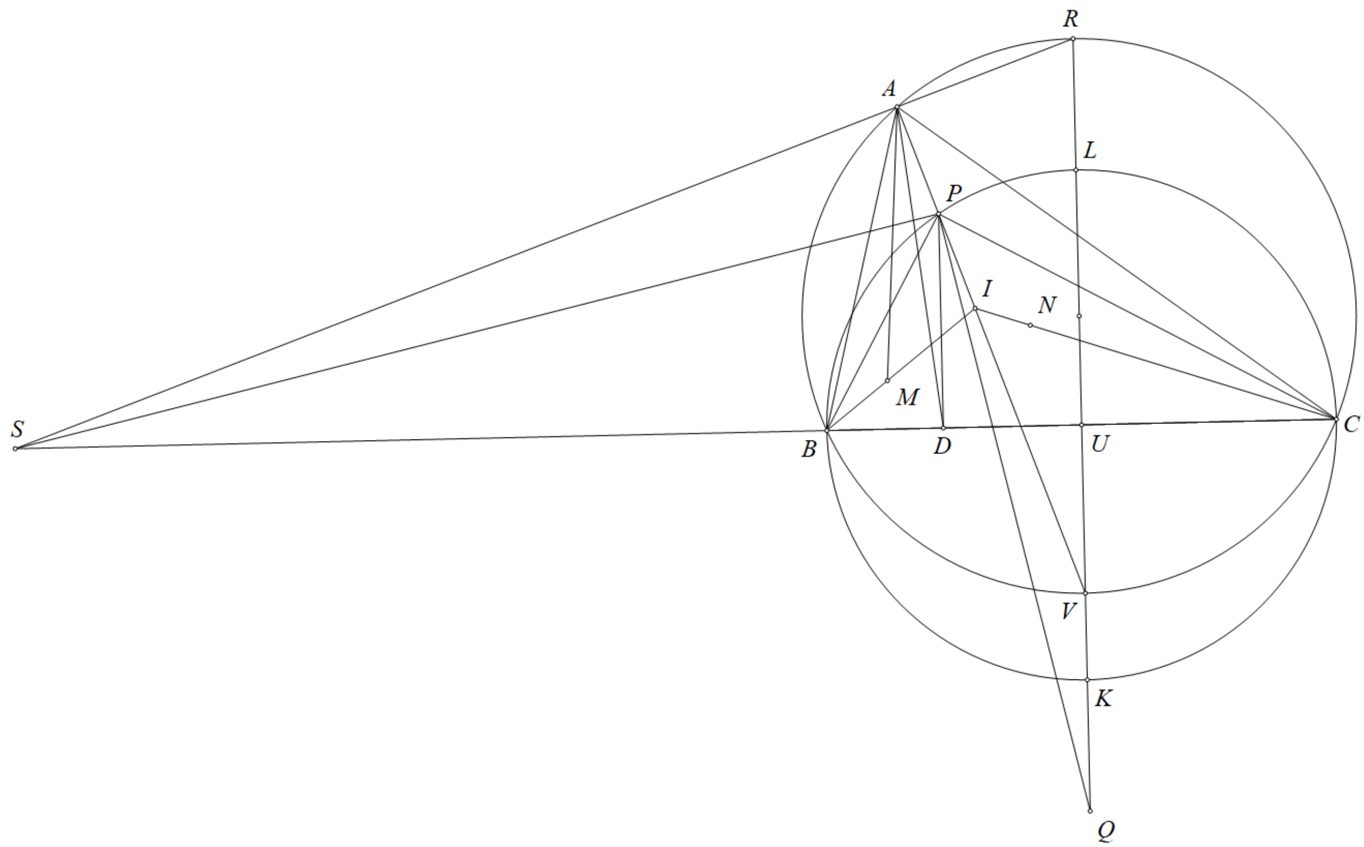
Problem 11

Ha Vu Anh



Proof: Let I be the incenter of $\triangle ABC$. Then B, M, I and C, N, I are collinear.
Let AI meet (O) again at N, and let U be the midpoint of BC. Then A, P, N are collinear and $UP = UB = UC$.
Let R be the midpoint of the arc BC containing A on (O), and let Q be the reflection of R across U. Then $UP^2 = UB^2 = UN \cdot UR = UN \cdot UQ$, so $\angle PNU = \angle QPU$.

Let AR meet BC at S. Then $\angle QSU = \angle ASU = \angle ANU = \angle QPU$, so the quadrilateral $SPUQ$ is cyclic. Since $APDS$ is cyclic, we have $\angle PQU = \angle PSD = \angle PAD$, and because $\angle APD = \angle PNQ$ it follows that $\angle ADP = \angle NPQ$.

Let UN meet the circle with diameter BC again at K, L. Then $UL^2 = UK^2 = UN \cdot UQ$, hence $(LK, NQ) = -1$.
Since $\angle KPL = 90^\circ$, line PK is the internal bisector of $\angle NPQ$.
Therefore

$$\angle ADP = \angle NPQ = 2\angle NPK = 2(\angle BPN - \angle BPK) = 2(\angle BPN - 45^\circ) = 2\angle BPN - 90^\circ.$$

Hence

$$\angle BPN = \frac{\angle ADP + 90^\circ}{2} = 90^\circ - \frac{\angle ADB}{2} = \angle AMI,$$

so $APMB$ is cyclic. Consequently $IP \cdot IA = IM \cdot IB$.

Similarly, $IM \cdot IB = IN \cdot IC = IP \cdot IA$, therefore $BMNC$ is cyclic, as desired. Hence, the problem is proven.