Problem 9

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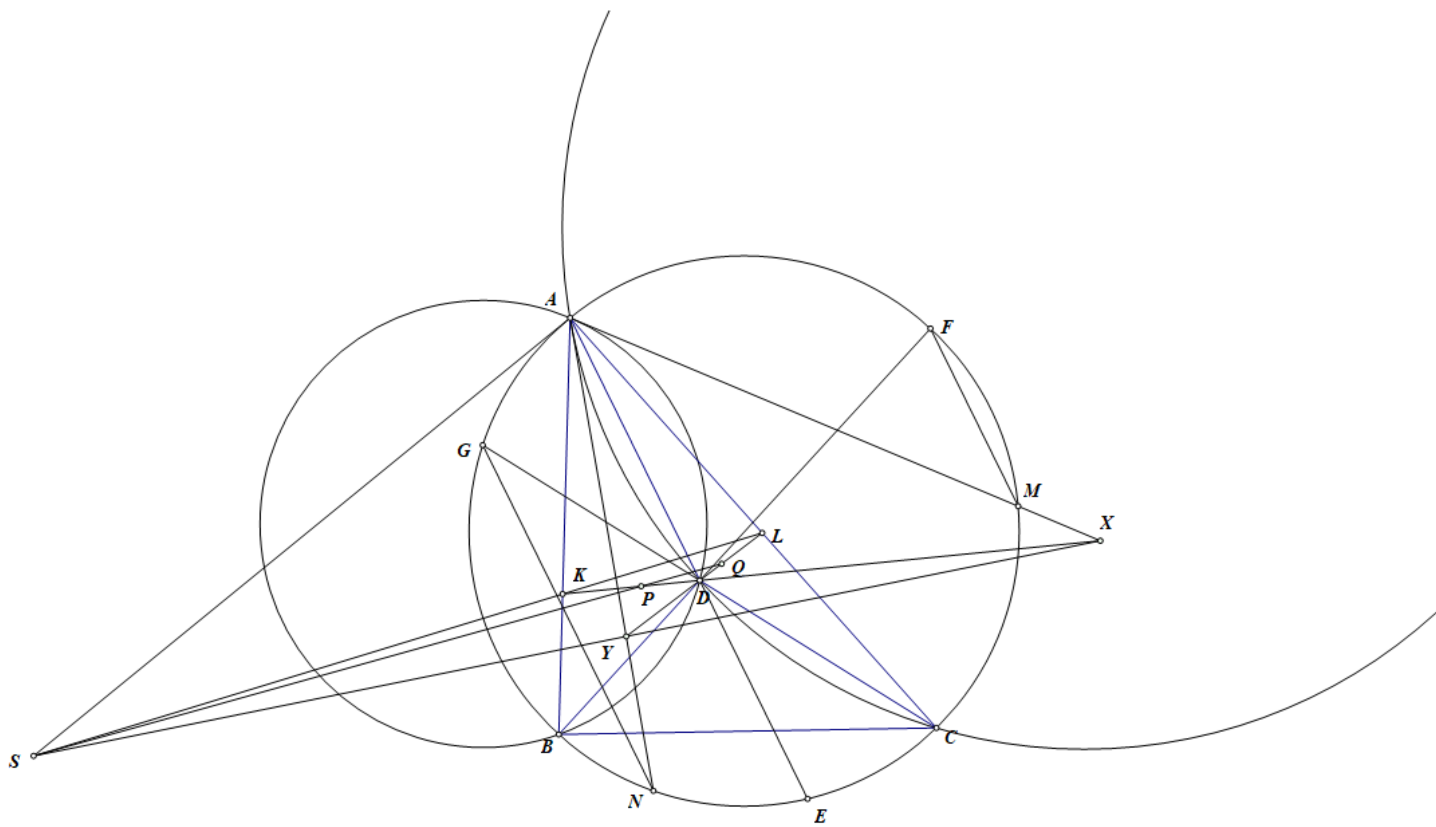
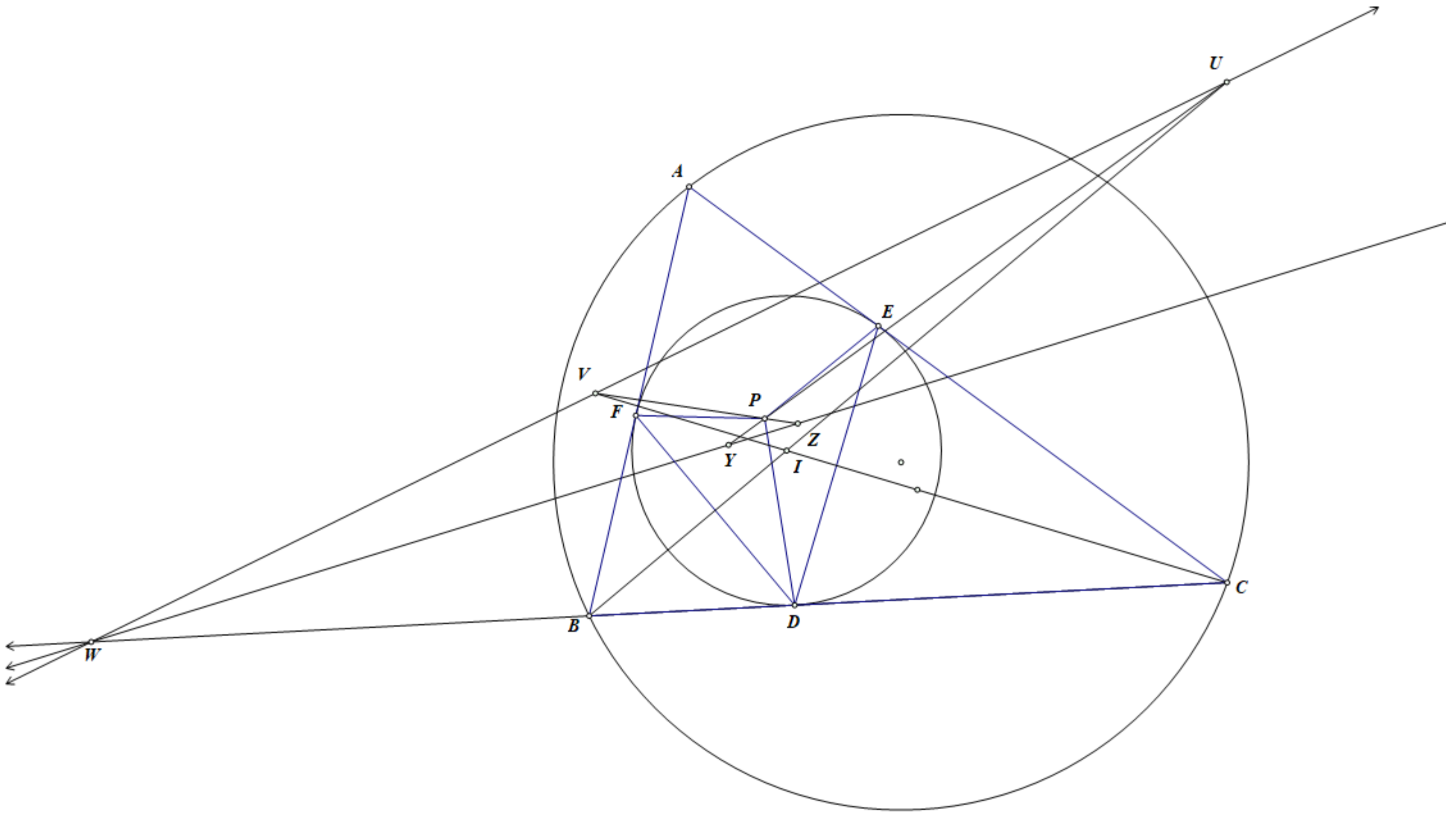
We will prove the following lemma: Let ABC be a triangle with an arbitrary point D lies inside ABC. Let tangents at A, B of (DAB) intersects at X, tangents at A, C of (DAC) intersects at Y, P, Q be the Lemoine point of DAB, DAC respectively. Prove that: the tangent at A of (ABC), PQ, XY concurrent at a point.



