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Maximal consistent blocks-based optimistic and pessimistic probabilistic rough fuzzy sets and their applications in three-way multiple attribute decision-making ☆

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ABSTRACT

The integration of three-way decision (3WD) into multiple attribute decision-making (MADM) problems has emerged as a pivotal research area. 3WD can effectively manage the inherent uncertainty within the decision-making process. Additionally, it offers a semantic interpretation of the outcomes. In this paper, we introduce two innovative 3WD-MADM approaches, with a focus on granule selection and the handling of multi-type information in the framework of three-way decisions. Firstly, we construct maximal consistent blocks (MCBs)-based pessimistic and optimistic probabilistic rough fuzzy set (RFS) models and investigate their properties to ascertain their efficacy and reliability in decision-making contexts. Then, we define relative loss functions associated with "good state" and "bad state" scenarios. Building on this, we introduce four types of 3WDs based on our newly proposed optimistic and pessimistic probabilistic RFSs. Furthermore, we integrate the 3WDs information from both scenarios to formulate optimistic and pessimistic 3WD-MADM approaches, handling both single-valued fuzzy and intuitionistic fuzzy information. Finally, we contrast our proposed methodologies with the current MADM methods, and demonstrate their validity, significance and generalization ability.

1. Introduction

Multiple attribute decision-making (MADM) [14,41], as a pivotal component of modern decision-making science, is tasked with the responsibility of prioritizing a finite number of alternatives according to evaluation information of various attributes provided by decision-makers. Addressing the fuzziness and uncertainty in information to achieve effective group decision-making has long been a focal point in decision analysis. Rough set [32] and fuzzy set [48] theories, as two of the most powerful and significant tools of handling with imprecise, insufficient, uncertain and vague information, have been extensively applied in solving decision-making problems. Recently, Chang and Fu [3] developed a consistency mechanism based on interval-valued distribution preference relations

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regarding interval-valued fuzzy information. They compared alternatives by calculating the mean of odd-order moment intervals, effectively avoiding information loss. Bilişk et al. [2] proposed an improved TOPSIS method for interval-valued intuitionistic fuzzy information and applied it to the selection of optimal transportation modes during the product delivery phase. Qi and Atef et al. [37] constructed a Fermatean fuzzy β-covering rough set model regarding Fermatean fuzzy information, proposed the selection principles for the precision parameter β, and applied the newly established group decision-making method to the location selection of electric vehicle charging stations in a city in India. However, when confronted with the current mixed and diversified types of information, the applicability of aforementioned MADM methods, which are tailored for single information types, may be constrained. In this study, we will propose an innovative MADM method with strong generalization ability, which can handle not only single-valued but also intuitionistic fuzzy information.

Choosing a suitable granule is a key step and core element in establishing an expected rough set model. Under a non-equivalent relation, the obtained granules no longer form a partition of the universe but rather a covering. This can lead to misjudgments among objects between granules and even among objects within the same granule. One strategic approach to addressing this challenge involves employing the concept of maximal consistent blocks (MCBs) from the realm of discrete mathematics. Leung and Li [19] developed a rough set model based on MCBs, which achieves higher approximation accuracy and provides a new approach for addressing incomplete information system problems. Then, Liu and Shao [27], Leung et al. [20] and Mroczek [31] proposed several efficient algorithms for MCBs. Clark et al. [6] examined the use of MCBs in rule induction for incomplete datasets, showing that rule complexity varied with the interpretation of missing values. Sun and Wu et al. [35] proposed the use of MCBs as substitutes for common similarity classes for optimal scale selection in multi-scale (decision) information systems with numerical incompleteness. Li et al. [23] presented a rough-gain anomaly detection algorithm using MCBs to detect telecom fraud in incomplete data, achieving improved efficiency and data quality. Clark et al. [7] explored the application of MCBs in constructing probabilistic approximations for handling missing attribute values, compared eight approaches, and highlighted their varying effectiveness across different datasets. The aforementioned achievements have demonstrated the advantages of MCBs as fundamental granules and their broad applicability. However, their use has been largely confined to incomplete information systems. Sun and Mi [34] extended this approach to general information systems, initially exploring the application of rough set models based on MCBs in MADM and achieving promising results. Despite these advancements, the MADM method lacks interpretability, and the rough set models they proposed are unable to effectively partition the universe of discourse into three mutually exclusive regions. To address these issues, we will construct a new probabilistic rough fuzzy set (RFS) model and, with the aid of three-way decision (3WD) theory, establish a novel MADM method.

Since the introduction of 3WD theory by Yao [46,47] in 2010, which has rapidly gained traction and evolved significantly within the realms of rough set theory, granular computing, and their applications. For instance, in feature selection, Chen et al. [4] categorized samples into three decision types: acceptance, rejection, and non-commitment. By optimizing feature weights to maximize the weighted fuzzy margin, they proposed a method called robust weighted fuzzy margin-based feature selection. In clustering, Zhao et al. [53] divided samples into three domains: core, fringe, and trivial. They introduced a hybrid machine learning-based three-way soft clustering ensemble method. In data analysis, Li et al. [15] developed a novel data-driven approach based on the semantics of the advantageous, boundary, and disadvantageous regions of 3WD. Owing to its capacity to provide rational semantic interpretations for decision analysis, 3WD theory has been extensively applied to address MADM problems [8,18,24,51]. As the three core components of 3WD theory, the definition of loss functions, the selection of granules and the calculation of conditional probabilities are all necessary for scholars to be explored and discovered in the existing MADM methods with 3WD. However, these existing methods have some shortcomings: the six loss values in the classical theory [46] are subjectively arranged. In order to reduce human interference, Yao [47] introduced relative loss functions. Subsequently, Liu and Jia [16] provided an innovative framework to define relative losses by using evaluation values of alternatives with the attribute set in the MADM problem, which greatly reduces the number of subjectively arranged parameters. However, conditional probabilities in [16] are subjectively arranged. For this, Ye and Zhan [50] introduced a determination definition, which also appeared in the existing MADM models [33,34]. The fuzzy ϵ-neighborhood class provided by Ye and Zhan [50], as a granule, is actually the ϵ-cut set of a fuzzy β-minimal description. Practically, they are a kind of similarity classes, which are lower classification accuracy than MCBs. Recently, Wang et al. [40] proposed a novel loss function to avoid directly setting negative deviations to zero, taking into account the regret psychology of decision-makers. Jia et al. [15] introduced optimistic, pessimistic, and compromise strategies based on multiple equivalence relations and developed a multi-granulation variable-precision fuzzy rough set model. By integrating these strategies with classical 3WD theory, they explored a new MADM method. Although these methods have made progress in addressing the aforementioned issues, they are all very limited in the types of information systems they can handle. Inspired by these existing methods, we will provide a new three-way decision-based MADM (3WD-MADM) approach to overcome the above shortcomings.

From the above analysis, the highlight of this article has been summarized as follows:

1. The MCBs-based optimistic and pessimistic probabilistic RFS models are presented, which effectively optimize the unclear demarcation of positive, negative, and boundary regions.

2. Leveraging the cognitive traits of human perception in discerning superior and inferior objects, a data preprocessing method is developed that bifurcates data from both positive and negative perspectives.

3. The limitations of existing MADM methods are overcome by our proposed MCBs-based 3WD-MADM approaches, which boast stronger validity, significance, and generalization ability.

The rest of the research for this paper is organized as follows: In Section 2, we will revisit the concepts of MCBs and relative loss functions within the framework of 3WD theory. Moving on to Section 3, we introduce pessimistic and optimistic probabilistic RFS models, and delve into their respective properties. In Section 4, we investigate 3WD of good and bad states generated from these new rough sets. In Section 5, using two practical MADM problems as examples, we described describe procedures of optimistic and

Table 1
Matrix of loss functions corresponding to three actions.

	$X(P)$	$\neg X(N)$
$\tilde{a_P}$	$\hbar_{PP}$	$\hbar_{PN}$
$\tilde{a_B}$	$\hbar_{BP}$	$\hbar_{BN}$
$\tilde{a_N}$	$\hbar_{NP}$	$\hbar_{NN}$

Table 2
Matrix of relative loss functions.

	$X(P)$	$\neg X(N)$
$\tilde{a_P}$	0	$\hbar^{\star}_{PN}$
$\tilde{a_B}$	$\hbar^{\star}_{BP}$	$\hbar^{\star}_{BN}$
$\tilde{a_N}$	$\hbar^{\star}_{NP}$	0

pessimistic 3WD-MADM approaches. And then, we conduct comparisons between our methods and the existent MADM methods for handling single-valued and intuitionistic fuzzy information in Section 6. In the concluding section, we provide a summary of the research findings and explore potential directions for future work.

2. Preliminaries

2.1. Generalized maximal consistent blocks

Throughout this paper, let U be a nonempty finite set and let $\mathcal{P}(U)$ and $\mathcal{F}(U)$ be the powerset of U and the set of all fuzzy sets on U, respectively. Suppose that $R \subseteq U \times U$ is a tolerance relation on U, that is, a reflexive and symmetric relation on U. For each $\varpi \in U$, let $R(\varpi)$ denote the tolerance class of ϖ, that is, $R(\varpi) = \{\varrho \in U \mid (\varpi, \varrho) \in R\}$. In this paper, the pair (U, R) is designated as a tolerance approximation space (TAS).

Definition 2.1 *([19,34]).* Let (U, R) be a TAS, $B \subseteq U$. Then Y is called a consistent block provided that

$$\forall \varpi, \varrho \in B \Longrightarrow (\varpi, \varrho) \in R. \tag{2.1}$$

Further, a consistent block B is called maximal if there does not exist X, which is also a consistent block, such that $B \subset X$. A maximal consistent block is usually shortened by MCB.

The set of all MCBs on U is denoted by $MCB(R)$ and for each $\varpi \in U$, $MCB_{\varpi}(R) = \{Y_{\varpi} \in MCB(R) \mid \varpi \in Y_{\varpi}\}$. That is, the set of all MCBs containing ϖ is denoted by $MCB_{\varpi}(R)$.

Example 2.2. Let $U = \{\varpi_1, \varpi_2, \varpi_3\}$, where $\varpi_1 = 0.5$, $\varpi_2 = 0.7$ and $\varpi_3 = 0.9$, and let $R = \{(\varpi, \varrho) \in U \times U \mid |\varpi - \varrho| \leq 0.2\}$. Then R is a tolerance relation on U. For $\varpi_2 \in U$, its tolerance class is $R(\varpi_2) = \{\varpi_1, \varpi_2, \varpi_3\}$ and the set of its maximal consistent blocks is $MCB_{\varpi_2}(R) = \{\{\varpi_1, \varpi_2\}, \{\varpi_2, \varpi_3\}\}$. By $(\varpi_1, \varpi_3) \notin R$, $\{\varpi_1, \varpi_2, \varpi_3\}$ is not a correct classification with respect to R. This demonstrates that the classification accuracy of tolerance classes is not precise. By using MCB, such situations can be avoided.

2.2. The relative loss function in the theory of three-way decision (3WD)

Yao [46] introduced loss functions to 3WD theory, considering expert preferences. Yao [47] explored the membership of an object ϖ on U by setting the action set $\Lambda = \{\tilde{a_P}, \tilde{a_B}, \tilde{a_N}\}$ and the state set $\Omega = \{X, \neg X\}$. Here, $\varpi \in X$ and $\varpi \in \neg X$ indicate good and bad states, respectively, while $\tilde{a_P}$, $\tilde{a_B}$ and $\tilde{a_N}$ represent acceptance, waiting, and rejection actions. The loss functions regarding the risk (or cost) of three actions in two states are given by a 3×2 matrix, which is shown in Table 1. $\hbar_{\blacksquare P}$ is the loss by $a_{\blacksquare}$, when $\varpi \in X$. $\hbar_{\blacksquare N}$ is the loss by $a_{\blacksquare}$ when $\varpi \in \neg X$, where $\blacksquare$ represents P, B and N, and they satisfy $\hbar_{NN} \leq \hbar_{BN} < \hbar_{PN}$ and $\hbar_{PP} \leq \hbar_{BP} < \hbar_{NP}$. Subsequently, in order to reduce artificially given parameters, Yao [47] provided the relative loss functions as shown in Table 2. When an object is in X, the relative losses $\hbar^{\star}_{PP} = 0$, $\hbar^{\star}_{BP} = \hbar_{BP} - \hbar_{PP}$ and $\hbar^{\star}_{NP} = \hbar_{NP} - \hbar_{PP}$; When an object is in $\neg X$, the relative losses are $\hbar^{\star}_{PN} = \hbar_{PN} - \hbar_{NN}$, $\hbar^{\star}_{BN} = \hbar_{BN} - \hbar_{NN}$ and $\hbar^{\star}_{NN} = 0$. The expected loss $\tilde{\mathcal{L}}(a_{\blacksquare} | R(\varpi))$ is shown as follows:
$\tilde{\mathcal{L}}(\tilde{a_P} | R(\varpi)) = \hbar^{\star}_{PN} P(\neg X | R(\varpi))$;
$\tilde{\mathcal{L}}(\tilde{a_B} | R(\varpi)) = \hbar^{\star}_{BP} P(X | R(\varpi)) + \hbar^{\star}_{BN} P(\neg X | R(\varpi))$;
$\tilde{\mathcal{L}}(\tilde{a_N} | R(\varpi)) = \hbar^{\star}_{NP} P(X | R(\varpi))$,
where $P(X | R(\varpi))$ (resp. $P(\neg X | R(\varpi))$) denotes the conditional probability defined by

$$P(X | R(\varpi)) = \frac{|X \cap R(\varpi)|}{|R(\varpi)|}, \; P(\neg X | R(\varpi)) = \frac{|\neg X \cap R(\varpi)|}{|R(\varpi)|}, \tag{2.2}$$

satisfying $P(X|R(\varpi))+P(\neg X|R(\varpi))=1$. According to the minimum-risk decision rules induced by the Bayesian decision procedure, three decision rules are simplified as follows:

(P) If $P(X|R(\varpi))\geq\alpha$ and $P(X|R(\varpi))\geq\gamma$, then $\varpi\in POS(X)$;

(B) If $P(X|R(\varpi))<\alpha$ and $P(X|R(\varpi))>\beta$, then $\varpi\in BND(X)$;

(N) If $P(X|R(\varpi))\leq\beta$ and $P(X|R(\varpi))\leq\gamma$, then $\varpi\in ENG(X)$,

where α, β and γ are determined by

$$\alpha=\frac{\hbar^\star_{PN}-\hbar^\star_{BN}}{(\hbar^\star_{PN}-\hbar^\star_{BN})+\hbar^\star_{BP}}; \tag{2.3}$$

$$\beta=\frac{\hbar^\star_{BN}}{\hbar^\star_{BN}+(\hbar^\star_{NP}-\hbar^\star_{BP})}; \tag{2.4}$$

$$\gamma=\frac{\hbar^\star_{PN}}{\hbar^\star_{PN}+\hbar^\star_{NP}}. \tag{2.5}$$

3. MCBs-based optimistic and pessimistic probabilistic RFSs

According to MCB-based technique decision rules, Ma et al. [29] and Sun et al. [34] introduced the optimistic decision-theoretic rough set (DTRS) model and the pessimistic DTRS model, respectively. In this section, we will generalize these two models to the fuzzy setting.

For $\tilde{\mathcal{A}}\in\mathcal{F}(U)$ and $B\in\mathcal{P}(U)$, let

$$P(\tilde{\mathcal{A}}|B)=\frac{\sum_{\varrho\in B}\tilde{\mathcal{A}}(\varrho)}{|B|}, \tag{3.1}$$

where $|B|$ denotes the cardinality of B.

Definition 3.1. Suppose that (U,R) is a TAS and $0\leq\beta<\alpha\leq 1$. Define a lower approximation operator $\underline{R}^O_{MCB}:\mathcal{F}(U)\longrightarrow\mathcal{P}(U)$ and an upper approximation operator $\overline{R}^O_{MCB}:\mathcal{F}(U)\longrightarrow\mathcal{P}(U)$ by

$$\underline{R}^O_{MCB}(\tilde{\mathcal{A}})=\{\varpi\in U\mid P(\tilde{\mathcal{A}}|B^{max}_\varpi)\geq\alpha\},\ \forall\tilde{\mathcal{A}}\in\mathcal{F}(U), \tag{3.2}$$

$$\overline{R}^O_{MCB}(\tilde{\mathcal{A}})=\{\varpi\in U\mid P(\tilde{\mathcal{A}}|B^{max}_\varpi)>\beta\},\ \forall\tilde{\mathcal{A}}\in\mathcal{F}(U), \tag{3.3}$$

where

$$B^{max}_\varpi=\arg\max_{B_\varpi\in MCB_\varpi(R)}\{P(\tilde{\mathcal{A}}|B_\varpi)\}.$$

Then an optimistic probabilistic RFS of $\tilde{\mathcal{A}}$ based on MCBs is denoted by the pair $(\underline{R}^O_{MCB}(\tilde{\mathcal{A}}),\overline{R}^O_{MCB}(\tilde{\mathcal{A}}))$.

Remark 3.2. (1) When $A,B\in\mathcal{P}(U)$, it is readily confirmed that

$$P(A|B)=\frac{|A\cap B|}{|B|}.$$

Then the optimistic probabilistic RFSs in Definition 3.1 degenerates to optimistic DTRs in [29]. This demonstrates our optimistic probabilistic RFSs are reasonable extensions of optimistic DTRs in [29].

(2) When $\alpha=1$, $\beta=0$ and $A\in\mathcal{P}(U)$, then $\underline{R}^O_{MCB}(A)$ and $\overline{R}^O_{MCB}(A)$ in Definition 3.1 coincide with that in [5,19]. So our optimistic probabilistic RFS can also be considered as an extension of [5,19].

According to Definition 3.1, U is partitioned into the following distinct decision regions, which are delineated as follows:

$POS^O(\tilde{\mathcal{A}})=\underline{R}^O_{MCB}(\tilde{\mathcal{A}})=\{\varpi\in U\mid P(\tilde{\mathcal{A}}|B^{max}_\varpi)\geq\alpha\}$;

$BND^O(\tilde{\mathcal{A}})=\overline{R}^O_{MCB}(\tilde{\mathcal{A}})-\underline{R}^O_{MCB}(\tilde{\mathcal{A}})=\{\varpi\in U\mid\beta<P(\tilde{\mathcal{A}}|B^{max}_\varpi)<\alpha\}$;

$ENG^O(\tilde{\mathcal{A}})=U-\overline{R}^O_{MCB}(\tilde{\mathcal{A}})=\{\varpi\in U\mid P(\tilde{\mathcal{A}}|B^{max}_\varpi)\leq\beta\}$.

Thus, we can obtain the following three decision rules:

(P_1) If $P(\tilde{\mathcal{A}}|B^{max}_\varpi)\geq\alpha$, then $\varpi\in POS^O(\tilde{\mathcal{A}})$;

(B_1) If $\beta<P(\tilde{\mathcal{A}}|B^{max}_\varpi)<\alpha$, then $\varpi\in BND^O(\tilde{\mathcal{A}})$;

(N_1) If $P(\tilde{\mathcal{A}}|B^{max}_\varpi)\leq\beta$, then $\varpi\in ENG^O(\tilde{\mathcal{A}})$.

