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The properties of 3-valued formal contexts in a cognitive viewpoint

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ABSTRACT

3-valued formal contexts are abstracted from various types of applications such as incomplete formal context based data mining, shadow sets based knowledge discovery and conflict analysis. 3-valued formal contexts differ from binary-valued formal contexts in many aspects, and many distinguished details have not been investigated. To this end, some of the most important properties of 3-valued formal contexts are systematically explored in a cognitive viewpoint based on formal concept analysis. At first, 3-valued concept lattices and formal concept lattices are compared from multiple perspectives, including the connections between formal concepts and 3-valued concepts, and the meet-preserving mappings from formal concept lattices to 3-valued concept lattices. After that, based on the completions of 3-valued contexts, the connections between 3-valued concept lattices and three-way concept lattices are explored. Finally, it is proved that a 3-valued concept lattice is the minimum closure that contains formal concept lattices, and there is an order-preserving mapping from formal concepts to equivalence classes of 3-valued concepts.

1. Introduction

Information loss is inevitable in the process of data collection. Exploring incomplete data from a concept cognitive perspective is a feasible method to improve the interpretability of data analysis. Formal concept analysis [5,27], as a conceptual modeling theory, has become an effective tool for uncertainty analysis. As an extension of the original setup of formal concept analysis, *i.e.*, formal context, incomplete formal context is proposed to deal with uncertain situations [11,44]. Burmeister and Holzer [2] defined incomplete formal context and proposed modal operators, thus constructing a type of partially known concepts. Obiedkov et al. [17] defined a type of 3-valued modal logic based on the completion of incomplete formal contexts. Ren et al. [23] revealed the structural properties, similarities, and differences of three types of partially-known formal concepts of incomplete formal contexts. Zhi et al. [44] investigated granule description with undetermined values in a three-way epistemic perspective.

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Formal concept analysis has a distinguished ability for information processing in knowledge-based systems, such as data mining [12,25,26], social network analysis [34], conflict analysis [14,33,42,43], concept cognitive learning [30,31] and so forth. Besides, many theoretical results have been obtained, such as the construction of concept lattices [15,45], attribute reduction [3,29], and granule description [13,44]. The classical theory mainly concerns the positive attributes of objects. However, studies have repeatedly shown that negative attributes play equally important roles in data analysis as positive attributes [1,28]. To this end, *three-way concept analysis* [18,19] was proposed by extending formal concepts with three-way decision [35,36,40]. At present, three-way concept analysis has obtained a lot of developments, such as the construction methods [21,37,39], knowledge discovery [12,22], and three-way network concept analysis [32].

An incomplete information system can be treated as a 3-valued information system on most occasions. For instance, conflict analysis adopts a 3-valued information system to describe the attitudes of participants in conflict situations, and divides the relationships among participants into three types, i.e., alliance, neutrality, and conflict [38]. By this way, Sun et al. [24] proposed a conflict analysis method based on probabilistic rough set model under the framework of three-way decision. Zhi et al. [41] established a hierarchical framework with approximate three-way concepts for conflict resolution. Lang et al. [9,10] took a decision theoretic view to classify participants into different groups in conflict situations.

3-valued concept analysis is a concept analysis method oriented for 3-valued information systems, by summarizing the commonalities of several important applications, including formal concept analysis with undetermined values, shadow set theory and three-way conflict analysis [20]. 3-valued formal context and 3-valued concept lattice are the fundamentals of 3-valued concept analysis. As 3-valued concept lattice can neatly deal with uncertain information, it has become an effective tool for uncertainty analysis.

3-valued formal context differs from binary-valued formal context in many aspects, and many intrinsic properties have not been discussed. Thus, it is necessary to explore 3-valued concept analysis in multiple perspectives, especially by comparing it with the existing model to discover its uniqueness. Specifically, 3-valued concept lattices can be investigated both in a concept viewpoint and a structural viewpoint when discussing their connection with the existing models. Besides, it is also important to answer whether a 3-valued concept lattice can be built based on the existing models, and whether there exist order-preserving mappings from concept lattices to 3-valued concept lattices.

Section 2 provides some basic notions in three-way concept analysis and 3-valued concept analysis. Section 3 explores the properties of 3-valued concept lattices from a concept viewpoint, by discussing the connections between 3-valued concept lattices and fundamental concept lattices. Section 4 investigates the connections between 3-valued concepts and three-way concepts. Section 5 discusses the structural properties of 3-valued concept lattices. Finally, the contributions are summarized in Section 6.

2. Preliminaries

The basic notions of formal concept analysis can be found in [27]. A formal context (U, V, I) is composed of two finite and nonempty sets U and V, and a binary-valued function $I : U \times V \to \{0, 1\}$. In applications, U usually represents the set of objects, and V usually denotes the set of attributes.

Let (U, V, I) be a formal context, $O \in 2^U$ and $A \in 2^V$. Two associated concept-forming operators $*: 2^U \to 2^V$ and $\bullet : 2^V \to 2^U$ are defined as:

$$O^* = \{v \in V \mid \forall u \in O, I(u, v) = 1\} \text{ and } A^\bullet = \{u \in U \mid \forall v \in A, I(u, v) = 1\}.$$

Moreover, if $O^* = A$ and $A^\bullet = O$, then (O, A) is called a formal concept of K, and O and A are called the extent and the intent of (O, A).

In addition, all the formal concepts of (U, V, I) form a lattice structure, i.e., formal concept lattice $\mathrm{L}(U, V, I)$, whose hierarchical structure is determined by

$$(O_1, A_1) \wedge (O_2, A_2) = (O_1 \cap O_2, (A_1 \cup A_2)^{\bullet *}),$$
$$(O_1, A_1) \vee (O_2, A_2) = ((O_1 \cup O_2)^{*\bullet}, A_1 \cap A_2).$$

In three-way concept analysis, '$*$' and '$\bullet$' are known as the positive operators, while '$\overline{*}$' and '$\overline{\bullet}$' are defined as the negative operators.

Definition 1. ([18,19]) Let (U, V, I) be a formal context, $O \in 2^U$, and $A \in 2^V$. Negative operators $\overline{*} : 2^U \to 2^V$ and $\overline{\bullet} : 2^V \to 2^U$ are defined as:

$$O^{\overline{*}} = \{v \in V \mid \forall u \in O, I(u, v) = 0\} \text{ and } A^{\overline{\bullet}} = \{u \in U \mid \forall v \in A, I(u, v) = 0\}.$$

If $O^{\overline{*}} = A$ and $A^{\overline{\bullet}} = O$, then (O, A) is called a negative concept.

Definition 2 *([18,19])*. Let (U, V, I) be a formal context, $O \in 2^U$, and $(A, B) \in 2^V \times 2^V$. Two associated three-way operators $\rhd : 2^U \to 2^V \times 2^V$ and $\lhd : 2^V \times 2^V \to 2^U$ are defined as:

$$O^{\rhd} = (O^*, O^{\overline{*}}) \text{ and } (A, B)^{\lhd} = A^\bullet \cap B^{\overline{\bullet}}.$$

If $O^{\rhd} = (A, B)$ and $(A, B)^{\lhd} = O$, then $(O, (A, B))$ is called a three-way concept, and call O and (A, B) the extent and the intent of $(O, (A, B))$.

By ordering the three-way concepts of (U, V, I), a lattice structure can be derived, i.e., three-way concept lattice $\mathrm{OEL}(U, V, I)$, whose structure is determined by

$$(O_1, (A_1, B_1)) \wedge (O_2, (A_2, B_2)) = (O_1 \cap O_2, ((A_1, B_1) \cup (A_2, B_2))^{\lhd\rhd}),$$
$$(O_1, (A_1, B_1)) \vee (O_2, (A_2, B_2)) = ((O_1 \cup O_2)^{\rhd\lhd}, (A_1, B_1) \cap (A_2, B_2)).$$

By abstracting the common features of incomplete formal contexts and shadow sets, Qi et al. [20] introduced 3-valued formal contexts.

Definition 3 *([20]).* A 3-valued formal context is a quadruple $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$, where U and V are two finite and nonempty sets, and $J : U \times V \to \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}$ is a 3-valued function.

In order to introduce 3-valued concepts, it is necessary to define the positive operator P, the negative operator N, and the zero operator Z. For simplicity, # is adopted to represent any one of $\{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}$.

Definition 4 *([20]).* Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$, $O \in 2^U$, and $A \in 2^V$. Two associated operators $\# : 2^U \to 2^V$ and $\widetilde{\#} : 2^V \to 2^U$ are defined as:

$$O^{\#} = \{v \in V \mid \forall u \in O, J(u, v) = \#\} \text{ and } A^{\widetilde{\#}} = \{u \in U \mid \forall v \in A, J(u, v) = \#\}.$$

If $O^{\#} = A$ and $A^{\widetilde{\#}} = O$, we call (O, A) a #-concept of K, and call O and A the extent and the intent of (O, A). Specifically, we have P-concepts, N-concepts and Z-concepts.

In addition, for a given 3-valued formal context K, we denote the set of #-concepts as $\#\mathrm{L}(K)$. The subsequent Theorem 1 shows that $\#\mathrm{L}(K)$ is a complete lattice.

Theorem 1 *([19,20]). Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$. Then, $\#\mathrm{L}(K)$ is a complete lattice, and the infimum and supremum are given by*

$$(O_1, A_1) \wedge (O_2, A_2) = \left(O_1 \cap O_2, (A_1 \cup A_2)^{\widetilde{\#}\#}\right),$$
$$(O_1, A_1) \vee (O_2, A_2) = \left((O_1 \cup O_2)^{\#\widetilde{\#}}, A_1 \cap A_2\right).$$

Then, by replacing # with P, N, *and* Z, *we have complete lattices $PL(K)$, $NL(K)$, and $ZL(K)$, i.e.,* P*-concept lattice,* N*-concept lattice, and* Z*-concept lattice.*

Proposition 1 *([20]). For $O, O_1, O_2 \subseteq U$, $A, A_1, A_2 \subseteq V$, and $\# \in \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}$, the following statements hold.*

(i) $O \subseteq O^{\#\widetilde{\#}}$, $A \subseteq A^{\widetilde{\#}\#}$.
(ii) $O_1 \subseteq O_2 \Rightarrow O_2^{\#} \subseteq O_1^{\#}$, $A_1 \subseteq A_2 \Rightarrow A_2^{\widetilde{\#}} \subseteq A_1^{\widetilde{\#}}$.
(iii) $O^{\#} = O^{\#\widetilde{\#}\#}$, $A^{\#} = A^{\widetilde{\#}\#\widetilde{\#}}$.
(iv) $O \subseteq A^{\widetilde{\#}} \Leftrightarrow A \subseteq O^{\#}$.
(v) $(O_1 \cup O_2)^{\#} = O_1^{\#} \cap O_2^{\#}$, $(A_1 \cup A_2)^{\widetilde{\#}} = A_1^{\widetilde{\#}} \cap A_2^{\widetilde{\#}}$.
(vi) $(O_1 \cap O_2)^{\#} \supseteq O_1^{\#} \cup O_2^{\#}$, $(A_1 \cap A_2)^{\widetilde{\#}} \supseteq A_1^{\widetilde{\#}} \cup A_2^{\widetilde{\#}}$.

Definition 5 *([20]).* Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$, $O \in 2^U$, and $(A, B, C) \in 2^V \times 2^V \times 2^V$. Two associated 3-valued operators $\trianglerighteq : 2^U \to 2^V \times 2^V \times 2^V$ and $\trianglelefteq : 2^V \times 2^V \times 2^V \to 2^U$ are defined as:

$$O^{\trianglerighteq} = \left(O^{\mathrm{P}}, O^{\mathrm{N}}, O^{\mathrm{Z}}\right) \text{ and } (A, B, C)^{\trianglelefteq} = A^{\widetilde{\mathrm{P}}} \cap B^{\widetilde{\mathrm{N}}} \cap C^{\widetilde{\mathrm{Z}}}.$$

Moreover, if $O^{\trianglerighteq} = (A, B, C)$ and $(A, B, C)^{\trianglelefteq} = O$, we call $(O, (A, B, C))$ a 3-valued concept of K, and call O and (A, B, C) the extent and the intent of $(O, (A, B, C))$.

