International Journal of Approximate Reasoning 187 (2025) 109551

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

Investigation of semantic behavior in probabilistic argumentation

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A R T I C L E I N F O

Keywords:
Probabilistic argumentation
Argumentation semantics
Argumentation dynamics

A B S T R A C T

Probabilistic Argumentation Frameworks (PAFs) extend Abstract Argumentation Frameworks (AAFs) by incorporating probabilistic measures to evaluate argument acceptability. While acceptability evaluations are determined by semantics in both AAFs and PAFs, some key properties underlying semantic behavior in PAFs remain underexplored. This paper systematically investigates directionality, skepticism adequacy, and dynamic monotony in PAFs, establishing their satisfiability across classical semantics. Importantly, we demonstrate that under any semantics, the satisfiability of directionality and skepticism adequacy from the perspective of individual argument acceptability is equivalent between AAFs and PAFs. Besides, for dynamics, we characterize how argument acceptabilities change with structural changes in PAFs, affected by the parity of attack paths. These theoretical insights advance the understanding of argumentation semantics under uncertainty, thereby providing guidance for adapting semantics in probabilistic environments.

1. Introduction

Evaluating the acceptability of an argument is a critical consideration in both expert systems and multi-agent systems [1–3], where it frequently serves as the bedrock for decision-making processes. The abstract argumentation framework (AAF) provides an efficient logical tool that evaluates arguments based on their interactions within the framework, as determined by a specific semantics function. Since Dung's seminal introduction of AAFs [4], the theoretical landscape has matured through systematic investigations into semantic properties, including directionality [5], composability [5], SCC-recursiveness [6], dynamics [7–9], and adequacy [5,10]. These foundational ideas, each with significant theoretical and practical value, have enabled the development of efficient algorithms for argument evaluation. They have also supported enforcement mechanisms and deepened our understanding of the philosophical and epistemological connections within AAFs.

However, because traditional AAFs lack quantitative representation, they struggle to precisely evaluate argument acceptability in practical debate contexts, which undermines their effectiveness in reasoning under uncertainty [11,12]. To address this limitation, several extended formalisms based on AAFs have been proposed, integrating probability calculus with argumentation semantics to create probabilistic argumentation frameworks (PAFs) [13–15]. By quantifying argument acceptability through continuous probabilistic measures, PAFs enable more nuanced evaluations in practical applications. Specifically, PAFs build upon the evaluation of acceptability across possible worlds, each represented as an AAF. Within each AAF, the acceptability of arguments is evaluated using semantics in combination with acceptance modes, namely credulous and skeptical modes. Then, the acceptability degree of a specific argument is computed as the sum of probabilities assigned to the AAFs in which the argument is deemed acceptable, known

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https://doi.org/10.1016/j.ijar.2025.109551
Received 21 March 2025; Received in revised form 11 August 2025; Accepted 13 August 2025

Available online 22 August 2025

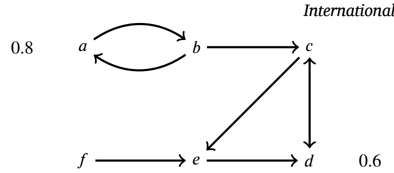

Fig. 1. The PAF of Example 1.

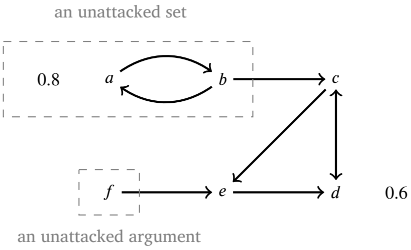

Fig. 2. In Example 1, $\{a, b\}$ is an unattacked set, and f is an unattacked argument.

as constellation approach [16,17]. To illustrate how PAFs evaluate the acceptability degree of arguments, we provide the following example.

Example 1. Consider the PAF in Fig. 1, where arguments a and d are assigned probabilities 0.8 and 0.6, respectively, while all other arguments have a probability of 1. This configuration induces a probability distribution over the possible worlds:

- Both a and d appear, with a probability of $0.8 \times 0.6 = 0.48$;
- Only a appears, with a probability of $0.8 \times (1 - 0.6) = 0.32$;
- Only d appears, with a probability of $(1 - 0.8) \times 0.6 = 0.12$
- Neither a nor d appears, with a probability of $(1 - 0.8) \times (1 - 0.6) = 0.08$.

Suppose we evaluate argument c using preferred semantics and credulous acceptance mode. We observe that c is credulously accepted under preferred semantics in two possible worlds: (i) both a and d appear; and (ii) only a appears. Consequently, the acceptability degree of credulous acceptance of c under preferred semantics is $0.48 + 0.32 = 0.8$ (The calculation is given in Eq. (1)).

It can be observed that semantics and acceptance modes retain their fundamental importance in PAFs, as they dictate the evaluation outcomes of argument acceptability. Consequently, investigating the mechanisms underlying the diverse semantics and acceptance modes in PAFs proves beneficial for implementing the formalism in decision-making processes. In PAFs, semantics are typically derived or extended from their counterparts in AAFs, which raises the question of whether their properties are still applicable in the probabilistic setting. Regarding argument acceptability degree, we propose three fundamental research questions for studying the operational behavior of the semantics as well as the acceptance modes in PAFs:

1. At the individual level, given an argument, is its degree of acceptability influenced by all other arguments, or only a specific subset? If the latter is true, which arguments specifically determine the acceptability of the target argument?
2. At the global level, what is the relationship between the directed/symmetric attack relations and the acceptability of arguments within the framework? More precisely, does more symmetric attack relations negatively correlate with the acceptability degrees of arguments?
3. From a dynamic perspective, what are the effects of adding a new argument (along with its associated attack relations) on the acceptability of existing arguments in the framework?

We have identified several key notions that are instrumental in studying the behavior of semantics in PAFs. These notions offer critical perspectives for tackling the research questions outlined above.

Directionality. A semantics is said to satisfy PAF directionality (under a specific acceptance mode) if, for any PAFs, the acceptability degrees of every argument in an unattacked set are not influenced by the arguments outside. This principle reflects a natural perspective in decision-making: the acceptability of an argument should depend only on the challenges it faces (defeaters) and not on the arguments it challenges. For instance, in Example 1, the argument set $\{a, b\}$ is unattacked, as illustrated in Fig. 2. Thus, the acceptability of a is not affected by c, d, e and f.

Unattacked Innocence. A semantics is said to satisfy PAF unattacked innocence (under a specific acceptance mode) if, for any PAF, the acceptability degree of an unattacked argument equals its initial probability. This implies that the acceptability degree of an argument remains unaffected by any factors other than its attackers. For example, Fig. 2 illustrates that the argument f is unattacked.

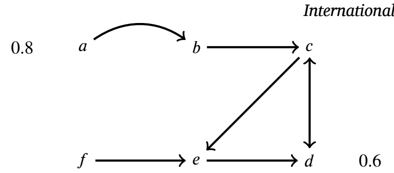

Fig. 3. A less skeptical PAF compared to Example 1.

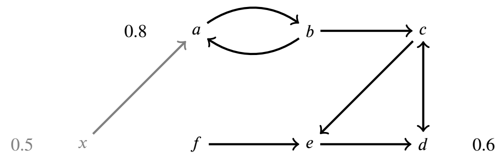

Fig. 4. Under preferred semantics and skeptical mode, the addition of x increases b's acceptability degree, since every path from x to b is of even length. It decreases c's acceptability degree, since every path is of odd length.

As a result, it retains an acceptability degree of 1 under preferred semantics and credulous acceptance mode, the same with its initial probability.

Skepticism Adequacy. Skepticism describes how cautiously a semantics accepts arguments, with more skeptical semantics making fewer committed choices about which arguments are justified. Intuitively, in comparing two PAFs with identical argument sets, a PAF is considered more skeptical than another if it contains symmetric attack relations where the other PAF maintains directed attacks. A semantics is said to satisfy PAF skepticism adequacy under a given acceptance mode if, for any two PAFs $\mathtt{PrF}$ and $\mathtt{PrF}'$, if $\mathtt{PrF}$ is more skeptical than $\mathtt{PrF}'$, then the acceptability degree of every argument in $\mathtt{PrF}$ does not exceed that of the corresponding argument in $\mathtt{PrF}'$. For instance, the acceptability degree of every argument in the framework of Fig. 3 does not exceed its corresponding acceptability degree in the framework of Fig. 1.

Dynamic Monotony. From the dynamic perspective, many semantics satisfy this property: adding a new argument x that has only odd path of attack to an existing argument a will not increase a's acceptability degree. And adding a new argument x that has only even path of attack to an existing argument a will not reduce a's acceptability degree. To illustrate this, see Fig. 4 for an illustration.

To provide a comprehensive analysis of these concepts in PAFs, we first introduce several principles in AAFs from an argument-centered perspective and examining their underlying contexts. We further develop corresponding principles in PAFs to bridge the connections, highlighting the strong relationships between the two frameworks. In this work, we often consider the complete, grounded, preferred, and stable semantics, denoted by com, grd, prf, and stb, respectively. Focusing on the satisfiability of these properties by the semantics, our main contributions can be summarized as follows:

- We define the principle of directionality in PAF and prove that, under both credulous and skeptical modes, com, grd, and prf satisfy directionality in PAF, while stb does not. Importantly, we establish the equivalence of directionality satisfaction between AAF and PAF for any semantics and acceptance mode. We emphasize the generality of the contribution.
- We propose the principle of unattacked innocence in PAF and prove that $\mathsf{com}, \mathsf{grd}$, and prf satisfy this principle in PAF, while stb does not. We generalize our findings by proving the equivalence between AAF and PAF for any semantics and acceptance mode. Besides, for stb, we identify the sufficient and necessary conditions under which an unattacked argument fails to retain its initial probability as its acceptability degree.
- We investigate skepticism in PAF, proposing the principles of skepticism adequacy and resolution adequacy in PAF. We prove that com, grd, and stb satisfy skepticism adequacy, while prf does not. Furthermore, we show that prf and stb satisfy PAF resolution adequacy, whereas com and grd do not. More broadly, we establish the equivalence of skepticism adequacy and resolution adequacy between AAF and PAF for any semantics under skeptical acceptance mode.
- We introduce the concepts of x-expansion and monotonicity in PAF to investigate the semantics-specific impact of adding a new argument x on the acceptability degrees of certain arguments. By analyzing odd and even paths from x to existing arguments, we characterize the behavior of changes in acceptability degrees, whether they increase, decrease, or remain unchanged.

The remainder of this paper is structured as follows. In Section 2, we present the necessary background concepts. Section 3 introduces argument-centered principles in AAF and establishes their corresponding principles in PAF. Sections 4 and 5 focus on two fundamental concepts in the context of argumentation frameworks: directionality and adequacy, providing detailed discussions on the theoretical underpinnings of these concepts. Dynamic properties are examined in Section 6, with a focus on how the acceptability degrees of arguments vary under specific conditions. Section 7 reviews related works and highlights their connections to our study. Finally, we provide concluding remarks in Section 8.

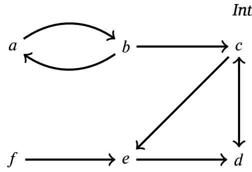

Fig. 5. The visualization of $\mathcal{F}_0$.

2. Preliminaries

2.1. Abstract argumentation

Abstract argumentation views arguments as atomic entities without internal structure, focusing solely on their interactions. The framework emphasizes determining the acceptability of arguments through a binary attack relation [4].

Definition 2 *(Abstract argumentation framework (AAF)).* An abstract argumentation framework is a tuple $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ where $\mathcal{A}$ is a finite set of arguments and $\mathcal{R}$ is a binary relation $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$.

Consider an argumentation framework $\mathcal{F}_0 = (\{a, b, c, d, e, f\}, \{(a, b), (b, a), (b, c), (c, d), (c, e), (d, c), (e, d), (f, e)\})$. This framework is visually represented in Fig. 5.

When no ambiguity arises, we refer to the abstract argumentation framework simply as "argumentation framework". For two arguments $a, b \in \mathcal{A}$, we use $\mathcal{R}(a, b)$ to denote the relation $(a, b) \in \mathcal{R}$, indicating that argument a attacks argument b. An argument is unattacked if no other argument in $\mathcal{A}$ attacks it and the set of unattacked arguments of $\mathcal{F}$ is denoted as $\mathcal{UA}(\mathcal{F}) \subseteq \mathcal{A}$.

For an argument set S and an argument a, we say that S attacks a if there exists $b \in S$ such that $\mathcal{R}(b, a)$ and we denote $S^+ = \{a \in \mathcal{A} \mid S \text{ attacks } a\}$. Similarly, a is said to attack S if there exists $b \in S$ such that $\mathcal{R}(a, b)$. An argument a is defended by a set S if, for every $b \in \mathcal{A}$ such that $\mathcal{R}(b, a)$, there exists $c \in S$ such that $\mathcal{R}(c, b)$. Additionally, we state that S_1 attacks a set S_2 if there exist $a \in S_1$ and $b \in S_2$ such that $\mathcal{R}(a, b)$.

A set S is conflict-free if, for every $a, b \in S$, we have $(a, b) \notin \mathcal{R}$. A set S is admissible in $\mathcal{F}$, denoted as $S \in ad(\mathcal{F})$, if S is conflict-free and defends all its arguments.

Definition 3 *(Subframework).* Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $X \subseteq \mathcal{A}$, the subframework of $\mathcal{F}$ on X is defined as $\mathcal{F}\downarrow_X = (X, \mathcal{R} \cap (X \times X))$. $\mathcal{F}\downarrow_X$ is also called the projection of $\mathcal{F}$ on X.

AAFs can be concisely represented by directed graphs, where arguments are depicted as nodes and edges represent the attack relation, and a subframework can be regarded as a subgraph. We denote the set of all subframeworks of an AAF $\mathcal{F}$ as $\wp(\mathcal{F})$, *i.e.*, $\wp(\mathcal{F}) = \{\mathcal{F}\downarrow_X \mid X \subseteq \mathcal{A}\}$.

Acceptability is prescribed by argumentation semantics.

Definition 4 *(Semantics).* A semantics is a function σ s.t. for every $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, we have $\sigma(\mathcal{F}) \in 2^{2^{\mathcal{A}}}$. The elements of $\sigma(\mathcal{F})$ are called extensions.

In this work, we often consider the complete, grounded, preferred, and stable semantics, denoted by com, grd, prf, and stb, respectively.

Definition 5 *(Complete, stable, grounded, preferred semantics).* Given an argumentation framework $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and an argument set $S \subseteq \mathcal{A}$.

