Problem 1

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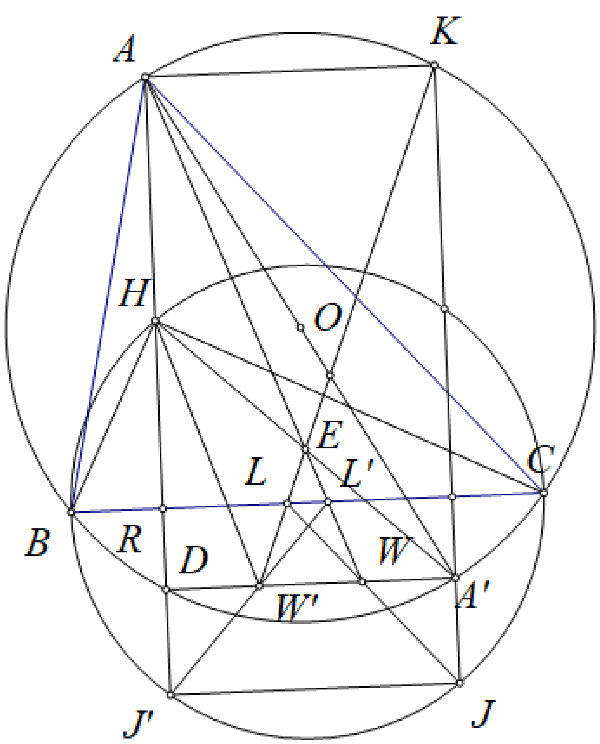
Lemma : Let ABC be a triangle with circumcircle (O), H be an arbitrary point that lies on the line from A perpendicular to BC. Let AH, AO cut (O) at D, A', HJ be the diameter of (BHC), E be the midpoint of HA', JA' cut (O) at K, EK cut BC at L, JL cut $A'D$ at W; I be an arbitrary point that lies on the line from E parallel to BC. Let X, Y be the points lies on AB, AC such that IX, IY is parallel to HB, HC respectively. Prove that W lies on the radical axis of (X, XB) and (Y, YC).

Claim: $\dfrac{WD}{WA'} = \dfrac{AD}{AH}(*)$



Redefine W as a point lies on $A'D$ such that $\dfrac{WD}{WA'} = \dfrac{AD}{AH}$, we will prove W lies on JL.

Applying Menelaus theorem for triangle $HA'R$ with $\dfrac{WA'}{WD} \cdot \dfrac{AD}{AH} \cdot \dfrac{EH}{EA'} = 1$ we get A, E, W are collinear.

Let J' be the projection of J on AD we get that B, H, C, J', J are concyclic therefore $RH \cdot RJ' = RB \cdot RC$ $= RA \cdot RD \iff \dfrac{RH}{RA} = \dfrac{RD}{RJ'} \iff \dfrac{AH}{AR} = \dfrac{J'D}{J'R}$.

Let AW cut BC at L', $J'L'$ cut $A'D$ at W' .

Applying Menelaus theorem for triangle WLW' with A, D, J' are three points collinear we get

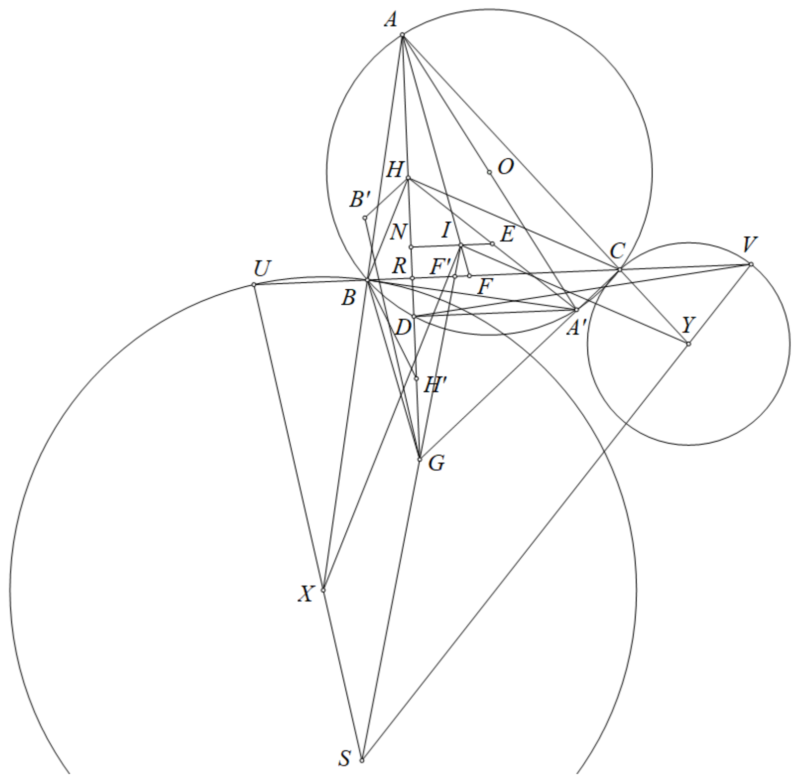
$\dfrac{DW'}{DW} \cdot \dfrac{AW}{AL'} \cdot \dfrac{JL'}{JW'} = 1$ therefore

