Problem 8

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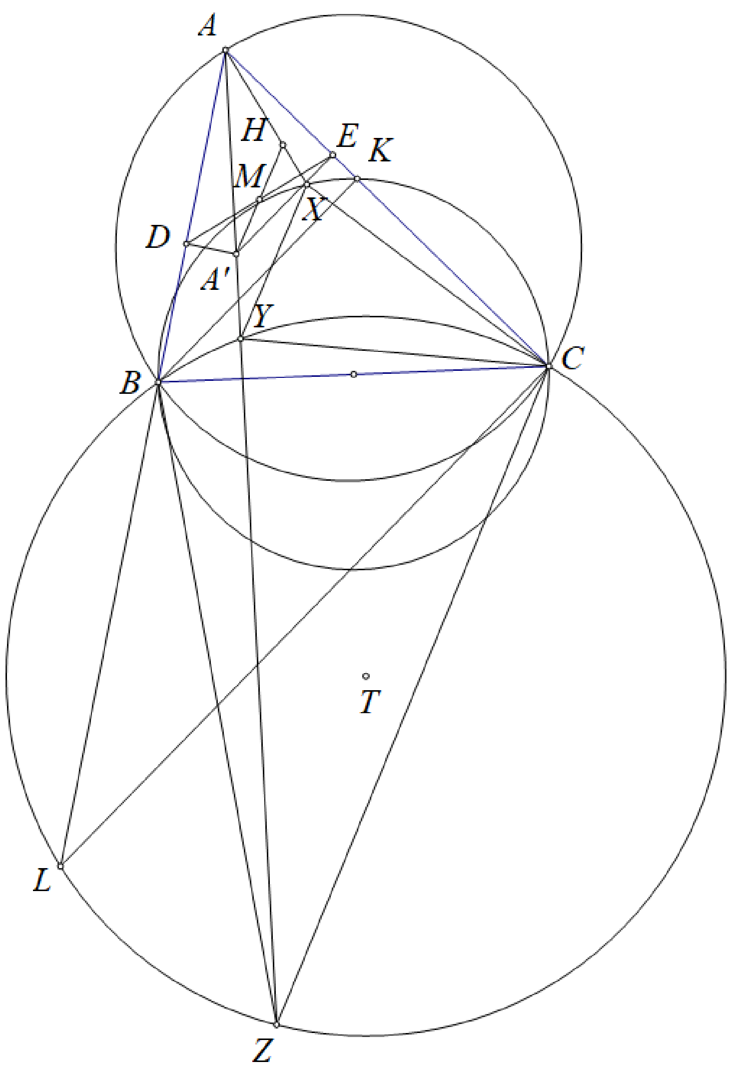
Lemma: Let triangle ABC have arbitrary point D, E lies on AB, AC respectively. Construct diameter AA' of (ADE), let M be the midpoint of DE. The line from A perpendicular to DE cut (BC) at X such that X lies inside triangle ABC. Let Y be the isogonal conjugate of X wrt ABC. Prove that $XY \parallel A'M$.



Proof: Let H be the orthocenter of ADE we got H is the intersection of AX and $A'M$.

Let T the intersection of tangents from B, C of (ABC). Let AB cut (T, TB) at L, K be the projection of B on AC we get $\angle BLC = \angle BTC/2 = 90^\circ - \angle BAC = \angle ABK$ therefore $BK \parallel CL$ therefore we get $\dfrac{AK}{AC} = \dfrac{AB}{AL}$ therefore $AB.AC = AK.AL$. Consider an inversion about a circle at A with radius $\sqrt{AB \cdot AC}$, followed by a reflection across the bisector of $\angle BAC$. It swaps B and C, K and L therefore it swaps (T, TB) with (BC) and therefore it swaps X with Z which is the intersection of AY and (T, TB).

