International Journal of Approximate Reasoning 187 (2025) 109532

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

Triadic data: Representation and reduction

Léa Aubin Kouankam Djouohou [a,], Blaise Blériot Koguep Njionou [a,], Leonard Kwuida [b,*,]

[a] University of Dschang, Faculty of Sciences, Department of Mathematics and Computer Science, Cameroon
[b] Bern University of Applied Sciences, School of Business, Brückenstrasse 73, Bern, 3005, Switzerland

ARTICLE INFO

Keywords:
Formal concept analysis
Triadic concept analysis
Subcontext
Data reduction

ABSTRACT

Triadic Concept Analysis (TCA) is an extension of Formal Concept Analysis (FCA) for handling data represented as a set of objects described by attributes and conditions via a ternary relation. However, the intuition to go from FCA to TCA is not always straightforward. In this paper we discuss some FCA notions from dyadic to triadic. Although some ideas admit straightforward adaptation, most do not. In particular, we address the representation problem, the notion of redundant attributes and subcontexts in the triadic setting.

1. Introduction

Formal Concept Analysis (FCA) is an important tool for qualitative data analysis, and has been used in various domains such as data mining and knowledge discovery [1–3], databases or information retrieval [4], social and neural networks [5–7], software engineering [8], machine learning [9], etc. Knowledge is extracted in the form of clusters and organized hierarchically, or in form of rules. Driven by practical requirements, and the pragmatic philosophy of Charles Peirce with his three universal categories, FCA were extended to Triadic Concept Analysis (TCA) by considering conditions in addition to objects and attributes [10].

The size of data increases steadily, making their treatment difficult or time-intensive. Data may contain redundant information. Preprocessing can then remove redundant objects or attributes, reducing complexity in cluster and rule extraction while preserving the conceptual structure and relevant knowledge. However, the conceptual structure can remain large even for a non-reducible data set. In FCA, a set of methods has been proposed in order to generate a subset of concepts [11], attribute reduction [12], generalized attributes [13], graded attributes [14], atlas decomposition [12], nested line diagrams [12,15], etc.

Working on subcontexts is one of these methods, which is well understood in dyadic FCA, but not yet investigated in TCA (to the best of our knowledge).

Our goal is to extend methods developed for dyadic data to triadic data. We will start with context clarification, data reduction and subsetting. We will discuss existing methods and propose some improvements. The rest of this paper is organized as follows: Section 2 contains some preliminaries as well as a discussion on the graphical representation of the set of triadic concepts. We propose an improvement of the geometrical representation of [10]. The Hasse diagram representation of [16] is also improved into a nested diagram. Section 3 compares and contrasts existing reductions of triadic contexts. Section 4 addresses subcontexts, with a special attention to the compatible ones. Section 5 concludes the paper and gives some hints for future works.

* Corresponding author.
E-mail address: leonard.kwuida@bfh.ch (L. Kwuida).

https://doi.org/10.1016/j.ijar.2025.109532
Received 16 March 2025; Received in revised form 29 June 2025; Accepted 19 July 2025

Available online 29 July 2025

$\mathbb{K}$	m	a	g	s	e	p	n	l
1	×		×	×	×		×	
2		$\boxed{\times}$				×		×
3	×			×				
4		×	×		×			
5	×			×		×	×	
6	×			×			×	
7		×				×		×

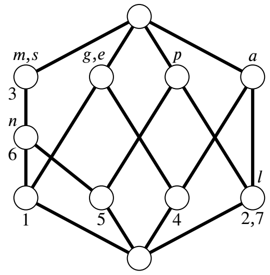

Fig. 1. A formal context (top) and its concept lattice (bottom).

2. From dyadic to triadic data: visualization

2.1. Visualizing dyadic data: the FCA approach

A **formal context** is a triple $\mathbb{K} := (G, M, I)$ with $I \subseteq G \times M$. G is the set of objects, M the set of attributes and I the incidence relation that describes if an object in G has an attribute in M. We write xIy for $(x, y) \in I$ and say that the object x has the attribute y. Fig. 1 gives a formal context (top); Its objects are students: "1: Amelia", "2: Betina", "3: Charles", "4: Declan", "5: Emma", "6: Freddy" and "7: Grace"; its attributes are subjects: "m: Mechanics", "a: Algebra", "g: Geometry", "s: Statistics", "e: Ethics and Development", "p: Programming", "n: Network" and "l: Language"; its incidence relation indicates whether a student has passed a subject, or not. For example, the highlighted cross indicates that Betina (2) has passed Algebra (a).

To formalize the notion of concept, **derivation operators** are defined for $A \subseteq G$ and $B \subseteq M$ by $A' := \{y \in M : \forall x \in A, xIy\}$ and $B' := \{x \in G : \forall y \in B, xIy\}$. A' is the set of attributes common to all objects in A and B' the set of all objects that share all attributes in B. A **formal concept** of $\mathbb{K}$ is a pair (A, B), where $A \subseteq G$ and $B \subseteq M$, such that $A' = B$ and $B' = A$. That is, a maximal rectangle $A \times B$ contained in I. A is called the **extent** and B the **intent** of the formal concept (A, B). Concepts are ordered by the subconcept-superconcept relation defined by

$$(A, B) \leq (C, D) :\iff A \subseteq C \ (\iff D \subseteq B).$$

When $(A, B) \leq (C, D)$, we say that (A, B) is a **subconcept** of (C, D) and (C, D) is a **superconcept** of (A, B)). The set $\mathfrak{B}(\mathbb{K})$ of all formal concepts of $\mathbb{K}$, ordered by $\leq$, forms a complete lattice, called the **concept lattice** of $\mathbb{K}$, denoted by $\underline{\mathfrak{B}}(\mathbb{K})$. The line diagram (aka Hasse diagram) of $\underline{\mathfrak{B}}(\mathbb{K})$ with reduced labeling is on Fig. 1 (bottom). Each node represents a formal concept, whose extent is obtained by collecting all objects from this node downwards, and whose intent is obtained by collecting all attributes from this node upwards.

For $x \in G \cup M$, we write x' as shorthand for $\{x\}'$. A context may contain redundant elements; for example, two objects or attributes may carry the same information. In this case, duplicates can be removed from the context without affecting the concepts extracted. The resulting context is called clarified. Formally, a context (G, M, I) is **clarified** if $x' = y' \implies x = y$, for all x, y objects or attributes. In this case, each node of the line diagram is labeled by at most one object or one attribute. On Fig. 1, $m' = s'$ and $2' = 7'$. The students have developed the same skills in mechanics and statistics, and Betina and Grace have the same status with respect to courses attended and exams taken. Such elements can be spotted immediately on the line diagram.

Another (nontrivial) case of redundant information occurs when an attribute is equivalent to a combination of other attributes. For example, in the context of Fig. 1, students succeeding in "p:Programming" and "a:Algebra" also succeed in "l:Language", and vice-versa. In this case l is equivalent to $\{p, a\}$, and can be removed from the context without changing the structure of the concepts extracted. Formally, an element $x \in G \cup M$ is said **reducible** if there is $X \subseteq G$ or $X \subseteq M$, such that $x' = X'$ and $x \notin X$. A **reduced context** is obtained by first removing duplicates of objects, attributes, and then removing reducible objects and attributes. Its concept lattice is still isomorphic to that of the initial context. On Fig. 1, l and 6 are reducible ($6' = \{1, 5\}'$ and $l' = \{a, p\}'$).

Duplicated objects or attributes are depicted in line diagrams as labels of a single node. A reducible attribute (resp. object) labels a node with at least two upper (resp. lower) neighbors. Implications among attributes can be directly inferred from the line diagram: e.g. m and g imply n, denoted by $\{m, g\} \to \{n\}$, and means that every object in the present context that has the attributes m and g also has the attribute n. Note that 6 has the attribute n but not the attribute g; therefore $\{n\}' \neq \{m, g\}'$. How does this apply to triadic data?

Table 1
A triadic context.

