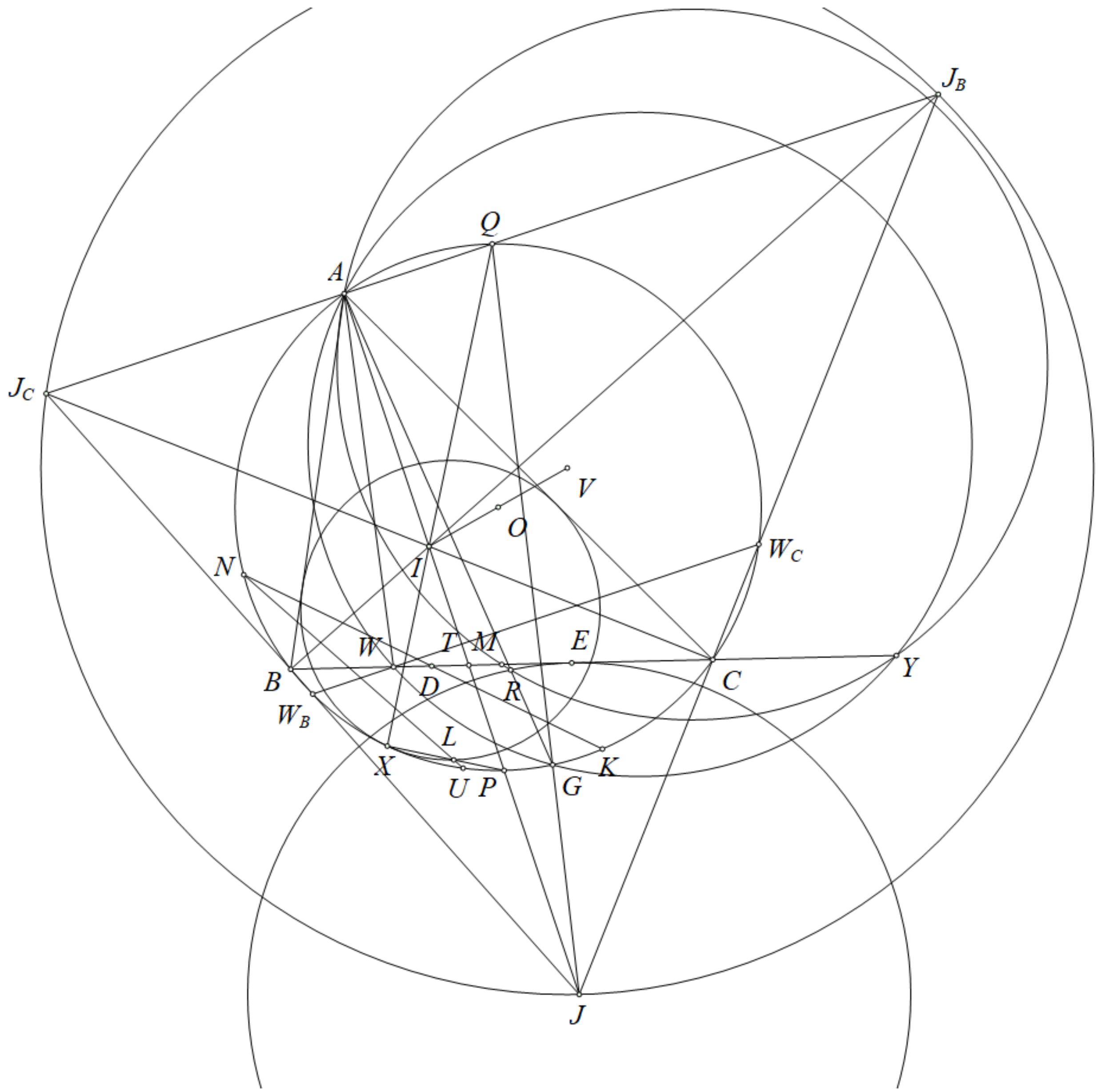
Inversion Illustrative Problem

Ha Vu Anh

Solution.

