Problem 3

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(Figure 1) **Claim**: $\angle XLO = \angle AOH + 2\angle(OL, BC)$

Let M be the midpoint of BC, and D the foot of the perpendicular from L onto OM.
The line ML bisects the altitude AV at U. From the given conditions, G is the midpoint of OX,
so let F be the midpoint of OL, then $\angle GFO = \angle XLO$, $\angle OFD = 2(OL, BC)$, so we need to prove that $\angle GFD = \angle AOG$.
Let AL intersect (O) and OM at E and T, respectively; then T is the intersection of the two tangents from B and C to (O).
Let N be the midpoint of AE; then ON is perpendicular to AT, so $BNCT$ is cyclic with circumcircle (OT).
Let AL intersect BC at I; we have

$TE \cdot TA = TB^2 = TI \cdot TN \Rightarrow \dfrac{TI \cdot TN}{TA} = TE$, and $IN \cdot IT = IE \cdot IA \Rightarrow$
$\dfrac{IN}{IA} = \dfrac{IE}{IT} \Rightarrow \dfrac{AN}{AI} = \dfrac{TE}{TI} \Rightarrow \dfrac{TE}{AN} = \dfrac{IT}{IA}$. Since $(TL, IA) = -1$, we have
$\dfrac{TI}{TA} = \dfrac{LI}{LA} \Rightarrow LI = \dfrac{TI \cdot AL}{AT}$.

We now prove that $\overline{N, G, D}$ are collinear.

$$\iff 1 = \frac{DM}{DT} \cdot \frac{NT}{NA} \cdot \frac{GA}{GM} = \frac{LI}{LT} \cdot \frac{NT}{NA} \cdot 2$$
$$= \frac{2 \cdot TI \cdot TN \cdot AL}{AT \cdot TL \cdot AN} = \frac{2 \cdot TI \cdot TN}{TA} \cdot \frac{AL}{TL \cdot AN} = \frac{2 \cdot TE \cdot AL}{AN \cdot TL} = \frac{2 \cdot IT}{IA} \cdot \frac{AL}{TL} = 2M(TA, IL) = 2M(TA, VU)$$
$= 2 \cdot \dfrac{AU}{AV} = 1$ (true). Hence N, G, D are indeed collinear.

Since $ODNL$ is cyclic with circumcircle (ON), we have $\angle GDO = \angle ALO \iff \angle GDF + \angle ODF = \angle OTA + \angle FOD \iff \angle GDF = \angle ATO = \angle OAG$.

Also, $\dfrac{GD}{GM} = \dfrac{\sin\angle GMO}{\sin\angle GDO} = \dfrac{\sin\angle OAL}{\sin\angle ALO} = \dfrac{OL}{OA} \iff \dfrac{GD}{OL} = \dfrac{GM}{OA} \iff \dfrac{GD}{2DF} = \dfrac{GA}{2OA}$,

hence $\triangle DGF \sim \triangle AGO$, and thus $\angle GFD = \angle AOG$ (Q.E.D.).

