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Some fuzzy neighborhood operators on fuzzy β-covering approximation space and their application in user preference evaluation

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ABSTRACT

As a generalization of covering, fuzzy β-covering provides a more accurate and practical representation for incomplete information. This paper primarily proposes several fuzzy neighborhood operators based on diverse aggregation functions in an fuzzy β-covering approximation space (FβCAS) and develops a novel TOPSIS method to address the decision-making problem related to user preference factors. First, two classes of fuzzy neighborhood operators are introduced, derived from t-norms, overlap functions and their residual implications in an FβCAS, with their properties thoroughly analyzed. In addition, multiple fuzzy β-coverings are generated from the original fuzzy β-covering, and the classifications of fuzzy neighborhood operators, along with their partial order relationships, are examined. Based on these operators, two kinds of fuzzy β-covering-based rough sets (FβCRS) are established. Finally, an FβCRS-based fuzzy TOPSIS method is developed to evaluate user preference factors for fresh fruit, thereby demonstrating the rationality and feasibility of the proposed approach.

1. Introduction

1.1. A short review on rough sets

To address the imprecision and uncertainty inherent in data analysis, rough set theory (RST) was introduced by Pawlak [21], with its core concept revolving around lower and upper approximation operators derived from equivalence relations. However, the reliance on equivalence relations imposes a restrictive condition, thereby limiting the applicability of Pawlak's RST [3,29]. Consequently, numerous scholars have extended Pawlak's rough set from diverse perspectives. For instance, from the perspective of elements, equivalence relations can be generalized to binary relations or neighborhood operators [27,30,36]; from the perspective of granules, partitions can be generalized to coverings [15,22,33,34]; and from the perspective of subsystems, the σ-algebra of subset can be generalized to pairs of systems [37,38].

Despite its widespread recognition in both theoretical and applied domains, RST is predominantly suited for qualitative (discrete) data, which restricts its effectiveness when dealing with symbolic and real-valued datasets. To overcome this limitation, fuzzy set theory (FST) [40] was introduced to handle graded indiscernibility and vague concepts. Building on this, Dubois and Prade [9]

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proposed fuzzy rough set and rough fuzzy set models, incorporating approximation operators into the fuzzy set framework. As a result, fuzzy rough sets have garnered considerable interest, leading to extensive research [7,10,19].

It is noteworthy that covering represents an extension of partition, making it natural to explore RST from a covering-based perspective. In 1983, Zakowski [41] introduced the concept of the covering-based rough set (CRS). However, the generalized approximation operators in Zakowski's model are not dual. To overcome this limitation, Pomykala [22] constructed two classes of dual approximation operators. Later, numerous extended models related to CRS were further explored within the fuzzy covering approximation space (FCAS) [4,7,14,34]. Meanwhile, Yao [36] proposed the notion of neighborhood-related CRS by utilizing neighborhood systems derived from predecessor and successor neighborhoods. This foundation has inspired further developments by various researchers. For instance, D'eer et al. [6] extended Pawlak's RST by replacing equivalence relations with neighborhood operators, achieving a specific classification of neighborhood operators for different coverings. Additionally, D'eer et al. [5] extended classical neighborhood operators [38] to fuzzy coverings using t-norm, and examined several new fuzzy coverings. From the perspective of overlap functions, Qi et al. [23] extended fuzzy neighborhood operators, thereby investigated their corresponding fuzzy covering-based rough set models (FCRS).

However, FCRS models may be overly restrictive in practical application, as it necessitates that each element must have an evaluation value of 1 under at least one criterion. To address this limitation, Ma [17] proposed the notion of fuzzy β-covering, replacing the requirement of 1 with $\beta \in (0,1]$. In contrast to FCAS model, FβCAS model offers a broader range of practical applicability. For example, consider a scenario where a company must select one interviewee from four candidates based on scores evaluated by three experts. Let the interviewee set be $\Omega = \{a_1, a_2, a_3\}$ and the expert set $\Psi = \{\psi_1, \psi_2, \psi_3\}$, where

$$\psi_1 = \frac{0.9}{a_1} + \frac{0.6}{a_2} + \frac{0.8}{a_3};$$
$$\psi_2 = \frac{0.8}{a_1} + \frac{0.8}{a_2} + \frac{0.7}{a_3};$$
$$\psi_3 = \frac{0.7}{a_1} + \frac{0.5}{a_2} + \frac{0.7}{a_3}.$$

Clearly, fuzzy covering is inapplicable to this decision-making problem, as none of the expert scores for interviewee a_i $(i = 1, 2, 3)$ equals 1, thus failing to meet the requirement of fuzzy covering. However, fuzzy β-covering provides a solution to this issue. For $\beta \in (0, 0.8]$, (Ω, Ψ) forms an FβCAS, enabling the selection of the optimal interviewee though a fuzzy β-covering-based rough set model (FβCRS). Subsequently, Yang and Hu [34,35] proposed concepts including fuzzy β-maximum and fuzzy β-minimum descriptions, and further investigated various FβCRS models along with their properties. Following this, various extensions of FβCRS models have been developed, such as lattice value-based FβCRS [15], multi-granulation CRS [11,12] and so on [32].

1.2. A short review on multi-attribute decision-making

As society continues to evolve, decision-making has become an indispensable aspect of daily life, with multi-attribute decision-making (MADM) serving as a critical tool for addressing complex decision problems. Numerous effective MADM methods have been proposed, including the aggregation operator method [31], the TOPSIS method [28], and the TODIM method [16]. However, the classical TOPSIS method exhibits limitations when handling complex data, prompting its extension to fuzzy environments. Specifically, the adaptation of the TOPSIS method to fuzzy environments was initially introduced in [2]. Subsequently, Zhang et al. [42] developed a new fuzzy TOPSIS method based on FCRS model, utilizing t-norms and their implications. Building on this work, Qi et al. [23] proposed the fuzzy TOPSIS based on FCRS model, leveraging overlap functions and their implications. However, the methods proposed in [23,42] rely on fuzzy covering and are unable to process data that fails to meet the requirements of fuzzy covering. To overcome this limitation, this paper introduces a novel class of TOPSIS methods under fuzzy β-covering, thereby expanding the applicability of the TOPSIS method.

1.3. The motivation and main work of this paper

To generalize RST and broaden its application in MADM, this paper focuses on extending the four types of classical neighborhood operators [36] to fuzzy β-covering by employing t-norms, overlap functions and their respective implications. Furthermore, leveraging the newly introduced fuzzy β-neighborhood operators, several CRS models are established in an FβCAS. The specific research motivations are listed as follows:

(1) This paper is a continuation of the classical neighborhood operators [36]. D'eer et al. [5] generalized the classical neighborhood operators to fuzzy covering. Notably, fuzzy covering is a special case of fuzzy β-covering, as the latter degenerates to the former when $\beta = 1$. Therefore, investigating neighborhood operators in an FβCAS holds significant importance. Moreover, there exist no essential inclusion relationships between the fuzzy neighborhood system (resp. the fuzzy minimum and fuzzy maximum descriptions) induced by fuzzy covering and the fuzzy β-neighborhood system (resp. the fuzzy β-minimum and fuzzy β-maximum descriptions) induced by fuzzy β-covering, respectively. Consequently, the study of neighborhood operators in an FβCAS is not merely an extension of the existing research on FCAS, but also an independent exploration, differing from [5,23] in both forms and properties.

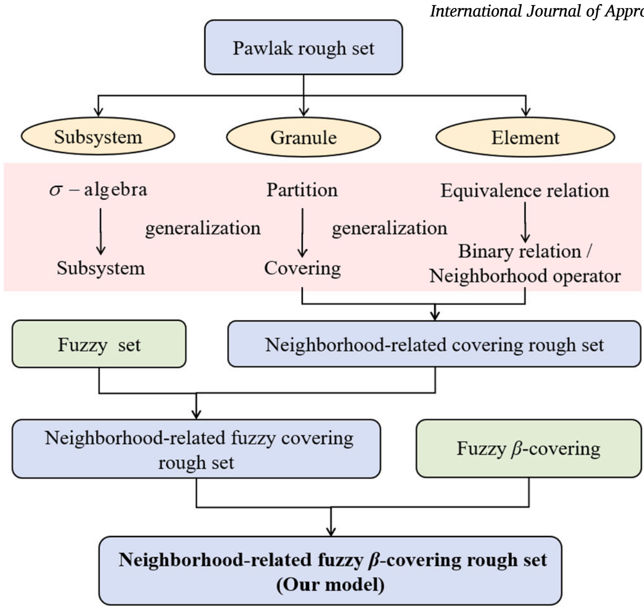

Fig. 1. The relationships among different rough set models.

(2) An investigation of neighborhood operators is conducted from the standpoint of fuzzy logical operator. Overlap functions and t-norms serve distinct roles in constructing neighborhood operators [5,23]. On one hand, neighborhood operators constructed with t-norms exhibit richer properties, due to their special properties such as the neutral element and associative law. On the other hand, the overlap function, rooted in practical applications, offers greater flexibility in handling data. This raises a natural question: what are the properties of neighborhood operators derived from different aggregation functions in an FβCAS, and how are they related? In this paper, we construct neighborhood operators using t-norms, overlap functions and their implications to explore the influence of aggregation functions on these operators. Additionally, we redefine several fuzzy β-coverings and analyze the relationships among neighborhood operators derived from different fuzzy β-coverings.

(3) A novel extension of the fuzzy TOPSIS method is introduced. Since fuzzy β-covering imposes less stringent requirements than fuzzy covering, its integration with TOPSIS method provides an effective solution for a broader range of MADM problems. Consequently, leveraging the fuzzy neighborhood operators defined above, two FβCRS models are established. To explicitly illustrate the relationships among the FβCRS models presented in this paper, Pawlak's rough set and its generalizations, Fig. 1 is provided. Furthermore, the FβCRS model is integrated with the TOPSIS method and applied to decision-making problems regarding user preference for fresh fruit. By analyzing online review data, the ranking of user preference factors for fresh fruits is determined using the TOPSIS method.

The outline of this paper is arranged as follows. Section 2 provides an overview of fundamental notions, including fuzzy logical operator, fuzzy covering and other related terminologies. Section 3 introduces several fuzzy neighborhood operators derived from t-norms, overlap functions and their implications, and examines their corresponding properties. Section 4 presents new fuzzy β-coverings generated from the given fuzzy β-covering, and explores the classifications and partial order relationships among fuzzy neighborhood operators. Section 5 constructs two FβCRS models using the aforementioned fuzzy neighborhood operators. Section 6 presents the fuzzy TOPSIS method and demonstrates its application in evaluating user preference factors for fresh fruits. Finally, Section 8 concludes the key contributions of this paper.

2. Preliminaries

This section presents essential concepts that will be utilized in the following section.

2.1. Fuzzy logical operators

In this subsection, the notions of t-norms, overlap functions, and fuzzy implications are presented.

Definition 2.1. [13] A binary function $T : [0,1]^2 \longrightarrow [0,1]$ is a t-norm if it fulfills commutativity, associativity, non-decreasing in each variable, and possesses the neutral element 1.

A t-norm T is idempotent if $T(\pi,\pi) = \pi$ for each $\pi \in [0,1]$.

Table 1
Common t-norms and their corresponding R-implications.

t-norm	R-implication I_T
$T_M(\pi,\varpi)=\min(\pi,\varpi)$	$I_{T_M}(\pi,\varpi)=\begin{cases}1, & \pi\le\varpi;\\ \varpi, & \pi>\varpi.\end{cases}$
$T_P(\pi,\varpi)=\pi*\varpi$	$I_{T_P}(\pi,\varpi)=\begin{cases}1, & \pi\le\varpi;\\ \frac{\varpi}{\pi}, & \pi>\varpi.\end{cases}$
$T_L(\pi,\varpi)=\max(\pi+\varpi-1,0)$	$I_{T_L}(\pi,\varpi)=\min(1-\pi+\varpi,1)$

Definition 2.2. [1] A binary function $O:[0,1]^2\longrightarrow[0,1]$ is an overlap function if it fulfills the following statements:

(O1) $O(\pi,\varpi)=O(\varpi,\pi)$;
(O2) $O(\pi,\varpi)=0$ iff $\pi\varpi=0$;
(O3) $O(\pi,\varpi)=1$ iff $\pi\varpi=1$;
(O4) If $\varpi\le\varsigma$, then $O(\pi,\varpi)\le O(\pi,\varsigma)$;
(O5) O is continuous in both variables.

Furthermore, the overlap function O meets the following properties [8]:

(O6) O fulfills the exchange principle if $O(\pi,O(\varpi,a))=O(\varpi,O(\pi,a))$ for any $\pi,\varpi,a\in[0,1]$;
(O7) O fulfills the property of 1-section deflation if $O(1,\pi)\le\pi$ for each $\pi\in[0,1]$;
(O8) O fulfills the property of 1-section inflation if $O(1,\pi)\ge\pi$ for each $\pi\in[0,1]$;
(O9) O fulfills the property of diagonal inflation if $O(\pi,\pi)\ge\pi$ for each $\pi\in[0,1]$.

Note that O fulfills (O6) iff O is associative.

Definition 2.3. [13] A binary function $I:[0,1]^2\longrightarrow[0,1]$ is a fuzzy implication, if it fulfills $I(0,0)=I(0,1)=I(1,1)=1$, $I(1,0)=0$ and I is monotonically decreasing in the first argument and monotonically increasing in the second argument.

An important class of fuzzy implications is the residual implication (R-implication) [26]. Specifically, the R-implication induced by a t-norm T is defined as

$$I_T(\pi,\varpi)=\sup\{\varsigma\in[0,1]\mid T(\pi,\varsigma)\le\varpi\}.$$

Consequently, T and I_T form an adjoint pair, satisfying the property that $T(\pi,\varsigma)\le\varpi$ iff $I_T(\pi,\varpi)\ge\varsigma$ for any $\pi,\varpi,\varsigma\in[0,1]$.

Additionally, I_O is the R-implication [8] induced by an overlap function O, defined as

$$I_O(\pi,\varpi)=\max\{\varsigma\in[0,1]\mid O(\pi,\varsigma)\le\varpi\}.$$

Common t-norms and their R-implications, along with overlap functions and their R-implications, are presented in Tables 1 and 2, respectively.

2.2. Fuzzy covering approximation spaces

Throughout this discussion, the universe Ω is assumed to be non-empty and finite unless otherwise specified. The collection of all fuzzy sets on Ω is called fuzzy power set of Ω, denoted as $\mathscr{F}(\Omega)$.

Definition 2.4. [7] A collection $\Psi=\{\psi_1,\psi_2,\dots,\psi_m\}$ with $\psi_i\in\mathscr{F}(\Omega)$ $(i=1,2,\dots,m)$ is termed a fuzzy covering, if there exists a $\psi_i\in\Psi$ such that $\psi_i(\pi)=1$ for all $\pi\in\Omega$. Further, the pair (Ω,Ψ) forms a fuzzy covering approximation space (FCAS).

Next, we introduce the fuzzy neighborhood systems, fuzzy maximum and minimum descriptions.

Definition 2.5. [5] Let (Ω,Ψ) be an FCAS. For any $\pi\in\Omega$, the fuzzy neighborhood system of π is defined as

$$\mathbb{C}(\Psi,\pi)=\{\psi_i\in\Psi\mid\psi_i(\pi)>0,\ i=1,2,\dots,m\}.$$

Furthermore, the fuzzy minimal description of π is defined as

$$md(\Psi,\pi)=\{\psi_i\in\mathbb{C}(\Psi,\pi)\mid(\forall\psi_j\in\mathbb{C}(\Psi,\pi))\wedge(\psi_j(\pi)=\psi_i(\pi))\wedge(\psi_j\subseteq\psi_i)\Rightarrow\psi_j=\psi_i\}.$$

The fuzzy maximal description of π is defined as

Table 2
Common overlap functions and their R-implications.

Overlap function / R-implication I_O	Related properties
$O_2^V(\pi,\varpi)=\begin{cases}\frac{1+(2\pi-1)^2(2\varpi-1)^2}{2}, & \text{if } \pi,\varpi\in(0.5,1];\\ \min\{\pi,\varpi\}, & \text{otherwise.}\end{cases}$ $I_{O_2^V}(\pi,\varpi)=\begin{cases}\min\left\{1,\frac{\sqrt{2\varpi-1}}{2(2\pi-1)}+\frac{1}{2}\right\}, & \text{if } \pi\in(0.5,1],\varpi\in[0.5,1];\\ \varpi, & \text{if } \varpi\in[0,0.5),\pi>\varpi;\\ 1, & \text{if } \pi\in[0,0.5],\pi\leqslant\varpi.\end{cases}$	(O7)
$O_{m_{\frac{1}{2}}}(\pi,\varpi)=\min\{\sqrt{\pi},\sqrt{\varpi}\}$ $I_{O_{m_{\frac{1}{2}}}}(\pi,\varpi)=\begin{cases}1, & \text{if } \sqrt{\pi}\leqslant\varpi;\\ \varpi^2, & \text{if } \sqrt{\pi}>\varpi.\end{cases}$	(O8), (O9)
$O_{mM}^V(\pi,\varpi)=\begin{cases}\frac{1+\min\{2\pi-1,2\varpi-1\}\max\{(2\pi-1)^2,(2\varpi-1)^2\}}{2}, & \text{if } \pi,\varpi\in(0.5,1];\\ \min\{\pi,\varpi\}, & \text{otherwise.}\end{cases}$ $I_{O_{mM}^V}(\pi,\varpi)=\begin{cases}\min\left\{1,\min\left\{\frac{\sqrt{2\varpi-1}}{2\sqrt{2\pi-1}},\frac{2\varpi-1}{2(2\pi-1)^2}\right\}+\frac{1}{2}\right\}, & \text{if } \pi\in(0.5,1],\varpi\in[0.5,1];\\ \varpi, & \text{if } \varpi\in[0,0.5),\pi>\varpi;\\ 1, & \text{if } \pi\in[0,0.5],\pi\leqslant\varpi.\end{cases}$	(O7), (O8)
$O_2(\pi,\varpi)=\pi^2\varpi^2$ $I_{O_2}(\pi,\varpi)=\begin{cases}\frac{\sqrt{\varpi}}{\pi}, & \text{if } \varpi<\pi^2;\\ 1, & \text{if } \varpi\geq\pi^2.\end{cases}$	(O7)

$$MD(\Psi,\pi)=\{\psi_i\in\mathbb{C}(\Psi,\pi)\mid(\forall\psi_j\in\mathbb{C}(\Psi,\pi))\wedge(\psi_j(\pi)=\psi_i(\pi))\wedge(\psi_j\supseteq\psi_i)\Rightarrow\psi_j=\psi_i)\}.$$

Note that there exists at least one $\psi_i\in\Psi$ such that $\psi_i(\pi)=1$, implying that $\mathbb{C}(\Psi,\pi)$ is non-empty. In order to establish a relationship between $\pi\in\Omega$ and fuzzy set $N(\pi)$, the fuzzy neighborhood operator $N:\Omega\longrightarrow\mathscr{F}(\Omega)$ is defined. Based on this, by generalizing the classical neighborhood operators to fuzzy set, D'eer et al. introduced the fuzzy neighborhood operators [5].

Definition 2.6. [5] Let (Ω,Ψ) be an FCAS, I an implication and T a t-norm. N_1^Ψ, N_2^Ψ, N_3^Ψ, $N_4^\Psi:\Omega\longrightarrow\mathscr{F}(\Omega)$ are four types of fuzzy neighborhood operators. The fuzzy neighborhoods $N_i^\Psi(\pi)$ $(i=1,2,3,4)$ are constructed as

$$N_1^\Psi(\pi)(\varpi)=\bigwedge_{\psi_i\in\Psi}I(\psi_i(\pi),\psi_i(\varpi)),$$

$$N_2^\Psi(\pi)(\varpi)=\bigvee_{\psi_i\in\mathrm{md}(\Psi,\pi)}T(\psi_i(\pi),\psi_i(\varpi)),$$

$$N_3^\Psi(\pi)(\varpi)=\bigwedge_{\psi_i\in\mathrm{MD}(\Psi,\pi)}I(\psi_i(\pi),\psi_i(\varpi)),$$

$$N_4^\Psi(\pi)(\varpi)=\bigvee_{\psi_i\in\Psi}T(\psi_i(\pi),\psi_i(\varpi)).$$

Definition 2.7. [5] Let (Ω,Ψ) be an FCAS and N a fuzzy neighborhood operator.

(1) N is reflexive iff $N(\pi)(\pi)=1$ for each $\pi\in\Omega$.
(2) N is symmetric iff $N(\pi)(\varpi)=N(\varpi)(\pi)$ for each $\pi,\varpi\in\Omega$.
(3) N is T-transitive iff $T(N(\pi)(\varpi),N(\varpi)(\varsigma))\leq N(\pi)(\varsigma)$ for each $\pi,\varpi,\varsigma\in\Omega$.

Furthermore, the fuzzy neighborhood operators N_k^Ψ $(k=1,2,3,4)$ in Definition 2.6 are all reflexive, and N_1^Ψ and N_3^Ψ are T-transitive, and N_4^Ψ is symmetric.

To overcome the limitation of fuzzy covering, which requires that each element at least have an evaluation value 1, the concept of a fuzzy β-covering is proposed.

Definition 2.8. [17] For each $\beta\in(0,1]$, $\Psi=\{\psi_1,\psi_2,\dots,\psi_m\}$ with $\psi_i\in\mathscr{F}(\Omega)$ $(i=1,2,\dots,m)$ is termed a fuzzy β-covering of Ω, if $\bigvee_{i=1}^{m}\psi_i(\pi)\geq\beta$ for any $\pi\in\Omega$. The pair (Ω,Ψ) represents a fuzzy β-covering approximation space (FβCAS).

3. Fuzzy neighborhood operators derived from fuzzy logical operators

3.1. Four kinds of fuzzy neighborhood operators derived from t-norms

In this subsection, several fuzzy neighborhood operators are introduced using t-norms T and their R-implications I_T, and their properties are examined. For clarity, the notions of fuzzy β-neighborhood system, fuzzy β-minimal and fuzzy β-maximal descriptions are introduced.

According to [33,35], the fuzzy β-neighborhood system of $\pi \in \Omega$ is given by $\mathbb{C}_\beta(\Psi,\pi)=\{\psi \in \Psi, \psi(\pi) \geq \beta\}$. Furthermore, the fuzzy β-minimal and fuzzy β-maximal descriptions, $md_{\Psi,\beta}(\pi)$ and $MD_{\Psi,\beta}(\pi)$, are successively introduced.

$$md_{\Psi,\beta}(\pi) = \{\psi_i \in \mathbb{C}_\beta(\Psi,\pi) : \forall \psi_j \in \mathbb{C}_\beta(\Psi,\pi) \wedge \psi_j \subseteq \psi_i \Rightarrow \psi_j = \psi_i\};$$

$$MD_{\Psi,\beta}(\pi) = \{\psi_i \in \mathbb{C}_\beta(\Psi,\pi) : \forall \psi_j \in \mathbb{C}_\beta(\Psi,\pi) \wedge \psi_j \supseteq \psi_i \Rightarrow \psi_j = \psi_i\}.$$

Remark 3.1. For each $\pi \in \Omega$ and $\beta = 1$, it holds that

(1) $\mathbb{C}_\beta(\Psi,\pi) \subseteq \mathbb{C}(\Psi,\pi)$.
(2) $md_{\Psi,\beta}(\pi) \subseteq md(\Psi,\pi)$.
(3) $MD_{\Psi,\beta}(\pi) \subseteq MD(\Psi,\pi)$.

It is noted that when $\beta = 1$, the fuzzy β-covering becomes a fuzzy covering. However, when $\beta = 1$, no equivalence relationship exists between the fuzzy β-neighborhood system (resp. fuzzy β-maximum and fuzzy β-minimum descriptions) in an FβCAS and fuzzy neighborhood system (resp. fuzzy maximum and fuzzy minimum descriptions) in FCAS. Therefore, it is necessary to generalize the neighborhood operators in Definition 2.6 to fuzzy β-covering.

Definition 3.1. Let (Ω,Ψ) be an FβCAS. The fuzzy neighborhood operator based on R-implication I_T, denoted as $TN^1_{\Psi,\beta} : \Omega \longrightarrow \mathscr{F}(\Omega) : \pi \mapsto TN^1_{\Psi,\beta}(\pi)$, and the fuzzy neighborhood $TN^1_{\Psi,\beta}(\pi)$ is constructed as

$$TN^1_{\Psi,\beta}(\pi)(\varpi) = \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} I_T(\psi(\pi),\psi(\varpi)).$$

Remark 3.2. The relationships between $TN^1_{\Psi,\beta}$ and existing neighborhood operators are examined below.

(1) When fuzzy β-covering reduces to a covering (i.e., fuzzy β-neighborhood system becomes a classical neighborhood system), $TN^1_{\Psi,\beta}$ simplifies to the classical neighborhood operator. The detailed proof can be found in [5].
(2) For $\beta = 1$, fuzzy β-covering degenerates into fuzzy covering, and fuzzy β-neighborhood system is a subset of fuzzy neighborhood system. Then it holds that

$$TN^1_{\Psi,\beta}(\pi)(\varpi) = \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} I_T(\psi(\pi),\psi(\varpi)) \geq \bigwedge_{\psi\in\Psi} I_T(\psi(\pi),\psi(\varpi)) = N^\Psi_1(\pi)(\varpi).$$

Hence, we have that $N^\Psi_1(\pi) \subseteq TN^1_{\Psi,\beta}(\pi)$.

(3) Since I_T satisfies (NP), i.e., $I_T(1,\pi) = \pi$ $(\forall \pi \in (0,1])$, for any $\beta \in (0,1]$, the relationship between $TN^1_{\Psi,\beta}$ and $\widetilde{FN^\beta_\Psi}$ in [35] is represented as

$$\begin{aligned} TN^1_{\Psi,\beta}(\pi)(\varpi) &= \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} I_T(\psi(\pi),\psi(\varpi)) \\ &\geq \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} I_T(1,\psi(\varpi)) \\ &= \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} \psi(\varpi) \\ &= \widetilde{FN^\beta_\Psi}(\pi)(\varpi). \end{aligned}$$

Therefore, we have that $\widetilde{FN^\beta_\Psi}(\pi) \subseteq TN^1_{\Psi,\beta}(\pi)$.

Proposition 3.1. *Let (Ω,Ψ) be an FβCAS and I_T an R-implication induced by t-norm T. For each $\pi,\varpi \in \Omega$, it holds that*

$$\bigwedge_{\psi\in\Psi} I_T(\psi(\pi),\psi(\varpi)) = \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} I_T(\psi(\pi),\psi(\varpi)).$$

Proof. Since I_T is monotonically decreasing with respect to the first variable, it holds that

$$\begin{aligned}\bigwedge_{\psi\in\Psi} I_T(\psi(\pi),\psi(\varpi)) &= \min\left\{\bigwedge_{\psi\in\Psi,\psi(\pi)\geq\beta} I_T(\psi(\pi),\psi(\varpi)), \bigwedge_{\psi\in\Psi,\psi(\pi)<\beta} I_T(\psi(\pi),\psi(\varpi))\right\}\\ &= \bigwedge_{\psi\in\Psi,\psi(\pi)\geq\beta} I_T(\psi(\pi),\psi(\varpi)).\end{aligned}$$

Hence, we have that $\bigwedge_{\psi\in\Psi} I_T(\psi(\pi),\psi(\varpi)) = \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} I_T(\psi(\pi),\psi(\varpi))$. □

In [5], D'eer proved that $\bigwedge_{\psi\in\Psi} I_T(\psi(\pi),\psi(\varpi)) = \bigwedge_{\psi\in\mathbb{C}(\Psi,u)} I_T(\psi(\pi),\psi(\varpi)) = \bigwedge_{\psi\in md(\Psi,\pi)} I_T(\psi(\pi),\psi(\varpi))$ in an FCAS. However, in an FβCAS,

$$\bigwedge_{\psi\in\Psi} I_T(\psi(\pi),\psi(\varpi)) = \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} I_T(\psi(\pi),\psi(\varpi)) = \bigwedge_{\psi\in md_{\Psi,\beta}(\pi)} I_T(\psi(\pi),\psi(\varpi))$$

does not hold. Next, we present the following counterexample.

Example 3.1. Suppose that $\Omega = \{a_1, a_2, a_3\}$ and $\Psi = \{\psi_1, \psi_2\}$ is the family of fuzzy subsets on Ω, defined as

$$\psi_1 = \frac{0.7}{a_1} + \frac{0.6}{a_2} + \frac{0.9}{a_3};$$
$$\psi_2 = \frac{0.6}{a_1} + \frac{0.6}{a_2} + \frac{0.6}{a_3}.$$

Note that Ψ forms a fuzzy β-covering of Ω when $0 < \beta \leq 0.6$. For $\beta = 0.6$, we obtain that $\mathbb{C}_{0.6}(\Psi,\, a_1) = \{\psi_1, \psi_2\}$ and $md_{\Psi,0.6}(a_1) = \{\psi_2\}$ with $\psi_2 \subseteq \psi_1$. It holds that

$$\begin{aligned}\bigwedge_{\psi\in\mathbb{C}_{0.6}(\Psi,a_1)} I_T(\psi(a_1),\psi(a_2)) &= I_T(\psi_1(a_1),\psi_1(a_2)) \bigwedge I_T(\psi_2(a_1),\psi_2(a_2))\\ &= I_T(0.7, 0.6) \bigwedge I_T(0.6, 0.6)\\ &= I_T(0.7, 0.6)\\ &\leq I_T(0.6, 0.6)\\ &= I_T(\psi_2(a_1),\psi_2(a_2))\\ &= \bigwedge_{\psi\in md_{\Psi,0.6}(a_1)} I_T(\psi(a_1),\psi(a_2)).\end{aligned}$$

Therefore, it holds that $\bigwedge_{\psi\in\Psi} I_T(\psi(\pi),\psi(\varpi)) = \bigwedge_{\psi\in\mathbb{C}_{0.6}(\Psi,\pi)} I_T(\psi(\pi),\psi(\varpi)) \neq \bigwedge_{C\in md_{\Psi,0.6}(\pi)} I_T(\psi(\pi),\psi(\varpi))$ under the condition of fuzzy 0.6-covering.

Definition 3.2. Let (Ω,Ψ) be an FβCAS. The fuzzy neighborhood operator based on t-norm T, denoted as $TN^2_{\Psi,\beta} : \Omega \longrightarrow \mathscr{F}(\Omega) : \pi \mapsto TN^2_{\Psi,\beta}(\pi)$, and the fuzzy neighborhood $TN^2_{\Psi,\beta}(\pi)$ is constructed as

$$TN^2_{\Psi,\beta}(\pi)(\varpi) = \bigvee_{\psi\in md_{\Psi,\beta}(\pi)} T(\psi(\pi),\psi(\varpi)).$$

Remark 3.3. The relationships between $TN^2_{\Psi,\beta}$ and existing neighborhood operators are discussed below.

(1) When fuzzy β-covering reduces to a covering (i.e., fuzzy β-minimal description becomes minimal description), $TN^2_{\Psi,\beta}$ simplifies to the classical neighborhood operator. The detailed proof can be found in [5].
(2) For $\beta = 1$, fuzzy β-covering degenerates into fuzzy covering, and fuzzy β-minimal description is a subset of fuzzy minimal description. Then it holds that

$$TN^2_{\Psi,\beta}(\pi)(\varpi) = \bigvee_{\psi\in md_{\Psi,\beta}(\pi)} T(\psi(\pi),\psi(\varpi)) \leq \bigvee_{\psi\in md(\Psi,\pi)} T(\psi(\pi),\psi(\varpi)) = N^\Psi_2(\pi)(\varpi).$$

Hence, we have that $TN^2_{\Psi,\beta}(\pi) \subseteq N^\Psi_2(\pi)$.

(3) For any $\beta \in (0,1]$, the connection between $TN^2_{\Psi,\beta}$ and $\widetilde{SN^{\beta}_{\psi}}$ in [35] are represented as

$$\begin{aligned} TN^2_{\Psi,\beta}(\pi)(\varpi) &= \bigvee_{\psi \in md_{\Psi,\beta}(\pi)} T(\psi(\pi),\psi(\varpi)) \\ &\leq \bigvee_{\psi \in md_{\Psi,\beta}(\pi)} T(1,\psi(\varpi)) \\ &= \bigvee_{\psi \in md_{\Psi,\beta}(\pi)} \psi(\varpi) \\ &= \widetilde{SN^{\beta}_{\psi}}(\pi)(\varpi). \end{aligned}$$

Therefore, we have that $TN^2_{\Psi,\beta}(\pi) \subseteq \widetilde{SN^{\beta}_{\psi}}(\pi)$.