Example 3.3. Suppose that $U=\{\varpi_1,\varpi_2,\varpi_3,\varpi_4,\varpi_5,\varpi_6,\varpi_7\}$, $MCB_{\varpi_1}(R)=\{B_1=\{\varpi_1,\varpi_2\},B_2=\{\varpi_1,\varpi_3,\varpi_4,\varpi_5\},B_3=\{\varpi_1,\varpi_6,\varpi_7\}\}$ and $A=\{\varpi_1,\varpi_2,\varpi_4,\varpi_5\}$. When $\alpha=0.7$ and $\beta=0.4$, then $P(A|B_1)=1$, $P(A|B_2)=0.75$ and $P(A|B_3)=0.33$. According to the model in [29], ϖ_1 belongs to the positive region and negative region. However, in our model of Definition 3.1, ϖ_1 belongs to the positive region.

This example clearly reveals the limitations of the model in [29] in partitioning the universe, as it fails to accurately divide the universe into positive, negative and boundary regions. In contrast, our model effectively overcomes this issue and achieves precise decision classification.

Next, we will design the pessimistic case. That is to say, we will introduce a pessimistic probabilistic RFS based on MCBs.

Definition 3.4. Suppose that (U, R) is a TAS and $0 \le \beta < \alpha \le 1$. Define a lower approximation operator $\underline{R}^P_{MCB} : \mathcal{F}(U) \longrightarrow \mathcal{P}(U)$ and an upper approximation operator $\overline{R}^P_{MCB} : \mathcal{F}(U) \longrightarrow \mathcal{P}(U)$ by

$$\underline{R}^P_{MCB}(\tilde{\mathcal{A}}) = \{\varpi \in U \mid P(\tilde{\mathcal{A}}|B^{min}_{\varpi}) \ge \alpha\}, \forall \tilde{\mathcal{A}} \in \mathcal{F}(U), \tag{3.4}$$

$$\overline{R}^P_{MCB}(\tilde{\mathcal{A}}) = \{\varpi \in U \mid P(\tilde{\mathcal{A}}|B^{min}_{\varpi}) > \beta\}, \forall \tilde{\mathcal{A}} \in \mathcal{F}(U), \tag{3.5}$$

where

$$B^{min}_{\varpi} = \arg \min_{B_{\varpi} \in MCB_{\varpi}(R)} \{P(\tilde{\mathcal{A}}|B_{\varpi})\}.$$

Then a pessimistic probabilistic RFS of $\tilde{\mathcal{A}}$ based on MCBs id denoted by the pair $(\underline{R}^P_{MCB}(\tilde{\mathcal{A}}), \overline{R}^P_{MCB}(\tilde{\mathcal{A}}))$.

Remark 3.5. In particular, when $A \in \mathcal{P}(U)$, it can be easily confirmed that $\underline{R}^P_{MCB}(A)$ and $\overline{R}^P_{MCB}(A)$ coincide with Sun et al. [34]. So Definition 3.4 can be restricted to pessimistic DTRSs in the sense of [34]. This shows our pessimistic probabilistic RFSs can be treated as reasonable extensions of pessimistic DTRSs.

Similarly to the optimistic model in Definition 3.1, U is partitioned into the following distinct decision regions derived from Definition 3.4, which are delineated as follows:

$POS^P(\tilde{\mathcal{A}}) = \underline{R}^P_{MCB}(\tilde{\mathcal{A}}) = \{\varpi \in U \mid P(\tilde{\mathcal{A}}|B^{min}_{\varpi}) \ge \alpha\}$;

$BND^P(\tilde{\mathcal{A}}) = \overline{R}^P_{MCB}(\tilde{\mathcal{A}}) - \underline{R}^P_{MCB}(\tilde{\mathcal{A}}) = \{\varpi \in U \mid \alpha < P(\tilde{\mathcal{A}}|B^{min}_{\varpi}) < \beta\}$;

$ENG^P(\tilde{\mathcal{A}}) = U - \overline{R}^P_{MCB}(\tilde{\mathcal{A}}) = \{\varpi \in U \mid P(\tilde{\mathcal{A}}|B^{min}_{\varpi}) \le \beta\}$.

Thus, we can obtain another three decision rules:

(P_2) If $P(\tilde{\mathcal{A}}|B^{min}_{\varpi}) \ge \alpha$, then $\varpi \in POS^P(\tilde{\mathcal{A}})$;

(B_2) If $\beta < P(\tilde{\mathcal{A}}|B^{min}_{\varpi}) < \alpha$, then $\varpi \in BND^P(\tilde{\mathcal{A}})$;

(N_2) If $P(\tilde{\mathcal{A}}|B^{min}_{\varpi}) \le \beta$, then $\varpi \in ENG^P(\tilde{\mathcal{A}})$.

According to Example 3.3, in the model of [34], ϖ_1 does not belong to any of the three regions. However, in the model of Definition 3.4, ϖ_1 belongs to the negative region. This fully demonstrates the necessity of proposing our optimistic and pessimistic probabilistic RFS models in solving decision classification problems.

Proposition 3.6. *Suppose that (U, R) is a TAS and $0 \le \alpha < \beta \le 1$. For each $\tilde{\mathcal{A}} \in \mathcal{F}(U)$, we have*

(1) $\underline{R}^P_{MCB}(\tilde{\mathcal{A}}) \subseteq \underline{R}^O_{MCB}(\tilde{\mathcal{A}})$;
(2) $\overline{R}^P_{MCB}(\tilde{\mathcal{A}}) \subseteq \overline{R}^O_{MCB}(\tilde{\mathcal{A}})$.

Proof. It is easy to be verified and omitted. □

In accordance with Proposition 3.6, the optimistic model and the pessimistic model each possess distinct advantages. The optimistic model is characterized by a larger lower approximation, which enables it to offer more inclusive solutions in certain application scenarios and thus more comprehensively cover potential possibilities. In contrast, the pessimistic model is distinguished by a smaller upper approximation, making it particularly effective in situations where precise definition and strict risk control are required, as it can effectively avoid potential issues caused by overestimation. Therefore, these two models each play a different key role in practical applications, providing diverse support for decision-making in various needs and objectives.

Theorem 3.7. *Suppose that (U, R) is a TAS and $0 \le \alpha < \beta \le 1$. Then the following properties hold:*

(L1) *If $\tilde{\mathcal{A}}_1 \le \tilde{\mathcal{A}}_2$, then $\underline{R}^O_{MCB}(\tilde{\mathcal{A}}_1) \subseteq \underline{R}^O_{MCB}(\tilde{\mathcal{A}}_2)$ and $\underline{R}^P_{MCB}(\tilde{\mathcal{A}}_1) \subseteq \underline{R}^P_{MCB}(\tilde{\mathcal{A}}_2)$;*
(U1) *If $\tilde{\mathcal{A}}_1 \le \tilde{\mathcal{A}}_2$, then $\overline{R}^O_{MCB}(\tilde{\mathcal{A}}_1) \subseteq \overline{R}^O_{MCB}(\tilde{\mathcal{A}}_2)$ and $\overline{R}^P_{MCB}(\tilde{\mathcal{A}}_1) \subseteq \overline{R}^P_{MCB}(\tilde{\mathcal{A}}_2)$;*
(L2) $\underline{R}^O_{MCB}(\bigcap_{i\in\Gamma} \tilde{\mathcal{A}}_i) \subseteq \bigcap_{i\in\Gamma} \underline{R}^O_{MCB}(\tilde{\mathcal{A}}_i)$, $\underline{R}^P_{MCB}(\bigcap_{i\in\Gamma} \tilde{\mathcal{A}}_i) \subseteq \bigcap_{i\in\Gamma} \underline{R}^P_{MCB}(\tilde{\mathcal{A}}_i)$ *for each subfamily $\{\tilde{\mathcal{A}}_i \mid i \in \Gamma\} \subseteq \mathcal{F}(U)$;*

(U2) $\overline{R}^O_{MCB}(\bigcap_{i\in\Gamma}\tilde{\mathcal{A}}_i) \subseteq \bigcap_{i\in\Gamma}\underline{R}^O_{MCB}(\tilde{\mathcal{A}}_i)$, $\overline{R}^P_{MCB}(\bigcap_{i\in\Gamma}\tilde{\mathcal{A}}_i) \subseteq \bigcap_{i\in\Gamma}\overline{R}^P_{MCB}(\tilde{\mathcal{A}}_i)$ *for each subfamily* $\{\tilde{\mathcal{A}}_i \mid i\in\Gamma\}\subseteq\mathcal{F}(U)$*;*

(L3) $\bigcup_{i\in\Gamma}\underline{R}^O_{MCB}(\tilde{\mathcal{A}}_i) \subseteq \underline{R}^O_{MCB}(\bigcup_{i\in\Gamma}\tilde{\mathcal{A}}_i)$, $\bigcup_{i\in\Gamma}\underline{R}^P_{MCB}(\tilde{\mathcal{A}}_i) \subseteq \underline{R}^P_{MCB}(\bigcup_{i\in\Gamma}\tilde{\mathcal{A}}_i)$ *for each subfamily* $\{\tilde{\mathcal{A}}_i \mid i\in\Gamma\}\subseteq\mathcal{F}(U)$*;*

(U3) $\bigcup_{i\in\Gamma}\overline{R}^O_{MCB}(\tilde{\mathcal{A}}_i) \subseteq \overline{R}^O_{MCB}(\bigcup_{i\in\Gamma}\tilde{\mathcal{A}}_i)$, $\bigcup_{i\in\Gamma}\overline{R}^P_{MCB}(\tilde{\mathcal{A}}_i) \subseteq \overline{R}^P_{MCB}(\bigcup_{i\in\Gamma}\tilde{\mathcal{A}}_i)$ *for each subfamily* $\{\tilde{\mathcal{A}}_i \mid i\in\Gamma\}\subseteq\mathcal{F}(U)$*;*

(L4) *If* $\beta = 1-\alpha$*, then* $\underline{R}^O_{MCB}(\tilde{\mathcal{A}}) = \neg\overline{R}^P_{MCB}(\neg\tilde{\mathcal{A}})$ *and* $\underline{R}^P_{MCB}(\tilde{\mathcal{A}}) = \neg\overline{R}^O_{MCB}(\neg\tilde{\mathcal{A}})$ *for each* $\tilde{\mathcal{A}}\in\mathcal{F}(U)$*;*

(U4) *If* $\beta = 1-\alpha$*, then* $\overline{R}^O_{MCB}(\tilde{\mathcal{A}}) = \neg\underline{R}^P_{MCB}(\neg\tilde{\mathcal{A}})$ *and* $\overline{R}^P_{MCB}(\tilde{\mathcal{A}}) = \neg\underline{R}^O_{MCB}(\neg\tilde{\mathcal{A}})$ *for each* $\tilde{\mathcal{A}}\in\mathcal{F}(U)$*.*

Proof. (L1) Take each $\tilde{\mathcal{A}}_1, \tilde{\mathcal{A}}_2 \in \mathcal{F}(U)$ such that $\tilde{\mathcal{A}}_1 \le \tilde{\mathcal{A}}_2$, i.e., $\tilde{\mathcal{A}}_1(\varrho) \le \tilde{\mathcal{A}}_2(\varrho)$ for each $\varrho\in U$. This implies that for each $B\in\mathcal{P}(U)$,

$$\sum_{\varrho\in B}\tilde{\mathcal{A}}_1(\varrho) \le \sum_{\varrho\in B}\tilde{\mathcal{A}}_2(\varrho).$$

According to formula (3.1), we have

$$P(\tilde{\mathcal{A}}_1|B) = \frac{\sum_{\varrho\in B}\tilde{\mathcal{A}}_1(\varrho)}{|B|} \le \frac{\sum_{\varrho\in B}\tilde{\mathcal{A}}_2(\varrho)}{|B|} = P(\tilde{\mathcal{A}}_2|B).$$

This shows $P(\tilde{\mathcal{A}}_1|B) \le P(\tilde{\mathcal{A}}_2|B)$, for each $\tilde{\mathcal{A}}_1 \le \tilde{\mathcal{A}}_2$.

Let $\varpi \in \underline{R}^O_{MCB}(\tilde{\mathcal{A}}_1)$. According to formula (3.2) in Definition 3.1, we have $P(\tilde{\mathcal{A}}_1|B^{max}_\varpi) \ge \alpha$, where $B^{max}_\varpi = \arg\max_{B_\varpi\in MCB_\varpi(R)}\{P(\tilde{\mathcal{A}}_1|B_\varpi)\}$. Thus, we obtain that $P(\tilde{\mathcal{A}}_2|B^{max}_\varpi) \ge P(\tilde{\mathcal{A}}_1|B^{max}_\varpi)$. Let $C^{max}_\varpi = \arg\max_{B_\varpi\in MCB_\varpi(R)}\{P(\tilde{\mathcal{A}}_2|B_\varpi)\}$. Then

$$P(\tilde{\mathcal{A}}_2|C^{max}_\varpi) \ge P(\tilde{\mathcal{A}}_2|B^{max}_\varpi).$$

So we can obtain that

$$P(\tilde{\mathcal{A}}_2|C^{max}_\varpi) \ge P(\tilde{\mathcal{A}}_2|B^{max}_\varpi) \ge P(\tilde{\mathcal{A}}_1|B^{max}_\varpi) \ge \alpha.$$

Because of $\underline{R}^O_{MCB}(\tilde{\mathcal{A}}_2) = \{\mu\in U \mid P(\tilde{\mathcal{A}}_2|C^{max}_\mu) \ge \alpha\}$, we have $\varpi \in \underline{R}^O_{MCB}(\tilde{\mathcal{A}}_2)$. Thus, we can obtain that $\underline{R}^O_{MCB}(\tilde{\mathcal{A}}_1) \subseteq \underline{R}^O_{MCB}(\tilde{\mathcal{A}}_2)$. Similarly, $\underline{R}^P_{MCB}(\tilde{\mathcal{A}}_1) \subseteq \underline{R}^P_{MCB}(\tilde{\mathcal{A}}_2)$.

(U1) The proof of (L1) can be adopted.

(L2) The proof can be given applying (L1).

(U2) The proof can be given applying (U1).

(L3) The proof can be given applying (L1).

(U3) The proof can be given applying (U1).

(L4) For each $\tilde{\mathcal{A}}\in\mathcal{F}(U)$, $\varpi\in U$ and $B\in MCB_\varpi(R)$, we have

$$\begin{aligned} P(\tilde{\mathcal{A}}|B) &= \frac{\sum_{\varrho\in B}\tilde{\mathcal{A}}(\varrho)}{|B|} \\ &= \frac{|B| - \sum_{\varrho\in B}\neg\tilde{\mathcal{A}}(\varrho)}{|B|} \\ &= 1 - P(\neg\tilde{\mathcal{A}}|B). \end{aligned}$$

Then, it can be easy to confirm that

$$\begin{aligned} \max\{P(\neg\tilde{\mathcal{A}}|B) \mid B \in MCB_\varpi(R)\} &= \max\{1 - P(\tilde{\mathcal{A}}|B) \mid B\in MCB_\varpi(R)\} \\ &= 1 - \min\{P(\tilde{\mathcal{A}}|B) \mid B\in MCB_\varpi(R)\}. \end{aligned}$$

Let $B^{max}_\varpi = \arg\max_{B\in MCB_\varpi(R)}\{P(\neg\tilde{\mathcal{A}}|B)\}$ and $B^{min}_\varpi = \arg\min_{B\in MCB_\varpi(R)}\{P(\tilde{\mathcal{A}}|B)\}$. Then we have $P(\neg\tilde{\mathcal{A}}|B^{max}_\varpi) + P(\tilde{\mathcal{A}}|B^{min}_\varpi) = 1$. Hence, we can obtain that

$$\begin{aligned} \neg\overline{R}^O_{MCB}(\neg\tilde{\mathcal{A}}) &= U - \{\varpi\in U \mid P(\neg\tilde{\mathcal{A}}|B^{max}_\varpi) > \beta\} \\ &= U - \{\varpi\in U \mid 1 - P(\tilde{\mathcal{A}}|B^{min}_\varpi) > \beta\} \\ &= U - \{\varpi\in U \mid P(\tilde{\mathcal{A}}|B^{min}_\varpi) < \alpha\} \qquad (\text{By } \beta = 1-\alpha) \\ &= \{\varpi\in U \mid P(\tilde{\mathcal{A}}|B^{min}_\varpi) \ge \alpha\} \\ &= \underline{R}^P_{MCB}(\tilde{\mathcal{A}}). \end{aligned}$$

Similarly, $\underline{R}^P_{MCB}(\tilde{\mathcal{A}}) = \neg\overline{R}^O_{MCB}(\neg\tilde{\mathcal{A}})$.

(U4) The proof of (L2) can be adopted. □

Remark 3.8. In the case that $\beta = 1-\alpha$, the lower and upper approximation operators w.r.t. optimistic probabilistic RFSs become

$$\underline{VR}^{O}_{MCB}(\tilde{\mathcal{A}}) = \{\varpi \in U \mid P(\tilde{\mathcal{A}}|B^{max}_{\varpi}) \geq \alpha\}, \forall \tilde{\mathcal{A}} \in \mathcal{F}(U), \tag{3.6}$$

$$\overline{VR}^{O}_{MCB}(\tilde{\mathcal{A}}) = \{\varpi \in U \mid P(\tilde{\mathcal{A}}|B^{max}_{\varpi}) > 1-\alpha\}, \forall \tilde{\mathcal{A}} \in \mathcal{F}(U). \tag{3.7}$$

Then an optimistic variable precision RFS based on MCBs is denoted by the pair $(\underline{VR}^{O}_{MCB}(\tilde{\mathcal{A}}), \overline{VR}^{O}_{MCB}(\tilde{\mathcal{A}}))$. Similarly, the pessimistic model in Definition 3.4 becomes

$$\underline{VR}^{P}_{MCB}(\tilde{\mathcal{A}}) = \{\varpi \in U \mid P(\tilde{\mathcal{A}}|B^{min}_{\varpi}) \geq \alpha\}, \forall \tilde{\mathcal{A}} \in \mathcal{F}(U), \tag{3.8}$$

$$\overline{VR}^{P}_{MCB}(\tilde{\mathcal{A}}) = \{\varpi \in U \mid P(\tilde{\mathcal{A}}|B^{min}_{\varpi}) > 1-\alpha\}, \forall \tilde{\mathcal{A}} \in \mathcal{F}(U). \tag{3.9}$$

Then a pessimistic variable precision RFS based on MCBs is denoted by the pair $(\underline{VR}^{P}_{MCB}(\tilde{\mathcal{A}}), \overline{VR}^{P}_{MCB}(\tilde{\mathcal{A}}))$.

4. Two kinds of 3WDs generated from positive and negative perspectives

From the positive perspective and the negative perspective. We will plumb a complete framework for formulating a set of 3WD rules, and investigate the relationship between these two kinds of 3WD rules under single valued and intuitionistic fuzzy information systems. First of all, we will introduce a pair of definitions, which are a positive ideal decision objects named by a qualified fuzzy set and a negative ideal decision objects named by an unqualified fuzzy set, respectively.

