Problem 2

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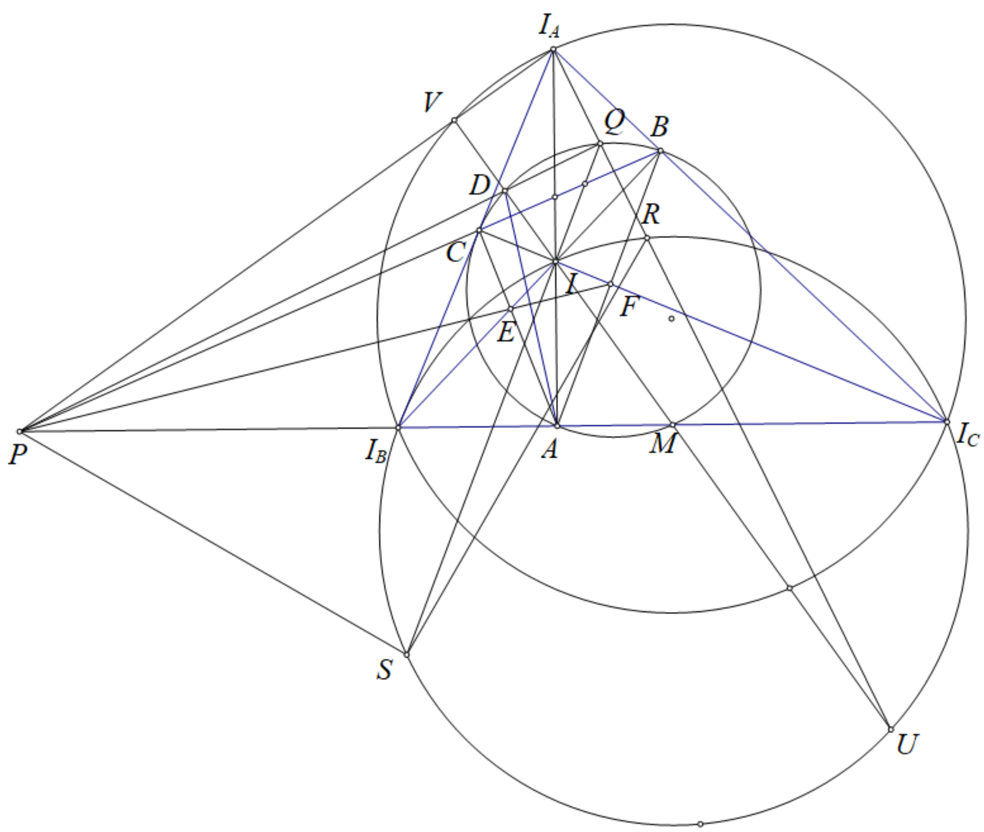
Let I_A, I_B, I_C be the $A, B, C - Excenter$ of ABC respectively, W be the circumcenter of (II_BI_C).
We will prove: QT is the radical axis of (W) and (APD).
Claim 1: Q lies on the radical axis of (W) and (APD).



Let QI cut (W) at S, the claim is equivalent to $DISP$ being cyclic. Let I_AQ cut (W) at R, U such that R is nearer to I_A than U.
We can easily see that P lies on I_BI_C.
Let I_AP cut $(I_AI_BI_C)$ at VM be the midpoint of I_BI_C then it is well known that M, I, D, V are collinear and $\angle MVI_A = 90°$

