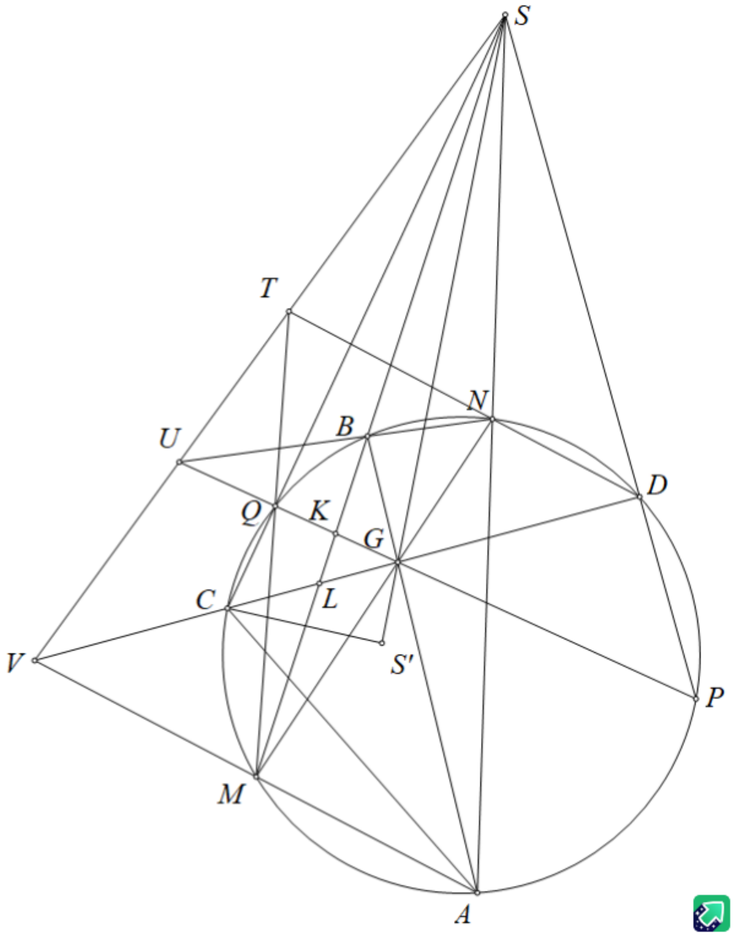
Problem 1

Ha Vu Anh

Lemma: Given circle (O), Let B, C, D, N, P, Q be arbitrary points lies on (O) such that CQ, BM, PD are concurrent at a point S (Q near S than C, D near S than P, B near S than M. Let PQ cut CD at G, GB, GM cut (O) at A, N respectively. Prove that: $(AB, CD) = (MN, PQ)$.



Proof:

We will prove A, N, S are collinear. Let SG cut (CBG) at S' we get $\angle SS'C = \angle SQG = \angle SDC$ therefore $SDS'C$ is cyclic therefore $GS \cdot GS' = GC \cdot GD$ and $SG \cdot SS' = SQ \cdot SC = SB \cdot SM$ therefore $BGS'M$ is cyclic.

Consider an inversion about a circle with center G and radius $\sqrt{GC \cdot GD}$. It sends $S \mapsto S'$, $N \mapsto M$, $A \mapsto B$ and since $GBMS'$ is cyclic we get S, N, A are collinear.

