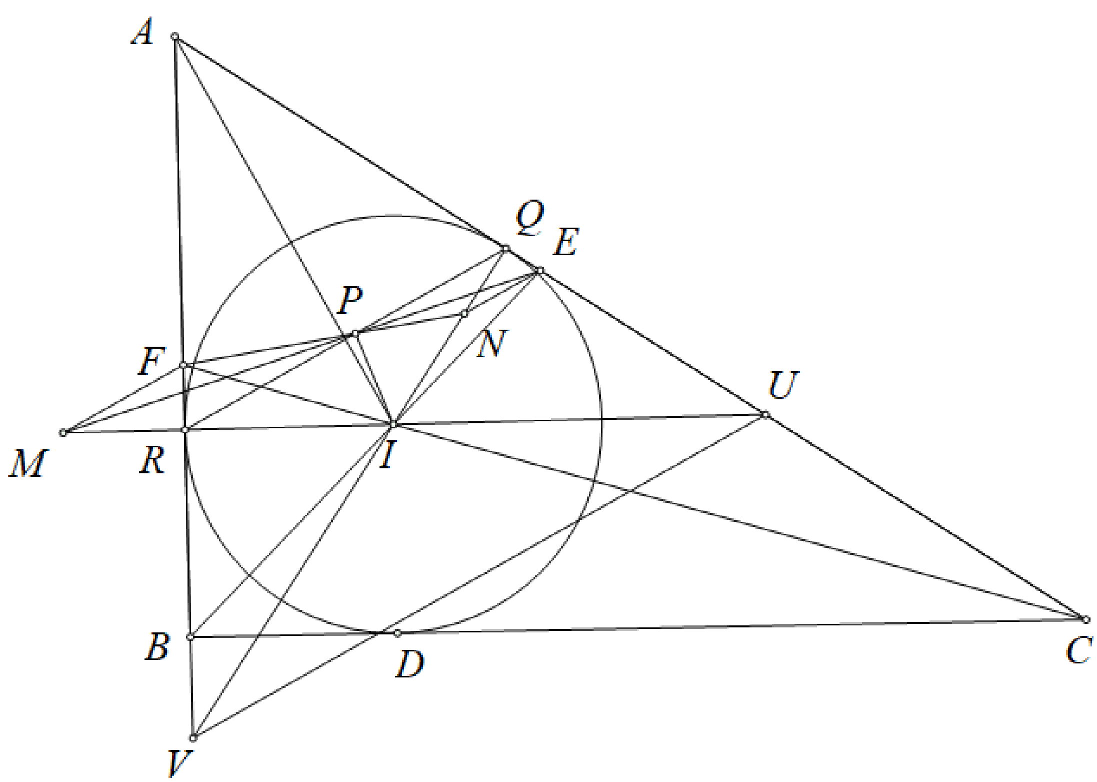
Problem 10

Ha Vu Anh

Let (I) touch AB, AC at R, Q; K be the orthocenter of triangle BIC.

