International Journal of Approximate Reasoning 187 (2025) 109543

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

Optimizations of approximation operators in covering rough set theory

Shizhe Zhang [a,b], Liwen Ma [a,b,*]

[a] School of Mathematical Sciences, Beijing University of Posts and Telecommunications, Beijing 100876, PR China

[b] Key Laboratory of Mathematics and Information Networks (Beijing University of Posts and Telecommunications), Ministry of Education, PR China

ARTICLE INFO

Keywords:
Covering rough set
Approximation operator optimization
Stable operator
Boolean matrix

ABSTRACT

Classical rough set theory fundamentally requires upper and lower approximations to be definite sets for precise knowledge representation. However, a significant problem arises as many widely used approximation operators inherently produce rough approximations (with non-empty boundaries), contradicting this core theoretical intent and undermining practical applicability. To resolve this core discrepancy, we introduce stable approximation operators and stable sets, and develop an optimization method that transforms unstable operators into stable ones, ensuring definite approximations. This method includes detailing the optimization process with algorithmic implementation, analyzing the topological structure of resulting approximation spaces and connections between optimized operators, and enhancing computational efficiency via matrix-based computation. This work may strengthen rough set theory's foundation by bridging the gap between theory and practice while enhancing its scope for practical applications.

1. Introduction

Since rough set theory was first proposed by Pawlak in 1982 [7], it has been widely used in different fields. Initially, the theory uses an operator model based on equivalence relation to characterize the indistinguishability between elements, providing a big data processing technique for dealing with uncertain information. However, when dealing with more complex real-world scenarios, the limitations of this classical model become apparent because it fails to integrate key information such as inclusivity and overlap between sets. To address these limitations, researchers have extended the original equivalence-based framework to more versatile non-equivalence-based models. After decades of development, the theory has diversified into multiple branches incorporating coverings [10], binary relations [13], fuzzy sets [11], and lattice structures, each of which demonstrating significant practical utility.

Covering rough set theory, as one of the main branches of rough set theory, uses overlapping covering sets to generate the neighborhoods of each element as the most basic descriptors to define the operator models. Contemporary research has revealed multiple neighborhood types, giving rise to various approximation operator models. Zhu has systematically established the theoretical framework of covering rough sets. In 2003, Zhu [12] proposed replacing the partition in classical rough sets with a covering, completed the theoretical system by introducing covering reduction algorithms and an axiomatic framework, demonstrating compatibility with Pawlak's rough set axioms. In 2006, Zhu [14] further analyzed the topological properties of covering rough sets, linking them to closure and interior operators. Together, these works laid the foundation for subsequent extensions and applications. In 2012, Yao

* Corresponding author at: School of Mathematical Sciences, Beijing University of Posts and Telecommunications, Beijing 100876, PR China.
E-mail addresses: zhangshizhe02@163.com (S. Zhang), maliwenmath@sina.com (L. Ma).

https://doi.org/10.1016/j.ijar.2025.109543
Received 1 May 2025; Received in revised form 3 July 2025; Accepted 1 August 2025

Available online 7 August 2025

and Yao [10] established a more comprehensive framework for covering rough sets, and their three-tiered framework categorizes approximation operators into element-based, granule-based, and subsystem-based ones.

However, we find a serious limitation from these studies, that is, the approximate sets of many operators are no longer definite sets, but still rough sets, thus deviating from Pawlak's basic principle, and we call such operators unstable operators. The instability of approximate operators may cause two primary consequences in practice. Firstly, computational complexity is significantly increased. The failure of the initial approximation to provide a precise description necessitates iterative approximations on the results to achieve higher accuracy. This not only substantially increases the computational load but also complicates algorithm design. Secondly, information loss and a decline in precision are exacerbated. Each approximation operation inherently constitutes an uncertain representation of the set: the lower approximation excludes elements potentially belonging to the target set, while the upper approximation includes elements potentially not belonging to it. Applying approximation again to such a result amounts to a secondary blurring of already imprecise information. This process may cause the final result to deviate considerably from the true information of the original set, leading to a marked reduction in precision. The purpose of this work is to optimize these unstable operators into stable operators, make their approximate sets be definite sets, so as to restore the original intention of rough set theory.

In our study, we identified that the fundamental cause of the instability in approximation operators lies in the lack of transitivity in the neighborhoods generating these operators. Consequently, we first introduce the concepts of stable approximate sets and stable operators, and then, the optimization method, theoretical basis, optimization steps, optimization algorithms and properties of unstable operators are introduced in detail. Moreover, we rise to the framework of topology theory to explore the essence of related theories. Through rigorous theoretical development, we demonstrate that each optimized operator induces a topological structure on the space, and conversely, that these topological structures can be derived from both the optimized approximation operators and their corresponding neighborhood systems. In particular, building upon previous research, Gao [2] introduced a modular computational framework for approximation operations, while Ma [4] developed an innovative matrix-based methodology. In the current study, we integrate and employ both the matrix approach and modular techniques to further optimize the calculation methods of the optimized operators, thereby enhancing computational efficiency and theoretical rigor.

This paper is organized as follows. Some fundamental concepts about rough set and covering approximation space will be introduced in Section 2. Some generalized results about optimized operators will be proposed in Section 3, and the optimization of known operators and relative topological properties will be studied in Section 4. Finally, the matrix and modular approaches have been used to compute optimized operators in Section 5.

2. Fundamental knowledge

First we recall the Pawlak's classical definition of rough set for subsequent need.

Definition 2.1 *([7])*. Let U be a universe and $\mathcal{P}$ be a partition of U. Then for each $X \subseteq U$, the lower and upper approximations of X are defined as

$$L(X) = \bigcup\{P \in \mathcal{P} : P \subseteq X\}, \quad H(X) = \bigcup\{P \in \mathcal{P} : P \cap X \neq \emptyset\}.$$

If $L(X) \neq H(X)$, X is called a *rough set*. Otherwise, if $L(X) = X = H(X)$, X is called a *definite set*.

In the sense of Definition 2.1, all sets in the family

$$\left\{ \bigcup_{P \in \mathcal{P}_1} P : \ \mathcal{P}_1 \subseteq \mathcal{P} \right\}$$

are definite sets, and the rough sets are approximated by the definite sets.

For practical application, the classical rough set theory is generalized to covering rough set theory. We then recall the basic knowledge of the covering rough set theory.

Definition 2.2 *([14,15])*. Let U be a universe and $\mathcal{C}$ be a family of non-empty subsets of U. If $U = \bigcup_{C \in \mathcal{C}} C$, $\mathcal{C}$ is called a *covering* of U and the ordered pair $(U, \mathcal{C})$ is called a *covering approximation space*.

Definition 2.3 *([4,5,10])*. Let $(U, \mathcal{C})$ be a covering approximation space. The families

$$\underline{\mathcal{M}}(x) = \{C \in \mathcal{C}(x) : \forall S \in \mathcal{C}(x)(S \subseteq C \rightarrow C = S)\},$$

$$\overline{\mathcal{M}}(x) = \{C \in \mathcal{C}(x) : \forall S \in \mathcal{C}(x)(S \supseteq C \rightarrow C = S)\},$$

are called the *minimal* and *maximal descriptions* of x respectively, where

$$\mathcal{C}(x) = \{C \in \mathcal{C} : x \in C\}.$$

For each $x \in U$, the following four subsets of U

$$N_1(x) = \bigcap \underline{\mathcal{M}}(x) = \bigcap\{C : C \in \underline{\mathcal{M}}(x)\}, \ N_2(x) = \bigcup \underline{\mathcal{M}}(x) = \bigcup\{C : C \in \underline{\mathcal{M}}(x)\},$$

$$N_3(x) = \bigcap \overline{\mathcal{M}}(x) = \bigcap \{C : C \in \overline{\mathcal{M}}(x)\},\ N_4(x) = \bigcup \overline{\mathcal{M}}(x) = \bigcup \{C : C \in \overline{\mathcal{M}}(x)\},$$

are called the *neighborhoods* of x and N_i $(i = 1, 2, 3, 4)$ are called the *neighborhood operators* on U. Based on these neighborhoods, the *co-neighborhood operators* M_i $(i = 1, 2, 3, 4)$ on U and the corresponding *co-neighborhoods* $M_i(x)$ $(i = 1, 2, 3, 4)$ of x are defined as

$$M_i(x) = \{y \in U : x \in N_i(y)\}.$$

Moreover, for every subset $A \subseteq U$, $N_i(A)$ and $M_i(A)$ for $i = 1, 2, 3, 4$ are defined as

$$N_i(A) = \bigcup \{N_i(x) : x \in A\},\quad M_i(A) = \bigcup \{M_i(x) : x \in A\}.$$

Example 2.4. Let $U = \{x_1, x_2, x_3, x_4\}$ and $C = \{\{x_1, x_2\}, \{x_1, x_3, x_4\}, \{x_1, x_2, x_4\}, \{x_1, x_3\}, \{x_3\}, \{x_4\}, \{x_3, x_4\}\}$. Then we have that

$$\begin{array}{ll}
N_1(x_1) = \{x_1\}, & M_1(x_1) = \{x_1, x_2\}, \\
N_2(x_1) = \{x_1, x_2, x_3\}, & M_2(x_1) = \{x_1, x_2\}, \\
N_3(x_1) = \{x_1, x_4\}, & M_3(x_1) = \{x_1, x_2, x_3, x_4\}, \\
N_4(x_1) = \{x_1, x_2, x_3, x_4\}, & M_4(x_1) = \{x_1, x_2, x_3, x_4\}.
\end{array}$$

For the convenience of discussion, we will introduce a neighborhood operator t as follows, which, as a representative, unifies the characteristics of these neighborhood operators in Definition 2.3.

Definition 2.5 *([8,9]).* Let (U, C) be a covering approximation space. An operator $t : U \to \mathscr{P}(U)$ is called

(1) a *reflexive neighborhood operator*, if for any $x \in U$, $x \in t(x)$;
(2) a *transitive operator*, if for any $x \in U$ and $y \in t(x)$, $t(y) \subseteq t(x)$.