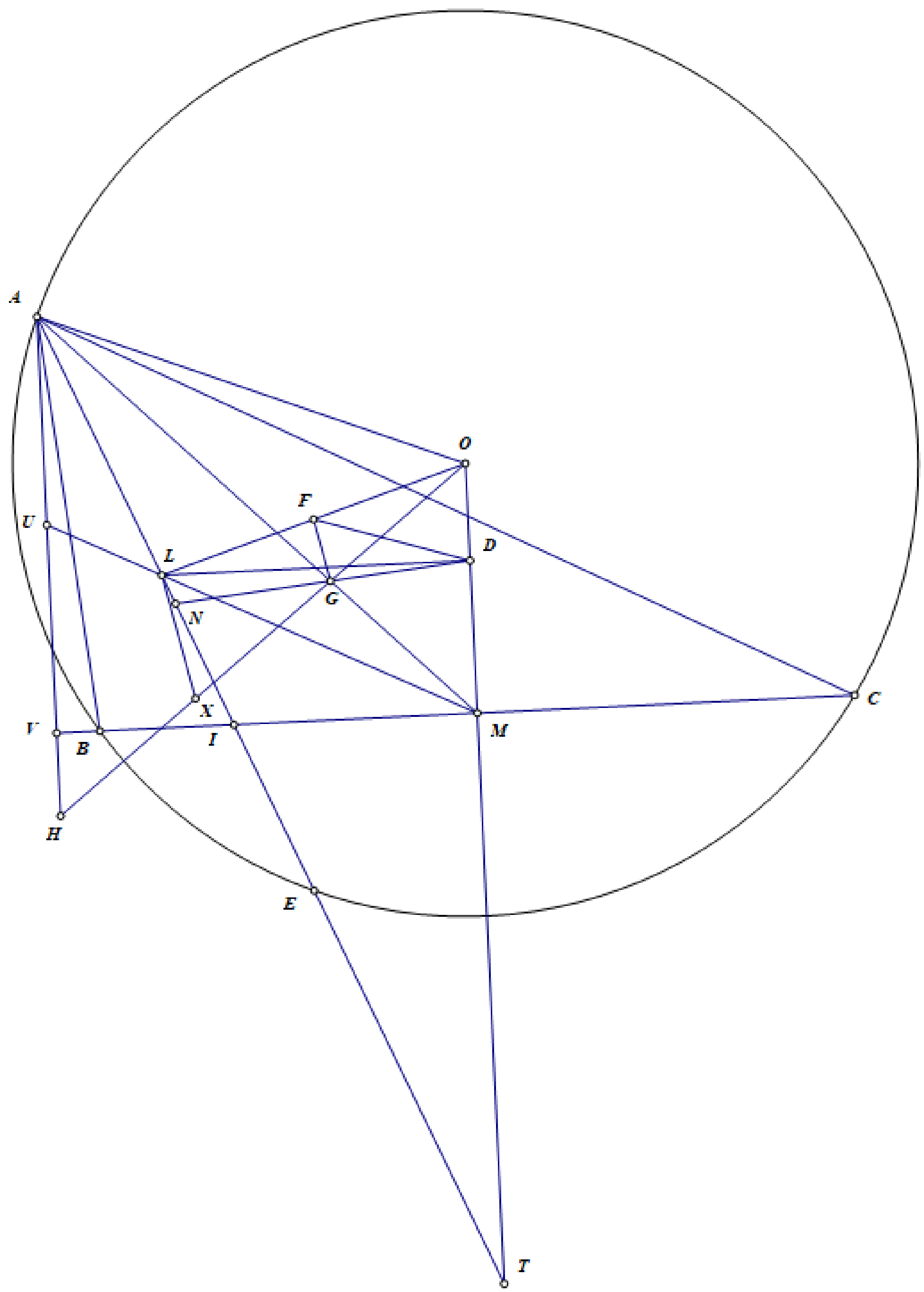

Figure 1:

Back to the main problem,

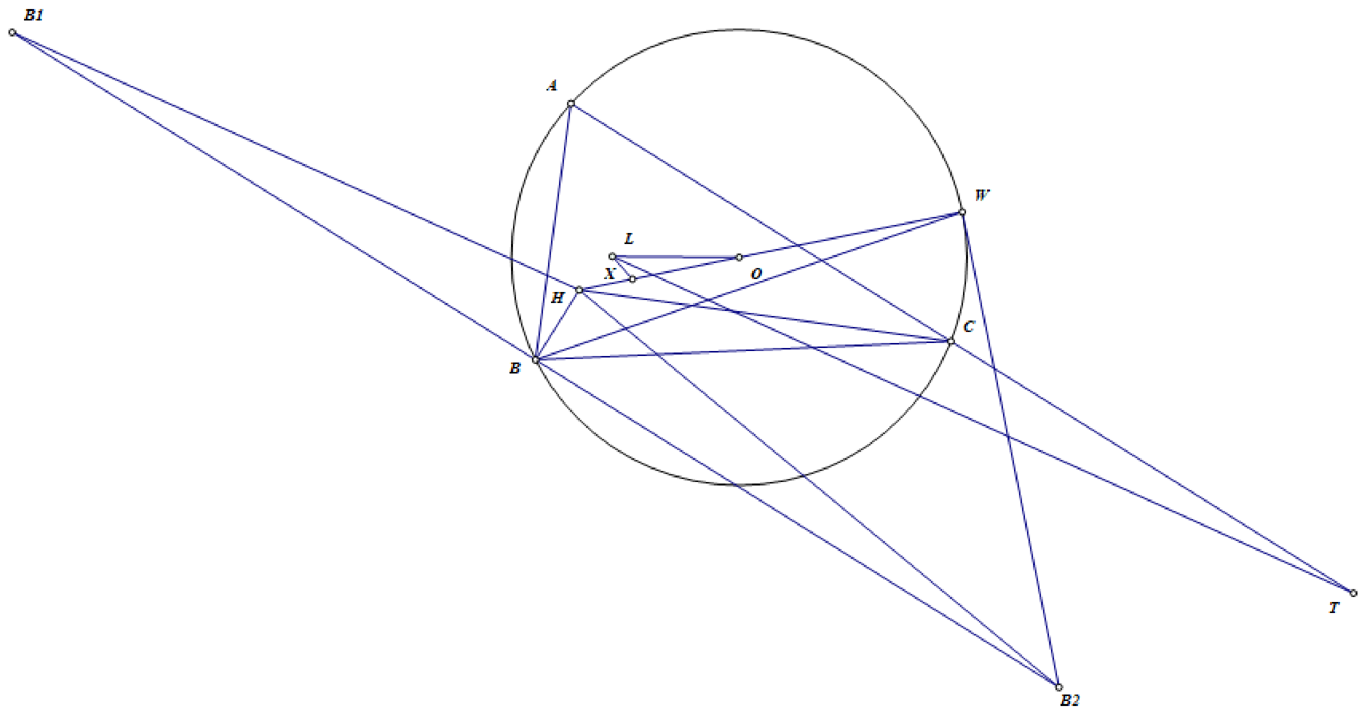

Figure 2:

(Figure 2) Let HO intersect (O) at W (with W and B lying on opposite sides of AC). We prove that B_2 lies on the tangent to (O) at W. This is equivalent to $BHWB_2$ being a cyclic quadrilateral.
$\Longleftrightarrow \angle BWH = \angle HB_2B$
$\Longleftrightarrow \angle AWB - \angle AWO = \angle HB_1B = \angle LTA$ (the bisector of angle XLO meets AC at T)
$= 180° - \angle XLO/2 - \angle ALO - \angle LAC$
$\Longleftrightarrow \angle ACB - \angle AOH/2 = 180° - \angle XLO/2 - \angle ALO - \angle LAC$
$\Longleftrightarrow \angle ACB - \angle AOH/2 = \angle ALC + \angle LCA - \angle XLO/2 - \angle ALO$
$\Longleftrightarrow \angle LCB = \angle CLO + (\angle AOH - \angle XLO)/2$
$\Longleftrightarrow 2\angle(OL, BC) = \angle AOH - \angle XLO$
$\Longleftrightarrow 2\angle(OL, BC) = |\angle AOH - \angle XLO|$ (which is true according to the claim).
Hence B_2 lies on the tangent of (O) at W. We can prove similarly that C_2 also lies on the tangent of (O) at W hence B_2C_2 tangents to (O), as desired.

Hence the problem is proved