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Three types of internal conflict and its measurement in Dempster-Shafer theory

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ABSTRACT

In this paper, we extend the ideas of measuring conflict among belief functions based on solving Kantorovich problems to the case of measuring conflict within a belief function. We consider three possible interpretations of conflict-free information and propose the functionals for measuring conflict, which can be considered as counterparts of known functionals like the auto-conflict measure, the measure of dissonance, and the measure of logical inconsistency.

1. Introduction

Belief function theory or Dempster-Shafer theory (DS-theory) [28,57] is a generalization of probability theory that allows us to represent uncertain and imprecise knowledge [58,60–63] and to choose optimal decisions under incomplete information [29,64,68]. The difference between belief function theory and probability theory is that belief functions, in contrast to probability measures, do not distinguish which elementary event occurred. In this case, we can register events, which are not singletons in general. This situation can be explained by imperfect measurements, by incomplete information, by imprecision in our beliefs, etc. One possible representation of information in DS-theory is a random set [52]. If we use probabilistic interpretation [3] of belief functions, then a belief function being a set function gives us lower bounds of probabilities, and its value on a set A gives us the probability of the event that the corresponding random set is a subset of A. However, this probabilistic interpretation is appropriate if a belief function represents the disjunctive information [35]. If a belief function represents the conjunctive information [35], then a random set does not describe the imprecision in information, and its values formally show that all elements in the set define exactly the situation. Let us describe these possible interpretations by the following example.

Example 1. There was a theft in a warehouse. The police officer after collecting pieces of evidence found that there were three suspects, which are in the set $X = \{x_1, x_2, x_3\}$. The first piece of evidence $B_1 = \{x_2\}$ certifies that x_2 committed the crime, the second one $B_2 = \{x_1, x_2\}$ certifies that the crime was committed by x_1 or x_2, and the third one $B_3 = \{x_2, x_3\}$ certifies that the crime is committed by x_2 or x_3. After combining pieces of evidences accounting their reliabilities, we can describe this information by a

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random set Ξ defined by probabilities $\Pr(\Xi = B_i) = m(B_i)$, $i = 1,2,3$, where $\sum_{i=1}^{3} m(B_i) = 1$ and $m(B_i)$ can be viewed as the reliability of B_i.

Let us discuss the difference between two possible interpretations of Ξ. If we assume that Ξ describes the disjunctive information, then only one suspect committed the crime, i.e. according to our assumption the crime is committed by x_1, x_2, or x_3, and probabilities $\Pr(\Xi \subseteq A)$ and $\Pr(\Xi \cap A \neq \emptyset)$ give us lower and upper bounds of the probability that one suspect is in A. If Ξ contains the conjunctive information, then we assume that a group of suspects is committed the crime and $\Pr(\Xi = A)$ is the probability of joint guilty of persons in A.

If you use belief functions, then several questions are raised:

1. What is the amount of imprecision within a belief function?
2. What is the amount of conflict within a belief function?

In our opinion, the above questions have solid answers for belief functions describing the disjunctive information. There are several works [1,2,6,7,16,34,36] providing a solid background measuring imprecision (non-specificity) for belief functions and more general models of imprecise probabilities. The measures of conflict like the Shannon and Rényi entropies [55,59] can be extended from probability measures to belief functions using lower bounds functionals [2,6,7,14] defined on the corresponding credal sets. For other types of conflict, namely, I2 and I3 considered in Section 5, the situation seems to be not very clear and requires further investigation [48].

In the paper, we continue to develop our approach of analyzing conflict related to belief functions started in our previous paper [21], where we study ways of measuring three types of external conflict among belief functions based on solving Kantorovich problems. This way assumes that the internal conflict within a belief function should accumulate conflicts among its focal elements. Following this approach, we will use the constructions like inclusion measures of random sets [15,20,21], distances between random sets [15,20,21], and generalize well-known functionals for measuring three types of internal conflict like the auto-conflict measure [53], the measure of dissonance [66], and the measure of logical inconsistency [27].

The evaluation of internal conflict was studied in many works (see the reviews of them, for example, in [17,48]). The following approaches can be identified for solving this problem:

1) the entropy-based approach, in which it is necessary to extend the Shannon entropy from probability measures to belief functions (the detailed discussion on this topic can be found in [43]);
2) the auto-conflict-based approach [53], in which the internal conflict is associated with the external conflict between duplicates of the analyzed belief function;
3) the focal element inconsistency-based approach [27] based on evaluating inconsistencies of focal elements within the body of evidence;
4) the decomposition-based approach [45,46,48,56], in which we assume that an information source is not homogeneous, i.e. a belief function is the result of combining several, in some sense, more simple belief functions. Therefore, for evaluating internal conflict, we need to decompose the belief function on simplest ones and to evaluate the external conflict among them.
5) the distance-based approach [14,11], in which a measure of internal conflict represents the distance between the analyzed belief function and conflict-free belief functions;
6) the axiomatic approach [6,17,43,48] that assumes the justification of some requirements or properties for measures of internal conflict.

The presented research combines many of the above approaches. The main feature of it is that we try to construct measures of internal conflict agreed with the type of analyzed information. Doing in such a way, we formalize, investigate and exemplify several generalizations of popular conflict measures mentioned above associated with 1) – 3) and 5) approaches.

The paper has the following structure. In Sections 2, 3, 4, we introduce the basic constructions of DS-theory and the inclusion measures and metrics introduced on random sets considered in the companion paper [21]. After that, in Section 5 we analyze known measures of conflict within a belief function and show how they can be generalized for the considered types of internal conflict. The paper ends with an illustrative example and the concluding remarks. We give in Appendix A some methods for verifying the specialization relation used in the proofs.

2. The basic notions and constructions from DS-theory

Let X be a finite non-empty set (sometimes called possibility space or sample space) and let 2^X be the algebra of all its subsets. Consider a random set Ξ[1] with values in 2^X. Then we consider the following set functions [28,57]:

1) the *belief function*: $Bel(A) = \Pr(\Xi \subseteq A)$, $A \in 2^X$;

[1] A random set can be viewed as a random variable whose values are sets. Thus, considering random sets, we can use usual constructions and definitions from probability theory.

2) the *plausibility function*: $Pl(A) = \Pr(\Xi \cap A \neq \emptyset)$, $A \in 2^X$;
3) the *basic probability assignment* (bpa): $m(A) = \Pr(\Xi = A)$, $A \in 2^X$.

These functions can be viewed as different representations of the same information, since Bel is connected with Pl by the *dual relation*: $Pl = Bel^d$, where $Bel^d(A) = 1 - Bel(A^c)$, $A \in 2^X$, and A^c denotes the complement of A; $Bel(A) = \sum_{B\in 2^X | B\subseteq A} m(B)$, $A \in 2^X$, and the inverse transformation: $m(B) = \sum_{A\in 2^X | A\subseteq B} (-1)^{|B\setminus A|} Bel(A)$, $B \in 2^X$, is called the *Möbius transform* [25]. A subset $B \in 2^X$ is called a *focal element* for a belief function Bel with bpa m if $m(B) > 0$.

The set of all focal elements for Bel is called the *body of evidence*. There are special classes of belief functions. Let Bel be a belief function with a body of evidence $\mathcal{F}$, then it is called:

1) *categorical* if $|\mathcal{F}| = 1$, and the categorical belief function with $\mathcal{F} = \{B\}$ is denoted by $\eta_{\langle B\rangle}$, and, obviously, $\eta_{\langle B\rangle}(A) = \begin{cases} 1, & B \subseteq A, \\ 0, & \text{otherwise;} \end{cases}$
2) *consistent* if $\bigcap_{i=1}^K B_i \neq \emptyset$, where $\mathcal{F} = \{B_1, \dots, B_K\}$;
3) *consonant* or a *necessity measure* if the focal elements in the body of evidence $\mathcal{F} = \{B_1, \dots, B_K\}$ can be enumerated such that $B_1 \subset B_2 \subset \ldots \subset B_K$;
4) *normalized* if $Bel(\emptyset) = 0$ (or, equivalently, $m(\emptyset) = 0$, where m is the bpa of Bel); otherwise, Bel is called *contradictory*, and $Con(Bel) = Bel(\emptyset)$ is the *amount of contradiction* in Bel;
5) *Bayesian* or a *probability measure* if $|B| = 1$ for every $B \in \mathcal{F}$.

Every belief function Bel is represented by a convex sum of categorical belief functions as $Bel = \sum_{B\in 2^X} m(B)\eta_{\langle B\rangle}$. For the introduced classes of belief functions, we use the following notations:

1) $M_{pr}(X)$ is the set of all normalized probability measures on 2^X;
2) $M_{nec}(X)$ is the set of all normalized necessity measures on 2^X;
3) $M_{cons}(X)$ is the set of all consistent belief functions on 2^X;
4) $M_{bel}(X)$ is the set of all normalized belief functions on 2^X.

We use the upper bar like in $\overline{M}_{bel}(X)$ to indicate that $\overline{M}_{bel}(X)$ contains all belief functions (including contradictory ones). Let $\mathcal{M} \subseteq \overline{M}_{bel}(X)$, then we denote $\mathcal{M}^d = \{\mu^d | \mu \in \mathcal{M}\}$. In such a way, $M^d_{bel}(X)$ denotes the set of all normalized plausibility functions on 2^X.

2.1. Probabilistic interpretation of belief functions

Let $Bel \in M_{bel}(X)$, then we can assume that values of Bel give us the exact lower bounds of probabilities, i.e. Bel defines the *credal set* $\mathbf{P}(Bel) = \{P \in M_{pr}(X) | P \geq Bel\}$ and $Bel(A) = \inf_{P\in\mathbf{P}(Bel)} P(A)$,[2] $A \in 2^X$. Since $P^d = P$ for $P \in M_{pr}(X)$ and $P \geq Bel$ implies $P^d \leq Bel^d$, we have $\mathbf{P}(Bel) = \{P \in M_{pr}(X) | P \leq Pl\}$, where $Pl = Bel^d$, i.e. plausibility functions give us the exact upper bounds of probabilities. Let us consider the probabilistic interpretation [5,38] of belief functions from $M_{nec}(X)$ and $M_{cons}(X)$:

1) let $Nec \in M_{nec}(X)$ and let $\mathcal{F}$ be its body of evidence, then $\mathbf{P}(Nec) = \{P \in M_{pr}(X) | \forall A \in \mathcal{F} : Nec(A) \leq P(A)\}$;
2) let $Bel \in M_{bel}(X)$, then $Bel \in M_{cons}(X)$ iff there is an $x \in X$ such that $\eta_{\langle\{x\}\rangle} \in \mathbf{P}(Bel)$.

2.2. Some notions and results from possibility theory

Consider a necessity measure $Nec \in \overline{M}_{nec}(X)$, then its dual $Pos = Nec^d$ is called a *possibility measure* [33]. Every possibility measure $Pos \in \overline{M}^d_{nec}(X)$ can be defined by its possibility distribution $\pi(x) = Pos(\{x\})$, $x \in X$, by $Pos(A) = \max_{x\in A} \pi(x)$, $A \in 2^X \setminus \{\emptyset\}$, $Pos(\emptyset) = 0$. Moreover, any $Nec \in \overline{M}_{nec}(X)$ and $Pos \in \overline{M}^d_{nec}(X)$ are characterized by:

1) $Nec(A \cap B) = \min\{Nec(A), Nec(B)\}$, $A, B \in 2^X$;
2) $Pos(A \cup B) = \max\{Pos(A), Pos(B)\}$, $A, B \in 2^X$.

Let $Nec \in \overline{M}_{nec}(X)$, then it is explicitly defined by the values on its focal elements in $\mathcal{F}$ as $Nec(A) = \max_{B\in\mathcal{F} | B\subseteq A} Nec(B)$. Therefore, necessity measures are also called *chain measures* [22].

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[2] Every $P \in M_{pr}(X)$ can be considered as a point $\left(P(\{x_1\}), \dots, P(\{x_n\})\right)$ in $\mathbb{R}^n$. In this sense, every $\mathbf{P}(Bel)$ is a closed and convex subset of $\mathbb{R}^n$. Thus, in the expression for $Bel(A)$, we can exchange inf to min.

3. Three types of external conflict

In this section, we present results from our companion paper [21], where we analyze three types of external conflict and introduce functionals for their evaluation, which will be used later in this paper. Assume that we have k sources of information described by categorical belief functions $\eta_{\langle B_i \rangle}$, $i = 1, \ldots, k$, then there are three interpretation of conflict among them. The information sources are assumed to be as non-conflicting ones iff one of the following conditions holds:

C1) $\bigcap_{i=1}^{k} B_i \neq \emptyset$;
C2) sets B_i can be indexed such that $\emptyset \subset B_1 \subseteq B_2 \subseteq \ldots \subseteq B_k$;
C3) $\emptyset \subset B_1 = B_2 = \ldots = B_k$.

These types of external conflict have been introduced in [17] and analyzed in a detailed way in our companion paper [21]. Note that the condition C3) implies C2) and the condition C2) implies C1), i.e. sources of information being non-conflicting by C3) are also non-conflicting ones by C2). In other words, C3) is the strongest type of conflict and C1) is the weakest one, and C2) is located between them.

