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Unimodular triangulations in Łukasiewicz logic: Complexity bounds of probabilistic coherence

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ABSTRACT

A proof for the NP-containment for the probabilistic coherence problem over events represented by formulas of the infinite-valued Łukasiewicz logic was proposed in [1]. The geometric and combinatorial argument to prove that complexity bound contains a mistake that is fixed in the present paper. Actually we present two ways to restore that imprecise claim and, by doing so, we show that the main result of that paper is indeed valid.

1. Introduction

Suppose that we are given a finite set of events $\varphi_1, \ldots, \varphi_t$ represented as formulas of classical logic, together with a rational-valued function $\beta : \varphi_i \mapsto \beta_i \in [0,1] \cap \mathbb{Q}$. The problem of deciding if the assignment β is *consistent* with the Kolmogorov axioms for probability theory is known to be decidable, and indeed NP-complete [20]. The probability consistency problem is equivalent to de Finetti's foundations of subjective probability theory based on *coherence* in which approach an assignment β as the above is said to be *coherent* if (by de Finetti's theorem) it extends to a finitely additive probability function on the Boolean algebra of events.

De Finetti's foundational work has been approached outside the borders of classical logic and, at best of authors knowledge, the paper [21] sets a general ground for it. In the present paper we will be particularly concerned with assignments on events being represented by formulas of the infinite-valued Łukasiewicz logic [19]. In that setting, finitely additive probability functions are replaced by *states of MV-algebras* [17,18].

From the computational complexity perspective, the coherence problem on formulas of Łukasiewicz logic (*i.e.*, on *Łukasiewicz events*) has been shown to be decidable by Mundici in [18], proved to be in PSPACE in [11], and NP-complete in [1]. After those results more in this direction followed, see for instance [5,7,6].

The present paper stems from the need of fixing a flaw in a proof of a result used in [1] to prove the NP-membership for the coherence problem on Łukasiewicz events. To better justify the present research it is convenient to give some more details in this direction and, to properly understand the next statements, it is opportune to think about the *size* of a set of formulas, or of a rational-valued function, as a reasonably compact binary encoding of the description of such an object. This function, denoted by size, will be defined in precise terms in Section 2.2 below. In general, we refer the reader to Section 2 for the notions that are not defined in this introduction.

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Let us consider a rational-valued assignment $\beta : \varphi_i \mapsto \beta_i \in [0,1] \cap \mathbb{Q}$ where $\varphi_1, \ldots, \varphi_t$ are Łukasiewicz events on variables from a finite set X. The NP-containment for the problem of deciding the coherence of β is based on the following.

Lemma 1.1 *([1, Lemma 2.7]). Let $\varphi_1, \ldots, \varphi_t$ be Łukasiewicz events and $\beta : \varphi_i \mapsto \beta_i \in [0,1] \cap \mathbb{Q}$ be an assignment. The following are equivalent:*

(i) β is coherent and hence it extends to a state on the MV-algebra $\mathbf{F}_X$ generated by X.
(ii) There exists a unary polynomial $p : \mathbb{N} \to \mathbb{N}$ and $l \leq t+1$ homomorphisms h_i of $\mathbf{F}_X$ to $[0,1]$ such that each h_i ranges over the finite set $\{0, 1/d_i, 2/d_i, \ldots, (d_i-1)/d_i, 1\}$ where $d_i \leq 2^{p(\mathsf{size}(\beta))}$ and β is a convex combination of the h_i's.

The proof of the above, and precisely the direction from (ii) to (i), is based on the next proposition that we will better explain in the next pages.

Proposition 1.2 *([1, Proposition 2.6]). Let $\varphi_1, \ldots, \varphi_t$ be finitely many Łukasiewicz events on a finite set of variables X. Then there exists a unary polynomial $p : \mathbb{N} \to \mathbb{N}$ and a unimodular triangulation $\mathcal{U}$ that linearizes the McNaughton functions $f_{\varphi_1}, \ldots, f_{\varphi_t}$ such that for every rational vertex $\mathbf{x}$ of $\mathcal{U}$, the denominator* $\mathrm{den}(\mathbf{x})$ *of* $\mathbf{x}$ *satisfies that* $\mathrm{den}(\mathbf{x}) \leq 2^{p(\mathsf{size}(\varphi_1,\ldots,\varphi_t))}$.

The argument proposed in [1] to prove the above proposition contains a mistake that we are now going to explain in some details. The idea of the proof of Proposition 1.2 goes along the following steps that we invite the reader to read focusing on the overall proof strategy while the notions they are not familiar with can be skimmed through as that will be precisely defined in the next sections.[1]

(1) To every Łukasiewicz event φ_i on variables from X we associate a (unique) *McNaughton function* $f_{\varphi_i} : [0,1]^X \to [0,1]$, i.e. a continuous and piecewise linear function with integer coefficients (see, e.g. [3, Sections 3 and 4]). McNaughton functions are for Łukasiewicz logic what Boolean functions are for classical logic.
(2) Given the McNaughton functions $f_{\varphi_1}, \ldots, f_{\varphi_t}$ corresponding to the previous Łukasiewicz events one can determine a set of simplices C such that their union covers $[0,1]^X$, i.e., C is a *triangulation* of the cube $[0,1]^X$, such that each f_{φ_i} is linear on each simplex of C.
(3) By [3, Proposition 9.3.3] one can assume that the triangulation C satisfies that there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ such that the denominator $\mathrm{den}(\mathbf{x})$ of each of its (necessarily rational) vertex $\mathbf{x}$ satisfies that $\mathrm{den}(\mathbf{x}) \leq 2^{p(\mathsf{size}(\varphi_1,\ldots,\varphi_t))}$, where $\mathsf{size}(\cdot)$ is an encoding function that will be precisely defined in Section 2.2 below.

Comparing with the statement of Proposition 1.2, the triangulation C that appeared in the above point (2) needs not to be *unimodular* and the argument provided in [1], while proving (3), it actually fails to validate Proposition 1.2. On the other hand, unimodularity cannot get rid of as it is key to prove Lemma 1.1 and this is the gap we need to fill to ensure the validity of the NP-containment for assignments on Łukasiewicz events.

Indeed, there are two ways to fix the key fact that has been overlooked in [1], and those are the following:

- The first one consists in providing an alternative proof of Lemma 1.1 that is based on the above (3), and hence without directly assuming the unimodularity of C. Yet, the bounds on the complexity of the denominators of its vertices are satisfied. This is what we will prove in Section 3.
- The second one is to prove a variant of Proposition 1.2 that ensures the existence of a polynomial bound to the denominator of the vertices of a unimodular refinement of the triangulation C. This second way will be explored in full details in Section 4.

That second solution constitutes the core of the present paper and it will establish a new complexity bound for the encoding of unimodular triangulations that linearize a finite set of McNaughton function. Methodologically, this second bound will be found by adapting results by W. Bruns and M. von Thaden, originally formulated for simplicial cones in [2,22], to our framework of simplicial complexes.

2. Preliminaries

This section is meant to provide an introduction to the basic notions and results that will be used along the paper.

