International Journal of Approximate Reasoning 187 (2025) 109535

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

Optimizing connectivity in fuzzy graphs for resilient disaster response networks

P Sujithra [a,*], Sunil Mathew [a], J.N. Mordeson [b]

[a] *Department of Mathematics, National Institute of Technology Calicut, 673601, India*
[b] *Department of Mathematics, Creighton University, NE, 68178, USA*

ARTICLE INFO

MSC:
05C72
05C40

Keywords:
Connectivity
Strong edge
Block
Critical block
Complete graph

ABSTRACT

Despite significant technological advances in recent years, communication challenges still persist. These issues are especially evident during crises, where system failures, network overloads, and incompatibilities among the communication technologies used by different organizations create major obstacles. Catastrophe scenarios are marked by high information uncertainty and limited control, which raises challenges for crisis communication. However, these aspects remain underexplored from a network-theoretic perspective. This study investigates the (x, y)-connectivity parameter between two nodes in a fuzzy graph, offering insights into network structure, robustness, and performance. We introduce a novel classification of nodes and edges into three categories: enhancing, eroded, and persisting, based on their impact on node-to-node connectivity. The behavior of these classifications is analyzed across different classes of fuzzy graphs. Furthermore, we establish upper and lower bounds for the (x, y)-connectivity under two graph operations. An efficient algorithm is proposed to identify and categorize nodes and edges accordingly. The practical relevance of our classification is illustrated through its application to disaster response communication networks, where maintaining resilient and adaptive communication is critical.

1. Introduction

Within the realm of disaster management, communication is an essential factor. Congested networks or communication breakdowns are the last things that emergency communicators need to be concerned about, regardless of whether they are reacting to a crisis that affects a large number of people or an incident that is more localized. When it comes to emergency response, having access to communications that are both dependable and secure is a lifeline between all of the individuals who play a role in the response, and it frequently determines whether or not someone lives or dies.

Fuzzy graphs are the most effective method for modeling a communication network. The foundation of fuzzy graphs by Kauffmann in 1970 was formed by the idea of fuzzy sets proposed by Zadeh in 1965 [34] to handle the notion of uncertainty. The nodes and edges of fuzzy graphs are assigned values ranging from 0 to 1. The membership values of the edges correspond to the relationships between nodes. The advantage of a fuzzy graph is in the membership values of nodes and edges, which enable the model to be more accurate and reliable than a classical graph. Connectivity characteristics may be utilized to analyze network susceptibility, hence enhancing system dependability and fault tolerance. Therefore, research on connectivity factors in fuzzy network models is

* Corresponding author.
E-mail addresses: sujisrigopal95@gmail.com (P. Sujithra), sm@nitc.ac.in (S. Mathew), mordes@creighton.edu (J.N. Mordeson).

https://doi.org/10.1016/j.ijar.2025.109535
Received 24 March 2025; Received in revised form 7 July 2025; Accepted 23 July 2025

Available online 29 July 2025

essential and pertinent. Further development in fuzzy graphs was led by Yeh and Bang [3]. The concepts of fuzzy trees [26], blocks, bridges, and cut nodes in fuzzy graphs have been studied in [30]. Various connectedness concepts in fuzzy graphs were in [3]. Yeh and Bang's methodology for examining fuzzy graphs was driven by its relevance to pattern classification and clustering analysis. Bhattacharya [4] expanded the notions of eccentricity and center based on metrics in fuzzy graphs. Bhutani and Rosenfeld defined fuzzy end nodes [10] and strong arcs [9] in fuzzy graphs. Strong arcs were further classified as α, β, and δ arcs by Mathew and Sunitha in [19]. Fuzzy trees were further studied by Sunitha and Vijayakumar in 1998 [33]. A new depiction of connectivity was introduced by Mathew and Sunitha [20]. The authors also reformed Menger's theorem and concept of k-connected fuzzy graphs [21]. The authors examined cycles in fuzzy graphs and cycle connectivity in [23,22,17,27,25]. Various connectivity parameters were studied in [31,13,12,14,11,29,18,32]. Fuzzy graphs have a wide range of applications in many fields, which can be seen in [7,8].

Traditional graph theoretic approaches to network analysis often fall short in modeling the uncertainty and vagueness inherent in real-world communication networks. Fuzzy graphs offer a natural and flexible framework to handle such imprecision. However, there remains a gap in understanding how the connectivity between specific nodes behaves under uncertain conditions and how individual elements of the network influence this connectivity. This motivated the study of connectivity between a pair of nodes in a fuzzy network. The definition of connectivity between a pair of nodes was given in [1], but was not studied in detail. The notion of connectedness between pairs quantifies the strength and number of disjoint paths between two particular nodes, revealing critical insights into the dependability, redundancy, and resilience of the connecting paths. This provides a more sophisticated comprehension of connectedness in contrast to conventional graph models, which just account for the number of disjoint routes. Our focus is on identifying the nodes that have an effect on the connectivity between pairs when they are removed. In this work, Section 2 addresses the foundational concepts necessary for this study. Section 3 illustrates and examines the connectivity between pairs of nodes. Section 4 gives the bounds for the two significant fuzzy graph operations: Cartesian product and Strong product. An application of the concept in the area of disaster response communication networks is being carried out.

2. Preliminaries

A fuzzy set ρ on a set X is a function that maps X into $[0,1]$. A fuzzy relation $\wp$ on a fuzzy set ρ is a fuzzy set on $X \times X$ with $\wp(x,y) \leq \rho(x) \wedge \rho(y)$ for all $x, y \in X$. A fuzzy graph Γ is a pair $(\rho, \wp)$, where $\wp$ is a fuzzy relation on ρ. Here ρ is the fuzzy node set and $\wp$ is the fuzzy edge strength. All nodes in the fuzzy graphs mentioned in this study have node strength 1 unless otherwise indicated. A fuzzy graph $H = (\gamma, \varsigma)$ is a partial fuzzy subgraph if for all pairs of nodes u, v in X, $\gamma(v) \leq \rho(v)$ and $\varsigma(u,v) \leq \wp(u,v)$. It is a spanning fuzzy subgraph if $\rho = \gamma$.

A sequence of nodes $(x_1, x_2, \ldots, x_n)$ is a path in a fuzzy graph Γ if $\wp(x_i, x_{i+1}) > 0$, for $i = 1, 2, \ldots, n-1$. The strength of a path is the weight of the weakest edge in it. The maximum strength of all paths between two nodes x and y is the strength of connectedness between x and y, denoted by $CONN_\Gamma(x,y)$. Any path of strength $CONN_\Gamma(x,y)$ between x and y is said to be the strongest path. A fuzzy graph is said to be connected if $CONN_\Gamma(x,y) > 0$ for all pairs of nodes in Γ. A node in a fuzzy graph is said to be a fuzzy cutnode if removal of the node decreases the strength of connectedness between some pair of nodes. Similarly, an edge is a fuzzy bridge if removal of the edge results in a decrease in connectivity between some other pair of nodes. Edges in a fuzzy graph are classified as strong edges. An edge xy is a strong edge in Γ, if $\wp(xy) > 0$ and $CONN_{\Gamma-xy} \leq \wp(xy)$. Strong edges are further classified as α,β and δ edges. An edge xy is α-strong if $CONN_{\Gamma-xy} < \wp(xy)$, β-strong if $CONN_{\Gamma-xy} = \wp(xy)$ and δ-edge if $CONN_{\Gamma-xy} > \wp(xy)$. A fuzzy graph $\Gamma = (\rho, \wp)$ is a fuzzy tree, if it contains a spanning subgraph $F = (\rho, \varrho)$ which is a tree and $\wp(xy) < CONN_F(x,y)$ for all edges xy not in F. A cycle with more than two weakest edges is a fuzzy cycle. A fuzzy graph is considered edge-disjoint if no two cycles in it share a common edge. A fuzzy graph $\Gamma = (\rho, \wp)$ is a complete fuzzy graph if $\wp(x,y) = \rho(x) \wedge \rho(y)$, for all $x, y \in \Gamma$. A node-strength sequence is a sequence of real numbers $(p_1, p_2, \ldots, p_n)$, where p_i denotes the strength of the node x_i of a fuzzy graph Γ.

Connectivity in fuzzy graphs is a major area of research over the past years. A fuzzy node cut is a set of nodes S whose removal decreases the connectivity between some pair of nodes not in S or $\Gamma - S$ is trivial, that is, $CONN_{\Gamma-S}(x,y) < CONN_\Gamma(x,y)$, for some pair of nodes x and y such that $x, y \notin S$. Similarly, a fuzzy edge cut $E = \{e_1, e_2, \ldots e_n\}$, $e_i = x_i y_i$ is a set of strong edges such that $CONN_{\Gamma-E}(u,v) < CONN_\Gamma(u,v)$ for u and v with at least one different from both x_i and y_i, $i = 1, 2, \ldots, n$ or $\Gamma - E$ is disconnected. The fuzzy node connectivity $\kappa(\Gamma)$ of a connected fuzzy graph Γ is defined as the minimum strong weight of fuzzy node cuts of Γ. The fuzzy edge connectivity $\kappa'(\Gamma)$ of a connected fuzzy graph Γ is defined as the minimum strong weight of fuzzy edge cuts of Γ. A fuzzy graph without fuzzy cutnodes is said to be a fuzzy block. Critical blocks are those blocks Γ for which, for each node x in Γ, $\Gamma - x$ is not a block.

Kim and Suil proposed the concept of average connectivities in graphs. Shanookha and Mathew redefined the concept in relation to fuzzy graphs. This work was initiated upon the realization that the elimination of each node in a fuzzy network distinctly affects the connectedness value. Specifically, the effect of node removal is not uniform; some nodes may significantly weaken the connection between two nodes, while others might unexpectedly enhance it or have no impact at all. This behavioral variety led us to investigate the specific functions of certain nodes in the graph's overall connection, resulting in the creation of a more sophisticated framework that encapsulates these varied relationships. Comprehending the influence of individual nodes on the connection parameter enhances our ability to assess, forecast, and optimize the resilience and efficacy of intricate networks.

3. The (x, y)-connectivity of fuzzy graphs

The studies have focused solely on the connectedness of the entire fuzzy graph. Limited research has been conducted by considering a pair of nodes. In [2], the authors investigated the collection of paths between a pair of nodes that are internally disjoint. Inspired

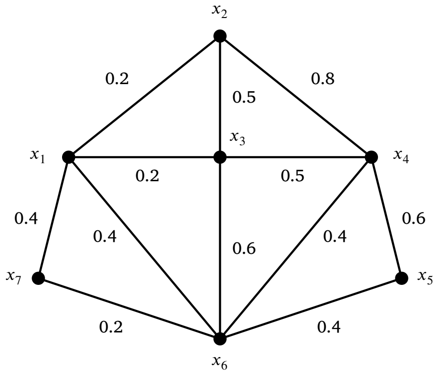

Fig. 1. Fuzzy graph in Example 3.2.

by this, this research examines the connectivity between a pair of nodes in a fuzzy graph. Let Γ be a fuzzy graph and x, y be nodes in Γ. In [1], the authors have defined the (x, y)-connectivity as in Definition 3.1.

Definition 3.1. The (x, y)-connectivity, $\kappa_\Gamma(x, y)$, between a pair of nodes x and y is defined by $\kappa_\Gamma(x, y) = mCONN_\Gamma(x, y)$, where m is the maximum number of internally disjoint $x - y$ strongest paths in Γ and $CONN_\Gamma(x, y)$, is the strength of connectedness between x and y.

Example 3.2. Let Fig. 1 represents a fuzzy graph Γ with node set $\{x_1, x_2, \dots, x_7\}$. Consider the nodes x_2 and x_3. The strongest paths are x_2x_3 and $x_2x_4x_3$ of strength 0.5. Then, $\kappa_\Gamma(x_2, x_3) = 2 \times 0.5 = 1$. But in the case of x_2 and x_6, $x_2x_3x_6$ is the only strongest path of strength 0.5 and hence $\kappa_\Gamma(x_2, x_6) = 0.5$.

Remark 3.3. For a fuzzy tree, the (x, y)-connectivity will be equal to the strength of the connectedness between them.

The further studies on the (x, y)-connectivity can be seen in [1]. This study examines the changes in (x, y)-connectivity that result from the removal of an edge or a node. In Example 3.2, consider the node x_3. When x_3 is removed, the (x_4, x_6)-connectivity will get increased to 0.8 from 0.5 while (x_2, x_6)-connectivity gets reduced to 0.4 from 0.5 and the (x_1, x_4)-connectivity remains the same. Thus, there are instances where the (x, y)-connectivity of fuzzy graphs may either decrease, increase, or remain the same when certain nodes or edges are removed. So, we have the following classifications for those types of nodes.

- If $\kappa_{\Gamma - v}(x, y) < \kappa_\Gamma(x, y)$, then v is an (x, y)-eroded node.
- If $\kappa_{\Gamma - v}(x, y) > \kappa_\Gamma(x, y)$, then v is an (x, y)-enhancing node.
- If $\kappa_{\Gamma - v}(x, y) = \kappa_\Gamma(x, y)$, then v is an (x, y)-persisting node.