Remark 2.6. Clearly, the neighborhood operators N_i and co-neighborhood operators M_i $(i = 1, 2, 3, 4)$ defined in Definition 2.3 are all reflexive neighborhood operators on U.

Definition 2.7. Let (U, C) be a covering approximation space and t be a reflexive neighborhood operator on U. Then, for any subset $X \subseteq U$, the operators defined by

$$O^-(X) = \{x : t(x) \subseteq X\}$$

and

$$O^+(X) = \{x : t(x) \cap X \neq \emptyset\}$$

are called the *lower* and *upper approximate operator induced by* t, respectively.

Remark 2.8. It is easy to verify that the upper operator defined in Definition 2.7 is equivalent to

$$O^+(X) = -O^-(-X).$$

Proposition 2.9 *([5]). Let (U, C) be a covering approximation space. Then, for every $x \in U$, it holds that $N_4(x) = M_4(x)$.*

The following proposition can be obtained easily, which shows the relationships between neighborhoods and co-neighborhoods.

Proposition 2.10. *Let (U, C) be a covering approximation space. Then, for $i = 1, 2, 3, 4$, the following statements hold for any $x, y \in U$:*

(1) $y \in N_i(x)$ if and only if $x \in M_i(y)$;
(2) $M_i(x) = \{y : x \in N_i(y)\}$ and $N_i(x) = \{y : x \in M_i(y)\}$.

We will focus on the following common approximation operators to explore the optimization topic [5,10], where $-X$ represents the set $U - X$ of $X \subset U$ in this paper.

Definition 2.11 *([5,6]).* Let (U, C) be a covering approximation space, and X be a subset of U. For $i = 1, 2, 3, 4$, the lower approximations and upper approximations of X are defined as follows:

$$\begin{array}{ll}
PN_i^-(X) = \{x : N_i(x) \subseteq X\}, & PN_i^+(X) = \{x : N_i(x) \cap X \neq \emptyset\}, \\
RN_i^-(X) = -RN_i^+(-X), & RN_i^+(X) = \bigcup \{N_i(x) : N_i(x) \cap X \neq \emptyset\}, \\
PM_i^-(X) = \{x : M_i(x) \subseteq X\}, & PM_i^+(X) = \{x : M_i(x) \cap X \neq \emptyset\},
\end{array}$$

$$RM_i^-(X) = -RM_i^+(-X), \qquad RM_i^+(X) = \bigcup\{M_i(x) : M_i(x) \cap X \neq \emptyset\}.$$

From the above definition we can see that the two pairs $(PN_i^-,\ PN_i^+)$ and $(PM_i^-,\ PM_i^+)$ are element-based operators. The following proposition shows that the two pairs $(RN_i^-,\ RN_i^+)$ and $(RM_i^-,\ RM_i^+)$ are also element-based operators. In fact, we will learn in the next section that, when the neighborhoods are changed appropriately, all the operators above can be treated as granule-based operators.

Proposition 2.12. *Let (U, C) be a covering approximation space, and X be a subset of U. Then the following formulas can be derived:*

$$RN_i^-(X) = \{x : N_i(M_i(x)) \subseteq X\}, \quad RN_i^+(X) = \{x : N_i(M_i(x)) \cap X \neq \emptyset\},$$
$$RM_i^-(X) = \{x : M_i(N_i(x)) \subseteq X\}, \quad RM_i^+(X) = \{x : M_i(N_i(x)) \cap X \neq \emptyset\}.$$

Proof. We first prove that $RN_i^-(X) = -RN_i^+(-X) = \{x : N_i(M_i(x)) \subseteq X\}$. By using Definition 2.11, we have that

$$-RN_i^+(-X) = -\bigcup\left\{N_i(X) : N_i(X) \cap (-X) \neq \emptyset\right\} = \bigcap\left\{-N_i(X) : N_i(X) \cap (-X) \neq \emptyset\right\}.$$

Then, for each $y \in U$, it follows that

$$\begin{aligned}
& y \in RN_i^-(X) \\
\Leftrightarrow\ & y \in \bigcap\left\{-N_i(x) : N_i(x) \cap (-X) \neq \emptyset\right\} \\
\Leftrightarrow\ & \forall x \left[N_i(x) \cap (-X) \neq \emptyset \to y \notin N_i(x)\right] \\
\Leftrightarrow\ & \forall x \left[y \in N_i(x) \to N_i(x) \subset X\right] \\
\Leftrightarrow\ & \forall x \left[x \in M_i(y) \to N_i(x) \subset X\right] \\
\Leftrightarrow\ & N_i(M_i(y)) \subseteq X \\
\Leftrightarrow\ & y \in \{x : N_i(M_i(x)) \subseteq X\}.
\end{aligned}$$

Next, we prove that $RN_i^+(X) = \{x : N_i(M_i(x)) \cap X \neq \emptyset\}$. By using Definition 2.11 again, we have that

$$\begin{aligned}
& y \in RN_i^+(X) \\
\Leftrightarrow\ & \exists x \left[y \in N_i(x) \wedge N_i(x) \cap X \neq \emptyset\right] \\
\Leftrightarrow\ & \exists x \left[x \in M_i(y) \wedge N_i(x) \cap X \neq \emptyset\right] \\
\Leftrightarrow\ & N_i(M_i(y)) \cap X \neq \emptyset \\
\Leftrightarrow\ & y \in \{x : N_i(M_i(x)) \cap X \neq \emptyset\}.
\end{aligned}$$

Therefore, the first two equations hold. Similarly, we can verify the last two equations. □

At the end of this section, we briefly introduce some definitions about topology to facilitate subsequent discussions on the topological properties induced by operators.

Definition 2.13 *([1,3])*. Let X be a nonempty set and $\mathcal{T}$ be a family of subsets of X. The pair $(X, \mathcal{T})$ is called a *topological space* if the following conditions are satisfied:

(1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$;
(2) $A \cap B \in \mathcal{T}$, when $A, B \in \mathcal{T}$;
(3) $\bigcup \mathcal{T}_1 \in \mathcal{T}$ for any subfamily $\mathcal{T}_1 \subseteq \mathcal{T}$.

Moreover, $\mathcal{T}$ is called a *topology* on X, and each element of $\mathcal{T}$ is called an *open set* in the topological space $(X, \mathcal{T})$, and if an open set contains x, it is called a *neighborhood* of x.

Definition 2.14 *([1,3])*. Let $(X, \mathcal{T})$ be a topological space, and $\mathcal{B} \subseteq \mathcal{T}$ be a family of open sets in X. If for each $x \in X$ and each open set U that contains x, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$, then $\mathcal{B}$ is called a *basis* of the topology $\mathcal{T}$.

Definition 2.15 *([1,3])*. Let X be a nonempty set. The map $int : \mathscr{P}(X) \to \mathscr{P}(X)$ is called an *interior operator* on X, if for any $A \subseteq X$, int satisfies that

(1) $int(X) = X$;
(2) $int(A) \subseteq A$;
(3) $int(int(A)) = int(A)$.

Definition 2.16 *([1,3]).* Let X be a nonempty set. The map $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ is called a *closure operator* on X, if for any $A \subseteq X$, cl satisfies that

(1) $cl(\emptyset) = \emptyset$;
(2) $cl(A) \supseteq A$;
(3) $cl(cl(A)) = cl(A)$.

Covering rough set theory naturally extends Pawlak's classical rough set theory, but it is worth noting that many of the approximate sets commonly used today are not definite sets. This is far from the original intention of Pawlak's classical rough set theory.

For a simple example, let $U = \{x_1, x_2, x_3, x_4\}$ and $C = \left\{\{x_n, x_{n+1}\} : n = 1, 2, 3\right\}$. Then for $X = \{x_1, x_2\}$, we have $PN_4^+(X) = \{x_1, x_2, x_3\}$, and $PN_4^+(PN_4^+(X)) = \{x_1, x_2, x_3, x_4\}$. Thus the approximate set $PN_4^+(X)$ of X is not a definite set.

Therefore, this paper aims to restore Pawlak's original idea of rough set theory and optimize some common operators so that their approximate sets are definite sets.

3. The basic idea of operator optimization

In order to solve the optimization problem, we need to introduce new concepts of the stable operator and stable set as follows.

Definition 3.1. Let (U, C) be a covering approximation space and O be an operator on U, that is, a mapping $O : \mathscr{P}(U) \to \mathscr{P}(U)$.

(1) A subset $X \subseteq U$ is called a *stable set* under O, if it satisfies that $O(X) = X$.
(2) The operator O is called a *stable operator*, if for each subset $X \subseteq U$, it holds that $O(O(X)) = O(X)$.

That is to say, O is a stable operator if and only if $O(X)$ is stable for each subset $X \subseteq U$.

Obviously, both of Palwak's classical lower and upper operators L and H are stable operators, and all approximate sets $L(X)$ and $H(X)$ are stable sets under their corresponding stable operators. In fact, in the original Pawlak's classical rough set theory, the pure intention is to study the rough sets by transforming them into the stable sets called lower and upper approximations using stable operators. However, after the classical rough set theory is extended to the covering rough set theory, the upper and lower approximations cannot be guaranteed to be completely stable sets, that is to say, they are still rough sets. This is undoubtedly contrary to the original intention of the Pawlak's classical rough set theory. In the theory of covering approximation spaces, we find that the non-transitivity of the neighborhood operator is the primary cause of the instability of the approximation operator induced by this neighborhood operator.

