Problem 4

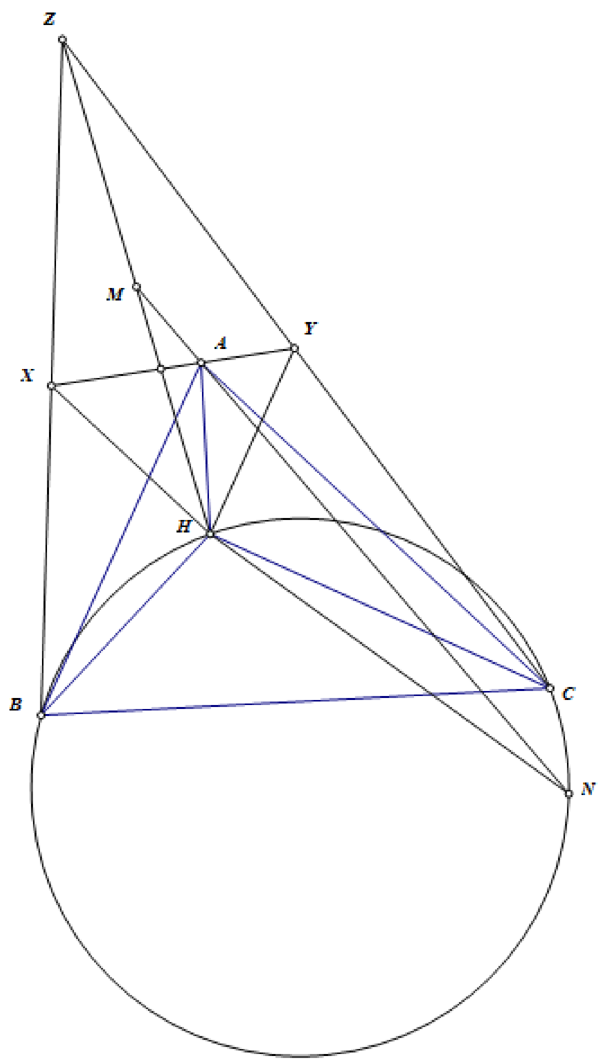
Ha Vu Anh

ERIQ lemma: Let $ABCD$ be a quadrilateral, and let E be a point on AB, F be a point on CD. Let X be a point on AB, Y a point on CD, and Z a point on EF such that

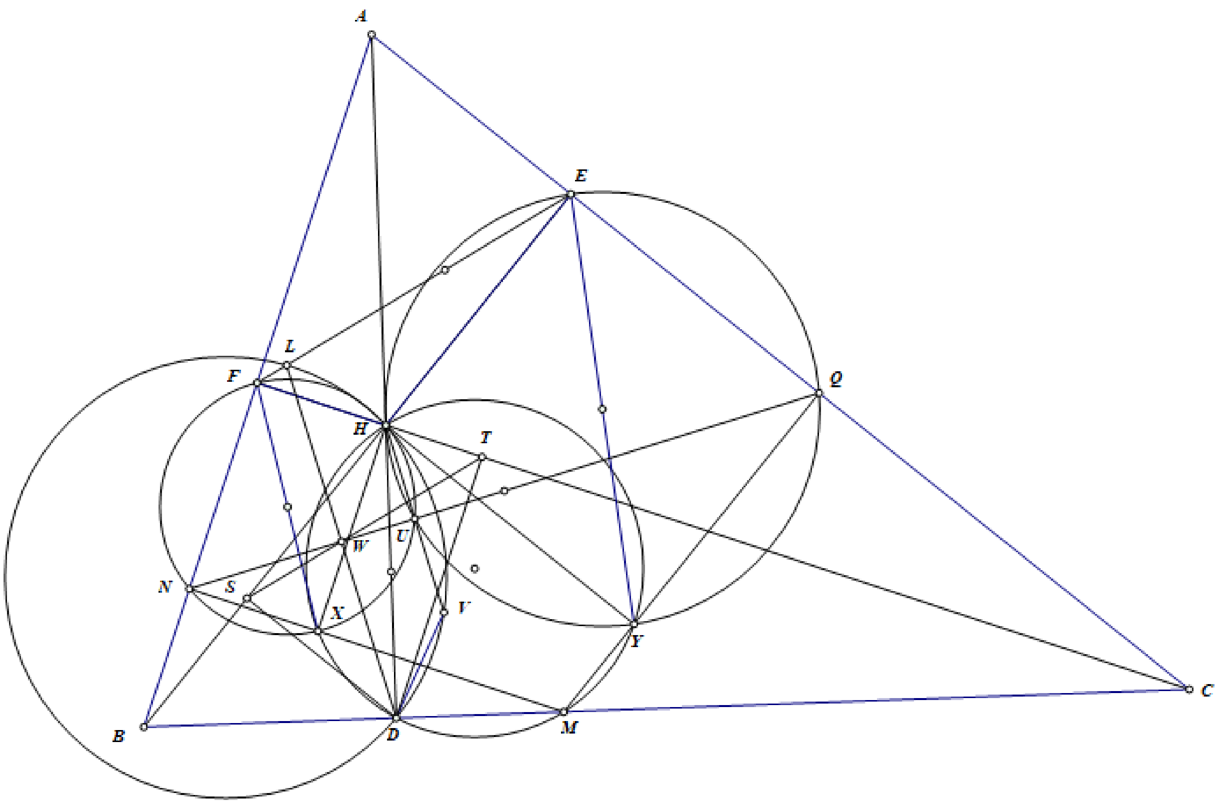
$$\frac{AX}{XD} = \frac{BY}{YC} = \frac{EZ}{ZF}.$$

Then the points X, Y, Z are collinear.