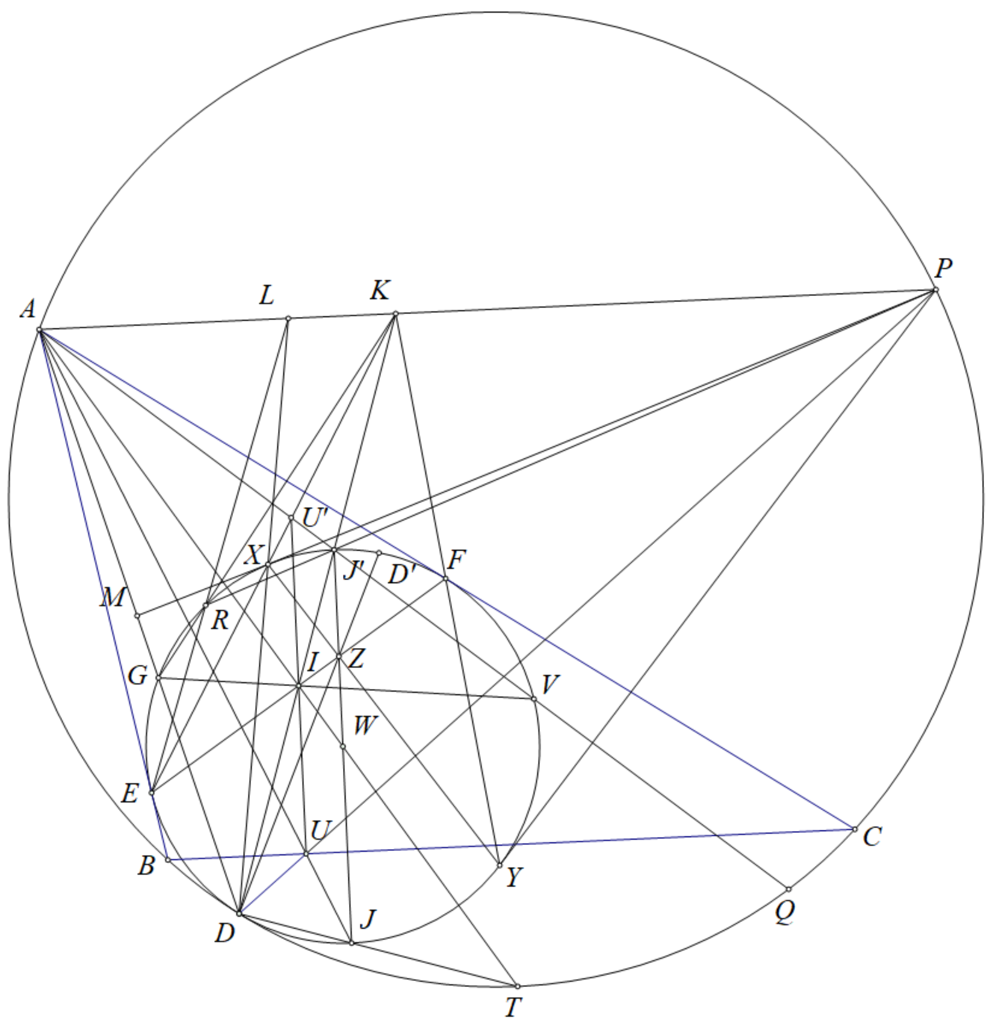
Let BN cut PQ at U, CD cut AM at V, MQ cut ND at T, Applying Pascal theorem for $\begin{pmatrix} B & Q & D \\ P & N & M \end{pmatrix}$ we get S, T, U are collinear.

Also applying Pascal theorem for $\begin{pmatrix} A & D & Q \\ C & M & N \end{pmatrix}$ we get V, T, S are collinear combine with above we get S, T, V, U is collinear.

Let SM cut PQ, CD at K, L. Since UV, CQ, KL, PD concurrent at a point S we get $(UK, QP) = (VL, CD)$. Also since $B(UK, QP) = (NM, QP)$ and $M(VL, CD) = (AB, CD)$ we get $(NM, QP) = (AB, CD)$.

Hence the lemma is prove.

Claim: Let (I) touch BC at U then $P(DA, NM) = \dfrac{UB}{UC}$.



Denote W as the center of (Ω), (Ω) touch AB, AC at E, F respectively. It is well known that P, D, U are collinear, the midpoint of EF is also the incenter of triangle ABC which we will denote as I, DI cut AP at K, EK, FK cut (Ω) at X, Y. We will prove PX and PY similiarly tangent to (Ω).

Let AD cut (Ω) at G, GK cut (Ω) at R, the line from W perpendicular to BC cut (Ω) at J', J respectively such that J' is nearer to A. We will prove P, R, J' are collinear.

Since D is the exsimilicenter of (Ω) and (O) we get DJ' pass through the midpoint of arc BAC of (O) therefore D, I, J' are collinear.

Let GI cut (Ω) at V, since A is the exsimilicenter of (I) and (Ω) we get AJ pass through U and AJ' pass through U' which is the reflection of U through I therefore AJ', AD are reflections through AI therefore G, J' are reflections through AI therefore $\angle AJ'I = \angle AGI = 180^\circ - \angle DGI = 180^\circ - \angle DJ'V$ therefore A, J', V are collinear.

Therefore $(J'K, J'x \parallel BC, J'A, J'R) = (DJ', VR) = G(DJ', VR) = (DJ', IK)(1)$.

Let AJ' cut (O) at Q since AD, AQ are reflections through AI, DI is the bisector of $\angle ADP$ we get $A(DJ', IK) = \dfrac{AD}{AQ} = \dfrac{DA}{DP} = \dfrac{KA}{KP} = (J'K, J'x \parallel BC, J'A, J'P)(2)$.

From $(1), (2)$ we get P, R, J' are collinear

Let ER cut DX at L. Applying Pascal theorem for $\begin{pmatrix} E & G & X \\ D & E & R \end{pmatrix}$ we get A, K, L are collinear.

