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Superhedging supermartingales

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ABSTRACT

Supermartingales are here defined in a non-probabilistic setting and can be interpreted solely in terms of superhedging operations. The classical expectation operator is replaced by a pair of subadditive operators: one defines a class of null sets, and the other acts as an outer integral. These operators are motivated by a financial theory of no-arbitrage pricing. Such a setting extends the classical stochastic framework by replacing the path space of the process by a trajectory set, while also providing a financial/gambling interpretation based on the notion of superhedging. The paper proves analogues of the following classical results: Doob's supermartingale decomposition and Doob's pointwise convergence theorem for non-negative supermartingales. The approach shows how linearity of the expectation operator can be circumvented and how integrability properties in the proposed setting lead to the special case of (hedging) martingales while no integrability conditions are required for the general supermartingale case.

1. Introduction

The paper introduces a class of non-probabilistic supermartingales in a setting where a set of price scenarios (also called trajectories) is given along with the possibility to trade as price trajectories unfold over time. Trajectories are sequences $S = (S_i)_{i\geq 0} \in \mathscr{S}$ in infinite discrete time with a common origin $S_0 = s_0$, where the set $\mathscr{S}$ substitutes the abstract sample space Ω of the probabilistic setting. Following ideas of the theory of *non-lattice* integration developed by Leinert [16] and König [14], one can construct an outer integral operator, denoted by $\overline{\sigma}$, which corresponds to the superhedging price when trading takes place by means of some idealized class of linear combinations of buy-and-hold strategies, see [10]. The idealization facilitates to establish an analogue of Daniell's continuity-from-below-condition for this outer superhedging integral operator, which is a standing assumption for the main results of this paper. Considering an investor, who enters the market at any later time, we can naturally introduce a conditional version of this superhedging outer integral operator, denoted by $\overline{\sigma}_j$, which gives rise to the notion of a superhedging supermartingale via the relation $\overline{\sigma}_j f_{j+1} \leq f_j$ where $(f_j)_{j\geq 0}$ is a sequence of real valued functions with domain $\mathscr{S}$. More precisely, the latter relation is only required to hold outside a null set – and it is an important subtlety of Leinert's integration theory, that the null sets are determined by a countably sub-additive operator, $\overline{I}$ say, which is closely related but, in general, different from the outer integral $\overline{\sigma}$.

While the definitions of the operators $\overline{\sigma}$ and $\overline{I}$ are motivated by Leinert's [16] theory of non-lattice integration, they closely connect to other theories that employ subadditive operators. If the continuity-from-below condition holds, then $\overline{\sigma}$ and $\overline{I}$ can be shown to satisfy

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the conditions of a sublinear expectation in the sense of Peng [20]. The operator $\bar{I}$ is defined on the cone of non-negative functions and is always countably subadditive. Therefore, its restriction to the set of indicator functions of subsets of $\mathscr{S}$ defines an outer probability measure and constitutes a discrete-time analogue to Vovk's [28] outer measure in the theory of continuous-time robust finance. We only rely on $\bar{I}$ for defining null sets – and we frequently exploit the countable sub-additivity of $\bar{I}$ in our infinite discrete-time framework to aggregate a countable family of null sets into a null set. In contrast, $\overline{\sigma}$ has an unrestricted domain, but fails to be countably sub-additive in general (and, thus, is less suitable for defining null sets). Both operators $\overline{\sigma}$ and $\bar{I}$ can also be interpreted as upper previsions in the sense of Walley [29]; cp. also the monograph [26]. The corresponding offer sets consists of all payoffs of financial derivatives (or, gambles) which can be superhedged with zero initial endowment, where different trading idealizations are used in the definitions of $\overline{\sigma}$ and $\bar{I}$, respectively. While $\overline{\sigma}$ can always be shown to be a coherent upper prevision, if continuity from below holds, $\bar{I}$ may fail to satisfy the coherence axiom. Therefore, superhedging prices in terms of $\bar{I}$ may violate the rationality requirement encoded in Walley's notion of coherence, explaining the need to work with two different subadditive operators in our framework; see Remark 3.4 for more details on these relations.

Proofs of classical results (i.e., in a stochastic setting) that involve supermartingales rely, at one point or another, on properties of conditional expectation operators as well as on some measure theory. In our non-probabilistic setting, also referred to as *trajectorial setting*, it turns out that the space of integrable functions (restricted on which the superhedging outer integral operator acts linearly) could be inconveniently small, see [2] for details. This fact obstructs the naive strategy of emulating classical proofs by replacing the expectation operator with the non-classical superhedging integral, but suggests to work with the superhedging outer integral instead. This is a subadditive operator with unrestricted domain but its definition allows to bypass the need for the linearity of the expectation as well as the non availability of some classical limit theorems.

In this paper, we prove analogues of several classical results (see for example [17] and [12]) for supermartingales in the trajectorial setting. In particular, we derive a representation theorem for superhedging supermartingales. Our representation is of a similar type as the uniform Doob decomposition in discrete time (Theorem 7.5 in [12]) or the optional decomposition in continuous time, see [9,15] in the classical setting or [19,18] for non-dominated versions. As illustrated by an example, our Doob decomposition can also be applied to trajectory sets which do not have any martingale measure and, thus, cannot be recovered by classical robust supermartingale decompositions.

Combining the supermartingale representation theorem with a convergence result for martingale transforms in the trajectorial setting (derived in [10]), we can, moreover, prove an analogue of Doob's a.e. pointwise convergence theorem for non-negative supermartingales.

As another application of the supermartingale representation theorem, we clarify the role of the two superhedging operators $\overline{\sigma}$ and $\overline{I}$. Theorem 8.1 shows that the superhedging outer integral $\overline{\sigma}$ indeed provides the 'correct' superhedging price in the sense that for payoffs of finite maturity it coincides with the infimal superhedging cost within the class of linear combinations of buy-and-hold strategies up to the null sets induced by the $\overline{I}$ operator.

Our work in the Leinert-König setting provides an independent meaning, purely financially motivated, to the results listed above. An inspection of our proof techniques shows also the need to rely on new and independent proof arguments.

1.1. Relation to the literature

The paper could be loosely considered as being part of the literature on robust financial mathematics that weakens a-priori probabilistic modeling hypothesis, or dispenses with them altogether. This literature ranges from discrete-time model-free superhedging dualities (e.g., [3–5]) to extending stochastic calculus beyond its original settings (e.g., [28,21,1]).

Our setting is, however, more closely related to the game theoretic approach to probability initiated by Shafer, Vovk and coauthors (see e.g. [23,24] and the references therein), which has been related to Walley's [29] notion of coherent (lower and upper) previsions by de Cooman and coauthors [6,25]. On a technical level, a key difference between the Shafer/Vovk approach and our setting is that their conditional global upper expectation operator $\mathbb{E}_s$ satisfies the axiomatic properties of an outer expectation in every situation (Proposition 8.3 in [24]), while our conditional outer integral operator $\overline{\sigma}_j$ may assign value $-\infty$ to any bounded function on a null set, on which the conditional version of the continuity from below property fails. In view of Corollary 3.14 below such a failure of the conditional continuity from below property can have two origins: a) The trajectory set may run in a situation (or, node, as we call it), in which the stock price will move upwards for sure (or will move downwards for sure) leading to an obvious arbitrage opportunity; b) A sure arbitrage situation arises at some node by trading up to an unbounded investment horizon, when the trajectory set turns out to be trajectorially incomplete (in the sense of Section 3 below). To the best of our knowledge, the aforementioned types of arbitrage situations are not presently accommodated into the abstract game-theoretic setting (as presented in Chapter 7 of [24]), but they are detected as null sets by our $\bar{I}$-operator (as should be); see also Section 4 for a detailed comparison to the game-theoretic approach. Dealing with these additional null sets does not only lead to significant technical difficulties, but, in view of Theorem 8.4, we are required to work with the two different families of conditional superhedging operators $\overline{\sigma}_j$ and $\overline{I}_j$ (as opposed to the single family of global conditional upper expectations $\mathbb{E}_s$ in the game-theoretic approach). Our developments, originally based on Leinert's and König's work on non-lattice integration, were developed independently of the game theoretic approach to probability, in particular, our proof techniques are new as well as the main results we provide. In this sense, our paper builds a bridge between the game-theoretic approach of Shafer and Vovk and the theory of non-lattice integration developed by Leinert and König.

1.2. Structure of the paper

The paper is organized as follows; Section 2 introduces the trajectorial setting, provides the definitions of the basic superhedging operators, and clarifies the relation of our constructions to classical integration. The crucial continuity-from-below property of the superhedging outer integral as well as easy-to-check sufficient conditions for continuity from below at almost every node are discussed in Section 3. A detailed comparison of our setting to the game-theoretic approach is carried out in Section 4. Section 5 defines supermartingales and stopping times and provides some examples. Section 6 proves our supermartingale representation theorem. Doob's pointwise convergence result for non-negative supermartingales is derived in Section 7. The relation of the two families of superhedging operators $\overline{\sigma}_j$ and $\overline{I}_j$ is discussed in Section 8. In particular, we show that these operators actually differ, if the conditional continuity from below property is only asked to hold almost everywhere (Theorem 8.4) and that $\overline{\sigma}_j$ provides the 'correct' superhedging prices for derivatives with finite maturity (see Theorem 8.1 for a precise statement). A concluding discussion can be found in Section 9. Some technical ramifications and the proofs of the results of Sections 3 and 4 are provided in the Appendices.

2. Basic setting and fundamental operators

2.1. Trajectorial setting

Definition 2.1 *(Trajectory set)*. [10, Definition 1] Given a real number s_0, a *trajectory set*, denoted by $\mathscr{S} = \mathscr{S}(s_0)$, is a subset of

$$\mathscr{S}_\infty(s_0) = \{S = (S_i)_{i\in\mathbb{N}\cup\{0\}} : S_i \in \mathbb{R},\ S_0 = s_0\}.$$

We make fundamental use of the following *conditional spaces*; for $S \in \mathscr{S}$ and $j \geq 0$ set:

$$\mathscr{S}_{(S,j)} \equiv \{\tilde{S} \in \mathscr{S} : \tilde{S}_i = S_i,\ \ 0 \leq i \leq j\},$$

the notation (S,j), henceforth referred as a *node*, will be used as a shorthand for $\mathscr{S}_{(S,j)}$.

We interpret $\mathscr{S}$ as a fixed market price model and $S \in \mathscr{S}$ as a possible stock price scenario unfolding in infinite discrete time. The units of the variables S_j are in terms of a bank account units i.e., the latter acts as a numeraire. From a mathematical point of view $\mathscr{S}$ is given and fixed. Then, the conditional space $\mathscr{S}_{(S,j)}$ models the set of stock price evolutions, if the investor enters the market at time $j \in \mathbb{N}$ and the stock prices $(S_0,\dots,S_j)$ have been realized at times $0,\dots,j$.

Notice $\mathscr{S}_{(S,0)} = \mathscr{S}$ and, if $\tilde{S} \in \mathscr{S}_{(S,j)}$, then $\mathscr{S}_{(\tilde{S},j)} = \mathscr{S}_{(S,j)}$. Moreover for $j \leq k$ it follows that $\mathscr{S}_{(S,k)} \subseteq \mathscr{S}_{(S,j)}$. On the other hand, for any fixed pair $j < k$, one can write $\mathscr{S}_{(S,j)}$ as a disjoint union of sets $\mathscr{S}_{(\tilde{S},k)}$ with $\tilde{S} \in A$, for some $A \subseteq \mathscr{S}_{(S,j)}$.

Local properties are relative to a given node. The classification of distinct nodes is presented in the following definition:

Definition 2.2 *(Types of nodes)*. Given a trajectory space $\mathscr{S}$ and a node (S,j):

- (S,j) is called an *up-down node* if

$$\sup_{\tilde{S}\in\mathscr{S}_{(S,j)}} (\tilde{S}_{j+1} - S_j) > 0 \quad \text{and} \quad \inf_{\tilde{S}\in\mathscr{S}_{(S,j)}} (\tilde{S}_{j+1} - S_j) < 0. \tag{1}$$

- (S,j) is called a *flat node* if

$$\sup_{\tilde{S}\in\mathscr{S}_{(S,j)}} (\tilde{S}_{j+1} - S_j) = 0 = \inf_{\tilde{S}\in\mathscr{S}_{(S,j)}} (\tilde{S}_{j+1} - S_j). \tag{2}$$

(S,j) is called an arbitrage-free node if (1) or (2) hold, otherwise it is called an *arbitrage node*. An arbitrage node (S,j) is said to be of *type I*, if there exists $\hat{S} \in \mathscr{S}_{(S,j)}$ such that $\hat{S}_{j+1} = S_j$; otherwise it is said to be of *type II*.

In practice, the coordinates S_i are multidimensional in order to allow for multiple sources of uncertainty. For simplicity we restrict to $S_i \in \mathbb{R}$, but one can also extend the framework to allow for several coordinates S_i^k (see [7], [8], and [11]). In particular, [7] presents a method to build trajectory sets from historical stock price data, which can lead to arbitrage nodes due to a pruning mechanism.

Besides the set $\mathscr{S}$, the other basic component are the *portfolios* defined as follows.

Definition 2.3 *(Conditional portfolio set)*. For any fixed $S \in \mathscr{S}$ and $j \geq 0$, $\mathscr{H}_{(S,j)}$ will be the set of all sequences of functions $H = (H_i)_{i\geq j}$, where $H_i : \mathscr{S}_{(S,j)} \to \mathbb{R}$ are non-anticipative in the sense: for all $\tilde{S}, \hat{S} \in \mathscr{S}_{(S,j)}$ such that $\tilde{S}_k = \hat{S}_k$ for $j \leq k \leq i$, then $H_i(\tilde{S}) = H_i(\hat{S})$ (i.e., $H_i(\tilde{S}) = H_i(\tilde{S}_0,\dots,\tilde{S}_i)$). Again, we introduce the shorthand notation $\mathscr{H} = \mathscr{H}_{(S,0)}$.

Given a *conditional portfolio* $(H_i)_{i\geq j} \in \mathscr{H}_{(S,j)}$, the function H_i represents the number of shares of stock S held by an investor at time i and who entered the market at time $i \geq j$. Notice that by assuming $H_i(S) \in \mathbb{R}$ we allow for $H_i(S) < 0$, an operation that is called short selling in finance. The notion of non-anticipativeness ensures that the portfolio position only depends on the past stock prices and does not make use of future information. Otherwise we do not impose any trading restrictions, but refer to [2] for the modeling of trading restrictions in the trajectorial framework.

Remark 2.4. Note that we do not impose any measurability condition on the functions $H_i : \mathbb{R}^{i+1} \to \mathbb{R}$ in the representation $H_i(\tilde{S}) = H_i(\tilde{S}_0, \dots \tilde{S}_i)$ of a portfolio position. The main reason is that we will work with a subadditive outer integral operator instead of a linear integral operator. On the one hand, this can be viewed in analogy to the use of the outer expectation operator in probability and statistics (see, e.g., [27]), which does not require any measurability properties of the integrands. On the other hand, this is in line with the protocols used in the discrete time game-theoretic approach of Shafer and Vovk [23], where no measurability conditions are imposed on the functions announced by the Skeptic.

For a node (S, j), $H \in \mathscr{H}_{(S,j)}$, $V \in \mathbb{R}$ and $n \geq j$ we define $\Pi^{V,H}_{j,n} : \mathscr{S}_{(S,j)} \to \mathbb{R}$, as:

$$\Pi^{V,H}_{j,n}(\tilde{S}) \equiv V + \sum_{i=j}^{n-1} H_i(\tilde{S})\, \Delta_i \tilde{S}, \quad \text{where } \Delta_i \tilde{S} = \tilde{S}_{i+1} - \tilde{S}_i, \;\; i \geq j, \;\; \tilde{S} \in \mathscr{S}_{(S,j)}.$$

This expression equals the wealth at time n of the self-financing portfolio with initial endowment V at time j, when H_i represents the number of shares of the stocks held by the investor at time i. We recall that 'self-financing' means that the remaining capital $\Pi^{V,H}_{j,i}(S) - H_i(S)S_i$, which is not invested in the stock, is put into the bank account at time i. Notice that V is assumed to be constant on $\mathscr{S}_{(S,j)}$ and so its value could change with S, i.e., $V = V(S)$ (depending on the past stock price evolution up to time j).

In the sequel, being $\mathscr{A}$ a set of real valued functions, $\mathscr{A}^+$ will denote the set of its non-negative elements.

Definition 2.5 *(Elementary vector spaces).* For a fixed node (S, j) set

$$\mathscr{E}_{(S,j)} = \{f = \Pi^{V,H}_{j,n_f} : H \in \mathscr{H}_{(S,j)}, \;\; V \in \mathbb{R} \;\; \text{and} \;\; n_f \in \mathbb{N}\}.$$

Observe that $\mathscr{E}_{(S,j)}$ is a real vector space. Its elements are called *elementary functions*.

Thus, elementary functions are nothing but the payoff functions of financial derivatives that can be perfectly hedged in the conditional stock price model by finite linear combinations of buy-and-hold strategies.

Let also define

$$\mathscr{E}_j = \{f : \mathscr{S} \to \mathbb{R} : f|_{\mathscr{S}_{(S,j)}} \in \mathscr{E}_{(S,j)} \;\; \forall S \in \mathscr{S}\},$$

where the notation $f|_{\mathscr{S}_{(S,j)}}$ means that the global domain of f, namely $\mathscr{S}$, is being restricted to the subset $\mathscr{S}_{(S,j)}$. We note in passing some abuse of notation as the same symbol $\Pi^{V,H}_{j,n_f}$ is used to denote elements from $\mathscr{E}_{(S,j)}$ and $\mathscr{E}_j$ (in particular the implicit dependence on S is not made explicit in the case when $\Pi^{V,H}_{j,n_f} \in \mathscr{E}_{(S,j)}$). More details on global versus local portfolios are provided in [2].

2.2. Fundamental operators and almost everywhere notions

Let Q denote the set of all functions from $\mathscr{S}$ to $[-\infty, \infty]$ and $P \subseteq Q$ denote the set of non-negative functions. The following conventions are in effect: $0\,\infty = 0$, $\infty + (-\infty) = \infty$, $u - v \equiv u + (-v)$ $\forall\, u, v \in [-\infty, \infty]$, and $\inf \emptyset = \infty$ (unless indicated otherwise). An inequality $a \leq b$ in $[-\infty, \infty]$ is read to be as $b - a \geq 0$ (i.e., $b - a$ is non-negative) and is, thus, valid, if $a = b = +\infty$ or $a = b = -\infty$.

We say that $f \in Q$ has *maturity* $n_f \in \mathbb{N}$, if $f(S) = f(\tilde{S})$ for every $S \in \mathscr{S}$ and $\tilde{S} \in \mathscr{S}_{(S,n_f)}$, i.e., if f depends on S only through the first $n_f + 1$ coordinates $S_0, \dots, S_{n_f}$. In this case, we sometimes write $f(S_0, \dots, S_{n_f})$ in place of $f(S)$. If f has maturity n_f for some $n_f \in \mathbb{N}$, we will speak of a function f with finite maturity.

We define next the operator $\overline{I}_j : P \to \mathscr{E}^+_j$, which is a conditional extension of the operator $\overline{I}$ defined in [10] and it is used to define null sets.

Definition 2.6. For a given node (S, j) and a general $f \in P$ define

$$\overline{I}_j f(S) \equiv \inf \left\{ \sum_{m \geq 1} V^m : \;\; f \leq \sum_{m \geq 1} \liminf_{n \to \infty} \;\; \Pi^{V^m,H^m}_{j,n} \;\; \text{on } \mathscr{S}_{(S,j)}, \; \Pi^{V^m,H^m}_{j,n} \in \mathscr{E}^+_{(S,j)} \;\; \forall \;\; n \geq j \right\}.$$

We will use the notation $\overline{I} f \equiv \overline{I}_0 f$. We also set, for a general $f \in Q$:

$$||f||_j(S) \equiv \overline{I}_j |f|(S) \;\; \text{and} \;\; ||f|| \equiv ||f||_0(S).$$

Notice that $\overline{I}_j f(S) = \overline{I}_j f(S_0, \dots, S_j)$, i.e., $\overline{I}_j f(\cdot)$ is constant on $\mathscr{S}_{(S,j)}$. Moreover $\sum_{m \geq 1} V^m \geq 0$ hence, $\overline{I}_j f \geq 0$, so $||0||_j = 0$. $|| \cdot ||_j(S)$ will be called a *conditional norm*.

The operator $\overline{I}_j$ is defined as the infimal superhedging cost with the following two idealizations: On the one hand, a superposition of wealth processes of countably many portfolios is utilized. On the other hand (and in contrast to the portfolios applied in the definition of elementary functions), each portfolio can be re-balanced infinitely many times. Note, however, that the portfolio wealth $\Pi^{V^m,H^m}_{j,n}$ of each of the individual portfolios must be non-negative at any time. This restriction on the portfolio wealth is crucial to ensure the countable sub-additivity of $\overline{I}_j$, see Proposition 2.8. The key role of the operators $\overline{I}_j$ is to detect subsets of trajectories, on which arbitrage opportunities exist, as (conditional) null sets. While the countable superposition is required to detect, e.g., the obvious arbitrage opportunities at arbitrage nodes of type II as null sets, the idealization of re-balancing a portfolio infinitely often is used to find further null sets that appear 'at infinite time'.

Our notions of *conditional null set* and the *conditional a.e. property* are introduced next.

Definition 2.7 *(Conditional a.e. notions).* Given a node (S, j), a function $g \in Q$ is a *conditionally null function at* (S, j) if:

$$\|g\|_j(S) = 0.$$

A subset $E \subseteq \mathscr{S}$ is a *conditionally null set at* (S, j) if $\|\mathbf{1}_E\|_j(S) = 0$. A property is said to hold conditionally a.e. at (S, j) (or equivalently: the property holds "a.e. on $\mathscr{S}_{(S,j)}$") if the subset of $\mathscr{S}_{(S,j)}$ where it does not hold is a conditionally null set at (S, j). In particular, the latter definition applies to $g = f$ a.e. on $\mathscr{S}_{(S,j)}$, which also will be noted with $g \doteq f$ when $j = 0$.

Notice that when $j = 0$, the previous notions do not depend on S and we apply the abbreviation "a.e." for "a.e. at $(S, 0)$". Moreover, $E \subseteq \mathscr{S}$ is called a *null set* and g is called a *null function*, if $\|\mathbf{1}_E\| = 0$ and $\|g\| = 0$, respectively.

The next results, from [10], give properties of null functions and null sets that are widely used.

Proposition 2.8. *[10, Proposition 1] $\overline{I}$ is isotone, positive homogeneous, countable subadditive and $\overline{I}(1_{\mathscr{S}}) \leq 1$.*

Proposition 2.9. *[10, Proposition 2] Consider $f, g : \mathscr{S} \to [-\infty, \infty]$, then*

1. $\|g\| = 0$ *iff* $g = 0$ *a.e.*
2. *The countable union of null sets is a null set.*

All appearing equalities and inequalities are valid for all points in the spaces where the functions are defined unless qualified by an explicit a.e.

We introduce next the operator $\overline{\sigma}_j : Q \to \mathscr{E}_j$, which we will call a *conditional superhedging outer integral* (or conditional outer integral); it is the key tool to define the notion of *trajectorial supermartingales*, the main object of study in our paper. The only difference compared to the superhedging operator $\overline{I}_j$ is that we relax the non-negativity assumption on the portfolio wealth and, in this way, enlarge the set of hedging strategies. While this relaxation may look harmless, it can, in general, destroy the countable sub-additivity of the operator. Moreover, it turns out to be crucial for computing reasonable superhedging prices which are compatible with the null sets determined by $\overline{I}$, see Theorem 8.1 and Example 1 in Section 4.

Definition 2.10 *(Conditional Outer Integral).* For a node (S, j) and a general $f \in Q$,

$$\overline{\sigma}_j f(S) \equiv \inf \left\{ \sum_{m\geq 0} V^m : \quad f \leq \sum_{m\geq 0} f_m \text{ on } \mathscr{S}_{(S,j)} \right\},$$

where $f_0 = \Pi^{V^0,H^0}_{j,n_0} \in \mathscr{E}_{(S,j)}$ and for $m \geq 1$,

$$f_m = \liminf_{n\to\infty} \ \Pi^{V^m,H^m}_{j,n}, \ \text{ and } \Pi^{V^m,H^m}_{j,n} \in \mathscr{E}^+_{(S,j)} \ \ \forall \ \ n \geq j.$$

Define also $\underline{\sigma}_j f(S) \equiv -\overline{\sigma}_j(-f)(S)$. We will also set $\overline{\sigma} f \equiv \overline{\sigma}_0 f$.

In some cases we may use the notation $\overline{\sigma}_{(S,j)} f$ to make clear that the quantity $\overline{\sigma}_{(S,j)} f$ keeps (S, j) fixed. More common and useful is our reliance on the defining notation $\overline{\sigma}_j f(S)$ treating $\overline{\sigma}_j f$ as a function on $\mathscr{S}$. Note that the initial endowments V^m at node (S, j) may depend on S through $(S_0, \dots, S_j)$ in all appearances.

Remark 2.11. Note that $\overline{\sigma}_j f(S) = \overline{\sigma}_j f(S_0, \dots, S_j)$. Also $\overline{\sigma}_j \leq \overline{I}_j$ on the set of non-negative functions (Example 1 provides a case where the inequality is strict). Also, and as a side remark, f_0 can also be written in a similar form as the f_m, $m \geq 1$, for notational convenience, by means of $f_0 = \liminf_{n\to\infty} \ \Pi^{V^0,H^0}_{j,n}$ with $H^0_i \equiv 0$ for $i \geq n_0$.

