Problem 6

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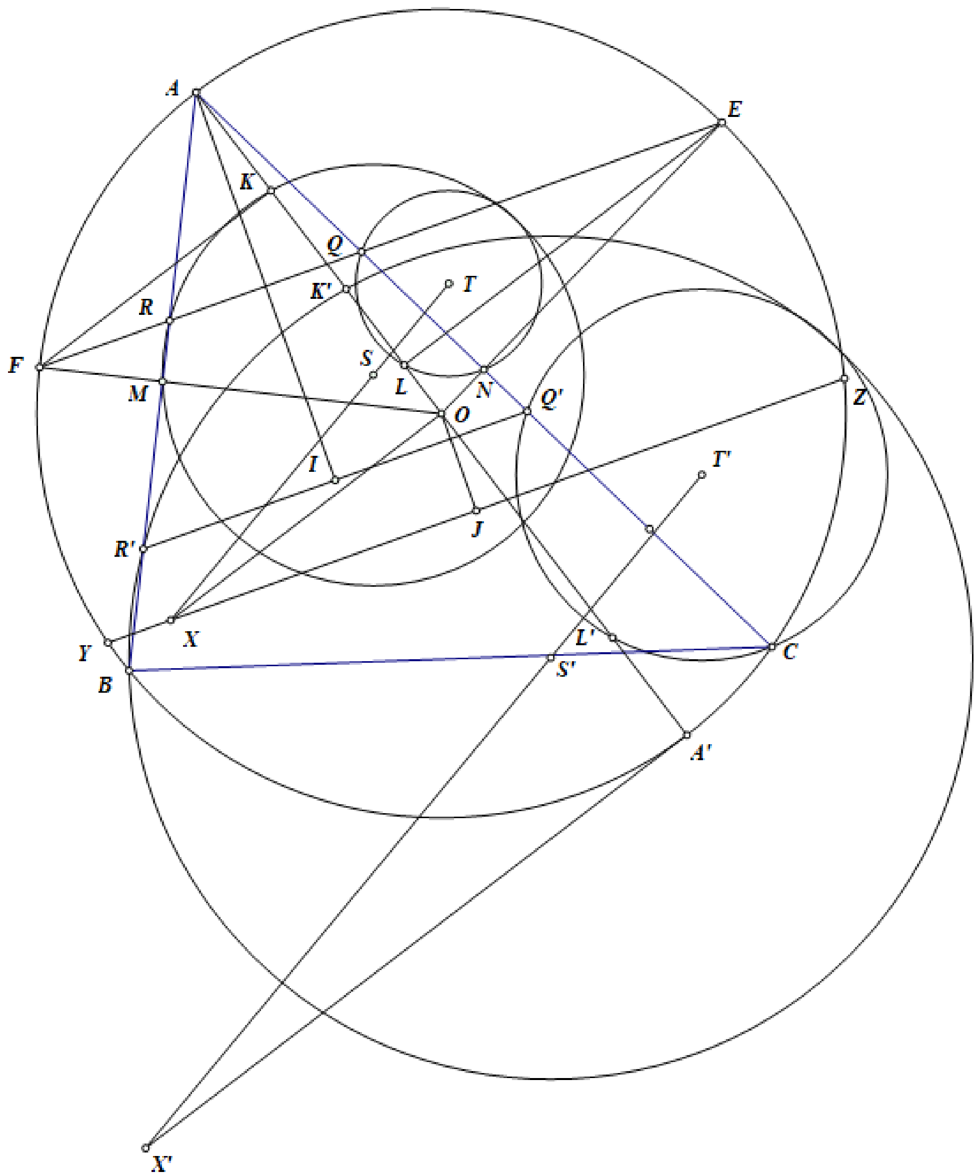
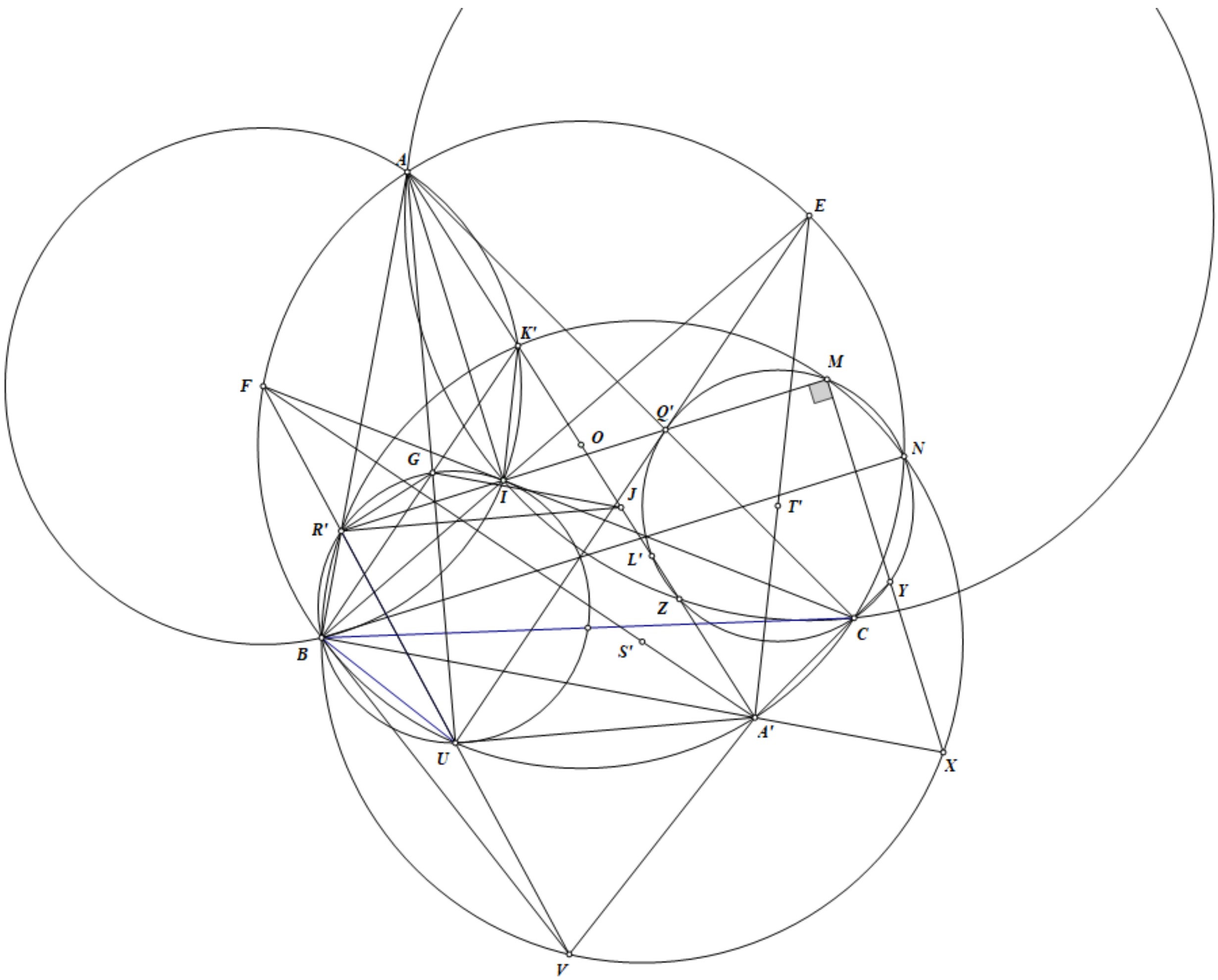