Obviously, the C1-type of external conflict is observed, when each belief function $\eta_{\langle B_i \rangle}$ describes the disjunctive information; the conjunctive information in $\eta_{\langle B_i \rangle}$, $i = 1, \ldots, k$, implies the C3-type of external conflict. The C2-type of external conflict can describe the situation if we have information from different experts, and the inclusions of sets reflects their expertise in presented forecasts.

The conditions C1), C2), C3) can be generalized for arbitrary belief functions $Bel_i \in M_{bel}(X)$, $i = 1, \ldots, k$, represented by random sets $\Xi_1, \ldots, \Xi_k$ as follows [15,21,31]. There is no conflict among $Bel_1, \ldots, Bel_k$ iff there is a joint probability distribution of $\Xi_1, \ldots, \Xi_k$ such that it holds one of the following properties:

E1) $\Pr\left(\bigcap_{i=1}^{k} \Xi_i \neq \emptyset\right) = 1$;
E2) random sets Ξ_i can be indexed such that $\Pr\left(\Xi_1 \subseteq \Xi_2 \subseteq \ldots \subseteq \Xi_k\right) = 1$;
E3) $\Pr\left(\Xi_1 = \Xi_2 = \ldots = \Xi_k\right) = 1$.

Note that the condition E1) is equivalent to $\bigcap_{i=1}^{k} \mathbf{P}(Bel_i) \neq \emptyset$. The condition E2) can be also represented by the inclusion relation $\subseteq$ on random sets. We write $\Xi_1 \subseteq \Xi_2$ if there is a joint probability distribution of Ξ_1 and Ξ_2 such that $\Pr(\Xi_1 \subseteq \Xi_2) = 1$. It is possible to show [15,35] that the relation $\subseteq$ is transitive and $\Pr\left(\Xi_1 \subseteq \Xi_2 \subseteq \ldots \subseteq \Xi_k\right) = 1$ from E3) is equivalent to $\Xi_1 \subseteq \Xi_2 \subseteq \ldots \subseteq \Xi_k$. Assume that $\Xi_1 \subseteq \Xi_2$, then this relation on the corresponding belief functions Bel_1 and Bel_2 is denoted by $Bel_1 \sqsubseteq Bel_2$ and called the *specialization relation*. Thus, E2) is equivalent to $Bel_1 \sqsubseteq Bel_2 \sqsubseteq \ldots \sqsubseteq Bel_k$. Analogously, E3) in terms of belief functions is represented by $Bel_1 = Bel_2 = \ldots = Bel_k$.

Remark 1. The consistency conditions C1-C3 were considered in [17], but without extending them to the set of all belief functions. The idea of measuring conflict by defining it initially on categorical belief functions and extending it to general belief functions based on Kantorovich problems was firstly formulated in [15]. The consistency condition C1) was considered in [30,54], some authors [39,40] proposed to use metrics on belief functions for measuring external conflict, but without connection to conjunctive or disjunctive information.

4. Functionals for measuring external conflict

For defining them, we will use the following constructions. Assume that Ξ_1 and Ξ_2 are random sets, m_J is their *joint bpa*, i.e. the set function of two arguments defined by $m_J(B, C) = \Pr(\Xi_1 = B, \Xi_2 = C)$, where $B, C \in 2^X$, then their intersection is denoted by $\Xi_3 = \Xi_1 \underset{m_J}{\cap} \Xi_2$. Clearly, with the help of m_J, we can calculate the probability:

$$\Pr(\Xi_3 = A) = \sum_{B,C \in 2^X | B \cap C = A} m_J(B, C), \quad A \in 2^X.$$

Analogously, the union of random sets is introduced: $\Xi_4 = \Xi_1 \underset{m_J}{\cup} \Xi_2$ is a random set such that

$$\Pr(\Xi_4 = A) = \sum_{B,C \in 2^X | B \cup C = A} m_J(B, C), \quad A \in 2^X.$$

If we represent such operations by the corresponding belief functions Bel_1, ..., Bel_4, then they are called [35,61,67] the *conjunction* and the *disjunction*, respectively; and they are denoted by $Bel_3 = Bel_1 \underset{m_J}{\wedge} Bel_2$, $Bel_4 = Bel_1 \underset{m_J}{\vee} Bel_2$. We write $Bel_3 = Bel_1 \wedge Bel_2$, $Bel_4 = Bel_1 \vee Bel_2$ if the corresponding random sets are independent. The cardinality of a random set Ξ is defined as [15,36] $|\Xi| = \sum_{B \in 2^X} |B| \Pr(\Xi = B)$. $C(\Xi_1, \Xi_2)$ denotes the set of all possible joint bpas for Ξ_1 and Ξ_2.

4.1. Inclusion measures

Inclusion measures indicate whether one random set Ξ_1 includes to another Ξ_2.

1) $\tilde{\varphi}_{\subseteq}(\Xi_1,\Xi_2)=\dfrac{1}{|\Xi_1|}\max\limits_{m_J\in C(\Xi_1,\Xi_2)}\left|\Xi_1\underset{m_J}{\cap}\Xi_2\right|$ if $|\Xi_1|>0$.

This measure extends the Dice coefficient [32] $\varphi_{\subseteq}(B,C)=|B\cap C|/|B|$, where $B,C\in 2^X$ and $B\neq\emptyset$, from usual sets to random sets. $\tilde{\varphi}_{\subseteq}$ has the following properties:

a) $\tilde{\varphi}_{\subseteq}(\Xi_1,\Xi_2)\in[0,1]$;
b) $\tilde{\varphi}_{\subseteq}(\Xi_1,\Xi_2)=1$ iff $\Xi_1\subseteq\Xi_2$;
c) $\tilde{\varphi}_{\subseteq}(\Xi_1,\Xi_2)=0$ iff $B\cap C=\emptyset$ for every $B,C\in 2^X$ with $\Pr(\Xi_1=B)>0$ and $\Pr(\Xi_1=C)>0$.

2) $\tilde{\varphi}_I(\Xi_1,\Xi_2)=\min\limits_{m_J\in C(\Xi_1,\Xi_2)}\left|\Xi_1\underset{m_J}{\cup}\Xi_2\right|-|\Xi_2|$.

This measure has the following properties:

a) $0\le\tilde{\varphi}_I(\Xi_1,\Xi_2)\le|\Xi_1|$;
b) $\tilde{\varphi}_I(\Xi_1,\Xi_2)=0$ iff $\Xi_1\subseteq\Xi_2$;
c) $\tilde{\varphi}_I(\Xi_1,\Xi_2)=|\Xi_1|$ iff $B\cap C=\emptyset$ for every $B,C\in 2^X$ with $\Pr(\Xi_1=B)>0$ and $\Pr(\Xi_1=C)>0$.

3) $\tilde{\varphi}'_I(\Xi_1,\Xi_2)=1-\dfrac{|\Xi_2|}{\min\limits_{m_J\in C(\Xi_1,\Xi_2)}\left|\Xi_1\underset{m_J}{\cup}\Xi_2\right|}$ if $|\Xi_2|>0$.

This measure has the following properties:

a) $0\le\tilde{\varphi}'_I(\Xi_1,\Xi_2)\le\dfrac{|\Xi_1|}{|\Xi_1|+|\Xi_2|}$;
b) $\tilde{\varphi}'_I(\Xi_1,\Xi_2)=0$ iff $\Xi_1\subseteq\Xi_2$;
c) $\tilde{\varphi}'_I(\Xi_1,\Xi_2)=\dfrac{|\Xi_1|}{|\Xi_1|+|\Xi_2|}$ iff $B\cap C=\emptyset$ for every $B,C\in 2^X$ with $\Pr(\Xi_1=B)>0$ and $\Pr(\Xi_1=C)>0$.

4.2. Metrics on random sets

We considered two metrics from [21] based on inclusion measures. They are:

1) $\tilde{d}_s(\Xi_1,\Xi_2)=\tilde{\varphi}_I(\Xi_1,\Xi_2)+\tilde{\varphi}_I(\Xi_2,\Xi_1)=2\min\limits_{m_J\in C(\Xi_1,\Xi_2)}\left|\Xi_1\underset{m_J}{\cup}\Xi_2\right|-|\Xi_1|-|\Xi_2|$.

This is the extension of the metric $d_s(B,C)=|B\backslash C|+|C\backslash B|$, $B,C\in 2^X$, on usual sets. This metric has the following properties:

a) $0\le\tilde{d}_s(\Xi_1,\Xi_2)\le|\Xi_1|+|\Xi_2|$;
b) $\tilde{d}_s(\Xi_1,\Xi_2)=0$ iff $\Xi_1=\Xi_2$;
c) $\tilde{d}_s(\Xi_1,\Xi_2)=|\Xi_1|+|\Xi_2|$ iff $B\cap C=\emptyset$ for every $B,C\in 2^X$ with $\Pr(\Xi_1=B)>0$ and $\Pr(\Xi_1=C)>0$.

2) $\tilde{d}_{Jac}(\Xi_1,\Xi_2)=\tilde{\varphi}'_I(\Xi_1,\Xi_2)+\tilde{\varphi}'_I(\Xi_2,\Xi_1)=2-\dfrac{|\Xi_1|+|\Xi_2|}{\min\limits_{m_J\in C(\Xi_1,\Xi_2)}\left|\Xi_1\underset{m_J}{\cup}\Xi_2\right|}$ if $|\Xi_1|>0$ and $|\Xi_2|>0$.

This is the extension of the Jaccard metric [49]:

$$d_{Jac}(B,C)=(|B\backslash C|+|C\backslash B|)/|B\cup C|,\ B,C\in 2^X\backslash\{\emptyset\},$$

from usual sets to random sets.

This metric has the following properties:

a) $0\le\tilde{d}_{Jac}(\Xi_1,\Xi_2)\le 1$;
b) $\tilde{d}_{Jac}(\Xi_1,\Xi_2)=0$ iff $\Xi_1=\Xi_2$;
c) $\tilde{d}_{Jac}(\Xi_1,\Xi_2)=1$ iff $B\cap C=\emptyset$ for every $B,C\in 2^X$ with $\Pr(\Xi_1=B)>0$ and $\Pr(\Xi_1=C)>0$.

Remark 2. It is possible to extend $\tilde{d}_{Jac}$ to the set of all possible random sets with values in 2^X, assuming that

1) $\tilde{d}_{Jac}(\emptyset,\Xi)=\tilde{d}_{Jac}(\Xi,\emptyset)=1$ for every random set Ξ with $|\Xi|>0$;
2) $\tilde{d}_{Jac}(\emptyset,\emptyset)=0$.

It is easy to check that the triangle inequality is fulfilled for this extension. Actually, it is sufficient to check the following cases:

a) $\tilde{d}_{Jac}(\emptyset,\Xi_1)+\tilde{d}_{Jac}(\Xi_1,\Xi_2)-\tilde{d}_{Jac}(\emptyset,\Xi_2)=\tilde{d}_{Jac}(\Xi_1,\Xi_2)\geqslant 0$ for every Ξ_1 and Ξ_2 with $|\Xi_1|>0$ and $|\Xi_2|>0$;
b) $\tilde{d}_{Jac}(\Xi_1,\emptyset)+\tilde{d}_{Jac}(\emptyset,\Xi_2)-\tilde{d}_{Jac}(\Xi_1,\Xi_2)\geqslant\tilde{d}_{Jac}(\Xi_1,\emptyset)\geqslant 0$ for every Ξ_1 and Ξ_2 with $|\Xi_2|>0$.

Note that for this extension the boundary condition c) is not fulfilled, since $\tilde{d}_{Jac}(\emptyset,\emptyset)=0$.

4.3. Evaluation of external conflict

The evaluation of E1-type of external conflict is thoroughly studied and discussed in [8,14,24,30,47]. One and the most justified way is based on the functional:

$$Con(Bel_1, Bel_2) = \inf_{m_J \in C(Bel_1, Bel_2)} Con(Bel_1 \underset{m_J}{\wedge} Bel_2),$$

where $Bel_1, Bel_2 \in M_{bel}(X)$ and $C(Bel_1, Bel_2)$ is the set of all joint bpas of Bel_1 and Bel_2.

See also the generalization of this measure of external conflict for several sources of information, and its interpretation through generalized credal sets in [9,10,12,13,18,19].

The reader can find ways of evaluating the other two types of external conflict between two sources of information described by random sets Ξ_1 and Ξ_2 in [21] by using functionals:

E2-type: $\min\left\{1-\tilde{\varphi}_{\subseteq}(\Xi_1,\Xi_2), 1-\tilde{\varphi}_{\subseteq}(\Xi_2,\Xi_1)\right\}$, $\min\left\{\tilde{\varphi}_I(\Xi_1,\Xi_2), \tilde{\varphi}_I(\Xi_2,\Xi_1)\right\}$, $\min\left\{\tilde{\varphi}'_I(\Xi_1,\Xi_2), \tilde{\varphi}'_I(\Xi_2,\Xi_1)\right\}$;

E3-type: $\max\left\{1-\tilde{\varphi}_{\subseteq}(\Xi_1,\Xi_2), 1-\tilde{\varphi}_{\subseteq}(\Xi_2,\Xi_1)\right\}$, $\tilde{d}_s(\Xi_1,\Xi_2)$, $\tilde{d}_{Jac}(\Xi_1,\Xi_2)$.