Proposition 3.2. *Let (U, C) be a covering approximation space, and t be a reflexive neighborhood operator on U. If t is also transitive, then, for any subset $X \subseteq U$, the approximate operators O^- and O^+ induced by t are stable operators.*

Proof. It is obvious that for any $X \subseteq U$, $O^-(O^-(X)) \subseteq O^-(X)$. Now we prove the converse. For any $y \in U$, we have that

$$
\begin{aligned}
& y \in O^-(X) \\
\Rightarrow\ & t(y) \subseteq X \\
\Rightarrow\ & \forall z[z \in t(y) \to t(z) \subseteq t(y) \subseteq X] \\
\Rightarrow\ & \forall z[z \in t(y) \to z \in O^-(X)] \\
\Rightarrow\ & t(y) \subseteq O^-(X) \\
\Rightarrow\ & y \in O^-(O^-(X)).
\end{aligned}
$$

Thus we have proved that $O^-(O^-(X)) = O^-(X)$, i.e., O^- is a stable operator on U, and by duality, O^+ is also stable on U. □

Proposition 3.3. *Let (U, C) be a covering approximation space. Then, for any subset $X \subseteq U$ and $i = 1, 3$, the following equations hold:*

$$PN_i^-(PN_i^-(X)) = PN_i^-(X), \quad PN_i^+(PN_i^+(X)) = PN_i^+(X),$$
$$PM_i^-(PM_i^-(X)) = PM_i^-(X), \quad PM_i^+(PM_i^+(X)) = PM_i^+(X).$$

Therefore, all the operators PN_i^-, PN_i^+, PM_i^-, PM_i^+ $(i = 1, 3)$ *are stable operators.*

Proof. We only need to prove that the reflexive neighborhood operators N_i and M_i, $i = 1, 3$, are transitive. For any $x \in U$ and $y \in N_1(x)$, by Definition 2.3, we have that

$$C \in \underline{\mathcal{M}}(x) \text{ implies } y \in C,$$

which means that

$$\underline{\mathcal{M}}(y) \supseteq \underline{\mathcal{M}}(x),$$

and thus

$$\bigcap \underline{\mathcal{M}}(y) \subseteq \bigcap \underline{\mathcal{M}}(x).$$

Therefore, it holds that $N_1(y) \subseteq N_1(x)$, i.e., N_1 is transitive.

For any $y \in M_1(x)$ and $z \in M_1(y)$ we have that

$$x \in N_1(y) \text{ and } y \in N_1(z).$$

By using the transitivity of N_1, we have that $x \in N_1(z)$, and equivalently, $z \in M_1(x)$, which shows that M_1 is also transitive.

The proof for the transitivity of N_3 and M_3 follows a similar argument as above. □

Next we give an example to show that all the operators PN_i^-, PN_i^+, PM_i^-, PM_i^+ $(i = 2, 4)$ and RN_i^-, RN_i^+, RM_i^-, $RM_i^+ (i = 1, 2, 3, 4)$ may not be stable operators in general.

Example 3.4. Let $U = \{x_1, x_2, \ldots, x_6\}$, $C = \left\{\{x_2, x_4, x_5\}, \{x_3, x_5, x_6\}, \{x_1, x_6\}, \{x_2, x_4\}\right\}$, $X_1 = \{x_1, x_2, x_3, x_5, x_6\}$, $X_2 = \{x_5\}$, $X_3 = \{x_1, x_3, x_6\}$, and $X_4 = \{x_4\}$. Then we have that

$$\begin{aligned}
&PN_2^-(X_1) = \{x_1, x_3, x_6\}, && PN_2^-\left(PN_2^-(X_1)\right) = \{x_1\},\\
&PN_4^-(X_3) = \{x_1\}, && PN_4^-\left(PN_4^-(X_3)\right) = \emptyset,
\end{aligned}$$

$$\begin{aligned}
&PN_2^+(X_2) = \{x_3, x_5, x_6\}, && PN_2^+\left(PN_2^+(X_2)\right) = \{x_1, x_3, x_5, x_6\},\\
&PN_4^+(X_3) = \{x_1, x_3, x_5, x_6\}, && PN_4^+\left(PN_4^+(X_3)\right) = U,
\end{aligned}$$

$$\begin{aligned}
&PM_2^-(X_3) = \{x_1\}, && PM_2^-\left(PM_2^-(X_3)\right) = \emptyset,\\
&PM_4^-(X_3) = \{x_1\}, && PM_4^-\left(PM_4^-(X_3)\right) = \emptyset,
\end{aligned}$$

$$\begin{aligned}
&PM_2^+(X_2) = U - \{x_1\}, && PM_2^+\left(PM_2^+(X_2)\right) = U,\\
&PM_4^+(X_2) = U - \{x_1\}, && PM_4^+\left(PM_4^+(X_2)\right) = U,
\end{aligned}$$

$$\begin{aligned}
&RN_1^-(X_3) = \{x_1\}, && RN_1^-\left(RN_1^-(X_3)\right) = \emptyset,\\
&RN_2^-(X_1) = \{x_1\}, && RN_2^-\left(RN_2^-(X_1)\right) = \emptyset,\\
&RN_3^-(X_1) = \{x_1, x_3, x_6\}, && RN_3^-\left(RN_3^-(X_1)\right) = \{x_1\},\\
&RN_4^-(X_1) = \{x_1\}, && RN_4^-\left(RN_4^-(X_1)\right) = \emptyset,
\end{aligned}$$

$$\begin{aligned}
&RN_1^+(X_2) = \{x_3, x_5, x_6\}, && RN_1^+\left(RN_1^+(X_2)\right) = \{x_1, x_3, x_5, x_6\},\\
&RN_2^+(X_2) = U - \{x_1\}, && RN_2^+\left(RN_2^+(X_2)\right) = U,\\
&RN_3^+(X_2) = U - \{x_1\}, && RN_3^+\left(RN_3^+(X_2)\right) = U,\\
&RN_4^+(X_2) = U - \{x_1\}, && RN_4^+\left(RN_4^+(X_2)\right) = U,
\end{aligned}$$

$$
\begin{aligned}
&RM_1^-(X_3)=\{x_1,x_6\}, && RM_1^-\left(RM_1^-(X_3)\right)=\emptyset,\\
&RM_2^-(X_1)=\{x_1,x_3,x_6\}, && RM_2^-\left(RM_2^-(X_1)\right)=\emptyset,\\
&RM_3^-(X_3)=\{x_1,x_6\}, && RM_3^-\left(RM_3^-(X_3)\right)=\emptyset,\\
&RM_4^-(X_1)=\{x_1\}, && RM_4^-\left(RM_4^-(X_1)\right)=\emptyset,
\end{aligned}
$$

$$
\begin{aligned}
&RM_1^+(X_2)=\{x_3,x_5\}, && RM_1^+\left(RM_1^+(X_2)\right)=\{x_1,x_3,x_5,x_6\},\\
&RM_2^+(X_3)=\{x_1,x_3,x_5,x_6\}, && RM_2^+\left(RM_2^+(X_3)\right)=U,\\
&RM_3^+(X_2)=\{x_2,x_3,x_4,x_5\}, && RM_3^+\left(RM_3^+(X_2)\right)=U,\\
&RM_4^+(X_4)=U-\{x_1\}, && RM_4^+\left(RM_4^+(X_4)\right)=U.
\end{aligned}
$$

If an approximation operator is not stable, using it to compute the lower and upper approximations of a rough set are meaningless because the approximation sets are still rough sets. Therefore, we need to optimize these operators so that the approximate sets under the optimized operators are stable sets.

Definition 3.5. Let $(U,\mathcal{C})$ be a covering approximation space, O^- and O^+ be dual lower and upper approximate operators on U. Let

$$\mathcal{A}^O=\{A\subseteq U : O^-(A)=A\},\ \ \mathcal{B}^O=\{B\subseteq U : O^+(B)=B\}.$$

The *optimized operators* O^{-*} of O^- and O^{+*} of O^+ are defined as follows:

$$O^{-*}(X)=\bigcup\{A\in\mathcal{A}^O : A\subseteq X\},\ \ O^{+*}(X)=\bigcap\{B\in\mathcal{B}^O : X\subseteq B\}.$$

We can prove that the optimized operators O^{-*} and O^{+*} are stable operators, and they share all important properties of Pawlak's classical operators.

Proposition 3.6. *Let $(U,\mathcal{C})$ be a covering approximation space, O^-, O^+ be lower and upper approximate operators on U, and O^{-*}, O^{+*} be their optimized operators respectively. Then both O^{-*} and O^{+*} are stable operators. Moreover, the following statements hold:*

(1) $O^{-*}(X)\subseteq O^-(X)\subseteq X\subseteq O^+(X)\subseteq O^{+*}(X)$;
(2) $O^{-*}(\emptyset)=\emptyset=O^{+*}(\emptyset)$;
(3) $O^{-*}(U)=U=O^{+*}(U)$;
(4) $O^{-*}(X\cap Y)=O^{-*}(X)\cap O^{-*}(Y)$, $O^{+*}(X\cup Y)=O^{+*}(X)\cup O^{+*}(Y)$;
(5) $X\subseteq Y$ *implies that* $O^{-*}(X)\subseteq O^{-*}(Y)$ *and* $O^{+*}(X)\subseteq O^{+*}(Y)$;
(6) $O^{-*}(X\cup Y)\supseteq O^{-*}(X)\cup O^{-*}(Y)$, $O^{+*}(X\cap Y)\subseteq O^{+*}(X)\cap O^{+*}(Y)$;
(7) $O^{-*}(-X)=-O^{+*}(X)$, $O^{+*}(-X)=-O^{-*}(X)$;
(8) $O^{-*}(O^{-*}(X))=O^{-*}(X)$, $O^{+*}(O^{+*}(X))=O^{+*}(X)$.

Proof. The verification of (2), (3), (5), and (6) are straightforward. We only verify the remaining formulas concerning the optimized lower operator O^{-*}. The corresponding results for the optimized upper operator O^{+*} follow by duality.

(1) According to the property of upper and lower operators, we have

$$O^-(X)\subseteq X,$$

so we only need to check that $O^{-*}(X)\subseteq O^-(X)$.