4.1. A pair of ideal decision object sets derived from positive and negative perspectives

Before introducing definitions of the two ideal decision object sets, we recall notions of a fuzzy information system and a multi-attribute decision-making (MADM) information system.

Definition 4.1 *([9,38]).* Suppose that $U = \{\varpi_1, \varpi_2, ..., \varpi_n\}$ is a nonempty finite set of objects, $C = \{C_1, C_2, ..., C_m\} \subseteq \mathcal{F}(U)$ is a nonempty finite set of attributes and $V = \bigcup_{j=1,2,...,m} V_{C_j} = \{\varpi_{ij} \mid i = 1,2,...,n,\ j = 1,2,...,m\} \subseteq [0,1]$, where V_{C_j} is the value domain of C_j. Then (U, C) is called a fuzzy information system.

The MADM problem refers to a decision maker who evaluates some alternative objects based on several evaluation criteria and uses this evaluation information to choose the optimal alternative objects.

Definition 4.2. Suppose that (U, AT, V, ω) denotes an MADM information system (also called a data table). That is, $U = \{\varpi_i \mid i = 1,2,...,n\}$ is the nonempty universe, $AT = \{a_j \mid j = 1,2,...,m\}$ denotes an evaluation criterion (attribute) set, $\omega = \{\omega_1, \omega_2, ..., \omega_m\}$ denotes the weight vector of AT and $V = \{u_{ij} \mid i = 1,2,...,n,\ j = 1,2,...,m\}$, where u_{ij} denotes the evaluation value (benefit value[1]) of ϖ_i with respect to a_j.

In an MADM information system (U, AT, V, ω), decision makers usually cannot directly determine the optimal object without data processing. However, they can find some objects in the universe that are better than the rest and some objects that are worse than the rest. According to this actual situation, we call the corresponding sets positive and negative ideal decision object sets, respectively. In reality, accurately ascertaining whether an object ϖ_i belongs to positive (negative) ideal decision object set is a formidable challenge. Actually, in practical problems, a fuzzy set can better reflect the uncertainty description of the positive (negative) ideal decision object set.

Explicitly, let $\tilde{Q}^{+}_{j}(\varpi_i)$ ($\tilde{Q}^{-}_{j}(\varpi_i)$) represent the degree to which ϖ_i belongs to a positive (negative) ideal decision object set with respect to a_j. This determines a fuzzy set $\tilde{Q}^{+}_{j}(\tilde{Q}^{-}_{j}) : U \longrightarrow [0,1]$, which is called a qualified (an unqualified) fuzzy set with respect to single attribute a_j. By aggregating different weights of attributes in (U, AT, V, ω), define a qualified fuzzy set $\tilde{Q}^{+} : U \longrightarrow [0,1]$ and an unqualified fuzzy set $\tilde{Q}^{-} : U \longrightarrow [0,1]$ as follows:

$$\tilde{Q}^{+}(\varpi_i) = \sum_{j=1}^{m} \omega_j \tilde{Q}^{+}_{j}(\varpi_i), \forall \varpi_i \in U, \tag{4.1}$$

$$\tilde{Q}^{-}(\varpi_i) = \sum_{j=1}^{m} \omega_j \tilde{Q}^{-}_{j}(\varpi_i), \forall \varpi_i \in U, \tag{4.2}$$

where $\tilde{Q}^{+}(\varpi_i)$ and $\tilde{Q}^{-}(\varpi_i)$ are denoted as degrees to which ϖ_i belongs to positive and negative ideal decision object sets with respect to AT, respectively.

[1] The benefit attribute is defined as the positive characteristics associated with the decision-making process. It emphasizes benefits derived from this particular attribute. In contrast, the cost attribute denotes the negative characteristic.

Table 3
An example of medical diagnosis in [49].

U/AT	a_1	a_2	a_3	a_4
ϖ_1	0.7	0.6	0.5	0.4
ϖ_2	0.8	1	0.65	0.8
ϖ_3	0.6	0.25	0.3	0.3
ϖ_4	0.45	0.6	0.7	0.9
ϖ_5	0.6	0.9	0.8	0.6
ϖ_6	0.75	0.4	0.7	0.5

Table 4
(U, C^+) in Example 4.3.

U/AT	a_1	a_2	a_3	a_4
ϖ_1	0.7	0.6	0.5	0.4
ϖ_2	0.8	1	0.65	0.8
ϖ_3	0.6	0.25	0.3	0.3
ϖ_4	0.45	0.6	0.7	0.9
ϖ_5	0.6	0.9	0.8	0.6
ϖ_6	0.75	0.4	0.7	0.5

Table 5
(U, C^-) in Example 4.3.

U/AT	a_1	a_2	a_3	a_4
ϖ_1	0.3	0.4	0.5	0.6
ϖ_2	0.2	0	0.35	0.2
ϖ_3	0.4	0.75	0.7	0.7
ϖ_4	0.55	0.4	0.3	0.1
ϖ_5	0.4	0.1	0.2	0.4
ϖ_6	0.25	0.6	0.3	0.5

Table 6
$\tilde{Q}^+$ and $\tilde{Q}^-$ in Example 4.3.

U/AT	$\tilde{Q}^+$	$\tilde{Q}^-$
ϖ_1	0.555	0.445
ϖ_2	0.805	0.195
ϖ_3	0.363	0.637
ϖ_4	0.653	0.347
ϖ_5	0.735	0.265
ϖ_6	0.597	0.403

Example 4.3. Suppose that $U = \{\varpi_1, \varpi_2, ..., \varpi_6\}$ denotes the universe consisting of six patients and $AT = \{a_1, a_2, a_3, a_4\}$ represents an attribute set of four flu symptoms which cough, body temperature, muscle pain and represents headache in turn. Table 3 derived from Ye et al. [49] shows that a doctor provided evaluation information u_{ij} for each patient ϖ_i $(i = \{1, 2, ..., 6\})$ with each symptom a_j $(j = \{1, 2, 3, 4\})$. We assume the weight vector of attributes $\omega = \{0.25, 0.25, 0.3, 0.2\}$. It is an MADM information system (U, AT, V, ω) with single-valued fuzzy information on $[0, 1]$. Then for each symptom a_j, we can give a qualified fuzzy set $\tilde{Q}_j^+$, which is defined by

$$\tilde{Q}_j^+(\varpi_i) = u_{ij}, \forall \varpi_i \in U, \tag{4.3}$$

and an unqualified fuzzy set $\tilde{Q}_j^-$, which is defined by

$$\tilde{Q}_j^-(\varpi_i) = 1 - u_{ij}, \forall \varpi_i \in U. \tag{4.4}$$

Let C^+ and C^- be denoted as the sets of all $\tilde{Q}_j^+$ and $\tilde{Q}_j^-$, $j = \{1, 2, 3, 4\}$. Thus, we can obtain two fuzzy information systems (U, C^-) and (U, C^+) as shown in Tables 4 and 5, where the evaluation information values of ϖ_i under a_j in (U, C^-) and (U, C^+) are $\varpi_{ij}^- = \tilde{Q}_j^-(\varpi_i) = 1 - u_{ij}$ and $\varpi_{ij}^+ = \tilde{Q}_j^+(\varpi_i) = u_{ij}$, respectively.

In Example 4.3, it is shown that an MADM information system (U, AT, V, ω) can be divided into two fuzzy information systems (U, C^+) and (U, C^-) and $\tilde{Q}_j^+(\varpi_i) + \tilde{Q}_j^-(\varpi_i) = 1$ for each $\varpi_i \in U$, that is, $\tilde{Q}_j^- = \neg\tilde{Q}_j^+$. Then $\tilde{Q}^+$ and $\tilde{Q}^-$ with respect to AT are shown in Table 6. This shows that $\tilde{Q}^+(\varpi_i) + \tilde{Q}^-(\varpi_i) = 1$ for each $\varpi_i \in U$, that is, $\tilde{Q}^- = \neg\tilde{Q}^+$. However, the qualified and unqualified fuzzy sets in an MADM information system with intuitionistic fuzzy information do not necessarily satisfy the above equations.

Table 7
An example of investment supplier in [28].

U/AT	a_1	a_2	a_3	a_4
ϖ_1	(0.6,0.3)	(0.2,0.4)	(0.3,0.3)	(0.1,0.5)
ϖ_2	(0.4,0.5)	(0.7,0.2)	(0.6,0.4)	(0.5,0.3)
ϖ_3	(0.5,0.3)	(0.1,0.6)	(0.4,0.2)	(0.3,0.4)
ϖ_4	(0.7,0.2)	(0.9,0.1)	(0.3,0.5)	(0.8,0.2)
ϖ_5	(0.6,0.1)	(0.6,0.2)	(0.3,0.3)	(0.5,0.2)
ϖ_6	(0.1,0.8)	(0.3,0.5)	(0.2,0.4)	(0.5,0.1)
ϖ_7	(0.4,0.5)	(0.3,0.6)	(0.4,0.2)	(0.3,0.5)
ϖ_8	(0.2,0.6)	(0.6,0.3)	(0.8,0.1)	(0.4,0.2)
ϖ_9	(0.3,0.4)	(0.8,0.1)	(0.5,0.4)	(0.6,0.3)
ϖ_{10}	(0.4,0.1)	(0.1,0.3)	(0.3,0.4)	(0.5,0.5)

Table 8
(U, C^+) in Example 4.6.

U/AT	a_1	a_2	a_3	a_4
ϖ_1	0.6	0.2	0.3	0.1
ϖ_2	0.4	0.7	0.6	0.5
ϖ_3	0.5	0.1	0.4	0.3
ϖ_4	0.7	0.9	0.3	0.8
ϖ_5	0.6	0.6	0.3	0.5
ϖ_6	0.1	0.3	0.2	0.5
ϖ_7	0.4	0.3	0.4	0.3
ϖ_8	0.2	0.6	0.8	0.4
ϖ_9	0.3	0.8	0.5	0.6
ϖ_{10}	0.4	0.1	0.3	0.5

Now, let us recall the concept of an intuitionistic fuzzy set.

Definition 4.4 *([1,43,44]).* An intuitionistic fuzzy set $\tilde{\mathscr{A}}$ is

$$\tilde{\mathscr{A}} = \{\langle \varpi, u_{\tilde{\mathscr{A}}}(\varpi), v_{\tilde{\mathscr{A}}}(\varpi)\rangle, \varpi \in U\}, \tag{4.5}$$

where $v_{\tilde{\mathscr{A}}}$: $U \longrightarrow [0,1]$ denotes non-membership of ϖ, $u_{\tilde{\mathscr{A}}}$: $U \longrightarrow [0,1]$ denotes degrees of membership of ϖ with respect to $\tilde{\mathscr{A}}$ and satisfy the condition:

$$\forall \varpi \in U, u_{\tilde{\mathscr{A}}}(\varpi) + v_{\tilde{\mathscr{A}}}(\varpi) \leq 1. \tag{4.6}$$

In this case, the pair $(u_{\tilde{\mathscr{A}}}(\varpi), v_{\tilde{\mathscr{A}}}(\varpi))$ is called an intuitionistic fuzzy number.

Definition 4.5. Suppose that (U, AT, V, ω) denotes an MADM intuitionistic fuzzy information system. That is, $U = \{\varpi_i \mid i = 1, 2, ..., n\}$ is the nonempty universe, $AT = \{a_j \mid j = 1, 2, ..., m\}$ denotes an attribute (evaluation criterion) set, $\omega = \{\omega_1, \omega_2, ..., \omega_m\}$ denotes the weight vector of AT and $V = \{(u_{ij}, v_{ij}) \mid i = 1, 2, ..., n,\ j = 1, 2, ..., m\}$, where (u_{ij}, v_{ij}) is an intuitionistic fuzzy number and denotes positives and negative evaluation values of ϖ_i with respect to a_j, respectively.

Example 4.6. Suppose that (U, AT, V, ω) denotes an MADM intuitionistic fuzzy information system, where $U = \{\varpi_1, \varpi_2, ...\varpi_{10}\}$ is the universe consisting of ten suppliers, $AT = \{a_1, a_2, a_3, a_4\}$ is an attribute set of four evaluation indicators consisting of the technical level, the service level, the business ability and the enterprise environment, and the weight vector of the attribute set AT is $\omega = \{0.22, 0.22, 0.36, 0.2\}$. The data given by Liu et al. [28] in Table 7 are those the manufacturer provides the pair of evaluation values (u_{ij}, v_{ij}) for each supplier ϖ_i $(i = \{1, 2, ..., 10\})$ with each indicator a_j $(j = \{1, 2, 3, 4\})$, where u_{ij} represents the positive evaluation value and v_{ij} represents the negative evaluation value. Then for each a_j, we can get a qualified fuzzy set $\tilde{Q}_j^+$, which is defined by

$$\tilde{Q}_j^+(\varpi_i) = u_{ij}, \forall \varpi_i \in U, \tag{4.7}$$

and an unqualified fuzzy set $\tilde{Q}_j^-$, which is defined by

$$\tilde{Q}_j^-(\varpi_i) = v_{ij}, \forall \varpi_i \in U. \tag{4.8}$$

Then $\tilde{Q}_j^+(\varpi_i) + \tilde{Q}_j^-(\varpi_i) \leq 1$ for each $\varpi_i \in U$. Let $C^- = \{\tilde{Q}_1^-, \tilde{Q}_2^-, \tilde{Q}_3^-, \tilde{Q}_4^-\}$ and $C^+ = \{\tilde{Q}_1^+, \tilde{Q}_2^+, \tilde{Q}_3^+, \tilde{Q}_4^+\}$. Then we obtain two fuzzy information systems (U, C^-) and (U, C^+), which are shown in Tables 8 and 9, where the evaluation information values of ϖ_i under a_j in (U, C^-) and (U, C^+) are $\varpi_{ij}^+ = \tilde{Q}_j^+(\varpi_i) = u_{ij}$ and $\varpi_{ij}^- = \tilde{Q}_j^-(\varpi_i) = v_{ij}$, respectively. According to Table 10, it shows that $\tilde{Q}^+(\varpi_i) + \tilde{Q}^-(\varpi_i) \leq 1$ for each $\varpi_i \in U$, which means that $\tilde{Q}^+ \neq \neg\tilde{Q}^-$.

Table 9
(U, C^-) in Example 4.6.

U/AT	a_1	a_2	a_3	a_4
ϖ_1	0.3	0.4	0.3	0.5
ϖ_2	0.5	0.2	0.4	0.3
ϖ_3	0.3	0.6	0.2	0.4
ϖ_4	0.2	0.1	0.5	0.2
ϖ_5	0.1	0.2	0.3	0.2
ϖ_6	0.8	0.5	0.4	0.1
ϖ_7	0.5	0.6	0.2	0.5
ϖ_8	0.6	0.3	0.1	0.2
ϖ_9	0.4	0.1	0.4	0.3
ϖ_{10}	0.1	0.3	0.4	0.5

Table 10
$\tilde{Q}^+$ and $\tilde{Q}^-$ in Example 4.6.

U/AT	$\tilde{Q}^+$	$\tilde{Q}^-$
ϖ_1	0.304	0.362
ϖ_2	0.558	0.358
ϖ_3	0.336	0.35
ϖ_4	0.62	0.286
ϖ_5	0.472	0.214
ϖ_6	0.26	0.45
ϖ_7	0.358	0.414
ϖ_8	0.544	0.274
ϖ_9	0.542	0.314
ϖ_{10}	0.318	0.332

Table 11
Relative losses of x_i with respect to a_j.

ϖ_i / a_j	$X(P)$	$\neg X(N)$
$\tilde{a_P}$	$\hbar^{ij}_{PP}$	$\hbar^{ij}_{PN}$
$\tilde{a_B}$	$\hbar^{ij}_{BP}$	$\hbar^{ij}_{BN}$
$\tilde{a_N}$	$\hbar^{ij}_{NP}$	$\hbar^{ij}_{NN}$

Table 12
Relative losses of ϖ_i with respect to AT.

ϖ_i	$X(P)$	$\neg X(N)$
$\tilde{a_P}$	$\hbar^{i}_{PP}$	$\hbar^{i}_{PN}$
$\tilde{a_B}$	$\hbar^{i}_{BP}$	$\hbar^{i}_{BN}$
$\tilde{a_N}$	$\hbar^{i}_{NP}$	$\hbar^{i}_{NN}$

4.2. 3WD derived from optimistic probabilistic RFS models based on MCBs

Traditionally, the loss function of the 3WD problem is determined by personal preference. Li and Zhou [25] first blended risk avoidance coefficients into the relative loss function. Recently, Liu and Jia [16] provided a novel notion of calculating the relative losses (see Table 11) by using both of u_{ij} of ϖ_i with respect to a_j and risk avoidance coefficients $\sigma_j \in [0,1]$. Specifically, let $u^+_j = \max_i u_{ij}$ and $u^-_j = \min_i u_{ij}$. Then $\hbar^{ij}_{PP} = 0$, $\hbar^{ij}_{NN} = 0$, $\hbar^{ij}_{BP} = \sigma_j(u_{ij} - u^-_j)$, $\hbar^{ij}_{NP} = u_{ij} - u^-_j$, $\hbar^{ij}_{PN} = u^+_j - u_{ij}$ and $\hbar^{ij}_{BN} = \sigma_j(u^+_j - u_{ij})$. Considering the different weights of different attributes in MADM problems, Jia and Liu [16] provided aggregated relative loss functions with the weight vector in Table 12, where $\hbar^{i}_{PP} = 0$, $\hbar^{i}_{NN} = 0$, $\hbar^{i}_{BP} = \sum_{j=1}^{m} \omega_j\sigma_j(u_{ij} - u^-_j)$, $\hbar^{i}_{NP} = \sum_{j=1}^{m} \omega_j(u_{ij} - u^-_j)$, $\hbar^{i}_{PN} = \sum_{j=1}^{m} \omega_j(u^+_j - u_{ij})$ and $\hbar^{i}_{BN} = \sum_{j=1}^{m} \omega_j\sigma_j(u^+_j - u_{ij})$.

Considering MCBs-based optimistic probabilistic RFS models in Definition 3.1, in order to estimate losses that an object $\varpi_i \in U$ is in good state X with respect to AT, the decision maker takes three actions, which classify ϖ_i into $POS^O(X)$, $BND^O(X)$ and $ENG^O(X)$, respectively. In "good state" case, we present the decision rules with respect to optimistic probabilistic RFS models. $\tilde{\mathcal{L}}(\tilde{a_N}|B^{max}_{\varpi_i})$, $\tilde{\mathcal{L}}(\tilde{a_B}|B^{max}_{\varpi_i})$ and $\tilde{\mathcal{L}}(\tilde{a_P}|B^{max}_{\varpi_i})$ are three optimistic expected losses with respect to "good state", which are shown as follows:

$\tilde{\mathcal{L}}(\tilde{a_P}|B^{max}_{\varpi_i}) = \hbar^{i}_{PN}(1 - P(X|B^{max}_{\varpi_i}))$;

Table 13
Relative losses of ϖ_i under $\tilde{Q}^+$.

