Problem 4

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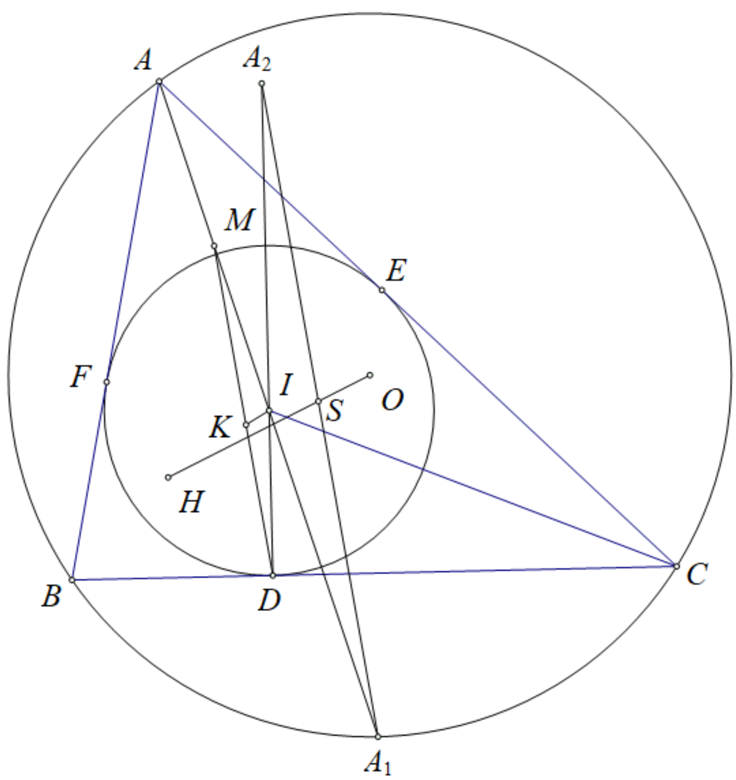
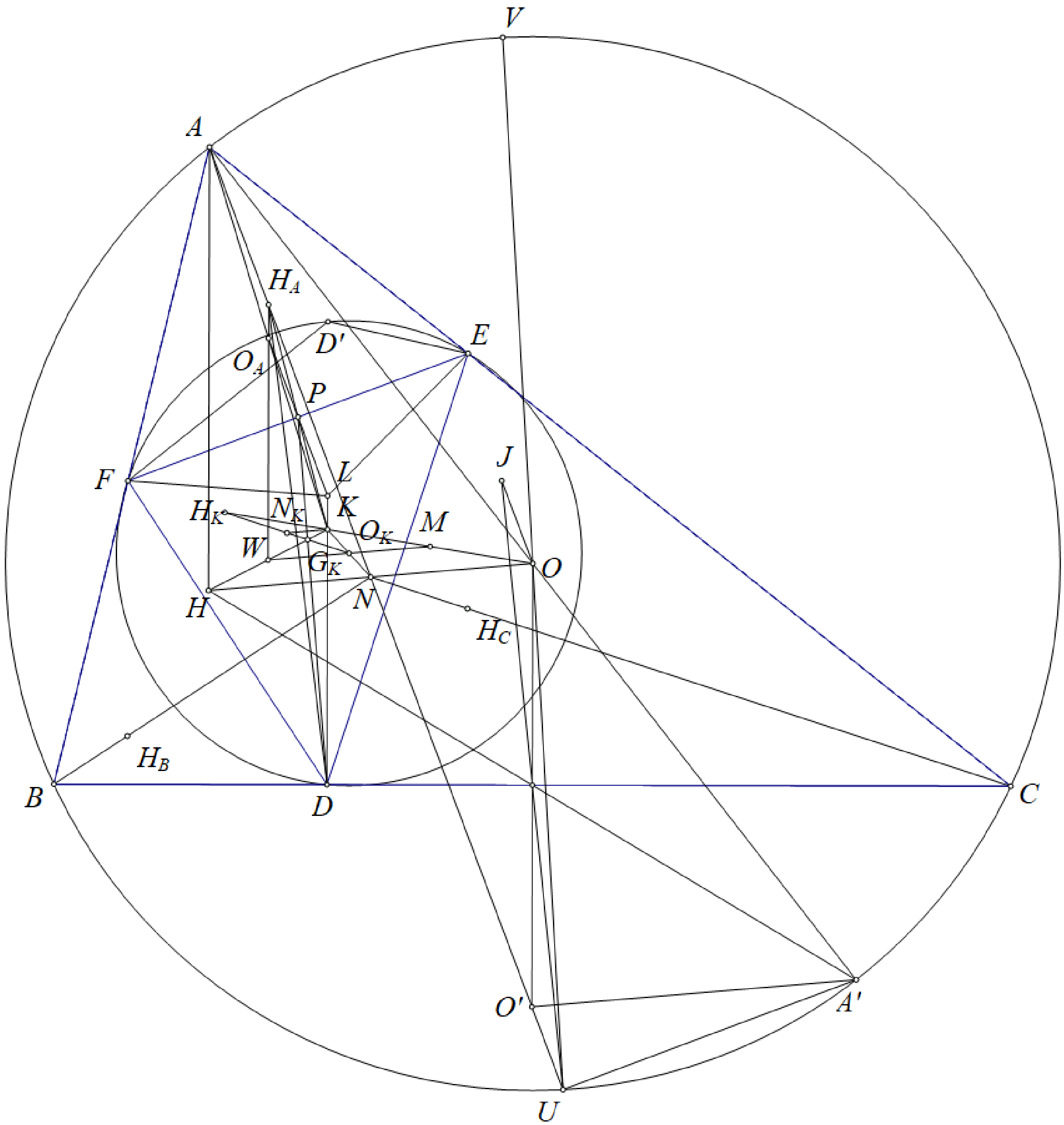
Lemma: Let ABC be a triangle with incenter I, DEF be the pedal triangle of I wrt ABC, K be the Kosnita point of DEF
then IK is parallel to the euler line of the orthic triangle of DEF.