Proposition 2 *([20]). For $O, O_1, O_2 \in 2^U$, and $\mathbf{B}, \mathbf{B}_1, \mathbf{B}_2 \in 2^V \times 2^V \times 2^V$, the following statements hold.*

(i) $O \subseteq O^{\trianglerighteq\trianglelefteq}$, $\mathbf{B} \subseteq \mathbf{B}^{\trianglelefteq\trianglerighteq}$.
(ii) $O_1 \subseteq O_2 \Rightarrow O_2^{\trianglerighteq} \subseteq O_1^{\trianglerighteq}$, $\mathbf{B}_1 \subseteq \mathbf{B}_2 \Rightarrow \mathbf{B}_2^{\trianglelefteq} \subseteq \mathbf{B}_1^{\trianglelefteq}$.
(iii) $O^{\trianglerighteq} = O^{\trianglerighteq\trianglelefteq\trianglerighteq}$, $\mathbf{B}^{\trianglelefteq} = \mathbf{B}^{\trianglelefteq\trianglerighteq\trianglelefteq}$.
(iv) $O \subseteq \mathbf{B}^{\trianglelefteq} \Leftrightarrow \mathbf{B} \subseteq O^{\trianglerighteq}$.
(v) $(O_1 \cup O_2)^{\trianglerighteq} = O_1^{\trianglerighteq} \cap O_2^{\trianglerighteq}$, $(\mathbf{B}_1 \cup \mathbf{B}_2)^{\trianglelefteq} = \mathbf{B}_1^{\trianglelefteq} \cap \mathbf{B}_2^{\trianglelefteq}$.
(vi) $(O_1 \cap O_2)^{\trianglerighteq} \supseteq O_1^{\trianglerighteq} \cup O_2^{\trianglerighteq}$, $(\mathbf{B}_1 \cap \mathbf{B}_2)^{\trianglelefteq} \supseteq \mathbf{B}_1^{\trianglelefteq} \cup \mathbf{B}_2^{\trianglelefteq}$.

Let K be a 3-valued formal context. We denote the set of 3-valued concepts of K as $\mathrm{TVL}(K)$. The subsequent Theorem 2 shows that $\mathrm{TVL}(K)$ is a complete lattice.

Theorem 2 *([20]). Let K be a 3-valued formal context. Then, TVL(K) is a complete lattice, and the infimum and supremum are given by*

$$(O_1,(A_1,B_1,C_1))\wedge(O_2,(A_2,B_2,C_2))=\left(O_1\cap O_2,((A_1,B_1,C_1)\cup(A_2,B_2,C_2))^{\trianglelefteq\trianglerighteq}\right),$$
$$(O_1,(A_1,B_1,C_1))\vee(O_2,(A_2,B_2,C_2))=\left((O_1\cup O_2)^{\trianglerighteq\trianglelefteq},(A_1,B_1,C_1)\cap(A_2,B_2,C_2)\right).$$

Hereinafter, we call TVL(K) a 3-valued concept lattice of K.

Before proceeding, the following symbols are given to prepare the subsequent discussion:

(i) Let $K=(U,V,\{\mathrm{P},\mathrm{N},\mathrm{Z}\},J)$ be a 3-valued context, and let $K'=(U\cup\{e\},V,\{\mathrm{P},\mathrm{N},\mathrm{Z}\},J')$ be the 3-valued context after adding a new object e into K.

(ii) Let TVL(K) and TVL(K') be the 3-valued concept lattices of K and K', respectively.

Proposition 3. *Let $(O,(A,B,C))$ be a 3-valued concept of TVL(K). If $(A,B,C)\cap\{e\}^{\trianglerighteq_{J'}}=(A,B,C)$, then $(O\cup\{e\},(A,B,C))$ is a new concept of TVL(K').*

Proof. On one hand, we have $(O\cup\{e\})^{\trianglerighteq_{J'}}=O^{\trianglerighteq_{J'}}\cap\{e\}^{\trianglerighteq_{J'}}$. By the fact that $O^{\trianglerighteq_J}$ and $O^{\trianglerighteq_{J'}}$ are the attributes processed by O in TVL(K) and TVL(K') respectively, and there is no change made on O after adding the new object e, it can be deduced that $O^{\trianglerighteq_{J'}}=O^{\trianglerighteq_J}$. Moreover, by $(A,B,C)\cap\{e\}^{\trianglerighteq_{J'}}=(A,B,C)$, it follows that $(O\cup\{e\})^{\trianglerighteq_{J'}}=O^{\trianglerighteq_{J'}}\cap\{e\}^{\trianglerighteq_{J'}}=(A,B,C)\cap\{e\}^{\trianglerighteq_{J'}}=(A,B,C)$.

On the other hand, noting that $(A,B,C)^{\trianglelefteq}$ and $(A,B,C)^{\trianglelefteq_{J'}}$ are the sets of objects possessing (A,B,C) in TVL(K) and TVL(K') respectively, and the only change is the addition of the new object e, we can obtain $(A,B,C)^{\trianglelefteq_{J'}}=(A,B,C)^{\trianglelefteq}\cup\{e\}$.

To sum up, and given the fact that there is no 3-valued concept in TVL(K) whose extent containing an object e, it follows that $(O\cup\{e\},(A,B,C))$ is a new concept of TVL(K'). $\square$

Proposition 4. *Let $(O,(A,B,C))$ be a 3-valued concept of TVL(K). If there is no 3-valued concept with an intent $(A,B,C)\cap\{e\}^{\trianglerighteq_{J'}}$, then $\left(O\cup\{e\},(A,B,C)\cap\{e\}^{\trianglerighteq_{J'}}\right)$ is a new 3-valued concept of TVL(K').*

Proof. On one hand, we have $(O\cup\{e\})^{\trianglerighteq_{J'}}=O^{\trianglerighteq_{J'}}\cap\{e\}^{\trianglerighteq_{J'}}$. Noting that $O^{\trianglerighteq_J}$ and $O^{\trianglerighteq_{J'}}$ are the attributes processed by O in TVL(K) and TVL(K') respectively, and there is no change made on O after adding the new object e, we can deduce that $O^{\trianglerighteq_{J'}}=O^{\trianglerighteq_J}$. Then, it follows that

$$(O\cup\{e\})^{\trianglerighteq_{J'}}=O^{\trianglerighteq_{J'}}\cap\{e\}^{\trianglerighteq_{J'}}=O^{\trianglerighteq_J}\cap\{e\}^{\trianglerighteq_{J'}}=(A,B,C)\cap\{e\}^{\trianglerighteq_{J'}}.$$

On the other hand, noting that $(A,B,C)^{\trianglerighteq_J}$ and $(A,B,C)^{\trianglerighteq_{J'}}$ are the sets of objects possessing (A,B,C) in TVL(K) and TVL(K') respectively, and the only change made to the 3-valued context is the arrival of e, it can be deduced that $(A,B,C)^{\trianglerighteq_{J'}}=(A,B,C)^{\trianglerighteq_J}\cup\{e\}=O\cup\{e\}$.

To sum up, and by considering that there is no 3-valued concept of TVL(K) with an intent $(A,B,C)\cap\{e\}^{\trianglerighteq_{J'}}$, it can be concluded that $\left(O\cup\{e\},(A,B,C)\cap\{e\}^{\trianglerighteq_{J'}}\right)$ is a new 3-valued concept of TVL(K'). $\square$

Inspired by the existing incremental methods [4,7,16], based on Proposition 3 and Proposition 4, Algorithm 1 builds the 3-valued concept lattice of K in an incremental way. When dealing with a new object, it is necessary to visit all the existing 3-valued concepts, and whether to generate a 3-valued concept is determined by the relationship between the current visited 3-valued concept and this object. The complexity of the algorithm is $O(|U|^2|V|M)$, where $|\cdot|$ represents the cardinality of a set, and M is the number of 3-valued concepts in TVL(K).

It should be pointed out that step 7 in Algorithm 1 is not deterministic because this ordering need not be strictly decreasing, but its result is the same for each ordering. Concretely, the ordering can affect the generation of 3-valued concepts in step 13 and step 14. Let $(O_i,(A_i,B_i,C_i))$ and $(O_j,(A_j,B_j,C_j))$ be two 3-valued concepts with $|O_i|\geq|O_j|$. If we visit $(O_i,(A_i,B_i,C_i))$ first, then by the condition in step 13, 3-valued concept $\left(O_j\cup\{e\},(A_j,B_j,C_j)\cap\{e\}^{\trianglerighteq}\right)$ can not be generated when $(A_i,B_i,C_i)\cap\{e\}^{\trianglerighteq}=(A_j,B_j,C_j)\cap\{e\}^{\trianglerighteq}$. In other words, if we follow the order that established in step 7, then it can be assured that if $(A_i,B_i,C_i)\cap\{e\}^{\trianglerighteq}=(A_j,B_j,C_j)\cap\{e\}^{\trianglerighteq}$, only one of $\left(O_i,(A_i,B_i,C_i)\cap\{e\}^{\trianglerighteq}\right)$ and $\left(O_j\cup\{e\},(A_j,B_j,C_j)\cap\{e\}^{\trianglerighteq}\right)$ can be generated. Hence, the correctness is guaranteed.

3. The properties of 3-valued formal contexts in a formal concept viewpoint

This section explores properties of 3-valued formal contexts in a formal concept viewpoint.

Theorem 3. *Let $K=(U,V,\{\mathrm{P},\mathrm{N},\mathrm{Z}\},J)$. The following statements hold.*

(i) If (O,A) is a P*-concept, then $\left(O,\left(A,O^{\mathrm{N}},O^{\mathrm{Z}}\right)\right)$ is a 3-valued concept.*

(ii) If (O,B) is a N*-concept, then $\left(O,\left(O^{\mathrm{P}},B,O^{\mathrm{Z}}\right)\right)$ is a 3-valued concept.*

(iii) If (O,C) is a Z-concept, then $\left(O,\left(O^{\mathrm{P}},O^{\mathrm{N}},C\right)\right)$ is a 3-valued concept.

Algorithm 1 Building a 3-valued concept lattice.

Input: $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$.
Output: $\mathrm{TVL}(K)$.
Initialize $\mathrm{TVL}(K) = \emptyset$.
$U \leftarrow U - \{e\}$. // e is an object of U
$\mathrm{TVL}(K) \leftarrow \mathrm{TVL}(K) \cup \left\{ \left(\{e\}, \{e\}^{\trianglerighteq} \right), \left((V, V, V)^{\trianglelefteq}, (V, V, V) \right) \right\}$.
While $U \neq \emptyset$
  $U \leftarrow U - \{e\}$. // e is an object of U
  Let $n = |\mathrm{TVL}(K)|$.
  Sort $\mathrm{TVL}(K)$ into a sequence $(O_1, (A_1, B_1, C_1)), \cdots, (O_n, (A_n, B_n, C_n))$ such that $|O_1| \geq \cdots \geq |O_n|$.
  Initialize $k = 1$.
  While $k \leq n$
    Suppose the current visited 3-valued concept is $(O_k, (A_k, B_k, C_k))$.
    If $(A_k, B_k, C_k) \cap \{e\}^{\trianglerighteq} = (A_k, B_k, C_k)$
      Then $\mathrm{TVL}(K) \leftarrow \mathrm{TVL}(K) \cup \{(O_k \cup \{e\}, (A_k, B_k, C_k))\}$.
    Else If there does not exist a 3-valued concept with an intent $(A_k, B_k, C_k) \cap \{e\}^{\trianglerighteq}$
      Then $\mathrm{TVL}(K) \leftarrow \mathrm{TVL}(K) \cup \left\{ \left(O_k \cup \{e\}, (A_k, B_k, C_k) \cap \{e\}^{\trianglerighteq} \right) \right\}$.
    End If
  End While
End While
Return $\mathrm{TVL}(K)$.