ϖ_i	$\tilde{Q}^+$	$\neg\tilde{Q}^+$
$\tilde{a_P}$	0	$\sum_{j=1}^m \omega_j(\varpi_j^{++} - \varpi_{ij}^+)$
$\tilde{a_B}$	$\sum_{j=1}^m \omega_j\sigma_j(\varpi_{ij}^+ - \varpi_j^{+-})$	$\sum_{j=1}^m \omega_j\sigma_j(\varpi_j^{++} - \varpi_{ij}^+)$
$\tilde{a_N}$	$\sum_{j=1}^m \omega_j(\varpi_{ij}^+ - \varpi_j^{+-})$	0

$\tilde{\mathcal{L}}(\tilde{a_B}|B_{\varpi_i}^{max}) = \hbar_{BP}^i P(X|B_{\varpi_i}^{max}) + \hbar_{BN}^i(1 - P(X|B_{\varpi_i}^{max}))$;
$\tilde{\mathcal{L}}(\tilde{a_N}|B_{\varpi_i}^{max}) = \hbar_{NP}^i(1 - P(X|B_{\varpi_i}^{max}))$.

Then the three optimistic decision rules can be written as follows:

(P_O^+) If $\tilde{\mathcal{L}}(\tilde{a_P}|B_{\varpi_i}^{max}) \le \tilde{\mathcal{L}}(\tilde{a_B}|B_{\varpi_i}^{max})$ and $\tilde{\mathcal{L}}(\tilde{a_P}|B_{\varpi_i}^{max}) \le \tilde{\mathcal{L}}(\tilde{a_N}|B_{\varpi_i}^{max})$, then $\varpi_i \in POS^O(X)$;
(B_O^+) If $\tilde{\mathcal{L}}(\tilde{a_B}|B_{\varpi_i}^{max}) < \tilde{\mathcal{L}}(\tilde{a_P}|B_{\varpi_i}^{max})$ and $\tilde{\mathcal{L}}(\tilde{a_B}|B_{\varpi_i}^{max}) < \tilde{\mathcal{L}}(\tilde{a_N}|B_{\varpi_i}^{max})$, then $\varpi_i \in BND^O(X)$;
(N_O^+) If $\tilde{\mathcal{L}}(\tilde{a_N}|B_{\varpi_i}^{max}) \le \tilde{\mathcal{L}}(\tilde{a_P}|B_{\varpi_i}^{max})$ and $\tilde{\mathcal{L}}(\tilde{a_N}|B_{\varpi_i}^{max}) \le \tilde{\mathcal{L}}(\tilde{a_B}|B_{\varpi_i}^{max})$, then $\varpi_i \in ENG^O(X)$.

Furthermore, they can be rewritten as follows:

(P_O^+) If $P(X|B_{\varpi_i}^{max}) \ge \alpha_i$, $P(X|B_{\varpi_i}^{max}) \ge \gamma_i$, then $\varpi_i \in POS^O(X)$;
(B_O^+) If $P(X|B_{\varpi_i}^{max}) < \alpha_i$, $P(X|B_{\varpi_i}^{max}) > \beta_i$, then $\varpi_i \in BND^O(X)$;
(N_O^+) If $P(X|B_{\varpi_i}^{max}) \le \beta_i$, $P(X|B_{\varpi_i}^{max}) \le \gamma_i$, then $\varpi_i \in ENG^O(X)$,

where the three thresholds are shown as follows:

$$\alpha_i = \frac{\sum_{j=1}^m \omega_j(1-\sigma_j)(u_j^+ - u_{ij})}{\sum_{j=1}^m \omega_j(1-\sigma_j)(u_j^+ - u_{ij}) + \sum_{j=1}^m \omega_j\sigma_j(u_{ij} - u_j^-)}; \tag{4.9}$$

$$\beta_i = \frac{\sum_{j=1}^m \omega_j\sigma_j(u_j^+ - u_{ij})}{\sum_{j=1}^m \omega_j\sigma_j(u_j^+ - u_{ij}) + \sum_{j=1}^m \omega_j(1-\sigma_j)(u_{ij} - u_j^-)}; \tag{4.10}$$

$$\gamma_i = \sum_{j=1}^m \omega_j \frac{\max_i u_{ij} - u_{ij}}{\max_i u_{ij} - \min_i u_{ij}}. \tag{4.11}$$

According to Theorem 5 in [16], if $\sigma_j \ne 0.5$ for some $a_j \in AT$, then $0 \le \beta_i < \gamma_i < \alpha_i \le 1$. The optimistic 3WD rules are rewritten as follows:

(P_O^+) If $P(X|B_{\varpi_i}^{max}) \ge \alpha_i$, then $\varpi_i \in POS^O(X)$;
(B_O^+) If $\beta_i < P(X|B_{\varpi_i}^{max}) < \alpha_i$, then $\varpi_i \in BND^O(X)$;
(N_O^+) If $P(X|B_{\varpi_i}^{max}) \le \beta_i$, then $\varpi_i \in ENG^O(X)$.

In the following, according to definition of qualified fuzzy set $\tilde{Q}^+$ with respect to AT, we can regard it as the good state X, and establish 3WDs of $\tilde{Q}^+$.

Suppose that $\Omega = \{\tilde{Q}^+, \neg\tilde{Q}^+\}$[2] is the set of good and bad states under AT, ${}^+Y_{\varpi_i}$ denotes a MCB of ϖ_i with respect to AT in (U, C^+). Table 13 (consistent with Table 12) illustrates the relative loss function for $\tilde{Q}^+$, where $\varpi_j^{+-} = \min_i \varpi_{ij}^+$, $\varpi_j^{++} = \max_i \varpi_{ij}^+$. This determines optimistic 3WD rules of $\tilde{Q}^+$ as follows:

(P_O^+) If $P(\tilde{Q}^+|{}^+Y_{\varpi_i}^{max}) \ge \alpha_i$, $P(\tilde{Q}^+|{}^+Y_{\varpi_i}^{max}) \ge \gamma_i$, then $\varpi_i \in POS^O(\tilde{Q}^+)$;
(B_O^+) If $P(\tilde{Q}^+|{}^+Y_{\varpi_i}^{max}) < \alpha_i$, $P(\tilde{Q}^+|{}^+Y_{\varpi_i}^{max}) > \beta_i$, then $\varpi_i \in BND^O(\tilde{Q}^+)$;
(N_O^+) If $P(\tilde{Q}^+|{}^+Y_{\varpi_i}^{max}) \le \beta_i$, $P(\tilde{Q}^+|{}^+Y_{\varpi_i}^{max}) \le \gamma_i$, then $\varpi_i \in ENG^O(\tilde{Q}^+)$,

where the three thresholds are shown as follows:

$$\alpha_i = \frac{\sum_{j=1}^m \omega_j(1-\sigma_j)(\varpi_j^{++} - \varpi_{ij}^+)}{\sum_{j=1}^m \omega_j(1-\sigma_j)(\varpi_j^{++} - \varpi_{ij}^+) + \sum_{j=1}^m \omega_j\sigma_j(\varpi_{ij}^+ - \varpi_j^{+-})}; \tag{4.12}$$

$$\beta_i = \frac{\sum_{j=1}^m \omega_j\sigma_j(\varpi_j^{++} - \varpi_{ij}^+)}{\sum_{j=1}^m \omega_j\sigma_j(\varpi_j^{++} - \varpi_{ij}^+) + \sum_{j=1}^m \omega_j(1-\sigma_j)(\varpi_{ij}^+ - \varpi_j^{+-})}. \tag{4.13}$$

$$\gamma_i = \sum_{j=1}^m \omega_j \frac{\varpi_j^{++} - \varpi_{ij}^+}{\varpi_j^{++} - \varpi_j^{+-}}. \tag{4.14}$$

When $\sigma_j \ne 0.5$ for some $a_j \in AT$, we have $0 \le \beta_i < \gamma_i < \alpha_i \le 1$. This determines optimistic three-way decision rules with respect to $\tilde{Q}^+$ as follows:

(P_O^+) If $P(\tilde{Q}^+|{}^+Y_{\varpi_i}^{max}) \ge \alpha_i$, then $\varpi_i \in POS^O(\tilde{Q}^+)$;

[2] It is important to observe that $\neg\tilde{Q}^+ = \tilde{Q}^-$ may not be valid in the context of an MADM intuitionistic fuzzy information system (U, AT, V, ω).

Table 14
Relative losses of ϖ_i with respect to $\tilde{Q}^-$.

ϖ_i	$\tilde{Q}^-$	$\neg\tilde{Q}^-$
$\tilde{a_P}$	$\sum_{j=1}^m \omega_j(\varpi_{ij}^- - \varpi_j^{--})$	0
$\tilde{a_B}$	$\sum_{j=1}^m \omega_j\sigma_j(\varpi_{ij}^- - \varpi_j^{--})$	$\sum_{j=1}^m \omega_j\sigma_j(\varpi_j^{-+} - \varpi_{ij}^-)$
$\tilde{a_N}$	0	$\sum_{j=1}^m \omega_j(\varpi_j^{-+} - \varpi_{ij}^-)$

(B_O^+) If $\beta_i < P(\tilde{Q}^+|^+Y_{\varpi_i}^{max}) < \alpha_i$, then $\varpi_i \in BND^O(\tilde{Q}^+)$;
(N_O^+) If $P(\tilde{Q}^+|^+Y_{\varpi_i}^{max}) \leq \beta_i$, then $\varpi_i \in ENG^O(\tilde{Q}^+)$.
According to the three decision rules, the universe U is partitioned into $POS^O(\tilde{Q}^+)$, $BND^O(\tilde{Q}^+)$ and $ENG^O(\tilde{Q}^+)$.

In order to estimate losses that $\varpi_i \in U$ is in bad state $\neg X$ with respect to AT, the decision maker takes three actions: $\tilde{a_N}$, $\tilde{a_B}$ and $\tilde{a_P}$, which classify ϖ_i to three distinct and individual decision regions. Now, in "bad state" case, there are $\tilde{\mathcal{L}}(\tilde{a_P}|B_{\varpi_i}^{max})$, $\tilde{\mathcal{L}}(\tilde{a_B}|B_{\varpi_i}^{max})$ and $\tilde{\mathcal{L}}(\tilde{a_N}|B_{\varpi_i}^{max})$ with respect to $\neg X$ as follows:
$\tilde{\mathcal{L}}(\tilde{a_P}|B_{\varpi_i}^{max}) = \hbar_{PN}^i P(\neg X|B_{\varpi_i}^{max})$;
$\tilde{\mathcal{L}}(\tilde{a_B}|B_{\varpi_i}^{max}) = \hbar_{BP}^i(1 - P(\neg X|B_{\varpi_i}^{max})) + \hbar_{BN}^i P(\neg X|B_{\varpi_i}^{max})$;
$\tilde{\mathcal{L}}(\tilde{a_N}|B_{\varpi_i}^{max}) = \hbar_{NP}^i(1 - P(\neg X|B_{\varpi_i}^{max}))$.
Then we present three decision rules of $\neg X$ as follows:
(P_O^-) If $\tilde{\mathcal{L}}(\tilde{a_N}|B_{\varpi_i}^{max}) \leq \tilde{\mathcal{L}}(\tilde{a_P}|B_{\varpi_i}^{max})$, $\tilde{\mathcal{L}}(\tilde{a_N}|B_{\varpi_i}^{max}) \leq \tilde{\mathcal{L}}(\tilde{a_B}|B_{\varpi_i}^{max})$, then $\varpi_i \in POS^O(\neg X)$;
(B_O^-) If $\tilde{\mathcal{L}}(\tilde{a_B}|B_{\varpi_i}^{max}) < \tilde{\mathcal{L}}(\tilde{a_P}|B_{\varpi_i}^{max})$, $\tilde{\mathcal{L}}(\tilde{a_B}|B_{\varpi_i}^{max}) < \tilde{\mathcal{L}}(\tilde{a_N}|B_{\varpi_i}^{max})$, then $\varpi_i \in BND^O(\neg X)$;
(N_O^-) If $\tilde{\mathcal{L}}(\tilde{a_P}|B_{\varpi_i}^{max}) \leq \tilde{\mathcal{L}}(\tilde{a_B}|B_{\varpi_i}^{max})$, $\tilde{\mathcal{L}}(\tilde{a_P}|B_{\varpi_i}^{max}) \leq \tilde{\mathcal{L}}(\tilde{a_N}|B_{\varpi_i}^{max})$, then $\varpi_i \in ENG^O(\neg X)$.
Since three thresholds ξ_i, η_i and θ_i are calculated as follows:

$$\xi_i = \frac{\sum_{j=1}^m \omega_j(1-\sigma_j)(u_{ij} - u_j^-)}{\sum_{j=1}^m \omega_j\sigma_j(u_j^+ - u_{ij}) + \sum_{j=1}^m \omega_j(1-\sigma_j)(u_{ij} - u_j^-)} = 1 - \beta_i; \tag{4.15}$$

$$\eta_i = \frac{\sum_{j=1}^m \omega_j\sigma_j(u_{ij} - u_j^-)}{\sum_{j=1}^m \omega_j(1-\sigma_j)(u_j^+ - u_{ij}) + \sum_{j=1}^m \omega_j\sigma_j(u_{ij} - u_j^-)} = 1 - \alpha_i; \tag{4.16}$$

$$\theta_i = \sum_{j=1}^m \omega_j \frac{u_{ij} - u_j^-}{u_j^+ - u_j^-} = 1 - \gamma_i. \tag{4.17}$$

Three optimistic decision rules are rewritten as follows:
(P_O^-) If $P(\neg X|B_{\varpi_i}^{max}) \geq \xi_i$, $P(\neg X|B_{\varpi_i}^{max}) \geq \theta_i$, then $\varpi_i \in POS^O(\neg X)$;
(B_O^-) If $P(\neg X|B_{\varpi_i}^{max}) < \xi_i$, $P(\neg X|B_{\varpi_i}^{max}) > \eta_i$, then $\varpi_i \in BND^O(\neg X)$;
(N_O^-) If $P(\neg X|B_{\varpi_i}^{max}) \leq \eta_i$, $P(\neg X|B_{\varpi_i}^{max}) \leq \theta_i$, then $\varpi_i \in ENG^O(\neg X)$.
If $\sigma_j \neq 0.5$ for some $a_j \in AT$, then $0 \leq \eta_i < \theta_i < \xi_i \leq 1$. The optimistic 3WD rules are rewritten as:
(P_O^-) If $P(\neg X|B_{\varpi_i}^{max}) \geq \xi_i$, then $\varpi_i \in POS^O(\neg X)$;
(B_O^-) If $\eta_i < P(\neg X|B_{\varpi_i}^{max}) < \xi_i$, then $\varpi_i \in BND^O(\neg X)$;
(N_O^-) If $P(\neg X|B_{\varpi_i}^{max}) \leq \eta_i$, then $\varpi_i \in ENG^O(\neg X)$.

Next, we regard the unqualified fuzzy set $\tilde{Q}^-$ as the bad state $\neg X$. Suppose that $\Omega = \{\neg\tilde{Q}^-, \tilde{Q}^-\}$ is the set of good and bad states with respect to AT. $^-Y_{\varpi_i}$ denotes a MCB of ϖ_i with respect to AT in (U, C^-). Table 14 shows the relative loss function of $\tilde{Q}^-$, where $\varpi_j^{--} = \min_i \varpi_{ij}^-$ and $\varpi_j^{-+} = \max_i \varpi_{ij}^-$.

There are three optimistic decision rules with respect to $\tilde{Q}^-$ as follows:
(P_O^-) If $P(\tilde{Q}^-|^-Y_{\varpi_i}^{max}) \geq \xi_i$ and $P(\tilde{Q}^-|^-Y_{\varpi_i}^{max}) \geq \theta_i$, then $\varpi_i \in POS^O(\tilde{Q}^-)$;
(B_O^-) If $P(\tilde{Q}^-|^-Y_{\varpi_i}^{max}) < \xi_i$ and $P(\tilde{Q}^-|^-Y_{\varpi_i}^{max}) > \eta_i$, then $\varpi_i \in BND^O(\tilde{Q}^-)$;
(N_O^-) If $P(\tilde{Q}^-|^-Y_{\varpi_i}^{max}) \leq \eta_i$ and $P(\tilde{Q}^-|^-Y_{\varpi_i}^{max}) \leq \theta_i$, then $\varpi_i \in \tilde{Q}NG^O(\tilde{Q}^-)$,
where three thresholds ξ_i η_i and θ_i are calculated as follows:

$$\xi_i = \frac{\sum_{j=1}^m \omega_j(1-\sigma_j)(\varpi_j^{-+} - \varpi_{ij}^-)}{\sum_{j=1}^m \omega_j(1-\sigma_j)(\varpi_j^{-+} - \varpi_{ij}^-) + \sum_{j=1}^m \omega_j\sigma_j(\varpi_{ij}^- - \varpi_j^{--})}; \tag{4.18}$$

$$\eta_i = \frac{\sum_{j=1}^m \omega_j\sigma_j(\varpi_j^{-+} - \varpi_{ij}^-)}{\sum_{j=1}^m \omega_j\sigma_j(\varpi_j^{-+} - \varpi_{ij}^-) + \sum_{j=1}^m \omega_j(1-\sigma_j)(\varpi_{ij}^- - \varpi_j^{--})}; \tag{4.19}$$

$$\theta_i = \sum_{j=1}^m \omega_j \frac{\varpi_j^{-+} - \varpi_{ij}^-}{\varpi_j^{-+} - \varpi_j^{--}}. \tag{4.20}$$

If $\sigma_j \neq 0.5$ for some $a_j \in AT$, then $0 \le \eta_i < \theta_i < \xi_i \le 1$. The optimistic 3WD rules with respect to $\tilde{Q}^-$ are rewritten as:
(P_O^-) If $P(\tilde{Q}^-|^-Y_{\varpi_i}^{max}) \ge \xi_i$, then $\varpi_i \in POS^O(\tilde{Q}^-)$;
(B_O^-) If $\eta_i < P(\tilde{Q}^-|^-Y_{\varpi_i}^{max}) < \xi_i$, then $\varpi_i \in BND^O(\tilde{Q}^-)$;
(N_O^-) If $P(\tilde{Q}^-|^-Y_{\varpi_i}^{max}) \le \eta_i$, then $\varpi_i \in ENG^O(\tilde{Q}^-)$.
According to the three rules, the universe U is partitioned into $POS^O(\tilde{Q}^-)$, $BND^O(\tilde{Q}^-)$ and $ENG^O(\tilde{Q}^-)$.