(Using the first figure) Redefine point L as the intersection of AD and the A-mixtilinear circle. Let NL cut (O) at U.



Let J be the excenter of triangle ABC tangent to BC at E; then AX and AE are isogonal. Let line XI meet (O) at Q, the midpoint of arc BAC of (O). Then JQ meets (O) at G, and AD and AG are isogonal.

Denote the three excenters of triangle ABC by J, J_B, J_C. Let JB, JC cut (O) again at W_b, W_c, the midpoints of JJ_B and JJ_C respectively. Let W_bW_c, which is also the perpendicular bisector of AJ, meet BC at W, and let V be the center of (JJ_BJ_C). Then W_B, W_C, J, V lie on the circle with diameter JV.

By the power of a point:

$$P(W,(JI)) = WB \cdot WC = WW_B \cdot WW_C = P(W,(JV)),$$

so W lies on the radical axis of (JV) and (JI). Hence $JW \perp IV$. Note that O is the nine-point center of triangle JJ_BJ_C, so O is the midpoint of IV. Therefore $JW \perp OI$, implying $JW \parallel AK$. On the other hand, $WJ = WA$, so AW and AK are isogonal.

Consider inversion centered at A with power $AB \cdot AC$, combined with reflection across the bisector of $\angle BAC$.
It maps:
- The internal A-mixtilinear circle $\mapsto (J)$ (the A-excenter circle).
- $K \mapsto W$.
- $D \mapsto G$.
- $X \mapsto E$.
- $P \mapsto T$ (intersection of AI with BC).

Hence, it maps:

- $N \mapsto Y$ (intersection of (AGW) with BC).
- $L \mapsto R$ (intersection of AG with (J), closer to A).
- $U \mapsto M$ (intersection of (ARY) with BC).

From the inversion transformation above, the problem is equivalent to proving:
(a) X, L, P are collinear, equivalent to showing A, T, R, E are concyclic.
(b) AU passes through the intersection of the two tangents at B and C to (O), equivalent to AU being a symmedian line, or that M (intersection of (ARY) with BC) is the midpoint of BC, since AU, AM are isogonal with respect to $\angle BAC$.
(Using the second figure) Performing extraversion on the above problem, transforming the excenter configuration into the incircle configuration, the problem is restated as below:
Triangle ABC is inscribed in (O) and has incircle (I). Line AI meets BC at T, and (I) is tangent to BC at D. Let G be the tangency point of the A-mixtilinear circle with (O). Let AG meets (I) at R (closer to G), (AGW) meets BC at Y.
Prove that:
(a) A, R, D, T are concyclic.
(b) (ARY) bisects segment BC.
Proof (a). Let AG meet (I) a second time at H. Since GI is the bisector of $\angle AGD$, quadrilateral $IHGD$ is cyclic. Therefore:

$$\angle ARD = 90^\circ + \angle IHD = 90^\circ + \angle IGA = 90^\circ + \angle DIT = 180^\circ - \angle ATD,$$

hence A, R, D, T are concyclic, as desired.
Proof (b). It is clear that R is the reflection of D across line GI, so $GR = GD$. Let MI meet BC at N, and let GI meet (O) at M. Let AI meet (O) at X. Then:

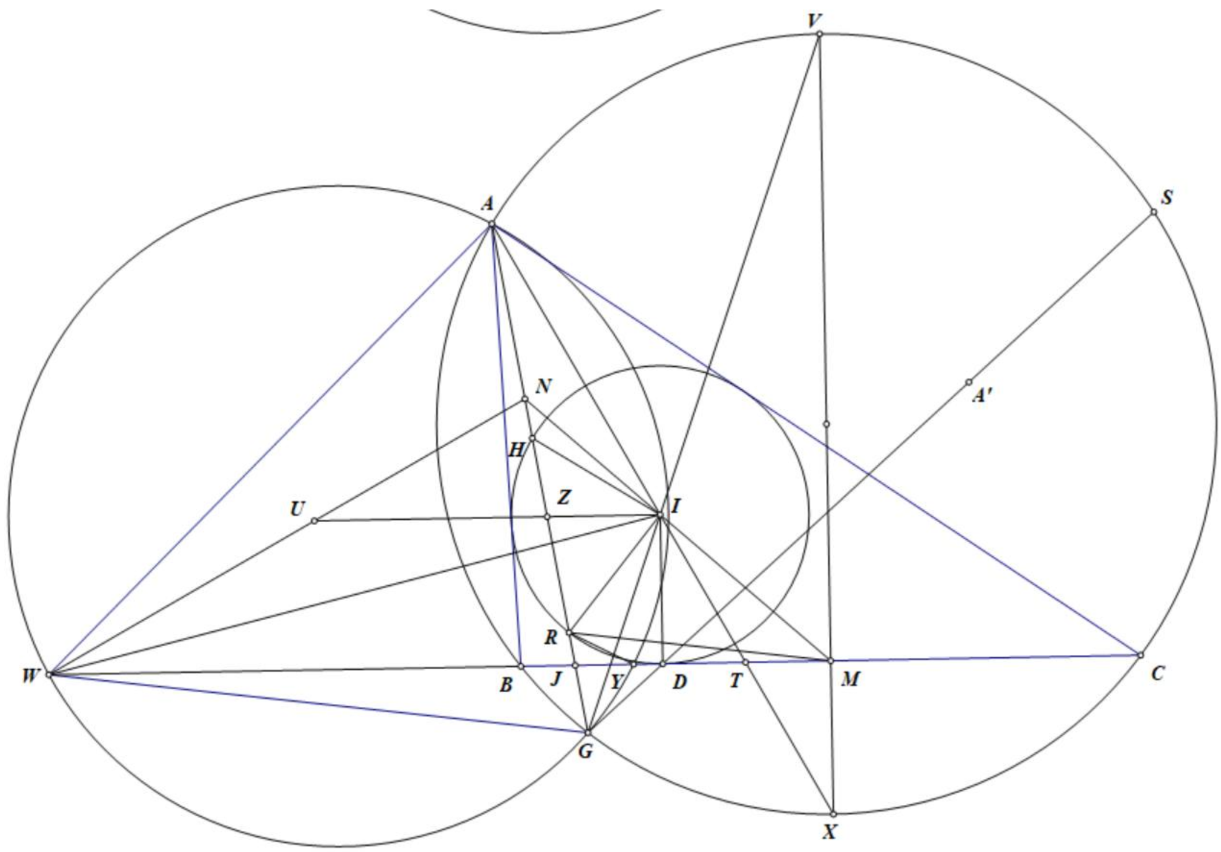
$$\angle AIN = \angle MIT = \angle IVX = \angle NAI,$$

so $NA = NI$. Let U be the center of (AIG); then U lies on MN, and:

$$\angle UIA = 90^\circ - \angle AGI = \angle ACX = \angle ATB,$$

so $UI \parallel BC$. Let AG meet UI and BC at Z and J respectively; then:

$$\frac{WJ}{WM} = \frac{UZ}{UI}.$$



Let GD meet (O) at S, and choose A' on GS with $GA' = GA$. Then:

$$\triangle ZUA \sim \triangle AGS, \quad \triangle AUI \sim \triangle AGA',$$

which yields:

$$\frac{WJ}{WM} = \frac{UZ}{UI} = \frac{GA'}{GS} = \frac{GA}{GS} = \frac{GJ}{GD} = \frac{GJ}{GR}.$$

Hence $MR \parallel WG$, so:

$$\angle RMY = \angle YWG = \angle YAR,$$

and therefore A, R, Y, M are concyclic, as desired.
Hence, the problem is proven.