Problem 2

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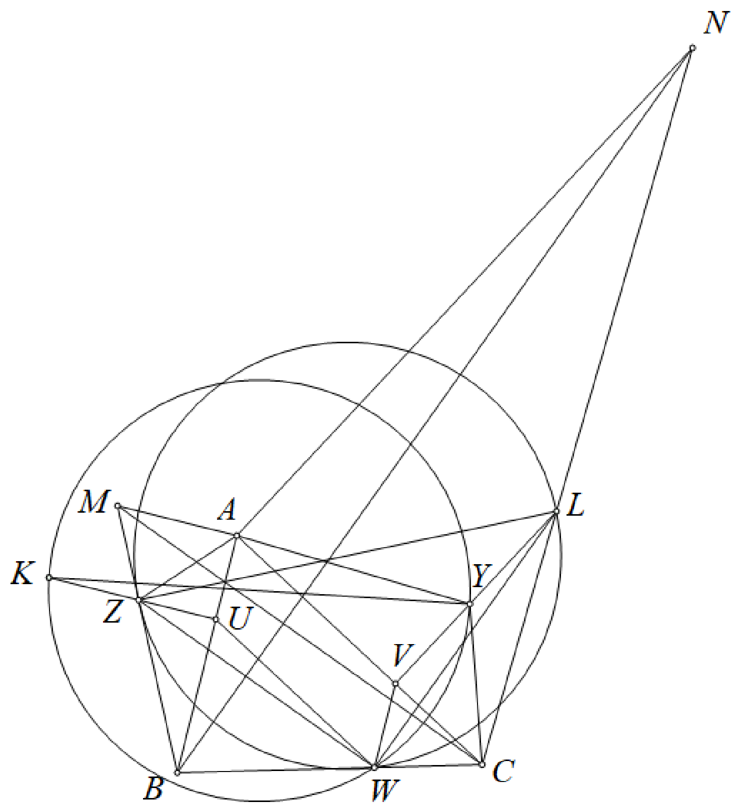
a) KZ cuts AB at U, LY cuts AC at V.



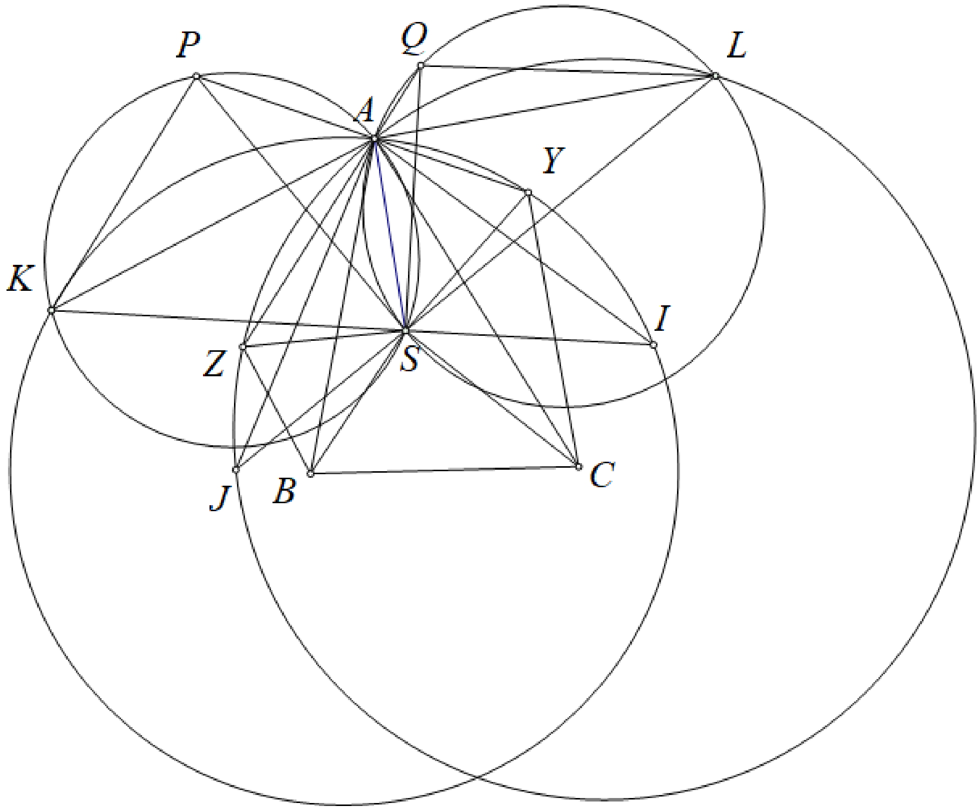
Since $\triangle AZB \sim \triangle CYA$, we have $\dfrac{UA}{UB} = \dfrac{VC}{VA}$.
By Thales' theorem, the line through U parallel to AC and the line through V parallel to AB intersect at point W on BC.
We will prove that W lies on both (KY) and (ZL).