Proof. (i) As (O, A) is a P-concept, we can obtain that $O^{\mathrm{P}} = A$ and $A^{\widetilde{\mathrm{P}}} = O$. By $O \subseteq O^{\mathrm{N}\widetilde{\mathrm{N}}}$, we have $O \cap O^{\mathrm{N}\widetilde{\mathrm{N}}} = O$, which leads to $A^{\widetilde{\mathrm{P}}} \cap O^{\mathrm{N}\widetilde{\mathrm{N}}} = O$. Similarly, we have $A^{\widetilde{\mathrm{P}}} \cap O^{\mathrm{Z}\widetilde{\mathrm{Z}}} = O$. To sum up, we can obtain $A^{\widetilde{\mathrm{P}}} \cap O^{\mathrm{N}\widetilde{\mathrm{N}}} \cap O^{\mathrm{Z}\widetilde{\mathrm{Z}}} = O$. Then, it can be concluded that $\left(O, \left(A, O^{\mathrm{N}}, O^{\mathrm{Z}} \right) \right)$ is a 3-valued concept.

Statements (ii) and (iii) can be analogously proved. □

Theorem 4. *Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$, and $(O, (A, B, C))$ be a 3-valued concept. The following statements hold.*

(i) $\left(A^{\widetilde{\mathrm{P}}}, A \right)$ is a P-*concept.*
(ii) $\left(B^{\widetilde{\mathrm{N}}}, B \right)$ is a N-*concept.*
(iii) $\left(C^{\widetilde{\mathrm{Z}}}, C \right)$ is a Z-*concept.*

Proof. (i) As $(O, (A, B, C))$ is a 3-valued concept, it follows $O^{\mathrm{P}} = A$. As a consequence, $\left(A^{\widetilde{\mathrm{P}}}, A \right) = \left(O^{\mathrm{P}\widetilde{\mathrm{P}}}, O^{\mathrm{P}} \right)$ is a P-concept.

The rest two statements can be proved in a similar way. □

For the sake of discussion, the following symbols are defined before proceeding. Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$. We define

$$\mathrm{PL}_{\mathrm{E}}(K) = \{O \mid (O, A) \in \mathrm{PL}(K)\},$$

$$\mathrm{NL}_{\mathrm{E}}(K) = \{O \mid (O, A) \in \mathrm{NL}(K)\},$$

$$\mathrm{ZL}_{\mathrm{E}}(K) = \{O \mid (O, A) \in \mathrm{ZL}(K)\},$$

$$\mathrm{PL}_{\mathrm{I}}(K) = \{A \mid (O, A) \in \mathrm{PL}(K)\},$$

$$\mathrm{NL}_{\mathrm{I}}(K) = \{A \mid (O, A) \in \mathrm{NL}(K)\},$$

$$\mathrm{ZL}_{\mathrm{I}}(K) = \{A \mid (O, A) \in \mathrm{ZL}(K)\},$$

$$\mathrm{TVL}_{\mathrm{E}}(K) = \{O \mid (O, (A, B, C)) \in \mathrm{TVL}(K)\},$$

$$\mathrm{TVL}_{\mathrm{I}_1}(K) = \{A \mid (O, (A, B, C)) \in \mathrm{TVL}(K)\},$$

$$\mathrm{TVL}_{\mathrm{I}_2}(K) = \{B \mid (O, (A, B, C)) \in \mathrm{TVL}(K)\},$$

$$\mathrm{TVL}_{\mathrm{I}_3}(K) = \{C \mid (O, (A, B, C)) \in \mathrm{TVL}(K)\}.$$

Theorem 5. *Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$. The following statements hold.*

(i) $PL_{\mathrm{E}}(K) \subseteq TVL_{\mathrm{E}}(K)$.
(ii) $NL_{\mathrm{E}}(K) \subseteq TVL_{\mathrm{E}}(K)$.
(iii) $ZL_{\mathrm{E}}(K) \subseteq TVL_{\mathrm{E}}(K)$.
(iv) $PL_{\mathrm{I}}(K) = TVL_{\mathrm{I}_1}(K)$.
(v) $NL_{\mathrm{I}}(K) = TVL_{\mathrm{I}_2}(K)$.
(vi) $ZL_{\mathrm{I}}(K) = TVL_{\mathrm{I}_3}(K)$.

Table 1
The 3-valued formal context K of Example 1.

	a	b	c	d	e
1	−	+	0	0	−
2	+	0	−	−	+
3	0	+	+	−	−
4	−	−	0	+	−
5	+	0	−	+	0

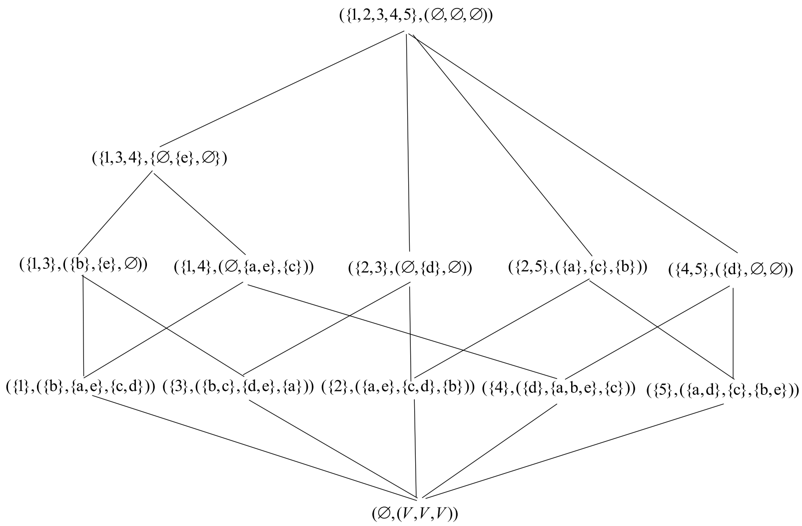

Fig. 1. TVL(K) of Example 1.

Proof. We just prove (i) and (iv), and the rest can be analogously proved.

(i) Let $O \in \mathrm{PL_E}(K)$. Then, we have $\left(O, O^{\mathrm{P}}\right) \in \mathrm{PL}(K)$. Based on Theorem 3, it follows $\left(O, \left(O^{\mathrm{P}}, O^{\mathrm{N}}, O^{\mathrm{Z}}\right)\right) \in \mathrm{TVL}(K)$, which means $O \in \mathrm{TVL_E}(K)$. As a consequence, $\mathrm{PL_E}(K) \subseteq \mathrm{TVL_E}(K)$ is at hand.

(iv) On one hand, for $A \in \mathrm{PL_I}(K)$, we have $\left(A^{\widetilde{\mathrm{P}}}, A\right) \in \mathrm{PL}(K)$. Based on Theorem 3, we can obtain $\left(A^{\widetilde{\mathrm{P}}}, \left(A, A^{\widetilde{\mathrm{P}}\mathrm{N}}, A^{\widetilde{\mathrm{P}}\mathrm{Z}}\right)\right) \in \mathrm{TVL}(K)$, which implies $A \in \mathrm{TVL}_{\mathrm{I}_1}(K)$. Therefore, we have $\mathrm{PL_I}(K) \subseteq \mathrm{TVL}_{\mathrm{I}_1}(K)$.

On the other hand, for $A \in \mathrm{TVL}_{\mathrm{I}_1}(K)$, there exists $O \subseteq U$, $B, C \subseteq V$ such that $(O, (A, B, C)) \in \mathrm{TVL}(K)$. Based on Theorem 4, we have $\left(A^{\widetilde{\mathrm{P}}}, A\right) \in \mathrm{PL}(K)$, which leads to $A \in \mathrm{PL_I}(K)$. As a consequence, we cam obtain $\mathrm{TVL}_{\mathrm{I}_1}(K) \subseteq \mathrm{PL_I}(K)$.

To sum up, $\mathrm{PL_I}(K) = \mathrm{TVL}_{\mathrm{I}_1}(K)$ is proved. □

Example 1. Table 1 shows a 3-valued formal context K which characterizes 5 objects with 5 attributes. In keeping with existing research, in this example as well as the ones in the rest discussion, we use +, -, and 0 to represent P, N, and Z. The 3-valued concept lattice $\mathrm{TVL}(K)$, P-concept lattice $\mathrm{PL}(K)$, N-concept lattice $\mathrm{NL}(K)$, and Z-concept lattice $\mathrm{ZL}(K)$ are shown in Figs. 1 to 4, respectively.

In conflict analysis, conflict situations can be represented by 3-valued formal contexts [41,43]. Concretely, let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$ be a 3-valued formal context. For $u \in U$ and $v \in V$, $J(u, v) = \mathrm{P}$ shows that u support the proposal v, $J(u, v) = \mathrm{N}$ indicates that u opposes the proposal v, while $J(u, v) = \mathrm{Z}$ represents that u retains neutral attitude towards the proposal v.

Then, a set $X \subseteq U$ with $|X| \geq 2$ can represent a clique, whose state can be analyzed based on 3-valued formal contexts. The consistency measurement of X with respect to $v \in V$ is defined by

$$c_v(X) = \begin{cases} 1, & \text{if } \forall u_i, u_j \in X, J(u_i, v) = J(u_j, v), \\ 0, & \text{otherwise.} \end{cases}$$

Then, the inconsistency measurement of X with respect to V is defined as:

$$m(X) = \frac{\sum\limits_{v \in V} (1 - c_v(X))}{|V|}.$$

By setting $0 \leq t_1 \leq t_2 \leq 1$, the state of X can be determined as follows.

(i) X is in a conflict state if $m(X) > t_2$;

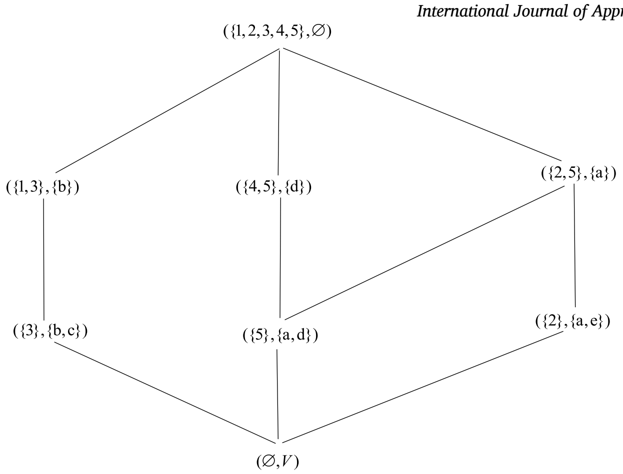

Fig. 2. PL(K) of Example 1.

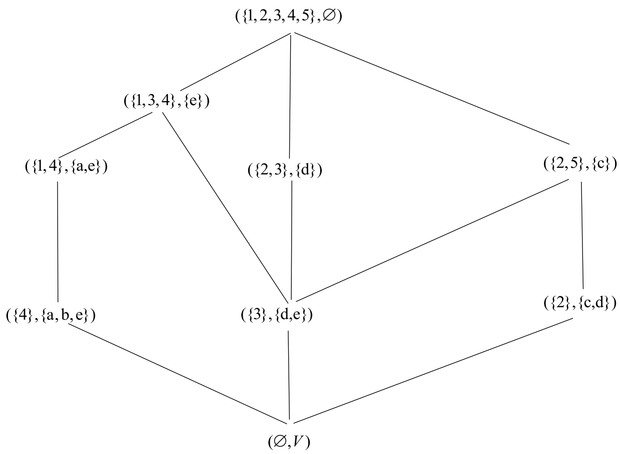

Fig. 3. NL(K) of Example 1.

