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Generalized conjunction and disjunction of two conditional events in the setting of conditional random quantities

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ABSTRACT

In recent papers, notions of conjunction and disjunction of two conditional events as suitable conditional random quantities, which satisfy basic probabilistic properties, have been deepened in the setting of coherence. In this framework, the conjunction and the disjunction of two conditional events are defined as five-valued objects, among which are the values of the (subjectively) assigned probabilities of the two conditional events. In the present paper we propose a generalization of these structures, where these new objects, instead of depending on the probabilities of the two conditional events, depend on two arbitrary values a, b in the unit interval. We show that they are connected by a generalized version of the De Morgan's law and, by means of a geometrical approach, we compute the lower and upper bounds on these new objects both in the precise and the imprecise case. Moreover, some particular cases, obtained for specific values of a and b or in case of some logical relations, are analyzed. The results of this paper lead to the conclusion that the only objects satisfying all the logical and the probabilistic properties already valid for the operations between events are the ones depending on the probabilities of the two conditional events.

1. Introduction

Conditionals play a central role in many fields, from probability to psychology, passing through artificial intelligence, and they have been widely studied in scientific literature (see, e.g., [1,15,32,34,41]). Traditionally, conditional events and logical operations among them have been defined, and deeply analyzed, in the context of three-valued logics (see, e.g., [3,8,11,12,29]). Another approach is given by the more general Boolean algebra of conditionals studied in [15], a super-structure that contains the three-valued algebra as a substructure. However, three-values approaches lead to some problems in preserving classical logical and probabilistic properties, for example the Fréchet-Hoeffding bounds are not satisfied ([7,27]). In recent literature, the study of conjunction and disjunction of conditionals, seen as suitable conditional random quantities which satisfy classical probabilistic properties, has been deepened in the setting of coherence (see e.g. [23,25]). Based on the notion of conjunction, a suitable notion of iterated conditionals has been defined in [21]. For some applications of compound and iterated conditionals see e.g. [6,37,40]. In the framework of conditional random quantities, the conjunction of two conditional events $A|H$ and $B|K$ was defined as the five-valued object $(A|H) \wedge (B|K) =$

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$[AHBK + x\overline{H}BK + yAH\overline{K}]|(H \vee K) \in [0,1]$, where $x = P(A|H), y = P(B|K)$. The latter definition was also extended to the case of the conjunction of n conditional events.

The choice of going from definitions of conjunctions in three-valued logics to five-valued objects is justified by the search for a structure able to generalize the conjunction of two events by also maintaining similar properties. However, one might wonder whether the choice to take the values of the probabilities of the two conditional events when one of them is true and the other is undefined is actually too restrictive. In other words, could the object $(A|H) \wedge (B|K)$ be just a particular case of a more general definition of conjunction that allows more values in the 'partially' true cases, obtained when a conjunct is true and the other is void ([4]), while also preserving all the logic and probabilistic properties that we would like to have? In this paper, to answer this question we consider what we called "generalized conjunction" $(A|H) \wedge_{a,b} (B|K)$ that, instead of being dependent on the probabilities x and y of the two conditional events, is defined by means of two arbitrary values $a, b \in [0,1]$, which may be related to x and y. Then, we define the negation of the generalized conjunction and the generalized disjunction $(A|H) \vee_{a,b} (B|K)$, showing that these new structures are linked by a generalization of De Morgan's law $A \vee B = \overline{(\overline{A} \wedge \overline{B})}$.

Our main results on these generalized structures are Theorem 5 and Theorem 6 where, by exploiting a geometrical approach, we compute the sets of all coherent assessments on the family $\{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$ and $\{A|H, B|K, (A|H) \vee_{a,b} (B|K)\}$, respectively. Thanks to these results, we obtain that the only objects that satisfy the Fréchet-Hoeffding bounds for the conjunction and the disjunction for every coherent assessment on $\{A|H, B|K\}$ are, respectively, $(A|H) \wedge (B|K)$ and $(A|H) \vee (B|K)$. Moreover, to deepen the analysis of these generalized objects, we analyze the case in which the conditioning events have some logical relations, we study these structures for particular values of a and b and we consider the imprecise case for interval-valued prevision assessments.

The paper is organized as follows. After briefly recalling the basics on coherence and on the conjunction and disjunction of conditional events (Section 2), in Section 3 we give the definition of generalized conjunction of conditional events $(A|H) \wedge_{a,b} (B|K)$, its negation and the generalized disjunction $(A|H) \vee_{a,b} (B|K)$. Then we introduce the generalized De Morgan's law and we analyze the Sum rule for these new objects by also proving that it is satisfied only for $a = P(A|H)$ and $b = P(B|K)$. In Section 4 we present the main results which characterize the set of all coherent prevision assessments, first on the family $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$, and then on the family $\mathcal{F} = \{A|H, B|K, (A|H) \vee_{a,b} (B|K)\}$. In Corollary 2 and Corollary 4 we show the relation between the satisfiability of the Fréchet-Hoeffding bounds and the choice of $a = x = P(A|H)$ and $b = y = P(B|K)$. In Section 5 both for $\wedge_{a,b}$ and $\vee_{a,b}$ we analyze the case in which H and K are incompatible, i.e. $HK = \emptyset$, and then we consider specific choices for the parameters a and b in which the multi-valued objects reduce to trivalent ones. In particular, we take into account as parameters' values the pairs $(a,b) = (0,0)$, $(a,b) = (1,1)$ and $(a,b) = (p,p)$ with $p = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ for the conjunction and $p = \mathbb{P}[(A|H) \vee_{a,b} (B|K)]$ for the disjunction, for which we find well-known definitions of conjunctions and disjunctions in three-valued logics, further emphasizing the idea that our structures provide a more general setting for the study of operations among conditionals events. Then, in Section 6 for a more general analysis of the coherence on the generalized conjunction and disjunction, we study the case of interval-valued prevision assessments. We deepen further aspects on the generalized conjunction in Section 7, in particular concerning a possible interpretation and the analysis of the inclusion relation $A|H \subseteq B|K$. Finally, in Section 8, we conclude and we point out future work.

This is a revised and extended version of the conference paper [5]. We reorganized the structure of the conference paper, focusing more on the idea that among these generalized operations between conditional events the only ones that can really represent a generalization of the ones for basic events are $(A|H) \wedge (B|K)$ and $(A|H) \vee (B|K)$, because they satisfy the same logic and probabilistic properties. We added some proofs and new results on the generalized conjunction, in particular, in Section 3 we talk about its negation and the way to extend De Morgan's Law, in Section 4 we completed the proof of the upper bound of the current Theorem 5, we added the Corollary 2 and in Section 5 we also consider the analysis of the particular case in which a and b coincide with the prevision of the object. Moreover we introduce the definition of the generalized disjunction, not considered in [5], analyzing its connection with the generalized conjunction, studying the Sum rule, finding the set of coherent assessments both in the precise (Section 4.2) and imprecise case (Theorem 11 and Theorem 12), and considering the object for same particular values of the parameters analyzed for $\wedge_{a,b}$. We illustrate some results by means of two new figures.

2. Preliminary notions and results

In this section we recall some preliminaries on events, conditional events, the notion of indicator of a conditional event, and the geometrical characterization of coherence. Then, we recall the notion of coherence in the case of imprecise assessments and the definitions of conjunction and disjunction of conditionals in the context of conditional random quantities.

In what follows we use the same symbol to refer to an event A, a two-valued logical entity which can be *true*, or *false*, and its indicator, which takes value 1 in the first case and 0 in the second. We use the symbols Ω and $\emptyset$ to refer to the sure event and the impossible event, respectively. The negation of an event A is denoted by $\overline{A}$. Given two events A and B, we denote by $A \wedge B$ (resp., $A \vee B$), or simply by AB, their conjunction (resp., disjunction). When, an event A logically implies an event B, i.e., $A\overline{B} = \emptyset$, we write $A \subseteq B$. We say that n events $E_1, \dots, E_n$ are logically independent when there are no logical relations among them. Given two events A and H, with $H \neq \emptyset$, the conditional event $A|H$ is a three-valued logical entity which is *true*, or *false*, or *void*, according to whether AH is true, or $\overline{A}H$ is true, or $\overline{H}$ is true, respectively. The negation of a conditional event $A|H$ is defined as $\overline{A|H} = \overline{A}|H$. We recall the relation of logical implication (also called Goodman-Nguyen inclusion relation) between two conditional events ([28], see also [35]).

Definition 1. Given two conditional events $A|H$ and $B|K$, we say that $A|H$ logically implies $B|K$, denoted by $A|H \subseteq B|K$, if and only if $AH \subseteq BK$ and $\overline{B}K \subseteq \overline{A}H$.

2.1. Conditional prevision and coherence

We denote by X a random quantity and by $\mathbb{P}(X)$ its prevision. In the betting scheme, given any event $H \neq \emptyset$, agreeing to the betting metaphor, if you assess that $\mu = \mathbb{P}(X|H)$ then for any given $s \in \mathbb{R}$ you are willing to pay an amount $s\mu$ and to receive sX, or $s\mu$, according to whether H is true, or false (bet called off), respectively. As we will see the conjunction of two conditional events can be seen as a conditional random quantity with a finite number of possible (numerical) values. Then, in what follows, for any given conditional random quantity $X|H$, we assume that, when H is true, the set of possible values of X is a finite set of real numbers and we say that $X|H$ is a finite conditional random quantity.

Given a prevision function $\mathbb{P}$ defined on an arbitrary family $\mathcal{K}$ of finite conditional random quantities, consider a finite subfamily $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\} \subseteq \mathcal{K}$ and the vector $\mathcal{P} = (\mu_1, \dots, \mu_n)$, where $\mu_i = \mathbb{P}(X_i|H_i)$ is the assessed prevision for the conditional random quantity $X_i|H_i$, $i \in \{1, \dots, n\}$. With the pair $(\mathcal{F}, \mathcal{P})$ we associate the random gain $G = \sum_{i=1}^{n} s_i H_i (X_i - \mu_i)$. We denote by $\mathcal{G}_{\mathcal{H}_n}$ the set of values of G restricted to $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$.

Definition 2. The function $\mathbb{P}$ defined on $\mathcal{K}$ is *coherent* if and only if, $\forall n \geqslant 1$, $\forall s_1, \dots, s_n$, $\forall \mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\} \subseteq \mathcal{K}$, it holds that: $\min \mathcal{G}_{\mathcal{H}_n} \leqslant 0 \leqslant \max \mathcal{G}_{\mathcal{H}_n}$.

In other words, $\mathbb{P}$ on $\mathcal{K}$ is incoherent if and only if there exists a finite combination of n bets such that, after discarding the case where all the bets are called off, the values of the random gain are all positive or all negative. In the particular case where $\mathcal{K}$ is a family of conditional events, then Definition 2 becomes the well known definition of coherence for a conditional probability function, denoted as P. Notice that, in the general case where the conditional random quantities in $\mathcal{K}$ are bounded but possibly infinitely valued, the condition $\min \mathcal{G}_{\mathcal{H}_n} \leqslant 0 \leqslant \max \mathcal{G}_{\mathcal{H}_n}$ in Definition 2 becomes $\inf \mathcal{G}_{\mathcal{H}_n} \leqslant 0 \leqslant \sup \mathcal{G}_{\mathcal{H}_n}$.

Following the approach given in [9,23,33], once a coherent assessment $\mu = \mathbb{P}(X|H)$ is specified, the conditional random quantity $X|H$ is not looked at as the restriction of X to H, but is defined as X, or μ, according to whether H is true, or $\overline{H}$ is true; that is,

$$X|H = XH + \mu\overline{H}. \tag{1}$$

Notice that the representation (1) is not circular. Once the value $\mu = \mathbb{P}(X|H)$ is (coherently) specified by the betting scheme, the object $X|H$ in (1) is (subjectively) determined. We observe that $\mathbb{P}(XH + \mu\overline{H}) = \mathbb{P}(X|H)$. Indeed,

$$\mathbb{P}(XH + \mu\overline{H}) = \mathbb{P}(X|H)P(H) + \mu P(\overline{H}) = \mu. \tag{2}$$

Remark 1. Given any (finite) conditional random quantity $X|H$, with $\{x_1, \dots, x_r\}$ set of possible values of X when H is true, and a prevision assessment μ on $X|H$, by Definition 2 coherence requires that $\mu \in [\min\{x_1, \dots, x_r\}, \max\{x_1, \dots, x_r\}]$. Indeed, if you pay $\mu < \min\{x_1, \dots, x_r\}$ in a bet on $X|H$, when H is true it holds that $X - \mu > 0$ and hence $\min \mathcal{G}_H > 0$. Likewise, if you pay $\mu > \max\{x_1, \dots, x_r\}$ in a bet on $X|H$, when H is true it holds that $X - \mu < 0$ and hence $\max \mathcal{G}_H < 0$. Thus, $X|H = XH + \mu\overline{H} \in [\min\{x_1, \dots, x_r\}, \max\{x_1, \dots, x_r\}]$.

In the particular case when X is (the indicator of) an event A, then $\mathbb{P}(X|H) = P(A|H)$, where $P(A|H)$ is the conditional probability on $A|H$. Given a conditional event $A|H$ and a probability assessment $P(A|H) = x$, the indicator of $A|H$, denoted by the same symbol, is

$$A|H = AH + x\overline{H} = AH + x(1-H) = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ 0, & \text{if } \overline{A}H \text{ is true,} \\ x, & \text{if } \overline{H} \text{ is true.} \end{cases} \tag{3}$$

Notice that $\mathbb{P}(AH + x\overline{H}) = xP(H) + xP(\overline{H}) = x$. For the indicator of the negation of $A|H$ it holds that $\overline{A}|H = 1 - A|H$. Given two conditional events $A|H$ and $B|K$, for every coherent assessment (x, y) on $\{A|H, B|K\}$, it holds that ([26, formula (15)])

$$AH + x\overline{H} \leqslant BK + y\overline{K} \Longleftrightarrow A|H \subseteq B|K, \text{ or } AH = \emptyset, \text{ or } K \subseteq B,$$

that is, between the numerical values of $A|H$ and $B|K$, under coherence it holds that

$$A|H \leqslant B|K \iff A|H \subseteq B|K, \text{ or } AH = \emptyset, \text{ or } K \subseteq B. \tag{4}$$

We conclude this subsection by recalling a result ([23, Theorem 4]), needed later in Section 3, showing that if two conditional random quantities $X|H$, $Y|K$ coincide when $H \vee K$ is true, then $X|H$ and $Y|K$ also coincide when $H \vee K$ is false, and hence $X|H$ coincides with $Y|K$ in all cases.

Theorem 1. *Given any events $H \neq \emptyset$ and $K \neq \emptyset$, and any random quantities X and Y, let Π be the set of the coherent prevision assessments $\mathbb{P}[X|H] = \mu$ and $\mathbb{P}[Y|K] = \nu$.*
(i) Assume that, for every $(\mu, \nu) \in \Pi$, $X|H = Y|K$ when $H \vee K$ is true; then $\mu = \nu$ for every $(\mu, \nu) \in \Pi$.
(ii) For every $(\mu, \nu) \in \Pi$, $X|H = Y|K$ when $H \vee K$ is true if and only if $X|H = Y|K$.

2.2. Geometrical characterization of coherence

Given a family $\mathcal{F}=\{X_1|H_1,\dots,X_n|H_n\}$, for each $i\in\{1,\dots,n\}$ we denote by $\{x_{i1},\dots,x_{ir_i}\}$ the set of possible values of X_i when H_i is true; then, we set $A_{ij}=(X_i=x_{ij})$, $i=1,\dots,n$, $j=1,\dots,r_i$. We set $C_0=\overline{H}_1\cdots\overline{H}_n$ (it may be $C_0=\emptyset$) and we denote by $C_1,\dots,C_m$ the constituents contained in $\mathcal{H}_n=H_1\vee\cdots\vee H_n$. Hence $\bigwedge_{i=1}^n(A_{i1}\vee\cdots\vee A_{ir_i}\vee\overline{H}_i)=\bigvee_{h=0}^m C_h$. With each C_h, $h\in\{1,\dots,m\}$, we associate a vector $Q_h=(q_{h1},\dots,q_{hn})$, where $q_{hi}=x_{ij}$ if $C_h\subseteq A_{ij}$, $j=1,\dots,r_i$, while $q_{hi}=\mu_i$ if $C_h\subseteq\overline{H}_i$; with C_0 we associate $Q_0=\mathcal{P}=(\mu_1,\dots,\mu_n)$. Denoting by $\mathcal{I}$ the convex hull of $Q_1,\dots,Q_m$, the condition $\mathcal{P}\in\mathcal{I}$ amounts to the existence of a vector $(\lambda_1,\dots,\lambda_m)$ such that: $\sum_{h=1}^m\lambda_hQ_h=\mathcal{P}$, $\sum_{h=1}^m\lambda_h=1$, $\lambda_h\geqslant 0$, $\forall h$; in other words, $\mathcal{P}\in\mathcal{I}$ is equivalent to the solvability of the system (Σ), associated with $(\mathcal{F},\mathcal{P})$,

$$(\Sigma)\quad\begin{cases}\sum_{h=1}^m\lambda_hq_{hi}=\mu_i\,,\; i\in\{1,\dots,n\}\,,\\ \sum_{h=1}^m\lambda_h=1,\;\;\lambda_h\geqslant 0,\;\; h\in\{1,\dots,m\}\,.\end{cases}\tag{5}$$

Given the assessment $\mathcal{P}=(\mu_1,\dots,\mu_n)$ on $\mathcal{F}=\{X_1|H_1,\dots,X_n|H_n\}$, let S be the set of solutions $\Lambda=(\lambda_1,\dots,\lambda_m)$ of system (Σ). We point out that the solvability of system (Σ) is a necessary (but not sufficient) condition for coherence of $\mathcal{P}$ on $\mathcal{F}$. When (Σ) is solvable, and hence $S\neq\emptyset$, we define:

$$\begin{aligned}&\Phi_i(\Lambda)=\Phi_i(\lambda_1,\dots,\lambda_m)=\textstyle\sum_{r:C_r\subseteq H_i}\lambda_r,\;;\;\Lambda\in S\,,\\&M_i=\max_{\Lambda\in S},\Phi_i(\Lambda),\; i\in\{1,\dots,n\}\,;\\&I_0=\{i\;:\;M_i=0\}\,,\mathcal{F}_0=\{X_i|H_i\,,i\in I_0\},\\&\mathcal{P}_0=(\mu_i,\, i\in I_0).\end{aligned}\tag{6}$$

Then, the following theorem can be proved ([2, Thm 3]):

Theorem 2. A conditional prevision assessment $\mathcal{P}=(\mu_1,\dots,\mu_n)$ on the family $\mathcal{F}=\{X_1|H_1,\dots,X_n|H_n\}$ is coherent if and only if the following conditions are satisfied: (i) the system (Σ) defined in (5) is solvable; (ii) if $I_0\neq\emptyset$, then $\mathcal{P}_0$ is coherent.

Remark 2. We observe that if Λ is a solution of System (Σ), associated with the pair $(\mathcal{F},\mathcal{P})$, such that $\Phi_j(\Lambda)>0$, $j=1,\dots,n$, then it holds that $I_0=\emptyset$ and hence by Theorem 2 the assessment $\mathcal{P}$ on $\mathcal{F}$ is coherent.