Similarly, those edges whose removal makes a change to the (x, y)-connectivity can be classified as follows.

- If $\kappa_{\Gamma - uv}(x, y) < \kappa_\Gamma(x, y)$, then uv is an (x, y)-eroded edge.
- If $\kappa_{\Gamma - uv}(x, y) > \kappa_\Gamma(x, y)$, then uv is an (x, y)-enhancing edge.
- If $\kappa_{\Gamma - uv}(x, y) = \kappa_\Gamma(x, y)$, then uv is an (x, y)-persisting edge.

In Example 3.4, we discuss some of the examples of nodes and edges that fall into these categories.

Example 3.4. Consider the fuzzy graph Γ in Example 3.2. Clearly, Γ contains all types of edges. The node x_3 exhibits the characteristics of being a (x_4, x_6)-enhancing node, a (x_2, x_6)-eroded node and a (x_1, x_4)-persisting node. The node, x_4 is an (x_2, x_5)-eroded node and the edges x_2x_4,x_4x_5 are (x_2, x_5)-eroded edges. The edge x_1x_6 is a (x_1, x_4)-persisting edge. The edges x_3x_6 and x_3x_4 are enhancing edges corresponding to the nodes x_4 and x_6. In addition to these examples, there are also other nodes and edges in Γ that belongs to the classification.

The categorization of edges and nodes enables us to analyze graphs in more depth. It evaluates the behavior of each pair of nodes and the level of connectedness between them on an individual basis.

Theorem 3.5. *Isomorphism preserves the structural characteristics of nodes.*

Proof. Let Γ and Γ' be two isomorphic graphs. Then there exists a bijection h between Γ and Γ', $h : \Gamma^* \to \Gamma'^*$ such that $\rho(x) = \rho'(h(x))$, for $x \in \rho^*$ and $\wp(xy) = \wp'(h(x)h(y))$, for $xy \in \wp^*$. Thus, $CONN_\Gamma(x, y) = CONN_{\Gamma'}(h(x), h(y))$, for all $x, y \in \Gamma$ and $h(x), h(y) \in \Gamma'$.

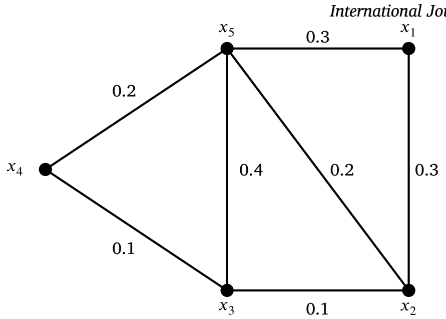

Fig. 2. Fuzzy graph in Example 3.8.

Since isomorphism preserves structure and adjacency, the number of strongest paths between a pair of nodes in Γ will be equal to that of the corresponding pairs in Γ'. Let v be an (x, y)-enhancing node in Γ, then $\kappa_{\Gamma-v}(x, y) > \kappa_{\Gamma}(x, y)$. By isomorphism, $\kappa_{\Gamma}(x, y) = \kappa_{\Gamma'}((h(x), h(y))$ and $\kappa_{\Gamma-v}(x, y) = \kappa_{\Gamma'-h(v)}(h(x), h(y))$. Thus, $\kappa_{\Gamma'-h(v)}(h(x), h(y)) > \kappa_{\Gamma'}((h(x), h(y))$ and $h(v)$ is an $(h(x), h(y))$-enhancing node of Γ'. Similar, to the case of eroded and persisting nodes. Thus, we can say that isomorphism preserves the characterization of nodes. $\square$

By the above theorem, we can also conclude that isomorphism preserves characterization of edges.

Theorem 3.6. *All the enhancing nodes in a connected fuzzy graph Γ are fuzzy cutnodes.*

Proof. Let Γ be a fuzzy graph and v be an (x, y)-enhancing node. That is, $\kappa_{\Gamma-v}(x, y) > \kappa_{\Gamma}(x, y)$. Also, let m and m' be the number of strongest $x - y$ paths in Γ and $\Gamma - v$. Since $CONN_{\Gamma}(x, y)$ being the maximum strength of all strongest path between x and y, $CONN_{\Gamma-u}(x, y) \leq CONN_{\Gamma}(x, y)$ for all nodes u in Γ. Suppose $m \neq 1$. Then, at least two strongest paths exist between the nodes x and y. If v belongs to any of the strongest paths, then $m' = m - 1$ and $(m - 1)CONN_{\Gamma-v}(x, y) < mCONN_{\Gamma}(x, y)$, a contradiction since v is an enhancing node. Similarly, if v does not belongs to any of the strongest $x - y$ paths, then $\kappa_{\Gamma-v}(x, y) = \kappa_{\Gamma}(x, y)$, again a contradiction. Thus, $m = 1$ and hence v is a cutnode. $\square$

Corollary 3.7. *Let Γ be a fuzzy graph. Then, every enhancing node is an internal node of every maximum spanning tree of Γ.*

Since a fuzzy block has no fuzzy cutnodes, it has no enhancing nodes. But, it can have enhancing edges. There are fuzzy graphs other than fuzzy blocks having no enhancing nodes. Example 3.8 is one among them.

Example 3.8. Consider the fuzzy graph Γ in the given Fig. 2 with node set $\{x_1, x_2, x_3, x_4, x_5\}$. This fuzzy graph is not a fuzzy block since v_1 is a fuzzy cutnode. Also, for each pair of nodes x, y in Γ, we have, $\kappa_{\Gamma-v}(x, y) \leq \kappa_{\Gamma}(x, y)$ for all v not equal to x and y. Hence, there are no enhancing nodes in Γ.

If v is an (x, y)-eroded node, we can infer directly that either v is a fuzzy cutnode or there exists more than one strongest path between x and y, one of which contains v. In the later case, $\kappa_{\Gamma}(x, y) = mCONN_{\Gamma}(x, y)$ and $\kappa_{\Gamma-v}(x, y) = (m-1)CONN_{\Gamma}(x, y)$. Similarly, if v is an (x, y)-persisting node, either v is a fuzzy cutnode or the node v does not belongs to any strongest $x - y$ paths. While considering the edges, if xy is an α-strong edge, all the other nodes in the fuzzy graph become (x, y)-persisting nodes. Whereas if xy is a β-strong edge, the nodes are either (x, y)-eroded or (x, y)-persisting nodes. In the case of a δ-edge, it can be either of the three.

Theorem 3.9. *For a connected fuzzy graph all the enhancing edges are fuzzy bridges.*

Proof. Let Γ be a fuzzy graph and for nodes u, v, x, y in Γ, let the edge uv be an (x, y)-enhancing edge. Then, $\kappa_{\Gamma-uv}(x, y) > \kappa_{\Gamma}(x, y)$. To show that uv is a fuzzy bridge. Assume that there is a strongest $x - y$ path not containing the edge uv. Then, removal of the edge uv does not affect the connectivity $CONN_{\Gamma}(x, y)$ between x and y. However, it may eliminate one of the strongest paths connecting x and y, hence reducing the (x, y)-connectivity. This is not possible, since uv is an enhancing edge. Hence, all the internally disjoint strongest path between x and y contains the edge uv, indicating uv is a fuzzy bridge. $\square$

Corollary 3.10. *Let Γ be a fuzzy graph. Then, every enhancing edge is in every maximum spanning tree of Γ.*

The proof applies when x and y differ from u and v, or when x or y equals u or v, including the case where $xy = uv$. From the above theorem, we can also infer that all the enhancing edges are α-strong edges. The converse of the above Theorems 3.6 and 3.9 are not always true. That is, all the fuzzy cutnodes need not be enhancing nodes and all fuzzy bridges need not be enhancing edges.

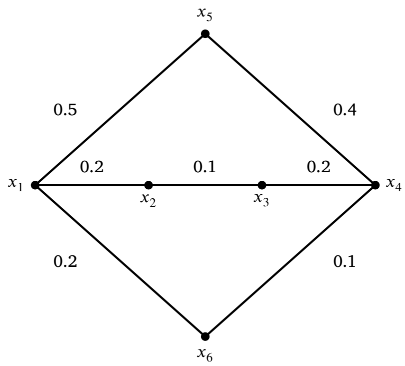

Fig. 3. Fuzzy graph in Example 3.11.

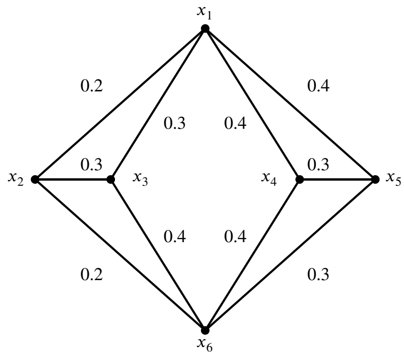

Fig. 4. Fuzzy graph in Example 3.13.

Example 3.11. Consider the fuzzy graph Γ given in Fig. 3 with node set $\rho^* = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Here, the node x_5 is a fuzzy cutnode as it reduces the connectivity between the nodes x_1 and x_4 from 0.4 to 0.1. Also, the edge x_1x_5 is a fuzzy bridge. For any pair of nodes x and y, $\kappa_{\Gamma - x_5}(x, y) \leq \kappa_\Gamma(x, y)$ and $\kappa_{\Gamma - x_1x_5}(x, y) \leq \kappa_\Gamma(x, y)$, which implies that both the node x_5 and the edge x_1x_5 are not an enhancing node nor an enhancing edge.

Let Γ be a fuzzy graph and xy be an edge in Γ. If an edge xy is an α-strong edge, it can either be an (x, y)-enhancing or an eroded edge. Also, in this case, all other edges are (x, y)-persisting edges. If the edge xy is a β-strong edge, then it is an (x, y)-eroded edge. If xy is a δ edge, it will always be an (x, y)-persisting edge. It is to be observed that all eroded edges need not be strong. If uv is an (x, y)-eroded edge such that it is a fuzzy bridge, then it does not hold. But for a case when uv is an (x, y)-eroded edge and $CONN_\Gamma(u, v) > \wp(uv) = CONN_\Gamma(x, y)$, uv is not a strong edge. We can obtain a fuzzy graph exhibiting this characteristic. Theorem 3.12 addresses one of the cases.

Theorem 3.12. *If an edge xy is an (x, y)-eroded edge, then it is a strong edge.*

Proof. Given xy is an (x, y)-eroded edge. Then, $\kappa_{\Gamma - xy}(x, y) < \kappa_\Gamma(x, y)$. If $\kappa_\Gamma(x, y) = CONN_\Gamma(x, y)$, then it equals $\wp(xy)$, otherwise, removing the edge xy would not impact the (x, y)-connectivity. Thus, xy is an α-strong edge. On the other hand, let $\kappa_\Gamma(x, y) \neq CONN_\Gamma(x, y)$. Then there exist at least two strongest paths between x and y. If the edge xy is not a strongest path, then its removal will not affect any strongest $x - y$ path and hence, it cannot be an eroded node. Thus, the edge xy is also a strongest $x - y$ path and hence xy is a β-strong edge. $\square$

If there exists an (x, y)-enhancing node, then there exists (x, y)-enhancing edges. It is because the edges incident with the enhancing node become the enhancing edges for x and y. Conversely, for any x, y with the edge xy, not a strong edge, if there exists an (x, y)-enhancing edge then there exists (x, y)-enhancing nodes. If xy is an (x, y)-enhancing edge then there are no enhancing nodes. Similarly, if xy is a β-strong edge, there are no enhancing edges and hence no enhancing nodes. For a connected fuzzy graph, the existence of enhancing nodes implies the existence of enhancing edges. But the converse is not always true and is depicted in Example 3.13. The converse is true only in the case of a fuzzy graph with an (x, y)-enhancing edge uv such that at least one of u or v is different from both x and y.

Example 3.13. Consider the fuzzy graph Γ in the given Fig. 4 with node set $\{x_1, x_2, x_3, x_4, x_5, x_6\}$. Here, the edge x_2x_3 is an (x_2, x_3)-enhancing edge. But, there are no (x_2, x_3)-enhancing nodes in Γ. While considering the nodes x_4 and x_5, the edge x_1x_5 is an enhancing edge and the node x_1 is an enhancing node. Thus, we can infer that the existence of enhancing edges implies the existence of enhancing nodes only if at least one of the end node of the enhancing edge is different from the pair of nodes we considered.

The common node of two enhancing edges need not be an enhancing node. It is given in Example 3.14. Hence, we can only conclude that for any nodes x, y in Γ, the common node of two (x, y)-enhancing edges is an (x, y)-enhancing node.

Example 3.14. Let Γ be a fuzzy graph given in Fig. 5 with node set $\{x_1, x_2, \ldots, x_7\}$. Here, the edge x_1x_4 is an enhancing edge, as $\kappa_{\Gamma - x_1x_4}(x_1, x_4) = 2 \times 0.2 = 0.4 > \kappa_\Gamma(x_1, x_4) = 0.3$. Similarly, $\kappa_{\Gamma - x_4x_7}(x_4, x_7) = 2 \times 0.3 = 0.6 > \kappa_\Gamma(x_4, x_7) = 0.4$. But, x_4 is a common node of two enhancing edges which is not an enhancing node.