$\mathbb{K}$	P				F				S			
	a	b	c	d	a	b	c	d	a	b	c	d
1	×					×	$\boxed{\times}$			×		×
2	×	×	×		×		×		×		×	×
3		×						×		×		
4				×		×		×	×		×	
5	×		×	×	×		×			×		×

Table 2
Dyadic context $\mathbb{K}^{(1)}$ extracted from triadic context $\mathbb{K}$ of Table 1.

$\mathbb{K}^{(1)}$	(a,P)	(b,P)	(c,P)	(d,P)	(a,F)	(b,F)	(c,F)	(d,F)	(a,S)	(b,S)	(c,S)	(d,S)
1	×					×	×			×		×
2	×	×	×		×		×		×	×	×	
3		×						×		×		
4				×		×		×	×		×	
5	×		×	×	×		×			×		×

2.2. Visualizing triadic data

In many real-world situations, objects may exhibit certain attributes under specific conditions and not the same attributes without those conditions. A **triadic context** is a quadruple $\mathbb{K} := (K_1, K_2, K_3, \mathcal{R})$ where K_1, K_2 and K_3 are sets of objects, attributes and conditions, respectively, and $\mathcal{R} \subseteq K_1 \times K_2 \times K_3$. Triadic contexts were proposed by Lehmann and Wille to generalize dyadic contexts [10,17]. Any triadic context $\mathbb{K}$ gives rise to dyadic contexts: $\mathbb{K}^{(i)} := (K_i, K_j \times K_k, \mathcal{R}^{(i)})$, with $\{i,j,k\} = \{1,2,3\}$ and $j < k$, and for any $(o,a,c) \in K_1 \times K_2 \times K_3$,

$$o\mathcal{R}^{(1)}(a,c) \iff a\mathcal{R}^{(2)}(o,c) \iff c\mathcal{R}^{(3)}(o,a) \iff (o,a,c) \in \mathcal{R}.$$

Each dyadic context $\mathbb{K}^{(i)}$ encodes, from the relational point of view, the same information as the triadic context $\mathbb{K}$. The derivation in $\mathbb{K}^{(i)}$ is called (i)-**derivation** in $\mathbb{K}$, and is given for $X \subseteq K_i$ and $Z \subseteq K_j \times K_k$, with $\{i,j,k\} = \{1,2,3\}$ and $j < k$ by,

- $X^{(i)} = \{(a_j, a_k) \in K_j \times K_k \mid \forall a_i \in X, (a_1, a_2, a_3) \in \mathcal{R}\}$,
- $Z^{(i)} = \{a_i \in K_i \mid \forall (a_j, a_k) \in Z, (a_1, a_2, a_3) \in \mathcal{R}\}$.

Further dyadic contexts are defined for $k \in \{1,2,3\}$ and $A_k \subseteq K_k$ by $\mathbb{K}^{ij}_{A_k} := (K_i, K_j, \mathcal{R}^{ij}_{A_k})$, where for all $(x_i, x_j) \in K_i \times K_j$, $x_i \mathcal{R}^{ij}_{A_k} x_j$ iff for all $x_k \in A_k$, $(x_1, x_2, x_3) \in \mathcal{R}$. The derivation operators defined on $\mathbb{K}^{ij}_{A_k}$ are referred to as (i,j,A_k)**-derivations**. For each $l \in \{i,j\}$, given a subset $X_l \subseteq K_l$, and denoting $r = \{i,j\} \setminus \{l\}$, the (i,j,A_k)-derivation of X_l is defined as:

$$X_l^{(i,j,A_k)} = \left\{a_r \in K_r \,\middle|\, \forall a_l \in X_l,\ \forall a_k \in A_k,\ (a_1, a_2, a_3) \in \mathcal{R}\right\}.$$

Similar to dyadic contexts, a triadic context can be represented by a cross table.

Table 1 presents a triadic context $\mathbb{K} = (K_1, K_2, K_3, \mathcal{R})$, where:

- $K_1 = \{1,2,3,4,5\}$ represents five students,
- $K_2 = \{a,b,c,d\}$ denotes four subjects,
- $K_3 = \{P,F,S\}$ corresponds to three exam sessions (Preliminary, First, Second),
- $\mathcal{R} \subseteq K_1 \times K_2 \times K_3$ encodes the assessment relations.

The incidence relation $(1,c,F) \in \mathcal{R}$ is marked by a shaded cell, indicating that Student 1 passed Subject c in Exam Session F. Table 2 gives the corresponding context $\mathbb{K}^{(1)}$ and Table 3 shows the formal context $\mathbb{K}^{12}_{A_3}$, with $A_3 = \{P,S\}$. We can easily check $\{2\}^{(1,2,A_3)} = \{a,c\}$ and $\{b\}^{(1,2,A_3)} = \{3\}$.

For convenience, throughout the text, we will often adopt a simplified set notation, in which brackets are omitted, and write for example $2 \times ac \times PF$ instead of $\{2\} \times \{a,c\} \times \{P,F\}$, or $(2, ac, PF)$ instead of $(\{2\}, \{a,c\}, \{P,F\})$.

A **triadic concept** (or **triconcept**) of $\mathbb{K}$ is a maximal cuboid in $\mathcal{R}$; i.e., $A_1 \times A_2 \times A_3 \subseteq \mathcal{R}$ and maximal with respect to inclusion. Formally, a triple (A_1, A_2, A_3) with $A_i \subseteq K_i$ is a triadic concept if $A_i = (A_j \times A_k)^{(i)}$, $\{i,j,k\} = \{1,2,3\}$ with $j < k$. A_1 is called extent, A_2 intent and A_3 modus. For example, $2 \times ac \times PF \subsetneq 25 \times ac \times PF \subsetneq \mathcal{R}$, thus $(2, ac, PF)$ is not a triconcept. However, $(25, ac, PF)$

Table 3
Formal context $\mathbb{K}^{12}_{A_3}$ from Table 1, with $A_3 = \{PS\}$.

$\mathbb{K}^{12}_{\{PS\}}$	a	b	c	d
1				
2	×		×	
3		×		
4				
5				×

is a triconcept since it is maximal. The triadic context given by Table 1 has 24 triconcepts,[1] while the dyadic contexts $\mathbb{K}^{(i)}$ have respectively 16, 11, and 8 concepts for $i = 1, 2, 3$, respectively. The first question is, how are triconcepts of $\mathbb{K}$ related to concepts of $\mathbb{K}^{(i)}$?

The **triadic concept generated** by (X_i, X_k), where $X_i \subseteq K_i$ and $X_k \subseteq K_k$, is denoted by $b_{ik}(X_i, X_k)$ and defined as (B_1, B_2, B_3), where:

$$B_j = X_i^{(i,j,X_k)}, \quad B_i = X_i^{(i,j,X_k)(i,j,X_k)} \quad \text{and} \quad B_k = \left(B_i \times B_j\right)^{(k)}.$$

Let α and β be sets of triconcepts of a triadic context $\mathbb{K}$. Let $i, k \in \{1, 2, 3\}$. We set $X_i := \bigcup\{A_i : (A_1, A_2, A_3) \in \alpha\}$ and $X_k := \bigcup\{A_k : (A_1, A_2, A_3) \in \beta\}$. The ik**-join** of (α, β), denoted by $\nabla_{ik}(\alpha, \beta)$, is the triadic concept generated by (X_i, X_k), i.e., $\nabla_{ik}(\alpha, \beta) = b_{ik}(X_i, X_k)$.

