Problem 5

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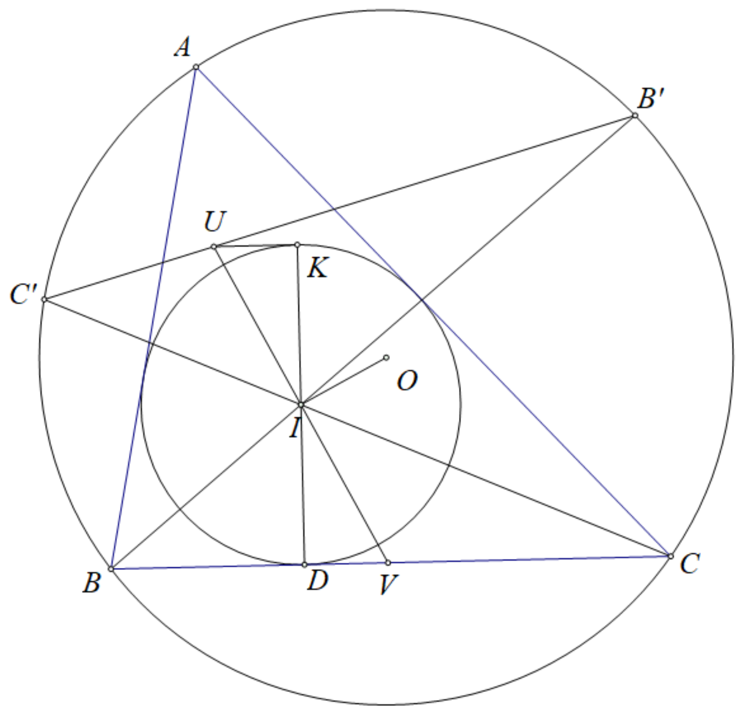
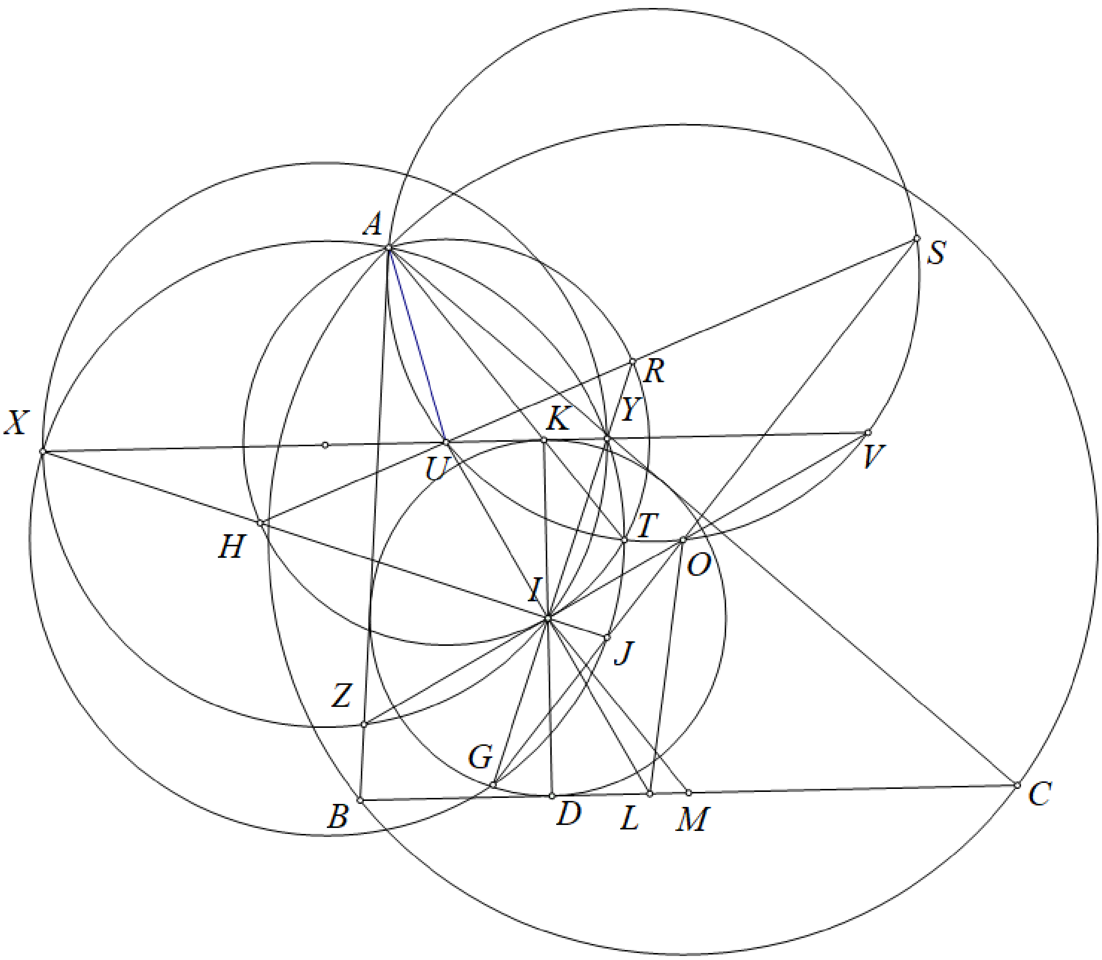
Through I, draw a line perpendicular to OI meeting XY at U
Claim 1: $UA = UI$.