4.3. 3WD derived from pessimistic probabilistic RFS models based on MCBs

In this subsection, we discuss 3WDs of $\tilde{Q}^+$ and $\tilde{Q}^-$ with respect to AT in the pessimistic case, respectively.
Suppose that $\Omega = \{\tilde{Q}^+, \neg\tilde{Q}^+\}$. The decision rules with respect to pessimistic probabilistic RFS models are shown as:
(P_P^+) If $P(\tilde{Q}^+|^+Y_{\varpi_i}^{min}) \ge \alpha_i$, $P(\tilde{Q}^+|^+Y_{\varpi_i}^{min}) \ge \gamma_i$, then $\varpi_i \in POS^P(\tilde{Q}^+)$;
(B_P^+) If $P(\tilde{Q}^+|^+Y_{\varpi_i}^{min}) < \alpha_i$, $P(\tilde{Q}^+|^+Y_{\varpi_i}^{min}) > \beta_i$, then $\varpi_i \in BND^P(\tilde{Q}^+)$;
(N_P^+) If $P(\tilde{Q}^+|^+Y_{\varpi_i}^{min}) \le \beta_i$, $P(\tilde{Q}^+|^+Y_{\varpi_i}^{min}) \le \gamma_i$, then $\varpi_i \in ENG^P(\tilde{Q}^+)$,
where the three thresholds α_i, β_i and γ_i are shown by three Equations (4.12), (4.13) and (4.14).
If $\sigma_j \neq 0.5$ for some $a_j \in AT$, then we have $0 \le \beta_i < \gamma_i < \alpha_i \le 1$. This determines pessimistic three-way decision rules with respect to $\tilde{Q}^+$ as follows:
(P_P^+) If $P(\tilde{Q}^+|^+Y_{\varpi_i}^{min}) \ge \alpha_i$, then $\varpi_i \in POS^P(\tilde{Q}^+)$;
(B_P^+) If $\beta_i < P(\tilde{Q}^+|^+Y_{\varpi_i}^{min}) < \alpha_i$, then $\varpi_i \in BND^P(\tilde{Q}^+)$;
(N_P^+) If $P(\tilde{Q}^+|^+Y_{\varpi_i}^{min}) \le \beta_i$, then $\varpi_i \in ENG^P(\tilde{Q}^+)$.
According to the three decision rules, the universe U can be partitioned into $POS^P(\tilde{Q}^+)$, $BND^P(\tilde{Q}^+)$ and $ENG^P(\tilde{Q}^+)$.
Suppose that $\Omega = \{\neg\tilde{Q}^-, \tilde{Q}^-\}$. The decision rules with respect to pessimistic probabilistic RFS models are shown as follows:
(P_P^-) If $P(\tilde{Q}^-|^-Y_{\varpi_i}^{min}) \ge \xi_i$, $P(\tilde{Q}^-|^-Y_{\varpi_i}^{min}) \ge \theta_i$, then $\varpi_i \in POS^P(\tilde{Q}^-)$;
(B_P^-) If $P(\tilde{Q}^-|^-Y_{\varpi_i}^{min}) < \xi_i$, $P(\tilde{Q}^-|^-Y_{\varpi_i}^{min}) > \eta_i$, then $\varpi_i \in BND^P(\tilde{Q}^-)$;
(N_P^-) If $P(\tilde{Q}^-|^-Y_{\varpi_i}^{min}) \le \eta_i$, $P(\tilde{Q}^-|^-Y_{\varpi_i}^{min}) \le \theta_i$, then $\varpi_i \in ENG^P(\tilde{Q}^-)$,
where three thresholds ξ_i, η_i and θ_i are shown by three Equations (4.18), (4.19) and (4.20).
If $\sigma_j \neq 0.5$ for some $a_j \in AT$, then we have $0 \le \eta_i < \theta_i < \xi_i \le 1$. This determines pessimistic 3WD rules with respect to $\tilde{Q}^-$ as follows:
(P_P^-) If $P(\tilde{Q}^-|^-Y_{\varpi_i}^{min}) \ge \xi_i$, then $\varpi_i \in POS^P(\tilde{Q}^-)$;
(B_P^-) If $\eta_i < P(\tilde{Q}^-|^-Y_{\varpi_i}^{min}) < \xi_i$, then $\varpi_i \in BND^P(\tilde{Q}^-)$;
(N_P^-) If $P(\tilde{Q}^-|^-Y_{\varpi_i}^{min}) \le \eta_i$, then $\varpi_i \in ENG^P(\tilde{Q}^-)$.
According to the three rules, the universe U can be partitioned into $POS^P(\tilde{Q}^-)$, $BND^P(\tilde{Q}^-)$ and $ENG^P(\tilde{Q}^-)$.
In what follows, we explore several properties of the above 3WDs in an MADM information system (U, AT, V, ω) with single-valued fuzzy information.

Theorem 4.7. *Suppose that (U, AT, V, ω) denotes an MADM information system. If $\neg\tilde{Q}_j^+ = \tilde{Q}_j^-$ for each $a_j \in AT$, then $\xi_i = 1-\beta_i$, $\eta_i = 1-\alpha_i$ and $\theta_i = 1-\gamma_i$.*

Proof. Take each $a_j \in AT$ such that $\varpi_{ij}^+ = \tilde{Q}_j^+(\varpi_i)$ and $\varpi_{ij}^- = \tilde{Q}_j^-(\varpi_i)$ for each $\varpi_i \in U$. Then it follows from $\tilde{Q}_j^- = \neg\tilde{Q}_j^+$ that $\varpi_{ij}^- = 1-\varpi_{ij}^+$, which implies $\varpi_j^{-+} = 1-\varpi_j^{+-}$ and $\varpi_j^{--} = 1-\varpi_j^{++}$. This implies

$$
\begin{aligned}
\xi_i &= \frac{\sum_{j=1}^m \omega_j(1-\sigma_j)(\varpi_j^{-+}-\varpi_{ij}^-)}{\sum_{j=1}^m \omega_j\sigma_j(\varpi_{ij}^- - \varpi_j^{--}) + \sum_{j=1}^m \omega_j(1-\sigma_j)(\varpi_j^{-+}-\varpi_{ij}^-)} \\
&= \frac{\sum_{j=1}^m \omega_j(1-\sigma_j)((1-\varpi_j^{+-})-(1-\varpi_{ij}^+))}{\sum_{j=1}^m \omega_j\sigma_j((1-\varpi_{ij}^+)-(1-\varpi_j^{++})) + \sum_{j=1}^m \omega_j(1-\sigma_j)((1-\varpi_j^{+-})-(1-\varpi_{ij}^+))} \\
&= \frac{\sum_{j=1}^m \omega_j(1-\sigma_j)(\varpi_{ij}^+-\varpi_j^{+-})}{\sum_{j=1}^m \omega_j\sigma_j(\varpi_j^{++}-\varpi_{ij}^+) + \sum_{j=1}^m \omega_j(1-\sigma_j)(\varpi_{ij}^+-\varpi_j^{+-})} \\
&= 1 - \frac{\sum_{j=1}^m \omega_j\sigma_j(\varpi_j^{++}-\varpi_{ij}^+)}{\sum_{j=1}^m \omega_j(1-\sigma_j)(\varpi_{ij}^+-\varpi_j^{+-}) + \sum_{j=1}^m \omega_j\sigma_j(\varpi_j^{++}-\varpi_{ij}^+)} \\
&= 1-\beta_i
\end{aligned}
$$

and

$$
\eta_i = \frac{\sum_{j=1}^m \omega_j\sigma_j(\varpi_j^{-+}-\varpi_{ij}^-)}{\sum_{j=1}^m \omega_j\sigma_j(\varpi_j^{-+}-\varpi_{ij}^-) + \sum_{j=1}^m \omega_j(1-\sigma_j)(\varpi_{ij}^- - \varpi_j^{--})}
$$

$$= \frac{\sum_{j=1}^{m}\omega_j\sigma_j((1-\varpi_j^{+-})-(1-\varpi_{ij}^{+}))}{\sum_{j=1}^{m}\omega_j\sigma_j((1-\varpi_j^{+-})-(1-\varpi_{ij}^{+}))+\sum_{j=1}^{m}\omega_j(1-\sigma_j)((1-\varpi_{ij}^{+})-(1-\varpi_j^{++}))}$$
$$= \frac{\sum_{j=1}^{m}\omega_j\sigma_j(\varpi_{ij}^{+}-\varpi_j^{+-})}{\sum_{j=1}^{m}\omega_j\sigma_j(\varpi_{ij}^{+}-\varpi_j^{+-})+\sum_{j=1}^{m}\omega_j(1-\sigma_j)(\varpi_j^{++}-\varpi_{ij}^{+})}$$
$$= 1-\frac{\sum_{j=1}^{m}\omega_j(1-\sigma_j)(\varpi_j^{++}-\varpi_{ij}^{+})}{\sum_{j=1}^{m}\omega_j(1-\sigma_j)(\varpi_j^{++}-\varpi_{ij}^{+})+\sum_{j=1}^{m}\omega_j\sigma_j(\varpi_{ij}^{+}-\varpi_j^{+-})}$$
$$= 1-\alpha_i.$$

Similarly, we can obtain that $\theta_i = 1-\gamma_i$. □

In the following, we will provide MCBs based on the attribute set AT in (U, C^+) and (U, C^-).

Definition 4.8. Suppose that (U, AT, V, ω) denotes a fuzzy information system. Define a binary relation $R \subseteq U \times U$ as follows:

$$R = \{(\varpi_i, \varpi_k) \in U \times U \mid \sum_{j=1}^{m}\omega_j d_j(\varpi_i, \varpi_k) \leq \phi\},$$

where $d_j(\varpi_i, \varpi_k) = |u_{ij} - u_{kj}|$, $\phi_j = \frac{\sum_{i=1}^{n}\sum_{k=1}^{n} d_j(\varpi_i,\varpi_k)}{n\times(n-1)}$ and $\phi = \sum_{j=1}^{m}\omega_j\phi_j$.

According to Definition 4.8, R is also a tolerance relation. Further, all MCBs, denoted by $MCB(R)$, can be derived based on the attribute set AT.

Theorem 4.9. *Suppose that (U, C^+) and (U, C^-) denotes two fuzzy information systems. Define $R^+ \subseteq U \times U$ and $R^- \subseteq U \times U$ as follows*

$$R^+ = \{(\varpi_i, \varpi_k) \in U \times U \mid \sum_{j=1}^{m}\omega_j|\varpi_{kj}^{+} - \varpi_{ij}^{+}| \leq \frac{2\sum_{j=1}^{m}\sum_{k=i+1}^{n}\sum_{i=1}^{n}\omega_j|\varpi_{ij}^{+}-\varpi_{kj}^{+}|}{n\times(n-1)}\}$$

and

$$R^- = \{(\varpi_i, \varpi_k) \in U \times U \mid \sum_{j=1}^{m}\omega_j|\varpi_{kj}^{+} - \varpi_{ij}^{+}| \leq \frac{2\sum_{j=1}^{m}\sum_{k=i+1}^{n}\sum_{i=1}^{n}\omega_j|\varpi_{ij}^{-}-\varpi_{kj}^{-}|}{n\times(n-1)}\},$$

respectively. For each $i = 1,2,...,n$ and $j = 1,2,...,m$, if $\varpi_{ij}^{+} + \varpi_{ij}^{-} = 1$, then $MCB(R^+) = MCB(R^-)$, where $MCB_{\varpi}(R^+) = \{^{+}Y_{\varpi} \in MCB(R^+) \mid \varpi \in {}^{+}Y_{\varpi}\}$ and $MCB_{\varpi}(R^-) = \{^{-}Y_{\varpi} \in MCB(R^-) \mid \varpi \in {}^{-}Y_{\varpi}\}$.

Proof. It is straightforward and omitted. □

An MADM information system (U, AT, V, ω) can be divided into two fuzzy information systems (U, C^+) and (U, C^-) induced from qualified and unqualified fuzzy sets $\tilde{Q}_j^+$ and $\tilde{Q}_j^-$, the sets of all MCBs based on AT in (U, C^+) and (U, C^-) satisfy $MCB(R^+) = MCB(R^-)$ while $\tilde{Q}_j^- = \neg\tilde{Q}_j^+$.

Theorem 4.10. *Suppose that (U, AT, V, ω) denotes an MADM information system and $\sigma_j \neq 0.5$ for some $a_j \in AT$. If $\tilde{Q}_j^- = \neg\tilde{Q}_j^+$ for each $a_j \in AT$, then $POS^O(\tilde{Q}^+) = ENG^P(\tilde{Q}^-)$, $BND^O(\tilde{Q}^+) = BND^P(\tilde{Q}^-)$ and $ENG^O(\tilde{Q}^+) = POS^P(\tilde{Q}^-)$.*

Proof. Since $\tilde{Q}_j^- = \neg\tilde{Q}_j^+$ for each $a_j \in AT$, we have $MCB(R^+) = MCB(R^-)$, which implies $^{+}Y_{\varpi_i} = {}^{-}Y_{\varpi_i}$ for each $\varpi_i \in U$. Then it follows from $\tilde{Q}^- = \neg\tilde{Q}^+$ and (L4) of Theorem 3.7 that for each $\varpi_i \in U$,

$$P(\tilde{Q}^+|^{+}Y_{\varpi_i}^{max}) + P(\tilde{Q}^-/^{-}Y_{\varpi_i}^{min}) = 1,$$

where $^{+}Y_{\varpi_i}^{max} = \arg\max_{^{+}Y_{\varpi_i}\in MCB_{\varpi_i}(R^+)}\{P(\tilde{Q}^+|^{+}Y_{\varpi_i})\}$ and $^{-}Y_{\varpi_i}^{min} = \arg\min_{^{-}Y_{\varpi_i}\in MCB_{\varpi_i}(R^-)}\{P(\tilde{Q}^-|^{-}Y_{\varpi_i})\}$. According to Definitions 3.1 and 3.4, we have

$$\begin{aligned}\varpi_i \in POS^O(\tilde{Q}^+) &\Longleftrightarrow P(\tilde{Q}^+|^{+}Y_{\varpi_i}^{max}) \geq \alpha_i\\ &\Longleftrightarrow 1 - P(\tilde{Q}^-|^{-}Y_{\varpi_i}^{min}) \geq \alpha_i\\ &\Longleftrightarrow P(\tilde{Q}^-|^{-}Y_{\varpi_i}^{min}) \leq 1-\alpha_i\\ &\Longleftrightarrow P(\tilde{Q}^-|^{-}Y_{\varpi_i}^{min}) \leq \eta_i \qquad \text{(By Theorem 4.7)}\end{aligned}$$

$$\begin{aligned}
&\Longleftrightarrow \varpi_i \in ENG^P(\tilde{Q}^-),\\
\varpi_i \in BND^O(\tilde{Q}^+) &\Longleftrightarrow \beta_i < P(\tilde{Q}^+|^+Y^{max}_{\varpi_i}) < \alpha_i\\
&\Longleftrightarrow \beta_i < 1 - P(\tilde{Q}^-|^-Y^{min}_{\varpi_i}) < \alpha_i\\
&\Longleftrightarrow 1-\alpha_i < P(\tilde{Q}^-|^-Y^{min}_{\varpi_i}) < 1-\beta_i\\
&\Longleftrightarrow \eta_i < P(\tilde{Q}^-|^-Y^{min}_{\varpi_i}) < \xi_i \qquad \text{(By Theorem 4.7)}\\
&\Longleftrightarrow \varpi_i \in BND^P(\tilde{Q}^-),
\end{aligned}$$

and

$$\begin{aligned}
\varpi_i \in ENG^O(\tilde{Q}^+) &\Longleftrightarrow P(\tilde{Q}^+|^+Y^{max}_{\varpi_i}) \le \beta_i\\
&\Longleftrightarrow 1 - P(\tilde{Q}^-|^-Y^{min}_{\varpi_i}) \le \beta_i\\
&\Longleftrightarrow P(\tilde{Q}^-|^-Y^{min}_{\varpi_i}) \ge 1-\beta_i\\
&\Longleftrightarrow P(\tilde{Q}^-|^-Y^{min}_{\varpi_i}) \ge \xi_i \qquad \text{(By Theorem 4.7)}\\
&\Longleftrightarrow \varpi_i \in POS^P(\tilde{Q}^-). \quad \square
\end{aligned}$$

Theorem 4.11. *Suppose that (U, AT, V, ω) denotes an MADM information system and $\sigma_j \neq 0.5$ for some $a_j \in AT$. If $\tilde{Q}^-_j = \neg\tilde{Q}^+_j$ for each $a_j \in AT$, then $POS^P(\tilde{Q}^+) = ENG^O(\tilde{Q}^-)$, $BND^P(\tilde{Q}^+) = BND^O(\tilde{Q}^-)$ and $ENG^P(\tilde{Q}^+) = POS^O(\tilde{Q}^-)$.*

Proof. Adopting the proof of Theorem 4.10, it is straightforward and is omitted. $\square$

5. Optimistic and pessimistic 3WD-MADM approaches and their illustrative examples

In this section, we will introduce two novel 3WD-MADM approaches to solve MADM problems with single-valued fuzzy information and intuitionistic fuzzy information.

5.1. The description of the optimistic 3WD-MADM approach

In what follows, we will utilize fusion of the optimistic 3WD-rules with respect to qualified and unqualified fuzzy sets to provide a novel 3WD-MADM approach.

Firstly, we establish the two criteria for the optimistic selection of optimal ideal objects in relation to the pair of $\tilde{Q}^+$ in (U, C^+) and $\tilde{Q}^-$ in (U, C^-).

According to the three rules (P^+_O)–(N^+_O), the universe U is partitioned into $POS^O(\tilde{Q}^+)$, $BND^O(\tilde{Q}^+)$ and $ENG^O(\tilde{Q}^+)$. Minimum risk cost of an object ϖ_i in each region is shown as follows:

$$MRC^{O+}(\varpi_i) = \begin{cases} \tilde{\mathcal{L}}(\tilde{a_P}|^+Y^{max}_{\varpi_i}), \text{ if } \varpi_i \in POS^O(\tilde{Q}^+),\\ \tilde{\mathcal{L}}(\tilde{a_B}|^+Y^{max}_{\varpi_i}), \text{ if } \varpi_i \in BND^O(\tilde{Q}^+),\\ \tilde{\mathcal{L}}(\tilde{a_N}|^+Y^{max}_{\varpi_i}), \text{ if } \varpi_i \in ENG^O(\tilde{Q}^+). \end{cases} \tag{5.1}$$

To facilitate the comparison of objects within the same region, we provide an optimistic ranking function by mean of the minimum risk cost, which is generated by the qualified fuzzy set $\tilde{Q}^+$.

Definition 5.1. Suppose that (U, AT, V, ω) denotes an MADM information system and $\sigma_j \neq 0.5$ for some $a_j \in AT$. Define an optimistic ranking function $d^{O+}\colon U \longrightarrow [0,1]$ by

$$d^{O+}(\varpi_i) = \begin{cases} MRC^{O+}(\varpi_i), \text{ if } \varpi_i \in POS^O(\tilde{Q}^+),\\ MRC^{O+}(\varpi_i)+1, \text{ if } \varpi_i \in BND^O(\tilde{Q}^+),\\ MRC^{O+}(\varpi_i)+2, \text{ if } \varpi_i \in ENG^O(\tilde{Q}^+). \end{cases} \tag{5.2}$$

The optimistic ranking function d^{O+} is decreasing, that is, the better the object ϖ_i is, the smaller the value of $d^{O+}(\varpi_i)$ is. Following this way, we can give an order of optimal ideal object selection with respect to $\tilde{Q}^+$ to rank all objects of the universe U in the optimistic case.