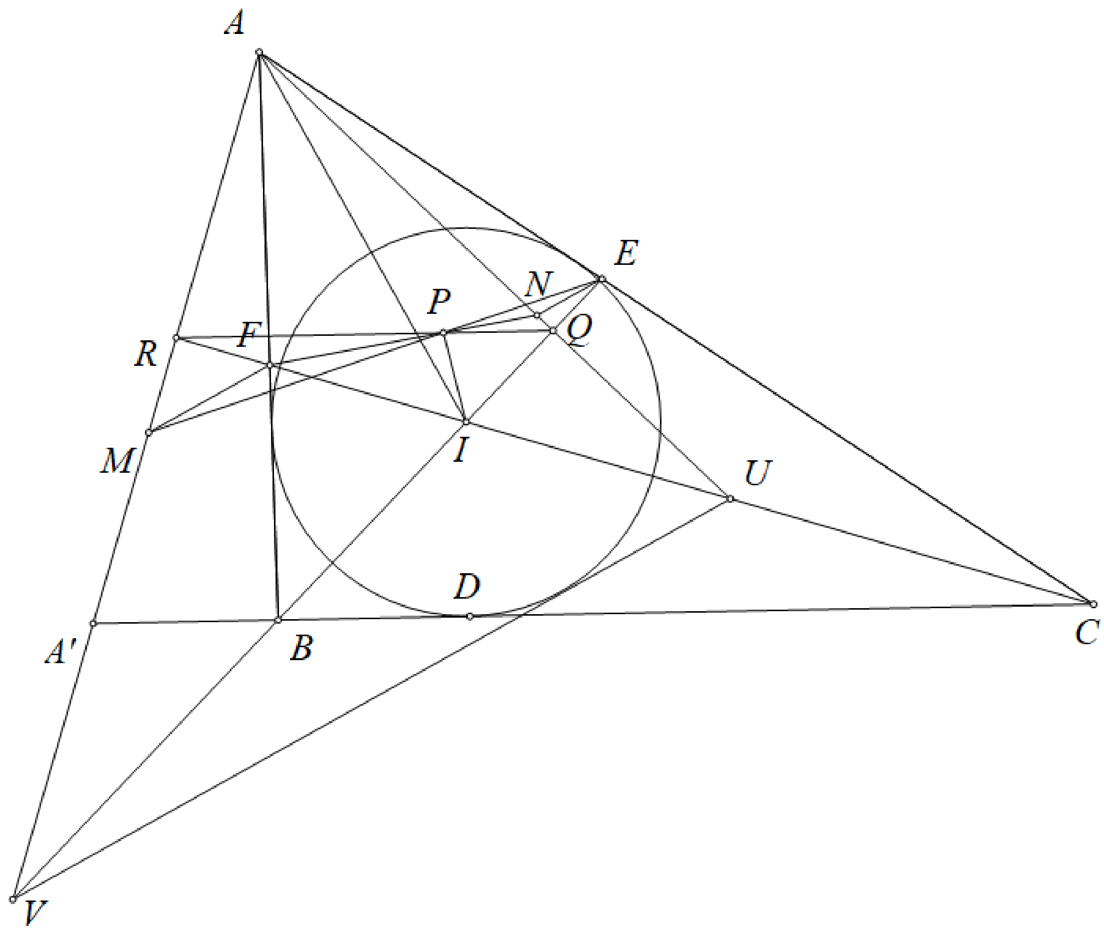
Claim 1: P lies on RQ.



It is obvious that I, M, R and I, N, Q are collinear, $MF \parallel NE$ since both of them are parallel to AI.
Let IR, IQ cut AC, AB at U, V; we get that I is the orthocenter of triangle AUV, hence $AI \perp UV$.
The claim is equivalent to proving that RQ, FN, EM are concurrent.

Consider triangle FRM and triangle NQE, we have: FR cuts NQ at V; RM cuts QE at U; and $FM \parallel NE \parallel UV$.
Hence, applying Desargues' theorem for these two triangles, we get that FN, RQ, ME are concurrent, as desired.
Hence, the claim is proven.