Theorem 3.15. *If u is a common node of at least two (x, y)-enhancing edges then it is an (x, y)-enhancing node.*

Proof. Let u be a common node of two (x, y)-enhancing edges, say, x_1u and ux_2. As enhancing edges, ux_1 and ux_2 are part of the strongest paths between x and y. Then the subpath x_1ux_2 will be contained in any strongest $x - y$ paths. Otherwise, let the scenario

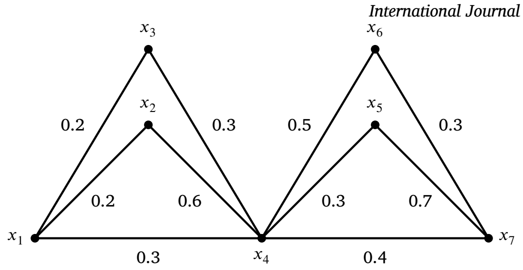

Fig. 5. Fuzzy graph in Example 3.14.

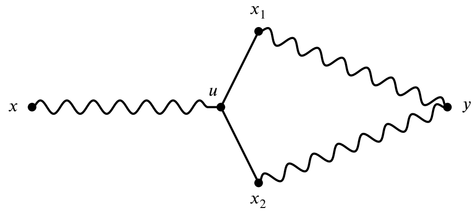

Fig. 6. Case of a fuzzy graph in Theorem 3.15.

be given in Fig. 6. Upon removal of the edge ux_1, there is no change in connectivity between x and y as the strongest path is taken by the edge ux_2. Similarly, when ux_2 is removed in this case, it will not then become an enhancing edge. Hence, the subpath x_1ux_2 is always contained in a strongest $x - y$ path. When the node u is deleted, it results in deletion of the edges ux_1 and ux_2, implying $\kappa_{\Gamma-u}(x, y) > \kappa_\Gamma(x, y)$. Thus, u becomes an (x, y)-enhanced node. □

Now, let us study about fuzzy cycles. Fuzzy cycles are those fuzzy graphs whose underlying graph is a cycle and contain more than one weakest edge. Theorem 3.16 and 3.18 discuss about the properties of nodes in a fuzzy cycle.

Theorem 3.16. *Let Γ be a fuzzy cycle and $x, y \in \Gamma$. Then, no node in Γ is an (x, y)-enhancing node.*

Proof. Let Γ be a fuzzy cycle. If xy is an α-strong edge, then all the other nodes are (x, y)-persisting nodes. Suppose the nodes x and y are joined by an α-strong path. Then any node in the $x - y$ strongest path is an (x, y)-eroded node, as removal of any nodes reduces the connectivity of x and y to $\wp(zw)$, where zw is the weakest edge in Γ. All the other nodes that do not belong to $x - y$ strongest path are (x, y)-persisting nodes as removal of those nodes will not affect the (x, y)-connectivity. Similarly, if x and y are not joined by an α-strong path, then the removal of any node in Γ will reduce the (x, y)-connectivity, as it reduces the number of strongest $x - y$ paths. □

All nodes in a fuzzy cycle are eroded nodes, except for those pairs which are joined by an α-strong path, for which each node is a persisting node.

Corollary 3.17. *Let Γ be a saturated fuzzy cycle. Then for any two nodes x and y with xy, not an α-strong edge, all nodes in Γ are (x, y)-eroded nodes.*

Proof. In a saturated fuzzy cycle, the connectivity between each pair of nodes is the weight of the weakest edge. Also, for x and y with xy not an α-strong edge, $\kappa_\Gamma(x, y) = 2\wp(zw)$, where zw is the weight of the weakest edge. Thus, removing any node other than u and v reduces the (x, y)-connectivity to $\wp(xy)$. Thus, all the nodes are (x, y)-eroded nodes. □

Since there are no enhancing nodes in a fuzzy cycle there are no enhancing edges. In case of a saturated fuzzy cycle, all edges are eroded edges of some pair of nodes.

Theorem 3.18. *If all the nodes in a fuzzy cycle are (x, y)-eroded, then x and y are not joined by an α-strong path.*

Proof. Let x and y be nodes in a fuzzy graph Γ. Suppose x and y are joined by an α-strong path, say xu_1y, where xu_1 and u_1y are α-strong edges. Then, $\kappa_\Gamma(x, y) = \min\{\wp(xu_1), \wp(u_1y)\}$. Thus, upon removal of all nodes other than u_1, there is no change in $\kappa_\Gamma(x, y)$. That is, all nodes other than u_1 are (x, y)-persisting nodes, a contradiction. Similar is the case when x and y are joined by α-strong path containing more than two nodes. Thus, x and y are not joined by an α strong path. □

A fuzzy graph Γ is a block if it has no fuzzy cutnodes. Critical blocks is a fuzzy graph Γ such that Γ is a block and $\Gamma - x$ is not a block for any node x in Γ. Theorem 3.19 discusses on critical blocks.

Theorem 3.19. *In a critical block Γ, every node z is an (x, y)-eroded node.*

Proof. For a critical block Γ all the nodes z are internal nodes of $x - y$ strongest paths. For if, $z \in \Gamma$ is not an internal node of some $x - y$ path, then removal of z from Γ will not affect any strongest path between x and y. Thus, $\Gamma - z$ is also a block, a contradiction. Also, between each pair of nodes x and y, there are at least two strongest paths implying $\kappa_\Gamma(x, y) \geq 2 \times CONN_\Gamma(x, y)$. Using the claim, we can assert that upon removal of any node z in Γ, $\kappa_\Gamma(x, y)$ decreases, and hence each node z is an (x, y)-eroded node. □

Theorem 3.20. *For a fuzzy graph Γ, either there exist strongest $x - y$ paths such that none of them contain a persisting node or all the strongest $x - y$ paths contain a persisting node.*

Proof. Let v be an (x, y)-persisting node. Then $\kappa_{\Gamma - v}(x, y) = \kappa_\Gamma(x, y)$. Let m' and m be the number of strongest $x - y$ paths in $\Gamma - v$ and Γ respectively. Then, $m' CONN_{\Gamma - v}(x, y) = m CONN_\Gamma(x, y)$. If $m \geq 1$ and $CONN_\Gamma(x, y) = CONN_{\Gamma - v}(x, y)$, then the value of m' is equal to m. That is, there exists m internally disjoint strongest paths not containing v. Similarly, if $m = 1$ and $CONN_\Gamma(x, y) > CONN_{\Gamma - v}(x, y)$, then v is a cutnode, and between x and y, there exists a unique strongest path containing v. □

From the below theorems, we can infer that a complete fuzzy graph has no enhancing nodes.

Theorem 3.21. *For a complete graph Γ and node set $\{x_1, x_2, x_3, \dots, x_n\}$, with distinct node membership values there exist $(n - (i + 1))$ number of $(x_i, x_j)_{j>i}$-eroded nodes and $(i - 1)$ number of $(x_i, x_j)_{j>i}$-persisting nodes for each i and j.*

Proof. Suppose Γ is a fuzzy graph on n nodes with each node having distinct node strength, say, $p_1, p_2, \dots, p_n$. Then from [1], we get the (x, y)-connectivities as,

$$\begin{aligned}
\kappa_\Gamma(x_1, x_i) &= (n-1)p_1, \quad i = 2, 3, \dots, n \\
\kappa_\Gamma(x_2, x_i) &= (n-2)p_2, \quad i = 3, 4, \dots, n \\
&\dots\dots \\
\kappa_\Gamma(x_{n-1}, x_n) &= p_{n-1}.
\end{aligned}$$

Now, consider the pair (x_1, x_i), $i > 1$. For each fixed $i > 1$, all the other nodes except x_1 and x_i is an (x_1, x_i)-eroded node. That is, $\kappa_{\Gamma - x_j}(x_1, x_i) = ((n-1) - 1)p_1 < (n-1)p_1 = \kappa_\Gamma(x_1, x_i)$, for all $j \neq 1, i$. Hence, there exist $(n-2)$ number of $(x_1, x_i)_{i>1}$-eroded nodes. Considering $(x_2, x_i)_{i>2}$, all the nodes except x_1 is in the strongest $x_2 - x_i$ paths. Thus, every node except x_1 is an $(x_2, x_i)_{i>2}$-eroded node. Since removal of x_1 does not affect any $x_2 - x_i$ strongest path, x_1 is a $(x_2, x_i)_{i>2}$-persisting node. Thus, $(n-3)$ number of $(x_2, x_i)_{i>2}$-eroded nodes and a $(x_2, x_i)_{i>2}$-persisting node. Proceeding like this, there exist $(n-4)$ number of $(x_3, x_i)_{i>3}$-eroded nodes and two $(x_3, x_i)_{i>3}$-persisting nodes.

$$\begin{aligned}
(x_3, x_i)_{i>3} &\to (n-4) - \text{eroded nodes} \\
&\qquad 2 - \text{persisting nodes} \\
(x_4, x_i)_{i>4} &\to (n-5) - \text{eroded nodes} \\
&\qquad 3 - \text{persisting nodes} \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
(x_{n-2}, x_i)_{i>n-2} &\to (n - (n-2+1)) - \text{eroded nodes} \\
&\qquad (n-2-1) - \text{persisting nodes.}
\end{aligned}$$

Considering the pair (x_{n-1}, x_n), as $x_{n-1}v_n$ is an α-strong edge, all the other nodes are (x_{n-1}, x_n)-persisting nodes. Thus, $n-2$ number of (x_{n-1}, x_n)-persisting nodes. □

Consider the pair x_1 and x_2. The strongest internally disjoint $x_1 - x_2$ paths are the edge x_1x_2 and the paths $x_1x_jx_2$, $j = 3, 4, \dots, n$. Thus upon removal of any of the edges x_1x_2, x_1x_j, x_jx_2, $j = 3, 4, \dots, n$, the (x_1, x_2)-connectivity decreases which implies there exist $1 + 2(n-2)$ number of (x_1, x_2)-eroded edges. Thus, for any pair $(x_1, x_i)_{i>1}$, there exist $1 + 2(n-2)$ number of $(x_1, x_i)_{i>1}$ -eroded edges. Similarly, considering the other pairs, $(x_2, x_i)_{i>2}$, there exists $1 + 2(n-3)$ number of $(x_2, x_i)_{i>2}$- eroded edges. Thus,

$$\begin{aligned}
(x_3, x_i)_{i>3} &\to 1 + 2(n-4) - \text{eroded edges} \\
(x_4, x_i)_{i>4} &\to 1 + 2(n-5) - \text{eroded edges}
\end{aligned}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(x_{n-2}, x_i)_{i>n-2} \rightarrow 1 + 2(n - (n - 2 + 1)) - \text{eroded edges.}$$

For the pair, (x_{n-1}, x_n), the removal of the α-strong edge $x_{n-1}x_n$, results in the path $x_{n-1}x_{n-2}x_n$. Thus, $x_{n-1}x_n$ is the only (x_{n-1}, x_n)-eroded edge. In this case, all other edges that are not eroded are persisting edges of the corresponding pair of nodes. In particular, since for a complete graph there exist $\frac{n(n-1)}{2}$ number of edges, the number of persisting edges for a pair (x, y) is the difference $\frac{n(n-1)}{2}$ − no. of (x, y)-eroded edges. From this, we can state that,

Theorem 3.22. *For a complete fuzzy graph Γ with each node having distinct membership value, there exist $1 + 2(n - (i + 1))$ number of $(x_i, x_j)_{j>i}$ -eroded edges and $\frac{n(n-1)}{2} - [1 + 2(n - (i + 1))]$ number of $(x_i, x_j)_{j>i}$ -persisting edges.*

Corollary 3.23. *For a complete fuzzy graph Γ with each node having the same strength, there exist $(n - 2)$ number of eroded nodes and $1 + 2(n - 2)$ number of eroded edges for each pair of nodes.*

Proof. Let Γ be a complete graph with n nodes such that each node has the same strength. Then, between any two nodes x_i and x_j, $n - 2$ number of internally disjoint strongest paths not including the edge x_ix_j exists. Upon removal of any node, one of the strongest paths gets eliminated, and the (x_i, x_j)-connectivity will be reduced. Thus, $(n - 2)$ number of eroded nodes exists for each pair of nodes. Similarly, the two edges in each of these $n - 2$ strongest paths and the edge x_ix_j form the set of (x_i, x_j)-eroded edges. Thus, $1 + 2(n - 2)$ number of eroded edges exists for each pair of nodes. □

In case of a complete fuzzy graph with node-strength sequence, $(p_1^{r_1}, p_2^{r_2}, \dots, p_k^{r_k})$, we have the following,

$$\begin{aligned}
(x_1, x_i)_{i>1} \rightarrow \quad & (n - 2) - \text{eroded nodes} \\
& \dots\dots \\
(x_{r_1}, x_i)_{i>r_1} \rightarrow \quad & (n - 2) - \text{eroded nodes} \\
(x_{r_1+1}, x_i)_{i>r_1+1} \rightarrow \quad & (n - (r_1 + 1)) - \text{eroded nodes} \\
& (r_1 - 1) - \text{persisting node} \\
& \dots\dots \\
(x_{r_1+r_2}, x_i)_{i>r_1+r_2} \rightarrow \quad & (n - (r_1 + 1)) - \text{eroded nodes} \\
& (r_1 - 1) - \text{persisting node.}
\end{aligned}$$

In general, the pair $(x_{r_1+r_2+\dots+r_{j-1}+t}, x_i)$, $i > r_1 + r_2 + \dots + r_{j-1} + t$ has $(n-(r_1+r_2+\ldots+r_{j-1}+1))$ number of eroded and $((r_1+r_2+\ldots+r_{j-1})-1)$ number of persisting nodes. About the edges we can find that for the pairs $(x_{r_1+r_2+\dots+r_{j-1}+t}, x_i)$, $i > r_1 + r_2 + \dots + r_{j-1} + t$, there exists $1 + 2(n - (r_1 + r_2 + \ldots + r_{j-1} + 1))$ number of eroded and $\frac{n(n-1)}{2} - [1 + 2(n - (r_1 + r_2 + \ldots + r_{j-1} + 1))]$ number of persisting edges. We have proved that there are no enhancing nodes in a complete graph, but it may have enhancing edges.