Definition 3.3. Let (Ω,Ψ) be an FβCAS. The fuzzy neighborhood operator based on R-implication I_T, denoted as $TN^3_{\Psi,\beta} : \Omega \longrightarrow \mathscr{F}(\Omega) : \pi \mapsto TN^3_{\Psi,\beta}(\pi)$, and the fuzzy neighborhood $TN^3_{\Psi,\beta}(\pi)$ is constructed as

$$TN^3_{\Psi,\beta}(\pi)(\varpi) = \bigwedge_{\psi \in MD_{\Psi,\beta}(\pi)} I_T(\psi(\pi),\psi(\varpi)).$$

Remark 3.4. The relationships between $TN^3_{\Psi,\beta}$ and existing neighborhood operators are discussed below.

(1) When fuzzy β-covering reduces to a covering (i.e., fuzzy β-maximal description becomes maximal description), $TN^3_{\Psi,\beta}$ simplifies to the classical neighborhood operator. The detailed proof can be found in [5].
(2) For $\beta = 1$, fuzzy β-covering degenerates into fuzzy covering, and the fuzzy β-maximal description is a subset of the fuzzy maximal description. Then it holds that

$$TN^3_{\Psi,\beta}(\pi)(\varpi) = \bigwedge_{\psi \in MD_{\Psi,\beta}(\pi)} I_T(\psi(\pi),\psi(\varpi)) \geq \bigwedge_{\psi \in MD(\Psi,\pi)} I_T(\psi(\pi),\psi(\varpi)) = N^{\Psi}_3(\pi)(\varpi).$$

Hence, we have that $N^{\Psi}_3(\pi) \subseteq TN^3_{\Psi,\beta}(\pi)$.

(3) Since I_T satisfies (NP), i.e., $I_T(1,\pi) = \pi$ $(\forall \pi \in (0,1])$, for any $\beta \in (0,1]$, the relationship between $TN^3_{\Psi,\beta}$ and $\widetilde{TN^{\beta}_{\psi}}$ in [35] is represented as

$$\begin{aligned} TN^3_{\Psi,\beta}(\pi)(\varpi) &= \bigwedge_{\psi \in MD_{\Psi,\beta}(\pi)} I_T(\psi(\pi),\psi(\varpi)) \\ &\geq \bigwedge_{\psi \in MD_{\Psi,\beta}(\pi)} I_T(1,\psi(\varpi)) \\ &= \bigwedge_{\psi \in MD_{\Psi,\beta}(\pi)} \psi(\varpi) \\ &= \widetilde{TN^{\beta}_{\psi}}(\pi)(\varpi). \end{aligned}$$

Therefore, we have that $\widetilde{TN^{\beta}_{\psi}}(\pi) \subseteq TN^3_{\Psi,\beta}(\pi)$.

Definition 3.4. Let (Ω,Ψ) be an FβCAS. The fuzzy neighborhood operator based on t-norm T, denoted as $TN^4_{\Psi,\beta} : \Omega \longrightarrow \mathscr{F}(\Omega) : \pi \mapsto TN^4_{\Psi,\beta}(\pi)$, and the fuzzy neighborhood $TN^4_{\Psi,\beta}(\pi)$ is constructed as

$$TN^4_{\Psi,\beta}(\pi)(\varpi) = \bigvee_{\psi \in \mathbb{C}_{\beta}(\Psi,\pi)} T(\psi(\pi),\psi(\varpi)).$$

Remark 3.5. The relationships between $TN^4_{\Psi,\beta}$ and existing neighborhood operators are discussed below.

(1) When fuzzy β-covering reduces to a covering (i.e., the fuzzy β-neighborhood system becomes a classical neighborhood system), $TN^4_{\Psi,\beta}$ simplifies to the classical neighborhood operator. The detailed proof can be found in [5].
(2) For $\beta = 1$, fuzzy β-covering degenerates into fuzzy covering, and the fuzzy β-neighborhood system is a subset of the fuzzy neighborhood system. Then it holds that

$$TN^4_{\Psi,\beta}(\pi)(\varpi) = \bigvee_{\psi \in \mathbb{C}_{\beta}(\Psi,\pi)} T(\psi(\pi),\psi(\varpi)) \leq \bigvee_{\psi \in \Psi} T(\psi(\pi),\psi(\varpi)) = N^{\Psi}_4(\pi)(\varpi).$$

Hence, we have that $TN^4_{\Psi,\beta}(\pi) \subseteq N^{\Psi}_4(\pi)$.

(3) For any $\beta \in (0,1]$, the connection between $TN^4_{\Psi,\beta}$ and $\widetilde{RN^{\beta}_{\psi}}$ in [35] is represented as

$$TN^4_{\Psi,\beta}(\pi)(\varpi) = \bigvee_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} T(\psi(\pi),\psi(\varpi)) \le \bigvee_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} \psi(\varpi) = \widetilde{RN^{\beta}_{\psi}}(\pi)(\varpi).$$

Therefore, we have that $TN^4_{\Psi,\beta}(\pi) \subseteq \widetilde{RN^{\beta}_{\psi}}(\pi)$.

Proposition 3.2. *Let (Ω,Ψ) be an FβCAS and T a t-norm. For each $\pi,\varpi\in\Omega$, it holds that*

$$\bigvee_{\psi\in\Psi} T(\psi(\pi),\psi(\varpi)) = \bigvee_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} T(\psi(\pi),\psi(\varpi)) = \bigvee_{\psi\in MD_{\Psi,\beta}(\pi)} T(\psi(\pi),\psi(\varpi)).$$

Proof. Since T is monotonically increasing in both variables, it follows that

$$\begin{aligned}\bigvee_{\psi\in\Psi} T(\psi(\pi),\psi(\varpi)) &= \max\left\{\bigvee_{\psi\in\Psi,\psi(\pi)\ge\beta} T(\psi(\pi),\psi(\varpi)), \bigvee_{\psi\in\Psi,\psi(\pi)<\beta} T(\psi(\pi),\psi(\varpi))\right\}\\ &= \bigvee_{\psi\in\Psi,\psi(\pi)\ge\beta} T(\psi(\pi),\psi(\varpi)).\end{aligned}$$

On the other end of the spectrum, we can obtain that

$$\bigvee_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} T(\psi(\pi),\psi(\varpi)) = \max\left\{\bigvee_{\psi\in MD_{\Psi,\beta}(\pi)} T(\psi(\pi),\psi(\varpi)), \bigvee_{C\in\mathbb{C}_\beta(\Psi,\pi)\setminus MD_{\Psi,\beta}(\pi)} T(\psi(\pi),\psi(\varpi))\right\}.$$

For $\psi\in\mathbb{C}_\beta(\Psi,\pi)\setminus MD_{\Psi,\beta}(\pi)$, there exists $\psi'\in MD_{\Psi,\beta}(\pi)$ such that $\psi\subseteq\psi'$, $\psi(\pi)\ge\beta$ and $\psi'(\pi)\ge\beta$. For any $\varpi\in\Omega$, we obtain that $T(\psi(\pi),\psi(\varpi)) \le T(\psi'(\pi),\psi'(\varpi))$, and

$$\bigvee_{\psi\in MD_{\Psi,\beta}(\pi)} T(\psi(\pi),\psi(\varpi)) \ge \bigvee_{C\in\mathbb{C}_\beta(\Psi,\pi)\setminus MD_{\Psi,\beta}(\pi)} T(\psi(\pi),\psi(\varpi)),$$

so $\bigvee_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} T(\psi(\pi),\psi(\varpi)) = \bigvee_{\psi\in MD_{\Psi,\beta}(\pi)} T(\psi(\pi),\psi(\varpi))$.

Hence, it holds that $\bigvee_{\psi\in\Psi} T(\psi(\pi),\psi(\varpi)) = \bigvee_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} T(\psi(\pi),\psi(\varpi)) = \bigvee_{\psi\in MD_{\Psi,\beta}(\pi)} T(\psi(\pi),\psi(\varpi))$. □

3.2. The characteristics of fuzzy neighborhood operators $TN^k_{\Psi,\beta}$

In [35], Yang and Hu defined β-reflexive for fuzzy neighborhood operators, where N is β-reflexive if $N(\pi)(\pi)\ge\beta$ for any $\pi\in\Omega$. The following propositions are established.

Proposition 3.3. *Let (Ω,Ψ) be an FβCAS, T a t-norm with R-implication I_T. The fuzzy neighborhood operators $TN^1_{\Psi,\beta}$ and $TN^3_{\Psi,\beta}$ defined by I_T are reflexive. If T is idempotent, $TN^2_{\Psi,\beta}$ and $TN^4_{\Psi,\beta}$ defined by T are β-reflexive.*

Proof. For an R-implication I_T, $I_T(\pi,\pi)=1$ holds for all $\pi\in\Omega$. Thus, it holds that

$$TN^1_{\Psi,\beta}(\pi)(\pi) = \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} I_T(\psi(\pi),\psi(\pi)) = 1.$$

Similarly, $TN^3_{\Psi,\beta}(\pi)(\pi)=1$, implying that $TN^1_{\Psi,\beta}$ and $TN^3_{\Psi,\beta}$ are reflexive.

If T is idempotent, then

$$TN^2_{\Psi,\beta}(\pi)(\pi) = \bigvee_{\psi\in md_{\Psi,\beta}(\pi)} T(\psi(\pi),\psi(\pi)) \ge T(\beta,\beta) = \beta.$$

Similarly, $TN^4_{\Psi,\beta}(\pi)(\pi)\ge\beta$, implying that $TN^2_{\Psi,\beta}$ and $TN^4_{\Psi,\beta}$ are β-reflexive. □

To demonstrate the T-transitivity of fuzzy neighborhood operators, the following lemma is introduced.

Lemma 3.1. *[24] For a left-continuous t-norm T with R-implication I_T, $T(I_T(\pi,\varpi),I_T(\varpi,\varsigma)) \le I_T(\pi,\varsigma)$ holds for any $\pi,\varpi,\varsigma\in[0,1]$.*

Proposition 3.4. *Let (Ω, Ψ) be an FβCAS, T a left-continuous t-norm with R-implication I_T. The fuzzy neighborhood operators $TN^1_{\Psi,\beta}$ and $TN^3_{\Psi,\beta}$ induced by I_T are T-transitive.*

Proof. From Lemma 3.1, the T-transitivity of $TN^1_{\Psi,\beta}$ is represented as

$$T(TN^1_{\Psi,\beta}(\pi)(\varpi), TN^1_{\Psi,\beta}(\varpi)(\varsigma)) \leq TN^1_{\Psi,\beta}(\pi)(\varsigma),$$

where $\pi, \varpi, \varsigma \in \Omega$. Furthermore, it follows that

$$\begin{aligned} T\left(\bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} I_T(\psi(\pi),\psi(\varpi)), \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} I_T(\psi(\varpi),\psi(\varsigma))\right) &\leq \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} T\left(I_T(\psi(\pi),\psi(\varpi)), I_T(\psi(\varpi),\psi(\varsigma))\right) \\ &\leq \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} I_T(\psi(\pi),\psi(\varsigma)). \end{aligned}$$

Thus, $TN^1_{\Psi,\beta}$ is T-transitivity. In a similar way, $TN^3_{\Psi,\beta}$ is also T-transitive. □

Next, the symmetry of fuzzy neighborhood operators is examined.

Proposition 3.5. *Let (Ω, Ψ) be an FβCAS and T a t-norm. The fuzzy neighborhood operator $TN^4_{\Psi,\beta}$ induced by T is symmetric.*

Proof. According to Proposition 3.2, $\bigvee_{\psi\in\Psi} T(\psi(\pi),\psi(\varpi)) = \bigvee_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} T(\psi(\pi),\psi(\varpi))$. Thus, it follows that

$$TN^4_{\Psi,\beta}(\pi)(\varpi) = TN^4_{\Psi,\beta}(\varpi)(\pi),$$

which holds by the symmetry of T. □

Since the fuzzy β-minimal descriptions of π and ϖ may differ, $TN^2_{\Psi,\beta}$ is not symmetric. Next, we explore the variations of fuzzy neighborhood operators with respect to β.

Proposition 3.6. *Let (Ω, Ψ) be an FβCAS and $0 < \beta_1 \leq \beta_2 \leq 1$. For any $\pi \in \Omega$, the following statements hold.*

$$TN^1_{\Psi,\beta_1}(\pi) \subseteq TN^1_{\Psi,\beta_2}(\pi),$$
$$TN^3_{\Psi,\beta_1}(\pi) \subseteq TN^3_{\Psi,\beta_2}(\pi),$$
$$TN^4_{\Psi,\beta_1}(\pi) \supseteq TN^4_{\Psi,\beta_2}(\pi).$$

Proof. For $0 < \beta_1 \leq \beta_2 \leq 1$ and $\pi, \varpi \in \Omega$, it follows that

$$\begin{aligned} TN^1_{\Psi,\beta_1}(\pi)(\varpi) &= \bigwedge_{\psi\in\mathbb{C}_{\beta_1}(\Psi,\pi)} I_T(\psi(\pi),\psi(\varpi)) \\ &= \bigwedge_{C\in\{\psi\in\Psi,\psi(\pi)\geq\beta_1\}} I_T(\psi(\pi),\psi(\varpi)) \\ &\leq \bigwedge_{C\in\{\psi\in\Psi,\psi(\pi)\geq\beta_2\}} I_T(\psi(\pi),\psi(\varpi)) \\ &= TN^1_{\Psi,\beta_2}(\pi)(\varpi), \end{aligned}$$

and

$$\begin{aligned} TN^3_{\Psi,\beta_1}(\pi)(\varpi) &= \bigwedge_{\psi\in MD_{\Psi,\beta_1}(\pi)} I_T(\psi(\pi),\psi(\varpi)) \\ &= \bigwedge_{\psi\in\{\psi\in\Psi:(\psi(\pi)\geq\beta_1)\wedge(\forall\psi'\in\Psi\wedge\psi'(\pi)\geq\beta_1\wedge\psi\subseteq\psi'\Rightarrow\psi'=\psi\}} I_T(\psi(\pi),\psi(\varpi)) \\ &\leq \bigwedge_{\psi\in\{\psi\in\Psi:(\psi(\pi)\geq\beta_2)\wedge(\forall\psi'\in\Psi\wedge\psi'(\pi)\geq\beta_2\wedge\psi\subseteq\psi'\Rightarrow\psi'=\psi\}} I_T(\psi(\pi),\psi(\varpi)) \\ &= TN^3_{\Psi,\beta_2}(\pi)(\varpi). \end{aligned}$$

Therefore, $TN^1_{\Psi,\beta_1}(\pi) \subseteq TN^1_{\Psi,\beta_2}(\pi)$ and $TN^3_{\Psi,\beta_1}(\pi) \subseteq TN^3_{\Psi,\beta_2}(\pi)$. Similarly, it holds that $TN^4_{\Psi,\beta_1}(\pi) \supseteq TN^4_{\Psi,\beta_2}(\pi)$. □

Remark 3.6. Note that when $0 < \beta_1 \leq \beta_2 \leq 1$, there is no monotonic relationship between fuzzy β-minimal descriptions md_{Ψ,β_1} and md_{Ψ,β_2}, so $TN^2_{\Psi,\beta_1}(\pi)$ and $TN^2_{\Psi,\beta_2}(\pi)$ do not exhibit monotonicity with respect to β.

This section concludes by exploring the conditions under which two fuzzy β-coverings construct identical fuzzy neighborhood operators. Since the aforementioned operators rely on fuzzy β-neighborhood system, fuzzy β-minimum and fuzzy β-maximum descriptions, respectively, the problem is transformed into identifying when two fuzzy β-coverings produce same fuzzy β-neighborhood, fuzzy β-maximum and fuzzy β-minimum descriptions. This issue has been studied by Yang and Hu in [35].

Definition 3.5. [35] Let (Ω, Ψ) be an FβCAS. For $\psi \in \Psi$, if $\psi(\pi) < \beta$ for any $\pi \in \Omega$, then ψ is a β-independent element of Ψ.

Furthermore, for each $\mathbb{B} \subseteq \Psi$, if $\Psi - \mathbb{B}$ consists of all β-independent elements of Ψ, then $\mathbb{B}$ is referred to as the β-base of Ψ, denoted as $\mathbb{B}^\beta(\Psi)$.

Proposition 3.7. *[35] For fuzzy β-coverings Ψ and Ψ', $\mathbb{C}_\beta(\Psi,\pi) = \mathbb{C}_\beta(\Psi',\pi)$ iff $\mathbb{B}^\beta(\Psi) = \mathbb{B}^\beta(\Psi')$ for each $\pi \in \Omega$.*

Subsequently, we introduce the notions of β-reducible element and β-dispensable element as follows.

Definition 3.6. [35] Let (Ω, Ψ) be an FβCAS and $\psi \in \Psi$. The element ψ is a β-reducible element of Ψ if either of the following conditions holds:

(1) ψ is a β-independent element of Ψ,
(2) for $\pi \in \Omega$, $\psi(\pi) \geq \beta$ means that there exists a $\psi' \in \Psi - \{\psi\}$ such that $\psi' \subseteq \psi$ and $\psi'(\pi) \geq \beta$.

Furthermore, for each $\mathbb{R} \subseteq \Psi$, if $\Psi - \mathbb{R}$ contains all β-reducible elements of Ψ, then $\mathbb{R}$ is termed the β-reduct of Ψ, denoted by $\mathbb{R}^\beta(\Psi)$.

Proposition 3.8. *[35] For fuzzy β-coverings Ψ and Ψ', $md_{\Psi,\beta}(\pi) = md_{\Psi',\beta}(\pi)$ iff $\mathbb{R}^\beta(\Psi) = \mathbb{R}^\beta(\Psi')$ for each $\pi \in \Omega$.*

Definition 3.7. [35] Let (Ω, Ψ) be an FβCAS and $\psi \in \Psi$. The element ψ is a β-dispensable element of Ψ if either of the following conditions holds:

(1) ψ is a β-independent element of Ψ,
(2) for $\pi \in \Omega$, $\psi(\pi) \geq \beta$ means that there exists a $\psi' \in \Psi - \{\psi\}$ such that $\psi \subseteq \psi'$.

Furthermore, for each $\mathbb{K} \subseteq \Psi$, if $\Psi - \mathbb{K}$ contains all β-dispensable elements of Ψ, then $\mathbb{K}$ is termed the kernel of Ψ, denoted by $\mathbb{K}^\beta(\Psi)$.

Proposition 3.9. *[35] For fuzzy β-coverings Ψ and Ψ', $MD_{\Psi,\beta}(\pi) = MD_{\Psi',\beta}(\pi)$ iff $\mathbb{K}^\beta(\Psi) = \mathbb{K}^\beta(\Psi')$ for each $\pi \in \Omega$.*

Theorem 3.1. *Let Ψ, Ψ' be two fuzzy β-coverings of Ω. For any $\pi \in \Omega$, it holds that*

(1) $TN^1_{\Psi,\beta}(\pi) = TN^1_{\Psi',\beta}(\pi)$ *and* $TN^4_{\Psi,\beta}(\pi) = TN^4_{\Psi',\beta}(\pi)$ *iff* $\mathbb{B}^\beta(\Psi) = \mathbb{B}^\beta(\Psi')$.
(2) $TN^2_{\Psi,\beta}(\pi) = TN^2_{\Psi',\beta}(\pi)$ *iff* $\mathbb{R}^\beta(\Psi) = \mathbb{R}^\beta(\Psi')$.
(3) $TN^3_{\Psi,\beta}(\pi) = TN^3_{\Psi',\beta}(\pi)$ *iff* $\mathbb{K}^\beta(\Psi) = \mathbb{K}^\beta(\Psi')$.

Proof. In light of Propositions 3.7, 3.8 and 3.9, it can be proved directly. □

3.3. Four kinds of fuzzy neighborhood operators derived from overlap functions

This subsection introduces four fuzzy neighborhood operators derived from overlap function O and its R-implication I_O, and explores their corresponding properties.

Definition 3.8. Let (Ω, Ψ) be an FβCAS. The fuzzy neighborhood operator based on R-implication I_O, denoted as $ON^1_{\Psi,\beta} : \Omega \longrightarrow \mathscr{F}(\Omega) : \pi \mapsto ON^1_{\Psi,\beta}(\pi)$, and the fuzzy neighborhood $ON^1_{\Psi,\beta}(\pi)$ is defined as

$$ON^1_{\Psi,\beta}(\pi)(\varpi) = \bigwedge_{\psi \in \mathbb{C}_\beta(\Psi,\pi)} I_O(\psi(\pi), \psi(\varpi)).$$

Remark 3.7. The relationships between $ON^1_{\Psi,\beta}$ and existing neighborhood operators are discussed below.

(1) When overlap function O fulfills the exchange principle, it follows that O and I_O become T and I_T. Therefore, $ON^1_{\Psi,\beta}$ is equivalent to $TN^1_{\Psi,\beta}$ in Definition 3.1.
(2) If $\beta = 1$, then $\mathbb{C}_\beta(\Psi,\pi) \subseteq \mathbb{C}(\Psi,\pi)$. Similar to Remark 3.2, we obtain that $N^\Psi_1(\pi) \subseteq ON^1_{\Psi,\beta}(\pi)$, where $N^\Psi_1(\pi)$ defined in [23].
(3) When the fuzzy β-covering degenerates into a crisp covering, it follows that fuzzy β-neighborhood system becomes neighborhood system and $ON^1_{\Psi,\beta}$ degenerates into a neighborhood operator. In fact,

$$\begin{aligned} ON^1_{\Psi,\beta}(\pi)(\varpi) = 1 \Longleftrightarrow & \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} I_O(\psi(\pi),\psi(\varpi)) = 1 \\ & \Longleftrightarrow \forall \psi \in \mathbb{C}_\beta(\Psi,\pi) : I_O(\psi(\pi),\psi(\varpi)) = 1 \\ & \Longleftrightarrow \forall \psi \in \mathbb{C}_\beta(\Psi,\pi) : \psi(\pi) = 1 \Longrightarrow \psi(\varpi) = 1 \\ & \Longleftrightarrow \forall \psi \in \mathbb{C}_\beta(\Psi,\pi) : \pi \in \psi \Longrightarrow \varpi \in \psi \\ & \Longleftrightarrow \varpi \in \cap\mathbb{C}_\beta(\Psi,\pi). \end{aligned}$$

(4) Since I_O satisfies (NP), i.e., $I_O(1,\pi) = \pi$ ($\forall \pi \in (0,1]$), the connection between $ON^1_{\Psi,\beta}$ and $\widetilde{FN^\beta_\Psi}$ (defined in [35]) is represented as follows:

$$\begin{aligned} ON^1_{\Psi,\beta}(\pi)(\varpi) &= \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} I_O(\psi(\pi),\psi(\varpi)) \\ &\geq \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} I_O(1,\psi(\varpi)) \\ &= \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} \psi(\varpi) \\ &= \widetilde{FN^\beta_\Psi}(\pi)(\varpi). \end{aligned}$$

Therefore, $\widetilde{FN^\beta_\Psi}(\pi) \subseteq ON^1_{\Psi,\beta}(\pi)$.

Proposition 3.10. *Let (Ω,Ψ) be an FβCAS and I_O be an R-implication. For each $\pi,\varpi \in \Omega$, it holds that*

$$\bigwedge_{\psi\in\Psi} I_O(\psi(\pi),\psi(\varpi)) = \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} I_O(\psi(\pi),\psi(\varpi)).$$

Proof. It can be proved in a similar way as Proposition 3.1. □

Next, an example is provided to illustrate that

$$\bigwedge_{\psi\in\Psi} I_O(\psi(\pi),\psi(\varpi)) = \bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} I_O(\psi(\pi),\psi(\varpi)) \neq \bigwedge_{\psi\in md_{\Psi,\beta}(\pi)} I_O(\psi(\pi),\psi(\varpi)).$$

Example 3.2. By applying Definition 3.8 to Example 3.1, we can observe that

$$\bigwedge_{\psi\in\mathbb{C}_{0.6}(\Psi,\pi)} I_T(\psi(\pi),\psi(\varpi)) \neq \bigwedge_{\psi\in md_{\Psi,0.6}(\pi)} I_T(\psi(\pi),\psi(\varpi))$$

is independent of the selection of t-norm and overlap function, thus $\bigwedge_{\psi\in\mathbb{C}_{0.6}(\Psi,\pi)} I_O(\psi(\pi),\psi(\varpi)) \neq \bigwedge_{\psi\in md_{\Psi,0.6}(\pi)} I_O(\psi(\pi),\psi(\varpi))$ still holds in an FβCAS.

Definition 3.9. Let (Ω,Ψ) be an FβCAS. The fuzzy neighborhood operator based on overlap function O, denoted as $ON^2_{\Psi,\beta} : \Omega \longrightarrow \mathscr{F}(\Omega) : \pi \mapsto ON^2_{\Psi,\beta}(\pi)$, and the fuzzy neighborhood $ON^2_{\Psi,\beta}(\pi)$ is defined as

$$ON^2_{\Psi,\beta}(\pi)(\varpi) = \bigvee_{\psi\in md_{\Psi,\beta}(\pi)} O(\psi(\pi),\psi(\varpi)).$$

Remark 3.8. The relationships between $ON^2_{\Psi,\beta}$ and existing neighborhood operators are discussed below.

(1) When overlap function O fulfills the exchange principle, $ON^2_{\Psi,\beta}$ is equivalent to $TN^2_{\Psi,\beta}$ in Definition 3.2.
(2) If $\beta = 1$, then $md_{\Psi,\beta}(\pi) \subseteq md(\Psi,\pi)$. Similar to Remark 3.2, we obtain that $ON^2_{\Psi,\beta} \subseteq N^\Psi_2(\pi)$, where $N^\Psi_2(\pi)$ defined in [23].

(3) When the fuzzy β-covering degenerates into a crisp covering, it follows that fuzzy β-minimal description becomes minimal description, and $ON^2_{\Psi,\beta}$ reduces to a neighborhood operator. In fact,

$$\begin{aligned} ON^2_{\Psi,\beta}(\pi)(\varpi)=1 &\Longleftrightarrow \bigvee_{\psi\in md_{\Psi,\beta}(\pi)} O(\psi(\pi),\psi(\varpi))=1 \\ &\Longleftrightarrow \bigvee_{\psi\in md(\Psi,\pi)} O(\psi(\pi),\psi(\varpi))=1 \\ &\Longleftrightarrow \exists\psi\in md(\Psi,\pi), \psi(\pi)=1 \bigwedge \psi(\varpi))=1 \\ &\Longleftrightarrow \exists\psi\in md(\Psi,\pi), \pi\in\psi \bigwedge \varpi\in\psi \\ &\Longleftrightarrow \varpi\in\cup md(\Psi,\pi). \end{aligned}$$

(4) For any $\beta\in(0,1]$ and overlap function O with (O7), the connection between fuzzy neighborhood operators $ON^2_{\Psi,\beta}$ and $\widetilde{SN^\beta_\Psi}$ (defined in [35]) is represented as follows:

$$\begin{aligned} ON^2_{\Psi,\beta}(\pi)(\varpi) &= \bigvee_{\psi\in md_{\Psi,\beta}(\pi)} O(\psi(\pi),\psi(\varpi)) \\ &\le \bigvee_{\psi\in md_{\Psi,\beta}(\pi)} O(1,\psi(\varpi)) \\ &\le \bigvee_{\psi\in md_{\Psi,\beta}(\pi)} \psi(\varpi) \\ &= \widetilde{SN^\beta_\Psi}(\pi)(\varpi). \end{aligned}$$

Therefore, $ON^2_{\Psi,\beta}(\pi)\subseteq\widetilde{SN^\beta_\Psi}(\pi)$.

Definition 3.10. Let (Ω,Ψ) be an FβCAS. The fuzzy neighborhood operator based on R-implication I_O, denoted as $ON^3_{\Psi,\beta}:\Omega\longrightarrow\mathscr{F}(\Omega):\pi\mapsto ON^3_{\Psi,\beta}(\pi)$, and the fuzzy neighborhood $ON^3_{\Psi,\beta}(\pi)$ is defined as

$$ON^3_{\Psi,\beta}(\pi)(\varpi)=\bigwedge_{\psi\in MD_{\Psi,\beta}(\pi)} I_O(\psi(\pi),\psi(\varpi)).$$

Remark 3.9. The relationships between $ON^3_{\Psi,\beta}$ and existing neighborhood operators are discussed below.

(1) When the overlap function O fulfills the exchange principle, $ON^3_{\Psi,\beta}$ is equivalent to $TN^3_{\Psi,\beta}$ in Definition 3.3.
(2) If $\beta=1$, then $MD_{\Psi,\beta}(\pi)\subseteq MD(\Psi,\pi)$. Similar to Remark 3.2, we obtain that $N^\Psi_3(\pi)\subseteq ON^3_{\Psi,\beta}(\pi)$, where $N^\Psi_3(\pi)$ in [23].
(3) When the fuzzy β-covering degenerates into a crisp covering, it follows that the fuzzy β-maximal description becomes a maximal description, and $ON^3_{\Psi,\beta}$ reduces to a neighborhood operator. In fact,

$$\begin{aligned} ON^3_{\Psi,\beta}(\pi)(\varpi)=1 &\Longleftrightarrow \bigwedge_{\psi\in MD_{\Psi,\beta}(\pi)} I_O(\psi(\pi),\psi(\varpi)) \\ &\Longleftrightarrow \bigwedge_{\psi\in MD(\Psi,\pi)} I_O(\psi(\pi),\psi(\varpi)) \\ &\Longleftrightarrow \forall\psi\in MD(\Psi,\pi): \psi(\pi)=1\Longrightarrow\psi(\varpi)=1 \\ &\Longleftrightarrow \forall\psi\in MD(\Psi,\pi): \pi\in\psi\Longrightarrow\varpi\in\psi \\ &\Longleftrightarrow \varpi\in\cap MD(\Psi,\pi). \end{aligned}$$

(4) Since I_O satisfies (NP), i.e., $I_O(1,\pi)=\pi$ $(\forall\pi\in(0,1])$, the connection between fuzzy neighborhood operators $ON^3_{\Psi,\beta}$ and $\widetilde{TN^\beta_\Psi}$ (defined in [35]) is represented as follows:

$$\begin{aligned} ON^3_{\Psi,\beta}(\pi)(\varpi) &= \bigwedge_{\psi\in MD_{\Psi,\beta}(\pi)} I_O(\psi(\pi),\psi(\varpi)) \\ &\ge \bigwedge_{\psi\in MD_{\Psi,\beta}(\pi)} I_O(1,\psi(\varpi)) \\ &= \bigwedge_{\psi\in MD_{\Psi,\beta}(\pi)} \psi(\varpi) \end{aligned}$$

$$= \widetilde{TN^{\beta}_{\Psi}}(\pi)(\varpi).$$

Therefore, $\widetilde{TN^{\beta}_{\Psi}}(\pi) \subseteq ON^{3}_{\Psi,\beta}(\pi)$.

Definition 3.11. Let (Ω, Ψ) be an FβCAS. The fuzzy neighborhood operator based on overlap function, denoted as $ON^{4}_{\Psi,\beta} : \Omega \longrightarrow \mathscr{F}(\Omega) : \pi \mapsto ON^{4}_{\Psi,\beta}(\pi)$, and the fuzzy neighborhood $ON^{4}_{\Psi,\beta}(\pi)$ is defined as

$$ON^{4}_{\Psi,\beta}(\pi)(\varpi) = \bigvee_{\psi \in \mathbb{C}_{\beta}(\Psi,\pi)} O(\psi(\pi), \psi(\varpi)).$$

Remark 3.10. The relationships between $ON^{4}_{\Psi,\beta}$ and existing neighborhood operators are discussed below.

(1) When the overlap function O fulfills the exchange principle, $ON^{4}_{\Psi,\beta}$ is equivalent to $TN^{4}_{\Psi,\beta}$ in Definition 3.4.
(2) If $\beta = 1$, then $\mathbb{C}_{\beta}(\Psi,\pi) \subseteq \mathbb{C}(\Psi,\pi)$. Similar to Remark 3.2, we obtain that $N^{\Psi}_{4}(\pi) \subseteq ON^{4}_{\Psi,\beta}(\pi)$, where $N^{\Psi}_{4}(\pi)$ defined in [23].
(3) When the fuzzy β-covering degenerates into a crisp covering, it follows that the fuzzy β-neighborhood system becomes a neighborhood system, and $ON^{4}_{\Psi,\beta}$ reduces into a neighborhood operator. In fact,

$$\begin{aligned}
ON^{4}_{\Psi,\beta}(\pi)(\varpi) = 1 &\Longleftrightarrow \bigvee_{\psi \in \mathbb{C}_{\beta}(\Psi,\pi)} O(\psi(\pi), \psi(\varpi)) = 1 \\
&\Longleftrightarrow \bigvee_{\psi \in \mathbb{C}_{\beta}(\Psi,\pi)} O(\psi(\pi), \psi(\varpi)) = 1 \\
&\Longleftrightarrow \exists \psi \in \mathbb{C}_{\beta}(\Psi,\pi) : \psi(\pi) = 1 \wedge \psi(\varpi) = 1 \\
&\Longleftrightarrow \exists \psi \in \mathbb{C}_{\beta}(\Psi,\pi) : \pi \in \psi \wedge \varpi \in \psi \\
&\Longleftrightarrow \varpi \in \cup \mathbb{C}_{\beta}(\Psi,\pi).
\end{aligned}$$

(4) For any $\beta \in (0,1]$, and overlap function O with $(O7)$, the connection between fuzzy neighborhood operators $ON^{4}_{\Psi,\beta}$ and $\widetilde{RN^{\beta}_{\Psi}}$ in [35] is defined as follows:

$$\begin{aligned}
ON^{4}_{\Psi,\beta}(\pi)(\varpi) &= \bigvee_{\psi \in \mathbb{C}_{\beta}(\Psi,\pi)} O(\psi(\pi), \psi(\varpi)) \\
&\le \bigvee_{\psi \in \mathbb{C}_{\beta}(\Psi,\pi)} \psi(\varpi) \\
&= \widetilde{RN^{\beta}_{\Psi}}(\pi)(\varpi).
\end{aligned}$$

Therefore, $ON^{4}_{\Psi,\beta}(\pi) \subseteq \widetilde{RN^{\beta}_{\Psi}}(\pi)$.