2.3. Relation to classical integration

In this subsection, we briefly explain that the construction of the conditional outer integral in Definition 2.10 is analogous to Daniells' approach to classical integration with respect to a measure, see Chapter 16 in Royden's textbook [22] for a detailed account on Daniell integration. To this end, let $\Omega \neq \emptyset$ be a set, $\mathscr{R}$ be an algebra of subsets of Ω and $\mu_0 : \mathscr{R} \to [0, \infty)$ be a measure on the algebra $\mathscr{R}$, and write

$$\mathscr{E}(\mathscr{R}) = \{ \sum_{i=1}^{n} \alpha_i \mathbf{1}_{A_i} \,:\, \alpha_i \in \mathbb{R},\ A_i \in \mathscr{R} \}$$

for the *set of step functions* over $\mathscr{R}$. Then, define the *elementary integral* via

$$I^{(\mu_0)} : \mathscr{E}(\mathscr{R}) \to \mathbb{R}, \qquad \sum_{i=1}^{n} \alpha_i \mathbf{1}_{A_i} \mapsto \sum_{i=1}^{n} \alpha_i \mu_0(A_i).$$

It is well-known that $\mathscr{E}(\mathscr{R})$ is a vector lattice, i.e., a vector space which is closed under the operation of taking the positive part, that the elementary integral is well-defined, and that the following *continuity-from-below* property is satisfied: If $(f_m)_{m\in\mathbb{N}}$ is a sequence in $\mathscr{E}^+(\mathscr{R})$ and $f \in \mathscr{E}(\mathscr{R})$, then

$$f \le \sum_{m=1}^{\infty} f_m \ \Rightarrow\ I^{(\mu_0)}(f) \le \sum_{m=1}^{\infty} I^{(\mu_0)}(f_m); \tag{3}$$

see, e.g., [22, p. 420]. For a function $f : \Omega \to [0, \infty)$, resp. $f : \Omega \to \mathbb{R}$, write

$$\bar{I}^{(\mu_0)}(f) = \inf \left\{ \sum_{m=1}^{\infty} I^{(\mu_0)}(f_m) \,:\, f_m \in \mathscr{E}^+(\mathscr{R}),\ f \le \sum_{m=1}^{\infty} f_m \right\}, \text{ resp.}$$

$$\overline{\sigma}^{(\mu_0)}(f) = \inf \left\{ \sum_{m=0}^{\infty} I^{(\mu_0)}(f_m) \,:\, f_0 \in \mathscr{E}(\mathscr{R}),\ f_m \in \mathscr{E}^+(\mathscr{R})\,(m \ge 1),\ f \le \sum_{m=0}^{\infty} f_m \right\}.$$

Then, a function $f : \Omega \to \mathbb{R}$ is called *Daniell integrable* ([22, p. 425]), if $\overline{\sigma}^{(\mu)} f + \overline{\sigma}^{(\mu)}(-f) = 0$. Denoting the space of Daniell integrable functions by $\mathscr{L}_1$, a function $f : \Omega \to \mathbb{R}$ is said to be *Daniell measurable* ([22, p. 429]), if $\min\{f, g\} \in \mathscr{L}_1$ for every $g \in \mathscr{L}_1$. Then, the system $\mathscr{A} = \{A \subseteq \Omega \,:\, \mathbf{1}_A \text{ is Daniell measurable}\}$ of subsets of Ω is a σ-field ([22, Lemma 16.19]) and

$$\mu(A) \equiv \overline{\sigma}^{(\mu_0)}(\mathbf{1}_A), \quad A \in \mathscr{A}$$

is a measure ([22, Lemma 16.21]), which extends the original measure μ_0 (which was defined on the algebra $\mathscr{R}$). As usual, we write $L_1(\Omega, \mathscr{A}, \mu)$ for the space of $\mathscr{A}$-measurable functions $f : \Omega \to \mathbb{R}$, which are integrable with respect to the measure μ. Then:

Proposition 2.12. *Let $f : \Omega \to \mathbb{R}$. Then, the following assertions are equivalent:*

(i) $f \in L_1(\Omega, \mathscr{A}, \mu)$.
(ii) *There is a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{E}(\mathscr{R})$ such that $\lim_{n\to\infty} \bar{I}^{(\mu)}(|f - f_n|) = 0$.*
(iii) $\overline{\sigma}^{(\mu)}(f) + \overline{\sigma}^{(\mu)}(-f) = 0$, *i.e., f is Daniell integrable.*

If one (and, then, all of these equivalent) conditions are satisfied, then

$$\int f\, d\mu = \overline{\sigma}^{(\mu)}(f) = \lim_{n\to\infty} I^{(\mu_0)}(f_n).$$

Proof. The equivalence of (i) and (ii) and the identity $\int f\, d\mu = \overline{\sigma}^{(\mu)}(f)$ for $f \in L_1(\Omega, \mathscr{A}, \mu)$ are justified by the formulation and the proof of Stone's theorem in [22, Theorem 16.22]. Continuity from below implies that $I^{(\mu_0)}(f) = \bar{I}^{(\mu_0)}(f)$ for every $f \in \mathscr{E}^+(\mathscr{R})$. By the lattice property of $\mathscr{E}(\mathscr{R})$ and the linearity of the elementary integral we observe that, for every $f \in \mathscr{E}(\mathscr{R})$,

$$\bar{I}^{(\mu_0)}(f_+) = I^{(\mu_0)}(f_+) = I^{(\mu_0)}(f) + I^{(\mu_0)}(f_-) = I^{(\mu_0)}(f) + \bar{I}^{(\mu_0)}(f_-).$$

Here, f_+ and f_- denote the positive and the negative part of f, respectively. Hence, the continuity property (*) imposed in [14, p. 449], is satisfied. The equivalence of (ii) and (iii) and the identity $\overline{\sigma}^{(\mu)}(f) = \lim_{n\to\infty} I^{(\mu_0)}(f_n)$ for Daniell integrable f now follows from [14, pp. 453–454]. □

Roughly speaking (i.e., ignoring the limit inferior in time for the moment), in our construction of the conditional superhedging outer integral $\overline{\sigma}_j(\cdot)(S)$, the step functions are replaced by the vector space $\mathscr{E}_{(S,j)}$ of elementary functions, see Definition 2.5, which model the terminal portfolio wealth $\Pi^{V,H}_{j,n_f}$ of finite linear combinations of buy-and-hold strategies. Moreover, in the definition of $\bar{I}_j(\cdot)(S)$ and $\overline{\sigma}_j(\cdot)(S)$, the elementary integral is replaced by the initial endowment V required to set up the portfolio.

Note, however, that the vector space $\mathscr{E}_{(S,j)}$, in general, fails to be a vector lattice except in simple special cases such as binomial tree models. Therefore, we are outside the classical Daniell integration theory, but closely follow the constructions of the general theory of non-lattice integration developed by Leinert [16] and König [14]. In contrast to [14,16], our framework incorporates the notion of conditioning, which is not present in their theory. Otherwise, the only way, in which we deviate from their constructions, is that we have added the limit inferior of the portfolio wealth of the individual portfolios (as time goes to infinity) in the definitions of $\bar{I}_j(\cdot)(S)$ and $\overline{\sigma}_j(\cdot)(S)$. This modification is required to obtain a sufficiently rich class of null sets 'at infinity' and plays an important role in deriving an analogue of the supermartingale convergence theorem in Section 7 below; compare also the discussion in [10].

Remark 2.13. a) Analogously to condition (iii) in Proposition 2.12, we say that $f \in Q$ is *integrable* in our framework, if $\underline{\sigma}_0 f = \overline{\sigma}_0 f$. Moreover, $f \in Q$ is called a *conditionally integrable* function at j if it satisfies: $\underline{\sigma}_j f = \overline{\sigma}_j f$, $\overline{I} - a.e.$ As emphasized in [2], this notion of integrability is rather restrictive and, therefore, it will play a side role in our work.

b) In the classical integration theory, the identity $\overline{\sigma}^{(\mu)}(f) = \bar{I}^{(\mu)}(f)$ holds for every $f \in L_1^+(\Omega, \mathscr{A}, \mu)$, see [13, p. 98]. In contrast, Example 1 below illustrates that the identity $\overline{\sigma} = \bar{I}$ may fail, in general, on the set of non-negative integrable functions in our non-lattice framework.

While the theory of Leinert [16] and König [14] shows how to dispense with the lattice property of the space of elementary functions in the integral construction, the continuity-from-below property is still crucial for deriving an (outer) integral with reasonable properties. Therefore, we will discuss continuity-from-below in our framework in the next section.

Remark 2.14. In some of the examples below, we will contrast our trajectorial approach to the classical approach to mathematical finance, in which pricing is linked to the concept of martingale measures by the fundamental theorem of asset pricing and the superhedging duality theorem, see [12] in a model-based context or [5] for a model-free (or, more precisely, a pointwise) theory in finite discrete time. Whenever comparing to the classical probabilistic literature, in order to avoid subtle measurability issues, cp. [5], we will only consider illustrative examples with a countable trajectory set $\mathscr{S}$. Writing $T = (T_j)_{j\geq 0}$ for the coordinate process, defined via $T_j(S) = S_j$ for every $S \in \mathscr{S}$ and $j \geq 0$, we denote by $\mathscr{T} = (\mathscr{T}_j)_{j\geq 0}$ the filtration generated by T, i.e., $\mathscr{T}_j = \sigma(T_1 \dots, T_j)$. Then, a sequence $(f_j)_{j\geq 0}$ of functions from $\mathscr{S}$ to $\mathbb{R}$ is non-anticipative, if and only if it is $\mathscr{T}$-adapted. Indeed, if it is non-anticipative, then for every $j \geq 1$, there is a function $F_j : \mathbb{R}^j \to \mathbb{R}$ such that $f_j = F_j(T_1, \dots, T_j)$ and F_j can be chosen Borel-measurable, because $(T_1, \dots, T_j)$ takes at most countably many values. The converse is implied by the factorization lemma of Doob and Dynkin [13, Corollary 1.97]. Given a probability measure $\mathbf{Q}$ on the power set $2^{\mathscr{S}}$ of $\mathscr{S}$, we, thus, say (following the standard definition [13, Definition 9.24]) that a non-anticipative sequence $(f_j)_{j\geq 0}$ of functions from $\mathscr{S}$ to $[-\infty, \infty]$ is a *classical probabilistic martingale* (with respect to $\mathbf{Q}$), if for every $j \geq 0$ and $B \in \mathscr{T}_j$,

$$\int |f_j| d\mathbf{Q} < \infty \quad \text{and} \quad \int (f_{j+1} - f_j)\mathbf{1}_B \, d\mathbf{Q} = 0. \tag{4}$$

The probability measure $\mathbf{Q}$ is said to be a *martingale measure* for the trajectory set $\mathscr{S}$, if the coordinate process $(T_j)_{j\geq 0}$ is a classical probabilistic martingale with respect to $\mathbf{Q}$.

3. Continuity from below

We now explain, how to phrase and check the crucial continuity-from-below condition in our framework. In view of the recap of classical integration in the previous subsection, the following property is completely analogous to the formulation of continuity from below in (3).

Definition 3.1 *(Property $(L_{(S,j)})$).* For a fixed node (S, j), $f = \Pi^{V,H}_{j,n_f} \in \mathscr{E}_{(S,j)}$ and $f_m = \liminf_{n\to\infty} \Pi^{V^m,H^m}_{j,n}$ with $\Pi^{V^m,H^m}_{j,n} \in \mathscr{E}^+_{(S,j)}$ for all $n \geq j$ and $m \geq 1$, define property $(L_{(S,j)})$ by

$$(L_{(S,j)}): \quad f \leq \sum_{m\geq 1} f_m \text{ on } \mathscr{S}_{(S,j)} \implies V \leq \sum_{m\geq 1} V^m.$$

If $(L_{(S,j)})$ holds for $S \in \mathscr{S}$ $\overline{I} - a.e.$, it will be written (L_j) holds $a.e.$ Since $(L_{(S,0)})$ does not depend on S, it will be denoted by (L).

We apply the letter 'L' for this property, as (up to our use of the limit inferior in the definitions of $\bar{I}$ and $\overline{\sigma}$) this is the continuity condition imposed in theory of non-lattice integration by Leinert in [16].

Remark 3.2. If $\mathscr{S}_{(S,j)}$ contains a trajectory S^0, which remains constant after time j, i.e., $S^0_i = S_j$ for every $i \geq j$, then $f_m(S^0) = V^m$ for every $m \geq 0$ and $f(S^0) = V$, and, thus, $(L_{(S,j)})$ holds.

The following proposition contains some equivalent formulations of property $(L_{(S,j)})$. Its proof is based on standard arguments in the theory of non-lattice integration and is postponed to Appendix A.3.

Proposition 3.3. *For a fixed node (S, j), the following items are equivalent:*

1. $\overline{\sigma}_j 0(S) = 0$.
2. $\underline{\sigma}_j f(S) \leq \overline{\sigma}_j f(S)$ *for any* $f \in Q$.
3. *Property* $(L_{(S,j)})$.
4. $\underline{\sigma}_j f(S) = V(S) = \overline{\sigma}_j f(S)$ *for every* $f = \Pi^{V,H}_{j,n_f} \in \mathscr{E}_j$.

Remark 3.4. Suppose $(L_{(S,j)})$ holds at a (fixed) node (S, j).

a) Then $\overline{\sigma}_j(\cdot)(S)$ satisfies the following properties:

(S1) *Isotone*: If $f, g \in Q$ and $f \leq g$, then $\overline{\sigma}_j f(S) \leq \overline{\sigma}_j f(S)$.
(S2) *Constant preserving:* $\overline{\sigma}_j c(S) = c$ for every constant $c \in \mathbb{R}$.
(S3) *Subadditive:* For every $f, g \in Q$, $\overline{\sigma}_j(f + g)(S) \leq \overline{\sigma}_j f(S) + \overline{\sigma}_j g(S)$.
(S4) *Positive homogeneous:* For every $f \in Q$ and every non-negative constant λ, $\overline{\sigma}_j(\lambda f)(S) = \lambda \overline{\sigma}_j(f)(S)$.

Here, (S1) and (S3) are implied by item b) and c) in Proposition A.2.1, (S2) is a consequence of Proposition 3.3–4., and (S4) holds by Proposition 3.3–1. for $\lambda = 0$ and by Proposition A.2.1–d) for $\lambda > 0$. Hence, $\overline{\sigma}_j(\cdot)(S)$ is a *sublinear expectation* in the sense of Peng [20, Definition 1.1.1]. Moreover, (S1) and (S2) imply that $\overline{\sigma}_j f(S) \leq \sup_{S' \in \mathscr{S}} f(S')$ for every $f \in Q$. Thus, by Theorem 4.15 in [26], the restriction of $\overline{\sigma}_j(\cdot)(S)$ to the vector space of bounded functions from $\mathscr{S}$ to $\mathbb{R}$ is a coherent upper prevision.

b) Moreover, one can easily check that $\bar{I}_j(\cdot)(S)$ satisfies (S1)–(S4) on the cone P of non-negative functions (i.e., Q must be replaced by P in (S1)–(S4) and (S2) is only supposed to hold for constants $c \geq 0$). These properties follow from the conditional version of Proposition 2.8 and noting that, for every constant $c \geq 0$, $c = \overline{\sigma}_j c(S) \leq \bar{I}_j c(S) \leq c$ by property (S2) for $\overline{\sigma}_j(\cdot)(S)$, Remark 2.11, and the definition of $\bar{I}_j$. Since the bounded functions in P do not form a vector space, Theorem 4.15 in [26] cannot be applied to check whether $\bar{I}_j(\cdot)(S)$ is a coherent upper prevision. We will show in Example 4 below, that $\bar{I}_j(\cdot)(S)$ may fail to satisfy, in general, the coherence property (D) in [26, Definition 4.10]. The operator $\bar{I}_j(\cdot)(S)$ is, however, always countably subadditive (Proposition 2.8), while $\overline{\sigma}_j(\cdot)(S)$ may fail to satisfy the latter property. Again, Example 4 serves as a counterexample, as detailed in [2].

Remark 3.5. Suppose the function f describes the payoff of a financial product. If $f \in \mathscr{E}_{(S,j)}$, then f has the form $\Pi^{V,H}_{j,n}$ and, thus, the payoff can be perfectly replicated by the portfolio value of a finite linear combination of buy-and-hold strategies with initial endowment $V(S)$ at time j. Hence, no investor entering the market at time j would be willing to buy the financial product for a higher price than $V(S)$ or to sell it for a lower price than $V(S)$, which makes $V(S)$ the only candidate for a rational price of f. Thus, the equivalent condition 4. for $(L_{(S,j)})$ in Proposition 3.3 is a minimal condition for $\overline{\sigma}_j$ to be a reasonable pricing operator.

Remark 3.6. Note that the conditional outer integral $\overline{\sigma}_j f(S)$ is defined at any node (S, j) and for any function $f \in Q$. However, if $(L_{(S,j)})$ fails, then, $\overline{\sigma}_j f(S) \in \{-\infty, +\infty\}$ for every $f \in Q$. Indeed, in this case, by Proposition 3.3, $\overline{\sigma}_j 0(S) < 0$. Consequently, by Proposition A.2.1–d),

$$\overline{\sigma}_j 0(S) = \lim_{N\to\infty} \overline{\sigma}_j(N \cdot 0)(S) \leq \lim_{N\to\infty} N\,\overline{\sigma}_j 0(S) = -\infty,$$

which in turn implies, by the subadditivity of $\overline{\sigma}_j$ in Proposition A.2.1–c),

$$\overline{\sigma}_j f(S) \leq \overline{\sigma}_j f(S) + \overline{\sigma}_j 0(S) = \overline{\sigma}_j f(S) + (-\infty)$$

for every $f \in Q$. Hence, $\overline{\sigma}_j f(S) \in \{-\infty, +\infty\}$.

By the previous two remarks, the failure of the continuity-from-below property $(L_{(S,j)})$ at a node (S, j) should be exceptional in order to come up with a reasonable theory of robust pricing by trajectorial superhedging. Therefore, we will assume, for most of the upcoming results, that property $(L_{(S,j)})$ holds outside an $\bar{I}$-null set in the sense of the following definition.

Definition 3.7 *(Assumption $(L) - a.e.$).* The following two properties will be referred as *the assumption* (L)-a.e.:

i) (L) (i.e., $(L_{(S,0)})$) holds,

ii)

$$\mathscr{N}^{(L)} \equiv \{S \in \mathscr{S} : \exists j \geq 0 \; s.t.\, (L_{(S,j)}) \text{ fails}\} \tag{5}$$

is a null set (in particular, (L_j) holds a.e. for every j).

Remark 3.8. a) If (L) fails, then the set $\mathcal{N}^{(L)}$ in display (5) equals the whole trajectory set $\mathcal{S}$. Therefore, in order to avoid trivialities, we include property (L) in the definition of the assumption (L)-a.e.
b) The continuity property (L) implies that $\overline{I}(\mathbf{1}_{\mathcal{S}})=1$ by Proposition 3.3–4), and, thus,

$$1=\overline{I}(\mathbf{1}_{\mathcal{S}})\leq \overline{I}(\mathbf{1}_{\mathcal{S}\setminus\mathcal{M}})+\overline{I}(\mathbf{1}_{\mathcal{M}})\leq \overline{I}(\mathbf{1}_{\mathcal{S}\setminus\mathcal{M}})\leq 1,$$

where $\mathcal{M}$ is any arbitrary null set. In particular, $\overline{I}(\mathbf{1}_{\mathcal{S}\setminus\mathcal{N}^{(L)}})=1$ whenever (L)-a.e. holds.

Example 5 below provides a counterexample in which $\mathcal{N}^{(L)}$ is not an $\bar{I}$-null set and, hence, property (L)-a.e. fails. For the remainder of this section, we will, therefore, discuss, how to check this crucial property in our framework.

The strategy is to find an as large as possible subset of $\mathcal{N}^{(L)}$, which can be shown to be a null set, and, then, to identify sufficient conditions under which this null set actually coincides with $\mathcal{N}^{(L)}$. This strategy leads to the following notion of a 'bad' node.

Definition 3.9 *(Bad nodes).* For a fixed node (S,j), let

$$N(S,j)=\{\tilde{S}\in\mathcal{S}_{(S,j)}\,:\,(\tilde{S},k)\text{ is arbitrage node and }\tilde{S}_{k+1}\neq\tilde{S}_k\text{ for some }k\geq j\}. \tag{6}$$

A node (S,j) is called *bad*, if $\mathcal{S}_{(S,j)}=N(S,j)$. Otherwise, (S,j) is said to be *good*.

Proposition 3.10. *For every node (S,j) the following chain of implications holds:*

$$(S,j)\textit{ is an arbitrage node of type II}\quad\Rightarrow\quad(S,j)\textit{ is bad}\quad\Rightarrow\quad(L_{(S,j)})\textit{ fails.}$$

Moreover, the set

$$\mathcal{N}^{bad}=\{S\in\mathcal{S}\,:\,(S,j)\textit{ is bad for some }j\geq 0\}$$

is an $\bar{I}$-null set.

The proof is provided in Appendix A.3.

We will next identify sufficient conditions for the converse implications in Proposition 3.10. These sufficient conditions will be based on two types of hypotheses. The first one is concerned with the possibility to construct trajectories iteratively (and relaxes the notion of trajectorial completeness from [10]), while the second one imposes some restrictions on the successors of up-down nodes.

Definition 3.11 *(Trajectorial completeness).* Suppose $(S^n)_{n\geq 0}$ is a sequence in $\mathcal{S}$ satisfying

$$S_i^n=S_i^{n+1},\quad 0\leq i\leq n, \tag{7}$$

for all $n\in\mathbb{N}_0$. Then, its *limit* is defined as

$$\lim_{n\to\infty}S^n\equiv\overline{S}\equiv(\overline{S}_i)_{i\geq 0},\quad\text{wherein}\quad\overline{S}_i\equiv S_i^i.$$

Denote by $\overline{\mathcal{S}}$ the set of all such limits $\overline{S}$. Then, $\overline{\mathcal{S}}$ is called the *trajectorial completion* of $\mathcal{S}$ and the trajectory set $\mathcal{S}$ is said to be *trajectorially complete*, (TC) for short, if $\mathcal{S}=\overline{\mathcal{S}}$.

Clearly, $\mathcal{S}\subseteq\overline{\mathcal{S}}$ given that for $\tilde{S}\in\mathcal{S}$ we can take $\tilde{S}^n=\tilde{S}$ for all $n\geq 0$. As argued in Proposition 13 of [10], $\overline{\mathcal{S}}$ is always trajectorially complete. Moreover, the completion process, i.e., passing from $\mathcal{S}$ to $\overline{\mathcal{S}}$ does not alter the type of the nodes (being up-down, no arbitrage, etc.), but, importantly, it can change a null node (i.e., a node which constitutes an $\overline{I}$-null set) into a non-null node, see Example 6 below.

We now list the hypotheses which will be applied in the upcoming results:

- ($\mathrm{TC}_{\mathrm{bad}}$) If $(S^n)_{n\geq 0}$ is a sequence in $\mathcal{S}$ satisfying (7) and if there is an $n_0\geq 0$ such that (S^n,n) is a good node for every $n\geq n_0$, then $\lim_{n\to\infty}S^n\in\mathcal{S}$.
- (TC_{II}) If $(S^n)_{n\geq 0}$ is a sequence in $\mathcal{S}$ satisfying (7) and if there is an $n_0\geq 0$ such that (S^n,n) is not an arbitrage node of type II for every $n\geq n_0$, then $\lim_{n\to\infty}S^n\in\mathcal{S}$.
- ($\mathrm{H}_{\mathrm{bad}}$) If (S,j) is a good up-down node, then for every $\varepsilon>0$ there are $S^{\varepsilon,1},S^{\varepsilon,2}\in\mathcal{S}_{(S,j)}$ such that $(S^{\varepsilon,1},j+1)$ and $(S^{\varepsilon,2},j+1)$ are good nodes satisfying

$$S_{j+1}^{\varepsilon,1}-S_j\geq-\varepsilon,\quad S_{j+1}^{\varepsilon,2}-S_j\leq\varepsilon.$$

- (H_{II}) If (S,j) is an arbitrage node of type II, then $j\geq 1$ and $(S,j-1)$ is an up-down node and for every $\varepsilon>0$ there are $S^{\varepsilon,1},S^{\varepsilon,2}\in\mathcal{S}_{(S,j-1)}$ such that

$$S_j^{\varepsilon,1} - S_{j-1} \geq -\varepsilon, \quad S_j^{\varepsilon,2} - S_{j-1} \leq \varepsilon$$

and such that $(S^{\varepsilon,1}, j)$, $(S^{\varepsilon,2}, j)$ are not type II arbitrage nodes.

(H_{II}') If (S,j) is an up-down node, then for every $\varepsilon > 0$ there are $S^{\varepsilon,1}, S^{\varepsilon,2} \in \mathscr{S}_{(S,j)}$ such that

$$S_{j+1}^{\varepsilon,1} - S_j \geq -\varepsilon, \quad S_{j+1}^{\varepsilon,2} - S_j \leq \varepsilon$$

and $(S^{\varepsilon,1}, j+1)$ and $(S^{\varepsilon,2}, j+1)$ are not arbitrage nodes of type II.

The interplay between these conditions and their relation to the converse implications in Proposition 3.10 are clarified in the following theorem, which we prove in Appendix A.3.

Theorem 3.12.

1. *(TC)* $\Rightarrow$ *(TC_{II})* $\Rightarrow$ *($\mathrm{TC}_{\mathrm{bad}}$)*
2. *(H_{II})* $\Rightarrow$ *(H_{II}')*
3. *[(TC_{II}) and (H_{II})]* $\Rightarrow$ *[(S,j) is an arbitrage node of type II* $\Leftrightarrow$ *(S,j) is bad]*
4. *[($\mathrm{TC}_{\mathrm{bad}}$) and ($\mathrm{H}_{\mathrm{bad}}$)]* $\Rightarrow$ *[(S,j) is bad* $\Leftrightarrow$ *$(L_{(S,j)})$ fails]*

As corollaries we obtain the following sufficient conditions for (L)-a.e. The first one, formulated in terms of bad nodes, is more general; the second one, formulated in terms of arbitrage nodes of type II, is easier to check.