We denote by $\mathfrak{T}(\mathbb{K})$ the set of all triconcepts of $\mathbb{K}$; then $\mathfrak{T}(\mathbb{K})$ is equipped with three quasi-order relations $\lesssim_i$, $i \in \{1, 2, 3\}$, and defined by

$$(A_1, A_2, A_3) \lesssim_i (B_1, B_2, B_3) :\iff A_i \subseteq B_i$$

$\lesssim_i$ induces an equivalence relation $\sim_i := \lesssim_i \cap \gtrsim_i$, given by

$$(A_1, A_2, A_3) \sim_i (B_1, B_2, B_3) \iff A_i = B_i.$$

A **triordered set** is a relational structure $(T, \lesssim_1, \lesssim_2, \lesssim_3)$, for which $\lesssim_i$ is a quasi-order, $i \in \{1, 2, 3\}$ and for all $x, y \in T$ and $\{i, j, k\} = \{1, 2, 3\}$, the following is satisfied,

- $x \lesssim_i y$ and $x \lesssim_j y$, imply $x \gtrsim_k y$, **(antiordinality)**
- $x \sim_i y$ and $x \sim_j y$, imply $x = y$. **(uniqueness condition)**

A triordered set $(T, \lesssim_1, \lesssim_2, \lesssim_3)$ in which the ik-join exists, for any arbitrary pairs of subsets of T, is called a **complete trilattice**.

The triordered set $\underline{\mathfrak{T}}(\mathbb{K}) := (\mathfrak{T}(\mathbb{K}), \lesssim_1, \lesssim_2, \lesssim_3)$ is a complete trilattice, called the **concept trilattice** of $\mathbb{K}$. We refer to [18] for more details.

Let $(U, \lesssim_1, \lesssim_2, \lesssim_3)$ and $(T, \lesssim_1, \lesssim_2, \lesssim_3)$ be triordered sets. A map $f : U \to T$ is a **triorder embedding** if for all $x, y \in U$, $x \lesssim_i y$ if and only if $f(x) \lesssim_i f(y)$, $i \in \{1, 2, 3\}$. The uniqueness condition guaranties that such a function is injective. A **triorder-isomorphism** is a surjective triorder-embedding.

It is well known that finite lattices can be graphically represented by means of a Hasse diagram. This representation is used in the case of dyadic context and provides information about the concepts and the corresponding formal context. In the case of trilattices, a graphical representation is a challenging task. The first representation is the one initiated by Lehmann and Wille [10] by means of a partial 3-net in which the three equivalence relations $\sim_1$, $\sim_2$, and $\sim_3$ are represented by three systems of parallel lines in the plane, locating the elements of one equivalence class on one line of the corresponding parallel system.

Fig. 2 below presents the (two dimensional) ordinal scale $\mathbb{O}_3$ with the graphical representation of its concept trilattice. $K_1 = K_2 = K_3 = \{1, 2, 3\}$ and $(i, j, k) \in \mathcal{R}$ iff $i \leq j \leq k$. This context has 9 triconcepts (one can observe that the two dimensional ordinal scale n, with $n \in \mathbb{N} \setminus \{0, 1\}$ has $\frac{n^2+n+6}{2}$ triconcepts): $(123, 3, 3)$; $(12, 23, 3)$; $(1, 12, 23)$; $(12, 2, 23)$; $(1, 1, 123)$; $(1, 123, 3)$; $(123, 123; \emptyset)$; $(123, \emptyset, 123)$; $(\emptyset, 123, 123)$. On the right and upper part of the figure are the Hasse diagrams of extents and modi, respectively and on the left part the upside-down Hasse diagram of intents. The dotted lines represent the equivalent classes which allows to read the triconcepts in the center diagram by making projections on the side diagrams. The extents of triconcepts are obtained by making projections on the right diagram through horizontal lines, collecting the corresponding object and all the objects below it. Similarly, one can get intents of triconcepts by projections on the left diagram through the lines descending to the left, collecting all the attributes attached to and above the discontinuous line, since the Hasse diagram of the attributes is upside-down. Modi of a triconcept can be obtained similarly by projections on the top diagram through the lines ascending to the left. For example, the triconcept represented by the black node of the triangular pattern in Fig. 2 is $(12, 23, 3)$.

[1] Computed using *FCA Tools Bundle*.

$\mathbb{O}_3$	1			2			3		
	1	2	3	1	2	3	1	2	3
1	×			×	×		×	×	×
2					×			×	×
3									×

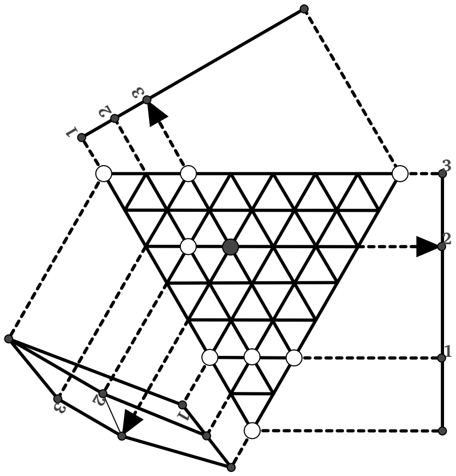

Fig. 2. A triadic context (top) and a 3-net diagram of its concept trilattice (bottom).

	P		F	
	a	b	a	b
1		×	×	
2	×			×

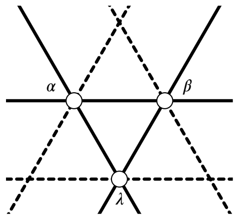

Fig. 3. Left: Small triadic context verifying the tetrahedron condition. **Right:** Illustration of the non representation of triconcepts which verifies this condition.

Although informative, this representation is often difficult to construct, even for small datasets, complicating data exploration. To mitigate this challenge, some authors have proposed local navigation strategies by reducing the problem to the classical setting (see [19]). Moreover, this representation (using exclusively straight lines) is not always possible.

The table in Fig. 3 is an example of a triadic context whose concepts cannot be represented using exclusively straight lines, as in Fig. 2. Its triconcepts are $c_1 = (\emptyset, ab, PF)$, $c_2 = (12, \emptyset, PF)$, $c_3 = (12, ab, \emptyset)$, $\alpha = (2, a, P)$, $\beta = (2, b, F)$, $\lambda = (1, b, P)$ and $\gamma = (1, a, F)$. Note that

$$\alpha \sim_1 \beta,\ \beta \sim_2 \lambda \text{ and } \lambda \sim_3 \alpha.$$

We say that they form a **cycle**, denoted by $\{\alpha, \beta, \lambda\}$, which corresponds to a triangle in the 3-net representation. In addition, the triconcept γ satisfies

$$\gamma \sim_1 \lambda,\ \gamma \sim_3 \beta \text{ and } \gamma \sim_2 \alpha.$$

Once the triconcepts α, β, and λ are represented on the geometrical structure (Fig. 3, right), it is no longer possible to represent γ, since there is no common point to the three dotted lines. The triconcept γ together with the cycle $\{\alpha, \beta, \lambda\}$ verifies the so-called **tetrahedron condition** [20]. The context in Fig. 3 is the smallest one satisfying the tetrahedron condition.

Another representation is proposed in [16] for visualizing and navigating through triadic concepts. The idea is to focus, for example, on extents (or intents or modi) and draw the corresponding Hasse diagram, and label the nodes with sets of triconcepts with the same extent. This representation corresponds to a side diagram of the partial 3-net structure, where each line represents a set of triconcepts with the same extent, intent or modus.

