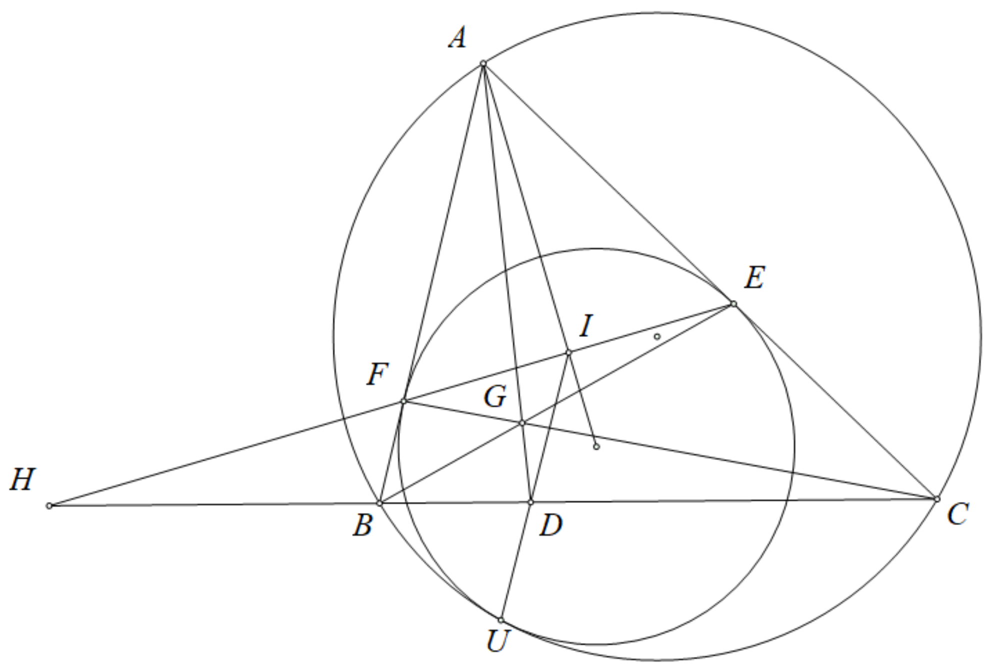
Problem 7

Ha Vu Anh

Lemma 1: Let triangle ABC be circumscribed about the incircle (I). The A-mixcircle is tangent to $AC, AB, (O)$ at points E, F, U, respectively. Let BE and CF intersect at G. Then lines AG, UI, BC are concurrent.



Proof: Let AG intersect BC at D, and EF intersect BC at H. Then $(H, D; B, C) = -1$, hence

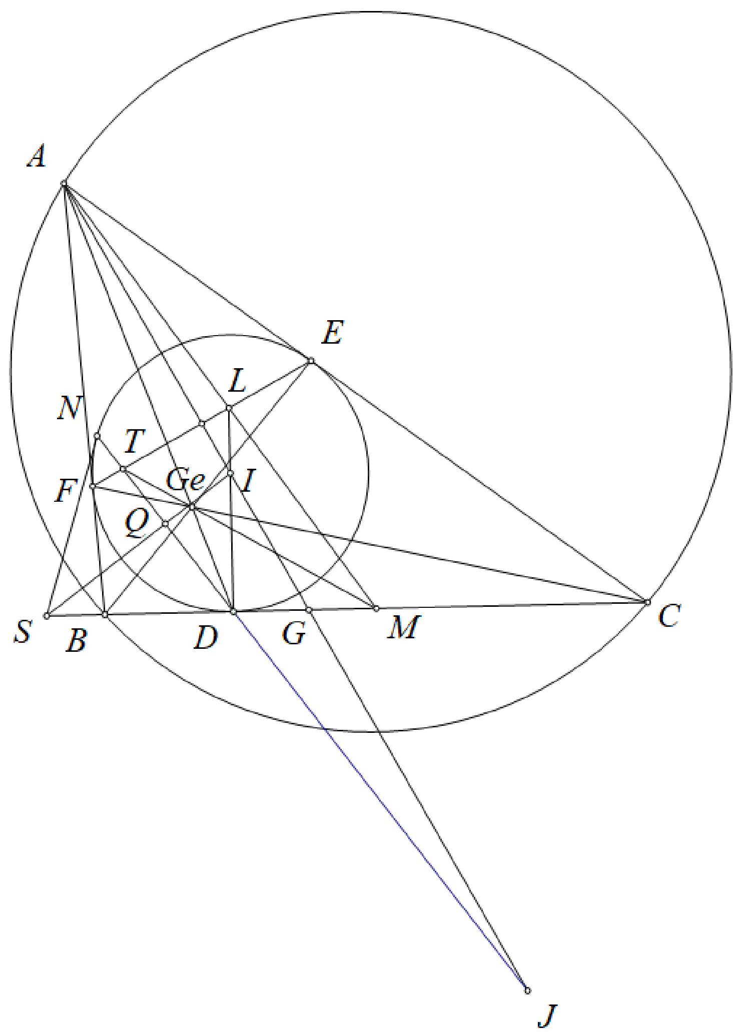
$$\frac{DB}{DC} = \frac{HB}{HC} = \frac{BF}{AF} \cdot \frac{EA}{EC} = \frac{UB}{UA} \cdot \frac{UA}{UC}$$

(since UF is the bisector of $\angle AUB$ and UE is the bisector of $\angle AUC$)

$$= \frac{UB}{UC}.$$

Therefore, UD is the bisector of $\angle BUC$. As UI is also the bisector of $\angle BUC$, it follows that UI passes through D, as desired.

