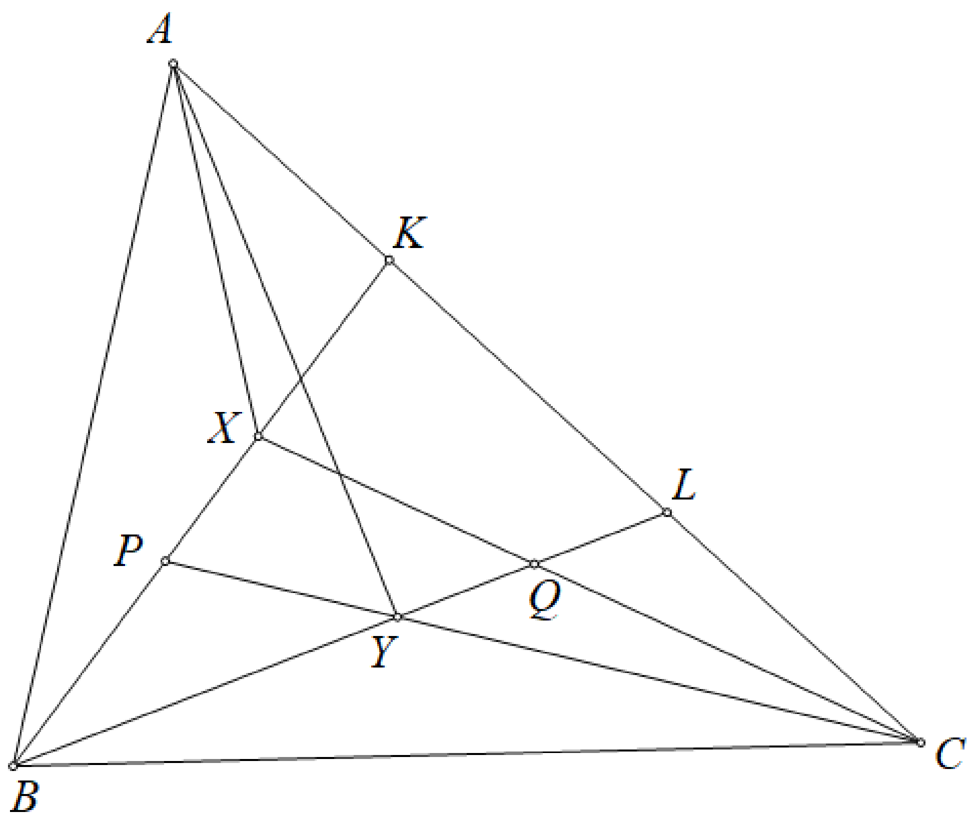
# Problem 1

Ha Vu Anh

Lemma: $ABC$, $P, Q$ be two arbitrary points. Let $BP$ cut $CQ$ at $X$, $BQ$ cut $CP$ at $Y$. Then $AP, AQ$ are reflections through the angle bisector of $\angle BAC$ if and only if $AX, AY$ are reflections through the angle bisector of $\angle BAC$. Proof:




Assume that $AP, AQ$ are reflections across the angle bisector of $\angle BAC$. Let $BP, BQ$ intersect $AC$ at $K, L$, respectively.

It is known that reflection of lines through a fixed line preserves the cross ratio. Hence, let $Y'$ be a point on $BQ$ such that $AY'$ is the reflection of $AX$ across the angle bisector of $\angle BAC$. Then we have

