increases the allocated demand and decreases the random demand of the others. This means that competition takes place on the level of demand. In turn, this issue is empirically evidenced in fashion item inventory [33] where the unit sales are proportional to the amount of inventory displayed (see, e.g., [30] for a literature review). We further highlight that classical newsvendor models work under *risk-neutrality* (i.e., taking a linear utility function), while [3,29,32] encode different behavioral features through suitable utility functions.

The underlying assumption of the classical one-player newsvendor model and the related games is that the demand (whether it is global or not) is endowed with a known probability measure. However, the assumption of a completely known probability measure can be limiting, especially when historical data are not available or products have variable demand over time. The quoted situation involving incomplete knowledge of a probability measure is referred to as *ambiguity* in decision theory (see, e.g., [14,10]).

The largely used approach to overcome this issue has been the use of a set of probability measures in which the problem is reduced to the selection of the quantity that maximizes the worst expected profit computed among the reference set of probability measures (see, e.g., [26]), implementing a *maximin* approach.

In [6], an ambiguous version of the one-player newsvendor problem is introduced, recurring to the use of an ϵ-contamination of a prior probability measure P_0 with respect to a suitable class of probability measures, constrained by their first moment. Working with the quoted class of ϵ-contaminated probabilities [18] translates in working with its lower envelope, which turns out to be a belief function in Dempster-Shafer theory [8,28].

In this paper we propose a single period two-player newsvendor game with competition on demand, where the uncertainty that characterizes the discrete random market demand is given by a non-additive measure defined, following [6], as the ϵ-contamination of a prior probability distribution, referring to a specific contaminating class. We notice that the reference to a discrete random demand is motivated by the fact that many goods can be bought and sold only in integral units in some markets.

More in details, we consider a market involving two newsvendors (modeled as players) who sell the same perishable product with same cost and revenue. We suppose a discrete random market demand that is split among the two players with a proportional allocation rule, i.e., the share of market demand allocated to each player depends on his/her share of ordering quantity. Before demand is realized, players have to settle their optimal order quantity so as to realize an equilibrium with respect to their lower expected profit.

Following [6] we assume that there is a prior probability distribution P_0 of the market demand, determining a corresponding expected value μ, but both newsvendors are not completely convinced about that. Thus, they replace P_0 with the class of probability measures obtained by ϵ-contaminating P_0 with a suitable class of probability measures, and work with the corresponding lower envelope, that is a belief function. Each newsvendor has his/her own parameter ϵ, that can be interpreted as a measure of ambiguity, i.e., a degree of confidence of each newsvendor in P_0: a larger ϵ means that the newsvendor is less convinced that P_0 is the "real" probability distribution of the market demand.

The class of probability measures used to contaminate P_0 is defined, in agreement with [6], as follows: starting with the class of probability measures with fixed expected value μ, denoted as $\mathcal{P}$, its lower envelope is a belief function even though the Choquet expectation of the market demand with respect to it is generally not equal to μ [7]. Thus, we define a new belief function ν^* that inner approximates the lower envelope with the constraint that the Choquet expectation with respect to it preserves μ. Finally, we ϵ-contaminate P_0 with the class of probability measures dominating ν^*, getting a new ϵ-contaminated class of probability measures whose lower envelope is a belief function, assuring that the Choquet expected demand is μ.

We assume that both newsvendors have a pessimistic attitude towards ambiguity. Thus, we work with the lower expected profit, defined as the Choquet expectation of the profit function with respect to the previously defined lower envelope. The optimization problem of each player reduces to settle the optimal order quantity that maximizes his/her own lower expected profit, i.e., it reduces to a maximin problem, given the opponent choice.

We prove the existence of a Nash equilibrium that, in general, due to the discrete demand assumption, may not be unique. Nevertheless, we provide a complete characterization of the set of all Nash equilibria by relying on a closed-form expression of players' best response functions. Finally, we show that the impact of the ambiguity parameter of each newsvendor, having the other's fixed, is not uniform since it depends on the market condition, given by the unit sale revenue and the unit purchase cost.

The paper is structured as follows. Section 2 recalls the preliminaries on the class of ϵ-contamination models with constrained first moment. Section 3 introduces the ϵ-contaminated two-player newsvendor game with proportional allocation rule. Section 4 defines the notion of Nash equilibrium and characterizes the set of all Nash equilibria. Finally, Section 5 draws our conclusions and future perspectives.

2. ϵ-contamination models with constrained first moment

We refer to a measurable space $(\Omega, \mathcal{F})$ where $\Omega = \{1, \dots, n\}$ is a finite non-empty set of states of the world with $\mathcal{F} = 2^\Omega$, and $\mathbb{R}^\Omega$ denotes the set of all random variables on Ω.

A *belief function* [8,28] is a mapping $\nu : \mathcal{F} \to [0, 1]$ satisfying:

(i) $\nu(\emptyset) = 0$ and $\nu(\Omega) = 1$;
(ii) for every $k \geq 2$ and every $A_1, \ldots, A_k \in \mathcal{F}$,

$$\nu\left(\bigcup_{i=1}^{k} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} A_i\right),$$

where *(ii)* is called *complete monotonicity*. If *(ii)* holds as an equality, ν is an *additive* belief function, i.e., it reduces to a probability measure and is customarily denoted by P. We denote by $\mathbb{B}(\Omega, \mathcal{F})$ the set of all belief functions on $\mathcal{F}$ and by $\mathbb{P}(\Omega, \mathcal{F})$ the subset of all probability measures on $\mathcal{F}$. As usual, every $P \in \mathbb{P}(\Omega, \mathcal{F})$ can be identified with the vector $P \equiv (P(\{1\}), \ldots, P(\{n\}))^T$.

Every belief function $\nu \in \mathbb{B}(\Omega, \mathcal{F})$ induces a non-empty, closed and convex set of probability measures, called *core*, defined as [15]

$$\mathbf{core}(\nu) = \{P \in \mathbb{P}(\Omega, \mathcal{F}) \,:\, P \geq \nu\},$$

such that ν is its lower envelope, that is, for all $A \in \mathcal{F}$,

$$\nu(A) = \min_{P \in \mathbf{core}(\nu)} P(A).$$

Every belief function ν is completely characterized by its *Möbius inverse* $m_\nu : \mathcal{F} \to [0,1]$ that satisfies:

(i) $m_\nu(\emptyset) = 0$;
(ii) $\sum_{A \in \mathcal{F}} m_\nu(A) = 1$;
(iii) for every $A \in \mathcal{F}$,

$$\nu(A) = \sum_{B \subseteq A} m_\nu(B).$$

For every random variable $X \in \mathbb{R}^\Omega$, we define the *Choquet expectation* of X with respect to ν through the Choquet integral [5,9]

$$\mathbb{C}_\nu[X] = \sum_{i=1}^{n} (X(\sigma(i)) - X(\sigma(i+1)))\nu(E_i^\sigma),$$

where σ is a permutation of Ω such that $X(\sigma(1)) \geq \cdots \geq X(\sigma(n))$, $E_i^\sigma = \{\sigma(1), \ldots, \sigma(i)\}$ for $i = 1, \ldots, n$, and $X(\sigma(n+1)) = 0$. The Choquet expectation operator $\mathbb{C}_\nu[\cdot]$ is completely monotone and, if ν is additive, it reduces to the classical expectation operator $\mathbb{E}_\nu[\cdot]$.

Belief functions are strictly connected with probability measures since the Choquet expectation with respect to ν is the *lower* expectation with respect to $\mathbf{core}(\nu)$ [27],

$$\mathbb{C}_\nu[X] = \min_{P \in \mathbf{core}(\nu)} \mathbb{E}_P[X].$$

Furthermore, the Choquet expectation with respect to ν can be equivalently expressed by relying on the Möbius inverse of ν, since

$$\mathbb{C}_\nu[X] = \sum_{B \in \mathcal{F} \setminus \{\emptyset\}} m_\nu(B) \left(\min_{i \in B} X(i)\right).$$

Given a prior probability measure P_0, a contamination parameter $\epsilon \in (0,1)$ and a contaminating belief function ν, ambiguity can be introduced through the *ϵ-contamination of P_0 with respect to ν*, which is the set of probability measures

$$\mathcal{P}_{P_0,\epsilon,\nu} = \{P' = (1-\epsilon)P_0 + \epsilon P'' \,:\, P'' \in \mathbf{core}(\nu)\}. \tag{1}$$

Denoting by ν_Ω the *vacuous* belief function whose Möbius inverse satisfies $m_{\nu_\Omega}(\Omega) = 1$ and is zero elsewhere, then $\mathcal{P}_{P_0,\epsilon,\nu_\Omega}$ is usually referred to as *linear-vacuous mixture model* in [31] or, simply, *ϵ-contamination model* in [18]. The belief function ν_Ω is called *vacuous* since it represents complete ignorance: it assigns full belief to the whole space Ω and 0 to all proper subsets. Therefore, the notion of ϵ-contamination in (1) generalizes what is normally encountered in the literature.

It is readily verified that the lower envelope of $\mathcal{P}_{P_0,\epsilon,\nu}$ defined, for all $A \in \mathcal{F}$, as

$$\nu_\epsilon(A) = \min_{P \in \mathcal{P}_{P_0,\epsilon,\nu}} P(A) = (1-\epsilon)P_0(A) + \epsilon\nu(A),$$

is a belief function. Equation (1) can be extended to the case $\epsilon = 0$, in which case $\nu_0 = P_0$, and to the case $\epsilon = 1$, in which case $\nu_1 = \nu$.

Let X be a discrete non-negative and non-degenerate random variable X taking values in $\mathcal{X} = \{x_1, \ldots, x_n\} \subseteq \mathbb{R}$ with $x_1 > \cdots > x_n$ and $X(i) = x_i$ for $i = 1, \ldots, n$. Given a reference probability measure P_0 with $\mathbb{E}_{P_0}[X] = \mu$, the class of probability measures on $(\Omega, \mathcal{F})$ with fixed expected value equal to μ is

$$\mathcal{P} = \{P \in \mathbb{P}(\Omega, \mathcal{F}) \,:\, \mathbb{E}_P[X] = \mu\}. \tag{2}$$

The lower envelope of $\mathcal{P}$, that is $\nu(A) = \min_{P \in \mathcal{P}} P(A)$, for every $A \in \mathcal{F}$, is a belief function with a closed-form Möbius inverse (see Lemma 1 and Theorem 1 in [7]), but it holds that $\mathcal{P} \subseteq \mathbf{core}(\nu)$ which generally implies

$$\mathbb{C}_\nu[X] \leq \min_{P \in \mathcal{P}} \mathbb{E}_P[X] = \mu.$$

For this reason, a new belief function ν^* is introduced in [6], whose Möbius inverse, given $x_s \leq \mu < x_{s-1}$ with $s \in \{2, \ldots, n\}$, is

$$m_{\nu^*}(A) = \begin{cases} \frac{\mu - x_n}{x_1 - x_n}, & \text{if } A = \{1\}, \\ \frac{x_{s-1} - \mu}{x_{s-1} - x_n}, & \text{if } A = \{s, \ldots, n\}, \\ \frac{x_{k-1} - \mu}{x_{k-1} - x_n} - \frac{x_k - \mu}{x_k - x_n}, & \text{if } A = \{k, \ldots, n\} \text{ and } \{s, \ldots, n\} \subset A \neq \Omega, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

The belief function ν^* satisfies the following properties:

(i) it is an inner approximation of ν, that is, $\nu^*(A) \geq \nu(A)$, for every $A \in \mathcal{F}$;
(ii) for every non-decreasing function $\varphi : \mathcal{X} \to \mathbb{R}$, setting $\alpha := \frac{\mu - x_n}{x_1 - x_n}$, it holds that

$$\mathbb{C}_{\nu^*}[\varphi(X)] = (1 - \alpha)\varphi(x_n) + \alpha\varphi(x_1). \tag{4}$$

In particular, it means that ν^* satisfies the expected value constraint, i.e., $\mathbb{C}_{\nu^*}[X] = \mathbb{E}_{P_0}[X] = \mu$;
(iii) $\mathbf{core}(\nu^*) \subseteq \mathbf{core}(\nu)$, and $\mathbf{core}(\nu^*) \cap \mathcal{P} = \{P_{1,n}\}$, where

$$P_{1,n} \equiv \left(\frac{\mu - x_n}{x_1 - x_n}, 0, \ldots, 0, \frac{x_1 - \mu}{x_1 - x_n} \right)^T.$$

Remark 1. In this paper we refer to the class $\mathcal{P}$ defined as in equation (2) since its lower envelope ν is a belief function, for which we can immediately derive the inner approximating belief function ν^* through equation (3). The quoted ν^* allows us to match the constraint on μ, moreover, lower expectations with respect to $\mathbf{core}(\nu^*)$ reduce to Choquet expectations: this last fact plays a crucial role in the game theoretic model we face in this paper.