Theorem 3.24. *Let Γ be a complete graph on n nodes. If Γ has an enhancing edge, then it corresponds to the pair $x_{n-1}x_n$ with $r_k = r_{k-1} = 1$ and $r_{k-2} \geq 2$.*

Proof. Let Γ be a complete graph. Suppose there exists an edge xy which is an enhancing edge. Being an enhancing edge, it must be a fuzzy bridge and in turn an α-strong edge. For this to happen, $r_k = r_{k-1} = 1$. Considering all other pairs of nodes, except x_{n-1} and x_n, each has more than one strongest path between them. For those pairs, there does not exist enhancing edges. Thus, xy must be $x_{n-1}x_n$. Being an enhancing edge, $\kappa_{\Gamma-xy}(x_{n-1}, x_n) > \kappa_\Gamma(x_{n-1}, x_n)$. Let m' denote the total number of internally disjoint strongest paths of strength p_{k-2}, then,

$$m' p_{k_2} > p_{k-1} \Rightarrow m' > \frac{p_{k-1}}{p_{k-2}} > 1.$$

The total number of paths of strength p_{k-2} is greater than 1 implies that r_{k-1} must be at least 2. □

4. Bounds on (x, y)-connectivity of graph operations

In this section, we discuss the bounds of the parameter for two different fuzzy graph operations: the Cartesian product and the Strong product.

4.1. Cartesian product

The Cartesian product of two fuzzy graphs was introduced in [24]. Let $\Gamma_i = (V_i, X_i)$ with the fuzzy subsets ρ_i and $\wp_i$ on V_i and X_i for $i = 1, 2$. The Cartesian product, $\Gamma_1 \times \Gamma_2$ is given by, $V = V_1 \times V_2$, $X = \{(x_1, x_2)(x_1, y_2) | x_1 \in V_1, x_2 y_2 \in X_2\} \cup \{(x_1, x_2)(y_1, x_2) | x_1 y_1 \in X_1, x_2 \in V_2\}$ and their corresponding fuzzy subsets

$$\rho_1 \times \rho_2(x_1, x_2) = \rho_1(x_1) \wedge \rho_2(x_2), \text{for all}(x_1, x_2) \in \rho_1^* \times \rho_2^*.$$

$$\wp_1 \wp_2((x_1, x_2)(y_1, y_2)) = \begin{cases} \rho_1(x_1) \wedge \wp_2(x_2 y_2), & x_1 \in \rho_1^*, x_2 y_2 \in \wp_2^* \\ \rho_2(x_2) \wedge \wp_1(x_1 y_1), & x_2 \in \rho_2^*, x_1 y_1 \in \wp_1^*. \end{cases}$$

Here, we denote $N_\Gamma(x) = \{w : \wp(xw) > 0\}$ as the neighbors of x. Theorems 4.1 and 4.2 give the bounds for (x, y)-connectivity of each pair of nodes in the cartesian product of two fuzzy graphs Γ_1 and Γ_2.

Theorem 4.1. *Let $\Gamma_1 = (\rho_1, \wp_1)$ and $\Gamma_2 = (\rho_2, \wp_2)$ be two connected fuzzy graphs with at least two nodes, then for any distinct nodes x_i, y_i in $V(\Gamma_i)$,*

(i) $\kappa_{\Gamma_2}(x_2, y_2) \le \kappa_{\Gamma_1 \times \Gamma_2}((x_1, x_2), (x_1, y_2)) \le (1 + deg_{\Gamma_1}(x_1))\kappa_{\Gamma_2}(x_2, y_2)$.
(ii) $\kappa_{\Gamma_1}(x_1, y_1) \le \kappa_{\Gamma_1 \times \Gamma_2}((x_1, x_2), (y_1, x_2)) \le (1 + deg_{\Gamma_2}(x_2))\kappa_{\Gamma_1}(x_1, y_1)$.

Proof. By the commutativity property of the cartesian product of two fuzzy graphs, it is enough to prove (i). Let $x_1 \in \rho_1^*$ and $x_2, y_2 \in \rho_2^*$ and let us denote $\kappa_{\Gamma_2}(x_2, y_2) = k_2 CONN_{\Gamma_2}(x_2, y_2)$. Let $P_j = a_0^j a_1^j \dots a_{r_j-1}^j, a_{r_j}^j$, $a_0^j = x_2$, $a_{r_j}^j = y_2$ and $j = 1, \dots, k_2$ be the k_2 strongest $x_2 - y_2$ paths in Γ_2. Now, we construct $(x_1, x_2) - (x_1, y_2)$ paths in $\Gamma_1 \times \Gamma_2$. Consider the path

$$P_j^{x_1} : (x_1, x_2), (x_1, a_1^j), \dots, (x_1, y_2).$$

Here, $P_j^{x_1}$ is a path of strength,

$$s(P) = \wedge\{\wp_2(a_i^j, a_{i+1}^j) : i = 0, 1, \dots r_j - 1\} = CONN_{\Gamma_2}(x_2, y_2). \tag{1}$$

Thus, there exists an $(x_1, x_2) - (x_1, y_2)$ path in $\Gamma_1 \times \Gamma_2$ of strength $CONN_{\Gamma_2}(x_2, y_2)$ and hence any strongest $(x_1, x_2) - (x_1, y_2)$ path must have strength greater than or equal to $CONN_{\Gamma_2}(x_2, y_2)$. Now, consider an arbitrary strongest $(x_1, x_2) - (x_1, y_2)$ path, $R : (x_1, x_2), (u_1, v_1), (u_2, v_2), \dots, (u_t, v_t)(x_1, y_2)$. Then, consider the set of nodes $\{x_2, v_1, v_2, \dots, v_t, y_2\}$. Any $x_2 - y_2$ path induced by this set will have strength less than or equal to $CONN_{\Gamma_2}(x_2, y_2)$. Thus, $s(R) \le \wp_1(x_1, a_1) \wedge \wp_1(a_1, a_2) \wedge \dots \wedge \wp_1(a_n, x_1) \wedge CONN_{\Gamma_2}(x_2, y_2)$. But, by Equation (1) and R being arbitrary,

$$CONN_{\Gamma_1 \times \Gamma_2}((x_1, x_2), (x_1, y_2)) = CONN_{\Gamma_2}(x_2, y_2).$$

We get that, $P_1^{x_1}, \dots, P_{k_2}^{x_1}$ are paths of strength $CONN_{\Gamma_2}(x_2, y_2)$. Hence,

$$\kappa_{\Gamma_2}(x_2, y_2) \le \kappa_{\Gamma_1 \times \Gamma_2}((x_1, x_2), (y_1, x_2)). \tag{2}$$

Consider $w \in N_{\Gamma_1}(x_1)$ and the paths formed by using the node w,

$$R_j^w : (x_1, x_2)(w, x_2)(w, a_1^j) \dots (w, y_2)(x_1, y_2).$$

Thus, R_j^w is again an $(x_1, x_2) - (x_1, y_2)$ path of strength $\wp(x_1, w) \wedge CONN_{\Gamma_2}(x_2, y_2)$. Now, any path between (x_1, x_2) and (x_1, y_2) will intersect with any of $P_j^{x_1}$ or P_j^w. Thus, there exists a maximum of $k_2 + deg_{\Gamma_1}(x_1)k_2$ paths and

$$\begin{aligned} \kappa_{\Gamma_1 \times \Gamma_2}((x_1, x_2), (x_1, x_2)) &\le [k_2 + deg_{\Gamma_1}(x_1)k_2]CONN_{\Gamma_2}(x_2, y_2) \\ &= (1 + deg_{\Gamma_1}(x_1))\kappa_{\Gamma_2}(x_2, y_2). \end{aligned} \tag{3}$$

We get the desired result from Equations (2) and (3). □

Theorem 4.2. *Let $\Gamma_1 = (\rho_1, \wp_1)$ and $\Gamma_2 = (\rho_2, \wp_2)$ be two connected fuzzy graphs with at least two nodes, then for any distinct nodes $x_1, y_1 \in V(\Gamma_1)$ and $x_2, y_2 \in V(\Gamma_2)$,*

$$\begin{aligned} \kappa_{\Gamma_i}(x_i, y_i) + \kappa_{\Gamma_j}(x_j, y_j)\frac{CONN_{\Gamma_i}(x_i, y_i)}{CONN_{\Gamma_j}(x_j, y_j)} &\le \kappa_{\Gamma_1 \times \Gamma_2}((x_1, x_2), (y_1, y_2)) \\ &\le \kappa_{\Gamma_i}(x_i, y_i) + (1 + \deg_{\Gamma_j}(x_j))CONN_{\Gamma_i}(x_i, y_i) \end{aligned}$$

where, $\begin{cases} i = 1 \text{ and } j = 2, & \text{if } CONN_{\Gamma_1 \times \Gamma_2}\big((x_1, x_2), (y_1, y_2)\big) = CONN_{\Gamma_1}(x_1, y_1), \\ i = 2 \text{ and } j = 1, & \text{if } CONN_{\Gamma_1 \times \Gamma_2}\big((x_1, x_2), (y_1, y_2)\big) = CONN_{\Gamma_2}(x_2, y_2). \end{cases}$

Proof. Let Γ denote the Cartesian product $\Gamma_1 \times \Gamma_2$. Suppose, for $x_1, y_1 \in V(\Gamma_1)$ and $x_2, y_2 \in V(\Gamma_2)$, $\kappa_{\Gamma_1}(x_1, y_1) = k_1 CONN_{\Gamma_1}(x_1, y_1)$ and $\kappa_{\Gamma_2}(x_2, y_2) = k_2 CONN_{\Gamma_2}(x_2, y_2)$. Let $P_i = a_0^i a_1^i \dots a_{r_i-1}^i, a_{r_i}^i$, $a_0^i = x_1$, $a_{r_i}^i = y_1$ and $i = 1, \dots, k_1$ be the k_1 strongest $x_1 - y_1$ paths in Γ_1. Let $Q_j = b_0^j b_1^j \dots b_{s_j-1}^j, b_{s_j}^j$, $b_0^j = x_2$, $b_{s_j}^j = y_2$ and $j = 1, \dots, k_2$ be the k_2 strongest $x_2 - y_2$ paths in Γ_2. Then, the $(x_1, x_2) - (y_1, y_2)$ path formed by the paths P_1 and Q_1

$$R' : (x_1, x_2)(a_1^1, x_2) \dots (y_1, x_2)(y_1, b_1^1) \dots (y_1, y_2),$$

is of strength $CONN_{\Gamma_1}(x_1, y_1) \wedge CONN_{\Gamma_2}(x_2, y_2)$. By considering any strongest path between (x_1, x_2) and (y_1, y_2), restriction of it into a path of $x_1 - y_1$ will be of strength atmost $CONN_{\Gamma_1}(x_1, y_1)$. Similarly, the restriction into a path of $x_2 - y_2$ will be of strength atmost $CONN_{\Gamma_2}(x_2, y_2)$. Thus, from the above discussions we can conclude that any path between (x_1, x_2) and (y_1, y_2) will be of strength $CONN_{\Gamma_1}(x_1, y_1) \wedge CONN_{\Gamma_2}(x_2, y_2)$.