According to Definition 5.1, the idea of our proposed optimistic ranking function d^{O+} is shown as follows: the order of optimistic optimal ideal object selection is $POS^O(\tilde{Q}^+) \succ BND^O(\tilde{Q}^+) \succ ENG^O(\tilde{Q}^+)$. If two objects of the universe belong to the same region, for example, $\varpi_i, \varpi_k \in POS^O(\tilde{Q}^+)$ and $MRC^+(\varpi_i) \le MRC^+(\varpi_k)$, then ϖ_i is better than ϖ_k, which is denoted by $\varpi_i \succ \varpi_k$. This is consistent with the principle of existing methods [13,49,50]. Our proposed ranking function can also provide an certain value for each object in this process of optimistic optimal ideal object selection.

Then, according to the three rules (P_O^-)–(N_O^-), the universe U is partitioned into $POS^O(\tilde{Q}^-)$, $BND^O(\tilde{Q}^-)$ and $ENG^O(\tilde{Q}^-)$. Minimum risk cost of an object ϖ_i in each region is shown as follows:

$$MRC^{O-}(\varpi_i) = \begin{cases} \tilde{\mathcal{L}}(\tilde{a_N}|^{-}Y^{max}_{\varpi_i}), \text{ if } \varpi_i \in POS^O(\tilde{Q}^-), \\ \tilde{\mathcal{L}}(\tilde{a_B}|^{-}Y^{max}_{\varpi_i}), \text{ if } \varpi_i \in BND^O(\tilde{Q}^-), \\ \tilde{\mathcal{L}}(\tilde{a_P}|^{-}Y^{max}_{\varpi_i}), \text{ if } \varpi_i \in ENG^O(\tilde{Q}^-). \end{cases} \tag{5.3}$$

Definition 5.2. Suppose that (U, AT, V, ω) denotes an MADM information system and $\sigma_j \neq 0.5$ for some $a_j \in AT$. Define an optimistic ranking function $d^{O-}: U \longrightarrow [0,1]$ by

$$d^{O-}(\varpi_i) = \begin{cases} 1 - MRC^{O-}(\varpi_i), \text{ if } \varpi_i \in POS^O(\tilde{Q}^-), \\ 2 - MRC^{O-}(\varpi_i), \text{ if } \varpi_i \in BND^O(\tilde{Q}^-), \\ 3 - MRC^{O-}(\varpi_i), \text{ if } \varpi_i \in ENG^O(\tilde{Q}^-). \end{cases} \tag{5.4}$$

The optimistic ranking function d^{O-} is monotone increasing, that is, the better the object ϖ_i is, the greater the value of $d^{O-}(\varpi_i)$ is.

According to Definition 5.2, the idea of our proposed optimistic ranking function d^{O-} is: the order of optimistic optimal ideal object selection is $ENG^O(\tilde{Q}^-) \succ BND^O(\tilde{Q}^-) \succ POS^O(\tilde{Q}^-)$. If two objects of the universe belong to the same region, for example, $\varpi_i, \varpi_k \in POS^O(\tilde{Q}^-)$ and $MRC^{O-}(\varpi_i) \leq MRC^{O-}(\varpi_k)$, then ϖ_i is worst than ϖ_k, which denotes $\varpi_i \succ \varpi_k$. Thus, we can give an order of optimistic optimal ideal object selection by mean of the minimum risk cost, which is generated by $\tilde{Q}^-$.

Next, we will introduce the notion of an optimistic sort function induced by aggregating optimistic order functions d^{O+} and d^{O-}, respectively.

Definition 5.3. Suppose that (U, AT, V, ω) denotes an MADM information system. Define an optimistic sort function $\delta^O: U \longrightarrow [0,1]$ by

$$\delta^O(\varpi_i) = \frac{d^{O+}(\varpi_i)}{d^{O+}(\varpi_i) + d^{O-}(\varpi_i)}. \tag{5.5}$$

$\delta^O(\varpi_i)$ represents the optimal ideal object ordering index of ϖ_i. The ranking function δ^O is decreasing, that is, the better the object ϖ_i is, the smaller the value of $\delta^O(\varpi_i)$ is.

Finally, we can obtain an optimal ideal object selection be means of the optimistic sort function δ^O.

In the following, we summarize the processing of the above optimistic 3WD-MADM approach and give its algorithm.

Step 1 Divide an MADM information system (U, AT, V, ω) into two fuzzy information systems (U, C^+) and (U, C^-).

Step 2 Identify a pair of ideal decision object sets in relation to the attribute set AT, denoted by $\tilde{Q}^-$ for the negative ideal and $\tilde{Q}^+$ for the positive ideal.

Step 3 Enumerate the two sets of all MCBs $MCB(R^+)$ and $MCB(R^-)$ with respect to AT.

Step 4 Compute conditional probability $P(\tilde{Q}^+|^{-}Y^{max}_{\varpi_i})$ and $P(\tilde{Q}^-|^{-}Y^{max}_{\varpi_i})$, the thresholds α_i, β_i, ξ_i and η_i.

Step 5 Enumerate $POS^O(\tilde{Q}^+)$, $BND^O(\tilde{Q}^+)$ $ENG^O(\tilde{Q}^+)$, $POS^O(\tilde{Q}^-)$, $BND^O(\tilde{Q}^-)$ and $ENG^O(\tilde{Q}^-)$.

Step 6 Calculate $d^{O+}(\varpi_i)$ and $d^{O-}(\varpi_i)$.

Step 7 Calculate $\delta^O(\varpi_i)$.

Remark 5.4. The time complexity of the optimistic 3WD-MADM approach is analyzed as follows: **Step 1** and **Step 2** obtain two fuzzy information systems (U, C^+) and (U, C^-), the complexity of these steps is $O(2n)$. **Step 3** calculates the two sets of all MCBs based on AT in (U, C^+) and (U, C^-) by using the existing method in [39], the complexity of this step is $O(n^3)$. In the next step, it calculates conditional probability and thresholds for each $x_i \in U$, the complexity of **Step 4** is $O(n)$. The complexity of enumerating the three regions with respect to $\tilde{Q}^+$ and $\tilde{Q}^-$ is $O(mn)$ in **Step 5**. **Step 6** and **Step 7** calculate the ranking functions and the complexity is $O(n)$. **Step 8** has a complexity of $O(n)$. Altogether, the overall complexity of the process is dominated by the most resource-intensive step, which is $O(n^3)$.

Drawing inspiration from the optimistic 3WD-MADM approach, we derive two pessimistic ranking functions, d^{P+} and d^{P-}, based on Equations (5.2) and (5.4). We also obtain a pessimistic sorting function, δ^P, from Equation (5.5). This establishes the pessimistic 3WD-MADM approach within the pessimistic probabilistic RFS model framework.

In the following theorem and corollary, we explore the relationships between d^{P-} (d^{P+}) and d^{O+} (d^{O-}) in an MADM single-valued fuzzy information system.

Algorithm 1

Input: An MADM information system (U, AT, V, ω) and for each $a_j \in AT$, σ_j, risk avoidance coefficient σ_j.
Output: The ranking δ for all objects.
1. **Get** two fuzzy information systems (U, C^+) and (U, C^-).
2. **for** j = 1:m
    Compute qualified and unqualified fuzzy set $\tilde{Q}_j^+$ and $\tilde{Q}_j^-$;
  end
Compute qualified and unqualified fuzzy set $\tilde{Q}^+$ and $\tilde{Q}^-$ based on the attribute set AT.
3. **Compute** all MCBs $MCB(R^+)$ in (U, C^+) and $MCB(R^-)$ in (U, C^-);
4. **Compute** conditional probabilities of all MCBs $P(\tilde{Q}^+|^+Y_{\varpi_i})$ in (U, C^+) and $P(\tilde{Q}^-|^-Y_{\varpi_i})$ in (U, C^-);
  Get $P(\tilde{Q}_j^+|^+Y_{\varpi_i}^{max})$ and $P(\tilde{Q}_j^-|^-Y_{\varpi_i}^{max})$.
5. **for** 1:n
    Compute thresholds α_i, β_i, ξ_i and η_i;
  end
6. **if** $P(\tilde{Q}^+|^+Y_{\varpi_i}^{max}) \geq \alpha_i \left(P(\tilde{Q}^-|^-Y_{\varpi_i}^{max}) \geq \xi_i\right)$
    Decide $\varpi_i \in POS^O(\tilde{Q}^+) \left(\varpi_i \in POS^O(\tilde{Q}^-)\right)$;
  elseif $P(\tilde{Q}^+|^+Y_{\varpi_i}^{max}) < \alpha_i$ and $P(\tilde{Q}^+|^+Y_{\varpi_i}^{max}) > \beta_i \left(P(\tilde{Q}^-|^-Y_{\varpi_i}^{max}) < \xi_i \text{ and } P(\tilde{Q}^-|^-Y_{\varpi_i}^{max}) > \eta_i\right)$
    Decide $\varpi_i \in BND^O(\tilde{Q}^+) \left(\varpi_i \in BND^O(\tilde{Q}^-)\right)$;
  else
    Decide $\varpi_i \in ENG^O(\tilde{Q}^+) \left(\varpi_i \in ENG^O(\tilde{Q}^-)\right)$;
  end
7. **if** $\varpi_i \in POS^O(\tilde{Q}^+) \left(\varpi_i \in POS^O(\tilde{Q}^-)\right)$
    Compute $d^{O+}(\varpi_i) = \tilde{\mathcal{L}}(\tilde{a_P}|^+Y_{\varpi_i}^{max}) \left(d^{O-}(\varpi_i) = 1 - \tilde{\mathcal{L}}(\tilde{a_N}|^-Y_{\varpi_i}^{max})\right)$;
  elseif $\varpi_i \in BND^O(\tilde{Q}^+) \left(\varpi_i \in BND^O(\tilde{Q}^-)\right)$
    Compute $d^{O+}(\varpi_i) = \tilde{\mathcal{L}}(\tilde{a_B}|^+Y_{\varpi_i}^{max}) + 1 \left(d^{O-}(\varpi_i) = 2 - \tilde{\mathcal{L}}(\tilde{a_B}|^-Y_{\varpi_i}^{max})\right)$;
  else
    Compute $d^{O+}(\varpi_i) = \tilde{\mathcal{L}}(\tilde{a_B}|^+Y_{\varpi_i}^{max}) + 2 \left(d^{O-}(\varpi_i) = 3 - \tilde{\mathcal{L}}(\tilde{a_P}|^-Y_{\varpi_i}^{max})\right)$;
  end
8. **for** 1:n
    Compute $\delta^O(\varpi_i)$
  end

Theorem 5.5. *Suppose that (U, AT, V, ω) denotes an MADM information system and $\sigma_j \neq 0.5$ for some $a_j \in AT$. If $\tilde{Q}_j^- = \neg\tilde{Q}_j^+$ for each $a_j \in AT$, then $d^{O+}(\varpi_i) \leq d^{O+}(\varpi_k)$ if and only if $d^{P-}(\varpi_i) \geq d^{P-}(\varpi_k)$ for each $\varpi_i, \varpi_k \in U$.*

Proof. For each $\varpi_i, \varpi_k \in U$, let $d^{O+}(\varpi_i) \leq d^{O+}(\varpi_k)$. Then there are two cases that may occur:

One case is that ϖ_i and ϖ_k are in the same region. We assume $d^{O+}(\varpi_i), d^{O+}(\varpi_k) \in [0, 1)$. Then it follows from Definition 5.1 that $\varpi_i, \varpi_k \in POS^O(\tilde{Q}^+)$ and $MRC^{O+}(\varpi_i) \leq MRC^{O+}(\varpi_k)$, i.e. $\tilde{\mathcal{L}}(\tilde{a_P}|^+Y_{\varpi_i}^{max}) \leq \tilde{\mathcal{L}}(\tilde{a_P}|^+Y_{\varpi_k}^{max})$. By Theorem 4.10, we have $\varpi_i, \varpi_k \in ENG^P(\tilde{Q}^-)$. Furthermore, we can obtain that

$$
\begin{aligned}
\tilde{\mathcal{L}}(\tilde{a_P}|^-Y_{\varpi_i}^{min}) &= (\sum_{j=1}^{m} \omega_j(\varpi_{ij}^- - \varpi_j^{--}))P(\tilde{Q}^-|^-Y_{\varpi_i}^{min}) \\
&= (\sum_{j=1}^{m} \omega_j((1 - \varpi_{ij}^+) - (1 - \varpi_j^{++})))P(\tilde{Q}^-|^-Y_{\varpi_i}^{min}) \\
&= (\sum_{j=1}^{m} \omega_j(\varpi_j^{++} - \varpi_{ij}^+))(1 - P(\tilde{Q}^+|^+Y_{\varpi_i}^{max})) \quad \text{(By Theorem 4.10)} \\
&= \tilde{\mathcal{L}}(\tilde{a_P}|^+Y_{\varpi_i}^{max}).
\end{aligned}
$$

Similarly, the expected loss relationship for ϖ_k can be expressed as $\tilde{\mathcal{L}}(\tilde{a_P}|^-Y_{\varpi_k}^{min}) = \tilde{\mathcal{L}}(\tilde{a_P}|^+Y_{\varpi_k}^{max})$. This implies that $\tilde{\mathcal{L}}(\tilde{a_P}|^-Y_{\varpi_i}^{min}) \leq \tilde{\mathcal{L}}(\tilde{a_P}|^-Y_{\varpi_k}^{min})$. We know $MRC^{P-}(\varpi_i) \leq MRC^{P-}(\varpi_k)$, which implies $d^{P-}(\varpi_i) \geq d^{P-}(\varpi_k)$. The case $d^{O+}(\varpi_i), d^{O+}(\varpi_k) \in [1, 2)$ and the case $d^{O+}(\varpi_i), d^{O+}(\varpi_k) \in [2, 3)$ can be proved similarly.

The other case is that ϖ_i and ϖ_k are in the different regions. We assume $d^{O+}(\varpi_i) \in [0, 1)$ and $d^{O+}(\varpi_k) \in [1, 2)$. Then we have $\varpi_i \in POS^O(\tilde{Q}^+)$ and $\varpi_k \in BND^O(\tilde{Q}^+)$. According to Theorem 4.10, we can obtain that $\varpi_i \in ENG^P(\tilde{Q}^-)$ and $\varpi_k \in BND^P(\tilde{Q}^-)$. It follows that $d^{P-}(\varpi_i) \geq d^{P-}(\varpi_k)$. The case $d^{O+}(\varpi_i) \in [0, 1)$, $d^{O+}(\varpi_k) \in [2, 3)$ and the case $d^{O+}(\varpi_i) \in [1, 2)$, $d^{O+}(\varpi_k) \in [2, 3)$ can be proved similarly.

Conversely, the proof of sufficiency can also be proved. □

Corollary 5.6. *Suppose that (U, AT, V, ω) denotes an MADM information system and $\sigma_j \neq 0.5$ for some $a_j \in AT$. If $\tilde{Q}_j^- = \neg\tilde{Q}_j^+$ for each $a_j \in AT$, then $d^{P+}(\varpi_i) \leq d^{P+}(\varpi_k)$ if and only if $d^{O-}(\varpi_i) \geq d^{O-}(\varpi_k)$ for each $\varpi_i, \varpi_k \in U$.*

Table 15
Thresholds α_i, β_i, ξ_i and η_i in Example 4.3.

U/AT	α_i	β_i	ξ_i	η_i
ϖ_1	0.6726	0.4773	0.5227	0.3274
ϖ_2	0.1688	0.0828	0.9172	0.8312
ϖ_3	0.9531	0.9002	0.0998	0.0469
ϖ_4	0.499	0.3069	0.6931	0.501
ϖ_5	0.3306	0.18	0.82	0.6694
ϖ_6	0.6	0.4	0.6	0.4

Table 16
Conditional probabilities in Example 4.3.

U/AT	$P(\tilde{Q}^+\|^+Y^{max}_{\varpi_i})$	$P(\tilde{Q}^+\|^+Y^{min}_{\varpi_i})$	$P(\tilde{Q}^-\|^-Y^{max}_{\varpi_i})$	$P(\tilde{Q}^-\|^-Y^{min}_{\varpi_i})$
ϖ_1	0.635	0.505	0.495	0.365
ϖ_2	0.6975	0.6975	0.3025	0.3025
ϖ_3	0.505	0.505	0.495	0.495
ϖ_4	0.6975	0.635	0.365	0.3025
ϖ_5	0.6975	0.635	0.365	0.3025
ϖ_6	0.6975	0.505	0.495	0.3025

Table 17
Evaluation indices in Example 4.3.

U/AT	$d^{O+}(\varpi_i)$	$d^{O-}(\varpi_i)$	$d^{P+}(\varpi_i)$	$d^{P-}(\varpi_i)$	$\delta^O(\varpi_i)$	$\delta^P(\varpi_i)$
ϖ_1	1.1044	1.1088	1.8912	1.8956	0.3687	0.3691
ϖ_2	0.0197	0.0197	2.9803	2.9803	0.0066	0.0066
ϖ_3	2.0189	2.0189	0.9811	0.9811	0.673	0.673
ϖ_4	0.0658	0.0794	2.9206	2.9342	0.022	0.0263
ϖ_5	0.0408	0.0493	2.9507	2.9592	0.0137	0.0164
ϖ_6	0.0824	1.109	1.891	2.9176	0.0418	0.2754

Proof. The proof can be given applying Definition 5.2 and Theorem 4.11. $\square$

Remark 5.7. When (U, AT, V, ω) is an MADM information system with single-valued fuzzy information on $[0,1]$ and $\sigma_j \neq 0.5$ for some $a_j \in AT$, the two orders of optimistic optimal ideal object selection generated by d^{O+} and d^{P-} are consistent, and the two orders of optimistic optimal ideal object selection generated by d^{O-} and d^{P+} are consistent.

5.2. *Illustrative examples of optimistic and pessimistic 3WD-MADM approaches*

In this subsection, we will apply our proposed optimistic and pessimistic 3WD-MADM approaches into the medical diagnosis in Example 4.3 and the selection of the optimal supplier in Example 4.6.

5.2.1. *Analysis of an MADM information system with single-valued fuzzy information*

We initially illustrate the decision-making process for medical diagnosis as detailed in Example 4.3.

Firstly, the two fuzzy information systems (U, C^+) and (U, C^-) obtained from Table 3 are shown in Tables 4 and 5. $\tilde{Q}^+$ and $\tilde{Q}^-$ are shown in Table 6.

Secondly, according to Definition 4.8, the MCBs generated by (U, C^+) and (U, C^-) are calculated as follows:

$$MCB(R^+) = MCB(R^-) = \{\{\varpi_1, \varpi_3, \varpi_6\}, \{\varpi_2, \varpi_4, \varpi_5, \varpi_6\}, \{\varpi_1, \varpi_4, \varpi_5, \varpi_6\}\}.$$

Thirdly, we fix on risk avoidance coefficients $\sigma = \{0.4, 0.4, 0.4, 0.4\}$. Three thresholds α_i, β_i, ξ_i and η_i are calculated via the Equations (4.12), (4.13), (4.18) and (4.19), as shown in Table 15. And computational results of the conditional probability $P(\tilde{Q}^+|^+Y^{max}_{\varpi_i})$, $P(\tilde{Q}^+|^+Y^{min}_{\varpi_i})$, $P(\tilde{Q}^-|^-Y^{max}_{\varpi_i})$ and $P(\tilde{Q}^-|^-Y^{min}_{\varpi_i})$ are given in Table 16.