$\dfrac{DW}{DW'} = \dfrac{AW}{AL'} \cdot \dfrac{JL'}{JW'} = \dfrac{AD}{AR} \cdot \dfrac{J'R}{J'D} = \dfrac{AD}{AR} \cdot \dfrac{AR}{AH} = \dfrac{AD}{AH} = \dfrac{WD}{WA'}$.

therefore we get $DW' = WA'$ therefore W are reflection of W' through the perpendicular bisector of BC and since J and L are reflections of J', L' through the perpendicular bisector of BC respectively combine with J', W', L' are collinear we get W, J, L are collinear.

Hence the claim is proved.

Back to proving the lemma:
Let G be the reflection of A through EI, (Y, YC), (X, XB) cut BC at V, U respectively, we will prove YV, XU, IG concurrent at a point.



Let B' be the reflection of B through EI, XU cut IG at S, H' be the reflection of H through BC.

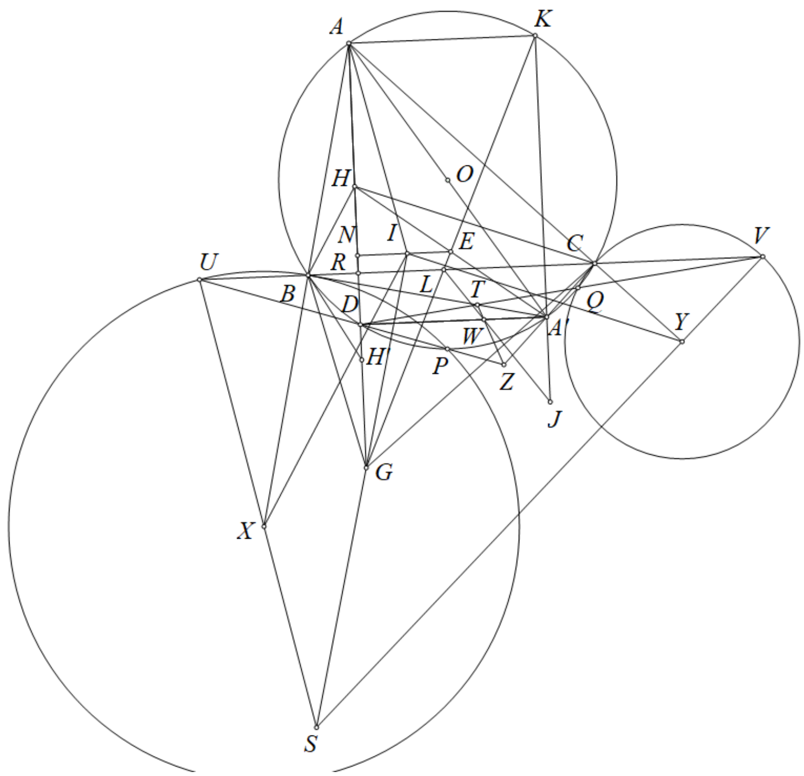
Since $\angle B'GA = \angle BAG$ we get $B'G \perp CD$, $\angle XUB = \angle UBX = \angle ABC = 90^\circ - \angle DCB$ therefore $UX \perp CD$ therefore $B'G \parallel XU$ and so $\angle XSI = \angle B'GI = \angle BAI$.