- S is a complete extension of $\mathcal{F}$, denoted as $S \in \mathsf{com}(\mathcal{F})$, iff S is admissible and contains all the arguments it defends.
- S is a stable extension of $\mathcal{F}$, denoted as $S \in \mathsf{stb}(\mathcal{F})$, iff S is conflict-free and attacks all the arguments of $\mathcal{A} \setminus S$.
- S is the grounded extension of $\mathcal{F}$, denoted as $S \in \mathsf{grd}(\mathcal{F})$, iff S is the minimal wrt. $\subseteq$ complete extension of $\mathcal{F}$.
- S is a preferred extension of $\mathcal{F}$, denoted as $S \in \mathsf{prf}(\mathcal{F})$, iff S is a maximal wrt. $\subseteq$ admissible set of $\mathcal{F}$.

For $\mathcal{F}_0$ in Fig. 5, the complete extensions are $\{f\}$, $\{f, a\}$, $\{f, a, c\}$, $\{f, a, d\}$, and $\{f, b, d\}$. The stable extensions are $\{f, a, c\}$, $\{f, a, d\}$, and $\{f, b, d\}$. The grounded extension is $\{f\}$. The preferred extensions are $\{f, a, c\}$, $\{f, a, d\}$, and $\{f, b, d\}$.

In Section 3, we also mentioned CF2 semantics. CF2 semantics is defined via an SCC-recursive procedure: in each SCC, it removes arguments attacked by earlier choices and selects a maximal conflict-free set; the union forms a CF2 extension. Since the underlying formal definition is quite complex and slightly beyond the scope of this paper, please refer to [18,6,19].

In AAFs, an acceptance mode is a function on extensions that describes the criterion for selecting accepted arguments. It reflects a way to interpret which arguments are considered "accepted".

Definition 6 *(Acceptance mode)*. An acceptance mode Ω is a function such that for every abstract argumentation $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\Omega : 2^{2^{\mathcal{A}}} \to 2^{\mathcal{A}}$ maps extensions to an argument set.

Intuitively, an acceptance mode aggregates a family of extensions into the set of arguments it deems accepted. We now introduce the distributive property of intersection for acceptance modes.

Definition 7 *(Distributive property of intersection)*. An acceptance mode Ω satisfies the distributive property of intersection if for any sets A and B, it holds that $\Omega(A \cap B) = \Omega(A) \cap \Omega(B)$.

Common acceptance modes are credulous acceptance, which refers to accepting arguments if they appear in at least one extension, and skeptical acceptance, which refers to accepting arguments if they appear in all extensions. Clearly, they both satisfy the distributive property of intersection.

Definition 8 *(Credulous and skeptical acceptance)*. An argument $a \in \mathcal{A}$ is credulously accepted in $\mathcal{F}$ wrt. σ iff $a \in \bigcup \sigma(\mathcal{F})$. An argument $a \in \mathcal{A}$ is skeptically accepted in $\mathcal{F}$ wrt. σ iff $a \in \bigcap \sigma(\mathcal{F})$. Here, we stipulate that $\bigcap \emptyset = \bigcup \emptyset = \emptyset$. We use s, c as symbols to refer to these two acceptance modes respectively.

We define acceptance modes abstractly for ease of presentation. In practice, we focus on credulous and skeptical acceptance rather than building a full taxonomy of modes. The credulous and skeptical acceptance correspond, in fact, the union and intersection functions of $\sigma(\mathcal{F})$. For convenience, we denote $\mathcal{CA}_\sigma(\mathcal{F}), \mathcal{SA}_\sigma(\mathcal{F})$ the credulously accepted argument set and the skeptically accepted argument set respectively: $\mathcal{CA}_\sigma(\mathcal{F}) = \bigcup_{\mathcal{E} \in \sigma(\mathcal{F})} \mathcal{E}$, $\mathcal{SA}_\sigma(\mathcal{F}) = \bigcap_{\mathcal{E} \in \sigma(\mathcal{F})} \mathcal{E}$. For a semantics σ and an acceptance mode Ω, an argument a is accepted in $\mathcal{F}$ wrt. Ω and σ is written as $\mathcal{F} \mid\!\sim^{\Omega}_{\sigma} a$.

For $\mathcal{F}_0$ in Fig. 5, we have $\mathcal{CA}_{\mathsf{prf}}(\mathcal{F}_0) = \{a, b, c, d, f\}$ and $\mathcal{SA}_{\mathsf{prf}}(\mathcal{F}_0) = \{f\}$.

2.2. Probabilistic argumentation

There are primarily two methods for integrating probabilities into abstract argumentation: the epistemic approach [20,21] and the constellation approach [16]. This paper examines the constellation approach, which considers possible projections of an argumentation framework onto sets of arguments. In the seminal work on the constellation approach [16], probabilities are assigned to both arguments and attacks. In contrast, in [22], only arguments are represented probabilistically. Notably, the former approach can be transformed into the latter [23], resulting in equivalent admissible extensions and probabilistic distributions. Accordingly, we focus on the latter approach, in which only arguments are assigned probabilities. The constellation approach allows us to represent the uncertainty over the topology of the graph by assigning initial probability. Each subgraph of the original graph is assigned a probability which is understood as the chances of it being the actual argument graph of one agent. We now define the probabilistic argumentation framework.

Definition 9 *(Probabilistic argumentation framework)*. A probabilistic argumentation framework is a triple $\mathtt{PrF} = (\mathcal{A}, \mathcal{R}, \mathcal{P})$ where $(\mathcal{A}, \mathcal{R})$ is an argumentation framework and $\mathcal{P}$ is a function $\mathcal{P} : \mathcal{A} \to [0, 1]$.

We also use the pair $(\mathcal{F}, \mathcal{P})$ to represent the triple $(\mathcal{A}, \mathcal{R}, \mathcal{P})$ by specifically identifying $\mathcal{F} = (\mathcal{A}, \mathcal{R})$. For each argument $a \in \mathcal{A}$ within a probabilistic argumentation framework, $\mathcal{P}(a)$ denotes the initial probability that a is present in the framework. Assuming probabilistic independence among the presence of different arguments, we derive a probability distribution over sets of arguments. This distribution is denoted as $P : 2^{\mathcal{A}} \to [0, 1]$, and is defined by:

$$P(X) = \prod_{a \in X} \mathcal{P}(a) \prod_{a \notin X} (1 - \mathcal{P}(a)) \tag{1}$$

for all $X \subseteq \mathcal{A}$. It can be demonstrated that P is indeed a probability distribution:

$$\begin{aligned}\sum_{X \subseteq \mathcal{A}} P(X) &= \sum_{X \subseteq \mathcal{A}} \left(\prod_{a \in X} \mathcal{P}(a) \prod_{a \notin X} (1 - \mathcal{P}(a)) \right) \\ &= \prod_{X \subseteq \mathcal{A}} \sum_{a \in X} (\mathcal{P}(a) + (1 - \mathcal{P}(a))) \\ &= 1\end{aligned} \tag{2}$$

The assumption of probabilistic independence between arguments in PAFs is often restrictive in applications. This paper adopts the independence assumption to isolate and analyze foundational properties of PAFs. Discussions on modeling sub-argument relations in PAFs appear in [14], and analyses of independence assumptions are provided in [17]. Developing PAF variants that relax independence and capture such dependencies is left for future work.

In PAF, the acceptability degree of an argument a is described by the probability $P(a)$ and it is evaluated in subgraphs across different scenarios.

Definition 10 *(Probability of acceptance)*. Given a probabilistic argumentation framework $\mathtt{PrF} = (\mathcal{F}, \mathcal{P})$, let σ be a semantics and Ω be an acceptance mode. The probability of acceptance of a is defined as:

$$P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(a) = \sum_{a \in X \subseteq \mathcal{A}, \mathcal{F}\downarrow_X \mid\!\sim^{\Omega}_{\sigma} a} P(X) \tag{3}$$

In other words, $P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(a)$ is the sum of the probabilities of the subgraphs of $\mathcal{F}$ where a is accepted wrt. σ and Ω. To facilitate computational representation, we introduce the truth function $\mathbb{1}$:

$$\mathbb{1}(s) = \begin{cases} 1, & \text{if the assertion } s \text{ is true} \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Using this truth function, we can reformulate $P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(a)$ as:

$$P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(a) = \sum_{a \in X \subseteq \mathcal{A}, \mathcal{F}\downarrow_X \mid\!\sim^{\Omega}_{\sigma} a} P(X) = \sum_{X \in 2^{\mathcal{A}}} \mathbb{1}(\mathcal{F}\downarrow_X \mid\!\sim^{\Omega}_{\sigma} a) P(X) \tag{5}$$

3. Basic principles

In this section, we revisit several basic principles from [10] in the context of AAF, adapting them to an argument-centric perspective. We transfer these principles into PAF by formalizing key concepts of equivalence and redundancy, while also defining and extending principles of anonymity and succinctness. Anonymity means invariance under argument renaming, whereas succinctness requires that no attacks are redundant. Notably, we demonstrate the alignment of these principles across both frameworks, ensuring consistency and coherence in the transfer.

3.1. Basic principles in AAF

We begin by examining foundational principles related to “equivalence” in AAF. These principles define the criteria for determining when different argumentation structures can be considered equivalent and identify the notion of redundancy. To formalize this idea, we introduce the concept of isomorphism [10].

Definition 11 *(Isomorphic argumentation frameworks)*. Two argumentation frameworks $\mathcal{F}_1 = (\mathcal{A}_1, \mathcal{R}_1)$ and $\mathcal{F}_2 = (\mathcal{A}_2, \mathcal{R}_2)$ are isomorphic iff there exists a bijective function $m : \mathcal{A}_1 \to \mathcal{A}_2$, such that $(a, b) \in \mathcal{R}_1 \Leftrightarrow (m(a), m(b)) \in \mathcal{R}_2$. This is denoted by $\mathcal{F}_1 \doteq_m \mathcal{F}_2$.

The first property ensures that semantics depend solely on the structure of the graph and are independent of the names of arguments, a property often referred to as anonymity [10].

Principle 1 *(Anonymity)*. A semantics σ satisfies the anonymity principle iff for every two argumentation frameworks $\mathcal{F}_1$ and $\mathcal{F}_2$, if $\mathcal{F}_1 \doteq_m \mathcal{F}_2$ then $\sigma(\mathcal{F}_2) = \{m(\mathcal{E}) \mid \mathcal{E} \in \sigma(\mathcal{F}_1)\}$ where $m(\mathcal{E}) = \{m(a) \mid a \in \mathcal{E}\}$.

For two AAFs $\mathcal{F}_1 = (\mathcal{A}_1, \mathcal{R}_1)$ and $\mathcal{F}_2 = (\mathcal{A}_2, \mathcal{R}_2)$, we write $\mathcal{F}_1 \subseteq \mathcal{F}_2$ to denote that $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{R}_1 \subseteq \mathcal{R}_2$. Additionally, $\mathcal{F} \setminus \{(a, b)\}$ denotes the argumentation framework $(\mathcal{A}, \mathcal{R} \setminus \{(a, b)\})$. To identify attacks that do not affect the evaluation of arguments under a given semantics, we introduce the concept of redundant attack [10].

Definition 12 *(Redundant attack)*. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation framework and σ a semantics. Attack $(a, b) \in \mathcal{R}$ is said to be redundant in $\mathcal{F}$ wrt. σ iff for all argumentation frameworks $\mathcal{F}'$ such that $\mathcal{F} \subseteq \mathcal{F}'$ we have $\sigma(\mathcal{F}') = \sigma(\mathcal{F}' \setminus \{(a, b)\})$.

In other words, an attack is considered redundant in $\mathcal{F}$ if its removal does not affect the extensions of any $\mathcal{F}'$ that contains $\mathcal{F}$. For CF2 semantics, no redundant attacks exist [19], and this property is referred to as the succinctness principle.

Principle 2 *(Succinctness)*. A semantics σ satisfies the succinctness principle iff no argumentation framework contains a redundant attack wrt. σ.

We now define redundancy from the perspective of argument acceptance and introduce succinctness from this viewpoint. An attack is deemed redundant in $\mathcal{F}$ if its removal does not influence the accepted arguments in any AAF that extends $\mathcal{F}$.

Definition 13 *(Acceptance-based redundant attack)*. Given $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, a semantics σ, and an acceptance mode Ω, an attack $(a, b) \in \mathcal{R}$ is said to be redundant in $\mathcal{F}$ wrt. σ and Ω iff for all argumentation frameworks $\mathcal{F}'$ such that $\mathcal{F} \subseteq \mathcal{F}'$, we have $\Omega(\sigma(\mathcal{F}')) = \Omega(\sigma(\mathcal{F}' \setminus \{(a, b)\}))$.

Building on the succinctness principle, we now capture an analogous concept at the level of acceptance by redefining succinctness in PAF.

Principle 3 *(Succinctness$^{\Omega}_{arg}$).* Given an acceptance mode Ω, a semantics σ satisfies the succinctness$^{\Omega}_{arg}$ principle iff no argumentation framework contains an acceptance-based redundant attack wrt. σ and Ω.

The succinctness$^{\Omega}_{arg}$ principle ensures that no redundancies exist that preserves argument acceptance.

3.2. Adapting principles to PAFs

We aim to reinterpret principles to account for the probabilistic characteristics of PAFs, ensuring that they remain meaningful in this context. Below, we formalize these basic principles within the framework of probabilistic argumentation, providing a foundation for the semantics analysis. We define the PAF anonymity principle, which is satisfied by all semantics, as follows:

Definition 14 *(Isomorphic PAFs).* Two probabilistic argumentation frameworks $\mathtt{PrF}_1 = (\mathcal{A}_1, \mathcal{R}_1, \mathcal{P}_1)$ and $\mathtt{PrF}_2 = (\mathcal{A}_2, \mathcal{R}_2, \mathcal{P}_2)$ are isomorphic iff there exists a bijective function $m : \mathcal{A}_1 \to \mathcal{A}_2$, such that $(\mathcal{A}_1, \mathcal{R}_1) \doteq_m (\mathcal{A}_2, \mathcal{R}_2)$ and $\mathcal{P}_2(m(a)) = \mathcal{P}_1(a)$ for all $a \in \mathcal{A}_1$. This is denoted by $\mathtt{PrF}_1 \doteq_m \mathtt{PrF}_2$.

Principle 4 *(PAF anonymity).* A semantics σ satisfies the PAF anonymity principle iff for any two probabilistic argumentation frameworks $\mathtt{PrF}_1 = (\mathcal{F}_1, \mathcal{P}_1)$ and $\mathtt{PrF}_2 = (\mathcal{F}_2, \mathcal{P}_2)$ with $\mathcal{F}_1 = (\mathcal{A}_1, \mathcal{R}_1)$, whenever $\mathtt{PrF}_1 \doteq_m \mathtt{PrF}_2$, it holds that $P^{\mathcal{F}_2,\mathcal{P}}_{\sigma,\Omega}(a) = P^{\mathcal{F}_1,\mathcal{P}}_{\sigma,\Omega}(m(a))$ for all $a \in \mathcal{A}_1$ and for any acceptance mode Ω.