Such representations apply to all trilattices, including those with triconcepts satisfying the tetrahedron condition. Fig. 4 displays the Hasse diagram of the trilattice of the triadic concepts of the context in Table 1. In addition, the nodes with same extent/intent/modus can be ordered. This information could be added to the diagram, resulting in a slight improvement, similar to a nested line diagram. Each square box in Fig. 4 represents a set of triadic concepts having the same modus. The triconcepts inside the square boxes are ordered with respect to intents and extents. The extent and intent of each triconcept can be read as in the dyadic case. For example, the black node in the figure represents the triadic concept $(14, b, F)$. This representation allows reduced labeling within each squared

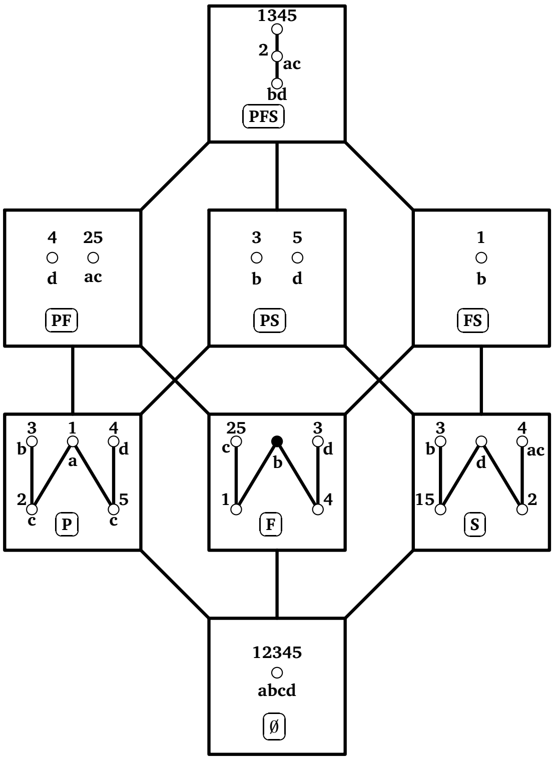

Fig. 4. Hasse diagram of the concept trilattice of the context from Table 1.

box. It should be investigated whether reduced labeling could be applied to the entire diagram and whether rules could be derived from this "nested" Hasse diagram.

Going from 3-nets diagrams to the representation in [16] is straightforward. Conversely, the question remains, whether there are efficient methods to construct 3-net diagrams from one, two or three diagrams from [16].

To conclude this section, we outline a geometric representation of triadic concepts satisfying the tetrahedron condition. Note that a cycle corresponds to a triangle in the 3-net diagram. A cycle whose triangle is the median triangle[2] of another is called the **median cycle** of the latter.

Proposition 2.1. *Let $\mathbb{K}$ be a triadic context and $\alpha \sim_1 \beta \sim_2 \lambda \sim_3 \alpha$ be a cycle.*

(i) *If γ_1 is a triadic concept that, together with $\{\alpha, \beta, \lambda\}$ satisfies the tetrahedron condition, then γ_1 is unique.*
(ii) *For any cycle, its median cycle never satisfies the tetrahedron condition with an arbitrary triconcept.*
(iii) *If the cycle $\{\alpha, \beta, \lambda\}$ satisfies the tetrahedron condition with a triadic concept γ, then $\{\eta, \mu, \nu\}$ is also a cycle and satisfies the tetrahedron condition with ρ, where $\{\eta, \mu, \nu, \rho\} = \{\alpha, \beta, \lambda, \gamma\}$.*
(iv) *If the cycle $\{\alpha, \beta, \lambda\}$ satisfies the tetrahedron condition with a triadic concept γ, then any cycle containing two triadic concepts from this tetrahedron must be a cycle of this tetrahedron.*

Proof. (i) If γ_1 satisfies the tetrahedron condition with the cycle $\alpha \sim_1 \beta \sim_2 \lambda \sim_3 \alpha$, then $\gamma_1 \sim_1 \lambda$, $\gamma_1 \sim_3 \beta$, and $\gamma_1 \sim_2 \alpha$. If a triconcept γ_2 satisfies the tetrahedron condition with the same cycle, then $\gamma_2 \sim_1 \lambda$, $\gamma_2 \sim_3 \beta$, $\gamma_2 \sim_2 \alpha$. It follows that $\gamma_1 \sim_1 \gamma_2$ and $\gamma_1 \sim_3 \gamma_2$. Thus $\gamma_1 = \gamma_2$, by the uniqueness condition.

(ii) Suppose that $\{\alpha', \beta', \lambda'\}$ is the median cycle of a cycle $\{\alpha, \beta, \lambda\}$. Without loss of generality, assume that $\alpha \sim_1 \alpha' \sim_1 \beta \sim_2 \beta' \sim_2 \lambda \sim_3 \lambda' \sim_3 \alpha$.
If $\{\alpha', \beta', \lambda'\}$ satisfies the tetrahedron condition, let's say with a triconcept γ_2, then $\gamma_2 \sim_1 \alpha'$, $\gamma_2 \sim_2 \beta'$, $\gamma_2 \sim_3 \lambda'$. Thus $\gamma_2 \sim_1 \alpha' \sim_1 \beta$ and $\gamma_2 \sim_2 \beta' \sim_2 \beta$, that is $\gamma_2 \sim_1 \beta$ and $\gamma_2 \sim_2 \beta$. By uniqueness condition, $\gamma_2 = \beta$ which is a contradiction because $\gamma_2 \sim_3 \lambda' \sim_3 \lambda$, i.e., $\gamma_2 \sim_3 \lambda$, while $\beta \not\sim_3 \lambda$.

[2] In any triangle, a median is a line segment connecting a vertex to the midpoint of the opposite side. The median triangle is the triangle formed by connecting the midpoints of the three medians of the original triangle.

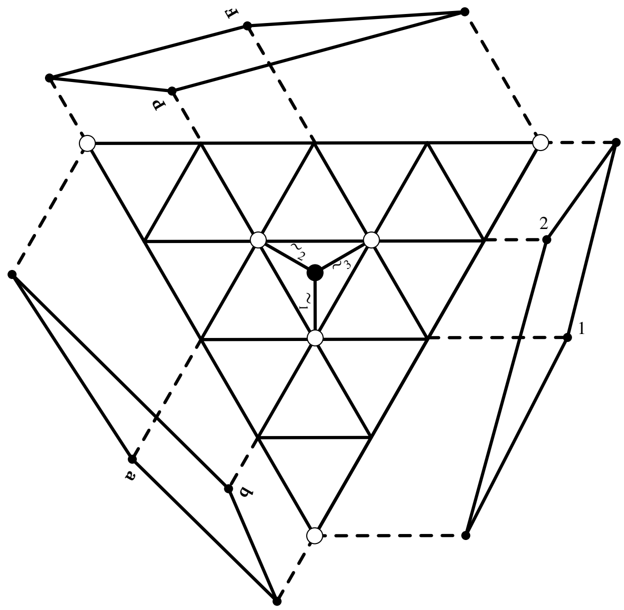

Fig. 5. Concept trilattice diagram of the context in Fig. 3. The black node represents the triconcept $(1, a, F)$.

(iii) Assume that γ together with a cycle $\{\alpha, \beta, \lambda\}$ satisfies the tetrahedron condition. Then, we have $\alpha \sim_1 \beta \sim_2 \lambda \sim_3 \alpha$ and $\gamma \sim_1 \lambda$, $\gamma \sim_3 \beta$, $\gamma \sim_2 \alpha$. This implies that $\alpha \sim_1 \beta \sim_3 \gamma \sim_2 \alpha$ and also $\lambda \sim_1 \gamma$, $\lambda \sim_2 \beta$, $\lambda \sim_3 \alpha$. It follows that $\{\alpha, \beta, \gamma\}$ is a cycle and satisfy the tetrahedron condition with λ. The proof is similar for the other cases.

(iv) Assume $\alpha \sim_1 \beta \sim_2 \lambda \sim_3 \alpha$, with γ satisfying $\gamma \sim_1 \lambda$, $\gamma \sim_3 \beta$, and $\gamma \sim_2 \alpha$. Without loss of generality, let ζ be a triadic concept forming a cycle with α and β. Then either $\alpha \sim_1 \beta \sim_3 \zeta \sim_2 \alpha$ holds, which implies $\gamma \sim_3 \zeta$ and $\gamma \sim_2 \zeta$, forcing $\zeta = \gamma$ by uniqueness, or $\alpha \sim_1 \beta \sim_2 \zeta \sim_3 \alpha$ holds, which implies $\lambda \sim_3 \zeta$ and $\lambda \sim_2 \zeta$, forcing $\zeta = \lambda$ by uniqueness. $\square$

We have proved that, given a triadic concept, there is at most one triconcept verifying the tetrahedron condition with a given cycle. Hence, the four triconcepts verifying the tetrahedron condition form a regular tetrahedron. This allows us to have a geometrical representation (Fig. 5) of the concept trilattice diagram of the triadic context in Fig. 3, which was not possible with only parallel straight lines. The extent of the black node is that of the node below, its intent is that of the node on the left and its modus is that of the node on the right.

