Problem 5

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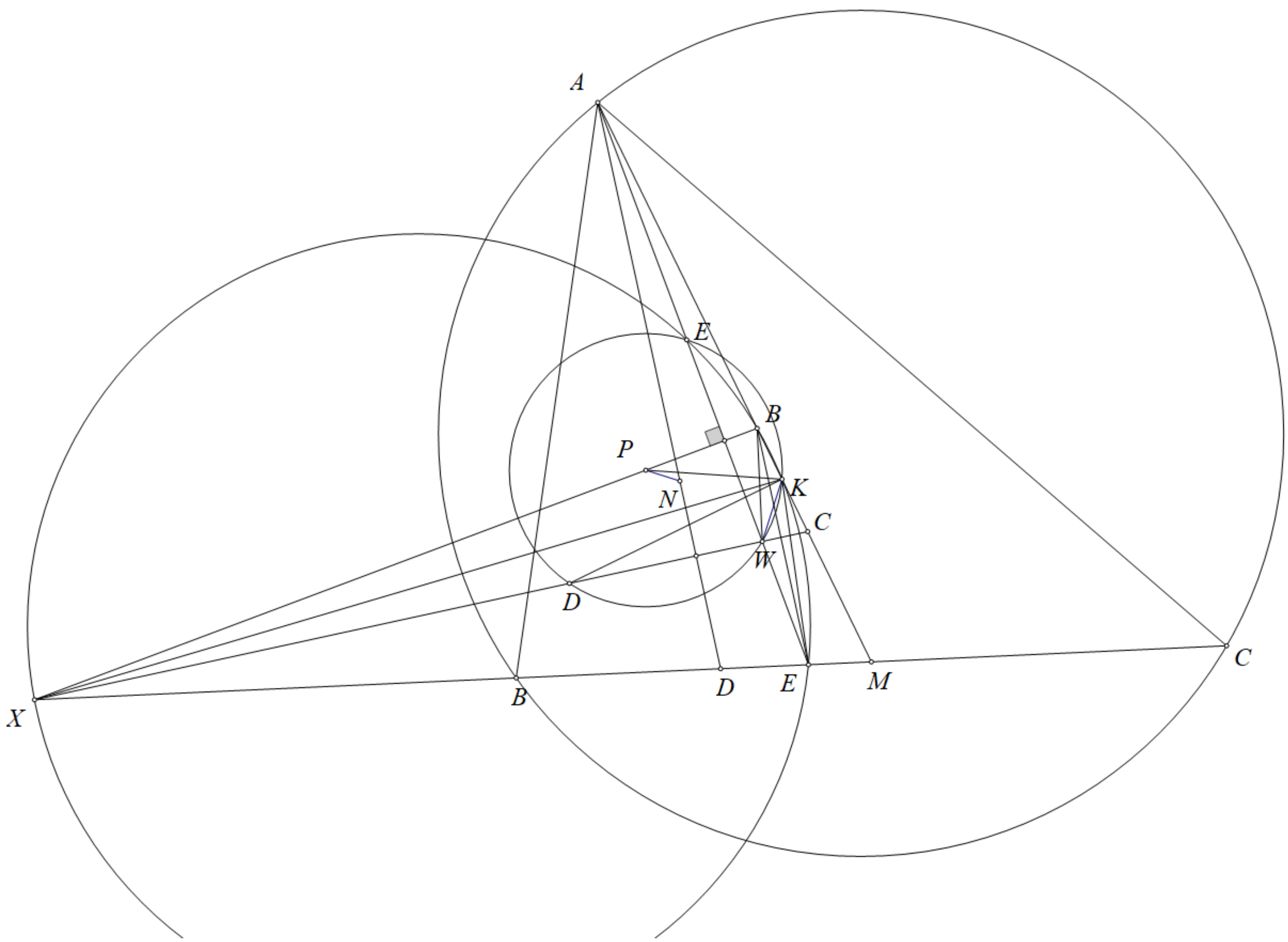

Let E be the intersection of the angle bisector of $\angle A$ of $\triangle ABC$ with BC.

Since K is the A-humpty point of $\triangle ABC$, we have

$$MB^2 = MC^2 = MK \cdot MA,$$

hence

$$\triangle MBK \sim \triangle MAB, \quad \triangle MCK \sim \triangle MAC \implies \frac{KB}{KC} = \frac{AB}{AC} = \frac{EB}{EC},$$

hence KE is the angle bisector of $\angle BKC$.

We also have

$$\angle EKM = \angle EKC - \angle MKC = \frac{\angle BKC}{2} - \angle ACB = \frac{180^\circ - \angle BAC}{2} - \angle ACB = 90^\circ - \angle AEB = \angle PXE.$$

Let PX intersect AM at S. Then

$$\angle SXE = \angle EKM \implies SKEX \text{ is cyclic } \implies MK \cdot MS = ME \cdot MX.$$

Since

$$MK \cdot MA = MB^2 = MD \cdot MX \quad (\text{since } (XD, BC) = -1),$$

we get that $AKDX$ is cyclic, and

$$\frac{MK \cdot MS}{MK \cdot MA} = \frac{ME \cdot MX}{MD \cdot MX} \implies \frac{MS}{MA} = \frac{ME}{MD} \implies SE \parallel AD \implies SE \perp XW.$$

Since $EW \perp SX$, W is the orthocenter of $\triangle SXE$.

Let (P, PW) intersect AE at U such that $UW \perp PS$. Then U is the reflection of W across XS. Since W is the orthocenter of $\triangle XSE$, we have

$$\angle XUS = \angle XWS = 180^\circ - \angle XES \implies U \in (XSE),$$

so X, U, S, K, E are concyclic. Then

$$\angle NPK = 90^\circ - \angle PKW = \angle KUW = \angle KUE = \angle KXE = \angle KAD \, (AKDX \text{ cyclic }) = \angle KAN,$$

hence $APNK$ is cyclic, as desired.
Hence the problem is proved.