Part (a): Deriving Picard's Method

To derive Picard's method for solving the initial-value problem:

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

we start by integrating both sides of the differential equation with respect to t:

$$\int_a^t y'(\tau)\, d\tau = \int_a^t f(\tau, y(\tau))\, d\tau.$$

This yields:

$$y(t) - y(a) = \int_a^t f(\tau, y(\tau))\, d\tau.$$

Using the initial condition $y(a) = \alpha$, we get:

$$y(t) = \alpha + \int_a^t f(\tau, y(\tau))\, d\tau.$$

Picard's method involves iteratively improving the approximation to the solution. We define a sequence of functions $\{y_k(t)\}$ as follows:

$$y_0(t) = \alpha,$$

$$y_k(t) = \alpha + \int_a^t f(\tau, y_{k-1}(\tau))\, d\tau, \quad k = 1, 2, \ldots.$$

Part (b): Generating $y_0(t), y_1(t), y_2(t)$, and $y_3(t)$ for $y' = -y + t + 1,\ 0 \le t \le 1,\ y(0) = 1$

We will use Picard's method to generate the approximations.

1. **Initial approximation $y_0(t)$:**

$$y_0(t) = 1.$$

2. **First iteration $y_1(t)$:**

$$y_1(t) = 1 + \int_0^t (-y_0(\tau) + \tau + 1)\, d\tau.$$

Since $y_0(t) = 1$:

$$y_1(t) = 1 + \int_0^t (-1 + \tau + 1)\, d\tau = 1 + \int_0^t \tau\, d\tau = 1 + \left[\frac{\tau^2}{2}\right]_0^t = 1 + \frac{t^2}{2}.$$

3. **Second iteration $y_2(t)$:**

$$y_2(t) = 1 + \int_0^t (-y_1(\tau) + \tau + 1)\, d\tau.$$

Since $y_1(t) = 1 + \frac{t^2}{2}$:

$$y_2(t) = 1 + \int_0^t \left(-\left(1 + \frac{\tau^2}{2}\right) + \tau + 1\right)\, d\tau = 1 + \int_0^t \left(-1 - \frac{\tau^2}{2} + \tau + 1\right)\, d\tau.$$

Simplify the integrand:

$$y_2(t) = 1 + \int_0^t \left(-\frac{\tau^2}{2} + \tau\right)\, d\tau = 1 + \left[-\frac{\tau^3}{6} + \frac{\tau^2}{2}\right]_0^t = 1 + \left(-\frac{t^3}{6} + \frac{t^2}{2}\right).$$

$$y_2(t) = 1 + \frac{t^2}{2} - \frac{t^3}{6}.$$

4. **Third iteration $y_3(t)$:**

$$y_3(t) = 1 + \int_0^t (-y_2(\tau) + \tau + 1)\, d\tau.$$

Since $y_2(t) = 1 + \frac{t^2}{2} - \frac{t^3}{6}$:

$$y_3(t) = 1 + \int_0^t \left(-\left(1 + \frac{\tau^2}{2} - \frac{\tau^3}{6}\right) + \tau + 1\right)\, d\tau.$$

Simplify the integrand:

$$y_3(t) = 1 + \int_0^t \left(-1 - \frac{\tau^2}{2} + \frac{\tau^3}{6} + \tau + 1\right)\, d\tau = 1 + \int_0^t \left(-\frac{\tau^2}{2} + \frac{\tau^3}{6} + \tau\right)\, d\tau.$$

$$y_3(t) = 1 + \left[-\frac{\tau^3}{6} + \frac{\tau^4}{24} + \frac{\tau^2}{2}\right]_0^t = 1 + \left(-\frac{t^3}{6} + \frac{t^4}{24} + \frac{t^2}{2}\right).$$

$$y_3(t) = 1 + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24}.$$

Part (c): Comparing Picard's Method with the Actual Solution

The actual solution to the differential equation $y' = -y + t + 1$ with $y(0) = 1$ is given by:

$$y(t) = t + e^{-t}.$$

The Maclaurin series for the actual solution is:

$$y(t) = t + e^{-t} = t + 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots = 1 + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} + \cdots .$$

The approximations generated by Picard's method up to $y_3(t)$ are:

$$y_0(t) = 1,$$

$$y_1(t) = 1 + \frac{t^2}{2},$$

$$y_2(t) = 1 + \frac{t^2}{2} - \frac{t^3}{6},$$

$$y_3(t) = 1 + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24}.$$

Comparing these with the Maclaurin series of the actual solution, we observe that the approximations from Picard's method are consistent with the terms of the Maclaurin series of the actual solution, confirming the accuracy and convergence of Picard's method.