Problem 8

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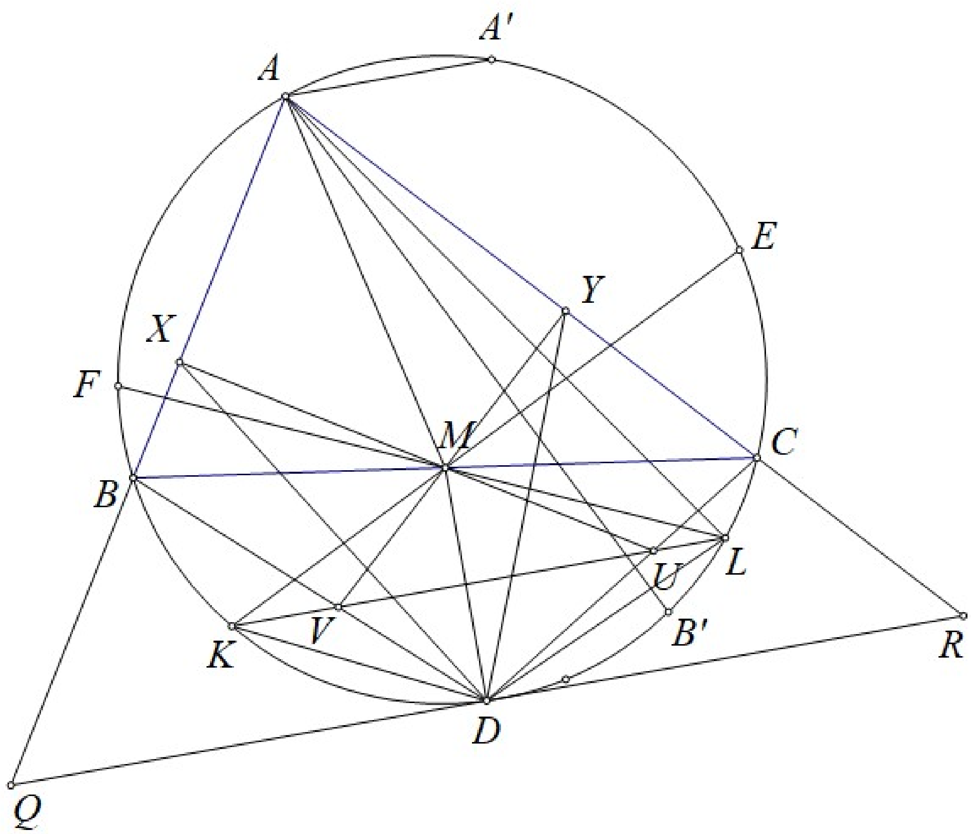

Proof: Remark: Each small arc BC, CA, AB of (O) has a unique "balanced" point.

Let D be the "balanced" point on the small arc BC; let X, Y be the perpendicular foots from M to AB, AC respectively.
The line through D perpendicular to DM meets AB, AC at Q, R respectively, then $MQ = MR$ so $\angle MXD = \angle MQD = \angle MRD = \angle MYD$.
If there existed another "balanced" point D' on the small arc BC, similarly we would have $\angle MXD' = \angle MYD'$, but D' must lie outside triangle DXY while D lies inside $D'XY$, contradiction.
The same argument applied to the other small arcs shows the remark is true.

Let MX, MY meet DC, DB at U, V respectively.
The line through A perpendicular to MD meets (O) again at A'.

Then $A(A'D, BC) = A(A'D, QR) = -1 \quad (1)$,
hence $\angle MDB = \angle A'DC = \angle A'AC = \angle VMD$, so $VM = VD$ and similarly $UM = UD$.
Therefore UV is the perpendicular bisector of DM.

Let UV meet the small arcs DB, DC of (O) at L, K respectively; let (ML, MK) meet (O) again at F, E respectively.
Then M is the incenter of triangle DEF.

The line through A perpendicular to ME meets (O) again at B'.
We have $\angle CEB' = \angle CAB' = \angle YME = \angle KMV = \angle BDK = \angle BEM$,
so $A(EB', BC) = -1$.
Thus the line through E parallel to AB' meets AB, AC at two points equidistant from E, and since $AB' \perp ME$ those two points are also equidistant from M.
Therefore E is a "balanced" point.

By the same argument F is also a "balanced" point.
As D, E, F lie respectively on the small arcs BC, CA, AB, by the uniqueness remark we conclude these D, E, F coincide with the D, E, F of the problem, hence M is the incenter of triangle DEF.