2.1. Basics of polyhedral geometry

We assume the reader to be familiar with the basic notions of polyhedral geometry. For the unexplained notions we refer to [4].

All the polyhedra we will consider in the present paper are thought as embedded into a finite dimensional space $\mathbb{R}^n$. By a *polyhedral complex* we understand a set of polyhedra C satisfying the following properties:

[1] For instance the definitions of *triangulation* and *unimodular* triangulation that are key for the proof on our main result will be recalled in the next Section 2.1.

- the faces of the polyhedra from C belong to C;
- two polyhedra from C intersect in a common face.

A polyhedral complex is called a *simplicial complex* if its polyhedra are simplices, that is to say they are convex hulls of (finitely many) affinely independent points.

The support of a simplicial complex C, denoted by $|C|$, is the union of all the simplices $T \in C$. A simplicial complex C is called a *triangulation* of a polyhedron P if $|C| = P$. A *refinement* of a simplicial complex C is a simplicial complex C' in which for any $P' \in C'$ there exists $P \in C$ such that $P' \subseteq P$.

For every $V \subseteq \mathbb{R}^n$, the *positive hull* of V is the set of all non-negative linear combinations of all finite subsets of V. A *polyhedral cone* (or, simply, a *cone*) is the positive hull of a finite set of points $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, in symbols,

$$\mathbb{R}_{\geq 0}\mathbf{v}_1 + \ldots + \mathbb{R}_{\geq 0}\mathbf{v}_k = \left\{\lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k \mid \lambda_i \geq 0,\ i = 1, \ldots, k\right\}.$$

The cone is called *simplicial* if the vectors in V are linearly independent.

A point $\mathbf{x} \in \mathbb{R}^n$ is said to be *rational* if all of its coordinates are rational numbers. For each rational point $\mathbf{x}$, the integer $\mathsf{den}(\mathbf{x})$ is defined as the least common multiple of the denominators of the coordinates of $\mathbf{x}$. Given a point $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Q}^n$, we denote by

$$\widetilde{\mathbf{x}} := (\mathsf{den}(\mathbf{x})x_1, \ldots, \mathsf{den}(\mathbf{x})x_n, \mathsf{den}(\mathbf{x})) \in \mathbb{Z}^{n+1}$$

the *homogeneous correspondent*, or *homogeneization*, of $\mathbf{x}$.

Polytopes and polyhedral cones in $\mathbb{R}^n$ are called *rational* if their vertices are rational. A polyhedron is rational if each of the simplices of which it is union is rational. Along the present paper polyhedra and polyhedral cones will always be rational.

Let T be a simplex and denote by M_T the matrix whose columns are the homogeneous correspondents of the vertices V_T of T. The simplex T is said to be *unimodular* (or *regular*) if the determinant of M_T is equal to ± 1. A rational polyhedron is unimodular if each of its simplices is unimodular. Any (rational) polyhedron can be triangulated and any triangulation of a rational polyhedron can be refined into a unimodular triangulation, see [19, Theorem 2.8, Appendix B-21.52]. In Section 2.3 we describe a procedure for refining a triangulation into a unimodular one. The refinement is carried out by adding new vertices, and, a priori, the number of such vertices may be exponential with respect to the number of vertices we started from. In Section 4 we show that it is nevertheless possible to add vertices while keeping control of their denominators, which remain bounded by an integer depending only on the original (non-unimodular) triangulation.

2.2. Łukasiewicz logic, MV-algebras and states

An *MV-algebra* is a structure $\mathbf{A} = (A, \oplus, \neg, 0)$ where $(A, \oplus, 0)$ forms a commutative monoid, $\neg$ is an involutive negation and the identity $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ holds for all elements $x, y \in A$. Further operations can be defined within the basic signature of MV-algebras; among them it is convenient to recall the following ones: $\top = \neg\bot$; $x \odot y = \neg(\neg x \oplus \neg y)$, $x \to y = \neg x \oplus y$.

MV-algebras form a variety that is the equivalent algebraic semantics for Łukasiewicz propositional logic Ł, a prominent many-valued logic, see [3]. The syntax of Ł is built as in classical logic starting from a countable (finite or infinite) set of propositional variables Var and the primitive connectives $\{\to, \neg\}$. For $X \subseteq \mathsf{Var}$, we denote by $\mathsf{Fm}(X)$ the set of formulas of Ł that only contain propositional variables in X.

For $X \subseteq \mathsf{Var}$ we denote by $\mathbf{F}_X$ the Lindenbaum-Tarski algebra of (equivalence classes of) formulas from $\mathsf{Fm}(X)$. It can be proved that $\mathbf{F}_X$ is the free MV-algebra generated by the set X. Moreover, it is known that, for any cardinal κ, the free κ-generated MV-algebra is isomorphic with the algebra of continuous and piecewise (affine) linear functions $f : [0,1]^\kappa \to [0,1]$ with the additional requirement that the linear pieces have integer coefficients. Such functions are also called *McNaughton functions*; further details can be found in [3, Section 9.1]. If φ is a formula in Łukasiewicz logic, we denote by f_φ the McNaughton function that correspond, via the above mentioned isomorphism, to the equivalence class of φ in the Lindenbaum-Tarski algebra of Łukasiewicz logic.

Take a set X of n variables. Homomorphisms of $\mathbf{F}_X$ to the MV-algebra on $[0,1]$ are in 1-1 correspondence with points $\mathbf{x} \in [0,1]^n$. Indeed, take $\mathbf{x} \in [0,1]^n$. The map $h_\mathbf{x} : \mathbf{F}_X \to [0,1]$ that assigns to every formula $\varphi \in \mathsf{Fm}(X)$ the value $h_\mathbf{x}(f_\varphi) = f_\varphi(\mathbf{x})$ is a homomorphism and all homomorphisms from $\mathbf{F}_X$ to $[0,1]$ arise in this way.

Finally, for a formula τ let $c(\tau)$ stand for the numbers of connectives occurring in it. We follow [10, Subsection 5.1] and assume a binary coding of τ with the following property:

- for any formula τ, in the language of Łukasiewicz logic, the number of bits of the encoding of τ, denoted by $\mathsf{size}(\tau)$, is at most $p_1(c(\tau))$, where p_1 is a polynomial;
- the number of bits needed to encode a finite set of formulas $\tau_1, \ldots, \tau_m$, denoted by $\mathsf{size}(\{\tau_1, \ldots, \tau_m\})$, is at most $p_2(\mathsf{size}(\tau_1) + \ldots + \mathsf{size}(\tau_m))$, where p_2 is a polynomial.

MV-algebras are among the most adequate frameworks to discuss probability in algebraic term. This is mainly due to the fact that they allow to codify the notion of a *state*, introduced by Mundici in [17]. For an MV-algebra $\mathbf{A}$, a *state of* $\mathbf{A}$ is a function $s : \mathbf{A} \to [0,1]$, that is normalized and additive on disjoint events. That is, $s(1) = 1$, and $s(a \oplus b) = s(a) \oplus s(b)$, whenever $a \odot b = 0$. State theory is rich and we invite the reader to consult, [10,19,18] for further details. For what concerns the present paper it is important to recall from [18] that states are the models of coherent assignments over Łukasiewicz events.