Note that $A\subseteq X$ implies $O^-(A)\subseteq O^-(X)$, so we have that

$$O^{-*}(X)=\bigcup\{A\in\mathcal{A}^O : A\subseteq X\}=\bigcup\{O^-(A) : A\subseteq X, A\in\mathcal{A}^O\}\subseteq O^-(X).$$

(4) It is clear that

$$O^{-*}(X\cap Y)\subseteq O^{-*}(X)\cap O^{-*}(Y).$$

Thus we only need to prove that $O^{-*}(X)\cap O^{-*}(Y)\subseteq O^{-*}(X\cap Y)$.

For any $x\in O^{-*}(X)\cap O^{-*}(Y)$, by Definition 3.5, we have that

$$x\in\bigcup\{A\in\mathcal{A}^O : A\subseteq X\},$$

and

$$x\in\bigcup\{A\in\mathcal{A}^O : A\subseteq Y\}.$$

Suppose $A, B \in \mathcal{A}^O$ such that $x \in A \subseteq X$ and $x \in B \subseteq Y$. It suffices to verify that $A \cap B \in \mathcal{A}^O$ as O^- is a lower approximate operator on U. So $x \in O^{-*}(X \cap Y)$, which shows that $O^{-*}(X \cap Y) = O^{-*}(X) \cap O^{-*}(Y)$.

(7) Given any subset $X \subseteq U$, we have that

$$\begin{aligned} & O^{-*}(-X) \\ =& \bigcup \{A : A \subseteq (-X), A = O^-(A)\} \\ =& -\bigcap \left\{-A : X \subseteq (-A), -A = O^+(-A)\right\} \\ =& -\bigcap \left\{B : X \subseteq B, B = O^+(B)\right\} \\ =& -O^{+*}(X). \end{aligned}$$

(8) We first prove that for any subfamily $\mathcal{A}_1 \subseteq \mathcal{A}^O$, if $A = \bigcup \mathcal{A}_1$, we have $O^{-*}(A) = A$.

In fact, this is because

$$A = \bigcup \mathcal{A}_1 \subseteq \bigcup \{B \subseteq A : O^-(B) = B\} = O^{-*}(A) \subseteq A.$$

In addition, from Definition 3.5, we know that for any subset $X \subseteq U$, $O^{-*}(X)$ is the union of a subfamily of $\mathcal{A}^O$, and thus we have that

$$O^{-*}(O^{-*}(X)) = O^{-*}(X),$$

which completes the proof. □

The following example shows how to obtain optimized operators through Definition 3.5.

Example 3.7. Let $(U, \mathcal{C})$ be the covering approximation space defined in Example 3.4. We first calculate N_2:

$$\begin{aligned} &N_2(x_1) = \{x_1, x_6\}, && N_2(x_2) = \{x_2, x_4\}, \\ &N_2(x_3) = \{x_3, x_5, x_6\}, && N_2(x_4) = \{x_2, x_4\}, \\ &N_2(x_5) = \{x_2, x_3, x_4, x_5, x_6\}, && N_2(x_6) = \{x_1, x_3, x_5, x_6\}. \end{aligned}$$

Then, we have that

$$\mathcal{A}^{PN_2} = \left\{\emptyset, \{x_2, x_4\}, U\right\}.$$

Let $X = \{x_1, x_2, x_4\}$, and we have that

$$PN_2^{-*}(X) = \{x_2, x_4\}.$$

In the same way, we can compute that

$$\begin{aligned} &\mathcal{A}^{PM_2} = \{\emptyset, \{x_1, x_3, x_5, x_6\}, U\}, && \mathcal{A}^{PM_4} = \{\emptyset, U\}, \\ &\mathcal{A}^{RN_1} = \{\emptyset, \{x_2, x_4\}, \{x_1, x_3, x_5, x_6\}, U\}, && \mathcal{A}^{RN_2} = \{\emptyset, U\}, \\ &\mathcal{A}^{RN_3} = \{\emptyset, U\}, && \mathcal{A}^{RN_4} = \{\emptyset, U\}, \\ &\mathcal{A}^{RM_1} = \{\emptyset, \{x_2, x_4\}, \{x_1, x_3, x_5, x_6\}, U\}, && \mathcal{A}^{RM_2} = \{\emptyset, U\}, \\ &\mathcal{A}^{RM_3} = \{\emptyset, U\}, && \mathcal{A}^{RM_4} = \{\emptyset, U\}, \\ &\mathcal{A}^{PN_4} = \{\emptyset, U\}, \end{aligned}$$

and thus we have that

$$\begin{aligned} PM_2^{-*}(X) &= PM_4^{-*}(X) = RN_2^{-*}(X) = RN_3^{-*}(X) = RN_4^{-*}(X) \\ &= RM_2^{-*}(X) = RM_3^{-*}(X) = RM_4^{-*}(X) = PN_4^{-*}(X) = \emptyset, \\ RN_1^{-*}(X) &= RM_1^{-*}(X) = \{x_2, x_4\}. \end{aligned}$$

From the previous concepts and discussions, the next property is evident.

Proposition 3.8. *Let $(U, \mathcal{C})$ be a covering approximation space, O^-, O^+ be dual lower and upper approximate operators on U which are not stable, and O^{-*}, O^{+*} be their optimized operators respectively.*

(1) *If Y is a stable set under operator O^- and $Y \subseteq X$, then $Y \subseteq O^{-*}(X)$;*
(2) *if Z is a stable set under operator O^+ and $Z \supseteq X$, then $Z \supseteq O^{+*}(X)$.*

Therefore, under the operators O^- and O^+, $O^{-*}(X)$ is a maximal stable set contained in X and $O^{+*}(X)$ is a minimal stable set containing X.

According to the conclusion in paper [6], the optimized operator pair has the following six sets approximation property.

Proposition 3.9 *([6]). Let $(U, \mathcal{C})$ be a covering approximation space. O^-, O^+ are dual lower and upper approximate operators on U, and O^{-*}, O^{+*} are their optimized operators respectively. Then for each $X \subset U$, using any composition of these two optimized operators by any number of times, there are at most six sets obtained:*

$$O^{-*}(X),\ O^{+*}(X),\ O^{+*}(O^{-*}(X)),\ O^{-*}(O^{+*}(X)),\ O^{-*}(O^{+*}(O^{-*}(X))),\ O^{+*}(O^{-*}(O^{+*}(X))).$$

Proposition 3.10. *Let $(U, \mathcal{C})$ be a covering approximation space, O^{-*} and O^{+*} are the optimized operators of the unstable dual lower and upper approximate operators O^- and O^+ on U respectively. Then O^{-*} and O^{+*} are interior operator and closure operator respectively. Therefore, $\mathcal{T}^O = \{O^{-*}(X) : X \subseteq U\} = \{-O^{+*}(Y) : Y \subseteq U\}$ is a topology of U.*

Proof. Since U is a finite set, to show that $\mathcal{T}^O$ is a topology, we only need to prove that $\mathcal{T}^O$ is closed under finite union, according to Proposition 3.6. Let A, B be two subsets of U, we have that

$$\begin{aligned} O^{-*}(A) \cup O^{-*}(B) &= O^{-*}(O^{-*}(A)) \cup O^{-*}(O^{-*}(B)) \\ &\subseteq O^{-*}(O^{-*}(A) \cup O^{-*}(B)) \\ &\subseteq O^{-*}(A) \cup O^{-*}(B), \end{aligned}$$

and thus we have that

$$O^{-*}(A) \cup O^{-*}(B) = O^{-*}(O^{-*}(A) \cup O^{-*}(B)).$$

This completes the proof. $\square$

Next we discuss the meaning of the optimized operators from a topological point of view.

Definition 3.11. The topology $\mathcal{T}^O$ defined in Proposition 3.10 is called the *induced topology* of U by the operator pair (O^-, O^+), and $(U, \mathcal{T}^O)$ is called topological space induced by the operator pair (O^-, O^+).

Proposition 3.12. *Let $(U, \mathcal{C})$ be a covering approximation space, O^-, O^+ be dual lower and upper approximate operators on U which are not stable, and O^{-*}, O^{+*} be their optimized operators respectively. If there is a topology τ of U satisfying $O^-(T) = T$ (or $O^+(-T) = -T$) for each $T \in \tau$, then $\tau \subseteq \mathcal{T}^O$, i.e., the topology $\mathcal{T}^O$ is the finest topology which is composed by some stable sets under operator O^-.*

Proof. According to Definition 3.5, it's easy to show that

$$\forall T \in \tau,\ T = O^{-*}(T),$$

which means $T \in \mathcal{T}^O$, completing the proof. $\square$

Remark 3.13. If O^-, O^+ are lower and upper approximation operators which are not stable on a covering approximation space $(U, \mathcal{C})$, then we can find a maximal topological structure $\mathcal{T}^O$ of U such that all the sets in $\mathcal{T}^O$ and their complementary sets are stable. So the topological structure of the approximation space is very important for the study of indeterminate set theory, because the topological structure is a stable structure and seeking certainty from uncertainty is the original intention of the theory.

4. Optimizations of some known operators

In this section, we will optimize the above unstable operators PN_i^-, PM_i^+, PN_i^-, $PM_i^+ (i = 2, 4)$ and RN_i^-, RN_i^+, RM_i^-, $RM_i^+ (i = 1, 2, 3, 4)$ into stable operators, thereby making the corresponding approximation sets definite sets.