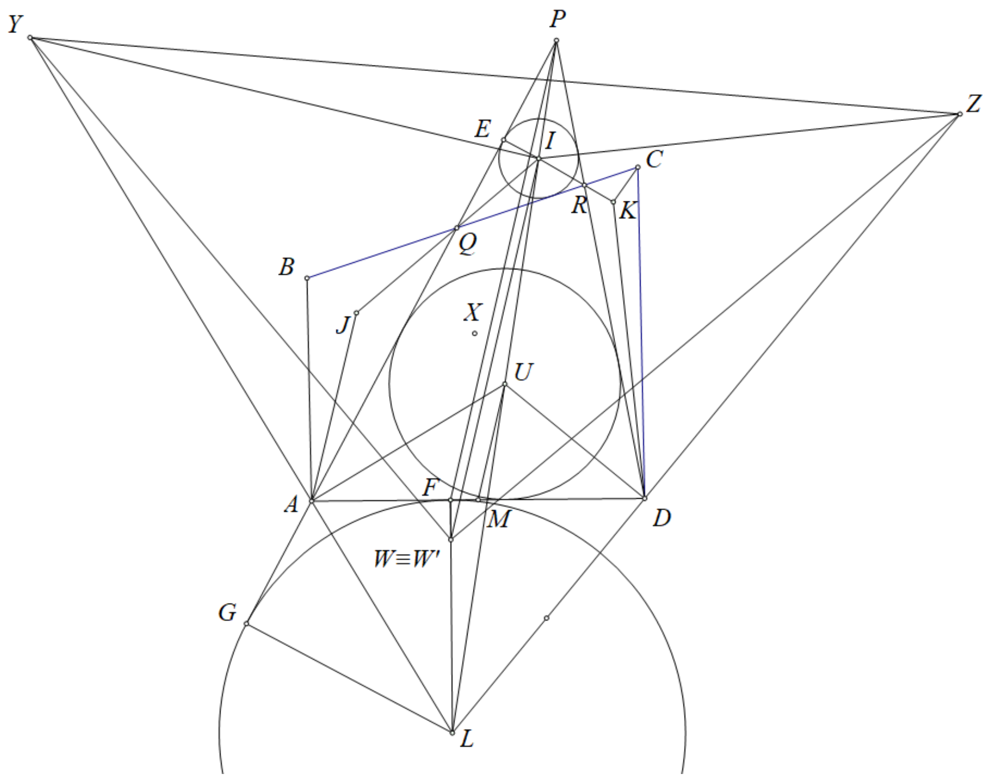
Therefore $\triangle ABZ \sim \triangle AXC$ therefore $\angle AZB = \angle ACX = \angle YCB$ therefore Y lies on (T, TB).

We have $AX \cdot AZ = AB \cdot AC$ therefore $\dfrac{AX}{AY} = \dfrac{AX \cdot AZ}{AY \cdot AZ} = \dfrac{AB \cdot AC}{AB \cdot AL} = \dfrac{AC}{AL} = \cos\angle BAC$.

Since AA' is the diameter of (ADE), H is the orthocenter of ADE we have $\dfrac{AH}{AA'} = \cos\angle DAE = \dfrac{AX}{AY}$

Therefore $XY \parallel A'M$ hence the lemma is proved.

Back to the main problem,



Let W' be the isogonal conjugate of I wrt $\triangle LYZ$ we got

$$\angle W'YZ + \angle W'ZY = \angle IYA + \angle IZD = 90^\circ - \angle YAJ + 90^\circ - \angle ZDK = \frac{\angle BAD + \angle CDA}{2} = \angle 90^\circ$$

therefore W' lies on (YZ).

Let (L) which is the $P-excenter$ of $\triangle PAD$ touches AD, AP at F, G respectively we got $\angle FLA = \angle PLD$ therefore W' lies on LF.

Let U be the incenter of PAD we get LU is the diameter of (PAD), M be the midpoint of AD. It is well known that $PF \parallel UM$.

Applying the lemma for triangle LYZ with A, D be arbitrary point lies on LY, LZ respectively, LU is the diameter of (PAD), M be the midpoint of AD, W' lies on (YZ) and the line from L perpendicular to AD, I be the isogonal conjugate of I wrt LYZ we get $W'I \parallel UM \parallel PF$ therefore $\dfrac{PI}{PL} = \dfrac{FW'}{FL}$

Let $P - excenter$ of PAD touches PA at F, (I) touch PA at E we get

$\dfrac{IE}{LG} = \dfrac{PI}{PL} = \dfrac{FW'}{FL}$ therefore $IE = FW'$ which is the inradius of PQR.

Therefore $W' \equiv W$ therefore W is the isogonal of I wrt LYZ therefore we get $\angle WYZ = \angle IYL = \dfrac{\angle BAD}{2} = \angle XAD$ and similiarly $\angle WZY = \angle XDA$ therefore two triangles WYZ and XAD are similiar.

Therefore, the problem is proved.