By using P_i's and Q_j's, we are going to construct strongest paths as follows. We can construct one more path using P_1 and Q_1 by fixing the node x_1, that is,

$$R'' : (x_1, x_2)(x_1, b_1^1) \dots (x_1, y_2)(a_1^1, y_2) \dots (y_1, y_2).$$

Secondly, consider P_i, $i \neq 1$ and Q_1,

$$R_{i1} : (x_1, x_2)(a_1^i, x_2) \dots (a_{r_i-1}^i, x_2)(a_{r_i-1}^i, v_1^1) \dots (a_{r_i-1}^i, y_2)(y_1, y_2).$$

Also, the strength of R_{i1},

$$S(R_{i1}) = CONN_{\Gamma_1}(x_1, y_1) \wedge CONN_{\Gamma_2}(x_2, y_2).$$

Thus, we get a total of $k_1 - 1$ strongest $(x_1, x_2) - (y_1, y_2)$ paths. Thirdly, considering the paths P_1 and Q_j, $j \neq 1$, we get

$$R_{1j} : (x_1, x_2)(x_1, v_1^j) \dots (x_1, v_{s_j-1}^j)(u_1^1, v_{s_j-1}^j) \dots (y_1, v_{s_j-1}^j)(y_1, y_2).$$

The strength of R_{1j} is again $CONN_{\Gamma_1}(x_1, y_1) \wedge CONN_{\Gamma_2}(x_2, y_2)$. Now, strongest $(x_1, x_2) - (y_1, y_2)$ paths formed by using the paths P_i's and Q_j's will intersect with any of the paths R', R'', R_{i1} and R_{1j}. Hence, from our construction we have a maximum of $2 + (k_1 - 1) + (k_2 - 1)$ strongest $(x_1, x_2) - (y_1, y_2)$ paths.

Case 1: If $CONN_{\Gamma_1}(x_1, y_1) \wedge CONN_{\Gamma_2}(x_2, y_2) = CONN_{\Gamma_1}(x_1, y_1)$, then $\kappa_{\Gamma_1 \times \Gamma_2}((x_1, x_2), (y_1, y_2)) \geq (k_1 + k_2) CONN_{\Gamma_1}(x_1, y_1) = \kappa_{\Gamma_1}(x_1 y_1) + \kappa_{\Gamma_2}(x_2, y_2) \dfrac{CONN_{\Gamma_1}(x_1, y_1)}{CONN_{\Gamma_2}(x_2, y_2)}$. Now, there may exist paths between x_2 and y_2 that is not stronger but of strength greater than $CONN_{\Gamma_1}(x_1, y_1)$. Let $x_2, v_1, \dots, v_{s-1}, y_2$ be such a path. Then,

$$R''' : (x_1, x_2)(x_1, v_1) \dots (x_1, v_{s-1})(a_1^i, v_{s-1}) \dots (y_1, v_{s-1})(y_1, y_2)$$

is a strongest $(x_1, x_2) - (y_1, y_2)$ path. Thus, the total number of such paths is a maximum of $\deg_{\Gamma_2}(x_2)$. Thus, the maximum value of the $((x_1, x_2), (y_1, y_2)$-connectivity is

$$[2 + (k_1 - 1) + \deg_{\Gamma_2}(x_2)] CONN_{\Gamma_1}(x_1, y_1) = \kappa_{\Gamma_1}(x_1, y_1) + (1 + \deg_{\Gamma_2}(x_2)) CONN_{\Gamma_1}(x_1, y_1).$$

Case 2: If $CONN_{\Gamma_1}(x_1, y_1) \wedge CONN_{\Gamma_2}(x_2, y_2) = CONN_{\Gamma_2}(x_2, y_2)$, then proceeding as in Case 1, $\kappa_{\Gamma_1 \times \Gamma_2}((x_1, x_2), (y_1, y_2)) \geq \kappa_{\Gamma_2}(x_2 y_2) + \kappa_{\Gamma_1}(x_1, y_1) \dfrac{CONN_{\Gamma_2}(x_2, y_2)}{CONN_{\Gamma_1}(x_1, y_1)}$. Let $x_1, u_1, \dots, u_{t-1}, y_1$ be an $x_1 - y_1$ path of strength less than $CONN_{\Gamma_1}(x_1, y_1)$ but not less than $CONN_{\Gamma_2}(x_2, y_2)$. Then,

$$R'''' : (x_1, x_2)(u_1, x_2) \dots (u_{t-1}, x_2)(u_{t-1}, b_1^j) \dots (u_{t-1}, y_2)(y_1, y_2)$$

is a path of strength $CONN_{\Gamma_2}(x_2, y_2)$. Thus, the maximum value becomes

$$[2 + (k_2 - 1) + \deg_{\Gamma_1}(x_1)] CONN_{\Gamma_2}(x_2, y_2) = \kappa_{\Gamma_2}(x_2, y_2) + (1 + \deg_{\Gamma_1}(x_1)) CONN_{\Gamma_2}(x_2, y_2).$$

Thus, by using Cases 1 and 2, we can conclude that,

$$\begin{aligned}\kappa_{\Gamma_i}(x_i, y_i) + \kappa_{\Gamma_j}(x_j, y_j) \frac{CONN_{\Gamma_i}(x_i, y_i)}{CONN_{\Gamma_j}(x_j, y_j)} &\leq \kappa_{\Gamma_1 \times \Gamma_2}((x_1, x_2), (y_1, y_2)) \\ &\leq \kappa_{\Gamma_i}(x_i, y_i) + (1 + \deg_{\Gamma_j}(x_j)) CONN_{\Gamma_i}(x_i, y_i)\end{aligned}$$

where, $\begin{cases} i = 1 \text{ and } j = 2, & \text{if } CONN_{\Gamma_1 \times \Gamma_2}\big((x_1, x_2), (y_1, y_2)\big) = CONN_{\Gamma_1}(x_1, y_1), \\ i = 2 \text{ and } j = 1, & \text{if } CONN_{\Gamma_1 \times \Gamma_2}\big((x_1, x_2), (y_1, y_2)\big) = CONN_{\Gamma_2}(x_2, y_2). \end{cases}$ $\square$

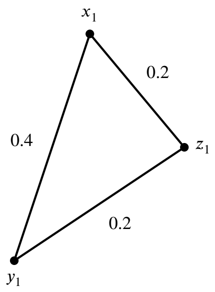

Fig. 7. Fuzzy graph Γ_1 in Example 4.3.

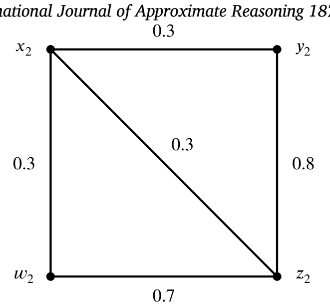

Fig. 8. Fuzzy graph Γ_2 in Example 4.3.

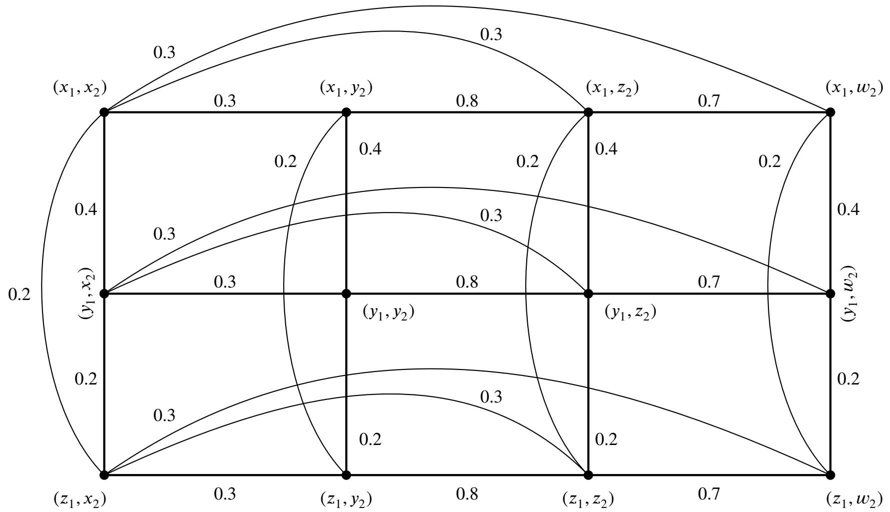

Fig. 9. Fuzzy graph $\Gamma_1 \times \Gamma_2$ in Example 4.3.

To be more accurate, we can use $\wedge\{\deg_{\Gamma_j}(x_j), \deg_{\Gamma_j}(y_j)\}$ instead of $\deg_{\Gamma_j}(x_j)$ in expression in the above theorem. Given below is an example for Theorems 4.1 and 4.2.

Example 4.3. Consider the fuzzy graph $\Gamma_1 = (\rho_1, \wp_1)$ in Fig. 7 and $\Gamma_2 = (\rho_2, \wp_2)$ in Fig. 8 with $\wp_1(x_1y_1) = 0.4$, $\wp_1(x_1z_1) = \wp_1(z_1y_1) = 0.2$, $\wp_2(x_2w_2) = \wp_2(z_2w_2) = \wp_2(y_2w_2) = 0.3$, $\wp_2(x_2y_2) = 0.8$ and $\wp_2(z_2y_2) = 0.7$. Let Fig. 9 represent the Cartesian product $\Gamma_1 \times \Gamma_2$. Here, in this example, we find some pair of nodes that satisfies the inequalities in Theorem 4.1 and 4.2. Consider the pair (x_1, x_2) and (x_1, z_2). The strongest paths are

$$(x_1, x_2)(x_1, z_2),$$

$$(x_1, x_2)(x_1, w_2)(x_1, z_2),$$

$$(x_1, x_2)(x_1, y_2)(x_1, z_2),$$

and

$$(x_1, x_2)(y_1x_2)(y_1, y_2)(y_1, z_2)(x_1, z_2).$$

Thus, for the pairs (x_1, x_2) and (x_1, z_2), we have

$$\kappa_\Gamma((x_1, x_2), (x_1, z_2)) = 4(0.3) = 1.2 < (1 + \deg_{\Gamma_1}(x_1))\kappa_{\Gamma_2}(x_2, z_2).$$

But, for the node pair (x_1, y_2) and (x_1, w_2), the only strongest path is $(x_1, y_2)(x_1, z_2)(x_1, w_2)$ and then $\kappa_\Gamma((x_1, y_2), (x_1, w_2)) = 0.7 = \kappa_{\Gamma_2}(y_2, w_2)$. Now, consider the pairs (y_1, x_2) and (z_1, x_2). The strongest paths corresponding to these pairs are

$$(y_1, x_2)(y_1, y_2)(z_1, y_2)(z_1, x_2),$$

$$(y_1, x_2)(y_1, z_2)(z_1, z_2)(z_1, x_2),$$

$$(y_1,x_2)(y_1,w_2)(z_1,w_2)(z_1,x_2),$$

and

$$(y_1,x_2)(x_1,x_2)(z_1,x_2).$$

Hence, we get $\kappa_\Gamma((y_1,x_2),(z_1,x_2)) = 4(0.2) = (1+3)(0.2) = (1+\deg_{\Gamma_2}(x_2))\kappa_{\Gamma_1}(y_1,z_1)$.

As an example for Theorem 4.2, consider the pairs (x_1,x_2) and (z_1,z_2). The strongest $(x_1,x_2)-(z_1,z_2)$ paths are

$$(x_1,x_2)(x_1,y_2)(x_1,z_2)(y_1,z_2)(z_1,z_2),$$

$$(x_1,x_2)(y_1,x_2)(z_1,x_2)(z_1,y_2)(z_1,z_2),$$

$$(x_1,x_2)(x_1,z_2)(z_1,z_2),$$

$$(x_1,x_2)(x_1,w_2)(z_1,w_2)(z_1,z_2),$$

$$(x_1,x_2)(z_1,x_2)(z_1,y_2)(z_1,z_2),$$

and $\kappa_\Gamma((x_1,x_2)(z_1,z_2)) = 5(0.2) = 0.2 + (1+3)(0.2) = \kappa_{\Gamma_1}(x_1,z_1) + (1+\deg_{\Gamma_2}(x_2))\kappa_{\Gamma_1}(x_1,z_1)$.

For the pairs (x_1,x_2) and (y_1,z_2), the strongest paths are

$$(x_1,x_2)(x_1,y_2)(x_1,z_2)(y_1,z_2),$$

$$(x_1,x_2)(y_1,x_2)(y_1,y_2)(y_1,z_2),$$

and

$$(x_1,x_2)(x_1,w_2)(y_1,w_2)(y_1,z_2).$$

Hence, $\kappa_\Gamma((x_1,x_2)(y_1,z_2)) = 3(0.3) < \kappa_{\Gamma_2}(x_2,z_2) + (1+\deg_{\Gamma_1}(x_1))\kappa_{\Gamma_2}(x_2,z_2)$.