Definition 4.1. Let $(U, \mathcal{C})$ be a covering approximation space, and the following families be set:

$$\mathcal{A}^{PN_i} = \{A \subseteq U : PN_i^-(A) = A\}, \qquad \mathcal{B}^{PN_i} = \{B \subseteq U : PN_i^+(B) = B\}\ (i = 2, 4),$$
$$\mathcal{A}^{PM_i} = \{A \subseteq U : PM_i^-(A) = A\}, \qquad \mathcal{B}^{PM_i} = \{B \subseteq U : PN_i^+(B) = B\}\ (i = 2, 4),$$

$$\mathcal{A}^{RN_i} = \{A \subseteq U : RM_i^-(A) = A\}, \qquad \mathcal{B}^{RN_i} = \{B \subseteq U : RN_i^+(B) = B\}\ (i = 1,2,3,4),$$
$$\mathcal{A}^{RM_i} = \{A \subseteq U : RM_i^-(A) = A\}, \qquad \mathcal{B}^{RM_i} = \{B \subseteq U : RM_i^+(B) = B\}\ (i = 1,2,3,4).$$

Then, for each $X \subseteq U$, its corresponding lower and upper approximations are defined as:

$$PN_i^{-*}(X) = \bigcup\{A \in \mathcal{A}^{PN_i} : A \subseteq X\}, \qquad PN_i^{+*}(X) = \bigcap\{B \in \mathcal{B}^{PN_i} : X \subseteq B\} \qquad (i = 2,4),$$
$$PM_i^{-*}(X) = \bigcup\{A \in \mathcal{A}^{PM_i} : A \subseteq X\}, \qquad PM_i^{+*}(X) = \bigcap\{B \in \mathcal{B}^{PM_i} : X \subseteq B\} \qquad (i = 2,4),$$
$$RN_i^{-*}(X) = \bigcup\{A \in \mathcal{A}^{RN_i} : A \subseteq X\}, \qquad RN_i^{+*}(X) = \bigcap\{B \in \mathcal{B}^{RN_i} : X \subseteq B\} \qquad (i = 1,2,3,4),$$
$$RM_i^{-*}(X) = \bigcup\{A \in \mathcal{A}^{RM_i} : A \subseteq X\}, \qquad RM_i^{+*}(X) = \bigcap\{B \in \mathcal{B}^{RM_i} : X \subseteq B\} \qquad (i = 1,2,3,4).$$

Thus, the operators PN_i^{-*}, PN_i^{+*}, PM_i^{-*}, $PM_i^{+*}(i = 2,4)$ and RN_i^{-*}, RN_i^{+*}, RM_i^{-*}, RM_i^{+*} $(i = 1,2,3,4)$ are called the *optimized operators* of PN_i^-, PN_i^+, PM_i^-, PM_i^+ $(i = 2,4)$ and RN_i^-, RN_i^+, RM_i^-, RM_i^+ $(i = 1,2,3,4)$ respectively.

The above definition is directly based on the optimization theory discussed in the previous section. However, for such unstable operators PN_i^-, PM_i^+, etc., it is not easy to obtain such family $\mathcal{A}^{PN_i}$, $\mathcal{B}^{PN_i}$, etc., so it is difficult to optimize corresponding operators directly with this definition. Therefore, our next important task is to find a simpler method of operator optimization. In fact, the optimized operators can be obtained by modifying the neighborhoods.

For convenience, in the following manuscript, for any operator $t : U \to \mathscr{P}(U)$, and any subset $X \subseteq U$, $t(X)$ is defined as follows:

$$t(X) = \bigcup\{t(x) : x \in X\}.$$

Definition 4.2. Let $(U, \mathcal{C})$ be a covering approximation space and $t : U \to \mathscr{P}(U)$ be a reflexive neighborhood operator on U. Then the *core* of $t(X)$ is defined as:

$$t^o(X) = \bigcup\{I \subseteq X : \forall y \in I, t(y) \subseteq I\}.$$

The operator t^o is called the *core operator* induced by t.

Remark 4.3. In Definition 4.2, for any $y \in I$, it holds that $t(y) \subseteq I$. Then, we have

$$I = \bigcup_{y\in I} y \subseteq \bigcup_{y\in I} t(y) \subseteq I,$$

and thus

$$I = \bigcup_{y\in I} t(y).$$

So, we have that,

$$t^o(X) = \bigcup\{I \subseteq X : \forall y \in I, t(y) \subseteq I\} = \bigcup_{I\subseteq X}\bigcup_{y\in I} t(y).$$

According to Definition 4.2, we give the following concepts.

Definition 4.4. Let $(U, \mathcal{C})$ be a covering approximation space and $N_i(x)$ and $M_i(x)$ $(i = 1,2,3,4)$ be the neighborhood operators and co-neighborhood operators on U defined in Definition 2.3. Then for each $i = 1,2,3,4$, the cores of $N_i(X)$ and $M_i(X)$ are defined as:

$$N_i^o(X) = \bigcup\{I \subseteq X : \forall y \in I, N_i(y) \subseteq I\} = \bigcup\{I \subseteq X : I = \bigcup_{y\in I} N_i(y)\},$$
$$M_i^o(X) = \bigcup\{I \subseteq X : \forall y \in I, M_i(y) \subseteq I\} = \bigcup\{I \subseteq X : I = \bigcup_{y\in I} M_i(y)\},$$
$$MN_i^o(X) = \bigcup\{I \subseteq X : \forall y \in I, M_i(N_i(y)) \subseteq I\} = \bigcup\{I \subseteq X : I = \bigcup_{y\in I} M_i(N_i(y))\},$$
$$NM_i^o(X) = \bigcup\{I \subseteq X : \forall y \in I, N_i(M_i(y)) \subseteq I\} = \bigcup\{I \subseteq X : I = \bigcup_{y\in I} N_i(M_i(y))\}.$$

Based on Definition 4.4, the next proposition gives the equivalent forms of all optimization operators in Definition 4.1.

Proposition 4.5. *Let $(U, \mathcal{C})$ be a covering approximation space and $X \subseteq U$. Then, for $i = 1,2,3,4$, the following formulas can be obtained:*

$$PN_i^{-*}(X) = N_i^o(X), \qquad PN_i^{+*}(X) = -PN_i^{-*}(-X),$$

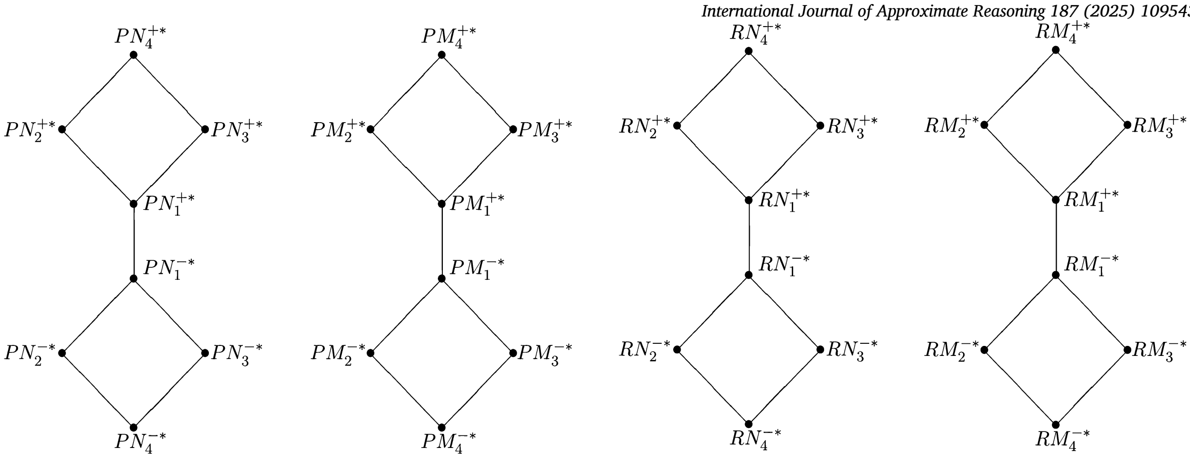

Fig. 1. The partial order among optimized operators.

$$PM_i^{-*}(X) = M_i^o(X), \qquad PM_i^{+*}(X) = -PM_i^{-*}(-X),$$
$$RN_i^{-*}(X) = NM_i^o(X), \qquad RN_i^{+*}(X) = -RN_i^{-*}(-X),$$
$$RM_i^{-*}(X) = MN_i^o(X), \qquad RM_i^{+*}(X) = -RM_i^{-*}(-X).$$

Proof. We only prove the first one:

$$PN_i^{-*}(X) = \bigcup\{A \in \mathcal{A}_i^{PN} : A \subseteq X\} = N_i^o(X),$$

and the rest can be proved similarly.

We reformulate this goal as proving the equality:

$$PN_i^-(N_i^o(X)) = N_i^o(X).$$

By Definition 4.2, we have that

$$PN_i^-(N_i^o(X)) = \{y : N_i(y) \subseteq N_i^o(X)\} = N_i^o(X).$$

Hence, the proposition holds. $\square$

The next proposition can be directly derived from Proposition 3.10.

Proposition 4.6. *Let $(U, \mathcal{C})$ be a covering approximation space. Then for each $i = 1,2,3,4$, PN_i^{-*}, PM_i^{-*}, RN_i^{-*}, RM_i^{-*} $(i = 1,2,3,4)$ are interior operators on U, and PN_i^{+*}, PM_i^{+*}, RN_i^{+*}, RM_i^{+*} $(i = 1,2,3,4)$ are closure operators on U.*

Definition 4.7. Let $(U, \mathcal{C})$ be a covering approximation space, then a partial order $\leq$ between any two operators F and G on U is defined as $F \leq G$, if

$$\forall X \subseteq U, F(X) \subseteq G(X).$$

Proposition 4.8. *The partial order of all the optimized operators are shown as Fig. 1, where the lower operator is smaller ($\leq$) than the upper operator which is connected to it by a line segment.*

Proof. From Proposition 4.5, it can be seen that the formulas of aforementioned approximation operators have the same forms, differing only in the core operators used within the formulas. Therefore, we first discuss the connection between the partial order relations among neighborhood operators and the partial order relations among approximation operators.