We recall the following extension theorem for conditional previsions, which is a generalization of de Finetti's fundamental theorem of probability to conditional random quantities (see, e.g., [30,38,44])

Theorem 3. *Let $\mathcal{P}=(\mu_1,\dots,\mu_n)$ be a coherent prevision assessment on a family of bounded conditional random quantities $\mathcal{F}=\{X_1|H_1,\dots,X_n|H_n\}$. Moreover, let $X|H$ be a further bounded conditional random quantity. Then, there exists a suitable closed interval $[\mu',\mu'']$ such that the extension $\mu=\mathbb{P}(X|H)$ is coherent if and only if $\mu\in[\mu',\mu'']$.*

2.3. Imprecise assessments

We recall below the notions of coherence, for imprecise, or set-valued, prevision assessments. As in this paper we limit our analysis to conditional random quantities with possible values in the unit interval, in our case the imprecise assessments are subsets of the unitary hypercube.

Definition 3. Let $\mathcal{F}=\{X_1|H_1,\dots,X_n|H_n\}$ be a family of n conditional random quantities. An *imprecise, or set-valued, assessment* $\mathcal{A}\subseteq[0,1]^n$ on $\mathcal{F}$ is a set of precise assessments $\mathcal{P}$ on $\mathcal{F}$.

Let $\mathcal{A}$ be an imprecise assessment on $\mathcal{F}$. For each $j\in\{1,2,\dots,n\}$, the projection $\rho_j(\mathcal{A})$ of $\mathcal{A}$ onto the j-th coordinate, is an imprecise assessment on $X_j|H_j$ and is defined as

$$\rho_j(\mathcal{A})=\{x_j\,:\,\mu_j=x_j,\text{ for some }(\mu_1,\dots,\mu_n)\in\mathcal{A}\}.$$

Definition 4. Let $\mathcal{F}=\{X_1|H_1,\dots,X_n|H_n\}$ be a family of n conditional random quantities. An imprecise assessment $\mathcal{A}$ on $\mathcal{F}$ is *coherent* if and only if, for every $j\in\{1,\dots,n\}$ and for every $x_j\in\rho_j(\mathcal{A})$, there exists a coherent precise assessment $\mathcal{P}=(\mu_1,\dots,\mu_n)$ on $\mathcal{F}$, such that $\mathcal{P}\in\mathcal{A}$ and $\mu_j=x_j$.

In the context of imprecise assessments the notions of g-coherence and total coherence have been also introduced ([17,20]).

Remark 3. We observe that an interval-valued prevision assessment $\mathcal{A}=[l_1,u_1]\times\cdots\times[l_n,u_n]$ is an imprecise prevision assessment. In this case it holds that $\rho_j(\mathcal{A})=[l_j,u_j]$. Then, based on Definition 4, the imprecise prevision assessment $\mathcal{A}$ on $\mathcal{F}$ is coherent if and only if, given any $j\in\{1,\dots,n\}$ and any $x_j\in[l_j,u_j]$, there exists a coherent precise prevision assessment $\mathcal{P}=(\mu_1,\dots,\mu_n)$ on $\mathcal{F}$, with $l_i\leqslant\mu_i\leqslant u_i$, $i=1,\dots,n$, and such that $\mu_j=x_j$.

2.4. Conjunction and disjunction of two conditional events

We recall below the notions of conjunction and disjunction of two conditional events in the framework of conditional random quantities ([23], see also [31,34]).

Definition 5. Given two conditional events $A|H$, $B|K$ and a (coherent) probability assessment $P(A|H) = x$, $P(B|K) = y$, the conjunction $(A|H) \wedge (B|K)$ is defined as the following conditional random quantity

$$(A|H) \wedge (B|K) = (AHBK + x\overline{H}BK + yAH\overline{K})|(H \vee K). \tag{7}$$

Remark 4. Notice that the conjunction in (7) can be represented as $X|H$ in (1) and, once the (coherent) assessment (x, y, z), where $z = \mathbb{P}[(A|H) \wedge (B|K)]$, is given, the conjunction is (subjectively) determined by

$$(A|H) \wedge (B|K) = AHBK + x\overline{H}BK + yAH\overline{K} + z\overline{H}\,\overline{K}.$$

Then, the set of possible values of $(A|H) \wedge (B|K)$, i.e. $\{1, 0, x, y, z\}$, is associated to a given (coherent) subjective assessment (x, y, z).

Within the betting scheme, by starting with a coherent assessment (x, y) on $\{A|H, B|K\}$, if you extend (x, y) (in a coherent way) by adding the assessment $\mathbb{P}[(A|H) \wedge (B|K)] = z$, then you agree to pay z, by receiving the random amount

$$(A|H) \wedge (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}H \vee \overline{B}K \text{ is true,} \\ x, & \text{if } \overline{H}BK \text{ is true,} \\ y, & \text{if } AH\overline{K} \text{ is true,} \\ z, & \text{if } \overline{H}\,\overline{K} \text{ is true.} \end{cases}$$

In other words, you receive:

- 1, if both conditional events $A|H$ and $B|K$ are true;
- 0, if $A|H$ or $B|K$ is false; $x = P(A|H)$, if $A|H$ is void and $B|K$ is true;
- $y = P(B|K)$, if $A|H$ is true and $B|K$ is void;
- z, that is the paid amount, if both conditional events $A|H$ and $B|K$ are void.

Notice that, in some particular case, the conjunction $(A|H) \wedge (B|K)$, which is a five-valued object, reduces to a conditional event, that is a three-valued object.

We recall that the Fréchet-Hoeffding bounds for the conjunction, i.e., the lower and upper bounds

$$z' = \max\{x + y - 1, 0\}, z'' = \min\{x, y\} \tag{8}$$

obtained under logical independence in the unconditional case for the coherent extensions $z = P(A \wedge B)$ of $P(A) = x$ and $P(B) = y$, are still valid when $P(A)$, $P(B)$, and $P(A \wedge B)$ are replaced by $P(A|H)$, $P(B|K)$, and $\mathbb{P}[(A|H) \wedge (B|K)]$ ([23]). We recall that the Fréchet-Hoeffding bounds are not satisfied by other notions of conjunction, e.g., the quasi conjunction, defined in suitable trivalent logics ([39]).

Definition 6. Given two conditional events $A|H$, $B|K$ and a (coherent) probability assessment $P(A|H) = x$, $P(B|K) = y$, the disjunction $(A|H) \vee (B|K)$ is defined as the following conditional random quantity

$$(A|H) \vee (B|K) = ((AH \vee BK) + x\overline{H}\,\bar{B}K + y\bar{A}H\bar{K})|(H \vee K). \tag{9}$$

Within the betting scheme, by starting with a coherent assessment (x, y) on $\{A|H, B|K\}$, if you extend (x, y) (in a coherent way) by adding the assessment $\mathbb{P}[(A|H) \vee (B|K)] = w$, then you agree to pay w, by receiving the random amount

$$(A|H) \vee (B|K) = \begin{cases} 1, & \text{if } AH \vee BK \text{ is true,} \\ 0, & \text{if } \overline{A}H \wedge \overline{B}K \text{ is true,} \\ x, & \text{if } \overline{H}\,\overline{B}K \text{ is true,} \\ y, & \text{if } \overline{A}H\overline{K} \text{ is true,} \\ w, & \text{if } \overline{H}\,\overline{K} \text{ is true.} \end{cases}$$

We recall that De Morgan's Laws are satisfied and hence it holds that

$$\overline{(A|H) \vee (B|K)} = (\bar{A}|H) \wedge (\bar{B}|K) \tag{10}$$

and

$$\overline{(A|H) \wedge (B|K)} = (\bar{A}|H) \vee (\bar{B}|K), \tag{11}$$

where the negations $\overline{(A|H) \vee (B|K)}$ and $\overline{(A|H) \wedge (B|K)}$ are defined as $\overline{(A|H) \vee (B|K)} = 1 - (A|H) \vee (B|K)$ and $\overline{(A|H) \wedge (B|K)} = 1 - (A|H) \wedge (B|K)$, respectively. We also observe that the prevision sum rule is satisfied, that is

$$A|H + B|K = (A|H) \wedge (B|K) + (A|H) \vee (B|K)$$

and hence

$$P(A|H) + P(B|K) = \mathbb{P}[(A|H) \wedge (B|K)] + \mathbb{P}[(A|H) \vee (B|K)]. \tag{12}$$

From (12), by exploiting the Fréchet-Hoeffding bounds for the conjunction (8), we obtain the Fréchet-Hoeffding bounds for the disjunction:

$$w' = \max\{x, y\},\ w'' = \min\{x + y, 1\}. \tag{13}$$

Quasi Conjunction and Quasi Disjunction. The quasi conjunction ([1,10]), or Sobocinski conjunction, ($\wedge_S$) of two conditional events $A|H$ and $B|K$ is defined, in a trivalent logic, as the following conditional event

$$(A|H) \wedge_S (B|K) = [(AH \vee \overline{H}) \wedge (BK \vee \overline{K})]|(H \vee K).$$

In terms of conditional random quantity, it holds that

$$(A|H) \wedge_S (B|K) = (AHBK + \overline{H}BK + AH\overline{K})|(H \vee K). \tag{14}$$

We recall that, by setting $x = P(A|H)$, $y = P(B|K)$, under logical independence, the lower and upper bounds z'_S, z''_S for $(A|H) \wedge_S (B|K)$ are ([16])

$$z'_S = \max\{x + y - 1, 0\}, \quad z''_S = \begin{cases} \frac{x+y-2xy}{1-xy}, \text{ if } (x, y) \neq (1, 1) \\ 1, \text{ if } (x, y) = (1, 1). \end{cases} \tag{15}$$

The quasi disjunction ($\vee_S$) of two conditional events $A|H$ and $B|K$ is defined as the following conditional event

$$(A|H) \vee_S (B|K) = (AH \vee BK)|(H \vee K). \tag{16}$$

We recall that, by setting $x = P(A|H)$, $y = P(B|K)$, under logical independence, the lower and upper bounds w'_S, w''_S for $(A|H) \vee_S (B|K)$ are ([16])

$$w'_S = \begin{cases} \dfrac{xy}{x + y - xy}, \text{ if } (x, y) \neq (0, 0), \\ 0, \text{ if } (x, y) = (0, 0), \end{cases} \qquad w''_S = \min\{x + y, 1\}. \tag{17}$$

3. Generalized operations for two conditional events

In this section we first generalize the notion of conjunction between two conditional events recalled in Definition 5 by replacing $x = P(A|H)$ and $y = P(B|K)$ with two arbitrary values a, b in $[0, 1]$, then we analyze it in the framework of the betting scheme, and we define its negation. Then, following the same idea we generalize the disjunction in Definition 6 and we study it in a coherent betting framework. Moreover, in order to check if these objects are consistent with classical probabilistic and logical properties, we give a generalization of De Morgan's laws and the Sum rule for these new random quantities.

Definition 7. Given four events A, B, H, K, with $H \neq \emptyset$ and $K \neq \emptyset$, and two values $a, b \in [0, 1]$, we define the *generalized conjunction* w.r.t. a and b of the conditional events $A|H$ and $B|K$ as the following conditional random quantity

$$(A|H) \wedge_{a,b} (B|K) = (AHBK + a\overline{H}BK + bAH\overline{K})|(H \vee K). \tag{18}$$

By the linearity of prevision, from (18), it follows that

$$\mathbb{P}[(A|H) \wedge_{a,b} (B|K)] = P(AHBK|(H \vee K)) + aP(\overline{H}BK|(H \vee K)) + bP(AH\overline{K}|(H \vee K)). \tag{19}$$

By exploiting (1), by setting $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$, it holds that

$$(A|H) \wedge_{a,b} (B|K) = AHBK + a\overline{H}BK + bAH\overline{K} + z(H \vee K). \tag{20}$$

In the betting framework, if you assess $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$, then you agree to pay z, by receiving the random amount given in (20), that is

$$(A|H) \wedge_{a,b} (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}H \vee \overline{B}K \text{ is true,} \\ a, & \text{if } \overline{H}BK \text{ is true,} \\ b, & \text{if } AH\overline{K} \text{ is true,} \\ z, & \text{if } \overline{H}\,\overline{K} \text{ is true.} \end{cases}$$

In other words, you agree to pay z in order to receive:

- 1, if both conditional events $A|H$ and $B|K$ are true;
- 0, if $A|H$ or $B|K$ is false;
- a, if $A|H$ is void and $B|K$ is true;
- b, if $A|H$ is true and $B|K$ is void;
- z, that is the paid amount, if both conditional events $A|H$ and $B|K$ are void.

We observe that $(A|H) \wedge_{a,b} (B|K)$, when $H \vee K$ is true, assumes values in $[0,1]$. Then, by coherence (see Remark 1) it must be $z \in [0,1]$ and hence $(A|H) \wedge_{a,b} (B|K) \in [0,1]$. Of course, $(A|H) \wedge_{a,b} (B|K) = (B|K) \wedge_{b,a} (A|H)$. We also observe that, if $H = K$, then $\overline{H}BK = AH\overline{K} = \emptyset$, and hence, for each pair (a,b), it holds that

$$(A|H) \wedge_{a,b} (B|H) = AB|H.$$

Remark 5. When we assess $P(A|H) = x = a$ and $P(B|K) = y = b$, from Definition 5 and Definition 7, and using that $(A|H) \wedge (B|K) = (B|K) \wedge (A|H)$, it holds that

$$(A|H) \wedge_{x,y} (B|K) = (A|H) \wedge (B|K) = (B|K) \wedge (A|H) = (B|K) \wedge_{y,x} (A|H),$$

that is both $(A|H) \wedge_{a,b} (B|K)$ and $(B|K) \wedge_{b,a} (A|H)$ reduces to $(A|H) \wedge (B|K)$, when $a = x$ and $b = y$. Moreover,

$$\mathbb{P}[(A|H) \wedge_{x,y} (B|K)] = P(AHBK|(H \vee K)) + P(A|H)P(\overline{H}BK|(H \vee K)) + P(B|K)P(AH\overline{K}|(H \vee K)). \tag{21}$$

We also notice that, when $x \leqslant a, y \leqslant b$ it holds that $(A|H) \wedge (B|K) - (A|H) \wedge_{a,b} (B|K) = (x-a)(\overline{H}BK|H \vee K) + (y-b)(\bar{K}AH|H \vee K) \leqslant 0$, and hence, by coherence, $\mathbb{P}[(A|H) \wedge (B|K)] \leqslant \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$.

We want now to define the negation for this new structure. Basic probability teaches us that, given an event A, the probability of its negation $\overline{A}$, is $P(\overline{A}) = 1 - P(A)$. The idea is to leave this relationship unchanged while also exploiting the linearity of the prevision. Then we have the following

Definition 8. Let us consider four events A, B, H, K, with $H \neq \emptyset$ and $K \neq \emptyset$, and two values $a, b \in [0,1]$. Given the generalized conjunction $(A|H) \wedge_{a,b} (B|K)$ of the conditional events $A|H$ and $B|K$, we define its negation as the following conditional random quantity

$$\overline{(A|H) \wedge_{a,b} (B|K)} = 1 - (A|H) \wedge_{a,b} (B|K).$$

We observe that

$$\mathbb{P}[\overline{(A|H) \wedge_{a,b} (B|K)}] = 1 - \mathbb{P}[(A|H) \wedge_{a,b} (B|K)].$$

Following the same philosophy used in Definition 7 to generalize to conjunction in Definition 5, considering the disjunction of two conditional events in the framework of conditional random quantities recalled in Definition 6, we define the generalized disjunction as follows

Definition 9. Given four events A, B, H, K, with $H \neq \emptyset$ and $K \neq \emptyset$, and two values $a, b \in [0,1]$, we define the *generalized disjunction w.r.t. a and b* of the conditional events $A|H$ and $B|K$ as the following conditional random quantity

$$(A|H) \vee_{a,b} (B|K) = (AH \vee BK + a\overline{H}\,\overline{B}K + b\overline{A}H\overline{K})|(H \vee K). \tag{22}$$

By exploiting (1), by setting $w = \mathbb{P}[(A|H) \vee_{a,b} (B|K)]$, it holds that

$$(A|H) \vee_{a,b} (B|K) = (AH \vee BK) + a\overline{H}\,\overline{B}K + b\overline{A}H\overline{K} + w(H \vee K). \tag{23}$$

In the betting framework, if you assess $w = \mathbb{P}[(A|H) \vee_{a,b} (B|K)]$, then you agree to pay w, by receiving the random amount

Table 1

Numerical values of the events $A|K, B|K$ and of the conditional random quantities $(A|H) \wedge_{a,b} (B|K)$, $(\overline{A}|H) \wedge_{a,b} (\overline{B}|K)$, $1 - (\overline{A}|H) \wedge_{a,b} (\overline{B}|K)$ and $(A|H) \vee_{c,d} (B|K)$, where $x = P(A|H)$, $y = P(B|K)$, $z_{a,b} = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$, $z = \mathbb{P}[(\overline{A}|H) \wedge_{a,b} (\overline{B}|K)]$, and $w = \mathbb{P}[(A|H) \vee_{c,d} (B|K)]$.

Const.	$A\|H$	$B\|K$	$(A\|H) \wedge_{a,b} (B\|K)$	$(\overline{A}\|H) \wedge_{a,b} (\overline{B}\|K)$	$1-(\overline{A}\|H) \wedge_{a,b} (\overline{B}\|K)$	$(A\|H) \vee_{c,d} (B\|K)$
$AHBK$	1	1	1	0	1	1
$AH\overline{B}K$	1	0	0	0	1	1
$AH\overline{K}$	1	y	b	0	1	1
$\overline{A}HBK$	0	1	0	0	1	1
$\overline{A}H\overline{B}K$	0	0	0	1	0	0
$\overline{A}H\overline{K}$	0	y	0	b	$1-b$	d
$\overline{H}BK$	x	1	a	0	1	1
$\overline{H}\,\overline{B}K$	x	0	0	a	$1-a$	c
$\overline{H}\,\overline{K}$	x	y	$z_{a,b}$	z	$1-z$	w

$$(A|H) \vee_{a,b} (B|K) = \begin{cases} 1, & \text{if } AH \vee BK \text{ is true,} \\ 0, & \text{if } \overline{A}H\overline{B}K \text{ is true,} \\ a, & \text{if } \overline{H}\,\overline{B}K \text{ is true,} \\ b, & \text{if } \overline{A}H\overline{K} \text{ is true,} \\ w, & \text{if } \overline{H}\,\overline{K} \text{ is true.} \end{cases}$$

In other words, you agree to pay w in order to receive:

- 1, if one of the conditional events $A|H$ or $B|K$ is true;
- 0, if both $A|H$ and $B|K$ are false;
- a, if $A|H$ is void and $B|K$ is false;
- b, if $A|H$ is false and $B|K$ is void;
- w, that is the paid amount, if both conditional events $A|H$ and $B|K$ are void.

Therefore

$$\mathbb{P}[(A|H) \vee_{a,b} (B|K)] = P((AH \vee BK)|(H \vee K)) + aP(\overline{H}\,\overline{B}K|(H \vee K)) + bP(\overline{A}H\overline{K}|(H \vee K)). \tag{24}$$

We observe that $(A|H) \vee_{a,b} (B|K)$, when $H \vee K$ is true, assumes values in $[0,1]$. Then, by coherence (see Remark 1) it must be $w \in [0,1]$ and hence $(A|H) \vee_{a,b} (B|K) \in [0,1]$. Of course it holds that $(A|H) \vee_{a,b} (B|K) = (B|K) \vee_{b,a} (A|H)$.

Remark 6. It can be easily observed that when we take $(a,b) = (x,y)$, it holds that $(A|H) \vee_{a,b} (B|K) = (B|K) \vee_{b,a} (A|H) = (A|H) \vee (B|K)$, where the latter disjunction is the one defined in Definition 6.

3.1. Generalized De Morgan's Law and Sum Rule

Two classical relations linking the conjunction and disjunction of events are De Morgan's Laws and the so called Sum Rule. We want now to check if some generalizations of these rules are satisfied also by the new structures $\wedge_{a,b}$ and $\vee_{c,d}$.

