To find the constants c_0, c_1, and x_1 for the quadrature formula

$$\int_0^1 f(x)\,dx = c_0 f(0) + c_1 f(x_1)$$

such that it has the highest possible degree of precision, we need to ensure that the formula integrates polynomials up to the highest possible degree exactly. The goal is to determine these constants so that the quadrature formula is exact for polynomials of the highest possible degree.

Step-by-Step Solution

1. **Polynomial Degree of Precision**:
   To achieve the highest degree of precision, we start by requiring that the quadrature formula be exact for polynomials of degree 0, 1, and possibly 2.

2. **Exactness for $f(x) = 1$**:
   For $f(x) = 1$,

   $$\int_0^1 1\,dx = 1$$

   The quadrature formula should give the same result:

   $$c_0 \cdot 1 + c_1 \cdot 1 = c_0 + c_1 = 1$$

3. **Exactness for $f(x) = x$**:
   For $f(x) = x$,

   $$\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

   The quadrature formula should give the same result:

   $$c_0 \cdot 0 + c_1 \cdot x_1 = c_1 x_1 = \frac{1}{2}$$

4. **Exactness for $f(x) = x^2$**:
   For $f(x) = x^2$,

   $$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

   The quadrature formula should give the same result:

$$c_0 \cdot 0^2 + c_1 \cdot x_1^2 = c_1 x_1^2 = \frac{1}{3}$$

Now, we have three equations:

$$\begin{cases} c_0 + c_1 = 1 \\ c_1 x_1 = \frac{1}{2} \\ c_1 x_1^2 = \frac{1}{3} \end{cases}$$

From the second equation:

$$c_1 = \frac{1}{2x_1}$$

Substitute c_1 in the third equation:

$$\frac{1}{2x_1} x_1^2 = \frac{1}{3}$$

Simplify:

$$\frac{x_1}{2} = \frac{1}{3}$$

$$x_1 = \frac{2}{3}$$

Now substitute x_1 back into $c_1 = \frac{1}{2x_1}$:

$$c_1 = \frac{1}{2 \cdot \frac{2}{3}} = \frac{3}{4}$$

From the first equation:

$$c_0 + c_1 = 1$$

$$c_0 + \frac{3}{4} = 1$$

$$c_0 = 1 - \frac{3}{4} = \frac{1}{4}$$

Summary

The constants are:

$$c_0 = \frac{1}{4}, \quad c_1 = \frac{3}{4}, \quad x_1 = \frac{2}{3}$$

Therefore, the quadrature formula

$$\int_0^1 f(x)\,dx = \frac{1}{4}f(0) + \frac{3}{4}f\left(\frac{2}{3}\right)$$

has the highest possible degree of precision, which is exact for polynomials up to degree 2.