Remark 3. Every functional defined in Subsections 4.1 and 4.2 can be calculated by solving the following Kantorovich (optimal transport) problem: $\left|\Xi_1 \underset{m_J}{\cap} \Xi_2\right| \to \max$, where the maximum is taken over all joint bpas m_J in $C(\Xi_1,\Xi_2)$. Note that every possible joint bpa m_J from $C(\Xi_1,\Xi_2)$ is defined by the following system of linear inequalities:

$$\begin{cases} \sum_{C\in 2^X} m_J(B,C) = m_1(B), B \in 2^X, \\ \sum_{B\in 2^X} m_J(B,C) = m_2(C), C \in 2^X, \\ \quad m_J(B,C) \geqslant 0, B, C \in 2^X, \end{cases}$$

where m_i is the bpa of Ξ_i, $i=1,2$. Note that

$$\min_{m_J \in C(\Xi_1,\Xi_2)} \left|\Xi_1 \underset{m_J}{\cup} \Xi_2\right| = \left|\Xi_1\right| + \left|\Xi_2\right| - \max_{m_J \in C(\Xi_1,\Xi_2)} \left|\Xi_1 \underset{m_J}{\cap} \Xi_2\right|.$$

There are several extensions of optimal transport problems for more general uncertainty theories. The reader can find them in [23, 42,50,65].

Remark 4. To preserve notations from our companion paper, set functions like $d_s : 2^X \times 2^X \to [0,+\infty)$ are denoted without tilde, and their extensions to random sets are denoted with tilde like $\tilde{d}_s$.

5. Evaluation of internal conflict within a belief function

5.1. Three types of conflict-free information

Many researchers (see the discussion on this topic in [43]) define the conflict within a belief function $Bel \in M_{bel}(X)$ as a conflict among its focal elements in its body of evidence $\mathcal{F} = \{B_1, \ldots, B_K\}$. Thus, possible interpretations of external conflict among categorical belief functions imply the following possible interpretations of conflict-free information:

I1) $\bigcap_{i=1}^{K} B_i \neq \emptyset$;
I2) the sets can be indexed in $\mathcal{F}$ such that $B_1 \subset B_2 \subset \ldots \subset B_K$;
I3) $|\mathcal{F}|=1$, i.e. Bel describes the conflict-free information iff there is a $B \in 2^X \backslash \{\emptyset\}$ such that $Bel = \eta_{\langle B\rangle}$.

As we mentioned above, belief functions satisfying I1) are called consistent, I2) are called necessity measures and I3) are called categorical.

5.2. Known functionals for measuring internal conflict

Let us notice that I1, I2 and I3-types of internal conflict become the equivalent ones $M_{pr}(X)$; and $P \in M_{pr}(X)$ describes the conflict-free information iff there is an $x \in X$ such that $P = \eta_{\langle\{x\}\rangle}$. There are many works [14,17,26,51,55,59] with suggestions of uncertainty measures on $M_{pr}(X)$ and their axiomatic, and the Shannon entropy denoted by S became very popular among researchers. Let $P \in M_{pr}(X)$ and $X = \{x_1, \ldots, x_n\}$, then

$$S(P) = -\sum_{i=1}^{n} P(\{x_i\}) \log_2 P(\{x_i\}),$$

where $0\log_2 0 = 0$ by convention. In this functional, the value $-\log_2 P(\{x_i\})$ measures the conflict between two hypotheses: the true alternative is x_i against the hypothesis that a true alternative is in the set $X\backslash\{x_i\}$. Therefore, $S(P)$ gives us the mean value of conflict, when we choose one alternative and do not consider other possibilities. Its extension on $M_{bel}(X)$ justified in the theory of imprecise probabilities is the minimal entropy functional:

$$S_{\min}(Bel) = \inf_{P\in\mathbf{P}(Bel)} S(P),\ Bel \in M_{bel}(X).$$

Since according to the I1-type of conflict-free information $\bigcap_{i=1}^{K} B_i \neq \emptyset$, where $\mathcal{F} = \{B_1, \dots, B_K\}$ is the body of evidence for Bel, and the last condition is equivalent to $\exists x \in X : \eta_{\langle\{x\}\rangle} \in \mathbf{P}(Bel)$, we see that $S_{\min}$ corresponds to the I1-type of conflict-free information. Axiomatics of $S_{\min}$ and analogous functionals providing the measuring of the I1-type of internal conflict are given in [17]. The measurement of internal conflict can be also based on:

1) the *auto-conflict measure* [53]:

$$U_{aut}(Bel) = (Bel \wedge Bel)(\emptyset) = \sum_{B\in 2^X} m(B)(1 - Pl(B));$$

2) the *measure of dissonance* [66]:

$$U_{dis}(Bel) = -\sum_{B\in 2^X} m(B)\log_2(Pl(B));$$

3) the *measure of logical inconsistency* [27]:

$$K_{pl}(Bel) = 1 - \max_{x\in X} Pl(\{x\}).$$

Note that the conflict-free information by U_{aut} and U_{dis} describes the case, when every pair of focal elements in $\mathcal{F}$ has non-empty intersection and this condition differs from the definition of the I1-type of conflict-free information. While K_{pl} satisfies the axiomatics given in [17], i.e. it can be used for measuring the I1-type of internal conflict. In addition, we see that U_{dis} is an extension of the Shannon entropy on $M_{bel}(X)$, and U_{aut}, K_{pl} are not. The auto-conflict can be generalized to the s-auto-conflict $U_{aut,s}(Bel) = \underbrace{(Bel \wedge \dots \wedge Bel)}_{s\text{ times}}(\emptyset)$ of order $s \in \{2, 3, \dots\}$ [53], and the values $1 - U_{aut,s}(Bel)$, $s = 2, 3, \dots$, are called shades of consistency in [54].

5.3. The generalizations of auto-conflict measure

After introducing some special measures of internal conflict in the previous subsection, our next goal is to analyze several possible ways of generalizing them to evaluate the internal conflict according to its possible interpretations. Let us observe that measures of conflict U_{aut}, U_{dis}, K_{pl} use the plausibility function in their construction, and we can express it through the function

$$\varphi(A, B) = \begin{cases} 1, & A \cap B \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

that describes the I1-type of conflict-free information, as

$$Pl^{(\varphi)}(A) = \sum_{B\in 2^X} m(B)\varphi(A, B),\ \ A \in 2^X. \tag{1}$$

Then the value $1 - Pl^{(\varphi)}(A) = Bel^{(\varphi)}(A^c)$, where $Bel^{(\varphi)} = (Pl^{(\varphi)})^d$, gives us the amount of external conflict between two sources of information described by $Bel^{(\varphi)}$ and $\eta_{\langle A\rangle}$. If we exchange Pl to $Pl^{(\varphi)}$ in expressions for U_{aut}, U_{dis} or K_{pl} with a certain φ like the Jaccard index $I_{Jac}(A, B) = |A \cap B|/|A \cup B|$, then we can obtain another measure of internal conflict that uses other interpretation of conflict-free information.

Remark 5. The construction of set functions by formula (1) is used in several works [41,44,47]. Functions like φ are known as *consistency indices* [30,41]. Some examples of them are: I_{Jac}, $\varphi_{\subseteq}$,

$$\varphi_b(A, B) = \begin{cases} 1, & B \subseteq A, \\ 0, & \text{otherwise}, \end{cases} \quad \varphi_q(A, B) = \begin{cases} 1, & A \subseteq B, \\ 0, & \text{otherwise}, \end{cases}$$

$$\varphi_m(A, B) = \begin{cases} 1, & A = B, \\ 0, & \text{otherwise}, \end{cases} \quad \varphi_{kr}(A, B) = |A \cap B|/|A|.$$

Note that if we take $\varphi = \varphi_b$ in (1), then Pl^{φ_b} is the plausibility function, analogously, Pl^{φ_q} is the commonality function, and Pl^{φ_m} is the basic probability assignment.

Such indices on the set $2^X\backslash\{\emptyset\}$ have the following properties:

1) $\varphi(A,B)=1$ if $A=B$;
2) $\varphi(A,B)=0$ if $A\cap B=\emptyset$.
   If we additionally require that
3) $\varphi(A,B)=\varphi(B,A)$ for all $A,B\in 2^X\backslash\{\emptyset\}$,

then φ is called a *similarity index* [44,47]. In [41], the reader can find the universal formula that captures many known measures of internal conflict, in which φ serves as a parameter.

The generalization of the auto-conflict measure is described in the following proposition.

Proposition 1. *Assume that $Bel\in M_{bel}(X)$, m denotes its the bpa, and $\mathcal{F}$ is its body of evidence. Consider the following functional on $Bel\in M_{bel}(X)$:*

$$U_{aut}^{(\psi)}(Bel)=\sum_{B\in\mathcal{F}}\sum_{A\in\mathcal{F}}m(B)m(A)\psi(A,B),$$

where the set function $\psi:2^X\times 2^X\to[0,+\infty)$ measures a certain relation on subsets in 2^X. Then:

1) *$U_{aut}^{(\psi)}(Bel)=0$ iff $A\subseteq B$ or $B\subseteq A$ for every $A,B\in\mathcal{F}$, when ψ has the following property: $\psi(A,B)=0$ iff $A\subseteq B$ or $B\subseteq A$;*
2) *$U_{aut}^{(\psi)}(Bel)=0$ iff $|\mathcal{F}|=1$, i.e. $Bel=\eta_{\langle B\rangle}$ for some $B\in 2^X\backslash\{\emptyset\}$, when ψ has the following property: $\psi(A,B)=0$ iff $A=B$.*

Let us discuss a number of possible choices of ψ in $U_{aut}^{(\psi)}$. The choice of ψ with the properties, described in the item 1) of Proposition 1 should be based on inclusion measures and they provide the desirable ψ as follows:

1) $\psi(A,B)=\min\{1-\varphi_{\subseteq}(A,B),1-\varphi_{\subseteq}(B,A)\}$, $A,B\in 2^X\backslash\{\emptyset\}$;
2) $\psi(A,B)=\min\{\varphi_I(A,B),\varphi_I(B,A)\}$, $A,B\in 2^X\backslash\{\emptyset\}$;
3) $\psi(A,B)=\min\{\varphi'_I(A,B),\varphi'_I(B,A)\}$, where $\varphi'_I(A,B)=|A\backslash B|/|A\cup B|$, $A,B\in 2^X\backslash\{\emptyset\}$;

Analogously, ψ with the properties in item 2) can be defined as follows:

4) $\psi(A,B)=\varphi_I(A,B)+\varphi_I(B,A)=d_s(A,B)$, where $A,B\in 2^X\backslash\{\emptyset\}$;
5) $\psi(A,B)=\varphi'_I(A,B)+\varphi'_I(B,A)=d_{Jac}(A,B)$, where $A,B\in 2^X\backslash\{\emptyset\}$.

5.4. The generalizations of dissonance measure

Proposition 2. *Assume that we use notations and definitions from Proposition 1. Consider the following functional defined on $M_{bel}(X)$ as*

$$U_{dis}^{(\varphi)}(Bel)=-\sum_{B\in\mathcal{F}}m(B)\log_2(Pl^{(\varphi)}(B)),$$

where $Pl^{(\varphi)}$ is defined by (1) and $\varphi:2^X\times 2^X\to[0,1]$. Then:

1) *$U_{dis}^{(\varphi)}(Bel)=0$ iff $A\subseteq B$ or $B\subseteq A$ for every $A,B\in\mathcal{F}$, when φ has the following property: $\varphi(A,B)=1$ iff $A\subseteq B$ or $B\subseteq A$;*
2) *$U_{dis}^{(\varphi)}(Bel)=0$ iff $|\mathcal{F}|=1$, i.e. $Bel=\eta_{\langle B\rangle}$ for some $B\in 2^X\backslash\{\emptyset\}$, when φ has the following property: $\varphi(A,B)=1$ iff $A=B$.*

In addition, if φ satisfies either the property in 1) or 2) and $\varphi(A,B)=0$ for every pair of disjoint subsets $A,B\in 2^X\backslash\{\emptyset\}$, then $U_{dis}^{(\varphi)}$ coincides with the Shannon entropy on $M_{pr}(X)$.

Proof. Assume that φ has the property described in 1). Then, obviously, $Pl^{(\varphi)}(B)=1$ for $B\in\mathcal{F}$ iff $A\subseteq B$ or $B\subseteq A$ for every $A\in\mathcal{F}$, i.e. $U_{dis}^{(\varphi)}(Bel)=0$ if $A\subseteq B$ or $B\subseteq A$ for every $A,B\in\mathcal{F}$. If the condition $A\subseteq B$ or $B\subseteq A$ for every $A,B\in\mathcal{F}$ is not fulfilled, then there is a $B\in\mathcal{F}$ such that $Pl^{(\varphi)}(B)<1$, i.e. $U_{dis}^{(\varphi)}(Bel)\geq -m(B)\log_2(Pl^{(\varphi)}(B))>0$.