2.3. Polyhedral geometry and Łukasiewicz logic

Let $\varphi_1,\dots,\varphi_t$ be a set of formulas in Łukasiewicz logic on variables from a finite set X and assume that X has cardinality n. In [19, Example 2.1] one can find a way to obtain a triangulation of the unit cube with the property that all functions $f_{\varphi_1},\dots,f_{\varphi_t}$ are linear in each simplex of the triangulation. We will now recall two basic procedures that are of key importance for our methodology: the *stratification* and the *blow-up* procedures. Our presentation does not dig into full details that can be found in [3,19].

The stratification procedure. Take an enumeration of the linear components of the McNaughton functions $f_{\varphi_1},\dots,f_{\varphi_t}$, namely $l_1,\dots,l_k$ and take, for any permutation π of the indices $1,\dots,k$,

$$P_\pi = \{\mathbf{x}\in[0,1]^n \mid l_{\pi(1)}(\mathbf{x}) \le \dots \le l_{\pi(k)}(\mathbf{x})\}.$$

The collection of all P_π and their faces gives the required triangulation of $[0,1]^n$. Moreover, as stated in [3, pp 63-64], all P_π can be taken to be n-dimensional.

The next example shows that, in general, the triangulation obtained with this procedure is not unimodular.

Example 2.1. Consider the following McNaughton function.

$$f(x) := \begin{cases} 1-3x & 0\le x\le \frac{1}{3}\\ 0 & \frac{1}{3}\le x\le 1\end{cases}$$

Notice that $1 \ge 1-3x$ for all $x\in[0,1]$. Hence, using the stratification procedure, we obtain the two 1-simplices (i.e. segments) that correspond to the regions $x\le\frac{1}{3}$ and $x\ge\frac{1}{3}$. Hence, the vertices of this triangulation are $x_1=0, x_2=\frac{1}{3}$ and $x_3=1$. Notice that the simplex $[1/3,1]$ is not unimodular. Indeed, the homogeneous correspondents of x_2 and x_3 are $\widetilde{x}_2=(1,3)$ and $\widetilde{x}_3=(1,1)$ and

$$\det\begin{pmatrix}1 & 3\\ 1 & 1\end{pmatrix} = -2.$$

This is the only triangulation that one can obtain with the stratification procedure and to make it unimodular one needs to refine the simplex $\left[\frac{1}{3},1\right]$ in more segments, hence adding new vertices. In particular, one has to add at least the vertex $\frac{1}{2}$.

The blow-up procedure. This procedure is also known as *stellar subdivision*, as described in [19, Section 2.3]. Informally, given a point $c\in|\mathcal{C}|$, the blow-up $\mathcal{C}'$ of a triangulation $\mathcal{C}$ at the point c is obtained by replacing each simplex C of $\mathcal{C}$ that contains c with the collection of simplices of the form $\mathsf{conv}(F\cup\{c\})$, where F ranges over the faces of C that do not contain c. It is straightforward to verify that $\mathcal{C}'$ constitutes a refinement of $\mathcal{C}$. This procedure is used in [19, Theorem 2.8] to obtain a unimodular refinement of a rational triangulation.

We refer the reader to [3] and [19, Section 1 and 10] for all undefined notions.

3. An alternative proof of Lemma 1.1

In this section we provide an alternative proof of Lemma 1.1 that does not rely on Proposition 1.2 but it does rely on the next proposition, that does not assume the unimodularity for the needed triangulation $\mathcal{C}$.

Let us start by giving the consequence of [3, Proposition 9.3.3] mentioned in item (3) of the Introduction. With respect to what is already proved in [1], we make explicit the additional bound (B2) that is used in Section 4.

Proposition 3.1. *Let $\varphi_1,\dots,\varphi_t$ be finitely many Łukasiewicz events defined on a finite set of variables X. Then there exists a triangulation $\mathcal{C}$ of $[0,1]^X$ that linearizes $f_{\varphi_1},\dots,f_{\varphi_t}$ and such that, for each $\mathbf{x}\in V_{\mathcal{C}}$ and for every simplex $T\in\mathcal{C}$,*

$$\mathsf{den}(\mathbf{x}) \le 2^{4\cdot\mathsf{size}(\varphi_1,\dots,\varphi_t)^2}; \tag{B1}$$

$$\det(M_T) \le (n+1)^{\frac{n+1}{2}}\cdot 2^{4\cdot(n+1)\cdot\mathsf{size}(\varphi_1,\dots,\varphi_t)^2}. \tag{B2}$$

Proof. Let $\mathcal{C}$ be the triangulation obtained using the stratification process, as defined in the previous section.

First recall Hadamard's inequality, see [14, 2.1.P23]: for each matrix $K=(k_{ij})$ of size $m\times m$, the following holds.

$$|\det(K)| \le \prod_{i=1}^{m}\left(\sum_{j=1}^{m}k_{ij}^2\right)^{1/2}.$$

Let P be the set of linear components of the functions $f_{\varphi_1},\dots,f_{\varphi_t}$, in the variables $x_1,\dots,x_n$. Let $p(x_1,\dots,x_n)=a_0^p+a_1^p x_1+\dots+a_n^p x_n$ be an element of P. Then, there exists a non-empty $F\subseteq\{\varphi_1,\dots,\varphi_t\}$ such that p is a linear component for f_φ for any $\varphi\in F$. By [3, Corollary 9.3.2] it follows that $|a_l^p|\le\min_{\varphi\in F}(\mathsf{size}(\varphi))\le\mathsf{size}(\varphi_1,\dots,\varphi_t)$ for $0\le l\le n$. The vertices of $\mathcal{C}$ can be obtained as solutions of a linear system of n equations in n unknowns, each equation is of one of the following types:

- $p(x_1,\dots,x_n)=q(x_1,\dots,x_n)$, for $p,q\in P$ such that $p\neq q$,

- $p(x_1, \dots x_n) = 0$ for nonzero $p \in P$,
- $p(x_1, \dots x_n) = 1$ for $1 \neq p \in P$,
- $x_k = 0$ or $x_k = 1$ for $k \in \{1, \dots, n\}$.

Hence, the ith equation of the system is in the form $a_{i1}x_1 + \cdots + a_{in}x_n = b_i$, with $|a_{ij}| \leq 2 \cdot \mathsf{size}(\varphi_1, \dots, \varphi_t)$ for all $1 \leq j \leq n$. Let $A = (a_{ij})$ be the matrix of coefficients of the system. By elementary linear algebra, it follows that $\mathsf{den}(\mathbf{x}) \leq |\det(A)|$. Hence, by Hadamard's inequality,

$$\begin{aligned}
&\mathsf{den}(\mathbf{x}) \leq |\det(A)| \leq \\
&\quad \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}^2 \right)^{1/2} \leq \\
&\quad \prod_{i=1}^{n} \left(\sum_{i=1}^{n} (2 \cdot \mathsf{size}(\varphi_1, \dots, \varphi_t))^2 \right)^{1/2} = \\
&\quad \prod_{i=1}^{n} (4 \cdot n \cdot \mathsf{size}(\varphi_1, \dots, \varphi_t)^2)^{1/2} = \\
&\quad 2^n \cdot \mathsf{size}(\varphi_1, \dots, \varphi_t)^n \cdot n^{n/2} = \\
&\quad 2^{(n + n \cdot \log(\mathsf{size}(\varphi_1, \dots, \varphi_t)) + n \cdot \log(n)/2)} \leq \\
&\quad 2^{4 \cdot \mathsf{size}(\varphi_1, \dots, \varphi_t)^2}.
\end{aligned}$$

Thus, (B1) is proved. It also follows that the homogenization $\widetilde{\mathbf{x}}$ has all entries smaller that $2^{4 \cdot \mathsf{size}(\varphi_1, \dots, \varphi_t)^2}$.