3. Reduction in triadic setting

In this section we assume that our contexts are finite, even if some of the results stated hold in general. We have recalled above how to handle redundant information in dyadic setting and we have seen that the concept lattice structure is preserved when removing duplicated or reducible attributes. Redundant attributes and clarification were explored in triadic contexts in [21]. In this section, we investigate reducible attributes and those that their omission do not change the number of triconcepts of the context.

As in the dyadic case two objects will be considered as duplicated if they have the same attributes under the same conditions. The first step in a reduction is to remove duplicates (objects, attributes and conditions). The resulting context is said to be clarified. Indeed, a triadic context $\mathbb{K} := (K_1, K_2, K_3, \mathcal{R})$ is **clarified** if for each $i \in \{1, 2, 3\}$ and for $u, v \in K_i$, we have $u^{(i)} = v^{(i)} \implies u = v$. For two such elements, the clarification process consists of removing one or merging the two.

Proposition 3.1. *If $\mathbb{H} = (H_1, H_2, H_3, S)$ is a clarified triadic context resulting from $\mathbb{K} = (K_1, K_2, K_3, \mathcal{R})$ by clarification, then $\underline{\mathfrak{T}}(\mathbb{K})$ is isomorphic to $\underline{\mathfrak{T}}(\mathbb{H})$.*

Proof. Suppose that $\mathbb{K} = (K_1, K_2, K_3, \mathcal{R})$ contains attributes x and y such that $x^{(2)} = y^{(2)}$ and that $\mathbb{H} = (H_1, H_2, H_3, S)$ result from $\mathbb{K}$ by removing y, i.e., $H_2 = K_2 \setminus \{y\}$, $H_1 = K_1$ and $H_3 = K_3$. Define the map $h : \underline{\mathfrak{T}}(\mathbb{H}) \to \underline{\mathfrak{T}}(\mathbb{K})$ such that $h((X_1, Y_1, Z_1)) = (X_2, Y_2, Z_2)$ where $X_2 = X_1$, $Z_2 = Z_1$ and

$$Y_2 = \begin{cases} Y_1 & \text{if } x \notin Y_1, \\ Y_1 \cup \{y\} & \text{if } x \in Y_1. \end{cases}$$

It is easy to see that (X_1, Y_1, Z_1) is a triconcept of $\mathbb{H}$ iff (X_2, Y_2, Z_2) is a triconcept of $\mathbb{K}$, i.e., h is a bijective map. Moreover, for all triconcepts (A_1, B_1, C_1) and (X_1, Y_1, Z_1) of $\mathbb{H}$ and for $i \in \{1, 2, 3\}$,

$$(A_1, B_1, C_1) \lesssim_i (X_1, Y_1, Z_1) \iff h((A_1, B_1, C_1)) \lesssim_i h((X_1, Y_1, Z_1)).$$

Then h preserves the quasiorders. Similarly, h^{-1} also preserves the quasiorders. It follows that h is an isomorphism as required. This justifies the claim for removing one attribute y such that there exists another attribute x with $x^{(2)} = y^{(2)}$. The same is true for objects and conditions. Repeated application gives the proposition. $\square$

Definition 3.2. Two contexts $\mathbb{K} := (K_1, K_2, K_3, \mathcal{R})$ and $\mathbb{H} := (H_1, H_2, H_3, \mathcal{S})$ are **isomorphic** if there are bijective maps $\phi_i : K_i \to H_i$, $1 \leq i \leq 3$, such that

$$(x, y, z) \in \mathcal{R} \iff (\phi_1(x), \phi_2(y), \phi_3(z)) \in \mathcal{S}, \text{ for all } (x, y, z) \in K_1 \times K_2 \times K_3.$$

The tuple (ϕ_1, ϕ_2, ϕ_3) is called an **isomorphism** between $\mathbb{K}$ and $\mathbb{H}$.

If two triadic contexts $\mathbb{K} = (K_1, K_2, K_3, \mathcal{R})$ and $\mathbb{H} = (H_1, H_2, H_3, \mathcal{S})$ are isomorphic via the maps $\phi_i : K_i \to H_i$ ($i \in \{1, 2, 3\}$), then the map $\phi : \underline{\mathfrak{T}}(\mathbb{K}) \to \underline{\mathfrak{T}}(\mathbb{H})$, defined by $\phi((X, Y, Z)) = (\phi_1(X), \phi_2(Y), \phi_3(Z))$, is an isomorphism between the concept trilattices $\underline{\mathfrak{T}}(\mathbb{H})$ and $\underline{\mathfrak{T}}(\mathbb{K})$. Then isomorphic contexts have isomorphic concept trilattices.

Proposition 3.3. *Let $\mathbb{H}$ and $\mathbb{G}$ be clarified contexts of a triadic context $\mathbb{K}$. Then $\mathbb{H}$ and $\mathbb{G}$ are isomorphic triadic contexts.*

Proof. Set $\mathbb{K} = (K_1, K_2, K_3, \mathcal{R})$, $\mathbb{H} = (H_1, H_2, H_3, \mathcal{S})$ and $\mathbb{G} = (G_1, G_2, G_3, \mathcal{I})$. Suppose that $\mathbb{K}$ contains attributes u and v such that $u^{(2)} = v^{(2)}$ and that $H_1 = B_1 = K_1$, $H_3 = B_3 = K_3$, $H_2 = K_2 \setminus \{u\}$ and $B_2 = K_2 \setminus \{v\}$. Taking ϕ_1 and ϕ_3 to be identity maps on K_1 and K_3, respectively, and defining

$$\phi_2 : H_2 \to B_2 \text{ by } \phi_2(x) = \begin{cases} x & \text{if } x \neq u \\ v & \text{if } x = u \end{cases},$$

we obtain bijective maps satisfying $(x, y, z) \in \mathcal{S} \iff (\phi_1(x), \phi_2(y), \phi_3(z)) \in \mathcal{I}$. According to Definition 3.2, $\mathbb{H}$ and $\mathbb{G}$ are isomorphic triadic contexts. The same is true with objects and conditions. Repeated application gives the desired conclusion. $\square$

The next step consists in removing attributes/objects/conditions that are obtained as combinations of other attributes/objects/conditions. An element $a \in K_i$ is **reducible** if there exists a subset $Y \subseteq K_i$ such that $a \notin Y$ and $a^{(i)} = Y^{(i)}$. Removing such elements does not change the concept lattice structure of the dyadic context $\mathbb{K}^{(i)}$. A clarified context is **reduced** if every element is irreducible. This is also true for the concept trilattice of $\mathbb{K}$ as shown in [21] and recalled below.

Proposition 3.4. *[21] Let $a \in K_2$ be an attribute and $Y \subseteq K_2$. If $a \notin Y$ and $a^{(2)} = Y^{(2)}$, then*

$$\underline{\mathfrak{T}}(K_1, K_2, K_3, \mathcal{R}) \cong \underline{\mathfrak{T}}\left(K_1, K_2 \setminus \{a\}, K_3, \mathcal{R} \cap (K_1 \times (K_2 \setminus \{a\}) \times K_3)\right).$$

From the above formalization, reducible elements are defined as reducible objects in the dyadic contexts $\mathbb{K}^{(i)}$. Therefore, the methods developed for dyadic contexts such as arrow-relations can be applied to identify and remove reducible elements in a triadic context [21]. In dyadic contexts, removing an attribute results in a suborder of the initial concept lattice that preserves meets. In finite case, each suborder of the concept lattice with the same cardinality would be isomorphic to the concept lattice. Thus we can define reducible attributes in finite case as an attribute $m \in M$ such that $|\mathfrak{B}(G, M, I)| = |\mathfrak{B}(G, M \setminus \{m\}, J)|$, where J is the restriction of I on $G \times M \setminus \{m\}$. How does this idea carry out to triadic data? Are attribute $a \in K_2$ such that

$$|\mathfrak{T}(K_1, K_2, K_3, \mathcal{R})| = |\mathfrak{T}\left(K_1, K_2 \setminus \{a\}, K_3, \mathcal{R} \cap (K_1 \times (K_2 \setminus \{a\}) \times K_3)\right)|$$

reducible? Note that the “orderings” on extents, intents and modi in triadic setting are just quasi-orders, in contrast to the dyadic case, where they are partial orders.

