# Problem 12

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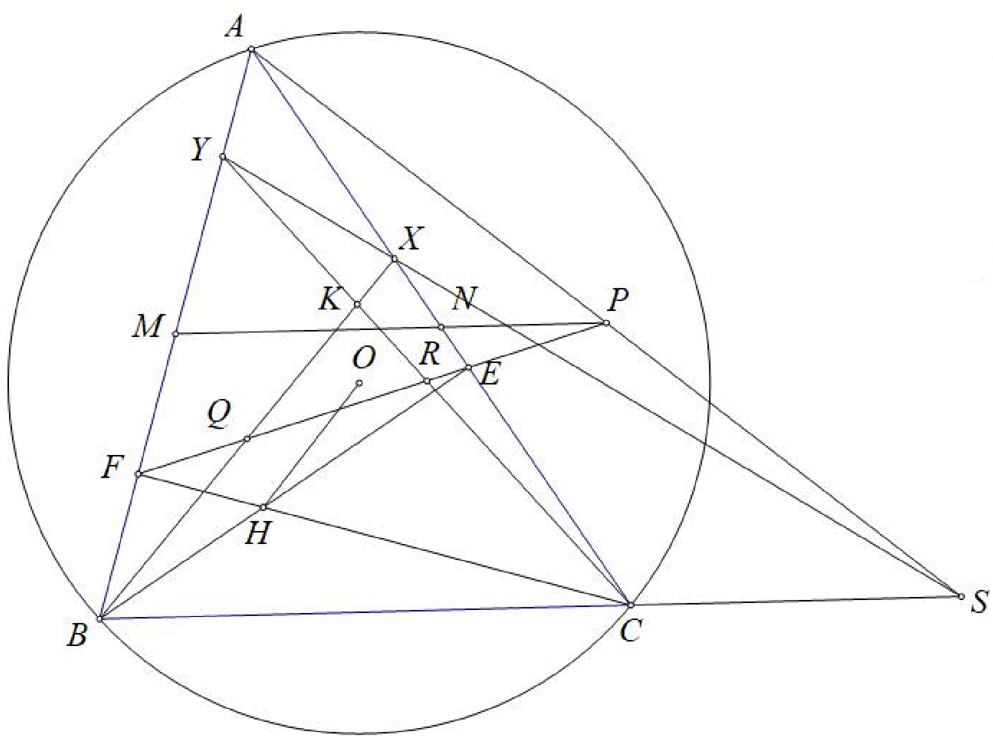
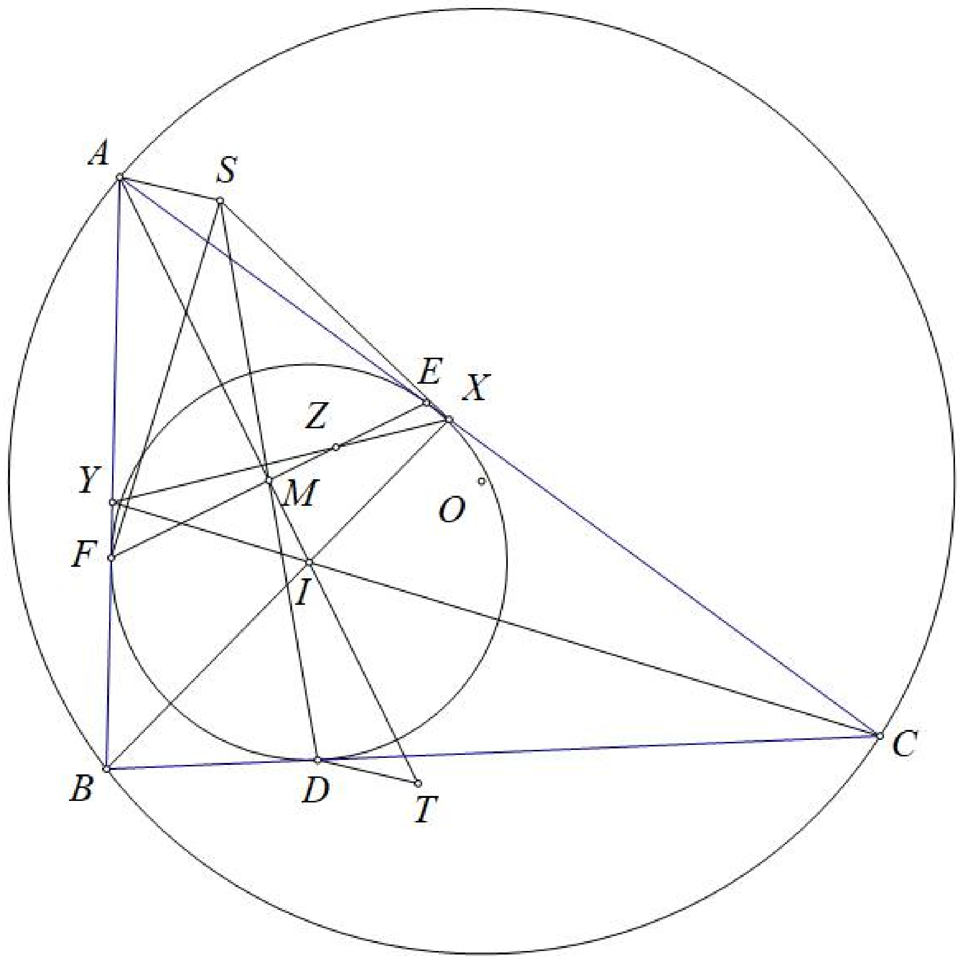
Lemma 1: Triangle $ABC$ is inscribed in $(O)$ with $K$ as the orthocenter of $\triangle BOC$, and $H$ as the orthocenter of $\triangle ABC$
Let the line through $A$ perpendicular to $OH$ be $Ax$, then $(AK, Ax, AB, AC) = -1$