To prove (B2), again by Hadamard's inequality,

$$\begin{aligned}
|\det(M_T)| &\leq \prod_{i=1}^{n+1} \left(\sum_{j=1}^{n+1} \left(2^{4 \cdot \mathsf{size}(\varphi_1, \dots, \varphi_t)^2} \right)^2 \right)^{1/2} \\
&= 2^{8 \cdot \mathsf{size}(\varphi_1, \dots, \varphi_t)^2 \cdot (n+1)/2} \cdot (n+1)^{\frac{n+1}{2}} \\
&= (n+1)^{\frac{n+1}{2}} \cdot 2^{4 \cdot (n+1) \cdot \mathsf{size}(\varphi_1, \dots, \varphi_t)^2}
\end{aligned}$$

settling the claim. □

Now, let us consider finitely many Łukasiewicz events $\varphi_1, \dots, \varphi_t$ and let C be the triangulation obtained by the stratification procedure as explained in Section 2.3. As we have already recalled it might be the case that C is not unimodular. However, by [3, Proposition 9.1.2], one can consider a unimodular refinement $\mathcal{U}_C$ of C by adding simplices to those in C, and hence more vertices to those in V_C. Precisely, let us consider the case in which $V_C = \{\mathbf{x}_1, \dots, \mathbf{x}_l\}$, $V_{\mathcal{U}_C} = \{\mathbf{y}_1, \dots, \mathbf{y}_m\}$ and $m > l$. Notice that we assume $\mathbf{x}_i = \mathbf{y}_i$ for all $i = 1, \dots, l$.

Consequently, for all $j = l+1, \dots, m$ there exists a simplex $T_j \in C$ such that $\mathbf{y}_j \in T_j$. Thus, $\mathbf{y}_j$ can be conveniently represented as convex combination of all the $\mathbf{x}_i$'s (possibly assigning 0 to some coordinate). In other words,

$$\mathbf{y}_j = \sum_{i=1}^{l} \lambda_i^j \cdot \mathbf{x}_i. \tag{1}$$

Now, let s be a state of the free n-generated MV-algebra $\mathbf{F}_X$ and consider C and $\mathcal{U}_C$ as above. Thanks to the unimodularity of $\mathcal{U}_C$, direct inspection on the proof of [1, Lemma 2.7] shows that the point $(s(\varphi_1), \dots, s(\varphi_t)) \in [0,1]^t$ actually is a convex combination of the sole vertices $\mathbf{x}_1, \dots, \mathbf{x}_l$ of C. To see this let us start by observing that [1, Lemma 2.7] yields that $(s(\varphi_1), \dots, s(\varphi_t))$ is a convex combination of all the points $\mathbf{x}_1, \dots, \mathbf{x}_l, \mathbf{y}_{l+1}, \dots, \mathbf{y}_m$, in other words, there are non-negative $\rho_1, \dots, \rho_l, \rho_{l+1}, \dots, \rho_m$ such that $\sum_{i=1}^{m} \rho_i = 1$ and

$$(s(\varphi_1), \dots, s(\varphi_t)) = \left(\sum_{r=1}^{l} \rho_r \cdot \mathbf{x}_r \right) + \left(\sum_{j=l+1}^{m} \rho_j \cdot \mathbf{y}_j \right). \tag{2}$$

Now, for all $j = l+1, \dots, m$, there exists $\lambda_1^j, \dots, \lambda_l^j$ such that $\sum_{i=1}^{l} \lambda_i^j = 1$ and $\mathbf{y}_j$ is as in (1). Thus, the right-hand side of (2) becomes

$$\begin{aligned}
&\left(\sum_{i=1}^{l} \rho_i \cdot \mathbf{x}_i \right) + \left(\sum_{j=l+1}^{m} \rho_j \cdot \left(\sum_{i=1}^{l} \lambda_i^j \cdot \mathbf{x}_i \right) \right) = \\
&\left(\sum_{i=1}^{l} \rho_i \cdot \mathbf{x}_i \right) + \left(\sum_{j=l+1}^{m} \left(\sum_{i=1}^{l} \rho_j \cdot \lambda_i^j \cdot \mathbf{x}_i \right) \right) =
\end{aligned}$$

$$\sum_{i=1}^{l}\left(\rho_i + \sum_{j=l+1}^{m} \rho_j \cdot \lambda_i^j\right)\mathbf{x}_i.$$

Hence, to see that $(s(\varphi_1), \ldots, s(\varphi_t))$ can be expressed as a convex combination of the sole vertices of C it is left to prove that the coefficients sum to 1. But this is a matter of computation:

$$\begin{aligned}\sum_{i=1}^{l}\rho_i + \sum_{j=l+1}^{m}\rho_j\left(\sum_{i=1}^{l}\lambda_i^j\right) &= \sum_{i=1}^{l}\rho_i + \sum_{j=l+1}^{m}\rho_j \cdot (1)\\ &= \sum_{i=1}^{l}\rho_i + \sum_{j=l+1}^{m}\rho_j\\ &= 1.\end{aligned}$$

Now, by Proposition 3.1, there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ such that for all $i = 1, \ldots, l$, $\mathrm{den}(\mathbf{x}_i) \le 2^{p(\mathsf{size}(\varphi_1,\ldots,\varphi_t))}$. Moreover, by Carathéodory's theorem there exists $k \le t+1$, among the vertices of C, such that $(s(\varphi_1), \ldots, s(\varphi_t))$ is a convex combination of the points $\{(h_i(\varphi_1), \ldots, h_i(\varphi_t)) \mid i = 1, \ldots, k\}$, where the maps $h_i : \mathbf{F}_X \to [0,1]$ denote the homomorphisms $h_{\mathbf{x}_i}$ as defined in Section 2.2.

In this way we prove the (1)⇒(2) direction of next result that is [1, Lemma 2.7].

Lemma 3.2 *([1, Lemma 2.7]). Let $\varphi_1, \ldots, \varphi_t$ be formulas of Łukasiewicz logic over a finite set of variables X. Let $\beta : \varphi_i \to \beta_i \in [0,1]$ be a Łukasiewicz assignment. The following are equivalent:*

(1) β is coherent and hence there exists a state of $\mathbf{F}_X$ extending β;
(2) There exists a unary polynomial $p : \mathbb{N} \to \mathbb{N}$ and $k \le t+1$ homomorphisms $h_i : \mathbf{F}_X \to \{0, 1/d_i, \ldots, (d_i-1)/d_i, 1\}$ where $d_i \le 2^{p(\mathsf{size}(\beta))}$ such that $(\beta(\varphi_1), \ldots, \beta(\varphi_t))$ is a convex combination of $\{(h_i(\varphi_1), \ldots, h_i(\varphi_t)) \mid i = 1, \ldots, k\}$.