We begin by recalling that, given two basic events A and B by De Morgan's law it holds that $A \vee B = \overline{(\overline{A} \wedge \overline{B})}$. On the basis of the previous relation, we seek to find a similar one linking $\wedge_{a,b}$, its negation and generalized disjunction $\vee_{c,d}$, for some $a,b,c,d \in [0,1]$. Then, also using Definition 8, we consider the following generalized De Morgan law

$$(A|H) \vee_{c,d} (B|K) = \overline{(\bar{A}|H) \wedge_{a,b} (\bar{B}|K)} = 1 - (\bar{A}|H) \wedge_{a,b} (\bar{B}|K). \tag{25}$$

By Definition 7, by replacing A with $\bar{A}$ and B with $\bar{B}$, if follows that

$$(\overline{A}|H) \wedge_{a,b} (\overline{B}|K) = (\overline{A}H\overline{B}K + a\overline{H}\,\overline{B}K + b\overline{A}H\overline{K})|(H \vee K).$$

Looking at Table 1 and setting $c = 1-a$ and $d = 1-b$, we obtain that $(A|H) \vee_{1-a,1-b} (B|K) = 1 - (\bar{A}|H) \wedge_{a,b} (\bar{B}|K)$, when $H \vee K$ is true. Then by Theorem 1, they also coincide when $\overline{H}\,\overline{K}$ is true, that is $w = 1 - z$, where $z = \mathbb{P}[(\overline{A}|H) \wedge_{a,b} (\overline{B}|K)]$. Therefore, formula

$$(A|H) \vee_{a,b} (B|K) = 1 - (\bar{A}|H) \wedge_{1-a,1-b} (\bar{B}|K), \tag{26}$$

holds in all cases and hence

$$\mathbb{P}[(A|H) \vee_{a,b} (B|K)] = 1 - \mathbb{P}[(\bar{A}|H) \wedge_{1-a,1-b} (\bar{B}|K)]. \tag{27}$$

Another important rule that characterizes the relation between the conjunction and the disjunction of two basic events is the *Sum Rule*. More precisely, given two events A and B it holds that $A+B=AB+A\vee B$ (and that $P(A)+P(B)=P(AB)+P(A\vee B)$). We want to generalize this relation to conditional events $A|H$ and $B|K$, that is, we want to verify if, for some values of $a,b,c,d\in[0,1]$, it happens that $A|H+B|K=(A|H)\wedge_{a,b}(B|K)+(A|H)\vee_{c,d}(B|K)$.

We have the following result.

Theorem 4. *Let $A,\ B,\ H,\ K$ be any logically independent events. Let $(x,y)\in[0,1]^2$ be a coherent assessment on $\{A|H,B|K\}$. Given any numbers $a,b,c,d\in[0,1]$ we have that*

$$A|H+B|K=(A|H)\wedge_{a,b}(B|K)+(A|H)\vee_{c,d}(B|K)\iff a=c=x \text{ and } b=d=y. \tag{28}$$

Proof. To prove relation (28), we compute the values of $A|H+B|K$ and $(A|H)\wedge_{a,b}(B|K)+(A|H)\vee_{c,d}(B|K)$ looking at Table 1. If we want that $A|H+B|K=(A|H)\wedge_{a,b}(B|K)+(A|H)\vee_{c,d}(B|K)$ then from the constituent $AH\bar{K}$, it must be $1+y=b+1\Leftrightarrow y=b$, from $\bar{A}H\bar{K}$, $0+y=0+d\Leftrightarrow y=d$, from $\overline{H}BK$, $x+1=a+1\Leftrightarrow x=a$, and from $\overline{H}\bar{B}K$, we have that $x+0=0+c\Leftrightarrow x=b$. Therefore, when $H\vee K$ is true, $A|H+B|K=(A|H)\wedge_{a,b}(B|K)+(A|H)\vee_{c,d}(B|K)$ if and only if $a=c=x$ and $b=d=y$. We observe that $A|H+B|K=(AH+x\overline{H}+BK+y\bar{K})|(H\vee K)$, with $\mathbb{P}[(AH+x\overline{H}+BK+y\bar{K})|(H\vee K)]=x+y$ (see [23, Theorem 2]). Moreover, as both the objects $(A|H)\wedge_{a,b}(B|K)$ and $(A|H)\vee_{c,d}(B|K)$ are conditional random quantities with the same conditioning event $H\vee K$, it holds that $(A|H)\wedge_{a,b}(B|K)+(A|H)\vee_{c,d}(B|K)$ is also a conditional random quantity conditioned to $H\vee K$. Moreover, when $a=c=x$ and $b=d=y$, by Theorem 1, it follows that $(AH+x\overline{H}+BK+y\bar{K})|(H\vee K)$ and $(A|H)\wedge_{a,b}(B|K)+(A|H)\vee_{c,d}(B|K)$, coincide when $H\vee K$ is false. Then eq. (28) is verified. □

Remark 7. From Theorem 4, we have that

$$A|H+B|K=(A|H)\wedge_{a,b}(B|K)+(A|H)\vee_{c,d}(B|K)\iff(\wedge_{a,b},\vee_{c,d})=(\wedge,\vee),$$

where $\wedge$ and $\vee$ are as in Definition 5 and Definition 6, respectively. Then, the pair $(\wedge,\vee)$ is the only one that satisfies the generalized version of the sum rule and we have that $P(A|H)+P(B|K)=\mathbb{P}[(A|H)\wedge(B|K)]+\mathbb{P}[((A|H)\vee(B|K)]$.

4. Sets of coherent assessments

In this section we study the sets of coherent assessments on the families $\{A|H,B|K,(A|H)\wedge_{a,b}(B|K)\}$ in Theorem 5, and $\{A|H,B|K,(A|H)\vee_{a,b}(B|K)\}$ in Theorem 6. Moreover, both for the generalized conjunction and for the generalized disjunction, we observe that the Fréchet-Hoeffding bounds are satisfied for every $(x,y)\in[0,1]^2$ only for $(a,b)=(x,y)$, that is for the conjunction and disjunction of conditional events defined in Definition 5 and Definition 6, respectively.

4.1. Generalized conjunction and coherence

We begin by analyzing the coherent assessments on the generalized conjunction.

Theorem 5. *Let $A,\ B,\ H,\ K$ be any logically independent events. A prevision assessment $\mathcal{P}=(x,y,z)$ on the family of conditional random quantities $\mathcal{F}=\{A|H,B|K,(A|H)\wedge_{a,b}(B|K)\}$ is coherent if and only if $(x,y)\in[0,1]^2$ and $z\in[z',z'']$, where*

$$z'=\begin{cases}(x+y-1)\cdot\min\{\frac{a}{x},\frac{b}{y},1\}, & \text{if } x+y-1>0,\\ 0, & \text{otherwise}\end{cases} \tag{29}$$

and

$$z''=\max\{z_1'',z_2'',\min\{z_3'',z_4''\}\}, \tag{30}$$

where

$$z_1''=\min\{x,y\},$$
$$z_2''=\begin{cases}\dfrac{x(b-ay)+y(a-bx)}{1-xy}, & \text{if } (x,y)\neq(1,1),\\ 1, & \text{if } (x,y)=(1,1),\end{cases}$$

$$z_3''=\begin{cases}\dfrac{x(1-a)+y(a-x)}{1-x}, & \text{if } x\neq 1,\\ 1, & \text{if } x=1,\end{cases}$$

$$z_4''=\begin{cases}\dfrac{x(b-y)+y(1-b)}{1-y}, & \text{if } y\neq 1,\\ 1, & \text{if } y=1.\end{cases}$$

Table 2
Values of the lower bound z' of $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ for different values of $x = P(A|H), y = P(B|K), a, b \in [0,1]$.

Case	z'
(A): $x + y - 1 \leqslant 0$	0
(B): $x + y - 1 > 0$	
Sub-cases:	
(B.1): $a \geqslant x$ and $b \geqslant y$	$x + y - 1$
(B.2): $a < x$ and $\frac{a}{x} \leqslant \frac{b}{y}$	$\frac{a}{x}(x + y - 1)$
(B.3): $b < y$ and $\frac{b}{y} \leqslant \frac{a}{x}$	$\frac{b}{y}(x + y - 1)$

Table 3
Values of the upper bound z'' of $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ for different values of $x = P(A|H), y = P(B|K), a, b \in [0,1]$.

Case	z''
(C)	
$a(1-y) + b(1-x) > 1 - xy$	$z''_2 = \frac{x(b-ay)+y(a-bx)}{1-xy}$
(D)	
$a(1-y) + b(1-x) \leqslant 1 - xy$	
Sub-cases:	
$x \leqslant y$ and $a \leqslant x$	$z''_1 = x$
$x \leqslant y$ and $a > x$	$z''_3 = \frac{x(1-a)+y(a-x)}{1-x}$
$y < x$ and $b \leqslant y$	$z''_1 = y$
$y < x$ and $b > y$	$z''_4 = \frac{x(b-y)+y(1-b)}{1-y}$

Proof. Due to its length, the proof has been moved to Appendix A.1 □

A summary of the different values of the lower and upper bounds z' and z'' for the prevision of $(A|H) \wedge_{a,b} (B|K)$ is given in Table 2 and in Table 3, respectively.

Moreover, from Theorem 5 it follows that

Corollary 1. *Let A, B, H, K be any logically independent events. Then, the set $\Pi_{a,b}$ of all coherent prevision assessments (x, y, z) on $\mathcal{F} = \{A|H, (B|K), (A|H) \wedge_{a,b} (B|K)\}$, is*

$$\Pi_{a,b} = \{(x, y, z) : (x, y) \in [0,1]^2, z \in [z', z'']\}, \tag{31}$$

where z' and z'' are defined in (29) and (30), respectively.

We recall that, under logical independence, the set of all coherent assessment (x, y, z) on $\{A|H, B|K, (A|H) \wedge (B|K)\}$ is the tetrahedron $\mathcal{T}$ of points $(1,1,1), (1,0,0), (0,1,0), (0,0,0)$, that is ([23]) $\mathcal{T} = \{(x, y, z) : (x, y) \in [0,1]^2, z \in [\max\{x + y - 1, 0\}, \min\{x, y\}]\}$. As, from (29) and (30), $z' \leqslant \max\{x + y - 1, 0\}$ and $z'' \geqslant \min\{x, y\}$ for every $(a, b) \in [0,1]^2$, it holds that (see Fig. 1)

$$\mathcal{T} \subseteq \Pi_{a,b} \ \forall (a, b) \in [0,1]^2. \tag{32}$$

From Theorem 5, we also have that the Fréchet-Hoeffding bounds are not satisfied by $(A|H) \wedge_{a,b} (B|K)$ for every $(x, y) \in [0,1]^2$, when a and b are arbitrarily chosen. For example, if we take $a > x$ and $b > y$, then from Equation (30), the upper bound of the coherent assessments on $(A|H) \wedge_{a,b} (B|K)$ is $z''_2 = \frac{x(b-ay)+y(a-bx)}{1-xy} \neq \min\{x, y\}$. In the following corollary we show that the only case the Fréchet-Hoeffding bounds for the conjunction are satisfied for every $(x, y) \in [0,1]^2$ is for $a = x$ and $b = y$, that is the only conjunction of two conditional events always coherent with these bounds is $(A|H) \wedge (B|K)$, explicitly depending on the probabilities of the conditionals.

Corollary 2. *Let A, B, H and K be any logically independent events. The generalized conjunction $(A|H) \wedge_{a,b} (B|K)$ satisfies the Fréchet-Hoeffding bounds for every coherent assessment $(x, y) \in [0,1]^2$ on $\{A|H, B|K\}$, if and only if $a = x$ and $b = y$.*

Proof. If $a = x$ and $b = y$, from Remark 5 we have that $(A|H) \wedge_{a,b} (B|K) = (A|H) \wedge (B|K)$ and this object satisfies the Fréchet-Hoeffding bounds for every $(x, y) \in [0,1]^2$.

Let us suppose now that $(A|H) \wedge_{a,b} (B|K)$ satisfies the Fréchet-Hoeffding bounds for every $(x, y) \in [0,1]^2$ but $a \neq x$. We observe from Table 3 that to have the Fréchet-Hoeffding upper bound $z'' = \min\{x, y\}$ we need to have $a(1-y) + b(1-x) \leqslant 1 - xy$ and $[(x \leqslant y \quad \wedge \quad a \leqslant x) \vee (x > y \quad \wedge \quad b \leqslant y)]$. Let us consider any coherent assessment (x, y) on $\{A|H, B|K\}$ such that $a(1-y) + b(1-y) \leqslant$

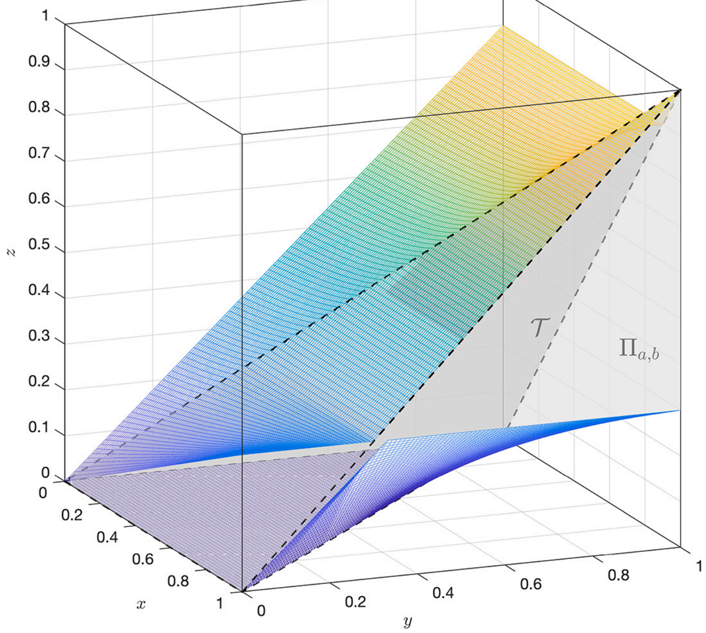

Fig. 1. The gray region enclosed between the two surfaces represents the set $\Pi_{a,b}$ of all coherent prevision assessments (x,y,z) on $\{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$, with $a = 0.9$ and $b = 0.3$. The tetrahedron $\mathcal{T}$ is the set of all coherent prevision assessments (x,y,z) on $\{A|H, B|K, (A|H) \wedge (B|K)\}$. It can be observed that $\mathcal{T} \subset \Pi_{a,b}$.

$1 - xy$, $x \leqslant y$ and $x + y - 1 > 0$. Then, $z'' = \min\{x, y\} = x$ and $a < x$, because we supposed $a \neq x$. Having that $x + y - 1 > 0$, the Fréchet-Hoeffding lower bound for these assessments would be $z' = x + y - 1$. Nevertheless, being $\frac{a}{x} < 1$, we have that $\min\{\frac{a}{x}, \frac{b}{y}, 1\} \neq 1$ and hence $z' = (x + y - 1)\min\{\frac{a}{x}, \frac{b}{y}, 1\} \neq x + y - 1$ so we arrive to a contradiction. Then it must be $a = x$. A similar contradiction arises if we suppose $b \neq y$, then for the Fréchet-Hoeffding bounds to be satisfied for every coherent assessment (x, y) on $\{A|H, B|K\}$ it is needed that $(a, b) = (x, y)$. □

In other words Corollary 2 amounts to say that $\Pi_{a,b} = \mathcal{T}$ if and only if $a = x$ and $b = y$, for every assessment $(x, y) \in [0, 1]^2$ on $\{A|H, B|K\}$.

4.2. Generalized disjunction and coherence

In the next result we compute the lower and upper bounds for the prevision of the generalized disjunction.

Theorem 6. *Let A, B, H, K be any logically independent events. A prevision assessment $\mathcal{P} = (x, y, w)$ on the family of conditional random quantities $\mathcal{F} = \{A|H, (B|K), (A|H) \vee_{a,b} (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $w \in [w', w'']$, where*

$$w' = \min\{w'_1, w'_2, \max\{w'_3, w'_4\}\}, \tag{33}$$

with

$$w'_1 = \max\{x, y\},$$
$$w'_2 = \begin{cases} \dfrac{ax + by - (a + b - 1)xy}{x + y - xy}, \text{ if } (x, y) \neq (0, 0), \\ 0, \text{ if } (x, y) = (0, 0), \end{cases}$$

$$w'_3 = \begin{cases} \dfrac{a(x - y) + xy}{x}, \text{if } x \neq 0, \\ 0, \text{ if } x = 0, \end{cases}$$

$$w'_4 = \begin{cases} \dfrac{b(y - x) + xy}{y}, \text{ if } y \neq 0, \\ 0, \text{ if } y = 0, \end{cases}$$

and

Table 4
Values of the lower bound w' of $w = \mathbb{P}[(A|H) \vee_{a,b} (B|K)]$ for different values of $x = P(A|H), y = P(B|K), a, b \in [0,1]$.

Case	w'
(C)	
$ax + by - xy < 0$	$w'_2 = \frac{ax+by-(a+b-1)xy}{x+y-xy}$
(D)	
$ax + by - xy \geqslant 0$	
Sub-cases:	
$x \geqslant y$ and $a \geqslant x$	$w'_1 = x$
$x \geqslant y$ and $a < x$	$w'_3 = \frac{a(x-y)+xy}{x}$
$y > x$ and $b \geqslant y$	$w'_1 = y$
$y > x$ and $b < y$	$w'_4 = \frac{b(y-x)+xy}{y}$

Table 5
Values of the upper bound w'' of $w = \mathbb{P}[(A|H) \vee_{a,b} (B|K)]$ for different values of $x = P(A|H), y = P(B|K), a, b \in [0,1]$.

Case	w''
(A): $x + y \geqslant 1$	1
(B): $x + y < 1$	
Sub-cases:	
(B.1): $a \leqslant x$ and $b \leqslant y$	$x + y$
(B.2): $a > x$ and $\frac{1-a}{1-x} \leqslant \frac{1-b}{1-y}$	$\frac{a(1-x)+(1-a)y}{1-x}$
(B.3): $b > y$ and $\frac{1-b}{1-y} \leqslant \frac{1-a}{1-x}$	$\frac{b(1-y)+(1-b)x}{1-y}$

$$w'' = \begin{cases} 1 - (1-x-y) \cdot \min\{\frac{1-a}{1-x}, \frac{1-b}{1-y}, 1\}, & \text{if } x + y < 1, \\ 1, & \text{otherwise.} \end{cases} \tag{34}$$

Proof. Due to its length, the proof has been moved to Appendix A.2 □

A summary of the different values of the lower bound w' for the prevision of $(A|H) \vee_{a,b} (B|K)$ can be found in Table 4. They are obtained from Table 3 by exploiting the relation $w' = 1 - z''(1-x, 1-y, 1-a, 1-b)$. Likewise, a summary of the different values of the upper bound w'' for the prevision of $(A|H) \vee_{a,b} (B|K)$ is given in Table 5. These values are obtained from Table 2 by exploiting the relation $w'' = 1 - z'(1-x, 1-y, 1-a, 1-b)$.

Moreover, from Theorem 6 it follows that

Corollary 3. *Let A, B, H, K be any logically independent events. Then, the set $\Pi'_{a,b}$ of all coherent prevision assessments (x, y, w) on $\mathcal{F} = \{A|H, (B|K), (A|H) \vee_{a,b} (B|K)\}$, is*

$$\Pi'_{a,b} = \{(x, y, w) : (x, y) \in [0,1]^2, w \in [w', w'']\}, \tag{35}$$

where w' and w'' are defined in eq. (33) and in eq. (34), respectively.

