Obizhaeva–Wang Model

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Abstract

In this research we provide and test our parameters fitting methodology and consider properties of real data. Also, we give some ideas of future research.

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Introduction

The Obyzhaeva—Wang model is a financial market trading model that is widely used in quantitative finance. Developed by Anna Obyzhaeva and Jiang Wang in 2013, the model is used to analyze the dynamics of financial markets and make trading decisions. The model has gained significant attention in the financial industry for its attention to the resiliency, that describes an important empirical fact: the supply/demand of financial securities is in general not perfectly elastic. Resiliency — the speed at which supply/demand recovers to its steady state after a trade — characterizes the beginning of a new stage in the development of optimal execution models. In our research we develope a practical way to utilize that object.

This fact is true even for liquid European markets, if we talk about much less liquid Russian markets, neglecting this fact can be disastrous. The main difference between OW model and others is precisely that resiliency plays a key role in it.

Our point of interest here is an optimal execution problem. If one wants to sell or buy an amount of an asset large enough to have a significant impact on the market, what should he do? To write the problem in the language of mathematics, we will need to study the basic principles of the exchange structure.

1 Basic concepts

We start with considering the Limit Order Book (LOB) structure. Within this paradigm of organizing an exchange, each person with access to trading has two opportunities:

- to express a desire to buy or sell a certain number of units of an asset at a price. In this case, the exchange will "remember" his desire. The set of price-quantity pairs of assets can be aggregated by price levels and depicted as at 1.

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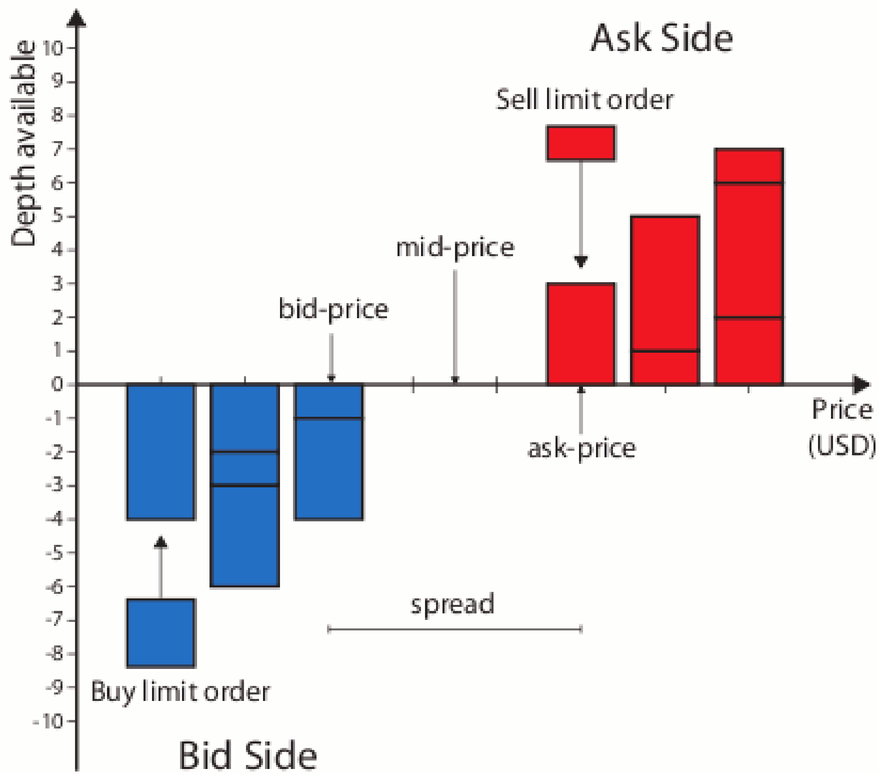

Figure 1: Graphical representation of the Limit Order Book

- to express a desire to buy or sell a certain number of units of an asset immediately. In this case, he will be offered the required number of shares at the best possible price. For example, in the case of a purchase, orders from the price level corresponding to the best price are first matched. If there are not enough shares at that level to fill the order, the next shares are taken from the next price level and so on.

So, in general, there exist two types of interaction with the exchange:

Definition 1. *A **limit order** is an order to buy or sell a security at a specific price or better. This type of order guarantees the execution price, but does not guarantee the execution itself.*

Definition 2. *A **market order** is an order to buy or sell a security immediately. This type of order guarantees that the order will be executed, but does not guarantee the execution price.*

Let us introduce several definitions that we will need later.

Definition 3. *A **bid** (price) is the highest price that a buyer (i.e., bidder) is willing to pay for the asset. We will denote bid at time t as B_t. An **ask** (price) is the price a seller states they will accept. We will denote ask at time t as A_t. The **bid–ask spread** s is the difference between the prices quoted for an immediate sale (ask) and an immediate purchase (bid): $s = A_t - B_t$. The **mid-quote price**: $V_t = \frac{A_t+B_t}{2}$.*

Now, it is clear that if one wants to sell or buy an amount of an asset large enough to have a significant impact on the market, he should not do it by one order: it would be very expensive, since a large order would remove all the upper levels in the limit order book. Therefore, in practice, all large orders are split into a large number of small ones. For example, one can simply divide an order into N equal parts and sell them at regular intervals (this is called TWAP). But is there a better solution?