Proof. Due to the argument developed above, it is left to prove that (2)⇒(1). To this end notice that it suffices to verify that the map $s : \mathbf{F}_X \to [0,1]$ defined below is a state. So let s be such that, for all $\varphi \in \mathbf{F}_X$,

$$s(\varphi) = \sum_{i=1}^{k} \lambda_i \cdot h_i(\varphi)$$

where the λ_i's are the parameters that presents $(\beta(\varphi_1), \ldots, \beta(\varphi_t))$ as the convex combination $(\beta(\varphi_1), \ldots, \beta(\varphi_t)) = \sum_{i=1}^{k} \lambda_i \cdot (h_i(\varphi_1), \ldots, h_i(\varphi_t))$ and as ensured by the hypothesis. Then by construction s extends β that is therefore coherent by [18, Theorem 2.1]. □

4. An alternative proof of Proposition 1.2

In this section we show how to adapt the strategy developed by Winfried Bruns and Michael von Thaden in [2] (see also [22]) to our setting. Specifically, we show how their approach– originally formulated in the context of simplicial cones –can be applied to simplices as well.

To start with, fix Łukasiewicz formulas $\varphi_1, \ldots, \varphi_t$ in n variables, and let C be a (rational) triangulation of the unit cube $[0,1]^n$ that linearizes the corresponding McNaughton functions $f_{\varphi_1}, \ldots, f_{\varphi_t}$ obtained via the stratification procedure as in Section 2.3. As previously noted, each simplex in C is an n-simplex. We assume C fixed throughout. Our goal is to construct a unimodular refinement $\mathcal{U}_C$ of C for which we can polynomially bound the denominators of all vertices in $V_{\mathcal{U}_C}$. In particular we show how to select suitable rational points for the blow-up procedure, in such a way that the denominators of the newly introduced vertices remain polynomially bounded.

Let T be a non-unimodular (but rational) n-simplex of C, with vertices $\mathbf{v}_i = (v_{i1}, \ldots, v_{in})$, for $i = 1, \ldots, n+1$. Recall that, for any $i = 1, \ldots, n+1$, $\widetilde{\mathbf{v}_i}$ denotes the point $(\mathsf{den}(\mathbf{v}_i)v_{i1}, \ldots, \mathsf{den}(\mathbf{v}_i)v_{in}, \mathsf{den}(\mathbf{v}_i))$ in $\mathbb{Z}^{n+1}$. Also note that, by construction, $\mathsf{den}(\mathbf{v}_i) > 0$ for all $i = 1, \ldots, n+1$ and therefore the last coordinate of $\widetilde{\mathbf{v}_i}$ is greater than zero.

Let $T^{\uparrow} := \mathbb{R}_{\ge 0}\widetilde{\mathbf{v}}_1 + \ldots + \mathbb{R}_{\ge 0}\widetilde{\mathbf{v}}_{n+1}$ be the cone spanned by the homogenization of the vertices of $T \in C$. It follows from [22, Lemma 1.2.4] that the multiplicity[2] $\mu(T^{\uparrow})$ of $T^{\uparrow}$ coincides with $\det(M_T)$, as defined in Section 2.

Thus, we are going to proceed as follows starting from any simplex T.

- We apply the **P2T** algorithm (*power 2 triangulation* algorithm) defined in [2,22] to obtain a triangulation of $T^{\uparrow}$ in which all points used in the blow-up have integer coordinates. In this triangulation, for all new simplices D, $\det(M_D)$ is a power of 2.
- Given a point $\widetilde{\mathbf{x}} = (x_1, \ldots, x_{n+1}) \in \mathbb{Z}^{n+1}$, with $x_{n+1} \neq 0$, take its *de-homogeneization*, i.e., the point $\mathbf{x} = \left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right)$. We show that the de-homogeneization of vertices added by the **P2T** algorithm belong to the original simplex T. Thus, by doing so we actually obtain a triangulation of T.

[2] The multiplicity of a cone $C \subseteq \mathbb{R}^n$ spanned by the primitive generators (i.e., vectors with relative prime coordinates) $\mathbf{w}_1, \ldots, \mathbf{w}_k$ is the index of the subgroup $\mathbb{Z}\mathbf{w}_1 + \ldots + \mathbb{Z}\mathbf{w}_k$ in the group generated by $C \cap \mathbb{Z}^n$. We invite the reader to consult [22,19] for more details. Although it is useful to introduce it here, we won't make use of this notion of multiplicity any longer in the paper.

- Finally, in Corollary 4.6 we give another ingredient needed to move between cones and simplices, allowing us to use again the results of [2] in order to obtain the needed unimodular triangulation and the bound that appear in Proposition 1.2.

Direct inspection on the functionality of the **P2T** algorithm, see [22, Section 1.2 and page 31], shows that at each step a point $\widetilde{\mathbf{x}'} \in \mathbb{Z}^{n+1}$ is chosen. Then, the algorithm refines the triangulation of $T^{\uparrow}$ by performing the blow-up procedure at $\widetilde{\mathbf{x}'}$. This process is iterated a finite number of times and hence, as shown in [2], the final output is produced in a finite number of steps. Regarding the termination of the algorithm, we also notice that, due to its structure, when considering adjacent simplices in a complex (or in a fan, in the case of cones), if the blow-up point selected for one simplex lies on their common face, then it can also be employed to triangulate the adjacent simplex, whose determinant cannot be a power of two by the very existence of this point on its face. When the algorithm stops, it outputs a triangulation $\mathcal{B}(T^{\uparrow})$ of the simplicial cone $T^{\uparrow}$, with the property that $\det(M_D)$ is a power of 2 for each simplicial cone $D \in \mathcal{B}(T^{\uparrow})$.

We also remark that the algorithm produces two sets of simplicial cones: $\mathcal{B}(T^{\uparrow})$ is the triangulation of $T^{\uparrow}$ that is the final output of the algorithm; $\mathcal{A}(T^{\uparrow})$ is the list of all the simplicial cones produced, including the ones that have been further subdivided (and replaced) in the run of the algorithm. Notice that these notations differ from the ones used in [2]. In particular, their $\widehat{T}(T^{\uparrow})$ is our $\mathcal{B}(T^{\uparrow})$ and their $\widehat{A}(T^{\uparrow})$ is our $\mathcal{A}(T^{\uparrow})$.

Now, in order to show that the above procedure applies also to our case, we need to prove that the de-homogeneization of each point like $\widetilde{\mathbf{x}'}$ used for the blow-up procedure belongs to the simplex T we started with. To do so, we prove first the next (more general) proposition where we adopt the notation used so far.

