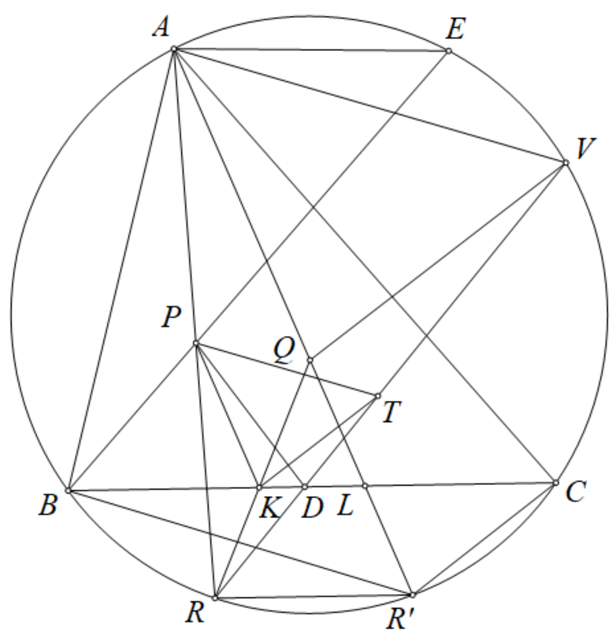
Problem 2

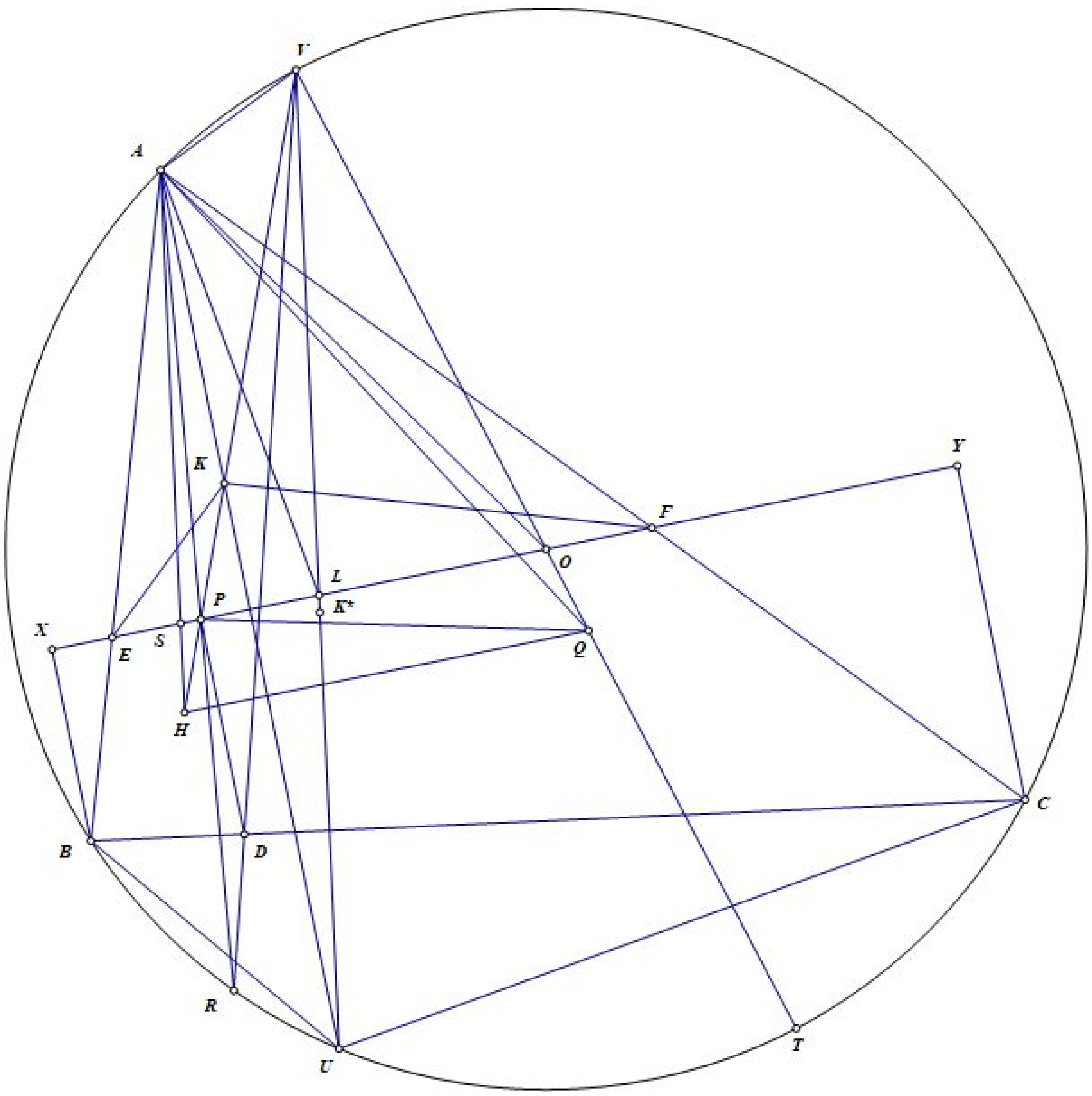
Ha Vu Anh

Lemma: Let ABC be a triangle, Let P, Q be 2 arbitrary points such that they are isogonal conjugates $W.R.T$ ABC, AP meets (O) again at R, for any point arbitrary V on (O), VR meets BC at D, then $\angle PDB = \angle QVA$



Proof:
Let AQ cut (ABC) at R' then $RR' \parallel BC$, BP cut (O) at E, QR cut BC at K, AQ cut BC at L.
By simple angle chasing, we get: $\triangle ABR \sim \triangle BLR'$ and $\triangle REA \sim \triangle BLQ$
Hence $\dfrac{LR'}{LQ} = \dfrac{LR'}{LB} \cdot \dfrac{LB}{LQ} = \dfrac{BR}{BA} \cdot \dfrac{ER}{EA} = \dfrac{PR}{PA}$.(1)
By Thales, we have $\dfrac{KR}{KQ} = \dfrac{LR'}{LQ}$.
Combine with (1) we get $\dfrac{PR}{PA} = \dfrac{LR'}{LQ} = \dfrac{KR}{KQ}$, hence $PK \parallel AQ$. Let the line through K parallel to QV cut RV at T then $\dfrac{PR}{PA} = \dfrac{KR}{KQ} = \dfrac{TR}{TV}$ hence $PT \parallel AV$.
We have: $\angle PKB = \angle ALB = \angle AR'R = \angle AVR = \angle PTD$ therefore $PKDT$ is a cyclic quadrilateral. Hence, $\angle PDK = \angle PTK = \angle QVA$, as desired.
Hence the claim is proved.

Back to the main problem:



Let AK cut (O) at U, and let UV be perpendicular to BC with V on (O). Let K^* be the orthocenter of triangle VBC. Then $VK^* = AH = 2OM$.
Since triangles VBC and AEF are similar, and triangles UBC and KEF are also similar,

we have $\dfrac{AK}{KU} = \dfrac{VK^*}{VU} = \dfrac{AH}{VU}$. Therefore, line HK passes through V.

Because P lies on HK, which is the radical axis of (BF) and (CE), drawing BX, CY perpendicular to EF gives $PX \cdot PF = PY \cdot PE$,

so $\dfrac{PE}{PF} = \dfrac{PX}{PY} = \dfrac{DB}{DC}$, where DP is perpendicular to EF and D lies on BC.

Since triangles AEF and VBC are similar, $\angle BVD = \angle EAP$, hence AP and VD intersect at R on (O).
Applying the lemma above: if P, Q are isogonal conjugates in ABC, AP meets (O) again at R, and for any point V on (O), VR meets BC at D, then $\angle PDB = \angle QVA$. Since $\angle OVA = 90^\circ - \angle AUV = 90^\circ - \angle HAK = \angle PDB = \angle QVA$, points Q, O, V are collinear.

Let UV cut EF at L, and AH cut EF at S. Then by angle chasing, triangles ASL and OVA are similar, and triangles APL and QVA are similar.

Therefore, $\dfrac{VO}{VQ} = \dfrac{LP}{SL} = \dfrac{VP}{VH}$, which implies $OP \parallel HQ$.

Hence the problem is proved.