Let I be the incenter of ABC, H, J be the projection of A, O on EF, YZ respectively.
It is well known that EF is the perpendicular bisector of AI therefore $AI = 2AH$.
Then H is the orthocenter of triangle AYZ if and only if
$\overrightarrow{AH} = 2.\overrightarrow{OJ}$.
since $AH \parallel OJ$ it is equivalent to $AI = 4OJ = 4.OX.sin\angle OXJ = 4.OX.sin\angle OAI$

therefore we will prove $OX = \dfrac{AI}{4.sin\angle OAI}$.
From I construct a line perpendicular to AI cut AB, AC at R', Q' respectively,
construct diameter AA' of (O), homothety at center A with scaling factor 2 sends
$M \to B, N \to C, R \to R', Q \to Q', K \to K', L \to L', S \to S', T \to T', O \to A', X \to X'$
we get S', T' is the center of $(BR'K'), (CQ'L')$ and X' is the intersection of $S'T'$
and the line from A perpendicular to AA'.
Since $AKMF$ is a cyclic quadrilateral, $OA = OF$ therefore $OK = OM$ and $OL = ON$ similiarly.
Therefore K' lies on AA' and $A'K' = A'B, A'L' = A'C$
and so $\angle AK'B = 90° + \angle AA'B/2 = \angle AIB$ and $AK'IB$ is a cylic quadrilateral and similiarly $AL'IC$.

Since $A'X' = 2OX$ we will prove $A'X' = \dfrac{AI}{2.sin\angle OAI}$.