We recall that, under logical independence, the set of all coherent assessment (x, y, w) on $\{A|H, B|K, (A|H) \vee (B|K)\}$ is the tetrahedron $\mathcal{T}'$ of points $(1,1,1), (1,0,1), (0,1,1), (0,0,0)$, that is $\mathcal{T}' = \{(x, y, w) : (x, y) \in [0,1]^2,\ w \in [\max\{x, y\}, \min\{x + y, 1\}]\}$. As, from (33) and (34), $w' \leqslant \max\{x, y\}$ and $w'' \geqslant \min\{x + y, 1\}$ for every $(a, b) \in [0,1]^2$, it holds that (see Fig. 2)

$$\mathcal{T}' \subseteq \Pi'_{a,b}\ \forall (a, b) \in [0,1]^2.$$

From Theorem 6, we have that in general the disjunction $(A|H) \vee_{a,b} (B|K)$ does not satisfy the Fréchet-Hoeffding bounds for every $(x, y) \in [0,1]^2$, when a and b are arbitrary chosen. For example, if we take $0 \neq b = x < y$, then from Equation (33) the lower bound of the coherent assessments on $(A|H) \vee_{a,b} (B|K)$ is $w'_4 = \frac{a(x-y)+xy}{x} \neq \max\{x, y\}$. In the following corollary we show that the only case the Fréchet-Hoeffding bounds are satisfied for every $(x, y) \in [0,1]^2$ is for $a = x$ and $b = y$, that is the only disjunction of two conditional events always coherent with these bounds is $(A|H) \vee (B|K)$, explicitly depending on the probabilities of the conditionals.

Corollary 4. *Let A, B, H and K be any logically independent events. The generalized disjunction $(A|H) \vee_{a,b} (B|K)$ satisfies the Fréchet-Hoeffding bounds for every coherent assessment $(x, y) \in [0,1]^2$ on $\{A|H, B|K\}$ if and only if $a = x$ and $b = y$.*

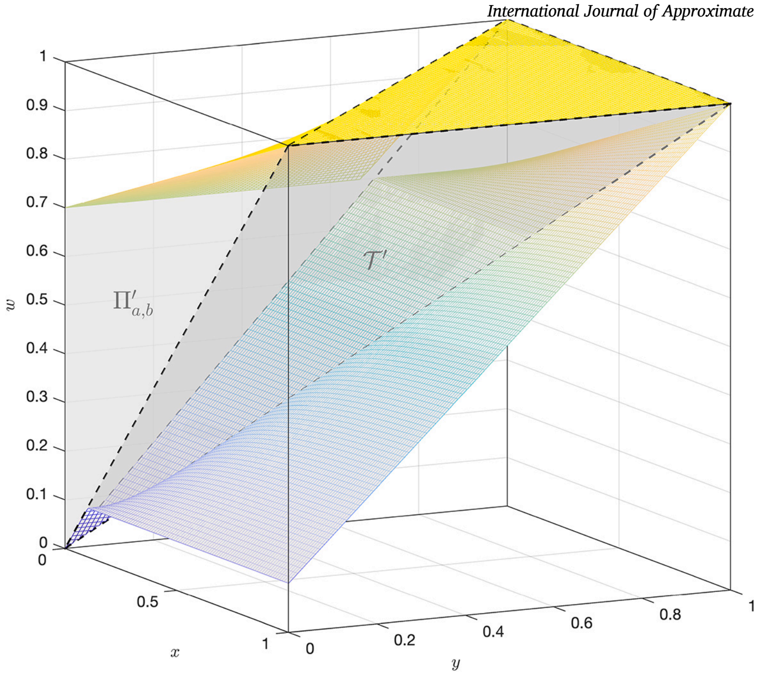

Fig. 2. The gray region enclosed between the two surfaces represents the set $\Pi'_{a,b}$ of all coherent prevision assessments (x, y, w) on $\{A|H, B|K, (A|H) \vee_{a,b} (B|K)\}$, with $a = 0.1$ and $b = 0.7$. The tetrahedron $\mathcal{T}'$ of points $(1,1,1), (1,0,1), (0,1,1), (0,0,0)$ is the set of all coherent prevision assessments (x, y, w) on $\{A|H, B|K, (A|H) \vee (B|K)\}$. It can be observed that $\mathcal{T}' \subset \Pi'_{a,b}$.

Proof. If $a = x$ and $b = y$, from Remark 6 we have that $(A|H) \vee_{a,b} (B|K) = (A|H) \vee (B|K)$ and this definition of disjunction satisfies the Fréchet-Hoeffding bounds for every $(x, y) \in [0,1]^2$.

Let us suppose now that $(A|H) \vee_{a,b} (B|K)$ satisfies the Fréchet-Hoeffding bounds for every $(x, y) \in [0,1]^2$ but $a \neq x$. Consider the set of coherent assessments on $\{A|H, B|K\}$ such that $x \geqslant y$ and $x + y < 1$. Then, it holds that $w' = \max\{x, y\} = x$ and $w'' = \min\{x + y, 1\} = x + y$. From Table 5 to have such a lower bound we need to have $ax + by - xy \geqslant 0$ and $a \geqslant x$. Then it must be $a > x$ using that $a \neq x$. However, to have that upper bound we need to have $a < x$, and hence we have a contradiction. Then it must be $a = x$. A similar contradiction arises if we suppose $b \neq y$. Then for the Fréchet-Hoeffding bounds to be satisfied for any coherent assessment (x, y) on $\{A|H, B|K\}$ the only choice for a and b is $(a, b) = (x, y)$. $\square$

By Corollary 4 it follows that $\Pi'_{a,b} = \mathcal{T}'$ if and only if $a = x$ and $b = y$, for every assessment $(x, y) \in [0,1]^2$ on $\{A|H, B|K\}$.

We conclude this section by underlining again that if we want to extend the operation of conjunction and disjunction from basic events A, B to conditionals $A|H, B|K$ in a way that preserve similar properties, a suitable choice is to consider the structures $(A|H) \wedge (B|K)$ (Definition 5) and $(A|H) \vee (B|K)$ (Definition 6).

5. Some particular cases

In this section, first we consider some particular cases of the generalized conjunction (Definition 7) and the generalized disjunction (Definition 9) obtained when the conditioning events H and K are incompatible. Then we analyze these structures for some given values of a and b in which the generalized objects reduce to conditional events. More precisely, considering $a, b \in \{0, 1, p\}$, with $p = z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ in the case of the conjunction, and $p = w = \mathbb{P}[(A|H) \vee_{a,b} (B|K)]$ in the case of the disjunction, we analyze the particular cases $(a, b) = (0, 0)$, $(a, b) = (1, 1)$, and $(a, b) = (p, p)$.

5.1. The case H and K incompatible

We analyze the generalized conjunction in the particular case where the conditioning events H and K are incompatible, i.e. $HK = \emptyset$.

Theorem 7. *Let $A|H$, $B|K$, be two conditional events with A, H, B logically independent, A, B, K logically independent, and $HK = \emptyset$. A prevision assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$ is coherent if and only if $(x, y) \in [0,1]^2$ and $z \in [z', z'']$, where*

$$z' = \min\{ay, bx\}, \quad z'' = \max\{ay, bx\}. \tag{36}$$

Proof. The proof can be found in Appendix A.3. $\square$

We observe that the bounds obtained in Theorem 7 (under the logical relation $HK = \emptyset$) are more restrictive than the bounds obtained in Theorem 5 (under the assumption of logical independence of A, H, B, K). That is, $[\min\{ay, bx\}, \max\{ay, bx\}] \subseteq [z', z'']$, where z' and z'' are given in (29) and (30), respectively. Of course, when $a < 1$ and $b < 1$, it holds that $z'' < 1$. This is in agreement to the fact that $(A|H) \wedge_{a,b} (B|K)$ can never be 1 in this case.

Remark 8. The result of Theorem 7 can be also obtained in a different way by exploiting the linearity of prevision. Indeed, when $HK = \emptyset$, it holds that $AHBK = \emptyset$, $\overline{H}BK = BK$ and $AH\overline{K} = AH$ and hence

$$(A|H) \wedge_{a,b} (B|K) = (aBK + bAH)|(H \vee K). \tag{37}$$

We observe that $P(BK|(H \vee K)) = P(B|K)P(K|(H \vee K))$ and $P[AH|(H \vee K)] = P(A|H)P[H|(H \vee K)]$. Then, from (37), by the linearity of prevision, it follows that

$$\begin{aligned} z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)] &= aP(\overline{H}BK|(H \vee K)) + bP(\overline{K}AH|(H \vee K)) = \\ &= aP(B|K)P(K|(H \vee K)) + bP(A|H)P(H|(H \vee K)). \end{aligned} \tag{38}$$

By setting $x = P(A|H)$, $y = P(B|K)$, $\alpha = P(H|(H \vee K)) = 1 - P(K|(H \vee K))$, formula (38) becomes

$$z = ay(1 - \alpha) + bx\alpha. \tag{39}$$

It can be proved[3] that the assessment (x, y, α) on $\{A|H, B|K, H|(H \vee K)\}$, with $HK = \emptyset$, is coherent for every $(x, y, \alpha) \in [0, 1]^3$. Then, from (39) we obtain that $z \in [\min\{ay, bx\}, \max\{ay, bx\}]$, because $\alpha \in [0, 1]$.

Remark 9. When $a = x$ and $b = y$, from (36) we obtain that $z' = z'' = ay = bx = xy$. That is, when $HK = \emptyset$, it holds that

$$\mathbb{P}[(A|H) \wedge_{x,y} (B|K)] = \mathbb{P}[(A|H) \wedge (B|K)] = P(A|H)P(B|K),$$

which is in agreement with the result given in [21] (see also [25]).

In the next theorem we analyze the generalized disjunction (Definition 9) when the conditioning events H and K are incompatible, i.e. $HK = \emptyset$. From Theorem 7, and by relation (26), it follows

Theorem 8. *Let $A|H$, $B|K$ be two conditional events with A, H, B logically independent, A, B, K logically independent, and $HK = \emptyset$. A prevision assessment $\mathcal{P} = (x, y, w)$ on $\mathcal{F} = \{A|H, B|K, (A|H) \vee_{a,b} (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $w \in [w', w'']$, where*

$$w' = \min\{a + y - ay,\ b + x - bx\},\ \ w'' = \max\{a + y - ay,\ b + x - bx\}. \tag{40}$$

Proof. From Equation (27), it holds that $w' = 1 - z''(1 - x, 1 - y, 1 - a, 1 - b)$, where $z''(1 - x, 1 - y, 1 - a, 1 - b)$ is the upper bound of the generalized conjunction valued in $1 - x, 1 - y, 1 - a$, and $1 - b$. Then

$$\begin{aligned} w' &= 1 - z''(1 - x, 1 - y, 1 - a, 1 - b) = 1 - \max\{(1 - a)(1 - y),\ (1 - b)(1 - x)\} = \\ &= \min\{1 - (1 - a)(1 - y),\ 1 - (1 - b)(1 - x)\} = \min\{a + y - ay,\ b + x - bx\}. \end{aligned}$$

In the same way, from (27), it follows that $w'' = 1 - z'(1 - x, 1 - y, 1 - a, 1 - b)$, where $z'(1 - x, 1 - y, 1 - a, 1 - b)$ is the lower bound of the generalized conjunction valued in $1 - x, 1 - y, 1 - a$, and $1 - b$. Then

$$\begin{aligned} w'' &= 1 - z'(1 - x, 1 - y, 1 - a, 1 - b) = 1 - \min\{(1 - a)(1 - y),\ (1 - b)(1 - x)\} = \\ &= \max\{1 - (1 - a)(1 - y),\ 1 - (1 - b)(1 - x)\} = \max\{a + y - ay,\ b + x - bx\}. \quad \square \end{aligned}$$

Remark 10. We observe that, when $a = x$ and $b = y$, from (40) we obtain that $w' = w'' = a + y - ay = b + x - bx = x + y - xy$. That is, when $HK = \emptyset$, it holds that

$$\mathbb{P}[(A|H) \vee_{x,y} (B|K)] = \mathbb{P}[(A|H) \vee (B|K)] = P(A|H) + P(B|K) - P(A|H)P(B|K).$$

5.2. *The case $a = b = 0$*

In the case where $a = b = 0$ the generalized conjunction reduces to a trivalent object, in particular it corresponds to the forth conjunction proposed by Walker in [42]. Indeed,

[3] For proving that (x, y, α) on $\{A|H, B|K, H|(H \vee K)\}$, with $HK = \emptyset$, is coherent for every $(x, y, \alpha) \in [0, 1]^3$ it is sufficient to check that each of the eight vertices of the unit cube is coherent ([17]).

$$(A|H) \wedge_{0,0} (B|K) = AHBK|(H \vee K). \tag{41}$$

Based on Theorem 5, under logical independence, it follows that the probability assessment (x, y, z) on $\{A|H, B|K, (A|H) \wedge_{0,0} (B|K)\} = \{A|H, B|K, AHBK|(H \vee K)\}$ is coherent if and only if $(x, y) \in [0,1]^2$ and $z \in [z', z'']$, where $z' = 0$ and $z'' = min\{x, y\}$.

For what concerns the generalized disjunction $(A|H) \vee_{a,b} (B|K)$, when $a = b = 0$ it reduces to a trivalent object which coincides with the quasi disjunction (eq. (16)). Indeed,

$$(A|H) \vee_{0,0} (B|K) = (AH \vee BK)|(H \vee K) = (A|H) \vee_S (B|K).$$

Then, when $H \vee K$ is true it holds that $(A|H) \vee_S (B|K) \leqslant (A|H) \vee_{a,b} (B|K)$. By applying [24, Theorem 6], it holds that $P((A|H) \vee_S (B|K)) \leqslant \mathbb{P}[(A|H) \vee_{a,b} (B|K)]$ and hence

$$(A|H) \vee_S (B|K) \leqslant (A|H) \vee_{a,b} (B|K)$$

in all cases. Moreover, based on Theorem 6 it follows that (x, y, w) on $\{A|H, B|K, (A|H) \vee_{0,0} (B|K)\}$ is coherent if and only if $(x, y) \in [0,1]^2$ and $w \in [w', w'']$, where

$$w' = w'_2 = \begin{cases} \dfrac{xy}{x+y-xy}, \text{ if } (x,y) \neq (0,0), \\ 0, \text{ if } (x,y) = (0,0), \end{cases}$$

because $ax + by - xy = -xy < 0$, and

$$w'' = \begin{cases} 1 - (1-x-y) \cdot \min\{\frac{1}{1-x}, \frac{1}{1-y}, 1\}, & \text{if } x + y < 1, \\ 1, & \text{otherwise.} \end{cases}$$
$$= \begin{cases} 1 - (1-x-y) = x + y, & \text{if } x + y < 1, \\ 1, & \text{otherwise.} \end{cases} = \min\{x+y, 1\},$$

because $\frac{1}{1-x} \geqslant 1$ and $\frac{1}{1-y} \geqslant 1$.

Then, the lower and upper bounds w' and w'' coincide with w'_S and w''_S given in (17). We observe that w'_S is the Hamacher t-norm with the parameter $\lambda = 0$. Thus, differently from the unconditional case where $P(A \vee B) = 1$ is the only coherent extension of $(1,0)$ on $\{A, B\}$, for the probability of the quasi disjunction $(A|H) \vee_S (B|K)$, any value $w_S \in [0,1]$ is a coherent extension of $(1,0)$ on $\{A|H, B|K\}$ because $w'_S = 0$ and $w''_S = 1$. From a probabilistic point of view this is not desirable because it allows to coherently assess probability 0 for the quasi disjunction even if one of the two conjuncts has probability 1 and the other one has probability zero. A similar comment can be also done in the particular case where H and K are incompatible. Indeed, when $HK = \emptyset$, by instantiating equation (40) with $a = b = 0$, the lower and upper bounds for $(A|H) \vee_S (B|K)$ become

$$w' = \min\{x, y\}, \;\; w'' = \max\{x, y\}. \tag{42}$$

5.3. The case $a = b = 1$

It is interesting to notice that quasi conjunction is a particular case of the generalized conjunction $(A|H) \wedge_{a,b} (B|K)$ where $a = 1$ and $b = 1$. Indeed, by recalling Equation (14)

$$(A|H) \wedge_{1,1} (B|K) = (AHBK + \overline{H}BK + AH\overline{K})|(H \vee K) = (A|H) \wedge_S (B|K).$$

Then, when $H \vee K$ is true it holds that $(A|H) \wedge_S (B|K) \geqslant (A|H) \wedge_{a,b} (B|K)$. By applying [24, Theorem 6], it holds that $P[(A|H) \wedge_S (B|K)] \geqslant \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ and hence

$$(A|H) \wedge_S (B|K) \geqslant (A|H) \wedge_{a,b} (B|K)$$

in all cases. Moreover, based on Theorem 5 it follows that (x, y, z) on $\{A|H, B|K, (A|H) \wedge_{1,1} (B|K)\}$ is coherent if and only if $(x, y) \in [0,1]^2$ and $z \in [z', z'']$, where

$$z' = \begin{cases} (x+y-1), & \text{if } x + y - 1 > 0, \\ 0, & \text{otherwise} \end{cases} = \max\{x+y-1, 0\},$$

and

$$z'' = \begin{cases} \dfrac{x(b-ay) + y(a-bx)}{1-xy}, \text{ if } (x,y) \neq (1,1), \\ 1, \text{ if } (x,y) = (1,1), \end{cases} = \begin{cases} \dfrac{x+y-2xy}{1-xy}, \text{ if } (x,y) \neq (1,1), \\ 1, \text{ if } (x,y) = (1,1), \end{cases}$$

because in this case it holds that $a(1-y) + b(1-x) + xy - 1 = 1 - y + 1 - x + xy - 1 = 1 - y - x + xy = 1 - y - x(1-y) = (1-y)(1-x) \geqslant 0$. Then, the lower and upper bounds z' and z'' coincide with z'_S and z''_S given in (15). We recall that z''_S is the Hamacher t-conorm with the parameter $\lambda = 0$. Then, for every $(x, y) \in [0,1]^2$, as $\max(x, y)$ is the smallest t-conorm, it holds that

$$z''_S \geqslant \max\{x, y\} \geqslant \min\{x, y\},$$

and hence the quasi conjunction do not preserve the bounds in (8). Thus, differently from the unconditional case where $P(AB) = 0$ is the only coherent extension of the assessment $(1,0)$ on $\{A, B\}$, for the probability of the quasi conjunction $(A|H) \wedge_S (B|K)$, any value $z_S \in [0,1]$ is a coherent extension of $(1,0)$ on $\{A|H, B|K\}$ because $z'_S = 0$ and $z''_S = 1$. This is not desirable from a probabilistic point of view because it allows to coherently assess probability 1 for the quasi conjunction even if one of the two conjuncts has probability 1 and the other one has probability zero. A similar comment can be also done in the particular case where H and K are incompatible. Indeed, when $HK = \emptyset$, by instantiating equation (36) with $a = b = 1$, the lower and upper bounds for $(A|H) \wedge_S (B|K)$ are

$$z' = \min\{x, y\}, \quad z'' = \max\{x, y\}. \tag{43}$$

Then, as $(A|H) \wedge_{1,1} (B|K) = (\bar{H}BK \vee AH\bar{K})|(H \vee K))$, it follows that, given two conditional events $A|H$ and $B|K$, with $HK = \emptyset$, a probability assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F} = \{A|H, B|K, (\bar{H}BK \vee AH\bar{K})|(H \vee K))\}$ is coherent if and only if $(x, y) \in [0,1]^2$ and $z \in [\min\{x, y\}, \max\{x, y\}]$.