In PAF, the notion of redundancy is formalized with respect to argument acceptance. Specifically, an attack is considered redundant in a PAF if its removal does not affect the final probability of acceptance of any argument in any possible world that includes it.

Definition 15 *(Acceptance-based redundant attack in PAF).* Let $\mathtt{PrF} = (\mathcal{A}, \mathcal{R}, \mathcal{P})$ be a probabilistic argumentation framework with $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, σ a semantics and Ω an acceptance mode. An attack $(a, b) \in \mathcal{R}$ is said to be redundant in $\mathtt{PrF}$ wrt. σ and Ω iff for all probabilistic argumentation frameworks $(\mathcal{F}', \mathcal{P}')$ such that $\mathcal{F} \subseteq \mathcal{F}'$ and $\forall c \in \mathcal{A}, \mathcal{P}'(c) = \mathcal{P}(c)$, we have

$$P^{\mathcal{F}',\mathcal{P}'}_{\sigma,\Omega}(e) = P^{\mathcal{F}'\setminus\{(a,b)\},\mathcal{P}'}_{\sigma,\Omega}(e)$$

for all $e \in \mathcal{A}'$.

We now demonstrate that the concept of succinctness$^{\Omega}_{arg}$ can be naturally extended to PAFs, preserving its meaning in probabilistic settings. To establish this, we first prove a proposition stating that acceptance-based redundant attacks have an analogous interpretation in both AAFs and PAFs.

Proposition 1. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation framework, σ a semantics, and Ω an acceptance mode. Let $\mathtt{PrF} = (\mathcal{F}', \mathcal{P}')$ be a probabilistic argumentation framework such that $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{P}'(c) = 1$ for all $c \in \mathcal{A}$. Then, an attack (a, b) is acceptance-based redundant in $\mathcal{F}$ wrt. σ and Ω iff (a, b) is acceptance-based redundant in $\mathtt{PrF}$ wrt. σ and Ω.*

Proof. Let us consider the attack $(a, b) \in \mathcal{R}$ and denote $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$, $\mathcal{F}'_- = \mathcal{F}' \setminus \{(a, b)\}$. We first prove the implication $\Rightarrow$. From $\forall c \in \mathcal{A}, \mathcal{P}'(c) = 1$, we deduce that each subframework of $\mathcal{F}'$ contains $\mathcal{F}$: $\forall X \subseteq \mathcal{A}', \mathcal{F} \subseteq \mathcal{F}' \downarrow_X$. If an attack (a, b) is redundant in $\mathcal{F}$, then $\sigma(\mathcal{F}' \downarrow_X) = \sigma(\mathcal{F}'_- \downarrow_X)$ and therefore $\Omega(\sigma(\mathcal{F}' \downarrow_X)) = \Omega(\sigma(\mathcal{F}'_- \downarrow_X))$, which implies

$$\mathcal{F}' \downarrow_X \mid\!\sim^{\Omega}_{\sigma} c \Leftrightarrow \mathcal{F}'_- \downarrow_X \mid\!\sim^{\Omega}_{\sigma} c$$

By the definition in Eq. (3), for every argument $e \in \mathcal{A}'$ we have:

$$\begin{aligned} P^{\mathcal{F}',\mathcal{P}'}_{\sigma,\Omega}(e) &= \sum_{X \subseteq \mathcal{A}', \mathcal{F}' \downarrow_X \mid\sim^{\Omega}_{\sigma} e} P(X) \\ &= \sum_{X \in 2^{\mathcal{A}'}} \mathbb{1}(\mathcal{F}' \downarrow_X \mid\!\sim^{\Omega}_{\sigma} e) P(X) \\ &= \sum_{X \in 2^{\mathcal{A}'}} \mathbb{1}(\mathcal{F}'_- \downarrow_X \mid\!\sim^{\Omega}_{\sigma} e) P(X) \\ &= \sum_{X \subseteq \mathcal{A}', \mathcal{F}'_- \downarrow_X \mid\sim^{\Omega}_{\sigma} e} P(X) \\ &= P^{\mathcal{F}'_-,\mathcal{P}'}_{\sigma,\Omega}(e) \end{aligned} \tag{6}$$

Next we prove the implication $\Leftarrow$. We just need to set $\mathcal{P}'(a) = 1, \forall a \in \mathcal{A}'$. In this case, if (a, b) is redundant in $\mathtt{PrF}$, then

$$P_{\sigma,\Omega}^{\mathcal{F}',\mathcal{P}'}(e) = \mathbb{1}(\mathcal{F}' \mid\!\sim_{\sigma}^{\Omega} e) = \mathbb{1}(\mathcal{F}'_{-} \mid\!\sim_{\sigma}^{\Omega} e) = P_{\sigma,\Omega}^{\mathcal{F}'_{-},\mathcal{P}'}(e)$$

for all $e \in \mathcal{A}'$. We then conclude $\Omega(\sigma(\mathcal{F}' \downarrow_X)) = \Omega(\sigma(\mathcal{F}'_{-} \downarrow_X))$. $\square$

This connection emphasizes the consistency of redundancy notions between deterministic and probabilistic frameworks, providing a foundation for seamlessly transferring principles from AAFs to PAFs. We now define the succinctness$^{\Omega}_{arg}$ principle in PAF, which corresponds directly to that in AAF that no attack is redundant for a semantics.

Principle 5 *(PAF succinctness$^{\Omega}_{arg}$).* Given an acceptance mode Ω, a semantics σ satisfies the PAF succinctness$^{\Omega}_{arg}$ principle iff no probabilistic argumentation framework contains an acceptance-based redundant attack wrt. σ and Ω.

We immediately obtain the following conclusion by the equivalence of attack redundancy from argument perspective in AAFs and PAFs:

Proposition 2. *Let Ω be an acceptance mode. A semantics σ satisfies succinctness$^{\Omega}_{arg}$ iff it satisfies PAF succinctness$^{\Omega}_{arg}$.*

4. Directionality

The notion of directionality in argumentation semantics was first introduced in [6], where a general framework was proposed to relate the topology of the defeat graph to the definition of extensions. In this section, we present the foundational notion of directionality along with its variants and propose similar principles from the perspective of argument acceptability. Subsequently, we extend this concept to PAFs and demonstrate the strong connection between them.

4.1. Directionality and its variants in AAF

Directionality posits that the acceptability of an argument a should be influenced solely by its defeaters and not by arguments that are only attacked by a. It reflects a natural perspective in decision-making: the justification status of an argument should depend only on the challenges it faces (defeaters) and not on the arguments it challenges. To formalize directionality, we first define unattacked sets, which collect arguments that receive no attacks from outside the set.

Definition 16 *(Unattacked set).* Given an argumentation framework $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, a set $U \subseteq \mathcal{A}$ is unattacked iff $\forall a \in \mathcal{A} \setminus U, \forall b \in U : \neg\mathcal{R}(a, b)$.

Note that an unattacked set U is different from a set of unattacked arguments, as U may contain internal attacks. The set of unattacked sets of $\mathcal{F}$ is denoted as $\mathcal{US}(\mathcal{F}) \subseteq 2^{\mathcal{A}}$. The directionality principle can then be defined by requiring that an unattacked set remains unaffected by the rest of the argumentation framework in terms of extensions.

Principle 6 *(Directionality).* A semantics σ satisfies the directionality principle iff for every argumentation framework $\mathcal{F}$, $\forall U \in \mathcal{US}(\mathcal{F})$, it holds that $\sigma(\mathcal{F}\downarrow_U) = \{\mathcal{E} \cap U \mid \mathcal{E} \in \sigma(\mathcal{F})\}$.

In words, the intersection of any σ-extension of $\mathcal{F}$ with an unattacked set U equals a σ-extension of the projection of $\mathcal{F}$ on U, and vice versa. Next, we introduce weak directionality and semi-directionality [10].

Principle 7 *(Weak directionality).* A semantics σ satisfies the weak directionality principle iff for every argumentation framework $\mathcal{F}$, $\forall U \in \mathcal{US}(\mathcal{F})$, it holds that $\sigma(\mathcal{F}\downarrow_U) \supseteq \{\mathcal{E} \cap U \mid \mathcal{E} \in \sigma(\mathcal{F})\}$.

Principle 8 *(Semi-directionality).* A semantics σ satisfies the semi-directionality principle iff for every argumentation framework $\mathcal{F}$, $\forall U \in \mathcal{US}(\mathcal{F})$, it holds that $\sigma(\mathcal{F}\downarrow_U) \subseteq \{\mathcal{E} \cap U \mid \mathcal{E} \in \sigma(\mathcal{F})\}$.

The aforementioned definitions pertain to properties concerning extensions. While these definitions capture the interaction between unattacked sets and the structure of extensions, they focus solely on extension-based reasoning and overlook the acceptability of individual arguments. To address this gap, we propose a refined concept of directionality that emphasizes argument-level acceptability, ensuring the principles of directionality apply not only to sets of arguments but also to individual arguments. The following definitions formalize this argument-focused perspective on directionality.

Definition 17 *(Directionality$^{\Omega}_{arg}$).* Given an acceptance mode Ω, a semantics σ satisfies the directionality$^{\Omega}_{arg}$ principle iff for every argumentation framework $\mathcal{F}$, $\forall U \in \mathcal{US}(\mathcal{F})$, it holds that $\Omega(\sigma(\mathcal{F}\downarrow_U)) = \Omega(\sigma(\mathcal{F})) \cap U$.

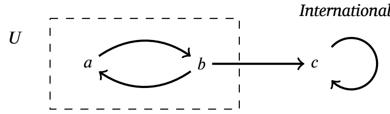

Fig. 6. Counterexample in directionality implication. The special semantics σ_{ct} is defined such that $\sigma_{ct}(\mathcal{F}\downarrow_U) = \{\{a,b\},\emptyset\}$ and $\sigma_{ct}(\mathcal{F}) = \{\{a,c\},\{b,c\}\}$.

Table 1
Properties of semantics.

Property	com	grd	prf	stb
Anonymity	✓	✓	✓	✓
Succinctness	×	×	×	×
Directionality	✓	✓	✓	×
Weak directionality	✓	✓	✓	✓
Semi-directionality	✓	✓	✓	×
Directionality$^{\Omega}_{arg}$	✓	✓	✓	×
Weak directionality$^{\Omega}_{arg}$	✓	✓	✓	✓
Semi-directionality$^{\Omega}_{arg}$	✓	✓	✓	×

Definition 18 *(Weak directionality$^{\Omega}_{arg}$).* Given an acceptance mode Ω, a semantics σ satisfies the weak directionality$^{\Omega}_{arg}$ principle iff for every AAF $\mathcal{F}$, $\forall U \in \mathcal{US}(\mathcal{F})$, it holds that $\Omega(\sigma(\mathcal{F}\downarrow_U)) \supseteq \Omega(\sigma(\mathcal{F})) \cap U$.

Definition 19 *(Semi-directionality$^{\Omega}_{arg}$).* Given an acceptance mode Ω, a semantics σ satisfies the semi-directionality$^{\Omega}_{arg}$ principle iff for every AAF $\mathcal{F}$, $\forall U \in \mathcal{US}(\mathcal{F})$, it holds that $\Omega(\sigma(\mathcal{F}\downarrow_U)) \subseteq \Omega(\sigma(\mathcal{F})) \cap U$.

Apart from the differing perspectives between arguments and extensions, we demonstrate that directionality$^{\Omega}_{arg}$ represents a weaker form of directionality under any acceptance mode satisfying the distributive property of intersection.

Proposition 3. *Let Ω be an acceptance mode satisfying the distributive property of intersection. If a semantics σ satisfies the directionality principle, then it also satisfies directionality$^{\Omega}_{arg}$. However, the converse does not hold, i.e., directionality$^{\Omega}_{arg}$ does not imply directionality.*

Proof. The implication is direct. In fact, directionality asserts that the extensions of $\mathcal{F}$ projected on an unattacked set U are equal to those of $\mathcal{F}\downarrow_U$. Consequently, the accepted arguments filtered by Ω using these extensions are also equal. Specifically, by definition:

$$\Omega(\sigma(\mathcal{F}\downarrow_U)) = \Omega(\{\mathcal{E} \mid \mathcal{E} \in \sigma(\mathcal{F}\downarrow_U)\})$$

Then use directionality,

$$\Omega(\{\mathcal{E} \mid \mathcal{E} \in \sigma(\mathcal{F}\downarrow_U)\}) = \Omega(\{\mathcal{E} \cap U \mid \mathcal{E} \in \sigma(\mathcal{F})\})$$

By distributive property of Ω over intersection:

$$\Omega(\{\mathcal{E} \cap U \mid \mathcal{E} \in \sigma(\mathcal{F})\}) = \Omega(\{\mathcal{E} \mid \mathcal{E} \in \sigma(\mathcal{F})\}) \cap U$$

Finally

$$\begin{aligned}\Omega(\sigma(\mathcal{F}\downarrow_U)) &= \Omega(\{\mathcal{E} \mid \mathcal{E} \in \sigma(\mathcal{F}\downarrow_U)\}) \\ &= \Omega(\{\mathcal{E} \cap U \mid \mathcal{E} \in \sigma(\mathcal{F})\}) \\ &= \Omega(\{\mathcal{E} \mid \mathcal{E} \in \sigma(\mathcal{F})\}) \cap U \\ &= \Omega(\sigma(\mathcal{F})) \cap U\end{aligned} \tag{7}$$

For the converse statement, see Fig. 6 for the counterexample. In the counterexample, the only unattacked set is $U = \{a,b\}$. We define a special semantics σ_{ct} such that $\sigma_{ct}(\mathcal{F}\downarrow_U) = \{\{a,b\},\emptyset\}$ and $\sigma_{ct}(\mathcal{F}) = \{\{a,c\},\{b,c\}\}$. Then $\mathcal{CA}_{\sigma_{ct}}(\mathcal{F}\downarrow_U) = \mathcal{CA}_{\sigma_{ct}}(\mathcal{F}) \cap U = \{a,b\}$ and $\mathcal{SA}_{\sigma_{ct}}(\mathcal{F}\downarrow_U) = \mathcal{SA}_{\sigma_{ct}}(\mathcal{F}) \cap U = \emptyset$, while $\{\mathcal{E} \cap U \mid \mathcal{E} \in \sigma(\mathcal{F})\} = \{\{a\},\{b\}\} \neq \sigma_{ct}(\mathcal{F}\downarrow_U)$. □

By similar proof, we can conclude that weak directionality implies weak directionality$^{\Omega}_{arg}$ and semi-directionality implies semi-directionality$^{\Omega}_{arg}$.