Claim: $(BR'K'), (CQ'L'), (O)$ have a common point N and let $(A - mix)$ tangents to (O) at U then $A'U = A'N(1)$. Let the second intersection of $(BR'K')$ and $(CQ'L')$ be M then M lies on $R'Q'(2)$



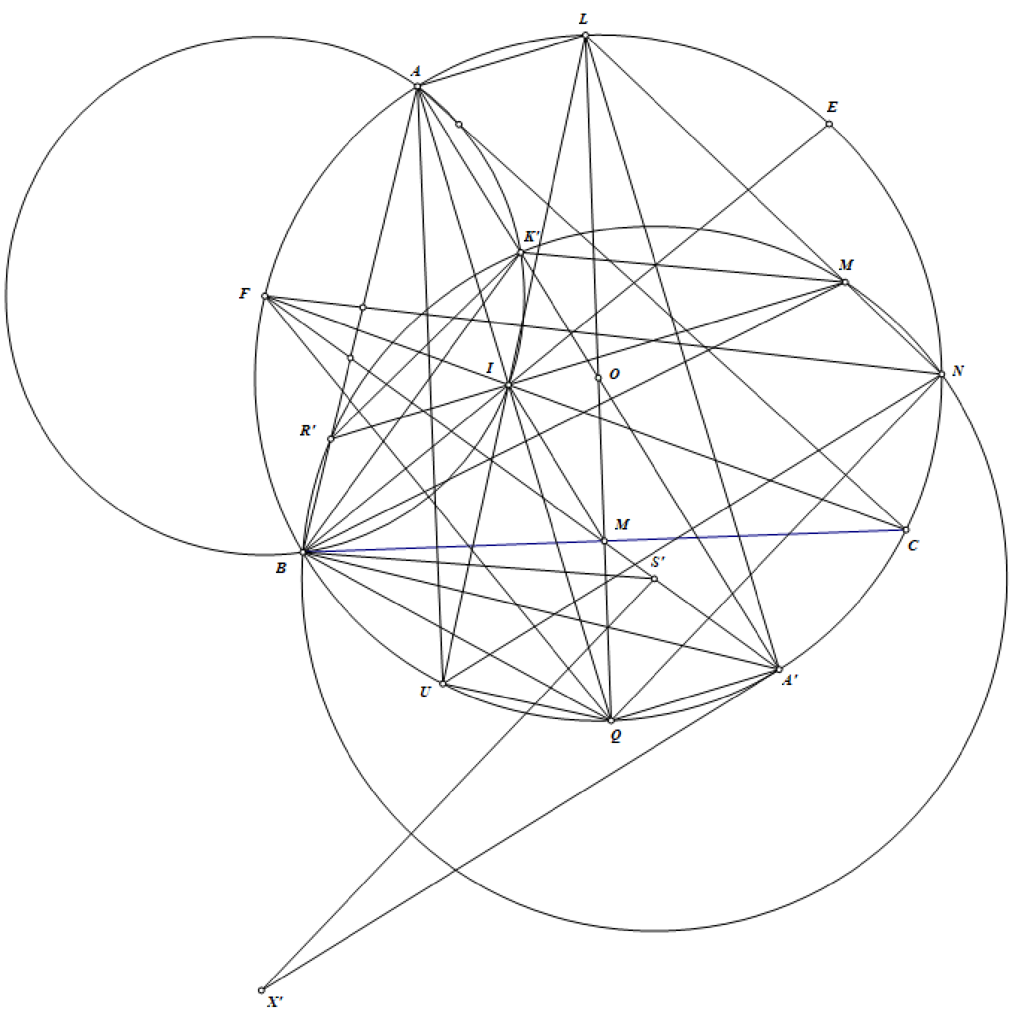
It is well known that $\overline{F, R', U}, \overline{E, Q', U}$ and $BR'IU$ is a cyclic quadrilateral,
$\angle ABK' = \angle ABI - \angle K'BI = \angle ABI - \angle OAI = \angle ACI = \angle AUF = \angle AUR'$
therefore let G be the intersection of BK' and AU then B, G, R', I, U lie on the same circle.
Let $(A', A'U)$ cut FU at V, perpendicular bisector of IG cut AO at J.
By simple angle chasing we get $\triangle JGK' \sim \triangle A'UV$, $JG \perp AN, R'G \perp AJ$