In the case where $a = b = 1$ the generalized disjunction reduces to a trivalent object, in particular it corresponds to the forth disjunction proposed by Walker in [42]. Indeed,

$$(A|H) \vee_{1,1} (B|K) = ((AH \vee BK) + \bar{H}\,\bar{B}K + \bar{A}H\bar{K})|(H \vee K).$$

Based on Theorem 6, under logical independence, it follows that (x, y, w) on $\{A|H, B|K, (A|H) \vee_{1,1} (B|K)\}$ is coherent if and only if $(x, y) \in [0,1]^2$ and $w \in [w', w'']$, where $w' = \max\{x, y\}$ and $w'' = 1$.

5.4. The case $a = b = p$

We analyze now the particular case for the generalized conjunction in which $a = b = z = \mathbb{P}[(A|H) \wedge_{z,z} (B|K)]$. From Equation (20), setting $a = b = z$ we have

$$(A|H) \wedge_{z,z} (B|K) = AHBK + z\overline{H}BK + zAH\overline{K} + z(H \vee K) = AHBK|(AHBK \vee \overline{A}H \vee \overline{B}K).$$

Moreover, for what concerns the generalized disjunction, by setting $a = b = w = \mathbb{P}[(A|H) \vee_{w,w} (B|K)]$ in Equation (23), we have that

$$(A|H) \vee_{w,w} (B|K) = (AH \vee BK) + w\overline{H}\,\overline{B}K + w\overline{A}H\overline{K} + w(H \vee K) = (AH \vee BK)|(\overline{A}H\overline{B}K \vee AH \vee BK).$$

Then for these choices of parameters the conditioning events change and the generalized conjunction and disjunction reduce, respectively, to the trivalent conjunction and disjunction of Kleene-de Finetti ([8,27]). The lower and upper bound for the coherent assessments on these objects are well-known and they are $z' = 0$ and $z'' = \min\{x, y\}$ for the conjunction, and $w' = \max\{x, y\}$ and $w'' = 1$ for the disjunction.

It worth notice that with the choices $(a, b) = (0, 0)$, $(a, b) = (1, 1)$, and $(a, b) = (p, p)$, we obtain three of the four associative conjunctions and disjunctions proposed by Walker ([42]) among the nine that were reasonable for the author to consider in a trivalent setting.

6. Interval-valued prevision assessments

In this section, to deepen the study on the generalized conjunction $\wedge_{a,b}$ and disjunction $\vee_{a,b}$, we analyze the case of interval-valued probability assessments. For both structures, given the imprecise assessment $\mathcal{A} = ([x_1, x_2] \times [y_1, y_2])$ on $\{A|H, B|K\}$, we study the coherent extensions to $(A|H) \wedge_{a,b} (B|K)$ (Theorem 9) and $(A|H) \vee_{a,b} (B|K)$ (Theorem 11), respectively. Moreover, we consider the case $HK = \emptyset$ both for the generalized conjunction (Theorem 10) and disjunction (Theorem 12).

We recall that under logical independence any assessment $(x, y) \in [0,1]^2$ on $\{A|H, B|K\}$ is coherent. Based on Theorem 5, we denote by $[z'(x, y), z''(x, y)]$ the interval of coherent prevision extensions of (x, y) to $(A|H) \wedge_{a,b} (B|K)$. Given the interval-valued assessment $[x_1, x_2] \times [y_1, y_2] \subseteq [0,1]^2$ on $\{A|H, B|K\}$, we set $[z^*, z^{**}]$ the interval of coherent extensions z on $(A|H) \wedge_{a,b} (B|K)$, where

$$z^* = \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z'(x, y)$$

and

$$z^{**} = \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z''(x, y).$$

The assessment $([x_1, x_2] \times [y_1, y_2] \times [z^*, z^{**}])$ on $\mathcal{F} = \{A|H, (B|K), (A|H) \wedge_{a,b} (B|K)\}$ is coherent w.r.t. Definition 4. Moreover, any assessment $[x_1, x_2] \times [y_1, y_2] \times [\alpha, \beta]$, with $[\alpha, \beta] \supset [z^*, z^{**}]$, is not coherent. Then, the interval $[z^*, z^{**}]$ is the least committal extension, that is the natural extension ([43], see also [36]), of the interval-valued assessment $[x_1, x_2] \times [y_1, y_2]$, which is equivalent to the lower probability assessment $(x_1, 1 - x_2, y_1, 1 - y_2)$ on $\{A|H, \overline{A}|H, B|K, \overline{B}|K\}$. In what follows we compute the lower and upper bounds, z^* and z^{**}, by also considering the case where $HK = \emptyset$.

Theorem 9. *Let A, B, H, K be any logically independent events and let $\mathcal{A} = ([x_1, x_2] \times [y_1, y_2])$ be an interval-valued assessment on $\{A|H, B|K\}$. Then, the interval of coherent extensions of $\mathcal{A}$ to $(A|H) \wedge_{a,b} (B|K)$ is the interval $[z^*, z^{**}] = [z'(x_1, y_1), z''(x_2, y_2)]$, where $z'(x, y)$ and $z''(x, y)$ are defined in formula (29) and formula (30), respectively.*

Proof. Due to its length, the proof has been moved to Appendix A.4 □

We now generalize Theorem 7 for interval-valued prevision assessments.

Theorem 10. *Let $\mathcal{A} = ([x_1, x_2] \times [y_1, y_2])$ be an interval-valued probability assessment on $\{A|H, B|K\}$ with $HK = \emptyset$. Then, the interval of coherent extensions of $\mathcal{A}$ to $(A|H) \wedge_{a,b} (B|K)$ is the interval $[z^*, z^{**}]$, where*

$$z^* = \min\{ay_1, bx_1\}, \;\; z^{**} = \max\{ay_2, bx_2\}. \tag{44}$$

Proof. The proof is straightforward by recalling that every assessment $(x, y) \in [0, 1]^2$ on $\{A|H, B|K\}$ is coherent when $HK = \emptyset$ and by observing that both the lower and upper bounds z' and z'' given in Theorem 7 are non-decreasing functions in the arguments x and y. □

Using the results obtained in Theorem 9, we can analyze the imprecise probabilities case also for the generalized disjunction $(A|H) \vee_{a,b} (B|K)$.

Theorem 11. *Let A, B, H, K be any logically independent events and let $\mathcal{A} = ([x_1, x_2] \times [y_1, y_2])$ be an interval-valued assessment on $\{A|H, B|K\}$. Then, the interval of coherent extensions of $\mathcal{A}$ to $(A|H) \vee_{a,b} (B|K)$ is the interval $[w^*, w^{**}] = [w'(x_1, y_1), w''(x_2, y_2)]$, where $w'(x, y)$ and $w''(x, y)$ are defined in formula (33) and formula (34), respectively.*

Proof. Given the interval-valued assessment $[x_1, x_2] \times [y_1, y_2] \subseteq [0, 1]^2$ on $\{A|H, B|K\}$, we set $[w^*, w^{**}]$ the interval of coherent extensions of w on $(A|H) \vee_{a,b} (B|K)$, with

$$w^* = \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} w'(x, y)$$

and

$$w^{**} = \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} w''(x, y),$$

where $w'(x, y)$ and $w''(x, y)$ are defined in formula (33) and formula (34), respectively. We recall that from Theorem 9, the lower and upper bounds for the generalized conjunction, $(A|H) \wedge_{a,b} (B|K)$, in the imprecise case where $(x, y) \in [x_1, x_2] \times [y_1, y_2]$, are

$$z^* = z'(x_1, y_1) = \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z'(x, y), \quad z^{**} = z''(x_2, y_2) = \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z''(x, y).$$

We observe that, as $(1 - x, 1 - y) \in [1 - x_2, 1 - x_1] \times [1 - y_2, 1 - y_1]$, then

$$z'(1 - x_2, 1 - y_2) = \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z'(1 - x, 1 - y),$$

and

$$z''(1 - x_1, 1 - y_1) = \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z''(1 - x, 1 - y).$$

From (27), we have that

$$w'(x, y) = 1 - z''(1 - x, 1 - y, 1 - a, 1 - b), \quad w''(x, y) = 1 - z'(1 - x, 1 - y, 1 - a, 1 - b),$$

and hence

$$\begin{aligned}
w^* &= \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} w'(x, y) = \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} (1 - z''(1 - x, 1 - y, 1 - a, 1 - b)) = \\
&= 1 - \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z''(1 - x, 1 - y) = 1 - z''(1 - x_1, 1 - y_1) = w'(x_1, y_1) \\
w^{**} &= \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} w''(x, y) = \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} (1 - z'(1 - x, 1 - y, 1 - a, 1 - b)) = \\
&= 1 - \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} z'(1 - x, 1 - y) = 1 - z'(1 - x_2, 1 - y_2) = w''(x_2, y_2). \quad \square
\end{aligned}$$

Moreover, considering the interval of coherent assessments obtained in the precise case (Theorem 8), we also find the interval of coherent prevision extensions for H and K incompatible.

Theorem 12. *Let $\mathcal{A} = ([x_1, x_2] \times [y_1, y_2])$ be an interval-valued probability assessment on $\{A|H, B|K\}$ with $HK = \emptyset$. Then, the interval of coherent extensions of $\mathcal{A}$ to $(A|H) \vee_{a,b} (B|K)$ is the interval $[w^*, w^{**}]$, where*

$$w^* = \min\{a + y_1 - ay_1, b + x_1 - bx_1\}, \;\; w^{**} = \max\{a + y_2 - ay_2, b + x_2 - bx_2\}. \tag{45}$$

Proof. From Theorem 8 we have that

$$w'(x, y) = \min\{a + y - ay,\ b + x - bx\}, \;\; w''(x, y) = \max\{a + y - ay,\ b + x - bx\}.$$

We recall that

$$w^* = \min_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} w'(x, y)$$

and

$$w^{**} = \max_{(x,y)\in[x_1,x_2]\times[y_1,y_2]} w''(x, y),$$

then, by observing that $b + x_1(1 - b) \leqslant b + x(1 - b) \leqslant b + x_2(1 - b)$ and $a + y_1(1 - a) \leqslant a + y(1 - a) \leqslant a + y_2(1 - a)$, it holds that

$$w^* = \min\{a + y_1 - ay_1, b + x_1 - bx_1\}, \;\; w^{**} = \max\{a + y_2 - ay_2, b + x_2 - bx_2\}. \quad \square$$

7. Further aspects on $(A|H) \wedge_{a,b} (B|K)$

In this section we deepen two further aspects of the generalized conjunction $(A|H) \wedge_{a,b} (B|K)$. We first give a subjective interpretation of $(A|H) \wedge_{a,b} (B|K)$ when we consider two individuals O and O'. Then, we examine $(A|H) \wedge_{a,b} (B|K)$, when $A|H \subseteq B|K$.

Let us consider two individuals O and O'. Suppose that O' asserts $P'(A|H) = a$ and $P'(B|K) = b$. Then, based on Remark 5, for O' the conjunction $(A|H) \wedge_{a,b} (B|K)$ coincides with its conjunction $(AHBK + P'(A|H)\overline{H}BK + P'(B|K)AH\overline{K})|(H \vee K)$, which we denote by $(A|H) \wedge' (B|K)$. Thus, by coherence, $\mathbb{P}'[(A|H) \wedge_{a,b} (B|K)]$ satisfies the Fréchet-Hoeffding bounds, that is:

$$\mathbb{P}'[(A|H) \wedge_{a,b} (B|K)] = \mathbb{P}'[(A|H) \wedge' (B|K)] \in [\max\{a + b - 1, 0\}, \min\{a, b\}].$$

Now, suppose that O asserts $P(A|H) = x$ and $P(B|K) = y$. Then, under logical independence, the lower and upper bounds z' and z'' on $(A|H) \wedge_{a,b} (B|K)$ computed in Theorem 5, for the individual O, represent the lower and upper bounds for the coherent extension $\mathbb{P}[(A|H) \wedge' (B|K)]$ of the assessment (x, y) on $\{A|H, B|K\}$. Thus, according to eq. (32), it generally holds that

$$\mathbb{P}[(A|H) \wedge_{a,b} (B|K)] = \mathbb{P}[(A|H) \wedge' (B|K)] \in [z', z''] \supseteq [\max\{x + y - 1, 0\}, \min\{x, y\}],$$

while

$$\mathbb{P}[(A|H) \wedge_{x,y} (B|K)] = \mathbb{P}[(A|H) \wedge (B|K)] \in [\max\{x + y - 1, 0\}, \min\{x, y\}].$$

A real world application of this interpretation could be given in the framework of a joint-bet in soccer betting (see [13,18]). The conditional events can be interpreted as $A|H$ = *The outcome of the first match is home win, given that the first match is valid*, and $B|K$ = *The outcome of the second match is draw, given that the second match is valid*. Suppose the bettor assesses $x = P(A|H)$, $y = P(B|K)$. However these are his personal evaluations, and when he bets on these matches he has to deal with the bookmaker ones. That is, bookmaker probabilities may be $a = P'(A|H) = \frac{1}{Q_1}$ and $b = P'(B|K) = \frac{1}{Q_2}$, where Q_1 and Q_2 are the assigned decimal odds on the two single bets. In a single bet the stake will be refunded if the match is not valid. A double bet on $A|H$ and $B|K$ in a linked series of the two single bets where the return of one bet is stacked to the other bet. Then, the bettor actually bets on the joint-bet $(A|H) \wedge_{a,b} (B|K)$, in the context where he has his own probabilities but the object is determined by those of the bookmaker. Moreover the bookmaker usually considers the two bets as "independent" and for him $(A|H) \wedge_{a,b} (B|K) = (A|H) \wedge' (B|K)$ with $\mathbb{P}'[(A|H) \wedge_{a,b} (B|K)] = ab = \frac{1}{Q_1Q_2}$, while the bettor could coherently assess $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)] \in [z', z'']$ as in Theorem 5. In particular, the bettor, paying ab, might feel he has an advantage when $ab < z'$, where z' is the lower bound in (29). This condition can only occur when $0 < ab < (x + y - 1) \min\{\frac{a}{x}, \frac{b}{y}\}$. Indeed, if $x + y - 1 \leqslant 0$ then $z' = 0 \leqslant ab$, and if $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = 1$ then $z' = x + y - 1 \leqslant a + b - 1 \leqslant ab$. Moreover, $0 < ab < (x + y - 1) \min\{\frac{a}{x}, \frac{b}{y}\}$ also requires that $0 < a < x$ and $0 < b < y$. Indeed, if $\frac{a}{x} \leqslant \frac{b}{y}$, then it must be $b < \frac{x+y-1}{x} \leqslant \frac{xy}{x} \leqslant y$. Similarly, if $\frac{b}{y} < \frac{a}{x}$, then it must be $a < \frac{x+y-1}{y} \leqslant \frac{xy}{y} \leqslant x$. As an example, consider the case where $x = 0.8$, $y = 0.9$, $a = 0.7$, and $b = 0.8$. In this case, $ab = 0.56 < z' = (0.8 + 0.9 - 1)\frac{7}{8} = 0.6125$.

Inclusion relation. We recall that, when $A|H \subseteq B|K$, it holds that $(A|H) \wedge (B|K) = A|H$ (see, e.g., [26, formula (16)]) and hence $\mathbb{P}[(A|H) \wedge (B|K)] = P(A|H)$. These relations are not preserved by $(A|H) \wedge_{a,b} (B|K)$. Indeed, if $A|H \subseteq B|K$, as $AHBK = AH$, $\overline{K}AH = \emptyset$, and $\overline{H}BK = \overline{H}K$, from (18) it holds that

$$(A|H) \wedge_{a,b} (B|K) = (AH + a\overline{H}K)|(H \vee K) \neq A|H, \tag{46}$$

because, when $\overline{H}K$ is true, it follows that $(A|H) \wedge_{a,b} (B|K) = a$, while $A|H = x$, where $x = P(A|H)$. Moreover, it holds that

$$\mathbb{P}[(A|H) \wedge_{a,b} (B|K)] = P(A|H)P(H|H \vee K) + aP(\overline{H}K|H \vee K) = \alpha x + (1-\alpha)a,$$

where $x = P(A|H)$, $\alpha = P(H|H \vee K)$. Thus, when $A|H \subseteq B|K$, it follows that

$$\mathbb{P}[(A|H) \wedge_{a,b} (B|K)] \in [\min\{x,a\}, \max\{x,a\}],$$

whereas

$$\mathbb{P}[(A|H) \wedge (B|K)] = P(A|H) = x.$$

In particular when $a = b = 1$, that is $(A|H) \wedge_{a,b} (B|K) = (A|H) \wedge_S (B|K)$, from (46) it follows that $(A|H) \wedge_{a,b} (B|K) = (AH \vee \overline{H}K)|(H \vee K) \supseteq (A|H)$ (see [22, Remark 4]) and $P((AH \vee \overline{H}K)|(H \vee K)) \in [x, 1]$.

Analogous observations from this section can be applied to $(A|H) \vee_{a,b} (B|K)$.

8. Conclusions

In the framework of conditional random quantities, conjunction and disjunction of two conditional events have been defined as structures which take among their values the probabilities of the conditionals involved. In this paper, we showed that this choice of values is not restrictive but even necessary if we want them to satisfy the same logic and probabilistic properties valid for conjunction and disjunction of basic events. To do so, we proposed a generalization of the notions of conjunction and disjunction for two conditional events, in which these structures take arbitrary values $a, b \in [0, 1]$ not necessarily equal to the probabilities $x = P(A|H)$ and $y = P(B|K)$, in the cases where one conditional event is void and the other is true (for the conjunction) or false (for the disjunction). After having defined the negation of the generalized conjunction, we found a generalization of De Morgan's law linking these new structures. Even though a generalized De Morgan's relation holds, the Sum rule, written for conditional events, is not satisfied, except in the case $a = x = P(A|H)$ and $b = y = P(B|K)$. Then, we computed the sets of coherent assessments on the families, $\{A|H, B|K, (A|H) \wedge_{a,b} (B|K)\}$ and $\{A|H, B|K, (A|H) \vee_{a,b} (B|K)\}$, and we proved that the Fréchet-Hoeffding bounds are satisfied for every coherent assessment (x, y) on $\{A|H, B|K\}$ only when we take $a = x$ and $b = y$. Moreover, to make the study of these objects more complete, both for the conjunction and the disjunction we studied these structures in the case of incompatibility between the antecedents and for particular values of a and b. In particular, we analyzed the case $(a, b) = (0, 0)$ for which the generalized conjunction reduces to one of the four associative conjunction proposed by Walker ([42]), and the generalized disjunction reduces to the quasi disjunction. Similarly, we analyzed the case $(a, b) = (1, 1)$ for which the generalized conjunction reduces to quasi conjunction and the generalized disjunction reduces to the fourth Walker's disjunction. Then, we studied the case $(a, b) = (p, p)$ in which the generalized five-valued structures coincide with the three-valued conjunction and disjunction in the trivalent logic of Kleene-de Finetti. Furthermore, we analyzed the coherent assessment on the conjunction and the disjunction in the case of interval-valued assessments and we gave an interpretation for the generalized conjunction in the setting of subjective probability theory. We also considered a possible real world application to soccer betting. Further work could concern the investigation of these generalizations within the theory of the Boolean algebra of conditionals ([14,15]), of Markov graphs ([45]) and the study of related generalized notions of iterated conditionals which could give further insights on the generalized structures and their applicability (see, e.g., [19]).

CRediT authorship contribution statement

Lydia Castronovo: Writing – original draft, Methodology, Formal analysis, Conceptualization. **Giuseppe Sanfilippo:** Writing – original draft, Methodology, Formal analysis, Conceptualization.