Recall that for a triadic concept $\mathfrak{a} := (A, B, C)$ of $\mathbb{K}$, we call A extent, B intent and C modus of $\mathfrak{a}$. In addition we call (B, C) feature or **1-feature** ([16,22]). Similarly, we call (A, C) a **2-feature** and (A, B) a **3-feature** of $\mathbb{K}$.

In general, an i-feature generates a triconcept whose i-component can be obtained by applying derivation in the formal context $\mathbb{K}^{(i)}$. For example, if (B, C) is a 1-feature of a triadic context $\mathbb{K}$, then $((B \times C)^{(1)}, B, C)$ is a triconcept of $\mathbb{K}$. Let $H_2 \subseteq K_2$ and $\mathbb{H}_2 := (K_1, H_2, K_3, \mathcal{R} \cap (K_1 \times H_2 \times K_3))$. If every 2-feature of $\mathbb{K}$ is a 2-feature of $\mathbb{H}_2$ we say that H_2 is an **attribute consistent subset** of K_2. An attribute $a \in K_2$ is said to be **2-feature reducible** if $K_2 \setminus \{a\}$ is a consistent subset of K_2; otherwise we say that a is **2-feature irreducible**. After removing reducible attributes, the next step would be to remove 2-feature reducible attributes. Note that, when a 2-feature reducible attribute is removed, the set of 2-features is preserved, but the structure of the triorder could be altered. If H_2 is an attribute consistent subset of K_2, and no proper subset of H_2 is an attribute consistent subset of K_2, then we say that $\mathbb{H}_2$ is **2-feature reduced**; in this case, H_2 contains exactly all 2-feature irreducible attributes of $\mathbb{K}$ and it is called an “attribute

Table 4
Triadic context with a 1-feature reducible object which is not reducible.

	P		F	
	a	b	a	b
1	×			
2		×	×	
3		×		

reduction set" of $\mathbb{K}$ in [22]. Similarly, 1-feature reducible objects and 3-feature reducible conditions are defined. A context is said to be **feature reduced** if it is i-feature reduced for all $i \in \{1, 2, 3\}$.

In the following proposition, we give some properties about reducible attributes and 2-feature reducible attributes.

Proposition 3.5. *Let $\mathbb{K}$ be a triadic context, $a \in K_2$ and $H_2 = K_2 \setminus \{a\}$.*

(i) *If a is a reducible attribute, then a is 2-feature reducible.*
(ii) *If a is 2-feature reducible then*

$$(A, B, C) \in \mathfrak{T}(\mathbb{K}) \implies (A, B \setminus \{a\}, C) \in \mathfrak{T}(\mathbb{H}_2).$$

(iii) *If $\mathbb{K}^\dagger$ and $\mathbb{K}^\ddagger$ are feature reduced contexts of $\mathbb{K}$ then they are isomorphic.*
(iv) *If $\mathbb{K}^\dagger$ and $\mathbb{K}^\ddagger$ are reduced contexts of $\mathbb{K}$ then they are isomorphic.*
(v) *a is 2-feature reducible iff $|\mathfrak{T}(\mathbb{K})| = |\mathfrak{T}(\mathbb{H}_2)|$.*

Proof. (i) Assume that a is a reducible attribute. Then, there exists a subset $Y \subseteq K_2$ such that $a \notin Y$ and $a^{(2)} = Y^{(2)}$. If (A, B, C) is a triconcept of $\mathbb{K}$, then $(A, B \setminus \{a\}, C)$ is a triconcept of $\mathbb{H}_2$. This implies that the 2-features of $\mathbb{K}$ are also 2-features of $\mathbb{H}_2$, hence a is 2-feature reducible.

(ii) Assume that a is 2-feature reducible and that $(A, B, C) \in \mathfrak{T}(\mathbb{K})$. Then (A, C) is a 2-feature of $\mathbb{K}$, but also a 2-feature of $\mathbb{H}_2$ because a is 2-feature reducible. Thus (A, C) generates in $\mathbb{H}_2$ a triconcept whose intent is $(A, C)^{(2)} \setminus \{a\} = B \setminus \{a\}$, hence $(A, B \setminus \{a\}, C)$ is a triconcept of $\mathbb{H}_2$.

(iii) To prove (iii), we consider the following two statements:
(S1): If a context $\mathbb{K}$ is clarified, then its feature reduced context is unique.
(S2): If $\mathbb{H}$ is a clarified context of $\mathbb{K}$, then $\underline{\mathfrak{T}}(\mathbb{H}) \cong \underline{\mathfrak{T}}(\mathbb{K})$.
It is easy to see that statement (S1) holds. Statement (S2) follows from Proposition 3.1. Now, let us prove (iii). We suppose that $\mathbb{K}^\dagger$ and $\mathbb{K}^\ddagger$ are feature reduced contexts of $\mathbb{K}$. If $\mathbb{K}$ is clarified, then $\mathbb{K}^\dagger = \mathbb{K}^\ddagger$ and the proof is completed. If $\mathbb{K}$ is not clarified, then $\mathbb{K}^\dagger$ and $\mathbb{K}^\ddagger$ are clarified contexts of $\mathbb{K}$. It follows from Proposition 3.3 that $\mathbb{K}^\dagger$ and $\mathbb{K}^\ddagger$ are isomorphic.

(iv) Knowing that every reduced context is clarified, the result follows from Proposition 3.3.

(v) Note that, if (A, B, C) is a triconcept of $\mathbb{H}_2$, then clearly, $(A, (A \times C)^{(2)}, C)$ is a triconcept of $\mathbb{K}$. This implies that every 2-feature of $\mathbb{H}_2$ is also a 2-feature of $\mathbb{K}$. Assume that a is 2-feature reducible. Then every 2-feature of $\mathbb{K}$ is a 2-feature of $\mathbb{H}_2$. Hence, $|\mathfrak{T}(\mathbb{K})| = |\mathfrak{T}(\mathbb{H}_2)|$.
Conversely, if $|\mathfrak{T}(\mathbb{K})| = |\mathfrak{T}(\mathbb{H}_2)|$, then the 2-features of $\mathbb{H}_2$ are exactly those of $\mathbb{K}$, i.e., a is 2-feature reducible. □

We end this section by showing that a 2-feature reducible attribute is not necessary reducible. We consider the context $\mathbb{K} = (K_1, K_2, K_3, \mathcal{R})$ with $K_1 = \{1, 2, 3\}$, $K_2 = \{a, b\}$ and $K_3 = \{P, F\}$ given in Table 4. This triadic context has exactly six triconcepts:

$$(\emptyset, ab, PF),\ (23, b, P),\ (123, ab, \emptyset),\ (2, a, F),\ (1, a, P) \text{ and } (123, \emptyset, PF).$$

The corresponding 3-net diagram is given on the left side of Fig. 6. The triconcepts of the context $\mathbb{H}_1 = (K_1 \setminus \{3\}, K_2, K_3, \mathcal{R} \cap ((K_1 \setminus \{3\}) \times K_2 \times K_3)$ are

$$(\emptyset, ab, PF),\ (2, b, P),\ (12, ab, \emptyset),\ (2, a, F),\ (1, a, P) \text{ and } (12, \emptyset, PF).$$

Every 1-feature of $\mathbb{K}$ is also a 1-feature of $\mathbb{H}_1$. Thus 3 is a 1-feature reducible object of K_1. However, 3 is not reducible because there is no subset $X \subseteq K_1$ with $3 \notin X$ and $3^{(1)} = X^{(1)}$. Indeed, $3^{(1)} = \{(b, P)\}$, but

$$1^{(1)} = \{(a, P)\}, \quad 2^{(1)} = \{(b, P), (a, F)\}, \quad \text{and} \quad \{1, 2\}^{(1)} = \emptyset.$$

The geometric line of $\underline{\mathfrak{T}}(\mathbb{H}_1)$ is given on Fig. 6 (right). Thus $\underline{\mathfrak{T}}(\mathbb{H}_1) \ncong \underline{\mathfrak{T}}(\mathbb{K})$. After removing object 3, the triconcept $(23, b, P)$ becomes $(2, b, P)$ and moves to the same object line as $(2, a, F)$.