2 Obizhaeva–Wang Framework

Trying to find better solution, we consider an Obizhaeva–Wang Model model, in which terms the problem has the following form:

$$J_0 = \min_{\{x_0 \cdots x_N\}} E_0 \left[\sum_{n=0}^{N} [A_{t_n} + x_n/(2q)] x_n \right],$$
$$A_{t_n} = F_{t_n} + \lambda(X_0 - X_{t_n}) + s/2 + \sum_{i=0}^{n-1} x_i \kappa e^{-\rho\tau(n-i)}.$$

Here:

- The trader has to buy $\mathbf{X_0}$ units of a security over a fixed time period $[0, T]$.
- x_{t_n} — the trade size at $t_n = \tau n$, where $\tau = T/N$.
- $X_{t_n} := X_0 - \sum_{t_k < t_n} x_{t_k}$.
- B_{t_n} and A_{t_n} — bid and ask prices at t_n.
- $V_{t_n} = \frac{A_{t_n} + B_{t_n}}{2}$ — the mid-quote price;
- s — the bid–ask spread.
- F_t — the fundamental price of the security.
- $q(P)$ — the density of limit orders to sell at price P.
- Parameter λ captures the permanent price impact.
- Parameter q is a LOB density.
- $\kappa = \frac{1}{q} - \lambda$
- Parameter ρ captures the resiliency.

The solution to this stochastic optimal control problem was found in the article [OW13]:

Theorem 1. *The solution to the optimal execution problem is*

$$x_n = -\frac{1}{2}\delta_{n+1}[D_{t_n}(1 - \beta_{n+1}e^{-\rho\tau} + 2\kappa\gamma_{n+1}e^{-2\rho\tau}) - X_{t_n}(\lambda + 2\alpha_{n+1} - \beta_{n+1}\kappa e^{-\rho\tau})],$$

with $x_N = X_N$ and $D_t = A_t - V_t - s/2$. The expected cost for future trades under the optimal strategy is determined according to

$$J_{t_n} = (F_{t_n} + s/2)X_{t_n} + \lambda X_0 X_{t_n} + \alpha_n X_{t_n}^2 + \beta_n D_{t_n} X_{t_n} + \gamma_n D_{t_n}^2,$$

where the coefficients α_{n+1}, β_{n+1}, γ_{n+1}, and δ_{n+1} are determined recursively as follows:

$$\alpha_n = \alpha_{n+1} - \frac{1}{4}\delta_{n+1}(\lambda + 2\alpha_{n+1} - \beta_{n+1}\kappa e^{-\rho\tau})^2,$$

$$\beta_n = \beta_{n+1}e^{-\rho\tau} + \frac{1}{2}\delta_{n+1}(1 - \beta_{n+1}e^{-\rho\tau} + 2\kappa\gamma_{n+1}e^{-2\rho\tau})(\lambda + 2\alpha_{n+1} - \beta_{n+1}\kappa e^{-\rho\tau}),$$

$$\gamma_n = \gamma_{n+1}e^{-2\rho\tau} - \frac{1}{4}\delta_{n+1}(1 - \beta_{n+1}e^{-\rho\tau} + 2\gamma_{n+1}\kappa e^{-2\rho\tau})^2,$$

with $\delta_{n+1} = [1/(2q) + \alpha_{n+1} - \beta_{n+1}\kappa e^{-\rho\tau} + \gamma_{n+1}\kappa^2 e^{-2\rho\tau}]^{-1}$ and terminal conditions

$$\alpha_N = 1/(2q) - \lambda, \qquad \beta_N = 1, \qquad \gamma_N = 0.$$

In our research we will consider the limit of this problem, because it requires less amount of factors:

Theorem 2. *As $N \to \infty$, the optimal execution strategy becomes:*

$$\lim_{N\to\infty} x_0 = x_{t=0} = \frac{X_0}{\rho T + 2},$$
$$\lim_{N\to\infty} x_n/(T/N) = \dot{X}_t = \frac{\rho X_0}{\rho T + 2}, \qquad t \in (0, T),$$
$$\lim_{N\to\infty} x_0 = x_{t=0} = \lim_{N\to\infty} x_n/(T/N) = x_{t=T} = \frac{X_0}{\rho T + 2}.$$

where x_0 is the trade at the beginning of trading period, x_N is the trade at the end of trading period, and $\dot{X}_t$ is the speed of trading in between these trades.

Thus, we have an explicit form of optimal execution strategy, but to use and implement it we need to find the factors ρ, κ and λ. And it is not obvious how to do that.