Proof: let BI, CI cut (O) at B', C' respectively, let IU cut BC at V, since $UK \parallel DV$ and I is the midpoint of DK we get $IU = IV$. Apply the Butterfly theorem, we get U lies on $B'C'$ and since $B'C'$ is the perpendicular bisector of AI we get $UA = UI$, as desired.

Let IX, IY intersect (AXY) at J, G.
Claim 2: G, O, J are collinear.



Proof: Let OI meet XY at V.
According to Claim 1, we have $UA = UI \Rightarrow UA^2 = UI^2 = UK \cdot UV$ hence $\triangle UAK \sim \triangle UVA$.
Let UI meet BC at L, and let M be the midpoint of BC. Then $IOML$ is cyclic, so $\angle UOI = \angle IOL = \angle IMB$.
Since AK pass through the touchpoint of $A - excenter$ and BC, we get $AK \parallel IM$ hence $\angle IMB = \angle AKU = \angle UAV$ (since $\triangle UAK \sim \triangle UVA$),
hence $\angle UOI = \angle IMB = \angle UAV$ or $AUOV$ is cyclic.

Let AK meet $(AUOV)$ at T. Then $\angle UAK = \angle UVA = \angle UTA$, so $UA = UT$, and $KA \cdot KT = KU \cdot KV = KI^2 = KY \cdot KX$, hence T lies on (AXY).
Therefore, (AXY), (AUO), and (U, UA) are coaxial with the line AT.

IY meet (U, UA) at R, and UR meets (AUO) at S.
A well known lemma: given $(A), (B), (C)$ are 3 coaxial circles, then for every point W on (C),

$$\frac{P(W,(A))}{P(W,(B))} = constant$$

Applying the lemma for 3 coaxial circle $(U, UA), (AUO), (AXY)$ with I, R lies on (U, UA), we get:

$$\frac{P(R/(AXY))}{P(R/(AUO))} = \frac{P(I/(AXY))}{P(I/(AUO))} = constant \Leftrightarrow \frac{RY \cdot RG}{RU \cdot RS} = \frac{IY \cdot IG}{IO \cdot IV} \Leftrightarrow \frac{IG \cdot IY}{IO \cdot YR} = \frac{RG \cdot IV}{RU \cdot RS} \quad (1)$$

Let IX meet (U, UA) at H, we get that R, U, H are collinear.
We will prove that $\triangle HUX \sim \triangle IOG(*)$
We have $\angle XHU = 180^\circ - \angle UIX = 180^\circ - \angle YIV = \angle OIG$,
thus proving $(*)$ is equivalent to proving $\dfrac{IG}{IO} = \dfrac{HX}{HU} = \dfrac{HX}{UI}(**)$.

By Menelaus' Theorem for triangle HRI with X, U, Y being collinear, we have

$$\frac{XH}{XI} = \frac{UH}{UR} \cdot \frac{YR}{YI} = \frac{YR}{YI},$$

hence $(**)$ is equivalent to

$$\frac{IG}{IO} = \frac{XH}{UI} = \frac{YR \cdot XI}{YI \cdot UI} \Leftrightarrow \frac{IG \cdot IY}{IO \cdot YR} = \frac{XI}{UI}.$$

Combining with (1), this is equivalent to proving

$$\frac{RG \cdot IV}{RU \cdot RS} = \frac{XI}{UI} \Leftrightarrow \frac{XI}{IV} = \frac{RG}{RS} = \frac{RT}{RS} \cdot \frac{RG}{RT} = \frac{\sin \angle RST}{\sin \angle RTS} \cdot \frac{\sin \angle RTG}{\sin \angle RGT} = \frac{\sin \angle ATU}{\sin \angle ATR} \cdot \frac{\sin \angle IAY}{\sin \angle YAK}.$$

(Since $\angle RTG = 180^\circ - \angle TRI - \angle TGY = 180^\circ - \angle TAI - \angle TAY = 180^\circ - \angle IAY$, and $UR = UA = UT$, R is the incenter of $\triangle AST$)

$$\begin{aligned}
&= \frac{\sin \angle AVU}{\sin \angle AIY} \cdot \frac{\sin \angle IAY}{\sin \angle YAK} = \frac{\sin \angle UAK}{\sin \angle YAK} \cdot \frac{YI}{YA} \\
&= \frac{KU}{KY} \cdot \frac{AY}{AU} \cdot \frac{YI}{YA} = \frac{UK}{UA} \cdot \frac{YI}{YK} = \frac{UK}{UI} \cdot \frac{YI}{YK} = \frac{\sin \angle UIK}{\sin \angle YIK} \\
&= \frac{\sin \angle UVI}{\sin \angle VXI} = \frac{XI}{IV} \quad \text{(true)}
\end{aligned}$$

Hence $(**)$ is true therefore $(*)$ is true or $\triangle HUX \sim \triangle IOG$ hence $\angle OGI = \angle HXU = \angle IXY = \angle JGI$.
Therefore G, J, O are collinear, as desired.
Therefore, the claim is proven.

According to Claim 2, G, J, O are collinear, hence
$\angle OGI = \angle IXY = \angle IZY$, so $OYZG$ is cyclic and $P(I/(AXY)) = IY \cdot IG = IO \cdot IZ = P(I/(AOZ))$.
Hence I lies on the radical axis of (AXY) and (AOZ), as desired.