Problem 3

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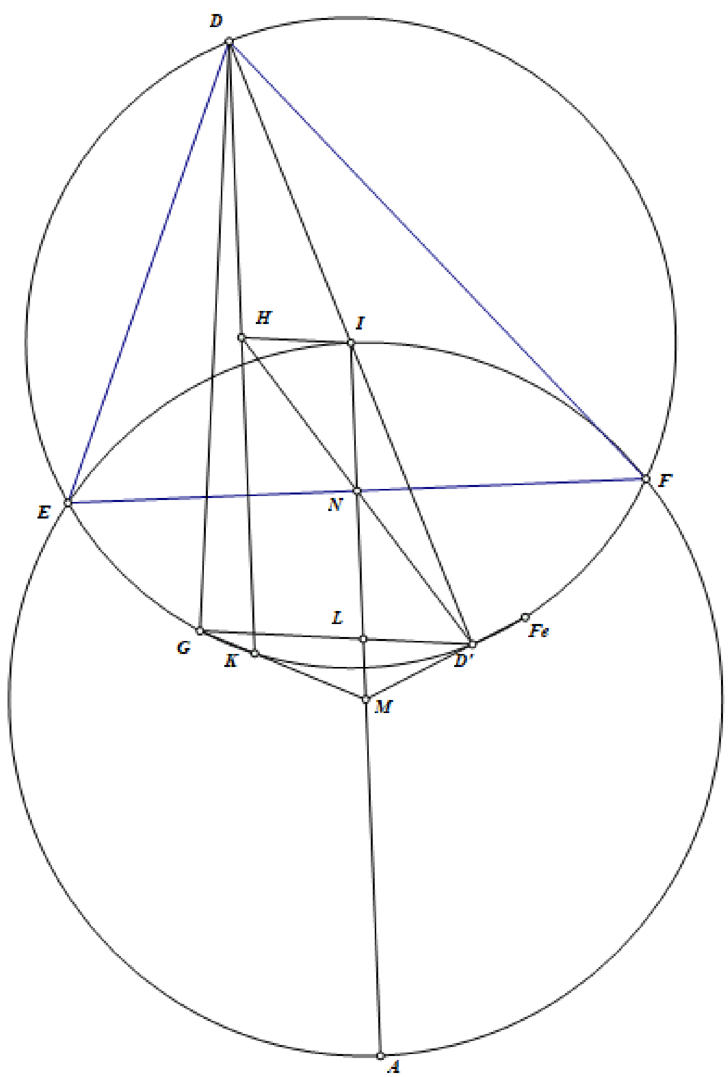
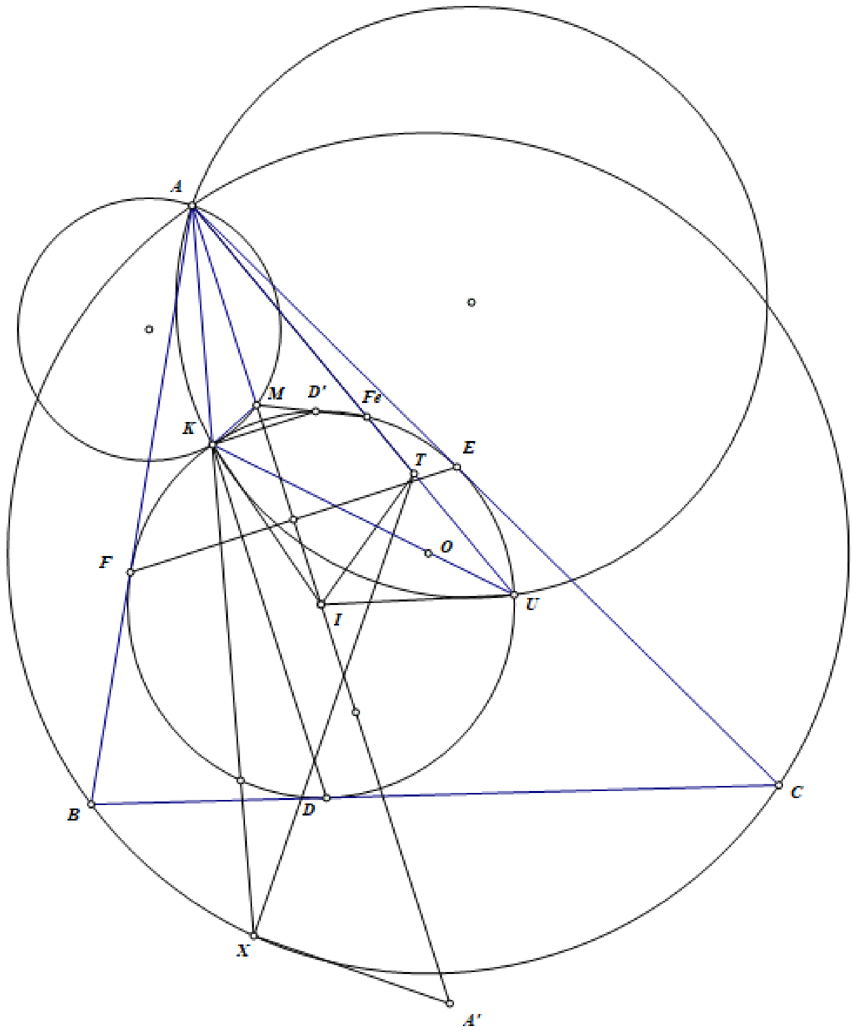
Let Fe be the Feuerbach point of triangle ABC, DD' be a diameter of (I), M is the midpoint of AI. Claim: M, D', Fe are collinear.



