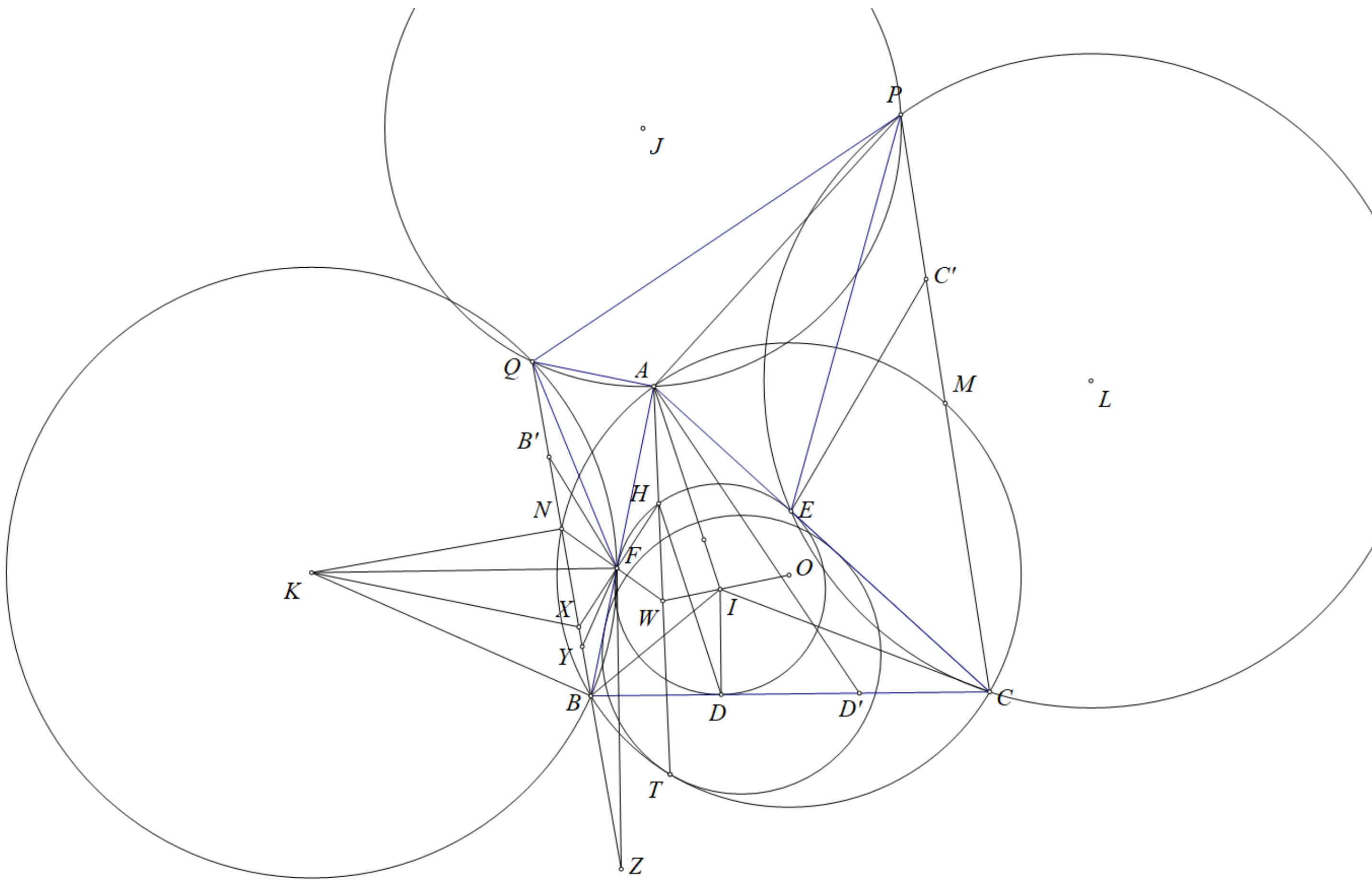
Problem 4

Ha Vu Anh



Let $A-mixtilinear$ circle which we denote as (Ω) tangent to (O) at T, we will prove A, J, T are collinear. Let (I) touch BC at D, the line from A perpendicular to PQ cut BC at D'

Since N is the center of (AIB), BQ is the diameter of (AIB) therefore $\triangle BAQ \sim \triangle CDI$ and $\triangle CAP \sim \triangle BDI$ therefore

$$\frac{AP}{AQ} = \frac{sin\angle CAD'}{sin\angle BAD'} = \frac{D'C}{D'B} \cdot \frac{AB}{AC} \Longleftrightarrow \frac{D'C}{D'B} = \frac{AC}{AP} \cdot \frac{AQ}{AB} = \frac{DB}{DI} \cdot \frac{DI}{DC} = \frac{DB}{DC}.$$

Hence we get D' are reflections of D through the midpoint of BC therefore AW, AD' are isgonal in $\angle BAC$ and since $AD' \perp PQ$ we get AT pass through J.

Let W be the insimilicenter of (O) and (I), since T is the exsimilicenter of (Ω) and (O), A is the exsimilicenter of (Ω) and (I) therefore applying the $Monge - D'Alembert$ theorem for $(\Omega), (O), (I)$ we get W lies on AT therefore NF, EM, AT concurrent at W.

Let the line from D parallel to AI cut (I) at H then it is well known that H lies on AT. We need to prove FK, EL, AT concurrent which is equivalent to $F(HW, AK) = E(HW, AL)(*)$.

Let the line from F perpendicular to KB, KF cut BQ at Y, Z respectively, FH cut BQ at X we get $\angle XFB = \angle AFH = \angle FDH = \angle ICB = \angle ABN$ therefore $XF = XB$.

Therefore $F(HW, AK) = F(XN, BK) = K(XN, BF) = (FB, Fx \parallel BQ, FY, FZ) = \dfrac{BY}{BZ}$

Let B' be the reflection of B through the line from F perpendicular to BQ we get $\triangle FB'Q \sim \triangle ZBF$, $\triangle BYF \sim \triangle BFQ$ therefore $F(HW, AK) = \dfrac{BY}{BZ} = \dfrac{BY}{BF} \cdot \dfrac{BF}{BZ} = \dfrac{BF}{BQ} \cdot \dfrac{B'Q}{B'F} = \dfrac{QB'}{QB}(1)$.

Similiarly denote C' as the point on CP such that $EC' = EC$ we get $E(HW, AL) = \dfrac{PC'}{PC}(2)$.

We have $\dfrac{CP}{BQ} = \dfrac{CM}{BN} = \dfrac{IC}{IB}$

$$\frac{CC'}{BB'} = \frac{CC'}{CE} \cdot \frac{BF}{BB'} \cdot \frac{DC}{DB} = \frac{2BD}{BI} \cdot \frac{IC}{2CD} \cdot \frac{DC}{DB} = \frac{IC}{IB}$$

Therefore $\dfrac{CP}{CC'} = \dfrac{BQ}{BB'}$ therefore $\dfrac{QB'}{QB} = \dfrac{PC'}{PC}$ therefore from (1) and (2) we get $F(HW, AK) = E(HW, AL)$ therefore $(*)$ is true.

Hence the problem is proved.