Assume that φ has the property described in 2). Then, obviously, $Pl^{(\varphi)}(B)=1$ for $B\in\mathcal{F}$ iff $\mathcal{F}=\{B\}$, i.e. $U_{dis}^{(\varphi)}(Bel)=0$ if $\mathcal{F}=\{B\}$ for some $B\in 2^X\backslash\{\emptyset\}$. Clearly, $Pl^{(\varphi)}(B)<1$ for every $B\in\mathcal{F}$ if $|\mathcal{F}|>1$, i.e. $U_{dis}^{(\varphi)}(Bel)>0$ if $|\mathcal{F}|>1$.

It remains to show that $U_{dis}^{(\varphi)}$ coincides with the Shannon entropy on $M_{pr}(X)$ if $\varphi(A,B)=0$ for every pair of disjoint subsets $A,B\in 2^X\backslash\{\emptyset\}$. Assume that $P\in M_{pr}(X)$, then $\mathcal{F}=\{\{x_i\}|P(\{x_i\})>0,x_i\in X\}$ and

$$Pl^{(\varphi)}(\{x_i\})=\sum_{B\in 2^X}m(B)\varphi(\{x_i\},B)=P(\{x_i\});$$

$$U_{dis}^{(\varphi)}(P)=-\sum_{\{x_i\}\in\mathcal{F}}P(\{x_i\})\log_2(P(\{x_i\}))=S(P).$$

The proposition is proved in the whole. □

A number of possible choices of φ that provides $U_{dis}^{(\varphi)}(P) = S(P)$ for all $P \in M_{pr}(X)$ are:

1) $\varphi(A, B) = \max\{\varphi_{\subseteq}(A, B), \varphi_{\subseteq}(B, A)\}$, $A, B \in 2^X \backslash \{\emptyset\}$;
2) $\varphi(A, B) = I_{Jac}(A, B)$, where $A, B \in 2^X \backslash \{\emptyset\}$ and $I_{Jac}(A, B) = 1 - d_{Jac}(A, B) = |A \cap B|/|A \cup B|$ is the Jaccard index.

The choice of 1) from above implies that the item 1) is fulfilled in Proposition 2, and the choice of 2) from above implies that the item 2) is fulfilled in Proposition 2.

5.5. The generalizations of logical inconsistency measure

Let us discuss possible ways of defining K_{pl} if we use other interpretations of conflict free information. Let us observe that according to the I1-type of conflict-free information, the amount of conflict in $Bel \in M_{bel}(X)$ with the body of evidence $\mathcal{F} = \{B_1, \dots, B_K\}$ is equal to zero iff there is an $x \in X$ such that $x \in \bigcap_{i=1}^{K} B_i$. This condition is equivalent to $Con(Bel, \eta_{\langle\{x\}\rangle}) = 0$. Since $Con(Bel, \eta_{\langle\{x\}\rangle}) = 1 - Pl(\{x\})$, where $Pl = Bel^d$, we can derive that

$$K_{pl}(Bel) = \min_{x \in X} Con(Bel, \eta_{\langle\{x\}\rangle}),$$

i.e. Bel does not contain the internal conflict iff it does not have the external conflict with some $\eta_{\langle\{x\}\rangle}$, $x \in X$.

The same idea can be exploited for measuring the I3-type internal conflict as shown in the following proposition.

Proposition 3. *Assume that $Bel \in M_{bel}(X)$ with the bpa m and $\mathcal{F}$ is its body of evidence. Consider the following functional on $M_{bel}(X)$:*

$$K_{pl}^{(\psi)}(Bel) = \min_{B \in \mathcal{F}} \sum_{A \in \mathcal{F}} m(A)\psi(A, B),$$

where the set function $\psi : 2^X \times 2^X \to [0, +\infty)$ is such that $\psi(A, B) = 0$ iff $A = B$. Then $K_{pl}^{(\psi)}(Bel) = 0$ iff $|\mathcal{F}| = 1$.

Proof. Obviously, $K_{pl}^{(\psi)}(Bel) = 0$ if $Bel = \eta_{\langle B\rangle}$ for some $B \in 2^X \backslash \{\emptyset\}$. If Bel is not categorical, then $|\mathcal{F}| \geq 2$ and for every $B \in \mathcal{F}$ there is $C \in \mathcal{F}$ with $C \neq B$. We see that

$$\sum_{A \in \mathcal{F}} m(A)\psi(A, B) \geq m(C)\psi(C, B) > 0.$$

The proposition is proved. □

A number of possible choices of ψ can be:

1) $\psi(A, B) = d_s(A, B)$, where $A, B \in 2^X \backslash \{\emptyset\}$;
2) $\psi(A, B) = d_{Jac}(A, B)$, where $A, B \in 2^X \backslash \{\emptyset\}$,

If $\psi : 2^X \times 2^X \to [0, 1]$, then $K_{pl}^{(\psi)}$ can be represented as

$$K_{pl}^{(\psi)}(Bel) = 1 - \max_{B \in \mathcal{F}} Pl^{(\varphi)}(B),$$

where $\varphi(A, B) = 1 - \psi(A, B)$, $A, B \in 2^X \backslash \{\emptyset\}$. In addition, if φ satisfies the conditions from Proposition 3, i.e. $\varphi(A, B) = 1$ iff $A = B$, and $\varphi(A, B) = 0$ for every pair of disjoint subsets $A, B \in 2^X \backslash \{\emptyset\}$, then $K_{pl}^{(\psi)}(P) = K_{pl}(P)$ for every $P \in M_{pr}(X)$.

Let us discuss, the sense of constructed uncertainty measures K_{pl} and $K_{pl}^{(\psi)}$. In the case of K_{pl}:

$$K_{pl}(Bel) = \min_{x \in X} \inf_{P \in \mathbf{P}(Bel)} d_1(P, \eta_{\langle\{x\}\rangle}),$$

where d_1 is a metric on probability measures defined as

$$d_1(P_1, P_2) = 0.5 \sum_{i=1}^{n} \left| P_1(\{x_i\}) - P_2(\{x_i\}) \right|, \; P_1, P_2 \in M_{pr}(X).$$

Notice that $2d_1(P_1, P_2)$ is the total variation distance [4] between probability measures P_1 and P_2. In the special case considered in the formula for K_{pl}, we have $d_1(P, \eta_{\langle\{x\}\rangle}) = 1 - P(\{x\})$.

Analogously, $K_{pl}^{(\psi)}$ can be viewed as the minimal distance between $Bel \in M_{bel}(X)$ and categorical belief functions $\eta_{\langle B\rangle}$, $B \in \mathcal{F}$:

1) $K_{pl}^{(\psi)}(Bel) = \min_{B\in\mathcal{F}} \tilde{d}_s(B,\Xi)$ if ψ is chosen as in item 1) given after Proposition 3 and Ξ is the random set that describes Bel;
2) $K_{pl}^{(\psi)}(Bel) = \min_{B\in\mathcal{F}} \tilde{d}_{Jac}(B,\Xi)$ if ψ is chosen as in item 2) after Proposition 3 and Ξ is the random set that describes Bel.

The above considerations allow us to propose the following idea of how K_{pl} can be constructed according to different interpretations of internal conflict. The values of such measures should reflect the distance between a given information described by $Bel \in M_{bel}(X)$, and possible representations of conflict-free information. This idea can be used for constructing analogs of K_{pl} if we consider the I2-type of internal conflict. We know that the conflict-free information that corresponds to the I2-type of internal conflict is represented by necessity measures. A $\mu \in M_{nec}(X)$ is called an *upper approximation* of $Bel \in M_{bel}(X)$ in the set $M_{nec}(X)$ if $Bel \sqsubseteq \mu$. Let $\tilde{\psi}$ be a metric on $M_{bel}(X)$, then the following functional gives us the distance between a given information described by $Bel \in M_{bel}(X)$ and its possible upper approximations in the set $M_{nec}(X)$:

$$\overline{K}_{pl}^{(\tilde{\psi})}(Bel) = \inf\{\tilde{\psi}(Bel,\mu) | \mu \in M_{nec}^{\sqsupseteq Bel}(X)\}, \tag{2}$$

where $M_{nec}^{\sqsupseteq Bel}(X) = \{\mu \in M_{nec}(X) | \mu \sqsupseteq Bel\}$. Next results allow us to propose how $\overline{K}_{pl}^{(\tilde{\psi})}$ can be computed in some special cases.

Lemma 1. *Let $Bel \in M_{bel}(X)$, $\mu \in M_{nec}(X)$ and $\mathcal{F}_\mu = \{C_1,\dots,C_N\}$, where $\emptyset \subset C_1 \subset \dots \subset C_N$, is the set of all focal elements for μ, then $Bel \sqsubseteq \mu$ iff $\mu(C_j) \le Bel(C_j)$, $j=1,\dots,N$.*

Proof. It is well known that $Bel \sqsubseteq \mu$ implies $\mu \le Bel$, i.e. it is sufficient to prove that $\mu(C_j) \le Bel(C_j)$, $j=1,\dots,N$, implies that $Bel \sqsubseteq \mu$. Assume that m_1 is the bpa of Bel, m_2 is the bpa of μ, and $\mathcal{F}_{Bel}$ is the set of all focal elements for Bel. Since $\mu(C_N) = Bel(C_N) = 1$ by our assumption, we can conclude that every $B \in \mathcal{F}_{Bel}$ is a subset of C_N. Consider the belief function μ' whose focal elements are in the set $\mathcal{F}_\mu$ with the bpa

$$m'(C_j) = \sum_{B\in\mathcal{F}_{Bel} | B\subseteq C_j \wedge B\not\subseteq C_{j-1}} m_1(B),\ j=1,\dots,N,$$

where $C_0 = \emptyset$ by convention. Obviously, $\mu' \in M_{nec}(X)$, and according to Method 3 from Appendix $Bel \sqsubseteq \mu'$, and $\mu'(C_j) = Bel(C_j)$, $j=1,\dots,N$. Since $\mu'(C_j) \ge \mu(C_j)$, $j=1,\dots,N$, we can consider $\mu^{(1)} \in M_{nec}(X)$ with the bpa

$$m^{(1)}(C_1) = m_2(C_1),\ m^{(1)}(C_2) = m_2(C_2) + m'(C_1) - m_2(C_1),$$
$$m^{(1)}(C_j) = m'(C_j),\ j=3,\dots,N.$$

Obviously, $\mu' \sqsubseteq \mu^{(1)}$ by Method 2 from Appendix and $\mu^{(1)}(C_j) \ge \mu(C_j)$, $j=1,\dots,N$. After that we can construct a $\mu^{(2)} \in M_{nec}(X)$, whose bpa is defined as

$$m^{(2)}(C_1) = m_2(C_1),\ m^{(2)}(C_2) = m_2(C_2),$$
$$m^{(2)}(C_3) = m^{(1)}(C_3) + m^{(1)}(C_2) - m_2(C_2),$$
$$m^{(2)}(C_j) = m^{(1)}(C_j),\ j=4,\dots,N,$$

and, obviously, $\mu^{(1)} \sqsubseteq \mu^{(2)}$ and $\mu^{(1)}(C_j) \ge \mu(C_j)$, $j=1,\dots,N$. Using the same constructions, we can generate the sequence $\{\mu^{(i)}\}_{i=1}^N \subseteq M_{nec}(X)$ such that

$$Bel \sqsubseteq \mu' \sqsubseteq \mu^{(1)} \sqsubseteq \mu^{(2)} \sqsubseteq \dots \sqsubseteq \mu^{(N)} = \mu.$$

Since the relation $\sqsubseteq$ is transitive, we derive that $Bel \sqsubseteq \mu$. The lemma is proved. $\square$

Remark 6. Lemma 1 establishes the fact that $Bel \sqsubseteq \mu$ for $Bel \in M_{bel}(X)$ and $\mu \in M_{nec}(X)$ iff $\mu \le Bel$. It is well known [15,35] that $Bel_1 \sqsubseteq Bel_2$ for $Bel_1, Bel_2 \in M_{bel}(X)$ implies $Bel_1 \ge Bel_2$, but the inverse is not fulfilled for this general case, i.e. $Bel_1 \ge Bel_2$ for $Bel_1, Bel_2 \in M_{bel}(X)$ does not imply $Bel_1 \sqsubseteq Bel_2$. In terms of credal sets, the order $\ge$ on $M_{bel}(X)$ means the following: $Bel_1 \ge Bel_2$ for $Bel_1, Bel_2 \in M_{bel}(X)$ iff $\mathbf{P}(Bel_1) \subseteq \mathbf{P}(Bel_2)$. In the paper [35], such an order $\ge$ on belief functions is called the *weak inclusion*, and the order $\sqsubseteq$ is called the *strong inclusion*. We see that by Lemma 1, $M_{nec}^{\sqsupseteq Bel}(X) = \{\mu \in M_{nec}(X) | \mu \le Bel\}$.

The following proposition describes the structure of the set $M_{nec}^{\sqsupseteq Bel}(X)$.

