Problem 9

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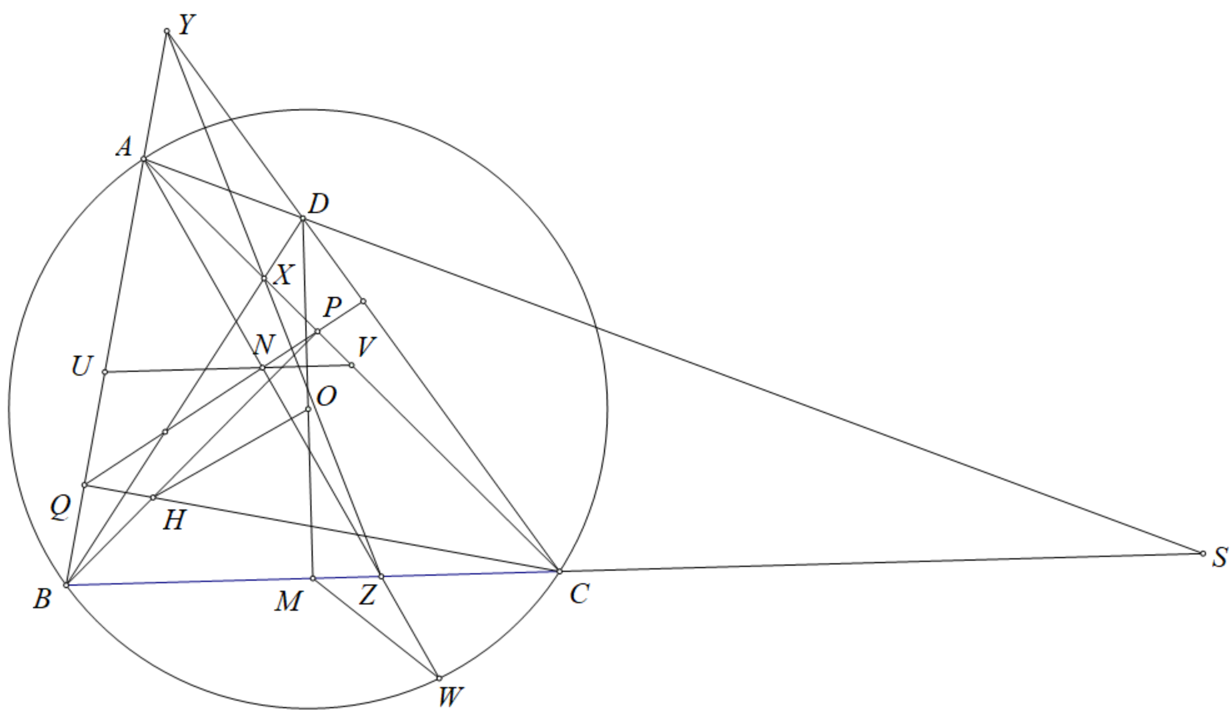
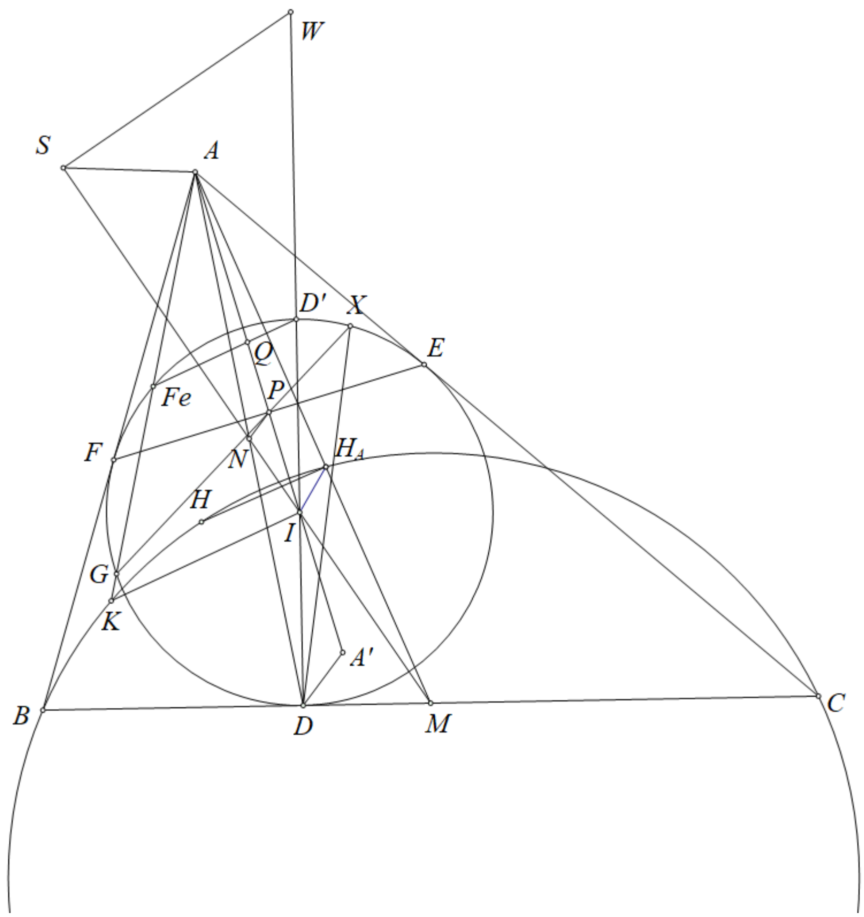
Lemma: Given triangle ABC, Let D be the point symmetric to the intersection of the two tangents at B and C of (O) across BC, M be the midpoint of BC, and W lie on (O) such that $AW \perp OH$. Then: $\angle AWM + 90^\circ = \angle ADM$