Proof: $BK$, $CK$ intersect $AB$, $AC$ at $X$, $Y$; $XY$ intersects $BC$ at $S$
Since $\angle AEF = \angle ABC = \angle BXC$, $EF$ bisects $BX$ at $Q$, and similarly $EF$ bisects $CY$ at $R$
Let $P$ be the midpoint of $AS$, then $P, R, Q$ are collinear, so $E, F, P$ are collinear
Let $M, N$ be midpoints of $AB, AC$, then $P \in MN$
Hence $P$ lies on the radical axis of $(AH)$ and $(AO)$, so $AS \perp OH$ or $Ax \equiv AS$
Therefore $-1 = A(KS, BC) = (AK, Ax, AB, AC)$, as desired.

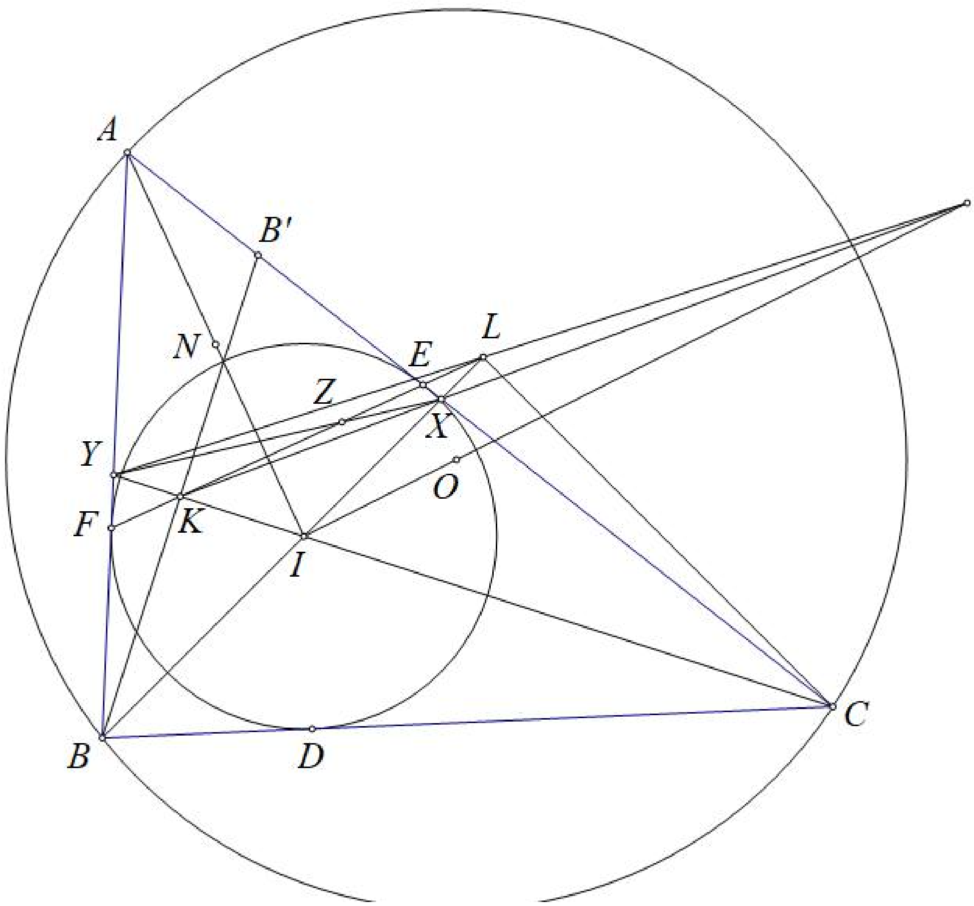
Lemma 2: Triangle $ABC$ is inscribed in $(O)$ and circumscribed about $(I)$, with $BI, CI$ intersecting $AB, AC$ at $X, Y$
$(I)$ touches $BC, CA, AB$ at $D, E, F$; $EF$ intersects $XY$ at $Z$, then $I(ZO, BC) = -1$