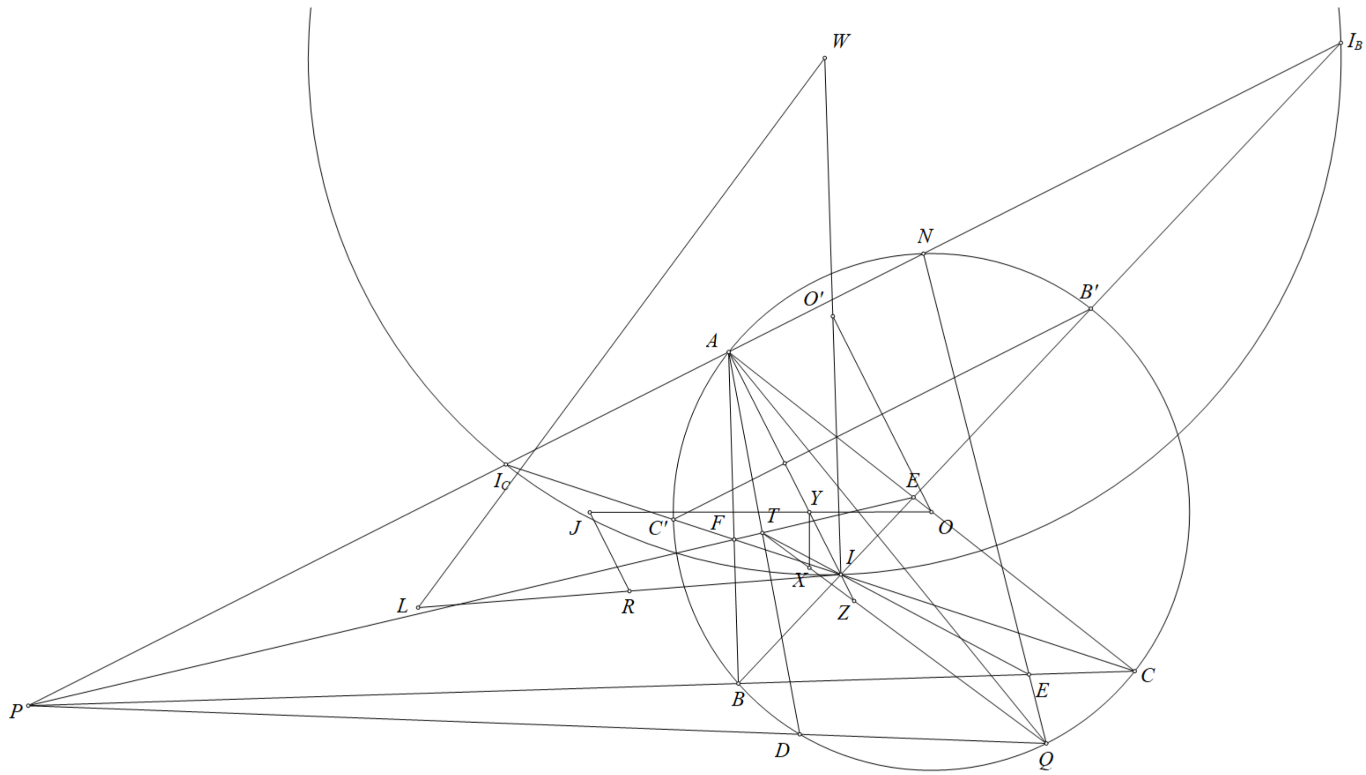
We have: Homothety center I_A, scaling factor 2 sends $(ABC)$to (W) therefore Q is the midpoint of I_AR and so $PI_A = PR$.

Also, consider an inversion about a circle at I_A with radius $\sqrt{I_AI \cdot I_AA}$.
It sends $(ABC) \mapsto (W)$.
Therefore, it also sends $Q \mapsto U$ or $I_AQ \cdot I_AU = I_AI \cdot I_AA = I_AV \cdot I_AP$,
which implies that $VQUP$ is cyclic.

From this, we get $\angle UVP = \angle UQP = 90^\circ = \angle IVP$ or V, D, I, M, U are collinear.

Also, we have $\angle QSR = \angle VUQ = \angle VPQ = \angle QPR$ or $QRSP$ is cyclic.
Therefore $\angle VDP = \angle PAQ = \angle PRQ = \angle PSQ$, which yields that $DISP$ being cyclic, as desired.
Hence the claim is proved

Back to the main problem,



Since $AIBI_C$ is cyclic then $\overline{FI} \cdot \overline{FI_C} = \overline{FA} \cdot \overline{FB}$.
Therefore, E lies on radical axis of (W) and (O), similarly we get F lies on radical axis of (W) and (O).
Therefore, EF is the radical axis of (W) and (O).
Therefore T lies on radical axis of $(W), (O)$ and $(APD), (O)$,
which yields T lies on the radical axis of $(W), (APD)$.
From the claim above we have Q lies on the radical of $(W), (APD)$
therefore QT is the radical axis of (W) and (APD).
This implies that $QT \perp WL$.
Let O' be the midpoint of IW then $O'R \parallel LW$, O' is the circumcenter of $(IB'C')$ with B', C' being the midpoint of II_B, II_C.
Simple angle chasing yields O, O' are reflections wrt $B'C'$, which is also the perpendicular bisector of AI.
Since R, J are also reflection wrt the perpendicular bisector of AI, we get $\angle(OJ, AI) = \angle(O'R, AI) = \angle(LW, AI) = 90^\circ - \angle(AI, QT) = 90^\circ - \angle XZY$.
Therefore we get $\angle XZY = 90^\circ - \angle(OJ, AI) = \angle XYZ$ or $XY = XZ$.
Hence the problem is proved.