Proposition 4. *The set $M_{nec}^{\sqsupseteq Bel}(X)$, $Bel \in M_{bel}(X)$, has a finite number of minimal elements, and every minimal element μ in $M_{nec}^{\sqsupseteq Bel}(X)$ is represented as*

$$\mu(A) = \max\{Bel(C_i) | C_i \subseteq A, i=0,\dots,n\},\ A \in 2^X, \tag{3}$$

where $\{C_i\}_{i=0}^n$ is a complete chain of subsets in 2^X, i.e. there is a permutation $(i_1,\dots,i_n)$ of $\{1,\dots,n\}$ such that

$$C_0 = \emptyset,\ C_1 = \{x_{i_1}\},\ C_2 = \{x_{i_1}, x_{i_2}\}, \dots, C_n = \{x_{i_1}, \dots, x_{i_n}\} = X.$$

In addition, for every $\nu \in M_{nec}^{\sqsupseteq Bel}(X)$ *there is a minimal element* $\mu \in M_{nec}^{\sqsupseteq Bel}(X)$ *defined by (3) such that* $\mu \sqsubseteq \nu$.

Proof. The expression (3) is the definition of a chain measure, and we know [22] that every chain measure is a necessity measure, i.e. $\mu \in M_{nec}(X)$. In addition, since $\mu \le Bel$, according to Lemma 1, we conclude that $\mu \in M_{nec}^{\sqsupseteq Bel}(X)$. To prove the proposition, it is sufficient to show that for any $\nu \in M_{nec}^{\sqsupseteq Bel}(X)$, we can construct a μ defined by (3) such that $\mu \sqsubseteq \nu$. Consider an arbitrary $\nu \in M_{nec}^{\sqsupseteq Bel}(X)$. Since it is also a chain measure, there is a complete chain of sets $\{C_i\}_{i=0}^n \subseteq 2^X$, such that

$$\nu(A) = \max\{\nu(C_i) | C_i \subseteq A, i = 0, \dots, n\},\ A \in 2^X,$$

where $\nu(C_i) \le Bel(C_i)$, $i = 0, \dots, n$. If we take the same complete chain of sets $\{C_i\}_{i=0}^n \subseteq 2^X$ and construct the chain measure μ by (3), then we have $\mu \ge \nu$, i.e. $\mu \sqsubseteq \nu$. The proposition is proved. □

The following proposition allows us to simplify the optimization problem of finding $K_{pl}^{(\psi)}(Bel)$, where $Bel \in M_{bel}(X)$, in some cases.

Proposition 5. *Assume that* $\tilde{\psi} : M_{bel}(X) \times M_{bel}(X) \to [0, +\infty)$ *has the following property:* $\tilde{\psi}(Bel_1, Bel_2) \le \tilde{\psi}(Bel_1, Bel_3)$ *for every* $Bel_1, Bel_2, Bel_3 \in M_{bel}(X)$ *with* $Bel_1 \sqsubseteq Bel_2 \sqsubseteq Bel_3$. *Then*

$$\overline{K}_{pl}^{(\tilde{\psi})}(Bel) = \min\{\tilde{\psi}(Bel, \mu) | \mu \in \min\{M_{nec}^{\sqsupseteq Bel}(X)\}\},\ Bel \in M_{bel}(X),$$

where $\min\{M_{nec}^{\sqsupseteq Bel}(X)\}$ *is the set of minimal elements in* $M_{nec}^{\sqsupseteq Bel}(X)$ *w.r.t.* $\sqsubseteq$.

Proof. According to Proposition 4, for every $\mu \in M_{nec}^{\sqsupseteq Bel}(X)$ there is a minimal element $\nu \in M_{nec}^{\sqsupseteq Bel}(X)$ such that $\nu \sqsubseteq \mu$, i.e. $Bel \sqsubseteq \nu \sqsubseteq \mu$, and the property given in the proposition implies that $\tilde{\psi}(Bel, \nu) \le \tilde{\psi}(Bel, \mu)$, i.e. in the expression (2) for $\overline{K}_{pl}^{(\tilde{\psi})}(Bel)$ the infimum can be taken over minimal elements of $M_{nec}^{\sqsupseteq Bel}(X)$. The proposition is proved. □

Let us check for which metrics $\tilde{\psi}$ on $M_{bel}(X)$ the conditions of Proposition 5 are satisfied.

1) Assume that $\tilde{\psi} = \tilde{d}_s$, i.e. $\tilde{d}_s(Bel_1, Bel_2) = \tilde{d}_s(\Xi_1, \Xi_2)$, where random sets Ξ_1, Ξ_2 describe the corresponding belief functions. We see that

$$\tilde{d}_s(\Xi_1, \Xi_2) = 2 \min_{m_J \in C(\Xi_1, \Xi_2)} \left| \Xi_1 \underset{m_J}{\cup} \Xi_2 \right| - |\Xi_1| - |\Xi_2| = |\Xi_2| - |\Xi_1| \text{ if } \Xi_1 \subseteq \Xi_2,$$

i.e. if $\Xi_1 \subseteq \Xi_2 \subseteq \Xi_3$, then

$$\tilde{d}_s(\Xi_1, \Xi_3) - \tilde{d}_s(\Xi_1, \Xi_2) = |\Xi_3| - |\Xi_1| - |\Xi_2| + |\Xi_1| = \tilde{d}_s(\Xi_2, \Xi_3) \ge 0,$$

i.e. $\tilde{d}_s$ satisfies the required property.

2) Assume that $\tilde{\psi} = \tilde{d}_{Jac}$, i.e. $\tilde{d}_{Jac}(Bel_1, Bel_2) = \tilde{d}_{Jac}(\Xi_1, \Xi_2)$, where random sets Ξ_1, Ξ_2 describe the corresponding belief functions. According to the definition,

$$\tilde{d}_{Jac}(\Xi_1, \Xi_2) = 2 - \frac{|\Xi_1| + |\Xi_2|}{\min\limits_{m_J \in C(\Xi_1, \Xi_2)} \left| \Xi_1 \underset{m_J}{\cup} \Xi_2 \right|}.$$

Assume that $\Xi_1 \subseteq \Xi_2$, then $\tilde{d}_{Jac}(\Xi_1, \Xi_2) = 1 - |\Xi_1| \big/ |\Xi_2|$ and if $\Xi_1 \subseteq \Xi_2 \subseteq \Xi_3$, then

$$\tilde{d}_{Jac}(\Xi_1, \Xi_3) - \tilde{d}_{Jac}(\Xi_1, \Xi_2) = \frac{|\Xi_1|}{|\Xi_2|} - \frac{|\Xi_1|}{|\Xi_3|} = \frac{|\Xi_1| \left(|\Xi_3| - |\Xi_2| \right)}{|\Xi_2| \, |\Xi_3|} \ge 0,$$

i.e. $\tilde{d}_{Jac}$ satisfies the required property.

Formally, we have shown that for both cases of $\tilde{\psi}$, when $\tilde{\psi} = \tilde{d}_s$ and $\tilde{\psi} = \tilde{d}_{Jac}$, for computing $\overline{K}_{pl}^{(\tilde{\psi})}(Bel)$, $Bel \in M_{bel}(X)$, it is sufficient to find $\mu \in M_{nec}^{\sqsupseteq Bel}(X)$, whose corresponding random set Ξ_μ has the minimal cardinality, and compute

1) $\overline{K}_{pl}^{(\tilde{\psi})}(Bel) = \left|\Xi_\mu\right| - |\Xi_{Bel}|$ for $\tilde{\psi} = \tilde{d}_s$;
2) $\overline{K}_{pl}^{(\tilde{\psi})}(Bel) = 1 - |\Xi_{Bel}| \big/ \left|\Xi_\mu\right|$ for $\tilde{\psi} = \tilde{d}_{Jac}$;

where the random set Ξ_{Bel} describes the belief function Bel.

Proposition 6. *Assume that $Bel \in M_{bel}(X)$ and $\mu \in M_{nec}^{\sqsupseteq Bel}(X)$ is a necessity measure with the minimal $\left|\Xi_\mu\right|$, then*

$$\left|\Xi_\mu\right| = \min_{(C_0,C_1,\dots,C_n)} \sum_{i=1}^{n} \left(Bel(C_i) - Bel(C_{i-1})\right) \left|C_i\right|, \tag{4}$$

where the minimum is taken over all possible complete chains in 2^X.

Proof. According to the Proposition 5, the minimum of $\left|\Xi_\mu\right|$ is achieved on chain measures, and every chain measure μ constructed by a complete chain $(C_0, C_1, \dots, C_n)$ has the bpa $m(C_i) = Bel(C_i) - Bel(C_{i-1})$, $i = 1, \dots, n$. This implies the formula (4). □

Corollary 1. *Assume that $Bel \in M_{bel}(X)$ with the bpa m and $\mathcal{F} = \{B_1, ..., B_K\}$ is its body of evidence. Let $\mu \in M_{nec}^{\sqsupseteq Bel}(X)$ be a possibility measure with the minimal $\left|\Xi_\mu\right|$, then*

$$\left|\Xi_\mu\right| = \min_{\gamma} \sum_{i=1}^{K} m(B_{\gamma(i)}) \left|\bigcup_{j=1}^{i} B_{\gamma(j)}\right|, \tag{5}$$

where the minimum is taken over all permutations $(\gamma(1), \dots, \gamma(K))$ of $\{1, \dots, K\}$.

Proof. Let us show that $\left|\Xi_\mu\right| = \sum_{i=1}^{K} m(B_{\gamma(i)}) \left|\bigcup_{j=1}^{i} B_{\gamma(j)}\right|$ for some chain measure $\mu \in M_{nec}^{\sqsupseteq Bel}(X)$. Let us consider the chain of sets: $D_0 = \emptyset$, $D_1 = B_{\gamma(1)}, \dots, D_i = \bigcup_{j=1}^{i} B_{\gamma(j)}, \dots, D_K = \bigcup_{j=1}^{K} B_{\gamma(j)}$ and consider a chain measure

$$\nu(A) = \max\{Bel(D_i) | D_i \subseteq A, i = 0, \dots, K\},\ \ A \in 2^X. \tag{6}$$

Clearly, $\nu \in M_{nec}^{\sqsupseteq Bel}(X)$, however, since the chain $(D_0, \dots, D_K)$ is not necessarily complete, it is not necessary that such a μ is a minimal element of $M_{nec}^{\sqsupseteq Bel}(X)$ w.r.t. $\sqsubseteq$. Let us show that a $\mu \in M_{nec}^{\sqsupseteq Bel}(X)$ with the minimal value $\left|\Xi_\mu\right|$ can be constructed using (6). According to Lemma 1, such a μ can be constructed by a complete chain $(C_0, C_1, \dots, C_n)$ as

$$\mu(A) = \max\{Bel(C_i) | C_i \subseteq A, i = 0, \dots, n\}.$$

Let us enumerate focal elements in $\mathcal{F} = \{B_{\gamma(1)}, ..., B_{\gamma(K)}\}$ such that for every $k \in \{1, \dots, n\}$, there is an index i_k such that

$$\mathcal{B}_k = \{B_{\gamma(i)} | B_{\gamma(i)} \subseteq C_k\} = \left\{B_{\gamma(i)}\right\}_{i=1}^{i_k}.$$

Then, obviously, a chain measure ν is constructed by a chain $(D_0, \dots, D_K)$ from (6) has the following property: $\nu(C_i) = Bel(C_i)$, $i = 1, \dots, n$. This implies $\nu \geq \mu$ or equivalently $\nu \sqsubseteq \mu$. Since μ is the minimal element of $M_{nec}^{\sqsupseteq Bel}(X)$ w.r.t. $\sqsubseteq$, we have the only possibility $\nu = \mu$, and for computing $\left|\Xi_\mu\right|$, we can consider only chain measures defined by (6), i.e. (5) is valid and the corollary is proved in the whole. □

Remark 7. We see that Proposition 6 and Corollary 1 give us two possible ways for computing $\inf\left\{\left|\Xi_\mu\right| | \mu \in M_{nec}^{\sqsupseteq Bel}(X)\right\}$. The optimal choice is obviously depends on $|X|$ and $|\mathcal{F}|$, i.e. if $|X| < |\mathcal{F}|$, then (4) provides less computations than (5), and (5) looks better otherwise. The problem of finding $\mu \in M_{nec}^{\sqsupseteq Bel}(X)$ with the minimal $\left|\Xi_\mu\right|$ was investigated in [37], where authors present an algorithm for finding an approximate solution. They also conclude that this problem is equivalent to a one-machine scheduling problem in operations research.

In the next example, we will show that both functionals $\overline{K}_{pl}^{(\tilde{d}_s)}$ and $\overline{K}_{pl}^{(\tilde{d}_{Jac})}$ do not coincide with K_{pl} on $M_{pr}(X)$.

Example 2. Assume that $P \in M_{pr}(X)$ and elements of $X = \{x_1, ..., x_n\}$ are indexed such that $P(\{x_1\}) \geq P(\{x_2\}) \geq ... \geq P(\{x_n\})$, then $\mu \in M_{nec}^{\sqsupseteq P}(X)$ with the minimal $\left|\Xi_\mu\right|$ has the bpa $m_\mu(\{x_1\}) = P(\{x_1\})$, $m_\mu(\{x_1, x_2\}) = P(\{x_2\}), \dots$, $m_\mu(\{x_1, \dots, x_n\}) = P(\{x_n\})$ and $\left|\Xi_\mu\right| = \sum_{i=1}^{n} \left|\{x_1, ..., x_i\}\right| P(\{x_i\}) = \sum_{i=1}^{n} i P(\{x_i\})$; $\left|\Xi_P\right| = \sum_{i=1}^{n} P(\{x_i\}) = 1$.