Lemma: Let triangle ABC have orthocenter H, and let X, Y be arbitrary points such that A, X, Y are collinear, $\angle BHX = \angle CHY = 90°$. Let BX intersect CY at Z, let M be the midpoint of ZH, and let MA intersect (BHC) at N. Then ZH is tangent to (AHN).



Proof: Let D, E, F be the feet of the three altitudes of triangle ABC.
Consider an inversion about a circle at H with radius $\sqrt{HA \cdot HD}$, the problem becomes:
Let triangle ABC have orthocenter H, and let X, Y be two arbitrary points that satisfy $\angle BHY = \angle CHX = 90^\circ$ and $HDXY$ is cyclic, (HXF) intersects (HEY) at U, V is symmetric to H with respect to U, (HVD) intersects EF at L.
Prove that $DL \parallel HU$.



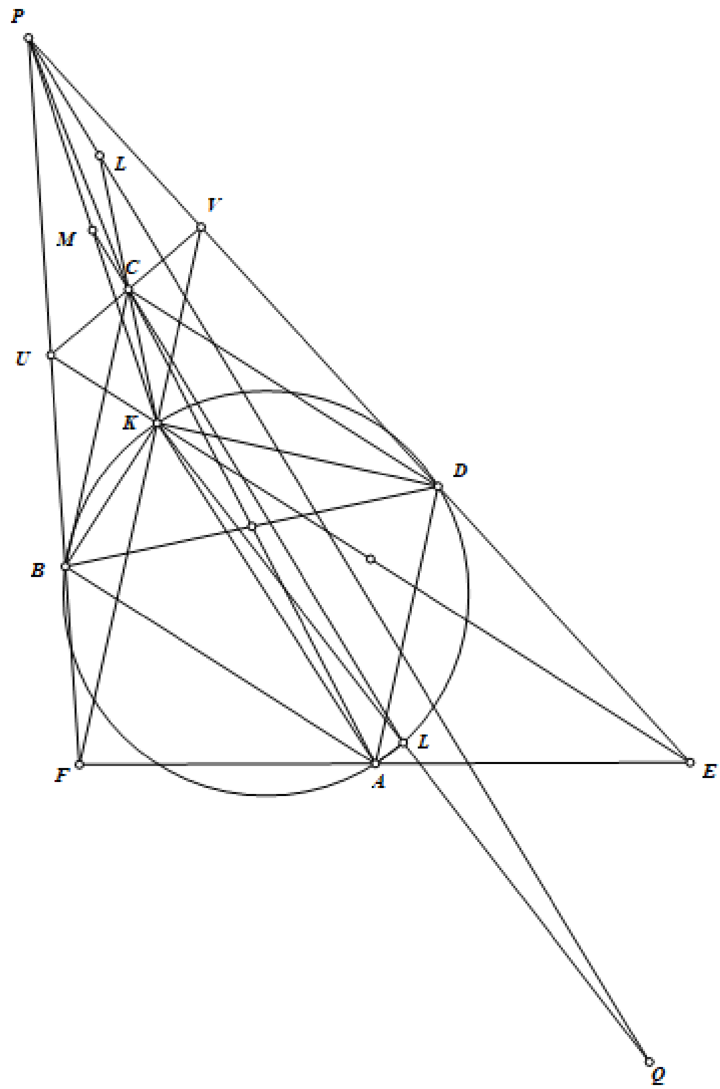
Indeed, let (HXY) intersect BC at M, and let N, Q be the projections of M onto AB, AC. Let L' as the reflection of D over NQ, we will first prove L lies on EF.
Let S, T, W be the projections of D onto BE, CF, NQ.

Since triangles DNQ, DBE, DFC are similar, we have $\dfrac{SB}{SE} = \dfrac{TF}{TC} = \dfrac{WN}{WQ}, \dfrac{NB}{NF} = \dfrac{MB}{MC} = \dfrac{QE}{QC}$.

Hence, by the ERIQ theorem, S, T, W are collinear. Moreover, the reflections of D across BE, CF lie on EF, therefore L' lies on EF.

It is clear that HN, HQ are diameters of (HXF) and (HEY), and since U lies on both circles, we get N, Q, U are collinear and $HU \perp NQ$. Therefore, L', H are reflections through NQ of D, V respectively, which yields $L'HVD$ is an isosceles trapezoid, so $(HVDL')$ is cyclic.
Hence, we have L' is also intersection of (HVD) and EF, therefore $L' \equiv L$. Since $DL' \parallel HU$(they are both perpendicular to NQ), we get $DL \parallel HU$, as desired.
Hence, the claim is proved.

Back to the main problem,



Redefine Q as a point on PL such that $AQ = AK$. The problem is equivalent to proving that PK is tangent to (LKQ).

Let M be the midpoint of PK, and let L be the intersection of CM with (KBD).
Using a homothety centered at K with ratio $1/2$, the problem reduces to proving that PK is tangent to (CKL).
Let EK, FK intersect BF, ED at U, V. Let C' be a point on UV such that $BC' \parallel FV \parallel AD$.
Then, by Thales' theorem, $\dfrac{C'U}{C'V} = \dfrac{BU}{BF} = \dfrac{AE}{AF} = \dfrac{DE}{DV}$, so $C'D \parallel AB$, which implies $C' \equiv C$.
Hence C lies on UV.
Applying the previous lemma to triangle CBD with orthocenter K, with three points U, C, V collinear, and $\angle BKU = \angle DKV = 90^\circ$, we obtain the desired result.
Hence, the problem is proved.