On the other hand, for the considered class of probability measures, other possibilities are available. If no constraint is imposed, then we refer to the class $\mathcal{P}' = \mathbb{P}(\Omega, \mathcal{F})$ of all probability measures on $\mathcal{F}$, whose lower envelope reduces to the vacuous belief function ν_Ω, for which $\mathbb{C}_{\nu_\Omega}[X] = x_n$.

Another possibility already considered in the literature (see, e.g., [26]) is to impose, besides a constraint on the mean, also a constraint on the variance of X, by considering the set

$$\mathcal{P}'' = \{P \in \mathbb{P}(\Omega, \mathcal{F}) \,:\, \mathbb{E}_P[X] = \mu, \mathbb{E}_P[(X - \mu)^2] = \sigma^2\}.$$

The problem connected to such $\mathcal{P}''$ is that its lower envelope $\nu'' = \min \mathcal{P}''$ is generally not a belief function. To see this, it is sufficient to take X ranging in $\mathcal{X} = \{40, 30, 20, 10\}$, and fix $\mu = 25$ and $\sigma^2 = 125$. Under this mean and variance constraints, we have that $\mathcal{P}'' = \{\alpha P^1 + (1 - \alpha)P^2 \,:\, \alpha \in [0, 1]\}$, with $P^1 \equiv \left(\frac{1}{6}, \frac{3}{6}, 0, \frac{2}{6}\right)^T$ and $P^2 \equiv \left(\frac{2}{6}, 0, \frac{3}{6}, \frac{1}{6}\right)^T$. Finally, the lower envelope ν'' of $\mathcal{P}''$ is not a belief function since

$$\nu''(\{1, 2, 3\}) = \frac{4}{6} < \frac{2}{6} + \frac{3}{6} - 0 = \nu''(\{1, 2\}) + \nu''(\{2, 3\}) - \nu''(\{2\}).$$

The previous example actually shows that ν'' generally fails 2-monotonicity (see, e.g., [15]), therefore, we cannot recur to Choquet expectations to compute lower expectations with respect to $\mathcal{P}''$. Moreover, to derive a belief function inner approximating ν'' we generally have to solve an optimization problem in the line of [21].

3. ϵ-contaminated two-player newsvendor game with proportional allocation rule

We consider a market composed by two competing newsvendors that sell the same product at a unit sale revenue r, by ordering it from a supplier at a unit purchase cost c, with $r > c > 0$. We assume that c and r are the same for both newsvendors and the unit unsold inventory has a cost $\upsilon = 0$.

The two players in the newsvendor game operate on a single period of time and have to satisfy a global random market demand X which takes values in a finite non-empty set $\mathcal{X} = \{x_1, \ldots, x_n\} \subseteq \mathbb{R}$ with $x_1 > \cdots > x_n > 0$ and $n > 1$. Uncertainty on X is quantified by a reference probability measure P_0 such that $P_0(X = x_i) = P_0(\{i\}) = p_i > 0$, for $i = 1, \ldots, n$. The reference probability measure P_0 determines the expected value of the demand $\mathbb{E}_{P_0}[X] = \mu \in (x_n, x_1)$.

We note that the use of same r and c for both newsvendors is consistent with the assumption of a global random market demand X. In the rest of the paper, each player is denoted by j, where $j \in \{1, 2\}$, and by $-j$ we denote the player that is not j.

We assume that the global random market demand is split among the two players according to a *proportional allocation rule*, that is, the share of market demand allocated to player j is proportional to his/her order quantity,

$$X_j = \frac{q_j}{q_1 + q_2} X,$$

where $q_j \in [0, +\infty)$ is the order quantity of player j. This allocation rule determines that an increase in the ordered quantity of newsvendor j raises his/her allocated demand and reduces the share of market random demand of newsvendor $-j$.

It follows that the profit function of player j is a function of quantities q_j and q_{-j},

$$\Pi_j(q_j, q_{-j}) = r \min\left(\beta_j X, q_j\right) - c q_j,$$

where, for $j \in \{1,2\}$,

$$\beta_j := \frac{q_j}{q_j + q_{-j}} = \frac{q_j}{q_1 + q_2},$$

and it holds that $\beta_j + \beta_{-j} = 1$.

Though X is endowed with the reference probability measure P_0, each of the two newsvendors may not be completely convinced of P_0, i.e., he/she faces a situation of ambiguity.

Remark 2. In game theory, incomplete information about other players' payoff functions is classically modeled as a Bayesian game [17]. In this line, a Bayesian newsvendor game appears, e.g., in [16]. A limitation of the Bayesian game theoretic formulation is due to the fact that each player has to work with a complete probability distribution of an unknown parameter for the competitor (see, e.g., [22]). Here, we propose a different approach to face incomplete information by referring to players with ambiguous beliefs.

In order to preserve the information on the expected demand μ, each newsvendor may consider the class of probability measures with expected value set equal to μ, that is the class $\mathcal{P}$ in (2). Nevertheless, as stated in Section 2, the lower envelope of $\mathcal{P}$ is the belief function $\nu = \min_{P \in \mathcal{P}} P$, for which it generally holds $\mathbb{C}_\nu[X] \leq \mu$.

Therefore, in order to keep the information incorporated in P_0 and preserve the expected value μ, each player j considers an ϵ-contamination of P_0 with respect to the belief function ν^*, defined through the Möbius inverse m_{ν^*} in (3), that is, for a fixed $\epsilon_j \in [0,1]$, he/she refers to the class of probability measures

$$\mathcal{P}_{P_0,\epsilon_j,\nu^*} = \{P' = (1-\epsilon_j)P_0 + \epsilon_j P'' \;:\; P'' \in \mathbf{core}(\nu^*)\}.$$

The lower envelope

$$\nu^*_{\epsilon_j}(A) = \min_{P \in \mathcal{P}_{P_0,\epsilon_j,\nu^*}} P(A) = (1-\epsilon_j)P_0(A) + \epsilon_j \nu^*(A),$$

for every $A \in \mathcal{F}$, is a belief function and it holds that

$$\mathbb{C}_{\nu^*_{\epsilon_j}}[X] = (1-\epsilon_j)\mathbb{E}_{P_0}[X] + \epsilon_j \mathbb{C}_{\nu^*}[X] = \mu.$$

We assume that each player is risk-neutral and adopts a pessimistic approach towards ambiguity by considering the lower expected profit with respect to $\mathcal{P}_{P_0,\epsilon_j,\nu^*}$, that is, for $j \in \{1,2\}$,

$$\underline{\pi}_j(q_j, q_{-j}) = \mathbb{C}_{\nu^*_{\epsilon_j}}[\Pi_j(q_j, q_{-j})] = \mathbb{C}_{\nu^*_{\epsilon_j}}\left[r \min\left(\beta_j X, q_j\right) - c q_j\right]. \tag{5}$$

Given the opponent's choice $q_{-j} \in [0, +\infty)$, the problem faced by player j is a *maximin* problem since player j aims to choose the order quantity $q_j \in [0, +\infty)$ that maximizes his/her lower expected profit, that is

$$\max_{q_j \geq 0} \underline{\pi}_j(q_j, q_{-j}) = \max_{q_j \geq 0} \min_{P \in \mathcal{P}_{P_0,\epsilon_j,\nu^*}} \mathbb{E}_P[\Pi_j(q_j, q_{-j})]. \tag{6}$$

We point out that we assume no cooperation among players and that they are taken to be *rational*, that is, each player j would not decrease his/her objective function (5) only to damage the competitor.

The maximin problem in (6) plays a crucial role in order to find the Nash equilibria of the game, that are discussed in the next Section 4.

Due to the properties of ν^* (see (4) and [6]) and of the Choquet integral (see [5,15]), the lower expected profit (5) can be written as

$$\begin{aligned} \underline{\pi}_j(q_j, q_{-j}) = {} & r(1-\epsilon_j)\mathbb{E}_{P_0}\left[\min\left(\beta_j X, q_j\right)\right] \\ & + r\epsilon_j\left[(1-\alpha)\min\left(\beta_j x_n, q_j\right) + \alpha \min\left(\beta_j x_1, q_j\right)\right] - c q_j. \end{aligned} \tag{7}$$

Remark 3. We notice that $\underline{\pi}_j(q_j, q_{-j})$ is not defined if $q_j = q_{-j} = 0$, which corresponds to a pathological situation for the proportional demand allocation rule.

Therefore, we set $\underline{\pi}_j(0,0) := \lim_{q_j \to 0^+} \underline{\pi}_j(q_j,0) = 0$, with $j \in \{1,2\}$, that assures continuity in the variable q_j, when $q_{-j} = 0$.

In Proposition 1 we provide an explicit expression of $\underline{\pi}_j(q_j,q_{-j})$, for $j \in \{1,2\}$.

Proposition 1. *The lower expected profit function $\underline{\pi}_j(q_j,q_{-j})$, for $j \in \{1,2\}$, is defined as*

$$\underline{\pi}_j(q_j,q_{-j}) = \begin{cases} (r-c)q_j, & \text{if } q_j + q_{-j} \in [0,x_n), \\ r(1-\epsilon_j)\left[q_j \sum_{k=1}^{i-1} p_k + \beta_j \sum_{k=i}^{n} x_k p_k\right], & \text{if } q_j + q_{-j} \in [x_i,x_{i-1}) \\ \quad +r\epsilon_j\left[(1-\alpha)\beta_j x_n + \alpha q_j\right] - c q_j & \quad \text{with } i = 2,\dots,n, \\ \beta_j r\mu - c q_j, & \text{if } q_j + q_{-j} \in [x_1,+\infty). \end{cases} \tag{8}$$

Proof. It holds that

$$\min(\beta_j X, q_j) = \min\left(\frac{q_j}{q_j+q_{-j}}X, q_j\right) = \begin{cases} \frac{q_j}{q_j+q_{-j}}X, & \text{if } q_j+q_{-j} \geq X, \\ q_j, & \text{if } q_j + q_{-j} < X. \end{cases}$$

Therefore, it follows that (7) can be written as:

(i) if $q_j + q_{-j} \in [0,x_n)$, equation (7) reduces to

$$\underline{\pi}_j(q_j,q_{-j}) = (r-c)q_j;$$

(ii) if $q_j + q_{-j} \in [x_i,x_{i-1})$, for all $i = 2,\dots,n$, equation (7) reduces to

$$\begin{aligned}\underline{\pi}_j(q_j,q_{-j}) = r(1-\epsilon_j)&\left[q_j \sum_{k=1}^{i-1} p_k + \beta_j \sum_{k=i}^{n} x_k p_k\right] \\ &+ r\epsilon_j\left[(1-\alpha)\beta_j x_n + \alpha q_j\right] - c q_j;\end{aligned}$$

(iii) if $q_j + q_{-j} \in [x_1,+\infty)$, equation (7) reduces to

$$\underline{\pi}_j(q_j,q_{-j}) = \beta_j r\mu - c q_j. \quad \square$$

For a fixed $q_{-j} \in [0,+\infty)$, we introduce the following $n+1$ functions of $q_j \in [0,+\infty)$:

$$f_{n+1}(q_j) = (r-c)q_j, \tag{9}$$

$$\begin{aligned} f_i(q_j) = r(1-\epsilon_j)&\left[q_j \sum_{k=1}^{i-1} p_k + \beta_j \sum_{k=i}^{n} x_k p_k\right] \\ &+ r\epsilon_j\left[(1-\alpha)\beta_j x_n + \alpha q_j\right] - c q_j, \qquad \text{for } i = 2,\dots,n, \end{aligned} \tag{10}$$

$$f_1(q_j) = \beta_j r\mu - c q_j. \tag{11}$$

In Proposition 2 we characterize the lower expected profit of player j for any fixed $q_{-j} \in [0,+\infty)$, with $j \in \{1,2\}$, as a piecewise continuous function formed by individual functions of types (9)–(11), the latter being continuous and concave functions.