Proof: Let $M$ be the midpoint of $EF$, $S$ the reflection of $D$ across $M$, then $SF \parallel DE$
Hence $SF \perp YI$, so $S$ lies on the polar of $Y$ with respect to $(I)$
Similarly, $S$ lies on the polar of $X$ with respect to $(I)$, so $XY$ is the polar of $S$ with respect to $(I)$
Therefore, $Z$ lies on the polar of $S$ with respect to $(I)$, by La Hire's theorem, $S$ lies on the polar of $Z$ with respect to $(I)$.
Similarly, since $EF$ is the polar of $A$ with respect to $(I)$, $A$ lies on the polar of $Z$ with respect to $(I)$.
Thus $AS$ is the polar of $Z$ with respect to $(I)$, or $AS \perp IZ$.
Let $T$ be the reflection of $A$ across $M$, then $T$ is the orthocenter of $\triangle IEF$
Since $OI$ is the Euler line of $\triangle DEF$, by Lemma: Let $Dx$ be the line perpendicular to $OI$
Then the four lines $DT, Dx, DF, DE$ through point $D$ form a harmonic bundle with two pairs of lines perpendicular
Hence $-1 = (DT, Dx, DF, DE) = I(ZO, BC)$, as desired.

Lemma 3: Triangle $ABC$ is inscribed in $(O)$ and circumscribed about $(I)$, with $BI, CI$ intersecting $AC, AB$ at $X, Y$

Claim: The median from $X$ of $\triangle AXI$, the median from $Y$ of $\triangle AYI$, and $OI$ are concurrent