Let X be a subset of the universe U, f, g be two reflexive neighborhood operators, f^o, g^o be the corresponding core operators, and f^{-*}, g^{-*} be the corresponding optimized lower approximation operators, respectively, where

$$f^{-*}(X) = f^o(X) = \bigcup\{I \subseteq X : I = f(I)\},$$
$$g^{-*}(X) = g^o(X) = \bigcup\{I \subseteq X : I = g(I)\}.$$

Suppose that $f \leq g$. If $I \subseteq U$ and $I = g(I)$, we have that

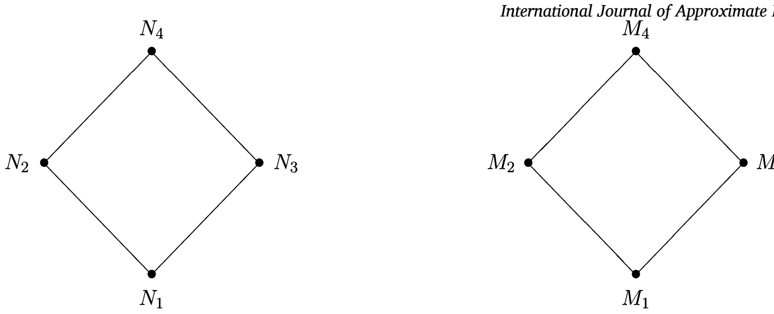

Fig. 2. The partial order between neighborhood and co-neighborhood operators.

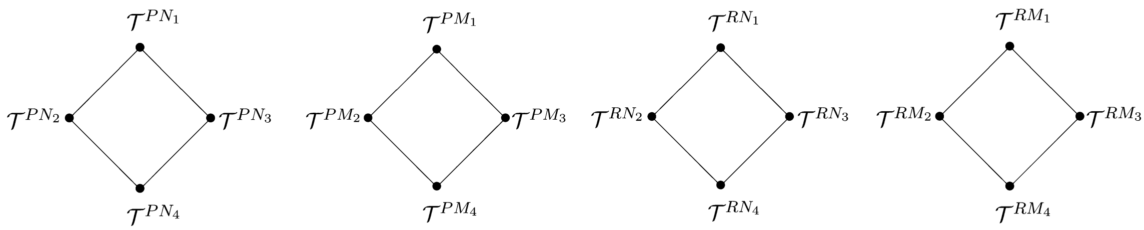

Fig. 3. The relations between induced topologies.

$$f(I) \subseteq g(I) = I \subseteq f(I).$$

So,

$$I = f(I),$$

which means that

$$g^{-*}(X) = \bigcup \{I \subseteq X : I = g(I)\} \subseteq \bigcup \{I \subseteq X : I = f(I)\} = f^{-*}(X).$$

Therefore, it is clear that $f \leq g$ can deduce $g^{-*} \leq f^{-*}$.

Now we just need to figure out the partial order of the neighborhood and co-neighborhood operators mentioned in Definition 2.3. In fact, this partial order is illustrated in Fig. 2, where the lower operator is smaller ($\leq$) than the upper operator which is connected to it by a line segment.

Now we prove the partial order in Fig. 2. Since the partial order represented by the left part of Fig. 2 can be easily proved from Definition 2.3, we need only to give the proof of the right part here.

Suppose that $N_i \leq N_j$. For all $x \in U$, if $y \in M_i(x)$, then $x \in N_i(y) = N_i(\{y\}) \subseteq N_j(\{y\}) = N_j(y)$, and thus $y \in M_j(x)$, therefore, we have $M_i(x) \subseteq M_j(x)$. Combining with Definition 2.3, for any subset $X \subseteq U$, we have $M_i(X) \subseteq M_j(X)$, i.e., $M_i \leq M_j$. Thus, the partial order on $\{M_i : i = 1,2,3,4\}$ is isomorphic to the partial order on $\{N_i : i = 1,2,3,4\}$.

Thus, the proof of the proposition is complete. $\square$

Considering that the stable lower approximation sets can form a topology of the approximation space, the following propositions are clearly valid.

Corollary 4.9. *Let $(U, \mathcal{C})$ be a covering approximation space. Then the relations of the induced topologies of all the operators in Definition 4.1 are shown as the partial order in Fig. 3, where the lower topology is a subset of the upper topology which is connected to it by a line segment.*

Corollary 4.10. *Let $(U, \mathcal{C})$ be a covering approximation space. Then the partial order among the relations of all the optimized lower operators and the partial order among their induced topologies are isomorphism.*

Remark 4.11. From the above propositions, it can be seen that, in a sense, seeking certainty from uncertainty is to seek the appropriate topological structure of the approximation space.

Furthermore, for any reflexive neighborhood operator t on U, the family $T = \{t(x_i) : x_i \in U\}$ may not be a basis of a topology, and the following algorithm can optimize the reflexive neighborhood operator and turn the family into a topological basis T^*.

Algorithm 1: The optimization t^* induced by t.

Input: $U = \{x_1, x_2, \ldots, x_n\}$, $T = \{t(x_i) : x_i \in U\}$
Let $T^* = \emptyset$, $T_k = \emptyset$, $k = 1, 2, \ldots, n$;
for $i = 1$ ***to*** n **do**
  $T_i \leftarrow \{x_i\}$;
  while $T_i \neq t(T_i)$ **do**
    $T_i \leftarrow t(T_i)$;
  end
  $T^* \leftarrow T^* \cup \{T_i\}$;
end
Output: The optimized family $T^* = \{T_k : k = 1, 2, ..., n\}$

Remark 4.12. The optimized neighborhood operator t^* is defined as follows:

$$t^*(x_i) = T_i,$$

where T_i is computed in Algorithm 1.

The optimized neighborhood operator t^* is readily verified to be reflexive and transitive, and the neighborhood family T^* constitutes a topological basis. The following proposition formalizes the relationship between t^* and the corresponding optimized approximation operator.

Proposition 4.13. *Let (U, C) be a covering approximation space, t be a reflexive neighborhood operator. The lower approximate operator O^- is induced by t, and O^{-*} is the optimized operator of O^-. Then the family T^* computed in Algorithm 1 is the topological basis of $\mathcal{T}^O$, i.e., the topology induced by O^{-*}.*

Proof. It suffices to verify that for any subset $X \subseteq U$,

$$O^{-*}(X) = \bigcup_{t^*(x) \subseteq X} t^*(x).$$

From Algorithm 1, it follows immediately that for any $x \in U$,

$$t^*(x) = \bigcup_{x \in t(x)} t(x).$$

Moreover, in conjunction with Definition 4.2, we obtain that

$$t^o(X) = \bigcup_{t^*(x) \subseteq X} t^*(x).$$

This completes the proof. $\square$

5. Matrix methods of the optimizations

This section examines efficient computation methods for lower and upper approximations using the optimized operators defined in Definition 4.1. Specifically, we derive computational approaches for determining neighborhood cores.

In addition, we write the composition $M_i \circ N_i$ of two operators as MN_i, and $N_i \circ M_i$ as NM_i $(i = 1, 2, 3, 4)$ for convenience.

Definition 5.1 *([5,6])*. Let $A_{n\times m} = (a_{ik})_{n\times m}$ and $B_{m\times l} = (b_{kj})_{m\times l}$ be two Boolean matrices. The Boolean products $C_{n\times l} = A \cdot B = (c_{ij})_{n\times l}$ and $D_{n\times l} = A * B = (d_{ij})_{n\times l}$ are defined as follows:

$$c_{ij} = \bigvee_{k=1}^{m} (\ a_{ik} \wedge b_{kj}\ ), i = 1, 2, \cdots, n, j = 1, 2, \cdots, l,$$

$$d_{ij} = \bigwedge_{k=1}^{m} [\ (1 - a_{ik}) \vee b_{kj}\ ], i = 1, 2, \cdots, n, j = 1, 2, \cdots, l,$$

where $\vee$ and $\wedge$ denote the max and the min operators, respectively.

Obviously, the Boolean product $A \cdot B$ and the product $A * B$ of two Boolean matrices A and B are still Boolean matrices.

Definition 5.2 *([5,6])*. Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ be two Boolean matrices. A is called *not greater than* B, denoted by $A \leq B$, if $a_{ij} \leq b_{ij}$, for all $i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n$.

The following properties are derived from two different types of matrix products.

Proposition 5.3. *Let $A = (a_{ij})_{n\times n}$ be a Boolean matrix and $\beta = (b_1, b_2, \cdots, b_n)^T$ and $\gamma = (c_1, c_2, \cdots, c_n)^T$ be two Boolean column vectors. If $\beta \preceq \gamma$, then it holds that*

$$A \cdot \beta \preceq A \cdot \gamma, \qquad A * \beta \preceq A * \gamma.$$

Proof. Suppose that

$$\begin{aligned} A \cdot \beta &= (s_1, s_2, \cdots, s_n)^T, \quad A \cdot \gamma = (t_1, t_2, \cdots, t_n)^T, \\ A * \beta &= (u_1, u_2, \cdots, u_n)^T, \quad A * \gamma = (v_1, v_2, \cdots, v_n)^T. \end{aligned}$$

Since $\beta \preceq \gamma$, it holds that for all $j = 1, 2, \cdots, n$, $b_j \le c_j$, and then, for all $i = 1, 2, ..., n$,

$$\begin{aligned} s_i &= \bigvee_{j=1}^{n} (a_{ij} \wedge b_j) \le \bigvee_{j=1}^{n} (a_{ij} \wedge c_j) = t_i, \\ u_i &= \bigwedge_{j=1}^{n} [(1 - a_{ij}) \vee b_j] \le \bigwedge_{j=1}^{n} [(1 - a_{ij}) \vee c_j] = v_i. \end{aligned}$$

Thus we have $A \cdot \beta \preceq A \cdot \gamma$ and $A * \beta \preceq A * \gamma$, completing the proof. □

Proposition 5.4. *Let $A = (a_{ij})_{m\times n}$, $B = (b_{jk})_{n\times p}$, and $C = (c_{ks})_{p\times q}$ be Boolean matrices. Then, the following equations hold:*

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C, \quad A * (B * C) = (A \cdot B) * C.$$

Proof. Suppose that $A \cdot (B \cdot C) = (r_{is})_{m\times q}, (A \cdot B) \cdot C = (r'_{is})_{m\times q}$ and $A \cdot B = (d_{ik})_{m\times p}, B \cdot C = (e_{js})_{p\times q}$. For each $i = 1, 2, \cdots, n$, and each $s = 1, 2, \cdots, q$, we have that

$$\begin{aligned} r_{is} &= \bigvee_{j=1}^{n} \left[a_{ij} \wedge e_{js}\right] \\ &= \bigvee_{j=1}^{n} \left[a_{ij} \wedge \left(\bigvee_{k=1}^{p} [b_{jk} \wedge c_{ks}]\right)\right] \\ &= \bigvee_{j=1}^{n} \left[\bigvee_{k=1}^{p} (a_{ij} \wedge b_{jk} \wedge c_{ks})\right] \\ &= \bigvee_{k=1}^{p} \left[\bigvee_{j=1}^{n} (a_{ij} \wedge b_{jk}) \wedge c_{ks}\right] \\ &= \bigvee_{k=1}^{p} \left(d_{ik} \wedge c_{ks}\right) \\ &= r'_{is}. \end{aligned}$$

So, the first equation holds.