Proof: Let A_1, B_1, C_1 be the midpoint of small arc BC, CA, AB of (O) respectively, let S be the Schiffler point of ABC point
then it is well known that S lies on the euler line of ABC.
Let A_2 be the orthocenter of BIC, M be the midpoint of AI
then D, K, M and A_1, S, A_2 are collinear.
Simple ratio chasing yield $A_1A_2 \parallel DM$ therefore $A_1S \parallel DK$.
Similiarly: $B_1S \parallel EK, C_1S \parallel FK$.
Combine this with the fact that DEF and $A_1B_1C_1$ are triangles with 3 pair of respective parallel sides
we get S is the Kosnita point of $A_1B_1C_1$ therefore $IK \parallel OS$ which is the euler line of ABC
which is also parallel to the euler line of the orthic triangle of DEF, as desired.

Claim: Let K be the Kosnita point of ABC, DEF be the pedal triangle of K wrt ABC.
Let H_K be the orthocenter of DEF, O, H be the circumcenter, orthocenter of ABC respectively, L be a point lies on DK such that $KH_K = LH_K$.
Then $H_K K$ is parallel to the euler line of the orthic triangle of ABC and $LE = LF$.

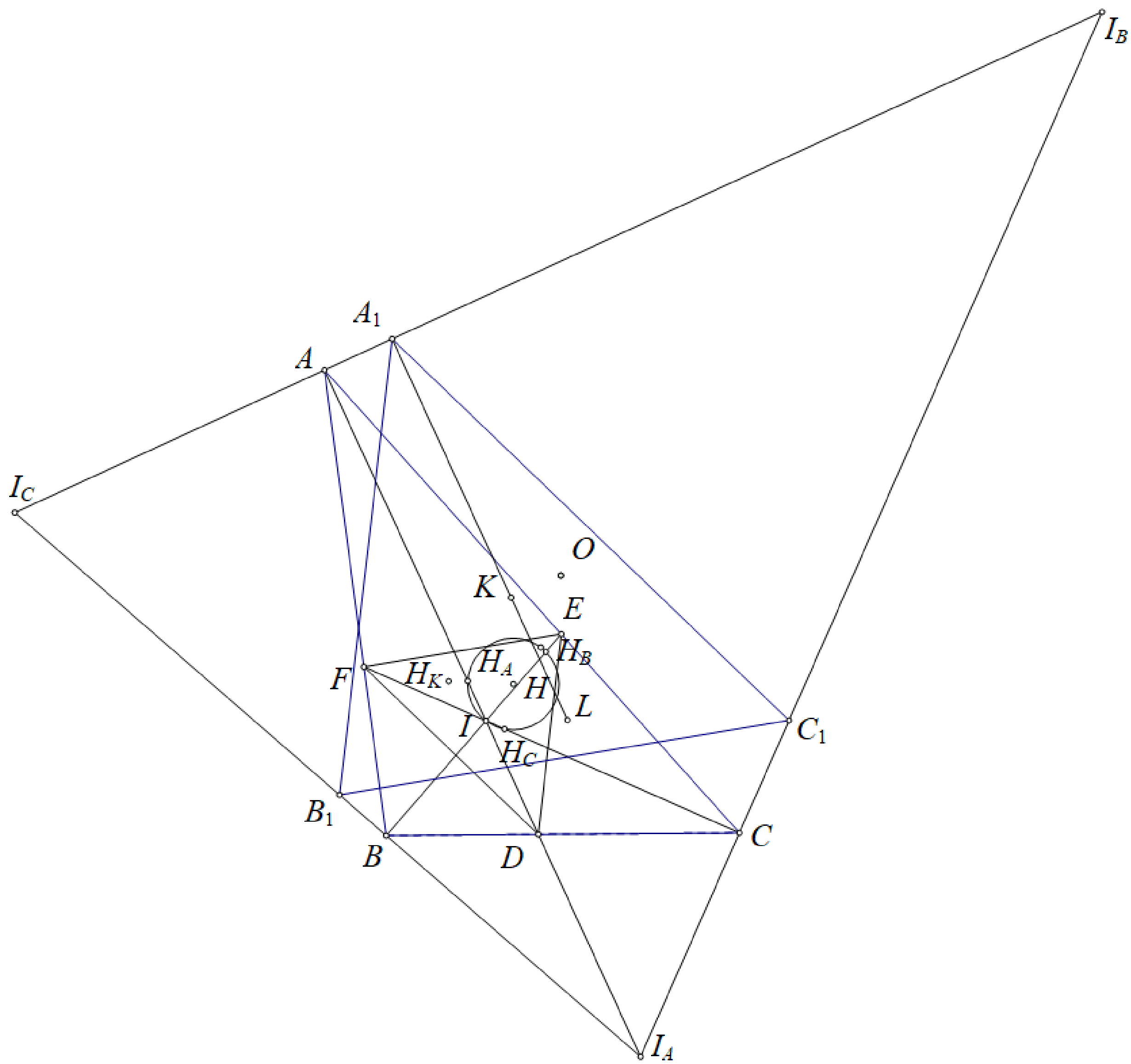


Proof: Let H_A, H_B, H_C be the orthocenter of AEF, BDF, CDE, O_A be the midpoint of AK which is circumcenter of AEF, we will prove $H_A O_A \perp BC$.
Let N be the nine point center of ABC, AN cut (O) at U and UV be the diameter of (O),
J be the orthocenter of VBC then $\triangle AEF \cup (H_A, O_A) \sim \triangle VBC \cup (J, O)$
therefore $\angle AO_A H_A = \angle VOJ$.
Let O' be the reflection of O through the midpoint of BC,

