Problem 2

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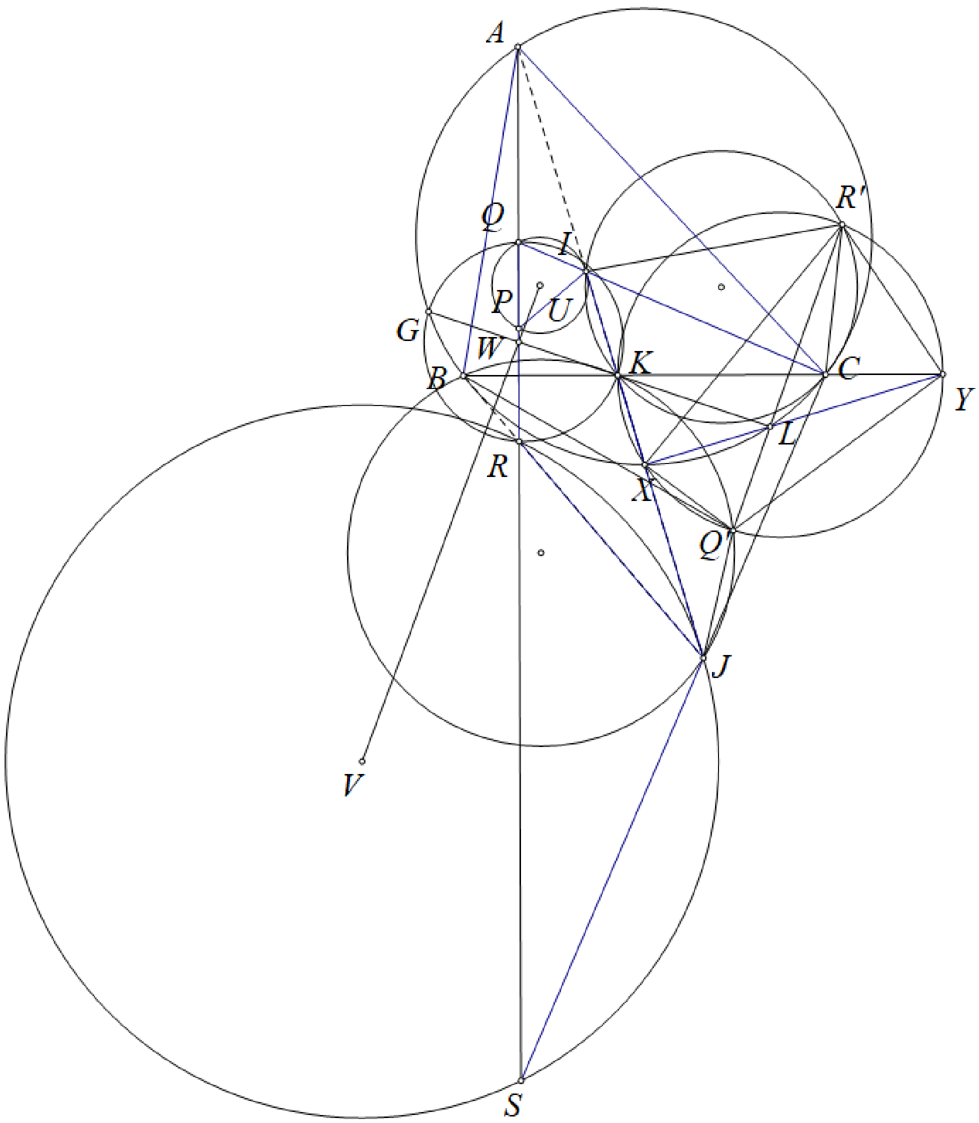

Let ℓ cut UV at W, denote U, V as the center of $(IPQ), (JRS)$ respectively then by simple angle chasing we get $UQ \parallel VS, UP \parallel VR$ therefore $\frac{WQ}{WS} = \frac{UQ}{VS} = \frac{UP}{VR} = \frac{WP}{WR}$ therefore $WP \cdot WS = WQ \cdot WR$ therefore W lies on the radical axis of (KQR) and (KPS).

We will prove KL is indeed the radical axis of (KQR), (KPS)

Let LK cut (O) at $G \neq L$, we will prove G lies on (KQR) and similiarly with (KPS). Let AI cut (O) at $X \neq A$, XL cut BC at Y. Consider an inversion about a circle at K with radius $\sqrt{KB \cdot KC}$, it sends $G \mapsto L$, $\ell \mapsto (KXY)$, $Q \mapsto Q'$, $R \mapsto R'$ then Q', R' is the intersection of $(KBJ), (KIC)$ with (KXY) respectively. We will need to prove R', Q', L are collinear.

We have $XI^2 = XB^2 = XL \cdot XY$, therefore $\dfrac{LX}{LY} = \dfrac{XL \cdot XY}{YL \cdot XY} = \dfrac{XI^2}{YB \cdot YC}(*)$

By simple angle chasing we get $\triangle R'IX \sim \triangle R'CY$ and $\triangle Q'JX \sim \triangle Q'BY$ therefore $\angle XR'Y = \angle IR'C = \angle JKC = 180^\circ - \angle XQ'Y$ therefore $XYR`Q'$ are cyclic.

Therefore let $R'Q'$ cut XY at L' then $\dfrac{L'X}{L'Y} = \dfrac{R'X}{R'Y} \cdot \dfrac{Q'X}{Q'Y} = \dfrac{XI}{CY} \cdot \dfrac{XJ}{YB} = \dfrac{XI^2}{YC \cdot YB}$. Combine with $(*)$ we get $\dfrac{L'X}{L`Y} = \dfrac{LX}{LY}$ therefore $L \equiv L'$ therefore L lies on $R'Q'$.

Hence the problem is proved.