In Table 1, we summarize the satisfiability of principles by four semantics. The complete, grounded, and preferred semantics satisfy directionality; therefore, they also satisfy directionality$^{\Omega}_{arg}$. However, for stable semantics, further discussion is needed. The example in Fig. 6 demonstrates that stable semantics does not satisfy directionality$^{s}_{arg}$. Specifically, we have $\mathsf{stb}(\mathcal{F}) = \{\{b\}\}$ and $\mathsf{stb}(\mathcal{F}\downarrow_U) = \{\{a,b\}\}$, indicating that directionality$^{\Omega}_{arg}$ and semi-directionality$^{\Omega}_{arg}$ do not hold in this case.

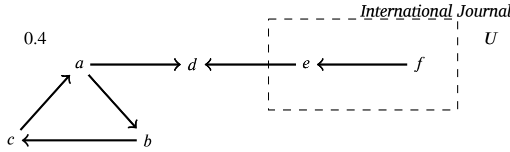

Fig. 7. Counterexample in PAF directionality$^{\Omega}_{arg}$. It holds that $P^{\mathcal{F},\mathcal{P}}_{\mathrm{stb},c}(f) = P^{\mathcal{F},\mathcal{P}}_{\mathrm{stb},s}(f) = 0.6$, while $P^{\mathcal{F}\downarrow_{\{e,f\}},\mathcal{P}}_{\mathrm{stb},c}(f) = P^{\mathcal{F}\downarrow_{\{e,f\}},\mathcal{P}}_{\mathrm{stb},s}(f) = 1$. As a result, $P^{\mathcal{F},\mathcal{P}}_{\mathrm{stb},c}(f) \neq P^{\mathcal{F}\downarrow_{\{e,f\}},\mathcal{P}}_{\mathrm{stb},c}(f)$, as well as $P^{\mathcal{F},\mathcal{P}}_{\mathrm{stb},s}(f) \neq P^{\mathcal{F}\downarrow_{\{e,f\}},\mathcal{P}}_{\mathrm{stb},s}(f)$.

4.2. Directionality in PAF

In PAF, the concept of directionality can be naturally extended, such that $P(a)$ is determined exclusively by the defeaters of a. Probabilistic directionality for subsets of arguments is introduced in [24]. Here, we define the argument-level directionality principle for PAFs, which ensures that the probability of an argument's acceptance depends only on its unattacked context.

Principle 9 *(PAF directionality$^{\Omega}_{arg}$).* Given an acceptance mode $\Omega \in \{c, s\}$, a semantics σ satisfies the PAF directionality$^{\Omega}_{arg}$ principle iff for every probabilistic argumentation framework $\mathtt{PrF} = (\mathcal{F}, \mathcal{P})$, $\forall U \in \mathcal{US}(\mathcal{F})$, it holds that

$$P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(a) = P^{\mathcal{F}\downarrow_U,\mathcal{P}}_{\sigma,\Omega}(a)$$

for every $a \in U$.

The directionality here states the same property of σ that the final probability of one argument a is affected only by its defeaters. The example in Fig. 7 shows that stable semantics does not satisfy PAF directionality$^{\Omega}_{arg}$ for $\Omega \in \{c, s\}$. Next, we prove that a semantics satisfies directionality$^{\Omega}_{arg}$ in an AAF is equivalent to it satisfying directionality$^{\Omega}_{arg}$ in a PAF.

Proposition 4. *Let σ be a semantics and Ω an acceptance mode that satisfies the distributive property of intersection. Then, σ satisfies directionality$^{\Omega}_{arg}$ in AAF iff it satisfies PAF directionality$^{\Omega}_{arg}$.*

Proof. We first prove that directionality$^{\Omega}_{arg}$ implies PAF directionality$^{\Omega}_{arg}$. Let $\mathtt{PrF} = (\mathcal{A}, \mathcal{R}, \mathcal{P})$ be a probabilistic argumentation framework with $\mathcal{F} = (\mathcal{A}, \mathcal{R})$. For every $U \in \mathcal{US}(\mathcal{F})$, denote $V = \mathcal{A} \setminus U$. By the definition in Eq. (3), for an argument $a \in U$:

$$P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(a) = \sum_{X \subseteq \mathcal{A}, \mathcal{F}\downarrow_X \mid\!\sim^{\Omega}_{\sigma} a} P(X) = \sum_{X \in 2^{\mathcal{A}}} \mathbb{1}(\mathcal{F}\downarrow_X \mid\!\sim^{\Omega}_{\sigma} a) P(X) \tag{8}$$

$$P^{\mathcal{F}\downarrow_U,\mathcal{P}}_{\sigma,\Omega}(a) = \sum_{X \subseteq U, \mathcal{F}\downarrow_U\downarrow_X \mid\!\sim^{\Omega}_{\sigma} a} P(X) = \sum_{X \in 2^{U}} \mathbb{1}(\mathcal{F}\downarrow_{X\cap U} \mid\!\sim^{\Omega}_{\sigma} a) P(X) \tag{9}$$

We need to prove Eq. (8) and Eq. (9) equal.

For an argument subset X and the corresponding projection $\mathcal{F}\downarrow_X$, we consider the set partition on U and V. In $\mathcal{F}\downarrow_X$, $X \cap U$ is unattacked because there is no attack from $X \cap V$ to $X \cap U$. By directionality$^{\Omega}_{arg}$ and the distributive property of intersection, we have

$$\Omega(\sigma(\mathcal{F}\downarrow_{X\cap U})) = \Omega(\sigma(\mathcal{F}\downarrow_X)) \cap U = \Omega(\sigma(\mathcal{F}\downarrow_X) \cap U)$$

Thus, for every $a \in U$,

$$\mathcal{F}\downarrow_X \mid\!\sim^{\Omega}_{\sigma} a \Leftrightarrow \mathcal{F}\downarrow_{X\cap U} \mid\!\sim^{\Omega}_{\sigma} a$$

It follows that

$$\mathbb{1}(\mathcal{F}\downarrow_X \mid\!\sim^{\Omega}_{\sigma} a) = \mathbb{1}(\mathcal{F}\downarrow_{X\cap U} \mid\!\sim^{\Omega}_{\sigma} a)$$

which holds for every argument subset X. Consequently,

$$\begin{aligned} P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(a) &= \sum_{X \in 2^{\mathcal{A}}} \mathbb{1}(\mathcal{F}\downarrow_X \mid\!\sim^{\Omega}_{\sigma} a) P(X) \\ &= \sum_{X \in 2^U, Y \in 2^V} \mathbb{1}(\mathcal{F}\downarrow_{X\cup Y} \mid\!\sim^{\Omega}_{\sigma} a) P(X \cup Y) \\ &= \sum_{X \in 2^U, Y \in 2^V} \mathbb{1}(\mathcal{F}\downarrow_{(X\cup Y)\cap U} \mid\!\sim^{\Omega}_{\sigma} a) P(X \cup Y) \end{aligned} \tag{10}$$

Notice that $(X \cup Y) \cap U = X \cap U$ for $X \in 2^U$, we continue the calculation by expanding $P(X \cup Y)$:

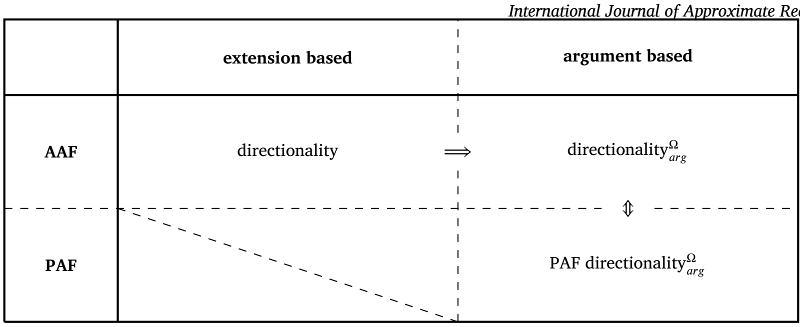

Fig. 8. Relationships between directionality concepts.

$$
\begin{aligned}
P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(a) &= \sum_{X\in 2^U, Y\in 2^V} \mathbb{1}(\mathcal{F}\downarrow_{X\cap U} \mathrel{|\!\sim}^{\Omega}_{\sigma} a) \prod_{b\in X}\mathcal{P}(b) \prod_{b\in U\setminus X}(1-\mathcal{P}(b)) \prod_{c\in Y}\mathcal{P}(c) \prod_{c\in V\setminus Y}(1-\mathcal{P}(c)) \\
&= \sum_{X\in 2^U} \mathbb{1}(\mathcal{F}\downarrow_{X\cap U} \mathrel{|\!\sim}^{\Omega}_{\sigma} a) \prod_{b\in X}\mathcal{P}(b) \prod_{b\in U\setminus X}(1-\mathcal{P}(b)) \left(\sum_{Y\in 2^V} \prod_{c\in Y}\mathcal{P}(c) \prod_{c\in V\setminus Y}(1-\mathcal{P}(c)) \right)
\end{aligned}
\tag{11}
$$

Since the sum of the probability distribution is 1, as shown in Eq. (2),

$$
\sum_{Y\in 2^V} \prod_{c\in Y}\mathcal{P}(c) \prod_{c\in V\setminus Y}(1-\mathcal{P}(c)) = 1
$$

we simplify to

$$
\begin{aligned}
P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(a) &= \sum_{X\in 2^U} \mathbb{1}(\mathcal{F}\downarrow_{X\cap U} \mathrel{|\!\sim}^{\Omega}_{\sigma} a) \prod_{b\in X}\mathcal{P}(b) \prod_{b\in U\setminus X}(1-\mathcal{P}(b)) \\
&= \sum_{X\in 2^U} \mathbb{1}(\mathcal{F}\downarrow_{X\cap U} \mathrel{|\!\sim}^{\Omega}_{\sigma} a) P(X) \\
&= P^{\mathcal{F}\downarrow_U,\mathcal{P}}_{\sigma,\Omega}(a)
\end{aligned}
\tag{12}
$$

We now prove that PAF directionality$^{\Omega}_{arg}$ implies directionality$^{\Omega}_{arg}$. This follows directly by setting $\mathcal{P}(c) = 1, \forall c \in \mathcal{A}$. And then for an argument $a \in U$:

$$
\begin{aligned}
P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(a) &= \sum_{X\subseteq \mathcal{A}, \mathcal{F}\downarrow_X \mathrel{|\!\sim}^{\Omega}_{\sigma} a} P(X) = \mathbb{1}(\mathcal{F} \mathrel{|\!\sim}^{\Omega}_{\sigma} a) \\
P^{\mathcal{F}\downarrow_U,\mathcal{P}}_{\sigma,\Omega}(a) &= \sum_{X\subseteq U, (\mathcal{F}\downarrow_U)\downarrow_X \mathrel{|\!\sim}^{\Omega}_{\sigma} a} P(X) = \mathbb{1}(\mathcal{F}\downarrow_U \mathrel{|\!\sim}^{\Omega}_{\sigma} a)
\end{aligned}
\tag{13}
$$

Using the principle of PAF directionality$^{\Omega}_{arg}$, the two equations above are equal, yielding

$$
\mathbb{1}(\mathcal{F} \mathrel{|\!\sim}^{\Omega}_{\sigma} a) = \mathbb{1}(\mathcal{F}\downarrow_U \mathrel{|\!\sim}^{\Omega}_{\sigma} a) \tag{14}
$$

which implies $\mathcal{F} \mathrel{|\!\sim}^{\Omega}_{\sigma} a \Leftrightarrow \mathcal{F}\downarrow_U \mathrel{|\!\sim}^{\Omega}_{\sigma} a, \forall a \in U$. Thus, we conclude

$$
\Omega(\sigma(\mathcal{F}\downarrow_U)) = \Omega(\sigma(\mathcal{F})) \cap U \quad \square
$$

The relationship between directionality and PAF directionality$^{\Omega}_{arg}$ is depicted in Fig. 8, which illustrates the conceptual connections and distinctions between these two properties under a given semantics σ. These relationships are summarized in the following proposition:

Proposition 5. *Given a semantics σ and an acceptance mode Ω satisfying the distributive property of intersection,*

(1) *σ satisfies directionality $\Rightarrow$ σ satisfies PAF directionality$^{\Omega}_{arg}$.*
(2) *σ satisfies PAF directionality$^{\Omega}_{arg}$ $\not\Rightarrow$ σ satisfies directionality.*

Proof. We can immediately obtain the first implication since directionality implies directionality$^{\Omega}_{arg}$. The second reverse implication is also straightforward, as directionality$^{\Omega}_{arg}$ is equivalent to PAF directionality$^{\Omega}_{arg}$ and direction-ality$^{\Omega}_{arg}$ cannot infer directionality. $\square$

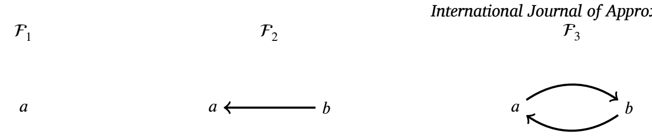

Fig. 9. Argumentation frameworks of three special cases.

4.3. Unattacked sets in PAF

The PAF directionality$^{\Omega}_{arg}$ principle states that the probability of an argument a within an unattacked set U is not influenced by arguments outside U. The role of acceptance mode Ω, however, has not been extensively studied. We refine this principle by considering both skeptical and credulous acceptance modes, showing that a reverse statement also holds for a semantics σ when admissibility is satisfied. In this context, we explore which semantics allow directionality to facilitate the identification of unattacked sets based on topology. To advance this analysis, we introduce two additional properties.

Principle 10 *(Unattacked innocence).* A semantics σ satisfies the unattacked innocence principle if, for every argumentation framework $\mathcal{F}$, $\forall a \in \mathcal{UA}(\mathcal{F}), a \in \mathcal{SA}_\sigma(\mathcal{F})$.

Unattacked innocence states that if an argument is not attacked in the framework, it must be universally accepted in every extension defined by the semantics. Not all semantics satisfy this principle. For instance, the stable semantics may fail to admit an extension in some frameworks, such as when cycles exist without unattacked arguments.

Principle 11 *(Admissibility).* A semantics σ satisfies the admissibility principle iff for every argumentation framework $\mathcal{F}$, every $\mathcal{E} \in \sigma(\mathcal{F})$ is admissible in $\mathcal{F}$.

The admissibility principle ensures that extensions prescribed by a semantics are conflict-free and can defend themselves against attacks. This property aligns with the foundational notion of rational acceptability in argumentation. All four specific semantics in this paper satisfy admissibility.

