Problem 11

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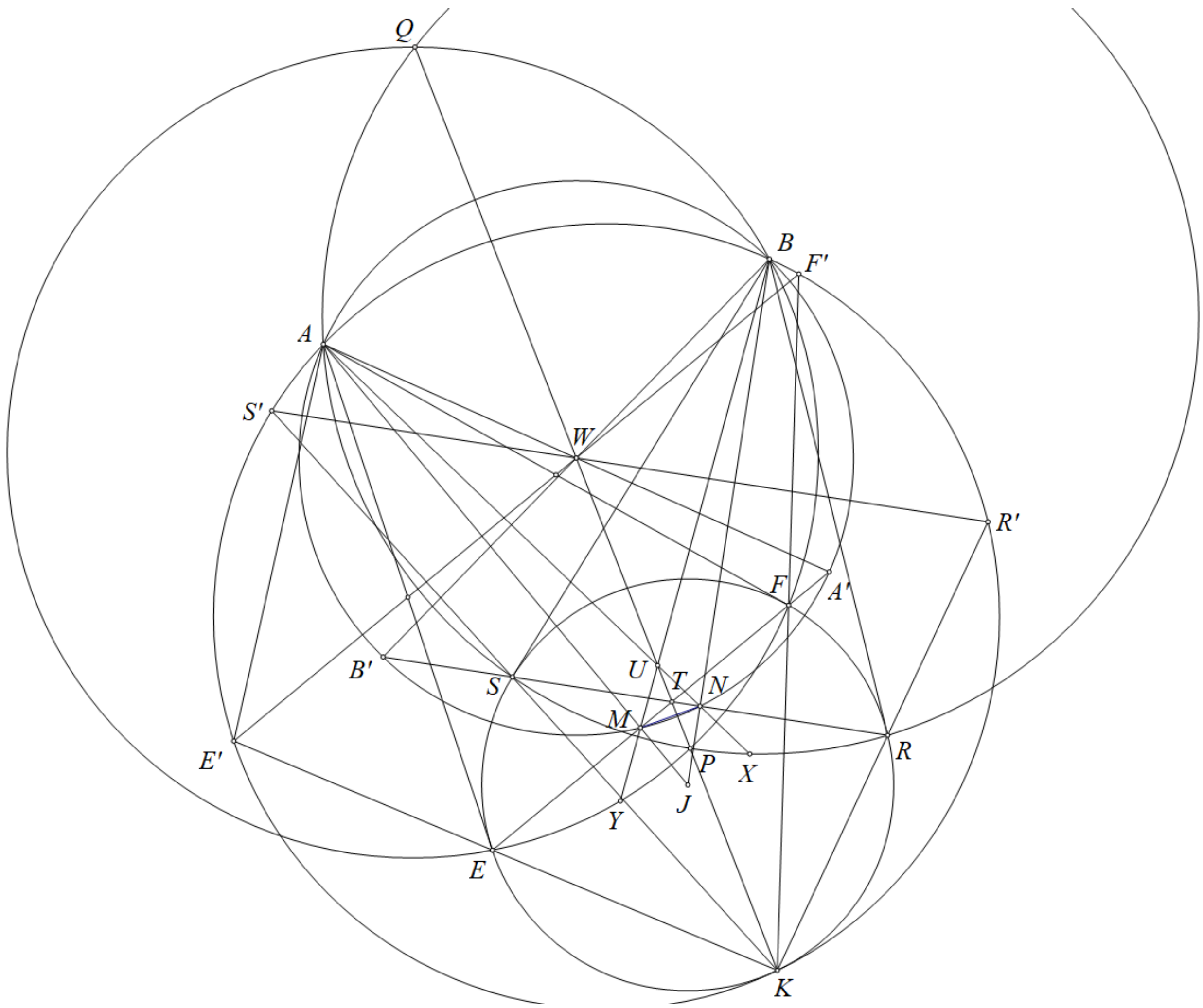

Let M, N be the midpoint of EF, SR. Denote J as the center of (Ω_2).

Let KE, KF, KS, KR cut (O) at E', F', S', R'; $E'F'$ cut $S'R'$ at W, EF cut SR at T. Since K is the exsimilicenter of (Ω_1) and (Ω_2) we get K, W, T are collinear.

Since $\angle E'AK = \angle E'F'K = \angle EFK = \angle AEE'$ we get $E'A^2 = E'E \cdot E'K$ which is equivalent to E' lies on the radical axis of $(A, 0)$ and (Ω_2).
Similiarly, we can prove that F' lies on the the radical axis of $(A, 0)$ and (Ω_2), therefore $E'F'$ is the radical axis of $(A, 0)$ and (Ω_2).
therefore $E'F'$ pass through the midpoint of AE, AF therefore $E'F'$ is the perpendicular bisector of AM.
and similiarly $S'R'$ is the perpendicular of BN therefore W is the intersection of perpendicular bisector of AM and $BN(1)$.
Since $JM \cdot JA = JN \cdot JB = R(\Omega_2)^2$ we get $AMNB$ are cyclic.

Combine this with (1), we get A, M, N, B lies on a circle with center W which we will denote as (Ω_3).

Let AA', BB' be the diameter of (Ω_3) we get A', B' lies on EF, SR respectively.
Let U be the intersection of AN and BM.

Applying Pascal theorem for: $\begin{pmatrix} A & M & B' \\ B & N & A' \end{pmatrix}$ we get T, U, W are collinear therefore K, W, T, U are collinear.

Let AN, BM cut $(ASR), (BEF)$ at X, Y respectively we get $MA \cdot MJ = ME \cdot MF = MB \cdot MY$ therefore $AYJB$ is cyclic and similiarly $AXJB$ is cyclic therefore $\angle AYB = \angle AJB = \angle AXB$ therefore $AYXB$ is cyclic therefore $\text{Pow}(U, (ASR)) = UA \cdot UX = UB \cdot UY = \text{Pow}(U, (BEF))$ therefore U lies on the radical axis of (ASR) and (BEF).
Also $TS \cdot TR = TE \cdot TF$ therefore T also lies on the radical axis of (ASR) and (BEF).
Therefore T, U lie on the radical axis of (ASR) and (BEF) and T, U, K are collinear we get K lies on the radical axis of (ASR) and (BEF), which is PQ, as desired.
Therefore, the problem is proved.