Proof: Let H be the orthocenter of triangle DEF, and let $GFe//EF$ such that G lies on (I). Then we get AG perpendicular to IH, since Fe is the Euler-reflection point of triangle DEF. Reflecting across MI, the problem is equivalent to proving that M, K, G are collinear.

Let GD' intersect MI at L. Since $G'D$ is perpendicular to AG, we have GD' parallel to HI. Moreover, H and D' are symmetric with respect to N, so N is the midpoint of IL. Then $IL \cdot IM = IN \cdot IA = IM^2$, so the quadrilateral $IGMD'$ is cyclic. Hence, $\angle MGD' = \angle MID' = \angle KAD' = \angle KGD'$, which implies that

M, K, G are collinear, as desired.
Back to the main problem,



Let AFe intersect (I) at U, and T be a point on AFe such that $AU \cdot AT = AI^2$.
Let AX intersect (I) at K (closer to A).
Consider an inversion about a circle at A with radius AI. It sends: $A - mixitilinear \mapsto (I)$, hence T lies on the $A - mixitilinear$. Hence, if XT is perpendicular to XA' then T coincides with T of the problem, and since T lies on AFe, T of the problem also lies on AFe, as desired.
In short, the problem is equivalent to proving that XT is perpendicular to XA'.
Furthermore, It also sends $A' \mapsto M$, and $X \mapsto K$. Hence, the inversion also sends $XT \mapsto (AKU)$, $XA' \mapsto (AKM)$.
Hence, $XT \perp XA'$ if and only if (AKM) and (AKU) are orthogonal $\Longleftrightarrow \angle AMK = 90^\circ + \angle AUK (*)$.
Since DD' be a diameter of (I); we have the familiar result that AD', AK are isogonal $W.R.T$ $\angle BAC$. Let K' be the reflection of D' across the perpendicular bisector of EF; then K' lies on (I) and AK', AD' are isogonal in $\angle BAC$, therefore $K' \equiv K$. Therefore, K and D are symmetric with respect to AI.
Applying the claim, we have M, D', Fe are collinear, hence $90^\circ + \angle AUK = 90^\circ + \angle MD'K = \angle AMD' = \angle AMK$.
Hence, $(*)$ is true, or the problem is proved.