Author contributions

Both authors contributed equally to the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

A.1. Proof of Theorem 5

Proof. First of all we observe that, by logical independence of A, H, B, K, the assessment (x, y) is coherent for every $(x, y) \in [0, 1]^2$. The constituents C_h's contained in $H \vee K$ and the points Q_h's associated with the assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F}$ are $C_1 = AHBK, C_2 = AH\overline{B}K, C_3 = \overline{A}HBK, C_4 = \overline{A}H\overline{B}K, C_5 = AH\overline{K}, C_6 = \overline{H}BK, C_7 = \overline{A}H\overline{K}, C_8 = \overline{H}\,\overline{B}K$ and $Q_1 = (1, 1, 1), Q_2 = (1, 0, 0), Q_3 = (0, 1, 0), Q_4 = (0, 0, 0), Q_5 = (1, y, b), Q_6 = (x, 1, a), Q_7 = (0, y, 0), Q_8 = (x, 0, 0)$. Considering the convex hull $\mathcal{I}$ of $Q_1, \dots, Q_8$, the coherence of $\mathcal{P}$ requires that the condition $\mathcal{P} \in \mathcal{I}$ be satisfied, that is

$$\mathcal{P} = \textstyle\sum_{h=1}^{8} \lambda_h Q_h,\ \sum_{h=1}^{8} \lambda_h = 1,\ \lambda_h \geqslant 0,\ h = 1, \dots, 8\,.$$

We observe that $Q_7 = yQ_3 + (1 - y)Q_4$ and $Q_8 = xQ_2 + (1 - x)Q_4$. Then, $\mathcal{I}$ is the convex hull of $Q_1, \dots, Q_6$. Thus, the condition $\mathcal{P} \in \mathcal{I}$ amounts to the solvability of the following system in the unknowns $\lambda_1, \dots, \lambda_6$

$$\mathcal{P} = \textstyle\sum_{h=1}^{6} \lambda_h Q_h,\ \sum_{h=1}^{6} \lambda_h = 1,\ \lambda_h \geqslant 0,\ h = 1, \dots, 6 \tag{A.1}$$

that is

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_5 + x\lambda_6 = x, \\ \lambda_1 + \lambda_3 + y\lambda_5 + \lambda_6 = y, \\ \lambda_1 + b\lambda_5 + a\lambda_6 = z, \\ \lambda_1 + \dots + \lambda_6 = 1,\ \lambda_i \geqslant 0,\ \forall i = 1, \dots, 6. \end{cases} \tag{A.2}$$

For each solution $\Lambda = (\lambda_1, \dots, \lambda_6)$ of system (A.2) we have that the functions Φ_j defined in (6) are

$$\begin{aligned} &\Phi_1(\Lambda) = \textstyle\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5, \\ &\Phi_2(\Lambda) = \textstyle\sum_{h:C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_6, \\ &\Phi_3(\Lambda) = \textstyle\sum_{h:C_h \subseteq H \vee K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6. \end{aligned} \tag{A.3}$$

For each given $(x, y) \in [0, 1]^2$, based on Theorem 3 we determine the lower and upper bounds z', z'' for the coherent extension $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$. *Lower Bound.* We distinguish two cases: **(A)** $x + y - 1 \leqslant 0$; **(B)** $x + y - 1 > 0$.

(A) We show that $z' = 0$ is the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ by proving that the assessment $(x, y, 0)$ on $\mathcal{F}$ is coherent. Let a pair $(x, y) \in [0, 1]^2$ be given. We first observe that $\mathcal{P} = (x, y, 0) = xQ_2 + yQ_3 + (1 - x - y)Q_4$. Then, $\mathcal{P} \in \mathcal{I}$, where $\mathcal{I}$ is the convex hull of $Q_1, \dots, Q_6$, with a solution of (A.2) given by $\Lambda = (0, x, y, 1 - x - y, 0, 0)$. For the functions Φ_j given in (A.3) it holds that $\Phi_1(\Lambda) = \Phi_2(\Lambda) = \Phi_3(\Lambda) = 1 > 0$. Then, from Remark 2, the assessment $(x, y, 0)$ on $\mathcal{F}$ is coherent. Thus, for every $(x, y) \in [0, 1]^2$ the assessment $(x, y, 0)$ is coherent and hence $z' = 0$.

(B) As $x + y - 1 > 0$, it holds that $x \neq 0$ and $y \neq 0$. We consider three subcases: *(B.1)* $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = 1$; *(B.2)* $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = \frac{a}{x}$; *(B.3)* $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = \frac{b}{y}$.

Case (B.1). We observe that $a \geqslant x$ and $b \geqslant y$. We first prove that the assessment $(x, y, x + y - 1)$ is coherent. Then, we show that $z' = x + y - 1$ is the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$. As $\mathcal{P} = (x, y, x + y - 1) = (x + y - 1)Q_1 + (1 - y)Q_2 + (1 - x)Q_3$, it follows that $\mathcal{P} \in \mathcal{I}$. Then, a solution of system (A.2) is given by $\Lambda = (x + y - 1, 1 - y, 1 - x, 0, 0, 0)$. From (A.3) it holds that $\Phi_1(\Lambda) = \Phi_2(\Lambda) = \Phi_3(\Lambda) = 1 > 0$. Then, from Remark 2, the assessment $(x, y, x + y - 1)$ on $\mathcal{F}$ is coherent.
In order to prove that $z' = x + y - 1$ is the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$, we verify that the assessment $\mathcal{P} = (x, y, z)$, with $(x, y) \in [0, 1]^2$ and $z < z' = x + y - 1$, is not coherent because $(x, y, z) \notin \mathcal{I}$. We observe that the points Q_1, Q_2, Q_3 belong to the plane $\pi : X + Y - Z = 1$. We set $f(X, Y, Z) = X + Y - Z - 1$ and we obtain $f(Q_1) = f(Q_2) = f(Q_3) = 0,\ \ f(Q_4) = -1 < 0,\ f(Q_5) = y - b \leqslant 0,\ f(Q_6) = x - a \leqslant 0$. Then, by considering $\mathcal{P} = (x, y, z)$, with $z < x + y - 1$, it holds that

$$f(\mathcal{P}) = x + y - 1 - z > 0 \geqslant f(Q_h),\ h = 1, \dots, 6,$$

and hence $\mathcal{P} = (x, y, z) \notin \mathcal{I}$. Indeed, if it were $\mathcal{P} \in \mathcal{I}$, that is $\mathcal{P}$ linear convex combination of $Q_1, \dots, Q_6$, it would follow that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leqslant 0$. Thus, the lower bound for z is $z' = x + y - 1$, for every $(x, y) \in [0, 1]^2$ such that $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = 1$.

Case (B.2). We notice that $a \leqslant x$ and $\frac{a}{x} \leqslant \frac{b}{y}$.
We show that in this case

$$z' = (x + y - 1)\min\{\frac{a}{x}, \frac{b}{y}, 1\} = \frac{a}{x}(x + y - 1).$$

We first prove that $(x, y, \frac{a}{x}(x+y-1))$ is coherent. Indeed, we observe that $\mathcal{P} = (x, y, \frac{a}{x}(x+y-1)) = (1-y)Q_2 + \frac{(1-x)(1-y)}{x}Q_3 + \frac{x+y-1}{x}Q_6$. Then, $\mathcal{P} \in \mathcal{I}$, where $\mathcal{I}$ is the convex hull of $Q_1, \dots, Q_6$, with a solution of (A.2) given by $\Lambda = (0, 1-y, \frac{(1-x)(1-y)}{x}, 0, 0, \frac{x+y-1}{x})$. By recalling (A.3), it holds that $\Phi_1(\Lambda) = \frac{1-y}{x}$, $\Phi_2(\Lambda) = \Phi_3(\Lambda) = 1 > 0$. We distinguish two cases: (i) $y \neq 1$, (ii) $y = 1$. In the case (i) we get $\Phi_j(\Lambda) > 0$, $j = 1,2,3$ and hence by Remark 2 it follows that the assessment $(x, y, \frac{a}{x}(x+y-1))$ is coherent. In case (ii), as $\Phi_1(\Lambda) = 0$, it holds that $\mathcal{I}_0 \subseteq \{1\}$, with the sub-assessment $\mathcal{P}_0 = x$ on $\mathcal{F}_0 = \{A|H\}$ coherent because $x \in [0,1]$. Then, by Theorem 2, the assessment $(x, y, \frac{a}{x}(x+y-1)) = (x, 1, a)$ on $\mathcal{F}$ is coherent. Thus, in this sub-case the assessment $(x, y, \frac{a}{x}(x+y-1))$ on $\mathcal{F}$ is coherent for every $(x, y) \in [0,1]^2$. In order to prove that $\frac{a}{x}(x+y-1)$ is the lower bound z' for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$, we verify that (x, y, z), with $(x, y) \in [0,1]^2$ and $z < \frac{a}{x}(x+y-1)$, is not coherent because $(x, y, z) \notin \mathcal{I}$. We observe that the points Q_2, Q_3, Q_6 belong to the plane $\pi : aX + aY - xZ = a$. We set $f(X, Y, Z) = a(X+Y-1) - xZ$ and we obtain $f(Q_2) = f(Q_3) = f(Q_6) = 0$, $f(Q_1) = a - x \leqslant 0$, $f(Q_4) = -a \leqslant 0$, $f(Q_5) = ay - bx \leqslant 0$. Then, by considering $\mathcal{P} = (x, y, z)$, with $z < \frac{a}{x}(x+y-1)$, it holds that $f(\mathcal{P}) = f(x, y, z) = a(x+y-1) - xz > 0 \geqslant f(Q_h)$, $h = 1, \dots, 6$, and hence $\mathcal{P} = (x, y, z) \notin \mathcal{I}$. Indeed, if it were $\mathcal{P} \in \mathcal{I}$, that is $\mathcal{P}$ linear convex combination of $Q_1, \dots, Q_6$, it would follow that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leqslant 0$. Thus, in this sub-case the lower bound for $z = \mathbb{P}((A|H) \wedge_{a,b} (B|K))$ is $z' = \frac{a}{x}(x+y-1)$, for every $(x, y) \in [0,1]^2$ such that $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = \frac{a}{x}$.

Case (B.3). We notice that $b \leqslant y$ and $\frac{a}{x} \geqslant \frac{b}{y}$. We prove that $(x, y, \frac{b}{y}(x+y-1))$ is coherent and that $z' = \frac{b}{y}(x+y-1)$ is the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$.
We observe that $\mathcal{P} = (x, y, \frac{b}{y}(x+y-1)) = \frac{(1-x)(1-y)}{y}Q_2 + (1-x)Q_3 + \frac{x+y-1}{y}Q_5$. Then, $\mathcal{P} \in \mathcal{I}$, with a solution of (A.2) given by $\Lambda = (0, \frac{(1-x)(1-y)}{y}, 1-x, 0, \frac{x+y-1}{y}, 0)$. It holds that $\Phi_1(\Lambda) = \Phi_3(\Lambda) = 1$, $\Phi_2(\Lambda) = \frac{1-x}{y}$. We distinguish two cases: (i) $x \neq 1$, (ii) $x = 1$. In the case (i) we get $\Phi_j(\Lambda) > 0$, $j = 1,2,3$, and hence by Remark 2, the assessment $(x, y, \frac{b}{y}(x+y-1))$ is coherent. In the case (ii) we get $\mathcal{I}_0 \subseteq \{2\}$, with the sub-assessment $\mathcal{P}_0 = y$ on $\mathcal{F}_0 = \{B|K\}$ coherent because $y \in [0,1]$. Then, by Theorem 2, the assessment $(x, y, \frac{b}{y}(x+y-1)) = (1, y, b)$ on $\mathcal{F}$ is coherent. Thus, the assessment $(x, y, \frac{b}{y}(x+y-1))$ on $\mathcal{F}$ is coherent for every $(x, y) \in [0,1]^2$ such that $\min\{\frac{a}{x}, \frac{b}{y}, 1\} = \frac{b}{y}$. In order to prove that $\frac{b}{y}(x+y-1)$ is the lower bound z' for $z = \mathbb{P}((A|H) \wedge_{a,b} (B|K))$, we verify that (x, y, z), with $(x, y) \in [0,1]^2$ and $z < \frac{b}{y}(x+y-1)$, is not coherent because $(x, y, z) \notin \mathcal{I}$. We observe that the points Q_2, Q_3, Q_5 belong to the plane $\pi : bX + bY - yZ = b$. We set $f(X, Y, Z) = b(X+Y-1) - yZ$ and we obtain $f(Q_2) = f(Q_3) = f(Q_5) = 0$, $f(Q_1) = b - y < 0$, $f(Q_4) = -b \leqslant 0$, $f(Q_5) = bx - ay \leqslant y$. Then, by considering $\mathcal{P} = (x, y, z)$, with $z < \frac{b}{y}(x+y-1)$, it holds that $f(\mathcal{P}) = f(x, y, z) = b(x+y-1) - yz > 0 \geqslant f(Q_h)$, $h = 1, \dots, 6$, and hence $\mathcal{P} = (x, y, z) \notin \mathcal{I}$. Indeed, if it were $\mathcal{P} \in \mathcal{I}$, that is $\mathcal{P}$ linear convex combination of $Q_1, \dots, Q_6$, it would follow that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leqslant 0$. Thus, the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$ is $z' = \frac{b}{y}(x+y-1)$.

Therefore, for every $(x, y) \in [0,1]^2$ the value z' given in formula (29) is the lower bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$.

Upper Bound. We distinguish two cases: **(C)** $a(1-y) + b(1-x) + xy - 1 > 0$; **(D)** $a(1-y) + b(1-x) + xy - 1 \leqslant 0$.

(C) We first prove that

$$\max\{z_1'', z_2'', \min\{z_3'', z_4''\}\} = z_2'', \tag{A.4}$$

where z_i'', $i = 1,2,3,4$, are given in (30). Then, we prove that (x, y, z_2'') is coherent. Finally we prove that z_2'' is the upper bound, by showing that (x, y, z) with $z > z_2''$ is not coherent. We observe that to be in case **(C)** we must have $a > x$ and $b > y$. Indeed, if it were $a \leqslant x$, as

$$\begin{aligned} &a(1-y) + b(1-x) + xy - 1 \leqslant x(1-y) + b(1-x) + xy - 1 \\ &= b(1-x) + x - 1 = (1-x)(b-1) \leqslant 0, \end{aligned}$$

we would be in case **(D)**. Likewise, if it were $b \leqslant y$, as

$$\begin{aligned} &a(1-y) + b(1-x) + xy - 1 \leqslant a(1-y) + y(1-x) + xy - 1 \\ &= a(1-y) + y - 1 = (1-y)(a-1) \leqslant 0, \end{aligned}$$

we would be in case **(D)**. As $a > x$ and $b > y$, it holds that $x \neq 1$ and $y \neq 1$. Moreover,

$$a - bx > x - bx = x(1-b) \geqslant 0, \quad b - ay > y(1-a) \geqslant 0. \tag{A.5}$$

We observe that

$$\begin{aligned} z_2'' - z_3'' &= \frac{x(b-ay) + y(a-bx)}{1-xy} - \frac{x(1-a) + y(a-x)}{1-x} \\ &= \frac{x(1-y)[a(1-y) + b(1-x) + xy - 1]}{(1-xy)(1-x)} \geqslant 0 \end{aligned} \tag{A.6}$$

and

$$z_2'' - z_4'' = \frac{x(b-ay)+y(a-bx)}{1-xy} - \frac{x(b-y)+y(1-b)}{1-y} = \frac{y(1-x)[a(1-y)+b(1-x)+xy-1]}{(1-xy)(1-y)} \geqslant 0. \tag{A.7}$$

Then $z_2'' \geqslant z_3''$ and $z_2'' \geqslant z_4''$. Concerning the relation between z_2'' and z_1'' we distinguish two cases: (i) $x \leqslant y$; (ii) $x \geqslant y$.
(i). In this case $z_1'' = \min\{x,y\} = x$. As $a(1-y)+b(1-x) > 1-xy$, it holds that $x(a(1-y)+b(1-x)) \geqslant x(1-xy)$. Then,

$$z_2'' = \frac{x(b-ay)+y(a-bx)}{(1-xy)} \geqslant \frac{x(b-ay)+x(a-bx)}{1-xy} = = \frac{x(a(1-y)+b(1-x))}{1-xy} \geqslant \frac{x(1-xy)}{1-xy} = x = z_1''.$$

(ii) In this case $z_1'' = \min\{x,y\} = y$. As $a(1-y)+b(1-x) > 1-xy$, it holds that $y(a(1-y)+b(1-x)) \geqslant y(1-xy)$. Then,

$$z_2'' = \frac{x(b-ay)+y(a-bx)}{(1-xy)} \geqslant \frac{y(b-ay)+y(a-bx)}{1-xy} = \frac{y(a(1-y)+b(1-x))}{1-xy} \geqslant \frac{y(1-xy)}{1-xy} = y = z_1''.$$

Thus, formula (A.4) holds.
We now prove that $(x,y,z_2'') = (x,y,\frac{x(b-ay)+y(a-bx)}{1-xy})$ is a coherent assessment on $\mathcal{F}$. We observe that $\mathcal{P} = (x,y,z_2'') = \frac{(1-x)(1-y)}{1-xy}Q_4 + \frac{x(1-y)}{1-xy}Q_5 + \frac{y(1-x)}{1-xy}Q_6$. Then, $\mathcal{P} \in \mathcal{I}$ with a solution of (A.2) given by $\Lambda = (0,0,0,\frac{(1-x)(1-y)}{1-xy},\frac{x(1-y)}{1-xy},\frac{y(1-x)}{1-xy})$. From (A.3) it follows

$$\Phi_1(\Lambda) = \frac{1-y}{1-xy},\ \Phi_2(\Lambda) = \frac{1-x}{1-xy},\ \Phi_3(\Lambda) = 1.$$

As $x < 1$ and $y < 1$, it holds that $\Phi_j(\Lambda) > 0$, $j = 1,2,3$. Then, by Remark 2 the assessment (x,y,z'') on $\mathcal{F}$ is coherent.
In order to prove that z_2'' is the upper bound z'' for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$, we verify that (x,y,z), with $z > z''$, is not coherent because $(x,y,z) \notin \mathcal{I}$. We observe that the points Q_4,Q_5,Q_6 belong to the plane $\pi : -(b-ay)X - (a-bx)Y + (1-xy)Z = 0$. We set $f(X,Y,Z) = -(b-ay)X - (a-bx)Y + (1-xy)Z$. It holds that $f(Q_4) = f(Q_5) = f(Q_6) = 0$, $f(Q_1) = -[a(1-y) + b(1-x) + xy - 1] < 0$, $f(Q_2) = -(b-ay) < 0$, $f(Q_3) = -(a-bx) < 0$. Then, for every $\mathcal{P} \in \mathcal{I}$, that is $\mathcal{P}$ is a linear convex combination of $Q_1,\dots,Q_6$, it would follow that $f(\mathcal{P}) = f(\sum_{h=1}^6 \lambda_h Q_h) = \sum_{h=1}^6 \lambda_h f(Q_h) \leqslant 0$. Then, by considering $\mathcal{P} = (x,y,z)$, with $z > z_2''$, it holds that $z(1-xy) > x(b-ay) + y(a-bx)$ and hence

$$f(\mathcal{P}) = -(b-ay)x - (a-bx)y + (1-xy)z > 0.$$

Then, $\mathcal{P} = (x,y,z) \notin \mathcal{I}$. Thus, in case **(C)** $z'' = \max\{z_1'', z_2'', \min\{z_3'', z_4''\}\} = z_2''$.