Proposition 3.11. *Let (Ω, Ψ) be an FβCAS. For each $\pi, \varpi \in \Omega$, it follows that*

$$\bigvee_{\psi \in \Psi} O(\psi(\pi), \psi(\varpi)) = \bigvee_{\psi \in \mathbb{C}_{\beta}(\Psi,\pi)} O(\psi(\pi), \psi(\varpi)) = \bigvee_{\psi \in MD_{\Psi,\beta}(\pi)} O(\psi(\pi), \psi(\varpi)).$$

Proof. The proof is similar to Proposition 3.11. $\square$

3.4. The characteristics of fuzzy neighborhood operators $ON^{k}_{\Psi,\beta}$

This subsection focuses on the reflexivity, symmetry and transitivity of $ON^{k}_{\Psi,\beta}$ $(k = 1,2,3,4)$. The specific conclusions are presented below.

Proposition 3.12. *Let (Ω, Ψ) be an FβCAS and O an overlap function with R-implication I_O. The fuzzy neighborhood operators $ON^{1}_{\Psi,\beta}$ and $ON^{3}_{\Psi,\beta}$ defined by I_O are reflexive if O satisfies (O7). The fuzzy neighborhood operators $ON^{2}_{\Psi,\beta}$ and $ON^{4}_{\Psi,\beta}$ defined by O are β-reflexive if O satisfies (O9).*

Proof. Since O satisfies (O7), $O(\pi, 1) \le \pi \le \varpi$ for any $\pi \le \varpi$. Thus, $I_O(\pi, \varpi) = \sup\{\varsigma \in [0,1] \mid O(\pi, \varsigma) \le \varpi\} = 1$, which implies that $I_O(\psi(\pi), \psi(\pi)) = 1$ for each $\pi \in \Omega$. Then, it holds that

$$ON^1_{\Psi,\beta}(\pi)(\pi) = \bigwedge_{\psi \in \mathbb{C}_\beta(\Psi,\pi)} I_O(\psi(\pi), \psi(\pi)) = 1.$$

Similarly, $ON^3_{\Psi,\beta}(\pi)(\pi) = 1$, i.e., both $ON^1_{\Psi,\beta}$ and $ON^3_{\Psi,\beta}$ are reflexive. In addition, if O satisfies that $O(\beta,\beta) \geq \beta$, then

$$ON^2_{\Psi,\beta}(\pi)(\pi) = \bigvee_{\psi \in md_{\Psi,\beta}(\pi)} O(\psi(\pi), \psi(\pi)) \geq O(\beta,\beta) \geq \beta.$$

Similarly, $ON^4_{\Psi,\beta}(\pi)(\pi) \geq \beta$, i.e., both $ON^2_{\Psi,\beta}$ and $ON^4_{\Psi,\beta}$ are β-reflexive. □

The following example demonstrates that (O7) is essential for the reflexivity of $ON^1_{\Psi,\beta}$ and $ON^3_{\Psi,\beta}$, while (O9) is required for the β-reflexivity of $ON^2_{\Psi,\beta}$ and $ON^4_{\Psi,\beta}$.

Example 3.3. Let $\Omega = \{a_1, a_2, a_3\}$ and $\Psi = \{\psi_1, \psi_2\}$, where

$$\psi_1 = \frac{0.5}{a_1} + \frac{0.9}{a_2} + \frac{0.7}{a_3} \text{ and } \psi_2 = \frac{0.8}{a_1} + \frac{0.6}{a_2} + \frac{0.8}{a_3}.$$

Then, Ψ forms a fuzzy β-covering for $\beta \in (0, 0.8]$. When $\beta = 0.6$, consider the overlap function $O_{m\frac{1}{2}}$, which satisfies (O8) and its R-implication $I_{O_{m\frac{1}{2}}}$, it holds that

$$ON^1_{\Psi,\beta}(a_1)(a_1) = ON^3_{\Psi,\beta}(a_1)(a_1) = 0.64 \neq 1.$$

Similarly, for $\beta = 0.6$, consider the overlap function O_2^V, which satisfies $O(\beta,\beta) \geq \beta$ and its R-implication $I_{O_2^V}$, it holds that

$$ON^2_{\Psi,\beta}(a_1)(a_1) = ON^4_{\Psi,\beta}(a_1)(a_1) = 0.5648 \ngeq 0.6.$$

D'eer et al. [5] verified that N_1^Ψ and N_3^Ψ are T-transitive fuzzy neighborhood operators in FCAS, i.e.,

$$T(N_1^\Psi(\pi)(\varpi), N_1^\Psi(\varpi)(\varsigma)) \leq N_1^\Psi(\pi)(\varsigma),$$
$$T(N_3^\Psi(\pi)(\varpi), N_3^\Psi(\varpi)(\varsigma)) \leq N_3^\Psi(\pi)(\varsigma).$$

Qi et al. [23] introduced the concept of O-transitivity and studied the O-transitivity of fuzzy neighborhood operators.

Definition 3.12. [23] Let N be a fuzzy neighborhood operator and O an overlap function. N is O-transitive iff $O(N(\pi)(\varpi), N(\varpi)(\varsigma)) \leq N(\pi)(\varsigma)$.

When O fulfills the exchange principle, $ON^1_{\Psi,\beta}$ and $ON^3_{\Psi,\beta}$ have O-transitivity. Otherwise, the following examples illustrate that $ON^1_{\Psi,\beta}$ and $ON^3_{\Psi,\beta}$ do not satisfy O-transitivity.

Example 3.4. Consider the FCAS (Ω, Ψ) in Example 3.3. Using the R-implication $I_{O_{m\frac{1}{2}}}$ to define $ON^1_{\Psi,\beta}$ and $ON^3_{\Psi,\beta}$, we obtain that

$$ON^1_{\Psi,\beta}(a_2)(a_3) = ON^3_{\Psi,\beta}(a_2)(a_3) = 0.49,$$
$$ON^1_{\Psi,\beta}(a_3)(a_1) = ON^3_{\Psi,\beta}(a_3)(a_1) = 0.25,$$
$$ON^1_{\Psi,\beta}(a_2)(a_1) = ON^3_{\Psi,\beta}(a_2)(a_1) = 0.25.$$

Thus, it holds that

$$O_{m\frac{1}{2}}(0.49, 0.25) = 0.5 > 0.25,$$

which implies that $ON^1_{\Psi,\beta}$ and $ON^3_{\Psi,\beta}$ do not satisfy the O-transitivity.

The following example further illustrates that O-transitivity does not hold when $ON^1_{\Psi,\beta}$ and $ON^3_{\Psi,\beta}$ fulfill (O7).

Example 3.5. Consider the FCAS (Ω, Ψ) in Example 3.3. Using the R-implication I_{O_2} to define $ON^1_{\Psi,\beta}$ and $ON^3_{\Psi,\beta}$, we obtain that

$$ON^1_{\Psi,\beta}(a_2)(a_3) = ON^3_{\Psi,\beta}(a_2)(a_3) = 0.93,$$
$$ON^1_{\Psi,\beta}(a_3)(a_1) = ON^3_{\Psi,\beta}(a_3)(a_1) = 1.00,$$
$$ON^1_{\Psi,\beta}(a_2)(a_1) = ON^3_{\Psi,\beta}(a_2)(a_1) = 0.79.$$

Therefore, we have that

$$O_2(0.93, 1.00) = 0.86 > 0.79,$$

which implies that $ON^1_{\Psi,\beta}$ and $ON^3_{\Psi,\beta}$ do not satisfy the O-transitivity.

Next, we examine the symmetry of $ON^4_{\Psi,\beta}$.

Proposition 3.13. *Let (Ω, Ψ) be an FβCAS. The fuzzy neighborhood operator $ON^4_{\Psi,\beta}$ is symmetric.*

Proof. According to Proposition 3.11, we have that

$$\bigvee_{\psi\in\Psi} O(\psi(\pi), \psi(\varpi)) = \bigvee_{\psi\in\mathbb{C}_\beta(\Psi,\pi)} O(\psi(\pi), \psi(\varpi)).$$

Due to the symmetry of O, it follows that

$$ON^4_{\Psi,\beta}(\pi)(\varpi) = ON^4_{\Psi,\beta}(\varpi)(\pi). \quad \square$$

Since fuzzy β-minimal descriptions of π and ϖ are not necessarily equal, $ON^2_{\Psi,\beta}$ is not symmetric.

Moreover, we present the variations of the fuzzy neighborhood operators $ON^k_{\Psi,\beta}$ $(k = 1, 2, 3, 4)$ with respect of β.

Proposition 3.14. *Let (Ω, Ψ) be an FβCAS. For any $0 < \beta_1 \le \beta_2 \le 1$ and $\pi \in \Omega$, the following statements hold.*

$$ON^1_{\Psi,\beta_1}(\pi) \subseteq ON^1_{\Psi,\beta_2}(\pi),$$
$$ON^3_{\Psi,\beta_1}(\pi) \subseteq ON^3_{\Psi,\beta_2}(\pi),$$
$$ON^4_{\Psi,\beta_1}(\pi) \supseteq ON^4_{\Psi,\beta_2}(\pi).$$

Proof. The proof is similar to Proposition 3.6. $\square$

Remark 3.11. Note that when $0 < \beta_1 \le \beta_2 \le 1$, there is no monotonic relationship between fuzzy β-minimal descriptions md_{Ψ,β_1} and md_{Ψ,β_2}, so $ON^2_{\Psi,\beta_1}(\pi)$ and $ON^2_{\Psi,\beta_2}(\pi)$ do not exhibit monotonicity with respect to β.

Based on Propositions 3.7, 3.8 and 3.9, we investigate the condition under which two fuzzy β-coverings produce identical fuzzy neighborhood operators $ON^k_{\Psi,\beta}$ $(k = 1, 2, 3, 4)$.

Theorem 3.2. *Let Ψ, Ψ' be two fuzzy β-coverings on Ω. For any $\pi \in \Omega$, it holds that*

(1) $ON^1_{\Psi,\beta}(\pi) = ON^1_{\Psi',\beta}(\pi)$ *and* $ON^4_{\Psi,\beta}(\pi) = ON^4_{\Psi',\beta}(\pi)$ *iff* $\mathbb{B}^\beta(\Psi) = \mathbb{B}^\beta(\Psi')$.
(2) $ON^2_{\Psi,\beta}(\pi) = ON^2_{\Psi',\beta}(\pi)$ *iff* $\mathbb{R}^\beta(\Psi) = \mathbb{R}^\beta(\Psi')$.
(3) $ON^3_{\Psi,\beta}(\pi) = ON^3_{\Psi',\beta}(\pi)$ *iff* $\mathbb{K}^\beta(\Psi) = \mathbb{K}^\beta(\Psi')$.

Note that when the β-base, β-reduct and kernel of two fuzzy β-coverings Ψ and Ψ' are equal, $ON^1_{\Psi,\beta}$ and $ON^4_{\Psi,\beta}$, $ON^2_{\Psi,\beta}$, $ON^3_{\Psi,\beta}$ induced by Ψ and Ψ' can conclude the same value, respectively.

4. Equalities among fuzzy neighborhood operators

4.1. Fuzzy β-coverings induced by a fuzzy β-covering

This subsection proposes several fuzzy β-coverings, denoted by Ψ^1_β, Ψ^2_β, Ψ^3_β, Ψ^4_β, Ψ^5_β, Ψ^6_β, $\Psi^\cap_\beta$ and $\Psi^\cup_\beta$, which are derived from a given fuzzy β-covering Ψ.

Definition 4.1. Let (Ω, Ψ) be an FβCAS. The fuzzy neighborhood operators $TN^4_{\Psi,\beta}(\pi)$ and $TN^1_{\Psi,\beta}(\pi)$ are based on T and I_T, while $ON^4_{\Psi,\beta}(\pi)$ and $ON^1_{\Psi,\beta}(\pi)$ are based on O and I_O. The collections of fuzzy sets are shown as follows:

$\Psi^1_\beta = \cup\{md_{\Psi,\beta}(\pi) : \pi \in \Omega\}$,
$\Psi^2_\beta = \cup\{MD_{\Psi,\beta}(\pi) : \pi \in \Omega\}$,
$\Psi^3_\beta = \{TN^1_{\Psi,\beta}(\pi) : \pi \in \Omega\}$,
$\Psi^4_\beta = \{TN^4_{\Psi,\beta}(\pi) : \pi \in \Omega\}$,
$\Psi^5_\beta = \{ON^1_{\Psi,\beta}(\pi) : \pi \in \Omega\}$,
$\Psi^6_\beta = \{ON^4_{\Psi,\beta}(\pi) : \pi \in \Omega\}$,
$\Psi^\cap_\beta = \Psi \setminus \{\psi \in \Psi : (\exists \Psi' \subseteq \Psi \setminus \{\psi\})(\psi = \cap\Psi')\}$,
$\Psi^\cup_\beta = \Psi \setminus \{\psi \in \Psi : (\exists \Psi' \subseteq \Psi \setminus \{\psi\})(\psi = \cup\Psi')\}$.

In what follows, we verify that these collections are all fuzzy β-coverings. Specifically, Ψ' is called a fuzzy β-subcovering of Ψ, if $\Psi' \subseteq \Psi$ and remains a fuzzy β-covering of Ω.

Proposition 4.1. *Let (Ω, Ψ) be an FβCAS. The collections Ψ^1_β, Ψ^2_β, $\Psi^\cap_\beta$ and $\Psi^\cup_\beta$ are all fuzzy β-subcoverings of Ψ.*

Proof. Since each of the four collections is a subset of the fuzzy β-covering Ψ, consisting of finite collections of non-empty fuzzy sets, we verify whether there exists $\psi \in \Psi^j_\beta$ such that $\psi(\pi) \geq \beta$ ($j = 1, 2, \cap, \cup$) for any $\pi \in \Omega$.

For Ψ^1_β and Ψ^2_β: Since Ψ is a fuzzy β-covering, there exists $\psi \in \Psi$ such that $\psi(\pi) \geq \beta$ for each $\pi \in \Omega$. Specifically, there exist $\psi_1 \in md_{\Psi,\beta}(\pi)$ and $\psi_2 \in MD_{\Psi,\beta}(\pi)$, where $\psi_1(\pi) \geq \beta$ and $\psi_2(\pi) \geq \beta$. Since $\psi_1 \in \Psi^1_\beta$ and $\psi_2 \in \Psi^2_\beta$, it follows that both Ψ^1_β and Ψ^2_β are fuzzy β-coverings.

For $\Psi^\cup_\beta$: assume that $\psi \notin \Psi^\cup_\beta$, implying that there exists $\Psi' \subseteq \Psi \setminus \{\psi\}$, $\psi = \cup\Psi'$. Furthermore, there exists $\psi' \in \Psi'$ such that $\psi'(\pi) \geq \beta$. As Ψ' can be chosen within $\Psi^\cup_\beta$, we conclude that there exists $\psi' \in \Psi^\cup_\beta$ such that $\psi'(\pi) \geq \beta$. Hence, $\Psi^\cup_\beta$ is a fuzzy β-covering.

For $\Psi^\cap_\beta$: assume that $\psi \notin \Psi^\cap_\beta$, implying that there exists $\Psi' \subseteq \Psi \setminus \{\psi\}$, $\psi = \cap\Psi'$. Moreover, for $\psi(\pi) \geq \beta$, there exists $\psi' \in \Psi'$ such that $\psi'(\pi) \geq \beta$. As Ψ' can be chosen within $\Psi^\cap_\beta$, we conclude that there exists $\psi' \in \Psi^\cap_\beta$ such that $\psi'(\pi) \geq \beta$. Hence, $\Psi^\cap_\beta$ is a fuzzy β-covering. $\square$

Proposition 4.2. *Let (Ω, Ψ) be an FβCAS. The collections Ψ^4_β and Ψ^3_β are constructed by T and I_T, respectively. Then, Ψ^3_β is a fuzzy β-covering, and Ψ^4_β is a fuzzy β-covering if T is idempotent.*

Proof. Since $TN^1_{\Psi,\beta}$ is reflexive, for any $\pi \in \Omega$, there exists $\psi = TN^1_{\Psi,\beta}(\pi) \in \Psi^3_\beta$ such that $\psi(\pi) = TN^1_{\Psi,\beta}(\pi)(\pi) = 1 \geq \beta$. Therefore, Ψ^3_β is a fuzzy β-covering.

Moreover, if T is idempotent, there exists $\psi = TN^4_{\Psi,\beta}(\pi) \in \Psi^4_\beta$ such that $\psi(\pi) = TN^4_{\Psi,\beta}(\pi)(\pi) \geq \beta$. Thus, Ψ^4_β is a fuzzy β-covering. $\square$

Proposition 4.3. *Let (Ω, Ψ) be an FβCAS. The collections Ψ^6_β and Ψ^5_β are constructed by O and I_O, respectively. Then, Ψ^5_β forms a fuzzy β-covering if O satisfies (O7) and Ψ^6_β forms a fuzzy β-covering if O satisfies (O9).*

Proof. Since $ON^1_{\Psi,\beta}$ is reflexive when O satisfies (O7), and $ON^4_{\Psi,\beta}$ is β-reflexive when O satisfies (O9), it can be proved that Ψ^5_β and Ψ^6_β are fuzzy β-coverings. $\square$

Proposition 4.4. *Let (Ω, Ψ) be an FβCAS. The collection Ψ^2_β is a fuzzy β-subcovering of $\Psi^\cap_\beta$.*

Proof. Suppose that $\psi \in \Psi^2_\beta$, there exists $\pi \in \Omega$ such that $\psi \in MD_{\Psi,\beta}(\pi)$. If $\psi \notin \Psi^\cap_\beta$, then there exists $\Psi' \subseteq \Psi \setminus \{\psi\}$ such that $\psi = \cap\Psi'$. As Ψ' can be chosen within $\Psi^\cap_\beta$, we get $\psi' \in \Psi'$ such that $\psi(\pi) = \psi'(\pi) \geq \beta$. Since $\psi \subseteq \psi'$, $\psi(\pi) = \psi'(\pi) \geq \beta$ and $\psi \in MD_{\Psi,\beta}(\pi)$, it holds that $\psi = \psi'$, which is a contradiction. Therefore, $\psi \in \Psi^\cap_\beta$ and $\Psi^2_\beta \subseteq \Psi^\cap_\beta$. $\square$

Proposition 4.5. *Let (Ω, Ψ) be an FβCAS. The collection Ψ^1_β is a fuzzy β-subcovering of $\Psi^\cup_\beta$.*

Proof. Suppose that $\psi \in \Psi^1_\beta$, there exists $\pi \in \Omega$ such that $\psi \in md_{\Psi,\beta}(\pi)$. If $\psi \notin \Psi^\cup_\beta$, then there exists $\Psi' \subseteq \Psi \setminus \{\psi\}$ such that $\psi = \cup\Psi'$. As Ψ' can be chosen within $\Psi^\cup_\beta$, we get $\psi' \in \Psi'$ such that $\psi(\pi) = \psi'(\pi) \geq \beta$. Since $\psi' \subseteq \psi$, $\psi(\pi) = \psi'(\pi) \geq \beta$ and $\psi \in md_{\Psi,\beta}(\pi)$, it holds that $\psi = \psi'$, which is contradiction. Therefore, $\psi \in \Psi^\cup_\beta$ and $\Psi^1_\beta \subseteq \Psi^\cup_\beta$. $\square$

Table 3
The values of $\mathbb{C}_\beta(\Psi, a_i)$, $\mathrm{md}_{\Psi,\beta}(a_i)$ and $\mathrm{MD}_{\Psi,\beta}(a_i)$.

	$\mathbb{C}_\beta(\Psi, a_i)$	$\mathrm{md}_{\Psi,\beta}(a_i)$	$\mathrm{MD}_{\Psi,\beta}(a_i)$
a_1	$\{\psi_1, \psi_2, \psi_4, \psi_5\}$	$\{\psi_1, \psi_2, \psi_4\}$	$\{\psi_5\}$
a_2	$\{\psi_1, \psi_2, \psi_5\}$	$\{\psi_1, \psi_2\}$	$\{\psi_5\}$
a_3	$\{\psi_2, \psi_3, \psi_4, \psi_5\}$	$\{\psi_3, \psi_4\}$	$\{\psi_5\}$

The following example illustrates the aforementioned conclusions.

Example 4.1. Suppose that $\Omega = \{a_1, a_2, a_3\}$ and $\Psi = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}$, where

$$\psi_1 = \frac{0.9}{a_1} + \frac{0.8}{a_2} + \frac{0.7}{a_3}, \psi_2 = \frac{0.8}{a_1} + \frac{0.8}{a_2} + \frac{0.8}{a_3}, \psi_3 = \frac{0.6}{a_1} + \frac{0.7}{a_2} + \frac{0.8}{a_3},$$
$$\psi_4 = \frac{0.8}{a_1} + \frac{0.6}{a_2} + \frac{0.9}{a_3}, \psi_5 = \frac{0.9}{a_1} + \frac{0.8}{a_2} + \frac{0.9}{a_3}.$$

Then, Ψ forms a fuzzy β-covering for $\beta \in (0, 0.8]$. For $\beta = 0.8$, the fuzzy β-neighborhood system, fuzzy β-minimal description and fuzzy β-maximal description of each $\pi \in \Omega$ are calculated as shown in Table 3.

By Definition 4.1, we obtain the following fuzzy β-coverings:

$$\Psi^1_\beta = \{\psi_1, \psi_2, \psi_3, \psi_4\}, \Psi^2_\beta = \{\psi_5\}, \Psi^\cap_\beta = \Psi, \Psi^\cup_\beta = \{\psi_1, \psi_2, \psi_3, \psi_4\}.$$

If $T = T_M$ and $I_T = I_{T_M}$, then

$$\Psi^3_\beta = \left\{ \frac{1}{a_1} + \frac{0.6}{a_2} + \frac{0.7}{a_3}, \frac{1}{a_1} + \frac{1}{a_2} + \frac{0.7}{a_3}, \frac{0.6}{a_1} + \frac{0.6}{a_2} + \frac{1}{a_3} \right\},$$

and

$$\Psi^4_\beta = \left\{ \frac{0.9}{a_1} + \frac{0.8}{a_2} + \frac{0.9}{a_3}, \frac{0.8}{a_1} + \frac{0.8}{a_2} + \frac{0.8}{a_3} \right\}.$$

Additionally, when $O = O_2$ and $I_O = I_{O_2}$, we have that

$$\Psi^5_\beta = \left\{ \frac{1}{a_1} + \frac{0.97}{a_2} + \frac{0.93}{a_3}, \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}, \frac{0.97}{a_1} + \frac{0.86}{a_2} + \frac{1}{a_3} \right\}.$$

And when $O = O_{m^{\frac{1}{2}}}$, it follows that

$$\Psi^6_\beta = \left\{ \frac{0.95}{a_1} + \frac{0.89}{a_2} + \frac{0.95}{a_3}, \frac{0.89}{a_1} + \frac{0.89}{a_2} + \frac{0.89}{a_3}, \frac{0.95}{a_1} + \frac{0.89}{a_2} + \frac{0.95}{a_3} \right\}.$$

Obviously, Ψ^2_β is a fuzzy β-subcovering of $\Psi^\cap_\beta$ and Ψ^1_β is a fuzzy β-subcovering of $\Psi^\cup_\beta$.

It is noteworthy that the inequality relationships among fuzzy neighborhood operators in covering approximation space remain valid in an FβCAS. To explore the equalities among fuzzy neighborhood operators in an FβCAS, we first establish key results concerning different fuzzy β-coverings that produce identical fuzzy β-maximum and fuzzy β-minimum descriptions.

Proposition 4.6. *Let (Ω, Ψ) be an FβCAS. For any $\pi \in \Omega$, it holds that*

(1) $md_{\Psi^1_\beta,\beta}(\pi) = md_{\Psi,\beta}(\pi)$.
(2) $MD_{\Psi^2_\beta,\beta}(\pi) = MD_{\Psi,\beta}(\pi)$.
(3) $MD_{\Psi^\cap_\beta,\beta}(\pi) = MD_{\Psi,\beta}(\pi)$.

Proof. According to Definition 2.5, the proof follows directly.

(1) For each $\pi \in \Omega$, if $\psi \in md_{\Psi,\beta}(\pi)$, then $\psi \in \Psi^1_\beta$ and $\psi(\pi) \geq \beta$. Let $\psi' \in \Psi^1_\beta$ with $\psi'(\pi) \geq \beta$ and $\psi' \subseteq \psi$. Since $\psi' \in \Psi$ and $\psi \in md_{\Psi,\beta}(\pi)$, it follows that $\psi' = \psi$. Thus, we conclude that $\psi \in md_{\Psi^1_\beta,\beta}(\pi)$.

Conversely, if $\psi \in md_{\Psi^1_\beta,\beta}(\pi)$, then $\psi \in \Psi^1_\beta \subseteq \Psi$ and $\psi(\pi) \geq \beta$. There exists $\psi' \in md_{\Psi,\beta}(\pi)$, $\psi'(\pi) \geq \beta$ and $\psi' \subseteq \psi$. Since $\psi' \in \Psi^1_\beta$ and $\psi \in md_{\Psi^1_\beta,\beta}(\pi)$, it holds that $\psi' = \psi$. Hence, we obtain that $\psi \in md_{\Psi,\beta}(\pi)$.

(2) For each $\pi \in \Omega$, if $\psi \in MD_{\Psi,\beta}(\pi)$, then $\psi \in \Psi^2_\beta$ and $\psi(\pi) \geq \beta$. Let $\psi' \in \Psi^2_\beta$ with $\psi'(\pi) \geq \beta$ and $\psi \subseteq \psi'$. Since $\psi' \in \Psi$ and $\psi \in MD_{\Psi,\beta}(\pi)$, it follows that $\psi' = \psi$. Thus, we conclude that $\psi \in md_{\Psi^2_\beta,\beta}(\pi)$.

Conversely, if $\psi \in MD_{\Psi^2_\beta,\beta}(\pi)$, then $\psi \in \Psi^2_\beta \subseteq \Psi$ and $\psi(\pi) \geq \beta$. There exists $\psi' \in MD_{\Psi,\beta}(\pi)$, $\psi'(\pi) \geq \beta$ and $\psi \subseteq \psi'$. Since $\psi' \in \Psi^2_\beta$ and $\psi \in MD_{\Psi^2_\beta,\beta}(\pi)$, it holds that $\psi' = \psi$. Hence, we obtain that $\psi \in MD_{\Psi,\beta}(\pi)$.

(3) By Proposition 4.4, Ψ^2_β is a fuzzy β-subcovering of $\Psi^\cap_\beta$. If $\psi \in MD_{\Psi,\beta}(\pi)$, then $\psi \in \Psi^2_\beta \subseteq \Psi^\cap_\beta$ and $\psi(\pi) \geq \beta$. Let $\psi' \in \Psi^\cap_\beta$, $\psi'(\pi) \geq \beta$ and $\psi \subseteq \psi'$. Since $\psi' \in \Psi$ and $\psi \in MD_{\Psi,\beta}(\pi)$, it holds that $\psi = \psi'$. Then, $\psi \in MD_{\Psi^\cap_\beta,\beta}(\pi)$.
Conversely, if $\psi \in MD_{\Psi^\cap_\beta,\beta}(\pi)$, then $\psi \in \Psi^\cap_\beta \subseteq \Psi$ and $\psi(\pi) \geq \beta$. Hence, there exists $\psi' \in MD_{\Psi,\beta}(\pi)$, $\psi'(\pi) \geq \beta$ and $\psi \subseteq \psi'$. Since $\psi' \in \Psi^2_\beta \subseteq \Psi^\cap_\beta$ and $\psi \in MD_{\Psi^\cap_\beta,\beta}(\pi)$, it holds that $\psi' = \psi$. Hence, we obtain that $\psi \in MD_{\Psi,\beta}(\pi)$. $\square$

Notably, the equalities $md_{\Psi^2_\beta,\beta}(\pi) = \mathbb{C}_\beta(\Psi,\pi)$ and $MD_{\Psi^2_\beta,\beta}(\pi) = \mathbb{C}_\beta(\Psi,\pi)$ do not hold in fuzzy β-covering, which are explained by the following counterexample.

Example 4.2. Suppose that $\Omega = \{a_1, a_2\}$ and $\Psi = \{\psi_1, \psi_2\}$, where

$$\psi_1 = \frac{0.7}{a_1} + \frac{0.9}{a_2} \text{ and } \psi_2 = \frac{0.6}{a_1} + \frac{0.8}{a_2}.$$

The collection Ψ forms a fuzzy β-covering for $\beta \in (0, 0.7]$. For $\beta = 0.5$, we obtain that

$$\mathbb{C}_\beta(\Psi, a_1) = \mathbb{C}_\beta(\Psi, a_2) = \{\psi_1, \psi_2\},$$
$$md_{\Psi,\beta}(a_1) = md_{\Psi,\beta}(a_2) = \{\psi_2\},$$
$$MD_{\Psi,\beta}(a_1) = MD_{\Psi,\beta}(a_2) = \{\psi_1\}.$$

By Definition 4.1, it follows that $\Psi^2_\beta = \{\psi_1\}$, and then $md_{\Psi^2_\beta,\beta}(a_1) = md_{\Psi^2_\beta,\beta}(a_2) = \{\psi_1\}$, $MD_{\Psi^2_\beta,\beta}(a_1) = MD_{\Psi^2_\beta,\beta}(a_2) = \{\psi_1\}$.

4.2. Equalities among fuzzy neighborhood operators derived from t-norms

Next, we analyze the first group: fuzzy neighborhood operators $TN^1_{\Psi,\beta}$, $TN^1_{\Psi^1_\beta,\beta}$, $TN^1_{\Psi^3_\beta,\beta}$, $TN^1_{\Psi^\cap_\beta,\beta}$ and $TN^2_{\Psi^3_\beta,\beta}$, where $TN^1_{\Psi,\beta}$, $TN^1_{\Psi^3_\beta,\beta}$, $TN^1_{\Psi^\cap_\beta,\beta}$ are equivalent, while the remaining are different.

Proposition 4.7. *Let (Ω, Ψ) be an FβCAS and I_T an R-implication to construct the fuzzy β-covering Ψ^3_β and fuzzy neighborhood operators $TN^1_{\Psi,\beta}$, $TN^1_{\Psi^3_\beta,\beta}$, $TN^1_{\Psi^\cap_\beta,\beta}$, it follows that*

(1) $TN^1_{\Psi,\beta} = TN^1_{\Psi^3_\beta,\beta}$, *if I_T is the R-implication of a left-continuous t-norm.*
(2) $TN^1_{\Psi,\beta} = TN^1_{\Psi^\cap_\beta,\beta}$.

Proof. (1) Since $TN^1_{\Psi,\beta}(\pi) \in \Psi^3_\beta$, for any $\varpi \in \Omega$, we have that

$$\begin{aligned} TN^1_{\Psi^3_\beta,\beta}(\pi)(\varpi) &= \bigwedge_{\psi \in \Psi^3_\beta} I_T(\psi(\pi), \psi(\varpi)) \\ &\leq I_T(TN^1_{\Psi,\beta}(\pi)(\pi), TN^1_{\Psi,\beta}(\pi)(\varpi)) \\ &= I_T(1, TN^1_{\Psi,\beta}(\pi)(\varpi)) \\ &= TN^1_{\Psi,\beta}(\pi)(\varpi). \end{aligned}$$

Conversely, $TN^1_{\Psi,\beta}$ is T-transitive. For any $\varsigma \in \Omega$, it follows that

$$\begin{aligned} T(TN^1_{\Psi,\beta}(\varsigma)(\pi), TN^1_{\Psi,\beta}(\pi)(\varpi)) \leq TN^1_{\Psi,\beta}(\varsigma)(\varpi) &\Longrightarrow T(TN^1_{\Psi,\beta}(\pi)(\varpi), TN^1_{\Psi,\beta}(\varsigma)(\pi)) \leq TN^1_{\Psi,\beta}(\varsigma)(\varpi) \\ &\Longrightarrow TN^1_{\Psi,\beta}(\pi)(\varpi) \leq I_T(TN^1_{\Psi,\beta}(\varsigma)(\pi), TN^1_{\Psi,\beta}(\varsigma)(\varpi)). \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} TN^1_{\Psi,\beta}(\pi)(\varpi) &\leq \bigwedge_{\varsigma \in \Omega} I_T(TN^1_{\Psi,\beta}(\varsigma)(\pi), TN^1_{\Psi,\beta}(\varsigma)(\varpi)) \\ &= \bigwedge_{\psi \in \Psi^3_\beta} I_T(\psi(\pi), \psi(\varpi)) \end{aligned}$$

$$= TN^1_{\Psi^3_\beta,\beta}(\pi)(\varpi).$$

(2) Since $\Psi^\cap_\beta \subseteq \Psi$, we conclude that $TN^1_{\Psi,\beta}(\pi) \subseteq TN^1_{\Psi^\cap_\beta,\beta}(\pi)$ for any $\pi \in \Omega$.