4.2. *Strong product*

The strong product on Fuzzy graphs is defined in [28]. Let $\Gamma_i = (V_i, X_i)$ with the fuzzy subsets ρ_i and $\wp_i$ on V_i and X_i for $i = 1,2$. The Strong product, $\Gamma_1 \boxtimes \Gamma_2$ is given by $V = V_1 \times V_2$, $X = \{(x_1,x_2)(x_1,y_2) | x_1 \in V_1, x_2y_2 \in X_2\} \cup \{(x_1,x_2)(y_1,x_2) | x_1y_1 \in X_1, x_2 \in V_2\} \cup \{(x_1,x_2)(y_1,y_2) | x_1y_1 \in X_1, x_2y_2 \in X_2\}$ and their corresponding fuzzy subsets

$$\rho_1 \times \rho_2(x_1,x_2) = \rho_1(x_1) \wedge \rho_2(x_2), \text{for all}(x_1,x_2) \in \rho_1^* \times \rho_2^*.$$

$$\wp_1\wp_2((x_1,x_2)(y_1,y_2)) = \begin{cases} \rho_1(x_1) \wedge \wp_2(x_2y_2), & x_1 \in \rho_1^*, x_2y_2 \in \wp_2^* \\ \rho_2(x_2) \wedge \wp_1(x_1y_1), & x_2 \in \rho_2^*, x_1y_1 \in \wp_1^* \\ \wp_1(x_1y_1) \wedge \wp_1(x_1y_1), & x_1y_1 \in \wp_1^*, x_2y_2 \in \wp_2^*. \end{cases}$$

Theorems 4.4 and 4.5 give the bounds for (x,y)-connectivity for each pair of nodes in the strong product of two fuzzy graphs Γ_1 and Γ_2.

Theorem 4.4. *Let $\Gamma_1 = (\rho_1, \wp_1)$ and $\Gamma_2 = (\rho_2, \wp_2)$ be two fuzzy graphs with at least two nodes, then for any distinct nodes x_i, y_i in $V(\Gamma_i)$,*

(i) $\kappa_{\Gamma_2}(x_2,y_2) \le \kappa_{\Gamma_1 \boxtimes \Gamma_2}((x_1,x_2),(x_1,y_2)) \le (1+\deg_{\Gamma_1}(x_1))\kappa_{\Gamma_2}(x_2,y_2) + \deg_{\Gamma_1}(x_1) CONN_{\Gamma_2}(x_2,y_2)$.
(ii) $\kappa_{\Gamma_1}(x_1,y_1) \le \kappa_{\Gamma_1 \boxtimes \Gamma_2}((x_1,x_2),(y_1,x_2)) \le (1+\deg_{\Gamma_2}(x_2))\kappa_{\Gamma_1}(x_1,y_1) + \deg_{\Gamma_1}(x_1) CONN_{\Gamma_1}(x_1,y_1)$.

Proof. Proceed the proof as in Theorem 4.1. That is, consider the $x_2 - y_2$ strongest paths P_j, $j = 1,\dots,k_2$ and construct the corresponding paths $P_j^{x_1}$. Thus, we get k_2 number of paths of strength $CONN_{\Gamma_2}(x_2,y_2)$.

Consider an arbitrary path $R : (x_1,x_2),(u_1,v_1),\dots,(u_t,v_t),(x_1,y_2)$. Then, the strength of R

$$s(R) = \wp_{\Gamma_1}(x_1,u_1) \wedge \wp_{\Gamma_1}(u_1,u_2) \wedge \cdots \wedge \wp_{\Gamma_1}(u_t,x_1) \wedge t,$$

where t is the strength of the $x_2 - y_2$ path in Γ_2 induced by the node set $\{x_2,v_1,v_2,\dots,v_t,y_2\}$. Also, $t \le CONN_\Gamma(x_2,y_2)$ implies R is a strongest path if $s(R) = CONN_{\Gamma_2}(x_2,y_2)$. Thus, any strongest $(x_1,x_2)-(x_1,y_2)$ path will be of strength $CONN_{\Gamma_2}(x_2,y_2)$. Hence,

$$\kappa_{\Gamma_2}(x_2,y_2) \le \kappa_{\Gamma_1 \boxtimes \Gamma_2}((x_1,x_2),(x_1,y_2)).$$

Now, consider $u \in N(x_1)$. Then

$$R_j^u : (x_1,x_2),(u,a_1^j),(u,a_2^j),\dots,(u,a_{r_j-1}^j)(x_1,y_2),$$

is a path of strength $\wp_{\Gamma_1}(x_1, u) \wedge CONN_{\Gamma_2}(x_2, y_2)$ and a total of $\deg_{\Gamma_1}(x_1)$ number of paths. But, two cases occur depending on whether x_2y_2 is a β-strong edge.

Case 1: x_2y_2 is a β-strong edge.

Let $Q_1 = x_2y_2$. Then consider the paths,

$$R' = (x_1, x_2), (u, x_2), (x_1, y_2),$$

$$R'' = (x_1, x_2), (u, y_2), (x_1, y_2).$$

These paths are in $\Gamma_1 \boxtimes \Gamma_2$ and are strongest if they are of strength $CONN_{\Gamma_2}(x_2, y_2)$.

Case 2: x_2y_2 is not a β-strong edge

Subcase 1: Since there exist at least three nodes, there may exist w such that $w \in N(u)\backslash N(x_1)$. Then the path

$$R_j^{uw} : (x_1, x_2), (u, x_2), (w, a_1^j)(w, a_2^j) \dots (w, a_{j-2}^j)(w, a_{j-1}^j)(w, y_2)(u, y_2)(x_1, y_2)$$

is of strength $\wp(x_1, u) \wedge \wp_{\Gamma_1}(u, w) \wedge CONN_{\Gamma_2}(x_2, y_2)$. The paths R_j^{uw} are at most $\deg_{\Gamma_1}(x_1)$.

Subcase 2: If the set $N(u)\backslash N(x_1)$ is empty, then only the paths $P_j^{x_1}$ and R_j^u, for $j = 1, 2, \dots, k_2$ and $u \in N(x_1)$ exists.

Now, any path between $(x_1, x_2) - (x_1, y_2)$ in $\Gamma_1 \boxtimes \Gamma_2$ will coincide with any of the abovementioned paths. Hence, from all the scenarios we can conclude that the maximum value of $\kappa_{\Gamma_1 \boxtimes \Gamma_2}((x_1, x_2), (x_1, y_2))$ paths is $(1 + \deg_{\Gamma_1}(x_1))\kappa_{\Gamma_2}(x_2, y_2) + \deg(\Gamma_1)CONN_{\Gamma_2}(x_2, y_2)$. Thus the desired result. $\square$

Theorem 4.5. *Let $\Gamma_1 = (\rho_1, \wp_1)$ and $\Gamma_2 = (\rho_2, \wp_2)$ be two connected fuzzy graphs with at least two nodes, then for any distinct nodes $x_1, y_1 \in V(\Gamma_1)$ and $x_2, y_2 \in V(\Gamma_2)$,*

$$\kappa_{\Gamma_i}(x_i, y_i) + \kappa_{\Gamma_j}(x_j, y_j) \left[\frac{CONN_{\Gamma_i}(x_i, y_i) + \kappa_{\Gamma_i}(x_i, y_i)}{CONN_{\Gamma_j}(x_j, y_j)} \right] \le \kappa_{\Gamma_1 \boxtimes \Gamma_2}((x_1, x_2), (y_1, y_2))$$
$$\le \kappa_{\Gamma_i}(x_i, y_i) + \deg_{\Gamma_j}(x_j)[\kappa_{\Gamma_i}(x_i, y_i) + CONN_{\Gamma_i}(x_i, y_i)],$$

where, $\begin{cases} i = 1 \text{ and } j = 2, & \text{if } CONN_{\Gamma_1 \times \Gamma_2}\big((x_1, x_2), (y_1, y_2)\big) = CONN_{\Gamma_1}(x_1, y_1), \\ i = 2 \text{ and } j = 1, & \text{if } CONN_{\Gamma_1 \times \Gamma_2}\big((x_1, x_2), (y_1, y_2)\big) = CONN_{\Gamma_2}(x_2, y_2). \end{cases}$

Proof. Proceed the proof as in 4.2. Define, P_i, $i = 1, 2, \dots, k_1$ and Q_j, $j = 1, 2, \dots, k_2$ as in 4.2. By combining the paths P_1 and Q_1, we get a path

$$R' : (x_1, x_2)(a_1^1, x_2) \dots (y_1, x_2)(y_1, b_1^1) \dots (y_1, y_2)$$

of strength $CONN_{\Gamma_1}(x_1, y_1) \wedge CONN_{\Gamma_2}(x_2, y_2)$. Now, consider any path $(x_1, x_2)(u_1, v_1)(u_2, v_2) \dots (u_t, v_t)(y_1, y_2)$ of strength greater than $CONN_{\Gamma_1}(x_1, y_1) \wedge CONN_{\Gamma_2}(x_2, y_2)$. Then, the $x_1 - y_1$ path $x_1u_1 \dots u_ty_1$ will also have strength greater than $CONN_{\Gamma_1}(x_1, y_1)$, similar to the case of $x_2 - y_2$ path. Thus, any strongest $((x_1, x_2) - (y_1, y_2))$ will have strength $CONN_{\Gamma_1}(x_1, y_1) \wedge CONN_{\Gamma_2}(x_2, y_2)$.

Now we construct the $(x_1, x_2) - (y_1, y_2)$ strongest paths as follows.

Case 1: Take P_1 and Q_1 R' and R'' defined in Theorem 4.2 are two such paths. Also, consider

$$\widetilde{R} : (x_1, x_2)(a_1^1, b_1^1)(a_1^1, b_2^2) \dots (a_1^1, b_{s_1-1}^1)(a_2^1, b_{s_1-1}^1) \dots (a_{r_1-1}^1, bs_1 - 1^1)(y_1, y_2).$$

Case 2: Fix P_1 and take $Q_2, Q_3, \dots, Q_{k_2}$. Consider R_{1j} and

$$\widetilde{R}_{1j} : (x_1, x_2)(a_1^1, b_1^j)(a_2^1, b_1^j) \dots (y_1, b_1^j)(y_1, b_2^j) \dots (y_1, y_2).$$

Case 3: Fix Q_1 and take $P_2, P_3, \dots$ Consider R_{i1} and

$$\widetilde{R}_{i1} : (x_1, x_2)(a_1^i, b_1^1) \dots (a_1^i, b_{s_1-1}^1)(a_2^i, y_2) \dots (y_1, y_2).$$

Case 4: Consider P_i, $i \ge 2$ and Q_j, $j \ge 2$

$$\widetilde{R}_{ij} : (x_1, x_2)(a_1^i, b_1^j)(a_1^i, b_2^j) \dots (a_1^i, b_{s_1-1}^j)(a_2^i, y_2) \dots (y_1, y_2).$$

Also, all the paths are $(x_1, x_2) - (y_1, y_2)$ strongest paths. Thus, if $CONN_{\Gamma_1}(x_1, y_1) \wedge CONN_{\Gamma_2}(x_2, y_2) = CONN_{\Gamma_1}(x_1, y_1)$, then

$$\kappa_{\Gamma_1 \boxtimes \Gamma_2}((x_1, x_2), (y_1, y_2)) \ge \kappa_{\Gamma_1}(x_1, y_1) + \kappa_{\Gamma_2}(x_2, y_2) \left[\frac{CONN_{\Gamma_1}(x_1, y_1) + \kappa_{\Gamma_1}(x_1, y_1)}{CONN_{\Gamma_2}(x_2, y_2)} \right].$$

Now, there may exist $x_2 - y_2$ paths of strength greater than $CONN_{\Gamma_1}(x_1, y_1)$ but less than $CONN_{\Gamma_2}(x_2, y_2)$. Then in Case 2, we get a maximum of $2(deg_{\Gamma_2}(x_2) - 1)$ and there exists $(k_1 - 1)(deg_{\Gamma_2}(x_2) - 1)$ paths in Case 4. Thus,

$$\kappa_{\Gamma_1 \boxtimes \Gamma_2}((x_1, x_2), (y_1, y_2)) \le \kappa_{\Gamma_1}(x_1, y_1) + deg_{\Gamma_2}(x_2)[\kappa_{\Gamma_1}(x_1, y_1) + CONN_{\Gamma_1}(x_1, y_1)].$$

Similarly if, $CONN_{\Gamma_1}(x_1,y_1) \wedge CONN_{\Gamma_2}(x_2,y_2) = CONN_{\Gamma_2}(x_2,y_2)$

$$\kappa_{\Gamma_1 \boxtimes \Gamma_2}((x_1,x_2),(y_1,y_2)) \geq \kappa_{\Gamma_2}(x_2,y_2) + \kappa_{\Gamma_1}(x_1,y_1)\left[\frac{CONN_{\Gamma_2}(x_2,y_2)+\kappa_{\Gamma_2}(x_2,y_2)}{CONN_{\Gamma_1}(x_1,y_1)}\right]$$

and

$$\kappa_{\Gamma_1 \boxtimes \Gamma_2}((x_1,x_2),(y_1,y_2)) \leq \kappa_{\Gamma_2}(x_2,y_2) + deg_{\Gamma_1}(x_1)[\kappa_{\Gamma_2}(x_2,y_2) + CONN_{\Gamma_2}(x_2,y_2)].$$

Hence the desired result. □

Example 4.6. Let Γ_1 and Γ_2 be as in Example 4.3. $\Gamma_1 \boxtimes \Gamma_2$ be their strong product. Consider the strongest paths between the node pairs (x_1,y_2) and (z_1,x_2),

$(x_1,y_2)(z_1,x_2)$,

$(x_1,y_2)(x_1,x_2)(z_1,x_2)$,

$(x_1,y_2)(x_1,z_2)(z_1,x_2)$,

$(x_1,y_2)(y_1,y_2)(z_1,x_2)$,

$(x_1,y_2)(z_1,y_2)(z_1,x_2)$,

$(x_1,y_2)(z_1,z_2)(z_1,x_2)$,

$(x_1,y_2)(y_1,z_2)(y_1,w_2)(z_1,x_2)$,

$(x_1,y_2)(y_1,x_2)(z_1,x_2)$.