Corollary 3.13. *Assume ($\mathrm{TC}_{\mathrm{bad}}$), ($\mathrm{H}_{\mathrm{bad}}$), and that $(S,0)$ is good. Then, $(L_{(S,j)})$ fails exactly at the bad nodes and (L)-a.e. holds.*

Proof. Since $(S,0)$ is good, Theorem 3.12–4) implies that (L) holds. Moreover, $\mathscr{N}^{(L)} = \mathscr{N}^{\mathrm{bad}}$ by Theorem 3.12–4), again. Since, $\mathscr{N}^{\mathrm{bad}}$ is an $\bar{I}$-null set by Proposition 3.10, the proof is complete. □

Corollary 3.14. *Assume (TC_{II}) and (H_{II}). Then, $(L_{(S,j)})$ fails exactly at the arbitrage nodes of type II and (L)-a.e. holds.*

Proof. By Theorem 3.12–3), the bad nodes are, then, exactly the arbitrage nodes of type II. Therefore,

$$(\mathrm{H}_{\mathrm{bad}}) \quad \Leftrightarrow \quad (\mathrm{H}_{II}').$$

Thus, in view Theorem 3.12–2) the assumptions imply that ($\mathrm{H}_{\mathrm{bad}}$) holds. Moreover ($\mathrm{TC}_{\mathrm{bad}}$) is satisfied by Theorem 3.12–1) and $(S,0)$ is not an arbitrage node of type II by assumption (H_{II}) (and, thus, good by Theorem 3.12–3), again). Therefore, Corollary 3.13 applies. □

4. Comparison to the game-theoretic framework

In this section, we provide a detailed comparison to the game-theoretic setting as described in the recent monograph of Shafer and Vovk [23]. We also refer to de Cooman and coauthors [6,25], who clarify the relation of the game-theoretic approach to Walley's behavioral notion of coherence, which can equivalently be formulated in terms of lower previsions or in terms of acceptance sets.

For the comparison, we consider the game-theoretic setting of Protocol 7.10 in [23]. Its starting point is a situation space $\mathbb{S}$, which is a set of finite sequences of elements of a nonempty set $\mathscr{Y}$. To accommodate the setting of Section 2, we let $\mathscr{Y} = \mathbb{R}$ and call $s = (s_0, \ldots, s_n)$ a *situation*, if $s = (S_0, \ldots, S_n)$ for some $S \in \mathscr{S}$, i.e., if s is the initial segment of some trajectory. We call (s_0) the *initial situation* and note that the initial situation is denoted by □ in [23]. Then, the set of all situations

$$\mathbb{S} = \left\{(S_0, S_1, \ldots S_n) : \; S \in \mathscr{S}, \, n \in \mathbb{N}\right\} \cup \{(s_0)\}$$

is called the *situation space*.

Note that the function, which maps the *situation* $s = (s_0, \ldots, s_n) \in \mathbb{S}$ to the node $\{S \in \mathscr{S} : \; (S_0, \ldots, S_n) = s\}$, provides a one-to-one correspondence between the game-theoretic situations and our nodes. For each $s = (s_0, \ldots, s_n) \in \mathbb{S}$ and $y \in \mathbb{R}$, the game-theoretic framework applies the notation $sy = (s_0, \ldots, s_n, y)$ and

$$\mathscr{Y}_s \equiv \{y \in \mathbb{R} : \; sy \in \mathbb{S}\}.$$

The assumption that $\mathscr{Y}_s$ is non-empty for every situation s ensures that one can continue from any situation of a given length n to a new one of length $n+1$ and so on. It is trivially satisfied in our specification: if $s = (S_0, \ldots, S_n)$ for some $S \in \mathscr{S}$ and $n \in \mathbb{N}_0$, then $S_{n+1} \in \mathscr{Y}_s$. Given a situation space $\mathbb{S}$, Shafer and Vovk introduce the *sample space*

$$\Omega = \{\omega \in \mathbb{R}^{\mathbb{N}} : \; \forall n \in \mathbb{N} \; (s_0, \omega_1, \ldots, \omega_n) \in \mathbb{S}\}.$$

The next lemma shows that the sample space of Shafer and Vovk coincides with the trajectorial completion of the trajectory set, see Definition 3.11.

Lemma 4.1. $\overline{\mathscr{S}} = \{(s_0, \omega_1, \omega_2, \dots) : \omega \in \Omega\}$.

The proof of Lemma 4.1 as well as the proofs of the other results of this section will be given in Appendix A.4.

Remark 4.2. By the previous lemma, the sample space of the game-theoretic framework always corresponds to a complete trajectory set, while no assumption on trajectorial completeness is made in the general setting of our trajectorial framework. Working with incomplete trajectory spaces provides more flexibility in the modeling. For example, let us consider

$$\mathscr{S} = \{(1, s_1, s_2, \dots) : s_j \in \{0, 1\},\ s_j = 1 \text{infinitely often and } s_j = 0 \text{ infinitely often}\}.$$

This trajectory space models a sequence of coin tosses, but rules out that heads ('0') or tails ('1') only come up finitely often. The corresponding sample space $\Omega = \{0, 1\}^{\mathbb{N}}$ cannot incorporate the a-priori belief on neither heads nor tails coming up only finitely often.

However, the lack of trajectorial completeness will lead to substantial differences in the theory to be developed in this paper compared to the game-theoretic framework.

To each situation $s = (s_0, \dots, s_j)$, one can associate a one-period financial model with initial stock price s_j, where $\mathscr{Y}_s$ denotes the set of possible stock prices one time step later. We may think of a financial derivative in this one-period model as a function $\mathfrak{f}_s : \mathscr{Y}_s \to [-\infty, \infty]$, where $\mathfrak{f}_s(y)$ describes the payoff of the derivative, if the stock price y realizes at the end of the one-period model. The set of payoffs offered to the investor in situation $s = (s_0, \dots, s_j)$ is denoted by $\mathscr{G}_s$. In the superhedging context, one may choose

$$\mathscr{G}_s = \{\mathfrak{f}_s : \mathscr{Y}_s \to [-\infty, \infty] : \exists h \in [-\infty, \infty] \forall y \in \mathscr{Y}_s : h \cdot (y - s_j) - \mathfrak{f}_s(y) \geq 0 \text{ and } h \cdot (y - s_j) > -\infty\} \tag{8}$$

as *offer sets*, for $s = (s_0, \dots, s_j) \in \mathbb{S}$. In the terminology of [23], the investor is also called 'Skeptic' and chooses the portfolio position h in the one-period model. The investor can be considered to play a game against an opponent called 'Reality' or 'World' who picks the stock price y at the end of the one-period model.

By (8), a payoff $\mathfrak{f}_s$ is offered at situation s, if and only if it can be superhedged with zero initial endowment in the one-period sub-model starting at situation s. Note that infinite portfolio positions are possible, provided the portfolio wealth cannot become $-\infty$, compare also Example 6.3 in [23] for a related definition. However, the infinite portfolio positions are only required to fully exploit the arbitrage opportunities that arise at arbitrage nodes. E.g., applying an infinite portfolio position at an arbitrage node of type II, leads to an infinite gain and, thus, all payoffs are offered at arbitrage nodes of type II. On the contrary, at up-down nodes finite portfolio positions can be applied only. These aspects are clarified by the following lemma.

Lemma 4.3. *Suppose* $s = (s_1, \dots, s_j) \in \mathbb{S}$. *Then,*

$$\mathscr{G}_s = \{\mathfrak{f}_s : \mathscr{Y}_s \to [-\infty, \infty] : \exists h \in (-\infty, \infty) \forall y \in \mathscr{Y}_s : h \cdot (y - s_j) \geq \mathfrak{f}_s(y)\},$$

if the associated node $\{S \in \mathscr{S} : (S_0, \dots, S_j) = s\}$ *is an up-down node;*

$$\mathscr{G}_s = \{\mathfrak{f}_s : \mathscr{Y}_s \to [-\infty, \infty] : \mathfrak{f}_s(s_j) \leq 0\},$$

if the associated node is an arbitrage-node of type I or a flat node;

$$\mathscr{G}_s = \{\mathfrak{f}_s : \mathscr{Y}_s \to [-\infty, \infty]\},$$

if the associated node is an arbitrage-node of type II.

The game-theoretic approach largely relies on the assumption that the acceptance sets $\mathscr{A}_s = \{-\mathfrak{f}_s : \mathfrak{f}_s \in \mathscr{G}_s\}$ constructed from the offer sets satisfy certain rationality assumptions related to Walley's behavioral notion of coherence. The latter include, see [6]:

(D.1) If $(-\mathfrak{f}_s)(y) \leq 0$ for every $y \in \mathscr{Y}_s$ and $(-\mathfrak{f}_s)(y') < 0$ for some $y' \in \mathscr{Y}_s$, then $(-\mathfrak{f}_s) \notin \mathscr{A}_s$.
(D.2) If $(-\mathfrak{f}_s)(y) \geq 0$ for every $y \in \mathscr{Y}_s$, then $(-\mathfrak{f}_s) \in \mathscr{A}_s$.
(D.3) If $(-\mathfrak{f}_s), (-\mathfrak{g}_s) \in \mathscr{A}_s$, then so is $(-\mathfrak{f}_s) + (-\mathfrak{g}_s)$.
(D.4) If $(-\mathfrak{f}_s) \in \mathscr{A}_s$ and $\lambda \geq 0$ is constant, then $\lambda(-\mathfrak{f}_s) \in \mathscr{A}_s$.

Axioms (D.2)–(D.4) are easily verified for the offer sets $\mathscr{G}_s$ introduced above. However, (D.1), which can be considered as a model-free no-arbitrage condition for the one-period models, fails, if for $s = (s_0, \dots, s_j) \in \mathbb{S}$ the associated node $\{S \in \mathscr{S} : (S_0, \dots, S_j) = s\}$ is an arbitrage node (as can be seen from Lemma 4.3). (D.1), at first glance, may look both natural and compelling as a starting axiom as one is not accepting a gamble, in which one cannot win, but one may lose. Nonetheless, this reasoning fails at arbitrage nodes: suffering a loss becomes irrelevant, if the loss can be compensated by exploiting the arbitrage that is available at an arbitrage node.

The setting of Shafer and Vovk [23], p. 121, weakens the rationality axiom (D.1) to

(D.1') If $(-\mathfrak{f}_s)(y) < 0$ for every $y \in \mathscr{Y}_s$, then $(-\mathfrak{f}_s) \notin \mathscr{A}_s$.

(D.1') can accommodate arbitrage nodes of type I, but is still in conflict with arbitrage nodes of type II (see Lemma 4.3, again).

Therefore, by considering arbitrage nodes of type II, our setting neither satisfies (D.1) nor the weaker assumption (D.1'), which crucially changes the analysis compared to the standard game-theoretic approach.

We next relate sequences $(g_j)_{j\geq 0}$ of non-anticipative functions (see Definition 2.3) with values in $[-\infty,\infty]$ to the notion of a *process* in the game-theoretic framework, which is (by definition) a mapping $\mathfrak{g}:\mathbb{S}\to[-\infty,+\infty]$. Given a sequence (g_j) of non-anticipative functions and a situation $s=(s_0,\dots,s_j)$, we choose some $S\in\mathscr{S}$ such that $(S_0,\dots,S_j)=s$ and define

$$\mathfrak{g}(s)\equiv g_j(S), \tag{9}$$

which does not depend on the choice of S, because the sequence (g_j) is non-anticipative. Clearly, this construction provides a one-to-one correspondence between the set of sequences of non-anticipative functions and the set of mappings from $\mathbb{S}$ to $[-\infty,\infty]$.

The game-theoretic approach applies the offer sets to define a notion of supermartingale. In view of Propositions 6.10 and 7.2 in [23], we say that a process $\mathfrak{g}:\mathbb{S}\to[-\infty,+\infty]$ is a *supermartingale with respect to the offer sets* $(\mathscr{G}_s)_{s\in\mathbb{S}}$ ($\mathscr{G}$-supermartingale, for short), if for every $s\in\mathbb{S}$

$$\inf\{\alpha\in\mathbb{R}:\ (\mathscr{Y}_s\to[-\infty,+\infty],\ y\mapsto\mathfrak{g}(sy)-\alpha)\in\mathscr{G}_s\}\leq\mathfrak{g}(s). \tag{10}$$

Accordingly, a sequence $(g_j)_{j\geq 0}$ of non-anticipative functions will be said to be a *$\mathscr{G}$-supermartingale sequence*, if the associated process via (9) is a $\mathscr{G}$-supermartingale. Note that this notion of a $\mathscr{G}$-supermartingale is 'local' in the sense that it only relies on superhedging in the one-period submodels that start at each node (or, equivalently, at each situation).

Based on the notion of $\mathscr{G}$-supermartingales, the game-theoretic approach defines a 'global' outer expectation operator, see p. 142 in [23]. In our notation it can be written as

$$\overline{\mathbb{E}}f=\inf\{g_0:\ (g_j)_{j\geq 0}\text{ is a }\mathscr{G}\text{-supermartingale sequence, }\inf_{j\geq 0,\,S\in\mathscr{S}}g_j(S)>-\infty,\text{ and }\liminf_{j\to\infty}g_j(S)\geq f(S)\text{ for every }S\in\mathscr{S}\}.$$

The following example shows that this global outer expectation operator $\overline{\mathbb{E}}$ does, in general, neither coincide with the superhedging outer integral $\overline{\sigma}$ nor with the operator $\overline{I}$, which determines the null sets in the trajectorial approach.

Example 1. We consider the trajectory set

$$\mathscr{S}=\{S^0,S^{-,-},S^{-,0},S^{-,+},S^{+,n}|n\in\mathbb{N}\},$$

wherein $S^0_j=4$ for every $j\geq 0$,

$$S^{-,0}_j=\begin{cases}4, & j=0\\ 2, & j\geq 1\end{cases},\quad S^{-,-}_j=\begin{cases}4, & j=0\\ 2, & j=1\\ 1, & j\geq 2\end{cases},\quad S^{-,+}_j=\begin{cases}4, & j=0\\ 2, & j=1\\ 3, & j=2\\ 2, & j\geq 3\end{cases},\quad S^{+,n}_j=\begin{cases}4, & j=0,\\ n+4, & j=1\\ n+7/2, & j\geq 2\end{cases},\quad n\in\mathbb{N}.$$

As illustrated in Fig. 1, this trajectory set features arbitrage nodes of type II. For this trajectory set, (L)-a.e. holds by virtue of Corollary 3.14. Moreover, the following will be shown in Appendix A.4:

a) $\overline{\sigma}(f)=f(S^0)=\int f\,d\mathbf{Q}$ for every $f:\mathscr{S}\to(-\infty,\infty)$, where $\mathbf{Q}$ denotes the Dirac measure on S^0;

b) $\mathbf{Q}$ is the unique martingale measure for $\mathscr{S}$, cp. Remark 2.14;

c) $\overline{\mathbb{E}}f=\max\{f(S^0),f(S^{-,0})\}$ for every $f:\mathscr{S}\to(-\infty,\infty)$;

d) $\overline{I}f=\max\{f(S^0),f(S^{-,0}),\frac{1}{2}f(S^{-,-})\}$ for every $f:\mathscr{S}\to[0,\infty)$.

Items a), c) and d) show that $\overline{\sigma}(f)<\overline{\mathbb{E}}f<\overline{I}f$, if $\frac{1}{2}f(S^{-,-})>f(S^{-,0})>f(S^0)\geq 0$. Thus, both operators $\overline{\sigma}$ and $\overline{I}$, which are constructed along the lines of the König-Leinert theory of non-lattice integration [14,16], are different from the global outer expectation operator of the game-theoretic approach of Shafer and Vovk [23]. Moreover, by item b), only the superhedging operator $\overline{\sigma}$ computes prices which are compatible with the paradigm of martingale pricing in mathematical finance. Theorem 8.1 below provides more evidence that the (conditional) superhedging outer integral $\overline{\sigma}_j$ has a proper interpretation as superhedging price operator, where superhedging takes place by trading with linear combinations of buy-and-hold strategies and the superhedge must hold up to an $\overline{I}$-null set. Note that, by item d), neither $\{S^{-,0}\}$ nor $\{S^{-,-}\}$ are $\overline{I}$-null sets, while both are null sets with respect to the unique martingale measure $\mathbf{Q}$. Hence, in this example, superhedges in our setting must hold on a larger set than superhedges constructed in the probabilistic framework.

Following the lines of Example 6 below, one can easily modify the trajectory set in such a way that it does not have arbitrage nodes of type II, but fails to be trajectorially complete, and properties a)–d) still hold. Summarizing, the example (as well as Theorem 8.1 below) illustrates that in our framework which encompasses trajectorially incomplete models and models with arbitrage nodes of type II, the outer integral operator $\overline{\mathbb{E}}$ of the game-theoretic approach might better be replaced by the superhedging outer integral $\overline{\sigma}$ for the computation of meaningful superhedging prices.

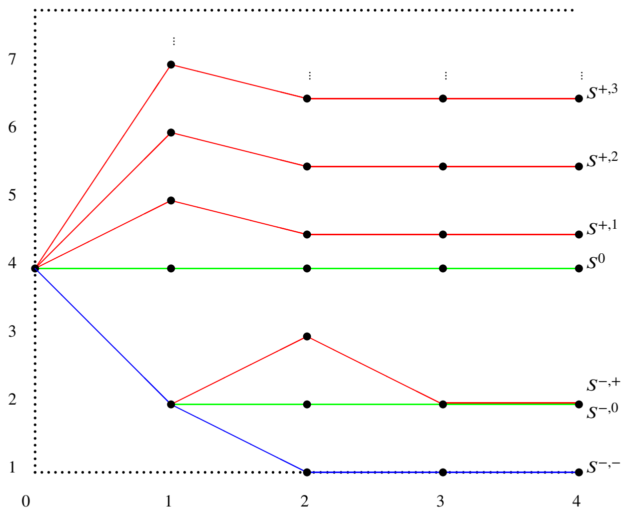

Fig. 1. Illustration of the trajectory set in Example 1.

The main object studied in this paper are supermartingales defined via the conditional superhedging outer integral $\overline{\sigma}_j$ by the property

$$\overline{\sigma}_j f_{j+1} \leq f_j \;\; \overline{I} - a.e. \;\; 0 \leq j < \infty.$$

Here, $(f_j)_{j\geq 0}$ is, of course, a sequence of non-anticipative functions. By the following proposition every $\mathscr{G}$-supermartingale sequence is also a supermartingale in our sense (defined via $\overline{\sigma}_j$).

Proposition 4.4. *If a sequence of non-anticipative* $[-\infty, +\infty]$*-valued functions* $(g_j)_{j\geq 0}$ *is a* $\mathscr{G}$*-supermartingale sequence, then, for every* $j \geq 0$ *and* $S \in \mathscr{S}$*,*

$$\overline{\sigma}_j g_{j+1}(S) \leq g_j(S).$$

It turns out, however, that there are supermartingales (based on $\overline{\sigma}_j$), which fail to be $\mathscr{G}$-supermartingale sequences. The intuitive reason is that $\overline{\sigma}_j$ is based on superhedging in infinite time, and so failure to superhedge in the one-period submodel can be compensated by exploiting arbitrage opportunities that may arise at later times. These arbitrage opportunities at later times and, thus, the new types of supermartingales appear because of trajectorial incompleteness or the presence of arbitrage nodes of type II, and, for these reasons, do not show up in the game-theoretic framework (compare Lemma 7.6 in [23]). Here is a simple example.

Example 2. In the framework of Example 1, consider the sequence $(f_j)_{j\geq 0}$ defined via $f_0 = 0$ and $f_j = \mathbf{1}_{T_1 \neq 4}$ for $j \geq 1$ – recalling $T_1(S) = S_1$ for every $S \in \mathscr{S}$. Then, by Proposition A.2.1–a), $\overline{\sigma}_j f_{j+1}(S) = \overline{\sigma}_j(\mathbf{1}_{T_1\neq 4})(S) \leq \mathbf{1}_{T_1\neq 4}(S) = f_j(S)$ for every $j \geq 1$ and $S \in \mathscr{S}$. Moreover, by item a) in Example 1, $\overline{\sigma}_0 f_1 = f_1(S^0) = 0 = f_0$. Hence, $(f_j)_{j\geq 0}$ satisfies $\overline{\sigma}_j f_{j+1}(S) \leq f_j(S)$ for every $j \geq 0$ and $S \in \mathscr{S}$. However, $(f_j)_{j\geq 0}$ is not a $\mathscr{G}$-supermartingale sequence for the offer sets introduced in (8). Otherwise, by Lemma 4.3 and the definition of a $\mathscr{G}$-supermartingale sequence in (10), for every $\varepsilon > 0$ we would find an $h \in \mathbb{R}$ such that $\varepsilon + h(S_1 - S_0) \geq f_1(S) = 1$ for every $S \in \mathscr{S} \setminus \{S^0\}$. This clearly leads to a contradiction, since, e.g., $S_1^{-,0} - S_0^{-,0} < 0$ and $S_1^{+,1} - S_0^{+,1} > 0$.

Summarizing, in the context of financial models in infinite discrete time, the game-theoretic framework can only be applied, if the trajectory set is trajectorially complete and has no arbitrage nodes of type II. In this case, the continuity condition $(L_{(S,j)})$ holds at every node (S, j) by Corollary 3.14 and the two families of superhedging operators $(\overline{\sigma}_j)_{j\geq 0}$ and $(\overline{I}_j)_{j\geq 0}$ coincide on the cone of non-negative functions by Theorem 8.4. More generally, our setting allows for the case where $(L_{(S,j)})$ fails on a null set; in this case the two families of superhedging operators $(\overline{\sigma}_j)_{j\geq 0}$ and $(\overline{I}_j)_{j\geq 0}$ differ. Moreover, $\overline{\sigma}$ has a proper interpretation as superhedging price by Theorem 8.1 and the role of $\overline{I}$ is to detect the arbitrage opportunities as null sets. In our more general financial setting, new types of supermartingales (defined via $\overline{\sigma}_j$) arise compared to the notion of supermartingales in the game-theoretic framework.

Remark 4.5. The game-theoretic approach to mathematical finance can also be applied in continuous time, as initiated by Vovk [28]. In this approach, the null sets of the model are determined by the outer measure introduced in [28] or variations thereof [1]. This outer measure, thus, plays a similar role as the superhedging functional $\overline{I}$ in our trajectorial approach, but we emphasize again that proper superhedging prices are computed by the outer superhedging integral $\overline{\sigma}$ in our approach; cp. also Section 2.5.2 in [2] for a more detailed comparison to the continuous-time game theoretic approach.

5. Supermartingales: definition and examples

Next is our definition of *trajectorial supermartingales, submartingales, and martingales* and of *stopping times*.

Definition 5.1. Consider a sequence $(f_j)_{j\geq 0}$ of non-anticipative functions $f_j : \mathscr{S} \to [-\infty, \infty]$, $j \geq 0$. We say,

(f_j) is a *supermartingale* if

$$\overline{\sigma}_j f_{j+1} \leq f_j \ \ a.e. \ 0 \leq j < \infty,$$

(f_j) is a *submartingale* if

$$f_j \leq \underline{\sigma}_j f_{j+1} \ \ a.e. \ 0 \leq j < \infty,$$

(f_j) is a *martingale* if

$$\underline{\sigma}_j f_{j+1} = \overline{\sigma}_j f_{j+1} = f_j \ \ a.e. \ 0 \leq j < \infty.$$

Remark 5.2. Notice that if $(f_j)_{j\geq 0}$ is a martingale, then according to Remark 2.13, f_{j+1} is conditionally integrable at j for any $j \geq 0$. Moreover, if $(f_j)_{j\geq 0}$ is a sub- and a supermartingale, then, under Assumption (L)-a.e., $(f_j)_{j\geq 0}$ is a martingale, since by Proposition 3.3 $\underline{\sigma}_j f \leq \overline{\sigma}_j f$ holds $a.e.$ for any $f \in Q$.

Definition 5.3 *(Stopping time, as per Definition 8 in [10]).* Given a trajectory space $\mathscr{S}$, a *trajectory based stopping time* (or *stopping time* for short) is a function $\tau : \mathscr{S} \to \mathbb{N} \cup \{\infty\}$ such that:

$$\text{for any } \ S,\ S' \in \mathscr{S} \quad \text{if } \ S_k = S'_k \ \text{ for } \ 0 \leq k \leq \tau(S), \quad \text{then } \ \tau(S) = \tau(S').$$

We next provide examples for the above definitions.

Example 3. a) Suppose that (L)-a.e. holds. If $V \in \mathbb{R}$ and $(H_j)_{j\geq 0}$ is a non-anticipative sequence, then

$$M_j(S) \equiv V + \sum_{i=0}^{j-1} H_i(S)\Delta_i S, \quad S \in \mathscr{S},\ j \geq 0,$$

satisfies the martingale property $\underline{\sigma}_j M_{j+1}(S) = \overline{\sigma}_j M_{j+1}(S) = M_j(S)$ whenever $(L_{(S,j)})$ holds (this is so by Proposition 3.3 item (4)). In particular, the coordinate process $(T_j)_{j\geq 0}$, $T_j(S) \equiv S_j$, forms a martingale sequence (with $H_i \equiv 1$ and $V = S_0$).

b) For any $f \in Q$, the sequence $(f_j)_{j\geq 0}$ defined by

$$f_j(S) \equiv \overline{\sigma}_j f(S), \quad S \in \mathscr{S},\ j \geq 0,$$

forms a supermartingale and the sequence $(f_j)_{j\geq 0}$ defined by

$$f_j(S) \equiv \underline{\sigma}_j f(S), \quad S \in \mathscr{S},\ j \geq 0,$$

forms a submartingale by the tower property in Proposition 5.5 below.

c) If $(f_j)_{j\geq 0}$ is a supermartingale and $(D_j)_{j\geq 0}$ is a non-anticipative sequence of non-negative functions, then the *supermartingale transform* $(g_j)_{j\geq 0}$

$$g_j(S) \equiv f_0 + \sum_{i=0}^{j-1} D_i(f_{i+1} - f_i), \quad S \in \mathscr{S},\ j \geq 0,$$

is again a supermartingale. This follows from subadditivity of $\overline{\sigma}_j$ and the remark that: $\overline{\sigma}_j g_{j+1}(S) \leq \overline{\sigma}_j g_j(S) + \overline{\sigma}_j(D_j(f_{j+1} - f_j))(S) \leq g_j(S) + D_j(S)\, \overline{\sigma}_j(f_{j+1} - f_j)(S) \leq g_j(S) + D_j(S)[\overline{\sigma}_j f_{j+1}(S) - f_j(S)] \leq g_j(S)$ (where we relied on Proposition A.2.1 for the second inequality and to conclude that $\overline{\sigma}_j(f_{j+1} - f_j)(S) \leq \overline{\sigma}_j(f_{j+1})(S) + \overline{\sigma}_j(-f_j)(S) \leq \overline{\sigma}_j(f_{j+1})(S) - f_j(S)$).