(D) First of all we observe that $z_2'' \leqslant \min\{z_3'', z_4''\}$. Indeed, if $x \neq 1$ and $y \neq 1$, from (A.6) and (A.7) it holds that $z_2'' \leqslant z_3''$ and $z_2'' \leqslant z_4''$. If $x = 1$, $y \neq 1$, of course $z_2'' \leqslant z_3'' = 1$ and from (A.7) it holds that $z_2'' \leqslant z_4''$. If $y = 1$, $x \neq 1$, of course $z_2'' \leqslant z_4'' = 1$ and from (A.6) it holds that $z_2'' \leqslant z_3''$. Finally, if $x = y = 1$, then $z_2'' = z_3'' = z_4'' = 1$. Then, $\max\{z_1'', z_2'', \min\{z_3'', z_4''\}\} = \max\{z_1'', \min\{z_3'', z_4''\}\}$. Moreover, as $\frac{x(1-a)+y(a-x)}{1-x} - \frac{x(b-y)+y(1-b)}{1-y} = \frac{(y-x)[a(1-y)+b(1-x)+xy-1]}{(1-y)(1-x)}$, we obtain

$$z_3'' - z_4'' = \begin{cases} \frac{(y-x)[a(1-y)+b(1-x)+xy-1]}{(1-y)(1-x)}, \text{ if } x \neq 1 \text{ and } y \neq 1, \\ 1-b, \text{ if } x = 1 \text{ and } y \neq 1, \\ a-1, \text{ if } x \neq 1 \text{ and } y = 1, \\ 0, \text{ if } x = y = 1. \end{cases} \tag{A.8}$$

We consider the following subcases: *(D.1)* $x \leqslant y$; *(D.2)* $y < x$;

Case (D.1). From (A.8) it holds that $\min\{z_3'', z_4''\} = z_3''$. Then, $\max\{z_1'', \min\{z_3'', z_4''\}\} = \max\{x, z_3''\}$. We distinguish two cases: (i) $a \leqslant x$; (ii) $a > x$.
(i). We observe that, if $x = 1$ (and hence $y = 1$) we have that $\max\{1, z_3''\} = 1 = z_1''$. If $x \neq 1$, as

$$z_3'' - z_1'' = \frac{x(1-a)+y(a-x)}{1-x} - x = \frac{x(1-a)+y(a-x)-x(1-x)}{1-x} = = \frac{a(y-x)-x(y-x)}{1-x} = \frac{(y-x)(a-x)}{1-x} \tag{A.9}$$

it holds that $z_3'' \leqslant z_1'' = x$. Then $\max\{z_1'', z_3''\} = z_1'' = x$.
Now we show that the assessment $\mathcal{P} = (x,y,z_1'') = (x,y,x)$ is coherent and that the upper bound z'' for z is $z_1'' = x$. We observe that $\mathcal{P} = (x,y,x) = xQ_1 + (y-x)Q_3 + (1-y)Q_4$, then, $\mathcal{P} \in \mathcal{I}$ and a solution of (A.2) is given by $\Lambda = (x,0,y-x,1-y,0,0)$.

It holds that $\Phi_1(\Lambda) = \Phi_2(\Lambda) = \Phi_3(\Lambda) = 1$, then by Remark 2 the probability assessment (x, y, x) on $\mathcal{F}$ is coherent. In order to prove that $z'' = z''_1 = x$ is the upper bound for z, we verify that $\mathcal{P} = (x, y, z)$, with $z > z'' = x$, is not coherent because $\mathcal{P} \notin \mathcal{I}$. We observe that the points Q_1, Q_3, Q_4 belong to the plane $\pi : X - Z = 0$. We set $f(X, Y, Z) = X - Z$ and we obtain $f(Q_1) = f(Q_3) = f(Q_4) = 0$, $f(Q_2) = 1 > 0$, $f(Q_5) = 1 - b \geqslant 0$, $f(Q_6) = x - a \geqslant 0$. Then, for every $\mathcal{P} \in \mathcal{I}$ it holds that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \geqslant 0$. Then, by considering $\mathcal{P} = (x, y, z)$, with $z > x$, it holds that $f(\mathcal{P}) = f(x, y, z) = x - z < 0$, and hence $\mathcal{P} \notin \mathcal{I}$. Thus, $z'' = z''_1 = x$.
(ii) As $x < a$, $x \neq 1$. In this case, from (A.9) it holds that

$$\max\{z''_1, z''_3\} = z''_3 = \frac{x(1-a) + y(a-x)}{1-x}.$$

We prove that $(x, y, z''_3) = (x, y, \frac{x(1-a)+y(a-x)}{1-x})$ is a coherent assessment on $\mathcal{F} = \{A|H, B|K, (A|H)|_{a,b}(B|K)\}$ and that $z'' = z''_3$. We observe that $\mathcal{P} = (x, y, \frac{x(1-a)+y(a-x)}{1-x}) = \frac{x(1-y)}{1-x} Q_1 + (1-y) Q_4 + \frac{y-x}{1-x} Q_6$, hence $\mathcal{P} \in \mathcal{I}$. A solution of (A.2) is given by $\Lambda = (\frac{x(1-y)}{1-x}, 0, 0, 1-y, 0, \frac{y-x}{1-x})$ and it holds that $\Phi_1(\Lambda) = \frac{1-y}{1-x}$, $\Phi_2(\Lambda) = \Phi_3(\Lambda) = 1$. We distinguish two cases: $y \neq 1$ and $y = 1$. If $y \neq 1$ we get $\Phi_1(\Lambda) > 0$ and $\Phi_2(\Lambda) = \Phi_3(\Lambda) = 1 > 0$; then $\mathcal{I}_0 = \emptyset$ and by Remark 2 the assessment (x, y, z''_3) is coherent. If $y = 1$ we get $\mathcal{I}_0 \subseteq \{1\}$, with the sub-assessment $\mathcal{P}_0 = x$ on $\mathcal{F}_0 = \{A|H\}$ coherent because $x \in [0, 1]$. Then, by Remark 2, the assessment (x, y, z''_3) on $\mathcal{F}$ is coherent. In order to prove that $z'' = z''_3$, we verify that (x, y, z), with $z > z''_3$ is not coherent because $(x, y, z) \notin \mathcal{I}$. We observe that the points Q_1, Q_4, Q_6 belong to the plane $\pi : (a-1)X + (x-a)Y + (1-x)Z = 0$. We set $f(X, Y, Z) = (a-1)X + (x-a)Y + (1-x)Z$ and we obtain $f(Q_1) = f(Q_4) = f(Q_6) = 0$, $f(Q_2) = a - 1 \leqslant 0$, $f(Q_3) = x - a < 0$, $f(Q_5) = a - 1 + y(x-a) + b(1-x) = a(1-y) + b(1-x) + xy - 1 \leqslant 0$. Then, for every $\mathcal{P} \in \mathcal{I}$ it follows that $f(Q) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leqslant 0$. Then, by considering $\mathcal{P} = (x, y, z)$, with $z > \frac{x(1-a)+y(a-x)}{1-x}$, i.e. $z(1-x) > x(1-a) + y(a-x)$, it holds that $f(\mathcal{P}) = f(x, y, z) = -[(1-a)x + (a-x)y] + (1-x)z > 0$ and hence $\mathcal{P} \notin \mathcal{I}$. Thus, the upper bound for $z = P((A|H) \wedge_{a,b} (B|K))$ is $z'' = z''_3 = \frac{x(1-a)+y(a-x)}{1-x}$.

Case (D.2) From (A.8) it holds that $\min\{z''_3, z''_4\} = z''_4$. Then, $\max\{z''_1, \min\{z''_3, z''_4\}\} = \max\{y, z''_4\}$. We distinguish two cases: (j) $b \leqslant y$; (jj) $b > y$.
(j). As $y < x$, it holds that $y \neq 1$. We observe that

$$z''_4 - z''_1 = \frac{y(1-b) + x(b-y)}{1-y} - y = \frac{(x-y)(b-y)}{1-y} \tag{A.10}$$

and hence, as $y < x$ and $b \leqslant y$, $z''_4 \leqslant z''_1$. Then $\max\{z''_1, z''_4\} = z''_1 = y$. We show that $\mathcal{P} = (x, y, z''_1) = (x, y, y)$ is coherent and that the upper bound z'' for z is $z''_1 = y$. We observe that $\mathcal{P} = (x, y, y) = yQ_1 + (x-y)Q_2 + (1-x)Q_4$, then, $\mathcal{P} \in \mathcal{I}$ and a solution of (A.2) is given by $\Lambda = (y, x-y, 0, 1-x, 0, 0)$. It holds that $\Phi_1(\Lambda) = \Phi_2(\Lambda) = \Phi_3(\Lambda) = 1$, then by Remark 2 the probability assessment (x, y, y) on $\mathcal{F}$ is coherent. In order to prove that $z'' = z''_1 = y$ is the upper bound for z, we verify that $\mathcal{P} = (x, y, z)$, with $z > z'' = y$, is not coherent because $\mathcal{P} \notin \mathcal{I}$. We observe that the points Q_1, Q_2, Q_4 belong to the plane $\pi : Y - Z = 0$. We set $f(X, Y, Z) = Y - Z$ and we obtain $f(Q_1) = f(Q_2) = f(Q_4) = 0$, $f(Q_3) = 1 > 0$, $f(Q_5) = y - b \geqslant 0$, $f(Q_6) = 1 - a \geqslant 0$. Then, for every $\mathcal{P} \in \mathcal{I}$ it holds that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \geqslant 0$. Then, by considering $\mathcal{P} = (x, y, z)$, with $z > y$, it holds that $f(\mathcal{P}) = f(x, y, z) = y - z < 0$, and hence $\mathcal{P} \notin \mathcal{I}$. Thus, $z'' = z''_1 = y$.
(jj) In this case, from (A.10) it holds that

$$\max\{z''_1, z''_4\} = z''_4 = \frac{x(b-y) + y(1-b)}{1-y}.$$

We prove that $(x, y, z''_4) = (x, y, \frac{x(b-y)+y(1-b)}{1-y})$ is a coherent assessment on $\mathcal{F}$ and that $z'' = z''_4$. We observe that $\mathcal{P} = (x, y, \frac{y(1-b)+x(b-y)}{1-y}) = \frac{y(1-x)}{1-y} Q_1 + (1-x) Q_4 + \frac{x-y}{1-y} Q_5$. Then, $\mathcal{P} \in \mathcal{I}$ with a solution of (A.2) given by $\Lambda = (\frac{y(1-x)}{1-y}, 0, 0, 1-x, \frac{x-y}{1-y}, 0)$. Based on (A.3) it holds that $\Phi_1(\Lambda) = \Phi_3(\Lambda) = 1 > 0$, and $\Phi_2(\Lambda) = \frac{1-x}{1-y}$. We distinguish two cases: $x \neq 1$; $x = 1$. If $x \neq 1$, we get $\Phi_j(\Lambda) > 0$, $j = 1, 2, 3$. Then $\mathcal{I}_0 = \emptyset$ and by Remark 2 the assessment (x, y, z''_4) on $\mathcal{F}$ is coherent. If $x = 1$, we get $\mathcal{I}_0 \subseteq \{2\}$, with the sub-assessment $\mathcal{P}_0 = y$ on $\mathcal{F}_0 = \{B|K\}$ coherent because $y \in [0, 1]$. Then, by Theorem 2, the assessment (x, y, z''_4) on $\mathcal{F}$ is coherent. In order to prove that $z'' = z''_4 = \frac{y(1-b)+x(b-y)}{1-y}$ is the upper bound for $z = P((A|H) \wedge_{a,b} (B|K))$, we verify that (x, y, z), with $(x, y) \in [0, 1]^2$ and $z > z''_4$, is not coherent because $(x, y, z) \notin \mathcal{I}$. We observe that the points Q_1, Q_4, Q_5 belong to the plane $\pi : (y-b)X + (b-1)Y + (1-y)Z = 0$. We set $f(X, Y, Z) = (y-b)X + (b-1)Y + (1-y)Z$ and we obtain $f(Q_1) = f(Q_4) = f(Q_5) = 0$, $f(Q_2) = y - b < 0$, $f(Q_3) = b - 1 \leqslant 0$, $f(Q_6) = x(y-b) + (b-1) + a(1-y) = a(1-y) + b(1-x) + xy - 1 \leqslant 0$. Then, for every $\mathcal{P} \in \mathcal{I}$, it holds that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leqslant 0$. Moreover, by considering $\mathcal{P} = (x, y, z)$, with $z > \frac{x(b-y)+y(1-b)}{1-y}$, i.e. $z(1-y) > x(b-y) + y(1-b)$, it holds that $f(\mathcal{P}) = f(x, y, z) = -[(b-y)x + (1-b)y] + (1-y)z > 0$ and hence $\mathcal{P} = (x, y, z) \notin \mathcal{I}$. Thus, the upper bound for $z = P((A|H) \wedge_{a,b} (B|K))$ is $z'' = z''_4 = \frac{x(b-y)+y(1-b)}{1-y}$.

Therefore, for every $(x, y) \in [0, 1]^2$ the value z'' given in formula (30) is the upper bound for $z = \mathbb{P}[(A|H) \wedge_{a,b} (B|K)]$. $\square$

A.2. Proof of Theorem 6

Proof. Of course the assessment (x,y) on $\{A|H, B|K\}$ is coherent for every $(x,y)\in[0,1]^2$. For each given $(x,y)\in[0,1]^2$, based on Theorem 3, we determine the interval of the coherent extensions on $w=\mathbb{P}[(A|H)\vee_{a,b}(B|K)]$. We observe that the lower and upper bounds, w' and w'', for w can be easily obtained from those of the conjunction $(\bar{A}|H)\wedge_{1-a,1-b}(\bar{B}|K)$ by exploiting formula (26). Indeed, since $\mathbb{P}[(A|H)\vee_{a,b}(B|K)]=1-\mathbb{P}[(\bar{A}|H)\wedge_{1-a,1-b}(\bar{B}|K)]$, it holds that $w'=1-\eta''$ and $w''=1-\eta'$, where η' and η'' are the lower and upper bounds for $\mathbb{P}[(\bar{A}|H)\wedge_{1-a,1-b}(\bar{B}|K)]$, respectively.

Lower bound. Let us consider first the lower bound w'. For each tuple $(x,y,a,b)\in[0,1]^4$ we denote by $z''(x,y,a,b)$ and by $z_i''(x,y,a,b)$, $i=1,2,3,4$ the values of z'' and z_i'', $i=1,2,3,4$, obtained by applying formula (30). We observe that $\eta''=z''(1-x,1-y,1-a,1-b)$ and hence $w'=1-z''(1-x,1-y,1-a,1-b)$. Then, as $1-\max\{x_1,\dots,x_n\}=\min\{1-x_1,\dots,1-x_n\}$, it holds that

$$\begin{aligned}
w' &= 1-z''(1-x,1-y,1-a,1-b)\\
&= 1-\max\Big\{z_1''(1-x,1-y,1-a,1-b), z_2''(1-x,1-y,1-a,1-b),\\
&\quad \min\{z_3''(1-x,1-y,1-a,1-b), z_4''(1-x,1-y,1-a,1-b)\}\Big\}=\\
&= \min\Big\{1-z_1''(1-x,1-y,1-a,1-b), 1-z_2''(1-x,1-y,1-a,1-b),\\
&\quad \max\{1-z_3''(1-x,1-y,1-a,1-b), 1-z_4''(1-x,1-y,1-a,1-b)\}\Big\},\\
&= \min\{w_1', w_2', \max\{w_3', w_4'\}\},
\end{aligned}$$

where

$$w_1'=1-z_1''(1-x,1-y,1-a,1-b)=1-\min\{1-x,1-y\}=\max\{x,y\},$$

$$w_2'=1-z_2''(1-x,1-y,1-a,1-b)=$$

$$=1-\begin{cases}\dfrac{(1-x)[1-b-(1-a)(1-y)]+(1-y)[1-a-(1-b)(1-x)]}{1-(1-x)(1-y)}, & \text{if } (1-x,1-y)\neq(1,1),\\ 1, \text{ if } (1-x,1-y)=(1,1),\end{cases}=$$

$$=\begin{cases}\dfrac{ax+by-(a+b-1)xy}{x+y-xy}, \text{ if } (x,y)\neq(0,0),\\ 0, \text{ if } (x,y)=(0,0),\end{cases}$$

$$w_3'=1-z_3''(1-x,1-y,1-a,1-b)=$$

$$=1-\begin{cases}\dfrac{(1-x)a+(1-y)(1-a-1+x)}{x}, \text{if } 1-x\neq1,\\ 1, \text{ if } 1-x=1,\end{cases}=\begin{cases}\dfrac{a(x-y)+xy}{x}, \text{if } x\neq0,\\ 0, \text{ if } x=0,\end{cases}$$

$$w_4'=1-z_4''(1-x,1-y,1-a,1-b)=$$

$$=1-\begin{cases}\dfrac{(1-x)(1-b-1+y)+(1-y)b}{y}, \text{if } 1-y\neq1,\\ 1, \text{ if } 1-y=1.\end{cases}=\begin{cases}\dfrac{b(y-x)+xy}{y}, \text{ if } y\neq0,\\ 0, \text{ if } y=0.\end{cases}$$

Therefore, formula (33) holds. *Upper Bound.* Now, let us consider the upper bound w''. For each tuple $(x,y,a,b)\in[0,1]^4$ we denote by $z'(x,y,a,b)$ the value of z' obtained by applying formula (29). We observe that the lower bound η' for $\mathbb{P}[(\bar{A}|H)\wedge_{1-a,1-b}(\bar{B}|K)]$ coincides with $z'(1-x,1-y,1-a,1-b)$. Then

$$\begin{aligned}
w'' &= 1-\eta''=1-z'(1-x,1-y,1-a,1-b)=\\
&= 1-\begin{cases}(1-x+1-y-1)\cdot\min\{\frac{1-a}{1-x},\frac{1-b}{1-y},1\}, & \text{if } 1-x+1-y-1>0,\\ 0, & \text{otherwise},\end{cases}=
\end{aligned}$$

$$=\begin{cases}1-(1-x-y)\cdot\min\{\frac{1-a}{1-x},\frac{1-b}{1-y},1\}, & \text{if } x+y<1,\\ 1, & \text{otherwise.}\end{cases}$$

Therefore, formula (34) holds. □

A.3. Proof of Theorem 7

Proof. The constituents and the points Q_h's associated with the pair $(\mathcal{F},\mathcal{P})$ are $C_1=AH\overline{K}$, $C_2=\overline{H}BK$, $C_3=\overline{A}H\overline{K}$, $C_4=\overline{H}\,\overline{B}K$, $C_0=\overline{H}\,\overline{K}$, and $Q_1=(1,y,b)$, $Q_2=(x,1,a)$, $Q_3=(0,y,0)$, $Q_4=(x,0,0)$, $Q_0=\mathcal{P}=(x,y,z)$. We observe that $\mathcal{H}_n=H\vee K$ and that

$\mathcal{I}$ is the convex hull of points $Q_1,\dots Q_4$. For each given assessment (x,y) based on Theorem 3 we determine the lower and upper bounds z',z'' for the coherent extension $z=\mathbb{P}[(A|H)\wedge_{a,b}(B|K)]$. We distinguish two cases: $(i) ay\leqslant bx$, $(ii) bx<ay$. *Case* (i). We show that the assessment (x,y,z) on $\mathcal{F}$ is coherent if and only if $(x,y)\in[0,1]^2$ and $z\in[z',z'']$, where $z'=ay$ and $z''=bx$.