Let $A * (B * C) = (t_{is})_{m\times q}, (A \cdot B) * C = (t'_{is})_{m\times q}$ and $A \cdot B = (d_{ik})_{m\times p}, B * C = (h_{js})_{p\times q}$. For each $i = 1, 2, \cdots, n$, and each $s = 1, 2, \cdots, q$, we have that

$$\begin{aligned} t_{is} &= \bigwedge_{j=1}^{n} \left[(1 - a_{ij}) \vee h_{js}\right] \\ &= \bigwedge_{j=1}^{n} \left[(1 - a_{ij}) \vee \left(\bigwedge_{k=1}^{p} [(1 - b_{jk}) \vee c_{ks}]\right)\right] \\ &= \bigwedge_{j=1}^{n} \left[\bigwedge_{k=1}^{p} ((1 - a_{ij}) \vee (1 - b_{jk}) \vee c_{ks})\right] \\ &= \bigwedge_{k=1}^{p} \left[\bigwedge_{j=1}^{n} [(1 - a_{ij}) \vee (1 - b_{jk})] \vee c_{ks}\right] \\ &= \bigwedge_{k=1}^{p} \left(\left[1 - \bigvee_{j=1}^{n} (a_{ij} \wedge b_{jk})\right] \vee c_{ks}\right) \end{aligned}$$

$$= \bigwedge_{k=1}^{p} \left([1 - d_{ik}] \vee c_{ks}\right)$$
$$= t'_{is}.$$

Thus, we have that $A * (B * C) = (A \cdot B) * C$, and this completes the proof. □

The next definition shows how to transform a subset X of the universe U into a Boolean vector.

Definition 5.5 *([5,6]).* Let $U = \{x_1, x_2, \cdots, x_n\}$ be a finite universe, and X be a subset of U. The vector $\chi_X^U = (a_1, a_2, \cdots, a_n)^T$ is called a *characteristic vector* of X if for any $i = 1, 2, \dots, n$,

$$a_i = \begin{cases} 1, & x_i \in X \\ 0, & x_i \notin X \end{cases}.$$

Let $\mathcal{F} = \{X_1, X_2, \cdots, X_m\}$ be a family of subset of U, and the matrix $F = (t_{ij})_{n\times m}$ is called the *matrix representation* of $\mathcal{F}$ if

$$t_{ij} = \begin{cases} 1, & x_i \in X_j \\ 0, & x_i \notin X_j \end{cases}, \quad i = 1, 2, \cdots, n, \ j = 1, 2, \cdots, m.$$

Remark 5.6. For convenience, if t is a reflexive neighborhood operator on U and

$$\mathcal{D} = \{t(x_1), t(x_2), \dots, t(x_n)\},$$

then the matrix representation D of $\mathcal{D}$ is also called the *matrix representation* of t.

The following propositions follows from Definitions 5.1 and 5.5.

Proposition 5.7. *Let $U = \{x_1, x_2, \cdots, x_n\}$ be a finite universe. If D is the matrix representation of a reflexive neighborhood operator t. Then, for any subset $X \subseteq U$,*

$$D * \chi_X^U = \chi^U_{\{y\in U : t(y)\subseteq X\}}.$$

Proof. Let $D * \chi_X^U = (c_1, c_2, \dots, c_n)^T$, then, for $i = 1, 2, \dots, n$, we have that

$$\begin{aligned} c_i = 1 &\Leftrightarrow \bigwedge_{k=1}^{n} \left[\left(1 - \chi^U_{t(x_i)}(x_k)\right) \vee \chi_X^U(x_k)\right] = 1 \\ &\Leftrightarrow \left[\left(\chi^U_{t(x_i)}(x_k) = 1\right) \to \left(\chi_X^U(x_k) = 1\right)\right] \\ &\Leftrightarrow \left[\left(x_k \in t(x_i)\right) \to \left(x_k \in X\right)\right] \\ &\Leftrightarrow t(x_i) \subseteq X \\ &\Leftrightarrow \chi^U_{\{y\in U : t(y)\subseteq X\}}(x_i) = 1, \end{aligned}$$

which completes the proof. □

For convenience, we use simple notations as:

$$D^0(X) = \chi_X^U,$$
$$D^k(X) = D * D^{k-1}(X) \ \text{ for } \ k \geq 1.$$

Proposition 5.8. *Let $U = \{x_1, x_2, \cdots, x_n\}$ be a finite universe, and D be the matrix representation of a reflexive neighborhood operator t. Then, for any subset $X \subseteq U$,*

$$D^T \cdot \chi_X^U = \chi^U_{\bigcup_{y\in X} t(y)}.$$

Now, the matrix method of computing optimized approximate sets and optimized neighborhoods will be given.

Proposition 5.9. *Let $U = \{x_1, x_2, \cdots, x_n\}$ be a finite universe, and D be the matrix representation of a reflexive neighborhood operator t. Then, for any subset $X \subseteq U$, there is a non-negative integer k such that*

$$D^k(X) = D^{k+1}(X),$$

and $D^k(X)$ is the characteristic vector of the core $t^\circ(X)$.

Proof. From Proposition 5.7, for any subset $X \subseteq U$, it follows that $D(X) \preceq \chi_X$. Since X is finite, there exists a minimal non-negative integer $k_0 \leq |X|$ such that the sequence of iterated derivatives stabilizes, i.e.,

$$D^{k_0}(X) = D^{k_0+1}(X).$$

Otherwise, the finite set X would admit infinitely many distinct subsets, leading to a contradiction.

Let $\{A_i : i = 1, 2, \dots, k_0 + 1\}$ be the sequence of sets satisfying that $\chi_{A_i} = D^i(X)$. From Proposition 5.7, we derive that

$$A_{k_0+1} = \{x \in U : t(x) \subseteq A_{k_0}\} = A_{k_0} \subseteq X,$$

which means that

$$A_{k_0} = t(A_{k_0}) \subseteq X,$$

and thus we have that

$$A_{k_0} \subseteq t^\circ(X).$$

Conversely, for any B such that $B = t(B) \subseteq X$, we have that

$$D * \chi_B^U = \chi_B^U.$$

Thus, we have

$$D^{k_0}(B) = \chi_B^U \preceq D^{k_0}(X) = \chi_{A_{k_0}}^U,$$

which means that $B \subseteq A_{k_0}$. Combining with Definition 4.2, we have that $t^o(X) \subseteq A_{k_0}$. This completes the proof. $\square$

Corollary 5.10. *Let $U = \{x_1, x_2, \cdots, x_n\}$ be a finite universe. If D is the matrix representation of a reflexive neighborhood operator t, then, there is a positive integer k such that*

$$\underbrace{D \cdot (D \cdot (D \cdot (\cdots \cdot D)))}_{k} = \underbrace{D \cdot (D \cdot (D \cdot (\cdots \cdot D)))}_{k+1}.$$

Furthermore, $\underbrace{D \cdot (D \cdot (D \cdot (\cdots \cdot D)))}_{k}$ is the matrix representation of the optimized neighborhood t^*.

Proof. It follows directly from Propositions 5.4 and 5.9. $\square$

Example 5.11. Let $U = \{x_1, x_2, \dots, x_{10}\}$ be the universe, and consider the covering

$$C = \left\{\{x_1, x_3, x_6, x_8\}, \{x_1, x_5, x_6\}, \{x_3, x_4, x_8\}, \{x_4, x_5, x_7, x_{10}\}, \{x_2, x_9\}, \{x_9\}\right\}.$$

The matrix representation of the reflexive neighborhood operator MN_1 is as follows:

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Subsequently, we observe that

$$D \cdot D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

$$D \cdot (D \cdot D) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Notably, the equation $D \cdot D = D \cdot (D \cdot D)$ holds. Consequently, the matrix representation of the optimized domain operator ${MN_1}^*$ can be expressed as

$$D^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The following proposition follows directly from Proposition 4.13.

Proposition 5.12. *Let $U = \{x_1, x_2, \cdots, x_n\}$ be a finite universe, and D^* be the matrix representation of an optimized neighborhood operator t^*. Then, for any subset $X \subseteq U$, it holds that*

$$D^* * \chi_X^U = \chi_{t^o(X)}^U.$$

The matrix representation of stable operators was first introduced by Ma in [5]. Building upon this work, next we will combine Gao's modular approach [2] to develop optimized module matrix calculation methods to implement the calculations of these optimized operators.

Proposition 5.13. *Let (U, C) be a covering approximation space, and let t be a reflexive neighborhood operator on U. Then, t induces an equivalence relation R_t on U, where xR_ty if and only if there exists a sequence of points $\{x_i : i = 1, 2, \ldots, n\} \subseteq U$ satisfying that*

(1) $x_1 = x$, $x_n = y$;
(2) $t(x_i) \cap t(x_{i+1}) \neq \emptyset$, $i = 1, 2, \ldots, n-1$.