On the other hand, let $\psi \in \Psi$ such that $TN^1_{\Psi,\beta}(\pi)(\varpi) = I_T(\psi(\pi),\psi(\varpi))$ for any $\varpi \in \Omega$. If $\psi \in \Psi^\cap_\beta$, then $TN^1_{\Psi^\cap_\beta,\beta}(\pi)(\varpi) \leq I_T(\psi(\pi),\psi(\varpi)) = TN^1_{\Psi,\beta}(\pi)(\varpi)$. If $\psi \notin \Psi^\cap_\beta$, then there exists $\Psi' \subseteq \Psi^\cap_\beta$ such that $\psi = \cap\Psi'$. Furthermore, there exists $\psi' \in \Psi'$ such that $\psi'(\varpi) = \psi(\varpi)$ and $\psi \subseteq \psi'$. Hence, $I_T(\psi(\pi),\psi(\varpi)) \geq I_T(\psi'(\pi),\psi'(\varpi))$. Since the infimum of $TN^1_{\Psi,\beta}(\pi)(\varpi)$ is got in ψ and $\psi' \in \Psi$, we have $I_T(\psi(\pi),\psi(\varpi)) \leq I_T(\psi'(\pi),\psi'(\varpi))$. Thus,

$$\begin{aligned} TN^1_{\Psi^\cap_\beta,\beta}(\pi)(\varpi) &\leq I_T(\psi'(\pi),\psi'(\varpi)) \\ &= I_T(\psi(\pi),\psi(\varpi)) \\ &= TN^1_{\Psi,\beta}(\pi)(\varpi), \end{aligned}$$

i.e., $TN^1_{\Psi^\cap_\beta,\beta}(\pi) \subseteq TN^1_{\Psi,\beta}(\pi)$. In summary, it concludes that $TN^1_{\Psi^\cap_\beta,\beta} = TN^1_{\Psi,\beta}$. □

In the following, we consider that $TN^1_{\Psi,\beta}$ and $TN^1_{\Psi^1_\beta,\beta}$, $TN^1_{\Psi,\beta}$ and $TN^2_{\Psi^3_\beta,\beta}$ are not equal in general.

Example 4.3. Let Ψ be a fuzzy 0.6-covering in Example 3.1. Consider $T = T_M$ and $I = I_{T_M}$, then $\Psi^1_\beta = \{\psi_2\}$ and

$$\Psi^3_\beta = \left\{ \frac{1}{a_1} + \frac{0.6}{a_2} + \frac{1}{a_3}, \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}, \frac{0.7}{a_1} + \frac{0.6}{a_2} + \frac{1}{a_3} \right\}.$$

Hence, it holds that

$$\begin{aligned} &TN^1_{\Psi,\beta}(a_1)(a_2) = 0.6 < 1 = TN^1_{\Psi^1_\beta,\beta}(a_1)(a_2), \\ &TN^1_{\Psi,\beta}(a_1)(a_1) = 1 > 0.7 = TN^2_{\Psi^3_\beta,\beta}(a_1)(a_2), \end{aligned}$$

i.e., $TN^1_{\Psi,\beta} \neq TN^1_{\Psi^1_\beta,\beta}$ and $TN^1_{\Psi,\beta} \neq TN^2_{\Psi^3_\beta,\beta}$.

Proposition 4.8. *Let (Ω,Ψ) be an FβCAS and T a t-norm to construct fuzzy neighborhood operators $TN^2_{\Psi,\beta}$ and $TN^2_{\Psi^1_\beta,\beta}$. Then, $TN^2_{\Psi,\beta} = TN^2_{\Psi^1_\beta,\beta}$.*

Proof. It can be directly proved by Proposition 4.6 (1). □

Furthermore, we explore the third group: fuzzy neighborhood operators $TN^3_{\Psi,\beta}$, $TN^3_{\Psi^2_\beta,\beta}$, $TN^3_{\Psi^\cap_\beta,\beta}$ and $TN^1_{\Psi^2_\beta,\beta}$ are equivalent.

Lemma 4.1. *Let (Ω,Ψ) be an FβCAS. For any $\psi \in \Psi$, $\psi \in \Psi^2_\beta(\pi)$ iff $\psi \in MD_{\Psi,\beta}(\pi)$.*

Proof. For any $\psi \in \Psi$, if $\psi \in MD_{\Psi,\beta}(\pi)$, then $\psi \in \Psi^2_\beta = \bigcup\{MD_{\Psi,\beta}(\pi) : \pi \in \Omega\}$ and $\psi(\pi) \geq \beta$, that is, $MD_{\Psi,\beta}(\pi) \subseteq \Psi^2_\beta(\pi)$. Conversely, for any $\psi \in \Psi^2_\beta(\pi)$, there exists $\varpi \in \Omega$ such that $\psi \in MD_{\Psi,\beta}(\varpi)$ and $\psi(\pi) \geq \beta$. If $\psi \notin MD_{\Psi,\beta}(\pi)$, then there exists $\psi' \in \mathbb{C}_\beta(\Psi,\pi), \psi' \neq \psi$ and $\psi' \supseteq \psi$. Since $\psi'(\varpi) \geq \psi(\varpi) \geq \beta$, it follows that $\psi' \in \mathbb{C}_\beta(\Psi,\varpi)$, which contradicts the assumption that $\psi \in MD_{\Psi,\beta}(\varpi)$. Therefore, $\psi \in MD_{\Psi,\beta}(\pi)$, implying that $\Psi^2_\beta(\pi) \subseteq MD_{\Psi,\beta}(\pi)$. □

Proposition 4.9. *Let (Ω,Ψ) be an FβCAS and I_T an R-implication to construct fuzzy neighborhood operators $TN^3_{\Psi,\beta}$, $TN^3_{\Psi^2_\beta,\beta}$ and $TN^3_{\Psi^\cap_\beta,\beta}$, it follows that*

(1) $TN^3_{\Psi,\beta} = TN^3_{\Psi^2_\beta,\beta}$.
(2) $TN^3_{\Psi,\beta} = TN^3_{\Psi^\cap_\beta,\beta}$.
(3) $TN^3_{\Psi,\beta} = TN^1_{\Psi^2_\beta,\beta}$.

Proof. In light of Proposition 4.6 (2) and (3), we can directly prove items (1) and (2), and item (3) can be proved by Lemma 4.1. □

Table 4
t-norm-based fuzzy neighborhood operators in an FβCAS.

Group	Operators	Group	Operators
$A1$	$TN^1_{\Psi,\beta}, TN^1_{\Psi^3_\beta,\beta}, TN^1_{\Psi^\cap_\beta,\beta}$	I	$TN^2_{\Psi^4_\beta,\beta}$
$A2$	$TN^1_{\Psi^1_\beta,\beta}$	J	$TN^3_{\Psi^4_\beta,\beta}$
$A3$	$TN^2_{\Psi^3_\beta,\beta}$	K	$TN^4_{\Psi^4_\beta,\beta}$
B	$TN^3_{\Psi^1_\beta,\beta}$	L	$TN^4_{\Psi^1_\beta,\beta}$
C	$TN^3_{\Psi^3_\beta,\beta}$	M	$TN^2_{\Psi^\cap_\beta,\beta}$
D	$TN^4_{\Psi^3_\beta,\beta}$	N	$TN^1_{\Psi^\cup_\beta,\beta}$
E	$TN^2_{\Psi,\beta}, TN^2_{\Psi^1_\beta,\beta}$	O	$TN^2_{\Psi^\cup_\beta,\beta}$
F	$TN^3_{\Psi,\beta}, TN^3_{\Psi^2_\beta,\beta}, TN^3_{\Psi^\cap_\beta,\beta}, TN^1_{\Psi^2_\beta,\beta}$	P	$TN^3_{\Psi^\cup_\beta,\beta}$
G	$TN^1_{\Psi^4_\beta,\beta}$	Q	$TN^4_{\Psi^\cup_\beta,\beta}$
H	$TN^4_{\Psi,\beta}, TN^4_{\Psi^2_\beta,\beta}, TN^4_{\Psi^\cap_\beta,\beta}, TN^2_{\Psi^2_\beta,\beta}$		

Finally, we present the group consisted of fuzzy neighborhood operators $TN^4_{\Psi,\beta}$, $TN^4_{\Psi^2_\beta,\beta}$, $TN^4_{\Psi^\cap_\beta,\beta}$ and $TN^2_{\Psi^2_\beta,\beta}$.

Proposition 4.10. *Let (Ω,Ψ) be an FβCAS and T a t-norm to construct fuzzy neighborhood operators $TN^4_{\Psi,\beta}$, $TN^4_{\Psi^2_\beta,\beta}$ and $TN^4_{\Psi^\cap_\beta,\beta}$, it follows that*

(1) $TN^4_{\Psi,\beta} = TN^4_{\Psi^2_\beta,\beta}$.
(2) $TN^4_{\Psi,\beta} = TN^4_{\Psi^\cap_\beta,\beta}$.
(3) $TN^4_{\Psi,\beta} = TN^2_{\Psi^2_\beta,\beta}$.

Proof. By Propositions 3.2 and 4.6 (2) and (3), we can immediately verify items (1) and (2).

(3) To prove $TN^4_{\Psi,\beta} = TN^2_{\Psi^2_\beta,\beta}$, we first demonstrate that $MD_{\Psi,\beta}(\pi) = md_{\Psi^2_\beta,\beta}(\pi)$. If $\psi \in md_{\Psi^2_\beta,\beta}(\pi)$, then it holds that $\psi \in \Psi^2_\beta(\pi)$. Furthermore, $\psi \in MD_{\Psi,\beta}(\pi)$ can be obtained by Lemma 4.1, implying that $md_{\Psi^2_\beta,\beta}(\pi) \subseteq MD_{\Psi,\beta}(\pi)$. Conversely, if $\psi \in MD_{\Psi,\beta}(\pi)$, then it must hold that $\psi \in \Psi^2_\beta$. For any $\psi' \in \Psi^2_\beta, \psi'(\pi) \geq \beta$ and $\psi' \subseteq \psi$, it follows that $\psi' = \psi$. Then, $\psi \in md_{\Psi^2_\beta,\beta}(\pi)$, implying that $MD_{\Psi,\beta}(\pi) \subseteq md_{\Psi^2_\beta,\beta}(\pi)$. Therefore, we conclude that $MD_{\Psi,\beta}(\pi) = md_{\Psi^2_\beta,\beta}(\pi)$ and $TN^4_{\Psi,\beta} = TN^2_{\Psi^2_\beta,\beta}$. □

In summary, within the framework of FβCAS, let T be a left-continuous t-norm used to construct Ψ^4_β, $TN^2_{\Psi^i_\beta,\beta}$ and $TN^4_{\Psi^i_\beta,\beta}$. Meanwhile, let its R-implication I_T be used to define Ψ^3_β, $TN^1_{\Psi^i_\beta,\beta}$ and $TN^3_{\Psi^i_\beta,\beta}$ $(i = 1,2,3,4,\cap,\cup)$, we obtain nineteen groups of fuzzy neighborhood operators, as shown in Table 4.

Next, we examine the partial order relationships "$\leq$" among different groups in Table 4. For fuzzy neighborhood operators N_1 and N_2, it holds that

$$N_1 \leq N_2 \Longleftrightarrow N_1(\pi)(\varpi) \leq N_2(\pi)(\varpi) \text{ for any } \pi, \varpi \in \Omega.$$

If neither $N_1 \leq N_2$ nor $N_2 \leq N_1$ holds, then N_1 and N_2 are incomparable. Notably, if two neighborhood operators are not comparable in covering approximation space, they remain incomparable in an FβCAS.

Proposition 4.11. *Let (Ω,Ψ) be an FβCAS, T a left-continuous t-norm to construct fuzzy neighborhood operators $TN^2_{\Psi,\beta}$ and $TN^4_{\Psi,\beta}$, I_T its R-implication to define $TN^1_{\Psi,\beta}$ and $TN^3_{\Psi,\beta}$, it follows that*

(1) $TN^1_{\Psi,\beta} \leq TN^3_{\Psi,\beta}$.
(2) $TN^2_{\Psi,\beta} \leq TN^4_{\Psi,\beta}$.

Proof. It can be proved by $MD_{\Psi,\beta}(\pi) \subseteq \mathbb{C}_\beta(\Psi,\pi)$ and $md_{\Psi,\beta}(\pi) \subseteq \mathbb{C}_\beta(\Psi,\pi)$, respectively. □

In covering approximation space [25], it holds that $N^\Psi_1 \leq N^\Psi_2 \leq N^\Psi_4$ and $N^\Psi_1 \leq N^\Psi_3 \leq N^\Psi_4$. However, these conclusions do not hold in an FβCAS. The following counterexample illustrates that $TN^1_{\Psi,\beta} \leq TN^2_{\Psi,\beta} \leq TN^4_{\Psi,\beta}$ and $TN^1_{\Psi,\beta} \leq TN^3_{\Psi,\beta} \leq TN^4_{\Psi,\beta}$ do not hold in an FβCAS.

Table 5
The fuzzy β-neighborhood system, fuzzy β-minimal (resp. maximal) descriptions of Ψ^1_β and Ψ^2_β.

	$\mathbb{C}_\beta(\Psi^1_\beta, a_i)$	$\mathrm{md}_{\Psi^1_\beta,\beta}(a_i)$	$\mathrm{MD}_{\Psi^1_\beta,\beta}(a_i)$	$\mathbb{C}_\beta(\Psi^2_\beta, a_i)$	$\mathrm{md}_{\Psi^2_\beta,\beta}(a_i)$	$\mathrm{MD}_{\Psi^2_\beta,\beta}(a_i)$
a_1	$\{\psi_1,\psi_2,\psi_4\}$	$\{\psi_1,\psi_2,\psi_4\}$	$\{\psi_1,\psi_2,\psi_4\}$	$\{\psi_5\}$	$\{\psi_5\}$	$\{\psi_5\}$
a_2	$\{\psi_1,\psi_2\}$	$\{\psi_1,\psi_2\}$	$\{\psi_1,\psi_2\}$	$\{\psi_5\}$	$\{\psi_5\}$	$\{\psi_5\}$
a_3	$\{\psi_2,\psi_3,\psi_4\}$	$\{\psi_3,\psi_4\}$	$\{\psi_2,\psi_4\}$	$\{\psi_5\}$	$\{\psi_5\}$	$\{\psi_5\}$

Example 4.4. Let Ψ be a fuzzy 0.8-covering in Example 4.1. Consider $T = T_M$ and $I = I_{T_M}$, we have that

$$TN^1_{\Psi,\beta}(a_1)(a_1) = 1 > 0.9 = TN^2_{\Psi,\beta}(a_1)(a_1),$$
$$TN^3_{\Psi,\beta}(a_1)(a_1) = 1 > 0.9 = TN^4_{\Psi,\beta}(a_1)(a_1).$$

Corollary 4.1. *Let (Ω, Ψ) be an FβCAS, T a left-continuous t-norm to construct Ψ^4_β and fuzzy neighborhood operators $TN^2_{\Psi^i_\beta,\beta}$ and $TN^4_{\Psi^i_\beta,\beta}$, and I_T its R-implication to define Ψ^3_β and fuzzy neighborhood operators $TN^1_{\Psi^i_\beta,\beta}$ and $TN^3_{\Psi^i_\beta,\beta}$ ($i = 1,2,3,4,\cap,\cup$).*

(1) *For Ψ, it holds that $A1 \le F$ and $E \le H$.*
(2) *For Ψ^1_β, it holds that $A2 \le B$ and $E \le L$.*
(3) *For Ψ^3_β, it holds that $A1 \le C$ and $A3 \le D$.*
(4) *For Ψ^4_β, it holds that $G \le J$ and $I \le K$.*
(5) *For $\Psi^\cap_\beta$, it holds that $A1 \le F$ and $M \le H$.*
(6) *For $\Psi^\cup_\beta$, it holds that $N \le P$ and $O \le Q$.*

Proof. It can be proved directly by Proposition 4.11. □

Note that if fuzzy β-coverings Ψ and Ψ' satisfy $\Psi \subseteq \Psi'$, then $\mathbb{C}'_\beta(\Psi, \pi) \subseteq \mathbb{C}_\beta(\Psi, \pi)$. However, the fuzzy β-minimum descriptions $md_{\Psi,\beta}(\pi)$ and $md_{\Psi',\beta}(\pi)$ may not necessarily satisfy $md_{\Psi,\beta}(\pi) \subseteq md_{\Psi',\beta}(\pi)$ or $md_{\Psi,\beta}(\pi) \supseteq md_{\Psi',\beta}(\pi)$, and similarly, fuzzy β-maximum descriptions $MD_{\Psi,\beta}(\pi)$ and $MD_{\Psi',\beta}(\pi)$ may not necessarily satisfy $MD_{\Psi,\beta}(\pi) \subseteq MD_{\Psi',\beta}(\pi)$ or $MD_{\Psi,\beta}(\pi) \supseteq MD_{\Psi',\beta}(\pi)$.

Example 4.5. Let Ψ be a fuzzy 0.8-covering in Example 4.1. Consider $T = T_M$ and $I = I_{T_M}$, we calculate that fuzzy β-coverings $\Psi^1_\beta = \{\psi_1,\psi_2,\psi_3,\psi_4\}$ and $\Psi^2_\beta = \{\psi_5\}$ by Definition 4.1. Moreover, the fuzzy β-neighborhood system, fuzzy β-minimum (resp. maximum) descriptions of Ψ^1_β and Ψ^2_β are presented in Table 5.

In light of Tables 3 and 5, for $\Psi^1_\beta \subseteq \Psi$, it holds that $\mathbb{C}_\beta(\Psi^1_\beta, a) \subseteq \mathbb{C}_\beta(\Psi, a)$. However, neither $MD_{\Psi^1_\beta,\beta}(\pi) \nsubseteq MD_{\Psi,\beta}(\pi)$ nor $MD_{\Psi,\beta}(\pi) \nsubseteq MD_{\Psi^1_\beta,\beta}(\pi)$ holds for any $\pi \in U$. Similar, for $\Psi^2_\beta \subseteq \Psi$, we have that $\mathbb{C}_\beta(\Psi^2_\beta, \pi) \subseteq \mathbb{C}_\beta(\Psi, \pi)$, but neither $md_{\Psi^2_\beta,\beta}(\pi) \nsubseteq md_{\Psi,\beta}(\pi)$ nor $md_{\Psi,\beta}(\pi) \nsubseteq md_{\Psi^2_\beta,\beta}(a)$ holds for any $\pi \in U$.

Since Ψ^1_β, Ψ^2_β, $\Psi^\cap_\beta$ and $\Psi^\cup_\beta$ are all fuzzy β-subcoverings of Ψ, the following properties hold.

Proposition 4.12. *Let (Ω, Ψ) be an FβCAS, T a t-norm to construct $TN^1_{\Psi^i_\beta,\beta}$, and I_T its R-implication to define $TN^4_{\Psi^i_\beta,\beta}$. For $TN^1_{\Psi^i_\beta,\beta}$, it holds that $TN^1_{\Psi,\beta} \le TN^1_{\Psi^1_\beta,\beta}$, $TN^1_{\Psi,\beta} \le TN^1_{\Psi^2_\beta,\beta}$ and $TN^1_{\Psi,\beta} \le TN^1_{\Psi^\cup_\beta,\beta}$, i.e., $A1 \le A2$, $A1 \le F$ and $A1 \le N$. For $TN^4_{\Psi^i_\beta,\beta}$, it holds that $TN^4_{\Psi^1_\beta,\beta} \le TN^4_{\Psi,\beta}$ and $TN^4_{\Psi^\cup_\beta,\beta} \le TN^4_{\Psi,\beta}$, i.e., $L \le H$ and $Q \le H$.*

Proof. It can be proved directly by Proposition 4.1. □

In a similar way, since $\Psi^1_\beta \subseteq \Psi^\cup_\beta$ and $\Psi^2_\beta \subseteq \Psi^\cap_\beta$, it holds that $TN^1_{\Psi^\cup_\beta,\beta} \le TN^1_{\Psi^1_\beta,\beta}$ and $TN^1_{\Psi^\cap_\beta,\beta} \le TN^1_{\Psi^2_\beta,\beta}$, that is, $N \le A2$ and $A1 \le F$. Meanwhile, $TN^4_{\Psi^1_\beta,\beta} \le TN^4_{\Psi^\cup_\beta,\beta}$ holds, that is, $L \le Q$.

Next, we discuss the partial order relationship $F \le G$.

Lemma 4.2. *[24] Let T be a left-continuous t-norm and I_T its R-implication. For any $\pi, \varpi, \varsigma \in \Omega$, it holds that*

$$I_T(\pi, \varpi) \le I_T(T(\pi, \varsigma), T(\varsigma, \varpi)).$$

Proposition 4.13. *Let (Ω, Ψ) be an FβCAS, T a left-continuous t-norm to construct Ψ_β^4, and I_T its R-implication to define fuzzy neighborhood operators $TN^1_{\Psi_\beta^4,\beta}$ and $TN^3_{\Psi,\beta}$. Then, $TN^3_{\Psi,\beta} \leq TN^1_{\Psi_\beta^4,\beta}$.*

Proof. For any $\pi, \varpi \in \Omega$, we have that

$$\begin{aligned}
TN^1_{\Psi_\beta^4,\beta}(\pi)(\varpi) &= \bigwedge_{\psi\in\Psi_\beta^4} I_T(\psi(\pi), \psi(\varpi)) \\
&= \bigwedge_{\varsigma\in\Omega} I_T(TN^4_{\Psi,\beta}(\varsigma)(\pi), TN^4_{\Psi,\beta}(\varsigma)(\varpi)) \\
&= \bigwedge_{\varsigma\in\Omega}\bigwedge_{\psi\in MD_{\Psi,\beta}(\pi)} I_T\left(T(\psi(\pi),\psi(\varsigma)), \bigvee_{\psi'\in\Psi} T(\psi'(\varsigma),\psi'(\varpi))\right) \\
&\geq \bigwedge_{\varsigma\in\Omega}\bigwedge_{\psi\in MD_{\Psi,\beta}(\pi)}\bigvee_{\psi'\in\Psi} I_T\left(T(\psi(\pi),\psi(\varsigma)), T(\psi'(\varsigma),\psi'(\varpi))\right) \\
&\geq \bigwedge_{\varsigma\in\Omega}\bigwedge_{\psi\in MD_{\Psi,\beta}(\pi)} I_T\left(T(\psi(\pi),\psi(\varsigma)), T(\psi(\varsigma),\psi(\varpi))\right) \\
&\geq \bigwedge_{\psi\in MD_{\Psi,\beta}(\pi)} I_T\left(\psi(\pi),\psi(\varpi)\right) \\
&= TN^3_{\Psi,\beta}(\pi)(\varpi).
\end{aligned}$$

Hence, $TN^3_{\Psi,\beta} \leq TN^1_{\Psi_\beta^4,\beta}$. □

The following counterexample illustrates that $TN^1_{\Psi_\beta^4,\beta} \leq TN^4_{\Psi,\beta}$ and $TN^4_{\Psi,\beta} \leq TN^2_{\Psi_\beta^4,\beta}$ do not hold in an FβCAS.

Example 4.6. Let Ψ be a fuzzy 0.8-covering in Example 4.1. Take $T = T_M$ and $I = I_{T_M}$, we have that

$$\Psi_\beta^4 = \{\frac{0.9}{a_1} + \frac{0.8}{a_2} + \frac{0.9}{a_3}, \frac{0.8}{a_1} + \frac{0.8}{a_2} + \frac{0.8}{a_3}\}.$$

Then, it can be calculated that

$$TN^4_{\Psi,\beta}(a_1)(a_1) = 0.9 < 1.0 = TN^1_{\Psi_\beta^4,\beta}(a_1)(a_1),$$

$$TN^4_{\Psi,\beta}(a_1)(a_1) = 0.9 > 0.8 = TN^2_{\Psi_\beta^4,\beta}(a_1)(a_1).$$

The next counterexample shows that $TN^4_{\Psi_\beta^3,\beta} \leq TN^4_{\Psi_\beta^1,\beta}$ ($i.e.$, $D \leq L$) does not hold.

Example 4.7. Let Ψ be a fuzzy 0.8-covering in Example 4.1. Take $T = T_M$ and $I = I_{T_M}$, we have that

$$\Psi_\beta^1 = \{\psi_1, \psi_2, \psi_3, \psi_4\} \text{ and } \Psi_\beta^3 = \{\frac{1}{a_1} + \frac{0.6}{a_2} + \frac{0.7}{a_3}, \frac{1}{a_1} + \frac{1}{a_2} + \frac{0.7}{a_3}, \frac{0.6}{a_1} + \frac{0.6}{a_2} + \frac{1}{a_3}\}.$$

Then, it can be calculated that

$$TN^4_{\Psi_\beta^3,\beta}(a_1)(a_1) = 1 > 0.9 = TN^4_{\Psi_\beta^1,\beta}(a_1)(a_1).$$

Finally, we verify that $TN^2_{\Psi_\beta^\cap,\beta} \leq TN^2_{\Psi_\beta^2,\beta}$ ($i.e.$, $M \leq H$) holds.

Proposition 4.14. *Let (Ω, Ψ) be an FβCAS, T a t-norm to construct fuzzy neighborhood operators $TN^2_{\Psi_\beta^\cap,\beta}$ and $TN^2_{\Psi_\beta^2,\beta}$. Then, $TN^2_{\Psi_\beta^\cap,\beta} \leq TN^2_{\Psi_\beta^2,\beta}$.*

Proof. According to Proposition 4.10 (3), it holds that $TN^4_{\Psi,\beta} = TN^2_{\Psi_\beta^2,\beta}$, which implies that $TN^2_{\Psi_\beta^\cap,\beta} \leq TN^2_{\Psi_\beta^2,\beta}$ is equivalent to $TN^2_{\Psi_\beta^\cap,\beta} \leq TN^4_{\Psi,\beta}$, i.e., $md_{\Psi_\beta^\cap,\beta}(\pi) \subseteq \mathbb{C}_\beta(\Psi,\pi)$. For any $\psi \in md_{\Psi_\beta^\cap,\beta}(\pi)$, we have that $\psi \in \Psi_\beta^\cap$, $\psi(\pi) \geq \beta$. It follows from $\Psi_\beta^\cap \subseteq \Psi$ that $\psi \in \Psi$ and $\psi(\pi) \geq \beta$, i.e., $\psi \in \mathbb{C}_\beta(\Psi,\pi)$. Consequently, we obtain that $md_{\Psi_\beta^\cap,\beta}(\pi) \subseteq \mathbb{C}_\beta(\Psi,\pi)$. □

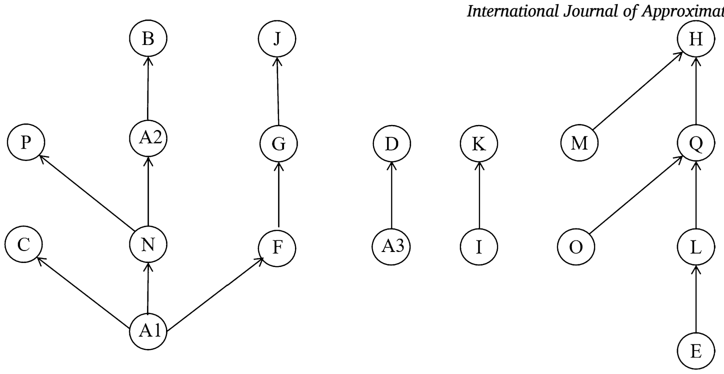

Fig. 2. Lattice of fuzzy neighborhood operators from Table 4.

In summary, the partial order relationships "$\leq$" among fuzzy neighborhood operators groups are illustrated in Fig. 2, which can be regarded as a lattice structure.

4.3. Equalities among fuzzy neighborhood operators derived from overlap functions

This subsection explores the equivalence relationships of fuzzy neighborhood operators generated by overlap functions and provide a grouping of these operators. The first group includes fuzzy neighborhood operators $ON^1_{\Psi,\beta}$, $ON^1_{\Psi^1_\beta,\beta}$, $ON^1_{\Psi^5_\beta,\beta}$ and $ON^1_{\Psi^\cap_\beta,\beta}$.

Proposition 4.15. *Let (Ω,Ψ) be an FβCAS and I_O an R-implication to construct fuzzy neighborhood operators $ON^1_{\Psi,\beta}$ and $ON^1_{\Psi^\cap_\beta,\beta}$. Then, $ON^1_{\Psi,\beta}=ON^1_{\Psi^\cap_\beta,\beta}$.*

Proof. Since $\Psi^\cap_\beta\subseteq\Psi$, it holds that $ON^1_{\Psi,\beta}(\pi)\subseteq ON^1_{\Psi^\cap_\beta,\beta}(\pi)$ for each $\pi\in\Omega$. Conversely, let $\varpi\in\Omega$, and consider $\psi\in\Psi$ such that $ON^1_{\Psi,\beta}(\pi)(\varpi)=I_O(\psi(\pi),\psi(\varpi))$. If $\psi\in\Psi^\cap_\beta$, then $ON^1_{\Psi^\cap_\beta,\beta}(\pi)(\varpi)\leq I_O(\psi(\pi),\psi(\varpi))=ON^1_{\Psi,\beta}(\pi)(\varpi)$. If $\psi\notin\Psi^\cap_\beta$, then there exists $\Psi'\subseteq\Psi^\cap_\beta$ such that $\psi=\cap\Psi'$. Furthermore, there exists $\psi'\in\Psi',\psi'(\varpi)=\psi(\varpi)$ and $\psi\subseteq\psi'$. Hence, $I_O(\psi(\pi),\psi(\varpi))\geq I_O(\psi'(\pi),\psi'(\varpi))$. Since the infimum of $ON^1_{\Psi,\beta}(\pi)(\varpi)$ is got in ψ and $\psi'\in\Psi$, we have $I_O(\psi(\pi),\psi(\varpi))\leq I_O(\psi'(\pi),\psi'(\varpi))$. Thus,

$$ON^1_{\Psi^\cap_\beta,\beta}(\pi)(\varpi)\leq I_O(\psi'(\pi),\psi'(\varpi))=I_O(\psi(\pi),\psi(\varpi))=ON^1_{\Psi,\beta}(\pi)(\varpi),$$

i.e., $ON^1_{\Psi^\cap_\beta,\beta}(\pi)\subseteq ON^1_{\Psi,\beta}(\pi)$. In summary, it concludes that $ON^1_{\Psi^\cap_\beta,\beta}=ON^1_{\Psi,\beta}$. $\square$

In the following, we verify that $ON^1_{\Psi,\beta}$ and $ON^1_{\Psi^1_\beta,\beta}$, $ON^1_{\Psi,\beta}$ and $ON^1_{\Psi^5_\beta,\beta}$, $ON^1_{\Psi,\beta}$ and $ON^2_{\Psi^5_\beta,\beta}$ are not equal in general.

Example 4.8. Suppose that $\Omega=\{a_1,a_2,a_3\}$ and $\Psi=\{\psi_1,\psi_2\}$, where

$$\psi_1=\frac{0.6}{a_1}+\frac{0.5}{a_2}+\frac{0.6}{a_3}\text{ and }\psi_2=\frac{0.5}{a_1}+\frac{0.5}{a_2}+\frac{0.5}{a_3}.$$

Then, Ψ forms a fuzzy β-covering for $\beta\in(0,0.5]$. For $\beta=0.5$, we obtain that $\mathbb{C}_\beta(\Psi,a_1)=\mathbb{C}_\beta(\Psi,a_2)=\mathbb{C}_\beta(\Psi,a_3)=\{\psi_1,\psi_2\}$ and $md_{\Psi,\beta}(a_1)=md_{\Psi,\beta}(a_2)=md_{\Psi,\beta}(a_3)=\{\psi_2\}$. By Definition 4.1, it holds that $\Psi^1_\beta=\{\psi_2\}$. Consider $O=O^V_2$ and $I_O=I_{O^V_2}$, we have that

$$ON^1_{\Psi,\beta}(a_1)(a_2)=0.5<1=ON^1_{\Psi^1_\beta,\beta}(a_1)(a_2),$$

i.e., $ON^1_{\Psi,\beta}\neq ON^1_{\Psi^1_\beta,\beta}$.