If $(f_j)_{j\geq 0}$ is a submartingale, then we call $(g_j)_{j\geq 0}$ (defined as above) a submartingale transform, which by the duality $f_j \to -f_j$, $\overline{\sigma}_j \to \underline{\sigma}_j$, is a submartingale.

d) If $(f_j)_{j\geq 0}$ is a supermartingale and τ is a stopping time, then the *stopped sequence* $(f_j^\tau)_{j\geq 0}$ defined by

$$f_j^\tau(S) \equiv f_{\tau(S)\wedge j}(S),$$

is a supermartingale. This is a consequence of the previous item with the choice

$$D_i(S) = \begin{cases} 1, & \tau(S) > i \\ 0, & \tau(S) \le i \end{cases}, \quad S \in \mathscr{S},\ j \ge 0,$$

which is non-anticipative by Lemma 5.4 below.

Lemma 5.4. *Let τ a stopping time and $H^k = (H_i^k)_{i\ge 0}$, $k = 1, 2$ sequences of non-anticipative functions. For $S \in \mathscr{S}$, $j \ge 0$ define the following functions on $\mathscr{S}_{(S,j)}$:*

$$H_i^\tau(\tilde S) = \begin{cases} H_i^1(\tilde S) & \text{if} \quad j \le i < \tau(\tilde S) \\ H_i^2(\tilde S) & \text{if} \quad \tau(\tilde S) \le i, \end{cases} \qquad i \ge j,\ \tilde S \in \mathscr{S}_{(S,j)}.$$

Then $H^\tau = (H_i^\tau)_{i\ge j}$ is a sequence of non-anticipative functions on $\mathscr{S}_{(S,j)}$.

Proof. Let $\tilde S, \hat S \in \mathscr{S}_{(S,j)}$ such that $\tilde S_k = \hat S_k$, $j \le k \le i$. If $j \le \tau(\hat S) \le i \Rightarrow \tau(\tilde S) = \tau(\hat S) \le i$ & $H_i^\tau(\tilde S) = H_i^2(\tilde S) = H_i^2(\hat S) = H_i^\tau(\hat S)$. While, by symmetry in previous reasoning, $i < \tau(\hat S) \Rightarrow i < \tau(\tilde S)$ & $H_i^\tau(\hat S) = H_i^1(\hat S) = H_i^1(\tilde S) = H_i^\tau(\tilde S)$. $\square$

Proposition 5.5 *(Tower Inequality). Let S be an arbitrary element of $\mathscr{S}$ and $j \le k$ non-negative integers; also let $f \in Q$. Then,*

$$\overline{\sigma}_j(\overline{\sigma}_k\ f)(S) \le \overline{\sigma}_j f(S).$$

Proof. In order to establish the desired result, it is enough to consider the case when the following inequality holds on $\mathscr{S}_{(S,j)}$ (otherwise $\overline{\sigma}_j f(S) = \infty$):
$f \le \Pi_{j,n_0}^{V^0,H^0} + \sum_{m\ge 1} \liminf_{n\to\infty} \Pi_{j,n}^{V^m,H^m}$ for some $(V^m, H^m)_{m\ge 0}$ such that $\Pi_{j,n_0}^{V^0,H^0} \in \mathscr{E}_{(S,j)}$ and $\Pi_{j,n}^{V^M,H^m} \in \mathscr{E}^+_{(S,j)}$ for every $n \ge j$. This inequality implies that $\overline{\sigma}_k f(\tilde S) \le \sum_{m\ge 0} \Pi_{j,k}^{V^m,H^m}(\tilde S)$ holds for all $\tilde S \in \mathscr{S}_{(S,j)}$, which in turn implies $\overline{\sigma}_j(\overline{\sigma}_k f)(S) \le \sum_{m\ge 0} V^m(S)$. Therefore the result follows by the definition of $\overline{\sigma}_j$ as an infimum. $\square$

We next present sufficient conditions for the tower inequality to turn into an equality (and so obtaining an analogous result to the classical tower property), which in turn implies that $(\overline{\sigma}_j f)_{j\ge 0}$ is a martingale sequence.

Corollary 5.6. *Assume (L)-a.e. and, for a fixed $f \in Q$, assume that either a) or b) below hold:*

a) *f is conditionally integrable for every $j \ge 0$;*
b) *f is integrable and $\overline{I} = \overline{\sigma}$ on non-negative functions with finite maturity.*

Then, $\overline{\sigma}_j(\overline{\sigma}_k\ f)(S) = \overline{\sigma}_j f(S)$ holds for every $j \le k$ and for a.e. S and $(\overline{\sigma}_j f)_{j\ge 0}$ is a martingale sequence.

Proof. We start with the following preliminary considerations: since (L)-a.e. holds, we conclude from Proposition 3.3 that $\underline{\sigma}_j g \le \overline{\sigma}_j g$ a.e. holds for every $g \in Q$ and $j \ge 0$. Applying this twice with $g = f$ and with $g = \overline{\sigma}_k f$, for $j \le k$, in view of Corollary A.2.2, the following chain of inequalities holds a.e.

$$\underline{\sigma}_j[\underline{\sigma}_k f] \le \underline{\sigma}_j[\overline{\sigma}_k f] \le \overline{\sigma}_j[\overline{\sigma}_k f].$$

Applying Proposition 5.5, we obtain,

$$\underline{\sigma}_j f \le \underline{\sigma}_j[\underline{\sigma}_k f] \le \underline{\sigma}_j[\overline{\sigma}_k f] \le \overline{\sigma}_j[\overline{\sigma}_k f] \le \overline{\sigma}_j f, \quad a.e. \tag{11}$$

a) The conditional integrability assumption now turns all a.e.-inequalities in (11) into equalities valid a.e. In particular, taking $k = j + 1$, it follows that $(\overline{\sigma}_j f)_{j\ge 0}$ is a martingale.

b) Let $j = 0$. Then, the integrability assumption turns all inequalities in (11) into identities. In particular, we obtain $\overline{\sigma}[\overline{\sigma}_k f] = \underline{\sigma}[\underline{\sigma}_k f]$, $\overline{\sigma}[\overline{\sigma}_k f] = \underline{\sigma}[\overline{\sigma}_k f]$ and also $\underline{\sigma}[\underline{\sigma}_k f] = \overline{\sigma}[\underline{\sigma}_k f]$ (with $-f$ in place of f). We then have access to Corollary A.2.4 (applied to the functions $\overline{\sigma}_k f$ and $-\underline{\sigma}_k f$ and $j = 0$) to compute

$$\overline{\sigma}[\overline{\sigma}_k f - \underline{\sigma}_k f] = \overline{\sigma}[\overline{\sigma}_k f] + \overline{\sigma}[-\underline{\sigma}_k f] = \overline{\sigma}[\overline{\sigma}_k f] - \underline{\sigma}[\underline{\sigma}_k f] = 0.$$

Therefore, given that $\overline{\sigma}_k f - \underline{\sigma}_k f \ge 0$ a.e. (as per Proposition 3.3) we have

$$\overline{I}[(\overline{\sigma}_k f - \underline{\sigma}_k f)_+] = \overline{\sigma}[(\overline{\sigma}_k f - \underline{\sigma}_k f)_+] = \overline{\sigma}[\overline{\sigma}_k f - \underline{\sigma}_k f] = 0,$$

which, by Proposition 2.9 item (1), implies $\overline{\sigma}_k f - \underline{\sigma}_k f \le 0$ a.e. The two inequalities together yield $\overline{\sigma}_k f = \underline{\sigma}_k f$ a.e., hence the conditional integrability of f at k. Since k is arbitrary, b) is reduced to a). $\square$

6. Supermartingale decomposition

In this section we prove a supermartingale representation theorem. It can be considered as an analogue of the uniform Doob decomposition in discrete time (see, e.g., Theorem 7.5 in [12]) or the optional decomposition theorem in continuous time [15], which apply to stochastic processes that are supermartingales simultaneously under a family of probability measures.

Theorem 6.1 *(Supermartingale decomposition). Under Assumption (L)-a.e., let $(f_j)_{j\geq 0}$ be a sequence of non-anticipative real-valued functions. Then, the following assertions are equivalent:*

(i) *$(f_j)_{j\geq 0}$ is a supermartingale.*
(ii) *For every sequence $(\delta_j)_{j\geq 0}$ of positive real numbers there are sequences $(H_j)_{j\geq 0}$ and $(A_j)_{j\geq 0}$, of non-anticipative real-valued functions defined on $\mathscr{S}$, such that $(A_j)_{j\geq 0}$ is nondecreasing, $A_0 = 0$, and*

$$f_i(S) = f_0 + \sum_{j=0}^{i-1} H_j(S)\Delta_j S - A_i(S) + \sum_{j=0}^{i-1} \delta_j,$$

for every $S \in \mathscr{S} \setminus N_f$ and $i \geq 0$. Here N_f is an $\overline{I}$-null set independent of $(\delta_j)_{j\geq 0}$.

Remark 6.2. Theorem 6.1 shows that up to the small δ-errors and null sets, supermartingales can be decomposed into a difference of a martingale (of the special form as in Example 3-(a)) and a non-anticipative, nondecreasing sequence. We will illustrate the supermartingale decomposition theorem, its assumptions, and its applicability beyond the classical probabilistic setting in a series of examples at the end of this section.

The proof of Theorem 6.1 relies on two lemmas, which we call Finite Maturity Lemma and Aggregation Lemma.

Lemma 6.3 *(Finite Maturity). Suppose $f : \mathscr{S} \to \mathbb{R}$ has maturity n_f for some $n_f \in \mathbb{N}$. Let $j \leq n_f$ and $S^* \in \mathscr{S}$ be such that the property $(L_{(S^*,n_f)})$ holds.*

If $f_m = \liminf_{n\to\infty} \Pi^{V^m,H^m}_{j,n}$, $\Pi^{V^m,H^m}_{j,n} \in \mathscr{E}^+_{(S^*,j)}$, where $m \geq 1$, and $f_0 = \Pi^{V^0,H^0}_{j,n_0} \in \mathscr{E}_{(S^*,j)}$ satisfy

$$f \leq \sum_{m=0}^{\infty} f_m, \quad \text{on } \mathscr{S}_{(S^*,j)}, \tag{12}$$

then,

$$f \leq \sum_{m=0}^{\infty} \Pi^{V^m,H^m}_{j,n_f} \quad \text{on } \mathscr{S}_{(S^*,n_f)} \ \text{(each side is constant on } \mathscr{S}_{(S^*,n_f)}). \tag{13}$$

Proof. Define for each $\tilde{S} \in \mathscr{S}_{(S^*,n_f)}$,

$$U^0(S^*) \equiv -f(S^*) + \Pi^{V^0,H^0}_{j,n_f\wedge n_0}(S^*) = -f(S^*) + \Pi^{V^0,H^0}_{j,n_f}(S^*),$$

$$g_0(\tilde{S}) \equiv U^0(S^*) + \sum_{i=n_f\wedge n_0}^{n_0-1} H^0_i(\tilde{S})\Delta_i\tilde{S} = U^0(S^*) + \sum_{i=n_f}^{n_0-1} H^0_i(\tilde{S})\Delta_i\tilde{S},$$

where we have used the fact that $H^0_i = 0$ for $i \geq n_0$, and for $m \geq 1$

$$U^m(S^*) \equiv \Pi^{V^m,H^m}_{j,n_f}(S^*), \qquad g_m(\tilde{S}) \equiv U^m(S^*) + \liminf_{n\to\infty} \sum_{i=n_f}^{n-1} H^m_i(\tilde{S})\Delta_i\tilde{S}.$$

It follows that

$$\Pi^{U^0(S^*),H^0}_{n_f,n} = -f(S^*) + \Pi^{V^0,H^0}_{j,n} \in \mathscr{E}_{(S^*,n_f)} \text{ for any } n \geq n_f,$$

and for $m \geq 1$

$$\Pi^{U^m(S^*),H^m}_{n_f,n} = \Pi^{V^m,H^m}_{j,n} \in \mathscr{E}^+_{(S^*,n_f)} \text{ for any } n \geq n_f.$$

Notice that (12) implies $0 \leq \sum_{m\geq 0} g_m$ holds on $\mathscr{S}_{(S^*,n_f)}$ and since property $(L_{(S^*,n_f)})$ holds, Proposition 3.3 yields

$$0 = (\overline{\sigma}_{n_f} 0)(S^*) \leq \sum_{m=0}^{\infty} U^m(S^*) = -f(S^*) + \sum_{m=0}^{\infty} \Pi_{j,n_f}^{V^m,H^m}(S^*),$$

from where (13) holds. □

The following aggregation lemma is proved in [10] (Lemma 3) under the assumption that all nodes are up-down nodes. The proof there can easily be adapted to our more general setting (which allows for any type of node).

Lemma 6.4 *(Aggregation Lemma). For $j \geq 0$ fixed let, for $m \geq 1$ $H^m = (H_i^m)_{i \geq j}$, be sequences of non-anticipative functions on $\mathscr{S}$, and V^m functions defined on $\mathscr{S}$, depending for each S only on $S_0, ..., S_j$.*

Fix a node (S, j) and assume for any $m \geq 1$, $n \geq j$, and $\tilde{S} \in \mathscr{S}_{(S,j)}$ that:

$$\Pi_{j,n}^{V^m,H^m}(\tilde{S}) = V^m(S) + \sum_{i=j}^{n-1} H_i^m(\tilde{S}) \Delta_i \tilde{S} \geq 0, \qquad \text{and} \quad \sum_{m \geq 1} V^m(S) < \infty.$$

Then, for every $\hat{S} \in \mathscr{S}_{(S,j)}$ and $k \geq j$ the following holds: if $(\hat{S}, k)$ is an up-down node and, for every $j \leq p < k$, $(\hat{S}, p)$ is an up-down node or $\hat{S}_{p+1} = \hat{S}_p$, then,

$$\sum_{m \geq 1} H_k^m(\hat{S}) \text{ converges in } \mathbb{R}.$$

Proof of Theorem 6.1. Let

$$\mathcal{N}^{(I)} = \{S \in \mathscr{S} : \exists j \geq 0 \; s.t. \; (S, j) \text{ is a type I arbitrage node and } S_{j+1} \neq S_j\},$$

$$\mathcal{N}^{(II)} = \{S \in \mathscr{S} : \exists j \geq 0 \; s.t. \; (S, j) \text{ is a type II arbitrage node}\},$$

and recall that $\mathcal{N}^{(L)}$, defined in (5), is a null set by assumption. Note that $\mathcal{N}^{(I)}$ and $\mathcal{N}^{(II)}$ are null sets by Lemma A.1.3 and that $\mathcal{N}^{(II)} \subseteq \mathcal{N}^{(L)}$ by Proposition 3.10.

(i) ⇒ (ii): By the supermartingale property of $(f_j)_{j \geq 0}$ we may fix an $\overline{I}$-null set $\mathcal{N}_f$ such that $\overline{\sigma}_j f_{j+1}(S) \leq f_j(S)$ for every $S \in \mathscr{S} \setminus \mathcal{N}_f$ and $j \geq 0$. Let $N_f \equiv \mathcal{N}^{(I)} \cup \mathcal{N}^{(L)} \cup \mathcal{N}_f$. We first introduce the stopping time

$$\tau^{\#}(S) = \inf\{k \geq 0 : \; (L_{(S,k)}) \text{ fails, or } \overline{\sigma}_k f_{k+1}(S) > f_k(S), \text{ or } [(S, k-1) \text{ is a type I arbitrage node and } S_k \neq S_{k-1}]\}, \tag{14}$$

and we recall the convention $\inf \emptyset = +\infty$. Note that $\tau^{\#}(S) = \infty$, if $S \notin N_f$.

Now we fix some $j \geq 0$ and choose a family $\{S^\lambda\}_{\lambda \in \Lambda_j}$ for some index set Λ_j so that $\{\mathscr{S}_{(S^\lambda,j)} : \lambda \in \Lambda_j\}$ is a partition of $\mathscr{S}$ (see Definition A.1.2 in Appendix A.1).

Step 1: Construction of H_j:

We now construct the function $H_j : \mathscr{S} \to \mathbb{R}$ in such a way that it is constant on the nodes of the partition (and, thus, non-anticipative). To this end, we consider an arbitrary but fixed node (S^λ, j) of the partition and proceed as follows:

If $\tau^{\#}(S^\lambda) \leq j$, then $\mathscr{S}_{(S^\lambda,j)} \subset N_f$ and we simply let $H_j(S) = 0$ for any $S \in \mathscr{S}_{(S^\lambda,j)}$.

If $\tau^{\#}(S^\lambda) \geq j+1$ note that, in particular, $(L_{(S^\lambda,j)})$ holds and $\overline{\sigma}_j f_{j+1}(S^\lambda) \leq f_j(S^\lambda) \in \mathbb{R}$. Applying the definition of $\overline{\sigma}_j$, we find g_m's such that

$$f_{j+1} \leq \sum_{m=0}^{\infty} g_m \text{ on } \mathscr{S}_{(S^\lambda,j)},$$

where $g_m = \liminf_{n\to\infty} \; \Pi_{j,n}^{V^m,H^m}$, $\; \Pi_{j,n}^{V^m,H^m} \in \mathscr{E}^+_{(S^\lambda,j)}$ for $m \geq 1$, $g_0 = \; \Pi_{j,n_0}^{V^0,H^0} \in \mathscr{E}_{(S^\lambda,j)}$ and

$$\sum_{m=0}^{\infty} V_m \leq f_j(S^\lambda) + \delta_j. \tag{15}$$

Notice that H_j^m is constant on $\mathscr{S}_{(S^\lambda,j)}$, and we write h_m for this constant value. If (S^λ, j) is an up-down node, then $\sum_{m=0}^{\infty} h_m$ converges in $\mathbb{R}$ by the Aggregation Lemma 6.4. We now define $H_j(S) \equiv H_j(S^\lambda)$ for $S \in \mathscr{S}_{(S^\lambda,j)}$ in the following way:

- If (S^λ, j) is an up-down node, then $H_j(S^\lambda) \equiv \sum_{m=0}^{\infty} h_m$,
- Otherwise $H_j(S^\lambda) \equiv 0$.

This completes the construction of H_j. For later use, we make the following observation: If $S \in (S^\lambda, j)$ satisfies $(L_{(S,j+1)})$, then, by the Finite Maturity Lemma 6.3 with $n_{f_{j+1}} = j+1$,

$$f_{j+1}(S) \leq \sum_{m=0}^{\infty}(V_m + h_m \Delta_j S). \tag{16}$$

Step 2: Construction of $A_{j+1} - A_j$:

We next construct the increment $\alpha_j \equiv A_{j+1} - A_j$. As in Step 1, we let $\alpha_j(S) \equiv 0$, if $S \in \mathscr{S}_{(S^\lambda, j)}$ and $\tau^\#(S^\lambda) \leq j$. Assuming now that $S \in \mathscr{S}_{(S^\lambda, j)}$ for some S^λ satisfying $\tau^\#(S^\lambda) \geq j+1$, we distinguish two cases. We say S belongs to *Case A*, if S satisfies $(L_{(S,j+1)})$ and $[(S^\lambda, j)$ is an up-down or $S_{j+1} = S_j]$. Otherwise, we say S belongs to *Case B*.

We first continue the construction of α_{j+1}, if S belongs to Case A. Then, by (15) and (16),

$$f_j(S) + \delta_j \quad = \quad f_j(S^\lambda) + \delta_j \geq \sum_{m=0}^{\infty} V_m \geq f_{j+1}(S) - \sum_{m=0}^{\infty} h_m \Delta_j S = f_{j+1}(S) - H_j(S)\Delta_j S.$$

Therefore,

$$\alpha_j(S) \equiv \delta_j + H_j(S)\Delta_j S - (f_{j+1}(S) - f_j(S)) \geq 0. \tag{17}$$

In order to complete the construction of α_j, we let

$$\alpha_j(S) \equiv 0,$$

if S belongs to Case B. Belonging to Case B means that $(L_{(S,j+1)})$ fails or $[(S^\lambda, j)$ is not an up-down and $S_{j+1} \neq S_j]$. If $(L_{(S,j+1)})$ fails, then $\tau^\#(S) = j+1$ by definition. If the other condition for case B holds, then (S^λ, j) can neither be an up-down node nor a flat node and, thus, must be an arbitrage node. However, by Proposition 3.10, (S^λ, j) can neither be an arbitrage node of type II, because $(L_{(S^\lambda, j)})$ holds by the assumption that $\tau^\#(S^\lambda) \geq j+1$. Therefore, the condition $[(S^\lambda, j)$ is not an up-down and $S_{j+1} \neq S_j]$ can be rephrased as $[(S, j)$ is an arbitrage node of type I and $S_{j+1} \neq S_j]$. Consequently, $\tau^\#(S) = j+1$.

Summarizing, if α_j is not defined via (17), then $\tau^\#(S) \leq j+1$, which implies $S \in N_f$.

Step 3: Finalizing the proof of (i) $\Rightarrow$ (ii):

We define $(A_j)_{j\geq 0}$ via $A_i \equiv \sum_{j=0}^{i-1} \alpha_j(S)$. Then, by Steps 1–2, the sequences $(A_j)_{j\geq 0}$ and $(H_j)_{j\geq 0}$ are non-anticipative, $(A_j)_{j\geq 0}$ is nondecreasing and $A_0 = 0$. Now let $S \in \mathscr{S} \setminus N_f$. Then, as emphasized at the end of Step 2, $\alpha_j(S)$ is defined by (17) for every $j \geq 0$. Hence, for $S \in \mathscr{S} \setminus N_f$ and every $i \geq 0$,

$$A_i(S) = \sum_{j=0}^{i-1}[\delta_j + H_j(S)\Delta_j S - (f_{j+1}(S) - f_j(S))] = f_0 - f_i(S) + \sum_{j=0}^{i-1} H_j(S)\Delta_j S + \sum_{j=0}^{i-1}\delta_j,$$

which provides the representation of $(f_j)_{j\geq 0}$ as asserted in (ii).

$(ii) \Rightarrow (i)$: We first fix some $j \geq 0$ and some positive integer K and let $\delta_i = 1/K$. Condition (ii) implies

$$f_{j+1}(S) \leq f_j(S) + 1/K + H_j(S)(S_{j+1} - S_j)$$

for every $S \in \mathscr{S} \setminus N_f$. We define

$$g_{j+1}(S) \equiv f_j(S) + 1/K + H_j(S)(S_{j+1} - S_j)$$

for every $S \in \mathscr{S}$. Then, $g_{j+1} \in \mathscr{E}_j$ and $\overline{\sigma}_j g_{j+1}(S) \leq f_j(S) + 1/K$ for every $S \in \mathscr{S}$. Consequently, by monotonicity and sub-additivity of $\overline{\sigma}_j$ (as per Proposition A.2.1),

$$\begin{aligned}\overline{\sigma}_j f_{j+1}(S) &\leq \overline{\sigma}_j(g_{j+1} + (f_{j+1} - g_{j+1})_+)(S) \leq \overline{\sigma}_j g_{j+1}(S) + \overline{\sigma}_j((f_{j+1} - g_{j+1})_+)(S) \\ &\leq f_j(S) + 1/K + \overline{\sigma}_j((f_{j+1} - g_{j+1})_+)(S).\end{aligned} \tag{18}$$

Noting that $(f_{j+1} - g_{j+1})_+$ can only be positive on the $\bar{I}$-null set N_f, we may deduce from Proposition A.2.1–e), that $\bar{I}_j(f_{j+1} - g_{j+1})_+ = 0$ $\bar{I}$-a.e. for every $j \geq 0$. Hence, there is an $\bar{I}$-null set $\mathscr{N}_f$ such that $\bar{I}_j(f_{j+1} - g_{j+1})_+(S) = 0$ for every $S \in \mathscr{S} \setminus \mathscr{N}_f$ and $j \geq 0$. In view of Remark 2.11, we conclude that $\overline{\sigma}_j((f_{j+1} - g_{j+1})_+)(S) \leq 0$ for every $S \in \mathscr{S} \setminus \mathscr{N}_f$ and $j \geq 0$. Inserting this identity into (18) and passing to the limit $K \to \infty$, yields (i). $\square$

We close this section with some illustrative examples. The first example shows that condition (L)-a.e. may hold in a situation when the model does not have a martingale measure, cp. Remark 2.14. Moreover, the example also shows that the δ-sequence in the supermartingale decomposition (Theorem 6.1) cannot be dispensed with.

Example 4. Let $\mathscr{S} = \mathscr{S}^+ \cup \mathscr{S}^-$, where $\mathscr{S}^\pm = \{S^{\pm,n} : n \in \mathbb{N}\}$ and

$$S_i^{+,n} = \begin{cases} 1, & i = 0 \\ 2, & i = 1 \\ 2 + \frac{1}{n}, & i \geq 2 \end{cases}, \qquad S_i^{-,n} = \begin{cases} 1, & i = 0 \\ 1 - \frac{1}{n^2}, & i \geq 1 \end{cases}.$$

Then, the node $(S^{+,1},1)=(S^{+,n},1)$, $n\geq 2$, is an arbitrage node of type II, the initial node $(S,0)$ is up-down, and all other nodes are flat. For an illustration of this trajectory set, we refer to [2], where different aspects of this example are discussed.

a) We first show that (L)-a.e. holds, but there is no martingale measure for this trajectory set.