$$(AB, AP, AX, AK) = (AL, AQ, AY', AB),$$

which is equivalent to

$$A(BP, XK) = A(LQ, Y'B).$$

Now consider the projection from $C$. We obtain

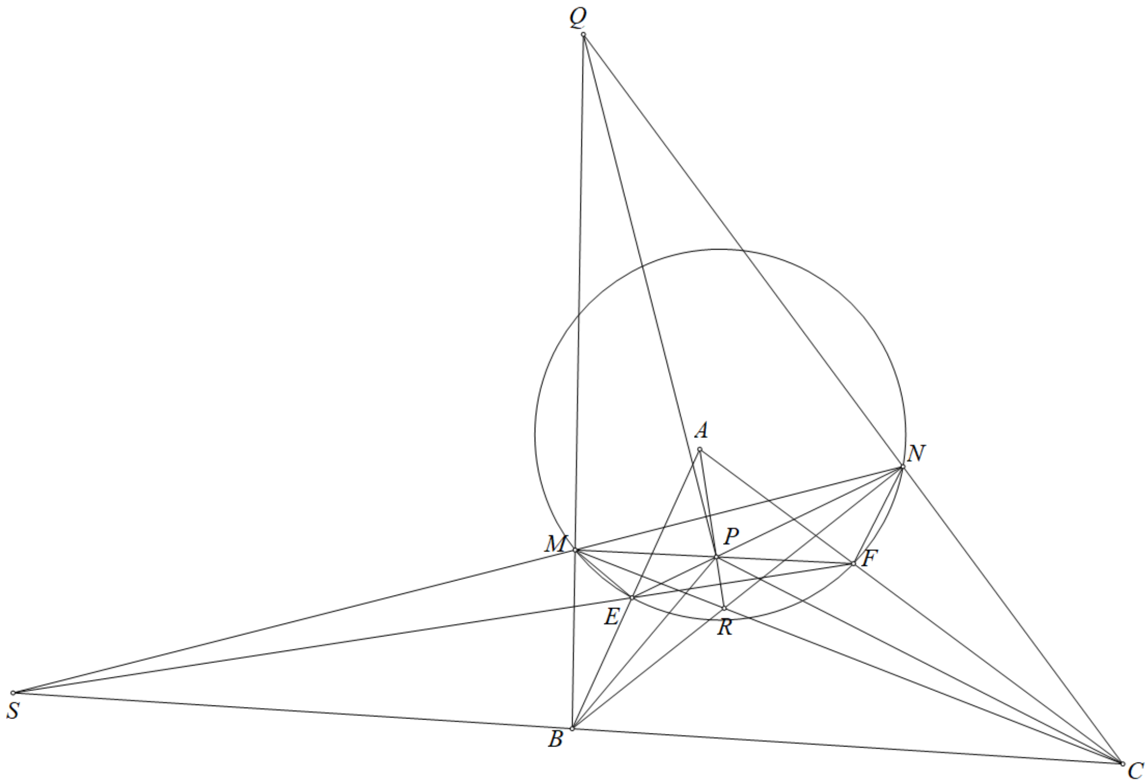
$$(CB, CY, CQ, CL) = (CB, CP, CX, CK),$$

which is equivalent to

$$(BY, QL) = (BP, XK) = (LQ, Y'B) = (BY', QL).$$

Hence, $Y' \equiv Y$, or equivalently, $AX$ and $AY$ are reflections across the angle bisector of $\angle BAC$, as desired.

Back to the main problem,




Claim: $MN, EF, BC$ are concurrent.
Since $\angle BPF = \angle CPE$, we have $\angle PFM = \angle PEN$, so $MFEN$ is a cyclic quadrilateral.

Let $MN$ intersect $EF$ at $S$. Then

$$\frac{SE}{SF} = \frac{SE}{SM} \cdot \frac{SM}{SF} = \frac{EN}{FM} \cdot \frac{SM}{SF} \ (1).$$

Let $BC$ intersect $EF$ at $S'$.

We have $\dfrac{BF}{BA} = \dfrac{\sin\angle BPF}{\sin\angle APB} \cdot \dfrac{PF}{PA}, \quad \dfrac{CE}{CA} = \dfrac{\sin\angle CPE}{\sin\angle CPA} \cdot \dfrac{PE}{PA}.$

Hence, by Menelaus' theorem,

$$\frac{S'E}{S'F} = \frac{CE}{CA} \cdot \frac{BA}{BF} = \frac{\sin\angle CPE}{\sin\angle BPF} \cdot \frac{\sin\angle APB}{\sin\angle CPA} \cdot \frac{PE}{PF} = \frac{\sin\angle MFE}{\sin\angle SMF} \cdot \frac{PE}{PF} = \frac{SM}{SF} \cdot \frac{EN}{FM}.$$

Combining with (1), we get $S' \equiv S$, so $MN, EF, BC$ are concurrent at $S$, or the claim is proved.
Therefore. we have that $S$-the intersection of $MN$ and $BC$; $E$-the intersection of $PM$ and $AC$; $F$-the intersection of $PN$ and $AB$ are collinear(2).

Hence, applying Desargues' theorem to $\triangle ABC$ and $\triangle PNM$, using (2) we obtain that $AP, BN, CM$ are concurrent at $R$.

Since $PB$ and $PC$ are isogonal lines in $\angle MPN$, and $Q$ is the intersection of $BM$ and $CN$, while $R$ is the intersection of $BN$ and $CM$, it follows that $PQ$ and $PR$ are isogonal lines in $\angle MPN$. Therefore, $PQ \perp MN$.