To show that the function $e(h) = \frac{\epsilon}{h} + \frac{h^2}{6}M$ has a minimum at $h = \sqrt[3]{\frac{3\epsilon}{M}}$, we need to find the critical points by setting the first derivative equal to zero and then verify that this critical point is a minimum using the second derivative test.

Step-by-Step Solution

1. **Function Definition**:

$$e(h) = \frac{\epsilon}{h} + \frac{h^2}{6}M$$

2. **First Derivative**:

   To find the critical points, we first compute the first derivative of $e(h)$ with respect to h:

$$e'(h) = -\frac{\epsilon}{h^2} + \frac{2h}{6}M$$

   Simplify the derivative:

$$e'(h) = -\frac{\epsilon}{h^2} + \frac{h}{3}M$$

3. **Setting the First Derivative to Zero**:

   Set $e'(h) = 0$ to find the critical points:

$$-\frac{\epsilon}{h^2} + \frac{h}{3}M = 0$$

   Solve for h:

$$\frac{h}{3}M = \frac{\epsilon}{h^2}$$

   Multiply both sides by h^2 to get rid of the denominator:

$$h^3 M = 3\epsilon$$

   Solve for h:

$$h^3 = \frac{3\epsilon}{M}$$

$$h = \sqrt[3]{\frac{3\epsilon}{M}}$$

4. **Second Derivative Test**:

   To confirm that this critical point is a minimum, we need to check the second derivative $e''(h)$:

$$e''(h) = \frac{d}{dh}\left(-\frac{\epsilon}{h^2} + \frac{h}{3}M\right)$$

   Compute the second derivative:

$$e''(h) = \frac{2\epsilon}{h^3} + \frac{M}{3}$$

   Evaluate $e''(h)$ at $h = \sqrt[3]{\frac{3\epsilon}{M}}$:

$$e''\left(\sqrt[3]{\frac{3\epsilon}{M}}\right) = \frac{2\epsilon}{\left(\sqrt[3]{\frac{3\epsilon}{M}}\right)^3} + \frac{M}{3}$$

   Simplify:

$$e''\left(\sqrt[3]{\frac{3\epsilon}{M}}\right) = \frac{2\epsilon}{\frac{3\epsilon}{M}} + \frac{M}{3}$$

$$e''\left(\sqrt[3]{\frac{3\epsilon}{M}}\right) = \frac{2\epsilon M}{3\epsilon} + \frac{M}{3}$$

$$e''\left(\sqrt[3]{\frac{3\epsilon}{M}}\right) = \frac{2M}{3} + \frac{M}{3}$$

$$e''\left(\sqrt[3]{\frac{3\epsilon}{M}}\right) = M$$

Since $M > 0$, the second derivative $e''(h)$ is positive at $h = \sqrt[3]{\frac{3\epsilon}{M}}$. Therefore, the function $e(h)$ has a minimum at this point.

Conclusion

The function $e(h) = \frac{\epsilon}{h} + \frac{h^2}{6}M$ has a minimum at $h = \sqrt[3]{\frac{3\epsilon}{M}}$, and this result is verified using the second derivative test, which shows that the second derivative is positive at this critical point.