Proposition 2. *For a fixed $q_{-j} \in [0,+\infty)$, the lower expected profit of player j, with $j \in \{1,2\}$, is a piecewise continuous function of $q_j \in [0,+\infty)$ such that $\lim_{q_j \to +\infty} \underline{\pi}_j(q_j,q_{-j}) = -\infty$, defined as:*

(i) if $q_{-j} \in [0,x_n)$, then

$$\underline{\pi}_j(q_j,q_{-j}) = \begin{cases} f_{n+1}(q_j), & \text{for } q_j \in [0, x_n - q_{-j}), \\ f_i(q_j), & \text{for } q_j \in [x_i - q_{-j}, x_{i-1} - q_{-j}), \\ & \quad \text{with } i = 2,\dots,n \\ f_1(q_j), & \text{for } q_j \in [x_1 - q_{-j}, +\infty), \end{cases} \tag{12}$$

(ii) *if* $q_{-j} \in [x_i, x_{i-1})$, *with* $i = 2, \dots, n$, *then*

$$\underline{\pi}_j(q_j, q_{-j}) = \begin{cases} f_i(q_j), & \text{for } q_j \in [0, x_{i-1} - q_{-j}), \\ f_l(q_j), & \text{for } i > 2 \text{ and } q_j \in [x_l - q_{-j}, x_{l-1} - q_{-j}), \\ & \quad \text{with } l = 2, \dots, i-1, \\ f_1(q_j), & \text{for } q_j \in [x_1 - q_{-j}, +\infty), \end{cases} \tag{13}$$

(iii) *if* $q_{-j} \in [x_1, +\infty)$, *then*

$$\underline{\pi}_j(q_j, q_{-j}) = f_1(q_j), \qquad \text{for } q_j \in [0, +\infty), \tag{14}$$

where $f_h(q_j)$, *for* $h = 1, \dots, n+1$, *defined as in* (9)–(11), *are continuous and concave functions.*

Proof. We firstly characterize $\underline{\pi}_j(q_j, q_{-j})$, for fixed q_{-j}.

(i) If $q_{-j} \in [0, x_n)$, it holds that $q_j + q_{-j} \geq 0$. From (8), for $q_j + q_{-j} \in [0, x_n)$, that is, $q_j < x_n - q_{-j}$, the lower expected profit function of player j is

$$(r - c)q_j = f_{n+1}(q_j).$$

In turn, for $q_j + q_{-j} \geq x_n$, we have that $q_j + q_{-j}$ can be in any $[x_i, x_{i-1})$ with $i = 2, \dots, n$, that is $q_j \in [x_i - q_{-j}, x_{i-1} - q_{-j})$ with $i = 2, \dots, n$. From (8) the lower expected profit function of player j, for $q_j \in [x_i - q_{-j}, x_{i-1} - q_{-j})$, with $i = 2, \dots, n$, is

$$r(1 - \epsilon_j)\left[q_j \sum_{k=1}^{i-1} p_k + \beta_j \sum_{k=i}^{n} x_k p_k\right] + r\epsilon_j\left[(1 - \alpha)\beta_j x_n + \alpha q_j\right] - cq_j = f_i(q_j).$$

Finally, for $q_j + q_{-j} \geq x_1$, that is $q_j \in [x_1 - q_{-j}, +\infty)$, the lower expected profit function of player j is

$$\beta_j r\mu - cq_j = f_1(q_j).$$

(ii) If $q_{-j} \in [x_i, x_{i-1})$, with $i = 2, \dots, n$, it holds that $q_j + q_{-j}$ can be in any $[x_h, x_{h-1})$, with $h = 2, \dots, i$.
For $q_j \in [0, x_{i-1} - q_{-j})$, we have that $q_j + q_{-j} \in [x_i, x_{i-1})$. From (8) the lower expected profit of player j is

$$r(1 - \epsilon_j)\left[q_j \sum_{k=1}^{i-1} p_k + \beta_j \sum_{k=i}^{n} x_k p_k\right] + r\epsilon_j\left[(1 - \alpha)\beta_j x_n + \alpha q_j\right] - cq_j = f_i(q_j).$$

In turn, for $q_j \in [x_l - q_{-j}, x_{l-1} - q_{-j})$, with $l = 2, \dots, i-1$, we have that $q_j + q_{-j} \in [x_l, x_{l-1})$ and the lower expected profit of player j is

$$r(1 - \epsilon_j)\left[q_j \sum_{k=1}^{l-1} p_k + \beta_j \sum_{k=l}^{n} x_k p_k\right] + r\epsilon_j\left[(1 - \alpha)\beta_j x_n + \alpha q_j\right] - cq_j = f_l(q_j).$$

For $q_j + q_{-j} \geq x_1$, that is $q_j \in [x_1 - q_{-j}, +\infty)$, the lower expected profit function of player j is

$$\beta_j r\mu - cq_j = f_1(q_j).$$

We note that, if $i = 2$, the second branch of (13) is not defined since we have only the cases $q_j \in [0, x_1 - q_{-j})$ and $q_j \in [x_1 - q_{-j}, +\infty)$.

(iii) If $q_{-j} \geq x_1$, for any $q_j \geq 0$ it holds that $q_j + q_{-j} \in [x_1, +\infty)$. From (8) the lower expected profit function of player j, for any $q_j \geq 0$, is

$$\beta_j r\mu - cq_j = f_1(q_j).$$

It can be straightforwardly seen that the functions $f_h(q_j)$, with $h = 1, \dots, n+1$, are continuous and concave. To see this, we compute the following second derivatives, with $i = 2, \dots, n$,

$$\frac{\mathrm{d}^2}{\mathrm{d}q_j^2} f_1(q_j) = \frac{-2q_{-j}}{(q_j + q_{-j})^3} r\mu < 0,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}q_j^2} f_i(q_j) = \frac{-2q_{-j}}{(q_j + q_{-j})^3} \gamma < 0,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}q_j^2} f_{n+1}(q_j) = 0,$$

where we set

$$\gamma := r\left[(1-\epsilon_j)\sum_{k=i}^{n} x_k p_k + \epsilon_j(1-\alpha)x_n\right] > 0.$$

Hence, it follows that $f_h(q_j)$, for $h = 1,\dots,n+1$ are concave functions, being strictly concave for $h \neq n+1$ and $q_{-j} \in (0,+\infty)$. Finally, we have that

$$\lim_{q_j\to+\infty} \underline{\pi}_j(q_j,q_{-j}) = \lim_{q_j\to+\infty} f_1(q_j) = -\infty. \quad \square$$

We note that, if $q_{-j} = 0$, problem (6) reduces to

$$\max_{q_j\geq 0} r\left[(1-\epsilon_j)\mathbb{E}_{P_0}[\min(X,q_j)] + \epsilon_j[(1-\alpha)\min(x_n,q_j) + \alpha\min(x_1,q_j)]\right] - cq_j, \tag{15}$$

that is the one-player model introduced in [6], involving a continuous and concave piecewise linear function of q_j.

In Proposition 3 we prove that, for a fixed $q_{-j} \in [0,+\infty)$, the lower expected profit of player j is the lower envelope of functions $f_h(q_j)$, with $h = 1,\dots,n+1$, restricting to those that are correctly defined, according to q_{-j}.

Proposition 3. *For a fixed $q_{-j} \in [0,+\infty)$, it holds that, for any q_j,*

(a) if $q_{-j} \in [0,x_n)$, then

$$\underline{\pi}_j(q_j,q_{-j}) = \min\{f_h(q_j) \,:\, h = 1,\dots,n+1\}; \tag{16}$$

(b) if $q_{-j} \in [x_i,x_{i-1})$, with $i = 2,\dots,n$, then

$$\underline{\pi}_j(q_j,q_{-j}) = \min\{f_h(q_j) \,:\, h = 1,\dots,i\}; \tag{17}$$

(c) if $q_{-j} \in [x_1,+\infty)$, then

$$\underline{\pi}_j(q_j,q_{-j}) = f_1(q_j). \tag{18}$$

Proof. *(a)* If $q_{-j} \in [0,x_n)$, the lower expected profit function of player j is given by (12).

(1.) For $q_j \in [0,x_n-q_{-j})$, the lower expected profit function is $\underline{\pi}_j(q_j,q_{-j}) = f_{n+1}(q_j)$.

(*) We compare $f_{n+1}(q_j)$ with any $f_i(q_j)$ with $i = 2,\dots,n$. We have that $f_{n+1}(q_j) - f_i(q_j)$ reduces to

$$1-(1-\epsilon_j)\left[1+\sum_{k=i}^{n} p_k\left(\frac{x_k}{q_j+q_{-j}}-1\right)\right] - \epsilon_j\left[(1-\alpha)\frac{x_n}{q_j+q_{-j}}+\alpha\right].$$

Condition $q_j + q_{-j} \in [0,x_n)$ assures that

$$\frac{x_1}{q_j+q_{-j}} > \cdots > \frac{x_n}{q_j+q_{-j}} > 1,$$

which implies that

$$(1-\epsilon_j)\left[1+\sum_{k=i}^{n} p_k\left(\frac{x_k}{q_j+q_{-j}}-1\right)\right] + \epsilon_j\left[(1-\alpha)\frac{x_n}{q_j+q_{-j}}+\alpha\right] \geq 1.$$

Thus, we conclude that $f_{n+1}(q_j) \leq f_i(q_j)$, for every $i = 2,\dots,n$.

(**) We compare $f_{n+1}(q_j)$ with $f_1(q_j)$. We have that $f_{n+1}(q_j) - f_1(q_j)$ reduces to

$$rq_j\left(1-\frac{\mu}{q_j+q_{-j}}\right).$$

Again, conditions $q_j + q_{-j} \in [0,x_n)$ and $\mu \in (x_n,x_1)$ assure that $\dfrac{\mu}{q_j+q_{-j}} > 1$, so, it follows that

$$rq_j\left(1-\frac{\mu}{q_j+q_{-j}}\right) < 0.$$

Thus, we conclude that $f_{n+1}(q_j) < f_1(q_j)$.

(2.) For $q_j \in [x_i - q_{-j}, x_{i-1} - q_{-j})$, the lower expected profit function of player j is $\underline{\pi}_j(q_j,q_{-j}) = f_i(q_j)$, with $i = 2,\dots,n$.

(*) We compare $f_i(q_j)$ with $f_1(q_j)$. We have that $f_i(q_j) - f_1(q_j)$ is

$$(1-\epsilon_j)\sum_{k=1}^{i-1} p_k(q_j - \beta_j x_k) + \epsilon_j \alpha(q_j - \beta_j x_1).$$

Condition $q_j + q_{-j} \in [x_i, x_{i-1})$ assures that, for every $i = 1, \dots, n$,

$$\beta_j x_{i-1} > q_j \geq \beta_j x_i,$$

so, it follows that $(q_j - \beta_j x_k) < 0$ for every $k = 1, \dots, i-1$.
Thus, we conclude that $f_i(q_j) < f_1(q_j)$ for every $i = 2, \dots, n$.

(**) We compare $f_i(q_j)$ with any $f_h(q_j)$ with $h = 2, \dots, n$, $h \neq i$. We have that $f_i(q_j) - f_h(q_j)$ is

$$q_j\left[\sum_{k=1}^{i-1} p_k - \sum_{k=1}^{h-1} p_k\right] + \beta_j\left[\sum_{k=i}^{n} x_k p_k - \sum_{k=h}^{n} x_k p_k\right].$$

It follows that

$$f_i(q_j) - f_h(q_j) = \begin{cases} \displaystyle\sum_{k=h}^{i-1}(q_j p_k - \beta_j x_k p_k), & \text{if } i > h, \\ \displaystyle\sum_{k=i}^{h-1}(\beta_j x_k p_k - q_j p_k), & \text{if } i < h. \end{cases}$$

Conditions $q_j + q_{-j} \in [x_i, x_{i-1})$ and $i > h$ assure that

$$\beta_j x_1 > \cdots > \beta_j x_h > \cdots > \beta_j x_{i-1} > q_j \geq \beta_j x_i > \cdots > \beta_j x_n,$$

so, it follows that

$$(q_j p_k - \beta_j x_k p_k) < 0, \quad \text{for any } k = h, \dots, i-1.$$

In turn, conditions $q_j + q_{-j} \in [x_i, x_{i-1})$ and $i < h$ assure that

$$\beta_j x_1 > \cdots > \beta_j x_{i-1} > q_j \geq \beta_j x_i > \cdots > \beta_j x_h > \cdots > \beta_j x_n,$$

so, it follows that

$$(\beta_j x_k p_k - q_j p_k) < 0, \quad \text{for any } k = i, \dots, h-1.$$

Thus, we conclude that $f_i(q_j) < f_h(q_j)$, for any $h = 2, \dots, n$ and $h \neq i$.