Example 4.9. Let Ψ be a fuzzy 0.8-covering in Example 4.1. Consider $O=O^V_2$ and $I_O=I_{O^V_2}$, it holds that

$$ON^1_{\Psi,\beta}(a_1)(a_2)=0.87<0.93=ON^1_{\Psi^5_\beta,\beta}(a_1)(a_2),$$

$$ON^1_{\Psi,\beta}(a_1)(a_2)=0.87>0.78=ON^2_{\Psi^5_\beta,\beta}(a_1)(a_2).$$

Obviously, $ON^1_{\Psi,\beta}\neq ON^1_{\Psi^5_\beta,\beta}$ and $ON^1_{\Psi,\beta}\neq ON^2_{\Psi^5_\beta,\beta}$.

Table 6
Overlap function-based fuzzy neighborhood operators in an FβCAS.

Group	Operators	Group	Operators
$A1$	$ON^1_{\Psi,\beta}, ON^1_{\Psi^{\cap}_\beta,\beta}$	H	$ON^4_{\Psi,\beta}, ON^4_{\Psi^2_\beta,\beta}, ON^4_{\Psi^{\cap}_\beta,\beta}, ON^2_{\Psi^2_\beta,\beta}$
$A2$	$ON^1_{\Psi^1_\beta,\beta}$	I	$ON^2_{\Psi^6_\beta,\beta}$
$A3$	$ON^1_{\Psi^5_\beta,\beta}$	J	$ON^3_{\Psi^6_\beta,\beta}$
$A4$	$ON^2_{\Psi^5_\beta,\beta}$	K	$ON^4_{\Psi^6_\beta,\beta}$
B	$ON^3_{\Psi^1_\beta,\beta}$	L	$ON^4_{\Psi^1_\beta,\beta}$
C	$ON^3_{\Psi^5_\beta,\beta}$	M	$ON^2_{\Psi^{\cap}_\beta,\beta}$
D	$ON^4_{\Psi^5_\beta,\beta}$	N	$ON^1_{\Psi^{\cup}_\beta,\beta}$
E	$ON^2_{\Psi,\beta}, ON^2_{\Psi^1_\beta,\beta}$	O	$ON^2_{\Psi^{\cup}_\beta,\beta}$
F	$ON^3_{\Psi,\beta}, ON^3_{\Psi^2_\beta,\beta}, ON^3_{\Psi^{\cap}_\beta,\beta}, ON^1_{\Psi^2_\beta,\beta}$	P	$ON^3_{\Psi^{\cup}_\beta,\beta}$
G	$ON^1_{\Psi^6_\beta,\beta}$	Q	$ON^4_{\Psi^{\cup}_\beta,\beta}$

Proposition 4.16. *Let (Ω,Ψ) be an FβCAS and O an overlap function to construct fuzzy neighborhood operators $ON^2_{\Psi,\beta}$ and $ON^2_{\Psi^1_\beta,\beta}$. Then, $ON^2_{\Psi,\beta} = ON^2_{\Psi^1_\beta,\beta}$.*

Proof. According to Proposition 4.6 (1), it can be proved directly. □

Further, we explore the third group: fuzzy neighborhood operators $ON^3_{\Psi,\beta}$, $ON^3_{\Psi^2_\beta,\beta}$, $ON^3_{\Psi^{\cap}_\beta,\beta}$ and $ON^1_{\Psi^2_\beta,\beta}$.

Proposition 4.17. *Let (Ω,Ψ) be an FβCAS and I_O an R-implication to construct fuzzy neighborhood operators $ON^3_{\Psi,\beta}$, $ON^3_{\Psi^2_\beta,\beta}$, $ON^3_{\Psi^{\cap}_\beta,\beta}$ and $ON^1_{\Psi^2_\beta,\beta}$, it follows that*

(1) $ON^3_{\Psi,\beta} = ON^3_{\Psi^2_\beta,\beta}$.
(2) $ON^3_{\Psi,\beta} = ON^3_{\Psi^{\cap}_\beta,\beta}$.
(3) $ON^3_{\Psi,\beta} = ON^1_{\Psi^2_\beta,\beta}$.

Proof. It can be verified directly by Proposition 4.9. □

Finally, we obtain the group consisted of fuzzy neighborhood operators $ON^4_{\Psi,\beta}$, $ON^4_{\Psi^2_\beta,\beta}$, $ON^4_{\Psi^{\cap}_\beta,\beta}$ and $ON^2_{\Psi^2_\beta,\beta}$.

Proposition 4.18. *Let (Ω,Ψ) be an FβCAS and O an overlap function to construct fuzzy neighborhood operators $ON^4_{\Psi,\beta}$, $ON^4_{\Psi^2_\beta,\beta}$ and $ON^4_{\Psi^{\cap}_\beta,\beta}$, it follows that*

(1) $ON^4_{\Psi,\beta} = ON^4_{\Psi^2_\beta,\beta}$.
(2) $ON^4_{\Psi,\beta} = ON^4_{\Psi^{\cap}_\beta,\beta}$.
(3) $ON^4_{\Psi,\beta} = ON^2_{\Psi^2_\beta,\beta}$.

Proof. It can be proved directly by Proposition 4.10. □

In summary, within the framework of FβCAS, let O be an overlap function to construct Ψ^6_β (where O satisfies (O7)), $ON^2_{\Psi^i_\beta,\beta}$ and $ON^4_{\Psi^i_\beta,\beta}$, and its R-implication I_O is used to define Ψ^5_β (where O satisfies (O9)), $ON^1_{\Psi^i_\beta,\beta}$ and $ON^3_{\Psi^i_\beta,\beta}$ ($i = 1,2,5,6,\cap,\cup$). Then, we can obtain twenty groups of fuzzy neighborhood operators as shown in Table 6.

In the following, we examine the partial order relationships among different groups in Table 6. First, the relationships among fuzzy neighborhood operators $ON^1_{\Psi,\beta}$, $ON^2_{\Psi,\beta}$, $ON^3_{\Psi,\beta}$ and $ON^4_{\Psi,\beta}$ are considered.

Proposition 4.19. *Let (Ω, Ψ) be an FβCAS, O an overlap function to construct fuzzy neighborhood operators $ON^2_{\Psi,\beta}$ and $ON^4_{\Psi,\beta}$, and I_O its R-implication to define $ON^1_{\Psi,\beta}$ and $ON^3_{\Psi,\beta}$.*

(1) $ON^1_{\Psi,\beta} \le ON^3_{\Psi,\beta}$.
(2) $ON^2_{\Psi,\beta} \le ON^4_{\Psi,\beta}$.

Proof. It can be verified similar to Proposition 4.11. □

The following counterexample illustrates that $ON^1_{\Psi,\beta} \le ON^2_{\Psi,\beta}$ and $ON^3_{\Psi,\beta} \le ON^4_{\Psi,\beta}$ do not hold in an FβCAS.

Example 4.10. Let Ψ be a fuzzy 0.8-covering in Example 4.1. Consider $O = O_2^V$ and $I_O = I_{O_2^V}$, we have that

$$ON^1_{\Psi,\beta}(a_1)(a_1) = ON^3_{\Psi,\beta}(a_1)(a_1) = 1,$$
$$ON^2_{\Psi,\beta}(a_1)(a_1) = ON^4_{\Psi,\beta}(a_1)(a_1) = 0.7048.$$

Therefore, $ON^1_{\Psi,\beta}(a_1)(a_1) > ON^2_{\Psi,\beta}(a_1)(a_1)$ and $ON^3_{\Psi,\beta}(a_1)(a_1) > ON^4_{\Psi,\beta}(a_1)(a_1)$ hold in fuzzy 0.8-covering.

Corollary 4.2. *Let (Ω, Ψ) be an FβCAS, O an overlap function to construct Ψ^6_β (where O satisfies (O7)) and fuzzy neighborhood operators $ON^2_{\Psi^i_\beta,\beta}$, $ON^4_{\Psi^i_\beta,\beta}$, and I_O its R-implication to define Ψ^5_β (where O satisfies (O9)) and fuzzy neighborhood operators $ON^1_{\Psi^i_\beta,\beta}$, $ON^3_{\Psi^i_\beta,\beta}$ ($i = 1, 2, 5, 6, \cap, \cup$).*

(1) *For Ψ, it holds that $A1 \le F$ and $E \le H$.*
(2) *For Ψ^1_β, it holds that $A2 \le B$ and $E \le L$.*
(3) *For Ψ^3_β, it holds that $A3 \le C$ and $A4 \le D$.*
(4) *For Ψ^4_β, it holds that $G \le J$ and $I \le K$.*
(5) *For $\Psi^\cap_\beta$, it holds that $A1 \le F$ and $M \le H$.*
(6) *For $\Psi^\cup_\beta$, it holds that $N \le P$ and $O \le Q$.*

Proof. It can be proved by Proposition 4.11. □

Since Ψ^1_β, Ψ^2_β, $\Psi^\cap_\beta$ and $\Psi^\cup_\beta$ are both fuzzy β-subcoverings of fuzzy β-covering Ψ, the following conclusions hold.

Proposition 4.20. *Let (Ω, Ψ) be an FβCAS, O an overlap function to construct $ON^1_{\Psi^i_\beta,\beta}$, and I_O its R-implication to define $ON^4_{\Psi^i_\beta,\beta}$. For $ON^1_{\Psi^i_\beta,\beta}$, we have $ON^1_{\Psi,\beta} \le ON^1_{\Psi^1_\beta,\beta}$, $ON^1_{\Psi,\beta} \le ON^1_{\Psi^2_\beta,\beta}$ and $ON^1_{\Psi,\beta} \le ON^1_{\Psi^\cup_\beta,\beta}$, i.e., $A1 \le A2$, $A1 \le F$ and $A1 \le N$. For $ON^4_{\Psi^i_\beta,\beta}$, we have $ON^4_{\Psi,\beta} \ge ON^4_{\Psi^1_\beta,\beta}$ and $ON^4_{\Psi,\beta} \ge ON^4_{\Psi^\cup_\beta,\beta}$, i.e., $L \le H$ and $Q \le H$.*

Proof. It can be proved by Proposition 4.19. □

Since $\Psi^1_\beta \subseteq \Psi^\cup_\beta$ and $\Psi^2_\beta \subseteq \Psi^\cap_\beta$, it holds that $ON^1_{\Psi^\cup_\beta,\beta} \le ON^1_{\Psi^1_\beta,\beta}$ and $ON^1_{\Psi^\cap_\beta,\beta} \le ON^1_{\Psi^2_\beta,\beta}$, that is, $N \le A2$ and $A1 \le F$. Meanwhile, $ON^4_{\Psi^1_\beta,\beta} \le ON^4_{\Psi^\cup_\beta,\beta}$ holds, that is, $L \le Q$. However, the partial order relationships $ON^3_{\Psi,\beta} \le ON^1_{\Psi^6_\beta,\beta} \le ON^4_{\Psi,\beta} \le ON^2_{\Psi^6_\beta,\beta}$ (*i.e.*, $F \le G \le H \le I$) do not hold.

Example 4.11. Let Ψ be a fuzzy 0.5-covering in Example 4.8. Consider $O = O_{DB}$ and $I_O = I_{O_{DB}}$, we have that

$$\Psi^6_\beta = \left\{ \frac{0.6}{a_1} + \frac{0.55}{a_2} + \frac{0.6}{a_3}, \frac{0.55}{a_1} + \frac{0.5}{a_2} + \frac{0.55}{a_3} \right\},$$

and $\mathbb{C}_\beta(\Psi^6_\beta, a_1) = \mathbb{C}_\beta(\Psi^6_\beta, a_2) = \mathbb{C}_\beta(\Psi^6_\beta, a_3) = \{\psi_1, \psi_2\}$ and $md_{\Psi^6_\beta,\beta}(a_1) = md_{\Psi^6_\beta,\beta}(a_2) = md_{\Psi^6_\beta,\beta}(a_3) = \{\psi_2\}$. Further, it holds that

$$ON^3_{\Psi,\beta}(a_1)(a_1) = 0.6 > 0.55 = ON^1_{\Psi^6_\beta,\beta}(a_1)(a_1),$$
$$ON^1_{\Psi^6_\beta,\beta}(a_2)(a_1) = 0.6 > 0.55 = ON^4_{\Psi,\beta}(a_2)(a_1),$$
$$ON^4_{\Psi,\beta}(a_1)(a_1) = 0.6 > 0.55 = ON^2_{\Psi^6_\beta,\beta}(a_1)(a_1).$$

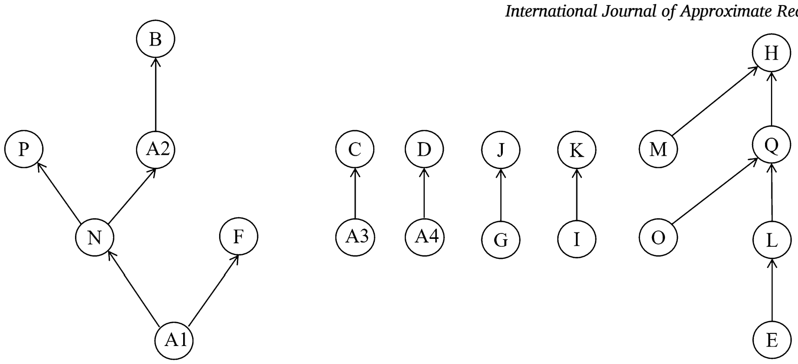

Fig. 3. Lattice of fuzzy neighborhood operators from Table 6.

Hence, $ON^3_{\Psi,\beta} \nleq ON^1_{\Psi^6_\beta,\beta} \nleq ON^4_{\Psi,\beta} \nleq ON^2_{\Psi^6_\beta,\beta}$.

Next, we provide the following counterexample to illustrate that $ON^4_{\Psi^5_\beta,\beta} \leq ON^4_{\Psi^1_\beta,\beta}$ (*i.e.*, $D \leq L$) does not hold.

Example 4.12. Let Ψ be a fuzzy 0.8-covering in Example 4.1. Consider $O = O^V_2$ and $I_O = I_{O^V_2}$, we have that

$$ON^4_{\Psi^5_\beta,\beta}(a_1)(a_1) = 1 > 0.7048 = ON^4_{\Psi^1_\beta,\beta}(a_1)(a_1).$$

In summary, the partial order relationships among different fuzzy neighborhood operators groups are illustrated in Fig. 3. These operators, ordered by the partial order relationship "$\leq$" can be regarded as a lattice structure.

5. Some neighborhood-related FβCRS models

This section proposes several types of FβCRS models using t-norms, overlap functions and their R-implications, respectively. Specifically, the FβCRS model derived from t-norms is denoted as TN-FβCRS, while the model based on overlap function is termed as ON-FβCRS. The primary distinction between TN-FβCRS and ON-FβCRS lies in the selection of aggregation functions.

For convenience, the fuzzy neighborhood operators induced by overlap function under both the original and new fuzzy β-coverings are uniformly denoted as $ON^i_{\Psi^k_\beta}$, where $i \in \{1,2,3,4\}$ represents the i-th type of fuzzy neighborhood operator and $k \in \{0,1,2,5,6,\cap,\cup\}$ represents the k-th type of fuzzy β-covering. Similarly, $TN^i_{\Psi^j_\beta}$ represents the fuzzy neighborhood operator based on t-norm, with $i \in \{1,2,3,4\}$ indicating the i-th type of fuzzy neighborhood operator and $j \in \{0,1,2,3,4,\cap,\cup\}$ representing the j-th type of fuzzy β-covering. Notably, ON^i_Ψ and TN^i_Ψ can be equivalently denoted as $ON^i_{\Psi^0_\beta}$ and $TN^i_{\Psi^0_\beta}$, respectively. Specifically, Ψ^5_β and Ψ^6_β are fuzzy β-coverings under the conditions that O satisfies (O7) and (O9), respectively. Additionally, Ψ^4_β qualifies as a fuzzy β-covering if T is idempotent. In the following discussion, we assume that Ψ^j_β $(j = 4,5,6)$ is a fuzzy β-covering, implying that O and T satisfy their corresponding conditions, respectively.

Next, we provide formal definitions for ON-FβCRS and TN-FβCRS as follows.

Definition 5.1. Let (Ω, Ψ) be an FβCAS. $TN^i_{\Psi^j_\beta}$ and $ON^i_{\Psi^k_\beta}$ are fuzzy neighborhood operators derived from t-norms and overlap functions, respectively. For any $\Gamma \in \mathscr{F}(\Omega)$, the lower and upper approximation operators of Γ based on $TN^i_{\Psi^j_\beta}$ are defined as

$$\underline{\Psi^T_{i,j}}(\Gamma)(\pi) = \bigwedge_{\varpi\in\Omega} \left((1 - TN^i_{\Psi^j_\beta}(\pi)(\varpi)) \vee \Gamma(\varpi) \right),$$
$$\overline{\Psi^T_{i,j}}(\Gamma)(\pi) = \bigvee_{\varpi\in\Omega} \left(TN^i_{\Psi^j_\beta}(\pi)(\varpi) \wedge \Gamma(\varpi) \right),$$

and the lower and upper approximation operators of Γ based on $ON^i_{\Psi^k_\beta}$ are defined as

$$\underline{\Psi^O_{i,k}}(\Gamma)(\pi) = \bigwedge_{\varpi\in\Omega} \left((1 - ON^i_{\Psi^k_\beta}(\pi)(\varpi)) \vee \Gamma(\varpi) \right),$$
$$\overline{\Psi^O_{i,k}}(\Gamma)(\pi) = \bigvee_{\varpi\in\Omega} \left(ON^i_{\Psi^k_\beta}(\pi)(\varpi) \wedge \Gamma(\varpi) \right),$$

Table 7
The classification of TN-FβCRS models.

Group	TN-FβCRS	Group	TN-FβCRS
$(\underline{\Psi^T_{A1}}(\Gamma),\overline{\Psi^T_{A1}}(\Gamma))$	$(\underline{\Psi^T_{1,0}}(\Gamma),\overline{\Psi^T_{1,0}}(\Gamma)),(\underline{\Psi^T_{1,3}}(\Gamma),\overline{\Psi^T_{1,3}}(\Gamma)),(\underline{\Psi^T_{1,\cap}}(\Gamma),\overline{\Psi^T_{1,\cap}}(\Gamma))$	$(\underline{\Psi^T_{I}}(\Gamma),\overline{\Psi^T_{I}}(\Gamma))$	$(\underline{\Psi^T_{2,4}}(\Gamma),\overline{\Psi^T_{2,4}}(\Gamma))$
$(\underline{\Psi^T_{A2}}(\Gamma),\overline{\Psi^T_{A2}}(\Gamma))$	$(\underline{\Psi^T_{1,1}}(\Gamma),\overline{\Psi^T_{1,1}}(\Gamma))$	$(\underline{\Psi^T_{J}}(\Gamma),\overline{\Psi^T_{J}}(\Gamma))$	$(\underline{\Psi^T_{3,4}}(\Gamma),\overline{\Psi^T_{3,4}}(\Gamma))$
$(\underline{\Psi^T_{A3}}(\Gamma),\overline{\Psi^T_{A3}}(\Gamma))$	$(\underline{\Psi^T_{2,3}}(\Gamma),\overline{\Psi^T_{2,3}}(\Gamma))$	$(\underline{\Psi^T_{K}}(\Gamma),\overline{\Psi^T_{K}}(\Gamma))$	$(\underline{\Psi^T_{4,4}}(\Gamma),\overline{\Psi^T_{4,4}}(\Gamma))$
$(\underline{\Psi^T_{B}}(\Gamma),\overline{\Psi^T_{B}}(\Gamma))$	$(\underline{\Psi^T_{3,1}}(\Gamma),\overline{\Psi^T_{3,1}}(\Gamma))$	$(\underline{\Psi^T_{L}}(\Gamma),\overline{\Psi^T_{L}}(\Gamma))$	$(\underline{\Psi^T_{4,1}}(\Gamma),\overline{\Psi^T_{4,1}}(\Gamma))$
$(\underline{\Psi^T_{C}}(\Gamma),\overline{\Psi^T_{C}}(\Gamma))$	$(\underline{\Psi^T_{3,3}}(\Gamma),\overline{\Psi^T_{3,3}}(\Gamma))$	$(\underline{\Psi^T_{M}}(\Gamma),\overline{\Psi^T_{M}}(\Gamma))$	$(\underline{\Psi^T_{2,\cap}}(\Gamma),\overline{\Psi^T_{2,\cap}}(\Gamma))$
$(\underline{\Psi^T_{D}}(\Gamma),\overline{\Psi^T_{D}}(\Gamma))$	$(\underline{\Psi^T_{4,3}}(\Gamma),\overline{\Psi^T_{4,3}}(\Gamma))$	$(\underline{\Psi^T_{N}}(\Gamma),\overline{\Psi^T_{N}}(\Gamma))$	$(\underline{\Psi^T_{1,\cup}}(\Gamma),\overline{\Psi^T_{1,\cup}}(\Gamma))$
$(\underline{\Psi^T_{E}}(\Gamma),\overline{\Psi^T_{E}}(\Gamma))$	$(\underline{\Psi^T_{2,0}}(\Gamma),\overline{\Psi^T_{2,0}}(\Gamma)),(\underline{\Psi^T_{2,1}}(\Gamma),\overline{\Psi^T_{2,1}}(\Gamma))$	$(\underline{\Psi^T_{O}}(\Gamma),\overline{\Psi^T_{O}}(\Gamma))$	$(\underline{\Psi^T_{2,\cup}}(\Gamma),\overline{\Psi^T_{2,\cup}}(\Gamma))$
$(\underline{\Psi^T_{F}}(\Gamma),\overline{\Psi^T_{F}}(\Gamma))$	$(\underline{\Psi^T_{3,0}}(\Gamma),\overline{\Psi^T_{3,0}}(\Gamma)),(\underline{\Psi^T_{3,2}}(\Gamma),\overline{\Psi^T_{3,2}}(\Gamma)),(\underline{\Psi^T_{3,\cap}}(\Gamma),\overline{\Psi^T_{3,\cap}}(\Gamma)),(\underline{\Psi^T_{1,2}}(\Gamma),\overline{\Psi^T_{1,2}}(\Gamma))$	$(\underline{\Psi^T_{P}}(\Gamma),\overline{\Psi^T_{P}}(\Gamma))$	$(\underline{\Psi^T_{3,\cup}}(\Gamma),\overline{\Psi^T_{3,\cup}}(\Gamma))$
$(\underline{\Psi^T_{G}}(\Gamma),\overline{\Psi^T_{G}}(\Gamma))$	$(\underline{\Psi^T_{1,4}}(\Gamma),\overline{\Psi^T_{1,4}}(\Gamma))$	$(\underline{\Psi^T_{Q}}(\Gamma),\overline{\Psi^T_{Q}}(\Gamma))$	$(\underline{\Psi^T_{4,\cup}}(\Gamma),\overline{\Psi^T_{4,\cup}}(\Gamma))$
$(\underline{\Psi^T_{H}}(\Gamma),\overline{\Psi^T_{H}}(\Gamma))$	$(\underline{\Psi^T_{4,0}}(\Gamma),\overline{\Psi^T_{4,0}}(\Gamma)),(\underline{\Psi^T_{4,2}}(\Gamma),\overline{\Psi^T_{4,2}}(\Gamma)),(\underline{\Psi^T_{4,\cap}}(\Gamma),\overline{\Psi^T_{4,\cap}}(\Gamma)),(\underline{\Psi^T_{2,2}}(\Gamma),\overline{\Psi^T_{2,2}}(\Gamma))$		

Table 8
The classification of ON-FβCRS models.

Group	ON-FβCRS	Group	ON-FβCRS
$(\underline{\Psi^O_{A1}}(\Gamma),\overline{\Psi^O_{A1}}(\Gamma))$	$(\underline{\Psi^O_{1,0}}(\Gamma),\overline{\Psi^O_{1,0}}(\Gamma)),(\underline{\Psi^O_{1,\cap}}(\Gamma),\overline{\Psi^O_{1,\cap}}(\Gamma))$	$(\underline{\Psi^O_{H}}(\Gamma),\overline{\Psi^O_{H}}(\Gamma))$	$(\underline{\Psi^O_{4,0}}(\Gamma),\overline{\Psi^O_{4,0}}(\Gamma)),(\underline{\Psi^O_{4,2}}(\Gamma),\overline{\Psi^O_{4,2}}(\Gamma)),$
$(\underline{\Psi^O_{A2}}(\Gamma),\overline{\Psi^O_{A2}}(\Gamma))$	$(\underline{\Psi^O_{1,1}}(\Gamma),\overline{\Psi^O_{1,1}}(\Gamma))$		$(\underline{\Psi^O_{4,\cap}}(\Gamma),\overline{\Psi^O_{4,\cap}}(\Gamma)),(\underline{\Psi^O_{2,2}}(\Gamma),\overline{\Psi^O_{2,2}}(\Gamma))$
$(\underline{\Psi^O_{A3}}(\Gamma),\overline{\Psi^O_{A3}}(\Gamma))$	$(\underline{\Psi^O_{1,3}}(\Gamma),\overline{\Psi^O_{1,3}}(\Gamma))$	$(\underline{\Psi^O_{I}}(\Gamma),\overline{\Psi^O_{I}}(\Gamma))$	$(\underline{\Psi^O_{2,4}}(\Gamma),\overline{\Psi^O_{2,4}}(\Gamma))$
$(\underline{\Psi^O_{A4}}(\Gamma),\overline{\Psi^O_{A4}}(\Gamma))$	$(\underline{\Psi^O_{2,3}}(\Gamma),\overline{\Psi^O_{2,3}}(\Gamma))$	$(\underline{\Psi^O_{J}}(\Gamma),\overline{\Psi^O_{J}}(\Gamma))$	$(\underline{\Psi^O_{3,4}}(\Gamma),\overline{\Psi^O_{3,4}}(\Gamma))$
$(\underline{\Psi^O_{B}}(\Gamma),\overline{\Psi^O_{B}}(\Gamma))$	$(\underline{\Psi^O_{3,1}}(\Gamma),\overline{\Psi^O_{3,1}}(\Gamma))$	$(\underline{\Psi^O_{K}}(\Gamma),\overline{\Psi^O_{K}}(\Gamma))$	$(\underline{\Psi^O_{4,4}}(\Gamma),\overline{\Psi^O_{4,4}}(\Gamma))$
$(\underline{\Psi^O_{C}}(\Gamma),\overline{\Psi^O_{C}}(\Gamma))$	$(\underline{\Psi^O_{3,3}}(\Gamma),\overline{\Psi^O_{3,3}}(\Gamma))$	$(\underline{\Psi^O_{L}}(\Gamma),\overline{\Psi^O_{L}}(\Gamma))$	$(\underline{\Psi^O_{4,1}}(\Gamma),\overline{\Psi^O_{4,1}}(\Gamma))$
$(\underline{\Psi^O_{D}}(\Gamma),\overline{\Psi^O_{D}}(\Gamma))$	$(\underline{\Psi^O_{4,3}}(\Gamma),\overline{\Psi^O_{4,3}}(\Gamma))$	$(\underline{\Psi^O_{M}}(\Gamma),\overline{\Psi^O_{M}}(\Gamma))$	$(\underline{\Psi^O_{2,\cap}}(\Gamma),\overline{\Psi^O_{2,\cap}}(\Gamma))$
$(\underline{\Psi^O_{E}}(\Gamma),\overline{\Psi^O_{E}}(\Gamma))$	$(\underline{\Psi^O_{2,0}}(\Gamma),\overline{\Psi^O_{2,0}}(\Gamma)),(\underline{\Psi^O_{2,1}}(\Gamma),\overline{\Psi^O_{2,1}}(\Gamma))$	$(\underline{\Psi^O_{N}}(\Gamma),\overline{\Psi^O_{N}}(\Gamma))$	$(\underline{\Psi^O_{1,\cup}}(\Gamma),\overline{\Psi^O_{1,\cup}}(\Gamma))$
$(\underline{\Psi^O_{F}}(\Gamma),\overline{\Psi^O_{F}}(\Gamma))$	$(\underline{\Psi^O_{3,0}}(\Gamma),\overline{\Psi^O_{3,0}}(\Gamma)),(\underline{\Psi^O_{3,2}}(\Gamma),\overline{\Psi^O_{3,2}}(\Gamma)),$	$(\underline{\Psi^O_{O}}(\Gamma),\overline{\Psi^O_{O}}(\Gamma))$	$(\underline{\Psi^O_{2,\cup}}(\Gamma),\overline{\Psi^O_{2,\cup}}(\Gamma)),$
	$(\underline{\Psi^O_{3,\cap}}(\Gamma),\overline{\Psi^O_{3,\cap}}(\Gamma)),(\underline{\Psi^O_{1,2}}(\Gamma),\overline{\Psi^O_{1,2}}(\Gamma))$	$(\underline{\Psi^O_{P}}(\Gamma),\overline{\Psi^O_{P}}(\Gamma))$	$(\underline{\Psi^O_{3,\cup}}(\Gamma),\overline{\Psi^O_{3,\cup}}(\Gamma))$
$(\underline{\Psi^O_{G}}(\Gamma),\overline{\Psi^O_{G}}(\Gamma))$	$(\underline{\Psi^O_{1,4}}(\Gamma),\overline{\Psi^O_{1,4}}(\Gamma))$	$(\underline{\Psi^O_{Q}}(\Gamma),\overline{\Psi^O_{Q}}(\Gamma))$	$(\underline{\Psi^O_{4,\cup}}(\Gamma),\overline{\Psi^O_{4,\cup}}(\Gamma))$

where $\pi \in \Omega, i \in \{1,2,3,4\}, j \in \{0,1,2,3,4,\cap,\cup\}$ and $k \in \{0,1,2,5,6,\cap,\cup\}$.

Building on the classifications of Tables 4 and 6, the classifications of TN-FβCRS and ON-FβCRS are shown in Tables 7 and 8, respectively. Furthermore, if $ON^i_{\Psi^k_\beta} \leq ON^{i'}_{\Psi^{k'}_\beta}$, then it follows from Definition 5.1 that $\underline{\Psi^O_{i',k'}}(\Gamma) \subseteq \underline{\Psi^O_{i,k}}(\Gamma)$ and $\overline{\Psi^O_{i,k}}(\Gamma) \subseteq \overline{\Psi^O_{i',k'}}(\Gamma)$. In a similar way as [23], we introduce the order of ON-FβCRS and TN-FβCRS models.

Definition 5.2. Let $(\underline{\Psi^O_{i,k}}(\Gamma),\overline{\Psi^O_{i,k}}(\Gamma))$ and $(\underline{\Psi^O_{i',k'}}(\Gamma),\overline{\Psi^O_{i',k'}}(\Gamma))$ be two ON-FβCRS models. If $\underline{\Psi^O_{i,k}}(\Gamma) \geq \underline{\Psi^O_{i',k'}}(\Gamma)$ and $\overline{\Psi^O_{i,k}}(\Gamma)) \leq \overline{\Psi^O_{i',k'}}(\Gamma)$ for any $\Gamma \in \mathscr{F}(\Omega)$, then the following order relation holds:

$$(\underline{\Psi^O_{i,k}}(\Gamma),\overline{\Psi^O_{i,k}}(\Gamma)) \preceq (\underline{\Psi^O_{i',k'}}(\Gamma),\overline{\Psi^O_{i',k'}}(\Gamma)).$$

Note that the construction of overlap function-based lower and upper approximation operators depends on fuzzy neighborhood operators, then the pair $\left(\left(\underline{\Psi^O_{i,k}}(\Gamma),\overline{\Psi^O_{i,k}}(\Gamma)\right),\preceq\right)$ still constitutes a partially ordered set. Since its Hasse diagram is identical to that in Fig. 3, the diagram description will not be repeated. Subsequently, we present some properties of ON-FβCRS models.