Let us calculate the values $\overline{K}_{pl}^{(\tilde{d}_s)}(P)$, $\overline{K}_{pl}^{(\tilde{d}_{Jac})}(P)$ and $K_{pl}(P)$ for $P \in M_{pr}(X)$ with $P(\{x_i\}) = 1/n$, $i = 1, \dots, n$. Then $K_{pl}(P) = (n-1)/n$, $\left|\Xi_\mu\right| = \frac{1}{n} \sum_{i=1}^{n} i = (n+1)/2$, i.e. $\overline{K}_{pl}^{(\tilde{d}_s)}(P) = (n-1)/2$ and $\overline{K}_{pl}^{(\tilde{d}_{Jac})}(P) = (n-1)/(n+1)$.

We see from Example 2, that the introduced functionals based on upper approximations of belief functions do not give us the proper extensions of K_{pl} from $M_{pr}(X)$ to $M_{bel}(X)$, since these functionals on $M_{bel}(X)$ do not coincide with K_{pl} on $M_{pr}(X)$. Therefore, we will investigate another possibility using lower approximations of belief functions by necessity measures. For this purpose, introduce the set of all consonant belief functions (necessity measures) $\overline{M}_{nec}(X)$[3] on 2^X, whose focal elements are linear ordered w.r.t. $\subseteq$ and their bodies of evidence may contain empty focal elements. Then the next functional measures the I2-type of internal conflict in $Bel \in M_{bel}(X)$:

$$\underline{K}_{pl}^{(\tilde{\psi})}(Bel) = \inf\{\tilde{\psi}(Bel,\mu)|\mu \in \overline{M}_{nec}^{\sqsubseteq Bel}(X)\}, \tag{7}$$

where $\overline{M}_{nec}^{\sqsubseteq Bel}(X) = \{\mu \in \overline{M}_{nec}(X)|\mu \sqsubseteq Bel\}$ and $\tilde{\psi}$ is a metric on $\overline{M}_{bel}(X)$.

For calculating the infimum in (7), we will establish first some properties of the set $\overline{M}_{nec}^{\sqsubseteq Bel}(X)$.

Proposition 7. *Assume that $Bel \in M_{bel}(X)$ with the bpa m and $\mathcal{F} = \{B_1, ..., B_K\}$ is its body of evidence. Then every maximal element in $\overline{M}_{nec}^{\sqsubseteq Bel}(X)$ w.r.t. $\sqsubseteq$ is in the set of necessity measures whose elements μ in $\overline{M}_{nec}^{\sqsubseteq Bel}(X)$ are defined by*

1) $C_0 = \emptyset$, $C_1 = \{x_{i_1}\}$, $C_2 = \{x_{i_1}, x_{i_2}\}, \dots$, $C_n = \{x_{i_1}, \dots, x_{i_n}\} = X$, *is a complete chain in* 2^X;
2) $\mu = \sum_{i=1}^{n} m_\mu(C_i)\eta_{\langle C_i\rangle}$ *with* $m_\mu(C_i) = \sum_{j|C_i\subseteq B_j \wedge C_{i+1}\nsubseteq B_j} m(B_j)$, $i = 0, \dots, n-1$, $m_\mu(X) = 1 - \sum_{i=0}^{n-1} m_\mu(C_i)$.

Proof. Obviously, every such a μ defined by 1) and 2) is in $\overline{M}_{nec}^{\sqsubseteq Bel}(X)$, since it is possible to define the joint bpa m_J of μ and Bel as

$$m_J(C_i, B_j) = \begin{cases} m(B_j), & i = \max\{k|C_k \subseteq B_j\}, \\ 0, & \text{otherwise}, \end{cases}$$

and $m_J(C_i, B_j) = 0$ if $C_i \nsubseteq B_j$ (see Method 1 in Appendix).

Let us show that every maximal element ν in $\overline{M}_{nec}^{\sqsubseteq Bel}(X)$ w.r.t. $\sqsubseteq$ can be constructed using 1) and 2). Assume that ν is the maximal element in $\overline{M}_{nec}^{\sqsubseteq Bel}(X)$ w.r.t. $\sqsubseteq$, then for some complete chain in 2^X defined like in 1), we have the following representation:

$$\nu = \sum_{i=1}^{n} m_\nu(C_i)\eta_{\langle C_i\rangle}.$$

Let us show that $\nu \sqsubseteq \mu$ if μ is constructed like in 2). Since $\nu \sqsubseteq Bel$, there is a joint bpa m_J of ν and Bel such that $m_J(C_i, B_j) = 0$ if $C_i \nsubseteq B_j$. Let us define the joint bpa m'_J of ν and μ as

$$m'_J(C_i, C_k) = \sum_{j|C_i\subseteq B_j \wedge C_k\nsubseteq B_j} m_J(C_i, B_j).$$

Since $\sum_{k=0}^{n} m'_J(C_i, C_k) = m_\nu(C_i)$, $i = 0, \dots, n$, and

$$\sum_{i=0}^{n} m'_J(C_i, C_k) = \sum_{i=0}^{n} \sum_{j|C_i\subseteq B_j \wedge C_k\nsubseteq B_j} m_J(C_i, B_j) =$$

$$\sum_{j|C_i\subseteq B_j \wedge C_k\nsubseteq B_j} \sum_{i=0}^{n} m_J(C_i, B_j) =$$

$$\sum_{j|C_k\subseteq B_j \wedge C_{k+1}\nsubseteq B_j} m(B_j) = m_\mu(C_k),\ k = 0, \dots, n-1,$$

this joint bpa is defined correctly and clearly, $m'_J(C_i, C_k) = 0$ if $C_i \nsubseteq C_k$, i.e. $\nu \sqsubseteq \mu$ and ν is the maximal element in $\overline{M}_{nec}^{\sqsubseteq Bel}(X)$ w.r.t. $\sqsubseteq$ iff $\nu = \mu$. The proposition is proved. □

Remark 8. The proof of Proposition 7 also shows that for every $\nu \in \overline{M}_{nec}^{\sqsubseteq Bel}(X)$ there is a maximal element μ in $\overline{M}_{nec}^{\sqsubseteq Bel}(X)$ such that $\nu \sqsubseteq \mu$, i.e.

$$\overline{M}_{nec}^{\sqsubseteq Bel}(X) = \left\{\nu \in \overline{M}_{nec}(X)|\exists\mu \in \max\{\overline{M}_{nec}^{\sqsubseteq Bel}(X)\} : \nu \sqsubseteq \mu\right\},$$

where $\max\{\overline{M}_{nec}^{\sqsubseteq Bel}(X)\}$ is the set of all maximal elements in $\overline{M}_{nec}^{\sqsubseteq Bel}(X)$.

[3] We introduce the set $\overline{M}_{nec}(X)$, since the set $M_{nec}^{\sqsubseteq Bel}(X) = \{\mu \in M_{nec}(X)|\mu \sqsubseteq Bel\}$ may be empty for many $Bel \in M_{bel}(X)$ (see the definition of $\underline{K}_{pl}^{(\tilde{\psi})}(Bel)$ given below).

The next proposition describes possible $\tilde{\psi}$, for which the infimum in (7) is achieved on maximal elements in $\overline{M}_{nec}^{\sqsubseteq Bel}(X)$ w.r.t. $\sqsubseteq$.

Proposition 8. *Assume that $\tilde{\psi}$ has the following property: $\tilde{\psi}(Bel_1, Bel_3) \leqslant \tilde{\psi}(Bel_2, Bel_3)$ for every $Bel_1, Bel_2 \in \overline{M}_{bel}(X)$ and $Bel_3 \in M_{bel}(X)$ with $Bel_1 \sqsubseteq Bel_2 \sqsubseteq Bel_3$. Then*

$$\underline{K}_{pl}^{(\tilde{\psi})}(Bel) = \min\{\tilde{\psi}(Bel, \mu) | \mu \in \max\{\overline{M}_{nec}^{\sqsubseteq Bel}(X)\}\}, \ \ \mu \in M_{bel}(X),$$

where $\max\{\overline{M}_{nec}^{\sqsubseteq Bel}(X)\}$ *is the set of all maximal elements in* $\overline{M}_{nec}^{\sqsubseteq Bel}(X)$ *w.r.t.* $\sqsubseteq$.

Proof. According to Proposition 7 (see also Remark 8), for every $\nu \in \overline{M}_{nec}^{\sqsubseteq Bel}(X)$ there is a maximal element $\mu \in \overline{M}_{nec}^{\sqsubseteq Bel}(X)$ such that $\nu \sqsubseteq \mu$, i.e. $\nu \sqsubseteq \mu \sqsubseteq Bel$, and the property of $\tilde{\psi}$ given in the proposition implies that $\tilde{\psi}(\nu, Bel) \geqslant \tilde{\psi}(\mu, Bel)$, i.e. we can compute $\underline{K}_{pl}^{(\tilde{\psi})}(Bel)$ taken the infimum over all maximal elements in $M_{nec}^{\sqsupseteq Bel}(X)$. The proposition is proved. $\square$

Let us check for which metrics $\tilde{\psi}$ on $\overline{M}_{bel}(X)$ the conditions of Proposition 8 are satisfied.

1) Assume that $\tilde{\psi} = \tilde{d}_s$, i.e. $\tilde{d}_s(Bel_1, Bel_2) = \tilde{d}_s(\Xi_1, \Xi_2)$, where random sets Ξ_1, Ξ_2 describe the corresponding belief functions. We see that

$$\tilde{d}_s(\Xi_1, \Xi_2) = 2 \min_{m_J \in C(\Xi_1, \Xi_2)} |\Xi_1 \cup \Xi_2| - |\Xi_1| - |\Xi_2| = |\Xi_2| - |\Xi_1| \ \text{ if } \Xi_1 \subseteq \Xi_2.$$

Therefore, for $\Xi_1 \subseteq \Xi_2 \subseteq \Xi_3$, we have

$$\tilde{d}_s(\Xi_1, \Xi_3) - \tilde{d}_s(\Xi_2, \Xi_3) = |\Xi_3| - |\Xi_1| - |\Xi_3| + |\Xi_2| = \tilde{d}_s(\Xi_2, \Xi_1) \geqslant 0,$$

i.e. $\tilde{d}_s$ satisfies the required property.

2) Assume that $\tilde{\psi} = \tilde{d}_{Jac}$, i.e. $\tilde{d}_{Jac}(Bel_1, Bel_2) = \tilde{d}_{Jac}(\Xi_1, \Xi_2)$, where random sets Ξ_1, Ξ_2 describe the corresponding belief functions. According to the definition (see also Remark 2),

$$\tilde{d}_{Jac}(\Xi_1, \Xi_2) = 2 - \frac{|\Xi_1| + |\Xi_2|}{\min\limits_{m_J \in C(\Xi_1, \Xi_2)} |\Xi_1 \cup \Xi_2|} \ \text{ if either } |\Xi_1| > 0 \text{ or } |\Xi_2| > 0.$$

Assume that $\Xi_1 \subseteq \Xi_2$, then $\tilde{d}_{Jac}(\Xi_1, \Xi_2) = 1 - |\Xi_1| / |\Xi_2|$, when $|\Xi_2| > 0$; and for $\Xi_1 \subseteq \Xi_2 \subseteq \Xi_3$ with $|\Xi_3| > 0$, we have

$$\tilde{d}_{Jac}(\Xi_1, \Xi_3) - \tilde{d}_{Jac}(\Xi_2, \Xi_3) = \frac{|\Xi_2|}{|\Xi_3|} - \frac{|\Xi_1|}{|\Xi_3|} = \frac{|\Xi_2| - |\Xi_1|}{|\Xi_3|} \geqslant 0,$$

i.e. $\tilde{d}_{Jac}$ satisfies the required property.

Formally, we have shown that for both cases of $\tilde{\psi}$, when $\tilde{\psi} = \tilde{d}_s$ and $\tilde{\psi} = \tilde{d}_{Jac}$, for computing $\underline{K}_{pl}^{(\tilde{\psi})}(Bel)$, $Bel \in M_{bel}(X)$, it is necessary to find $\mu \in \overline{M}_{nec}^{\sqsubseteq Bel}$, whose corresponding random set Ξ_μ has the maximal cardinality, and compute

1) $\underline{K}_{pl}^{(\tilde{\psi})}(Bel) = |\Xi_{Bel}| - |\Xi_\mu|$ for $\tilde{\psi} = \tilde{d}_s$;
2) $\underline{K}_{pl}^{(\tilde{\psi})}(Bel) = 1 - |\Xi_\mu| / |\Xi_{Bel}|$ for $\tilde{\psi} = \tilde{d}_{Jac}$;

where the random set Ξ_{Bel} describes the belief function Bel.

The search of $\mu \in \overline{M}_{nec}^{\sqsubseteq Bel}(X)$ with the maximal cardinality $|\Xi_\mu|$ can be produced using the following proposition.

Proposition 9. *Assume that $Bel \in M_{bel}(X)$ with the bpa m and $\mathcal{F} = \{B_1, ..., B_K\}$ is its body of evidence. Let $\nu \in \overline{M}_{nec}^{\sqsubseteq Bel}(X)$ be a necessity measure with the maximal $|\Xi_\nu|$, then*

$$|\Xi_\nu| = \max_\gamma \sum_{i=1}^{K} m(B_{\gamma(i)}) \left| \bigcap_{j=1}^{i} B_{\gamma(j)} \right|,$$

where the maximum is taken over all permutations $(\gamma(1), \dots, \gamma(K))$ of $\{1, \dots, K\}$.