Since $S_1^{-,n}-S_0^{-,n}=-1/n^2$ is arbitrarily close to zero for sufficiently large n, we observe that (H_{II}) in Corollary 3.14 is satisfied. Trajectorial completeness is obvious, because all trajectories stay constant after time $j=2$. Hence, by Corollary 3.14, this trajectory set satisfies (L)-a.e. Since all trajectories $S^{+,n}$ pass through the arbitrage node $(S^{+,1},1)$ of type II, $\mathscr{S}^+$ is a null set by Lemma A.1.3 and, hence, $\bar{I}(\mathbf{1}_{\mathscr{S}^-})=1$ by Remark 3.8.

It is also clear that there is no probability measure $\mathbf{Q}$ on the power set of $\mathscr{S}$, which turns the coordinate process $T_j:\mathscr{S}\to\mathbb{R}$, $S\mapsto S_j$ into a martingale in the classical probabilistic sense. Otherwise $\mathbf{Q}(\mathscr{S}^+)=0$ (because any martingale measure assigns probability zero to type II arbitrage nodes) and then $T_1<T_0$ $\mathbf{Q}$-almost surely – a contradiction.

Consequently, by Example 3–a), the coordinate process $(T_j)_{j\geq 0}$ is an example of a trajectorial martingale, which fails to be a martingale in the classical probabilistic setup.

b) We next construct a supermartingale, for which a decomposition as in Theorem 6.1–(ii) is not possible, if we let $\delta_j\equiv 0$ (and, hence, the small δ-errors cannot be avoided in the formulation of the supermartingale decomposition theorem).

To this end, we define $f_j:\mathscr{S}\to\mathbb{R}$ via

$$f_j(S)=\begin{cases}0, & S\in\mathscr{S}^+ \text{ or } j=0\\ \frac{1}{n}, & S=S^{-,n} \text{ and } j\geq 1\end{cases},$$

and consider the sequence $(f_j)_{j\geq 0}$. Since all of its 'paths' $j\mapsto f_j(S)$ are nondecreasing, $(f_j)_{j\geq 0}$ obviously is a submartingale. We claim that $(f_j)_{j\geq 0}$ is also a supermartingale (despite of the nondecreasing paths) and apply Theorem 6.1 to verify this. Given a sequence $(\delta_j)_{j\geq 0}$ of positive reals, let

$$H_0\equiv -\lceil \delta_0{}^{-1}\rceil,\quad H_j\equiv 0,\ j\geq 0,$$

and note that, for every $n\in\mathbb{N}$,

$$\delta_0+H_0(S_1^{-,n}-S_0^{-,n})=\delta_0+\lceil\delta_0{}^{-1}\rceil\,\frac{1}{n^2}\geq\frac{1}{n}=f_1(S^{-,n}), \tag{19}$$

by considering the cases $n\leq\lceil\delta_0{}^{-1}\rceil$ and $n>\lceil\delta_0{}^{-1}\rceil$ separately. Hence, we may define a nondecreasing, non-anticipative sequence (A_j) via $A_0\equiv 0$ and

$$A_j(S)-A_{j-1}(S)=\begin{cases}\delta_0+\lceil\delta_0{}^{-1}\rceil\cdot\frac{1}{n^2}-\frac{1}{n}, & j=1 \text{ and } S=S^{-,n}\\ \delta_{j-1}, & j\geq 2 \text{ or } S\in\mathscr{S}^+.\end{cases}$$

It is then straightforward to check that, for every $i\geq 0$ and $S\in\mathscr{S}^-$,

$$f_i(S)=f_0+\sum_{j=0}^{i-1}H_j(S)\Delta_jS-A_i(S)+\sum_{j=0}^{i-1}\delta_j.$$

Indeed, for $i=1$, this is a consequence of (19) and the definition of A_1, while, for $i\geq 2$, $f_i=f_1$, and the increments of A compensate the δ's. Hence, (f_j) is a supermartingale by Theorem 6.1. Note that, in view of Remark 5.2, $(f_j)_{j\geq 0}$ is a martingale.

It remains to prove that a decomposition as in Theorem 6.1–(ii) is not possible for this (super-)martingale $(f_j)_{j\geq 0}$, if $\delta_j=0$ for every $j\geq 0$. Precisely, we will argue that such a decomposition requires $\delta_0>0$. We first show that for every $H_0\in\mathbb{R}$ there is an $n_0\in\mathbb{N}$ such that for every $n\geq n_0$

$$f_1(S^{-,n})>f_0+H_0(S_1^{-,n}-S_0^{-,n}).$$

By inserting the definition of f_1 and f_0, this inequality is equivalent to

$$\frac{1}{n}\left(1+\frac{H_0}{n}\right)>0,$$

which trivially holds for sufficiently large n. Hence, no matter of the choice of (H_j) and (A_j), a decomposition as in Theorem 6.1–(ii) with $\delta_0=0$ will necessarily fail at $i=1$ on the set $\mathscr{S}^{-,\geq n_0}=\{S^{-,n}|n\geq n_0\}$ for some $n_0\in\mathbb{N}$. It, thus, remains to show that $\bar{I}(\mathbf{1}_{\mathscr{S}^{-,\geq n_0}})>0$ for every $n_0\in\mathbb{N}$. To this end, assume that for every $S\in\mathscr{S}$,

$$\mathbf{1}_{\mathscr{S}^{-,\geq n_0}}(S)\leq\sum_{m\geq 1}\liminf_{k\to\infty}\ \Pi_{0,k}^{V^m,H^m}(S),$$

where $\Pi_{0,k}^{V^m,H^m}\in\mathscr{E}_0^+$ for all $k\geq 0$ and $\sum_{m\geq 1}V_m<\infty$. Since the trajectories in $\mathscr{S}^-$ are constant after time $k=1$, the previous inequality and the Aggregation Lemma 6.4 imply

$$1 \le \sum_{m\ge 1} \Pi_{0,1}^{V^m,H^m}(S^{-,n}) = \sum_{m\ge 1}(V^m - H_0^m \frac{1}{n^2}) = \sum_{m\ge 1} V_m - \frac{1}{n^2}\sum_{m\ge 1} H_0^m$$

for every $n \ge n_0$. Passing with n to infinity, we observe that $\sum_{m\ge 1} V_m \ge 1$ and, hence, $\bar{I}(\mathbf{1}_{\mathscr{S}^{-,\ge n_0}}) = 1$ for every $n_0 \in \mathbb{N}$.

The foregoing also shows that $(f_j)_{j\ge 0}$ is an example of a martingale, which is not a.e. equal to a 'simple' martingale of the form discussed in Example 3–a).

c) We finally show, as asserted in Remark 3.4–b), that the restriction of $\bar{I}$ to the set of bounded functions in P fails to be a coherent upper prevision. Recall that coherence of $\bar{I}$ means that for every $m, n \in \mathbb{N}$ and for every bounded $g_0, \dots g_n$ in P,

$$\sup_{S\in\mathscr{S}} \left(\left(\sum_{k=1}^{n} \bar{I}(g_k) - g_k(S) \right) - m(\bar{I}(g_0) - g_0(S)) \right) \ge 0;$$

see condition (D) in [26, Definition 4.10]. To see that this property is violated, we choose $m = n = 1$ and $g_1 = 1 + g_0$ for

$$g_0(S) = \begin{cases} 0, & S \in \mathscr{S}^+ \\ \frac{1}{n^2}, & S = S^{-,n} \text{ for some } n \in \mathbb{N} \end{cases}, \qquad S \in \mathscr{S}.$$

Then,

$$\sup_{S\in\mathscr{S}} \left(\left(\sum_{k=1}^{n} \bar{I}(g_k) - g_k(S) \right) - m(\bar{I}(g_0) - g_0(S)) \right) = -1 + \bar{I}(g_1) - \bar{I}(g_0);$$

and it suffices to show that $\bar{I}(g_1) \le 1$ and $\bar{I}(g_0) > 0$. Suppose, to the contrary, that $\bar{I}(g_0) = 0$. Noting that $\mathbf{1}_{\mathscr{S}^-} \le \lim_{k\to\infty} k g_0 = \sum_{i=1}^{\infty} g_0$, we conclude that $\bar{I}(\mathbf{1}_{\mathscr{S}^-}) \le \sum_{i=1}^{\infty} \bar{I}(g_0) = 0$ by isotonicity and countable sub-additivity of $\bar{I}$ (see Proposition 2.8). Hence, $\mathbf{1}_{\mathscr{S}^-}$ is a null function, contradicting the property that $\bar{I}(\mathbf{1}_{\mathscr{S}^-}) = 1$, which has been derived in part a) of this example. To verify that $\bar{I}(g_1) \le 1$, consider the non-negative elementary function $\tilde{g}_1(S) \equiv 1 - (S_1 - S_0) = g_0(S) + \mathbf{1}_{\mathscr{S}^-}(S) = g_1(S) - \mathbf{1}_{\mathscr{S}^+}(S)$, which clearly satisfies $\bar{I}(\tilde{g}_1) \le 1$. Then, $\bar{I}(g_1) = \bar{I}(\tilde{g}_1 + \mathbf{1}_{\mathscr{S}^+}) \le \bar{I}(\tilde{g}_1) + \bar{I}(\mathbf{1}_{\mathscr{S}^+}) \le 1$, because $\mathbf{1}_{\mathscr{S}^+}$ is a null function by part a) of this example.

Remark 6.5. Consider the following variant of Example 4. Let $\mathscr{S} = \mathscr{S}^+ \cup \{S^0, S^-\}$, where the up-branch $\mathscr{S}^+$ is as in Example 4, $S^0 \equiv 1$, $S_0^- = 1$ and $S_j^- = 0$ for $j \ge 1$. Adapting the arguments in Example 4, one can check that: 1) The point mass $\mathbf{Q}$ on S^0 is the unique martingale measure of this model; 2) (L)-a.e. holds; 3) $\bar{I}(\mathbf{1}_{\{S^0\}}) = 1$ and $\bar{I}(\mathbf{1}_{\{S^-\}}) = 1/2$. Hence $\{S^-\}$ is a null set for $\mathbf{Q}$, but not w.r.t. $\bar{I}$. In such a situation our supermartingale decomposition (Theorem 6.1) holds on a larger set than the (uniform) Doob decomposition in the classical theory.

We finally present an example, in which (L)-a.e. fails, and demonstrate the importance of this assumption for our results.

Example 5. Let $\mathscr{S} = \{S^{+,-}, S^0, S^-\} \cup \{S^{+,n} : n \in \mathbb{N}\}$, where $S_0 = 1$ for every $S \in \mathscr{S}$,

$$S_1 = \begin{cases} 1, & S = S^0 \\ 2, & S \in \{S^{+,n}, S^{+,-} : n \in \mathbb{N}\} \\ 0, & S = S^- \end{cases}, \quad S_2 = \begin{cases} S_1, & S \in \{S^0, S^-\} \\ 3, & S \in \{S^{+,n} : n \in \mathbb{N}\} \\ 3/2, & S = S^{+,-} \end{cases}, \quad S_j = \begin{cases} S_2, & S \in \{S^0, S^-, S^{+,-}\} \\ 3 + 1/n, & S \in \{S^{+,n} : n \in \mathbb{N}\} \end{cases},$$

for $j \ge 3$. This trajectory space is illustrated in Fig. 2. Note that (L) holds by Remark 3.2, because the constant trajectory S^0 is included in the trajectory set.

a) We first show that $(L_{(S^{+,-},1)})$ fails and $\bar{I}(\mathbf{1}_{\{S^{+,-}\}}) \ge 1/6$, and, thus, condition (L)-a.e. is violated.

Suppose $f : \mathscr{S}_{(S^{+,-},1)} \to \mathbb{R}$. Define, for arbitrary, but fixed $k \in \mathbb{N}$,

$$g_0(S) = -k - 2(f(S^{+,-}) + k)(S_2 - S_1), \quad g_m(S) = S_3 - S_2, \quad S \in \mathscr{S}_{(S^{+,-},1)}, \; m \ge 1.$$

Then, $g_0 \in \mathscr{E}_{(S^{+,-},1)}$, $g_m \in \mathscr{E}^+_{(S^{+,-},1)}$ for $m \ge 1$, and, on $\mathscr{S}_{(S^{+,-},1)}$,

$$\sum_{m=0}^{\infty} g_m = f(S^{+,-}) + \infty \, \mathbf{1}_{\{S^{+,n} : \, n\in\mathbb{N}\}} \ge f.$$

Since k was arbitrary, we obtain $\overline{\sigma}_1 f(S^{+,-}) = -\infty$. By choosing $f \equiv 0$, we observe that $(\overline{\sigma}_1 0)(S^{+,-}) \ne 0$ and, thus, $(L_{(S^{+,-},1)})$ fails by Proposition 3.3.

We next show that $\bar{I}(\mathbf{1}_{\{S^{+,-}\}}) \ge 1/6$. To this end, assume that for every $S \in \mathscr{S}$,

$$\mathbf{1}_{\{S^{+,-}\}}(S) \le \sum_{m\ge 1} \liminf_{k\to\infty} \; \Pi_{0,k}^{V^m,H^m}(S),$$

where $\Pi_{0,k}^{V^m,H^m} \in \mathscr{E}_0^+$ for all $k \ge 0$ and $\sum_{m\ge 1} V_m < \infty$. Since $S^{+,-}$ stays constant after time 2 and the $\Pi_{0,k}^{V^m,H^m}$'s are non-negative at any time, we obtain

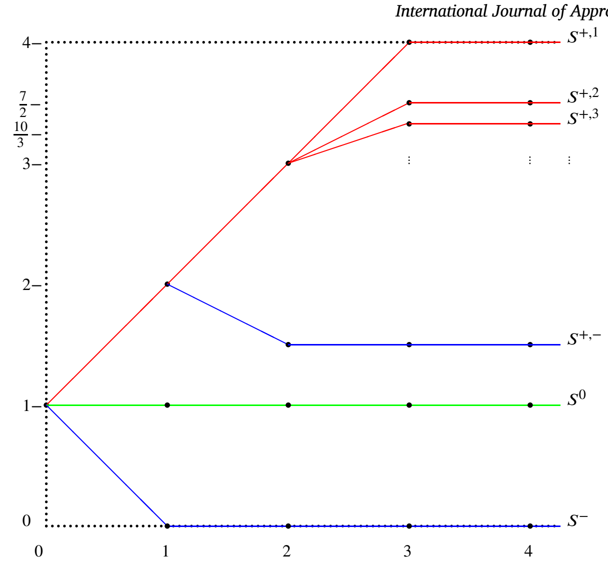

Fig. 2. Illustration of the trajectory set in Example 5.

$$\mathbf{1}_{\{S^{+,-}\}}(S) \le \sum_{m\ge 1} \Pi_{0,2}^{V^m,H^m}(S) \tag{20}$$

for every $S \in \mathscr{S}$. Writing $v = \sum_{m\ge1} V_m$, $a = \sum_{m\ge1} H_0^m$ and $b = \sum_{m\ge1} H_1^m(S^{+,-})$, we note that the series defining a and b converge in $\mathbb{R}$ by the Aggregation Lemma 6.4. Inserting $S = S^{+,-}$, $S = S^{+,1}$, and $S = S^-$ into (20) yields

$$(I):\ v + a - b/2 \ge 1, \quad (II):\ v + a + b \ge 0, \quad (III):\ v - a \ge 0.$$

Considering $2(I) + (II) + 3(III)$, we observe that $v \ge 1/6$ and, hence, $\bar{I}(\mathbf{1}_{\{S^{+,-}\}}) \ge 1/6$. Note that by similar, but easier arguments $\bar{I}(\mathbf{1}_{\{S^-\}}) \ge 1/2$ and $\bar{I}(\mathbf{1}_{\{S^0\}}) = 1$.

b) We now define the sequence $(f_j)_{j\ge0}$ via $f_0 = 1$ and $f_j \equiv f_\infty = \mathbf{1}_{\{S^0\}} + 2\,\mathbf{1}_{\mathscr{S}\setminus\{S^0\}}$ for $j \ge 1$. We claim that $(f_j)_{j\ge0}$ is a supermartingale, which does not have a decomposition as in Theorem 6.1–(ii) for sequences (δ_j) with $0 < \delta_0 < 1$. In particular, the assumption (L)-a.e. cannot be dropped in the latter theorem.

In our considerations, we may ignore the null set $\{S^{+,n} \,|\, n \in \mathbb{N}\}$ of those trajectories which pass through the arbitrage node $(S^{+,1}, 2)$ of type II. As the computation of $\bar{\sigma}_j$ is trivial after trajectories have become constant, we get

$$\bar{\sigma}_j f_{j+1}(S) = \bar{\sigma}_j f_\infty(S) = f_\infty(S) = f_j(S)$$

for $S \in \{S^0, S^-\}$ and $j \ge 1$ and for $S = S^{+,-}$ and $j \ge 2$. Moreover, by a),

$$\bar{\sigma}_1 f_2(S^{+,-}) = -\infty \le f_1(S^{+,-}).$$

For the supermartingale property at the initial node, consider

$$g_0(S) = 1 - (S_1 - S_0) - 4(S_2 - S_1), \quad g_m(S) = S_3 - S_2, \quad S \in \mathscr{S},\ m \ge 1.$$

Then, $g_0 \in \mathscr{E}$, $g_m \in \mathscr{E}^+$ for $m \ge 1$, and, on $\mathscr{S}$,

$$\sum_{m\ge0} g_m = f_1\,\mathbf{1}_{\{S^0,S^-,S^{+,-}\}} + \infty\,\mathbf{1}_{\{S^{+,n}|\,n\in\mathbb{N}\}} \ge f_1.$$

Hence $\bar{\sigma} f_1 \le 1 = f_0$. (Since the trajectory S^0 is constant, we also obtain $\bar{\sigma} f_1 \ge f_1(S^0) = 1$, i.e., $\bar{\sigma} f_1 = 1 = f_0$.)

We now fix a sequence $(\delta_j)_{j\ge0}$ of positive reals and assume existence of a representation for (f_j) as in Theorem 6.1–(ii). Then, at time $i = 1$,

$$f_1(S) \le (1 + \delta_0) + H_0(S_1 - S_0)$$

for $S \in \{S^0, S^-, S^{+,-}\}$ (where H_0 is a constant), because none of the singletons $\{S\}$, $S \in \{S^0, S^-, S^{+,-}\}$, is a null set by a). This leads to the three inequalities

$$1 \le (1 + \delta_0), \quad 2 \le (1 + \delta_0) - H_0, \quad 2 \le (1 + \delta_0) + H_0,$$

and combining the second and third inequality implies $\delta_0 \ge 1$.

7. Doob's pointwise supermartingale convergence

This section proves our version of Doob's pointwise convergence theorem for non-negative supermartingales.

Theorem 7.1 *(Supermartingale convergence). Suppose (L)-a.e. holds. Let $(f_i)_{i\geq 0}$ be a supermartingale with values in $[0,\infty)$ and impose the following assumption on the trajectory set:*

(P) *Whenever (S,j) is an up-down node such that $(L_{(S,j)})$ holds and $(L_{(S,j+1)})$ fails, then there are $S^1, S^2 \in \mathscr{S}_{(S,j)}$ such that $S^1_{j+1} > S_{j+1} > S^2_{j+1}$ and $(L_{(S^\iota,j+1)})$ holds for $\iota = 1,2$.*

Then, there exists a null set $\mathscr{N}_{div}$ such that $\lim_{i\to\infty} f_i(S)$ exists in $\mathbb{R}$ for every $S \in \mathscr{S} \setminus \mathscr{N}_{div}$.

In the game-theoretic setting, a version of Doob's convergence theorem has been established for non-negative $\mathscr{G}$-supermartingale sequences, see, e.g., Theorem 7.5 in [23]. By the discussion in Section 4, the game-theoretic approach can neither accommodate incomplete trajectory sets nor arbitrage nodes of type II. The following proposition provides easy-to-check sufficient conditions for the validity of our version of Doob's pointwise convergence theorem (Theorem 7.1) that can hold in trajectorially incomplete models and in the presence of arbitrage nodes of type II. It, thus, illustrates that Theorem 7.1 cannot be recovered as a special case of the game-theoretic Doob convergence theorem.

Proposition 7.2. *Suppose either of the following two sets of conditions holds:*

1. (TC$_{\text{bad}}$) and

(P$_{\text{bad}}$) *Whenever (S,j) is a good up-down node such that $(S,j+1)$ is bad, then there are $S^1, S^2 \in \mathscr{S}_{(S,j)}$ such that $S^1_{j+1} > S_{j+1} > S^2_{j+1}$ and $(S^\iota, j+1)$ are good for $\iota = 1,2$.*

Or

2. (TC$_{II}$) and

(P$_{II}$) *If (S,j) is an arbitrage node of type II, then $j \geq 1$ and $(S,j-1)$ is an up-down node and there are $S^1, S^2 \in \mathscr{S}_{(S,j-1)}$ such that $S^1_j > S_j > S^2_j$ and such that (S^1,j), (S^2,j) are not type II arbitrage nodes.*

Then, (L)-a.e. and condition (P) in Theorem 7.1 hold.

Proof. 1. Suppose that (TC$_{\text{bad}}$) and (P$_{\text{bad}}$) hold. We first show that (P$_{\text{bad}}$) implies (H$_{\text{bad}}$). To this end, suppose (S,j) is a good up-down node. If $(\tilde{S}, j+1)$ is good for every $\tilde{S} \in \mathscr{S}_{(S,j)}$, then, we find $S^1, S^2 \in \mathscr{S}_{(S,j)}$ such that $S^1_{j+1} > S_j > S^2_{j+1}$, because (S,j) is up-down – and $(S^\iota, j+1)$, $\iota = 1,2$, are automatically good. If $(\tilde{S}, j+1)$ is bad for some $\tilde{S} \in \mathscr{S}_{(S,j)}$, then, applying (P$_{\text{bad}}$) to $(\tilde{S}, j)$, we find $S^1, S^2 \in \mathscr{S}_{(S,j)} = \mathscr{S}_{(\tilde{S},j)}$ such that $S^1_{j+1} > \tilde{S}_j = S_j > S^2_{j+1}$. In both cases, letting $S^{\varepsilon,1} = S^1$ and $S^{\varepsilon,2} = S^2$ for every $\varepsilon > 0$, we obtain,

$$S^{\varepsilon,1}_{j+1} - S_j > 0 > -\varepsilon, \quad S^{\varepsilon,2}_{j+1} - S_j < 0 < \varepsilon,$$

i.e., condition (H$_{\text{bad}}$) is satisfied. Now, Corollary 3.13 applies and yields (L)-a.e. Moreover, that corollary implies that a node (S,j) is bad, if and only if $(L_{(S,j)})$ fails. Therefore, assuming (TC$_{\text{bad}}$), condition (P) is a consequence of condition (P$_{\text{bad}}$).
2. Now suppose that (TC$_{II}$) and (P$_{II}$) are in force. Analogously to part 1., we first show that (P$_{II}$) implies (H$_{II}$). Indeed, assuming that (S,j) is a type II arbitrage node and applying (P$_{II}$), we observe that $j \geq 1$ and $(S,j-1)$ is an up-down node. Consequently, there is a $\tilde{S} \in (S, j-1)$ such that $\tilde{S}_j > \tilde{S}_{j-1} = S_{j-1}$ and we may take $S^{\varepsilon,1} = \tilde{S}$ in (H$_{II}$), provided $(\tilde{S}, j)$ is not an arbitrage node of type II. Otherwise, we may apply (P$_{II}$) with $\tilde{S}$ in place of S and find some $S^1 \in (S, j-1) = (\tilde{S}, j-1)$ such that $S^1_j > \tilde{S}_j$ and (S^1, j) is not an arbitrage node of type II. Then, we may take $S^{\varepsilon,1} = S^1$. The construction of $S^{\varepsilon,2}$ follows by a 'symmetric' argument. Now that (H$_{II}$) is verified, Corollary 3.14 implies that (L)-a.e. holds – and that $L_{(S,j)}$ fails, if and only if (S,j) is an arbitrage node of type II. The latter implies (P). Indeed, if (S,j) is an up-down node and $L_{(S,j+1)}$ fails, then $(S,j+1)$ is a type II arbitrage node, and, thus, by (P$_{II}$) there are $S^1, S^2 \in \mathscr{S}_{(S,j)}$ such that

$$S^1_{j+1} > S_{j+1} > S^2_{j+1}$$

and such that $L_{(S^1,j+1)}$, $L_{(S^2,j+1)}$ hold, because (S^1,j), (S^2,j) are not type II arbitrage nodes. $\square$

The proof strategy of Theorem 7.1 is to apply the supermartingale decomposition in Theorem 6.1 and to pass to the limit separately for the various terms. For the martingale part $\sum_{j=0}^{i-1} H_j(S)\Delta_j S$, we can make use of Theorem 2 in [10]. However, this result requires that $\sum_{j=0}^{i-1} H_j(S)\Delta_j S$ is bounded from below by the same constant for every $S \in \mathscr{S}$, while Theorem 6.1 in conjunction with the nonnegativity of f only implies boundedness of $\sum_{j=0}^{i-1} H_j(S)\Delta_j S$ from below for a.e. $S \in \mathscr{S}$. In view of the following lemma, the required boundedness condition can be guaranteed under the additional assumption (P).

