Problem 6

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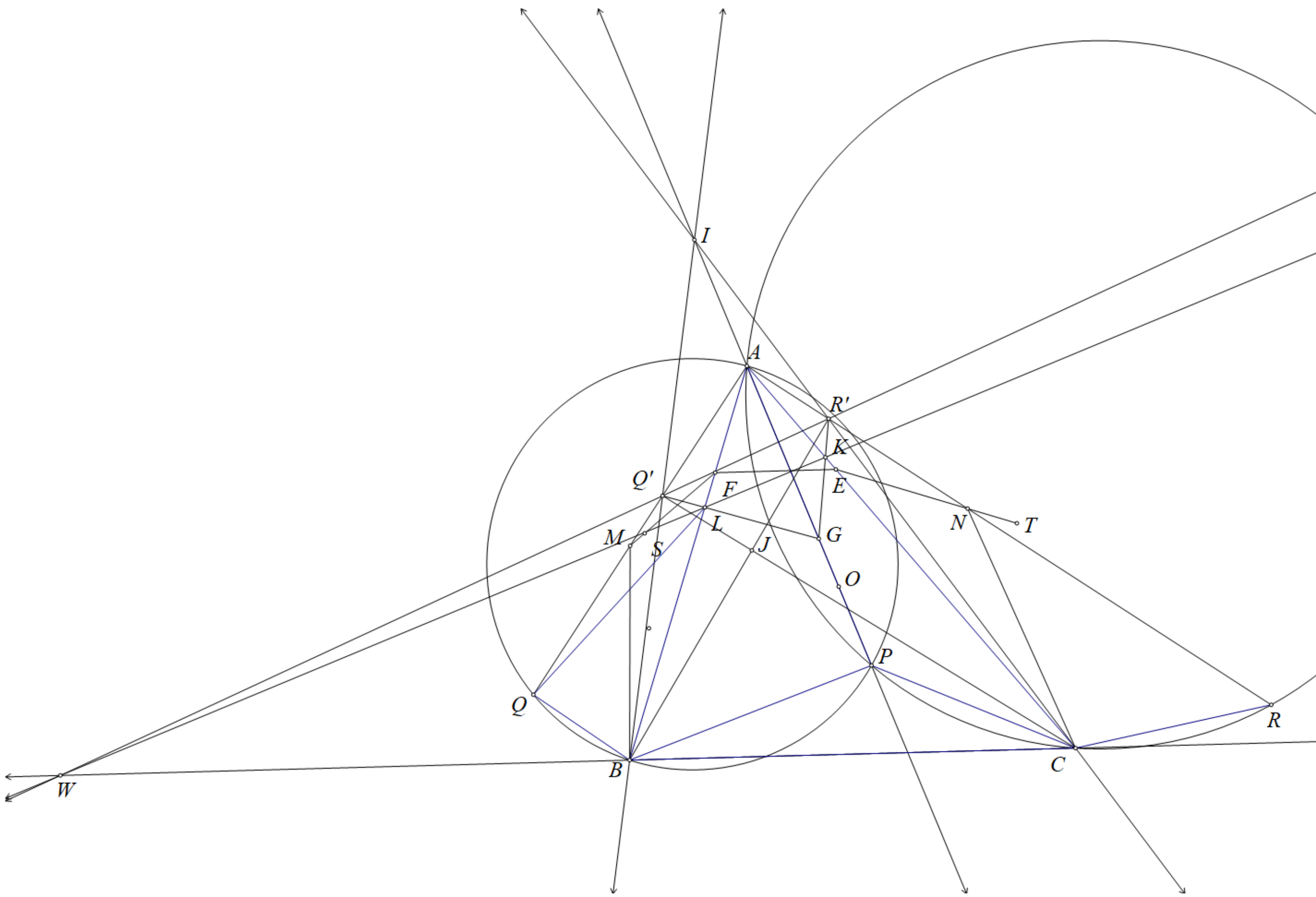

By simple angle chasing we get $\triangle AMF \sim \triangle ANE$ and $\triangle AMB \sim \triangle ANC$

therefore $\dfrac{AF}{AB} = \dfrac{AE}{AC}$ therefore $EF \parallel BC$.

Let K be the reflection of A through SF, L be the reflection of A through TE then K, L lies on AC, AB respectively
and we get S, T are the centers of $(AQK), (ARL)$ respectively and M, N are the centers of $(AKB), (ALC)$ respectively.
Therefore $\angle AKB = 90^\circ + \angle MAB = 90^\circ + \angle NAC = \angle ALC$ therefore $AL \cdot AB = AK \cdot AC$.
Consider an inversion about a circle at A with radius $\sqrt{AL \cdot AB}$. It sends $Q \mapsto Q', R \mapsto R', B \mapsto L, C \mapsto K$ hence it sends P which is the intersection of (AQB) and (ACR) to the intersection of LQ' and KR' which we will denote as G hence A, G, P, O are collinear.
Let J be the intersection of BR' and CQ' then the inversion above sends J to the intersection of (AQK)

and (ARL)
therefore AJ is the radical axis of (AQK) and (ARL) therefore $AJ \perp ST$.
We will prove $Q'R', KL, BC$ concurrent at a point.

Let KL cut BC at W we get $\dfrac{WL}{WK} = \dfrac{WB}{WK} \cdot \dfrac{WL}{WB} = \dfrac{sin\angle WKB}{sin\angle KBC} \cdot \dfrac{LC}{BK} = \dfrac{sin\angle LCB}{sin\angle KBC} \cdot \dfrac{AL}{AK}$.(1)

Let $Q'R'$ cut KL at W', we have $\angle Q'AG = \angle QAB + \angle BAO = 90^\circ - \angle BKC + 90^\circ - \angle ACB = 180^\circ - \angle BKC - \angle ACB = \angle KBC$ and similiarly $\angle R'AG = \angle LCB$.
Therefore, applying Menelaus theorem for triangle $GQ'R'$ with 3 collinear points W', L, K we get:

$$\frac{W'L}{W'K} = \frac{Q'L}{Q'G} \cdot \frac{R'G}{R'K} = \frac{sin\angle Q'AL}{sinQ'AG} \cdot \frac{AL}{AG} \cdot \frac{sin\angle R'AG}{sinR'AK} \cdot \frac{AG}{AK} = \frac{sin\angle Q'AL}{sin\angle R'AK} \cdot \frac{AL}{AK} \cdot \frac{sin\angle R'AG}{sin\angle Q'AG} = \frac{AL}{AK} \cdot \frac{sin\angle LCB}{sin\angle KBC}.$$

Combine this with (1) we get $W' \equiv W$ therefore $Q'R', KL, BC$ concurrent at W.
Applying Desargues theorem for triangle ALK and $IQ'R'$ we get AI, LQ', KR' concurrent at a point therefore AI pass through G therefore A, I, G, P, O are collinear.

Consider triangle ABC have AQ', AR' are isogonal wrt $\angle BAC$, I is the intersection of BQ' and CR', J is the intersection of BR' and CQ' therefore by the isogonal line lemma we get AI, AJ are isogonal wrt BAC and since AI pass through O we get $AJ \perp BC$.
Since $AJ \perp ST$ we get $ST \parallel BC$. Therefore, the problem is proved.