Problem 3

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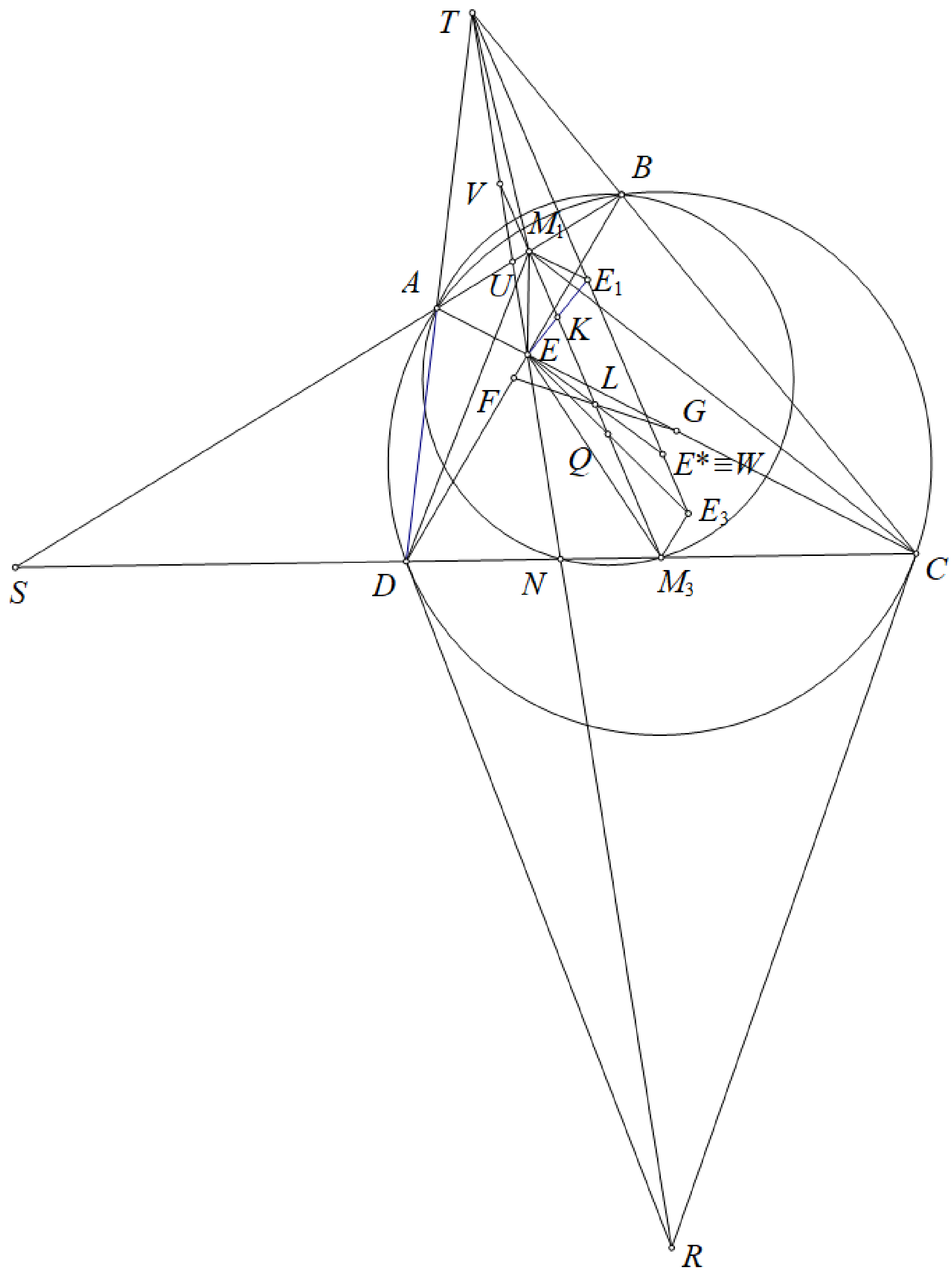

First, we will prove that the segment connecting the midpoints of EE_1 and EE_3 passes through the midpoint FG, where F and G are the midpoints of AC and BD, respectively.

Let AD intersect BC at T, AB intersect CD at S, and let TE intersect AB and CD at U and N. Let M_1M_3 be the Gauss line of the quadrilateral $AEBT$; then M_1M_3 bisects TE at V.

Since $(\mathrm{TE}, \mathrm{UN}) = -1$, we have

$$VE^2 = VU \cdot VN, \quad (SU, AB) = (SN, DC) = -1,$$

so

$$SU \cdot SM_1 = SA \cdot SB = SD \cdot SC = SN \cdot SM_3.$$

Hence UM_1M_3N is cyclic $\implies VT^2 = VU \cdot VN = VM_1 \cdot VM_3$, so

$$\triangle VM_1T \sim \triangle VTM_3 \implies \angle VM_1T = \angle VTM_3.$$

Let R be the intersection of the two tangents at C and D of the circumcircle of $ABCD$. By Pascal's theorem applied to

$$\begin{pmatrix} A & D & C \\ B & C & D \end{pmatrix},$$

we obtain that TE passes through R.

Since BR is the symmedian of $\triangle BCD$ (a well-known result), we have

$$\angle RBC = \angle M_3BD = \angle E_1BA$$

(as E_1 is the isogonal conjugate of E in $\triangle M_3AB$), so BE_1 and BR are symmetric with respect to the internal bisector of $\angle ABC$, which is the external bisector of $\angle TBA$.

Similarly, AE_1 and AR are symmetric with respect to the external bisector of $\angle TAB$, so E_1 and R are two isogonal conjugate points in $\triangle TAB$, hence TE_1 and TE are isogonal with respect to $\angle BTA$.

Also,

$$\triangle TAB \cup \{M_1\} \sim \triangle TCD \cup \{M_3\},$$

so $\angle M_1TA = \angle M_3TC$, implying TM_1 and TM_3 are isogonal in $\angle BTA \implies \angle VM_1T = \angle M_3TE = \angle M_1TE_1$, hence $VM_1 \parallel TE_1$.

Since V is the midpoint of TE, VM_1 bisects EE_1, so M_1M_3 bisects EE_1 at K. Similarly, M_1M_3 bisects EE_3 at Q.

We have

$$M_3G = AD/2 = M_1F, \quad M_3G \parallel AD \parallel M_1F,$$

so M_1FM_3G is a parallelogram, hence M_1M_3 bisects FG at L, and thus KQ also bisects FG (Q.E.D.).

Returning to the problem, let E^* be the reflection of E across the midpoint of FG.
Consider a homothety centered at E with ratio 2 then E_1E_3 passes through E^*. Similarly, E_2E_4 passes through E^*, so W coincides with E^*.
Therefore, FG bisects EW, and since FG is the Newton-Gauss line of quadrilateral $ABCD$, the problem is proved.