Lemma 7.3. *Under the assumptions of Theorem 7.1, fix a sequence $(\delta_j)_{j\geq 0}$ of summable positive reals and construct the supermartingale decomposition of $(f_j)_{j\geq 0}$ as in the proof of the implication $(i) \Rightarrow (ii)$ in Theorem 6.1. Then, for every $S \in \mathscr{S}$ and $i \geq 0$,*

$$f_0 + \sum_{j=0}^{\infty} \delta_j + \sum_{j=0}^{i-1} H_j(S)\Delta_j S \geq 0.$$

Proof. We construct $(H_j)_{j\geq 0}$ as in the proof of the implication $(i) \Rightarrow (ii)$ in Theorem 6.1 and recall that the stopping time $\tau^\#$ has been defined in (14) there. As emphasized at the end of Step 2 of the proof of Theorem 6.1; if $j \leq \tau^\#(S) - 2$, then, the inequality in (17) is valid, and, consequently

$$f_{j+1}(S) - f_j(S) \leq \delta_j + H_j(S)(S_{j+1} - S_j). \tag{21}$$

Thus, for $0 \leq i < \tau^\#(S)$,

$$f_0 + \sum_{j=0}^{\infty} \delta_j + \sum_{j=0}^{i-1} H_j(S)\Delta_j S \geq f_0 + \sum_{j=i}^{\infty} \delta_j + \sum_{j=0}^{i-1} f_{j+1}(S) - f_j(S) \geq f_i(S) \geq 0.$$

For the remainder of the proof, we consider the case $i \geq \tau^\#(S)$. Recalling that $H_j(S) = 0$ for $j \geq \tau^\#(S)$ by the beginning of Step 1 in the proof of Theorem 6.1 and inserting, again, (21) for $j \leq \tau^\#(S) - 2$, we obtain

$$\begin{aligned}
&f_0 + \sum_{j=0}^{\infty} \delta_j + \sum_{j=0}^{i-1} H_j(S)\Delta_j S \\
&= f_0 + \sum_{j=\tau^\#(S)}^{\infty} \delta_j + \sum_{j=0}^{\tau^\#(S)-2} \delta_j + H_j(S)(S_{j+1} - S_j) + \left(\delta_{\tau^\#(S)-1} + H_{\tau^\#(S)-1}(S)(S_{\tau^\#(S)} - S_{\tau^\#(S)-1})\right) \\
&\geq f_{\tau^\#(S)-1}(S) + \left(\delta_{\tau^\#(S)-1} + H_{\tau^\#(S)-1}(S)(S_{\tau^\#(S)} - S_{\tau^\#(S)-1})\right).
\end{aligned}$$

If $(S, \tau^\#(S) - 1)$ is not an up-down-node, then, $H_{\tau^\#(S)-1}(S) = 0$ by the construction in Step 1 of the proof of Theorem 6.1 and, thus,

$$f_0 + \sum_{j=0}^{\infty} \delta_j + \sum_{j=0}^{i-1} H_j(S)\Delta_j S \geq f_{\tau^\#(S)-1}(S) \geq 0.$$

If $(S, \tau^\#(S) - 1)$ is an up-down-node and $(L_{(S,\tau^\#(S))})$ holds, then, S belongs to Case A of Step 2 in the proof of Theorem 6.1 with $j = \tau^\#(S) - 1$. Hence, by (17),

$$f_0 + \sum_{j=0}^{\infty} \delta_j + \sum_{j=0}^{i-1} H_j(S)\Delta_j S \geq f_{\tau^\#(S)}(S) \geq 0.$$

If $(S, \tau^\#(S) - 1)$ is an up-down-node and $(L_{(S,\tau^\#(S))})$ fails, then (noting that property $(L_{(S,\tau^\#(S)-1)})$ holds by the definition of $\tau^\#$), we may apply assumption (P) with $j = \tau^\#(S) - 1$. Hence, there are $S^1, S^2 \in \mathscr{S}_{(S,\tau^\#(S)-1)}$ such that $(L_{(S^\imath,\tau^\#(S))})$ holds for $\imath = 1, 2$ and $S^2_{\tau^\#(S)} \leq S_{\tau^\#(S)} \leq S^1_{\tau^\#(S)}$. Then, S^2 and S^1 belong to Case A of Step 2 in the proof of Theorem 6.1 with $j = \tau^\#(S) - 1$. If $H_{\tau^\#(S)-1}(S) \leq 0$, then, by invoking (17) for S^1,

$$f_0 + \sum_{j=0}^{\infty} \delta_j + \sum_{j=0}^{i-1} H_j(S)\Delta_j S \geq f_{\tau^\#(S)-1}(S^1) + \left(\delta_{\tau^\#(S)-1} + H_{\tau^\#(S)-1}(S^1)(S^1_{\tau^\#(S)} - S^1_{\tau^\#(S)-1})\right) \geq f_{\tau^\#(S)}(S^1) \geq 0.$$

If $H_{\tau^\#(S)-1}(S) > 0$, then, the same argument with S^2 in place of S^1 yields

$$f_0 + \sum_{j=0}^{\infty} \delta_j + \sum_{j=0}^{i-1} H_j(S)\Delta_j S \geq f_{\tau^\#(S)}(S^2) \geq 0. \quad \square$$

Proof of Theorem 7.1. Fix a sequence $(\delta_j)_{j\geq 0}$ of positive reals with $\sum_j \delta_j < \infty$. By Theorem 6.1 there are sequences $(H_j)_{j\geq 0}$, $(A_j)_{j\geq 0}$ of non-anticipative real-valued functions such that $(A_j)_{j\geq 0}$ is nondecreasing, $A_0 = 0$, and

$$f_i(S) = f_0 + \sum_{j=0}^{i-1} H_j(S)\Delta_j S - A_i(S) + \sum_{j=0}^{i-1} \delta_j, \tag{22}$$

for every $S \in \mathscr{S} \setminus N_f$ and $i \geq 0$, with N_f a null set independent of $(\delta_j)_{j\geq 0}$. Besides, with $V \equiv f_0 + \sum_{j=0}^{\infty} \delta_j$ it follows by Lemma 7.3 that, for every $i \geq 0$ and $S \in \mathscr{S}$,

$$\Pi_i^{V,H}(S) = f_0 + \sum_{j=0}^{\infty} \delta_j + \sum_{j=0}^{i-1} H_j(S)\Delta_j S \geq 0.$$

Having in mind that there are no portfolio restrictions on $\mathscr{H}$, from [10, Theorem 2] it follows that there exists a null set N_0 such that $\lim_{i\to\infty} \Pi_i^{V,H}(S)$ exists and is finite for any $S \in \mathscr{S} \setminus N_0$. Consequently $(\Pi_i^{V,H}(S))_{i\geq 0}$ is bounded for those S. Let $\mathscr{N}_{div} \equiv N_0 \cup N_f$ and restrict the following arguments to $S \in \mathscr{S} \setminus \mathscr{N}_{div}$: From (22) and the nonnegativity of f, we obtain that $A_i(S) \leq \Pi_i^{V,H}(S)$ for every $i \geq 0$. Hence, $(A_i(S))_{i\geq 0}$ is bounded from above and since it is nondecreasing, $\lim_{i\to\infty} A_i(S)$ exists in $\mathbb{R}$. In view of (22), the convergences of $(A_i(S))_{i\geq 0}$ and of $(\Pi_i^{V,H}(S))_{i\geq 0}$ in $\mathbb{R}$ imply that $\lim_{i\to\infty} f_i(S)$ exists in $\mathbb{R}$, for any $S \in \mathscr{S} \setminus \mathscr{N}_{div}$. □

8. On the relation between the two superhedging operators

As another consequence of the supermartingale decomposition, we show, in this section, that $\overline{\sigma}$ is the 'correct' superhedging operator in the sense that, for bounded (from below) functions with finite maturity, it corresponds to the infimal superhedging price within the usual class of simple portfolios up to the null sets induced by $\overline{I}$. We also clarify the relation between the two (conditional) superhedging operators $\overline{I}_j$ and $\overline{\sigma}_j$.

Theorem 8.1. *Suppose that (L)-a.e. holds and that $f \in Q$ has maturity $n_f \in \mathbb{N}$, is bounded from below and satisfies $\overline{\sigma} f < \infty$. Let $0 \leq j < n_f$. Then: For every $\varepsilon > 0$, there are a null set N_f and a non-anticipative sequence $(H_i)_{i=j,\dots,n_f-1}$ such that for every $S \in \mathscr{S} \setminus N_f$*

$$f(S) \leq (\overline{\sigma}_j f(S) + \varepsilon) + \sum_{i=j}^{n_f-1} H_i(S)\Delta_i S.$$

Conversely, if there are a $V \in Q$ with maturity j, a non-anticipative sequence $(H_i)_{i=j,\dots,n_f-1}$ and a null set $\tilde{N}_f$ such that for every $S \in \mathscr{S} \setminus \tilde{N}_f$

$$f(S) \leq V(S_0,\dots,S_j) + \sum_{i=j}^{n_f-1} H_i(S)\Delta_i S,$$

then $\overline{\sigma}_j f \leq V$ a.e.

In particular,

$$\overline{\sigma} f = \quad \inf\{V \in \mathbb{R} \,|\, \exists (H_j)_{j=0,\dots,n_f-1} \text{ non-anticipative such that } V + \sum_{j=0}^{n_f-1} H_j(S)\Delta_j S \geq f(S) \text{ for a.e. } S \in \mathscr{S}\}.$$

The proof combines Theorem 6.1 with the following lemma, which deals with the issue that the supermartingale $(\overline{\sigma}_i f)_{i\geq 0}$ (see Example 3–b)) may take values $\pm\infty$.

Lemma 8.2. *Suppose (L)-a.e. If $f \in Q$ is bounded from below by some $c \in \mathbb{R}$ and satisfies $\overline{\sigma} f < \infty$, then there is a supermartingale $(f_i)_{i\geq 0}$ with values in $[c;+\infty)$ such that $\overline{\sigma}_j f = f_j$ a.e. for every $j \geq 0$.*

Proof. As in the proof of Theorem 6.1, we consider the stopping time

$$\tau^{\#}(S) = \inf\{k \geq 0 : (L_{(S,k)}) \text{ fails, or } [(S,k-1) \text{ is a type I arbitrage node and } S_k \neq S_{k-1}]\}.$$

Define

$$f_i(S) = \overline{\sigma}_i f(S) \mathbf{1}_{\{i<\tau^{\#}(S)\}} + c\mathbf{1}_{\{i\geq \tau^{\#}(S)\}}, \quad i \geq 0,\ S \in \mathscr{S},$$

and note that $(f_i)_{i\geq 0}$ is non-anticipative by Lemma 5.4. Recall that $\{S \in \mathscr{S} : \tau^{\#}(S) < \infty\}$ is a null set by Lemma A.1.3 and by assumption (L)-a.e. In view of Example 3–b), we conclude that (f_j) is a supermartingale and $\overline{\sigma}_j f = f_j$ a.e. for every $j \geq 0$.

For the lower bound, note that by Proposition 3.3,

$$f_i(S) = \overline{\sigma}_i f(S) \geq \overline{\sigma}_i(c)(S) = c,$$

whenever $i < \tau^{\#}(S)$. We finally need to verify that $f_i(S) < \infty$ for every $i \geq 0$ and $S \in \mathscr{S}$. Since $\overline{\sigma} f < \infty$, we find $\Pi_{0,n_0}^{V^0,H^0} \in \mathscr{E}_0$ and, for every $m \in \mathbb{N}$, $\Pi_{0,\cdot}^{V^m,H^m}$ such that $\Pi_{0,n}^{V^m,H^m} \in \mathscr{E}_0^+$ for every $n \geq 0$, $\sum_{m=1}^{\infty} V^m < \infty$ and

$$f(S) \leq \sum_{m=0}^{\infty} \liminf_{n\to\infty} \Pi_{0,n}^{V^m,H^m}(S), \quad S \in \mathscr{S}.$$

We now fix a node (S,i). The previous inequality implies

$$f(\tilde{S}) \leq \sum_{m=0}^{\infty} \liminf_{n\to\infty} \Pi_{i,n}^{\tilde{V}^m(S),\tilde{H}^m}(\tilde{S}), \quad \tilde{S} \in \mathscr{S}_{(S,i)}$$

where $\tilde{V}^m(S) = \Pi_{0,i}^{V^m,H^m}(S)$ and $\tilde{H}^m = H^m_{|\mathscr{S}_{(S,i)}}$. Hence, $\overline{\sigma}_i f(S) \leq \sum_{m=0}^{\infty} \Pi_{0,i}^{V^m,H^m}(S)$. We still need to check that the right-hand side is finite, if $i < \tau^{\#}(S)$. In this case, for every $j < i$, the node (S,j) is an up-down node or $S_{j+1} = S_j$. Indeed, if (S,j) were an arbitrage node of type II, then, by Proposition 3.10, $(L_{(S,j)})$ fails, which results in $\tau^{\#}(S) \leq j < i$; a contradiction. Similarly, we arrive at a contradiction, if (S,j) were an arbitrage node of type I and $S_{j+1} \neq S_j$, because, then, $\tau^{\#}(S) \leq j+1 \leq i$. Thus, we may apply the Aggregation Lemma 6.4 to conclude that, for every $0 \leq j < i$, the series $\sum_{m=0}^{\infty}(H_j^m(S)\Delta_j S)$, converges in $\mathbb{R}$. Consequently

$$\sum_{m=0}^{\infty} \Pi_{0,i}^{V^m,H^m}(S) = \sum_{m=0}^{\infty} V^m + \sum_{j=0}^{i-1}\sum_{m=0}^{\infty}(H_j^m(S)\Delta_j S) \in \mathbb{R}. \quad \square$$

Proof of Theorem 8.1. Fix some $\varepsilon > 0$ and choose a sequence $(\delta_i)_{i\geq 0}$ of positive reals such that $\sum_{i=j}^{n_f-1} \delta_j \leq \varepsilon$. In view of Lemma 8.2, we choose a real-valued supermartingale $(f_i)_{i\geq 0}$ such that $f_i = \overline{\sigma}_i f$ a.e. for every $i \geq 0$. Note that $f_{n_f} = \overline{\sigma}_{n_f} f = f$ a.e. by Lemma A.2.3 and assumption (L)-a.e., since f has maturity n_f. Hence, we may apply Theorem 6.1 to $(f_i)_{i\geq 0}$ in order to construct a non-anticipative sequence $(H_i)_{i=0,\ldots,n_f-1}$ such that

$$f(S) = f_{n_f}(S) \leq \left(f_j + \sum_{i=j}^{n_f-1}\delta_j\right) + \sum_{i=j}^{n_f-1} H_i(S)\Delta_i S \leq \left(\overline{\sigma}_j f + \varepsilon\right) + \sum_{i=j}^{n_f-1} H_i(S)\Delta_i S \tag{23}$$

for every $S \in \mathscr{S} \setminus N_f$, where N_f is a null set.

For the converse inequality, assume that there are $V \in Q$ with maturity j, $H = (H_i)_{i=j,\ldots,n_f-1}$ non-anticipative, and a null set $\tilde{N}_f$ such that for every $S \in \mathscr{S} \setminus \tilde{N}_f$

$$f(S) \leq V(S_0,\ldots,S_j) + \sum_{i=j}^{n_f-1} H_i(S)\Delta_i S.$$

If $V(S_0,\ldots,S_j) = +\infty$, the inequality $\overline{\sigma}_j f(S) \leq V(S)$ is trivial. Otherwise, we note that $g = \infty \mathbf{1}_{\tilde{N}_f}$ is a null function and that

$$f(S) \leq V(S_0,\ldots,S_j) + \sum_{i=j}^{n_f-1} H_i(S)\Delta_i S + g(S) \tag{24}$$

holds for every $S \in \mathscr{S}$. By Proposition A.2.1–e), there is a null set N_f such that $\overline{I}_j g(S) = 0$ for every $S \in \mathscr{S} \setminus N_f$. Consequently, for every $S \in \mathscr{S} \setminus N_f$ and $\varepsilon > 0$ there are sequences $(V_m)_{m\geq 1}$ of non-negative reals and $(H^m)_{m\geq 1}$ in $\mathscr{H}_{(S,j)}$ such that $\Pi_{j,n}^{V_m,H^m} \in \mathscr{E}^+_{(S,j)}$ for every $n \geq j$, $\sum_{m\geq 1} V_m \leq \varepsilon$ and $g \leq \sum_{m\geq 1} \liminf_{n\to\infty} \Pi_{j,n}^{V_m,H^m}$ on $\mathscr{S}_{(S,j)}$. Let $V^0 = V(S_0,\ldots,S_j)$ and define $H^0 \in \mathscr{H}_{(S,j)}$ via $H_i^0(\hat{S}) = H_i(\hat{S})$ for every $j \leq i \leq n_f - 1$ and $\hat{S} \in \mathscr{S}_{(S,j)}$. Then, $\Pi_{j,n_f}^{V^0,H^0}(\hat{S}) = V(S_0,\ldots,S_j) + \sum_{i=j}^{n_f-1} H_i(\hat{S})\Delta_i\hat{S}$ for every $\hat{S} \in \mathscr{S}_{(S,j)}$. In view of (24), we obtain

$$f \leq \Pi_{j,n_f}^{V^0,H^0} + \sum_{m\geq 1} \liminf_{n\to\infty} \Pi_{j,n}^{V_m,H^m} \text{ on } \mathscr{S}_{(S,j)},$$

which implies $\overline{\sigma}_j f(S) \leq V(S_0,\ldots,S_j) + \varepsilon$ for every $S \in \mathscr{S} \setminus N_f$. Passing with ε to zero, yields $\overline{\sigma}_j f \leq V$ a.e. $\square$

Remark 8.3. Example 1 in Section 4 illustrates how to construct non-negative, bounded functions f with finite maturity such that $\overline{I}(f) > \overline{\sigma} f$ on a trajectory set satisfying (L)-a.e. In light of the previous theorem, we may conclude that $\overline{I}$ cannot be applied for computing superhedging prices in general, but only serves as an auxiliary operator to determine the null sets of the model.

The following theorem shows that the validity of $(L_{(S,j)})$ at all nodes (S,j) implies equality for the two families of superhedging operators (for non-negative, finite maturity functions f). Therefore, the possibility that $\overline{\sigma}_j f < \overline{I}_j f$ does stem from cases when (L) only holds a.e.

Theorem 8.4. *The following assertions are equivalent:*
(i) For every $f \in P$ with finite maturity and every node (S, j),

$$\overline{I}_j f(S) = \overline{\sigma}_j f(S).$$

(ii) $(L_{(S,j)})$ holds at every node (S, j).

Proof. (i) ⇒ (ii): since $\overline{I}_j(0)(S) = 0$ always holds, we immediately obtain

$$\overline{\sigma}_j(0)(S) = \overline{I}_j(0)(S) = 0$$

at any node (S, j). (ii) then follows from Proposition 3.3.
(ii) ⇒ (i): *Step 1:* We first consider the initial node $(S, 0)$.

Noting that $\overline{\sigma} f \leq \overline{I} f$ by Remark 2.11, it suffices to show that $\overline{I} f \leq \overline{\sigma} f$, for which we may and do assume that $\overline{\sigma} f < \infty$. By Lemma 8.2, we find a supermartingale $(f_j)_{j\geq 0}$ with values in $[0, \infty)$ such that $f_j = \overline{\sigma}_j f$ a.e. for every $j \geq 0$. We fix some arbitrary $\varepsilon > 0$ and a sequence $(\delta_i)_{i\geq 0}$ of positive reals such that $\sum_{i=0}^{\infty} \delta_i \leq \varepsilon$. Applying the supermartingale decomposition in Theorem 6.1 to $(f_j)_{j\geq 0}$, we obtain the estimate

$$f(S) = f_{n_f}(S) \leq \left(\overline{\sigma} f + \varepsilon\right) + \sum_{i=0}^{n_f-1} H_i(S)\Delta_i S, \quad a.e., \tag{25}$$

for a sequence of non-anticipative functions $(H_i)_{i=0,\dots,n_f-1}$; cp. (23). As $(L_{(S,j)})$ holds at every node (S, j), condition (P) in Theorem 7.1 is trivially satisfied. Thus, in view of Lemma 7.3, we may choose the sequence $(H_i)_{i=0,\dots,n_f-1}$ in such a way that

$$\left(\overline{\sigma} f + \varepsilon\right) + \sum_{i=0}^{j-1} H_i(S)\Delta_i S \geq \overline{\sigma} f + \sum_{i=0}^{\infty} \delta_j + \sum_{i=0}^{j-1} H_i(S)\Delta_i S \geq 0$$

for every $0 \leq j \leq n_f$ and $S \in \mathscr{S}$. Let $V^0 = \overline{\sigma} f + \varepsilon$ and define H^0 via $H^0_j = H_j$ for $j \leq n_f - 1$ and $H^0_j = 0$ for $j \geq n_f$. Then, $\Pi^{V^0,H^0}_{0,j} \in \mathscr{E}^+_0$ for every $j \geq 0$, and, in view of (25), we obtain

$$f(S) \leq \Pi^{V^0,H^0}_{0,n_f} = \liminf_{n\to\infty} \Pi^{V^0,H^0}_{0,n}$$

for every $S \in \mathscr{S} \setminus N_f$, where N_f is a null set. Dealing with the null set N_f as in the second part of the proof of Theorem 8.1 (taking $j = 0$ there), we conclude that $\overline{I} f \leq V^0 + \varepsilon = \overline{\sigma} f + 2\varepsilon$. Letting ε tend to zero, the proof of Step 1 is complete.
Step 2: We now consider a generic node $(\tilde{S}, \tilde{j})$.

Define the auxiliary trajectory set

$$\widetilde{\mathscr{S}} = \{(S_{\tilde{j}+i})_{i\geq 0} | S \in \mathscr{S}_{(\tilde{S},\tilde{j})}\}.$$

Then, the $\overline{\sigma}$-operator and the $\overline{I}$-operator for $\widetilde{\mathscr{S}}$ at time 0 coincide with $\overline{\sigma}_{\tilde{j}}(\cdot)(\tilde{S})$ and $\overline{I}_{\tilde{j}}(\cdot)(\tilde{S})$. Moreover, each node (S, j) in $\widetilde{\mathscr{S}}$ corresponds to the node $((\tilde{S}_0, \dots, \tilde{S}_{\tilde{j}-1}, S_0, \dots), \tilde{j} + j)$ in $\mathscr{S}$. Hence, every node (S, j) in $\widetilde{\mathscr{S}}$ satisfies $(L_{(S,j)})$. These observations reduce the case of a generic node to the case of an initial node. □

It is important to connect the superhedging outer integral $\overline{\sigma}$ to the pricing paradigm based on martingale measures. While a general study of such relations is beyond the scope of the present paper, the following example provides one such connection. The example partially relies on the following general observation.

Remark 8.5. Suppose (L)-a.e., that the trajectory set $\mathscr{S}$ is countable, and that there is a probability measure $\mathbf{P}$ on $2^{\mathscr{S}}$ such that $\mathbf{P}$ has the same null sets as $\overline{I}$, i.e., $\mathbf{P}(A) = 0 \Leftrightarrow \overline{I}\mathbf{1}_A = 0$ for every $A \subseteq \mathscr{S}$. If $f \in Q$ has maturity $n_f < \infty$ and is bounded, then, by Theorem 8.1 and recalling Remark 2.14,

$$\overline{\sigma} f = \inf\{V \in \mathbb{R} | \ \exists (H_j)_{j=0,\dots,n_f-1} \ \mathscr{T}\text{-adapted such that } V + \sum_{j=0}^{n_f-1} H_j(T_{j+1} - T_j) \geq f \ \mathbf{P}\text{-almost surely}\}.$$

Writing $\mathscr{Q}_n(\mathbf{P})$ for the set of all probability measures $\mathbf{Q}$ equivalent to $\mathbf{P}$ such that $(T_0, \dots T_n)$ is a martingale with respect to $\mathbf{Q}$, i.e., (4) holds for $j = 0, \dots, n-1$, the classical superhedging duality [12, Corollary 7.18] yields,

$$\overline{\sigma} f = \sup_{\mathbf{Q}\in\mathscr{Q}_{n_f}(\mathbf{P})} \int f \, d\mathbf{Q}, \tag{26}$$

provided the set $\mathscr{Q}_{n_f}(\mathbf{P})$ is non-empty. (Recall here, that two probability measures $\mathbf{P}$ and $\mathbf{Q}$ are said to be equivalent, if they have the same null sets.)

Example 6. We consider a variant of Example 4, replacing the 'sure' arbitrage at the type II arbitrage node $(S^+, 1)$ by a 'sure' arbitrage opportunity by trading up to unbounded time. To this end, we replace the up-branch of the model by $\tilde{\mathscr{S}}^+ = \{\tilde{S}^{+,n} : n \in \mathbb{N}\}$, where now

$$\tilde{S}_i^{+,n} = \begin{cases} 1, & i = 0 \\ 2, & 1 \le i < n+1 \\ 4, & i \ge n+1 \end{cases},$$

and consider the trajectory set $\mathscr{S} \equiv \tilde{\mathscr{S}}^+ \cup \mathscr{S}^-$, with the lower branch $\mathscr{S}^-$ defined as in Example 4. This modified trajectory set fails to be trajectorial complete, since $S^* \equiv (1, 2, 2, 2, \dots) \notin \mathscr{S}$, but satisfies (L)-a.e. by Corollary 3.13. Its trajectorial completion $\overline{\mathscr{S}} = \mathscr{S} \cup \{S^*\}$ satisfies $(L_{(S,j)})$ at every node by Corollary 3.14. Note that all potential losses at the node $(\tilde{S}^{+,1}, 1)$ by trading between time 0 and 1 can be recuperated in the model $\mathscr{S}$ by buying the stock (at all times $i \ge 1$) and waiting until the stock price eventually increases to 4. All conclusions of Example 4 are, therefore, easily seen to remain valid for this variant of the trajectory set. In particular, $\mathscr{S}$ has no martingale measure. This reasoning fails in the completed model $\overline{\mathscr{S}}$, because the stock price may remain constant after time 1. We will next explain, how the procedure of trajectorial completion changes pricing via the superhedging outer integral in this example. To distinguish between the model $\mathscr{S}$ and its trajectorial completion $\overline{\mathscr{S}}$, we write $\overline{\sigma}^{(\mathscr{S})}$ and $\overline{\sigma}^{(\overline{\mathscr{S}})}$ for the superhedging outer integrals with respect to the trajectory sets $\mathscr{S}$ and $\overline{\mathscr{S}}$, respectively.