Proposition 5.1. *Let (Ω,Ψ) be an FβCAS and $ON^i_{\Psi^j_\beta}$ be reflexive fuzzy neighborhood operators derived from overlap function and its implication, respectively. For any $\Gamma \in \mathscr{F}(\Omega)$, the lower and upper approximation operator $\underline{\Psi^O_{i,j}}(\Gamma)$ and $\overline{\Psi^O_{i,j}}(\Gamma)$ satisfy the comparable property, that is,*

$$\underline{\Psi^O_{i,j}}(\Gamma) \subseteq \Gamma \subseteq \overline{\Psi^O_{i,j}}(\Gamma).$$

Proof. Since $ON^i_{\Psi^j_\beta}$ is reflexive, for any $\Gamma \in \mathscr{F}(\Omega)$, we have that

$$\begin{aligned}\underline{\Psi^O_{i,j}}(\Gamma)(\pi) &= \bigwedge_{\varpi\in\Omega}\left((1-ON^i_{\Psi^j_\beta}(\pi)(\varpi))\vee\Gamma(\varpi)\right)\\ &\leq (1-ON^i_{\Psi^j_\beta}(\pi)(\pi))\vee\Gamma(\pi))\\ &= \Gamma(\pi))\end{aligned}$$

and

$$\begin{aligned}\overline{\Psi^O_{i,j}}(\Gamma)(\pi) &= \bigvee_{\varpi\in\Omega}\left(ON^i_{\Psi^j_\beta}(\pi)(\varpi)\wedge\Gamma(\varpi)\right)\\ &\geq ON^i_{\Psi^j_\beta}(\pi)(\pi)\wedge\Gamma(\pi)\\ &= \Gamma(\pi).\end{aligned}$$

Hence, it holds that $\underline{\Psi^O_{i,j}}(\Gamma)\subseteq\Gamma\subseteq\overline{\Psi^O_{i,j}}(\Gamma)$. $\square$

Corollary 5.1. *As demonstrated in the proof of Proposition 3.3, the reflexivity of fuzzy neighborhood operators mainly depends on aggregation functions and is not affected by the type of fuzzy β-covering. Therefore, for the fuzzy neighborhood operator $ON^i_{\Psi^k_\beta}$, when O satisfies (O7), $i\in\{1,3\}$ and $k\in\{0,1,2,5,6,\cap,\cup\}$, $ON^i_{\Psi^k_\beta}$ satisfies reflexivity, and then $\underline{\Psi^O_{i,j}}(\Gamma)$ and $\overline{\Psi^O_{i,j}}(\Gamma)$ satisfy the comparable property. For the fuzzy neighborhood operator $TN^i_{\Psi^j_\beta}$, when $i\in\{1,3\}$, $j\in\{0,1,2,3,4,\cap,\cup\}$, $TN^i_{\Psi^j_\beta}$ satisfies reflexivity, and then $\underline{\Psi^T_{i,j}}(\Gamma)$ and $\overline{\Psi^T_{i,j}}(\Gamma)$ satisfy the comparable property.*

Proposition 5.2. *Let (Ω,Ψ) be an FβCAS. For any $\Gamma,\Upsilon\in\mathscr{F}(\Omega)$, it holds that*

(1) $\underline{\Psi^O_{i,k}}(\Gamma^c)=\left(\overline{\Psi^O_{i,k}}(\Gamma)\right)^c$, $\overline{\Psi^O_{i,k}}(\Gamma^c)=\left(\underline{\Psi^O_{i,k}}(\Gamma)\right)^c$ *with* $\Gamma^c(\pi)=1-\Gamma(\pi)$ $(\forall\pi\in\Omega)$;
(2) $\underline{\Psi^O_{i,k}}(\Omega)=\Omega$, $\overline{\Psi^O_{i,k}}(\emptyset)=\emptyset$;
(3) *If* $\Gamma\subseteq\Upsilon$, *then* $\underline{\Psi^O_{i,k}}(\Gamma)\subseteq\underline{\Psi^O_{i,k}}(\Upsilon)$ *and* $\overline{\Psi^O_{i,k}}(\Gamma)\subseteq\overline{\Psi^O_{i,k}}(\Upsilon)$;
(4) $\underline{\Psi^O_{i,k}}(\Gamma\cap\Upsilon)=\underline{\Psi^O_{i,k}}(\Gamma)\cap\underline{\Psi^O_{i,k}}(\Upsilon)$, $\overline{\Psi^O_{i,k}}(\Gamma\cup\Upsilon)=\overline{\Psi^O_{i,k}}(\Gamma)\cup\overline{\Psi^O_{i,k}}(\Upsilon)$;
(5) $\underline{\Psi^O_{i,k}}(\Gamma)\cup\underline{\Psi^O_{i,k}}(\Upsilon)\subseteq\underline{\Psi^O_{i,k}}(\Gamma\cup\Upsilon)$, $\overline{\Psi^O_{i,k}}(\Gamma\cap\Upsilon)\subseteq\overline{\Psi^O_{i,k}}(\Gamma)\cap\overline{\Psi^O_{i,k}}(\Upsilon)$;
(6) *If* $\Gamma\subseteq\Upsilon$, *then* $\underline{\Psi^O_{i,k}}(\Gamma\cup\Upsilon)=\underline{\Psi^O_{i,k}}(\Gamma)\cup\underline{\Psi^O_{i,k}}(\Upsilon)$ *and* $\overline{\Psi^O_{i,k}}(\Gamma\cap\Upsilon)=\overline{\Psi^O_{i,k}}(\Gamma)\cap\overline{\Psi^O_{i,k}}(\Upsilon)$;
(7) *If* $1-ON^i_{\Psi^k_\beta}(\pi)(\pi)\leq\Gamma(\pi)\leq ON^i_{\Psi^k_\beta}(\pi)(\pi)$ *for any* $\pi\in\Omega$, *then* $\underline{\Psi^O_{i,k}}(\underline{\Psi^O_{i,k}}(\Gamma))\subseteq\underline{\Psi^O_{i,k}}(\Gamma)\subseteq\Gamma\subseteq\overline{\Psi^O_{i,k}}(\Gamma)\subseteq\overline{\Psi^O_{i,k}}(\overline{\Psi^O_{i,k}}(\Gamma))$,

where $i\in\{1,2,3,4\}$ and $j\in\{0,1,2,5,6,\cap,\cup\}$.

Proof. According to Definition 5.1, we have the following statements.

(1) For any $\pi\in\Omega$, it holds that

$$\begin{aligned}\underline{\Psi^O_{i,k}}(\Gamma^c)(\pi) &= \bigwedge_{\varpi\in\Omega}\left((1-ON^i_{\Psi^k_\beta}(\pi)(\varpi))\vee(1-\Gamma(\varpi))\right)\\ &= 1-\bigvee_{\varpi\in\Omega}\left((ON^i_{\Psi^k_\beta}(\pi)(\varpi))\wedge\Gamma(\varpi)\right)\\ &= 1-\overline{\Psi^O_{i,k}}(\Gamma)(\pi)=\left(\overline{\Psi^O_{i,k}}(\Gamma)\right)^c(\pi).\end{aligned}$$

Therefore, $\underline{\Psi^O_{i,k}}(\Gamma^c)=\left(\overline{\Psi^O_{i,k}}(\Gamma)\right)^c$. In a similar way, $\overline{\Psi^O_{i,k}}(\Gamma^c)=\left(\underline{\Psi^O_{i,k}}(\Gamma)\right)^c$ holds.

(2) Since $\Omega(\pi)=1$ and $\emptyset(\pi)=0$, it follows that for any $\pi\in\Omega$,

$$\underline{\Psi^{O}_{i,k}}(\Omega)(\pi)=\bigwedge_{\varpi\in\Omega}\left((1-ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varpi))\vee\Omega(\varpi)\right)=1,$$

$$\overline{\Psi^{O}_{i,k}}(\emptyset)(\pi)=\bigvee_{\varpi\in\Omega}\left(ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varpi)\wedge\emptyset(\varpi)\right)=0.$$

(3) If $\Gamma\subseteq\Upsilon$, then $\underline{\Psi^{O}_{i,k}}(\Gamma)(\pi)=\bigwedge_{\varpi\in\Omega}\left((1-ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varpi))\vee\Gamma(\varpi)\right)\le\bigwedge_{\varpi\in\Omega}\left((1-ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varpi))\vee\Upsilon(\varpi)\right)=\underline{\Psi^{O}_{i,k}}(\Upsilon)(\pi)$. Similarly, $\overline{\Psi^{O}_{i,k}}(\Gamma)\subseteq\overline{\Psi^{O}_{i,k}}(\Upsilon)$ can be verified.

(4) For any $\pi\in\Omega$, it holds that

$$\begin{aligned}\underline{\Psi^{O}_{i,k}}(\Gamma\cap\Upsilon)(\pi)&=\bigwedge_{\varpi\in\Omega}\left((1-ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varpi))\vee(\Gamma\cap\Upsilon)(\varpi)\right)\\&=\bigwedge_{\varpi\in\Omega}\left(\left((1-ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varpi))\vee(\Gamma)(\varpi)\right)\wedge\left((1-ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varpi))\vee(\Upsilon)(\varpi)\right)\right)\\&=\bigwedge_{\varpi\in\Omega}\left((1-ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varpi))\vee(\Gamma)(\varpi)\right)\wedge\bigwedge_{\varpi\in\Omega}\left((1-ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varpi))\vee(\Upsilon)(\varpi)\right)\\&=\left(\underline{\Psi^{O}_{i,k}}(\Gamma)\cap\underline{\Psi^{O}_{i,k}}(\Upsilon)\right)(\pi).\end{aligned}$$

In a similar way, $\overline{\Psi^{O}_{i,k}}(\Gamma\cup\Upsilon)=\overline{\Psi^{O}_{i,k}}(\Gamma)\cup\overline{\Psi^{O}_{i,k}}(\Upsilon)$ holds.

(5) Since $\Gamma\subseteq\Gamma\cup\Upsilon$ and $\Upsilon\subseteq\Gamma\cup\Upsilon$, it follows item (3) that $\underline{\Psi^{O}_{i,k}}(\Gamma)\subseteq\underline{\Psi^{O}_{i,k}}(\Gamma\cup\Upsilon)$ and $\underline{\Psi^{O}_{i,k}}(\Upsilon)\subseteq\underline{\Psi^{O}_{i,k}}(\Gamma\cup\Upsilon)$. Hence, $\underline{\Psi^{O}_{i,k}}(\Gamma)\cup\underline{\Psi^{O}_{i,k}}(\Upsilon)\subseteq\underline{\Psi^{O}_{i,k}}(\Gamma\cup\Upsilon)$. Further, it holds that $\overline{\Psi^{O}_{i,k}}(\Gamma\cap\Upsilon)\subseteq\overline{\Psi^{O}_{i,k}}(\Gamma)\cap\overline{\Psi^{O}_{i,k}}(\Upsilon)$.

(6) It can be proved directly by items (3) and (5).

(7) For each $\pi\in\Omega$, if $1-ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\pi)\le\Gamma(\pi)\le ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\pi)$, then

$$\begin{aligned}\underline{\Psi^{O}_{i,k}}(\Gamma)(\pi)&=\bigwedge_{\varpi\in\Omega}\left((1-ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varpi))\vee\Gamma(\varpi)\right)\\&\le(1-ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\pi))\vee\Gamma(\pi)\\&=\Gamma(\pi)=ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\pi)\wedge\Gamma(\pi)\\&\le\bigvee_{\varpi\in\Omega}\left(ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varpi)\wedge\Gamma(\varpi)\right)\\&=\overline{\Psi^{O}_{i,k}}(\Gamma)(\pi).\end{aligned}$$

Hence, $\underline{\Psi^{O}_{i,k}}(\Gamma)\subseteq\Gamma\subseteq\overline{\Psi^{O}_{i,k}}(\Gamma)$. Furthermore, it follows item (3) that $\underline{\Psi^{O}_{i,k}}(\underline{\Psi^{O}_{i,k}}(\Gamma))\subseteq\underline{\Psi^{O}_{i,k}}(\Gamma)$ and $\overline{\Psi^{O}_{i,k}}(\Gamma)\subseteq\overline{\Psi^{O}_{i,k}}(\overline{\Psi^{O}_{i,k}}(\Gamma))$. □

Note that if $ON^{i}_{\Psi^{k}_{\beta}}$ is reflexive, then for any $\pi\in\Omega$, $1-ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\pi)\le\Gamma(\pi)\le ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\pi)$ always holds. In particular, when Γ represents special fuzzy sets, the following conclusions can be drawn.

Proposition 5.3. *Let (Ω,Ψ) be an FβCAS. For any $\Gamma\subseteq\Omega$, the following statements hold.*

(1) $\overline{\Psi^{O}_{i,k}}(1_{\varsigma})(\pi)=ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varsigma)$;

(2) $\underline{\Psi^{O}_{i,k}}(1_{\Omega\setminus\{\varsigma\}})(\pi)=1-ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varsigma)$;

(3) $\overline{\Psi^{O}_{i,k}}(1_{\Gamma})(\pi)=\bigvee_{y\in\Gamma}ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varpi)$;

(4) $\underline{\Psi^{O}_{i,k}}(1_{\Gamma})(\pi)=\bigwedge_{\varpi\in\Omega\setminus\Gamma}\left(1-ON^{i}_{\Psi^{k}_{\beta}}(\pi)(\varpi)\right)$,

where $1_{\varsigma}(\pi)=\begin{cases}1, & \pi=\varsigma,\\ 0, & otherwise,\end{cases}$ *and* $1_{\Gamma}(\pi)=\begin{cases}1, & \pi\in\Gamma,\\ 0, & otherwise.\end{cases}$

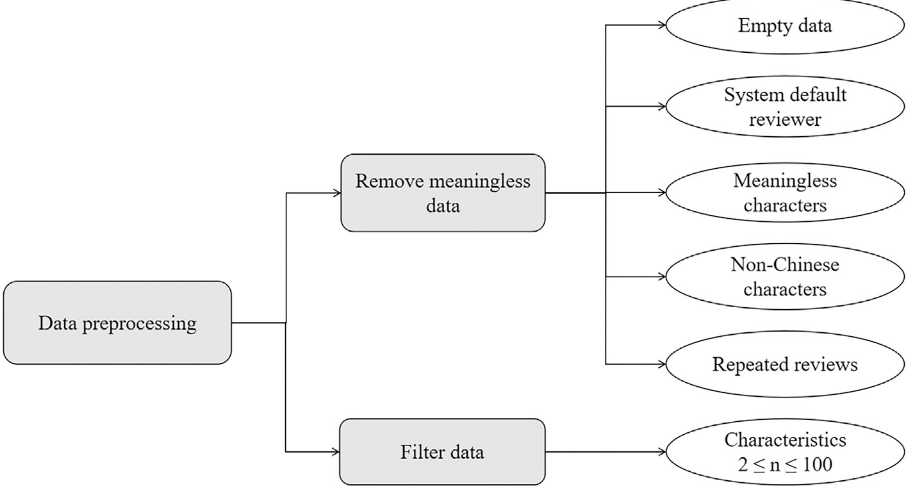

Fig. 4. The procedure of data preprocessing.

Proof. It can be proved by Definition 5.1. □

Since the distinction between TN-FβCRS and ON-FβCRS lies in the selection of fuzzy neighborhood operator, which does not affect the properties of lower and upper approximation operators discussed above. The aforementioned conclusions remain valid for TN-FβCRS models and will not be restated.

6. Application of user preference evaluation in fresh fruits

In MADM problem, the determination of weight is critical. Common approaches for determining weight include objective method, subjective method and comprehensive method. In this section, we propose a novel method for weight determination based on the FβCRS model. Building upon this foundation, the TOPSIS method is established and further used to address the decision-making problem regarding user preference factors for online purchases of fresh fruit.

6.1. Background description

With the advancement of technology, e-commerce platforms become increasingly integral to daily life and production activities due to their convenience. Consumers tend to share their shopping experiences through social media and browse online reviews prior to making purchases. Compared to traditional purchasing methods, online shopping transcends the limitations of personal relationships by utilizing online reviews, which offer a broader reach and a stronger influence on consumer behavior. Research has demonstrated that online reviews on e-commerce platforms provide a more accurate reflection of users' purchasing experience [39], effectively conveying their satisfaction levels. Furthermore, negative reviews tend to emphasize existing issues more than positive reviews [18], thereby offering better insight into users' preferences and needs in online shopping.

As one of the prominent e-commerce platforms, Jingdong (JD) not only operates its own stores, but also hosts a wide range of registered merchants and agricultural product suppliers, creating an extensive online trade network. Among the products sold, fresh fruits attract considerable consumers attention due to their appealing taste and nutritional value, making them a staple in daily life. This paper collects negative online reviews of fresh fruits from JD e-commerce platforms between January 1, 2020 and October 28, 2022, including user IDs, product titles, negative review texts, product descriptions and store IDs, totaling 158129 reviews for 1020 different types of fresh fruit.

It is important to note that user reviews effectively convey preference factors for fresh fruit shopping. However, these reviews are characterized by a large amount of information and a low value density, which complicates the process of quickly summarizing the preferences of most users. Given this challenge, it is necessary to perform extensive data processing and filtering to extract meaningful preference information from the collected data.

Initially, the online review data is preprocessed as shown in Fig. 4. Online reviews effectively reflect users' primary concerns during fresh fruits purchases. Specifically, keywords such as "logistics" and "taste" in the text indicate that users prioritize logistics transportation experience and taste attributes, thereby according higher importance to logistics and taste preference factors compared to others. Guided by these observations, we first preprocess the data to convert unstructured user reviews into a structured format, followed by the extraction and merging of preference factors. The detailed procedure is presented in Fig. 5.

Based on the above analysis, user preference factors regarding different types of fresh fruits were screened and consolidated, retaining useful information for m types of fresh fruits under n categories of preference factors. The set of m fresh fruit types is denoted as $\Psi = \{\psi_1, \psi_2, \dots, \psi_m\}$ and the set of n preference factor categories is denoted as $\Omega = \{\aleph_1, \aleph_2, \dots, \aleph_n\}$, where $\aleph_t \in \Omega$

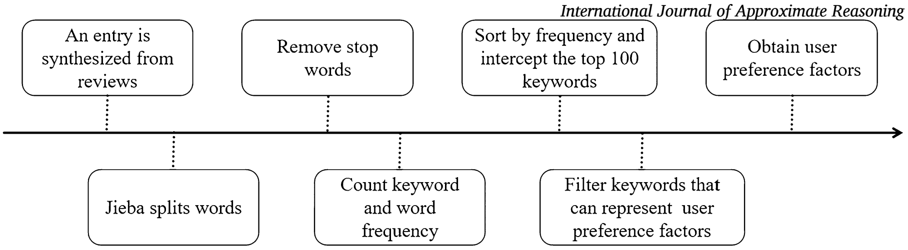

Fig. 5. The flow chart of data extraction.

Table 9
The matrix of MADM.

Ω \ Ψ	ψ_1	ψ_2	$\cdots$	ψ_m
$\aleph_1$	$\psi_1(\aleph_1)$	$\psi_2(\aleph_1)$	$\cdots$	$\psi_m(\aleph_1)$
$\aleph_2$	$\psi_1(\aleph_2)$	$\psi_2(\aleph_2)$	$\cdots$	$\psi_m(\aleph_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\aleph_n$	$\psi_1(\aleph_n)$	$\psi_2(\aleph_n)$	$\cdots$	$\psi_m(\aleph_n)$

represents distinct preference factors such as quality, price, after-sales service, packaging, taste, etc. The conformity degree of fresh fruit ψ_s to preference factor $\aleph_t$ is represented by $\psi_s(\aleph_t) \in [0,1]$, where a higher value indicates a greater degree of conformity to the preference factor, which is presented as an information table in Table 9.

In the following, the lower and upper approximation operators are utilized to formulate the approximation precision as the weight for each user preference. Additionally, the fuzzy TOPSIS method is used to rank the importance of user preference factors. The specific steps of MADM process are presented below.

6.2. Decision-making process

At first, we introduce the concept of approximation precision based on FβCRS, which is constructed as the ratio of the lower approximation operator to the upper approximation operator for each attribute ψ_s. For comparable ON-FβCRS in Table 8 (resp. comparable TN-FβCRS in Table 7), the $*$-based approximation precision of ψ_s is defined as

$$\mathbb{AP}_{\left(\underline{\Psi^*_{i,j}},\overline{\Psi^*_{i,j}}\right)}(\psi_s) = \frac{|\underline{\Psi^*_{i,j}}(\psi_s)|}{|\overline{\Psi^*_{i,j}}(\psi_s)|}, \tag{1}$$

where ψ_s represents the element of Ψ, $i \in \{1,3\}$, $j \in \{0,1,2,5,6,\cap,\cup\}$ when $*= O$ and $j \in \{0,1,2,3,4,\cap,\cup\}$ when $*= T$. $|\cdot|$ denotes the cardinality of fuzzy set.

Furthermore, the attribute weight vector ω^*_s based on $\left(\underline{\Psi^*_{i,j}},\overline{\Psi^*_{i,j}}\right)$ is defined as

$$\omega^*_s = \frac{\mathbb{AP}_{\left(\underline{\Psi^*_{i,j}},\overline{\Psi^*_{i,j}}\right)}(\psi_s)}{\sum_{s=1}^{m}\mathbb{AP}_{\left(\underline{\Psi^*_{i,j}},\overline{\Psi^*_{i,j}}\right)}(\psi_s)}, \tag{2}$$

where $s \in \{1,2,\ldots,m\}$.

To construct the TOPSIS method, the next step is to determine the fuzzy positive ideal solution (FPIS) ψ^+_s and the fuzzy negative ideal solution (FNIS) ψ^-_s for each attribute $\psi_s \in \Psi$. The FPIS ψ^+_s and FNIS ψ^-_s are defined as

$$\psi^+_s = \begin{cases} \max\limits_{1\le t\le n}\{\psi_s(\aleph_t)\}, & \psi_s \in \psi_\dagger, \\ \min\limits_{1\le t\le n}\{\psi_s(\aleph_t)\}, & \psi_s \in \psi_\ddagger, \end{cases} \quad and \quad \psi^-_s = \begin{cases} \min\limits_{1\le t\le n}\{\psi_s(\aleph_t)\}, & \psi_s \in \psi_\dagger, \\ \max\limits_{1\le t\le n}\{\psi_s(\aleph_t)\}, & \psi_s \in \psi_\ddagger, \end{cases} \tag{3}$$

where $\psi_\dagger$ and $\psi_\ddagger$ represent the set of benefit attribute and cost attribute.

By utilizing the one-dimensional Euclidean distance, the distance E^+_{st} between the score $\psi_s(\aleph_t)$ of ψ_s about $\aleph_t$ and FPIS ψ^+_s, the distance E^-_{st} between the score $\psi_s(\aleph_t)$ of ψ_s about $\aleph_t$ and FNIS ψ^-_s can be calculated as

$$E^+_{st} = |\psi_s(\aleph_t) - \psi^+_s| = \psi^+_s - \psi_s(\aleph_t), \tag{4}$$

$$E^-_{st} = |\psi_s(\aleph_t) - \psi^-_s| = \psi_s(\aleph_t) - \psi^-_s. \tag{5}$$

Table 10
The score of user preference factors in 19 fresh fruits.

Ψ \ Ω	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	$\aleph_6$	$\aleph_7$	$\aleph_8$
ψ_1	0.0541	0.5417	0.5000	0.0833	0.7500	0.5000	0.2500	0.7333
ψ_2	0.0541	0.3056	0.8333	0.5000	0.6786	0.7500	0.3750	0.9333
ψ_3	0.8649	0.3333	0.5000	0.0833	0.3929	0.5000	0.3750	0.1333
ψ_4	0.4054	0.5278	0.2222	0.5000	0.2857	0.6250	0.1250	0.0667
ψ_5	0.7027	0.8333	0.6111	0.4167	0.1786	0.6250	0.3750	0.6000
ψ_6	0.3514	0.6250	0.1667	0.2500	0.2143	0.5000	0.3750	0.1333
ψ_7	0.2703	0.4028	0.3333	0.2500	0.5714	0.6250	0.1250	0.4000
ψ_8	0.4865	0.0694	0.3333	0.9167	0.6786	0.6250	0.8750	0.7333
ψ_9	0.2162	0.3194	0.4444	0.7500	0.2500	0.7500	0.8750	0.7333
ψ_{10}	0.3514	0.2083	0.8889	0.5000	0.4286	0.1250	0.6250	0.4000
ψ_{11}	0.1351	0.1111	0.2778	0.6250	0.2500	0.1250	0.3750	0.2667
ψ_{12}	0.6216	0.1806	0.4444	0.8750	0.2143	0.1250	0.2500	0.3333
ψ_{13}	0.4595	0.4028	0.3889	0.5000	0.3929	0.1250	0.1250	0.2000
ψ_{14}	0.3514	0.5000	0.2778	0.1667	0.0357	0.5000	0.5000	0.0667
ψ_{15}	0.7297	0.0278	0.1667	0.4167	0.3214	0.1250	0.8750	0.7333
ψ_{16}	0.4054	0.4444	0.6667	0.2083	0.5714	0.7500	0.7500	0.4000
ψ_{17}	0.6216	0.2083	0.2222	0.3750	0.3571	0.5000	0.6250	0.0667
ψ_{18}	0.4324	0.0556	0.3889	0.6250	0.3214	0.3750	0.1250	0.2000
ψ_{19}	0.5946	0.4167	0.1667	0.4167	0.2143	0.5000	0.2500	0.2667

In accordance with the above steps, the closeness coefficient $\mathfrak{R}_t^*$ for each $\aleph_t$ can be obtained as follows.

$$\mathfrak{R}_t^* = \frac{\sum_{s=1}^{m} \omega_s^* E_{st}^-}{\max\limits_{1\le t\le n}\left(\sum_{s=1}^{m} \omega_s^* E_{st}^-\right)} - \frac{\sum_{s=1}^{m} \omega_s^* E_{st}^+}{\min\limits_{1\le t\le n}\left(\sum_{s=1}^{m} \omega_s^* E_{st}^+\right)}, \tag{6}$$

where $\mathfrak{R}_t^* \le 0$ for any $1 \le t \le n$. Moreover, the ranking of each $\aleph_t$ can be determined by the value of $\mathfrak{R}_t^*$, and the larger the value of $\mathfrak{R}_t^*$, the higher the ranking of $\aleph_t$, indicating that the user considers $\aleph_t$ to be more important.

Remark 6.1. The rationality and superiority of this method can be summarized as follows:

(1) The core idea of the classical TOPSIS method is that the optimal solution should approach the FPIS while distancing itself from the FNIS. Based on this, this paper further integrates the FβCRS model with TOPSIS method and proposes an objective way for determining attribute weights based on upper and lower approximation operators. Subsequently, the FPIS and FNIS are calculated, along with the distances from each attribute to these ideal solutions. Finally, the preference factors are ranked according to the closeness coefficient.
(2) Although [23] and [42] have established fuzzy rough set models based on fuzzy coverings, these models are restricted when data fail to satisfy the requirements of fuzzy coverings. To overcome this limitation, this paper extends the existing models from fuzzy covering to fuzzy β-covering, and proposes an objective method for determining weight in MADM problem.

6.3. The algorithms of decision-making process

In this subsection, we provide the algorithm for the upper and lower approximation operators $\left(\underline{\Psi_{i,j}^{O}}, \overline{\Psi_{i,j}^{O}}\right)$ and their corresponding approximation precisions. For $\left(\underline{\Psi_{i,j}^{T}}, \overline{\Psi_{i,j}^{T}}\right)$, the algorithm is analogous.

6.4. Numerical example

Through screening and merging, eight distinct preference factors are ultimately categorized, including customer service, taste, size, logistics, quality, after-sales, packaging, and platform in turn, denoted as $\Omega = \{\aleph_1, \aleph_2, \dots, \aleph_8\}$. Additionally, fresh fruits containing outliers are deleted under each preference factor, followed by Min-max normalization. To alleviate potential negative impacts from excessive data on subsequent analysis, only data points within the interval $(0,1)$ are retained to derive importance scores for the 8 preference factors across 19 fresh fruit categories, represented as $\Omega = \{\aleph_1, \aleph_2, \dots, \aleph_8\}$ and $\Psi = \{\psi_1, \psi_2, \dots, \psi_{19}\}$. The specific data is presented in Table 10. As indicated in Table 10, Ψ forms a fuzzy β-covering for $\beta \in (0, 0.75]$. In the following, $\beta = 0.75$ is adopted unless otherwise specified.

Without loss of generality, we take $\left(\underline{\Psi_{i,j}^{T}}, \overline{\Psi_{i,j}^{T}}\right)$ and $\left(\underline{\Psi_{i,j}^{O}}, \overline{\Psi_{i,j}^{O}}\right)$ as examples to give detailed steps for the above decision-making issues, and similar decision-making process are available for other models.

Algorithm 1 The algorithm for approximation precision $\mathbb{AP}_{\left(\underline{\Psi^O_{i,j}},\overline{\Psi^O_{i,j}}\right)}(\psi_s)$.

Input: The matrix ψ_{m*n} of fuzzy β-covering $\Psi=\{\psi_s \mid s=1,\dots,m\}$ about universe $\Omega=\{\aleph_t \mid t=1,\dots,n\}$.
Output: The attribute precision ω_s based on $(\underline{\Psi^O_{i,k}},\overline{\Psi^O_{i,k}})$.

Compute the fuzzy β-neighborhood system of each $\aleph_t$:
for $t=1:n$ **do**
  $\mathbb{C}_\beta(\Psi,\aleph_t)=\{\psi\in\Psi,\psi(\aleph_t)\geq\beta\}$
end for
Compute the values of $md_{\Psi,\beta}(\aleph_t)$ and $MD_{\Psi,\beta}(\aleph_t)$:
for $t=1:n$ **do**
  $md_{\Psi,\beta}(\aleph_t)=\{\psi\in\mathbb{C}_\beta(\Psi,\aleph_t):\forall\psi'\in\mathbb{C}_\beta(\Psi,\aleph_t)\wedge\psi'\subseteq\psi\Rightarrow\psi'=\psi\}$
  $MD_{\Psi,\beta}(\aleph_t)=\{\psi\in\mathbb{C}_\beta(\Psi,\aleph_t):\forall\psi'\in\mathbb{C}_\beta(\Psi,\aleph_t)\wedge\psi'\supseteq\psi\Rightarrow\psi'=\psi\}$
end for
Compute the fuzzy neighborhood operators $ON^i_{\Psi,\beta}$ $(i=1,2,3,4)$:
for $t=1:n$ **do**
  for $\varsigma=1:n$ **do**
    $ON^1_{\Psi,\beta}(\aleph_t)(\aleph_\varsigma)=\bigwedge_{\psi\in\mathbb{C}_\beta(\Psi,\aleph_t)}I_O(\psi(\aleph_t),\psi(\aleph_\varsigma))$
    $ON^2_{\Psi,\beta}(\aleph_t)(\aleph_\varsigma)=\bigvee_{\psi\in md_{\Psi,\beta}(\aleph_t)}O(\psi(\aleph_t),\psi(\aleph_\varsigma))$
    $ON^3_{\Psi,\beta}(\aleph_t)(\aleph_\varsigma)=\bigwedge_{\psi\in MD_{\Psi,\beta}(\aleph_t)}I_O(\psi(\aleph_t),\psi(\aleph_\varsigma))$
    $ON^4_{\Psi,\beta}(\aleph_t)(\aleph_\varsigma)=\bigvee_{\psi\in\mathbb{C}_\beta(\Psi,\aleph_t)}O(\psi(\aleph_t),\psi(\aleph_\varsigma))$
  end for
end for
Compute six kinds of fuzzy β-coverings:

$$\Psi^1_\beta=\cup\{md_{\Psi,\beta}(\pi):\pi\in\Omega\};\ \Psi^2_\beta=\cup\{MD_{\Psi,\beta}(\pi):\pi\in\Omega\}$$

$$\Psi^5_\beta=\{ON^1_{\Psi,\beta}(\pi):\pi\in\Omega\};\ \Psi^6_\beta=\{ON^4_{\Psi,\beta}(\pi):\pi\in\Omega\}$$

$$\Psi^\cap_\beta=\Psi\setminus\{\psi\in\Psi:(\exists\Psi'\subseteq\Psi\setminus\{\psi\})(\psi=\cap\Psi')\}$$

$$\Psi^\cup_\beta=\Psi\setminus\{\psi\in\Psi:(\exists\Psi'\subseteq\Psi\setminus\{\psi\})(\psi=\cup\Psi')\}$$

(O in Ψ^5_β satisfies (O7) and O in Ψ^6_β satisfies (O9))
Repeat the above steps to obtain different fuzzy neighborhood operators $ON^i_{\Psi^k_\beta}$ under fuzzy β-coverings Ψ^k_β.
Compute the ON-FβCRS $(\underline{\Psi^O_{i,k}},\overline{\Psi^O_{i,k}})$ and attribute weight ω_s:
for $k=0,1,2,5,6,\cap,\cup$ **do**
  for $i=1:4$ **do**
    for $s=1:m$ **do**
      for $t=1:n$ **do**
        $\underline{\Psi^O_{i,k}}(\psi_s)(\aleph_t)=\bigwedge_{\aleph_\varsigma\in\Omega}\left((1-ON^i_{\Psi^k_\beta}(\aleph_t)(\aleph_\varsigma))\vee\psi_s(\aleph_\varsigma)\right)$
        $\overline{\Psi^O_{i,k}}(\psi_s)(\aleph_t)=\bigvee_{\aleph_\varsigma\in\Omega}\left(ON^i_{\Psi^k_\beta}(\aleph_t)(\aleph_\varsigma)\wedge\psi_s(\aleph_\varsigma)\right)$
      end for
      $\mathbb{AP}_{\left(\underline{\Psi^O_{i,k}},\overline{\Psi^O_{i,k}}\right)}(\psi_s)=\frac{|\underline{\Psi^O_{i,k}}(\psi_s)|}{|\overline{\Psi^O_{i,k}}(\psi_s)|}$ and $\omega_s=\frac{\mathbb{AP}_{\left(\underline{\Psi^O_{i,k}},\overline{\Psi^O_{i,k}}\right)}(\psi_s)}{\sum_{s=1}^{m}\mathbb{AP}_{\left(\underline{\Psi^O_{i,k}},\overline{\Psi^O_{i,k}}\right)}(\psi_s)}$
    end for
  end for
end for

Table 11
The values of FPIS ψ_s^+ and FNIS ψ_s^-.