Proposition 4.1. *Let T be an n-simplex of C, with vertices $\mathbf{v}_i = (v_{i1}, \dots, v_{in})$, for $i = 1, \dots, n+1$. For any $\widetilde{\mathbf{u}} = \sum_{i=1}^{n+1} \beta_i \widetilde{\mathbf{v}}_i$ in $\mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$ with $0 \le \beta_i \in \mathbb{R}$ for all $1 \le i \le n+1$, its de-homogenization $\mathbf{u}$ is a rational point in $\mathsf{conv}(\mathbf{v}_1, \dots, \mathbf{v}_{n+1})$.*

Proof. Let $\widetilde{\mathbf{u}} = (u_1, \dots, u_{n+1})$. By hypothesis, $\widetilde{\mathbf{u}} \neq \mathbf{0}$ and $\mathsf{den}(\mathbf{v}_i) > 0$ for any $i = 1, \dots, n+1$. Also recall that the last coordinate of $\widetilde{\mathbf{v}}_i$ is $\mathsf{den}(\mathbf{v}_i)$. Hence, $u_{n+1} \neq 0$ since there exists an index i such that $\beta_i \neq 0$, and therefore we get that $\mathbf{u} := (u_1/u_{n+1}, \dots, u_n/u_{n+1})$ is well defined. Notice that

$$\frac{u_i}{u_{n+1}} = \sum_{j=1}^{n} \frac{\beta_j v_{ji} \mathsf{den}(\mathbf{v}_j)}{\beta_1 \cdot \mathsf{den}(\mathbf{v}_1) + \ldots + \beta_{n+1} \cdot \mathsf{den}(\mathbf{v}_{n+1})}.$$

Thus,

$$\mathbf{u} = \sum_{i=1}^{n+1} \frac{\beta_i \mathsf{den}(\mathbf{v}_i)}{\beta_1 \cdot \mathsf{den}(\mathbf{v}_1) + \ldots + \beta_{n+1} \cdot \mathsf{den}(\mathbf{v}_{n+1})} \mathbf{v}_i$$

and, for all $1 \le i \le n+1$,

$$\frac{\beta_i \mathsf{den}(\mathbf{v}_i)}{\beta_1 \cdot \mathsf{den}(\mathbf{v}_1) + \ldots + \beta_{n+1} \cdot \mathsf{den}(\mathbf{v}_{n+1})} \ge 0.$$

Moreover,

$$\sum_{i=1}^{n+1} \frac{\beta_i \mathsf{den}(\mathbf{v}_i)}{\beta_1 \cdot \mathsf{den}(\mathbf{v}_1) + \ldots + \beta_{n+1} \cdot \mathsf{den}(\mathbf{v}_{n+1})} = 1.$$

Consequently, $\mathbf{u} \in \mathsf{conv}(\mathbf{v}_1, \dots, \mathbf{v}_{n+1})$. □

Corollary 4.2. *Let $T \in C$ and $T^{\uparrow}$ as before. Then, for each point $\widetilde{\mathbf{x}'}$ obtained in the run of the **P2T** algorithm for $T^{\uparrow}$, its de-homogeneization $\mathbf{x}'$ belongs to T.*

Proof. It follows from the remark that the point added by the **P2T** algorithm satisfies the conditions of Proposition 4.1. □

Hence, let us denote by $\mathcal{A}(T)$ and $\mathcal{B}(T)$ the collections simplices that are obtained from $\mathcal{A}(T^{\uparrow})$ and $\mathcal{B}(T^{\uparrow})$ using, for the blow-up, the de-homogeneization of the points used to obtain $\mathcal{A}(T^{\uparrow})$ and $\mathcal{B}(T^{\uparrow})$. Note that $\mathcal{B}(T) \subseteq \mathcal{A}(T)$ and $\mathcal{B}(T)$ is a triangulation of T. We now focus on the bound of the denominator of the vertices added in this way.

Definition 4.3. Let T be a fixed simplex in C. With the notation of the **P2T** algorithm, for any $D \in \mathcal{A}(T)$, we define the set $\zeta_D(s)$ of rational points indexed by $s \in \mathbb{Z}$ and defined recursively as follows:

- We begin by defining $\zeta_T(-s) = \mathbf{v}_s$ if $s = 1, \dots, n+1$, otherwise $\zeta_T(s) = \mathbf{0}$;
- Let D be obtained as a blow-up of $T' \in \mathcal{A}(T)$ at the rational point $\mathbf{x}$. Let $\chi(T') = \max\{s \in \mathbb{Z} \mid \zeta_{T'}(s) \neq \mathbf{0}\}$, then

$$\zeta_D(s) := \begin{cases} \mathbf{x} & \text{if } s = \chi(T') + 1 \\ \zeta_{T'}(s) & \text{otherwise}. \end{cases}$$

Notice that, when the **P2T** algorithm stops, the set of vertices of any simplex $D \in \mathcal{A}(T)$ is, by definition, a subset of ζ_D and $\zeta_D(s) \in T \cup \{\mathbf{0}\}$ for any $s < 0$.

Notation 4.4. In the remaining part of this section, we fix a simplex $T \in C$ and we denote by maxden_T the natural number obtained as $\max\{\mathsf{den}(\mathbf{x}) \mid \mathbf{x} \in V_T\}$. We recall that V_T denotes the set of vertices of T.

To prove Proposition 1.2, we now need to give another result that allows to move back from cones to simplices and to use a language that is more suited to our purposes. Following the notations of [2], if C is a simplicial cone, Δ_C stands for the convex hull of the generators of C and the origin $\mathbf{0}$ and the set $k\Delta_C$ is the set of points of the form $k\mathbf{v}$, for $\mathbf{v} \in \Delta_C$.

Proposition 4.5. *The simplex $k\Delta_{T\uparrow}$ is generated by $k\widetilde{\mathbf{v}}_1, \dots, k\widetilde{\mathbf{v}}_{n+1}$ and the origin $\mathbf{0}$.*

Proof. Let $\mathbf{x} \in k\Delta_{T\uparrow}$. Then, by definition, $\mathbf{x} = k\mathbf{v}$, with $\mathbf{v}$ taken in $\Delta_{T\uparrow}$. Hence, by definition of $\Delta_{T\uparrow}$, there exists $\alpha_0, \dots, \alpha_{n+1}$ such that $\sum_{i=0}^{n+1} \alpha_i = 1$ and $\mathbf{v} = \sum_{i=1}^{n+1} \alpha_i \widetilde{\mathbf{v}}_i + \alpha_0 \mathbf{0}$. Then, it follows that $\mathbf{x} = k(\sum_{i=1}^{n+1} \alpha_i \widetilde{\mathbf{v}}_i + \alpha_0 \mathbf{0}) = \sum_{i=1}^{n+1} \alpha_i (k\widetilde{\mathbf{v}}_i) + \alpha_0 (k\mathbf{0}) = \sum_{i=1}^{n+1} \alpha_i (k\widetilde{\mathbf{v}}_i) + \alpha_0 \mathbf{0}$, hence the claim holds. □

Corollary 4.6. *If $\widetilde{\mathbf{x}}$ belongs to $k\Delta_{T\uparrow}$, then $\mathsf{den}(\mathbf{x}) \le k\mathsf{maxden}_T$.*

Proof. We first recall that the last coordinate of $\widetilde{\mathbf{x}}$ coincides with $\mathsf{den}(\mathbf{x})$. Then, by Proposition 4.5, $\widetilde{\mathbf{x}} = \sum_{i=1}^{n+1} \alpha_i (k\widetilde{\mathbf{v}}_i)$, with $\sum_{i=1}^{n+1} \alpha_i \le 1$, and

$$\mathsf{den}(\mathbf{x}) = \sum_{i=1}^{n+1} \alpha_i k \mathsf{den}(\mathbf{v}_i) \le \left(\sum_{i=1}^{n+1} \alpha_i \right) k\mathsf{maxden}_T \le k\mathsf{maxden}_T. \quad \square$$

Finally, Corollary 4.6 and [2, Theorem 4.3] yield the following results.