Thus, $\kappa_{\Gamma_1 \boxtimes \Gamma_2}((x_1,y_2),(z_1,x_2)) = 8(0.2) = \kappa_{\Gamma_1}(x_1,z_1) + deg_{\Gamma_2}(y_2)[\kappa_{\Gamma_1}(x_1,z_1) + CONN_{\Gamma_1}(x_1,z_1)]$.

In this work, we have introduced a theoretical and mathematical framework to analyze the (x,y)-connectivity in fuzzy graphs, where the strength of connection between two nodes x and y is defined as the maximum strength among all fuzzy paths connecting them, with each path's strength determined by the minimum membership value of its edges. Based on how the removal or addition of a node or edge affects this fuzzy connectivity, we classify network components into three categories: enhancing, eroded, and persisting. An enhancing element increases connectivity when added, an eroded element weakens connectivity when removed, and a persisting element does not affect the connectivity. This classification provides a systematic method to evaluate the structural importance of nodes and edges within uncertain or degraded networks. Applied to disaster response communication systems—where infrastructure may be partially damaged or overloaded—this framework enables planners to identify critical communication links, anticipate points of failure, and reinforce parts of the network that most significantly improve connectivity between emergency units. Thus, the model bridges theoretical fuzzy graph analysis with practical needs in crisis communication, offering actionable insights for designing resilient and adaptive communication networks.

5. Algorithm

Given a connected fuzzy graph $\Gamma = (\rho, \wp)$ with $|\rho| = n$. This algorithm is to check whether a pair of nodes is either enhancing, eroded or persisting. The strength of connectedness $CONN_\Gamma(x,y)$ can be found by using the algorithm in [6,5].

Algorithm 5.1.

(1) Find $CONN_\Gamma(x,y)$ and $CONN_{\Gamma-u}(x,y)$ using the algorithm in [6].
(2) Let $d_\Gamma(x) \leq d_\Gamma(y)$ and $N_\Gamma(x) = \{u_i : 1 \leq i \leq r_x\}$.
Initialize $COUNT = 0$ and $i = 1$;
  (2.1) if $\wp(x,u_i) \geq CONN_\Gamma(x,y)$. Find $CONN_{\Gamma-x}(u_i,y)$.
    (2.1.1) If $\wp(x,u_i) \wedge CONN_{\Gamma-x}(u_i,y) = CONN_\Gamma(x,y)$, then
      $COUNT = COUNT + 1$, $i = i + 1$.
    (2.1.2) Else, go to step 2.1 with $i = i + 1$.
  (2.2) Else, go to step 2.1 with $i = i + 1$.
(3) If every node in $N_\Gamma(x)$ has been marked visited, stop. Find $\kappa_\Gamma(x,y) = COUNT \times CONN_\Gamma(x,y)$.
(4) Repeat the steps (1), (2) and (3), for $\Gamma - u$. Find $\kappa_{\Gamma-u}(x,y)$.
(5) According to the inequality between $\kappa_\Gamma(x,y)$ and $\kappa_{\Gamma-u}(x,y)$, we can identify the node u as any of the three types.

With the help of connectivity matrix in fuzzy graphs we can infer the following.

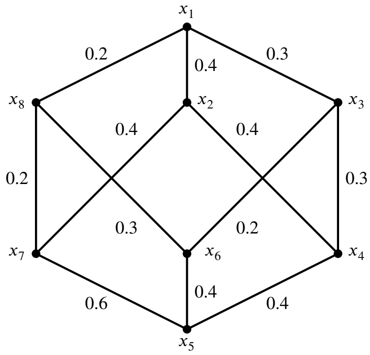

Fig. 10. Fuzzy graph Γ in illustration.

Algorithm 5.2. Let $A = [a_{ij}]$ and $A' = [a'_{ij}]$ be the connectivity matrix of a fuzzy graph Γ and $\Gamma - u$. Then, for the pair (x_i, x_j),

- If $a_{ij} > a'_{ij}$, then either u is an enhancing node or an eroded node.
- If $a_{ij} = a'_{ij}$, then u is either an eroded node or a persisting node.

The aforementioned algorithms may likewise be employed to describe edges by substituting the removal of a node with the removal of an edge.

Illustration of algorithm:

Consider the fuzzy graph $\Gamma = (\rho, \wp)$ in Fig. 10 with $\rho^* = \{x_1, x_2, \dots, x_8\}$, $\wp(x_1x_2) = \wp(x_2x_7) = \wp(x_2x_4) = \wp(x_4x_5) = \wp(x_5x_6) = 0.4$, $\wp(x_1x_8) = \wp(x_3x_6) = \wp(x_7x_8) = 0.2$, $\wp(x_1x_3) = \wp(x_3x_4) = \wp(x_6x_8) = 0.3$ and $\wp(x_5x_7) = 0.6$. The strength of connectedness between the nodes x_2 and x_5 is 0.4. Here, $d_\Gamma(x_5) = 3 = d_\Gamma(x_2)$ and $N_\Gamma(x_2) = \{x_1, x_4, x_7\}$. Let $COUNT = 0$ and $i = 1$. Consider $u_1 = x_1$, then $\wp(x_2, u_1) = 0.4 = CONN_\Gamma(x_2, x_5)$. Finding $CONN_{\Gamma - x_2}(u_1, x_5)$, we get the value 0.3. Now, $\wp(x_2, u_1) \wedge CONN_{\Gamma - x_2}(u_1, x_5) = 0.3$ implies $COUNT = 0$ and $i = 2$. Take $u_2 = x_4$, then $\wp(x_2, u_2) = 0.4 = CONN_\Gamma(x_2, x_5)$. Finding $CONN_{\Gamma - x_2}(u_2, x_5)$, we get the value 0.4. Now, $\wp(x_2, u_2) \wedge CONN_{\Gamma - x_2}(u_2, x_5) = 0.4$ implies $COUNT = 1$ and $i = 3$. Similarly, proceed for $u_3 = x_7$ and then we get $COUNT = 2$.

Now, consider the vertex x_4. Upon removal of x_6 and proceeding the steps of algorithm we get $COUNT' = 1$. Thus, $\kappa_\Gamma(x_2, x_5) = 2 \times 0.4$ and $\kappa_{\Gamma - x_4}(x_2, x_5) = 1 \times 0.4$. Thus x_4 is an (x_2, x_5)-eroded node.

6. Applications

Disasters are significant disturbances that inflict substantial losses on a community. The impacted community cannot manage it alone. It can be caused by various factors such as natural, man-made, and technological influences.

It is crucial to handle the disaster's aftermath once it has occurred. The magnitude and consequences of a disaster might differ, necessitating effective communication among first responders to execute relief and rescue activities quickly. Thus, communication has a vital role in disaster management. Communication during and shortly following a disaster is a critical element of response and recovery since it links impacted individuals, families, and communities with first responders, support networks, and other relatives. Dependable and accessible communication and information networks are essential to a community's resilience. The recent advancements in communication and information technology over the past two decades enable enhanced integration of various communication systems. The interoperability of many communication platforms, including the internet, mobile phones, fax, email, radio, and television, is progressively becoming operational. Consequently, the potential applications of communication technology in catastrophe mitigation and prevention are expanding. The formulation of a disaster communication strategy is the initial step following a natural disaster. These disaster management communication systems must prioritize reliability, particularly in distant and sometimes inaccessible regions such as deep oceans and mountain summits. Numerous individuals become disoriented while traveling in woods or mountains. Numerous incidents transpire during avalanches and landslides. A communication system must be dependable and operational at all times [35]. Numerous novel discoveries and user-friendly communication devices are periodically presented. Numerous research studies have been conducted to optimize disaster communication networks [15], [16].

Here, in this section, we will consider a method to optimize the disaster communication networks. The study was conducted with the objective that networks must remain operational in the event of a component failure. In a disaster communication network, there exists a critical pair of nodes whose communication is essential within the network. Identifying those critical pairs and the roles of the other nodes that act on those pairs enhances the resilience of the network. As a first step, model the communication network as a fuzzy graph by identifying nodes and the edge strength as the capacity of each link. The nodes can be a central communication hub, emergency response units, relief camps, evacuation points, or utility nodes that are essential for emergency operations. The edges represent the communication links, transportation links, relief supply routers, or even emergency power supply lines. The edge strength can be signal strength, response time of the communication, backup availability for the link or the stability of the power supply to the communication infrastructure.

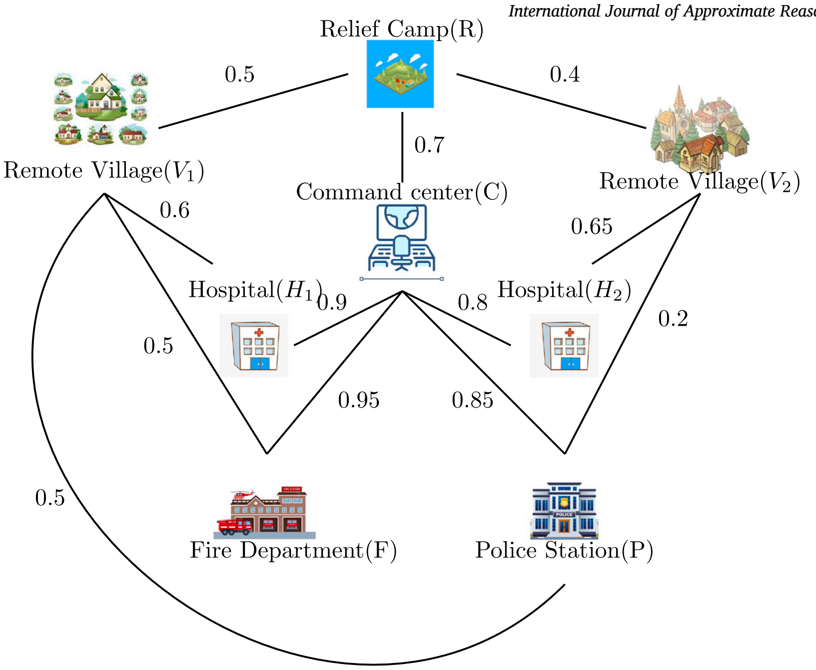

Fig. 11. Fuzzy graph in Application.

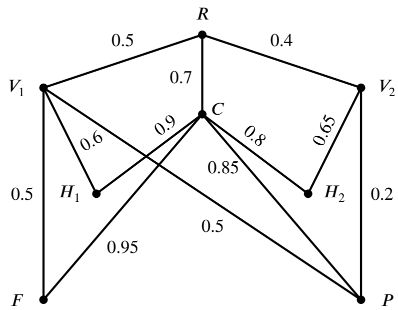

Fig. 12. Fuzzy graph of the outline 11(a).

By considering the fuzzy graph, one can identify the eroded, persisting, and enhancing nodes between each pair of nodes. If a node u is an (x, y)-eroded node, we can infer that there is a sudden decrease in the connectivity between x and y which shows that u must be prioritized for redundancy measures. If u were an enhancing node for (x, y), then it is possible that u is causing bottlenecks owing to inadequate routing or by carrying an excessive amount of load. If it was a persisting node, then u can be kept as a backup route. When it comes to nodes that perform several crucial functions, such as a node that is both an eroding node and a persistent node, it is important to prioritize redundancy and optimization techniques in order to prevent them from becoming single points of failure.

Consider the outline 11 of the disaster communication network consisting of Relief Camp, Villages, Hospitals, Fire departments, Police station and Command centers. As in the outline, the command center is the critical node coordinating all communication operation within the network. Medical care is provided mostly at hospitals, which have an easy connection to neighboring villages. Fire department and police stations works together with command center in case of emergencies. The villages have no direct access with the command center, they have to go through an intermediate. This outline can be depicted as a fuzzy graph in Fig. 12 consisting of the nodes and edges. Each node here has membership value 1 and the membership values of edges are as in the graph.