Ψ	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	ψ_6	ψ_7	ψ_8	ψ_9	ψ_{10}	ψ_{11}	ψ_{12}	ψ_{13}	ψ_{14}	ψ_{15}	ψ_{16}	ψ_{17}	ψ_{18}	ψ_{19}
ψ_s^+	0.750	0.933	0.865	0.625	0.833	0.625	0.625	0.917	0.875	0.889	0.625	0.875	0.500	0.500	0.875	0.750	0.625	0.625	0.595
ψ_s^-	0.054	0.054	0.083	0.067	0.179	0.133	0.125	0.069	0.216	0.125	0.111	0.125	0.125	0.036	0.028	0.208	0.067	0.056	0.167

Step1: Based on Table 10, we calculate the FPIS ψ_s^+ and FNIS ψ_s^- for each fresh fruit ψ_s using Eq. (3). The corresponding results are presented in Table 11.

Step2: Subsequently, the distances E_{st}^+ and E_{st}^- are computed using Eqs. (4) and (5), which are presented in Tables 12 and 13. As shown in Table 12, $E_{11}^+=0.6959$ indicates the distance from preference factor $\aleph_1$ to the FPIS ψ_1^+ for fresh fruit ψ_1, and $E_{11}^-=0$ in Table 13 indicates the distance from $\aleph_1$ to the FNIS ψ_1^- for fresh fruit ψ_1.

Step3: To illustrate attribute weights calculation, we exemplify $\left(\underline{\Psi^T_{A1}},\overline{\Psi^T_{A1}}\right)$ and $\left(\underline{\Psi^O_{A1}},\overline{\Psi^O_{A1}}\right)$, which satisfy comparable property. Specifically, the TN-FβCRS models are constructed using $T=T_P$ and its R-implication $I_T=I_{T_P}$, while the ON-FβCRS models employ

Algorithm 2 The algorithm for decision-making process.

Input: The matrix ψ_{m*n} of fuzzy β-covering $\Psi = \{\psi_s \mid s = 1, \ldots, m\}$ and $\Omega = \{\aleph_t \mid t = 1, \ldots, n\}$ and the attribute precision ω_s.

Output: The ranking of each element $\aleph_t$.

Compute the FPIS ψ_s^+ and FNIS ψ_s^- for each ψ_s:
for $s = 1 : m$ **do**
$$\psi_s^+ = \begin{cases} \max\limits_{1\le t\le n}\{\psi_s(\aleph_t)\}, & \psi_s \in \psi_\dagger, \\ \min\limits_{1\le t\le n}\{\psi_s(\aleph_t)\}, & \psi_s \in \psi_\ddagger, \end{cases} \quad and \quad \psi_s^- = \begin{cases} \min\limits_{1\le t\le n}\{\psi_s(\aleph_t)\}, & \psi_s \in \psi_\dagger, \\ \max\limits_{1\le t\le n}\{\psi_s(\aleph_t)\}, & \psi_s \in \psi_\ddagger, \end{cases}$$
where $\psi_\dagger$ and $\psi_\ddagger$ represent the sets of benefit and cost attributes, respectively.
end for
Compute the distance E_{st}^+ and E_{st}^-:
for $s = 1 : m$ **do**
  for $t = 1 : n$ **do**
    $E_{st}^+ = |\psi_s(\aleph_t) - \psi_s^+| = \psi_s^+ - \psi_s(\aleph_t)$
    $E_{st}^- = |\psi_s(\aleph_t) - \psi_s^-| = \psi_s(\aleph_t) - \psi_s^-$
  end for
end for
Compute the closeness coefficient $\mathfrak{R}_t$:
for $t = 1 : n$ **do**
$$\mathfrak{R}_t = \frac{\sum_{s=1}^{m}\omega_s E_{st}^-}{\max\limits_{1\le t\le n}\left(\sum_{s=1}^{m}\omega_s E_{st}^-\right)} - \frac{\sum_{s=1}^{m}\omega_s E_{st}^+}{\min\limits_{1\le t\le n}\left(\sum_{s=1}^{m}\omega_s E_{st}^+\right)}$$
end for
The larger the value of $\mathfrak{R}_t$, the higher the ranking of $\aleph_t$, that is, $\aleph_t$ is the most important factor.

Table 12
The distance E_{st}^+ between $\psi_s(\aleph_t)$ and ψ_s^+.

Ψ \ Ω	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	$\aleph_6$	$\aleph_7$	$\aleph_8$
ψ_1	0.6959	0.2083	0.2500	0.6667	0	0.2500	0.5000	0.0167
ψ_2	0.8793	0.6278	0.1000	0.4333	0.2548	0.1833	0.5583	0
ψ_3	0	0.5315	0.3649	0.7815	0.4720	0.3649	0.4899	0.7315
ψ_4	0.2196	0.0972	0.4028	0.1250	0.3393	0	0.5000	0.5583
ψ_5	0.1306	0	0.2222	0.4167	0.6548	0.2083	0.4583	0.2333
ψ_6	0.2736	0	0.4583	0.3750	0.4107	0.1250	0.2500	0.4917
ψ_7	0.3547	0.2222	0.2917	0.3750	0.0536	0	0.5000	0.2250
ψ_8	0.4302	0.8472	0.5833	0	0.2381	0.2917	0.0417	0.1833
ψ_9	0.6588	0.5556	0.4306	0.1250	0.6250	0.1250	0	0.1417
ψ_{10}	0.5375	0.6806	0	0.3889	0.4603	0.7639	0.2639	0.4889
ψ_{11}	0.4899	0.5139	0.3472	0	0.3750	0.5000	0.2500	0.3583
ψ_{12}	0.2534	0.6944	0.4306	0	0.6607	0.7500	0.6250	0.5417
ψ_{13}	0.0405	0.0972	0.1111	0	0.1071	0.3750	0.3750	0.3000
ψ_{14}	0.1486	0	0.2222	0.3333	0.4643	0	0	0.4333
ψ_{15}	0.1453	0.8472	0.7083	0.4583	0.5536	0.7500	0	0.1417
ψ_{16}	0.3446	0.3056	0.0833	0.5417	0.1786	0	0	0.3500
ψ_{17}	0.0034	0.4167	0.4028	0.2500	0.2679	0.1250	0	0.5583
ψ_{18}	0.1926	0.5694	0.2361	0	0.3036	0.2500	0.5000	0.4250
ψ_{19}	0	0.1779	0.4279	0.1779	0.3803	0.0946	0.3446	0.3279

$O = O_2^V$ and its R-implication $I_O = I_{O_2^V}$. For $TN^1_{\Psi^1_\beta}$, substituting the fuzzy set Γ in Definition 5.1 with $\psi_s \in \Psi$, the lower and upper approximation operators are depicted in Table 14, respectively. Analogously, the lower and upper approximation operators for $ON^1_{\Psi^1_\beta}$ are calculated in Table 15, respectively.

Step4: Calculate the T-based approximation precision and O-based approximation precision using Eq. (1). Based on these results, the attribute weight vectors ω_s^T and ω_s^O are derived from Eq. (2), which are presented in Table 16.

Step5: Moreover, the T-based closeness coefficient $\mathfrak{R}_t^T$ and O-based closeness coefficient $\mathfrak{R}_t^O$ of each preference factor $\aleph_t$ are calculated using Eq. (6), which are shown in Table 17.

Step6: Finally, the 8 user preference factors can be ranked based on the closeness coefficients $\mathfrak{R}_t^T$ and $\mathfrak{R}_t^O$. Specifically, a higher closeness coefficient corresponds to a higher ranking. Then the ranking based on $\left(\underline{\Psi^T_{A1}}, \overline{\Psi^T_{A1}}\right)$ is represented as

$$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2.$$

And the ranking based on $\left(\underline{\Psi^O_{A1}}, \overline{\Psi^O_{A1}}\right)$ is denoted as

$$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2.$$

Table 13
The distance E^-_{st} between $\psi_s(\aleph_t)$ and ψ^-_s.

Ψ \ Ω	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	$\aleph_6$	$\aleph_7$	$\aleph_8$
ψ_1	0	0.4876	0.4459	0.0293	0.6959	0.4459	0.1959	0.6793
ψ_2	0	0.2515	0.7793	0.4459	0.6245	0.6959	0.3209	0.8793
ψ_3	0.7815	0.2500	0.4167	0	0.3095	0.4167	0.2917	0.0500
ψ_4	0.3387	0.4611	0.1556	0.4333	0.2190	0.5583	0.0583	0
ψ_5	0.5241	0.6548	0.4325	0.2381	0	0.4464	0.1964	0.4214
ψ_6	0.2180	0.4917	0.0333	0.1167	0.0810	0.3667	0.2417	0
ψ_7	0.1453	0.2778	0.2083	0.1250	0.4464	0.5000	0	0.2750
ψ_8	0.4170	0	0.2639	0.8472	0.6091	0.5556	0.8056	0.6639
ψ_9	0	0.1032	0.2282	0.5338	0.0338	0.5338	0.6588	0.5171
ψ_{10}	0.2264	0.0833	0.7639	0.3750	0.3036	0	0.5000	0.2750
ψ_{11}	0.0240	0	0.1667	0.5139	0.1389	0.0139	0.2639	0.1556
ψ_{12}	0.4966	0.0556	0.3194	0.7500	0.0893	0	0.1250	0.2083
ψ_{13}	0.3345	0.2778	0.2639	0.3750	0.2679	0	0	0.0750
ψ_{14}	0.3156	0.4643	0.2421	0.1310	0	0.4643	0.4643	0.0310
ψ_{15}	0.7020	0	0.1389	0.3889	0.2937	0.0972	0.8472	0.7056
ψ_{16}	0.1971	0.2361	0.4583	0	0.3631	0.5417	0.5417	0.1917
ψ_{17}	0.5550	0.1417	0.1556	0.3083	0.2905	0.4333	0.5583	0
ψ_{18}	0.3769	0	0.3333	0.5694	0.2659	0.3194	0.0694	0.1444
ψ_{19}	0.4279	0.2500	0	0.2500	0.0476	0.3333	0.0833	0.1000

This analysis reveals that when purchasing fresh fruits online, users attach greater importance to the "after-sales" factor than to other preference factors, with logistics and other factors following in importance.

Analogously, the aforementioned decision-making procedure is implemented for the TN-FβCRS and ON-FβCRS models presented in Tables 7 and 8, and the detailed ranking results are shown in Table 18.

7. Comparative analysis

By establishing an MADM method based on fuzzy β-covering rough sets, e-commerce platforms can make more accurate assessments of user preference factors during online purchases of fresh fruits. In the preceding two subsections, we validated the stability of the proposed model regarding parameter β and the fuzzy logical operators.

7.1. The results of decision-making based on different parameter β

In light of the above decision-making procedure, it can be concluded that Ψ is a fuzzy β-covering for $\beta \in (0, 0.75]$. Without loss of generality, the ranking results of preference factors in online fresh fruit procurement are calculated for $\beta = 0.15, 0.35, 0.55, 0.75$, respectively.

As shown in Fig. 6, when β increases, the positions of preference factors $\aleph_1$ and $\aleph_4$ in the decision-making result derived from $(\underline{\Psi^T_{A1}}(\Gamma), \overline{\Psi^T_{A1}}(\Gamma))$ are reversed. This reversal implies that users gradually place more emphasis on "customer service" compared to "logistics". In Fig. 7, the ranking of preference factors based on $(\underline{\Psi^O_{A1}}(\Gamma), \overline{\Psi^O_{A1}}(\Gamma))$ is relatively stable and remains unaffected by variations in β.

7.2. The results of decision-making based on different aggregation functions

In this subsection, we take $(\underline{\Psi^T_{A1}}(\Gamma), \overline{\Psi^T_{A1}}(\Gamma))$ and $(\underline{\Psi^O_{A1}}(\Gamma), \overline{\Psi^O_{A1}}(\Gamma))$ as examples. We examine the values of the fuzzy neighborhood operators $TN^1_{\Psi^0_\beta}$ and $ON^1_{\Psi^0_\beta}$ under three distinct types of aggregation functions, respectively. Here, $I_T = I_{T_P}, I_{T_M}, I_{T_L}$ and $I_O = I_{O^V_2}, I_{O^V_{mM}}, I_{O_2}$, with the aim of assessing the impact of varying aggregation functions on the final decision-making outcomes.

As shown in Table 19, for the model $(\underline{\Psi^T_{A1}}(\Gamma), \overline{\Psi^T_{A1}}(\Gamma))$ when the aggregation functions are set as I_{T_P}, I_{T_M} and I_{T_L}, the rankings of the preference factors exhibit greater similarity. Notably, the positions of preference factors $\aleph_7$ and $\aleph_1$ are inverted in the ranking results based on I_{T_M}, while the ranking corresponding to the other two aggregation functions remain identical. For the $(\underline{\Psi^O_{A1}}(\Gamma), \overline{\Psi^O_{A1}}(\Gamma))$ model, the ranking results computed using $I_O = I_{O^V_2}, I_{O^V_{mM}}, I_{O_2}$ remain consistent, and all follow the order $\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$.

7.3. Cross-validation and hypothesis testing

The following further investigates the influence of the dataset on the final decision outcomes, including the cross-validation and hypothesis testing of the proposed method.

Table 14
The pair of approximation operators $(\underline{\Psi^T_{A1}}(\psi_s), \overline{\Psi^T_{A1}}(\psi_s))$ based on I_{T_p}.

$(\underline{\Psi^T_{A1}}(\psi_s), \overline{\Psi^T_{A1}}(\psi_s))$	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	$\aleph_6$	$\aleph_7$	$\aleph_8$
$(\underline{\Psi^T_{A1}}(\psi_1), \overline{\Psi^T_{A1}}(\psi_1))$	(0.0541, 0.5000)	(0.1568, 0.7200)	(0.4375, 0.5000)	(0.0833, 0.3810)	(0.5000, 0.7500)	(0.5000, 0.5333)	(0.2500, 0.5333)	(0.4643, 0.7333)
$(\underline{\Psi^T_{A1}}(\psi_2), \overline{\Psi^T_{A1}}(\psi_2))$	(0.0541, 0.5781)	(0.1568, 0.7500)	(0.5000, 0.8333)	(0.5000, 0.5000)	(0.3056, 0.9333)	(0.5000, 0.7500)	(0.3750, 0.5333)	(0.5000, 0.9333)
$(\underline{\Psi^T_{A1}}(\psi_3), \overline{\Psi^T_{A1}}(\psi_3))$	(0.5000, 0.8649)	(0.2800, 0.8432)	(0.4375, 0.5000)	(0.0833, 0.3636)	(0.1333, 0.5000)	(0.4667, 0.5000)	(0.3750, 0.3750)	(0.1333, 0.5000)
$(\underline{\Psi^T_{A1}}(\psi_4), \overline{\Psi^T_{A1}}(\psi_4))$	(0.4054, 0.5781)	(0.2667, 0.6250)	(0.2222, 0.5000)	(0.5000, 0.5000)	(0.0667, 0.6250)	(0.4074, 0.6250)	(0.1250, 0.2857)	(0.0667, 0.6250)
$(\underline{\Psi^T_{A1}}(\psi_5), \overline{\Psi^T_{A1}}(\psi_5))$	(0.5458, 0.7027)	(0.5000, 0.8333)	(0.4375, 0.6111)	(0.4167, 0.4167)	(0.1786, 0.7222)	(0.5000, 0.6250)	(0.3750, 0.5333)	(0.2730, 0.6250)
$(\underline{\Psi^T_{A1}}(\psi_6), \overline{\Psi^T_{A1}}(\psi_6))$	(0.3514, 0.5000)	(0.2667, 0.6250)	(0.1667, 0.3750)	(0.2500, 0.2883)	(0.1333, 0.6250)	(0.4074, 0.5000)	(0.3750, 0.3750)	(0.1333, 0.5000)
$(\underline{\Psi^T_{A1}}(\psi_7), \overline{\Psi^T_{A1}}(\psi_7))$	(0.2703, 0.5781)	(0.2703, 0.6250)	(0.3333, 0.4821)	(0.2500, 0.3810)	(0.3333, 0.6250)	(0.4074, 0.6250)	(0.1250, 0.4000)	(0.3333, 0.6250)
$(\underline{\Psi^T_{A1}}(\psi_8), \overline{\Psi^T_{A1}}(\psi_8))$	(0.4219, 0.5781)	(0.0694, 0.7200)	(0.3333, 0.5625)	(0.6364, 0.9167)	(0.2778, 0.7333)	(0.4074, 0.6250)	(0.7143, 0.8750)	(0.3333, 0.7333)
$(\underline{\Psi^T_{A1}}(\psi_9), \overline{\Psi^T_{A1}}(\psi_9))$	(0.2162, 0.5781)	(0.2162, 0.7500)	(0.4444, 0.5625)	(0.6364, 0.7500)	(0.2500, 0.7333)	(0.4444, 0.7500)	(0.7143, 0.8750)	(0.2730, 0.7500)
$(\underline{\Psi^T_{A1}}(\psi_{10}), \overline{\Psi^T_{A1}}(\psi_{10}))$	(0.3514, 0.5781)	(0.2083, 0.7333)	(0.5000, 0.8889)	(0.5000, 0.5000)	(0.2778, 0.6667)	(0.1250, 0.5926)	(0.4667, 0.6250)	(0.1964, 0.8889)
$(\underline{\Psi^T_{A1}}(\psi_{11}), \overline{\Psi^T_{A1}}(\psi_{11}))$	(0.1351, 0.3750)	(0.1111, 0.5000)	(0.2778, 0.5625)	(0.6190, 0.6250)	(0.2500, 0.3333)	(0.1250, 0.3750)	(0.3750, 0.3750)	(0.1964, 0.5357)
$(\underline{\Psi^T_{A1}}(\psi_{12}), \overline{\Psi^T_{A1}}(\psi_{12}))$	(0.4219, 0.6216)	(0.1806, 0.6216)	(0.4444, 0.5625)	(0.6190, 0.8750)	(0.2143, 0.4444)	(0.1250, 0.4444)	(0.2500, 0.3333)	(0.1964, 0.5357)
$(\underline{\Psi^T_{A1}}(\psi_{13}), \overline{\Psi^T_{A1}}(\psi_{13}))$	(0.4219, 0.4595)	(0.2500, 0.5000)	(0.3889, 0.5000)	(0.5000, 0.5000)	(0.2000, 0.4028)	(0.1250, 0.4028)	(0.1250, 0.2857)	(0.1964, 0.5000)
$(\underline{\Psi^T_{A1}}(\psi_{14}), \overline{\Psi^T_{A1}}(\psi_{14}))$	(0.3514, 0.5000)	(0.2778, 0.5000)	(0.2778, 0.4500)	(0.1667, 0.2883)	(0.0357, 0.5000)	(0.4074, 0.5000)	(0.4667, 0.5000)	(0.0667, 0.5000)
$(\underline{\Psi^T_{A1}}(\psi_{15}), \overline{\Psi^T_{A1}}(\psi_{15}))$	(0.4219, 0.7297)	(0.0278, 0.7297)	(0.1667, 0.4500)	(0.4167, 0.4167)	(0.2778, 0.7333)	(0.1250, 0.5333)	(0.7143, 0.8750)	(0.1667, 0.7333)
$(\underline{\Psi^T_{A1}}(\psi_{16}), \overline{\Psi^T_{A1}}(\psi_{16}))$	(0.4054, 0.5781)	(0.4000, 0.7500)	(0.4375, 0.6667)	(0.2083, 0.3810)	(0.4000, 0.6667)	(0.4667, 0.7500)	(0.4667, 0.7500)	(0.4000, 0.7500)
$(\underline{\Psi^T_{A1}}(\psi_{17}), \overline{\Psi^T_{A1}}(\psi_{17}))$	(0.4219, 0.6216)	(0.2083, 0.6216)	(0.2222, 0.4500)	(0.3750, 0.3750)	(0.0667, 0.5000)	(0.4074, 0.5000)	(0.4667, 0.6250)	(0.066, 0.5000)
$(\underline{\Psi^T_{A1}}(\psi_{18}), \overline{\Psi^T_{A1}}(\psi_{18}))$	(0.4219, 0.4324)	(0.0556, 0.5000)	(0.3889, 0.5625)	(0.6190, 0.6250)	(0.2000, 0.3889)	(0.3750, 0.3889)	(0.1250, 0.2857)	(0.2000, 0.5357)
$(\underline{\Psi^T_{A1}}(\psi_{19}), \overline{\Psi^T_{A1}}(\psi_{19}))$	(0.4219, 0.5946)	(0.2667, 0.5946)	(0.1667, 0.4167)	(0.4167, 0.4167)	(0.2143, 0.5000)	(0.4074, 0.5000)	(0.2500, 0.2778)	(0.1667, 0.5000)

Table 15
The pair of approximation operators $(\underline{\Psi^{O}_{A1}}(\psi_s), \overline{\Psi^{O}_{A1}}(\psi_s))$ based on $I_{O_2^{\gamma}}$.

$(\underline{\Psi^{O}_{A1}}(\psi_s), \overline{\Psi^{O}_{A1}}(\psi_s))$	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	$\aleph_6$	$\aleph_7$	$\aleph_8$
$(\underline{\Psi^{O}_{A1}}(\psi_1), \overline{\Psi^{O}_{A1}}(\psi_1))$	(0.0541, 0.5000)	(0.0541, 0.7333)	(0.5000, 0.5000)	(0.0833, 0.3333)	(0.5000, 0.7500)	(0.5000, 0.5000)	(0.2500, 0.4000)	(0.5000, 0.7500)
$(\underline{\Psi^{O}_{A1}}(\psi_2), \overline{\Psi^{O}_{A1}}(\psi_2))$	(0.0541, 0.5000)	(0.0541, 0.8354)	(0.5000, 0.8333)	(0.5000, 0.5000)	(0.3056, 0.9333)	(0.6250, 0.7500)	(0.3750, 0.4000)	(0.5000, 0.9333)
$(\underline{\Psi^{O}_{A1}}(\psi_3), \overline{\Psi^{O}_{A1}}(\psi_3))$	(0.5000, 0.8649)	(0.1646, 0.8649)	(0.5000, 0.5000)	(0.0833, 0.3333)	(0.1333, 0.5000)	(0.5000, 0.5000)	(0.3750, 0.3750)	(0.1333, 0.5000)
$(\underline{\Psi^{O}_{A1}}(\psi_4), \overline{\Psi^{O}_{A1}}(\psi_4))$	(0.4054, 0.5000)	(0.1646, 0.6250)	(0.2222, 0.5000)	(0.5000, 0.5000)	(0.0667, 0.5278)	(0.5556, 0.6250)	(0.1250, 0.2500)	(0.0667, 0.6250)
$(\underline{\Psi^{O}_{A1}}(\psi_5), \overline{\Psi^{O}_{A1}}(\psi_5))$	(0.6071, 0.7027)	(0.5833, 0.8333)	(0.5000, 0.6111)	(0.4167, 0.4167)	(0.1786, 0.7887)	(0.6000, 0.6250)	(0.3750, 0.4000)	(0.1786, 0.6250)
$(\underline{\Psi^{O}_{A1}}(\psi_6), \overline{\Psi^{O}_{A1}}(\psi_6))$	(0.3514, 0.5000)	(0.1646, 0.6250)	(0.1667, 0.3750)	(0.2500, 0.2500)	(0.1333, 0.6250)	(0.5000, 0.5000)	(0.3750, 0.3750)	(0.1333, 0.5000)
$(\underline{\Psi^{O}_{A1}}(\psi_7), \overline{\Psi^{O}_{A1}}(\psi_7))$	(0.2703, 0.5000)	(0.2703, 0.6250)	(0.3333, 0.4286)	(0.2500, 0.3333)	(0.4000, 0.5714)	(0.5556, 0.6250)	(0.1250, 0.4000)	(0.3333, 0.6250)
$(\underline{\Psi^{O}_{A1}}(\psi_8), \overline{\Psi^{O}_{A1}}(\psi_8))$	(0.4865, 0.5000)	(0.0694, 0.7333)	(0.3333, 0.5000)	(0.6667, 0.9167)	(0.2113, 0.7333)	(0.5556, 0.6250)	(0.7333, 0.8750)	(0.3333, 0.7333)
$(\underline{\Psi^{O}_{A1}}(\psi_9), \overline{\Psi^{O}_{A1}}(\psi_9))$	(0.2162, 0.5000)	(0.2162, 0.7500)	(0.4444, 0.5000)	(0.6667, 0.7500)	(0.2500, 0.7333)	(0.5556, 0.7500)	(0.7333, 0.8750)	(0.2500, 0.7500)
$(\underline{\Psi^{O}_{A1}}(\psi_{10}), \overline{\Psi^{O}_{A1}}(\psi_{10}))$	(0.3514, 0.5000)	(0.1250, 0.8536)	(0.5000, 0.8889)	(0.5000, 0.5000)	(0.2113, 0.5000)	(0.1250, 0.4444)	(0.6000, 0.6250)	(0.1250, 0.8889)
$(\underline{\Psi^{O}_{A1}}(\psi_{11}), \overline{\Psi^{O}_{A1}}(\psi_{11}))$	(0.1351, 0.3750)	(0.1111, 0.4167)	(0.2778, 0.5000)	(0.6250, 0.6250)	(0.2113, 0.2778)	(0.1250, 0.3750)	(0.3750, 0.3750)	(0.1250, 0.5000)
$(\underline{\Psi^{O}_{A1}}(\psi_{12}), \overline{\Psi^{O}_{A1}}(\psi_{12}))$	(0.5000, 0.6216)	(0.1250, 0.6216)	(0.4444, 0.5000)	(0.6667, 0.8750)	(0.2113, 0.4444)	(0.1250, 0.4444)	(0.2500, 0.3333)	(0.1250, 0.5000)
$(\underline{\Psi^{O}_{A1}}(\psi_{13}), \overline{\Psi^{O}_{A1}}(\psi_{13}))$	(0.4595, 0.4595)	(0.1250, 0.4595)	(0.3889, 0.5000)	(0.5000, 0.5000)	(0.2000, 0.4028)	(0.1250, 0.3889)	(0.1250, 0.2500)	(0.1250, 0.5000)
$(\underline{\Psi^{O}_{A1}}(\psi_{14}), \overline{\Psi^{O}_{A1}}(\psi_{14}))$	(0.3514, 0.5000)	(0.1646, 0.5000)	(0.2778, 0.3750)	(0.1667, 0.2778)	(0.0357, 0.5000)	(0.5000, 0.5000)	(0.5000, 0.5000)	(0.0667, 0.5000)
$(\underline{\Psi^{O}_{A1}}(\psi_{15}), \overline{\Psi^{O}_{A1}}(\psi_{15}))$	(0.5000, 0.7297)	(0.0278, 0.7333)	(0.1667, 0.4167)	(0.4167, 0.4167)	(0.2113, 0.7333)	(0.1250, 0.4000)	(0.7333, 0.8750)	(0.1250, 0.7333)
$(\underline{\Psi^{O}_{A1}}(\psi_{16}), \overline{\Psi^{O}_{A1}}(\psi_{16}))$	(0.4054, 0.5000)	(0.4000, 0.7500)	(0.5000, 0.6667)	(0.2083, 0.3333)	(0.4000, 0.5714)	(0.6000, 0.7500)	(0.6000, 0.7500)	(0.4000, 0.7500)
$(\underline{\Psi^{O}_{A1}}(\psi_{17}), \overline{\Psi^{O}_{A1}}(\psi_{17}))$	(0.5000, 0.6216)	(0.1646, 0.6216)	(0.2222, 0.3750)	(0.3750, 0.3750)	(0.0667, 0.5000)	(0.5000, 0.5000)	(0.6000, 0.6250)	(0.0667, 0.5000)
$(\underline{\Psi^{O}_{A1}}(\psi_{18}), \overline{\Psi^{O}_{A1}}(\psi_{18}))$	(0.4324, 0.4324)	(0.0556, 0.4324)	(0.3889, 0.5000)	(0.6250, 0.6250)	(0.2000, 0.3889)	(0.3750, 0.3889)	(0.1250, 0.2500)	(0.2000, 0.5000)
$(\underline{\Psi^{O}_{A1}}(\psi_{19}), \overline{\Psi^{O}_{A1}}(\psi_{19}))$	(0.5000, 0.5946)	(0.1667, 0.5946)	(0.1667, 0.4167)	(0.4167, 0.4167)	(0.2143, 0.5000)	(0.5000, 0.5000)	(0.2500, 0.2667)	(0.1667, 0.5000)

Table 16
The approximate precision and attribute weight of each ψ_s.

	$\mathbb{AP}_{\left(\underline{\Psi^T_{A1}},\overline{\Psi^T_{A1}}\right)}(\psi_s)$	ω_s^T	$\mathbb{AP}_{\left(\underline{\Psi^O_{A1}},\overline{\Psi^O_{A1}}\right)}(\psi_s)$	ω_s^O
ψ_1	0.5259	0.0502	0.5466	0.0496
ψ_2	0.4975	0.0475	0.5125	0.0465
ψ_3	0.5418	0.0517	0.5384	0.0489
ψ_4	0.4721	0.0451	0.5072	0.0461
ψ_5	0.6365	0.0608	0.6875	0.0624
ψ_6	0.5501	0.0525	0.5531	0.0502
ψ_7	0.5351	0.0511	0.6177	0.0561
ψ_8	0.5560	0.0531	0.6035	0.0548
ψ_9	0.5557	0.0531	0.5942	0.0540
ψ_{10}	0.4797	0.0458	0.4879	0.0443
ψ_{11}	0.5676	0.0542	0.5764	0.0523
ψ_{12}	0.5523	0.0528	0.5639	0.0512
ψ_{13}	0.6216	0.0594	0.5919	0.0537
ψ_{14}	0.5484	0.0524	0.5647	0.0513
ψ_{15}	0.4454	0.0425	0.4577	0.0416
ψ_{16}	0.6017	0.0575	0.6928	0.0629
ψ_{17}	0.5330	0.0509	0.6059	0.0550
ψ_{18}	0.6414	0.0613	0.6828	0.0620
ψ_{19}	0.6079	0.0581	0.6284	0.0571

Table 17
The closeness coefficient $\mathfrak{R}_t^T$ and $\mathfrak{R}_t^O$ of each $\aleph_t$.

	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	$\aleph_6$	$\aleph_7$	$\aleph_8$
$\mathfrak{R}_t^T$	−0.2187	−0.7601	−0.3350	−0.1020	−0.6104	0	−0.2334	−0.5200
$\mathfrak{R}_t^O$	−0.2678	−0.8065	−0.3797	−0.1523	−0.6417	0	−0.2566	−0.5572

Table 18
The results of ranking based on TN-FβCRS and ON-FβCRS models.