Theorem 4.7. *Let $D \in \mathcal{B}(T)$ be a simplex in the final triangulation obtained by the **P2T** algorithm. Then, for all $s \ge 0$,*

$$\mathsf{den}(\zeta_D(s)) \le \frac{n+1}{2} \cdot \det(M_T) \cdot 4^s \cdot \mathsf{maxden}_T.$$

To state next corollary, we need to define the function Φ as follows. Let $n \in \mathbb{N}$, and consider its prime factorization $n = \prod_{i=1}^{\infty} p_i^{\alpha_i}$. Denoted by $\eta(n) := \sum_{i=1}^{\infty} \alpha_i$, we define the function $\Phi(n) := 2(\log_2(n) - \eta(n))$.

Corollary 4.8. *Let $\mathbf{x}$ be any vertex in $\mathcal{B}(T)$. Then,*

$$\mathsf{den}(\mathbf{x}) \le \frac{n+1}{2} \cdot \det(M_T) \cdot 4^{\Phi(\det(M_T))} \cdot \mathsf{maxden}_T.$$

Proof. It follows from [2, Lemma 4.1] and Theorem 4.7. □

So far, we have started from a triangulation C of $[0,1]^n$ that linearizes a set of McNaughton functions and obtained by the stratification procedure. By the application of the **P2T** algorithm, we have refined each simplex of C obtaining a new triangulation $\mathcal{B}(T)$ (for each T in C) in which the denominator of each simplex is a power of 2. The following result shows that, for $n \ge 2$, $\mathcal{B}(T)$ can be further refined into a unimodular triangulation.

Theorem 4.9. *Let $T \subseteq [0,1]^n$, with $n \ge 2$, be a simplex such that $\det(M_T) = 2^l$, for some $l \in \mathbb{N}$. Then there exists a unimodular refinement $\mathcal{T}$ of T such that for any vertex $\mathbf{x}$ in $\mathcal{T}$,*

$$\mathsf{den}(\mathbf{x}) \le \frac{n+1}{2} \cdot \left(\frac{3}{2}\right)^l \cdot \mathsf{maxden}_T.$$

Proof. We apply [2, Theorem 2.1] (a more detailed version of it is [22, Theorem 3.1.1]) to $T^{\uparrow} \subseteq \mathbb{R}^{n+1}$. A direct inspection of the proof shows that:

- the vertices added to triangulate $T^{\uparrow}$ satisfy the hypothesis of Proposition 4.1. Consequently, their de-homogenaizations induce a triangulation of T;
- any added vertex $\widetilde{\mathbf{x}}$ belongs to $\frac{n+1}{2} \cdot \left(\frac{3}{2}\right)^l \Delta_{T\uparrow}$.

Hence, the claim follows from Corollary 4.6. □

Combining the **P2T** algorithm with the result given in Theorem 4.9, we obtain the following theorem. This is the analogous of [2, Theorem 4.5], where it is stated without proof and with a different language. For this reason, we prove it here.

Theorem 4.10. *Let T be an arbitrary non-unimodular simplex in $[0,1]^n$, with $n \geq 2$, and recall that maxden_T denotes $\max\{\mathsf{den}(\mathbf{x}) \mid \mathbf{x} \in V_T\}$. Then there exists a unimodular triangulation $\mathcal{T}$ of T such that*

$$\mathsf{den}(\mathbf{x}) \leq \left(\frac{n+1}{2}\right)^2 \cdot \det(M_T) \cdot 2^{\log_2(\det(M_T))^2+8} \cdot \mathsf{maxden}_T,$$

for any vertex $\mathbf{x}$ in $\mathcal{T}$.

Proof. Let $\mathcal{B}(T)$ be the triangulation of T obtained with the **P2T** algorithm and let $V_{\mathcal{B}}$ be its set of vertices. Notice that $V_{\mathcal{B}}$ is the union of all vertices of all $D \in \mathcal{B}(T)$.

By Theorem 4.9, for any $D \in \mathcal{B}(T)$, we obtain a unimodular triangulation $\mathcal{U}_D$ such that, for all vertices $\mathbf{x}$ in $\mathcal{U}_D$,

$$\mathsf{den}(\mathbf{x}) \leq \frac{n+1}{2} \cdot \left(\frac{3}{2}\right)^{\log_2(\det(M_D))} \cdot \mathsf{maxden}_D. \tag{3}$$

Furthermore, $\mathcal{T} := \{\mathcal{U}_D \mid D \in \mathcal{B}(T)\}$ is a unimodular triangulation of T whose set of vertices is obtained as the union of the sets of vertices of all simplices in the triangulation $\mathcal{U}_D$ of D. In symbols, $V_{\mathcal{T}} = \bigcup\{V_{\mathcal{U}_D} \mid D \in \mathcal{B}(T)\}$.

For better clarity, we set $b := \log_2(\det(M_T))$. By [2, Theorem 4.2] (the reader be aware that the notation is slightly different), for any $D \in \mathcal{B}(T)$ it holds that

$$\det(M_D) \leq 2^{\frac{b\cdot(b+3)}{2}},$$

and by Corollary 4.8,

$$\max\{\mathsf{den}(\mathbf{x}) \mid \mathbf{x} \in V_{\mathcal{B}}\} \leq \frac{n+1}{2} \cdot \det(M_T) \cdot 4^{\Phi(\det(M_T))} \cdot \mathsf{maxden}_T. \tag{4}$$

Moreover, by the definition of the function Φ, we have that $\Phi(\det(M_T)) \leq 2 \cdot \log_2(\det(M_T)) = 2b$. Furthermore, we remark that for any $D \in \mathcal{B}(T)$

$$\mathsf{maxden}_D = \max\{\mathsf{den}(\mathbf{x}) \mid \mathbf{x} \in V_D\} \leq \max\{\mathsf{den}(\mathbf{x}) \mid \mathbf{x} \in V_{\mathcal{B}}\}. \tag{5}$$

By definition of $\mathcal{T}$, since its vertices belong to $V_{\mathcal{U}_D}$, for some appropriate $\mathcal{U}_D$, for any vertex $\mathbf{x}$ of $\mathcal{T}$ the inequalities (3), and (5) and (4) yield the following.