Lower Bound. First we prove that (x,y,ay), is coherent, and then that $z'=ay$ is the lower bound for z. We observe that $\mathcal{P}=(x,y,ay)=yQ_2+(1-y)Q_4$. Then, $\mathcal{P}\in\mathcal{I}$ with a solution of (A.2) given by $\Lambda=(0,y,0,1-y)$. It follows that

$$\begin{aligned}&\Phi_1(\Lambda)=\textstyle\sum_{h:C_h\subseteq H}\lambda_h=\lambda_1+\lambda_3=0,\\&\Phi_2(\Lambda)=\textstyle\sum_{h:C_h\subseteq K}\lambda_h=\lambda_2+\lambda_4=1,\\&\Phi_3(\Lambda)=\textstyle\sum_{h:C_h\subseteq H\vee K}\lambda_h=\lambda_1+\lambda_2+\lambda_3+\lambda_4=1.\end{aligned}$$

As $\Phi_1(\Lambda)=0$, it holds that $\mathcal{I}_0\subseteq\{1\}$, with the sub-assessment $\mathcal{P}_0=x$ on $\mathcal{F}_0=\{A|H\}$ coherent because $x\in[0,1]$. Then, by Theorem 2 the assessment $\mathcal{P}=(x,y,ay)$ on $\mathcal{F}$ is coherent. Now we show that $z'=ay$ is the lower bound. Of course, if $x=0$, as $ay\leqslant bx=0$, it holds that $ay=0$. In this case we have that $z'=0$ is the lower bound because $(A|H)\wedge_{a,b}(B|K)\in[0,1]$. We assume now that $x\neq 0$. We observe that Q_2,Q_3,Q_4 belong to the plane $\pi: ayX+axY-xZ-axy=0$. By setting $f(X,Y,Z)=ayX+axY-xZ-axy$, it holds that

$$f(Q_1)=ay-bx\leqslant 0,\;\; f(Q_2)=f(Q_3)=f(Q_4)=0.$$

Then, for every $\mathcal{P}\in\mathcal{I}$ it holds that $f(\mathcal{P})=f(\sum_{h=1}^4\lambda_hQ_h)=\sum_{h=1}^4\lambda_hf(Q_h)\leqslant 0$. Then, by considering $\mathcal{P}=(x,y,z)$, with $z<ay$, it holds that $f(\mathcal{P})=f(x,y,z)=x(ay-z)>0$, and hence $\mathcal{P}\notin\mathcal{I}$. Thus, as (x,y,ay) is coherent and (x,y,z), with $z<ay$, is not coherent, it follows that $z'=ay$ is the lower bound for $(A|H)\wedge_{a,b}(B|K)$.

Upper Bound. We show that the assessment $\mathcal{P}=(x,y,bx)$ on $\mathcal{F}$ is coherent. We observe that $(x,y,bx)=xQ_1+(1-x)Q_3$. Then, $\mathcal{P}\in\mathcal{I}$ with a solution of (A.2) given by $\Lambda=(x,0,1-x,0)$. It follows that

$$\Phi_1(\Lambda)=\Phi_3(\Lambda)=1,\;\Phi_2(\Lambda)=0.$$

As $\Phi_2(\Lambda)=0$, it holds that $\mathcal{I}_0\subseteq\{2\}$, with the sub-assessment $\mathcal{P}_0=y$ on $\mathcal{F}_0=\{B|K\}$ coherent because $y\in[0,1]$. Then, by Theorem 2 the assessment $\mathcal{P}=(x,y,bx)$ on $\mathcal{F}$ is coherent. We show that $z''=bx$ is the upper bound for z. We distinguish two cases: $y>0$ and $y=0$. Let us suppose for now that $y>0$. We observe that Q_1,Q_3,Q_4 belong to the plane $\pi: byX+bxY-yZ=bxy$. Considering the function $f(X,Y,Z)=byX+bxY-yZ-bxy$, it holds that

$$f(Q_1)=f(Q_3)=f(Q_4)=0,\;\; f(Q_2)=bx-ay\geqslant 0.$$

Then, for every $\mathcal{P}\in\mathcal{I}$ it holds that $f(\mathcal{P})=f(\sum_{h=1}^6\lambda_hQ_h)=\sum_{h=1}^6\lambda_hf(Q_h)\geqslant 0$. Then, by considering $\mathcal{P}=(x,y,z)$, with $z>bx$, it holds that $f(\mathcal{P})=f(x,y,z)=y(bx-z)<0$ and hence $\mathcal{P}\notin\mathcal{I}$. Thus, the assessment $\mathcal{P}=(x,y,z)$ with $z>bx$ is not coherent and hence $z''=bx$.

If $y=0$, we observe that Q_1,Q_2,Q_3 belong to the plane $\pi: -bX+(bx-a)Y+Z=0$. We set $f(X,Y,Z)=-bX+(bx-a)Y+Z$ and it holds that $f(Q_1)=f(Q_2)=f(Q_3)=0$, $f(Q_4)=-bx\leqslant 0$. Then, for every $\mathcal{P}\in\mathcal{I}$ it holds that $f(\mathcal{P})=f(\sum_{h=1}^6\lambda_hQ_h)=\sum_{h=1}^6\lambda_hf(Q_h)\leqslant 0$. Then, by considering $\mathcal{P}=(x,0,z)$, with $z>bx$, it holds that $f(\mathcal{P})=f(x,0,z)=-bx+z>0$, and hence $\mathcal{P}\notin\mathcal{I}$. Thus, the assessment $\mathcal{P}=(x,0,z)$, with $z>bx$, is not coherent and hence $z''=bx$.

Case (ii). The proof can be obtained in a way similar to the proof in *Case* (i), when switching x with y and a with b. □

A.4. Proof of Theorem 9

Proof. We first want to observe that $z^*=z'(x_1,y_1)$ and that $z^{**}=z''(x_2,y_2)$, where $z'(x,y)$ and $z''(x,y)$ are given in Theorem 5. The proof is straightforward by observing that

$$z^*=z'(x_1,y_1)=\min_{(x,y)\in\mathcal{A}}z'(x,y),$$

where $z'(x,y)$ is given in (29), and that

$$z^{**}=z''(x_2,y_2)=\max_{(x,y)\in\mathcal{A}}z''(x,y),$$

where $z''(x,y)$ is given in (30).

We now show for the lower bound $z^*=z'(x_1,y_1)$ that $z'(x_1,y_1)=\min_{(x,y)\in\mathcal{A}}z'(x,y)$ and for the upper bound $z^{**}=z''(x_2,y_2)$ that $z''(x_2,y_2)=\max_{(x,y)\in\mathcal{A}}z''(x,y)$.

Lower bound. We distinguish the following cases: **(A)**: $x_1+y_1-1\leqslant 0$; **(B)**: $x_1+y_1-1>0$.
Case **(A)**. If $x_1+y_1-1\leqslant 0$ then $z'(x_1,y_1)=0=\min_{(x,y)\in\mathcal{A}}z'(x,y)$ and hence $z^*=z'(x_1,y_1)=0$.
Case **(B)**. We consider three subcases: *(B.1)*: $\min\{\frac{a}{x_1},\frac{b}{y_1},1\}=1$; *(B.2)*: $\min\{\frac{a}{x_1},\frac{b}{y_1},1\}=\frac{a}{x_1}$; *(B.3)*: $\min\{\frac{a}{x_1},\frac{b}{y_1},1\}=\frac{b}{y_1}$;
Case (B.1). If $x_1+y_1-1>0$ and $a\geqslant x_1$ and $b\geqslant y_1$, then $z'(x_1,y_1)=x_1+y_1-1$. In this case $z'(x_1,y_1)=\min_{(x,y)\in\mathcal{A}}z'(x,y)$. Indeed, as $x\geqslant x_1$ and $y\geqslant y_1$, it holds that $x+y-1\geqslant x_1+y_1-1>0$, from Table 2, we can have the following three cases: (i) $z'(x,y)=x+y-1$, (ii) $z'(x,y)=\frac{a}{x}(x+y-1)$, (iii) $z'(x,y)=\frac{b}{y}(x+y-1)$.

(i). Of course, $z'(x,y) = x + y - 1 \geqslant x_1 + y_1 - 1 = z'(x_1, y_1)$.
(ii). We observe that $\frac{a}{x}(x + y - 1)$ is non-decreasing in the arguments x and y. Then, as $a \geqslant x_1$, it holds that

$$z'(x,y) = \tfrac{a}{x}(x + y - 1) \geqslant \tfrac{a}{x}(x + y_1 - 1) \geqslant \tfrac{a}{x_1}(x_1 + y_1 - 1) \geqslant (x_1 + y_1 - 1) = z'(x_1, y_1).$$

(iii). Similar to the reasoning exploited in (ii) it can be easily proved that $z'(x,y) = \frac{b}{y}(x + y - 1) \geqslant (x_1 + y_1 - 1) = z'(x_1, y_1)$.
Case (B.2).
In this case $a < x_1$ and $\frac{a}{x_1} \leqslant \frac{b}{y_1}$. We show that $z'(x_1, y_1) = \frac{a}{x_1}(x_1 + y_1 - 1) = \min_{(x,y)\in\mathcal{A}} z'(x,y)$. As $x \geqslant x_1 > a$ and $y \geqslant y_1$, it holds that $x + y - 1 \geqslant x_1 + y_1 - 1 > 0$, and $a < x$. Then from Table 2, we can have the following cases: (i) $z'(x,y) = \frac{a}{x}(x + y - 1)$, (ii) $z'(x,y) = \frac{b}{y}(x + y - 1)$.
(i). As $\frac{a}{x}(x + y - 1)$ is non-decreasing in the arguments x and y it holds that $z'(x_1, y_1) = \frac{a}{x_1}(x_1 + y_1 - 1) \leqslant \frac{a}{x}(x + y - 1) = z'(x,y)$.
(ii). We observe that $\frac{b}{y}(x + y - 1)$ is non-decreasing in the arguments x and y. Then, as $\frac{b}{y_1} \geqslant \frac{a}{x_1}$, it holds that

$$\tfrac{b}{y}(x + y - 1) \geqslant \tfrac{b}{y_1}(x_1 + y_1 - 1) \geqslant \tfrac{a}{x_1}(x_1 + y_1 - 1) = z'(x_1, y_1).$$

Case (B.3). By applying a reasoning similar to the case *(B.2)* it can be shown that $z'(x_1, y_1) = \frac{b}{y_1}(x_1 + y_1 - 1) = \min_{(x,y)\in\mathcal{A}} z'(x,y)$.
Upper bound. We distinguish two cases: **(C)** $a(1 - y_2) + b(1 - x_2) + x_2 y_2 - 1 > 0$; **(D)** $a(1 - y_2) + b(1 - x_2) + x_2 y_2 - 1 \leqslant 0$.

Case (**C**). From Table 3 it holds that

$$z^{**} = z''(x_2, y_2) = \frac{x_2(b - a y_2) + y_2(a - b x_2)}{1 - x_2 y_2}. \tag{A.11}$$

We show that $z^{**} = z''(x_2, y_2) \geqslant z''(x,y)$ for every $(x,y) \in \mathcal{A}$. Let be given $(x,y) \in \mathcal{A}$. We distinguish the following cases: (i) $a(1 - y) + b(1 - x) + xy - 1 > 0$; (ii) $a(1 - y) + b(1 - x) + xy - 1 \leqslant 0$, $x \leqslant y$, $a \leqslant x$; (iii) $a(1 - y) + b(1 - x) + xy - 1 \leqslant 0$, $x \leqslant y$, $a > x$ (iv) $a(1 - y) + b(1 - x) + xy - 1 \leqslant 0$, $y < x$, $b \leqslant y$; (v) $a(1 - y) + b(1 - x) + xy - 1 \leqslant 0$, $y < x$, $b > y$.
(i) We have that $z''(x,y) = z_2''(x,y)$. From Equation (A.5), as $b - ay \geqslant 0$, it holds that

$$z_2''(x,y) - z_2''(x_2, y) = -\tfrac{(x_2 - x)(1 - y)(b - a y)}{(1 - x y)(1 - x_2 y)} \leqslant 0, \tag{A.12}$$

that is $z_2''(x,y) \leqslant z_2''(x_2, y)$. Moreover, as $a - b x_2 \geqslant 0$ (see Equation (A.5)), it holds that

$$z_2''(x_2, y) - z_2''(x_2, y_2) = -\tfrac{(y_2 - y)(1 - x_2)(a - b x_2)}{(1 - x_2 y)(1 - x_2 y_2)} \leqslant 0. \tag{A.13}$$

Then, $z''(x,y) = z_2''(x,y) \leqslant z_2''(x_2, y) \leqslant z_2''(x_2, y_2) = z^{**}$.
(ii) We have that $z''(x,y) = z_1''(x,y) = x$. Moreover, by recalling (30), it holds that

$$z''(x,y) = x = \min\{x, y\} \leqslant \min\{x_2, y_2\} == z_1''(x_2, y_2) \leqslant z_2''(x_2, y_2).$$

Then, $z''(x,y) \leqslant z_2''(x_2, y_2)$.
(iii) We have that $z''(x,y) = z_3''(x,y)$. We observe that

$$z_3''(x,y) - z_3''(x_2, y) = -\tfrac{(x_2 - x)(1 - a)(1 - y)}{(1 - x)(1 - x_2)} \leqslant 0 \tag{A.14}$$

and that

$$z_3''(x_2, y) - z_3''(x_2, y_2) = -\tfrac{(a - x_2)(y_2 - y)}{1 - x_2} \leqslant 0 \tag{A.15}$$

because $a > x_2$. Then, by recalling (30), it holds that $z''(x,y) = z_3''(x,y) \leqslant z_3''(x_2, y) \leqslant z_3''(x_2, y_2) \leqslant z_2''(x_2, y_2) = z^{**}$.
(iv) We have that $z''(x,y) = z_1''(x,y) = y$ and, as in case (ii), it can be shown that $z''(x,y) \leqslant z_2''(x_2, y_2) = z^{**}$.
(v) We have that $z''(x,y) = z_4''(x,y)$. As in case (iii), it can be shown that $z''(x,y) = z_4''(x,y) \leqslant z_4''(x_2, y) \leqslant z_4''(x_2, y_2) \leqslant z_2''(x_2, y_2) = z^{**}$.

Case **(D)**. From Table 3 we can distinguish the following situations: *(D.1)* $x_2 \leqslant y_2$ and $a \leqslant x_2$; *(D.2)* $x_2 \leqslant y_2$ and $a > x_2$; *(D.3)* $y_2 < x_2$ and $b \leqslant y_2$; *(D.4)* $y_2 < x_2$ and $b > y_2$.
(D.1) It holds that

$$z^{**} = z''(x_2, y_2) = z_1''(x_2, y_2) = x_2.$$

We show that $z^{**} = z''(x_2, y_2) \geqslant z''(x,y)$ for every $(x,y) \in \mathcal{A}$. Let $(x,y) \in \mathcal{A}$ be given. We distinguish the following cases: (i) $a(1 - y) + b(1 - x) + xy - 1 > 0$; (ii) $a(1 - y) + b(1 - x) + xy - 1 \leqslant 0$, $x \leqslant y$, $a \leqslant x$; (iii) $a(1 - y) + b(1 - x) + xy - 1 \leqslant 0$, $x \leqslant y$, $a > x$ (iv) $a(1 - y) + b(1 - x) + xy - 1 \leqslant 0$, $y < x$, $b \leqslant y$; (v) $a(1 - y) + b(1 - x) + xy - 1 \leqslant 0$, $y < x$, $b > y$.

(i) We have that $z''(x,y) = z_2''(x,y)$. Moreover,

$$z_2''(x,y) - a = -\tfrac{(1 - y)(a - b x)}{1 - x y} \leqslant 0$$

because $a - bx > 0$. Then,

$$z_2''(x,y) \leqslant a \leqslant x_2 = z''(x,y) = z^{**}.$$

(ii) We have that $z''(x,y) = z_1''(x,y) = x \leqslant x_2 = z''(x,y) = z^{**}$.
(iii) We have that $z''(x,y) = z_3''(x,y)$. We observe that

$$z_3''(x,y) - a = -\tfrac{(a-x)\,(1-y)}{1-x} \leqslant 0$$

because $a > x$. Then

$$z_3''(x,y) \leqslant a \leqslant x_2 = z''(x,y) = z^{**}.$$

(iv) We have that $z''(x,y) = z_1''(x,y) = y < x \leqslant x_2 = z_1''(x_2,y_2) = z^{**}$.
(v) We have that $z''(x,y) = z_4''(x,y)$. As $y < x$, we obtain

$$z_4''(x,y) - x = -\tfrac{(x-y)\,(1-b)}{1-y} \leqslant 0$$

Then

$$z_4''(x,y) \leqslant x \leqslant x_2 = z''(x,y) = z^{**}.$$

(D.2) It holds that

$$z^{**} = z''(x_2,y_2) = z_3''(x_2,y_2) = \frac{x_2(1-a) + y_2(a-x_2)}{1-x_2}.$$

We show that $z^{**} = z''(x_2,y_2) = \frac{x_2(1-a)+y_2(a-x_2)}{1-x_2} \geqslant z''(x,y)$ for every $(x,y) \in \mathcal{A}$. Let $(x,y) \in \mathcal{A}$ be given. We distinguish the following cases: (i) $a(1-y)+b(1-x)+xy-1 > 0$; (ii) $a(1-y)+b(1-x)+xy-1 \leqslant 0$, $x \leqslant y$, $a \leqslant x$; (iii) $a(1-y)+b(1-x)+xy-1 \leqslant 0$, $x \leqslant y$, $a > x$ (iv) $a(1-y)+b(1-x)+xy-1 \leqslant 0$, $y < x$, $b \leqslant y$; (v) $a(1-y)+b(1-x)+xy-1 \leqslant 0$, $y < x$, $b > y$.
(i) We have that $z''(x,y) = z_2''(x,y)$. As $b - ay > 0$ and $a > x_2 \geqslant bx_2$, it follows from (A.12) and (A.13) that $z_2''(x,y) \leqslant z_2''(x_2,y_2)$. By recalling from Theorem 5 that in case *(D.2)* $z_2''(x_2,y_2) \leqslant z_3''(x_2,y_2)$, we obtain that $z_2''(x,y) \leqslant z_3''(x_2,y_2) = z^{**}$.
(ii) This subcase is not satisfied by any $(x,y) \in \mathcal{A}$. Indeed, we are considering $a > x_2$, then in this case $a > x_2 \geqslant x$ and hence it is not possible to have $a \leqslant x$.
(iii) We have that $z''(x,y) = z_3''(x,y)$. From relations (A.14) and (A.15) it holds that

$$z_3''(x,y) \leqslant z_3''(x_2,y_2) = z^{**}.$$

(iv) We have that, from (30), $z''(x,y) = z_1''(x,y) = y < x \leqslant x_2 = z_1''(x_2,y_2) \leqslant z_3''(x_2,y_2) = z^{**}$.
(v) We have that $z''(x,y) = z_4''(x,y)$. As $y < x$, we obtain

$$z_4''(x,y) - x = -\tfrac{(x-y)\,(1-b)}{1-y} \leqslant 0$$

Then, by (30),

$$z_4''(x,y) \leqslant x \leqslant x_2 = z_1''(x_2,y_2) \leqslant z_3''(x_2,y_2) = z''(x,y) = z^{**}.$$

(D.3) In this case, by following a reasoning similar to the case *(D.1)*, it can be proved that $z^{**} = z''(x_2,y_2) = z_1''(x_2,y_2) = y_2$.

(D.4) In a way similar to the one in case *(D.2)*, it can be proved that $z^{**} = z''(x_2,y_2) = z_4''(x_2,y_2) = \frac{x_2(b-y)+y_2(1-b)}{1-y_2}$. □

Data availability

No data was used for the research described in the article.

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