Consider the connectivity matrices of Γ in (a) and $\Gamma - H_1$ in (b). We can see that the entries in the rows of V_1 in the matrix of Γ have decreased to 0.5 from 0.6. Hence, H_1 is either an enhancing node or an eroded node for the pairs (V_1, u), where u is a node other than V_1 and H_1. Since there is no change in strength of connectivity between other pairs of nodes, H_1 is either an eroded or a persisting node for those pairs. In the case of the nodes V_1 and C, the path through H_1 has the strongest communication. But upon removal of H_1, the paths through R, F, P get strengthened. Thus, H_1 is a (V_1, C)-enhancing node. Here, we get a backup route which is more efficient and reliable. While H_1 is an eroded node for the pairs (V_1, V_2) and (V_1, H_2). The node H_2, is an eroded node of the village 2 and the command center. Thus, H_2 should be given priority in backup planning since their failure will reduce the performance of the network. The node R is a persisting node of the pairs (V_1, C) and (V_2, C). Consequently, it is less imperative, facilitating network designers to concentrate on more essential links that influence total connectivity.

$$\begin{array}{c|c} & \begin{array}{cccccccc} R & V_1 & V_2 & C & H_1 & H_2 & F & P \end{array} \\ \begin{array}{c} R \\ V_1 \\ V_2 \\ C \\ H_1 \\ H_2 \\ F \\ P \end{array} & \begin{bmatrix} 0 & 0.6 & 0.65 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 0.6 & 0 & 0.6 & 0.6 & 0.6 & 0.6 & 0.6 & 0.6 \\ 0.65 & 0.6 & 0 & 0.65 & 0.65 & 0.65 & 0.65 & 0.65 \\ 0.7 & 0.6 & 0.65 & 0 & 0.9 & 0.8 & 0.95 & 0.85 \\ 0.7 & 0.6 & 0.65 & 0.9 & 0 & 0.8 & 0.9 & 0.85 \\ 0.7 & 0.6 & 0.65 & 0.8 & 0.8 & 0 & 0.8 & 0.8 \\ 0.7 & 0.6 & 0.65 & 0.95 & 0.9 & 0.8 & 0 & 0.85 \\ 0.7 & 0.6 & 0.65 & 0.85 & 0.85 & 0.8 & 0.85 & 0 \end{bmatrix} \end{array}$$

(a) Connectivity Matrix of Γ

$$\begin{array}{c|c} & \begin{array}{cccccccc} R & V_1 & V_2 & C & H_1 & H_2 & F & P \end{array} \\ \begin{array}{c} R \\ V_1 \\ V_2 \\ C \\ H_1 \\ H_2 \\ F \\ P \end{array} & \begin{bmatrix} 0 & 0.5 & 0.65 & 0.7 & 0 & 0.7 & 0.7 & 0.7 \\ 0.5 & 0 & 0.5 & 0.5 & 0 & 0.5 & 0.5 & 0.5 \\ 0.65 & 0.5 & 0 & 0.65 & 0 & 0.65 & 0.65 & 0.65 \\ 0.7 & 0.5 & 0.65 & 0 & 0 & 0.8 & 0.95 & 0.85 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7 & 0.5 & 0.65 & 0.8 & 0 & 0 & 0.8 & 0.8 \\ 0.7 & 0.5 & 0.65 & 0.95 & 0 & 0.8 & 0 & 0.85 \\ 0.7 & 0.5 & 0.65 & 0.85 & 0 & 0.8 & 0.85 & 0 \end{bmatrix} \end{array}$$

(b) Connectivity Matrix of $\Gamma - H_1$

By examining these data, a multi-layered strategy is developed in which nodes are assessed across several pairs and classified according to their overall influence. This helps communication networks in enhancing resilience, optimizing network structure and ensuring reliable communication networks.

6.1. Application in financial systems and systemic risk analysis

In financial systems, particularly in interbank lending or transaction networks, institutions are interconnected by a complex network of credit, payment, and liquidity interactions. These networks can be accurately represented as fuzzy graphs, with nodes symbolizing banks or financial institutions and edges indicating transactional interactions, characterized by fuzzy weights that reflect the extent of financial exposure, interaction stability, or liquidity strength. The (x, y)-connectivity between two institutions quantifies how reliably financial obligations or funds can be transferred under uncertain or stressed conditions.

Using the proposed classification:

- Enhancing elements may include strong, diversified financial ties or clearing houses that increase the reliability of transactions between institutions.
- Eroded elements are delicate connections, such as over-leveraged intermediaries or counterparties susceptible to default, which, if compromised or eliminated, substantially destabilize the network.
- Persisting elements are robust financial channels that sustain connectivity even under systemic stress, playing a critical role in maintaining liquidity and confidence in the market.

This classification can support regulators and central banks in identifying systemically important financial institutions (SIFIs), modeling contagion pathways, and planning intervention strategies. For instance, reinforcing enhancing elements (e.g., via central bank guarantees) and monitoring eroded links (e.g., by stress testing) can reduce the likelihood of cascading failures. Thus, your fuzzy graph-based framework offers a novel and mathematically rigorous approach to understanding, monitoring, and improving financial system resilience.

7. Conclusion

This study delves into the nuanced understanding of connectivity in fuzzy graphs by focusing on the behavior of individual nodes and edges and their impact on pairwise connectivity. Unlike traditional approaches that analyze the overall connectivity of the graph, we explore the effects of node and edge removal on the connectivity between specific node pairs, introducing classifications such as enhancing, eroded, and persisting nodes and edges. This approach broadens the theoretical understanding of fuzzy graph connectivity. It provides practical tools for analyzing and optimizing real-world networks, where the role of individual nodes and edges is critical. Future research can extend these findings to dynamic and multi-layered networks, further exploring the interplay between node roles and network resilience under evolving conditions. Bounds on the connectivity between nodes in a fuzzy graph are determined for the fuzzy graph operations such as Cartesian and Strong product. Disaster response communication networks are often characterized by unpredictability, partial infrastructure failure, and time-critical information flow. In such high-stakes environments, maintaining robust and adaptive communication channels is essential for coordinating rescue operations, allocating resources efficiently, and minimizing human and economic loss. The classification of nodes and edges into enhancing, eroded, and persisting elements, as developed in this study, provides a powerful tool for analyzing and improving the resilience and adaptability of these networks. Furthermore, by analyzing (x, y)-connectivity in fuzzy graphs, disaster planners can model varying degrees of uncertainty in communication availability, such as degraded signal strength, probabilistic link failures, or fluctuating node capacities. In summary, the theoretical contributions of this work directly support the practical goal of designing more resilient, responsive, and intelligent disaster communication infrastructures. By integrating fuzzy connectivity analysis and structural classification, emergency response agencies can better anticipate vulnerabilities, plan resource allocation, and adapt swiftly to evolving conditions on the ground.

As future work, we can extend the current classification framework to multi-layer fuzzy graphs and temporal fuzzy graphs. This can better model real-world crisis communication systems that evolve rapidly during emergencies. Also, we can develop optimization algorithms that maximize (x, y)- connectivity by strategically reinforcing enhancing elements and minimizing the influence of eroded ones under budget or resource constraints.

CRediT authorship contribution statement

P Sujithra: Writing – original draft, Conceptualization. **Sunil Mathew:** Writing – review & editing, Supervision. **J.N. Mordeson:** Writing – review & editing, Validation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The authors do not have permission to share data.

References

[1] S. Ali, S. Mathew, J.N. Mordeson, H. Rashmanlou, Vertex connectivity of fuzzy graphs with applications to human trafficking, New Math. Nat. Comput. 14 (2018) 457–485.
[2] S. Ali, S. Mathew, J.N. Mordeson, Containers and spanning containers in fuzzy graphs with application to human trafficking, New Math. Nat. Comput. 20 (2024) 103–128.
[3] S.Y. Bang, R.T. Yeh, Fuzzy relations, fuzzy graphs and their applications to clustering analysis, in: L.A. Zadeh, K.S. Fu, M. Shimura (Eds.), Fuzzy Sets and Their Applications, Academic Press, 1975, pp. 125–149.
[4] P. Bhattacharya, Some remarks on fuzzy graphs, Pattern Recognit. Lett. 6 (1987) 297–302.
[5] P. Bhattacharya, F. Suraweera, An algorithm to compute the supremum of max-min powers and a property of fuzzy graphs, Pattern Recognit. Lett. 12 (1991) 413–420.
[6] S. Banerjee, An optimal algorithm to find the degrees of connectedness in an undirected edge-weighted graph, Pattern Recognit. Lett. 12 (7) (1991) 421–424.
[7] A. Bhattacharya, M. Pal, Fuzzy tree covering number for fuzzy graphs with its real-life application in electricity distribution system, Sadhana 47 (280) (2022) 1–10.
[8] A. Bhattacharya, M. Pal, A fuzzy graph theory approach to the facility location problem: a case study in the Indian banking system, Mathematics 11 (2992) (2023).
[9] K.R. Bhutani, A. Rosenfeld, Strong arcs in fuzzy graphs, Inf. Sci. 152 (2003) 319–322.
[10] K.R. Bhutani, A. Rosenfeld, Fuzzy end nodes in fuzzy graphs, Inf. Sci. 152 (2003) 323–326.
[11] M. Binu, S. Mathew, J.N. Mordeson, Connectivity index of a fuzzy graph and its application to human trafficking, Fuzzy Sets Syst. 360 (2019) 117–136.
[12] M. Binu, S. Mathew, J.N. Mordeson, Wiener index of a fuzzy graph and application to illegal immigration networks, Fuzzy Sets Syst. 384 (2020) 132–147.
[13] M. Binu, S. Mathew, J.N. Mordeson, Cyclic connectivity index of fuzzy graphs, IEEE Trans. Fuzzy Syst. 29 (6) (2021) 1340–1349.
[14] M. Binu, S. Mathew, J.N. Mordeson, Connectivity status of fuzzy graphs, Inf. Sci. 573 (2021) 382–395.
[15] I. Chandran, K. Vipin, Multi-UAV networks for disaster monitoring: challenges and opportunities from a network perspective, SN Comput. Sci. 5 (2024).
[16] M. Dai, T.H. Luan, Z. Su, N. Zhang, Q. Xu, R. Li, Joint channel allocation and data delivery for UAV-assisted cooperative transportation communications in post-disaster networks, IEEE Trans. Intell. Transp. Syst. 23 (2022) 676–689.
[17] N. Jicy, S. Mathew, Connectivity analysis of cyclically balanced fuzzy graphs, Fuzzy Inf. Eng. 7 (2015) 245–255.
[18] S. Kosari, H. Jiang, A. Khan, M. Akhoundi, Properties of connectivity in vague fuzzy graphs with application in building university, J. Mult.-Valued Log. Soft Comput. 41 (2023) 463–482.
[19] S. Mathew, M.S. Sunitha, Types of arcs in a fuzzy graph, Inf. Sci. 179 (2009) 1760–1768.
[20] S. Mathew, M.S. Sunitha, Node connectivity and arc connectivity in fuzzy graphs, Inf. Sci. 180 (2010) 519–531.
[21] S. Mathew, M.S. Sunitha, Menger's theorem for fuzzy graphs, Inf. Sci. 222 (2013) 717–726.
[22] S. Mathew, M.S. Sunitha, Strongest strong cycles and θ-fuzzy graphs, IEEE Trans. Fuzzy Syst. 6 (2013) 1096–1104.
[23] S. Mathew, M.S. Sunitha, Cycle connectivity in fuzzy graphs, J. Intell. Fuzzy Syst. 24 (2013) 549–554.
[24] J.N. Mordeson, C.S. Peng, Operations on fuzzy graphs, Inf. Sci. 79 (1994) 159–170.
[25] J.N. Mordeson, P.S. Nair, Cycles and cocycles of fuzzy graphs, Inf. Sci. 90 (1996) 39–49.
[26] J.N. Mordeson, Y.Y. Yao, Fuzzy cycles and fuzzy trees, J. Fuzzy Math. 10 (2002) 189–202.
[27] A. Prabhath, S. Mathew, J.N. Mordeson, Generalized cycle connectivity in fuzzy graphs, Fuzzy Sets Syst. 482 (2024).
[28] K. Radha, S. Arumugam, On strong product of two fuzzy graphs, Int. J. Sci. Res. Publ. 4 (10) (2014) 275–280.
[29] Y. Rao, S. Kosari, S. Hameed, et al., Multi-attribute decision-making using q-rung orthopair fuzzy Zagreb index, Artif. Intell. Rev. 58 (2025) 153.
[30] A. Rosenfeld, Fuzzy graphs, in: L.A. Zadeh, K.S. Fu, M. Shimura (Eds.), Fuzzy Sets and Their Applications to Cognitive and Decision Processes, Academic Press, 1975, pp. 77–95.
[31] A. Sebastian, S. Mathew, J.N. Mordeson, A new fuzzy graph parameter for the comparison of human trafficking chains, Fuzzy Sets Syst. 450 (2022) 27–46.
[32] X. Shi, S. Kosari, S. Hameed, A.G. Shah, S. Ullah, Application of connectivity index of cubic fuzzy graphs for identification of danger zones of tsunami threat, PLoS ONE 19 (2024).
[33] M.S. Sunitha, A. Vijayakumar, A characterization of fuzzy trees, Inf. Sci. 113 (1999) 293–300.
[34] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (1965) 338–353.
[35] National Disaster Management Guidelines, National disaster management information and communication system (NDMICS), https://ndma.gov.in/Governance/Guidelines, 2011.