therefore G is the orthocenter of $\triangle AR'J$
therefore by angle chasing $\triangle R'GJ \sim \triangle BUA'$
combine with $\triangle JGK' \sim \triangle A'UV$
we get $\triangle R'GK' \sim \triangle BUV$
therefore $\angle R'K'G = \angle BVU$ and V lies on $(BR'K')$.
Redefine N as the intersection of $(BR'K')$ and (O)
we get $\angle A'UN = \angle A'AN = 90^\circ - \angle R'BN = 90^\circ - \angle UVN$
combine with $A'U = A'V$ we get A' is the center of (UVN) therefore $A'U = A'N$.
Let N' as the intersection of $(CQ'L')$ and (O) similiarly we get $A'U = A'N'$ therefore $N' \equiv N$ and so (1) is true.
Let X, Y be the intersection of $A'B, A'C$ with $(BR'K'), (CQ'L')$ respectively.
By simple angle chasing we get $\triangle NA'X \sim \triangle NAR', \triangle NA'Y \sim \triangle NAQ'$

therefore $\dfrac{AR'}{A'X} = \dfrac{NA}{NA'} = \dfrac{AQ'}{A'Y}$ and so $A'X = A'Y$.

Therefore $\angle A'XY = 90^\circ - \angle BAC/2 = 90^\circ - \angle(R'Q', A'B)$ and so $XY \perp R'Q'$. Therefore let XY intersect $R'Q'$ at M' then M' lies on $(BR'K')$ and $(CQ'L')$ and so $M' \equiv M$ which imply that (2) is true.

Back to the main problem, we will prove $A'X' = \dfrac{AI}{2.sin\angle OAI}(*)$.



Let Q be the midpoint of arc BC not containing A of (O), UI cut (O) at L we get L is the mid point of arc BAC of (O) therefore LQ is the diameter of (O).
We have $S'X'$ is the perpendicular bisector of MN, $\angle MNQ = \angle MNB + \angle BNQ = \angle AKM + \angle BAQ = 90°$ therefore $X'S' \parallel QN$. Also since $A'N = A'U$, $UN \perp AA'$
therefore $UN \parallel A'X'$ and so $\angle A'X'S' = \angle QNU = \angle IAU = \angle QIM$, $\angle X'A'S' = \angle FAO = 90° - \angle ACB/2 = \angle BIQ$
therefore $\angle A'S'X' = 180° - \angle A'X'S' - \angle X'A'S' = 180° - \angle QIM - \angle BIQ = 180° - \angle BIM$.
We have $\angle BS'A' = 180° - \angle BS'K'/2 = \angle BR'K', \angle BA'S' = \angle FCB = \angle R'BK'$ which is equivalent to

$$\triangle BS'A' \sim \triangle K'R'B \Longrightarrow \frac{A'S'}{A'B} = \frac{BR'}{BK'}.$$

By simple angle chasing, we get $\triangle BIR' \sim \triangle BCI, \triangle BK'A' \sim \triangle BIQ \Longrightarrow BI^2 = BR'.BC, \dfrac{BK'}{BA'} = \dfrac{BI}{BQ}$.

$$(*) \Longleftrightarrow \frac{A'X'}{A'S'}.\frac{A'S'}{A'B}.A'B = \frac{AI}{2.A'Q/A'A} = \frac{AI.A'A}{2.A'Q}$$
$$\Longleftrightarrow \frac{AI.A'A}{2.A'Q} = \frac{sin\angle A'S'X'}{sin\angle A'X'S'}.\frac{BR'}{BK'}.A'B = \frac{sin\angle BIM}{sin\angle IAU}.\frac{BR'}{BK'}.A'B$$
$$= \frac{sin\angle BIM}{sin\angle IMB}.\frac{sin\angle IMB}{sin\angle IAU}\frac{BA'}{BK'}.BR'$$
$$= \frac{BM}{BI}.\frac{sin\angle AQU}{sin\angle QAU}\frac{BQ}{BI}.BR'$$
$$= \frac{BM}{BI}.\frac{UA}{UQ}\frac{BQ}{BI}.BR' = \frac{BM}{BI^2}.\frac{UA}{UQ}.QI.BR' = \frac{BM}{BR'.BC}.\frac{UA}{UQ}.QI.BR' = \frac{UA.QI}{2UQ}$$
$$\Longleftrightarrow 1 = \frac{AI}{AU}.\frac{QU}{QI}.\frac{AA'}{A'Q} = \frac{LI}{LQ}.\frac{LA}{LI}.\frac{AA'}{A'Q}. \text{ (which is true since } AL = QA', AA' = LQ)$$

therefore $(*)$ is true and so the problem is proved.