(***) We compare $f_i(q_j)$ with $f_{n+1}(q_j)$. We have that $f_i(q_j) - f_{n+1}(q_j)$ reduces to

$$(1-\epsilon_j)\left[1 + \sum_{k=i}^{n} p_k\left(\frac{x_k}{q_j + q_{-j}} - 1\right)\right] + \epsilon_j\left[(1-\alpha)\frac{x_n}{q_j + q_{-j}} + \alpha \cdot 1\right] - 1.$$

Condition $q_j + q_{-j} \in [x_i, x_{i-1})$ assures that

$$\frac{x_1}{q_j + q_{-j}} > \cdots > \frac{x_{i-1}}{q_j + q_{-j}} > 1 \geq \frac{x_i}{q_j + q_{-j}} > \cdots > \frac{x_n}{q_j + q_{-j}},$$

so, it follows that, for any $k = i, \dots, n$,

$$\left(\frac{x_k}{q_j + q_{-j}} - 1\right) \leq 0.$$

This determines that

$$(1-\epsilon_j)\left[1 + \sum_{k=i}^{n} p_k\left(\frac{x_k}{q_j + q_{-j}} - 1\right)\right] + \epsilon_j\left[(1-\alpha)\frac{x_n}{q_j + q_{-j}} + \alpha \cdot 1\right] \leq 1.$$

Thus, we conclude that $f_i(q_j) \leq f_{n+1}(q_j)$.

(3.) For $q_j \in [x_1 - q_{-j}, +\infty)$, the lower expected profit function of player j is $\underline{\pi}_j(q_j, q_{-j}) = f_1(q_j)$.

(*) We compare $f_1(q_j)$ with any $f_i(q_j)$, with $i = 2, \dots, n$. We have that $f_1(q_j) - f_i(q_j)$ is

$$(1-\epsilon_j)\sum_{k=1}^{i-1} p_k(\beta_j x_k - q_j) + \epsilon_j \alpha(\beta_j x_1 - q_j).$$

Condition $q_j + q_{-j} \in [x_1, +\infty)$ assures that

$$q_j \geq \beta_j x_1 > \cdots > \beta_j x_n,$$

so, it follows that $(\beta x_k - q_j) < 0$ for every $k = 1, \dots, i-1$.
Thus, we conclude that $f_1(q_j) < f_i(q_j)$ for any $i = 2, \dots, n$.

(**) We compare $f_1(q_j)$ with $f_{n+1}(q_j)$. We have that $f_1(q_j) - f_{n+1}(q_j)$ reduces to

$$(1-\epsilon_j)\frac{\mu}{q_j + q_{-j}} + \epsilon_j \frac{\mu}{q_j + q_{-j}} - 1 = \frac{\mu}{q_j + q_{-j}} - 1.$$

Conditions $q_j + q_{-j} \in [x_1, +\infty)$ and $\mu \in (x_n, x_1)$ assure that $1 > \dfrac{\mu}{q_j + q_{-j}}$. Thus, we conclude that $f_1(q_j) \leq f_{n+1}(q_j)$.

(b) For $q_{-j} \in [x_i, x_{i-1})$, with $i = 2, \dots, n$, the lower expected profit function of player j is given by (13).

(1.) If $q_j \in [x_1 - q_{-j}, +\infty)$, that is $q_j + q_{-j} \geq x_1$, the lower expected profit function is $\underline{\pi}_j(q_j, q_{-j}) = f_1(q_j)$.

(*) We compare $f_1(q_j)$ with any $f_l(q_j)$ with $l = 2, \dots, i$. We have that $f_1(q_j) - f_l(q_j)$ is

$$(\beta_j x_1 - q_j)\left[(1-\epsilon_j)p_1 + \epsilon_j \alpha\right] + (1-\epsilon_j)\sum_{k=2}^{l-1} p_k(\beta_j x_k - q_j).$$

Condition $q_j + q_{-j} \geq x_1$ assures that, for any $k = 1, \dots, l-1$,

$$\beta_j x_k - q_j \leq 0.$$

Thus, it follows that $f_1(q_j) \leq f_l(q_j)$, for any $l = 2, \dots, i$.

(2.) If $q_j \in [x_l - q_{-j}, x_{l-1} - q_{-j})$, with $l = 2, \dots, i$, it holds that $q_j + q_{-j} \in [x_l, x_{l-1})$ and the lower expected profit function is $\underline{\pi}_j(q_j, q_{-j}) = f_l(q_j)$.

(*) We compare $f_l(q_j)$ with $f_1(q_j)$. We have that $f_l(q_j) - f_1(q_j)$ is

$$(1-\epsilon_j)\sum_{k=1}^{l-1} p_k\left(q_j + \beta_j x_k\right) + \epsilon_j \alpha (q_j - \beta_j x_1).$$

Condition $q_j + q_{-j} \in [x_l, x_{l-1})$ assures that, for any $l = 2, \dots, i$,

$$q_j - \beta_j x_1 < 0.$$

Thus, we conclude that $f_l(q_j) < f_1(q_j)$, for any $l = 2, \dots, i$.

(**) We compare $f_l(q_j)$ with any $f_h(q_j)$, with $h = 2, \dots, i$, $h \neq l$.
We have that $f_l(q_j) - f_h(q_j)$ is

$$q_j\left[\sum_{k=1}^{l-1} p_k - \sum_{k=1}^{h-1} p_k\right] + \beta_j\left[\sum_{k=l}^{n} x_k p_k - \sum_{k=h}^{n} x_k p_k\right].$$

It follows that

$$f_l(q_j) - f_h(q_j) = \begin{cases} \displaystyle\sum_{k=h}^{l-1}(q_j p_k - \beta_j x_k p_k), & \text{if } l > h, \\ \displaystyle\sum_{k=l}^{h-1}(\beta_j x_k p_k - q_j p_k), & \text{if } l < h. \end{cases}$$

The conditions $q_j + q_{-j} \in [x_l, x_{l-1})$ and $l > h$ assure that

$$(q_j - \beta_j x_k) < 0, \quad \text{for any } k = h, \dots, l-1.$$

In turn, the conditions $q_j + q_{-j} \in [x_l, x_{l-1})$ and $l < h$ assure that

$$(\beta_j x_k - q_j) < 0, \quad \text{for any } k = l, \dots, h-1.$$

Thus, we conclude that $f_l(q_j) < f_h(q_j)$, for any $h \in \{2, \dots, i\}$ and $h \neq l$.

(c) If $q_{-j} \in [x_1, +\infty)$, for any $q_j \geq 0$, the lower expected profit function of player j is (14). □

The following corollary investigates the concavity of the lower expected profit, seen as a function of q_j.

Corollary 1. *For a fixed $q_{-j} \in [0, +\infty)$, the lower expected profit of player j, where $j \in \{1, 2\}$, that is $\underline{\pi}_j(q_j, q_{-j})$ as defined in (12)–(14), is a concave function of q_j.*

Proof. From Proposition 3, the lower expected profit function of q_j for any fixed q_{-j} is the minimum of a class of concave functions of q_j, or it is equal to a concave function of q_j. This implies that $\underline{\pi}_j(q_j, q_{-j})$ is concave for any q_j (see [2]). $\square$

In Example 1 we show the lower expected profit function of q_j for an arbitrary choice of q_{-j}.

Example 1. Let $\mathcal{X} = \{x_1, x_2, x_3\}$ and take P_0 as reported below:

$\mathcal{X}$	100	50	30
P_0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

It follows that $\mu = 60$ and $\alpha = \dfrac{3}{7}$. We consider $r = 5$ and $c = 3$, and assume that both newsvendors are quite unconvinced of P_0, thus $\epsilon_1 = \epsilon_2 = 0.9$.

We compute, for a fixed set of q_2, the lower expected profit of player 1.

- For $q_2 = 20$, that is $q_2 \in [0, 30)$, we get

$$\underline{\pi}_1(q_1, 20) = \begin{cases} f_4(q_1) = 2q_1, & \text{if } q_1 \in [0, 10), \\ f_3(q_1) = -\frac{31}{42}q_1 + \frac{575}{7}\frac{q_1}{q_1+20}, & \text{if } q_1 \in [10, 30), \\ f_2(q_1) = -\frac{19}{21}q_1 + \frac{1900}{21}\frac{q_1}{q_1+20}, & \text{if } q_1 \in [30, 80), \\ f_1(q_1) = -3q_1 + 300\frac{q_1}{q_1+20}, & \text{if } q_1 \in [80, +\infty), \end{cases}$$

  whose graph is shown in Fig. 1a.
- For $q_2 = 35$, that is $q_2 \in [30, 50)$, we get

$$\underline{\pi}_1(q_1, 35) = \begin{cases} f_3(q_1) = -\frac{31}{42}q_1 + \frac{575}{7}\frac{q_1}{q_1+35}, & \text{if } q_1 \in [0, 15), \\ f_2(q_1) = -\frac{19}{21}q_1 + \frac{1900}{21}\frac{q_1}{q_1+35}, & \text{if } q_1 \in [15, 65), \\ f_1(q_1) = -3q_1 + 300\frac{q_1}{q_1+35}, & \text{if } q_1 \in [65, +\infty), \end{cases}$$

  whose graph is shown in Fig. 1b.
- For $q_2 = 80$, that is $q_2 \in [50, 100)$, we get

$$\underline{\pi}_1(q_1, 80) = \begin{cases} f_2(q_1) = -\frac{19}{21}q_1 + \frac{1900}{21}\frac{q_1}{q_1+80}, & \text{if } q_1 \in [0, 20), \\ f_1(q_1) = -3q_1 + 300\frac{q_1}{q_1+80}, & \text{if } q_1 \in [20, +\infty), \end{cases}$$

  whose graph is shown in Fig. 1c.
- For $q_2 = 105$, that is $q_2 \in [100, +\infty)$, we get

$$\underline{\pi}_1(q_1, 105) = f_1(q_1) = -3q_1 + 300\frac{q_1}{q_1 + 105}, \quad \text{if } q_1 \in [0, +\infty),$$

  whose graph is shown in Fig. 1d.

Finally, Corollary 2 shows that, for $q_{-j} \in (0, +\infty)$, the lower expected profit $\underline{\pi}_j(q_j, q_{-j})$ has a unique maximum point, seen as a function of q_j. We notice that the previous claim does not hold if we take $q_{-j} = 0$, since $\underline{\pi}_j(q_j, 0)$ reduces to (15) which can have infinitely many maximum points, as shown in Example 3 in [6].

Corollary 2. *For a fixed $q_{-j} \in (0, +\infty)$, the lower expected profit of player j, where $j \in \{1, 2\}$, that is $\underline{\pi}_j(q_j, q_{-j})$ as defined in (12)–(14), has a unique maximum point, seen as a function of q_j.*

Proof. The proof is immediate if $\underline{\pi}_j(q_j, q_{-j})$ can be expressed as in (17) or (18), since the pointwise minimum of a finite family of strictly concave functions is a strictly concave function. Hence, $\underline{\pi}_j(q_j, q_{-j})$ has a unique maximum point since $\lim_{q_j \to +\infty} \underline{\pi}_j(q_j, q_{-j}) = -\infty$.