Lemma 2: Let triangle ABC be circumscribed about the incircle (I). The incircle (I) is tangent to BC, CA, AB at D, E, F, respectively. Let BE and CF intersect at G_e. Let N be the point on (I) such that the circle (BNC) is tangent to (I) at N. Prove that lines ND, G_eM, EF are concurrent.



Proof: Let DJ intersect (I) again at N', and the line through I perpendicular to DN' intersect DN' and BC at Q and S, respectively. Since B, Q, I, C, J are concyclic on the circle with diameter IJ, we have

$$SB \cdot SC = SQ \cdot SI = SD^2 = SN'^2,$$

which implies that SN' is tangent to $(BN'C)$. Therefore, SN' is the common tangent of $(N'BC)$ and (I) at N', so $(BN'C)$ is tangent to (I) at N'. Hence $N' \equiv N$, and thus D, N, J are collinear.

It is a well-known result that if DI intersects EF at L, then points A, L, M are collinear.

Let DN intersect EF at T. Proving the lemma is equivalent to proving that T, G_e, M are collinear.

Let AI intersect BC at G. We have

$$-1 = D(JI, GA) = D(TL, MA) = T(DL, MA).$$

Moreover,

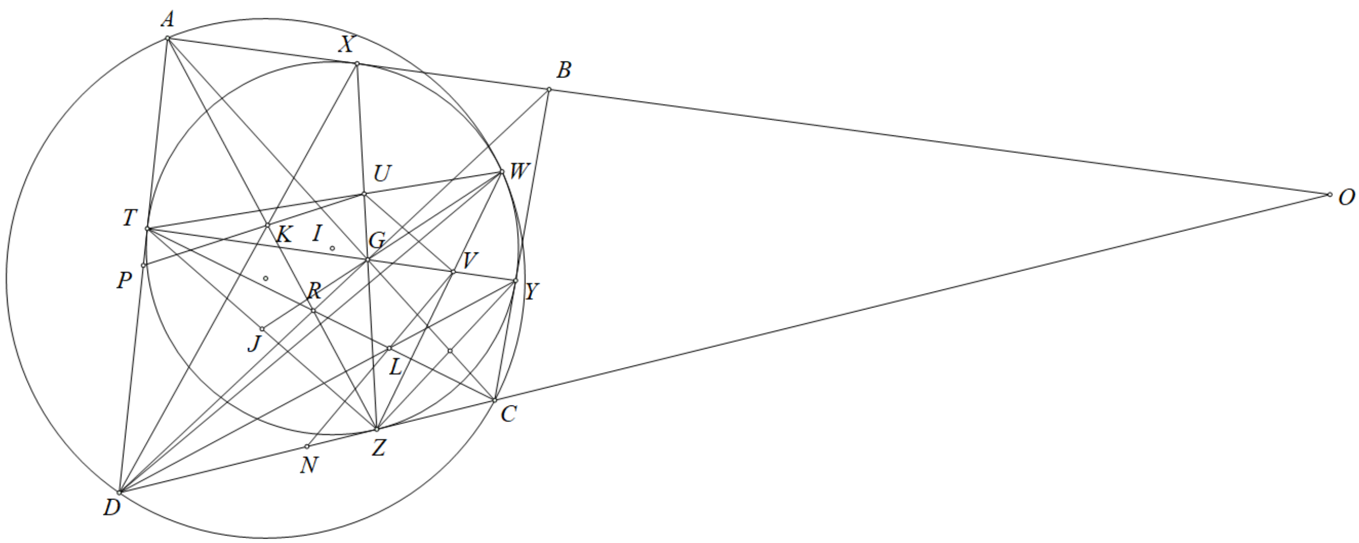
$$T(DL, G_eA) = F(DE, G_eA) = -1,$$

hence

$$T(DL, G_eA) = T(DL, MA) = -1.$$

Therefore, points T, G_e, M are collinear, as desired.
Hence, the claim is proved.

Back to the main problem,



We have a well-known result regarding tangential quadrilaterals: the lines AC, BD, XZ, YT are concurrent at G, and

$$\frac{GA}{GC} = \frac{AX}{CZ} = \frac{AT}{CZ},$$

with $AT = AZ$.
Using these ratios and applying Ceva's theorem to triangle ADC, we deduce that DG, CT, AZ are concurrent.
Let the circle (DAC) be tangent to (I) at W. Since (I) istangent to DA and DC at T and Z, respectively, it follows that (I) is the D-mix circle of triangle DAC.
Let J be the midpoint of TZ; then J is the incenter of triangle DAC. By Lemma 1, we obtain that J, G, W are collinear, which is equivalent to WG bisects TZ.
Let AB intersect CD at O. Then (AWD) is the circle passing through A, D and tangent to (I) at W, while (I) is the incircle of triangle OAD, tangent to AD, DO, OA at T, Z, X, respectively.
Since G is the intersection of AC and BD, applying Lemma 2 gives that TW, PK, XZ are concurrent. Let U be the intersection of PK and XZ, then T, U, W are collinear. Similarly, Z, V, W are collinear.
Therefore, in triangle WTZ, points U and V lie on WT and WZ, respectively, and line TV intersects ZU at G. Since WG bisects TZ, it follows that $UV \parallel TZ$.
Hence, by Reim's lemma, quadrilateral $UVYX$ is cyclic, as desired.
Therefore, the problem is proved