Problem 7

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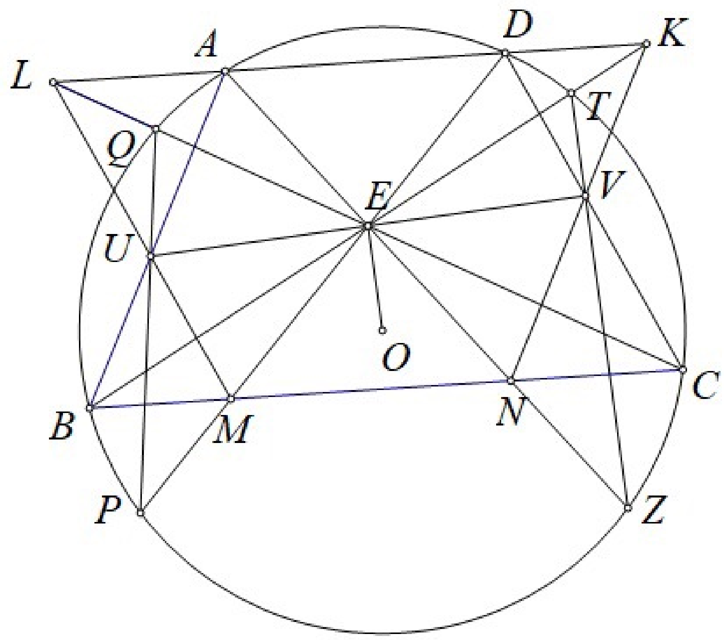
Lemma: Let $ABCD$ be a quadrilateral inscribed in (O) such that $AD \parallel BC$, E be an arbitrary point lies on the line connecting the midpoint of AB and CD. The line from E perpendicular to OE cut AB, CD at UV. Prove that E is the midpoint of UV.



Proof: Let AE, DE cut BC at N, M respectively, BE, CE cut AD at K, L respectively we got E is the midpoint of AN, DM, CL, BK.

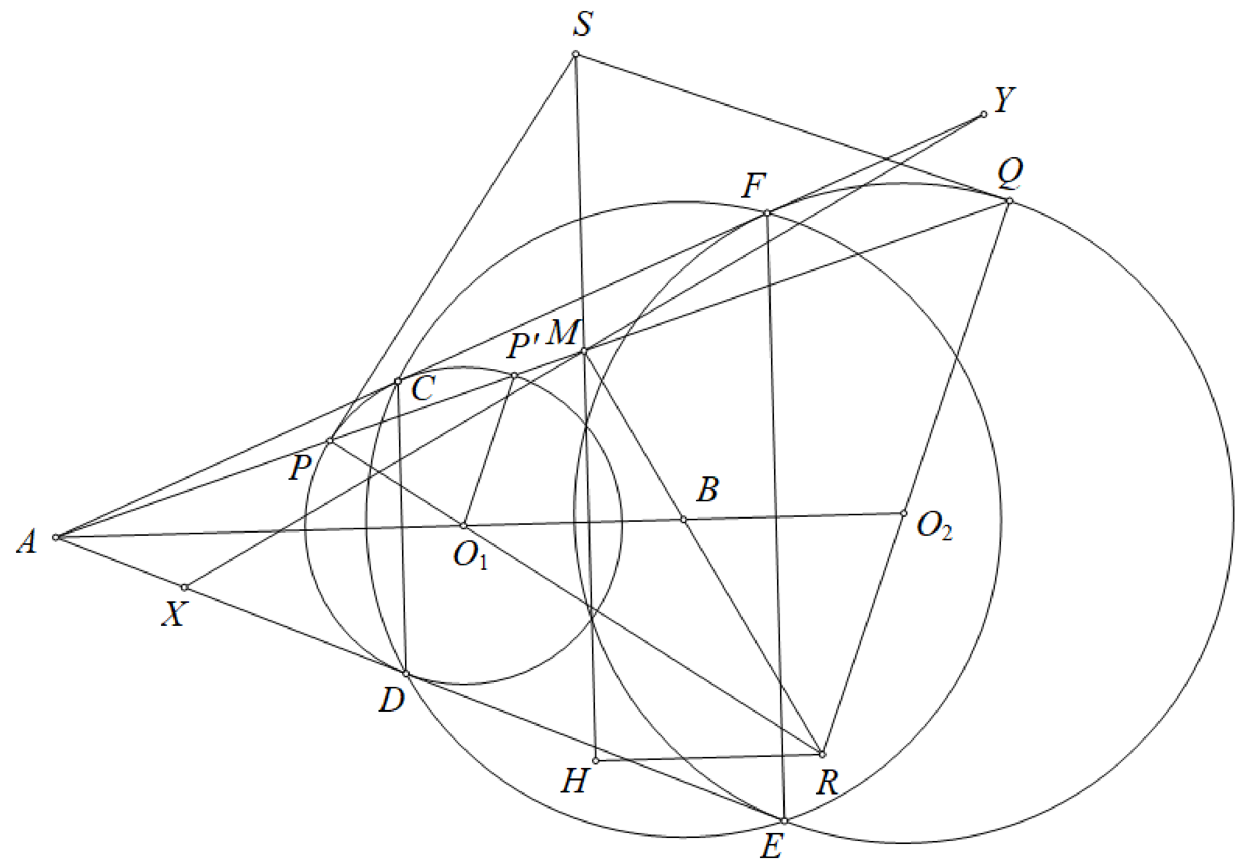
Let EA, EB, EC, ED cut (O) at Z, T, Q, P respectively. Since $\angle MLC = \angle QCD = \angle QPD$ we get that $LQMP$ are cyclic. By simple angle chasing we get $ALBM$ and $AQBP$ are also cyclic we get LM, AB, PQ concurrent at U'. Similiarly we get NK, CD, TZ concurrent at V'.

Since AB, LM are reflection of NK, DC through E respectively we get that U' are reflection of V' through E or E is the midpoint of $U'V'$.

Applying the butterfly theorem for quadrilateral $ABZT$ inscribed in (O) with E being the intersection of AZ and BT, U', V' lies on AB, ZT such that E is the midpoint of $U'V'$ we get OE is perpendicular to $U'V'$.

Therefore $U' \equiv U, V' \equiv V$ hence E is the midpoint of UV.

Back to the main problem,



Let AP at (O_1) at P' we get $O_1P' \parallel O_2Q$ therefore $\angle O_2QP = \angle O_1P'P = \angle O_1PP'$ therefore $RP = RQ$, Let S be the intersection of the line from P perpendicular to O_1P and the line from Q perpendicular to O_2Q since $RP = RQ$ we get $SP = SQ$ therefore S lies on radical axis of (O_1) and (O_2).

Let the line from S perpendicular to O_1O_2 cut PQ at M', cut the line from R parralel to O_1O_2 at H.

We have $-1 = H(RS, PQ) = R(HM', PQ)$ and $-1 = R(HB, O_1O_2)$ therefore M' lies on RB therefore $M' \equiv M$ therefore SM perpendicular O_1O_2 therefore M lies on radical axis of (O_1) and (O_2).

Let the comment tangent from A of $(O_1), (O_2)$ touch (O_1) at C, D, touch (O_2) at E, F such that A, D, E and A, C, F are collinear.

Since M lies on the radical axis (O_1), (O_2) which is the line connect midpoint DE and CF.

Applying the lemma for quadrilatetal $CDEF$ inscribed in the circle with center B we get M is the midpoint of XY, as desired.

Therefore, the problem is proved.