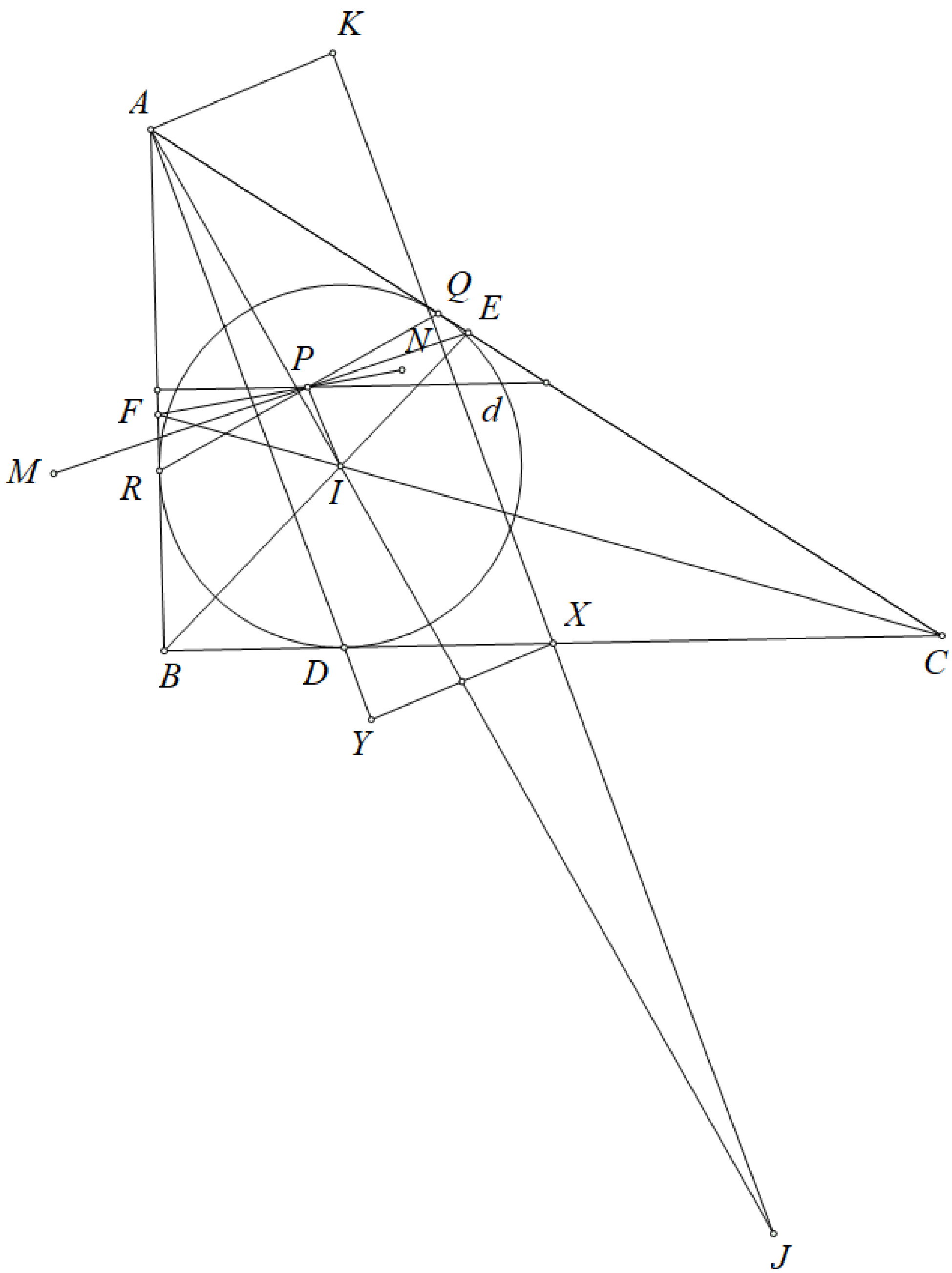
Claim 2: P lies on the line connecting midpoints of AB and AC.



Let CI cut AM, AN at R, U; BI cut AM, AN at V, Q; since $UR \perp AM, VQ \perp AN$, we get I is the orthocenter of AUV, hence $AI \perp UV$, hence $FM \parallel EN \parallel UV$.
Consider triangle FRM and triangle NQE, we have: FR cuts NQ at U; RM cuts QE at V; and $FM \parallel NE \parallel UV$.
Hence, applying Desargues for these two triangles, we get that FN, RQ, ME are concurrent, hence P lies on RQ.

Let AM cut BC at A', consider triangle $AA'C$, $CI \perp AA'$ and CI is the angle bisector of $\angle ACA'$, hence R is the midpoint of AA'.
Thus, R lies on the line connecting midpoints of AB and AC. Similarly, Q lies on this line, hence P lies on RQ and also lies on the line connecting midpoints of AB and AC, as desired.
Hence, the claim is proved.

Back to the main problem, from Claim 1 and Claim 2 we get that P is the intersection of RQ and the line connecting midpoints of AB, AC, which we will denote as d.



Lemma: K is the pole of d with respect to (I) — this is a well-known result.
Hence P lies on the polar of K with respect to (I), applying La Hire's theorem we get that K lies on the polar of P with respect to (I) (1).

Since P lies on RQ, which is the polar of A with respect to (I), applying La Hire's theorem, we get that A lies on the polar of P with respect to (I) (2).

From (1), (2) we get that AK is the polar of P with respect to (I), hence $AK \perp IP$, hence $AK \parallel XY$.

Denote J as the A-excenter of triangle ABC, it is well known that $XJ \parallel AD$ and $BKCJ$ is a parallelogram.
Hence X is the midpoint of KJ and $AKXY$ is a parallelogram, hence $\overrightarrow{AY} = \overrightarrow{KX} = \overrightarrow{XJ}$, hence AJ bisects XY, as desired.

Hence the problem is proven.