Before we establish the properties of the unattacked set, we first present an example analysis as a lemma, which will be used later. Consider the three special cases shown in Fig. 9.

Lemma 1. *Let $\mathcal{F}_1 = (\{a\}, \emptyset)$, $\mathcal{F}_2 = (\{a,b\}, \{(b,a)\})$, $\mathcal{F}_3 = (\{a,b\}, \{(a,b),(b,a)\})$, and let σ be a semantics. If σ satisfies the unattacked innocence principle and the admissibility principle, then the following holds:*

$$a \in \mathcal{SA}_\sigma(\mathcal{F}_1), \quad a \notin \mathcal{SA}_\sigma(\mathcal{F}_2), \quad a \notin \mathcal{CA}_\sigma(\mathcal{F}_2), \quad a \notin \mathcal{SA}_\sigma(\mathcal{F}_3).$$

Proof. The first three claims can be obtained directly by unattacked innocence and admissibility. For $\mathcal{F}_3$, $\{a,b\} \notin \sigma(\mathcal{F}_3)$ because a and b attack each other. If $\{a\} \in \sigma(\mathcal{F}_3)$, then according to symmetry (anonymity), $\{b\} \in \sigma(\mathcal{F}_3)$. So we have $a \notin \bigcap_{\mathcal{E}\in\sigma(\mathcal{F}_3)} \mathcal{E}$. □

Under skeptical acceptance mode, it is established that the probability of an argument a in an unattacked set U remains unaffected by external arguments under a semantics satisfying PAF directionality$^{\Omega}_{arg}$. We further demonstrate that for some semantics if the probability $P(a)$ remains invariant to external influences in a set S, regardless of changes in the initial probability distribution $\mathcal{P}$, then S must be unattacked.

Proposition 6. *Given a probabilistic argumentation framework $(\mathcal{A}, \mathcal{R}, \mathcal{P})$ and a semantics σ that satisfies unattacked innocence and admissibility, for any argument set $S \subseteq \mathcal{A}$, if under skeptical acceptance mode*

$$\forall \mathcal{P}, \forall a \in S, P^{\mathcal{F},\mathcal{P}}_{\sigma,s}(a) = P^{\mathcal{F}\downarrow_S,\mathcal{P}}_{\sigma,s}(a),$$

then S is unattacked.

Proof. We use proof by contradiction. Assume S is not unattacked, that is, there exists $a \in S, b \in \mathcal{A} \setminus S$ such that $\mathcal{R}(b,a)$. Consider the assignment $\mathcal{P}(a) = 1, \mathcal{P}(b) = 0.5$, and $\forall c \in \mathcal{A} \setminus \{a,b\}, \mathcal{P}(c) = 0$; this corresponds to the analysis of examples in Fig. 9. With this assignment, we obtain $P^{\mathcal{F}\downarrow_S}_{\sigma,s}(a) = 1$ by Lemma 1.

To calculate $P^{\mathcal{F}}_{\sigma,s}(a)$, consider the following cases:

- If $\neg\mathcal{R}(a,b)$, we need to consider two possible worlds $\mathcal{F}_1, \mathcal{F}_2$. Here, $P^{\mathcal{F}}_{\sigma,s}(a) = 0.5 \neq P^{\mathcal{F}\downarrow_S}_{\sigma,s}(a)$, leading to a contradiction.
- If $\mathcal{R}(a,b)$, we need to consider two possible worlds $\mathcal{F}_1, \mathcal{F}_3$. Here, $P^{\mathcal{F}}_{\sigma,s}(a) = 0.5 \neq P^{\mathcal{F}\downarrow_S}_{\sigma,s}(a)$, again leading to a contradiction.

Thus, S must be unattacked. □

The proposition above is valid for com, prf, grd. Nevertheless, the proposition does not hold under credulous acceptance mode, as demonstrated in Fig. 9 for $\mathcal{F}_3$, where $P^{\mathcal{F}}_{\mathsf{prf},c}(a) = P^{\mathcal{F}\downarrow_{\{a\}}}_{\mathsf{prf},c}(a) = \mathcal{P}(a)$, yet $\{a\}$ is attacked by b. This analysis highlights the distinction between these acceptance modes in the context of directionality within PAFs. However, we can refine the notion of "unattacked" to "semi-unattacked," as defined below:

Definition 20 *(Semi-unattacked).* Given an argumentation framework $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, a set $U \subseteq \mathcal{A}$ is semi-unattacked if $\forall a \in \mathcal{A} \setminus U, \forall b \in U, \neg\mathcal{R}(a,b) \vee (\mathcal{R}(a,b) \wedge \mathcal{R}(b,a))$.

It is evident that every unattacked set is also semi-unattacked. The semi-unattacked condition states that all attackers of an argument a outside the set S are also attacked by a itself. Then under some semantics, a may be considered not influenced by those attackers. We now propose the reverse statement concerning the credulous acceptance mode:

Proposition 7. *Given a probabilistic argumentation framework* $\mathtt{PrF} = (\mathcal{F}, \mathcal{P})$ *with* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *and a semantics* σ *satisfying unattacked innocence and admissibility, for an argument set* $S \subseteq \mathcal{A}$*, if* S *satisfies*

$$\forall \mathcal{P}, \forall a \in S, P^{\mathcal{F},\mathcal{P}}_{\sigma,c}(a) = P^{\mathcal{F}\downarrow_S,\mathcal{P}}_{\sigma,c}(a)$$

then S *is semi-unattacked.*

Proof. We use a similar strategy as in the previous proof. Assume S is not semi-unattacked, that is, there exists $a \in S, b \in \mathcal{A} \setminus S$ such that $\mathcal{R}(b,a) \wedge \neg\mathcal{R}(a,b)$. Set $\mathcal{P}(a) = 1, \mathcal{P}(b) = 0.5$, and $\forall c \in \mathcal{A} \setminus \{a,b\}, \mathcal{P}(c) = 0$. With this assignment, we have $P^{\mathcal{F}\downarrow_S}_{\sigma,c}(a) = 1$. While under overall framework, we need to consider two subframeworks $\mathcal{F}_1$ and $\mathcal{F}_2$ in Fig. 9. By Lemma 1, we have $P^{\mathcal{F}}_{\sigma,c}(a) = 0.5$, leading to a contradiction.

Thus, S must be semi-unattacked. □

4.4. Unattacked argument in PAF

The intuition that an unattacked argument should be fully acceptable is straightforward. In the context of PAFs, however, the probability of acceptance of an argument a cannot exceed its initially assigned probability $\mathcal{P}(a)$. Accordingly, this intuition can be expressed as follows: the value of $P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(a)$ should not be diminished by the framework and should therefore remain equal to its initial probability $\mathcal{P}(a)$. This idea is formalized in the following principle.

Principle 12 *(PAF unattacked innocence).* Let $\Omega \in \{s, c\}$. A semantics σ satisfies the PAF unattacked innocence principle wrt. Ω if for every probabilistic argumentation frameworks $\mathtt{PrF} = (\mathcal{F}, \mathcal{P})$ with $\mathcal{F} = (\mathcal{A}, \mathcal{R})$,

$$P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(a) = \mathcal{P}(a)$$

for all argument $a \in \mathcal{UA}(\mathcal{F})$.

We have the following theorem that allows us to infer that com, prf, and grd satisfy this principle under credulous and skeptical acceptance modes.

Proposition 8. *Let* $\Omega \in \{s, c\}$ *and* σ *be a semantics. If* σ *satisfies unattacked innocence, then it satisfies the PAF unattacked innocence principle wrt.* Ω*.*

Proof. Let $\mathcal{F}$ be an argumentation framework, and consider a subframework $\mathcal{F}\downarrow_X$, where $X \subseteq \mathcal{A}$. For the semantics σ satisfying unattacked innocence, it holds that every unattacked argument $a \in \mathcal{UA}(\mathcal{F}\downarrow_X)$ is accepted. Consequently, $P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(a)$ equals the summation of the probabilities of the subframeworks in which a appears, which is $\mathcal{P}(a)$. □

Among the four specific semantics we consider in this paper, stb is the only one violating this principle.

Proposition 9. *Stable semantics does not satisfy the PAF unattacked innocence principle.*

Proof. As a counterexample to the unattacked innocence principle under stb, consider a PAF that assigns a probability of 0.5 to both a and b in the argumentation framework depicted in Fig. 10. In this case, it follows that $P^{\mathcal{F},\mathcal{P}}_{\sigma,s}(a) = P^{\mathcal{F},\mathcal{P}}_{\sigma,c}(a) = 0.25 \neq 0.5$. □

For stable semantics, we are curious about why it behaves in this way. At the outset, we pinned our hopes on a sufficient and necessary condition for the existence of a stable extension in AAF, which is illustrated using the notion of cycle.

Definition 21 *(Path and cycle).* Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation framework.

Fig. 10. The AAF yields a unique complete extension, namely $\{a\}$, which is also the grounded extension and the unique preferred extension. The AAF has no stable extension.

- A finite sequence of arguments $a_1, a_2, \dots, a_n$ is called a path in $\mathcal{F}$ if
  1. $(a_i, a_{i+1}) \in \mathcal{R}$ for all $1 \le i < n$, and
  2. $a_i \neq a_j$ for all $1 \le i < j \le n$.

  If n is odd, the path is called an attack path; if n is even, it is called a support path.
- A finite sequence $a_1, a_2, \dots, a_n, a_1$ is called a cycle in $\mathcal{F}$ if $a_1, a_2, \dots, a_n$ forms a path with $n \ge 2$. A cycle is called odd if n is odd, and even if n is even.

The set of odd cycles in $\mathcal{F}$ is denoted as $OC(\mathcal{F})$. Now we can introduce the condition of existence of stable extension:

Lemma 2 *(from [9]). Given an argumentation framework $\mathcal{F}$, $\mathsf{stb}(\mathcal{F}) \neq \emptyset$ if and only if $\exists S \in ad(\mathcal{F})$ such that $S^+ \cap C \neq \emptyset$ for each $C \in OC(\mathcal{F})$.*

Lemma 3. *Given an argumentation framework $\mathcal{F}$, it holds that if stable extensions exist, then $a \in \bigcap_{\mathcal{E} \in \mathsf{stb}(\mathcal{F})} \mathcal{E}$ for all arguments $a \in \mathcal{UA}(\mathcal{F})$.*

It shows that the topological condition "there is an admissible set S in $\mathcal{F}$ s.t. $S^+ \cap C \neq \emptyset$ for each $C \in OC(\mathcal{F})$" makes AAF's stable extension exist. By combining Lemma 2 and Lemma 3, we obtain the following equivalent characterization: for any unattacked argument in a PAF, its acceptance probabilities under stable semantics coincide with its prior probability if and only if every positively probable realization admits an admissible set S whose attack range S^+ intersects every odd cycle.

Proposition 10. *Let $\mathtt{PrF} = (\mathcal{F}, \mathcal{P})$ be a probabilistic argumentation framework with $\mathcal{F} = (\mathcal{A}, \mathcal{R})$. $\forall a \in \mathcal{UA}(\mathcal{F})$,*

$$
\begin{gathered}
P^{\mathcal{F},\mathcal{P}}_{stb,s}(a) = \mathcal{P}(a) \\
\Updownarrow \\
P^{\mathcal{F},\mathcal{P}}_{stb,c}(a) = \mathcal{P}(a) \\
\Updownarrow \\
\forall X \subseteq \mathcal{A},\ \text{if } a \in X \text{ and } P(X) > 0,\ \text{then} \\
\exists S \in ad(\mathcal{F}\downarrow_X), \forall C \in OC(\mathcal{F}\downarrow_X), S^+ \cap C \neq \emptyset
\end{gathered}
$$

Proof. ($\Rightarrow$) Proof by contradiction. Assume that $\exists X \subseteq \mathcal{A}$, $a \in X$, $P(X) > 0$, and "$\exists S \in ad(\mathcal{F}\downarrow_X)$, $\forall C \in OC(\mathcal{F}\downarrow_X)$, $S^+ \cap C \neq \emptyset$" is false. According to Lemma 2, $\mathsf{stb}(\mathcal{F}\downarrow_X) = \emptyset$. Thus, $\mathcal{F}\downarrow_X \not\mathrel{|\!\sim}^s_{\mathsf{stb}} a$, and we have

$$
\begin{aligned}
P^{\mathcal{F},\mathcal{P}}_{\mathsf{stb},s}(a) &= \sum_{Y \subseteq \mathcal{A}} \mathbb{1}(\mathcal{F}\downarrow_Y \mathrel{|\!\sim}^s_{\mathsf{stb}} a) P(Y) \\
&= \sum_{Y \subseteq \mathcal{A}} \mathbb{1}(a \in Y \wedge \mathcal{F}\downarrow_Y \mathrel{|\!\sim}^s_{\mathsf{stb}} a) P(Y) \\
&= \sum_{Y \subseteq \mathcal{A}} \mathbb{1}(a \in Y) P(Y) - \sum_{Y \subseteq \mathcal{A}} \mathbb{1}(a \in Y \wedge \mathcal{F}\downarrow_Y \not\mathrel{|\!\sim}^s_{\mathsf{stb}} a) P(Y) \\
&= \mathcal{P}(a) - \sum_{Y \subseteq \mathcal{A}} \mathbb{1}(a \in Y \wedge \mathcal{F}\downarrow_Y \not\mathrel{|\!\sim}^s_{\mathsf{stb}} a) P(Y)
\end{aligned}
\tag{15}
$$

Because $a \in X$ and $\mathcal{F}\downarrow_X \not\mathrel{|\!\sim}^s_{\mathsf{stb}} a$, we have

$$
0 < P(X) \le \sum_{Y \subseteq \mathcal{A}} \mathbb{1}(a \in Y \wedge \mathcal{F}\downarrow_Y \not\mathrel{|\!\sim}^s_{\mathsf{stb}} a) P(Y)
$$

As a result, $P^{\mathcal{F},\mathcal{P}}_{\mathsf{stb},s}(a) \le \mathcal{P}(a) - P(X) < \mathcal{P}(a)$, leading to a contradiction.