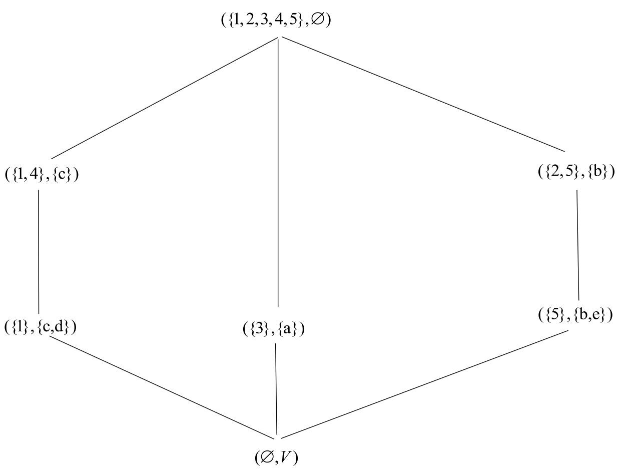

Fig. 4. ZL(K) of Example 1.

(ii) X is in a neutral state if $t_2 \geq m(X) \geq t_1$;
(iii) X is in an allied state if $m(X) < t_1$.
For $X \subseteq U$ and $X \notin \mathrm{TVL_E}(K)$, we define

$$t(X) = Y, Y \in \mathrm{TVL_E}(K), Y \supset X \text{ and for } Y' \in \mathrm{TVL_E}(K), Y' \subset Y \Rightarrow Y' \not\supset X.$$

Furthermore, for $X \subseteq U$, we define

$$f(X) = \begin{cases} X, & X \in \mathrm{TVL_E}(K), \\ t(X), & \text{otherwise.} \end{cases}$$

Then, the inconsistency degree of $X \subseteq U$ can be obtained by a 3-valued concept $\left(f(X), f(X)^{\trianglerighteq}\right)$. Suppose the intent of $\left(f(X), f(X)^{\trianglerighteq}\right)$ is (A, B, C), then we have $m(X) = \dfrac{|V - A - B - C|}{|V|}$.

Example 2. Considering the 3-valued formal context K in Table 1, we analyze the inconsistency degree of a clique $\{1,4\}$. Following the above discussion, we resort to the 3-valued concept $(\{1,4\}, (\emptyset, \{\mathrm{a}, \mathrm{e}\}, \{\mathrm{c}\}))$, and $m(\{1,4\}) = \dfrac{|V - \{\mathrm{a},\mathrm{e}\} - \{\mathrm{c}\}|}{|V|} = 0.4$.

Then, some meaningful questions arise below.

(i) If we have obtained P-concept lattice $\mathrm{PL}(K)$, N-concept lattice $\mathrm{NL}(K)$, and Z-concept lattice $\mathrm{ZL}(K)$, then whether conflict analysis can be done based on $\mathrm{PL}(K)$, $\mathrm{NL}(K)$ and $\mathrm{ZL}(K)$ rather than based on $\mathrm{TVL}(K)$.

(ii) If $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$ is considered as an incomplete context with $J(u, v) = \mathrm{Z}$ being viewed as an uncertain case, then whether there are connections between 3-valued concepts and the three-way concepts of the complements.

(iii) Since there is no one-to-one correspondence between $\mathrm{PL}(K)$ to $\mathrm{TVL}(K)$, whether a mapping can be established to make a one-to-one correspondence between them, and the same questions for $\mathrm{NL}(K)$ and $\mathrm{ZL}(K)$.

We will investigate problem (i) in the rest of this section, and discuss the rest two problems in Section 4 and Section 5.

Theorem 6. *Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$. Then the following statements hold.*
(i) There exists a meet-preserving mapping from $\mathrm{PL}(K)$ to $\mathrm{TVL}(K)$.
(ii) There exists a meet-preserving mapping from $\mathrm{NL}(K)$ to $\mathrm{TVL}(K)$.
(iii) There exists a meet-preserving mapping from $\mathrm{ZL}(K)$ to $\mathrm{TVL}(K)$.

Proof. (i) Let $\varphi((O, A)) = \left(O, \left(A, O^{\mathrm{N}}, O^{\mathrm{Z}}\right)\right)$. Then for $(O_1, A_1), (O_2, A_2) \in \mathrm{PL}(K)$, we have

$$\begin{aligned}
&\varphi((O_1, A_1) \wedge (O_2, A_2)) \\
&= \varphi\left(\left(O_1 \cap O_2, (A_1 \cup A_2)^{\tilde{\mathrm{P}}\mathrm{P}}\right)\right) \\
&= \varphi\left(\left(O_1 \cap O_2, \left(A_1^{\tilde{\mathrm{P}}} \cap A_2^{\tilde{\mathrm{P}}}\right)^{\mathrm{P}}\right)\right) \\
&= \varphi\left(\left(O_1 \cap O_2, (O_1 \cap O_2)^{\mathrm{P}}\right)\right) \\
&= \left(O_1 \cap O_2, \left((O_1 \cap O_2)^{\mathrm{P}}, (O_1 \cap O_2)^{\mathrm{N}}, (O_1 \cap O_2)^{\mathrm{Z}}\right)\right)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\varphi((O_1, A_1) \wedge (O_2, A_2)) \\
&= \left(O_1, \left(A_1, O_1^{\mathrm{N}}, O_1^{\mathrm{Z}}\right)\right) \wedge \left(O_2, \left(A_2, O_2^{\mathrm{N}}, O_2^{\mathrm{Z}}\right)\right) \\
&= \left(O_1 \cap O_2, \left(\left(A_1, O_1^{\mathrm{N}}, O_1^{\mathrm{Z}}\right) \cup \left(A_2, O_2^{\mathrm{N}}, O_2^{\mathrm{Z}}\right)\right)^{\trianglelefteq\trianglerighteq}\right) \\
&= \left(O_1 \cap O_2, \left(\left(A_1, O_1^{\mathrm{N}}, O_1^{\mathrm{Z}}\right)^{\trianglelefteq} \cap \left(A_2, O_2^{\mathrm{N}}, O_2^{\mathrm{Z}}\right)^{\trianglelefteq}\right)^{\trianglerighteq}\right) \\
&= \left(O_1 \cap O_2, \left(\left(O_1 \cap O_1^{\mathrm{N}\tilde{\mathrm{N}}} \cap O_1^{\mathrm{Z}\tilde{\mathrm{Z}}}\right) \cap \left(O_2 \cap O_2^{\mathrm{N}\tilde{\mathrm{N}}} \cap O_2^{\mathrm{Z}\tilde{\mathrm{Z}}}\right)\right)^{\trianglerighteq}\right) \\
&= \left(O_1 \cap O_2, (O_1 \cap O_2)^{\trianglerighteq}\right) \\
&= \left(O_1 \cap O_2, \left((O_1 \cap O_2)^{\mathrm{P}}, (O_1 \cap O_2)^{\mathrm{N}}, (O_1 \cap O_2)^{\mathrm{Z}}\right)\right).
\end{aligned}$$

As a consequence, $\varphi((O_1, A_1) \wedge (O_2, A_2)) = \varphi((O_1, A_1)) \wedge \varphi((O_2, A_2))$.
In addition, we have

$$\begin{aligned}
&((O_1, A_1) \leq (O_2, A_2)) \\
&\Leftrightarrow O_1 \subseteq O_2 \\
&\Leftrightarrow \left(O_1, \left(A_1, O_1^{\mathrm{N}}, O_1^{\mathrm{Z}}\right)\right) \leq \left(O_2, \left(A_2, O_2^{\mathrm{N}}, O_2^{\mathrm{Z}}\right)\right) \\
&\Leftrightarrow \varphi((O_1, A_1)) \leq \varphi((O_2, A_2)).
\end{aligned}$$

Then, this statement is proved. Statements (ii) and (iii) can be analogously proved. □

Theorem 7. *Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$. Then there exists a meet-preserving mapping from $TVL(K)$ to $PL(K) \times NL(K) \times ZL(K)$.*

Proof. Let $\varphi((O, (A, B, C))) = \left(\left(A^{\widetilde{\mathrm{P}}}, A\right), \left(B^{\widetilde{\mathrm{N}}}, B\right), \left(C^{\widetilde{\mathrm{Z}}}, C\right)\right)$. Then, based on Theorem 4, we have that $\left(A^{\widetilde{\mathrm{P}}}, A\right) \in \mathrm{PL}(K)$, $\left(B^{\widetilde{\mathrm{N}}}, B\right) \in \mathrm{NL}(K)$, and $\left(C^{\widetilde{\mathrm{Z}}}, C\right) \in \mathrm{ZL}(K)$.

For $(O_1, (A_1, B_1, C_1)), (O_2, (A_2, B_2, C_2)) \in \mathrm{TVL}(K)$, we can obtain

$$\begin{aligned}
&\varphi((O_1, (A_1, B_1, C_1)) \vee (O_2, (A_2, B_2, C_2))) \\
&= \varphi\left((O_1 \cup O_2)^{\trianglerighteq\trianglelefteq}, (A_1, B_1, C_1) \cap (A_2, B_2, C_2)\right) \\
&= \varphi\left((O_1 \cup O_2)^{\trianglerighteq\trianglelefteq}, (A_1 \cap A_2, B_1 \cap B_2, C_1 \cap C_2)\right) \\
&= \left(\left((A_1 \cap A_2)^{\widetilde{\mathrm{P}}}, A_1 \cap A_2\right), \left((B_1 \cap B_2)^{\widetilde{\mathrm{N}}}, B_1 \cap B_2\right), \left((C_1 \cap C_2)^{\widetilde{\mathrm{Z}}}, C_1 \cap C_2\right)\right) \\
&= \left(\left(\left(A_1^{\widetilde{\mathrm{P}}} \cup A_2^{\widetilde{\mathrm{P}}}\right)^{\mathrm{P}\widetilde{\mathrm{P}}}, A_1 \cap A_2\right), \left(\left(B_1^{\widetilde{\mathrm{N}}} \cup B_2^{\widetilde{\mathrm{N}}}\right)^{\mathrm{N}\widetilde{\mathrm{N}}}, B_1 \cap B_2\right), \left(\left(C_1^{\widetilde{\mathrm{Z}}} \cup C_2^{\widetilde{\mathrm{Z}}}\right)^{\mathrm{Z}\widetilde{\mathrm{Z}}}, C_1 \cap C_2\right)\right) \\
&= \left(\left(A_1^{\widetilde{\mathrm{P}}}, A_1\right) \vee \left(A_2^{\widetilde{\mathrm{P}}}, A_2\right), \left(B_1^{\widetilde{\mathrm{N}}}, B_1\right) \vee \left(B_2^{\widetilde{\mathrm{N}}}, B_2\right), \left(C_1^{\widetilde{\mathrm{Z}}}, C_1\right) \vee \left(C_2^{\widetilde{\mathrm{Z}}}, C_2\right)\right) \\
&= \left(\left(A_1^{\widetilde{\mathrm{P}}}, A_1\right), \left(B_1^{\widetilde{\mathrm{N}}}, B_1\right), \left(C_1^{\widetilde{\mathrm{Z}}}, C_1\right)\right) \vee \left(\left(A_2^{\widetilde{\mathrm{P}}}, A_2\right), \left(B_2^{\widetilde{\mathrm{N}}}, B_2\right), \left(C_2^{\widetilde{\mathrm{Z}}}, C_2\right)\right) \\
&= \varphi((O_1, (A_1, B_1, C_1))) \vee \varphi((O_2, (A_2, B_2, C_2)))
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&((O_1, (A_1, B_1, C_1)) \leq (O_2, (A_2, B_2, C_2)) \\
&\Leftrightarrow A_1 \subseteq A_2, B_1 \subseteq B_2, C_1 \subseteq C_2 \\
&\Leftrightarrow \left(A_1^{\mathrm{P}}, A_1\right) \leq \left(A_2^{\mathrm{P}}, A_2\right), \left(B_1^{\mathrm{N}}, B_1\right) \leq \left(B_2^{\mathrm{N}}, B_2\right), \left(C_1^{\mathrm{Z}}, C_1\right) \leq \left(C_2^{\mathrm{Z}}, C_2\right) \\
&\Leftrightarrow \left(\left(A_1^{\mathrm{P}}, A_1\right), \left(B_1^{\mathrm{N}}, B_1\right), \left(C_1^{\mathrm{Z}}, C_1\right)\right) \leq \left(\left(A_2^{\mathrm{P}}, A_2\right), \left(B_2^{\mathrm{N}}, B_2\right), \left(C_2^{\mathrm{Z}}, C_2\right)\right) \\
&\Leftrightarrow \varphi((O_1, (A_1, B_1, C_1))) \leq \varphi((O_2, (A_2, B_2, C_2)))
\end{aligned}$$