On the other hand, if $\underline{\pi}_j(q_j, q_{-j})$ can be expressed as in (16), then it holds that

$$\underline{\pi}_j(q_j, q_{-j}) = \min\{f_{n+1}(q_j), g(q_j)\},$$

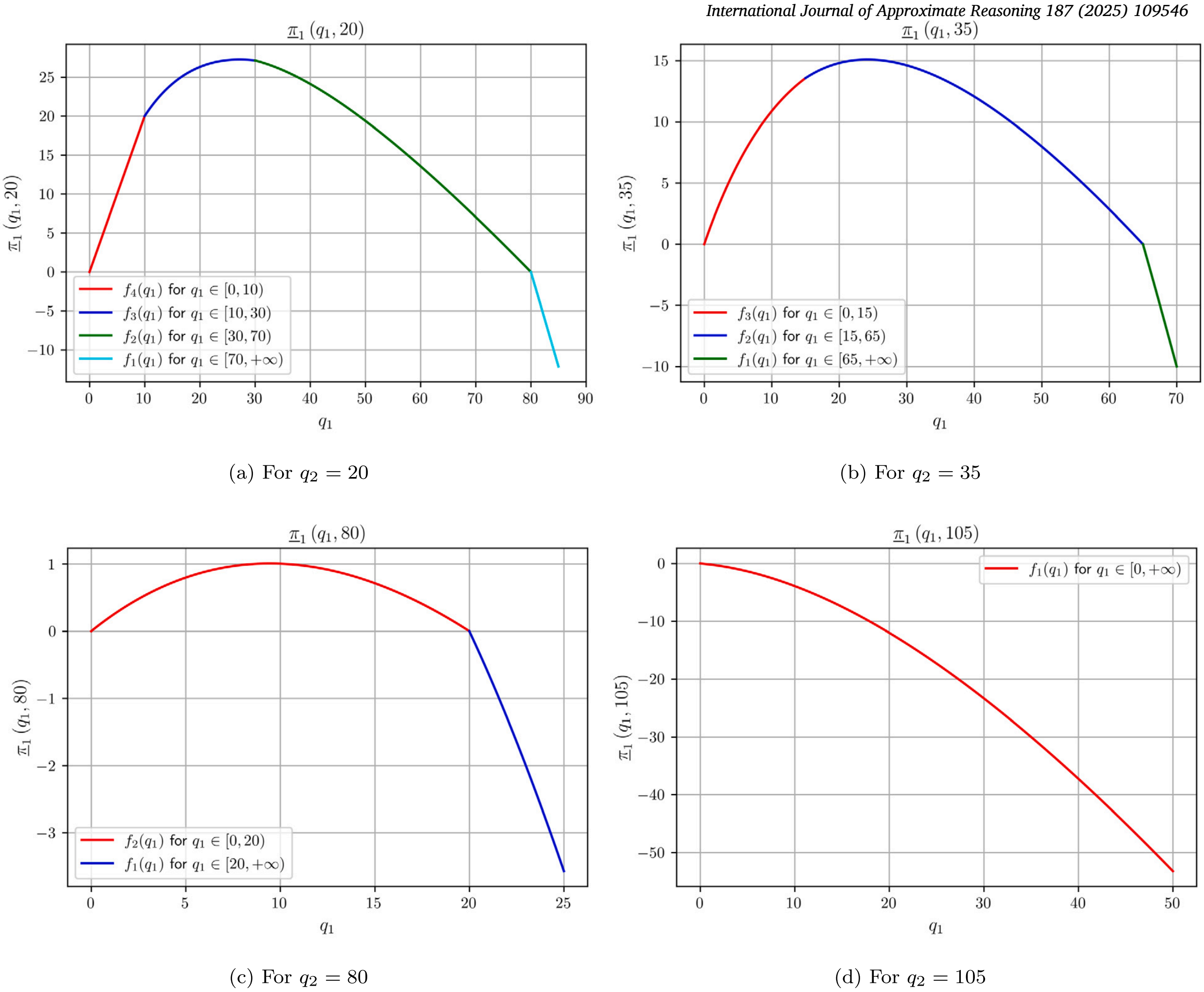


Fig. 1. Lower expected profit of player 1. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

where $f_{n+1}(q_j)$ is a strictly increasing and affine function, while $g(q_j) := \min\{f_h(q_j) : h = 1, \dots, n\}$ is a strictly concave function with $\lim_{q_j \to +\infty} g(q_j) = -\infty$. Therefore, $\underline{\pi}_j(q_j, q_{-j})$ has a unique maximum point either in the junction point between functions $f_{n+1}(q_j)$ and $g(q_j)$, or in the open interval $\{q_j \in [0, +\infty) : g(q_j) < f_{n+1}(q_j)\}$. $\square$

4. Nash equilibria

Following [12,23], we define a Nash equilibrium as a pair of quantities such that each newsvendor, given the opponent's quantity, does not deviate from it since he/she would get a smaller lower expected profit. In turn, the pairs of quantities among which the two newsvendors choose are referred to as *(pure) strategies* in game-theoretic jargon (see, e.g., [13]).

Definition 1. A pair $(q_1^*, q_2^*) \in [0, +\infty)^2$ is a **Nash equilibrium** for a two-player newsvendor game with lower expected profit functions as in (7) if and only if, for $j \in \{1, 2\}$, it holds that

$$\underline{\pi}_j(q_j^*, q_{-j}^*) \geq \underline{\pi}_j(q_j, q_{-j}^*) \quad \text{for all } q_j \in [0, +\infty).$$

Theorem 1 proves that there exists at least one Nash equilibrium.

Theorem 1. *Given a two-player newsvendor game with lower expected profit functions as in* (7)*, there exists at least one Nash equilibrium* $(q_1^*, q_2^*) \in [0, +\infty)^2$.

Proof. For $j \in \{1,2\}$ and a fixed $q_{-j} \in [0,+\infty)$, Proposition 2 implies that $\underline{\pi}_j(q_j,q_{-j}) = f_1(q_j)$, for $q_j \in [[x_1 - q_{-j}]^+,+\infty)$, where $[x]^+ := \max(x,0)$.

Suppose $q_{-j} = 0$. Since $f_1(q_j)$ is a strictly decreasing function with $\lim_{q_j \to +\infty} f_1(q_j) = -\infty$, then the maximum of $\underline{\pi}_j(q_j,0)$, seen as a function of q_j, is reached in the closed interval $[0,x_1]$.

On the other hand, suppose $q_{-j} \in (0,+\infty)$. Then, $f_1(q_j)$ attains its unique maximum at $q_j = \sqrt{\frac{r\mu}{c}q_{-j}} - q_{-j}$, which is an admissible quantity provided $q_{-j} \leq \frac{r\mu}{c}$. Moreover, $f_1(q_j)$ is strictly decreasing for $q_j \in \left[\sqrt{\frac{r\mu}{c}q_{-j}} - q_{-j}, 0\right)$ with $\lim_{q_j \to +\infty} f_1(q_j) = -\infty$. In turn, this implies that the maximum of $\underline{\pi}_j(q_j,q_{-j})$, seen as a function of q_j, is reached in the closed interval $\left[0, \max\left\{[x_1 - q_{-j}]^+, \sqrt{\frac{r\mu}{c}q_{-j}} - q_{-j}\right\}\right]$. Notice that the expression $\sqrt{\frac{r\mu}{c}q_{-j}} - q_{-j}$ is strictly concave as a function of q_{-j} and reaches its maximum value $\frac{r\mu}{4c}$ at $q_{-j} = \frac{r\mu}{4c}$.

Hence, the strategy space can be restricted to the compact and convex set $[0,M]\times[0,M]$, where $M := \max\left\{x_1, \frac{r\mu}{c}\right\}$, as candidate Nash equilibria cannot fall outside such a set. By Proposition 2 and Corollary 1, $\underline{\pi}_j(q_j,q_{-j})$ is continuous and concave in q_j, for $j \in \{1,2\}$. Therefore, by Theorem 1.2 in [12], there exists at least one Nash equilibrium. $\square$

In Example 2 we show that a Nash equilibrium is generally not unique.

Example 2. Let $\mathcal{X} = \{x_1,\dots,x_5\}$ and P_0 be as reported below:

$\mathcal{X}$	100	80	30	20	5
P_0	$\frac{2}{10}$	$\frac{4}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

It follows that $\mu = 60.5$ and $\alpha = \frac{111}{190}$. We consider $r = 10, c = 6$ and $\epsilon_1 = \epsilon_2 = 0.9$. The lower expected profit function of each player $j \in \{1,2\}$ is

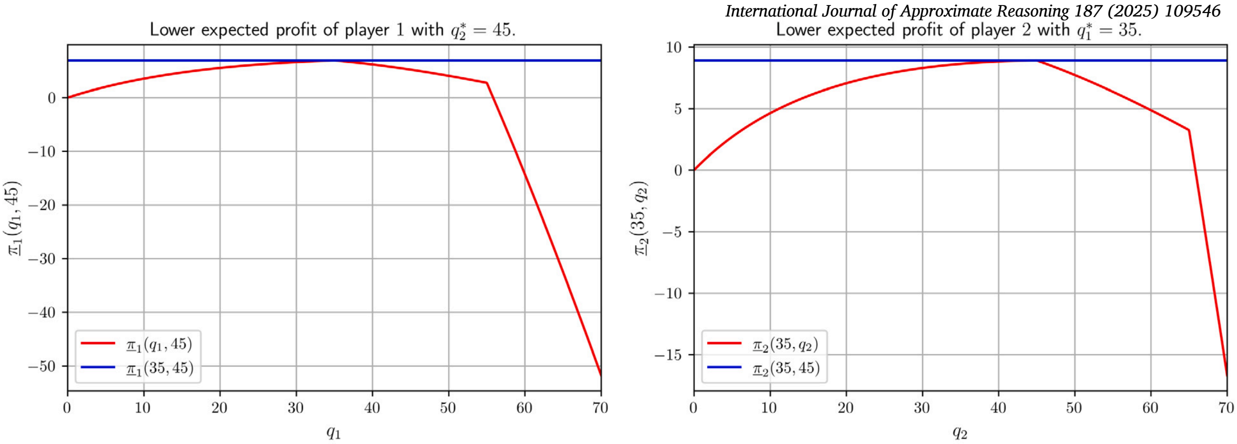
$$\underline{\pi}_j(q_j,q_{-j}) = \begin{cases} 4q_j, & \text{if } q_j + q_{-j} \in [0,5), \\ \frac{3}{19}q_j + \frac{365}{19}\beta_j, & \text{if } q_j + q_{-j} \in [5,20), \\ \frac{11}{190}q_j + \frac{403}{19}\beta_j, & \text{if } q_j + q_{-j} \in [20,30), \\ -\frac{27}{190}q_j + \frac{517}{19}\beta_j, & \text{if } q_j + q_{-j} \in [30,80), \\ -\frac{103}{190}q_j + \frac{1125}{19}\beta_j, & \text{if } q_j + q_{-j} \in [80,100), \\ -6q_j + 605\beta_j, & \text{if } q_j + q_{-j} \in [100,+\infty). \end{cases}$$

Let us consider $q_1^* = 35$ and $q_2^* = 45$, for which it holds that

$$\underline{\pi}_1(q_1^*,q_2^*) = \frac{2107}{304} \quad \text{and} \quad \underline{\pi}_2(q_1^*,q_2^*) = \frac{2709}{304}.$$

Given $q_2^* = 45$, we have that

$$\underline{\pi}_1(q_1,45) = \begin{cases} -\frac{27}{190}q_1 + \frac{517}{19}\frac{q_1}{q_1+45}, & \text{for } q_1 \in [0,35), \\ -\frac{103}{190}q_1 + \frac{1125}{19}\frac{q_1}{q_1+45}, & \text{for } q_1 \in [35,55), \\ -6q_1 + 605\frac{q_1}{q_1+45}, & \text{for } q_1 \in [55,+\infty). \end{cases}$$



(a) $\underline{\pi}_1(q_1, q_2^*)$ (b) $\underline{\pi}_2(q_1^*, q_2)$

Fig. 2. Lower expected profit of player j with fixed q_{-j}^*, for $j \in \{1,2\}$.

As Fig. 2a shows, it holds that

$$\underline{\pi}_1(q_1^*, q_2^*) \geq \underline{\pi}_1(q_1, q_2^*), \text{ for every } q_1 \in [0, +\infty).$$

In turn, given $q_1^* = 35$, we have that

$$\underline{\pi}_2(35, q_2) = \begin{cases} -\frac{27}{190} q_2 + \frac{517}{19} \frac{q_2}{35+q_2}, & \text{for } q_2 \in [0, 45), \\ -\frac{103}{190} q_2 + \frac{1125}{19} \frac{q_2}{35+q_2}, & \text{for } q_2 \in [45, 65), \\ -6q_2 + 605 \frac{q_2}{35+q_2}, & \text{for } q_2 \in [65, +\infty). \end{cases}$$

As Fig. 2b shows, it holds that

$$\underline{\pi}_2(q_1^*, q_2^*) \geq \underline{\pi}_2(q_1^*, q_2), \text{ for every } q_2 \in [0, +\infty).$$

Thus, $q_1^* = 35$, $q_2^* = 45$ is a Nash equilibrium.
Let us consider $q_1^{**} = 40$ and $q_2^{**} = 40$ for which it holds that

$$\underline{\pi}_1(q_1^{**}, q_2^{**}) = \underline{\pi}_2(q_1^{**}, q_2^{**}) = \frac{301}{38}.$$

For any $j \in \{1,2\}$ and $q_{-j}^{**} = 40$ we have that

$$\underline{\pi}_j(40, q_j^{**}) = \begin{cases} -\frac{27}{190} q_j + \frac{517}{19} \frac{q_j}{q_j+40}, & \text{for } q_j \in [0, 40), \\ -\frac{103}{190} q_j + \frac{1125}{19} \frac{q_j}{q_j+40}, & \text{for } q_j \in [40, 60), \\ -6q_j + 605 \frac{q_j}{q_j+40}, & \text{for } q_j \in [60, +\infty). \end{cases}$$

As Fig. 3 shows, it holds that

$$\underline{\pi}_j(q_1^{**}, q_2^{**}) \geq \underline{\pi}_j(q_j, q_{-j}^{**}), \text{ for every } q_j \in [0, +\infty).$$

Thus, $q_1^{**} = q_2^{**} = 40$ is a Nash equilibrium.

