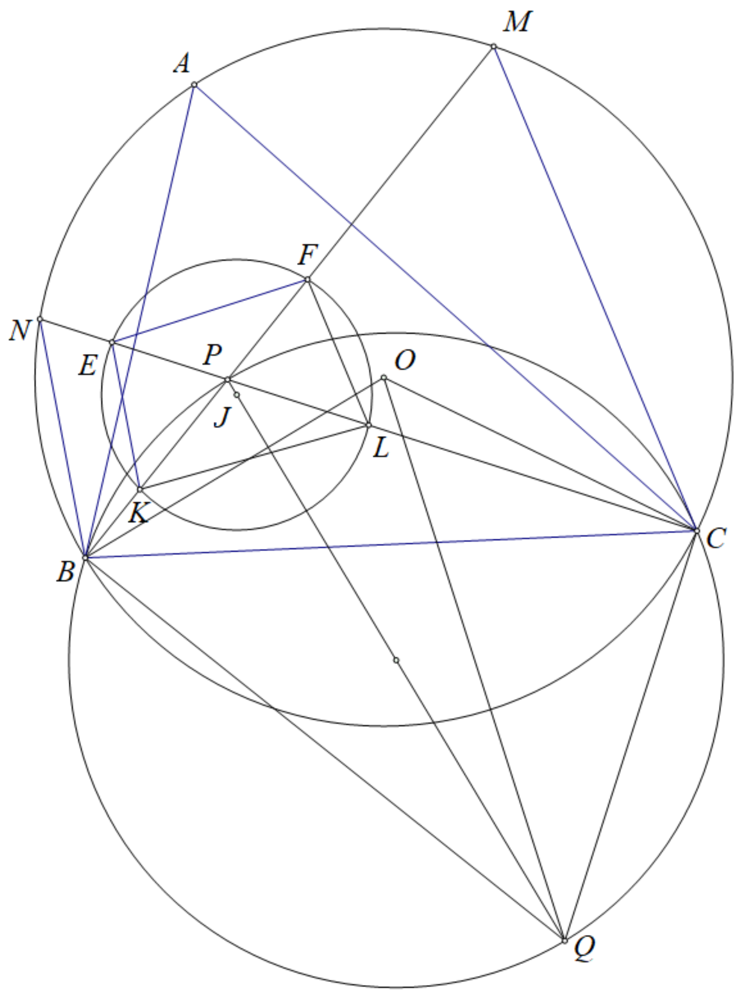
Problem 2

Ha Vu Anh



Claim: $\dfrac{PE}{PF} = \dfrac{BM}{CN}$.

Proof: Simple angle chasing yields $\angle PFE = \angle OQB, \angle OCQ = \angle NBC$ and similarly $\angle PEF = \angle OQC, \angle OBQ = \angle MCB$.

Therefore, we have:

$$\frac{PE}{PF} = \frac{sin\angle PFE}{sin\angle PEF} = \frac{sin\angle OQB}{sin\angle OQC} = \frac{sin\angle OQB}{sin\angle OBQ} \cdot \frac{sin\angle OCQ}{sin\angle OQC} \cdot \frac{sin\angle OBQ}{sin\angle OCQ}$$

$$= \frac{OB}{OQ} \cdot \frac{OQ}{OC} \cdot \frac{sin\angle BCM}{sin\angle NBC} = \frac{sin\angle BCM}{sin\angle NBC} = \frac{BM}{CN} (\text{Since } \frac{sin\angle BCM}{BM} = \frac{sin\angle NBC}{CN} = \frac{sin\angle BAC}{BC})$$

Back to the main problem,
$\angle PFL = \angle PNB = \angle PMC = \angle PEK$ yields $EFLK$ being cyclic. Since $PQ \perp MN$ the problem is equivalent to $PJ \perp MN$, or
$PM^2 - PN^2 = MJ^2 - NJ^2 = \overline{MF} \cdot \overline{MK} - \overline{NE} \cdot \overline{NL}$. This is equivalent to $\overline{MF} \cdot \overline{MK} - PM^2 = \overline{NE} \cdot \overline{NL} - PN^2 (*)$.
We have:

$\overline{MF} \cdot \overline{MK} - PM^2$
$= \overline{MF} \cdot (\overline{MP} + \overline{PK}) - \overline{MP} \cdot \overline{MP}$
$= \overline{MP} \cdot (\overline{MF} - \overline{MP}) + (\overline{MP} - \overline{FP}) \cdot \dfrac{\overline{PE} \cdot \overline{PC}}{\overline{PM}}$
$= \overline{MP} \cdot \overline{PF} - \overline{PE} \cdot \overline{PC} - x$
with $x = \dfrac{\overline{FP} \cdot \overline{PE} \cdot \overline{PC}}{\overline{PM}} = \dfrac{-\overline{PF} \cdot \overline{PE} \cdot P_{P/(O)}}{\overline{PM} \cdot \overline{PN}}$ then similarly, we get
$\overline{NE} \cdot \overline{NL} - PN^2 = \overline{NP} \cdot \overline{PE} - \overline{PF} \cdot \overline{PB} - x$.
Hence $(*)$ is equivalent to $\overline{MP} \cdot \overline{PF} - \overline{PE} \cdot \overline{PC} = \overline{NP} \cdot \overline{PE} - \overline{PF} \cdot \overline{PB} \Longleftrightarrow \overline{MB} \cdot \overline{PF} = \overline{PE} \cdot \overline{NC}$(which is true due to the claim above).
So $(*)$ is true hence the problem is proved.