Problem 7

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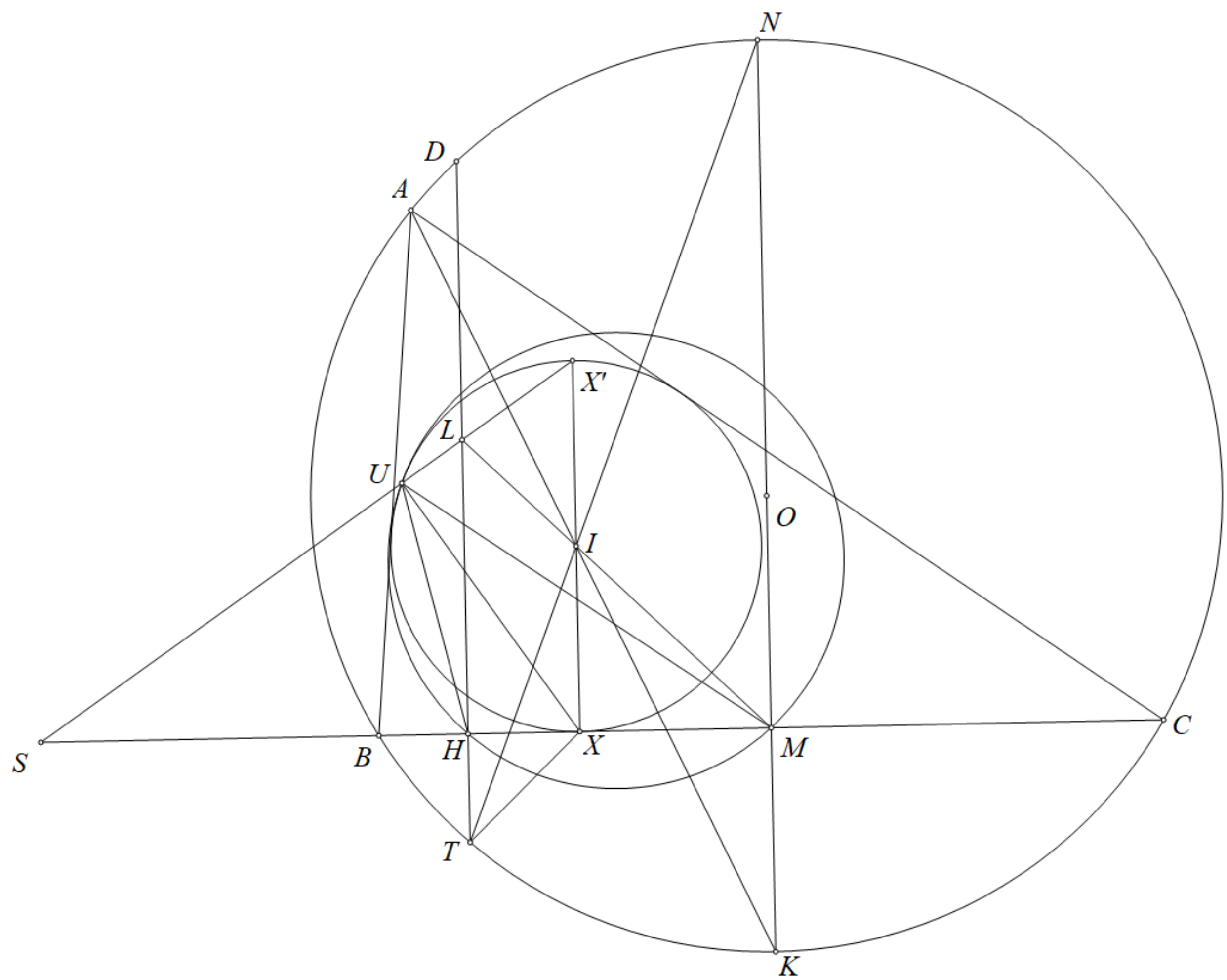

Let (I) tangents to BC at X. Draw the diameter XX' of (I). Let M be the midpoint of BC, and let MI intersect DT at L. The line LX' meets BC at S.

Let TI intersect (O) at N, which is the midpoint of arc $\widehat{BAC}$ of (O). Let H be the foot of the altitude from D in $\triangle DBC$.

The line NM intersects (O) again at K, which is the midpoint of the smaller arc $\widehat{BC}$ of (O). We have:

$$KI^2 = KB^2 = KM \cdot KN$$

thus:

$$\angle IMK = \angle KIN$$

and consequently:

$$\angle IMX = \angle ANI = \angle IKT$$

Therefore, $\triangle IMX \sim \triangle IKT$, and $\triangle IMK \sim \triangle IXT$, which implies:

$$\angle IXT = \angle IMK = \angle DLI$$

Hence, quadrilateral $LIXT$ is cyclic.

Let LX' intersect (I) again at U, then $\triangle SUX$ is right-angled. Since:

$$(SX, HM) = L(X'X, HI) = -1$$

it follows that XU is the bisector of $\angle HUM$.

Because $ULXH$ is cyclic, we have:

$$\angle HUX = \angle HLX = \angle TIX = \angle DTN$$

We also have:

$$\angle DTN = \angle DTC - \angle NTC = \angle DBC - \angle NBC = \angle DBC - \left(90^\circ - \frac{\angle BDC}{2}\right) = \frac{\angle DBC - \angle DCB}{2}$$

Therefore:

$$\angle HUM = 2\angle HUX = \angle DBC - \angle DCB$$

which means that U lies on the Euler circle of $\triangle DBC$.

Since (I) is tangent to BC at X and passes through U, and UX is the bisector of $\angle HUM$, simple angle chasing yields that (I) is tangent to (UHM) at U, with (UHM) being the Euler circle of $\triangle DBC$. Therefore, the problem is proven.