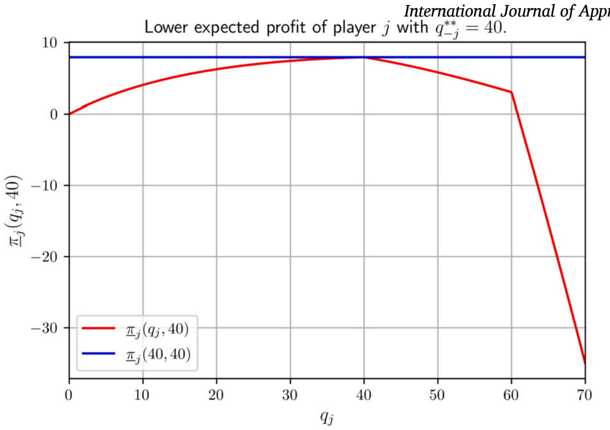


Fig. 3. Lower expected profit of player j with fixed $q^{**}_{-j} = 40$, for $j \in \{1,2\}$.

Though the two-player newsvendor game has generally not a unique equilibrium, here we provide a procedure to explicitly determine all Nash equilibria.

We define the *best response function* of player j to the strategy q_{-j}, for $j \in \{1,2\}$, i.e., the strategy of player j that maximizes his/her lower expected profit for a fixed q_{-j}, as

$$q_j^*(q_{-j}) = \arg\max_{q_j \geq 0} \underline{\pi}_j(q_j, q_{-j}).$$

Remark 4. As a consequence of Corollary 2, $q_j^*(q_{-j})$ is indeed a single-valued function for all $q_{-j} \in (0, +\infty)$ but it could be many-valued for $q_{-j} = 0$ (see [6]). In what follows, we specify a selection criterion in order to get a single-valued function on the whole $[0, +\infty)$ (see next Remark 5).

In the literature (see, e.g., [4]), assuming that $\underline{\pi}_j(q_j, q_{-j})$ is continuously differentiable and concave as a function of q_j, the best response function is uniquely defined by the first order condition on $\underline{\pi}_j(q_j, q_{-j})$,

$$I_j(q_j, q_{-j}) := \frac{\partial}{\partial q_j} \underline{\pi}_j(q_j, q_{-j}) = 0.$$

This means that a Nash equilibrium (q_j^*, q_{-j}^*) is such that q_j^* is a best response to q_{-j}^*, for $j \in \{1,2\}$, and that Nash equilibria can be characterized by solving the first order conditions, i.e., the intersection of the best response functions,

$$\begin{cases} I_1(q_1, q_2) = 0, \\ I_2(q_1, q_2) = 0. \end{cases}$$

In our two-player newsvendor game, since the lower expected profit functions $\underline{\pi}_j(q_j, q_{-j})$, for $j \in \{1,2\}$, are piecewise continuous functions, the first order conditions cannot be applied as differentiability fails. Nevertheless, we can provide an explicit expression of the best response function $q_j^*(q_{-j})$ that uses the first order condition $I_j(q_j, q_{-j}) = 0$, whenever it is meaningful.

As a preliminary to the next proposition, we define the following quantities: for $j \in \{1,2\}$, $B_{1,\epsilon_j} := 0$, $D_{1,\epsilon_j} := \frac{c}{r}$, $N_{1,\epsilon_j} := \mu$, while for $i = 2, \dots, n$,

$$B_{i,\epsilon_j} := \epsilon_j \alpha + (1 - \epsilon_j) \sum_{k=1}^{i-1} p_k, \tag{19}$$

$$D_{i,\epsilon_j} := \frac{c}{r} - B_{i,\epsilon_j}, \tag{20}$$

$$N_{i,\epsilon_j} := (1 - \epsilon_j) \sum_{k=i}^{n} x_k p_k + \epsilon_j (1 - \alpha) x_n. \tag{21}$$

Proposition 4. *For the best response function $q_j^*(q_{-j})$, the following statements hold:*

(i) $q_j^*(q_{-j})$ is a continuous function of $q_{-j} \in (0, +\infty)$;

(ii) *For all $q_{-j} \in (0, +\infty)$, let $\bar{i}$ be the maximum index $i \in \{1, \dots, n\}$ such that $B_{i,\epsilon_j} < \frac{c}{r}$ and denote $[x]^+ := \max(x, 0)$, then*

$$
q_j^*(q_{-j}) = \begin{cases}
x_{\bar{i}} - q_{-j}, & \text{if } q_{-j} \in \left(0, \frac{D_{\bar{i},\epsilon_j}}{N_{\bar{i},\epsilon_j}} x_{\bar{i}}^2\right], \\
\left[\sqrt{\frac{N_{i,\epsilon_j}}{D_{i,\epsilon_j}} q_{-j}} - q_{-j}\right]^+, & \text{if } q_{-j} \in \left(\frac{D_{i,\epsilon_j}}{N_{i,\epsilon_j}} x_i^2, \frac{D_{i,\epsilon_j}}{N_{i,\epsilon_j}} x_{i-1}^2\right), \\
& \quad \text{with } \bar{i} \geq 2 \text{ and } i = 2, \dots, \bar{i}, \\
& \text{or } q_{-j} \in \left(\frac{D_{i,\epsilon_j}}{N_{i,\epsilon_j}} x_i^2, \frac{r\mu}{c}\right], \text{ with } i = 1, \\
\left[x_{i-1} - q_{-j}\right]^+, & \text{if } q_{-j} \in \left[\frac{D_{i,\epsilon_j}}{N_{i,\epsilon_j}} x_{i-1}^2, \frac{D_{i-1,\epsilon_j}}{N_{i-1,\epsilon_j}} x_{i-1}^2\right], \\
& \quad \text{with } \bar{i} \geq 2 \text{ and } i = 2, \dots, \bar{i}, \\
0, & \text{if } q_{-j} \in \left(\frac{r\mu}{c}, +\infty\right).
\end{cases}
$$

Proof. Statement *(i)*. By Corollary 2, for $q_{-j} \in (0, +\infty)$, it holds that

$$
q_j^*(q_{-j}) = \arg\max_{q_j \geq 0} \underline{\pi}_j(q_j, q_{-j})
$$

is a single-valued function. Moreover, the lower expected profit $\underline{\pi}_j(q_j, q_{-j})$ is continuous as a function of two variables. Indeed, referring to Proposition 1, the single branches are continuous functions of two variables and continuity can be verified taking two-variable limits to points belonging to the lines $q_j + q_{-j} = x_i$, for $i = 1, \dots, n$. Hence, by the Berge's maximum theorem (see, e.g., [1]), it holds that $q_j^*(q_{-j})$ is a continuous function of $q_{-j} \in (0, +\infty)$.

Statement *(ii)*. It holds that

$$
I_j(q_j, q_{-j}) = \begin{cases}
r - c, & \text{if } q_j + q_{-j} \in (0, x_n), \\
\frac{q_{-j}}{(q_j + q_{-j})^2} r N_{i,\epsilon_j} - r D_{i,\epsilon_j}, & \text{if } q_j + q_{-j} \in (x_i, x_{i-1}), \\
& \quad \text{with } i = 2, \dots, n, \\
\frac{q_{-j}}{(q_j + q_{-j})^2} r N_{1,\epsilon_j} - r D_{1,\epsilon_j}, & \text{if } q_j + q_{-j} \in (x_1, +\infty).
\end{cases}
$$

Therefore, solving in q_j the first order condition $I_j(q_j, q_{-j}) = 0$ we get that:

- for the branch $q_j + q_{-j} \in (0, x_n)$, no solution exists;
- for the branch $q_j + q_{-j} \in (x_i, x_{i-1})$, with $i = 2, \dots, n$, it holds

$$
q_j = \sqrt{\frac{N_{i,\epsilon_j}}{D_{i,\epsilon_j}} q_{-j}} - q_{-j},
$$

  which is acceptable provided

$$
\begin{cases}
D_{i,\epsilon_j} = \frac{c}{r} - B_{i,\epsilon_j} > 0 \iff B_{i,\epsilon_j} < \frac{c}{r}, \\
x_i < \sqrt{\frac{N_{i,\epsilon_j}}{D_{i,\epsilon_j}} q_{-j}} < x_{i-1} \iff \frac{D_{i,\epsilon_j}}{N_{i,\epsilon_i}} x_i^2 < q_{-j} < \frac{D_{i,\epsilon_j}}{N_{i,\epsilon_i}} x_{i-1}^2;
\end{cases}
$$

- for the branch $q_j + q_{-j} \in (x_1, +\infty)$ it holds

$$
q_j = \sqrt{\frac{N_{1,\epsilon_j}}{D_{1,\epsilon_j}} q_{-j}} - q_{-j},
$$

  which is acceptable provided $\frac{D_{1,\epsilon_j}}{N_{1,\epsilon_j}} x_1^2 < q_{-j} \leq \frac{r\mu}{c}$.

We also notice that, when $q_{-j} > \frac{r\mu}{c}$, we have $I_j(q_j, q_{-j}) < 0$ for all $q_j > 0$, and $\underline{\pi}_j(q_j, q_{-j})$ has its maximum at $q_j = 0$.

Finally, the global expression of $q_j^*(q_{-j})$ is obtained by taking all the parts determined by $I_j(q_j, q_{-j}) = 0$ in their existence intervals, and connecting each of this pieces with the corresponding pieces of lines $q_j + q_{-j} = x_i$, where $i = 1, \dots, \bar{i}$, taking care of selecting only their non-negative parts. □

Remark 5. We have that $\lim_{q_{-j} \to 0^+} q_j^*(q_{-j}) = x_{\bar{i}}$ and by the proof of Theorem 5 in [6] we get that $x_{\bar{i}}$ is the maximum value of the set of maximizers of $\underline{\pi}_j(q_j, 0)$, seen as a function of q_j. Therefore, setting $q_j^*(0) := x_{\bar{i}}$ we obtain a continuous function of $q_{-j} \in [0, +\infty)$. We notice that this choice differs from the convention adopted in [6] where the minimum value of the set of maximizers is selected, in agreement with [25].

The following theorem characterizes the set of all Nash equilibria through the best response functions $q_j^*(q_{-j})$, with $j \in \{1, 2\}$. In particular, since the two best response functions have a closed-form expression, this allows to derive an operative procedure to extract all Nash equilibria by computing the intersection of two curves in the plane.

Theorem 2. *Given a two-player newsvendor game with lower expected profit functions as in (7), the set of Nash equilibria is*

$$\bigcap_{j=1,2} \{(q_j^*(q_{-j}), q_{-j}) \ : \ q_{-j} \in [0, +\infty)\}. \tag{22}$$

Proof. It holds that every Nash equilibrium (q_1^*, q_2^*) satisfies $q_1^*(q_2^*) = q_1^*$ and $q_2^*(q_1^*) = q_2^*$, thus all Nash equilibria can be found by taking the intersection of best response curves as in (22). □

Theorem 3 provides a condition on parameters, characterizing a Nash equilibrium (q_1^*, q_2^*) satisfying $q_1^* + q_2^* \in (x_i, x_{i-1})$, with $i = 1, \dots, n$. In turn, when such condition holds, the quoted Nash equilibrium turns out to be the unique Nash equilibrium whose sum is contained in an open interval (x_i, x_{i-1}). This highlights that Nash equilibria can be such that their sum does not match a discrete level of demand x_i.