TN-FβCRS	Ranking	ON-FβCRS	Ranking
$(\underline{\Psi^T_{A1}}(\Gamma), \overline{\Psi^T_{A1}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{A1}}(\Gamma), \overline{\Psi^O_{A1}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
$(\underline{\Psi^T_{A2}}(\Gamma), \overline{\Psi^T_{A2}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{A2}}(\Gamma), \overline{\Psi^O_{A2}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
$(\underline{\Psi^T_{A3}}(\Gamma), \overline{\Psi^T_{A3}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{A3}}(\Gamma), \overline{\Psi^O_{A3}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_3 > \aleph_1 > \aleph_8 > \aleph_5 > \aleph_2$
$(\underline{\Psi^T_{B}}(\Gamma), \overline{\Psi^T_{B}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{A4}}(\Gamma), \overline{\Psi^O_{A4}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
$(\underline{\Psi^T_{C}}(\Gamma), \overline{\Psi^T_{C}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{B}}(\Gamma), \overline{\Psi^O_{B}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
$(\underline{\Psi^T_{D}}(\Gamma), \overline{\Psi^T_{D}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{C}}(\Gamma), \overline{\Psi^O_{C}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_3 > \aleph_1 > \aleph_8 > \aleph_5 > \aleph_2$
$(\underline{\Psi^T_{E}}(\Gamma), \overline{\Psi^T_{E}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{D}}(\Gamma), \overline{\Psi^O_{D}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
$(\underline{\Psi^T_{F}}(\Gamma), \overline{\Psi^T_{F}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{E}}(\Gamma), \overline{\Psi^O_{E}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
$(\underline{\Psi^T_{G}}(\Gamma), \overline{\Psi^T_{G}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{F}}(\Gamma), \overline{\Psi^O_{F}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
$(\underline{\Psi^T_{H}}(\Gamma), \overline{\Psi^T_{H}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{G}}(\Gamma), \overline{\Psi^O_{G}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
$(\underline{\Psi^T_{I}}(\Gamma), \overline{\Psi^T_{I}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{H}}(\Gamma), \overline{\Psi^O_{H}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
$(\underline{\Psi^T_{J}}(\Gamma), \overline{\Psi^T_{J}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{I}}(\Gamma), \overline{\Psi^O_{I}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_2 > \aleph_5$
$(\underline{\Psi^T_{K}}(\Gamma), \overline{\Psi^T_{K}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{J}}(\Gamma), \overline{\Psi^O_{J}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
$(\underline{\Psi^T_{L}}(\Gamma), \overline{\Psi^T_{L}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{K}}(\Gamma), \overline{\Psi^O_{K}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_2 > \aleph_5$
$(\underline{\Psi^T_{M}}(\Gamma), \overline{\Psi^T_{M}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{L}}(\Gamma), \overline{\Psi^O_{L}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
$(\underline{\Psi^T_{N}}(\Gamma), \overline{\Psi^T_{N}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{M}}(\Gamma), \overline{\Psi^O_{M}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
$(\underline{\Psi^T_{O}}(\Gamma), \overline{\Psi^T_{O}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{N}}(\Gamma), \overline{\Psi^O_{N}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
$(\underline{\Psi^T_{P}}(\Gamma), \overline{\Psi^T_{P}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{O}}(\Gamma), \overline{\Psi^O_{O}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
$(\underline{\Psi^T_{Q}}(\Gamma), \overline{\Psi^T_{Q}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$(\underline{\Psi^O_{P}}(\Gamma), \overline{\Psi^O_{P}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
		$(\underline{\Psi^O_{Q}}(\Gamma), \overline{\Psi^O_{Q}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$

7.3.1. *Cross-validation*

In general, we take $(\underline{\Psi^T_{A1}}(\Gamma), \overline{\Psi^T_{A1}}(\Gamma))$ and $(\underline{\Psi^O_{A1}}(\Gamma), \overline{\Psi^O_{A1}}(\Gamma))$ as examples. The TN-FβCRS models are constructed using $T = T_P$ and its R-implication $I_T = I_{T_P}$, while the ON-FβCRS models employ $O = O_2^V$ and its R-implication $I_O = I_{O_2^V}$.

Firstly, the universe Ω is partitioned into four subsets, denoted as $\Omega_1 = \{\aleph_1, \aleph_2\}$, $\Omega_2 = \{\aleph_1, \aleph_2, \aleph_3, \aleph_4\}$, $\Omega_3 = \{\aleph_1, \aleph_2, \aleph_3, \aleph_4, \aleph_5, \aleph_6\}$, and $\Omega_4 = \{\aleph_1, \aleph_2, \aleph_3, \aleph_4, \aleph_5, \aleph_6, \aleph_7, \aleph_8\}$. For these subsets, if the ranking results and optimal solutions remain consistent with those

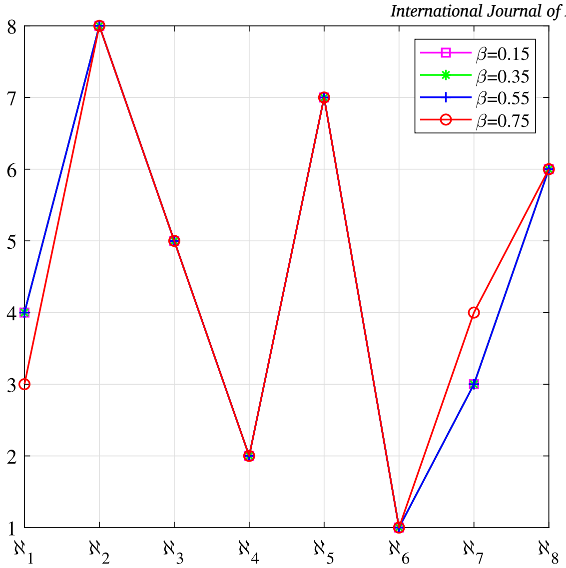

Fig. 6. The ranking of $\aleph_t$ based on $(\underline{\Psi^T_{A1}}(\Gamma), \overline{\Psi^T_{A1}}(\Gamma))$.

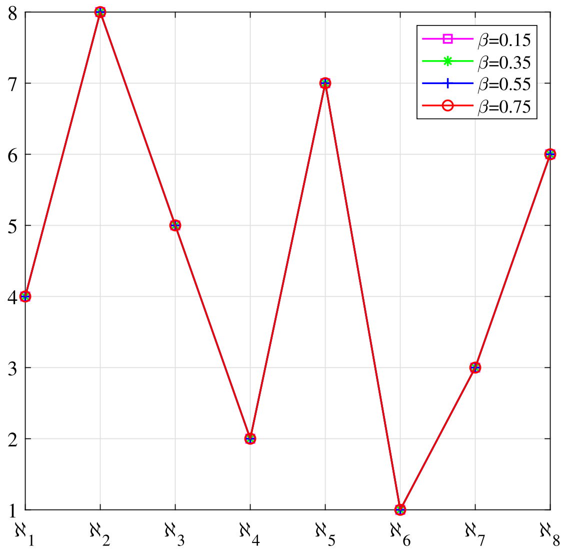

Fig. 7. The ranking of $\aleph_t$ based on $(\underline{\Psi^O_{A1}}(\Gamma), \overline{\Psi^O_{A1}}(\Gamma))$.

Table 19
The results of ranking based on different aggregation functions.

t-norm	Ranking based on $TN^1_{\Psi^0_\beta}$	Overlap function	Ranking based on $ON^1_{\Psi^0_\beta}$
I_{T_P}	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$I_{O_2^V}$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
I_{T_M}	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$I_{O_{mM}^V}$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
I_{T_L}	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	I_{O_2}	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$

obtained in Step 6, the proposed method demonstrates robustness against variations in specific datasets, indicating strong feasibility. The decision results based on different models and datasets are presented in Table 20 and Figs. 8 and 9.

As evidenced by Table 20 and Figs. 8 and 9, the ranking trends across different object sets exhibit consistent patterns. This indicates that the number of elements in Ω_i $(i = 1, 2, 3, 4)$ exerts minimal influence on the proposed method. For instance, the ranking of preference factors $\aleph_1$ and $\aleph_2$ maintains the relationship $\aleph_1 > \aleph_2$ across all subsets Ω_i. To demonstrate the feasibility of the proposed models, the concept of ranking similarity is shown as follows.

Table 20
The ranking of different datasets.

dataset	Ranking based on $TN^1_{\Psi^0_\beta}$	Ranking based on $ON^1_{\Psi^0_\beta}$
Ω_1	$\aleph_1 > \aleph_2$	$\aleph_1 > \aleph_2$
Ω_2	$\aleph_4 > \aleph_1 > \aleph_3 > \aleph_2$	$\aleph_4 > \aleph_1 > \aleph_3 > \aleph_2$
Ω_3	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_3 > \aleph_5 > \aleph_2$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_3 > \aleph_5 > \aleph_2$
Ω_4	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$

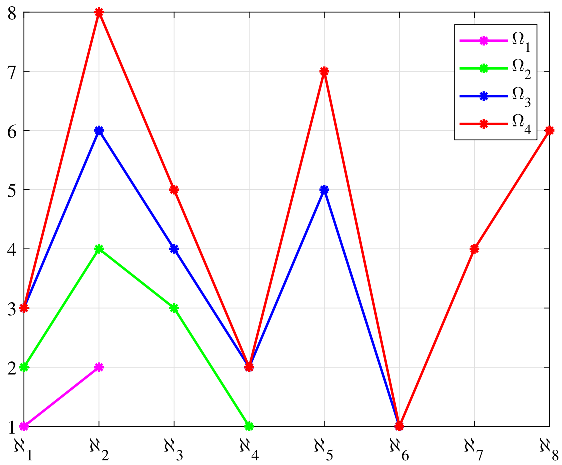

Fig. 8. The ranking of different datasets based on $(\underline{\Psi^T_{A1}}(\Gamma), \overline{\Psi^T_{A1}}(\Gamma))$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

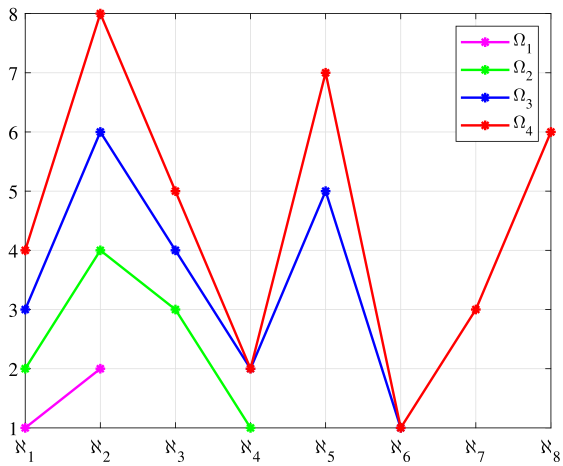

Fig. 9. The ranking of different datasets based on $(\underline{\Psi^O_{A1}}(\Gamma), \overline{\Psi^O_{A1}}(\Gamma))$.

Definition 7.1. [42] Let $\{\Omega_1, \Omega_2, \cdots, \Omega_l\}$ be a family of object sets, where $\Omega_1 \subseteq \Omega_2 \subseteq \cdots \subseteq \Omega_l$. If the number of ordered relation of Ω_i is a, and there are b $(b \le c)$ same ordered relations of Ω_i and Ω_j $(j \ge i, i, j = 1, 2, \cdots, l)$, then the sorting similarity between Ω_i and Ω_j is $\frac{b}{a}$.

Note that the sorting similarity between two object sets is higher, the performance of the sorting decision-making is better. In light of Table 20, it can be calculated that the sorting similarity of Ω_i and Ω_j $(i < j)$ are 100% in $(\underline{\Psi^T_{A1}}(\Gamma), \overline{\Psi^T_{A1}}(\Gamma))$. In a similar way, the sorting similarity of Ω_i and Ω_j $(i < j)$ are 100% in $(\underline{\Psi^O_{A1}}(\Gamma), \overline{\Psi^O_{A1}}(\Gamma))$. Hence, the performance of our MADM method is better.

7.3.2. Hypothesis testing

Analysis of the aforementioned ranking results indicates that preference factor $\aleph_3$ is the most critical factor in users' online purchasing for fresh fruits. To validate the performance of the proposed decision-making method under random object sets, a *p*-

Table 21
The ranking of sample sets based on $(\underline{\Psi^{T}_{A1}}(\Gamma), \overline{\Psi^{T}_{A1}}(\Gamma))$.

Sample set	Ranking	Sample set	Ranking
$\Omega_1 = \{\aleph_1, \aleph_2, \aleph_3, \aleph_4, \aleph_6\}$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_3 > \aleph_2$	$\Omega_{11} = \{\aleph_1, \aleph_3, \aleph_4, \aleph_5, \aleph_6\}$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_3 > \aleph_5$
$\Omega_2 = \{\aleph_1, \aleph_2, \aleph_6, \aleph_7, \aleph_8\}$	$\aleph_6 > \aleph_7 > \aleph_1 > \aleph_8 > \aleph_2$	$\Omega_{12} = \{\aleph_1, \aleph_3, \aleph_4, \aleph_6, \aleph_8\}$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_3 > \aleph_8$
$\Omega_3 = \{\aleph_1, \aleph_5, \aleph_6, \aleph_7, \aleph_8\}$	$\aleph_6 > \aleph_7 > \aleph_1 > \aleph_8 > \aleph_5$	$\Omega_{13} = \{\aleph_1, \aleph_3, \aleph_6, \aleph_7, \aleph_8\}$	$\aleph_6 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8$
$\Omega_4 = \{\aleph_2, \aleph_3, \aleph_4, \aleph_5, \aleph_6\}$	$\aleph_6 > \aleph_4 > \aleph_3 > \aleph_5 > \aleph_2$	$\Omega_{14} = \{\aleph_1, \aleph_4, \aleph_5, \aleph_6, \aleph_7\}$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_5$
$\Omega_5 = \{\aleph_2, \aleph_3, \aleph_6, \aleph_7, \aleph_8\}$	$\aleph_6 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_2$	$\Omega_{15} = \{\aleph_1, \aleph_4, \aleph_6, \aleph_7, \aleph_8\}$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_8$
$\Omega_6 = \{\aleph_2, \aleph_4, \aleph_5, \aleph_6, \aleph_7\}$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_5 > \aleph_2$	$\Omega_{16} = \{\aleph_2, \aleph_3, \aleph_4, \aleph_6, \aleph_7\}$	$\aleph_6 > \aleph_7 > \aleph_4 > \aleph_3 > \aleph_2$
$\Omega_7 = \{\aleph_3, \aleph_4, \aleph_5, \aleph_6, \aleph_8\}$	$\aleph_6 > \aleph_4 > \aleph_3 > \aleph_8 > \aleph_5$	$\Omega_{17} = \{\aleph_2, \aleph_3, \aleph_5, \aleph_6, \aleph_7\}$	$\aleph_6 > \aleph_7 > \aleph_3 > \aleph_5 > \aleph_2$
$\Omega_8 = \{\aleph_3, \aleph_5, \aleph_6, \aleph_7, \aleph_8\}$	$\aleph_6 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5$	$\Omega_{18} = \{\aleph_2, \aleph_4, \aleph_5, \aleph_6, \aleph_8\}$	$\aleph_6 > \aleph_4 > \aleph_8 > \aleph_5 > \aleph_2$
$\Omega_9 = \{\aleph_4, \aleph_5, \aleph_6, \aleph_7, \aleph_8\}$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_8 > \aleph_5$	$\Omega_{19} = \{\aleph_3, \aleph_4, \aleph_5, \aleph_6, \aleph_7\}$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_3 > \aleph_5$
$\Omega_{10} = \{\aleph_1, \aleph_2, \aleph_5, \aleph_6, \aleph_7\}$	$\aleph_6 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_5$	$\Omega_{20} = \{\aleph_1, \aleph_2, \aleph_3, \aleph_5, \aleph_6\}$	$\aleph_6 > \aleph_1 > \aleph_3 > \aleph_5 > \aleph_2$

Table 22
The ranking of sample sets based on $(\underline{\Psi^{O}_{A1}}(\Gamma), \overline{\Psi^{O}_{A1}}(\Gamma))$.

Sample set	Ranking	Sample set	Ranking
$\Omega_1 = \{\aleph_1, \aleph_2, \aleph_3, \aleph_4, \aleph_6\}$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_3 > \aleph_2$	$\Omega_{11} = \{\aleph_1, \aleph_3, \aleph_4, \aleph_5, \aleph_6\}$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_3 > \aleph_5$
$\Omega_2 = \{\aleph_1, \aleph_2, \aleph_6, \aleph_7, \aleph_8\}$	$\aleph_6 > \aleph_7 > \aleph_1 > \aleph_8 > \aleph_2$	$\Omega_{12} = \{\aleph_1, \aleph_3, \aleph_4, \aleph_6, \aleph_8\}$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_3 > \aleph_8$
$\Omega_3 = \{\aleph_1, \aleph_5, \aleph_6, \aleph_7, \aleph_8\}$	$\aleph_6 > \aleph_7 > \aleph_1 > \aleph_8 > \aleph_5$	$\Omega_{13} = \{\aleph_1, \aleph_3, \aleph_6, \aleph_7, \aleph_8\}$	$\aleph_6 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8$
$\Omega_4 = \{\aleph_2, \aleph_3, \aleph_4, \aleph_5, \aleph_6\}$	$\aleph_6 > \aleph_4 > \aleph_3 > \aleph_5 > \aleph_2$	$\Omega_{14} = \{\aleph_1, \aleph_4, \aleph_5, \aleph_6, \aleph_7\}$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_5$
$\Omega_5 = \{\aleph_2, \aleph_3, \aleph_6, \aleph_7, \aleph_8\}$	$\aleph_6 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_2$	$\Omega_{15} = \{\aleph_1, \aleph_4, \aleph_6, \aleph_7, \aleph_8\}$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_8$
$\Omega_6 = \{\aleph_2, \aleph_4, \aleph_5, \aleph_6, \aleph_7\}$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_5 > \aleph_2$	$\Omega_{16} = \{\aleph_2, \aleph_3, \aleph_4, \aleph_6, \aleph_7\}$	$\aleph_6 > \aleph_7 > \aleph_4 > \aleph_3 > \aleph_2$
$\Omega_7 = \{\aleph_3, \aleph_4, \aleph_5, \aleph_6, \aleph_8\}$	$\aleph_6 > \aleph_4 > \aleph_3 > \aleph_8 > \aleph_5$	$\Omega_{17} = \{\aleph_2, \aleph_3, \aleph_5, \aleph_6, \aleph_7\}$	$\aleph_6 > \aleph_7 > \aleph_3 > \aleph_5 > \aleph_2$
$\Omega_8 = \{\aleph_3, \aleph_5, \aleph_6, \aleph_7, \aleph_8\}$	$\aleph_6 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5$	$\Omega_{18} = \{\aleph_2, \aleph_4, \aleph_5, \aleph_6, \aleph_8\}$	$\aleph_6 > \aleph_4 > \aleph_8 > \aleph_5 > \aleph_2$
$\Omega_9 = \{\aleph_4, \aleph_5, \aleph_6, \aleph_7, \aleph_8\}$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_8 > \aleph_5$	$\Omega_{19} = \{\aleph_3, \aleph_4, \aleph_5, \aleph_6, \aleph_7\}$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_3 > \aleph_5$
$\Omega_{10} = \{\aleph_1, \aleph_2, \aleph_5, \aleph_6, \aleph_7\}$	$\aleph_6 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_5$	$\Omega_{20} = \{\aleph_1, \aleph_2, \aleph_3, \aleph_5, \aleph_6\}$	$\aleph_6 > \aleph_1 > \aleph_3 > \aleph_5 > \aleph_2$

Table 23
The ranking of different decision-making methods.

Decision-making method	Ranking
Our method based on $(\underline{\Psi^{T}_{A1}}(\Gamma), \overline{\Psi^{T}_{A1}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
Our method based on $(\underline{\Psi^{O}_{A1}}(\Gamma), \overline{\Psi^{O}_{A1}}(\Gamma))$	$\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$
The Vikor method ($v = 0.8$) [20]	$\aleph_6 > \aleph_1 > \aleph_4 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_2 > \aleph_5$
The Vikor method ($v = 0.9$) [20]	$\aleph_6 > \aleph_4 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_2 > \aleph_8 > \aleph_5$
The TOPSIS method based on ONRFRS [23]	$\aleph_6 > \aleph_5 > \aleph_1 > \aleph_4 > \aleph_7 > \aleph_2 > \aleph_8 > \aleph_3$
The TOPSIS method based on TNRFRS [23]	$\aleph_6 > \aleph_5 > \aleph_1 > \aleph_4 > \aleph_7 > \aleph_2 > \aleph_8 > \aleph_3$

value comparison method (at 0.05 confidence level) was employed through hypothesis testing, demonstrating that $\aleph_3$ is the optimal solution. Specifically, two complementary hypotheses are formulated:

- The test hypothesis H_0: $\aleph_3$ is not the optimal preference factor;
- The alternative hypothesis H_1: $\aleph_3$ is the optimal preference factor.

According to the random-number table, 20 subsets of preference factors $\Omega = \{\aleph_1, \aleph_2, \cdots, \aleph_8\}$ were generated based on Table 10. Each subset contained 5 distinct elements and included the element $\aleph_3$. Let t denote the number of subsets where $\aleph_3$ is the optimal decision result. Then the probability of the hypothesis H_0 holding true is $p = \frac{20-t}{20}$. Based on the p-value comparison method, if $p < 0.05$, the H_0 is rejected, and the hypothesis H_1 is accepted. Otherwise, the test hypothesis H_0 is accepted. The specific sorting result based on $(\underline{\Psi^{T}_{A1}}(\Gamma), \overline{\Psi^{T}_{A1}}(\Gamma))$ and $(\underline{\Psi^{O}_{A1}}(\Gamma), \overline{\Psi^{O}_{A1}}(\Gamma))$ are shown in Tables 21 and 22.

In light of Tables 21 and 22, both $(\underline{\Psi^{T}_{A1}}(\Gamma), \overline{\Psi^{T}_{A1}}(\Gamma))$ and $(\underline{\Psi^{O}_{A1}}(\Gamma), \overline{\Psi^{O}_{A1}}(\Gamma))$ yield the same optimal preference factor, $\aleph_6$, across different sample sets. This indicates that the probability of the test hypothesis H_0 holding true is 0, that is, $p < 0.5$ holds for each p. Consequently, the test hypothesis H_0 is rejected and the alternative hypothesis H_1 is accepted, indicating that "after-sales ($\aleph_6$)" is the most valued purchasing indicator for users when buying fresh fruits online.

7.4. The comparison of different decision-making results

In the previous subsection, a novel fuzzy TOPSIS method based on TN-FβCRS and ON-FβCRS models is introduced, and the effects of various models and fuzzy neighborhood operators on decision-making results are analyzed. To further validate the efficacy of the proposed model, this subsection compares it with existing decision-making methods, with the detailed results presented in Table 23.

In Table 23, we set $I_T = I_{T_P}$ in the model $(\underline{\Psi^{T}_{A1}}(\Gamma), \overline{\Psi^{T}_{A1}}(\Gamma))$ and $I_O = I_{O_2^V}$ in the model $(\underline{\Psi^{O}_{A1}}(\Gamma), \overline{\Psi^{O}_{A1}}(\Gamma))$. Overall, the decision-making results of the two models are largely similar, with the primary difference being the reversed position of preference factors $\aleph_1$

and $\aleph_7$. Specifically, fluctuations occur in the relative importance of "customer service" and "after-sales" when users purchase fresh fruits online.

Furthermore, the data is incorporated into the VIKOR method, another MADM method, which is an optimized compromise solution method. The core idea of VIKOR is to determine the coefficient of decision-making mechanism v, where $v > 0.5$ indicates decision-making based on the majority group opinion, $v = 0.5$ considers both group and minority opinions equally, and $v < 0.5$ represents decision-making based on minority opinion. In Table 23, by examining the values of v, we observe that the decision-making results for $v = 0.8$ and $v = 0.9$ are most similar to those derived from $(\underline{\Psi^T_{A1}}(\Gamma), \overline{\Psi^T_{A1}}(\Gamma))$ and $(\underline{\Psi^O_{A1}}(\Gamma), \overline{\Psi^O_{A1}}(\Gamma))$. This indicates that VIKOR method is well-suited for decision-making based on majority group opinion in this framework.

Subsequently, the ranking results of our methods are compared with the TOPSIS method based on ONRFRS and TNRFRS [23]. Qi et al. [23] explored several FCRS models based on neighborhood operators in the FCAS, where each alternative is required to have at least one attribute with a value of 1. This restriction is more stringent for the datasets in practical applications. Therefore, in the comparison experiment, the original dataset must first be normalized using the Min-max method to ensure that each preference factor $\aleph_t$ has at least one value 1 under 19 types of fresh fruits. Then, the decision-making results can be derived following the process outlined in [23]. Due to the normalization of the original dataset, the decision-making results exhibit considerable deviation. Only the preference factor $\aleph_6$ (the optimal preference factor) remains unchanged, while the rankings of the other preference factors exhibit varying degree of change.

8. Conclusion

The analysis of consumer review data on e-commerce platforms to sort out the preference factors when purchasing fresh fruits is significant for improving shopping services and enhancing user satisfaction. In view of this, we propose several types of fuzzy neighborhood operators based on aggregation function in an FβCAS, and present a novel fuzzy TOPSIS method to solve the problem of preference factor selection in MADM. The main contents are outlined below:

(1) Four fuzzy neighborhood operators are proposed, including $TN^k_{\Psi,\beta}$ $(k = 1, 2, 3, 4)$ derived from t-norm and its implication (short for T-FNO) and $ON^k_{\Psi,\beta}$ $(k = 1, 2, 3, 4)$ derived from overlap function and its implication (short for O-FNO). And then, properties of fuzzy neighborhood operators, such as reflexivity, symmetry, transitivity are examined. The conditions under which two fuzzy β-coverings yield identical T-FNO and O-FNO are also derived.
(2) Starting from a given fuzzy β-covering, eight fuzzy β-coverings, denoted as Ψ^i_β $(i = 1, 2, 3, 4, 5, 6, \cap, \cup)$, are derived. These derived fuzzy β-coverings generate novel fuzzy neighborhood operators. Furthermore, we verify equalities between fuzzy neighborhood operators, leading to 19 groups of T-FNOs and 20 groups of O-FNOs. Subsequently, the partial order relationships among T-FNOs (show in Table 7) and O-FNOs (show in Table 8) are established, which clearly demonstrate the relationship among different fuzzy neighborhood operators.
(3) Finally, TN-FβCRS and ON-FβCRS are proposed and applied in the construction of the fuzzy TOPSIS method. In the application to user preference evaluation for fresh fruits, the decision-making results for TN-FβCRS and ON-FβCRS are $\aleph_6 > \aleph_4 > \aleph_1 > \aleph_7 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$ and $\aleph_6 > \aleph_4 > \aleph_7 > \aleph_1 > \aleph_3 > \aleph_8 > \aleph_5 > \aleph_2$, respectively. This indicates that "after-sales" is the most important preference factor when users purchase fresh fruits online, followed by other factors such as logistics.

Theoretically, the structure of fuzzy neighborhood operators is relatively complex in an FβCAS, particularly for operators based on the newly generated fuzzy β-subcoverings. It is meaningful to further investigate their related properties, such as topological and lattice-valued properties. From an application perspective, although the proposed method demonstrates good feasibility and stability, its computational process is intricate, hindering its suitability for large-scale dataset. Consequently, deeper exploration into the theory related to FβCRS is necessary to develop MADM methods with superior performance, and apply them to practical problems, such as credit evaluation, medical diagnosis and other fields.

CRediT authorship contribution statement

Wei Li: Writing – original draft, Methodology, Conceptualization. **Xiaolei Wang:** Data curation. **Bin Yang:** Writing – review & editing, Methodology.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data will be made available on request.

References

[1] H. Bustince, J. Fernández, R. Mesiar, J. Montero, R. Orduna, Overlap index, overlap functions and migrativity, in: Joint 2009 International Fuzzy Systems Association World Congress and 2009 European Society of Fuzzy Logic and Technology Conference, Lisbon, Portugal, 2009, pp. 300–305.
[2] C.-T. Chen, Extensions of the topsis for group decision-making under fuzzy environment, Fuzzy Sets Syst. 114 (2000) 1–9.
[3] J. Dai, Q. Xu, Approximations and uncertainty measures in incomplete information systems, Inf. Sci. 198 (2012) 62–80.
[4] M. De Cock, C. Cornelis, E. Kerre, Fuzzy Rough Sets: Beyond the Obvious, 2004 IEEE International Conference on Fuzzy Systems (IEEE Cat. No. 04CH37542), vol. 1, IEEE, 2004, pp. 103–108.
[5] L. D'eer, C. Cornelis, L. Godo, Fuzzy neighborhood operators based on fuzzy coverings, Fuzzy Sets Syst. 312 (2017) 17–35.
[6] L. D'eer, M. Restrepo, C. Cornelis, J. Gómez, Neighborhood operators for covering-based rough sets, Inf. Sci. 336 (2016) 21–44.
[7] T. Deng, Y. Chen, W. Xu, Q. Dai, A novel approach to fuzzy rough sets based on a fuzzy covering, Inf. Sci. 177 (11) (2007) 2308–2326.
[8] G.P. Dimuro, B. Bedregal, On residual implications derived from overlap functions, Inf. Sci. 312 (2015) 78–88.
[9] D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, Int. J. Gen. Syst. 17 (2–3) (1990) 191–209.
[10] B.Q. Hu, Generalized interval-valued fuzzy variable precision rough sets determined by fuzzy logical operators, Int. J. Gen. Syst. 44 (7–8) (2015) 849–875.
[11] Z. Huang, L. Jinjin, Covering based multi-granulation rough fuzzy sets with applications to feature selection, Expert Syst. Appl. 238 (2024) 121908.
[12] P. Jain, T. Som, Multigranular rough set model based on robust intuitionistic fuzzy covering with application to feature selection, Int. J. Approx. Reason. 156 (2023) 16–37.
[13] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, vol. 8, Springer Science and Business Media, 2013.
[14] T.J. Li, Y. Leung, W.X. Zhang, Generalized fuzzy rough approximation operators based on fuzzy coverings, Int. J. Approx. Reason. 48 (3) (2008) 836–856.
[15] W. Li, B. Yang, J. Qiao, On three types of L-fuzzy β-covering-based rough sets, Fuzzy Sets Syst. 461 (2023) 108492.
[16] B. Llamazares, An analysis of the generalized todim method, Eur. J. Oper. Res. 269 (3) (2018) 1041–1049.
[17] L. Ma, Two fuzzy covering rough set models and their generalizations over fuzzy lattices, Fuzzy Sets Syst. 294 (2016) 1–17.
[18] S. Mankad, H.S. Han, J. Goh, S. Gavirneni, Understanding online hotel reviews through automated text analysis, Serv. Sci. 8 (2) (2016) 124–138.
[19] J.-S. Mi, W.-X. Zhang, An axiomatic characterization of a fuzzy generalization of rough sets, Inf. Sci. 160 (1–4) (2004) 235–249.
[20] S. Opricovic, G.-H. Tzeng, Compromise solution by mcdm methods: a comparative analysis of vikor and topsis, Eur. J. Oper. Res. 156 (2) (2004) 445–455.
[21] Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11 (5) (1982) 341–356.
[22] J.A. Pomykala, Approximation operations in approximation space, Bull. Pol. Acad. Sci., Math. 35 (9–10) (1987) 653–662.
[23] G. Qi, B. Yang, W. Li, Some neighborhood-related fuzzy covering-based rough set models and their applications for decision making, Inf. Sci. 621 (2023) 799–843.
[24] A.M. Radzikowska, E.E. Kerre, Characterisation of main classes of fuzzy relations using fuzzy modal operators, Fuzzy Sets Syst. 152 (2) (2005) 223–247.
[25] M. Restrepo, C. Cornelis, J. Gómez, Partial order relation for approximation operators in covering based rough sets, Inf. Sci. 284 (2014) 44–59.
[26] E. Sanchez, Resolution of composite fuzzy relation equations, Inf. Control 30 (1) (1976) 38–48.
[27] R. Slowinski, D. Vanderpooten, A generalized definition of rough approximations based on similarity, IEEE Trans. Knowl. Data Eng. 12 (2) (2000) 331–336.
[28] G.-H. Tzeng, J.-J. Huang, Multiple Attribute Decision Making: Methods and Applications, CRC Press, 2011.
[29] W. Wei, J. Liang, Y. Qian, A comparative study of rough sets for hybrid data, Inf. Sci. 190 (2012) 1–16.
[30] W.-Z. Wu, W.-X. Zhang, Neighborhood operator systems and approximations, Inf. Sci. 144 (1–4) (2002) 201–217.
[31] Z. Xu, W. Da, The ordered weighted geometric averaging operators, Int. J. Intell. Syst. 17 (7) (2002) 709–716.
[32] B. Yang, M. Atef, Novel classes of fuzzy β-covering-based rough set over two distinct universes, Fuzzy Sets Syst. 461 (2023) 108350.
[33] B. Yang, B.Q. Hu, A fuzzy covering-based rough set model and its generalization over fuzzy lattice, Inf. Sci. 367 (2016) 463–486.
[34] B. Yang, B.Q. Hu, On some types of fuzzy covering-based rough sets, Fuzzy Sets Syst. 312 (2017) 36–65.
[35] B. Yang, B.Q. Hu, Fuzzy neighborhood operators and derived fuzzy coverings, Fuzzy Sets Syst. 370 (2019) 1–33.
[36] Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, Inf. Sci. 111 (1–4) (1998) 239–259.
[37] Y. Yao, On generalizing rough set theory, in: International Workshop on Rough Sets, Fuzzy Sets, Data Mining, and Granular-Soft Computing, Springer, 2003, pp. 44–51.
[38] Y. Yao, B. Yao, Covering based rough set approximations, Inf. Sci. 200 (2012) 91–107.
[39] B.H. Ye, J.M. Luo, H.Q. Vu, Spatial and temporal analysis of accommodation preference based on online reviews, J. Destin. Mark. Manag. 9 (2018) 288–299.
[40] L.A. Zadeh, Fuzzy sets, in: Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems: Selected Papers by Lotfi A Zadeh, World Scientific, 1996, pp. 394–432.
[41] W. Zakowski, Approximations in the space (U, Π), Demonstr. Math. 16 (3) (1983) 761–770.
[42] K. Zhang, J. Zhan, Y. Yao, Topsis method based on a fuzzy covering approximation space: an application to biological nano-materials selection, Inf. Sci. 502 (2019) 297–329.