We first consider the completed model $\overline{\mathscr{S}}$. Analogously to the proof of assertion d) of Example 1 in Appendix A.4, it is not hard to verify that a singleton $\{S\}$, $S \in \overline{\mathscr{S}}$, is an $\bar{I}$-null set, if and only if $S \in \tilde{\mathscr{S}}^+$ (and, thus, passes through the up-branch of an arbitrage node of type I). Hence, every $f \in Q$ is $\bar{I}$-a.e. equal to $f^{(1)} \equiv f\mathbf{1}_{\overline{\mathscr{S}} \setminus \tilde{\mathscr{S}}^+}$, which has maturity 1, because all trajectories outside $\tilde{\mathscr{S}}^+$ remain constant after time 1. We now fix a probability measure $\mathbf{P}$ on $2^{\overline{\mathscr{S}}}$ such that $\mathbf{P}(A) = 0$, if and only if $A \subseteq \tilde{\mathscr{S}}^+$. Assuming that f is bounded, we may combine (26) with Proposition A.2.1–b) to conclude that

$$\overline{\sigma}^{(\overline{\mathscr{S}})} f = \overline{\sigma}^{(\overline{\mathscr{S}})} f^{(1)} = \sup_{\mathbf{Q}} \int f^{(1)} d\mathbf{Q} = \sup_{\mathbf{Q}} \int f \, d\mathbf{Q}, \tag{27}$$

where $\mathbf{Q}$ runs over the set of all probability measures on $2^{\overline{\mathscr{S}}}$ satisfying, for every $n \in \mathbb{N}$,

$$\mathbf{Q}(\{\tilde{S}^{+,n}\}) = 0, \ \mathbf{Q}(\{S^{-,n}\}) > 0, \ \mathbf{Q}(\{S^*\}) > 0, \text{ and } \mathbf{Q}(\{S^*\}) - \sum_{n=1}^{\infty} \frac{\mathbf{Q}(\{S^{-,n}\})}{n^2} = 0.$$

Here, the last identity rephrases the property that (T_0, T_1) is a martingale with respect to $\mathbf{Q}$. The supremum on the right-hand side of (27) does not change, if the positivity conditions $\mathbf{Q}(\{S^*\}) > 0$ and $\mathbf{Q}(\{S^{-,n}\}) > 0$ for every $n \in \mathbb{N}$ are skipped (e.g., by applying Fatou's lemma). Then, noting that all martingale measures assign zero probability to $\tilde{\mathscr{S}}^+$ and recalling that the other trajectories remain constant after time 1, we observe that the classical probabilistic martingale property of $(T_j)_{j \ge 0}$ is equivalent to that of (T_0, T_1). Hence,

$$\overline{\sigma}^{(\overline{\mathscr{S}})} f = \sup_{\mathbf{Q} \in \overline{\mathscr{Q}}} \int f \, d\mathbf{Q},$$

where $\overline{\mathscr{Q}}$ denotes the set of all martingale measures on $\overline{\mathscr{S}}$. Thus, pricing via the superhedging outer integral is in line with the model-free superhedging duality of [4,5] in this example. For later use, we note that, by the previous considerations, $\mathbf{Q} \in \overline{\mathscr{Q}}$, if and only if

$$\mathbf{Q}(\tilde{\mathscr{S}}^+) = 0 \ \text{ and } \ \mathbf{Q}(\{S^*\}) - \sum_{n=1}^{\infty} \frac{Q(\{S^{-,n}\})}{n^2} = 0 \tag{28}$$

Coming back to the original trajectory set $\mathscr{S} = \tilde{\mathscr{S}}^+ \cup \mathscr{S}^-$, we recall that there is no martingale measure in this model. We first note that

$$\overline{\sigma}^{(\mathscr{S})} f = \limsup_{n \to \infty} f(S^{-,n}), \quad f \in Q. \tag{29}$$

This identity has been shown in [2] for the trajectory set in Example 4 and their proof can easily been adapted to the present trajectory set (the only difference being, how exactly the arbitrage in the respective up-branch of the two trajectory sets is exploited). The pricing rule (29) can be related to the martingale measures of the completed model $\overline{\mathscr{S}}$ by observing that

$$\overline{\sigma}^{(\mathscr{S})} f = \lim_{\varepsilon \to 0} \sup_{\mathbf{Q} \in \overline{\mathscr{Q}}_\varepsilon} \int f \, d\mathbf{Q}, \quad f \in Q \text{ bounded from below}, \tag{30}$$

where $\overline{\mathscr{Q}}_\varepsilon$ denotes the set of martingale measures $\mathbf{Q}$ for $\overline{\mathscr{S}}$ such that $\mathbf{Q}(\overline{\mathscr{S}} \setminus \mathscr{S}) \le \varepsilon$. To verify (30), we write $f^* = \limsup_{n \to \infty} f(S^{-,n})$ and first show that $f^* \le \liminf_{\varepsilon \to 0} \sup_{\mathbf{Q} \in \overline{\mathscr{Q}}_\varepsilon} \int f \, d\mathbf{Q}$. For a fixed $\nu \ge 2$, define a probability measure $\mathbf{Q}^{(\nu)}$ on $2^{\overline{\mathscr{S}}}$ via

$$\mathbf{Q}^{(\nu)}(\{S\}) = \begin{cases} 2/(\nu^2+3), & S = S^* \\ 1/(\nu^2+3), & S = S^{-,1} \\ \nu^2/(\nu^2+3), & S = S^{-,\nu} \\ 0, & S \in \overline{\mathscr{S}} \setminus \{S^*, S^{-,1}, S^{-,\nu}\} \end{cases}.$$

In view of (28), $\mathbf{Q}^{(\nu)} \in \overline{\mathscr{Q}}_\varepsilon$ for $\varepsilon \geq 2/(\nu^2+3)$ and

$$\int f\, d\mathbf{Q}^{(\nu)} = f(S^{-,\nu})\frac{\nu^2}{\nu^2+3} + f(S^{-,1})\frac{1}{\nu^2+3} + f(S^*)\frac{2}{\nu^2+3}.$$

Passing with ν to infinity along a subsequence $(\nu_k)_{k\in\mathbb{N}}$ such that $f(S^{-,\nu_k})$ converges to f^*, we obtain,

$$f^* = \lim_{k\to\infty} \int f\, d\mathbf{Q}^{(\nu_k)} \leq \liminf_{\varepsilon\to 0} \sup_{\mathbf{Q}\in\overline{\mathscr{Q}}_\varepsilon} \int f\, d\mathbf{Q}.$$

In order to finish the proof of (30), it remains to show that $f^* \geq \limsup_{\varepsilon\to 0} \sup_{\mathbf{Q}\in\overline{\mathscr{Q}}_\varepsilon} \int f\, d\mathbf{Q}$. To this end, we may assume $f^* < \infty$. Fixing some $\mathbf{Q} \in \overline{\mathscr{Q}}_\varepsilon$ and $N_0 \in \mathbb{N}$, we decompose

$$\begin{aligned}\int f\, d\mathbf{Q} - f^* &= \sum_{n\geq N_0} (f(S^{-,n}) - f^*)\mathbf{Q}(\{S^{-,n}\}) + \sum_{n=1}^{N_0-1} (f(S^{-,n}) - f^*)\mathbf{Q}(\{S^{-,n}\}) + (f(S^*) - f^*)\mathbf{Q}(\{S^*\}) \\ &\leq \sup_{n\geq N_0} (f(S^{-,n}) - f^*) + \sup_{n=1,\dots,N_0-1} |f(S^{-,n}) - f^*| \sum_{n=1}^{N_0-1} \mathbf{Q}(\{S^{-,n}\}) + |f(S^*) - f^*|\mathbf{Q}(\{S^*\}).\end{aligned}$$

Noting that $\mathbf{Q}(\{S^*\}) = \mathbf{Q}(\overline{\mathscr{S}} \setminus \mathscr{S}) \leq \varepsilon$ and, then, by (28), $\sum_{n=1}^{N_0-1} \mathbf{Q}(\{S^{-,n}\}) \leq \varepsilon(N_0-1)^2$, we obtain

$$\limsup_{\varepsilon\to 0} \sup_{\mathbf{Q}\in\overline{\mathscr{Q}}_\varepsilon} \int f\, d\mathbf{Q} - f^* \leq \sup_{n\geq N_0} (f(S^{-,n}) - f^*).$$

Taking the infimum over $N_0 \in \mathbb{N}$, we finally get $\limsup_{\varepsilon\to 0} \sup_{\mathbf{Q}\in\overline{\mathscr{Q}}_\varepsilon} \int f\, d\mathbf{Q} - f^* \leq 0$.

9. Discussion

We introduce a framework, motivated by financial considerations, where the fundamental elements are future price scenarios and trading opportunities. This setting naturally fits within the Leinert-König integration framework, which generalizes Lebesgue integration beyond lattice structures. A key analytic tool in constructing the conditional outer integral operators $\overline{\sigma}_j$ (interpreted as superhedging operators in finance) is property (L), which we show is closely tied to no-arbitrage conditions.

Building on this foundation, it is possible to extend classical probabilistic results concerning supermartingales, the latter emerge naturally through $\overline{\sigma}_j$. In particular, we establish Doob's supermartingale decomposition and pointwise convergence theorems, using entirely new proof techniques that rely solely on our framework, without invoking classical results or constructions. We also provide extensive discussions and examples highlighting the novelty of our results and their dependence on property (L).

Our approach reveals previously unnoticed aspects of null sets and connects to the game-theoretic probability framework of Shafer and Vovk [23]. We clarify how our technical hypotheses differ from theirs, leading to distinct assumptions, results, and proof techniques. This establishes a link between the Leinert-König [16,14] and Shafer-Vovk [23] approaches which also bridges our framework with imprecise probability theory given the known relations between the latter and Shafer-Vovk theory.

CRediT authorship contribution statement

C. Bender: Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **S.E. Ferrando:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **K. Gajewski:** Investigation, Formal analysis. **A.L. González:** Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

A.1. Partitions, arbitrage nodes, and null sets

In view of Proposition 3.3, the following Lemma shows that property $(L_{(S,j)})$ fails at type II arbitrage nodes (S,j).

Lemma A.1.1. *Given a trajectory set $\mathscr{S}$ consider a node (S,j), $j \geq 0$, then: If (S,j) is a type II arbitrage node, then*

$$\overline{\sigma}_j f(S) = -\infty \text{ for any } \; f \in Q.$$

Proof. We may consider the case when $\tilde{S}_{j+1} > S_j$ for all $\tilde{S} \in \mathscr{S}_{(S,j)}$. Take then, for all $m \geq 1$: $H_j^m(\tilde{S}) = 1$ and $H_i^m(\tilde{S}) = 0$ for all $i > j$, $V^m = 0$. Also, $H_i^0 = 0$ for all $i \geq j$; then, for any $V^0 \in \mathbb{R}$:

$$f(\tilde{S}) \leq V^0 + \infty = V^0 + \sum_{m\geq 1} H_j^m(\tilde{S})\, \Delta_j \tilde{S} \quad \text{holds for any } \; \tilde{S} \in \mathscr{S}_{(S,j)} \; \text{ and } \; f \in Q.$$

Thus, the claim follows. □

In Lemma A.1.3 below, it will be proved that trajectories passing through arbitrage nodes of type II form a null set.

Define for $j \geq 0$,

$$N_j^\circ \equiv \{S \in \mathscr{S} : (S,j) \text{ is an arbitrage node, and } \Delta_j S \neq 0\}, \quad N_k \equiv \bigcup_{j\geq k} N_j^\circ, \text{ for } \; k \geq 0, \;\text{ and } \quad N(S,j) \equiv N_j \cap \mathscr{S}_{(S,j)} \text{ for } j \geq 0,$$

and recall that the set $N(S,j)$ has already been introduced in (6). Notice that

$$N_0 = \mathscr{N} \equiv \mathscr{N}^{(I)} \cup \mathscr{N}^{(II)} = \{S \in \mathscr{S} : \exists \;\; j \geq 0 \;\; \text{s.t.}\, (S,j) \; \text{is an arbitrage node and} \;\; S_{j+1} \neq S_j\} \tag{31}$$

where the sets $\mathscr{N}^{(I)}, \mathscr{N}^{(II)}$ were introduced in the proof of Theorem 6.1.

It will be shown that $\mathscr{N}$ is a null set. Whenever $S \notin \mathscr{N}$ it follows that $S \notin N_j^\circ$ for any $j \geq 0$, therefore such node (S,j) is: flat, or up-down, or type I arbitrage node with $S_{j+1} = S_j$. On the other hand if (S,j) is a type II arbitrage node then $\mathscr{S}_{(S,j)} \subseteq N_j^\circ$. Moreover, it can be that $S \in N_j^\circ$, but (S,k) is arbitrage free for some $k > j$.

Definition A.1.2. Since for any $j \geq 0$, $\mathscr{S}$ is a disjoint union of $\mathscr{S}_{(S,j)}$, let Λ_j be an index set, such that for $\lambda \in \Lambda_j$ there exists $S^\lambda \in \mathscr{S}$ such that

$$\lambda \neq \lambda' \;\Rightarrow S^{\lambda'} \notin \mathscr{S}_{(S^\lambda,j)}, \quad \mathscr{S} = \bigcup_{\lambda\in\Lambda_j} \mathscr{S}_{(S^\lambda,j)}, \quad \text{and} \quad \text{if } (S^\lambda,j) \text{ is an arbitrage node then } |\Delta_j S^\lambda| > 0.$$

For $\Gamma \subseteq \Lambda_j$ define $H^\Gamma = (H_i^\Gamma)_{i\geq 0}$, $H_i^\Gamma : \mathscr{S} \to \mathbb{R}$ by

$$H_i^\Gamma \equiv 0 \;\text{ if }\; i \neq j \;; \quad H_j^\Gamma \equiv \mathbf{1}_{\mathscr{S}^\Gamma}, \;\text{ where }\; \mathscr{S}^\Gamma \equiv \bigcup_{\lambda\in\Gamma} \mathscr{S}_{(S^\lambda,j)}.$$

H^Γ is non-anticipative: Let $\tilde{S}_k = S_k$, $0 \leq k \leq i$. If $i \neq j$ then $H_i^\Gamma(\tilde{S}) = H_i^\Gamma(S) = 0$. For $i = j$, $S \in \mathscr{S}_{(S^\lambda,j)}$ iff $\tilde{S} \in \mathscr{S}_{(S^\lambda,j)}$ so $H_j^\Gamma(\tilde{S}) = H_j^\Gamma(S)$.

Lemma A.1.3. *Consider $j \geq 0$ and $0 \leq k \leq j$, then N_j°, thus also N_j, are conditionally null sets at any (S,k).*

Proof. Define $N_j^{\circ,+} \equiv \{S \in N_j^\circ : \Delta_j S > 0\}$ and $\Lambda_j^+ \equiv \{\lambda \in \Lambda_j : S^\lambda \in N_j^{\circ,+}\}$, and consider for $m \geq 1$, $f_m = \Pi_{k,j+1}^{0,H^{\Lambda_j^+}} \in \mathscr{E}_{(S,k)}^+$ for any $S \in \mathscr{S}$. Then

$$\mathbf{1}_{N_j^{\circ,+}} \leq \sum_{m\geq 1} f_m, \;\text{ which implies that } \|\mathbf{1}_{N_j^{\circ,+}}\|_k = 0.$$

In a similar way it is shown that $N_j^{\circ,-} \equiv \{S \in N_j^\circ : \Delta_j S < 0\}$ is a conditionally null set at any (S,k), consequently $N_j^\circ = N_j^{\circ,+} \cup N_j^{\circ,-}$ is also conditionally null. □

A.2. Basic properties of superhedging functional $\overline{\sigma}_j$

We next provide some basic properties of the conditional outer integral. Related results can be found for the non-conditional outer integral in [10] and for a variant of the conditional outer integral (without the lim inf) in [2].

Proposition A.2.1 *(Basic Properties). The following properties hold for $f, g \in Q$*

a) $\overline{\sigma}_j f(S) \le f(S)$ *if f is constant on $\mathscr{S}_{(S,j)}$. (Implies $\overline{\sigma}_j 0 \le 0$ and $\underline{\sigma}_j f \ge f$.)*
b) $\overline{\sigma}_j f(S) \le \overline{\sigma}_j g(S)$, *if $f \le g$ a.e. on $\mathscr{S}_{(S,j)}$.*
c) $\overline{\sigma}_j[f+g] \le \overline{\sigma}_j f + \overline{\sigma}_j g$.
d) *Let $f \in Q$, $g \in P$, and g is constant on $\mathscr{S}_{(S,j)}$ then $\overline{\sigma}_j(gf)(S) \le g\overline{\sigma}_j f(S)$.*
e) *Let $f \in P$ and $k \ge 0$ then $0 \le \overline{I}(\overline{I}_k f) \le \overline{I} f$. Therefore if f is a null function we get $\overline{I}_k f$ is a null function.*

Proof. The proofs are immediate but we do indicate the arguments for item d).

Let $c \ge 0$ be a constant such that $g(\hat{S}) = c$ for all $\hat{S} \in \mathscr{S}_{(S,j)}$. If $c = 0$, $\overline{\sigma}_j(gf)(S) = \overline{\sigma}_j(0)(S) \le g\overline{\sigma}_j 0(S)$ (already covered by item a). So assume $c > 0$. Let $gf(\tilde{S}) \le \sum_{m\ge 0} \liminf_{n\to\infty} \Pi^{V^m,H^m}_{j,n}(\tilde{S})$, $\tilde{S} \in \mathscr{S}_{(S,j)}$, with $\Pi^{V^0,H^0}_{j,n_0} \in \mathscr{E}_{(S,j)}$ and, for $m \ge 1$, $\Pi^{V^m,H^m}_{j,n} \in \mathscr{E}^+_{(S,j)}$ for all $n \ge 0$. For each $\tilde{S} \in \mathscr{S}_{(S,j)}$ and $m \ge 0$ define

$$U^m(\tilde{S}) = \frac{V^m}{g(S)}, \text{ and } G^m_i(\tilde{S}) = \frac{H^m_i(\tilde{S})}{g(S)}, \text{ for } i \ge j.$$

It follows that $f(\tilde{S}) \le \sum_{m\ge 0} \liminf_{n\to\infty} \Pi^{U^m,G^m}_{j,n}(\tilde{S})$, $\tilde{S} \in \mathscr{S}_{(S,j)}$, with $\Pi^{U^0,G^0}_{j,n_0} \in \mathscr{E}_{(S,j)}$, and for $m \ge 1$, $\Pi^{U^m,G^m}_{j,n} \in \mathscr{E}^+_{(S,j)}$ for all $n \ge 0$. Thus

$$\overline{\sigma}_j f(S) \le \frac{\overline{\sigma}_j[gf](S)}{g(S)}.$$

Notice that one actually obtains $\overline{\sigma}_j(gf)(S) = g\overline{\sigma}_j f(S)$ if $g = c > 0$ on $\mathscr{S}_{(S,j)}$.

We also note that the proof of item e) is analogous to the one of Proposition 5.5. □

Corollary A.2.2. *Suppose $f, g \in Q$. If $f \le g$ a.e., then $\overline{\sigma}_j f \le \overline{\sigma}_j g$ a.e.*

Proof. Note that, for every $S \in \mathscr{S}$,

$$\overline{\sigma}_j f(S) \le \overline{\sigma}_j(\max\{f,g\})(S) = \overline{\sigma}_j(g + (f-g)_+)(S) \le \overline{\sigma}_j g(S) + \overline{I}_j(f-g)_+(S),$$

making use of Remark 2.11. As $(f-g)_+$ is a null function, we conclude by Proposition A.2.1 that $\overline{I}_j(f-g)_+$ is a null function. Hence, $\overline{\sigma}_j f \le \overline{\sigma}_j g$ a.e. □

Lemma A.2.3. *Let $f \in Q$, (S,j) a fixed node and $k \ge j$. If f is constant on $\mathscr{S}_{(S,j)}$ then $\overline{\sigma}_k f(S) \le f(S) \le \underline{\sigma}_k f(S)$. Moreover once $(L_{(S,k)})$ holds then*

1. *If f is constant on $\mathscr{S}_{(S,j)}$ then $\underline{\sigma}_k f(S) = f(S) = \overline{\sigma}_k f(S)$.*
2. *For a general $f \in Q$, $\overline{\sigma}_j f$ is constant on $\mathscr{S}_{(S,j)}$; hence: $\overline{\sigma}_k[\overline{\sigma}_j f](S) = \overline{\sigma}_j f(S) = \underline{\sigma}_k[\overline{\sigma}_j f](S)$ and $\overline{\sigma}_k[\underline{\sigma}_j f](S) = \underline{\sigma}_j f(S) = \underline{\sigma}_k[\underline{\sigma}_j f](S)$.*

Proof. If f is constant on $\mathscr{S}_{(S,j)}$, it is also constant on $\mathscr{S}_{(S,k)} \subseteq \mathscr{S}_{(S,j)}$, then $f \in \mathscr{E}_{(S,k)}$, so by Definition 2.10, $\overline{\sigma}_k f(S) \le f(S)$.

If, furthermore, also $(L_{(S,k)})$ holds, we have $\underline{\sigma}_k f(S) = f(S) = \overline{\sigma}_k f(S)$ by applying item (4) of Proposition 3.3 to $\Pi^{V,H}_{j,j}$ with $V = f\mathbf{1}_{\mathscr{S}_{(S,j)}}$ and $H = 0$. □

Corollary A.2.4. *Let $f, g \in Q$ and consider a fixed $S \in \mathscr{S}$. If for $j \ge 0$ property $(L_{(S,j)})$ holds and $\overline{\sigma}_j f(S) - \underline{\sigma}_j f(S) = 0 = \overline{\sigma}_j g(S) - \underline{\sigma}_j g(S)$, then all the involved quantities are finite and*

$$(a) \quad \overline{\sigma}_j(f+g)(S) = \overline{\sigma}_j f(S) + \overline{\sigma}_j g(S) = \underline{\sigma}_j f(S) + \underline{\sigma}_j g(S) = \underline{\sigma}_j(f+g)(S).$$

$$(b) \quad \overline{\sigma}_j(cf)(S) = c\overline{\sigma}_j f(S) = c\underline{\sigma}_j f(S) = \underline{\sigma}_j(cf)(S) \quad \forall c \in \mathbb{R}.$$

Proof. The finiteness claims follow from our conventions in the first paragraph of Section 2.2. We then see that the hypotheses imply that $\overline{\sigma}_j f(S) = \underline{\sigma}_j f(S)$ and $\overline{\sigma}_j g(S) = \underline{\sigma}_j g(S)$.

(a) holds from

$$\overline{\sigma}_j f(S) + \overline{\sigma}_j g(S) = \underline{\sigma}_j f(S) + \underline{\sigma}_j g(S) \le \underline{\sigma}_j[f+g](S) \le \overline{\sigma}_j[f+g](S) \le \overline{\sigma}_j f(S) + \overline{\sigma}_j g(S),$$

where we have relied on Proposition 3.3. For (b), if $c = 0$ or $c = -1$ the result is clear. For $c > 0$ it follows from the proof of item d) of Proposition A.2.1, from where, if $c < 0$

$$\overline{\sigma}_j(cf)(S) = \overline{\sigma}_j(-c(-f))(S) = -c\overline{\sigma}_j(-f)(S) = c\underline{\sigma}_j f(S). \quad \square$$

A.3. Proofs for Section 3

Proof of Proposition 3.3. The proof follows the lines of [2], where the analogous result is shown in a related setting (without the limit inferior in time in the definition of the superhedging operators).

From item 1. and item c) of Proposition A.2.1 it follows item 2., since $0 \le \overline{\sigma}_j f(S) + \overline{\sigma}_j(-f)(S)$.

Assumed item 2., it follows that $0 \le \underline{\sigma}_j 0(S) \le \overline{\sigma}_j 0(S) \le 0$, first and last inequalities from item a) of Proposition A.2.1, so item 1. holds. From here onwards we let $f = \Pi^{V,H}_{j,n_f}$.

Item 3. follows from item 1. as follows. Let $h_m = \liminf_{n\to\infty} \Pi^{V^m,H^m}_{j,n}$, $\Pi^{V^m,H^m}_{j,n} \in \mathscr{E}^+_{(S,j)}$ for every $n \ge j$ and $m \ge 1$, such that $f \equiv \Pi^{V,H}_{j,n_f} \le \sum_{m\ge1} h_m$ on $\mathscr{S}_{(S,j)}$. Then $0 \le -\Pi^{V,H}_{j,n_f} + \sum_{m\ge1} h_m$, thus (taking $f_0 \equiv -\Pi^{V,H}_{j,n_f}$, and $f_m = h_m$ for $m \ge 1$) by Definition 2.10, $0 = \overline{\sigma}_j(0) \le -V + \sum_{m\ge1} V^m$, which leads to $V \le \sum_{m\ge1} V^m$ as required.

Assumed item 3., let $f_0 = \Pi^{V^0,H^0}_{j,n_0} \in \mathscr{E}_{(S,j)}$ and $f_m = \liminf_{n\to\infty} \Pi^{V^m,H^m}_{j,n}$, $\Pi^{V^m,H^m}_{j,n} \in \mathscr{E}^+_{(S,j)}$ $\forall\, n \ge j$ and $m \ge 1$, such that $f \le \sum_{m\ge0} f_m$. Then $f - f_0 \le \sum_{m\ge1} f_m$ with $f - f_0 \in \mathscr{E}_{(S,j)}$, so $V(S) - V_0 \le \sum_{m\ge0} V^m$, and $V(S) \le \overline{\sigma}_j f(S)$. Since by Definition 2.10, $\overline{\sigma}_j f(S) \le V(S)$, item 4. holds, having in mind that $\underline{\sigma}_j f(S) = -\overline{\sigma}_j[-f](S) = V(S)$.

Finally, it is clear that item 1. follows from item 4. $\square$

Proof of Proposition 3.10. If (S,j) is an arbitrage node of type II, then, for every $\tilde{S} \in \mathscr{S}_{(S,j)}$, $(\tilde{S},j)$ is an arbitrage node and $\tilde{S}_{j+1} \neq \tilde{S}_j$. Hence, $\tilde{S} \in N(S,j)$ for every $\tilde{S} \in \mathscr{S}_{(S,j)}$, i.e., the node (S,j) is bad.