Particular attention should be paid to how reducible or feature-reducible elements can be identified in line diagrams.

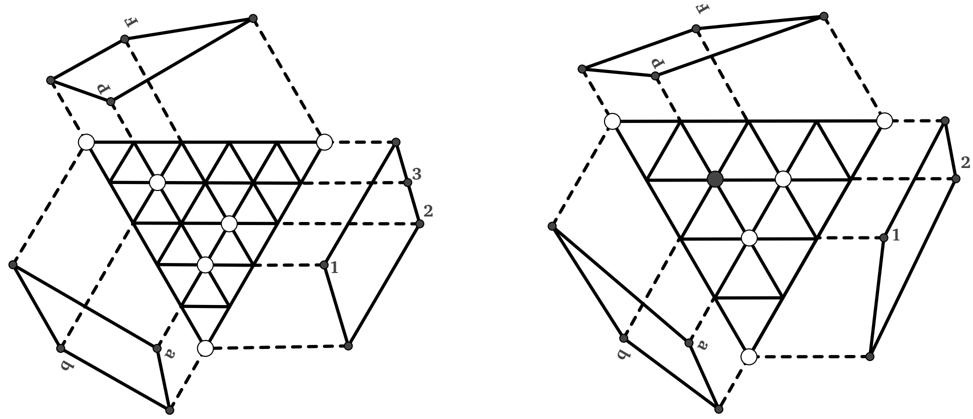

Fig. 6. Left: concept trilattice diagram of the triadic context in Table 4. **Right:** the trilattice diagram after removing object 3.

By removing 2-feature reducible or reducible attributes, we obtain a subcontext. The next section investigates subcontexts in triadic contexts more generally.

4. Subcontexts

In the previous sections, we have discussed how some attributes can be removed from the initial context. The case of reducible or 2-feature reducible attributes is just a special case. Given a dyadic context $\mathbb{K} := (G, M, I)$ a **subcontext** is a triple $\mathbb{H} := (H, N, J)$ with $H \subseteq G$, $N \subseteq M$ and $J = (H \times N) \cap I$. Similarly, a subcontext of a triadic context $\mathbb{K} := (K_1, K_2, K_3, \mathcal{R})$ is a context $\mathbb{H} := (H_1, H_2, H_3, \mathcal{S})$ such that $H_i \subseteq K_i$, $1 \leq i \leq 3$ and $\mathcal{S} = \mathcal{R} \cap (H_1 \times H_2 \times H_3)$. To avoid confusion, we will denote the derivation in a context by the corresponding incidence relation.

In dyadic case, if only attributes are removed, then the extents of the subcontext are extents of the initial context. The concept lattice of the subcontext naturally embeds in that of the initial context. Formally,

Proposition 4.1. *[12] If $N \subseteq M$, then every extent of $(G, N, I \cap (G \times N))$ is an extent of (G, M, I) and the correspondence $(A, B) \mapsto (A, A^I)$ is a $\bigwedge$-preserving order-embedding from $\underline{\mathfrak{B}}(G, N, I \cap (G \times N))$ to $\underline{\mathfrak{B}}(G, M, I)$.*

In general, the concepts of a subcontext can not simply be derived from those of the initial context by restricting extents and intents to the subcontext. Given a formal context $\mathbb{K} := (G, M, I)$, a subcontext $\mathbb{H} := (H, N, J)$ is said **compatible** if whenever (A, B) is a concept of $\mathbb{K}$, then $(A \cap H, B \cap N)$ is a concept of $\mathbb{H}$. Compatible subcontexts play an important role in congruences and decomposition of concept lattices.

Proposition 4.2. *[12] A subcontext (H, N, J) of (G, M, I) is compatible if and only if the correspondence $(A, B) \mapsto (A \cap H, B \cap N)$ defines a surjective complete homomorphism from $\underline{\mathfrak{B}}(G, M, I)$ to $\underline{\mathfrak{B}}(H, N, J)$.*

There are well established methods to find all compatible subcontexts of a given formal context, for example using the transitive closure of the arrow-relations [12]. The context in Fig. 1 has exactly one nontrivial compatible subcontext, obtained by removing object 3 and attribute n. How can we extend these ideas to triadic data? How are triconcepts of a subcontext $\mathbb{H}$ of a triadic context $\mathbb{K}$ related to those of $\mathbb{K}$? Unless otherwise stated, $\mathbb{K} := (K_1, K_2, K_3, \mathcal{R})$ will denote a triadic context and $\mathbb{H} := (H_1, H_2, H_3, \mathcal{S})$ a subcontext of $\mathbb{K}$.

Proposition 4.3. *Let $\mathbb{K}$ be a triadic context. If $\mathbb{H}$ is a subcontext with $H_1 = K_1$ and $H_2 = K_2$, then every extent (resp. intent) of $\mathbb{H}$ is an extent (resp. intent) of $\mathbb{K}$.*

Proof. If A is an extent of the subcontext $\mathbb{H}$, there exist $B \subseteq K_2$ and $C \subseteq H_3$ such that (A, B, C) is a triconcept of $\mathbb{H}$, i.e., $A = (B \times C)^{(1)_\mathcal{S}}$, $B = (A \times C)^{(2)_\mathcal{S}}$ and $C = (A \times B)^{(3)_\mathcal{S}}$. Let us show that $(A, B, (A \times B)^{(3)})$ is a triconcept of $\mathbb{K}$. To this end, it will be sufficient to show that $A = \left(B \times (A \times B)^{(3)}\right)^{(1)}$ and $B = \left(A \times (A \times B)^{(3)}\right)^{(2)}$. We will show the first equality and the second one can be obtained similarly.

Let $x \in A$. We want to show that $x \in \left(B \times (A \times B)^{(3)}\right)^{(1)}$. Let $(y, z) \in B \times (A \times B)^{(3)}$. Then $x \in A$, $y \in B$ and $z \in (A \times B)^{(3)}$, and $z\mathcal{R}^{(3)}(x, y)$, which gives $x\mathcal{R}^{(1)}(y, z)$ and then $x \in \left(B \times (A \times B)^{(3)}\right)^{(1)}$, i.e., $A \subseteq \left(B \times (A \times B)^{(3)}\right)^{(1)}$. For the reverse inclusion, as $(A \times B)^{(3)_\mathcal{S}} \subseteq (A \times B)^{(3)}$, we have $B \times (A \times B)^{(3)_\mathcal{S}} \subseteq B \times (A \times B)^{(3)}$, thus $\left(B \times (A \times B)^{(3)}\right)^{(1)} \subseteq \left(B \times (A \times B)^{(3)_\mathcal{S}}\right)^{(1)}$. Therefore, $\left(B \times (A \times B)^{(3)_\mathcal{S}}\right)^{(1)} = \left(B \times (A \times B)^{(3)_\mathcal{S}}\right)^{(1)_\mathcal{S}}$. In fact, if $a \in \left(B \times (A \times B)^{(3)_\mathcal{S}}\right)^{(1)}$, then for all $(b, c) \in B \times (A \times B)^{(3)_\mathcal{S}}$, $(a, b, c) \in \mathcal{R}$. As

$(b,c) \in B\times(A\times B)^{(3)_S} \subseteq K_2\times H_3$, we have $(a,b,c)\in\mathcal{S}$, i.e., $a\in\left(B\times(A\times B)^{(3)_S}\right)^{(1)_S}$. The other inclusion is trivial. Finally, we have: $\left(B\times(A\times B)^{(3)}\right)^{(1)} \subseteq \left(B\times(A\times B)^{(3)_S}\right)^{(1)} = \left(B\times(A\times B)^{(3)_S}\right)^{(1)_S} = (B\times C)^{(1)_S} = A$. □

If objects and attributes are kept in a subcontext, then all extents and intents of the subcontext are preserved in the original context. This implies that every 3-feature of the subcontext is also a 3-feature of the original context. The following proposition examines the case where only one dimension is preserved.