To sum up, this theorem is proved. □

Definition 6. Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$, $(X_1, A_1), (X_2, A_2) \in \mathrm{PL}(K)$, $(Y_1, B_1), (Y_2, B_2) \in \mathrm{NL}(K)$, and $(W_1, C_1), (W_2, C_2) \in \mathrm{ZL}(K)$. If $X_1 \cap Y_1 \cap W_1 = X_2 \cap Y_2 \cap W_2$, then this case is denoted as:

$$((X_1, A_1), (Y_1, B_1), (W_1, C_1))\mathrm{R}((X_2, A_2), (Y_2, B_2), (W_2, C_2)).$$

Proposition 5. *The* R *defined on $PL(K) \times NL(K) \times ZL(K)$ is an equivalence ration.*

Proof. Reflexivity. $\forall((X, A), (Y, B), (W, C)) \in \mathrm{PL}(K) \times \mathrm{NL}(K) \times \mathrm{ZL}(K)$, as $X \cap Y \cap W = X \cap Y \cap W$, it follows that $((X, A), (Y, B), (W, C))\mathrm{R}((X, A), (Y, B), (W, C))$.

Symmetry. If $((X_1, A_1), (Y_1, B_1), (W_1, C_1))\mathrm{R}((X_2, A_2), (Y_2, B_2), (W_2, C_2))$, then it follows $X_1 \cap Y_1 \cap W_1 = X_2 \cap Y_2 \cap W_2$, which implies that $((X_2, A_2), (Y_2, B_2), (W_2, C_2))\mathrm{R}((X_1, A_1), (Y_1, B_1), (W_1, C_1))$.

Transitivity. If $((X_1, A_1), (Y_1, B_1), (W_1, C_1))\mathrm{R}((X_2, A_2), (Y_2, B_2), (W_2, C_2))$ and $((X_2, A_2), (Y_2, B_2), (W_2, C_2))\mathrm{R}((X_3, A_3), (Y_3, B_3), (W_3, C_3))$, then we have that $X_1 \cap Y_1 \cap W_1 = X_2 \cap Y_2 \cap W_2$ and $X_2 \cap Y_2 \cap W_2 = X_3 \cap Y_3 \cap W_3$, which leads to $X_1 \cap Y_1 \cap W_1 = X_3 \cap Y_3 \cap W_3$. As a consequence, we have $((X_1, A_1), (Y_1, B_1), (W_1, C_1))\mathrm{R}((X_3, A_3), (Y_3, B_3), (W_3, C_3))$.

To sum up, this proposition is proved. □

Completing Definition 6, if $X_1 \cap Y_1 \cap W_1 = X_2 \cap Y_2 \cap W_2 = T$, then we denote the induced equivalence class by $[T]_{\mathrm{R}}$.

Theorem 8. *$((X, A), (Y, B), (W, C))$ is the least element of $[T]_{\mathrm{R}}$, if and only if $(X \cap Y \cap W)^{\mathrm{P}} = A$, $(X \cap Y \cap W)^{\mathrm{N}} = B$, and $(X \cap Y \cap W)^{\mathrm{Z}} = C$.*

Proof.

"⇒". As $T \subseteq X$ and $X^{\mathrm{P}\widetilde{\mathrm{P}}} = X$, it follows $T^{\mathrm{P}\widetilde{\mathrm{P}}} \subseteq X$. Similarly, we have $T^{\mathrm{N}\widetilde{\mathrm{N}}} \subseteq Y$ and $T^{\mathrm{Z}\widetilde{\mathrm{Z}}} \subseteq W$. As a consequence, we can obtain $T^{\mathrm{P}\widetilde{\mathrm{P}}} \cap T^{\mathrm{N}\widetilde{\mathrm{N}}} \cap T^{\mathrm{Z}\widetilde{\mathrm{Z}}} \subseteq X \cap Y \cap W = T$. Besides, we have that $T^{\mathrm{P}\widetilde{\mathrm{P}}} \supseteq T$, $T^{\mathrm{N}\widetilde{\mathrm{N}}} \supseteq T$, and $T^{\mathrm{Z}\widetilde{\mathrm{Z}}} \supseteq T$, which further implies $T^{\mathrm{P}\widetilde{\mathrm{P}}} \cap T^{\mathrm{N}\widetilde{\mathrm{N}}} \cap T^{\mathrm{Z}\widetilde{\mathrm{Z}}} \supseteq T$. To sum up, we can obtain $T^{\mathrm{P}\widetilde{\mathrm{P}}} \cap T^{\mathrm{N}\widetilde{\mathrm{N}}} \cap T^{\mathrm{Z}\widetilde{\mathrm{Z}}} = T$, which means $\left(\left(T^{\mathrm{P}\widetilde{\mathrm{P}}}, T\right), \left(T^{\mathrm{N}\widetilde{\mathrm{N}}}, T\right), \left(T^{\mathrm{Z}\widetilde{\mathrm{Z}}}, T\right)\right) \in [T]_{\mathrm{R}}$.

Table 2
The experimental results with a fill ratio $1:1:1$.

K	$(20,20)$	$(30,30)$	$(40,40)$	$(50,50)$	$(60,60)$
TVL	918	4899	20083	63896	176038
PL	192	638	2511	5375	13812
NL	151	721	1889	5391	12824
ZL	207	704	2317	6926	14023
r	59.91%	42.11%	33.45%	27.69%	23.10%

On the other hand, by the fact that $T^{\mathrm{P}\widetilde{\mathrm{P}}} \subseteq X$, $T^{\mathrm{N}\widetilde{\mathrm{N}}} \subseteq Y$ and $T^{\mathrm{Z}\widetilde{\mathrm{Z}}} \subseteq W$, we have $\left(\left(T^{\mathrm{P}\widetilde{\mathrm{P}}},T\right),\left(T^{\mathrm{N}\widetilde{\mathrm{N}}},T\right),\left(T^{\mathrm{Z}\widetilde{\mathrm{Z}}},T\right)\right) \leq ((X,A),(Y,B),(W,C))$. By considering the condition that $((X,A),(Y,B),(W,C))$ is the least element of $[T]_{\mathrm{R}}$, we can conclude that $T^{\mathrm{P}} = A$, $T^{\mathrm{N}} = B$, and $T^{\mathrm{Z}} = C$.

"$\Leftarrow$". For any $((X_i,A_i),(Y_j,B_j),(W_k,C_k)) \in [T]_{\mathrm{R}}$, it follows that $X_i \cap Y_j \cap W_k = T$, which leads to $A_i = X_i^{\mathrm{P}} \subseteq (X_i \cap Y_j \cap W_k)^{\mathrm{P}} = T^{\mathrm{P}} = A$. Similarly, we have $B_j \subseteq B$ and $C_k \subseteq C$. As a consequence, we have that $(X_i,A_i) \geq (X,B)$, $(Y_j,B_j) \geq (Y,B)$, and $(W_K,C_k) \geq (W,C)$, i.e., $((X,A),(Y,B),(W,C))$ is the least element of $[T]_{\mathrm{R}}$. □

Theorem 9. *Let $K = (U,V,\{\mathrm{P},\mathrm{N},\mathrm{Z}\},J)$, and Δ denotes minimal elements in $PL(K) \times NL(K) \times ZL(K)/R$. Then, $TVL(K) = \{(X \cap Y \cap W,(A,B,C)) \mid ((X,A),(Y,B),(W,C)) \in \Delta\}$.*

Proof. Denote $\{(X \cap Y \cap W,(A,B,C)) \mid ((X,A),(Y,B),(W,C)) \in \Delta\} = \Omega$. On one hand, for any $(X,(A,B,C)) \in \mathrm{TVL}(K)$, we have that $\left(A^{\widetilde{\mathrm{P}}},A\right) \in \mathrm{PL}(K)$, $\left(B^{\widetilde{\mathrm{N}}},B\right) \in \mathrm{NL}(K)$, and $\left(C^{\widetilde{\mathrm{Z}}},C\right) \in \mathrm{ZL}(K)$. Moreover, as $\left(A^{\widetilde{\mathrm{P}}} \cap B^{\widetilde{\mathrm{N}}} \cap C^{\widetilde{\mathrm{Z}}}\right)^{\mathrm{P}} = X^{\mathrm{P}} = A$, $\left(A^{\widetilde{\mathrm{P}}} \cap B^{\widetilde{\mathrm{N}}} \cap C^{\widetilde{\mathrm{Z}}}\right)^{\mathrm{N}} = X^{\mathrm{N}} = B$, and $\left(A^{\widetilde{\mathrm{P}}} \cap B^{\widetilde{\mathrm{N}}} \cap C^{\widetilde{\mathrm{Z}}}\right)^{\mathrm{Z}} = X^{\mathrm{Z}} = C$, by Theorem 8 we can conclude that $\left(\left(A^{\widetilde{\mathrm{P}}},A\right),\left(B^{\widetilde{\mathrm{N}}},B\right),\left(C^{\widetilde{\mathrm{Z}}},C\right)\right) \in \Delta$. Moreover, by the fact that $(X,(A,B,C)) = \left(A^{\widetilde{\mathrm{P}}} \cap B^{\widetilde{\mathrm{N}}} \cap C^{\widetilde{\mathrm{Z}}},(A,B,C)\right)$, we can obtain $(X,(A,B,C)) \in \Omega$, which implies $\mathrm{TVL}(K) \subseteq \Omega$.

On the other hand, for any $((X,A),(Y,B),(W,C) \in \Delta$, by Theorem 8 we have $(X \cap Y \cap W)^{\mathrm{P}} = A$, $(X \cap Y \cap W)^{\mathrm{N}} = B$, and $(X \cap Y \cap W)^{\mathrm{Z}} = C$. Then, by the definition of a 3-valued concept, $(X \cap Y \cap W,(A,B,C)) \in \mathrm{TVL}(K)$ is derived, which implies $\Omega \subseteq \mathrm{TVL}(K)$.

To sum up, this theorem is proved. □

Theorem 9 shows that for a given 3-valued formal context, $\mathrm{TVL}(K)$ can be derived based on fundamental lattices, i.e., $\mathrm{PL}(K)$, $\mathrm{NL}(K)$, and $\mathrm{ZL}(K)$.

