P5 VN TST 2024

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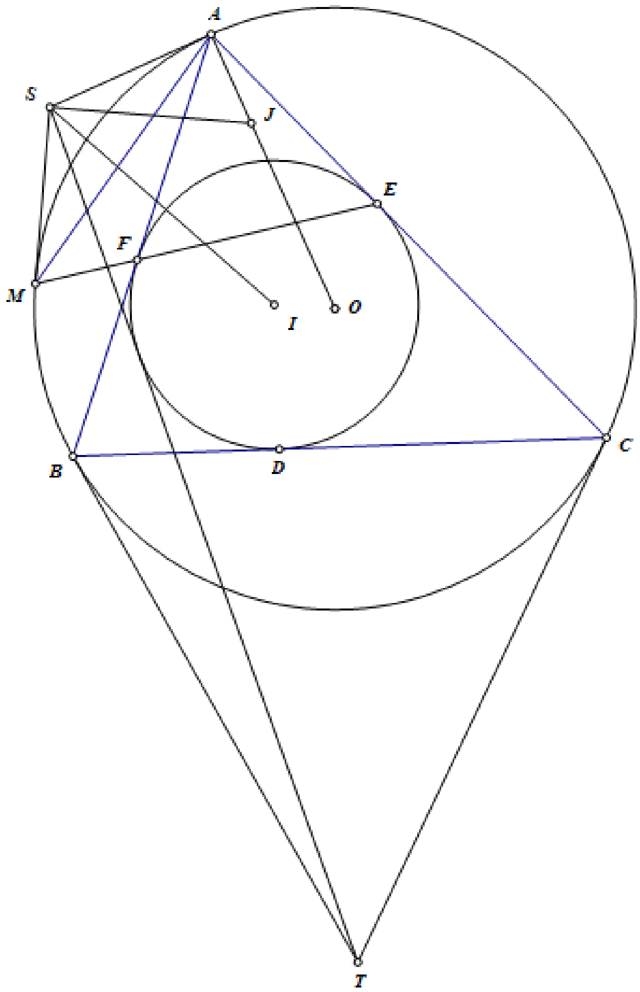
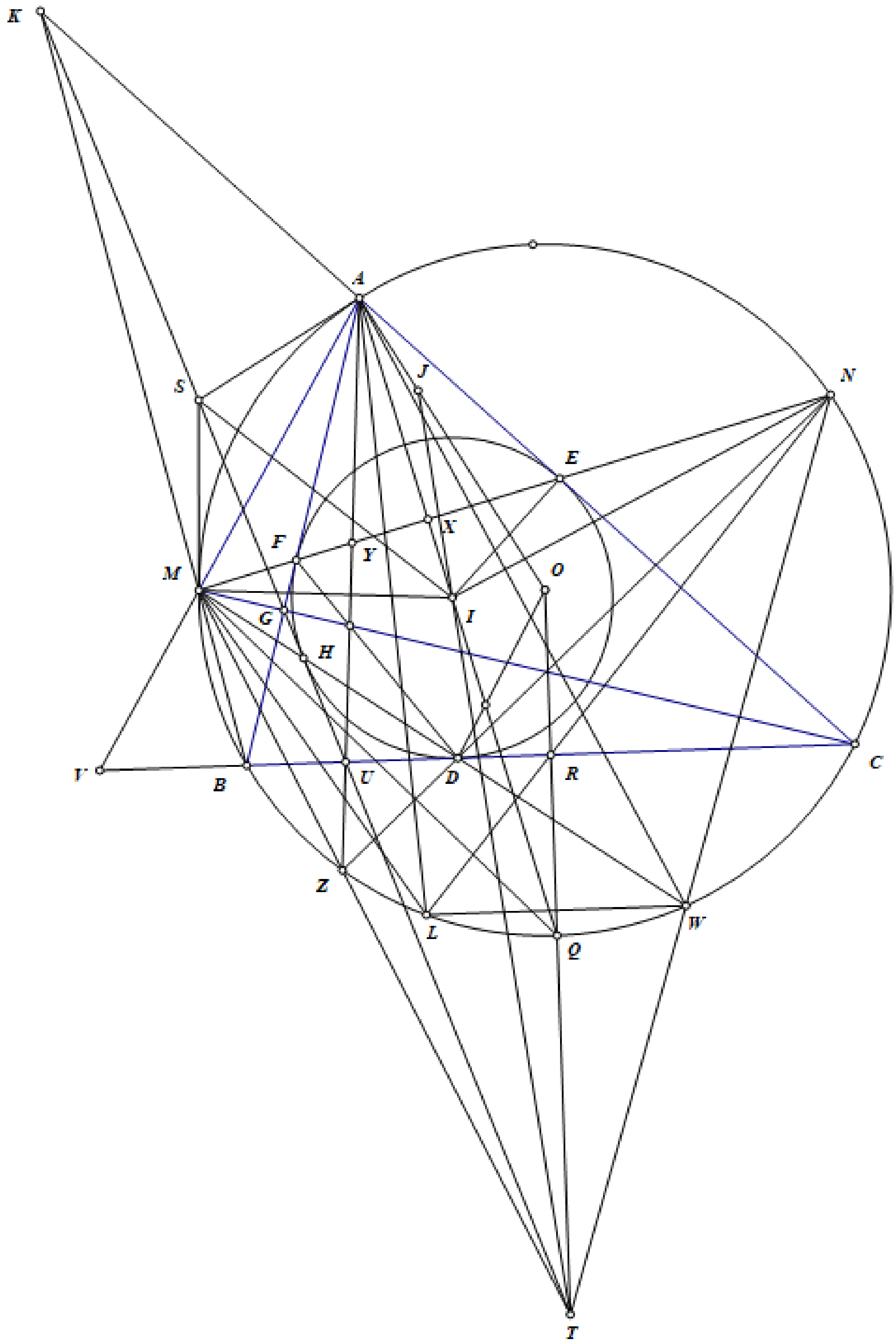
1 Problem Statement