Let P' be the intersection of RJ' and the tangent at X of (Ω). Also applying Pascal theorem for $\begin{pmatrix} R & X & D \\ X & J' & E \end{pmatrix}$ we get P', K, L are collinear therefore P' is the intersection of AP and RJ' therefore

$P' \equiv P$ therefore PX tangent to (Ω) at X therefore P, X, M are collinear, P, Y, N are collinear therefore from the claim we will prove $P(DA, YX) = \dfrac{UB}{UC}(*)$.

Let XY cut EF at Z we get Z is the polar of AP wrt (Ω) therefore $WZ \perp BC$ therefore Z lies on JJ'.

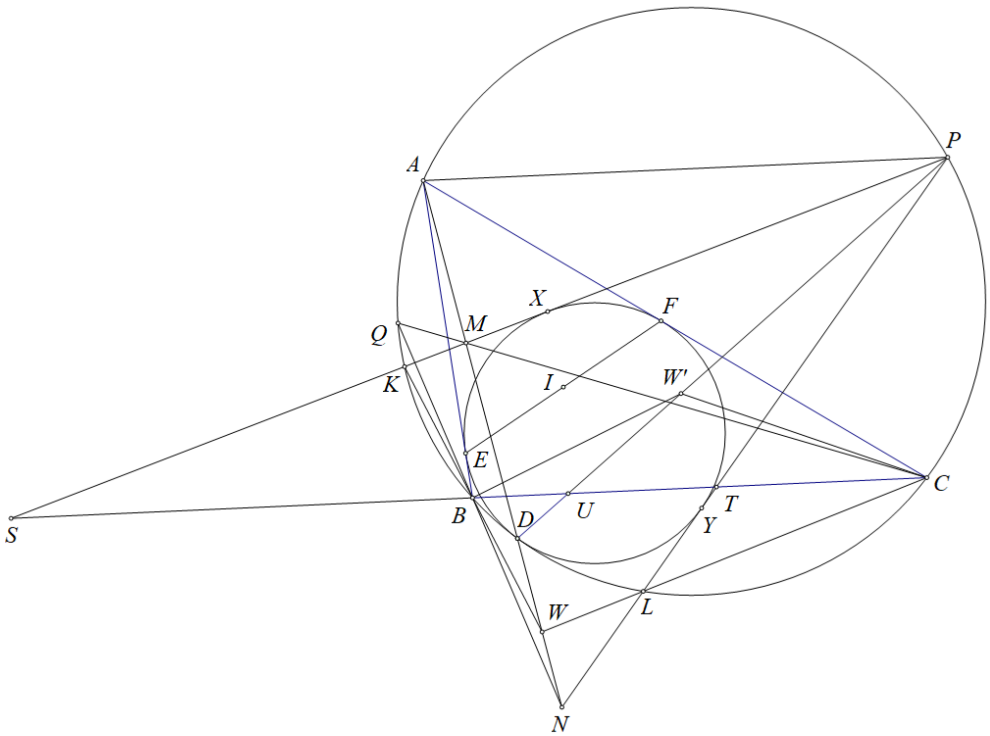
Let DZ cut (Ω) at D'. We have $P(DA, YX) = (Wx \perp PD, WZ, WY, WX) = D(DD', YX) = (DD', YX)(3)$

$\dfrac{UB}{UC} = A(UP, BC) = A(JP, BC) = (Wx \perp AJ, WZ, WE, WF) = J(JJ', EF)(4)$.

Applying the Lemma above for E, F, X, Y, D, J' be arbitrary points lies on (Ω) with EX, FY, DJ' concurrent at K, XY cut EF at Z, ZJ', ZD cut (Ω) at J, D' we get $(JJ', EF) = (DD', YX)$. Combine with $(3), (4)$ we get $P(DA, YX) = \dfrac{UB}{UC}$ therefore $(*)$ is proved.

Hence the claim is proved.

Back to the main problem,



Let PM cut (O) at K, cut BC at S, PN cut (O) at L, cut BC at T.

From the claim we get $\dfrac{UB}{UC} = P(DA, NM) = P(UA, TS) = \dfrac{UT}{US}$ therefore $UB \cdot US = UC \cdot UT$ therefore U lies on the radical axis of (PBS) and (PCT) therefore let (PBS) cut (PCT) at $W' \neq P$ we get W' lies on PD.

Let BK cut CL at W we get $\angle WBD = \angle SPU = \angle W'BC$ and similiarly $\angle WCD = \angle W'CB$ therefore W, W' are isogonal conjugates wrt triangle BDC.

Therefore DW, DP are isogonal wrt $\angle BDC$ therefore D, W, A are collinear hence BK, CL, AD concurrent at W. We need to prove BN cut CM on (O) which is equivalent to $B(NADP) = C(MADP) \iff P(NADB) = P(MADC) \iff C(LA, DB) = B(KA, DC) \iff C(WA, DB) = B(WA, DC)$ which is true since W, A, D are collinear. Hence the problem is proved.