Problem 3

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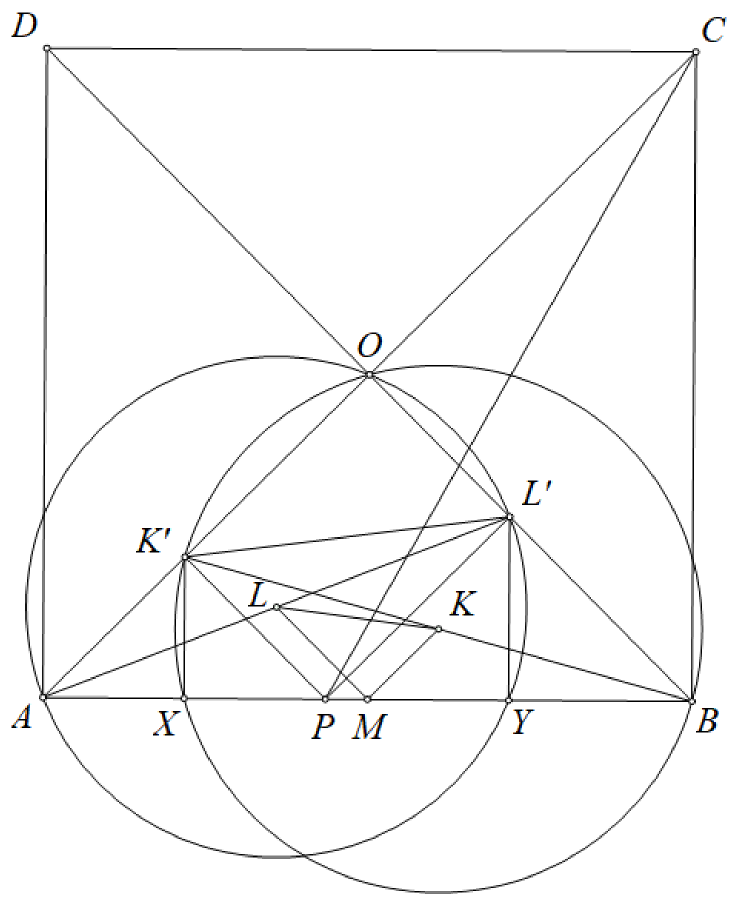

Let O be the center of $ABCD$, let X, Y be the midpoint of PA, PB respectively.
Then we get: (OAY) and (OBX) are the Euler circles of $\triangle PBD$ and $\triangle PAC$, respectively.
Therefore, K and L are the centers of (OBX) and (OAY), respectively.
Therefore, let K', L' be the reflections of B, A through K, L, respectively,
then we get $K' \in OA, K'X \perp AB$ and $L' \in OB, L'Y \perp AB$.
Let M the midpoint of AB then $\overrightarrow{ML} = \frac{1}{2} \cdot \overrightarrow{BL'}$ and $\overrightarrow{MK} = \frac{1}{2} \cdot \overrightarrow{AK'}$.
Therefore, $ML \perp MK \Longrightarrow KL^2 = ML^2 + MK^2 = \frac{1}{4} \cdot (BL'^2 + AK'^2)(1)$.
Let $PA = x, PB = y(x, y \in \mathbb{Z}^+$, then we have
$AL^4 = \frac{1}{16} \cdot AL'^2 = \frac{1}{16} \cdot (AY^2 + YL'^2)^2$
$= \frac{1}{16} \cdot [(x + \frac{y}{2})^2 + (\frac{y}{2})^2]^2$.
Similarly $BK^4 = \frac{1}{16} \cdot [(y + \frac{x}{2})^2 + (\frac{x}{2})^2]^2$.

From (1), we get $KL^4 = \frac{1}{16} \cdot (BL'^2 + AK'^2) = \frac{1}{16} \cdot (\frac{x^2}{2} + \frac{y^2}{2})$
Therefore:

$$S = \frac{32}{3} \cdot \big(AL^4 + BK^4 + KL^4\big)$$
$$= \frac{2}{3} \cdot \Big((x^2+y^2)^2 + [(x+\frac{y}{2})^2 + (\frac{y}{2})^2]^2 + [(y+\frac{x}{2})^2 + (\frac{x}{2})^2]^2\Big)$$
$$= x^4 + y^4 + 3x^2y^2 + 2x^3y + 2xy^3$$
$$= (x^2+y^2)^2 + x^2y^2 + 2xy(x^2+y^2)$$

$= (x^2+y^2+xy)^2$, which is a perfect square.

Hence, $S = \frac{32}{3} \cdot \big(AL^4 + BK^4 + KL^4\big)$ is a perfect square, as desired.