Let the acute, scalene triangle ABC be inscribed in the circle (O). The incircle (I) of triangle ABC touches the sides BC, CA, AB at D, E, F, respectively. The ray EF meets (O) again at M. The tangents to (O) at A and M intersect at S, and the tangents to (O) at B and C intersect at T. Suppose IT intersects OA at J. Prove that: $\angle ASJ = \angle TSI$.



Hình 1: Illustration

2 Solution



Hình 2: Figure 1

(Figure 1) Let AI intersect (O) again at Q, we get T, O, Q are collinear. Let R be the midpoint of BC, then triangles BQR and AIF are similar, hence $\frac{IA}{IF} = \frac{QB}{QR} = \frac{QI}{QR}$. Since $OQ^2 = OA^2 = OR.OT$, we have $\frac{OQ}{OR} = \frac{OT}{OQ}$, thus $\frac{QR}{QO} = \frac{TQ}{TO}$. Applying Menelaus' theorem to triangle AQO

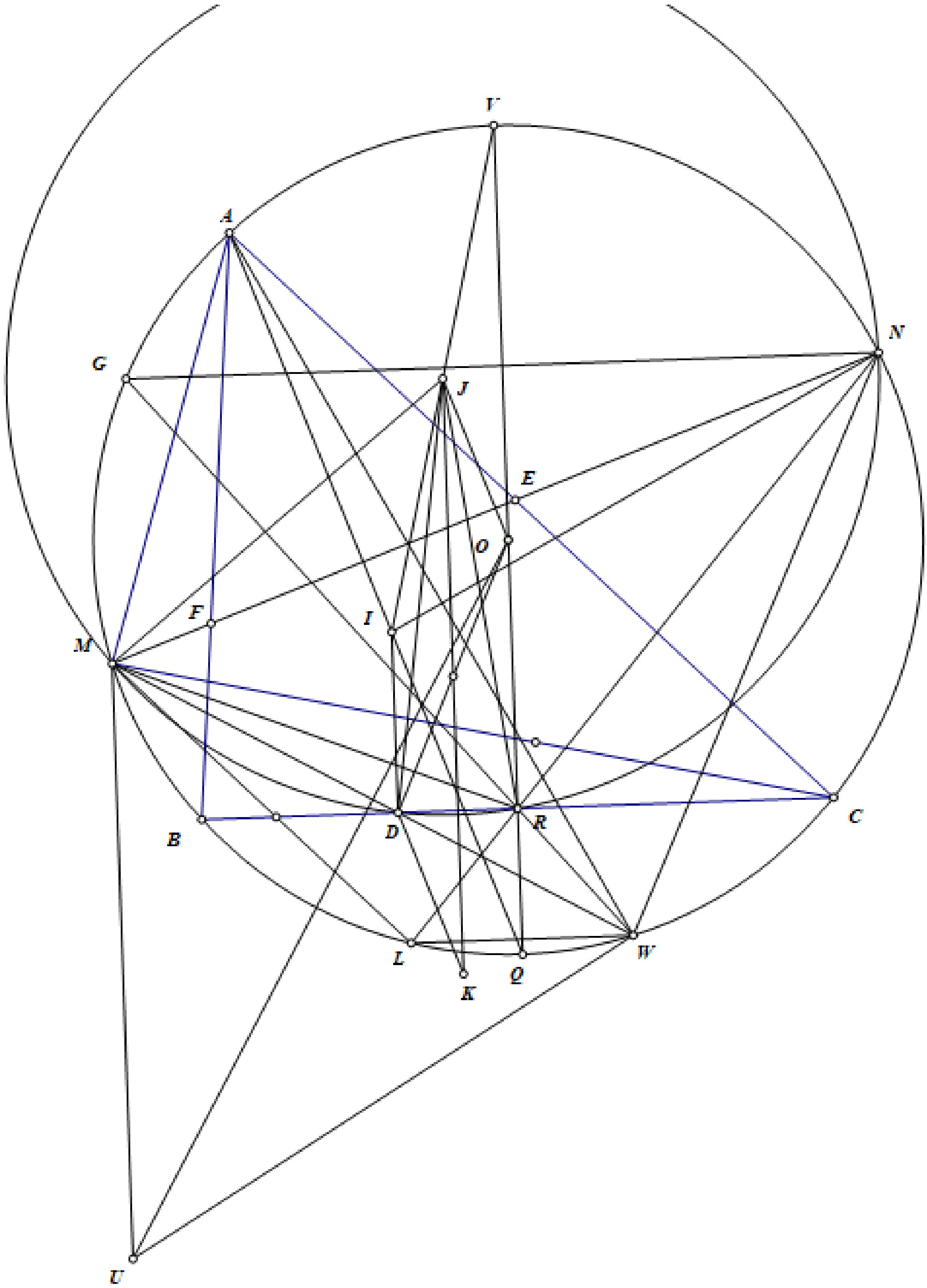
with T, I, J collinear, we get: $\dfrac{JA}{JO} = \dfrac{IA}{IQ}.\dfrac{TQ}{TO} = \dfrac{IA}{IQ}.\dfrac{QR}{QO} = \dfrac{IA}{QO}.\dfrac{QR}{QI} = \dfrac{IA}{QO}.\dfrac{IF}{IA} = \dfrac{IF}{QO} = \dfrac{r}{R}$ hence $\dfrac{AJ}{AO} = \dfrac{r}{R+r}$ and $\dfrac{AJ}{r} = \dfrac{R}{R+r}$.