If (S,j) is a bad node, then, by Lemma A.1.3, $\mathscr{S}_{(S,j)} = N(S,j)$ is a conditional null set at (S,j). Therefore, by Remark 2.11, $\overline{\sigma}_j 1(S) \le \bar{I}_j(\mathbf{1}_{\mathscr{S}_{(S,j)}})(S) = 0$. Thus, property 4. in Proposition 3.3 fails, resulting in the failure of $(L_{(S,j)})$.

If $S \in \mathscr{N}^{\text{bad}}$, then, (S,j) is bad for some $j \ge 0$, and, thus, there is a $k \ge j$ such that (S,k) is an arbitrage node and $S_{k+1} \neq S_k$. Therefore, $\mathscr{N}^{\text{bad}} \subset \mathscr{N}$, where the set $\mathscr{N}$, introduced in (31), is a null set by Lemma A.1.3. $\square$

Proof of Theorem 3.12. 1. The first implication is obvious, while the second one is an immediate consequence of the fact that any good node is not an arbitrage node of type II, see Proposition 3.10.

2. Assuming (H$_{II}$) we verify (H$_{II}$') as follows. Consider any up-down node (S,j). If $(\hat{S}, j+1)$ is not a type II arbitrage node for every $\hat{S} \in \mathscr{S}_{(S,j)}$, then we find $S^1, S^2 \in \mathscr{S}_{(S,j)}$ such that $S^1_{j+1} - S_j > 0$ and $S^2_{j+1} - S_j < 0$. We may choose $S^{\varepsilon,1} = S^1$ and $S^{\varepsilon,2} = S^2$ for every $\varepsilon > 0$. Otherwise, (H$_{II}$) directly applies.

3. Since arbitrage nodes of type II are always bad (Proposition 3.10), we only need to show that all other nodes are good. To this end, we fix a node (S,j) which is not arbitrage of type II and construct a trajectory $\tilde{S} \in \mathscr{S}_{(S,j)} \setminus N(S,j)$. We set $S^n = S$ for $n \le j$ and inductively construct S^n for $n > j$ in the following way, which guarantees that (S^n, n) is not a type II arbitrage node: if (S^n,n) is flat or an arbitrage node of type I, we choose $S^{n+1} \in \mathscr{S}_{(S^n,n)}$ such that $S^{n+1}_{n+1} = S^n_n$ and note that $(S^{n+1}, n+1)$ is not an arbitrage node of type II by (H$_{II}$). If (S^n,n) is an up-down node, then by (H$_{II}$) again, we find some $S^{n+1} \in \mathscr{S}_{(S^n,n)}$ such that $(S^{n+1}, n+1)$ is not an arbitrage node of type II. Since (S^n,n) is not an arbitrage node of type II for every $n \ge j$, condition (TC$_{II}$) implies $\lim_{n\to\infty} S^n \in \mathscr{S}$. By construction, $\tilde{S} \equiv \lim_{n\to\infty} S^n = (S^i_i)_{i\ge0} \in \mathscr{S}_{(S,j)} \setminus N(S,j)$.

4. In view of Proposition 3.10, we only need to show that $(L_{(S,j)})$ holds at every good node. Our proof extends a related argument in [10] beyond models that have up-down nodes only.

Fix a good node (S,j). Let $f = \Pi^{V^0,H^0}_{j,n_f} \in \mathscr{E}_{(S,j)}$ and $f_m = \liminf_{n\to\infty} \Pi^{V^m,H^m}_{j,n}$ with $\Pi^{V^m,H^m}_{j,n} \in \mathscr{E}^+_{(S,j)}$ for all $n \ge j$ and $m \ge 1$ such that

$$f \le \sum_{m\ge1} f_m \text{ on } \mathscr{S}_{(S,j)}.$$

We need to show that

$$V^0 \le \sum_{m\ge1} V^m \equiv V,$$

(and, hence, can and will assume that $V < \infty$). Recall that

$$N(S,j) = \{\tilde{S} \in \mathscr{S}_{(S,j)} |\, (\tilde{S},k) \text{ is an arbitrage node and } \tilde{S}_{k+1} \neq \tilde{S}_k \text{ for some } k \ge j\}.$$

If $\tilde{S} \in \mathscr{S}_{(S,j)} \setminus N(S,j)$, then, for every $k \ge j$, $(\tilde{S},k)$ is an up-down node or $\tilde{S}_{k+1} = \tilde{S}_k$. Hence, by the Aggregation Lemma 6.4, for every $\tilde{S} \in \mathscr{S}_{(S,j)} \setminus N(S,j)$ and $n \ge j$

$$\sum_{m\ge1} \Pi^{V^m,H^m}_{j,n}(\tilde{S}) = V + \sum_{i=j}^{n-1} H_i(\tilde{S})\Delta_i \tilde{S},$$

for the non-anticipative sequence

$$H_i(\hat{S}) = \begin{cases} \sum_{m=1}^{\infty} H_i^m(\hat{S}), & \text{if convergent in } \mathbb{R}, \\ 0, & \text{otherwise,} \end{cases} \qquad i \geq j,\ \hat{S} \in \mathscr{S}_{(S,j)}.$$

Then, by Fatou's lemma, for every $\hat{S} \in \mathscr{S}_{(S,j)}$

$$f \leq \liminf_{n\to\infty} \sum_{m\geq 1} \Pi_{j,n}^{V^m,H^m}(\hat{S}) \leq V + \liminf_{n\to\infty} \sum_{i=j}^{n-1} H_i(\hat{S})\Delta_i\hat{S} + \infty \mathbf{1}_{N(S,j)}(\hat{S}),$$

which we may rearrange into

$$V^0 \leq V + \liminf_{n\to\infty} \sum_{i=j}^{n-1} \tilde{H}_i(\hat{S})\Delta_i\hat{S} + \infty \mathbf{1}_{N(S,j)}(\hat{S})$$

where $\tilde{H}_i = H_i - H_i^0$ for $i < n_f$ and $\tilde{H}_i = H_i$ otherwise.

Thus, it is enough to show the following: for every $\delta > 0$ there is an $\tilde{S} \in \mathscr{S}_{(S,j)} \setminus N(S,j)$ such that

$$\liminf_{n\to\infty} \sum_{i=j}^{n-1} \tilde{H}_i(\tilde{S})\Delta_i\tilde{S} \leq \delta. \tag{32}$$

To this end, we construct a sequence (S^n) in $\mathscr{S}$ as follows: $S^n = S$ for $n \leq j$ and, inductively for $n > j$ in the following way, which guarantees that (S^n, n) is a good node for every $n \geq j$. Assume S^n has already been constructed for some $n \geq j$ and (S^n, n) is a good node.

If (S^n, n) is a flat node or an arbitrage node of type I, then choose $S^{n+1} \in \mathscr{S}_{(S^n,n)}$ such that $S^{n+1}_{n+1} = S^n_n$. We argue that $(S^{n+1}, n+1)$ is good in the type I case (the flat case being similar and easier). Suppose to the contrary that $(S^{n+1}, n+1)$ is bad. Then, for every $\hat{S} \in \mathscr{S}_{(S^{n+1},n+1)} \subseteq \mathscr{S}_{(S^n,n)}$ there is an $i \geq n+1$ such that $(\hat{S}, i)$ is an arbitrage node and $\hat{S}_{i+1} \neq \hat{S}_i$ – hence, $\hat{S} \in N(S^n, n)$. Moreover, any $\hat{S} \in \mathscr{S}_{(S^n,n)} \setminus \mathscr{S}_{(S^{n+1},n+1)}$ belongs to $N(S^n, n)$ because (S^n, n) is a type I arbitrage node. Thus, $\mathscr{S}_{(S^n,n)} = N(S, n)$ – a contradiction.

If (S^n, n) is an up-down node, then, by ($\mathrm{H}_{\mathrm{bad}}$), one can choose a sufficiently small $\varepsilon > 0$ and $S^{n+1} \in \mathscr{S}_{(S^n,n)}$ such that

$$\tilde{H}_n(S^n)(S^{n+1}_{n+1} - S^n_n) \leq |\tilde{H}_n(S^n)|\varepsilon \leq \delta 2^{-(n+1)} \tag{33}$$

and $(S^{n+1}, n+1)$ is a good node.

Note that (S^n, n) cannot be an arbitrage node of type II, because it is good by the inductive hypothesis. Hence, the construction of S^{n+1} is finished.

By ($\mathrm{TC}_{\mathrm{bad}}$), $\tilde{S} \equiv \lim_{n\to\infty} S^n = (S^i_i)_{i\in\mathbb{N}_0} \in \mathscr{S}$. Then, by construction, $\tilde{S} \in \mathscr{S}_{(S,j)}$ and either $(\tilde{S}, n) = (S^n, n)$ is an up-down node (and then (33) holds) or $\tilde{S}_{n+1} = S^{n+1}_{n+1} = S^n_n = \tilde{S}_n$, whenever $n \geq j$. Hence, by construction, $\tilde{S} \notin N(S,j)$ and

$$\liminf_{n\to\infty} \sum_{i=j}^{n-1} \tilde{H}_i(\tilde{S})\Delta_i\tilde{S} \leq \delta \sum_{i=j}^{\infty} 2^{-(i+1)} \leq \delta,$$

which establishes (32). Consequently, $(L_{(S,j)})$ holds. □

A.4. Proofs for Section 4

Proof of Lemma 4.1. Notice that

$$\begin{aligned} &\omega \in \Omega \Leftrightarrow \forall n \in \mathbb{N}\ \exists S^n \in \mathscr{S} : (s_0, \omega_1, \ldots, \omega_n) = (S^n_0, \ldots, S^n_n) \\ \Leftrightarrow\ &\exists (S^n)_{n\in\mathbb{N}} \subseteq \mathscr{S} : \quad [(\forall n \in \mathbb{N} : S^n_i = S^{n+1}_i,\ i = 0, \ldots, n) \wedge \lim_{n\to\infty} S^n = (s_0, \omega_1, \omega_2, \ldots)] \Leftrightarrow (s_0, \omega_1, \omega_2, \ldots) \in \overline{\mathscr{S}}. \quad \square \end{aligned}$$

Proof of Lemma 4.3. *Case 1:* The node associated to $s = (s_0, \ldots, s_j) \in \mathbb{S}$ is an up-down-node:

then, there are $y_1, y_2 \in \mathscr{Y}_s$ such that $y_1 < s_j < y_2$. If $|h| = +\infty$, then either $h(y_2 - s_j) = -\infty$ or $h(y_1 - s_j) = -\infty$, which violates the condition $h(y - s_j) > -\infty$ for every $y \in \mathscr{Y}_s$. If $|h| < +\infty$, then, the condition $h(y - s_j) > -\infty$ for every $y \in \mathscr{Y}_s$ is automatically satisfied. Hence,

$$h \in (-\infty, +\infty) \ \Leftrightarrow\ h(y - s_j) > -\infty \text{ for every } y \in \mathscr{Y}_s.$$

Case 2: The node associated to $s = (s_0, \ldots, s_j) \in \mathbb{S}$ is flat:

then $\mathscr{Y}_s = \{s_j\}$ and, hence, $h(y - s_j) = h \cdot 0 = 0$ for every $y \in \mathscr{Y}_s$ and $h \in [-\infty, +\infty]$.

Case 3: The node associated to $s = (s_0, \ldots, s_j) \in \mathbb{S}$ is an arbitrage node of type I:

we assume that the arbitrage node is of up-type, i.e., $s_j \in \mathscr{Y}_s$ and $y > s_j$ for every $y \in \mathscr{Y}_s \setminus \{s_j\}$. Take $h^* = +\infty$. Then, for every $y \in \mathscr{Y}_s$ and $h \in [-\infty, +\infty]$

$$h^*(y-s_j)=\left\{\begin{matrix} 0, & y=s_j \\ +\infty, & y\neq s_j \end{matrix}\right\} \geq h(y-s_j).$$

Hence,

$$\mathscr{G}_s=\{\mathfrak{f}_s:\mathscr{Y}_s\to[-\infty,\infty]\,:\,\forall y\in\mathscr{Y}_s:\ h^*\cdot(y-s_j)-\mathfrak{f}_s(y)\geq 0\}=\{\mathfrak{f}_s:\mathscr{Y}_s\to[-\infty,\infty]\,:\,\mathfrak{f}_s(s_j)\leq 0\}.$$

If the arbitrage node is of down type, the same argument works with $h^*=-\infty$.

Case 4: The node associated to $s=(s_0,\dots,s_j)\in\mathbb{S}$ is an arbitrage node of type II:

again, we only spell out the proof for an up-type arbitrage node, i.e., $y>s_j$ for every $y\in\mathscr{Y}_s$. Taking $h^*=+\infty$ again, we obtain $h^*(y-s_j)=+\infty$ for every $y\in\mathscr{Y}_s$, which completes the proof. $\square$

Proof of Items a)-d) in Example 1. Write $\mathscr{S}^+=\{S^{+,n}\,:\,n\in\mathbb{N}\}$ and note that all trajectories in $\mathscr{S}^+$ pass through an arbitrage node of type II at time $j=1$.

a) Fix $f:\mathscr{S}\to(-\infty,+\infty)$. Then, one can easily check that $f=\Pi^{V^f,H^f}_{n^f}$ for $V^f=f(S^0)$, $n^f=3$, $H_0=(f(S^0)-f(S^{-,0}))/2$,

$$H_1(S)=\begin{cases}2\left(f(S^0)(1+\frac{n}{2})-f(S^{-,0})\frac{n}{2}-f(S^{+,n})\right), & S=S^{+,n}\\ 0, & S=S^0,\\ f(S^{-,0})-f(S^{-,-}), & S\in\{S^{-,0},S^{-,-},S^{-,+}\}\end{cases},$$

and $H_2(S)=(2f(S^{-,0})-f(S^{-,-})-f(S^{-,+}))\mathbf{1}_{\{S^{-,+}\}}(S)$. Thus, $\overline{\sigma}(f)=V^f=f(S^0)$ by Proposition 3.3.

b) Suppose $\mathbf{Q}'$ is a martingale measure (see Remark 2.14). As in Example 4, we conclude that $\mathbf{Q}'(\mathscr{S}^+)=0$, because all trajectories in $\mathscr{S}^+$ pass through an arbitrage node of type II. This in turn implies $T_1\leq T_0$ $\mathbf{Q}'$-a.s. and the classical martingale property then results into $T_1=T_0$ $\mathbf{Q}'$-a.s. This shows that $\mathbf{Q}'$ must be the Dirac mass on S^0. Conversely, if $\mathbf{Q}'$ is the Dirac mass on S^0, then $T_j=T_0$ $\mathbf{Q}'$-almost surely for every $j\geq 0$, which implies that $(T_j)_{j\geq 0}$ is a classical martingale under $\mathbf{Q}'$.

c) Let $f:\mathscr{S}\to(-\infty,+\infty)$. We define the non-anticipative sequence $(g^*_j)_{j\geq 0}$ via

$$g^*_0(S)=g^*_1(S)=\max\{f(S^{-,0}),f(S^0)\},\quad S\in\mathscr{S}$$

$$g^*_2(S)=\begin{cases}+\infty, & S\in\mathscr{S}^+\\ g^*_1(S)+(g^*_1(S)-f(S^{-,-}))(S_2-S_1), & S\notin\mathscr{S}^+\end{cases},\quad g^*_j(S)=\begin{cases}+\infty, & S\in\mathscr{S}^+\cup\{S^{-,+}\}\\ g^*_2(S), & S\in\{S^0,S^{-,-},S^{-,0}\}\end{cases},\quad j\geq 3.$$

Noting that $(S,1)$ is an arbitrage node of type II for $S\in\mathscr{S}^+$ and $(S^{+,-},2)$ is an arbitrage node of type II (and that all other nodes are either up-down or flat), one can easily check by Lemma 4.3 that (g^*_j) is a $\mathscr{G}$-supermartingale sequence. Moreover, (g^*_j) obviously is bounded from below (uniformly in j and S). Since $g^*_j(S)\geq f(S)$ for every $j\geq 3$ and $S\in\mathscr{S}$, we conclude that $\overline{\mathbb{E}}(f)\leq g^*_0=\max\{f(S^{-,0}),f(S^0)\}$.

For the converse inequality suppose that $(g_j)_{j\geq 0}$ is any $\mathscr{G}$-supermartingale sequence, which is bounded from below and satisfies $\liminf_{j\to\infty}g_j\geq f$ on $\mathscr{S}$. Since the nodes (S^0,j) are flat for every $j\geq 1$ and the nodes $(S^{-,0},j)$ are flat for every $j\geq 2$, the sequences $(g_j(S^0))_{j\geq 1}$ and $(g_j(S^{-,0}))_{j\geq 2}$ are nonincreasing by Lemma 4.3 and the $\mathscr{G}$-supermartingale property. Therefore,

$$g_1(S^0)\geq f(S^0),\quad g_2(S^{-,0})\geq f(S^{-,0}). \tag{34}$$

The $\mathscr{G}$-supermartingale property at the node $(S^{-,0},1)$ implies (by Lemma 4.3) that

$$g_1(S^{-,0})\geq\inf\{\alpha\in\mathbb{R}\,:\,\exists\,h\in\mathbb{R}\,\forall\,S\in\{S^{-,-},S^{-,0},S^{-,+}\}\,:\,g_2(S)-\alpha\leq h(S_2-2)\}.$$

Since $S^{-,0}_2=2$, we observe that $g_1(S^{-,0})\geq g_2(S^{-,0})$, which, in view of (34), yields

$$g_1(S^0)\geq f(S^0),\quad g_1(S^{-,0})\geq f(S^{-,0}). \tag{35}$$

Now, the $\mathscr{G}$-supermartingale property at the initial node $(S,0)$ implies (by Lemma 4.3, again) that

$$g_0\geq\inf\{\alpha\in\mathbb{R}\,:\,\exists\,h\in\mathbb{R}\,\forall\,S\in\mathscr{S}\,:\,g_1(S)-\alpha\leq h(S_1-S_0)\}.$$

Note that for every $\alpha\in\mathbb{R}$ and $h<0$, $\inf_{n\in\mathbb{N}}(\alpha+h(S^{+,n}_1-S^{+,n}_0))=-\infty$. Hence, the requirement that g_1 is bounded from below leads to

$$g_0\geq\inf\{\alpha\in\mathbb{R}\,:\,\exists\,h\geq 0\,\forall\,S\in\mathscr{S}\,:\,g_1(S)-\alpha\leq h(S_1-S_0)\}.$$

Then, by (35),

$$g_0\geq\inf\{\alpha\in\mathbb{R}\,:\,\exists\,h\geq 0\,\forall\,S\in\{S^0,S^{-,0}\}\,:\,f(S)-\alpha\leq h(S_1-S_0)\}.$$

Noting that $h(S_1 - S_0) \le 0$ for every $h \ge 0$ and $S \in \{S^0, S^{-,0}\}$, we conclude that $g_0 \ge \max\{f(S^{-,0}), f(S^0)\}$. Since g was arbitrary, this in turn implies $\overline{\mathbb{E}}(f) \ge \max\{f(S^{-,0}), f(S^0)\}$.

d) Fix some $f : \mathscr{S} \to [0,\infty)$. We first prove $\overline{I}(f) \ge \max\{\frac{1}{2}f(S^{-,-}), f(S^{-,0}), f(S^0)\}$ and, hence, assume that $\overline{I}f < \infty$. Suppose we are given portfolios $(V^m, H^m)_{m\ge 1}$ such that $\Pi^{V^m,H^m}_{0,n}$ is non-negative for every $m \ge 1$ and $n \ge 0$, $v_0 \equiv \sum_{m=1}^{\infty} V_m < \infty$ (this is possible as $\overline{I}f < \infty$) and $f \le \sum_{m=1}^{\infty} \liminf_{n\to\infty} \Pi^{V^m,H^m}$ on $\mathscr{S}$. Applying the aggregation lemma (Lemma 6.4) to the initial node $(S, 0)$ and to the node $(S^{-,0}, 1)$, we find real numbers $h_0, h_1 \in \mathbb{R}$ such that

$$v_0 \ge f(S^0),\ v_0 - 2h_0 \ge f(S^{-,0}),\ v_0 - 2h_0 - h_1 \ge f(S^{-,-}),\ v_0 - 2h_0 + h_1 \ge 0.$$

Here, the first three inequalities arise from superhedging on the trajectories S^0, $S^{-,0}$, and $S^{-,-}$ (noting that these trajectories become constant after times $j = 0$, $j = 1$, and $j = 2$, respectively). The last inequality is implied by the nonnegativity of the portfolio wealth at time $j = 2$ on the trajectory $S^{-,+}$. Adding the last two inequalities, we get $v_0 - 2h_0 \ge \frac{1}{2}f(S^{-,-})$. Now the nonnegativity of the portfolio wealth at time $j = 1$ on the trajectories $S^{+,n}$, $n \in \mathbb{N}$, implies $h_0 \ge 0$. Hence, $v_0 \ge \max\{\frac{1}{2}f(S^{-,-}), f(S^{-,0}), f(S^0)\}$. By passing to the infimum in $v_0 \ge 0$, we obtain $\overline{I}(f) \ge \max\{\frac{1}{2}f(S^{-,-}), f(S^{-,0}), f(S^0)\}$.

For the converse inequality, consider the portfolio (V, H) defined via $V = \max\{\frac{1}{2}f(S^{-,-}), f(S^{-,0}), f(S^0)\}$ and

$$H_j(S) = \begin{cases} 0, & j \ne 1 \text{ or } S \notin \mathscr{S}_{(S^{-,0},1)} \\ -\frac{1}{2}f(S^{-,-}), & j = 1 \text{ and } S \in \mathscr{S}_{(S^{-,0},1)}. \end{cases}$$

Then, one easily observes that $\Pi^{V,H}_{0,n}(S) \ge 0$ for every $n \ge 0$ and $S \in \mathscr{S}$ and that $\liminf_{n\to\infty} \Pi^{V,H}_{0,n} \ge f\mathbf{1}_{\{S^0,S^{-,0},S^{-,-}\}}$ on $\mathscr{S}$. Noting that all trajectories $S \in \mathscr{S} \setminus \{S^0, S^{-,0}, S^{-,-}\}$ pass through an arbitrage node of type II and, thus, $\mathscr{S} \setminus \{S^0, S^{-,0}, S^{-,-}\}$ is an $\overline{I}$-null set by Lemma A.1.3, we conclude from Proposition 2.9–1. that

$$\overline{I}(f) \le \overline{I}(f\mathbf{1}_{\{S^0,S^{-,0},S^{-,-}\}}) + \overline{I}(f\mathbf{1}_{\mathscr{S}\setminus\{S^0,S^{-,0},S^{-,-}\}}) = I(f\mathbf{1}_{\{S^0,S^{-,0},S^{-,-}\}}) \le V = \max\{\frac{1}{2}f(S^{-,-}), f(S^{-,0}), f(S^0)\}. \quad \square$$

Proof of Proposition 4.4. We fix some time index $j \ge 0$ and some trajectory $S \in \mathscr{S}$. We may and do assume without loss of generality that $g_j(S) < +\infty$, since otherwise the inequality $\overline{\sigma}_j g_{j+1}(S) \le g_j(S)$ is obviously satisfied. We distinguish the following cases:

(S, j) is an arbitrage node of type II: then, $\overline{\sigma}_j g_{j+1}(S) = -\infty$ by Lemma A.1.1; and the inequality $\overline{\sigma}_j g_{j+1}(S) \le g_j(S)$ trivially holds.

(S, j) is an arbitrage node of type I or flat: we consider the case that $\tilde{S}_{j+1} \ge S_j$ for every $\tilde{S} \in \mathscr{S}_{(S,j)}$ and fix a trajectory $\bar{S} \in \mathscr{S}_{(S,j)}$ such that $\bar{S}_{j+1} = S_j$. Then, applying the same portfolio as in the proof of Lemma A.1.1, we obtain

$$\sum_{m=0}^{\infty} \liminf_{n\to\infty}\ \Pi^{V^m,H^m}_{j,n}(\tilde{S}) = V^0 + \infty\,\mathbf{1}_{\tilde{S}_{j+1}\ne S_j},\ \text{ for every } \tilde{S} \in \mathscr{S}_{(S,j)},$$

which is bigger than or equal to $g_{j+1}(\tilde{S})$ for every $\tilde{S} \in \mathscr{S}_{(S,j)}$, if and only if $V^0 \ge g_{j+1}(\bar{S})$. Passing to the infimum in $V^0 \in \mathbb{R}$, we obtain that $\overline{\sigma}_j g_{j+1}(S) \le g_{j+1}(\bar{S})$ (which, in fact, can easily be seen to be an equality). Moreover, by the $\mathscr{G}$-supermartingale property and Lemma 4.3, we observe that

$$g_{j+1}(\bar{S}) = \inf\{\alpha \in \mathbb{R} :\ g_{j+1}(\bar{S}) - \alpha \le 0\} \le g_j(S).$$

Combining both inequalities, we arrive at $\overline{\sigma}_j g_{j+1}(S) \le g_j(S)$.

(S, j) is an up-down node: if $g_j(S) \in \mathbb{R}$, then we conclude by the $\mathscr{G}$-supermartingale property and Lemma 4.3: for every $\varepsilon > 0$ there is an $h \in \mathbb{R}$ such that

$$g_{j+1}(\tilde{S}) \le (g_j(S) + \varepsilon) + h(\tilde{S}_{j+1} - S_j) = (g_j(\tilde{S}) + \varepsilon) + h(\tilde{S}_{j+1} - \tilde{S}_j)$$

for every $\tilde{S} \in \mathscr{S}_{(S,j)}$. Defining $V^0 = g_j(S) + \varepsilon$, $V^m = 0$ for $m \ge 1$, $H^0_j(S) = h$ and $H^m_i \equiv 0$ for ($m \ge 1$ and $i \ge j$) and for ($m = 0$ and $i > j$), we observe that $\overline{\sigma}_j g_{j+1}(S) \le g_j(S)$. If $g_j(S) = -\infty$, we may replace $g_j(S) + \varepsilon$ by $-\varepsilon^{-1}$ in the above argument and obtain $\overline{\sigma}_j g_{j+1}(S) = -\infty = g_j(S)$. $\square$

Data availability

No data was used for the research described in the article.

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