($\Leftarrow$) Assume that $\forall X \subseteq \mathcal{A}$, if $a \in X$ and $P(X) > 0$, then $\exists S \in ad(\mathcal{F}\downarrow_X)$ such that $\forall C \in OC(\mathcal{F}\downarrow_X)$, $S^+ \cap C \neq \emptyset$. Now suppose $P^{\mathcal{F},\mathcal{P}}_{\mathsf{stb},s}(a) \neq \mathcal{P}(a)$. Using Eq. (15), we have

$$
\sum_{Y \subseteq \mathcal{A}} \mathbb{1}(a \in Y \wedge \mathcal{F}\downarrow_Y \not\mathrel{|\!\sim}^s_{\mathsf{stb}} a) P(Y) > 0
$$

Thus, $\{Y \subseteq \mathcal{A} \mid a \in Y \wedge \mathcal{F}\downarrow_Y \not\mathrel{|\!\sim}^s_{\mathsf{stb}} a \wedge P(Y) > 0\} \neq \emptyset$, meaning $\exists Y \subseteq \mathcal{A}$ such that $a \in Y$, $P(Y) > 0$, and $\mathcal{F}\downarrow_Y \not\mathrel{|\!\sim}^s_{\mathsf{stb}} a$. On the other hand, because $a \in Y$ and $P(Y) > 0$, we have

$$
\exists S \in ad(\mathcal{F}\downarrow_Y), \forall C \in OC(\mathcal{F}\downarrow_Y), S^+ \cap C \neq \emptyset
$$

By Lemma 2, it follows that $\mathsf{stb}(\mathcal{F}\downarrow_Y) \neq \emptyset$. By Lemma 3, $\mathcal{F}\downarrow_Y \mid\!\sim^s_{\mathsf{stb}} a$, leading to a contradiction. Thus, the proof is complete. □

5. Skepticism adequacy and resolution adequacy

We consider the properties related to adequacy criteria [5], which are associated with orders in AAFs. These criteria evaluate whether semantics appropriately reflect skepticism relations and provide a structured approach to comparing extensions across frameworks, offering insights into how arguments are resolved under varying conditions. In this section, we follow a similar illustration style by first introducing the concepts in AAFs and then adapting them to PAFs.

5.1. Notions in AAF

The following definition establishes an order between extensions: a set of extensions $\mathtt{Ext}_1$ is considered more skeptical than $\mathtt{Ext}_2$ if the set of skeptically accepted arguments with respect to $\mathtt{Ext}_1$ is a subset of the set of skeptically accepted arguments with respect to $\mathtt{Ext}_2$.

Definition 22 *($\preceq^E_\cap$).* Let $\mathtt{Ext}_1$ and $\mathtt{Ext}_2$ be two sets of sets of arguments. We say that $\mathtt{Ext}_1 \preceq^E_\cap \mathtt{Ext}_2$ iff

$$\bigcap_{\mathcal{E}_1 \in \mathtt{Ext}_1} \mathcal{E}_1 \subseteq \bigcap_{\mathcal{E}_2 \in \mathtt{Ext}_2} \mathcal{E}_2. \tag{16}$$

Given an AAF $\mathcal{F}$, if $\mathtt{Ext}$ contains all extensions for a specific semantics σ, *i.e.*, $\mathtt{Ext} = \sigma(\mathcal{F})$, then $\bigcap_{\mathcal{E}\in\mathtt{Ext}} \mathcal{E}$ is exactly the skeptically accepted argument set $\mathcal{SA}_\sigma(\mathcal{F})$. In the works [25,5], this relation is further refined by introducing more stringent definitions, $\preceq^E_W$ and $\preceq^E_S$.

We now define a skepticism relation $\preceq^A$ between argumentation frameworks. It says that $\mathcal{F}_1 \preceq^A \mathcal{F}_2$ if $\mathcal{F}_1$ may have some symmetric attacks where $\mathcal{F}_2$ has a directed attack.

Definition 23 *($\preceq^A$).* Given $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, the conflict set is defined as $\mathtt{CONF}(\mathcal{F}) = \{(a,b) \in \mathcal{A}\times\mathcal{A} \mid \mathcal{R}(a,b) \vee \mathcal{R}(b,a)\}$. Given two argumentation frameworks $\mathcal{F}_1 = (\mathcal{A}_1, \mathcal{R}_1)$ and $\mathcal{F}_2 = (\mathcal{A}_2, \mathcal{R}_2)$, we say that $\mathcal{F}_1 \preceq^A \mathcal{F}_2$ iff $\mathtt{CONF}(\mathcal{F}_1) = \mathtt{CONF}(\mathcal{F}_2)$ and $\mathcal{R}_2 \subseteq \mathcal{R}_1$.

It can be observed that $\preceq^A$ defines a partial order, as it is grounded in equality and set inclusion relations. Within the set of argumentation frameworks comparable to a given framework $\mathcal{F}$, multiple maximal elements may exist with respect to $\preceq^A$, since symmetric attacks can be replaced by asymmetric ones in various ways.

Next, we introduce the principle of skepticism adequacy. The fundamental idea is that the skeptical order in AAFs induces an order on their extensions. Specifically, if $\mathcal{F}_1$ is more skeptical than $\mathcal{F}_2$, then the set of extensions of $\mathcal{F}_1$ must also be more skeptical than the set of extensions of $\mathcal{F}_2$.

Principle 13 *(Skepticism adequacy).* Given a skepticism relation $\preceq^E$ between sets of sets of arguments, a semantics σ satisfies the $\preceq^E$-skepticism adequacy principle iff for every two argumentation frameworks $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $\mathcal{F}_1 \preceq^A \mathcal{F}_2$ it holds that $\sigma(\mathcal{F}_1) \preceq^E \sigma(\mathcal{F}_2)$.

The skepticism adequacy principle asserts that an increase in symmetric attacks leads to a reduction in the number of accepted arguments. Under $\preceq^A$, a more skeptical framework includes all attacks of the other plus additional symmetric attacks. Adding attacks cannot create new defenses and may invalidate existing ones. Thus the resulting extension sets can only become (weakly) more skeptical under $\preceq^E$, which is precisely what the principle requires.

The resolution adequacy principle proposed in [5] considers all maximal frameworks of an AAF $\mathcal{F}$ with respect to $\preceq^A$.

Definition 24 *(RES).* We denote by $RES(\mathcal{F})$ the set of all argumentation frameworks that are comparable with $\mathcal{F}$ and maximal with respect to $\preceq^A$.

For ease of later illustration, we define a special semantics based on σ induced by $\preceq^A$.

Definition 25 *($\sigma^{\preceq^A}$).* Given a semantics σ, the $\preceq^A$-induced semantics $\sigma^{\preceq^A}$ is defined as:

$$\sigma^{\preceq^A}(\mathcal{F}) = \bigcup_{\mathcal{F}' \in RES(\mathcal{F})} \sigma(\mathcal{F}').$$

Resolution adequacy is a principle that makes a semantics behave coherently when comparing abstract argumentation frameworks. Intuitively, we resolve a framework into its maximally comparable variants and check whether their extensions agree with the original's under a chosen skepticism relation. Its key characteristic is conservative alignment: resolving should not make the semantics commit to more than the original framework.

Table 2
Properties of semantics.

Property	com	grd	prf	stb
$\preceq^E_\cap$-skepticism adequacy	✓	✓	×	✓
$\preceq^E_W$-skepticism adequacy	✓	✓	×	✓
$\preceq^E_S$-skepticism adequacy	×	✓	×	×
$\preceq^E_\cap$-resolution adequacy	×	×	✓	✓
$\preceq^E_W$-resolution adequacy	×	×	✓	✓
$\preceq^E_S$-resolution adequacy	×	×	✓	✓

Principle 14 *(Resolution adequacy).* Given a skepticism relation $\preceq^E$ between sets of sets of arguments, a semantics σ satisfies the $\preceq^E$-resolution adequacy principle iff for every argumentation framework $\mathcal{F}$ we have

$$\sigma^{\preceq^A}(\mathcal{F}) \preceq^E \sigma(\mathcal{F})$$

In Table 2, we summarize the satisfiability of various forms of skepticism adequacy and resolution adequacy for four different semantics.

5.2. Adequacy in PAF

Formally, the skepticism adequacy principle states that a more skeptical AAF has fewer accepted arguments. This concept can also be extended to PAFs, where an argument in a more skeptical AAF has a lower probability of being accepted. We define the skepticism adequacy principle for PAFs as follows:

Principle 15 *(PAF skepticism adequacy).* A semantics σ satisfies the PAF skepticism adequacy principle iff for every two probabilistic argumentation frameworks $\mathtt{PrF}_1 = (\mathcal{A}, \mathcal{R}_1, \mathcal{P})$ and $\mathtt{PrF}_2 = (\mathcal{A}, \mathcal{R}_2, \mathcal{P})$, with $\mathcal{F}_1 = (\mathcal{A}, \mathcal{R}_1)$ and $\mathcal{F}_2 = (\mathcal{A}, \mathcal{R}_2)$ such that $\mathcal{F}_1 \preceq^A \mathcal{F}_2$, it holds that

$$P^{\mathcal{F}_1,\mathcal{P}}_{\sigma,s}(a) \leq P^{\mathcal{F}_2,\mathcal{P}}_{\sigma,s}(a)$$

for all $a \in \mathcal{A}$.

The skepticism adequacy principle asserts that the final probability of an argument a is lower in a more skeptical argumentation framework. It is noteworthy that the definition does not specify a particular type of skepticism relation ($\preceq^E$), as skepticism is inherently reflected in the acceptance mode s. We prove that a semantics satisfying skepticism adequacy is equivalent to one satisfying skepticism adequacy in PAF.

Proposition 11. *A semantics σ satisfies $\preceq^E_\cap$-skepticism adequacy iff σ satisfies PAF skepticism adequacy.*

Proof. We first prove the implication $\Rightarrow$. By employing skepticism adequacy, where $\mathcal{F}_1{\downarrow_X} \preceq^A \mathcal{F}_2{\downarrow_X}$ implies $\sigma(\mathcal{F}_1{\downarrow_X}) \preceq^E_\cap \sigma(\mathcal{F}_2{\downarrow_X})$, *i.e.*, $\bigcap_{\mathcal{E}\in\sigma(\mathcal{F}_1\downarrow_X)} \mathcal{E} \subseteq \bigcap_{\mathcal{E}\in\sigma(\mathcal{F}_2\downarrow_X)} \mathcal{E}$, we establish that:

$$\mathbb{1}(a \in \bigcap_{\mathcal{E}\in\sigma(\mathcal{F}_1\downarrow_X)} \mathcal{E}) \leq \mathbb{1}(a \in \bigcap_{\mathcal{E}\in\sigma(\mathcal{F}_2\downarrow_X)} \mathcal{E})$$

Then, for the skeptical acceptance mode, we have:

$$\begin{aligned} P^{\mathcal{F}_1,\mathcal{P}}_{\sigma,s}(a) &= \sum_{X\in 2^{\mathcal{A}}} \mathbb{1}(\mathcal{F}_1{\downarrow_X} \mathrel{|\!\sim^s_\sigma} a) P(X) \\ &= \sum_{X\in 2^{\mathcal{A}}} \mathbb{1}(a \in \bigcap_{\mathcal{E}\in\sigma(\mathcal{F}_1\downarrow_X)} \mathcal{E}) P(X) \\ &\leq \sum_{X\in 2^{\mathcal{A}}} \mathbb{1}(a \in \bigcap_{\mathcal{E}\in\sigma(\mathcal{F}_2\downarrow_X)} \mathcal{E}) P(X) \\ &= \sum_{X\in 2^{\mathcal{A}}} \mathbb{1}(\mathcal{F}_2{\downarrow_X} \mathrel{|\!\sim^s_\sigma} a) P(X) \\ &= P^{\mathcal{F}_2,\mathcal{P}}_{\sigma,s}(a) \quad \square \end{aligned} \tag{17}$$

Next, we prove the implication $\Leftarrow$. This is straightforward by setting $\forall a \in \mathcal{A}, \mathcal{P}(a) = 1$. With this setting, we have

$$\begin{aligned} P^{\mathcal{F}_1,\mathcal{P}}_{\sigma,s}(a) &= \sum_{X\in 2^{\mathcal{A}}} \mathbb{1}(a \in \bigcap_{\mathcal{E}\in\sigma(\mathcal{F}_1)} \mathcal{E})P(X) = \mathbb{1}(a \in \bigcap_{\mathcal{E}\in\sigma(\mathcal{F}_1)} \mathcal{E}), \\ P^{\mathcal{F}_2,\mathcal{P}}_{\sigma,s}(a) &= \sum_{X\in 2^{\mathcal{A}}} \mathbb{1}(a \in \bigcap_{\mathcal{E}\in\sigma(\mathcal{F}_2)} \mathcal{E})P(X) = \mathbb{1}(a \in \bigcap_{\mathcal{E}\in\sigma(\mathcal{F}_2)} \mathcal{E}) \end{aligned} \tag{18}$$

Using PAF skepticism adequacy, we find that:

$$\mathbb{1}(a \in \bigcap_{\mathcal{E}\in\sigma(\mathcal{F}_1)} \mathcal{E}) \leq \mathbb{1}(a \in \bigcap_{\mathcal{E}\in\sigma(\mathcal{F}_2)} \mathcal{E})$$

which implies:

$$\bigcap_{\mathcal{E}_1\in\sigma(\mathcal{F}_1)} \mathcal{E}_1 \subseteq \bigcap_{\mathcal{E}_2\in\sigma(\mathcal{F}_2)} \mathcal{E}_2$$

Finally, we can conclude that $\sigma(\mathcal{F}_1\downarrow_X) \preceq^E_\cap \sigma(\mathcal{F}_2\downarrow_X)$.

PAF resolution adequacy states that the resolution process within a probabilistic argumentation framework should uphold certain probabilistic constraints to ensure coherent argument evaluation.

Principle 16 *(PAF resolution adequacy)*. A semantics σ satisfies the PAF resolution adequacy principle iff for every probabilistic argumentation framework $\mathtt{PrF} = (\mathcal{A}, \mathcal{R}, \mathcal{P})$ with $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, it holds that

$$P^{\mathcal{F},\mathcal{P}}_{\sigma^{\preceq^A},s}(a) \leq P^{\mathcal{F},\mathcal{P}}_{\sigma,s}(a)$$

for all $a \in \mathcal{A}$.

Similar to skepticism adequacy, we have the following relationship.

