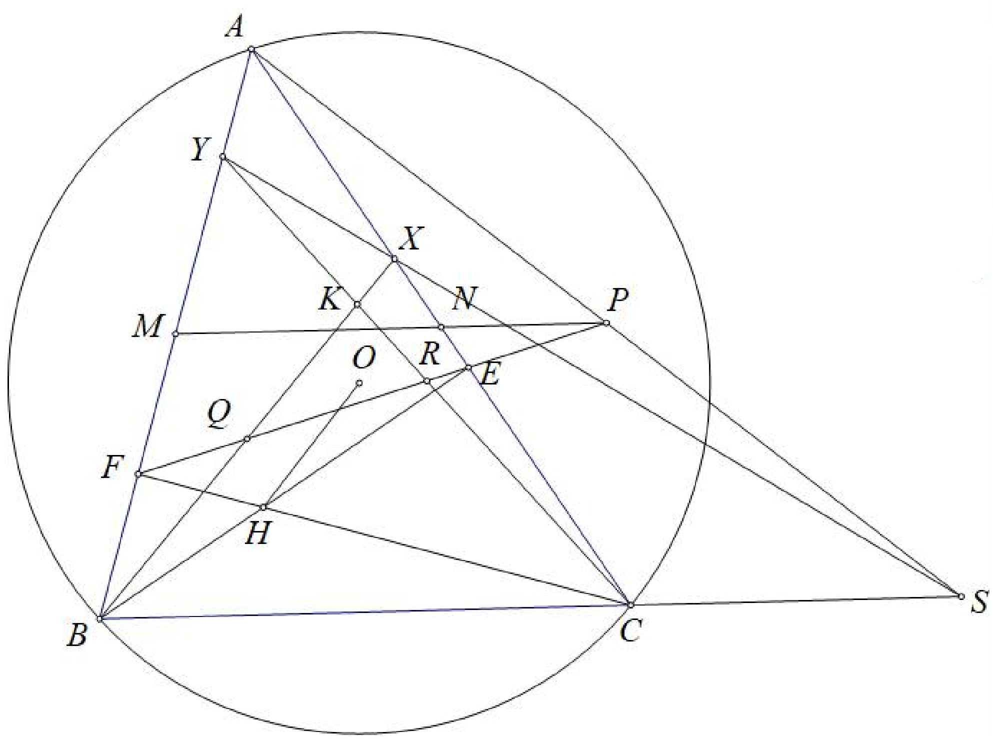
Problem 12

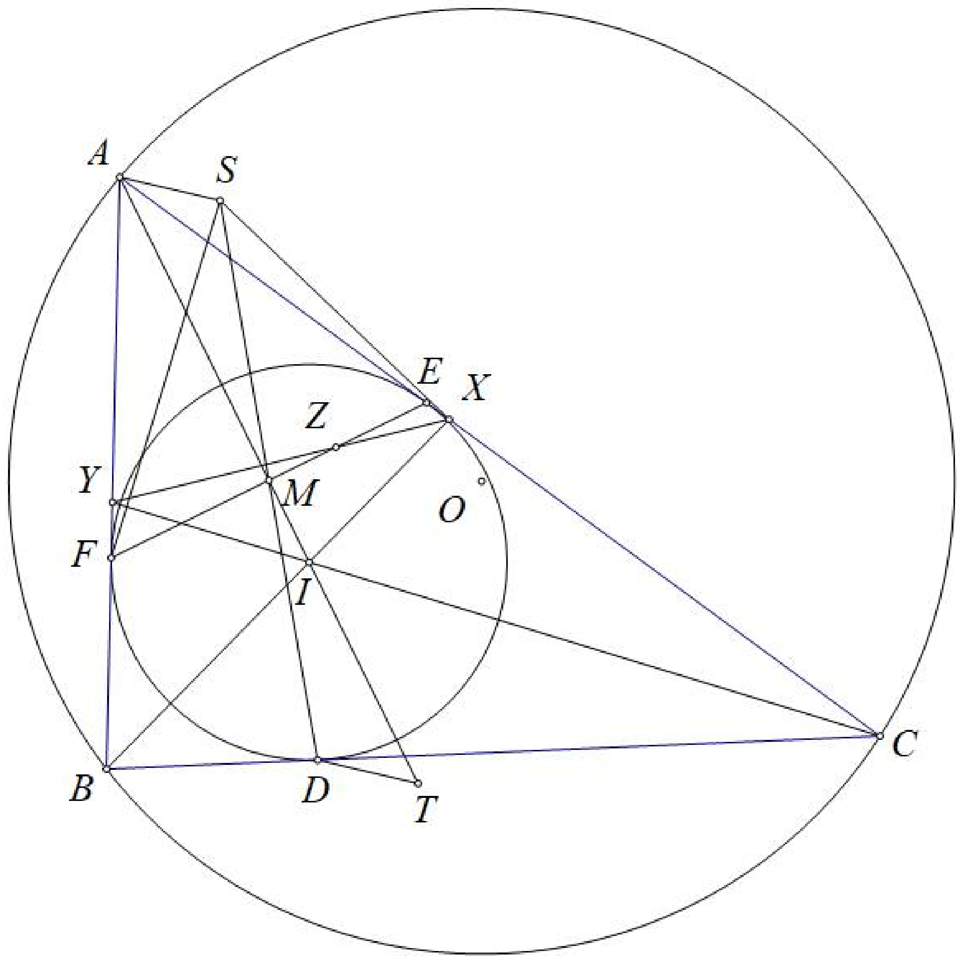
Ha Vu Anh

Lemma 1: Triangle ABC is inscribed in (O) with K as the orthocenter of $\triangle BOC$, and H as the orthocenter of $\triangle ABC$
Let the line through A perpendicular to OH be Ax, then $(AK, Ax, AB, AC) = -1$



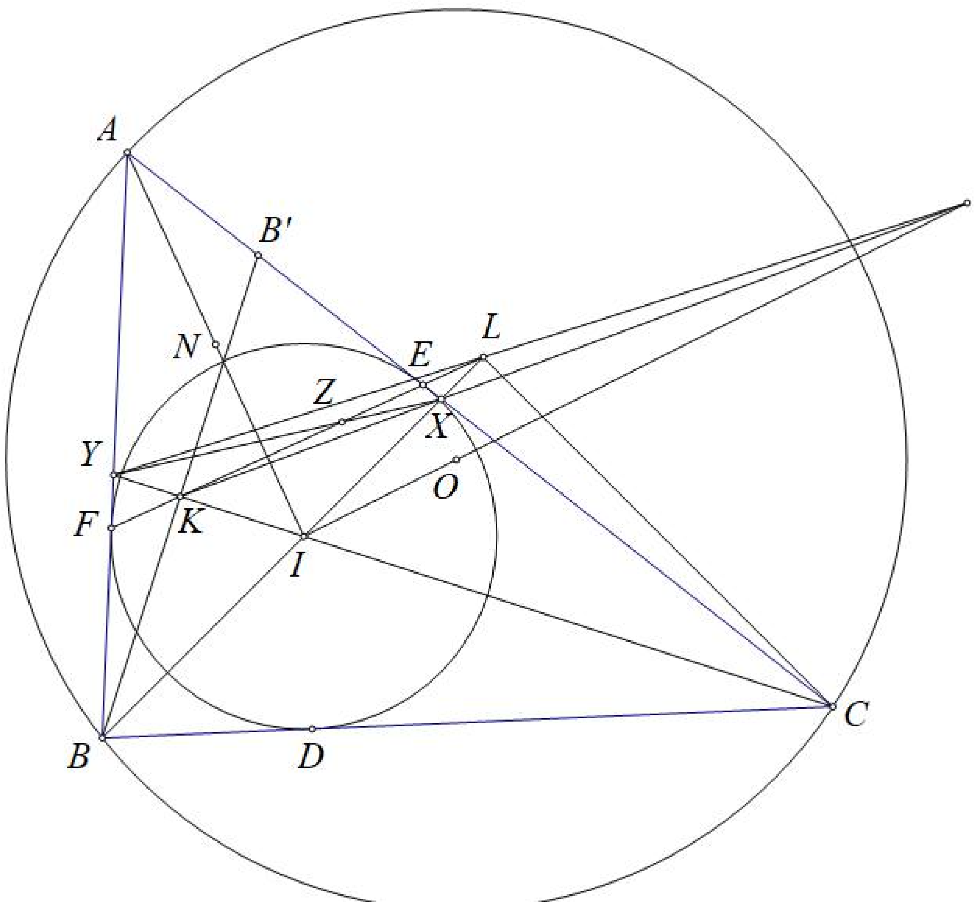
Proof: BK, CK intersect AB, AC at X, Y; XY intersects BC at S
Since $\angle AEF = \angle ABC = \angle BXC$, EF bisects BX at Q, and similarly EF bisects CY at R
Let P be the midpoint of AS, then P, R, Q are collinear, so E, F, P are collinear
Let M, N be midpoints of AB, AC, then $P \in MN$
Hence P lies on the perpendicular bisector of (AH) and (AO), so $AS \perp OH$ or $Ax \equiv AS$
Therefore $-1 = A(KS, BC) = (AK, Ax, AB, AC)$, as desired.