Proposition 4.4. *Let $\mathbb{K}$ be a triadic context. If $\mathbb{H}$ is a subcontext with $H_1 = K_1$ then every extent of $\mathbb{H}$ is an extent of $\mathbb{K}$.*

Proof. Assume (A,B,C) is a triconcept of $\mathbb{H}$. Then $(B\times C)^{(1)} = (B\times C)^{(1)_S} = A$. Hence, the triconcept $b_{23}(B,C)$ generated by B and C in $\mathbb{K}$ has A as extent. □

We define the maps $\varphi_j : \mathfrak{T}(\mathbb{H}) \to \mathfrak{T}(\mathbb{K})$ by $\varphi_j((A_1,A_2,A_3)) = b_{ik}(A_i,A_k)$, where $\{i,j,k\}=\{1,2,3\}$ and $i<k$. The triconcept $\varphi_j(A_1,A_2,A_3)$ is characterized as the triadic concept with greatest i-th component among all triadic concepts with greatest j-th component (and simultaneously smallest k-th component) having A_i in the i-th component and A_k in the k-th component. It intuitively corresponds to an enlargement of the maximal cuboid corresponding to the triconcept (A_1,A_2,A_3) in the large context. Thus a triconcept (A,B,C) of a subcontext $\mathbb{H}$ of $\mathbb{K}$ induce three triconcepts $\varphi_i(A,B,C)$ of $\mathbb{K}$, with $i\in\{1,2,3\}$ whose restrictions on $\mathbb{H}$ are equal to (A,B,C); i.e., if $\varphi_j(A,B,C)=(X_1,X_2,X_3)$, then $X_1\cap H_1 = A$, $X_2\cap H_2 = B$ and $X_3\cap H_3 = C$. In addition, a fix point of φ_i is also a fixed point of φ_j and φ_k, for $\{i,j,k\}=\{1,2,3\}$.

However, these maps are not triorder embeddings. Indeed, let us consider the triadic context of Table 1, and the subcontext $\mathbb{H}$ with $H_1=\{1,2,3\}$, $H_2=\{a,b\}$ and $H_3=\{P,F\}$ whose triconcepts are $(1,b,F)$, $(\emptyset,ab,PF)$, $c_1 := (2,ab,P)$, $(23,b,P)$, $(123,ab,\emptyset)$, $(2,a,PF)$, $c_2 := (12,a,P)$, $(123,\emptyset,PF)$.

The images of c_1 and c_2 by φ_1 are $\varphi_1(c_1) = (25,ac,PF)$ and $\varphi_1(c_2) = (125,a,P)$. Note that $c_1 \lesssim_2 c_2$, but $\varphi_1(c_1) \not\lesssim_2 \varphi_1(c_2)$, thus φ_1 is not $\lesssim_2$-isotone. Hence, φ_1 is not a triorder embedding.

Similar to dyadic case, compatible subcontexts are defined in triadic setting. A subcontext $\mathbb{H}$ of $\mathbb{K}$ is **compatible** if $(A\cap H_1, B\cap H_2, C\cap H_3)\in\mathfrak{T}(\mathbb{H})$ for any $(A,B,C)\in\mathfrak{T}(\mathbb{K})$. Each triadic context has two trivial compatible subcontexts which are the empty context with no object, no attribute, and no condition, and the initial context itself. Also note that removing reducible and feature reducible elements in a triadic context leads to compatible subcontexts.

By $\overline{X}$ we denote the set complement of X.

Proposition 4.5. *Let $\mathbb{K}$ be a triadic context, $o\in K_1$, $a\in K_2$ and $c\in K_3$ such that $(o,a,c)\notin\mathcal{R}$. If $(o,a,c)\in\left(A\times B\times\overline{C}\right)\cup\left(A\times\overline{B}\times C\right)\cup\left(\overline{A}\times B\times C\right)$ for all triconcepts (A,B,C) of $\mathbb{K}$, then $(\{o\},\{a\},\{c\},\emptyset)$ is a compatible subcontext of $\mathbb{K}$.*

Proof. First, observe that $(\{o\},\{a\},\{c\},\emptyset)$ is indeed a subcontext of $\mathbb{K}$ since $(o,a,c)\notin\mathcal{R}$. This subcontext has exactly three triconcepts:

$$(o,a,\emptyset),\quad (o,\emptyset,c),\quad \text{and}\quad (\emptyset,a,c).$$

Now, let (A,B,C) be any triconcept of $\mathbb{K}$. The condition

$$(o,a,c)\in(A\times B\times\overline{C})\cup(A\times\overline{B}\times C)\cup(\overline{A}\times B\times C)$$

ensures that the intersection $(A\cap\{o\}, B\cap\{a\}, C\cap\{c\})$ coincides with one of these three triconcepts. This establishes the compatibility of the subcontext. □

According to Proposition 4.5, $(\{2\},\{a\},\{P\},\emptyset)$ is a compatible subcontext of the triadic context $\mathbb{K}$ in Table 4. This small artificial example may lack practical significance, but it illustrates the challenge of identifying compatible subcontexts. Also, all subcontexts for which any component is empty should be called trivial. For the moment, it is not clear if tools like arrows relations could be of any help in finding compatible subcontexts. The first step would be to investigate how compatible subcontexts are related to compatible subcontexts of the dyadic contexts $\mathbb{K}^{(i)}$. The connection to congruence and morphism should be investigated. If $\mathbb{H}$ is a compatible subcontext of $\mathbb{K}$, then the projection $\Pi_{\mathbb{H}}$ from $\mathfrak{T}(\mathbb{K})$ to $\mathfrak{T}(\mathbb{H})$, with $\Pi_{\mathbb{H}}(A,B,C) = (A\cap H_1, B\cap H_2, C\cap H_3)$ is a surjective map such that $\pi_{\mathbb{H}}(\varphi_i(x)) = x$.

The algebraic structure of trilattices is defined by the ik-joins. If $\Pi_{\mathbb{H}}$ preserves these operations, then it is a morphism. However, this is not always the case, as illustrated by the example above. Take $x=(\emptyset,ab,PF)$, $y=(1,a,P)$ and $z=(123,ab,\emptyset)$. We have $\Pi_{\mathbb{H}}(y\nabla_{12}x) = \Pi_{\mathbb{H}}(z) = (2,a,\emptyset)$ and $\Pi_{\mathbb{H}}(y)\nabla_{12}\Pi_{\mathbb{H}}(x) = (\emptyset,a,P)$.

5. Conclusion

In this paper, we proposed a refinement of graphical representations for triadic data, with a particular focus on the case satisfying the tetrahedron condition. We have discussed redundant elements and shown how to identify them. We also introduced the notion of compatible subcontexts in a triadic framework, with illustrative examples and properties. The problem of identifying reducible attributes or 2-feature reducible attributes on the graphical representation of the trillatice remains open; this is an aspect we intend to address in future work. It is also worth noting that the behavior of subcontexts in the triadic case differs significantly from the dyadic case. Consequently, developing a new approach to better understand (compatible) subcontexts is also on our to-do list.

CRediT authorship contribution statement

Léa Aubin Kouankam Djouohou: Writing – original draft, Methodology, Conceptualization. **Blaise Blériot Koguep Njionou:** Writing – review & editing, Supervision, Methodology, Conceptualization. **Leonard Kwuida:** Writing – review & editing, Supervision, Methodology, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Reports a relationship with that includes:. Has patent pending to. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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