To demonstrate that a polynomial of degree 3 is its own clamped cubic spline but not its own natural cubic spline, let's delve into the properties and conditions of cubic splines.

Clamped Cubic Spline

A clamped cubic spline requires that the first derivatives at the endpoints match the given values. For a polynomial of degree 3, we can write:

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

The first derivative of $f(x)$ is:

$$f'(x) = 3a_3x^2 + 2a_2x + a_1$$

If we consider a clamped cubic spline that matches $f(x)$ at the endpoints x_0 and x_n, the conditions are:

$$f'(x_0) = 3a_3x_0^2 + 2a_2x_0 + a_1$$

$$f'(x_n) = 3a_3x_n^2 + 2a_2x_n + a_1$$

These conditions will be naturally satisfied because they are derived from the same polynomial $f(x)$. Thus, the polynomial $f(x)$ is its own clamped cubic spline because it inherently satisfies the conditions imposed by the clamped spline at the endpoints.

Natural Cubic Spline

A natural cubic spline requires that the second derivatives at the endpoints are zero. The second derivative of $f(x)$ is:

$$f''(x) = 6a_3x + 2a_2$$

For a natural cubic spline, the conditions are:

$$f''(x_0) = 6a_3x_0 + 2a_2 = 0$$

$$f''(x_n) = 6a_3x_n + 2a_2 = 0$$

For these conditions to be satisfied simultaneously, we need:

$$6a_3x_0 + 2a_2 = 0 \quad \text{and} \quad 6a_3x_n + 2a_2 = 0$$

This system of equations implies:

$$a_2 = -3a_3x_0 \quad \text{and} \quad a_2 = -3a_3x_n$$

Thus, x_0 and x_n must be the same for a_2 to be consistent. However, for different x_0 and x_n, this consistency does not hold, indicating that a polynomial of degree 3 cannot satisfy the conditions required for being its own natural cubic spline unless $x_0 = x_n$, which is not practical.

Conclusion

- **Clamped Cubic Spline**: A polynomial of degree 3 is its own clamped cubic spline because the conditions for the first derivatives at the endpoints match naturally with the derivatives of the polynomial.
- **Natural Cubic Spline**: A polynomial of degree 3 cannot be its own natural cubic spline unless x_0 and x_n are the same point, which is not feasible in practice.