since J is the reflection of U through the midpoint of BC,
we get $OJ \parallel O'U$ therefore $\angle AO_AH_A = \angle VOJ = \angle VUO' = \angle OAU = \angle HAK$.
Therefore, H_AO_A is parallel to AH therefore H_AW is parralel to KD with W being the midpoint of KH.
Similarly H_BW, H_CW is parallel to KE, KF respectively. Combine this with the fact that $\triangle DEF = \triangle H_AH_BH_C$ we get $\overrightarrow{H_AW} = \overrightarrow{KD}$
therefore $\overrightarrow{WD} = \overrightarrow{H_AK} = 2 \cdot \overrightarrow{PK}$ with P being the midpoint of EF
therefore let G_K be the centroid of DEF then since $\overrightarrow{G_KD} = 2 \cdot \overrightarrow{PG_K}$
we get $\overrightarrow{G_KW} = 2 \cdot \overrightarrow{KG_K}(1)$.
Denote N_K, O_K as the nine point center, circumcenter of DEF we get O_K is the midpoint of KN and $\overrightarrow{G_KO_K} = 2 \cdot \overrightarrow{N_KG_K}$
combine this with (1) we get $\overrightarrow{WO_K} = 2 \cdot \overrightarrow{N_KK}$.
Let M be the midpoint of OK then $\overrightarrow{O_KM} = \overrightarrow{WO_K} = 2 \cdot \overrightarrow{N_KK}$.
Combine this with N_K being the midpoint of H_KO_K we get H_K, K, M are collinear which imply H_K, K, O are collinear.
Applying the lemma above we get OK parallel to the euler line of the orthic triangle of ABC
therefore KH_K is parallel to the euler line of the orthic triangle of ABC, as desired.

Redefine L as the reflection of O_A through the midpoint of EF which is P,
we will prove $KH_K = LH_K$ and L lies on DK.
Since L, K are reflections of O_A, H_A through P respectively we get $LK \parallel O_AH_A \parallel DK$ therefore L lies on DK.
Since O, J, H are reflections of O', U, A' through the midpoint of BC with AA' being the diameter of (O), we get $\angle HJO = \angle A'UO = 90^\circ$ therefore $NJ = NO$.
Since $\triangle NBC \sim \triangle D'EF$ with DD' being the diameter of DEF
we get $\triangle AEF \cup D', H_A, O_A \sim \triangle VBC \cup N, J, O$ combine with $NJ = NO$ we get $D'H_A = D'O_A$.
Since D', H_A, O_A are reflections of H_K, K, L through P respectively we get $KH_K = LH_K$, as desired.

Back to the main problem,



Let H_A be the midpoint of small arc EF of (AEF) and denote H_B, H_C similiarly.
Since the vertices of $\mathcal{T}$ are the antipode of A, B, C in $(AEF), (BDF), (CDE)$
we get the 3 angle bisectors of $\mathcal{T}$ is the lines from H_A, H_B, H_C perpendicular to IA, IB, IC
therefore II_1 is the diameter of $(IH_AH_BH_C)$.

Denote I_A, I_B, I_C as the A, B, C excenter of ABC respectively, let H be the orthocenter of DEF.
Let K be the Kosnita point of $I_AI_BI_C$ and $A_1B_1C_1$ be the pedal triangle of K wrt $I_AI_BI_C$.
Let H_K be the orthocenter of $A_1B_1C_1$, L be a point lies on A_1K such that $H_KK = H_KL$.
Applying the claim above, we get KH_K is parallel to the euler line of the orthic triangle of $I_AI_BI_C$ which is ABC and $LB_1 = LC_1$.

Let O be the circumcenter of ABC which is also the nine point center of $I_AI_BI_C$.
It is well known that $I_AO \perp EF$ and since $I_AO \perp B_1C_1$ we get $EF \parallel B_1C_1$.
Similiarly, $A_1B_1C_1$ and DEF are triangles with 3 pairs of respective sides parallel to each other.

Let H'_A as a point lies on DI such that $HH'_A = HI$.
Since $A_1K \parallel DI$ and similiarly $B_1K \parallel EI, C_1K \parallel FI$,
we get $\triangle DEF \cup H, I, H'_A \sim \triangle A_1B_1C_1 \cup H_K, K, L$
therefore since $LB_1 = LC_1$ we get $H'_AE = H'_AF$
and since H'_A also lies on AI therefore $H'_A \equiv H_A$ therefore $HI = HH_A$
and similiarly we get $HI = HH_A = HH_B = HH_C$ and since II_1 is the diameter of $(IH_AH_BH_C)$ we get I, H, I_1 are collinear.
We also get $IH \parallel KH_K$ and since KH_K is parallel to the euler line of ABC,
we get IH is parallel to the euler line of ABC therefore II_1 is parallel to the euler line of ABC.
Hence the problem is proved.