Problem 5

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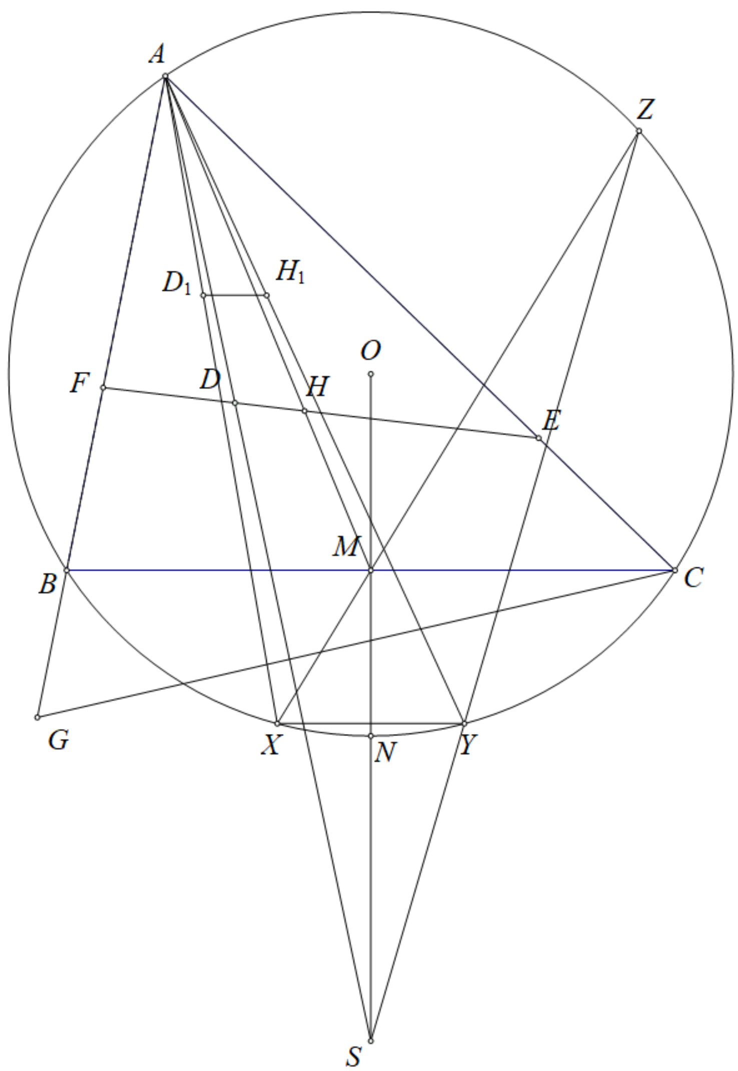

Denote H, D as the A-Humpty point and A-Dumpty point of ABC respectively, denote H_1, D_1 as the A-Humpty point and A-Dumpty point of AEF respectively.

Claim $\dfrac{AD_1}{AH_1} = \dfrac{AD}{AH} = 2 \cdot cos\angle BAC$.

Proof: Let G be the intersection of (BDC) and AB, since B, D, O, C, G lies on a circle we get $CA = CG$.

Denote O as the circumcenter of ABC, S be the intersection of tangents at B, C of (O) we get A, D, S are collinear and $BDCS$ are cyclic.

Combine with $\triangle AHB \sim \triangle ACS \Rightarrow AB \cdot AC = AH \cdot AS$ we get:

$$\frac{AD}{AH} = \frac{AD \cdot AS}{AH \cdot AS} = \frac{AB \cdot AG}{AB \cdot AC} = \frac{AG}{AC} = 2 \cdot cos\angle BAC.$$

Similiarly we get the claim above.
Back to the main problem, let AD_1, AH_1 cut (O) at X, Y respectively, the line from A parallel to HD cut (O) at Z. Let M be the midpoint of BC, since $-1 = A(H_1Z, FE) = (YZ, BC)$ therefore Z, Y, S are collinear, since $XY \parallel BC$ we get Z, M, X are collinear.
Let N be the midpoint of arc BC not containing A of (O). We have

$$\frac{AX}{AY} = Z(AN, XY) = Z(AN, MS) = A(ZN, MS) = A(ZN, HD) = \frac{AD}{AH}$$

(since AN is the bisector of $\angle DAH$ and $AZ \parallel DH$).

Therefore $\dfrac{AD_1}{AH_1} = \dfrac{AD}{AH} = \dfrac{AX}{AY}$ therefore $H_1D_1 \parallel XY \parallel BC$.

Hence the problem is proved.