Proof. The reflexivity and symmetry of R_t are straightforward. Next, we prove the transitivity.

Assume $x, y, z \in U$ such that xR_ty and yR_tz. By definition, there exist finite sequences $\{x_i : i = 1, 2, \ldots, n\}$ and $\{y_j : j = 1, 2, \ldots, m\}$ in U satisfying that

$$x_1 = x, \quad x_n = y = y_1, \quad y_m = z;$$

$$t(x_i) \cap t(x_{i+1}) \neq \emptyset, \quad t(y_j) \cap t(y_{j+1}), \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m.$$

Now, construct $\{z_k : k = 1, 2, \ldots, n+m\} \subseteq U$ such that

$$z_k = \begin{cases} x_k, & 1 \le k \le n; \\ y_{k-n}, & n+1 \le j \le n+m. \end{cases}$$

It is trivial to show that $\{z_k\}$ satisfies the required conditions for $xR_t z$, thereby the proof is completed. □

The equivalence relation above induced a partition $\mathcal{S}^t$ on U, and, for convenience, we also say that this partition is induced by the operator t. The following algorithm gives a method to calculate the partition.

Algorithm 2: The partition $\mathcal{S}^t$ on $(U, \mathcal{C})$ induced by t.

```
Input: U = {x_1, x_2, ..., x_n}, T = {t(x_i) : x_i ∈ U}
Let S^t = ∅, N = ∅, 𝒮^t = ∅;
for i = 1 to n do
    if x_i ∉ N then
        S^t ← {x_i};
        for j = 1 to n do
            if t(x_j) ∩ S^t ≠ ∅ then
                S^t ← S^t ∪ t(x_j);
            end
        end
        N ← N ∪ S^t, 𝒮^t ← 𝒮^t ∪ {S^t};
    end
end
Output: The partition 𝒮^t on U
```

Example 5.14. Consider the covering approximation space presented in Example 5.11. To determine the partition $\mathcal{S}^{MN_1}$, we begin by computing the MN_1 neighborhood for each element in U:

$$MN_1(x_1) = \{x_1, x_6\}, \qquad MN_1(x_2) = \{x_2, x_9\},$$
$$MN_1(x_3) = \{x_3, x_8\}, \qquad MN_1(x_4) = \{x_4, x_7, x_{10}\},$$
$$MN_1(x_5) = \{x_5, x_7, x_{10}\}, \qquad MN_1(x_6) = \{x_1, x_6\},$$
$$MN_1(x_7) = \{x_4, x_5, x_7, x_{10}\}, \qquad MN_1(x_8) = \{x_3, x_8\},$$
$$MN_1(x_9) = \{x_2, x_9\}, \qquad MN_1(x_{10}) = \{x_4, x_5, x_7, x_{10}\}.$$

Following Algorithm 2, we initialize $S^{MN_1} = \{x_1\}$ and iteratively compute:

$$MN_1(S^{MN_1}) = MN_1(x_1) = \{x_1, x_6\}, \quad MN_1(\{x_1, x_6\}) = \{x_1, x_6\}.$$

This yields the first equivalence class $\{x_1, x_6\}$. Applying the same procedure to the remaining elements, we obtain the complete partition:

$$\mathcal{S}^{MN_1} = \left\{\{x_1, x_6\}, \{x_2, x_9\}, \{x_3, x_8\}, \{x_4, x_5, x_7, x_{10}\}\right\}.$$

The following proposition concerning the partitions induced by t and t^* is an immediate consequence of their definitions.

Proposition 5.15. *The partition $\mathcal{S}^t$ induced by a reflexive neighborhood operator t is the same as the partition $\mathcal{S}^{t^*}$ induced by t^*.*

Now, given the partition $\mathcal{S}^t$ induced by t, the computation of approximation operators can be simplified.

Proposition 5.16 *([2]). Let $(U, \mathcal{C})$ be a covering approximation space equipped with a reflexive neighborhood operator t, O^- be the lower approximate operator induced by t, and O^{-*} denote the optimized lower approximate operator of O^-. For the partition $\mathcal{S}^t$ induced by t, the following statement holds for any subset $X \subseteq U$:*

$$O^{-*}(X) = \bigcup_{S^t \in \mathcal{S}^t} O^{-*}(X \cap S^t).$$

At the end of this section, we give an example to show that the matrix-based computation of approximation operators, implementing the previous propositions.

Example 5.17. Consider the covering approximation space presented in Example 5.11 with the partition $\mathcal{S}^{MN_1}$ induced by the neighborhood operator MN_1. Treating each equivalence class in $\mathcal{S}^{MN_1}$ as a subspace, we derive the corresponding neighborhood matrices of these subspaces as follows:

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ A_4 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

where these matrices correspond to the subspaces $S_1^{MN_1} = \{x_1, x_6\}$, $S_2^{MN_1} = \{x_2, x_9\}$, $S_3^{MN_1} = \{x_3, x_8\}$ and $S_4^{MN_1} = \{x_4, x_5, x_7, x_{10}\}$, respectively.

Given a subset $X = \{x_1, x_2, x_4, x_7, x_9, x_{10}\}$, we first compute the characteristic vectors representing the intersection of X with each subspace $S_i^{MN_1}$ within their respective subspaces:

$$\beta_1 = \chi^{S_1^{MN_1}}_{X\cap S_1^{MN_1}} = (1,0)^{\mathrm{T}},$$

$$\beta_2 = \chi^{S_2^{MN_1}}_{X\cap S_2^{MN_1}} = (1,1)^{\mathrm{T}},$$

$$\beta_3 = \chi^{S_3^{MN_1}}_{X\cap S_3^{MN_1}} = (0,0)^{\mathrm{T}},$$

$$\beta_4 = \chi^{S_4^{MN_1}}_{X\cap S_4^{MN_1}} = (1,0,1,1)^{\mathrm{T}}.$$

Consequently, we obtain the matrix-vector product:

$$A_1 * \beta_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$A_2 * \beta_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \beta_2,$$

$$A_3 * \beta_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} * \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$A_4 * \beta_4 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$A_4 * (A_4 * \beta_4) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Moreover, by converting the neighborhood matrix to its optimized neighborhood operator-induced counterpart, the computation on $S_4^{MN_1}$ admits further simplification.

The optimized neighborhood matrix A_4^* of A_4 is

$$A_4^* = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

and we have that

$$A_4^* * \beta_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, combining with Proposition 2.12, and let $X_i = X \cap S_i^{MN_1}$, $i = 1,2,3,4$, we can compute the following results:

$$RM^{-*}(X_1) = \emptyset, \quad RM^{-*}(X_2) = \{x_2, x_9\},$$

$$RM^{-*}(X_3) = \emptyset, \quad RM^{-*}(X_4) = \emptyset.$$

Thus we have that

$$RM^{-*}(X) = \bigcup_{i=1}^{4} RM^{-*}(X_i) = \{x_2, x_9\}.$$

6. Conclusions

In this paper, we propose a method for optimizing unstable approximation operators, addressing certain theoretical gaps that have persisted in covering rough set theory since its inception. Additionally, we present a computational approach to derive optimized operators using matrix and modular methods, providing a basis for computer operation. Therefore, this paper not only optimizes the unstable operator models, but also optimizes their calculation methods. The conclusions of this paper will significantly enhance the application ability of the covering rough set theory.

Building upon this work, we plan to extend the concept of stable operators to other domains, such as fuzzy rough sets. This expansion will fundamentally consolidate the theoretical foundation of uncertain sets and help to exert its application potential.

CRediT authorship contribution statement

Shizhe Zhang: Writing – original draft, Visualization, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Liwen Ma:** Writing – review & editing, Supervision, Resources, Project administration, Methodology, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

1. The authors are grateful to the anonymous reviewers and editors for their valuable comments which have enabled to improve the original version of this paper.
2. This work is supported by Beijing Natural Science Foundation (1222011).

Data availability

No data was used for the research described in the article.

References

[1] M.A. Armstrong, Basic Topology, Springer Science & Business Media, 2013.
[2] Q. Gao, L. Ma, A novel notion in rough set theory: invariant subspace, Fuzzy Sets Syst. 440 (2022) 90–111.
[3] J.L. Kelly, General Topology, Van Nostrand Company, New York, 1995.
[4] L. Ma, On some types of neighborhood-related covering rough sets, Int. J. Approx. Reason. 53 (2012) 901–911.
[5] L. Ma, The investigation of covering rough sets by Boolean matrices, Int. J. Approx. Reason. 100 (2018) 69–84.
[6] L. Ma, H. Lu, Six-set approximation theorem of neighborhood related rough sets, Int. J. Approx. Reason. 115 (2019) 180–193.
[7] Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11 (1982) 341–356.
[8] K. Qin, J. Yang, Z. Pei, Generalized rough sets based on reflexive and transitive relations, Inf. Sci. 178 (2008) 4138–4141.
[9] Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, Inf. Sci. 111 (1998) 239–259.
[10] Y. Yao, B. Yao, Covering based rough set approximations, Inf. Sci. 200 (2012) 91–107.
[11] J. Zhou, F. Xu, Y. Guan, H. Wang, Three types of fuzzy covering-based rough set models, Fuzzy Sets Syst. 423 (2021) 122–148.
[12] W. Zhu, F. Wang, Reduction and axiomization of covering generalized rough sets, Inf. Sci. 152 (2003) 217–230.
[13] W. Zhu, Relationship between generalized rough sets based on binary relation and covering, Inf. Sci. 179 (2009) 210–225.
[14] W. Zhu, Topological approaches to covering rough sets, Inf. Sci. 177 (6) (2007) 1499–1508.
[15] W. Zhu, F. Wang, On three types of covering based rough sets, IEEE Trans. Knowl. Data Eng. 19 (8) (2007) 1131–1144.