Lemma 2: Triangle ABC is inscribed in (O) and circumscribed about (I), with BI, CI intersecting AB, AC at X, Y
(I) touches BC, CA, AB at D, E, F; EF intersects XY at Z, then $I(ZO, BC) = -1$



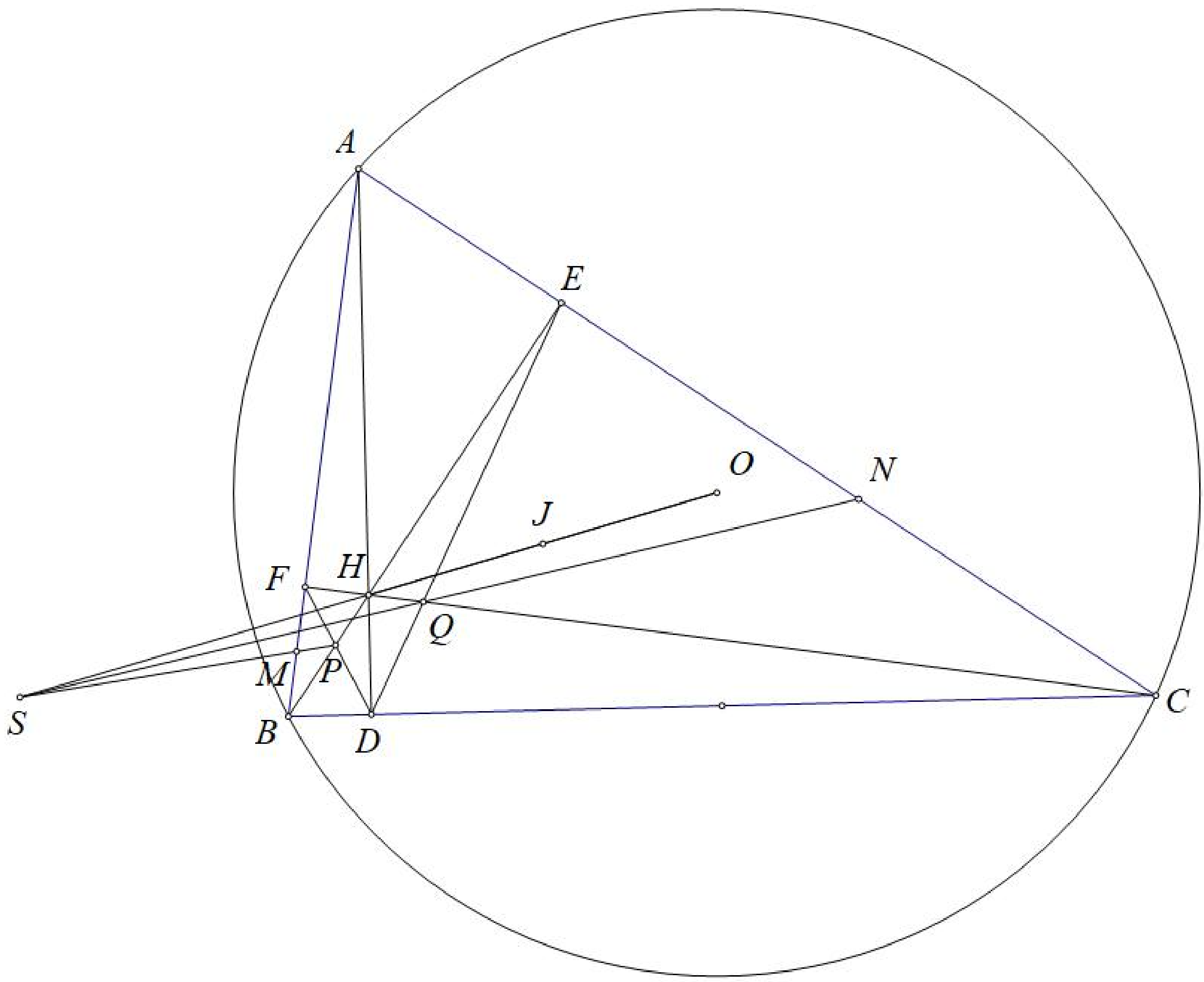
Proof: Let M be the midpoint of EF, S the reflection of D across M, then $SF \parallel DE$
Hence $SF \perp YI$, so S lies on the polar of Y with respect to (I)
Similarly, S lies on the polar of X with respect to (I), so XY is the polar of S with respect to (I)
Since EF is the polar of A with respect to (I), $AS \perp IZ$
Let T be the reflection of A across M, then T is the orthocenter of $\triangle IEF$
Since OI is the Euler line of $\triangle DEF$, by Lemma: Let Dx be the line perpendicular to OI
Then the four lines DT, Dx, DF, DE through point D form a harmonic bundle with two pairs of lines perpendicular
Hence $-1 = (DT, Dx, DF, DE) = I(ZO, BC)$, as desired.

Lemma 3: Triangle ABC is inscribed in (O) and circumscribed about (I), with BI, CI intersecting AC, AB at X, Y
Claim: The median from X of $\triangle AXI$, the median from Y of $\triangle AYI$, and OI are concurrent



Let (I) touch AC, AB at E, F, then E, F, K, L are collinear
EF intersects XY at Z, by Lemma 2 we have $-1 = I(ZO, BC) = I(ZO, XY)$, hence XK, YL, OI are concurrent(1)
Let N be the midpoint of AI, BK intersects AC at B', then K is the midpoint of BB'
Moreover, $\triangle XIA \sim \triangle XB'B$ (angle-angle), so XK is the median from X of $\triangle AXI$, similarly YL is the median from Y of $\triangle AYI$.
Hence, (1) is equivalent to the median from X of $\triangle AXI$, the median from Y of $\triangle AYI$, and OI are concurrent, as desired.

Back to the main problem,



To see that H is the incenter of $\triangle DEF$

PM is the median from P of $\triangle PHD$, QN is the median from Q of $\triangle QHD$

Let J be the circumcenter of (DEF), then J is the midpoint of HO

Applying Proposition 3 to $\triangle DEF$, we get HJ, PM, QN are concurrent

Hence HO, PM, QN are concurrent, or the problem is proven.