VEGA INSTITUTE
Science Promotion Foundation

Optimal execution problem in Obizhaeva–Wang framework

Vsevolod Zaostrovsky, Peter Shkenev

Student Research Group «Market Microstructure»

Supervisors: Anton Belyakov, Alexey Savin

Introduction

The introduction of resiliency — the speed at which supply/demand recovers to its steady state after a trade — characterizes the beginning of a new stage in the development of optimal execution models. In our research we develope a practical way to utilize that object.

The supply/demand of financial securities is in general not perfectly elastic. This fact is true even for liquid European markets, if we talk about much less liquid Russian markets, neglecting this fact can be disastrous. The main difference between OW model and others is precisely that resiliency plays a key role in it.

Optimal execution problem

If one wants to sell or buy an amount of an asset large enough to have a significant impact on the market, he, obviously, should not do it by one order: it would be very expensive, since a large order would remove all the upper levels in the limit order book. Therefore, in practice, all large orders are split into a large number of small ones. For example, one can simply divide an order into N equal parts and sell them at regular intervals (this is called TWAP). To find a better solution, we consider the OW model, in which terms the problem has the following form:

$$J_0 = \min_{\{x_0 \cdots x_N\}} E_0 \left[\sum_{n=0}^{N} [A_{t_n} + x_n/(2q)] x_n \right],$$

$$A_{t_n} = F_{t_n} + \lambda(X_0 - X_{t_n}) + s/2 + \sum_{i=0}^{n-1} x_i \kappa e^{-\rho\tau(n-i)}.$$

Here:

- The trader has to buy $\mathbf{X_0}$ units of a security over a fixed time period $[0, T]$.
- x_{t_n} — the trade size at $t_n = \tau n$, where $\tau = T/N$.
- $X_{t_n} := X_0 - \sum_{t_k < t_n} x_{t_k}$.
- B_{t_n} and A_{t_n} — bid and ask prices at t_n.
- $V_{t_n} = \frac{A_{t_n} + B_{t_n}}{2}$ — the mid-quote price;
- s — the bid–ask spread.z
- F_t — the fundamental price of the security.
- $q(P)$ — the density of limit orders to sell at price P.
- Parameter λ captures the permanent price impact.
- Parameter q is a LOB density.
- $\kappa = \frac{1}{q} - \lambda$
- Parameter ρ captures the resiliency.

The key question: how to find ρ?

We provide our methodology to find ρ. We find it, considering time series on elements of the model that can be calculated from market data. As an example, we are going to consider the regression:

Our method to find ρ

$$\frac{\Delta A_{k+2}}{\Delta t_{k+2}} - \frac{\Delta A_{k+1}}{\Delta t_{k+1}} = -\rho \Delta A_{k+1} + \rho\lambda x_{k+1} + (\kappa + \lambda)\left(\frac{x_{k+2}}{\Delta t_{k+2}} - \frac{x_{k+1}}{\Delta t_{k+1}}\right).$$

Where ΔA_{k+2} is an ask change after execution of the limit order with the depth x_k and Δt_{k+2} is a time time between two adjacent orders of dataset.

Optimal execution strategies

Using the fitted parameter ρ, now we are able to execute the limit of the following strategy. To fit the discrete variant one needs more parameters and more regression.

Discrete optimal execution strategy

The solution to the optimal execution problem is

$$x_n = -\frac{1}{2}\delta_{n+1}[D_{t_n}(1 - \beta_{n+1}e^{-\rho\tau} + 2\kappa\gamma_{n+1}e^{-2\rho\tau}) - \\ - X_{t_n}(\lambda + 2\alpha_{n+1} - \beta_{n+1}\kappa e^{-\rho\tau})],$$

with $x_N = X_N$ and $D_t = A_t - V_t - s/2$. The expected cost for future trades under the optimal strategy is determined according to

$$J_{t_n} = (F_{t_n} + s/2)X_{t_n} + \lambda X_0 X_{t_n} + \alpha_n X_{t_n}^2 + \beta_n D_{t_n} X_{t_n} + \gamma_n D_{t_n}^2,$$

where the coefficients α_{n+1}, β_{n+1}, γ_{n+1}, and δ_{n+1} are determined recursively as follows:

$$\alpha_n = \alpha_{n+1} - \frac{1}{4}\delta_{n+1}(\lambda + 2\alpha_{n+1} - \beta_{n+1}\kappa e^{-\rho\tau})^2,$$

$$\beta_n = \beta_{n+1}e^{-\rho\tau} + \\ + \frac{1}{2}\delta_{n+1}(1 - \beta_{n+1}e^{-\rho\tau} + 2\kappa\gamma_{n+1}e^{-2\rho\tau})(\lambda + 2\alpha_{n+1} - \beta_{n+1}\kappa e^{-\rho\tau}),$$

$$\gamma_n = \gamma_{n+1}e^{-2\rho\tau} - \frac{1}{4}\delta_{n+1}(1 - \beta_{n+1}e^{-\rho\tau} + 2\gamma_{n+1}\kappa e^{-2\rho\tau})^2,$$

with $\delta_{n+1} = [1/(2q) + \alpha_{n+1} - \beta_{n+1}\kappa e^{-\rho\tau} + \gamma_{n+1}\kappa^2 e^{-2\rho\tau}]^{-1}$ and terminal conditions

$$\alpha_N = 1/(2q) - \lambda, \qquad \beta_N = 1, \qquad \gamma_N = 0.$$

Limit of the discrete optimal execution strategy

As $N \to \infty$, the optimal execution strategy becomes:

$$\lim_{N\to\infty} x_n/(T/N) = \dot{X}_t = \frac{\rho X_0}{\rho T + 2}, \qquad t \in (0, T),$$

$$\lim_{N\to\infty} x_0 = x_{t=0} = \lim_{N\to\infty} x_n/(T/N) = x_{t=T} = \frac{X_0}{\rho T + 2}.$$

Problems

- The task formulated in the KPI is not directly related to the article [OW13]. [OW13] and [Vel20] pose the problem significantly differently. Similar terminology we have found in [Web23], but we did not find the theory to work with in that framework.
- The data we previously had did not have a sufficient level of detail to extract accurate model values. It was necessary to make assumptions that significantly distorted the final result. New data will require significant time to parse and research. Anyway, data work is very complicated.
- This area is very rich and complicated. It is very hard to do even easy steps in the theoretical research, because we did not have courses on that theory.

Purposes

- Develope methodology for fitting OWM factors and use it to get optimal execution strategy.
- Compare different approaches of measuring resiliency on l3 data.
- Compare discrete and limit OW execution strategy.
- Propose a backtest procedure for the optimal execution algorithm, implement it and compare the algorithm with TWAP on real market data.

References

[OW13] Anna A Obizhaeva and Jiang Wang. "Optimal trading strategy and supply/demand dynamics". In: *Journal of Financial markets* 16.1 (2013), pp. 1–32.

[Vel20] Raja Velu. *Algorithmic trading and quantitative strategies*. CRC Press, 2020.

[Web23] Kevin T Webster. *Handbook of Price Impact Modeling*. CRC Press, 2023.