Let CM intersect AB at G, and BM intersect AC at K. Using Pascal's theorem, it is easy to see that K, S, G, T are collinear. Let the tangent to (I) through G intersect AC at K', then BK', CG, EF are concurrent, hence K' coincides with K and KG is tangent to (I) at H.

On the other hand, let KG and AM intersect BC at U, V, respectively. We have $-1 = A(UV, BC) = (ZM, BC) = A(ZM, BC) = (YM, EF)$, so if X is the midpoint of EF, then $XA.XI = XF^2 = XY.XM$, hence Y is the orthocenter of triangle AMI, so IY is perpendicular to AM. Therefore, projecting the orthogonal bundle we get $-1 = I(YM, EF) = (AM, Ax, AC, AB) = A(MZ, CB)$, so AZ is perpendicular to IM. Moreover, $N(DM, BC) = -1 = N(ZM, BC)$ implies N, D, Z are collinear. Similarly, let MD intersect (O) again at W, then AW is perpendicular to IN.

Since M lies on the polar of A with respect to (I) and AU is perpendicular to IM, AU is the polar of M with respect to (I), hence M lies on the polar of U with respect to (I), which is HD.

Let L be the point on (O) such that $LW \parallel BC$, we get $\angle SMH = \angle UHD = \angle UDH = \angle MWL, \angle SMH = \angle MLW$, so triangles SMH and MLW are similar. It is easy to prove that $MDRN$ is cyclic, hence N, R, L are collinear. We need to prove $\angle ASJ = \angle TSI$, which reduces to proving that triangles SAJ and SHI are similar. Since $\angle SAJ = \angle SHI = 90°$, we only need to prove $\dfrac{SA}{SH} = \dfrac{AJ}{HI} \Longleftrightarrow \dfrac{ML}{MW} = \dfrac{R}{R+r}$.



Hình 3: Figure 2

(Figure 2)

We will prove separately as a lemma: Let the acute, scalene triangle ABC be inscribed in circle (O). The incircle (I) of triangle ABC touches sides BC, CA, AB at D, E, F, respectively. Let R be the midpoint of BC, the ray EF meets (O) again at M, the ray FE meets (O) again at N, MD meets (O) at W, and NR meets (O) at L. Prove that $\frac{ML}{MW} = \frac{R}{R+r}$. Let the line through O perpendicular to MD intersect the line through M perpendicular to BC at W. It is easy to prove that triangles UMW and OML are similar, hence $\frac{ML}{R} = \frac{ML}{MO} = \frac{MW}{MU}$, so we reduce to proving $MU = R + r$.

Let V be the midpoint of the arc BC containing A of (O). It is easy to prove that the midpoint J of IV is the center of the circumcircle of quadrilateral $MDRN$. Construct the parallelogram $OJDK$, we get $JK = R+r$, $DK \parallel OJ \parallel AI$, and since $\angle JMR = 90^\circ - \angle MNR = 90^\circ - \angle MDB = \angle WMU$, triangles JMR and UMW are similar, hence $\dfrac{MW}{MR} = \dfrac{MU}{MJ}$(1).

Moreover, $-1 = M(ED, BC) = (NW, BC)$, so let NR intersect (O) at G, then $NG \parallel BC$, and since $DK \parallel AI$, we have $\angle RWM = \angle GNM = \angle JKD$, and as J is the center of $(MDRN)$, $\angle KJD = \angle RMD$, so triangles MRW and JDK are similar, giving $\dfrac{MW}{MR} = \dfrac{JK}{JD}$. Combining with (1), we get $MU = JK = R + r$ (Q.E.D.).

Thus, the proof is completed.