Given the function $f(x) = \cos(\pi x)$, we want to approximate the second derivative $f''(0.5)$ using Eq. (4.9) with the values of $f(x)$ at $x = 0.25$, $x = 0.5$, and $x = 0.75$.

Eq. (4.9) typically refers to the centered finite difference formula for the second derivative:

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

In this case, we are using $x = 0.5$, $h = 0.25$, and the points $x - h = 0.25$, $x = 0.5$, and $x + h = 0.75$.

Step-by-Step Solution

1. **Calculate $f(0.25)$, $f(0.5)$, and $f(0.75)$**:

$$f(0.25) = \cos(\pi \cdot 0.25) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f(0.5) = \cos(\pi \cdot 0.5) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$f(0.75) = \cos(\pi \cdot 0.75) = \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

2. **Apply the centered finite difference formula**:

$$f''(0.5) \approx \frac{f(0.5 + 0.25) - 2f(0.5) + f(0.5 - 0.25)}{(0.25)^2}$$

   Substitute the values:

$$f''(0.5) \approx \frac{f(0.75) - 2f(0.5) + f(0.25)}{(0.25)^2}$$

$$f''(0.5) \approx \frac{-\frac{\sqrt{2}}{2} - 2 \cdot 0 + \frac{\sqrt{2}}{2}}{(0.25)^2}$$

$$f''(0.5) \approx \frac{-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}{(0.25)^2}$$

$$f''(0.5) \approx \frac{0}{(0.25)^2} = 0$$

Therefore, the approximation yields $f''(0.5) \approx 0$.

Explanation of Accuracy

The method is particularly accurate for this problem because the function $f(x) = \cos(\pi x)$ is a smooth, periodic function. The centered difference formula for the second derivative is generally more accurate for smooth functions because it uses points symmetrically around the point of interest, minimizing error due to the even-order derivatives of the function.

Error Bound

The error for the centered difference approximation of the second derivative is given by the error term:

$$E = -\frac{h^2}{12} f^{(4)}(\xi)$$

for some ξ in the interval $[x - h, x + h]$.

1. **Fourth Derivative of $f(x)$**:

$$f(x) = \cos(\pi x)$$

$$f''(x) = -\pi^2 \cos(\pi x)$$

$$f^{(4)}(x) = \pi^4 \cos(\pi x)$$

2. **Evaluate the fourth derivative at ξ**:
   The maximum value of $|f^{(4)}(x)|$ over any interval is π^4, since $\cos(\pi x)$ oscillates between -1 and 1.

3. **Calculate the error bound**:

$$|E| \leq \frac{h^2}{12} \pi^4$$

$$h = 0.25$$

$$|E| \leq \frac{(0.25)^2}{12} \pi^4$$

$$|E| \leq \frac{0.0625}{12} \pi^4$$

$$|E| \leq \frac{\pi^4}{192}$$

Therefore, the error bound for the approximation is $\frac{\pi^4}{192}$.

Summary

- The second derivative $f''(0.5)$ is approximated as 0.
- The method is particularly accurate for this problem because the function $\cos(\pi x)$ is smooth and periodic, which minimizes the error in the centered difference formula.
- The error bound for the approximation is $\frac{\pi^4}{192}$.