Proposition 6. *Let $((X,A),(Y,B),(W,C)) \in \Delta$ and $(X \cap Y \cap W,(A,B,C))$ be 3-valued concept. Then, $m(X \cap Y \cap W) = m_{\mathrm{P}}(X) + m_{\mathrm{N}}(Y) + m_{\mathrm{Z}}(W) - 2$, where $m_{\mathrm{P}}(X) = \dfrac{|V-A|}{|V|}$, $m_{\mathrm{N}}(X) = \dfrac{|V-B|}{|V|}$, and $m_{\mathrm{Z}}(W) = \dfrac{|V-C|}{|V|}$.*

Proof. $m(X \cap Y \cap W) = 1 - ((1-m_{\mathrm{P}}(X)) + (1-m_{\mathrm{N}}(Y)) + (1-m_{\mathrm{Z}}(W))) = m_{\mathrm{P}}(X) + m_{\mathrm{N}}(Y) + m_{\mathrm{Z}}(W) - 2$. □

Example 3. Completing Example 2, based on Theorem 9, the 3-valued concept $(\{1,4\},(\emptyset,\{\mathrm{a},\mathrm{e}\},\{\mathrm{c}\}))$ can be obtained based on P-concept $(\{1,2,3,4,5\},\emptyset)$, N-concept $(\{1,4\},\{\mathrm{a},\mathrm{e}\})$ and Z-concept $(\{1,4\},\{\mathrm{c}\})$. As $m_{\mathrm{P}}(\{1,2,3,4,5\}) = 1$, $m_{\mathrm{N}}(\{1,4\}) = 0.6$, and $m_{\mathrm{Z}}(\{1,4\}) = 0.8$, by Proposition 6, we can obtain that $m(\{1,4\}) = 1 + 0.6 + 0.8 - 2 = 0.4$.

In what follows, we perform numerical experiments to discuss the difference in the number of concepts between 3-valued concept lattices and fundamental lattices.

The experiment is divided into two steps: (1) Determine the number of objects, the number of attributes, and the proportion of values P, N and Z in the binary relation J, and randomly generate the 3-valued formal context K. (2) Build $\mathrm{TVL}(K)$, $\mathrm{PL}(K)$, $\mathrm{NL}(K)$, and $\mathrm{ZL}(K)$, and record their sizes.

In the following discussion, a binary group is used to record the number of objects and the number of attributes of K. For instance, $(20,30)$ shows that a 3-valued context has 20 objects and 30 attributes. Besides, a fill ratio shows the ratio of the numbers of $+$, $-$, and 0 filled in K. The experimental results are shown in Table 2, Table 3 and Table 4, where TVL, PL, NL and ZL represent the number of concepts in the corresponding 3-valued concept lattice, P-concept lattice, N-concept lattice, and Z-concept lattice, and $r = (\mathrm{PL} + \mathrm{NL} + \mathrm{ZL})/\mathrm{TVL}$.

The following statements can be concluded from the experimental results.

- With the increase of the number of objects and attributes, the sizes of the 3-valued concept lattices gradually increase.
- The ratio of values P, N and Z in the binary relation J is an important factor affecting the sizes of the 3-valued concept lattices, and when P, N and Z take equal proportions, the sizes of the 3-valued concept lattice tend to be smallest.
- The number of concepts in a 3-valued concept lattice is more than that in its fundamental lattices, and when P, N and Z take equal proportions, the value of r tends to be smallest.

Table 3
The experimental results with a fill ratio $2:1.5:1$.

K	$(20,20)$	$(30,30)$	$(40,40)$	$(50,50)$	$(60,60)$
TVL	1079	6847	27892	107159	308946
PL	483	2260	7648	33457	105938
NL	167	690	2751	6381	12528
ZL	68	251	498	1062	2230
r	66.54%	46.75%	39.07%	38.17%	39.07%

Table 4
The experimental results with a fill ratio $3:2:1$.

K	$(20,20)$	$(30,30)$	$(40,40)$	$(50,50)$	$(60,60)$
TVL	1137	9294	45739	162846	612879
PL	563	4326	25973	70092	255624
NL	261	626	1774	5964	16573
ZL	52	141	252	576	797
r	77.04%	54.80%	59.90%	47.06%	44.54%

4. The properties of 3-valued formal contexts in a three-way concept viewpoint

This section discusses the properties of 3-valued formal contexts in a three-way concept viewpoint. It should be noted that Kuznetsov investigated uncertain data with multiple perspectives [6], which relates to three-way concept analysis. In this study, three contexts are considered: the positive context, the negative context and the undefined one. After that, Kuznetsov revisited Galois connections in data analysis, and proposed four assessment methods of concepts [8].

Definition 7. Let $K=(U,V,\{\mathrm{P},\mathrm{N},\mathrm{Z}\},J)$. The least complement $K_\circ$ and the greatest complement K° of K are defined as:

$$K_\circ=(U,V,I_\circ) \text{ and } K^\circ=(U,V,I^\circ)$$

where

$$I_\circ(u,v)=\begin{cases}1, & \text{if } J(u,v)=\mathrm{P},\\ 0, & \text{otherwise.}\end{cases} \quad \text{and} \quad I^\circ(u,v)=\begin{cases}1, & \text{if } J(u,v)=\mathrm{P} \text{ or } J(u,v)=\mathrm{Z},\\ 0, & \text{otherwise.}\end{cases}$$

Theorem 10. *Let $K=(U,V,\{\mathrm{P},\mathrm{N},\mathrm{Z}\},J)$. If $(O,(A,B,C))$ is a 3-valued concept of K, then $\left(A^{\bullet},\left(A,A^{*\bar{\bullet}}\right)\right)$ is a three-way concept of $K_\circ$, and $A^{\bullet}\supseteq O$.*

Proof. As $(O,(A,B,C))$ is a 3-valued concept of K, it follows that $(A^{\bullet},A)$ is a formal concept of $K_\circ$, which implies that $\left(A^{\bullet},\left(A,A^{\bullet\bar{\bullet}}\right)\right)$ is a three-way concept of $K_\circ$. For any $e\in O$, by the fact that $O=A^{\widetilde{\mathrm{P}}}$, we can obtain $e\in A^{\widetilde{\mathrm{P}}}$, which leads to $A^{\bullet}\supseteq O$. $\square$

Theorem 11. *Let $K=(U,V,\{\mathrm{P},\mathrm{N},\mathrm{Z}\},J)$. If $(O,(A,B,C))$ is a 3-valued concept of K, and $\left(O,\left(A,O^{\bar{*}}\right)\right)$ is a three-way concept of $K_\circ$, then $O^{\bar{*}}\supseteq B\cup C$.*

Proof. As $(O,(A,B,C))$ is a 3-valued concept of K, it follows that $O^{\mathrm{P}}=A$, $O^{\mathrm{N}}=B$, and $O^{\mathrm{Z}}=C$, which leads to $B\cup C=O^{\mathrm{N}}\cup O^{\mathrm{Z}}$. As a consequence, for any $b\in B\cup C$, we have $b\in O^{\mathrm{N}}\cup O^{\mathrm{Z}}$. Moreover, as $K_\circ$ is a formal context, it is obvious that in this case operator $\bar{*}$ is equivalent to the union of operator N and operator Z. Therefore, we have $b\in O^{\bar{*}}$, which means $O^{\bar{*}}\supseteq B\cup C$. $\square$

Example 4. Table 5 shows a 3-valued formal context K, whose 3-valued concept lattice $\mathrm{TVL}(K)$ is shown in Fig. 5.

The least complement $K_\circ$ of K is shown in Table 6, and its three-way concept lattice $\mathrm{OEL}(K_\circ)$ is shown in Fig. 6.

Based on Fig. 5 and Fig. 6, it can be observed that $(\{3,4,5\},(\{\mathrm{d}\},\{\mathrm{b}\},\emptyset)),(\{4,5\},(\{\mathrm{d}\},\{\mathrm{b}\},\{\mathrm{c}\})),(\{5\},(\{\mathrm{d}\},\{\mathrm{b}\},\{\mathrm{c},\mathrm{e}\}))\in\mathrm{TVL}(K)$, and $(\{3,4,5\},(\{\mathrm{d}\},\{\mathrm{b}\}))\in\mathrm{OEL}(K_\circ)$, which is consistent with Theorem 10. In addition, Theorem 11 can be verified by the fact that $(\{1,3\},(\{\mathrm{c}\},\emptyset,\emptyset))\in\mathrm{TVL}(K)$, and $(\{1,3\},(\{\mathrm{c}\},\{\mathrm{b},\mathrm{e}\}))\in\mathrm{OEL}(K_\circ)$.

Theorem 12. *Let $K=(U,V,\{\mathrm{P},\mathrm{N},\mathrm{Z}\},J)$. If $(O,(A,B,C))$ is a 3-valued concept of K, then $\left(B^{\bar{\bullet}},\left(B^{\bar{\bullet}*},B\right)\right)$ is a three-way concept of K°, and $B^{\bar{\bullet}}\supseteq O$.*

Table 5
The 3-valued formal context K of Example 4.

	a	b	c	d	e
1	+	0	+	−	0
2	−	+	−	0	+
3	0	−	+	+	−
4	+	−	0	+	+
5	−	−	0	+	0

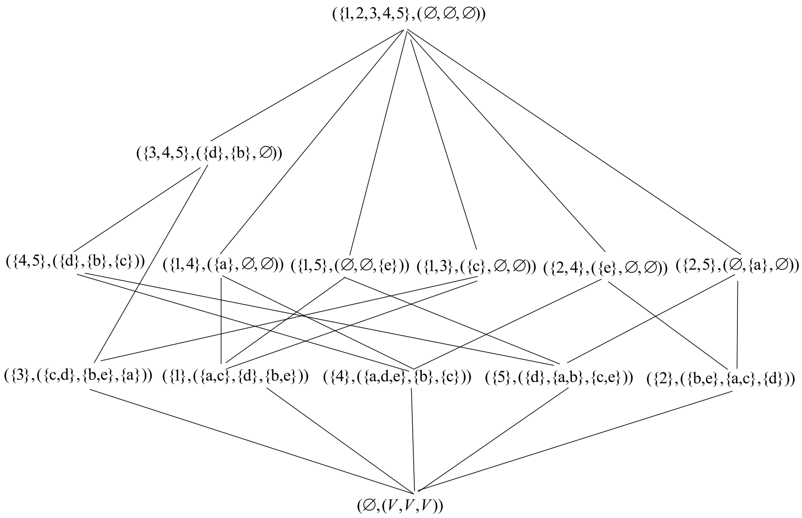

Fig. 5. TVL(K) of Example 4.