Proposition 12. *A semantics σ satisfies $\preceq^E_\cap$-resolution adequacy iff σ satisfies PAF resolution adequacy.*

Proof. From $\sigma^{\preceq^A}(\mathcal{F}) \preceq^E \sigma(\mathcal{F})$, we have

$$\mathbb{1}(a \in \bigcap_{\mathcal{E}\in\sigma^{\preceq^A}(\mathcal{F})} \mathcal{E}) \leq \mathbb{1}(a \in \bigcap_{\mathcal{E}\in\sigma(\mathcal{F})} \mathcal{E})$$

Then, by a similar calculation as in Eq. (17), we can obtain

$$P^{\mathcal{F},\mathcal{P}}_{\sigma^{\preceq^A},s}(a) \leq P^{\mathcal{F},\mathcal{P}}_{\sigma,s}(a)$$

The converse statement is also proved by setting $\forall a \in \mathcal{A}, \mathcal{P}(a) = 1$, where we can derive

$$\bigcap_{\mathcal{E}_1\in\sigma^{\preceq^A}(\mathcal{F})} \mathcal{E}_1 \subseteq \bigcap_{\mathcal{E}_2\in\sigma(\mathcal{F})} \mathcal{E}_2$$

and we can conclude that $\sigma^{\preceq^A}(\mathcal{F}) \preceq^E \sigma(\mathcal{F})$. □

5.3. Order implication in PAF

PAF skepticism adequacy describes a property of order implication, where the order in the framework implies an order in probability. In fact, the implication can be reversed under the assumption that the conflict sets are equal.

Proposition 13. *Let $\mathtt{PrF}_1 = (\mathcal{F}_1, \mathcal{P})$ and $\mathtt{PrF}_2 = (\mathcal{F}_2, \mathcal{P})$ be two probabilistic argumentation frameworks with $\mathcal{F}_1 = (\mathcal{A}, \mathcal{R}_1)$ and $\mathcal{F}_2 = (\mathcal{A}, \mathcal{R}_2)$, and suppose $\mathtt{CONF}(\mathcal{F}_1) = \mathtt{CONF}(\mathcal{F}_2)$. Let $\Omega \in \{s, c\}$ and σ be a semantics satisfying unattacked innocence and admissibility. If*

$$\forall \mathcal{P}, \forall a \in \mathcal{A},\ P^{\mathcal{F}_1}_{\sigma,\Omega}(a) \leq P^{\mathcal{F}_2}_{\sigma,\Omega}(a),$$

then $\mathcal{F}_1 \preceq^A \mathcal{F}_2$.

Proof. We prove the statement by contradiction. Suppose $\mathcal{F}_1 \npreceq^A \mathcal{F}_2$. Since $\mathtt{CONF}(\mathcal{F}_1) = \mathtt{CONF}(\mathcal{F}_2)$, this implies that there exist $a, b \in \mathcal{A}$ such that

$$\neg\mathcal{R}_1(a, b) \wedge \mathcal{R}_1(b, a) \wedge \mathcal{R}_2(a, b) \wedge \neg\mathcal{R}_2(b, a)$$

Set $\mathcal{P}(a) = 1$, $\mathcal{P}(b) = 0.5$, and $\forall c \in \mathcal{A} \setminus \{a, b\}, \mathcal{P}(c) = 0$. Under this configuration, we have $P^{\mathcal{F}_1}_{\sigma,\Omega}(b) = 0.5$ and $P^{\mathcal{F}_2}_{\sigma,\Omega}(b) = 0$, which is a contradiction. □

6. Dynamics

The study of properties of argumentation semantics includes both static and dynamic aspects. The previous content in this paper has mainly focused on the static semantic properties. Existing literature [10,9] also discusses the dynamic aspects of argumentation semantics. The study of dynamics focuses on how outcomes change when the internal structure of a system is altered or new arguments are introduced. In the context of PAFs, this involves examining how the probability of an argument, $P(a)$, changes in response to modifications in the framework. Such an analysis requires investigating the calculation of $P(a)$ in relation to structural changes or the addition of new arguments.

6.1. Dynamic properties for x-expansion

In the research on AAF dynamics, scholars primarily focus on a specific subclass of normal expansions [8], referred to as x-expansions, which were first introduced in [7]. Normal expansions involve the addition of new arguments and attacks to an argumentation framework $\mathcal{F}$, while ensuring no novel attacks are introduced among the existing arguments within $\mathcal{F}$. An x-expansion is a particular type of normal expansion, where precisely one new argument x is added to $\mathcal{F}$.

Definition 26 *(x-expansion).* An argumentation framework $\mathcal{F}_1 = (\mathcal{A}_1, \mathcal{R}_1)$ is an x-expansion of an argumentation framework $\mathcal{F}_2 = (\mathcal{A}_2, \mathcal{R}_2)$ iff $\mathcal{A}_1 = \mathcal{A}_2 \cup \{x\}$, and for every $(a, b) \in \mathcal{R}_1 \setminus \mathcal{R}_2$, it holds that $a = x$ or $b = x$.

One notable and representative dynamic property is monotony [7], which occurs when an argument is added to the framework.

Definition 27 *(AAF monotony).* Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation framework, σ a semantics, and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ an x-expansion of $\mathcal{F}$. The change from $\mathcal{F}$ to $\mathcal{F}'$ is said to satisfy monotony under σ iff

$$\forall \mathcal{E} \in \sigma(\mathcal{F}), \exists \mathcal{E}' \in \sigma(\mathcal{F}') \text{ such that } \mathcal{E} \subseteq \mathcal{E}'.$$

The authors [7] present a synthesis of the necessary and sufficient conditions for the grounded and preferred semantics. Now let's see what kind of dynamic properties in AAFs can be transferred in to PAFs. We define similar concepts in PAF as follows:

Definition 28 *(PAF x-expansion).* Let $\mathtt{PrF}_1 = (\mathcal{F}_1, \mathcal{P}_1)$ and $\mathtt{PrF}_2 = (\mathcal{F}_2, \mathcal{P}_2)$ where $\mathcal{F}_1 = (\mathcal{A}_1, \mathcal{R}_1)$ and $\mathcal{F}_2 = (\mathcal{A}_2, \mathcal{R}_2)$. $\mathtt{PrF}_1$ is an x-expansion of $\mathtt{PrF}_2$ iff $\mathcal{F}_1$ is an x-expansion of $\mathcal{F}_2$ and $\mathcal{P}_1(a) = \mathcal{P}_2(a), \forall a \in \mathcal{A}_2$.

Definition 29 *(PAF monotony).* Let $\mathtt{PrF}_1 = (\mathcal{F}_1, \mathcal{P}_1)$ be an x-expansion of $\mathtt{PrF}_2 = (\mathcal{F}_2, \mathcal{P}_2)$. The change from $\mathtt{PrF}_1$ to $\mathtt{PrF}_2$ is said to satisfy monotony under a semantics σ iff

$$P^{\mathcal{F}_1,\mathcal{P}_1}_{\sigma,s}(a) \leq P^{\mathcal{F}_2,\mathcal{P}_2}_{\sigma,s}(a)$$

for every $a \in \mathcal{SA}_\sigma(\mathcal{F})$.

In the context of argumentation dynamics, it is important to note that the implication of AAF monotony to PAF monotony does not generally hold. Specifically, even if a semantics satisfies monotony under certain structural constraints in an AAF, it may fail to satisfy monotony in the PAF after assigning initial probability. The work in [7] introduced concepts closely tied to AAF dynamics, but the satisfaction of these properties for some semantics does not necessarily transfer to PAFs. For instance, as shown in Fig. 11, introducing uncertainty in an argument b by assigning a probability $\mathcal{P}(b)$ can break monotony, even when the underlying topological structure remains unchanged, thereby demonstrating that AAF monotony does not imply PAF monotony. This discrepancy arises because these properties often rely on specific structural characteristics that are not inherently preserved in subgraphs: while these principles describe properties of semantics, the satisfaction of such properties for a given semantics may hold in every framework, including subframeworks, but their behavior in the probabilistic setting often deviates due to the introduction of uncertainty.

However, the following lemma, borrowed from [9], provides insights into how specific x-expansions influence the acceptability of arguments under common semantics such as complete, grounded, and preferred semantics. The lemma highlights two key cases: (1) when x purely attacks an argument a, and (2) when x purely supports a. Here, we say that a purely attacks (resp. purely supports) b if every path from a to b is an attack (resp. support) path. According to the above definitions, an argument a with no path to b both purely attacks and purely supports b.

Lemma 4 *(from [9]). Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation framework and $\mathcal{F}'$ an x-expansion of $\mathcal{F}$. For a semantics $\sigma \in \{\mathit{com}, \mathit{grd}, \mathit{prf}\}$ and an argument $a \in \mathcal{A}$, the following hold:*

1. *If $a \notin \mathcal{SA}_\sigma(\mathcal{F})$ and x purely attacks a, then $a \notin \mathcal{SA}_\sigma(\mathcal{F}')$.*

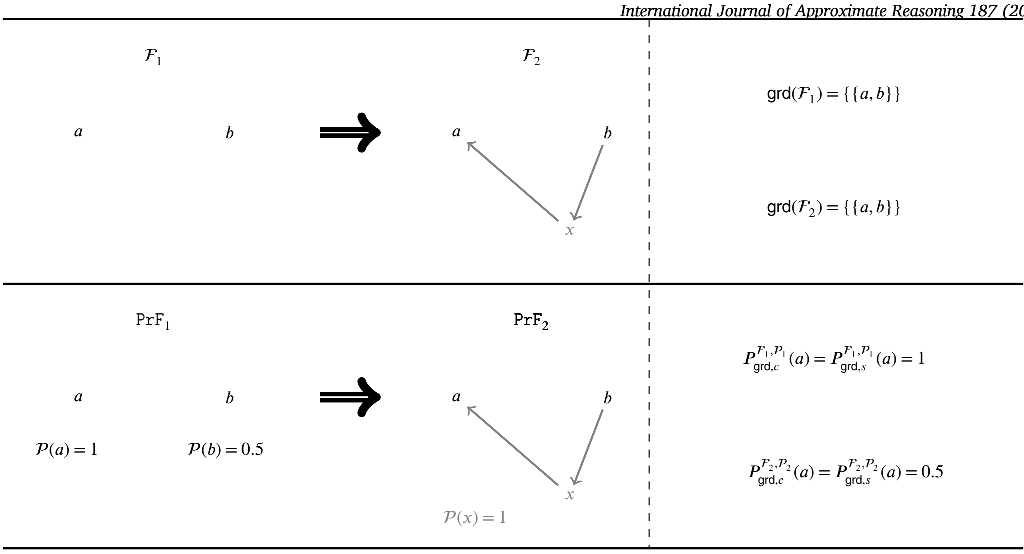

Fig. 11. Counterexample for transferring monotony. Monotony holds in the x-expansion of the AAFs but not in the x-expansion of the PAFs.

2. *If $a \in \mathcal{SA}_\sigma(\mathcal{F})$ and x purely supports a, then $a \in \mathcal{SA}_\sigma(\mathcal{F}')$.*

Inspired by the behavior of argument probabilities under system changes, we define two monotony-related principles: in a PAF, adding a pure attacker must not raise, and adding a pure supporter must not lower, the target's acceptance probability. This mirrors AAF monotonicity but uses probabilistic inequalities under uncertainty.

Principle 17 *(PAF attack monotony).* Let $\mathtt{PrF}_1 = (\mathcal{F}_1, \mathcal{P}_1)$ and $\mathtt{PrF}_2 = (\mathcal{F}_2, \mathcal{P}_2)$ be two probabilistic argumentation frameworks with $\mathcal{F}_1 = (\mathcal{A}_1, \mathcal{R}_1)$ and $\mathcal{F}_2 = (\mathcal{A}_2, \mathcal{R}_2)$ such that $\mathtt{PrF}_2$ is an x-expansion of $\mathtt{PrF}_1$. A semantics σ satisfies the PAF attack monotony principle iff for all $a \in \mathcal{A}_1$,

$$\text{If } x \text{ purely attacks } a, \text{ then } P^{\mathcal{F}_2,\mathcal{P}_2}_{\sigma,c}(a) \leq P^{\mathcal{F}_1,\mathcal{P}_1}_{\sigma,c}(a).$$

Principle 18 *(PAF support monotony).* Let $\mathtt{PrF}_1 = (\mathcal{F}_1, \mathcal{P}_1)$ and $\mathtt{PrF}_2 = (\mathcal{F}_2, \mathcal{P}_2)$ be two probabilistic argumentation frameworks with $\mathcal{F}_1 = (\mathcal{A}_1, \mathcal{R}_1)$ and $\mathcal{F}_2 = (\mathcal{A}_2, \mathcal{R}_2)$ such that $\mathtt{PrF}_2$ is an x-expansion of $\mathtt{PrF}_1$. A semantics σ satisfies the PAF support monotony principle iff for all $a \in \mathcal{A}_1$,

$$\text{If } x \text{ purely supports } a, \text{ then } P^{\mathcal{F}_2,\mathcal{P}_2}_{\sigma,c}(a) \geq P^{\mathcal{F}_1,\mathcal{P}_1}_{\sigma,c}(a).$$

These two principles formalize the relationship between argument probabilities and the expansion of a probabilistic argumentation framework. The PAF attack monotony principle ensures that the probability of an argument a does not increase when a new argument x is added that purely attacks a. Conversely, the PAF support monotony principle guarantees that the probability of a does not decrease when a new argument x is added that purely supports a. Together, these principles capture the intuitive idea that the addition of new arguments should affect existing argument probabilities in a consistent and predictable manner based on the nature of their interactions.

Proposition 14. *The semantics* com*,* grd*, and* prf *satisfy both the PAF attack monotony principle and the PAF support monotony principle.*

Proof. Let $\mathtt{PrF}_1 = (\mathcal{A}_1, \mathcal{R}_1, \mathcal{P}_1)$ and $\mathtt{PrF}_2 = (\mathcal{A}_2, \mathcal{R}_2, \mathcal{P}_2)$. As defined by the x-expansion, we have $\mathcal{A}_2 = \mathcal{A}_1 \cup \{x\}$. Since $\forall a \in \mathcal{A}_1, \mathcal{P}_1(a) = \mathcal{P}_2(a)$, we denote the probability of the argument set distribution by the same symbol $P : 2^{\mathcal{A}_1} \to [0, 1]$. For any argument subset X_1 containing a, where $a \in X_1 \subseteq \mathcal{A}_1$, let $X_2 = X_1 \cup \{x\} \subseteq \mathcal{A}_2$. Then $\mathcal{F}_2 \downarrow_{X_2}$ is an x-expansion of $\mathcal{F}_1 \downarrow_{X_1}$.
(1) From Lemma 4, we know that if $a \notin \mathcal{SA}_\sigma(\mathcal{F}_1 \downarrow_{X_1})$, then $a \notin \mathcal{SA}_\sigma(\mathcal{F}_2 \downarrow_{X_2})$. By the contrapositive, if $a \in \mathcal{SA}_\sigma(\mathcal{F}_2 \downarrow_{X_2})$, then it must hold that $a \in \mathcal{SA}_\sigma(\mathcal{F}_1 \downarrow_{X_1})$. This implies:

$$\mathbb{1}(a \in \mathcal{SA}_\sigma(\mathcal{F}_2 \downarrow_{X_2})) \leq \mathbb{1}(a \in \mathcal{SA}_\sigma(\mathcal{F}_1 \downarrow_{X_1}))$$

and therefore:

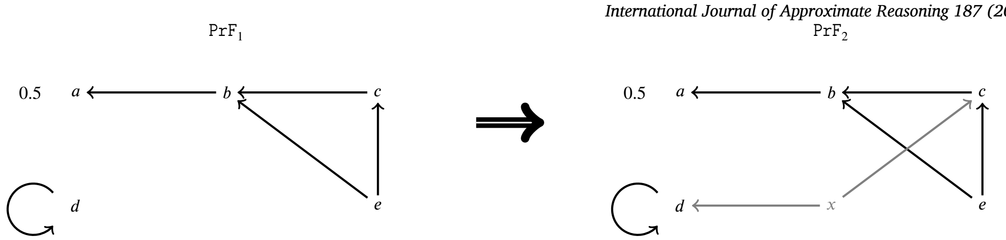

Fig. 12. The acceptability degree of a in $\mathtt{PrF}_1$, namely $P^{\mathcal{F}_1,\mathcal{P}_1}_{\mathsf{stb},c}(a)$, is 0. However, after adding an argument x that purely attacks a, the acceptability degree of a increases to $P^{\mathcal{F}_2,\mathcal{P}_2}_{\mathsf{stb},c}(a) = 0.5$. This demonstrates that stb does not satisfy the PAF attack monotonicity principle.