Proof. Clearly, every

$$\nu = \sum_{i=1}^{K} m_\nu(C_i) \eta_{\langle C_i \rangle}, \tag{8}$$

where $C_i = \bigcap_{j=1}^{i} B_{\gamma(j)}$ and $m_\nu(C_i) = m(B_{\gamma(i)})$, $i = 1, \dots, K$, is in $\overline{M}_{nec}^{\sqsubseteq Bel}(X)$, since $\nu \sqsubseteq Bel$. This can be shown by Method 1 from Appendix using the joint bpa:

$$m_J(C_i, B_{\gamma(j)}) = \begin{cases} m(B_{\gamma(i)}), & i = j, \\ 0, & \text{otherwise}. \end{cases}$$

Hence, according to the proposition we can compute $\sup\left\{ |\Xi_\nu| \,|\, \nu \in \overline{M}_{nec}^{\sqsubseteq Bel}(X) \right\}$ taken the maximum over necessity measures defined by (8). Consider an arbitrary maximal element μ in $\overline{M}_{nec}^{\sqsubseteq Bel}(X)$ w.r.t. $\sqsubseteq$, then by Proposition 7, it has the representation obeying 1) and 2). Consider the permutation γ of $\{1, \dots, K\}$ chosen such that

$$\max\{k | C_k \subseteq B_{j_1}\} \le \max\{k | C_k \subseteq B_{j_2}\} \text{ implies } \gamma(j_1) \ge \gamma(j_2)$$

for every $j_1, j_2 \in \{1, \dots, K\}$. Then a ν defined by (8) has

$$|\Xi_\nu| = \sum_{i=1}^{K} m(B_{\gamma(i)}) \left| \bigcap_{j=1}^{i} B_{\gamma(j)} \right|.$$

Since $\bigcap_{j=1}^{i} B_{\gamma(j)} \supseteq C_k$, where $k = \max\{l | C_l \subseteq B_{\gamma(i)}\}$, we can conclude that

$$|\Xi_\nu| \ge \sum_{k=0}^{n} \sum_{i|k=\max\{l|C_l \subseteq B_{\gamma(i)}\}} m(B_{\gamma(i)}) |C_k| = |\Xi_\mu|,$$

i.e. the maximal value $|\Xi_\nu|$, where $\nu \in \overline{M}_{nec}^{\sqsubseteq Bel}(X)$, can be searched in the set of necessity measures ν defined by (8). The proposition is proved. □

Example 3. Assume that $P \in M_{pr}(X)$, where $X = \{x_1, \dots, x_n\}$, then $\underline{K}_{pl}^{(\tilde{\psi})}(P) = K_{pl}(P) = 1 - \max_{x_i \in X} P(\{x_i\})$ if $\tilde{\psi} = \tilde{d}_s$ and $\tilde{\psi} = \tilde{d}_{Jac}$.

Proof. It is easy to show that maximal elements in $\overline{M}_{nec}^{\sqsubseteq P}(X)$ w.r.t. $\sqsubseteq$ are

$$\nu_i = P(\{x_i\}\eta_{\langle\{x_i\}\rangle} + (1 - P(\{x_i\})\eta_{\langle\emptyset\rangle} \text{ for every } x_i \in X \text{ with } P(\{x_i\}) > 0,$$

and $\left|\Xi_{\nu_i}\right| = P(\{x_i\})$, $i = 1, \dots, n$, $\left|\Xi_P\right| = 1$, therefore,

$$\underline{K}_{pl}^{(\tilde{d}_s)}(P) = \min_{i=1,\dots,n} \tilde{d}_s(\nu_i, P) = \min_{i=1,\dots,n} \left(|P| - |\nu_i|\right) = 1 - \max_{x_i \in X} P(\{x_i\}),$$

$$\underline{K}_{pl}^{(\tilde{d}_{Jac})}(P) = \min_{i=1,\dots,n} \tilde{d}_{Jac}(\nu_i, P) = \min_{i=1,\dots,n} \left(1 - |\nu_i| / |P|\right) = 1 - \max_{x_i \in X} P(\{x_i\}). \quad \square$$

Notice that according to Example 3, $\underline{K}_{pl}^{(\tilde{d}_s)}$ and $\underline{K}_{pl}^{(\tilde{d}_{Jac})}$ coincide on the set of probability measures $M_{pr}(X)$ and they are identical to K_{pl}, i.e. $\underline{K}_{pl}^{(\tilde{d}_s)}$ and $\underline{K}_{pl}^{(\tilde{d}_{Jac})}$ are proper extensions of K_{pl}. Note (cf. Remark 7) that Proposition 7 and Proposition 9 give us two possible ways of computing $\left|\Xi_\mu\right| = \left\{ |\Xi_\nu| \,|\, \nu \in \overline{M}_{nec}^{\sqsubseteq Bel}(X) \right\}$. The first way is to generate all necessity measures defined by complete chains like in Proposition 7, and compute the cardinalities of corresponding random sets. Proposition 9 gives us another formula that uses the permutations of a body of evidence. The choice between these two ways depends on the number of elements in X and $\mathcal{F}$ like we indicated in Remark 7. The following result [37] allows us to perceive the difference between two orders $\ge$ and $\sqsubseteq$ on $\overline{M}_{bel}(X)$.

Proposition 10. *Let $Bel \in \overline{M}_{bel}(X)$, $\overline{M}_{nec}^{\ge Bel}(X) = \{\nu \in \overline{M}_{nec}(X) | \nu \ge Bel\}$, then $\overline{M}_{nec}^{\ge Bel}(X)$ has an unique minimal element ν_0 w.r.t. $\ge$, whose possibility distribution function is defined by $\nu_0^d(\{x\}) = Bel^d(\{x\})$, $x \in X$.*

The sense of Proposition 10 is in the following. Since $\nu \sqsubseteq Bel$ implies $\nu \ge Bel$ for $\nu, Bel \in \overline{M}_{bel}(X)$, we can conclude that $\overline{M}_{nec}^{\ge Bel}(X) \supseteq \overline{M}_{nec}^{\sqsubseteq Bel}(X)$, and $\overline{M}_{nec}^{\ge Bel}(X) = \overline{M}_{nec}^{\sqsubseteq Bel}(X)$ iff ν_0 defined in Proposition 10 belongs to $\overline{M}_{nec}^{\sqsubseteq Bel}(X)$. In this special case, ν_0 is the single maximal element in $\overline{M}_{nec}^{\sqsubseteq Bel}(X)$ w.r.t. $\sqsubseteq$ and if $\tilde{\psi}$ satisfies the conditions from Proposition 5, then $\underline{K}_{pl}^{(\tilde{\psi})}(Bel) = \tilde{\psi}(Bel, \nu_0)$. Unfortunately, the measure ν_0 is useless for evaluating $\underline{K}_{pl}^{(\tilde{d}_s)}(Bel)$ and $\underline{K}_{pl}^{(\tilde{d}_{Jac})}(Bel)$, when $\nu_0 \notin \overline{M}_{nec}^{\sqsubseteq Bel}(X)$, as shown in the next examples.

Example 4. Assume that $P \in M_{pr}(X)$, $X = \{x_1, \ldots, x_n\}$ and $P(\{x_i\}) = 1/n$, $i = 1, \ldots, n$. Then $v_0 = \frac{n-1}{n}\eta_{\langle\emptyset\rangle} + \frac{1}{n}\eta_{\langle X\rangle}$,

$$\tilde{d}_s(v_0, P) = 2 \min_{m_J \in C(v_0,P)} \left|\Xi_{v_0} \underset{m_J}{\cup} \Xi_P\right| - \left|\Xi_{v_0}\right| - \left|\Xi_P\right| =$$

$$2\left(1 + \frac{n-1}{n}\right) - 1 - 1 = \frac{2n-2}{n},$$

$$\tilde{d}_{Jac}(v_0, P) = 2 - \frac{\left|v_0\right| + |P|}{\min\limits_{m_J \in C(v_0,P)} \left|v_0 \underset{m_J}{\cup} P\right|} = 2 - \frac{2n}{2n-1} = \frac{2n-2}{2n-1}$$

and since $\underline{K}_{pl}^{(\tilde{d}_s)}(P) = \underline{K}_{pl}^{(\tilde{d}_{Jac})}(P) = (n-1)/n$, we see that $\underline{K}_{pl}^{(\tilde{d}_s)}(P) < \tilde{d}_s(v_0, P)$ and $\underline{K}_{pl}^{(\tilde{d}_{Jac})}(P) < \tilde{d}_{Jac}(v_0, P)$.

Example 5. Assume that $X = \{x_1, x_2, x_3\}$ and $Bel = 0.4\eta_{\langle\{x_1,x_2\}\rangle} + 0.6\eta_{\langle\{x_2,x_3\}\rangle}$. Then the maximal elements of $\overline{M}_{nec}^{\sqsubseteq Bel}(X)$ w.r.t. $\sqsubseteq$ are $v_1 = 0.4\eta_{\langle\{x_2\}\rangle} + 0.6\eta_{\langle\{x_2,x_3\}\rangle}$ and $v_2 = 0.4\eta_{\langle\{x_1,x_2\}\rangle} + 0.6\eta_{\langle\{x_2\}\rangle}$,

$$\tilde{d}_s(v_1, Bel) = \left|\Xi_{Bel}\right| - \left|\Xi_{v_1}\right| = 2 - 1.6 = 0.4,$$

$$\tilde{d}_s(v_2, Bel) = \left|\Xi_{Bel}\right| - \left|\Xi_{v_2}\right| = 2 - 1.4 = 0.6.$$

Therefore, $\underline{K}_{pl}^{(\tilde{d}_s)}(Bel) = 0.4$. In this case, the necessity measure v_0 from Proposition 9 has the following possibility distribution: $v_0^d(\{x_1\}) = Bel^d(\{x_1\}) = 0.4$, $v_0^d(\{x_2\}) = Bel^d(\{x_2\}) = 1$, $v_0^d(\{x_3\}) = Bel^d(\{x_3\}) = 0.6$. Therefore, $v_0 = 0.4\eta_{\langle\{x_2\}\rangle} + 0.2\eta_{\langle\{x_2,x_3\}\rangle} + 0.4\eta_{\langle X\rangle}$. Let us compute the value $\left|\Xi_\mu\right| = \min\limits_{m_J \in C(v_0,Bel)} \left|\Xi_{v_0} \underset{m_J}{\cup} \Xi_{Bel}\right|$ needed for $\tilde{d}_s(v_0, Bel)$ calculation. This value can be found as a solution of the following linear programming problem, where $A = \{x_2\}$, $B = \{x_1, x_2\}$, $C = \{x_2, x_3\}$ are used as notations:

$$\begin{aligned}\left|\Xi_\mu\right| = {} & m_J(A,B)\,|B| + m_J(C,B)\,|X| + m_J(X,B)\,|X| + \\ & m_J(A,C)\,|C| + m_J(C,C)\,|C| + m_J(X,C)\,|X| \to \min\end{aligned}$$

$$\begin{cases} m_J(A,B) + m_J(A,C) = m_{v_0}(A), \\ m_J(C,B) + m_J(C,C) = m_{v_0}(C), \\ m_J(X,B) + m_J(X,C) = m_{v_0}(X), \\ m_J(A,B) + m_J(C,B) + m_J(X,B) = m_{Bel}(B), \\ m_J(A,C) + m_J(C,C) + m_J(X,C) = m_{Bel}(C), \\ m_J(D,F) \geq 0 \text{ for all } \; D, F \in 2^X. \end{cases}$$

The solution of this optimization problem is

$$m_J(A,B) = 0.4, \; m_J(C,C) = 0.2, \; m_J(X,C) = 0.4,$$

$$m_J(C,B) = m_J(X,B) = m_J(A,C) = 0,$$

and $\left|\Xi_\mu\right| = 2.4$. Therefore,

$$\tilde{d}_s(v_0, Bel) = 2 \min_{m_J \in C(v_0,Bel)} \left|\Xi_{v_0} \underset{m_J}{\cup} \Xi_{Bel}\right| - \left|\Xi_{v_0}\right| - \left|\Xi_{Bel}\right| = 2 \cdot 2.4 - 2 - 2 = 0.8.$$

Analogously, we can compute $\underline{K}_{pl}^{(\tilde{d}_{Jac})}(Bel)$. In this case,

$$\tilde{d}_{Jac}(v_1, Bel) = 1 - \left|\Xi_{v_1}\right| \Big/ \left|\Xi_{Bel}\right| = 1 - 1.6/2 = 0.2,$$

$$\tilde{d}_{Jac}(v_2, Bel) = 1 - \left|\Xi_{v_2}\right| \Big/ \left|\Xi_{Bel}\right| = 1 - 1.4/2 = 0.3.$$

Therefore, $\underline{K}_{pl}^{(\tilde{d}_{Jac})}(Bel) = 0.2$.

$$\tilde{d}_{Jac}(v_0, Bel) = 2 - \frac{\left|v_0\right| + |Bel|}{\min\limits_{m_J \in C(v_0,Bel)} \left|v_0 \underset{m_J}{\cup} Bel\right|} = 2 - (2+2)/2.4 = 1/3.$$

We see that also $\underline{K}_{pl}^{(\tilde{d}_{Jac})}(Bel) < \tilde{d}_{Jac}(v_0, Bel)$.