Table 6
The least complement $K_\circ$.

	a	b	c	d	e
1	+	−	+	−	−
2	−	+	−	−	+
3	−	−	+	+	−
4	+	−	−	+	+
5	−	−	−	+	−

Proof. As $(O,(A,B,C))$ is a 3-valued concept of K, it follows that $\left(B^{\bar{\bullet}}, B\right)$ is a negative concept of K°, which further implies that $\left(B^{\bar{\bullet}}, \left(B^{\bar{\bullet}*}, B\right)\right)$ is a three-way concept of K°. For any $e \in O$, by the condition $O = B^{\tilde{\mathrm{N}}}$, we have $e \in B^{\tilde{\mathrm{N}}}$. Moreover, in K°, operator $\tilde{\mathrm{N}}$ degrades to operator $\bar{\bullet}$. As a consequence, $B^{\bar{\bullet}} \supseteq O$ is at hand. □

Theorem 13. *Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$. If $(O,(A,B,C))$ is a 3-valued concept of K, and $(O,(O^*,B))$ is a three-way concept of K°, then $O^* \supseteq A \cup C$.*

Proof. As $(O,(A,B,C))$ is a 3-valued concept of K, it follows that $O^{\mathrm{P}} = A$, $O^{\mathrm{N}} = B$, and $O^{\mathrm{Z}} = C$. Then, we have $A \cup C = O^{\mathrm{P}} \cup O^{\mathrm{Z}}$. Moreover, for any $a \in A \cup C$, it is obvious that $a \in O^{\mathrm{P}} \cup O^{\mathrm{Z}}$. In K°, operator $*$ is equivalent to the union of operator P and operator Z. Then, it can be concluded that $O^* \supseteq A \cup C$. □

Example 5. Completing Example 4, Table 7 shows the greatest complement K° of K, and OEL(K°) is shown in Fig. 7. Then, it can be observed that $(\{3\},(\{\mathrm{c},\mathrm{d}\},\{\mathrm{b},\mathrm{e}\},\{\mathrm{a}\})) \in \mathrm{TVL}(K)$, and $(\{3\},(\{\mathrm{a},\mathrm{c},\mathrm{d}\},\{\mathrm{b},\mathrm{e}\})) \in \mathrm{OEL}(K^\circ)$, which is consistent with Theorem 12. Besides, we have $(\{2,5\},(\emptyset,\{\mathrm{a}\},\emptyset)) \in \mathrm{TVL}(K)$ and $(\{2,5\},(\{\mathrm{d},\mathrm{e}\},\{\mathrm{a}\})) \in \mathrm{OEL}(K^\circ)$, which is consistent with Theorem 13.

5. The structure of 3-valued concept lattices

This section studies the structure of 3-valued concept lattices.

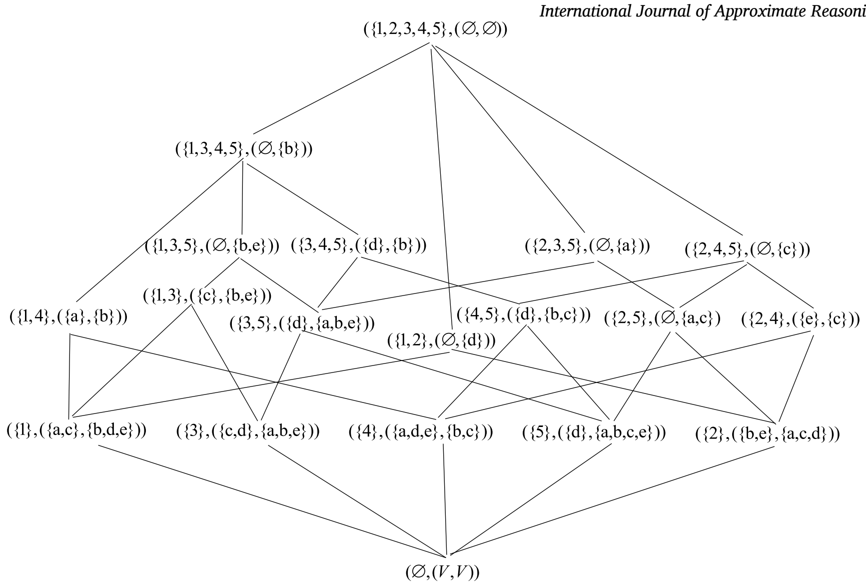

Fig. 6. OEL($K_{\circ}$) of Example 4.

Table 7
The greatest complement K°.

	a	b	c	d	e
1	+	+	+	−	+
2	−	+	−	+	+
3	+	−	+	+	−
4	+	−	+	+	+
5	−	−	+	+	+

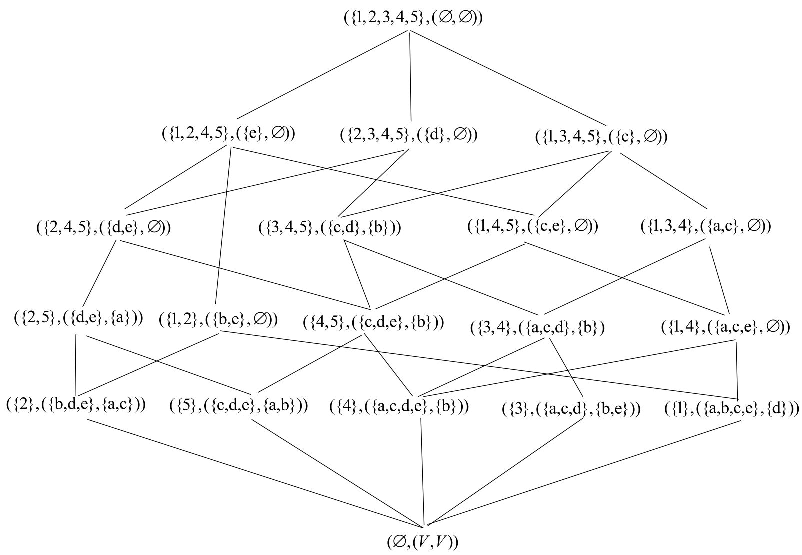

Fig. 7. OEL(K°) of Example 5.

Theorem 14. *Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$. Then, $TVL_{\mathrm{E}}(K)$ is the minimal closure that contains $PL_{\mathrm{E}}(K) \cup NL_{\mathrm{E}}(K) \cup ZL_{\mathrm{E}}(K)$.*

Proof. As TVL(K) is a complete lattice, and $\mathrm{TVL}_{\mathrm{E}}(K)$ is isomorphic to TVL(K), it can be concluded that $\mathrm{TVL}_{\mathrm{E}}(K)$ is a closure.

By Theorem 5, we have $\mathrm{PL}_{\mathrm{E}}(K) \cup \mathrm{NL}_{\mathrm{E}}(K) \cup \mathrm{ZL}_{\mathrm{E}}(K) \subseteq \mathrm{TVL}_{\mathrm{E}}(K)$. Next, we prove the minimality of $\mathrm{TVL}_{\mathrm{E}}(K)$. If $O^{\trianglerighteq} = (A, B, C)$, then it follows that $O = A^{\widetilde{\mathrm{P}}} \cap B^{\widetilde{\mathrm{N}}} \cap C^{\widetilde{\mathrm{Z}}} = \left(\bigcap_{a \in A} \{a\}^{\widetilde{\mathrm{P}}} \right) \cap \left(\bigcap_{b \in B} \{b\}^{\widetilde{\mathrm{N}}} \right) \cap \left(\bigcap_{c \in C} \{c\}^{\widetilde{\mathrm{Z}}} \right)$, where $\{a\}^{\widetilde{\mathrm{P}}} \in \mathrm{PL}_{\mathrm{E}}(K)$, $\{b\}^{\widetilde{\mathrm{N}}} \in \mathrm{NL}_{\mathrm{E}}(K)$, and $\{c\}^{\widetilde{\mathrm{Z}}} \in \mathrm{ZL}_{\mathrm{E}}(K)$.

As a consequence, for any $O \in \mathrm{TVL_E}(K) - \mathrm{PL_E}(K) \cup \mathrm{NL_E}(K) \cup \mathrm{ZL_E}(K)$, we have $O = \bigcap_i O_i$, where $O_i \in \mathrm{PL_E}(K) \cup \mathrm{NL_E}(K) \cup \mathrm{ZL_E}(K)$. Then, it can be concluded that any element of $\mathrm{TVL_E}(K)$ can be generated by the intersection of some elements of $\mathrm{PL_E}(K)$, $\mathrm{NL_E}(K)$, and $\mathrm{ZL_E}(K)$. Therefore, the minimality is proved. □

In what follows, the 3-valued concepts of TVL(K) are classified into five groups based on their connections with the related formal concepts.

Definition 8. Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$, and $(O, (A, B, C)) \in \mathrm{TVL}(K)$.
(i) If $(O, A) \in \mathrm{PL}(K)$, $(O, B) \in \mathrm{NL}(K)$, and $(O, C) \in \mathrm{ZL}(K)$, then $(O, (A, B, C))$ is called a positive 3-valued concept.
(ii) If $(O, A) \notin \mathrm{PL}(K)$, $(O, B) \notin \mathrm{NL}(K)$, and $(O, C) \notin \mathrm{ZL}(K)$, then $(O, (A, B, C))$ is called a negative 3-valued concept.
(iii) If $(O, A) \in \mathrm{PL}(K)$, then $(O, (A, B, C))$ is called a P 3-valued concept.
(iv) If $(O, B) \in \mathrm{NL}(K)$, then $(O, (A, B, C))$ is called a N 3-valued concept.
(v) If $(O, C) \in \mathrm{ZL}(K)$, then $(O, (A, B, C))$ is called a Z 3-valued concept.

Proposition 7. *Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$, and $(O, (A, B, C)) \in TVL(K)$. The following properties hold.*
(i) If $O = A^{\widetilde{\mathrm{P}}} = B^{\widetilde{\mathrm{N}}} = C^{\widetilde{\mathrm{Z}}}$, then $(O, (A, B, C))$ is a positive 3-valued concept.
(ii) If $O \subset A^{\widetilde{\mathrm{P}}}$, $O \subset B^{\widetilde{\mathrm{N}}}$, and $O \subset C^{\widetilde{\mathrm{Z}}}$, then $(O, (A, B, C))$ is a negative 3-valued concept.
(iii) If $O = A^{\widetilde{\mathrm{P}}}$, then $(O, (A, B, C))$ is a P *3-valued concept.*
(iv) If $O = B^{\widetilde{\mathrm{N}}}$, then $(O, (A, B, C))$ is a N *3-valued concept.*
(v) If $O = C^{\widetilde{\mathrm{Z}}}$, then $(O, (A, B, C))$ is a Z *3-valued concept.*

Example 6. Completing Example 1, the 3-valued concepts of TVL(K) can be classified as follows.

- Positive 3-valued concepts: $(\emptyset, (V, V, V))$, $(\{3\}, (\{\mathrm{b}, \mathrm{c}\}, \{\mathrm{d}, \mathrm{e}\}, \{\mathrm{a}\})$, and $(U, (\emptyset, \emptyset, \emptyset))$.
- P 3-valued concept: $(\emptyset, (V, V, V))$, $(\{2\}, (\{\mathrm{a}, \mathrm{e}\}, \{\mathrm{c}, \mathrm{d}\}, \{\mathrm{b}\}))$, $(\{3\}, (\{\mathrm{b}, \mathrm{c}\}, \{\mathrm{d}, \mathrm{e}\}, \{\mathrm{a}\}))$, $(\{5\}, (\{\mathrm{a}, \mathrm{d}\}, \{\mathrm{c}\}, \{\mathrm{b}, \mathrm{e}\}))$, $(\{1, 3\}, (\{\mathrm{b}\}, \{\mathrm{e}\}, \emptyset))$, $(\{2, 5\}, (\{\mathrm{a}\}, \{\mathrm{c}\}, \{\mathrm{b}\}))$, $(\{4, 5\}, (\{\mathrm{d}\}, \emptyset, \emptyset))$, and $(U, (\emptyset, \emptyset, \emptyset))$.
- N 3-valued concept: $(\emptyset, (V, V, V))$, $(\{2\}, (\{\mathrm{a}, \mathrm{e}\}, \{\mathrm{c}, \mathrm{d}\}, \{\mathrm{b}\}))$, $(\{3\}, (\{\mathrm{b}, \mathrm{c}\}, \{\mathrm{d}, \mathrm{e}\}, \{\mathrm{a}\}))$, $(\{4\}, (\{\mathrm{d}\}, \{\mathrm{a}, \mathrm{b}, \mathrm{e}\}, \{\mathrm{c}\}))$, $(\{1, 4\}, (\emptyset, \{\mathrm{a}, \mathrm{e}\}, \{\mathrm{c}\}))$, $(\{2, 5\}, (\{\mathrm{a}\}, \{\mathrm{c}\}, \{\mathrm{b}\}))$, $(\{2, 3\}, (\emptyset, \{\mathrm{d}\}, \emptyset))$, $(\{1, 3, 4\}, \emptyset, \{\mathrm{e}\}, \emptyset))$, and $(U, (\emptyset, \emptyset, \emptyset))$.
- Z 3-valued concept: $(\emptyset, (V, V, V))$, $(\{1\}, (\{\mathrm{b}\}, \{\mathrm{a}, \mathrm{e}\}, \{\mathrm{c}, \mathrm{d}\}))$, $(\{3\}, (\{\mathrm{b}, \mathrm{c}\}, \{\mathrm{d}, \mathrm{e}\}, \{\mathrm{a}\}))$, $(\{5\}, (\{\mathrm{a}, \mathrm{d}\}, \{\mathrm{c}\}, \{\mathrm{b}, \mathrm{e}\}))$, $(\{1, 4\}, (\emptyset, \{\mathrm{a}, \mathrm{e}\}, \{\mathrm{c}\}))$, $(\{2, 5\}, (\{\mathrm{a}\}, \{\mathrm{c}\}, \{\mathrm{b}\}))$, and $(U, (\emptyset, \emptyset, \emptyset))$.