Proof: Let DX, DY intersects AB, AC at K, L, S be the intersection of KL and the tangent of (ABC) at A. We will prove S, X, Y are collinear, which is equivalent to: $A(SD, XY) = D(SA, XY)$. Let AX, AY, AD, BD, CD intersects (ABC) at M, N, E, F, G respectively. Since $\angle MAB = \angle BDE$ we got $\angle MBA = \angle DBE = \angle FBE$ therefore $MF \parallel AE$ and similiarly: $NG \parallel AE$.
Therefore: $A(SD, XY) = (AE, MN) = (EA, FG) = (AE, BC) = A(AE, BC) = A(SD, KL) = D(SA, KL) = D(SA, XY)$ and so S, X, Y are collinear.

Since $(DK, PX) = (DL, QY) = -1$ we got XY, KL, PQ concurrent at a point and since XY intersects KL at S we got the tangent at A of (ABC), PQ, XY concurrent at S, as desired.

Back to the main problem,



We will prove BY, CZ, IP concurrent at a point.
Applying *Desargues* theorem for triangle PYZ and IBC, we need to prove U, V, W which are the intersections of PY, IB; PZ, IC and YZ, BC are collinear. It is trivial that U is the intersection of tangents at D, F of (PDF), V is the intersection of tangents at D, E of (PDE) therefore applying the lemma to the triangle DEF with the point P lies inside DEF we get BC, YZ, UV concurrent at a point therefore U, V, W are collinear.

Hence, BY, CZ, IP concurrent at a point and similiarly: AX, BY, CZ, IP concurrent at a point, as desired.
Therefore, the problem is proved.