$$\begin{aligned}
\mathsf{den}(\mathbf{x}) &\leq \frac{n+1}{2} \cdot \left(\frac{3}{2}\right)^{\max\{\log_2(\det(M_D)) \mid D \in \mathcal{B}(T)\}} \cdot \max\{\mathsf{den}(\mathbf{x}) \mid \mathbf{x} \in V_{\mathcal{B}}\} \\
&\leq \frac{n+1}{2} \cdot 2^{\frac{b\cdot(b+3)}{2}} \cdot \frac{n+1}{2} \cdot \det(M_T) \cdot 4^{2b} \cdot \mathsf{maxden}_T \\
&= \left(\frac{n+1}{2}\right)^2 \cdot \det(M_T) \cdot \mathsf{maxden}_T \cdot 2^{\frac{b(b+3)}{2}} \cdot 2^{4b} \\
&= \left(\frac{n+1}{2}\right)^2 \cdot \det(M_T) \cdot \mathsf{maxden}_T \cdot 2^{\frac{b(b+3)+8b}{2}} \\
&\leq \left(\frac{n+1}{2}\right)^2 \det(M_T) \cdot 2^{\log_2(\det(M_T))^2+8} \cdot \mathsf{maxden}_T,
\end{aligned}$$

using the fact that $2b^2 + 8 \geq \frac{b^2+11b}{2}$ for all $b \in \mathbb{R}$. $\square$

Finally, we prove Proposition 1.2.

Theorem 4.11 *(Proposition 1.2). For any tuple $\varphi_1, \dots, \varphi_t$ of Łukasiewicz formulas, there exists a unimodular triangulation $\mathcal{U}$ of $[0,1]^n$ and a polynomial p such that $\mathcal{U}$ linearizes $f_{\varphi_1}, \dots, f_{\varphi_t}$ and for every $\mathbf{x} \in V_{\mathcal{U}}$*

$$\mathsf{den}(\mathbf{x}) \leq 2^{O(p(\mathsf{size}(\varphi_1,\dots,\varphi_t)))}.$$

Proof. If $n = 1$, we consider the triangulation C obtained by the stratification procedure for $\varphi_1, \dots, \varphi_t$ and let M be the upper bound given in Equation (B1). Then, it is enough to consider the refinement of C given by the *Farey sequence* $F_M := \{\mathbf{x} \in [0,1] \mid \mathsf{den}(\mathbf{x}) \leq M\}$, as defined in [13, Section 1.2]. By [13, Proposition 1.1], this triangulation is unimodular.

If $n \geq 2$, we apply Theorem 4.10 to each simplex in the triangulation C given by the stratification procedure. In this way we obtain a refinement, denoted by $\mathcal{U}_C$, that is unimodular by construction. Thus, we only need to check the bound of the denominator of each vertex of $\mathcal{U}_C$. To simplify the notation, we set $s := \mathsf{size}(\varphi_1, \dots, \varphi_t)$.

We first notice that, for each simplex T in C, by Equation (B2) the following holds.

$$2^{\log_2(\det(M_T))^2+8} \leq 2^{\log_2\left((n+1)^{\frac{n+1}{2}}\cdot 2^{4\cdot(n+1)\cdot s^2}\right)^2+8}$$
$$= 2^{\left(\log_2\left((n+1)^{\frac{n+1}{2}}\right)+4\cdot(n+1)\cdot s^2\right)^2+8}.$$

Therefore, setting $k := \log_2((n+1)^{\frac{n+1}{2}})$, we have that

$$2^{\log_2(\det(M_T))^2+8} \leq 2^{16\cdot(n+1)^2\cdot s^4+8k\cdot(n+1)\cdot s^2+k^2+8}.$$

Hence, again by Equation (B2), we deduce that for each T in the triangulation C,

$$\begin{aligned}\det(M_T)\cdot 2^{\log_2(\det(M_T))^2+8} &\leq (n+1)^{\frac{n+1}{2}}\cdot 2^{4\cdot(n+1)\cdot s^2}\cdot 2^{16\cdot(n+1)^2\cdot s^4+8k\cdot(n+1)\cdot s^2+k^2+8}\\ &= (n+1)^{\frac{n+1}{2}}\cdot 2^{16\cdot(n+1)^2\cdot s^4+4(2k+1)\cdot(n+1)\cdot s^2+k^2+8}.\end{aligned}$$

Finally, Theorem 4.10 yields the following bound for any $\mathbf{x}\in V_{\mathcal{U}_C}$.

$$\begin{aligned}\mathsf{den}(\mathbf{x}) &\leq \left(\frac{n+1}{2}\right)^2\cdot\det(M_T)\cdot 2^{\log_2(\det(M_T))^2+4}\cdot\mathsf{maxden}_T \qquad (*)\\ &\leq \left(\frac{n+1}{2}\right)^2\cdot(n+1)^{\frac{n+1}{2}}\cdot 2^{16\cdot(n+1)^2\cdot s^4+4(2k+1)\cdot(n+1)\cdot s^2+k^2+8}\cdot 2^{4\cdot s^2}\\ &= \left(\frac{n+1}{2}\right)^2\cdot(n+1)^{\frac{n+1}{2}}\cdot 2^{16\cdot(n+1)^2\cdot s^4+4((2k+1)\cdot(n+1)+1)\cdot s^2+k^2+8}.\end{aligned}$$

It is worth mentioning that in ($*$) we have also used Proposition 3.1(B1).

Hence $\mathsf{den}(\mathbf{x})\leq 2^{O(s^4)}$ for any $\mathbf{x}\in V_{\mathcal{U}_C}$, since by hypothesis n (and therefore k) is a constant. □

Note that Proposition 1.2 holds as well when the number of variables is not constantly equal to n. This is because $n+1\leq \mathsf{size}(\varphi_1,\dots,\varphi_t)+1$ and the last part of the proof would yield $\mathsf{den}(\mathbf{x})\leq 2^{O(s^6)}$.

5. Conclusion and future work

The present paper offers two solutions to a claim that was made imprecisely in [1] and on which it depends the NP-containment of the probabilistic coherence test over Łukasiewicz formulas. The results presented here then show that the main claims of the aforementioned paper are correct.

Future work in this direction involves the following topics:

(1) The first one is to establish complexity bounds for assignment on other logic-based non-classical events. In this direction we recall the general setting developed in [12].
(2) A second one is to prove (or disprove) that the stronger notion of *strict* coherence can be decided by an NP-algorithm. The strict coherence problem has been fully described in [15] for the case of classical events and in [9] for the case of Łukasiewicz events. In [8] strict coherence has been characterized by geometric means and in terms of membership to the relative interior of a polytope and that makes that notion amenable to be studied in a framework similar to what we discussed in the present paper.
(3) A third one is an investigation into the computational complexity of the theory of equational states, a two-sorted generalization of states recently introduced by Kroupa and Marra [16]. One promising direction for future research is to develop the computational complexity of this theory. In particular we foresee applications to the (probabilistic) MAX-SAT problem within a two-sorted logic that is the syntactical counterpart of equational states.

CRediT authorship contribution statement

Tommaso Flaminio: Writing – review & editing, Writing – original draft, Conceptualization. **Serafina Lapenta:** Writing – review & editing, Writing – original draft, Conceptualization. **Sebastiano Napolitano:** Writing – review & editing, Writing – original draft, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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