$$P_n(x) = f[x_0] + f[x_0,x_1](x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}),$$

we aim to show that $a_2 = f[x_0,x_1,x_2]$ using $P_n(x_2)$.

Using $P_n(x_2)$

Let's plug x_2 into the polynomial:

$$P_n(x_2) = f[x_0] + f[x_0,x_1](x_2-x_0) + a_2(x_2-x_0)(x_2-x_1) + a_3(x_2-x_0)(x_2-x_1)(x_2-x_2) + \cdots + a_n(x_2-x_0)(x_2-x_1)\cdots(x_2-x_{n-1}).$$

Notice that the term containing a_3 and higher-order terms become zero because they include the factor $(x_2 - x_2)$.

Thus, we have:

$$P_n(x_2) = f[x_0] + f[x_0,x_1](x_2-x_0) + a_2(x_2-x_0)(x_2-x_1).$$

Divided Differences

We also know that $P_n(x)$ interpolates $f(x)$ at the given points. Therefore, $P_n(x_2) = f(x_2)$. This gives us:

$$f(x_2) = f[x_0] + f[x_0,x_1](x_2-x_0) + a_2(x_2-x_0)(x_2-x_1).$$

Now, we can solve for a_2:

$$f(x_2) - f[x_0] - f[x_0,x_1](x_2-x_0) = a_2(x_2-x_0)(x_2-x_1).$$

Dividing both sides by $(x_2-x_0)(x_2-x_1)$ yields:

$$a_2 = \frac{f(x_2) - f[x_0] - f[x_0,x_1](x_2-x_0)}{(x_2-x_0)(x_2-x_1)}.$$

Recall that $f[x_0,x_1,x_2]$ is defined as:

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2] - f[x_0,x_1]}{x_2-x_0}.$$

And,

$$f[x_1,x_2] = \frac{f(x_2)-f(x_1)}{x_2-x_1}.$$

Combining these, we get:

$$f[x_0,x_1,x_2] = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - f[x_0,x_1]}{x_2-x_0}.$$

Simplifying this:

$$f[x_0,x_1,x_2] = \frac{f(x_2)-f(x_1)-f[x_0,x_1](x_2-x_1)}{(x_2-x_1)(x_2-x_0)}.$$

Since:

$$f[x_0,x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0},$$

we have:

$$f[x_0, x_1, x_2] = \frac{f(x_2) - f(x_0) - f[x_0, x_1](x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}.$$

This matches the formula for a_2:

$$a_2 = f[x_0, x_1, x_2].$$

Thus, we've shown that $a_2 = f[x_0, x_1, x_2]$.