6. An illustrative example

Assume that we have three main parties in the parliament described by the set $X = \{x_1, x_2, x_3\}$ and possible coalitions of these parties are described by a belief function $Bel = \sum_{B\in 2^X} m(B)\eta_{\langle B\rangle}$ in $M_{bel}(X)$. The value $m(B)$ shows the relative frequency of support for accepted legislative projects by the non-empty coalition $B \in 2^X$.

Assume that $K_{pl}(Bel) = 1 - \max_{x\in X} Pl(\{x\}) = 0$, i.e. Bel describes I1-type of conflict-free information. For our example, it means that there is a party $x \in X$ with $Pl(\{x\}) = 1$, in other words, there is a party $x \in X$ with the total support for all accepted legislative projects. Assume that

$$Bel = 0.6\eta_{\langle\{x_1,x_3\}\rangle} + 0.4\eta_{\langle\{x_2,x_3\}\rangle},$$

then such a Bel can be obtained, when parties x_1 and x_2 have together the majority of votes in the parliament, for example, 45% and 35%, which always vote oppositely to each other, and the acceptance of a legislative project depends on the votes of the third party with 20% of votes. Let us calculate $U^{(\psi)}_{aut}(Bel)$ for different ψ defined earlier for measuring the I2-type of internal conflict. Consider the following possible choices of ψ:

1) if $\psi(A,B) = \min\{1 - \varphi_{\subseteq}(A,B), 1 - \varphi_{\subseteq}(B,A)\}$, then

$$\varphi_{\subseteq}(\{x_1,x_3\},\{x_2,x_3\}) = \varphi_{\subseteq}(\{x_2,x_3\},\{x_1,x_3\}) = 0.5,$$
$$\psi(\{x_1,x_3\},\{x_2,x_3\}) = \psi(\{x_2,x_3\},\{x_1,x_3\}) = 0.5,$$
$$U^{(\psi)}_{aut}(Bel) = m(B)m(A)\left(\psi(A,B) + \psi(B,A)\right)\big|_{A=\{x_1,x_3\},B=\{x_2,x_3\}} = 0.24.$$

2) if $\psi(A,B) = \min\{\varphi_I(A,B), \varphi_I(B,A)\}$, then

$$\varphi_I(\{x_1,x_3\},\{x_2,x_3\}) = \varphi_I(\{x_2,x_3\},\{x_1,x_3\}) = 1,$$
$$U^{(\psi)}_{aut}(Bel) = m(B)m(A)\left(\psi(A,B) + \psi(B,A)\right)\big|_{A=\{x_1,x_3\},B=\{x_2,x_3\}} = 0.48.$$

3) if $\psi(A,B) = \min\{\varphi'_I(A,B), \varphi'_I(B,A)\}$, then

$$\varphi'_I(\{x_1,x_3\},\{x_2,x_3\}) = \varphi'_I(\{x_2,x_3\},\{x_1,x_3\}) = 1/3,$$
$$U^{(\psi)}_{aut}(Bel) = m(B)m(A)\left(\psi(A,B) + \psi(B,A)\right)\big|_{A=\{x_1,x_3\},B=\{x_2,x_3\}} = 0.16.$$

Let us compute $U^{(\varphi)}_{dis}(Bel)$, when $\varphi(A,B) = \max\{\varphi_{\subseteq}(A,B), \varphi_{\subseteq}(B,A)\}$. In this case,

$$Pl^{(\varphi)}(A)\Big|_{A=\{x_1,x_3\}} = m(A)\varphi(A,A) + m(B)\varphi(A,B)\big|_{A=\{x_1,x_3\},B=\{x_2,x_3\}} = 0.6 + 0.4\cdot 0.5 = 0.8,$$
$$Pl^{(\varphi)}(B)\Big|_{B=\{x_2,x_3\}} = m(B)\varphi(B,B) + m(A)\varphi(B,A)\big|_{A=\{x_1,x_3\},B=\{x_2,x_3\}} = 0.4 + 0.6\cdot 0.5 = 0.7,$$
$$U^{(\varphi)}_{dis}(Bel) = -m(A)\log_2(Pl^{(\varphi)}(A))\Big|_{A=\{x_1,x_3\}} - m(B)\log_2(Pl^{(\varphi)}(B))\Big|_{B=\{x_2,x_3\}} = -0.6\log_2 0.8 - 0.4\log_2 0.7 = 0.398\ldots$$

We see that in our example of Bel, parties x_1 and x_2 looks like conflicting ones and only the support of the third party allows passing a legislation through parliament. This is the reason why we can conclude that we need to consider I2 or I3-types of internal conflict in this case.

Consider another belief function defined by

$$Bel = 0.4\eta_{\langle\{x_1,x_2\}\rangle} + 0.6\eta_{\langle X\rangle}.$$

We see that $Bel \in M_{nec}(X)$, i.e. Bel presents the I2-type of conflict-free information. If we assume the above quantitative representation of parties in parliament, then we can suppose that each legislation is accepted by two first parties, and the third party has no influence on passing the legislation through the parliament. Let us calculate the introduced functionals for evaluating the I3-type of conflict within Bel. Let us calculate $U^{(\psi)}_{aut}(Bel)$:

1) if $\psi = d_s$, then $d_s(\{x_1,x_2\},X) = 1$,

$$U^{(d_s)}_{aut} = 2m(A)m(B)d_s(A,B)\big|_{A=\{x_1,x_2\},B=X} = 2\cdot 0.4\cdot 0.6 = 0.48;$$

2) if $\psi = d_{Jac}$, then $d_{Jac}(\{x_1, x_2\}, X) = 1/3$,

$$U_{aut}^{(d_{Jac})} = 2m(A)m(B)d_{Jac}(A,B)\big|_{A=\{x_1,x_2\},B=X} = 0.16.$$

Let us calculate $U_{dis}^{(\varphi)}(Bel)$ if $\varphi = I_{Jac}$. In this case

$$Pl^{(\varphi)}(A)\Big|_{A=\{x_1,x_2\}} = m(A)\varphi(A,A) + m(B)\varphi(A,B)|_{A=\{x_1,x_2\},B=X} = 0.4 + 0.6 \cdot (2/3) = 0.8,$$

$$Pl^{(\varphi)}(B)\Big|_{B=X} = m(B)\varphi(B,B) + m(A)\varphi(B,A)|_{A=\{x_1,x_3\},B=\{x_2,x_3\}} = 0.6 + 0.4 \cdot (2/3) = 13/15,$$

$$U_{dis}^{(\varphi)}(Bel) = -m(A)\log_2(Pl^{(\varphi)}(A))\Big|_{A=\{x_1,x_2\}} - m(B)\log_2(Pl^{(\varphi)}(B))\Big|_{B=X} = -0.4\log_2 0.8 - 0.6\log_2(13/15) = 0.252\ldots$$

Let us calculate $K_{pl}^{(\psi)}$ for evaluating the I3-type of internal conflict:

1) if $\psi = d_s$, then

$$K_{pl}^{(\psi)}(Bel) = d_s(A,B)\min\{m(A), m(B)\}\big|_{A=\{x_1,x_2\},B=X} = 0.4;$$

2) if $\psi = d_{Jac}$, then

$$K_{pl}^{(\psi)}(Bel) = d_{Jac}(A,B)\min\{m(A), m(B)\}\big|_{A=\{x_1,x_2\},B=X} = 0.4/3 = 0.133\ldots$$

In our opinion, there is no large difference between the above consonant belief function Bel and $\eta_{\langle\{x_1,x_2\}\rangle}$ according to which each legislation was passed through the parliament due to the support of the first two parties, and none of legislations was supported by the third party. Since $\eta_{\langle\{x_1,x_2\}\rangle}$ presents the I3-type of conflict-free information, we can conclude that the evaluation of the I2-type of internal conflict is more appropriate in this case.

7. Conclusion

In the paper, we establish the novel approach to evaluating conflict within belief functions. Its application consists in two steps:

1) finding the description of conflict-free information given by belief functions;
2) the construction of functionals for measuring the internal conflict consistent with the functionals for measuring the external conflict.

The description of conflict-free information should reflect the external conflict among focal elements, and the measures of internal conflict for belief functions should be extensions of known measures of conflict for probability measures. The application of this approach for measuring I2 and I3-types of internal conflict is shown by defining counterparts of known measures of conflict such as the auto-conflict measure [53], the measure of dissonance [66], and the measure of logical inconsistency [27].

Our next goal in future research is to show how the introduced measures of external and internal conflict can be used in the analysis of information presented by belief functions, and in the conflict management for combining several information sources.

CRediT authorship contribution statement

Andrey G. Bronevich: Writing – original draft, Methodology, Investigation, Funding acquisition, Conceptualization. **Alexander E. Lepskiy:** Writing – review & editing, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Methods for verifying the specialization relation

There are several methods for checking the inclusion relation on random sets (or the specialization relation on belief functions):

Method 1. Let Ξ_1 and Ξ_2 be random sets with values in 2^X and their joint probability distribution is defined by the joint bpa $m_J(B,C) = \Pr(\Xi_1 = B, C = \Xi_2)$, $B, C \in 2^X$. Then $\Pr(\Xi_1 \subseteq \Xi_2) = \sum_{B,C\in 2^X | B\subseteq C} m_J(B,C)$. Thus, $\Xi_1 \subseteq \Xi_2$ iff there is a joint bpa m_J of Ξ_1 and Ξ_2 such that $m_J(B,C) = 0$ if $B \not\subseteq C$.

Method 2. $Bel_1 \sqsubseteq Bel_2$ for $Bel_1, Bel_2 \in \overline{M}_{bel}(X)$ iff there are representations: $Bel_1 = \sum_{i=1}^N a_i \eta_{\langle B_i\rangle}$, $Bel_2 = \sum_{i=1}^N a_i \eta_{\langle C_i\rangle}$, where $a_i \geqslant 0$, $i = 1, ..., N$, and $\sum_{i=1}^N a_i = 1$, such that $B_i \subseteq C_i$, $i = 1, ..., N$.

Proof. *Necessity.* Assume that $Bel_1 \sqsubseteq Bel_2$, then there is a joint bpa m_J such that $m_J(B,C) = 0$ if $B \not\subseteq C$. Let $\{(B,C) | m_J(B,C) > 0\} = \{(B_i, C_i)\}_{i=1}^N$, then $Bel_1 = \sum_{i=1}^N a_i \eta_{\langle B_i\rangle}$ and $Bel_2 = \sum_{i=1}^N a_i \eta_{\langle C_i\rangle}$, where $a_i = m_J(B_i, C_i)$, $i = 1, ..., N$.

Sufficiency. Assume that there are representations: $Bel_1 = \sum_{i=1}^N a_i \eta_{\langle B_i\rangle}$, $Bel_2 = \sum_{i=1}^N a_i \eta_{\langle C_i\rangle}$, where $a_i \geqslant 0$, $i = 1, ..., N$, and $\sum_{i=1}^N a_i = 1$, such that $B_i \subseteq C_i$, $i = 1, ..., N$. Then we define the joint bpa m_J of $Bel_1, Bel_2 \in \overline{M}_{bel}(X)$ as $m_J(B,C) = \sum_{i=1}^N a_i I(B_i = B) I(C_i = C)$, where

$$I(B_i = B) = \begin{cases} 1, & B_i = B, \\ 0, & \text{otherwise}, \end{cases} \quad I(C_i = C) = \begin{cases} 1, & C_i = C, \\ 0, & \text{otherwise}. \end{cases}$$

We see that $m_J(B,C) = 0$ for every $B, C \in 2^X$ with $B \not\subseteq C$, i.e. $Bel_1 \sqsubseteq Bel_2$. $\square$

Method 3. Assume that $Bel_1, Bel_2 \in \overline{M}_{bel}(X)$ are given by bpas m_1 and m_2 defined on the corresponding bodies of evidence $\mathcal{F}_1$ and $\mathcal{F}_2$, and the mapping $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ is such that $m_2(C) = \sum_{B\in\varphi^{-1}(C)} m_1(B)$, where $\varphi^{-1}(C) = \{B \in \mathcal{F}_1 | \varphi(B) = C\}$ and $C \in \mathcal{F}_2$, then

1) $Bel_1 \sqsubseteq Bel_2$ if $\varphi(B) \supseteq B$ for all $B \in \mathcal{F}_1$;
2) $Bel_2 \sqsubseteq Bel_1$ if $\varphi(B) \subseteq B$ for all $B \in \mathcal{F}_1$.

Proof. Obviously, $Bel_1 = \sum_{B\in\mathcal{F}_1} m_1(B)\eta_{\langle B\rangle}$ and $Bel_2 = \sum_{B\in\mathcal{F}_1} m_1(B)\eta_{\langle\varphi(B)\rangle}$, therefore, Method 3 follows from Method 2. $\square$

Data availability

No data was used for the research described in the article.

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