It should be noted that in this example, there is no negative 3-valued concept.

Definition 9. Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$, equivalence relations $\mathrm{R}_{\mathrm{I}_1}$, $\mathrm{R}_{\mathrm{I}_2}$, and $\mathrm{R}_{\mathrm{I}_3}$ on TVL(K) are respectively defined as follows:

$$(O, (A, B, C))\mathrm{R}_{\mathrm{I}_1}(O', (A', B', C')) \Leftrightarrow A = A',$$

$$(O, (A, B, C))\mathrm{R}_{\mathrm{I}_2}(O', (A', B', C')) \Leftrightarrow B = B',$$

$$(O, (A, B, C))\mathrm{R}_{\mathrm{I}_3}(O', (A', B', C')) \Leftrightarrow C = C'.$$

Theorem 15. *Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$. $\bigvee \mathrm{R}_{\mathrm{I}_1}[A] = \left(X, \left(A, X^{\mathrm{N}}, X^{\mathrm{Z}}\right)\right)$, $\bigvee \mathrm{R}_{\mathrm{I}_2}[B] = \left(Y, \left(Y^{\mathrm{P}}, B, Y^{\mathrm{Z}}\right)\right)$, $\bigvee \mathrm{R}_{\mathrm{I}_3}[C] = \left(W, \left(W^{\mathrm{P}}, W^{\mathrm{N}}, C\right)\right)$, if and only if $(X, A) \in PL(K)$, $(Y, B) \in NL(K)$, and $(W, C) \in ZL(K)$.*

Proof. "⇒". As $(X, A) \in \mathrm{PL}(K)$, we have $X = A^{\widetilde{\mathrm{P}}}$. Then, it follows that $\bigvee \mathrm{R}_{\mathrm{I}_1}[A] = \left(A^{\widetilde{\mathrm{P}}}, \left(A, A^{\widetilde{\mathrm{P}}\mathrm{N}}, A^{\widetilde{\mathrm{P}}\mathrm{Z}}\right)\right) = \left(X, \left(A, X^{\mathrm{N}}, X^{\mathrm{Z}}\right)\right)$. Similarly, we have $\bigvee \mathrm{R}_{\mathrm{I}_2}[B] = \left(Y, \left(Y^{\mathrm{P}}, B, Y^{\mathrm{Z}}\right)\right)$ and $\bigvee \mathrm{R}_{\mathrm{I}_3}[C] = \left(W, \left(W^{\mathrm{P}}, W^{\mathrm{N}}, C\right)\right)$.

"⇐". On one hand, for any $\left(A^{\widetilde{\mathrm{P}}}, \left(A, A^{\widetilde{\mathrm{P}}\mathrm{N}}, A^{\widetilde{\mathrm{P}}\mathrm{Z}}\right)\right) \in \mathrm{R}_{\mathrm{I}_1}[A]$, we have $X = \left(\bigcup_{(X,(A,B,C)) \in \mathrm{R}_{\mathrm{I}_1}[A]} X\right)^{\trianglerighteq\trianglelefteq} \supseteq \bigcup_{(X,(A,B,C)) \in \mathrm{R}_{\mathrm{I}_1}[A]} X \supseteq A^{\widetilde{\mathrm{P}}}$. On the other hand, for any $(X, (A, B, C)) \in \mathrm{R}_{\mathrm{I}_1}[A]$, we have that $X = A^{\widetilde{\mathrm{P}}} \cap B^{\widetilde{\mathrm{N}}} \cap C^{\widetilde{\mathrm{Z}}} \subseteq A^{\widetilde{\mathrm{P}}}$, which leads to $X = \left(\bigcup_{(X,(A,B,C)) \in \mathrm{R}_{\mathrm{I}_1}[A]} X\right)^{\trianglerighteq\trianglelefteq} \subseteq A^{\widetilde{\mathrm{P}}\trianglerighteq\trianglelefteq}$. Moreover, as $A^{\widetilde{\mathrm{P}}\trianglerighteq\trianglelefteq} = \left(A, A^{\widetilde{\mathrm{P}}\mathrm{N}}, A^{\widetilde{\mathrm{P}}\mathrm{Z}}\right)^{\trianglelefteq} = A^{\widetilde{\mathrm{P}}} \cap A^{\widetilde{\mathrm{P}}\mathrm{N}\widetilde{\mathrm{N}}} \cap A^{\widetilde{\mathrm{P}}\mathrm{Z}\widetilde{\mathrm{Z}}} = A^{\widetilde{\mathrm{P}}}$, we have $X \subseteq A^{\widetilde{\mathrm{P}}}$. To sum up, $X = A^{\widetilde{\mathrm{P}}}$ is at hand. Then, we have $(X, A) = \left(A^{\widetilde{\mathrm{P}}}, A\right) \in \mathrm{PL}(K)$. Similarly, $(Y, B) = \left(B^{\widetilde{\mathrm{N}}}, B\right) \in \mathrm{NL}(K)$ and $(W, C) = \left(C^{\widetilde{\mathrm{Z}}}, C\right) \in \mathrm{ZL}(K)$ can be proved. □

Definition 10. Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$, 3 mappings from fundamental concepts to equivalent 3-valued concepts $\mathrm{PT} : \mathrm{PL}(K) \to \mathrm{TVL}(K)/\mathrm{R}_{\mathrm{I}_1}$, $\mathrm{NT} : \mathrm{NL}(K) \to \mathrm{TVL}(K)/\mathrm{R}_{\mathrm{I}_2}$, and $\mathrm{ZT} : \mathrm{ZL}(K) \to \mathrm{TVL}(K)/\mathrm{R}_{\mathrm{I}_3}$ are respectively defined as follows:

$$\mathrm{PT}((X, A)) = \mathrm{R}_{\mathrm{I}_1}[A],$$

$$\mathrm{NT}((Y, B)) = \mathrm{R}_{\mathrm{I}_2}[B],$$

$\mathrm{ZT}((W,C)) = \mathrm{R}_{\mathrm{I}_3}[C]$.

Theorem 16. *Let $K = (U, V, \{\mathrm{P}, \mathrm{N}, \mathrm{Z}\}, J)$. Then the following statements hold.*
(i) PT is a join-preserving and meet-preserving from PL(K) to TVL$(K)/\mathrm{R}_{\mathrm{I}_1}$.
(ii) NT is a join-preserving and meet-preserving from NL(K) to TVL$(K)/\mathrm{R}_{\mathrm{I}_2}$.
(iii) ZT is a join-preserving and meet-preserving from ZL(K) to TVL$(K)/\mathrm{R}_{\mathrm{I}_3}$.

Proof. (i) For any $(X_1, A_1), (X_2, A_2) \in \mathrm{PL}(K)$, we have

$$\mathrm{PT}((X_1, A_1) \vee (X_2, A_2)) = \mathrm{PT}\left(\left((X_1 \cup X_2)^{\mathrm{P}\widetilde{\mathrm{P}}}, A_1 \cap A_2\right)\right) = \mathrm{R}_{\mathrm{I}_1}[A_1 \cap A_2],$$

and

$$\begin{aligned}
&\mathrm{PT}((X_1, A_1)) \vee \mathrm{PT}((X_2, A_2)) \\
&= \mathrm{R}_{\mathrm{I}_1}[A_1] \vee \mathrm{R}_{\mathrm{I}_1}[A_2] \\
&= \left(\bigvee \mathrm{R}_{\mathrm{I}_1}[A_1]\right) \vee \left(\bigvee \mathrm{R}_{\mathrm{I}_1}[A_2]\right) \\
&= \left(A_1^{\widetilde{\mathrm{P}}}, \left(A_1, A_1^{\widetilde{\mathrm{P}}\mathrm{N}}, A_1^{\widetilde{\mathrm{P}}\mathrm{Z}}\right)\right) \vee \left(A_2^{\widetilde{\mathrm{P}}}, \left(A_2, A_2^{\widetilde{\mathrm{P}}\mathrm{N}}, A_2^{\widetilde{\mathrm{P}}\mathrm{Z}}\right)\right) \\
&= \left(\left(A_1^{\widetilde{\mathrm{P}}} \cup A_2^{\widetilde{\mathrm{P}}}\right)^{\lhd\rhd}, \left(A_1 \cap A_2, A_1^{\widetilde{\mathrm{P}}\mathrm{N}} \cap A_2^{\widetilde{\mathrm{P}}\mathrm{N}}, A_1^{\widetilde{\mathrm{P}}\mathrm{Z}} \cap A_2^{\widetilde{\mathrm{P}}\mathrm{Z}}\right)\right).
\end{aligned}$$

As a consequence, we have $\mathrm{PT}((X_1, A_1) \vee (X_2, A_2)) = \mathrm{PT}((X_1, A_2)) \vee \mathrm{PT}((X_2, A_2))$.

Moreover, we also have

$$\begin{aligned}
&(X_1, A_1) \leq (X_2, A_2) \\
&\Leftrightarrow X_1 \subseteq X_2 \\
&= \left(X_1, \left(A_1, X_1^{\mathrm{N}}, X_1^{\mathrm{Z}}\right)\right) \leq \left(X_2, \left(A_2, X_2^{\mathrm{N}}, X_2^{\mathrm{Z}}\right)\right) \\
&\Leftrightarrow \mathrm{R}_{\mathrm{I}_1}[A_1] \leq \mathrm{R}_{\mathrm{I}_1}[A_2] \Leftrightarrow \mathrm{PT}(X_1, A_1) \leq \mathrm{PT}(X_2, A_2).
\end{aligned}$$

Then, this statement is proved. Similarly, statements (ii) and (iii) can be proved. □

6. Conclusion

In this study, the most important properties of 3-valued formal contexts are studied via the comparison of the existing types of concepts. It is revealed that 3-valued formal context as well as 3-valued formal concept lattices can provide more unique details, although they are closely related to formal contexts and the existing concept lattice models.

In order to effectively apply the 3-values concept analysis for large-scale data, several important problems should be taken into consideration, including efficient conversion mechanisms between 3-valued concept lattices and other types of concept lattices, attribute reduction of 3-valued formal contexts based on 3-valued concept lattices, cognitive learning mechanisms of 3-valued concepts, and so forth. In addition, in order to effectively deal with the rapid growth of multimodal data in applications, it is also a promising and interesting study to integrate multi-granularity analysis and multi-level analysis into 3-valued concept analysis.

CRediT authorship contribution statement

Huilai Zhi: Writing – review & editing, Writing – original draft, Methodology, Funding acquisition, Formal analysis, Conceptualization. **Qing Wan:** Supervision. **Ting Qian:** Software. **Yinan Li:** Software, Methodology, Investigation. **Jiang Yang:** Writing – review & editing, Writing – original draft, Methodology.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data will be made available on request.

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