Let $(I)$ touch $AC, AB$ at $E, F$, then $E, F, K, L$ are collinear

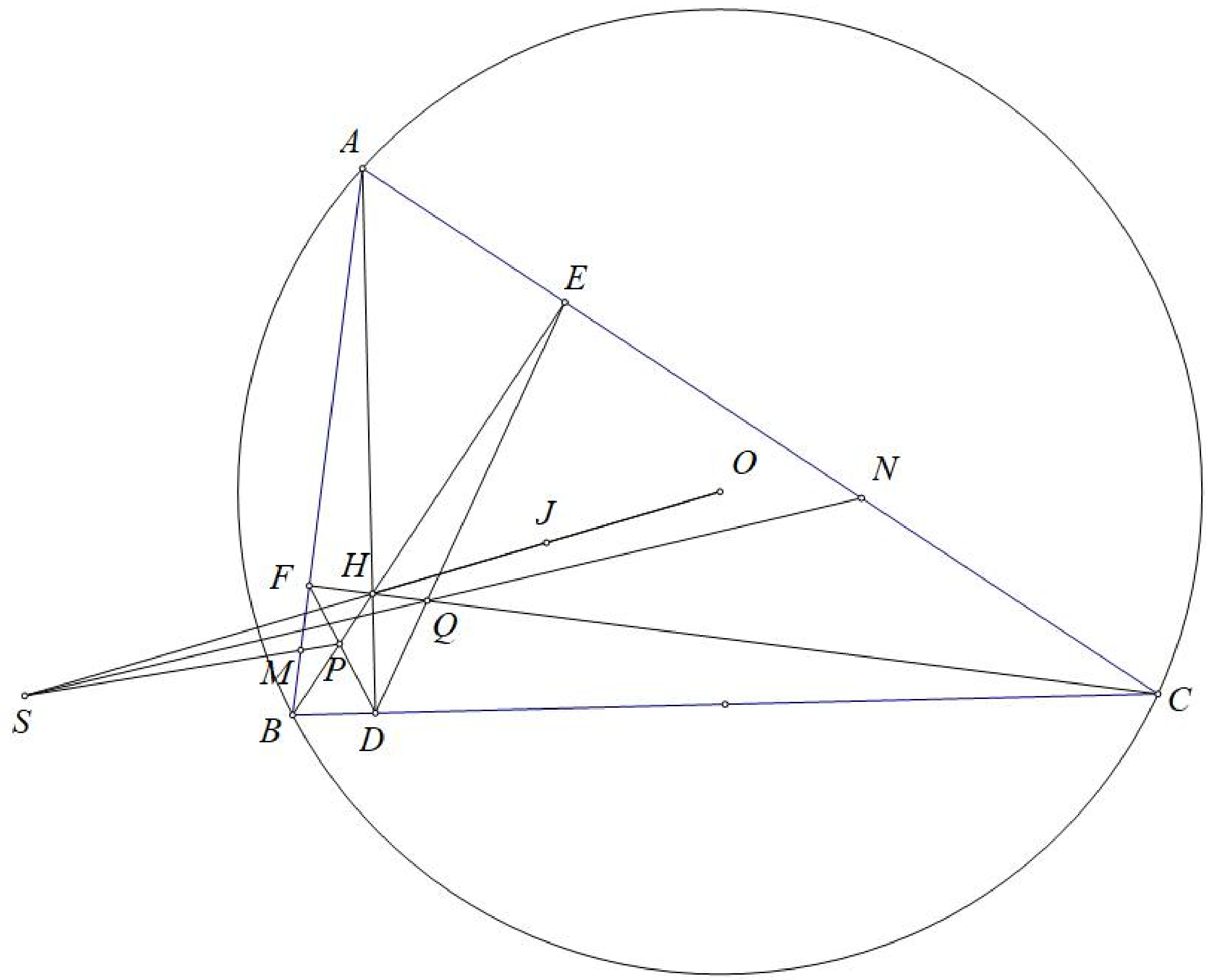
$EF$ intersects $XY$ at $Z$, by Lemma 2 we have $-1 = I(ZO, BC) = I(ZO, XY)$, hence $XK, YL, OI$ are concurrent(1)

Let $N$ be the midpoint of $AI$, $BK$ intersects $AC$ at $B'$, then $K$ is the midpoint of $BB'$

Moreover, $\triangle XIA \sim \triangle XB'B$ (angle-angle), so $XK$ is the median from $X$ of $\triangle AXI$, similarly $YL$ is the median from $Y$ of $\triangle AYI$.

Hence, (1) is equivalent to the median from $X$ of $\triangle AXI$, the median from $Y$ of $\triangle AYI$, and $OI$ are concurrent, as desired.

Back to the main problem,




To see that $H$ is the incenter of $\triangle DEF$
$PM$ is the median from $P$ of $\triangle PHD$, $QN$ is the median from $Q$ of $\triangle QHD$
Let $J$ be the circumcenter of $(DEF)$, then $J$ is the midpoint of $HO$
Applying Lemma 3 to $\triangle DEF$, we get $HJ, PM, QN$ are concurrent
Hence $HO, PM, QN$ are concurrent, or the problem is proven.