Fourthly, optimistic and pessimistic 3WD-results generated by $\tilde{Q}^+$ and $\tilde{Q}^-$ are illustrated via the rules (P^+_O)–(N^+_O), (P^+_P)–(N^+_P), (P^-_O)–(N^-_O) and (P^-_P)–(N^-_P), respectively, as shown in Tables 19 and 20.

Finally, results of evaluation indices $d^{O+}(\varpi_i)$, $d^{O-}(\varpi_i)$ $d^{P+}(\varpi_i)$, $d^{P-}(\varpi_i)$, $\delta^O(\varpi_i)$ and $\delta^P(\varpi_i)$ can be calculated as shown in Table 17. Rank results of these evaluation indices are plotted in Table 18. The order of the optimal ideal object selection in optimistic and pessimistic cases is:

$$\varpi_2 \succ \varpi_5 \succ \varpi_4 \succ \varpi_6 \succ \varpi_1 \succ \varpi_3.$$

Table 18
Rank results of evaluation indices in Example 4.3.

	Rank results
d^{O+}	$\varpi_2 \succ \varpi_5 \succ \varpi_4 \succ \varpi_6 \succ \varpi_1 \succ \varpi_3$
d^{O-}	$\varpi_2 \succ \varpi_5 \succ \varpi_4 \succ \varpi_1 \succ \varpi_6 \succ \varpi_3$
d^{P+}	$\varpi_2 \succ \varpi_5 \succ \varpi_4 \succ \varpi_1 \succ \varpi_6 \succ \varpi_3$
d^{P-}	$\varpi_2 \succ \varpi_5 \succ \varpi_4 \succ \varpi_6 \succ \varpi_1 \succ \varpi_3$
δ^O	$\varpi_2 \succ \varpi_5 \succ \varpi_4 \succ \varpi_6 \succ \varpi_1 \succ \varpi_3$
δ^P	$\varpi_2 \succ \varpi_5 \succ \varpi_4 \succ \varpi_6 \succ \varpi_1 \succ \varpi_3$

Table 19
Optimistic three-way decision results in Example 4.3.

	POS^O	BND^O	ENG^O
$\tilde{Q}^-$	$\{\varpi_3\}$	$\{\varpi_1, \varpi_6\}$	$\{\varpi_2, \varpi_4, \varpi_5\}$
$\tilde{Q}^+$	$\{\varpi_2, \varpi_4, \varpi_5, \varpi_6\}$	$\{\varpi_1\}$	$\{\varpi_3\}$

Table 20
Pessimistic three-way decision results in Example 4.3.

	POS^O	BND^O	ENG^O
$\tilde{Q}^-$	$\{\varpi_3\}$	$\{\varpi_1\}$	$\{\varpi_2, \varpi_4, \varpi_5, \varpi_6\}$
$\tilde{Q}^+$	$\{\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_1, \varpi_6\}$	$\{\varpi_3\}$

Table 21
Optimistic three-way decision results in Example 4.6.

	POS^O	BND^O	ENG^O
$\tilde{Q}^+$	$\{\varpi_4\}$	$\{\varpi_2, \varpi_5, \varpi_7, \varpi_8, \varpi_9\}$	$\{\varpi_1, \varpi_3, \varpi_6, \varpi_{10}\}$
$\tilde{Q}^-$	$\emptyset$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_6, \varpi_7, \varpi_9, \varpi_{10}\}$	$\{\varpi_4, \varpi_5, \varpi_8\}$

Table 22
Pessimistic three-way decision results in Example 4.6.

	POS^O	BND^O	ENG^O
$\tilde{Q}^+$	$\{\varpi_4\}$	$\{\varpi_2, \varpi_5, \varpi_8, \varpi_9\}$	$\{\varpi_1, \varpi_3, \varpi_6, \varpi_7, \varpi_{10}\}$
$\tilde{Q}^-$	$\emptyset$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_6, \varpi_7, \varpi_{10}\}$	$\{\varpi_4, \varpi_5, \varpi_8, \varpi_9\}$

5.2.2. Analysis of an MADM information system with intuitionistic fuzzy information

In the following, we will display the selection process of the optimal supplier in Example 4.6.

All MCBs generated by (U, C^+) and (U, C^-) in Tables 8 and 9 are:

$$MCB(R^+) = \{\{\varpi_1, \varpi_3, \varpi_5, \varpi_7, \varpi_{10}\}, \{\varpi_2, \varpi_8, \varpi_9\}, \{\varpi_2, \varpi_5, \varpi_7, \varpi_{10}\}, \{\varpi_2, \varpi_5, \varpi_7, \varpi_9\}, \{\varpi_4, \varpi_5, \varpi_9\},$$

$$\{\varpi_5, \varpi_6, \varpi_7, \varpi_{10}\}\},$$

$$MCB(R^-) = \{\{\varpi_1, \varpi_3, \varpi_7\}, \{\varpi_1, \varpi_5, \varpi_9, \varpi_{10}\}, \{\varpi_2, \varpi_6\}, \{\varpi_2, \varpi_5, \varpi_9, \varpi_{10}\}, \{\varpi_4, \varpi_5, \varpi_9, \varpi_{10}\},$$

$$\{\varpi_7, \varpi_8\}\}.$$

This shows that MCBs generated by (U, C^+) and (U, C^-) in an MADM intuitionistic fuzzy information system are inconsistent. Next, we fix on $\sigma = \{0.3, 0.1, 0.4, 0.2\}$, the three thresholds α_i, β_i, ξ_i and η_i can be calculated, and optimistic and pessimistic 3WD-results generated by $\tilde{Q}^+$ and $\tilde{Q}^-$ are shown in Tables 21 and 22. Rank results of these evaluation indices are plotted in Table 23. The order of the optimal ideal object selection in the optimistic case is:

$$\varpi_4 \succ \varpi_5 \succ \varpi_8 \succ \varpi_9 \succ \varpi_2 \succ \varpi_7 \succ \varpi_6 \succ \varpi_1 \succ \varpi_3 \succ \varpi_{10}.$$

The order of the optimal ideal object selection in the pessimistic case is:

$$\varpi_4 \succ \varpi_5 \succ \varpi_9 \succ \varpi_8 \succ \varpi_2 \succ \varpi_6 \succ \varpi_1 \succ \varpi_{10} \succ \varpi_3 \succ \varpi_7.$$

Table 23
Rank results of evaluation indices in Example 4.6.

	Rank results
d^{O+}	$\varpi_4 \succ \varpi_5 \succ \varpi_9 \succ \varpi_7 \succ \varpi_2 \succ \varpi_8 \succ \varpi_6 \succ \varpi_1 \succ \varpi_3 \succ \varpi_{10}$
d^{O-}	$\varpi_5 \succ \varpi_4 \succ \varpi_8 \succ \varpi_6 \succ \varpi_2 \succ \varpi_9 \succ \varpi_7 \succ \varpi_1 \succ \varpi_{10} \succ \varpi_3$
d^{P+}	$\varpi_4 \succ \varpi_2 \succ \varpi_9 \succ \varpi_8 \succ \varpi_5 \succ \varpi_6 \succ \varpi_1 \succ \varpi_{10} \succ \varpi_3 \succ \varpi_7$
d^{P-}	$\varpi_5 \succ \varpi_4 \succ \varpi_8 \succ \varpi_9 \succ \varpi_6 \succ \varpi_2 \succ \varpi_7 \succ \varpi_1 \succ \varpi_{10} \succ \varpi_3$
δ^{O}	$\varpi_4 \succ \varpi_5 \succ \varpi_8 \succ \varpi_9 \succ \varpi_2 \succ \varpi_7 \succ \varpi_6 \succ \varpi_1 \succ \varpi_3 \succ \varpi_{10}$
δ^{P}	$\varpi_4 \succ \varpi_5 \succ \varpi_8 \succ \varpi_9 \succ \varpi_2 \succ \varpi_6 \succ \varpi_1 \succ \varpi_{10} \succ \varpi_3 \succ \varpi_7$

Table 24
The ranking results of various MADM methods in Example 4.3.

Methods	Ranking results	Optimal
WAA [45]	$\varpi_2 \succ \varpi_5 \succ \varpi_4 \succ \varpi_6 \succ \varpi_1 \succ \varpi_3$	ϖ_2
TOPSIS [26]	$\varpi_2 \succ \varpi_5 \succ \varpi_4 \succ \varpi_6 \succ \varpi_1 \succ \varpi_3$	ϖ_2
EDAS [11]	$\varpi_2 \succ \varpi_5 \succ \varpi_4 \succ \varpi_6 \succ \varpi_1 \succ \varpi_3$	ϖ_2
VIKOR [36] ($\upsilon = 0.6$)	$\varpi_2 \succ \varpi_5 \succ \varpi_6 \succ \varpi_4 \succ \varpi_1 \succ \varpi_3$	ϖ_2
Ye et al. [49]	$\varpi_2 \succ \varpi_5 \succ \varpi_6 \succ \varpi_4 \succ \varpi_1 \succ \varpi_3$	ϖ_2
Our method (O & P)	$\varpi_2 \succ \varpi_5 \succ \varpi_4 \succ \varpi_6 \succ \varpi_1 \succ \varpi_3$	ϖ_2

6. Comparative analysis and discussions

In this section, in order to demonstrate the validity, applicability, and practicality of our proposed optimistic 3WD-MADM approach (method (O)) and pessimistic 3WD-MADM approach (method (P)), we compare them to some related MADM methods. Considering that existing MADM methods can only make available for one of MADM single-valued and intuitionistic fuzzy information systems, we will choose four kinds of traditional MADM methods and six newly MADM methods, namely, WAA [45], TOPSIS [26], EDAS [11] and VIKOR [36], Ye et al. [49], Xu and Member [42], Liu et al. [28], Jiang [17], Zhang [52] and Ye [50].

6.1. Comparative analysis in an MADM single-valued fuzzy information system

In the current subsection, we utilize a medical diagnosis in Example 4.3, and compare our proposed approaches to the four kinds of traditional MADM methods [11,26,36,45] and Ye et al.'s MADM approach [49] via the same risk avoidance coefficients $\sigma = \{0.4, 0.4, 0.4, 0.4\}$. Their ranking results of the optimal object selection are shown in Table 24.

According to Table 24, we can confirm that the optimal object selection of these seven methods is x_2, ranking results of WAA, TOPSIS, EDAS and our proposed methods are completely consistent and there is a high degree of similarity among our proposed methods, VIKOR and Ye et al.'s MADM method.

Through this case study, we aim to illustrate four key points. Firstly, our method is applicable to single-valued fuzzy decision information systems and is capable of efficiently identifying the optimal decision alternative. Secondly, in terms of consistency with the results of traditional methods, our approach significantly outperforms Ye et al.'s MADM method. Thirdly, despite a high degree of similarity in the overall ranking, there are still some differences. The reasons for these differences are twofold. On the one hand, our method conducts data processing and decision analysis from both positive and negative dimensions, whereas Ye et al.'s MADM method performs decision analysis only from the positive dimension. On the other hand, the use of MCBs in our approaches also contributes to the differences observed, which are considered to be within the normal range of expected results. Finally, the case study demonstrates that the results of our proposed methods (P) and (O) are consistent in an MADM single-valued fuzzy information system, thereby experimentally verifying Remark 5.7.

6.2. Comparative analysis in an MADM intuitionistic fuzzy information system

To show applicability of our proposed methods, we compare our methods (P) and (O) with the existing MADM methods [28,42] in an intuitionistic fuzzy environment.

According to the selection of the optimal supplier in Example 4.6, we compare our proposed methods with Xu and Member [42] under the same weight vector $\omega = \{0.22, 0.22, 0.36, 0.2\}$. Furthermore, we compare them with Liu [28] by using $\sigma = \{0.3, 0.1, 0.4, 0.2\}$. The ranking results are shown in Table 25, where Xu and Member's MADM method [42] denotes Xu, Liu et al.'s MADM method [28] denotes Liu.

From Table 25, it can be seen that the selection of the optimal supplier of the four methods is ϖ_4. In order to observe the correlation between these methods, we introduce the definition of the Spearman's rank correlation coefficient, which is referenced in [22,30] and usually shortened by SRCC, to assess the correlation between the rankings produced by our proposed methods and those of existing methods [28,42]. The equation of SRCC is defined by

$$\mathrm{SRCC} = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)}, \tag{6.1}$$

Table 25
The ranking results of various MADM methods in Example 4.6.

Methods	Ranking results	Optimal
Xu [42]	$\varpi_4 \succ \varpi_2 \succ \varpi_9 \succ \varpi_8 \succ \varpi_5 \succ \varpi_7 \succ \varpi_3 \succ \varpi_6 \succ \varpi_1 \succ \varpi_{10}$	ϖ_4
Liu [28]	$\varpi_4 \succ \varpi_8 \succ \varpi_9 \succ \varpi_5 \succ \varpi_2 \succ \varpi_3 \succ \varpi_{10} \succ \varpi_7 \succ \varpi_1 \succ \varpi_6$	ϖ_4
Our method (O)	$\varpi_4 \succ \varpi_5 \succ \varpi_8 \succ \varpi_9 \succ \varpi_2 \succ \varpi_7 \succ \varpi_6 \succ \varpi_1 \succ \varpi_3 \succ \varpi_{10}$	ϖ_4
Our method (P)	$\varpi_4 \succ \varpi_5 \succ \varpi_9 \succ \varpi_8 \succ \varpi_2 \succ \varpi_6 \succ \varpi_1 \succ \varpi_{10} \succ \varpi_3 \succ \varpi_7$	ϖ_4

Table 26
The SRCCs between our methods and existing MADM methods in Example 4.6.

Methods	Xu [42]	Liu [28]	method (O)	method (P)
Xu [42]	1	0.9677	0.9737	0.9495
Liu [28]	—	1	0.9616	0.9576
method (O)	—	—	1	0.9758
method (P)	—	—	—	1

Table 27
An example of investment projects.

U/AT	c_1	c_2	c_3	c_4	c_5
ϖ_1	0.8	0.4	0.3	0.8	0.9
ϖ_2	0.9	0.5	0.5	0.7	0.6
ϖ_3	0.3	0.4	0.6	0.4	0.3
ϖ_4	0.5	0.2	0.2	0.7	0.6
ϖ_5	0.7	0.6	0.6	0.5	0.8
ϖ_6	0.4	0.8	0.7	0.7	0.3
ϖ_7	0.9	0.5	0.1	0.8	0.7
ϖ_8	0.6	0.8	0.8	0.3	0.4

where n is the number of all objects, $d_i = r_i - s_i$, in which r_i and s_i denote the ranking positions of ϖ_i in the rankings of our proposed methods and the existing methods, respectively. The value range of SRCC is -1 to 1. Generally speaking, when SRCC ≥ 0.8, it shows that there is a strong similarity between the two ranking [12,50].

In Table 26, it is shown that the SRCCs between the ranking results of our methods and these existing methods [12,50] are greater than 0.9, which implies that the degrees of correlation among the four methods are very high.

This case study aims to highlight three critical points. Firstly, the fact that all four methods consistently selected ϖ_4 as the optimal decision alternative in Table 26 demonstrates the effectiveness of our proposed methods. Secondly, the differences in the ranking results between methods (P) and (O) validate the theoretical analysis of our methods when applied to intuitionistic fuzzy information systems. Thirdly, despite the high values of SRCCs, some discrepancies still exist. This is because the data analysis approaches differ: our methods on decomposing the data from both positive and negative perspectives for decision analysis, whereas Xu and Liu [28,42] integrate the data values as intuitionistic fuzzy numbers without fully analyzing the information from both angles. However, this analysis does not imply that one method is more representative than the other, nor does it illustrate the superiority of using MCBs. Therefore, in the following subsection, we will further elaborate on the differences by analyzing the selection of granularity.

6.3. Comparative analysis with three fuzzy β-neighborhood-based MADM methods

In this subsection, we provide another example to illustrate the practicality of our proposed methods, and show the superiority of our methods by comparing them with the existing methods [17,50,52], which are three types of fuzzy β-neighborhood-based MADM methods.

Suppose that $U = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5, \varpi_6, \varpi_7, \varpi_8\}$ denotes the set of eight investment projects, $AT = \{c_1, c_2, c_3, c_4, c_5\}$ denotes five evaluation criteria, where c_2 is environmental influence, c_4 is social benefit, c_1 is expected benefits, c_3 is market saturation and c_5 is energy conservation. Except that c_2 and c_3 are cost attributes, the rest are benefit attributes. $\omega = \{0.3, 0.1, 0.3, 0.2, 0.1\}$ is the weight vector of AT. The data in Table 27 is the evaluation values ${}^oV = \{{}^ou_{ij} \mid j = 1, 2, ..., 5, i = 1, 2, ..., 8\}$ of the eight investment projects with respect to the five attributes.

First of all, we have transformed the data presented in Table 27 into an MADM information system denoted as (U, AT, V, ω), as illustrated in Table 28, employing the following principles:

$$u_{ij} = \begin{cases} {}^ou_{ij}, \text{ if } a_j \text{ is a benefit attribute,} \\ 1 - {}^ou_{ij}, \text{ if } a_j \text{ is a cost attribute.} \end{cases}$$

Table 28
An MADM information system (U, AT, V, ω).

U/AT	c_1	c_2	c_3	c_4	c_5
ϖ_1	0.8	0.6	0.7	0.8	0.9
ϖ_2	0.9	0.5	0.5	0.7	0.6
ϖ_3	0.3	0.6	0.4	0.4	0.3
ϖ_4	0.5	0.8	0.8	0.7	0.6
ϖ_5	0.7	0.4	0.4	0.5	0.8
ϖ_6	0.4	0.2	0.3	0.7	0.3
ϖ_7	0.9	0.5	0.9	0.8	0.7
ϖ_8	0.6	0.2	0.2	0.3	0.4

Table 29
The ranking results of various MADM methods.