Theorem 3. *Denoting by $x_0 := +\infty$, there exists an index $i \in \{1, \dots, n\}$ such that*

$$\frac{c}{r} < \frac{\mu}{2x_i}, \qquad \text{if } i = 1 \tag{23}$$

$$\begin{cases} \dfrac{c}{r} > \dfrac{1}{N_{i,\epsilon_1} + N_{i,\epsilon_2}} \left[\dfrac{N_{i,\epsilon_1} N_{i,\epsilon_2}}{x_{i-1}} + N_{i,\epsilon_1} B_{i,\epsilon_2} + N_{i,\epsilon_2} B_{i,\epsilon_1} \right] \\ \dfrac{c}{r} < \dfrac{1}{N_{i,\epsilon_1} + N_{i,\epsilon_2}} \left[\dfrac{N_{i,\epsilon_1} N_{i,\epsilon_2}}{x_i} + N_{i,\epsilon_1} B_{i,\epsilon_2} + N_{i,\epsilon_2} B_{i,\epsilon_1} \right] \end{cases} \qquad \text{if } i = 2, \dots, n \tag{24}$$

if and only if there exists a Nash equilibrium (q_1^*, q_2^*) such that $q_1^* + q_2^* \in (x_i, x_{i-1})$, where

$$q_1^* = \begin{cases} \dfrac{\mu r}{4c} & \text{if } i = 1, \\ \dfrac{N_{i,\epsilon_2} (N_{i,\epsilon_1})^2 D_{i,\epsilon_2}}{(N_{i,\epsilon_1} D_{i,\epsilon_2} + N_{i,\epsilon_2} D_{i,\epsilon_1})^2} & \text{if } i = 2, \dots, n, \end{cases} \tag{25}$$

$$q_2^* = \begin{cases} \dfrac{\mu r}{4c} & \text{if } i = 1, \\ \dfrac{D_{i,\epsilon_1} (N_{i,\epsilon_2})^2 N_{i,\epsilon_1}}{(N_{i,\epsilon_1} D_{i,\epsilon_2} + N_{i,\epsilon_2} D_{i,\epsilon_1})^2} & \text{if } i = 2, \dots, n, \end{cases} \tag{26}$$

in which, for $j \in \{1, 2\}$ and $i = 2, \dots, n$, $B_{i,\epsilon_j}, D_{i,\epsilon_j}, N_{i,\epsilon_j}$ are defined as in equations (19)–(21).

Moreover, the Nash equilibrium in (25)–(26) is the unique Nash equilibrium such that $q_j^* + q_{-j}^* \in (x_i, x_{i-1})$, for an $i \in \{1, \dots, n\}$.

Proof. Let $q_j + q_{-j} \in (x_i, x_{i-1})$, for $i \in \{1, \dots, n\}$, where $(x_1, x_0) = (x_1, +\infty)$. From Proposition 2, $\underline{\pi}_j(q_j, q_{-j})$ for every $q_{-j} > 0$, where $j \in \{1, 2\}$, is strictly concave in (x_i, x_{i-1}) as a function of q_j.

We consider $q_1 + q_2 \in (x_1, +\infty)$, where the first order conditions are

$$\begin{cases} I_1(q_1, q_2) = \frac{q_2}{(q_1+q_2)^2} r\mu - c = 0, \\ I_2(q_1, q_2) = \frac{q_1}{(q_1+q_2)^2} r\mu - c = 0. \end{cases}$$

The system has the unique solution $q_1^* = q_2^* = \frac{\mu r}{4c}$, which is a Nash equilibrium if $q_1^* + q_2^* \in (x_1, +\infty)$, that is

$$\frac{1}{2}\frac{\mu r}{c} > x_1,$$

which reduces to condition (23).

Now, we consider $q_1 + q_2 \in (x_i, x_{i-1})$ for $i = 2, \ldots, n$, for which the first order conditions are

$$\begin{cases} I_1(q_1, q_2) = \frac{q_2}{(q_1+q_2)^2} r N_{i,\epsilon_1} - r D_{i,\epsilon_1} = 0, \\ I_2(q_1, q_2) = \frac{q_1}{(q_1+q_2)^2} r N_{i,\epsilon_2} - r D_{i,\epsilon_2} = 0. \end{cases}$$

The system has the unique solution q_1^* and q_2^* in (26), which is a Nash equilibrium if $q_1^* + q_2^* \in (x_i, x_{i-1})$, that is

$$x_i < \frac{N_{i,\epsilon_1} N_{i,\epsilon_2}}{N_{i,\epsilon_1} D_{i,\epsilon_2} + N_{i,\epsilon_2} D_{i,\epsilon_1}} < x_{i-1},$$

that, after some straightforward algebraic manipulations reduces to conditions in (24).

Condition (23) or (24) can be verified for at most one index $i \in \{1, \ldots, n\}$ and this means that the sum $q_1^* + q_2^*$ can belong to at most one interval (x_i, x_{i-1}). Therefore, if there exists an index i such that (23) or (24) holds, then the point defined in (25)–(26) is a Nash equilibrium.

The opposite implication is straightforward and comes from the fact that, if (q_1^*, q_2^*) is a Nash equilibrium defined as in (25)–(26) and $q_1^* + q_2^* \in (x_i, x_{i-1})$, with $i \in \{1, \ldots, n\}$, then condition (23) or (24) follows.

Finally, the uniqueness statement follows since conditions (23)–(24) assure that only a (unique) solution given by the first order conditions can be selected. □

Remark 6. If both newsvendors have the same contaminating parameter, i.e., $\epsilon_1 = \epsilon_2$, we get $N_i := N_{i,\epsilon_1} = N_{i,\epsilon_2}$, $D_i := D_{i,\epsilon_1} = D_{i,\epsilon_2}$ and $B_i := B_{i,\epsilon_1} = B_{i,\epsilon_2}$, for every $i = 2, \ldots, n$. This implies that condition (23) remains the same, while, for $i = 2, \ldots, n$, condition (24) simplifies in

$$\frac{1}{2x_{i-1}} N_i + B_i < \frac{c}{r} < \frac{1}{2x_i} N_i + B_i, \tag{27}$$

and the related Nash equilibrium is symmetric and reduces to

$$q_1^* = q_2^* = \frac{1}{4}\frac{N_i}{D_i}. \tag{28}$$

We stress that, if condition (23) holds, the Nash equilibrium $(q_1^*, q_2^*) = \left(\frac{\mu r}{4c}, \frac{\mu r}{4c}\right)$ does not depend on the contaminating parameters and it is symmetric, whether $\epsilon_1 = \epsilon_2$ or not.

In Example 3 we show the application of Theorem 3 considering newsvendors with different levels of ambiguity.

Example 3. Let $\mathcal{X} = \{x_1, x_2, x_3\}$ and take P_0 as in Example 1. It follows that $\mu = 60$ and $\alpha = \frac{3}{7}$. We consider $r = 2$, $c = 1$, and assume that newsvendor 1 is quite convinced of the probability P_0, while newsvendor 2 is very unconvinced about it, that is, $\epsilon_1 = 0.2$ and $\epsilon_2 = 0.9$.

Condition (24) is satisfied for $i = 2$, that is

$$\begin{cases} \frac{c}{r} = \frac{4945}{9890} > \frac{1}{N_{2,\epsilon_1} + N_{2,\epsilon_2}} \left[\frac{N_{2,\epsilon_1} N_{2,\epsilon_2}}{x_1} + N_{2,\epsilon_1} B_{2,\epsilon_2} + N_{2,\epsilon_2} B_{2,\epsilon_1} \right] = \frac{4900}{9890} \\ \frac{c}{r} = \frac{4945}{9890} < \frac{1}{N_{2,\epsilon_1} + N_{2,\epsilon_2}} \left[\frac{N_{2,\epsilon_1} N_{2,\epsilon_2}}{x_2} + N_{2,\epsilon_1} B_{2,\epsilon_2} + N_{2,\epsilon_2} B_{2,\epsilon_1} \right] = \frac{5934}{9890} \end{cases}$$

It follows that Theorem 3 holds and the Nash equilibrium (q_1^*, q_2^*) such that $q_1^* + q_2^* \in (x_i, x_{i-1})$, for $i \in \{1, 2, 3\}$ is unique and it is

$$q_1^* = \frac{9408}{229} \approx 41.08, \qquad q_2^* = \frac{3668}{67} \approx 54.75,$$

for which $q_1^* + q_2^* = \frac{3929}{41} \in (50, 100)$.

It holds that

$$\underline{\pi}_1(q_1^*, q_2^*) = \frac{3477}{382}, \quad \text{and} \quad \underline{\pi}_2(q_1^*, q_2^*) = \frac{6331}{536},$$

and

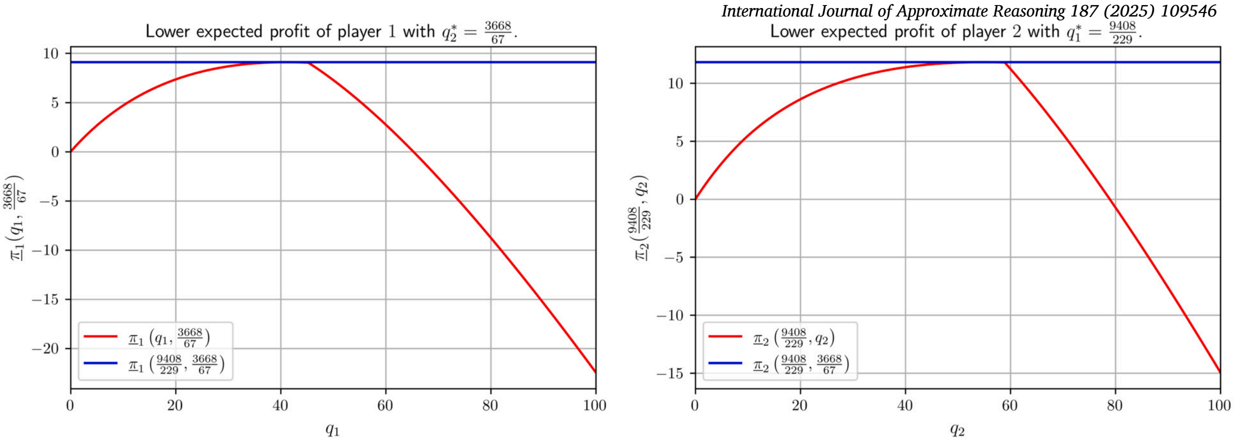


Fig. 4. Lower expected profit of player j with fixed q_{-j}^*, for $j \in \{1,2\}$.

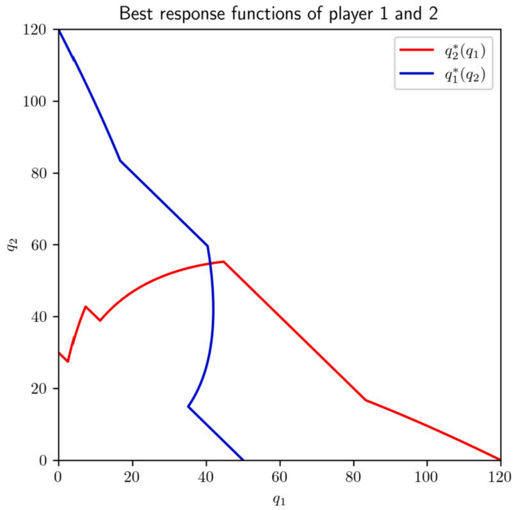


Fig. 5. Best response functions of player 1 and 2.

$$\underline{\pi}_1\left(q_1, \frac{3668}{67}\right) = \begin{cases} -\dfrac{31}{105}q_1 + \dfrac{1040}{21}\dfrac{q_1}{q_1 + \frac{3668}{67}}, & \text{for } q_1 \in \left[0, 100 - \dfrac{3668}{67}\right), \\ -q_1 + 120\dfrac{q_1}{q_1 + \frac{3668}{67}}, & \text{for } q_1 \in \left[100 - \dfrac{3668}{67}, +\infty\right), \end{cases}$$

$$\underline{\pi}_2\left(\frac{9408}{229}, q_2\right) = \begin{cases} -\dfrac{2}{21}q_2 + \dfrac{230}{7}\dfrac{q_2}{\frac{9408}{229} + q_2}, & \text{for } q_2 \in \left[0, 50 - \dfrac{9408}{229}\right), \\ -\dfrac{17}{105}q_2 + \dfrac{760}{21}\dfrac{q_2}{\frac{9408}{229} + q_2}, & \text{for } q_2 \in \left[50 - \dfrac{9408}{229}, 100 - \dfrac{9408}{229}\right), \\ -q_2 + 120\dfrac{q_2}{\frac{9408}{229} + q_2}, & \text{for } q_2 \in \left[100 - \dfrac{9408}{229}, +\infty\right). \end{cases}$$

Fig. 4 shows that (q_1^*, q_2^*) is indeed a Nash equilibrium since

$$\underline{\pi}_1(q_1^*, q_2^*) \geq \underline{\pi}_1(q_1, q_2^*), \text{ for every } q_1 \in [0, +\infty),$$
$$\underline{\pi}_2(q_1^*, q_2^*) \geq \underline{\pi}_2(q_1^*, q_2), \text{ for every } q_2 \in [0, +\infty).$$

We notice that, in this example, the Nash equilibrium (q_1^*, q_2^*) is the unique Nash equilibrium as can be verified from Fig. 5, where we observe that the two best response functions intersect only in the point (q_1^*, q_2^*), that satisfies $q_1^* + q_2^* \in (50, 100)$.

