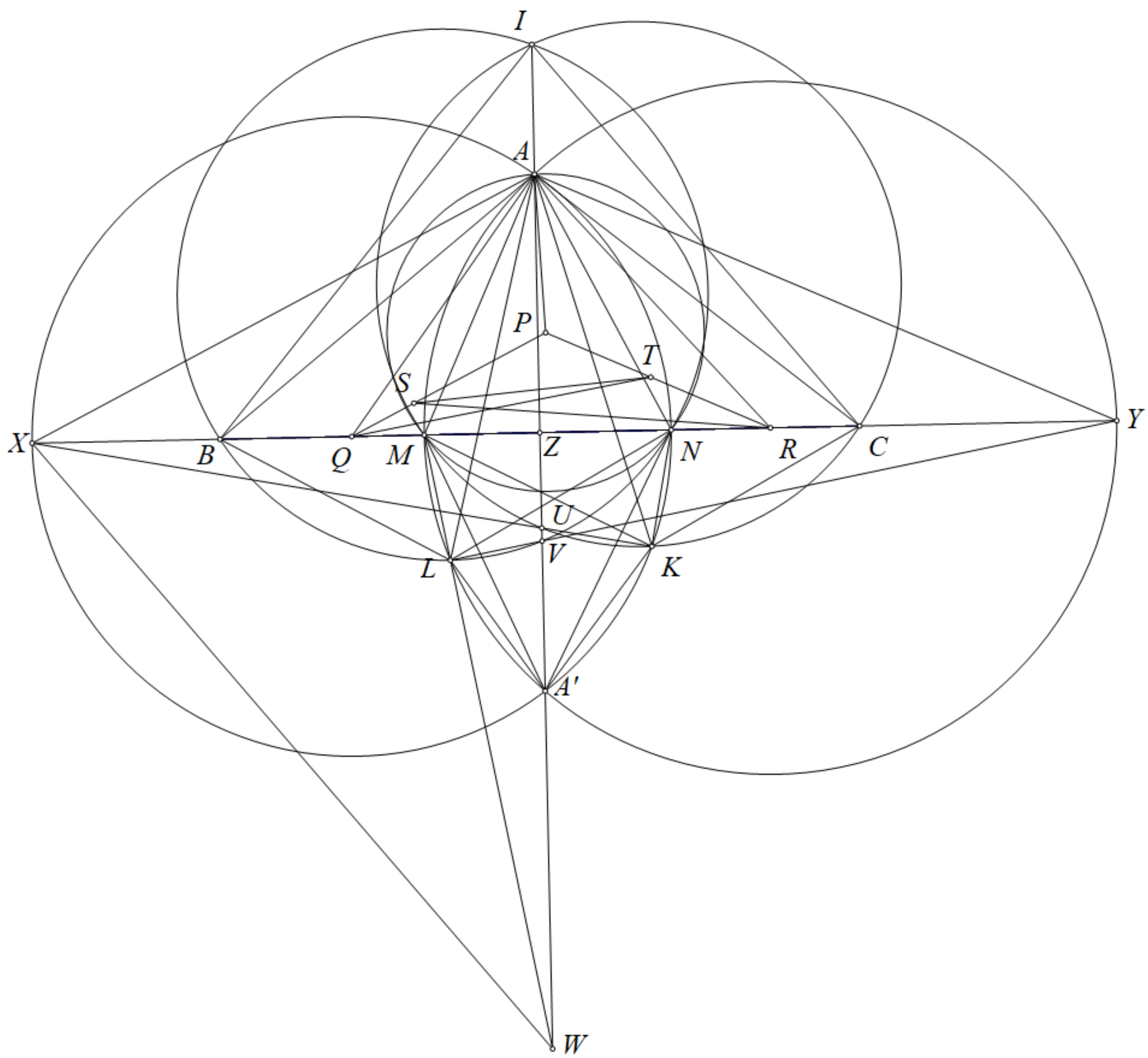
Problem 12

Ha Vu Anh



Proof: We have $ST \perp AP$ and $PS \perp AN$, so $\angle PST = \angle PAN = 90^\circ - \angle AMN = \angle PRQ$.
Hence $QSTR$ is cyclic, and thus $\angle SQT = \angle SRT$.
From the hypothesis, $QT \perp AK$, $QS \perp AN$, $RS \perp AL$, and $RT \perp AM$, so $\angle KAN = \angle SQT = \angle SRT = \angle LAM \quad (1)$.

Let A' be the reflection of A across BC. Then $A' \in (ANK)$ and $A' \in (AML)$.
Let BC meet (ANK) and (AML) at X, Y, respectively; then NX and MY are diameters of these circles.
Let KX and LY meet AA' at U, V. We prove that $V \in (BLN)$.

Let AA' meet ML at W. We have $ZV \cdot ZW = ZM \cdot ZY = ZA^2 = ZX \cdot ZN$, and since $WZ \perp XN$, it follows that V is the orthocenter of $\triangle XWN$, hence $NV \perp XW$.

From the hypothesis, $\angle BAM = 90^\circ - \angle ANM = \angle AXM$, so $MX \cdot MB = MA^2 = ML \cdot MW$.
Therefore, $\angle MLB = \angle MXW = 90^\circ - \angle VNX$.
This gives $180^\circ = \angle VNX + \angle MLB + 90^\circ = \angle VNX + \angle BLV$,
hence $VBLN$ is cyclic. Similarly, we also have $UCKM$ cyclic.

Let UV meet $(VBLN)$ at I. Then $\angle VIB = 180^\circ - \angle BLV = 90^\circ - \angle BLM = 90^\circ - \angle MXW = \angle ZNV$.
By similar angle chasing, we also obtain $180^\circ - \angle CKY = \angle ZMU$.

From (1), we have $\angle KAN = \angle LAM \iff \angle KXN = \angle LYM$,

thus $\triangle UZX \sim \triangle VZY$ (angle-angle) and $\dfrac{ZU}{ZV} = \dfrac{ZX}{ZY}$.

Moreover, since $ZM \cdot ZY = ZA^2 = ZN \cdot ZX$, we have

$$\frac{ZM}{ZN} = \frac{ZX}{ZY} = \frac{ZU}{ZV}.$$

Hence $\triangle ZUM \sim \triangle ZVN$, thus $\angle ZNV = \angle ZMU$.
Thus $\angle VIB = \angle ZNV = \angle ZMU$.

Since UV is the perpendicular bisector of BC, we have $\angle UIC = \angle VIB = \angle ZMU = 180^\circ - \angle CKY$, so I lies on (CKU), meaning that C, K, M, U, I are concyclic.
Therefore, I lies on both (CKM) and (BLN), and since I also lies on the perpendicular bisector of BC, as desired.