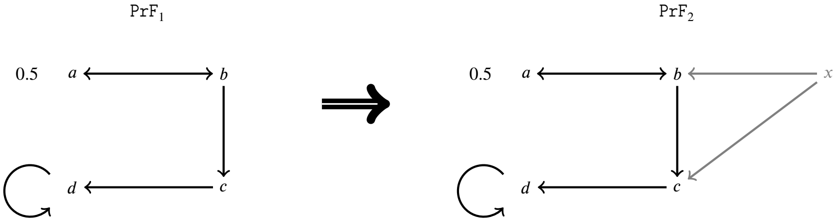

Fig. 13. The acceptability degree of a in $\mathtt{PrF}_1$, namely $P^{\mathcal{F}_1,\mathcal{P}_1}_{\mathsf{stb},c}(a)$, is 0.5. However, after adding an argument x that purely supports a, the acceptability degree of a decreases to $P^{\mathcal{F}_2,\mathcal{P}_2}_{\mathsf{stb},c}(a) = 0$. This demonstrates that stb does not satisfy the PAF support monotonicity principle.

$$\begin{aligned}
P^{\mathcal{F}_2,\mathcal{P}_2}_{\sigma,c}(a) &= \sum_{X_2 \in 2^{\mathcal{A}_2}} \mathbb{1}(a \in \mathcal{SA}_\sigma(\mathcal{F}_2{\downarrow}_{X_2}))P(X_2) \\
&= (1-\mathcal{P}_2(x)) \sum_{X_1 \in 2^{\mathcal{A}_1}} \mathbb{1}(a \in \mathcal{SA}_\sigma(\mathcal{F}_1{\downarrow}_{X_1}))P(X_1) \\
&\quad + \mathcal{P}_2(x) \sum_{X_1 \in 2^{\mathcal{A}_1}} \mathbb{1}(a \in \mathcal{SA}_\sigma(\mathcal{F}_2{\downarrow}_{X_1 \cup \{x\}}))P(X_1) \\
&\le (1-\mathcal{P}_2(x)) \sum_{X_1 \in 2^{\mathcal{A}_1}} \mathbb{1}(a \in \mathcal{SA}_\sigma(\mathcal{F}_1{\downarrow}_{X_1}))P(X_1) \\
&\quad + \mathcal{P}_2(x) \sum_{X_1 \in 2^{\mathcal{A}_1}} \mathbb{1}(a \in \mathcal{SA}_\sigma(\mathcal{F}_1{\downarrow}_{X_1}))P(X_1) \\
&= \sum_{X_1 \in 2^{\mathcal{A}_1}} \mathbb{1}(a \in \mathcal{SA}_\sigma(\mathcal{F}_1{\downarrow}_{X_1}))P(X_1) \\
&= P^{\mathcal{F}_1,\mathcal{P}_1}_{\sigma,c}(a)
\end{aligned} \tag{19}$$

(2) From Lemma 4, we know that if $a \in \mathcal{SA}_\sigma(\mathcal{F}_2{\downarrow}_X)$, then $a \in \mathcal{SA}_\sigma(\mathcal{F}_1{\downarrow}_X)$. This implies:

$$\mathbb{1}(a \in \mathcal{SA}_\sigma(\mathcal{F}_1{\downarrow}_X)) \le \mathbb{1}(a \in \mathcal{SA}_\sigma(\mathcal{F}_2{\downarrow}_X)).$$

Using a similar calculation as in Eq. (19), we have:

$$P^{\mathcal{F}_2,\mathcal{P}_2}_{\sigma,c}(a) \ge P^{\mathcal{F}_1,\mathcal{P}_1}_{\sigma,c}(a) \quad \square$$

For stb, the violations of the PAF attack and support monotonicity principle are demonstrated by the counterexamples in Fig. 12 and Fig. 13, respectively.

6.2. Discussion of initial probability change

For a probabilistic argumentation framework $\mathtt{PrF} = (\mathcal{A}, \mathcal{R}, \mathcal{P})$, we here focus on how $P(x)$ varies when the initial probability distribution $\mathcal{P}$ changes in $\mathtt{PrF}$. We expand the probability expression using the base probability as follows:

$$\begin{aligned}
P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(x) &= \sum_{X \subseteq \mathcal{A}, \mathcal{F}{\downarrow}_X \mathrel{|\!\sim}^{\Omega}_{\sigma} x} P(X) \\
&= \sum_{X \in 2^{\mathcal{A}}} \mathbb{1}(\mathcal{F}{\downarrow}_X \mathrel{|\!\sim}^{\Omega}_{\sigma} x)P(X) \\
&= \sum_{X \in 2^{\mathcal{A}}} \mathbb{1}(\mathcal{F}{\downarrow}_X \mathrel{|\!\sim}^{\Omega}_{\sigma} x) \prod_{a \in X} \mathcal{P}(a) \prod_{a \notin X} (1-\mathcal{P}(a)).
\end{aligned} \tag{20}$$

From the equation above, it can be observed that for every $a \in \mathcal{A}$, $P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(x)$ is a *linear function* of $\mathcal{P}(a)$. Consequently, the influence of $\mathcal{P}(a)$ on $P(x)$ can be evaluated by comparing the probability $P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(x)$ before and after removing a. This discussion can be categorized under the topic of x-expansion. We define the difference as:

$$D^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(x,a) = P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(x) - P^{\mathcal{F}_{-a},\mathcal{P}}_{\sigma,\Omega}(x), \tag{21}$$

where $\mathcal{F}_{-a} = (\mathcal{A} \setminus \{a\}, \mathcal{R})$ represents the framework after the removal of a, and $\mathcal{F}$ is an x-expansion of $\mathcal{F}_{-a}$. A positive $D^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(x,a)$ indicates that a acts as a defeater, meaning that an increase in $\mathcal{P}(a)$ results in a decrease in $P(x)$. Conversely, if $D^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(x,a)$ is negative, a acts as a defender.

We aim to identify a property concerning dynamics, specifically determining the positivity or negativity of $D^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}$, which holds for every initial probability distribution. Given the complexity of probabilistic computations involving all subgraphs, it appears that if a property holds in a PAF, it consequently holds in each subgraph. In fact, we have the following conclusion:

Proposition 15. *Given a probabilistic argumentation framework* $(\mathcal{F},\mathcal{P})$ *with* $\mathcal{F} = (\mathcal{A},\mathcal{R})$*, a semantics* σ*, and an acceptance mode* Ω*. Suppose* $|\mathcal{A}| = n$*; we denote* $\mathcal{A} = \{a_1, a_2, \dots, a_n\}$ *and* $\mathcal{P}_v = [\mathcal{P}(a_1), \mathcal{P}(a_2), \dots, \mathcal{P}(a_n)] \in [0,1]^n$*. Let* G *be a function involving a linear combination of probability computations,* $G : [0,1]^n \to \mathbb{R}$*:*

$$G^{\mathcal{F}}_{\sigma,\Omega}(\mathcal{P}_v) = \sum_{i=1}^{n} k_i P^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}(a_i),$$

where $k_i \in \mathbb{R}$*. Then:*

$$\forall \mathcal{P}_v \in [0,1]^n, G^{\mathcal{F}}_{\sigma,\Omega}(\mathcal{P}_v) \geq 0 \Leftrightarrow \forall X \subseteq \mathcal{A}, \forall \mathcal{P}_v \in [0,1]^n, G^{\mathcal{F}\downarrow_X}_{\sigma,\Omega}(\mathcal{P}_v) \geq 0.$$

Proof. We first prove $\Rightarrow$. Assume by contradiction that there exists X and $\mathcal{P}_v$ such that in $\mathcal{F}\downarrow_X$, $G^{\mathcal{F}\downarrow_X}_{\sigma,\Omega}(\mathcal{P}_v) < 0$. Then we can construct an initial probability distribution $\mathcal{P}'$ defined by $\mathcal{P}'(x) = \mathcal{P}(x), \forall x \in X$ and $\mathcal{P}'(x) = 0, \forall x \notin X$, resulting in $G^{\mathcal{F}}_{\sigma,\Omega}(\mathcal{P}'_v) < 0$ in $\mathcal{F}$, which contradicts the assumption.

The proof of $\Leftarrow$ is straightforward by choosing $X = \mathcal{A}$. □

As we can see, $D^{\mathcal{F},\mathcal{P}}_{\sigma,\Omega}$ is also captured within this function format. Therefore, in a PAF, if we want to establish a property G involving a linear combination of probabilities that holds for every $\mathcal{P}$, it is equivalent to showing that G holds in every subframework.

7. Related work

7.1. Probabilistic argumentation

The study of PAFs represents a significant advancement in the field of argumentation theory, merging AAFs with probability theory to handle uncertainties regarding the existence of arguments and attacks. The probabilistic argumentation approach proposed by [22] introduces uncertainty in argument graphs using a probability distribution over sets of arguments, each inducing a subgraph with a probability assignment. In [16], researchers expand on Dung's framework by assigning probabilities to each argument and attack. They define the probability of extensions for all semantics and address computational challenges by proposing a Monte Carlo simulation approach. After establishing these theoretical foundations, numerous studies contribute to the representation of probability in AAF. Hunter [26] explores argument strength within probabilistic frameworks, focusing on defeasible rules. This work contributes to understanding how probabilistic necessity and sufficiency influence argument acceptability and presentation in discussions. The study in [27] introduces a novel probabilistic explanation framework, exploring the computation of the probability of acceptance for a goal argument. It notes that traditional probabilistic credulous acceptance may not yield intuitive results. To forecast in jury-based contexts, JPAA and JPABA [28] apply probabilistic argumentation where arguments may be supported by assignments to random variables within the Bayesian network. This integration of quantitative probabilistic reasoning and qualitative argumentation allows for the combination of opinions from different reasoners, enhancing judgmental forecasting.

7.2. Semantic properties in argumentation

Baroni and Giacomin [5] examine principle-based evaluations of extension-based semantics in argumentation frameworks, introducing criteria for evaluating semantics. In this paper we demonstrate that some of them like directionality and skepticism can also be reflected in PAF. Van der Torre and Vesic [10] outline a principle-based approach for selecting and developing argumentation semantics, classifying them using twenty-seven principles and exploring applications across frameworks. Gradual argumentation, akin to PAF in evaluating argument value, assigns varying strengths to arguments, offering nuanced analysis beyond binary acceptance or rejection [29–31], crucial for complex decision-making where simple evaluations are insufficient [32–35]. Amgoud and Ben-Naim [36] explore evaluation in weighted bipolar argumentation graphs, introducing principles to assess and compare semantics. They also propose a new semantics of acyclic graphs which satisfies all the principles. Spaans et al. [37] investigate gradual semantics in extended frameworks using probabilities, introducing a technique to calculate each argument's overall strength against established principles. The probability in PAFs can be viewed as a type of gradual strength, with each node assigned a value from 0 to 1. For ranking-based semantics, various logical properties have been proposed to characterize different semantics. Amgoud and Ben-Naim [38] and Bonzon et al. [39] have systematically formalized and compared key properties such as abstraction, independence, void precedence, cardinality precedence, and defense principles. Mailly [40] investigates incomplete argumentation frameworks and introduces two grounded semantics that avoid completion-based reasoning, proving their polynomial computability. They also provide a principle-based analysis of these semantics and explore how completion-based semantics adhere to key argumentation principles.

8. Conclusion

In this paper, we have conducted a systematic investigation into the properties of argumentation semantics within PAFs, addressing a relatively underexplored area within the argumentation community. By focusing on principles derived through the constellation approach, we conduct a systematic investigation emphasizing argument-centered semantics properties. Specifically, we analyze whether foundational principles, such as directionality and skepticism adequacy, can be faithfully adapted to the probabilistic setting and explore the implications of these adaptations for argument evaluation. Notably, we have proven that the satisfiability of these principles remains equivalent between the two frameworks, which underscores the robustness and adaptability of classical principles when applied to reasoning under uncertainty.

Beyond extending classical principles, this work has explored novel dynamic properties unique to PAFs, such as PAF support monotony, which captures how argument probabilities respond to structural changes in the framework. Our findings highlight both the challenges and opportunities posed by probabilistic environments, particularly in cases where certain classical properties, like monotony, fail to hold. These insights contribute to a more nuanced understanding of how probabilistic measures interact with argumentation structures, offering new theoretical tools for reasoning in uncertain contexts.

Our future research directions include investigating the semantic behavior in alternative probabilistic argumentation approaches, such as those proposed in [27], and conducting a comparative analysis against the current framework. Additionally, further efforts will focus on developing enhanced explanation methods for PAFs based on the established properties.

Through this work, we have gained valuable insights into the behavior of argumentation semantics in probabilistic contexts, including the interplay between structural dynamics and probability distributions. These contributions pave the way for more flexible and realistic applications of probabilistic argumentation frameworks in areas such as expert systems, decision support, and multi-agent systems. By bridging the gap between classical and probabilistic frameworks, this work lays a foundation for further exploration of argumentation under uncertainty, enabling richer and more discriminative reasoning models.

CRediT authorship contribution statement

Zhaoqun Li: Writing – original draft, Methodology. **Beishui Liao:** Writing – review & editing, Formal analysis. **Chen Chen:** Writing – original draft, Methodology, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The research reported in this paper was partially supported by the National Social Science Foundation Major Project of China (20&ZD047).

Data availability

No data was used for the research described in the article.

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