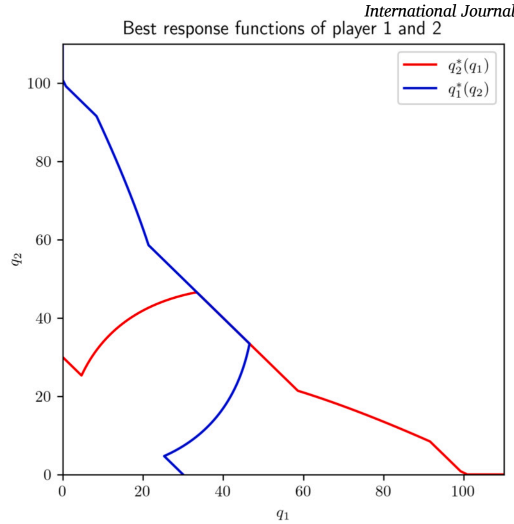


Fig. 6. Best response functions of player 1 and 2.

As a consequence of Theorem 3, if conditions (23)–(24) do not hold, every Nash equilibrium is such that $q_1^* + q_2^* = x_i$, with $i \in \{1, \ldots, n\}$.

In the following examples we show that if conditions (23)–(24) do not hold, we can have infinitely many Nash equilibria, as in Example 4, or a unique Nash equilibrium, as in Example 5. In turn, all such Nash equilibria satisfy $q_1^* + q_2^* = x_i$, with $i \in \{1, \ldots, n\}$.

Example 4. Let $\mathcal{X} = \{x_1, \ldots, x_5\}$, P_0, r, c and $\epsilon_1 = \epsilon_2$ as in Example 2.

Conditions (23)–(24) do not hold, for any $i = 1, \ldots, 5$, and the points found as (25)–(26), that are,

$$q_1 = q_2 = \begin{cases} \frac{605}{204} & \text{for } i = 1, \\ \frac{5625}{206} & \text{for } i = 2, \\ \frac{2585}{54} & \text{for } i = 3, \\ \frac{-2015}{22} & \text{for } i = 4, \\ \frac{-365}{12} & \text{for } i = 5, \end{cases}$$

are easily verified not to be Nash equilibria. In particular, for $i = 4, 5$ we get unfeasible quantities.

As Fig. 6 shows, the best response functions of player 1 and 2 meet in infinitely many points forming a segment. The quoted Nash equilibria satisfy $q_1^* + q_2^* = 80$ with $q_1^* \in [33.42, 46.58]$ and $q_2^* = 80 - q_1^*$, thus uniqueness fails in this example.

Incidentally, this example also shows that, despite $\epsilon_1 = \epsilon_2$, the newsvendor game can have also non-symmetric equilibria, i.e., for which $q_1^* \neq q_2^*$, though the symmetric Nash equilibrium $q_1^* = q_2^* = 40$ exists.

Example 5. Let $\mathcal{X}$, P_0, $r = 5$, $c = 3$, $\epsilon_1 = \epsilon_2$ as in Example 1.

The lower expected profit function of each player $j \in \{1, 2\}$ is

$$\underline{\pi}_j(q_j, q_{-j}) = \begin{cases} 2q_j, & \text{if } q_j + q_{-j} \in [0, 30), \\ -\frac{31}{42} q_j + \frac{575}{7} \beta_j, & \text{if } q_j + q_{-j} \in [30, 50), \\ -\frac{19}{21} q_j + \frac{1900}{21} \beta_j, & \text{if } q_j + q_{-j} \in [50, 100), \\ -3q_j + 300\beta_j, & \text{if } q_j + q_{-j} \in [100, +\infty). \end{cases}$$

Conditions (23)–(24) do not hold, therefore a Nash equilibrium is such that $q_1^* + q_2^* = x_i$, for $i \in \{1, 2, 3\}$.

As Fig. 7 shows, the best response functions of player 1 and 2 meet in only one point, which is the unique Nash equilibrium $q_1^* = q_2^* = 25$, that satisfies $q_1^* + q_2^* = 50$.

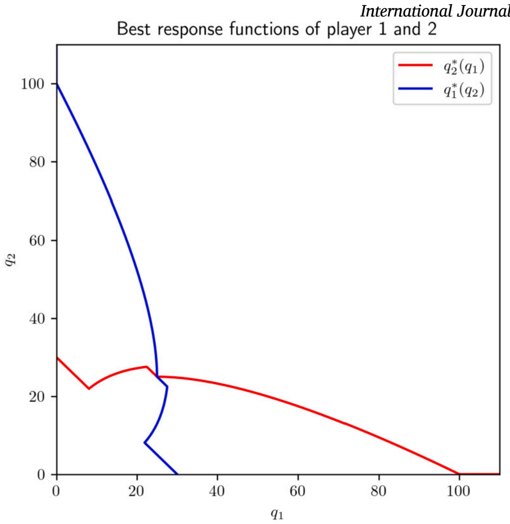


Fig. 7. Best response functions of player 1 and 2.

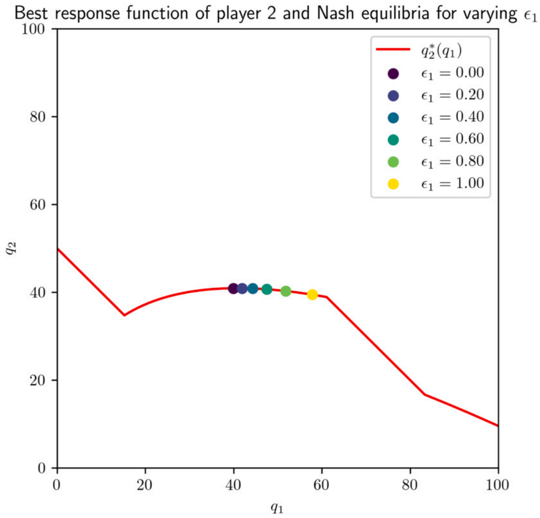


Fig. 8. Best response function of player 2 and Nash equilibria for $r = 2$, $c = 1$, $\epsilon_2 = 0.1$ and $\epsilon_1 \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$.

In Example 6 we show Nash equilibria for various values of the ambiguity parameter ϵ_1, for a fixed ϵ_2 and for two distinct pairs of values of r and c. This example illustrates that the impact of increasing ambiguity on the Nash equilibria is not uniform since it depends on the specific values chosen for the parameters r and c, that encode different market conditions.

Example 6. Let $\mathcal{X}$ and P_0 be as in Example 1. We find Nash equilibria for various values of ϵ_1, for a fixed value of $\epsilon_2 = 0.1$, which express the change in the ambiguity of player 1. Fig. 8 shows the best response function of player 2 and the intersection points with the best response function of player 1 (i.e., the Nash equilibria) for $\epsilon_1 \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$.

Under this choice of parameters r and c, we note that the Nash equilibrium is unique for every considered value of ϵ_1. Furthermore, it holds that an increase in the ambiguity of player 1 determines a right shift of Nash equilibria along the best response curve of player 2.

Table 1 reports the values of equilibria (q_1^*, q_2^*) and the corresponding proportions (β_1^*, β_2^*) of allocated demand, where $\beta_j^* = \dfrac{q_j^*}{q_1^* + q_2^*}$, for $j \in \{1, 2\}$. From the reported values we can observe that, for increasing ϵ_1, i.e., if player 1 becomes less and less confident of P_0, his/her optimal order quantity q_1^* increases. At the same time, q_2^* decreases but globally the total quantity $q_1^* + q_2^*$ increases and makes the proportion of demand allocated to player 1 pass from 49.4% to 59.5%.

Table 1
Nash equilibria (q_1^*, q_2^*) and corresponding demand proportions (β_1^*, β_2^*) for $r = 2$, $c = 1$, $\epsilon_2 = 0.1$ and $\epsilon_1 \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$.

ϵ_1	q_1^*	q_2^*	$q_1^* + q_2^*$	β_1^*	β_2^*
0	39.995	40.904	80.899	0.494	0.506
0.2	41.929	40.903	82.832	0.506	0.494
0, 4	44.368	40.839	85.207	0.521	0.479
0.6	47.535	40.661	88.196	0.539	0.461
0.8	51.805	40.267	92.072	0.563	0.437
1	57.852	39.445	97.297	0.595	0.405

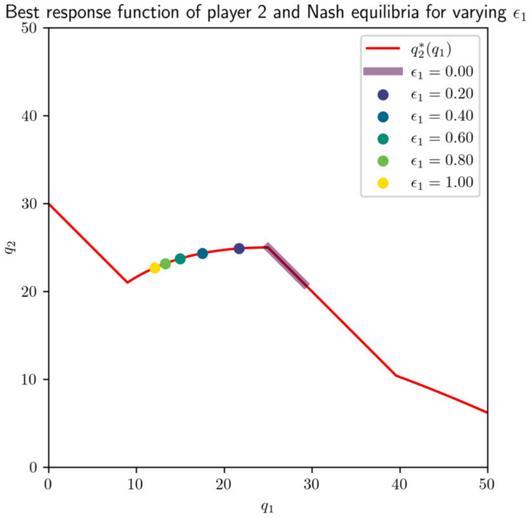


Fig. 9. Best response function of player 2 and Nash equilibria for $r = 4$, $c = 3$, $\epsilon_2 = 0.1$ and $\epsilon_1 \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$.

Table 2
Nash equilibria (q_1^*, q_2^*) and corresponding demand proportions (β_1^*, β_2^*) for $r = 4$, $c = 3$, $\epsilon_2 = 0.1$ and $\epsilon_1 \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$.

ϵ_1	q_1^*	q_2^*	$q_1^* + q_2^*$	β_1^*	β_2^*
0	$[25, 29.167]$	$50 - q_1^*$	50	$[0.500, 0.583]$	$1 - \beta_1^*$
0.2	21.717	24.885	46.602	0.466	0.534
0.4	17.523	24.337	41.860	0.419	0.581
0.6	14.984	23.725	38.709	0.387	0.613
0.8	13.296	23.168	36.464	0.365	0.635
1	12.098	22.684	34.782	0.348	0.652

Now, let us consider a different market condition given by $r = 4$ and $c = 3$. Fig. 9 shows the best response function of player 2 and the intersection points with the best response function of player 1 (i.e., the Nash equilibria) for $\epsilon_1 \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$, with fixed $\epsilon_2 = 0.1$.

In this case, we have infinitely many Nash equilibria for $\epsilon_1 = 0$, while uniqueness is obtained for other values of ϵ_1. Moreover, with this set of parameters, the increase of ϵ_1 determines a left shift of Nash equilibria along the best response curve of player 2.

Table 2 reports the values of equilibria (q_1^*, q_2^*) and the corresponding proportions (β_1^*, β_2^*) of allocated demand. In this second case we get that, for increasing ϵ_1, i.e., if player 1 becomes less and less confident of P_0, his/her optimal order quantity q_1^* decreases as well as q_2^* and the total quantity $q_1^* + q_2^*$. In turn, the proportion of demand allocated to player 1 passes from $[50.0, 58.3]\%$ to 34.8%.

Hence, the effect of ambiguity on Nash equilibria is essentially related to market conditions, expressed by r and c.

5. Conclusions

This paper introduces a game theoretic model for analyzing a market with two newsvendors competing on the global market demand of the same perishable product, the latter being sold only in a discrete set of units. We suppose that both newsvendors are risk-neutral, rational and operate under ambiguity, as we allow that they can be not completely convinced of the market demand probability measure. In turn, such ambiguity is modeled through two (possibly different) ϵ-contamination models induced by the reference probability measure of the demand and a suitable belief function whose Choquet expected demand matches the market expected demand.

Modeling both newsvendors as lower expected profit maximizers with respect to the corresponding ϵ-contamination model, we prove the existence of a (pure strategy) Nash equilibrium in the form of a pair of order quantities. We show that a Nash equilibrium is generally not unique, and provide a characterization of the set of all Nash equilibria by giving a closed-form expression for the players' best response functions.

As a line of future research we aim at providing a necessary and sufficient condition for uniqueness of Nash equilibria, besides the extension to $N \geq 2$ players. Another line of possible research concerns modeling the demand of the market not in aggregated way but through a random vector of individual demands as in [24]. We notice that, assuming a vector of individual demands endowed with marginal probability measures and modeling ambiguity with ϵ-contaminations, requires to suitably encode dependence in Dempster-Shafer theory (see, e.g., [11,20]). This task is particularly challenging since modeling dependence using copulas in this framework involves multiple and non-equivalent approaches that is, applying copulas to the non-additive measures, to their cores, or to their Möbius inverses can lead to different results.

CRediT authorship contribution statement

Andrea Cinfrignini: Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Silvia Lorenzini:** Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Davide Petturiti:** Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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