Proof: Let BD, CD intersect AC, AB at X, Y.
The line XY intersects BC at Z; BH, CH intersect AC, AB at P, Q.
Since $\angle DBC = \angle BAC$, we have $\angle BXC = \angle ABC = \angle APQ$.
Thus PQ bisects BX, similarly PQ bisects CY, so PQ is the Gauss line.
Therefore, PQ bisects AZ, so N is the intersection of PQ with UV with U, V being midpoint of AB, AC.
Since $NU \cdot NV = NP \cdot NQ$, AN is the radical axis of (AO) and (AH), therefore $AN \perp OH$, hence $AZ \perp OH$, and A, Z, W are collinear.
Moreover, $(BC, ZS) = -1$ therefore $ZM \cdot ZS = ZB \cdot ZC = ZA \cdot ZW$ hence $AMWS$ is cyclic.
Therefore, $\angle AWM = \angle ASM = \angle ADM - 90^\circ$, as desired
Hence the lemma is proved.

Back to the main problem,



Redefine K as the reflection of A through the Feuerbach point - Fe, the problem is equivalent to proving that K is the intersection other than H_A of (AIH_A) and (BHC).
Consider homothety center A with ratio 2, it sends $Fe \mapsto K$, the Euler circle of triangle ABC to (BHC). Since Fe lies on the Euler circle of triange ABC, K lies on (BHC).
We will prove K lies on $(AIH_A)(*)$.
Let MI intersects AD at N then it is well known that N is the midpoint of AD Let MI intersects (AIH_A) then $MI \cdot MS = MH_A \cdot MA = MB^2$
Let W be the orthocenter of $\triangle BIC$, since S is the $I - Humpty$ point of $\triangle IBC$, we get $WS \perp MI$.
Furthermore, it is well known that N lies on the polar of W with respect to (I), which is the line connect the midpoint of AB, AC.
Therefore, by La Hire Theorem, W also lies on the polar of N wrt (I), combine this with the fact that S is the projection of W on MI, we get $IN \cdot IS = ID^2$
Let P be the midpoint of EF, then $IN \cdot IS = ID^2 = IP \cdot IA$ therefore $\angle ASI = \angle IPN$

Thus we need to prove $\angle AKI = \angle ASI = \angle IPN \quad (**)$
Let Fe be the Euler reflection of $\triangle DEF$, and let Q be the midpoint of AI, DD' the diameter of (I), then D', Q, Fe are collinear
$\angle AKI = \angle AFeD'$
Let AFe intersect (I) at G, and X lie on (I) such that $FeX \parallel BC$, then X, P, G are collinear, $DX \perp$ the Euler line of $\triangle DEF$ hence $\angle AFeD' = \angle D'XG = 90^\circ - \angle DXP$
Let A' be the reflection of A across P. Applying the lemma above for triangle DEF, we get $\angle DA'A = 90^\circ + \angle DXP$

$\angle IPN = 180^\circ - \angle DA'A = 90^\circ - \angle DXP = \angle AFeD' = \angle AKI$
Hence $(**)$ is true, therefore K lies on (ASI), which is (AIH_A).
Hence $(*)$ is true, therefore K lies on (AIH_A) and (BHC), as desired.
Hence the problem is proven.