3 How to find the factors

We provide our methodology to find ρ. We find it, considering the time series of elements of the model that can be calculated from market data. As an example, we are going to consider the regression:

Theorem 3. *In the regression:*

$$\frac{\Delta A_{k+2}}{\Delta t_{k+2}} - \frac{\Delta A_{k+1}}{\Delta t_{k+1}} = -\rho \Delta A_{k+1} + \rho \lambda x_{k+1} + (\kappa + \lambda)\left(\frac{x_{k+2}}{\Delta t_{k+2}} - \frac{x_{k+1}}{\Delta t_{k+1}}\right).$$

the coefficients ρ, κ and λ the same as in OW model describes the market with dynamics describing by series $A_k, \Delta t_k, x_k$.

Here all the information needed can be extracted from the l3 data:

- ΔA_k is an ask change after execution of the limit order with the depth x_k.
- Δt_k is a time between k and $k+1$ orders of dataset.

Another idea is to consider leveling the dynamics-setting ask after a large order:

$$A_t = \overline{p}_t + \frac{s}{2} + x_1 \kappa e^{-\rho t}.$$

This idea looks good, but it is very hard numerically: we tested several different approaches related to this idea, but we were unable to extract adequate parameters.

4 Data preparing

Conclusion

References

[Alm+05] Robert Almgren et al. "Direct estimation of equity market impact". In: *Risk* 18.7 (2005), pp. 58–62.

[Gre05] Robin Greenwood. "Short-and long-term demand curves for stocks: theory and evidence on the dynamics of arbitrage". In: *Journal of Financial Economics* 75.3 (2005), pp. 607–649.

[OW13] Anna A Obizhaeva and Jiang Wang. "Optimal trading strategy and supply/demand dynamics". In: *Journal of Financial markets* 16.1 (2013), pp. 1–32.

[KO18] Albert S Kyle and Anna A Obizhaeva. "The market impact puzzle". In: *Anna A., The Market Impact Puzzle (February 4, 2018)* (2018).

[Vel20] Raja Velu. *Algorithmic trading and quantitative strategies*. CRC Press, 2020.

[Web23] Kevin T Webster. *Handbook of Price Impact Modeling*. CRC Press, 2023.

Appendix A Justification of the method

Theorem 4. *To simplify the notation, consider:*

$$\Delta t_{k+1} := t_{k+1} - t_k, \qquad D_k := D_{t_k}, \qquad x_k := x_{t_k}, \qquad \Delta D_{k+1} := D_{k+1} - D_k.$$

In the regression:

$$\frac{\Delta A_{k+2}}{\Delta t_{k+2}} - \frac{\Delta A_{k+1}}{\Delta t_{k+1}} = -\rho \Delta A_{k+1} + \rho\lambda x_{k+1} + (\kappa+\lambda)(\frac{x_{k+2}}{\Delta t_{k+2}} - \frac{x_{k+1}}{\Delta t_{k+1}}).$$

the coefficients ρ, κ and λ the same as in OW model describes the market with dynamics describing by series $A_k, \Delta t_k, x_k$.

Here:

- x_k *— the trade size at* t_k.
- A_k *— ask price at* t_k.

Proof. From the definitions of the model follows three equations:

$$A_k = V_k + \frac{s}{2} + \sum_{i=0}^{k} x_i \kappa e^{-\rho(k-i)} \tag{1}$$

$$V_{k+1} = V_k + \lambda x_k \to V_{k+1} - V_k = \lambda x_k \tag{2}$$

$$D_k = A_k - V_k - \frac{s}{2} \tag{3}$$

From (1) and (3):

$$D_k = \sum_{i=0}^{k} x_i \kappa e^{-\rho(k-i)}$$

$$\Delta D_k = \sum_{i=0}^{k} x_i \kappa e^{-\rho(k-i)} - \sum_{i=0}^{k} x_i \kappa e^{-\rho(k-i)}$$

$$D_{k+1} - D_k = A_{k+1} + V_{k+1} - A_k - V_k = -\rho D_k \Delta t_{k+1} + \lambda x_k$$

$$V_{k+1} - V_k = \lambda x_k \to \Delta D_{k+1} = \Delta A_{k+1} - \lambda x_k$$

$$\frac{\Delta D_{k+1}}{\Delta t_{k+1}} = -\rho D_k + \kappa \frac{x_{k+1}}{\Delta t_{k+1}}$$

$$\frac{\Delta D_{k+2}}{\Delta t_{k+2}} - \frac{\Delta D_{k+1}}{\Delta t_{k+1}} = -\rho \Delta D_{k+1} + \kappa(\frac{x_{k+2}}{\Delta t_{k+2}} - \frac{x_{k+1}}{\Delta t_{k+1}})$$

$$\frac{\Delta A_{k+2}}{\Delta t_{k+2}} - \frac{\Delta A_{k+1}}{\Delta t_{k+1}} = -\rho \Delta A_{k+1} + \rho\lambda x_{k+1} + (\kappa+\lambda)(\frac{x_{k+2}}{\Delta t_{k+2}} - \frac{x_{k+1}}{\Delta t_{k+1}})$$

□