Methods	Ranking results	Optimal
WAA [45]	$\varpi_7 \succ \varpi_1 \succ \varpi_2 \succ \varpi_4 \succ \varpi_5 \succ \varpi_6 \succ \varpi_3 \succ \varpi_8$	ϖ_7
TOPSIS [26]	$\varpi_7 \succ \varpi_1 \succ \varpi_2 \succ \varpi_4 \succ \varpi_5 \succ \varpi_6 \succ \varpi_8 \succ \varpi_3$	ϖ_7
EDAS [11]	$\varpi_7 \succ \varpi_1 \succ \varpi_4 \succ \varpi_2 \succ \varpi_5 \succ \varpi_6 \succ \varpi_3 \succ \varpi_8$	ϖ_7
VIKOR [36] ($\upsilon = 0.6$)	$\varpi_7 \succ \varpi_1 \succ \varpi_2 \succ \varpi_4 \succ \varpi_5 \succ \varpi_6 \succ \varpi_3 \succ \varpi_8$	ϖ_7
Ye et al. [50]	$\varpi_7 \succ \varpi_1 \succ \varpi_4 \succ \varpi_2 \succ \varpi_5 \succ \varpi_6 \succ \varpi_3 \succ \varpi_8$	ϖ_7
method (O & P)	$\varpi_7 \succ \varpi_1 \succ \varpi_4 \succ \varpi_2 \succ \varpi_5 \succ \varpi_8 \succ \varpi_3 \succ \varpi_6$	ϖ_7
Zhang et al. [52]	$\varpi_5 \succ \varpi_3 \succ \varpi_2 \approx \varpi_4 \approx \varpi_7 \succ \varpi_6 \succ \varpi_8 \succ \varpi_1$	ϖ_5
Jiang et al. [17]	$\varpi_1 \succ \varpi_7 \succ \varpi_4 \succ \varpi_2 \approx \varpi_5 \succ \varpi_3 \succ \varpi_8 \succ \varpi_6$	ϖ_1

Table 30
The SRCCs between our methods and four traditional MADM methods.

Methods	Zhang et al.	Jiang et al.	Ye et al.	method (O & P)
WAA [45]	0.0119	0.8690	0.9761	0.881
TOPSIS [26]	-0.1071	0.8452	0.9523	0.9048
EDAS [11]	0.0119	0.8928	1	0.9048
VIKOR [36]	0.0119	0.8690	0.9761	0.881

Table 31
The SRCCs between our methods and fuzzy β-neighborhood-based three MADM methods.

Methods	Jiang et al.	Zhang et al.	Ye et al.	method (O & P)
Jiang et al. [17]	1	0.0476	0.8929	0.9405
Zhang et al. [52]	—	1	0.0119	-0.0357
Ye et al. [50]	—	—	1	0.9048
method (O & P)	—	—	—	1

According to our proposed methods, Table 29 illustrates the ranking result of the eight investment projects, which also displays the ranking results by using four traditional MADM methods (WAA, TOPSIS, EDAS and VIKOR), Jiang et al.'s MADM approach [17] ($\beta = 0.5$, $k = 0$), Zhang et al.'s MADM approach [52] ($\beta = 0.5$, $q = 0.5$) and Ye et al.'s MADM approach [50] ($\alpha = 0.5$, $\varepsilon = 0.7$).

Except for Jiang [17] and Zhang [52], the selection of optimal investment project of the remaining six MADM approaches is ϖ_7. The ranking results of these MADM approaches are obviously inconsistent. To further explore correlation between our proposed MADM approaches and these existing MADM approaches, we adopt the SRCC to analyze the ranking decision-making results in Table 29. Firstly, Table 30 illustrates that values of SRCCs between our proposed MADM approaches and the four traditional MADM approaches (WAA, TOPSIS, EDAS and VIKOR) are greater than Jiang's MADM approach [17] and Zhang's MADM approach [52], and it is lower than the values of SRCCs between Ye's MADM approach [50] and the four traditional MADM approaches. Furthermore, we can learn from Tables 30 and 31 that the correlation between Zhang's MADM approach [52] and other MADM approaches is less than 0.8, which illustrates that the ranking result obtained by Zhang's MADM approach are of little reference value to our decision-makers in this MADM problem. Secondly, except for Zhang's MADM approach, the SRCCs among other MADM approaches are greater than 0.8 as shown in Table 31. Finally, in Fig. 1, the four bar charts in sequence represent the values of SRCCs between Jiang et al., Zhang et al., Ye et al., and method (O & P) and the methods indicated by the horizontal axis. It shows that the SRCCs between our MADM approaches and Jiang's MADM approach are higher than the SRCCs between Jiang's MADM approach and Ye's MADM approach, and the SRCCs between our methods and Ye's MADM approach is higher than the SRCCs between Ye's MADM approach and Jiang's MADM approach. This sufficiently shows that the ranking result of our MADM approach is highly similar to these existing MADM approaches and our 3WD-MADM approaches are reliable.

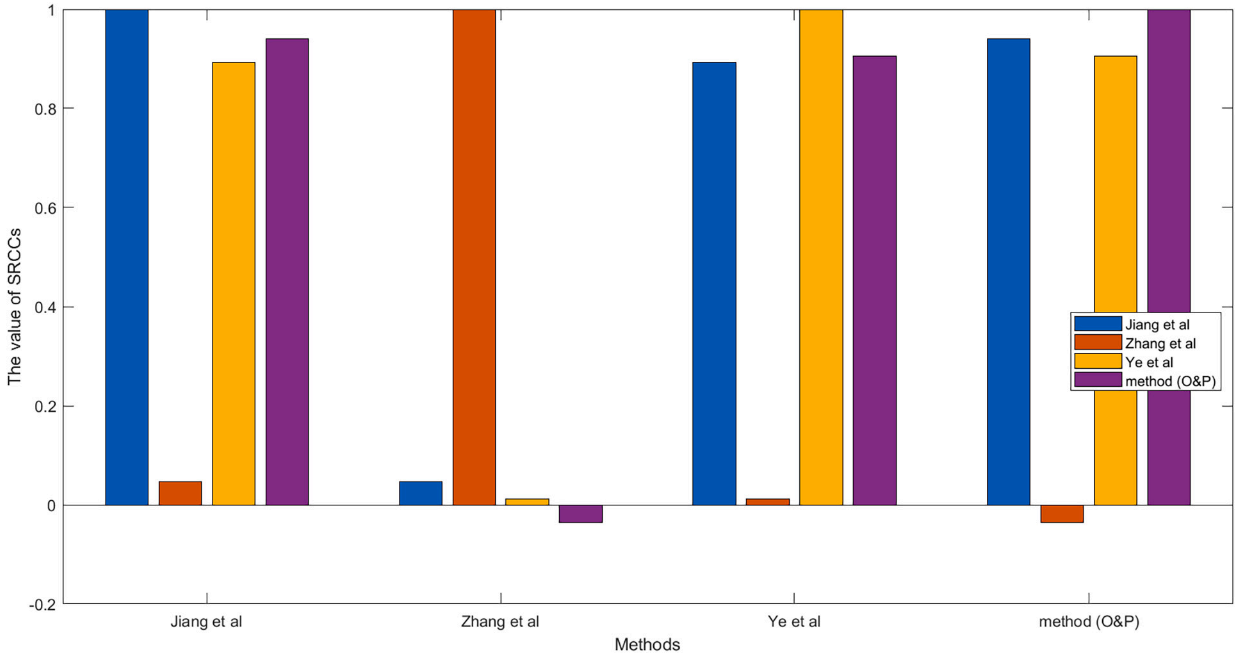

Fig. 1. Comparing the four MADM methods induced by the SRCCs. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

7. Conclusion

In this paper, we mainly proposed two novel 3WD-MADM approaches based on MCBs in the two types of MADM fuzzy information systems with single-valued and intuitionistic fuzzy information. The contributions are as follows:

(1) We generalized the existing MCBs-based rough set models to fuzzy situations and proposed optimistic and pessimistic MCBs-based probabilistic RFSs. Furthermore, we pointed out the necessity of their existence and got some properties.

(2) We provided two definitions of relative loss functions based on evaluation information values. Aim at a single-valued fuzzy information system, we got some interesting results for optimistic and pessimistic MCBs-based positive, negative and boundary regions of qualified and unqualified fuzzy sets.

(3) We tested our optimistic and pessimistic 3WD-MADM approaches in single-valued and intuitionistic fuzzy information systems, and compared them with some existing MADM methods suitable for single-valued fuzzy systems and some existing MADM methods suitable for intuitionistic fuzzy systems, demonstrating that our approaches can effectively solve the two types of MADM problems. Then our approaches were also compared with the methods of using fuzzy ε-neighborhood classes as granules, and we also concluded that it can effectively solve MADM problems.

The main idea of our proposed 3WD-MADM approaches is to divide the original complex evaluation information table into two new fuzzy information tables, and fusion the results of the 3WD rules obtained in the new tables. In this paper, we only considered single-valued and intuitionistic fuzzy information, the evaluation values of which is consisted of two values (membership value and non-membership value). In the further, we will improve and prefect our proposed 3WD-MADM approaches, and apply them to more types MADM fuzzy information systems such as interval-valued intuitionistic fuzzy [21], pythagorean fuzzy information [22,26] and interval-valued intuitionistic fuzzy soft sets information [10] and so on.

CRediT authorship contribution statement

Yan Sun: Writing – review & editing, Writing – original draft, Visualization, Resources, Methodology, Data curation, Conceptualization. **Bin Pang:** Writing – review & editing, Visualization, Supervision, Resources, Methodology, Investigation, Funding acquisition, Conceptualization. **Ju-Sheng Mi:** Writing – review & editing, Supervision, Methodology, Formal analysis, Funding acquisition. **Wei-Zhi Wu:** Writing – review & editing, Supervision, Methodology, Funding acquisition, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20 (1986) 87–96.
[2] Ö.N. Bilişk, N.H. Duman, E. Taş, A novel interval-valued intuitionistic fuzzy CRITIC-TOPSIS methodology: An application for transportation mode selection problem for a glass production company, Expert Syst. Appl. 235 (2024) 121134.
[3] W.J. Chang, C. Fu, A multi-criteria group decision-making method based on consistency of interval-valued distributed preference relations, Inf. Fusion 107 (2024) 102315.
[4] Z.X. Chen, G. Chen, C. Gao, J. Zhou, J.J. Wen, Robust weighted fuzzy margin-based feature selection with three-way decision, Int. J. Approx. Reason. 173 (2024) 109253.
[5] P.G. Clark, C. Gao, J.W. Grzymala-Busse, T. Mroczek, Characteristic sets and generalized maximal consistent blocks in mining incomplete data, Inf. Sci. 453 (2018) 66–79.
[6] P.G. Clark, C. Gao, J.W. Grzymala-Busse, T. Mroczek, R. Niemiec, Complexity of rule sets in mining incomplete data using characteristic sets and generalized maximal consistent blocks, Log. J. IGPL 29 (2) (2021) 124–137.
[7] P.G. Clark, J.W. Grzymala-Busse, Z.S. Hippe, T. Mroczek, Mining incomplete data using global and saturated probabilistic approximations based on characteristic sets and maximal consistent blocks, Inf. Sci. 662 (2024) 120287.
[8] J. Deng, J.M. Zhan, Z.S. Xu, E. Herrera-Viedma, Regret-theoretic multiattribute decision-making model using three-way framework in multiscale information systems, IEEE Trans. Cybern. 53 (6) (2023) 3988–4001.
[9] T. Feng, H.T. Fan, J.S. Mi, Uncertainty and reduction of variable precision multigranulation fuzzy rough sets based on three-way decisions, Int. J. Approx. Reason. 85 (2017) 36–58.
[10] H. Garg, R. Arora, A nonlinear-programming methodology for multi-attribute decision-making problem with interval-valued intuitionistic fuzzy soft sets information, Appl. Intell. 48 (2018) 2031–2046.
[11] M.K. Ghorabaee, E.K. Zavadskas, L. Olfat, Z. Turskis, Multi-criteria inventory classification using a new method of evaluation based on distance from average solution (EDAS), Informatica 26 (3) (2015) 435–451.
[12] X.F. Huang, J.M. Zhan, B.Z. Sun, A three-way decision method with pre-order relations, Inf. Sci. 595 (2022) 231–256.
[13] C.C. Hung, L.H. Chen, A fuzzy TOPSIS decision making model with entropy weight under intuitionistic fuzzy environment, Lect. Notes Eng. Comput. Sci. 2174 (1) (2009) 13–16.
[14] C.L. Hwang, K. Yoon, Multiple Attributes Decision Making Methods and Applications, Springer, Berlin, Heidelberg, 1981.
[15] C.Z. Jia, L.Q. Li, X.R. Li, A three-way decision combining multi-granularity variable precision fuzzy rough set and TOPSIS method, Int. J. Approx. Reason. 176 (2025) 109318.
[16] F. Jia, P.D. Liu, A novel three-way decision model under multiple-criteria environment, Inf. Sci. 471 (2019) 29–51.
[17] H.B. Jiang, J.M. Zhan, D.G. Chen, Covering-based variable precision (I,T)-fuzzy rough sets with applications to multiattribute decision-making, IEEE Trans. Fuzzy Syst. 27 (8) (2019) 1558–1572.
[18] W.J. Lei, W.M. Ma, X.N. Li, B.Z. Sun, Three-way group decision based on regret theory under dual hesitant fuzzy environment: an application in water supply alternatives selection, Expert Syst. Appl. 237 (2024) 121249.
[19] Y. Leung, D.Y. Li, Maximal consistent block technique for rule acquisition in incomplete information systems, Inf. Sci. 115 (2003) 85–106.
[20] Y. Leung, W.Z. Wu, W.X. Zhang, Knowledge acquisition in incomplete information systems: a rough set approach, Expert Syst. 168 (1) (2006) 164–180.
[21] D.-F. Li, Closeness coefficient based nonlinear programming method for interval-valued intuitionistic fuzzy multiattribute decision making with incomplete preference information, Appl. Soft Comput. 11 (4) (2011) 3402–3418.
[22] H.M. Li, Y.C. Cao, L.M. Su, Pythagorean fuzzy multi-criteria decision-making approach based on Spearman rank correlation coefficient, Soft Comput. 26 (2022) 3001–3012.
[23] R. Li, H.C. Chen, S.X. Liu, K. Wang, B. Wang, X.X. Hu, TFD-IIS-CRMCB: telecom fraud detection for incomplete information systems based on correlated relation and maximal consistent block, Entropy 25 (1) (2023) 112.
[24] R.C. Li, D. Liu, Y.J. Huang, RT-DIFTWD: a novel data-driven intuitionistic fuzzy three-way decision model with regret theory, Inf. Sci. 689 (2025) 121471.
[25] H.X. Li, X.Z. Zhou, Risk decision based on decision-theoretic rough set: a multi-view decision model, Int. J. Comput. Intell. Syst. 4 (1) (2011) 1–11.
[26] D.C. Liang, Z.S. Xu, D. Liu, Y. Wu, Method for three-way decisions using ideal TOPSIS solutions at Pythagorean fuzzy information, Inf. Sci. 435 (2018) 282–295.
[27] X.R. Liu, M.W. Shao, Approaches to computing maximal consistent block, Int. J. Mach. Learn. Cybern. 48 (2014) 264–274.
[28] P.D. Liu, Y.M. Wang, F. Jia, H. Fujita, A multiple attribute decision making three-way model for intuitionistic fuzzy numbers, Int. J. Approx. Reason. 119 (2020) 177–203.
[29] X.B. Ma, X.B. Yang, Y. Qi, X.N. Song, J.Y. Yang, Multicost decision-theoretic rough sets based on maximal consistent blocks, Rough Sets Knowl. Technol. 8818 (2014) 824–833.
[30] F. Mafakheri, L.M. Dai, D. Slezak, F. Nasiri, Project delivery system selection under uncertainty: multi-criteria multilevel decision aid model, J. Manag. Eng. 23 (2007) 200–206.
[31] T. Mroczek, Handling the complexity of computing maximal consistent blocks, Electronics 12 (10) (2023) 2295.
[32] Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11 (1982) 341–356.
[33] B.Z. Sun, W.M. Ma, Y. Qian, Multigranulation fuzzy rough set over two universes and its application to decision making, Knowl.-Based Syst. 123 (2017) 61–74.
[34] Y. Sun, J.-S. Mi, J.K. Chen, W. Liu, A new fuzzy multi-attribute group decision-making method with generalized maximal consistent block and its application in emergency, Knowl.-Based Syst. 215 (2021) 106594.
[35] Y. Sun, W.Z. Wu, X. Wang, Maximal consistent block based optimal scale selection for incomplete multi-scale information systems, Int. J. Mach. Learn. Cybern. 14 (5) (2023) 1797–1809.
[36] S. Opricovic, G.H. Tzeng, Compromise solution by MCDM methods: a comparative analysis of VIKOR and TOPSIS, Eur. J. Oper. Res. 156 (2) (2004) 445–455.
[37] G.A. Qi, M. Atef, B. Yang, Fermatean fuzzy covering-based rough set and their applications in multi-attribute decision-making, Eng. Appl. Artif. Intell. 127 (2024) 107181.
[38] C.Z. Wang, D.G. Chen, Q.H. Hu, Fuzzy information systems and their homomorphisms, Fuzzy Sets Syst. 249 (2014) 128–138.
[39] J.Q. Wang, X.H. Zhang, A new method of incomplete information processing based on maximal consistent block and its application, J. Nangjing Univ., Nat. Sci.) 58 (1) (2022) 83–93.
[40] W.D. Wang, K. Zhang, L.G. Zou, The multi-criteria ranking method for criterion-oriented regret three-way decision, Int. J. Approx. Reason. 179 (2025) 109374.
[41] Z.S. Xu, Uncertain Multi-Attribute Decision Making, Tsinghua University Press, Beijing, 2004.
[42] Z.S. Xu, S. Member, Intuitionistic fuzzy multiattribute decision making: an interactive method, IEEE Trans. Fuzzy Syst. 20 (3) (2012) 514–525.
[43] Z.S. Xu, M. Xia, Induced generalized intuitionistic fuzzy operators, Knowl.-Based Syst. 24 (2011) 197–209.

[44] Z.S. Xu, R.R. Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets, Int. J. Gen. Syst. 35 (4) (2006) 417–433.
[45] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decisionmaking, IEEE Trans. Syst. Man Cybern. 18 (1998) 183–190.
[46] Y.Y. Yao, Three-way decision: an interpretation of rules in rough set theory, in: Rough Sets and Knowledge Technology, Fourth International Conference, RSKT 2009, in: LNAI, vol. 5589, 2009, pp. 642–649.
[47] Y.Y. Yao, Three-way decisions with probabilistic rough sets, Inf. Sci. 180 (3) (2010) 341–353.
[48] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (1965) 338–353.
[49] J. Ye, J.M. Zhan, B.Z. Sun, A three-way decision method based on fuzzy rough set models under incomplete environments, Inf. Sci. 577 (2021) 22–48.
[50] J. Ye, J.M. Zhan, Z.S. Xu, A novel decision-making approach based on three-way decisions in fuzzy information systems, Inf. Sci. 541 (2020) 362–390.
[51] J.M. Zhan, J. Deng, Z.S. Xu, L. Martnez, A three-way decision methodology with regret theory via triangular fuzzy numbers in incomplete multiscale decision information systems, IEEE Trans. Fuzzy Syst. 31 (8) (2023) 2773–2787.
[52] K. Zhang, J.M. Zhan, W.Z. Wu, J.C.R. Alcantud, Fuzzy β-covering based (I,T)-fuzzy rough set models and applications to multi-attribute decision-making, Comput. Ind. Eng. 128 (2019) 605–621.
[53] X.X. Zhao, B.Z. Sun, X.D. Chu, J. Ye, X.L. Chu, A hybrid machine learning and three-way soft clustering integrated decision-making method with incomplete multi-source heterogeneous attribute information, Int. J. Approx. Reason. 186 (2025) 109491.