Let the line through A perpendicular to AB intersect BZ at M, and the line through A perpendicular to AC intersect CL at N.

We have $\dfrac{WB}{WC} = \dfrac{UB}{UA} = \dfrac{ZB}{ZM}$,

hence $ZW \parallel CM$.
Similarly, $LW \parallel BN$.
Since $\triangle AMB \sim \triangle ACN$ (by angle-angle), we have $\triangle AMC \sim \triangle ABN$.
Because $AM \perp AB$, it follows that $CM \perp BN$, thus $ZW \perp WL$, so W lies on (ZL).
By the same reasoning, W also lies on (KY). Hence (KY) and (ZL) both contain W, a fixed point lies on BC, as desired.

b)



Let S be the A–Dumpty point; then S lies on the A–symmedian (the median from A to the symmedian line).
We will prove that S lies on the radical axis of (AKY) and (AZL).
Since $\triangle SAB \sim \triangle SCA$,
S is the spiral similarity center that sends $\triangle AZB$ to $\triangle CYA$, and therefore it also maps KZ to YL.
Hence $\triangle SZK \sim \triangle SYL$.
Let SK intersect (AKY) at I, and SL intersect (AZL) at J.
Let AY intersect (ASK) at P, and AZ intersect (ASL) at Q.
Then by simple angle chasing, we get $\triangle KPY \sim \triangle ASI$ (by angle-angle), and $\triangle LQZ \sim \triangle ASJ$ (also by angle-angle).
We need to prove that $SK \cdot SI = SJ \cdot SL$,

which is equivalent to $SK \cdot \dfrac{SI}{SA} = SL \cdot \dfrac{SJ}{SA} \Leftrightarrow SK \cdot \dfrac{PY}{PK} = SL \cdot \dfrac{QZ}{QL}$.
That is,

$$\frac{KS}{KP} \cdot \frac{YP}{YS} \cdot YS = \frac{LS}{LQ} \cdot \frac{ZQ}{ZS} \cdot ZS \Leftrightarrow \frac{\sin\angle KPS}{\sin\angle KSP} \cdot \frac{\sin\angle YSP}{\sin\angle YPS} = \frac{\sin\angle LQS}{\sin\angle LSQ} \cdot \frac{\sin\angle ZSQ}{\sin\angle ZQS} \cdot \frac{SZ}{SY}.$$

Thus,

$$\frac{\sin\angle KAS}{\sin\angle KAP} \cdot \frac{\sin\angle YSP}{\sin\angle AKS} = \frac{\sin\angle LAS}{\sin\angle LAQ} \cdot \frac{\sin\angle ZSQ}{\sin\angle ALS} \cdot \frac{SZ}{SY}.(1)$$

Now note that $\angle KAP = \angle LAQ$, and

$$\angle YSP + \angle ZSQ = 180^\circ - \angle SPA - \angle AYS + 180^\circ - \angle SQA - \angle AZS = 360^\circ - \angle SKA - \angle AYS - \angle SLA - \angle AZS.$$

Since $\triangle SBK \sim \triangle SAL$ and $\triangle SAZ \sim \triangle SCY$, we have $\angle SKA = \angle SKB$, $\angle SLA = \angle SYC$.

Hence, we have:

$\angle YSP + \angle ZSQ = 360^\circ - \angle KAB - \angle AYC = 360^\circ - \angle ALC - \angle AYC = 180^\circ$.
Therefore $\sin \angle KAP = \sin \angle LAQ$, and $\sin \angle YSP = \sin \angle ZSQ$, so from (1) we obtain

$$\frac{\sin \angle KAS}{\sin \angle AKS} = \frac{\sin \angle LAS}{\sin \angle ALS} \cdot \frac{SZ}{SY} \Leftrightarrow \frac{SK}{SA} = \frac{SL}{SA} \cdot \frac{SK}{SL},$$

which is true. Hence the problem is proved