By simple angle chasing we get $\angle IXS = \angle ABH'$ therefore we get
$\dfrac{IA}{IS} = \dfrac{IA}{IX} \cdot \dfrac{IX}{IS} = \dfrac{sin\angle IXA}{sin\angle IAX} \cdot \dfrac{sin\angle ISX}{sin\angle IXS} = \dfrac{sin\angle HBA}{sin\angle IAB} \cdot \dfrac{sin\angle IAB}{sin\angle ABH'} = \dfrac{sin\angle ABH}{sin\angle ABH'} = \dfrac{AH}{AH'}$(since $BH = BH'$).

Let AI cut BC at F, IG cut BC at F', EI cut HD at N, AH cut BC at R then N is the midpoint of HD we get $IF = IF'$ therefore $\dfrac{IF'}{IS} = \dfrac{IF}{IA} \cdot \dfrac{IA}{IS} = \dfrac{NR}{NA} \cdot \dfrac{AH}{AH'}$.

Similiarly let YV cut IG at S' we get $\dfrac{IF'}{IS'} = \dfrac{NR}{NA} \cdot \dfrac{AH}{AH'}$ therefore $S' \equiv S$ therefore YV, XU, IG concurrent at S

Since $\triangle SUV \sim \triangle ABC$ we get
$\dfrac{UV}{BC} = \dfrac{d(S, UV)}{AR} = \dfrac{d(S, UV)}{d(I, BC)} \cdot \dfrac{d(I, BC)}{AR} = \dfrac{F'S}{F'I} \cdot \dfrac{RN}{RA} = \left(\dfrac{IS}{IF'} - 1\right) \cdot \dfrac{RN}{RA} = \left(\dfrac{NA}{NR} \cdot \dfrac{AH'}{AH} - 1\right) \cdot \dfrac{RN}{RA}$ (1).



Since the midpoint of AJ lies on the perpendicular bisector of BC we get $JA' \perp BC$

Let DU cut (X, XB) at P we get $\angle DPB = 90^\circ - \angle XBU = 90^\circ - \angle ABC = \angle DAB$ therefore P lies on (O) and similiarly DV cut (X, XC) at Q we get Q lies on (O).

Let BA' cut DV at T we get $\angle TBQ = \angle A'BQ = \angle A'DQ = \angle BVT$ therefore $TB^2 = TQ \cdot TV$ and since TB tangent to (X, XB) at B we get T lies on the radical axis of (X, XB) and (Y, YC). Similiarly let CA' cut DU at Z we get YZ is the radical axis of (X, XB) and (Y, YC). Let YZ cut $A'D$ at W^*.

We have $\dfrac{TD}{TA'} = \dfrac{sin\angle TA'D}{sin\angle TDA'} = \dfrac{sin\angle A'BC}{sin\angle DVC} = \dfrac{sin\angle DCV}{sin\angle DVC} = \dfrac{DV}{DC} = \dfrac{DV}{A'B}$.

$\dfrac{W^*D}{W^*A'} \cdot \dfrac{ZA'}{ZD} = \dfrac{sin\angle TZD}{sin\angle TZA'} = \dfrac{sin\angle TZD}{sin\angle TDZ} \cdot \dfrac{sin\angle TA'Z}{sin\angle TZA'} \cdot \dfrac{sin\angle TDZ}{sin\angle TA'Z} = \dfrac{TD}{TZ} \cdot \dfrac{TZ}{TA'} \cdot \dfrac{sin\angle UDV}{sin\angle BA'C}$

therefore $\dfrac{W^*D}{W^*A'} = \dfrac{TD}{TA'} \cdot \dfrac{sin\angle UDV}{sin\angle BA'C} \cdot \dfrac{ZD}{ZA'} = \dfrac{DV}{A'B} \cdot \dfrac{sin\angle UDV}{sin\angle BA'C} \cdot \dfrac{DU}{A'C} = \dfrac{areaof\triangle DUV}{areaof\triangle A'BC} = \dfrac{UV}{BC}$(2)(since $DA' \parallel BC$).

Let $AH = x$, $HD = y$, $HR = z$. From (1) and (2) we get

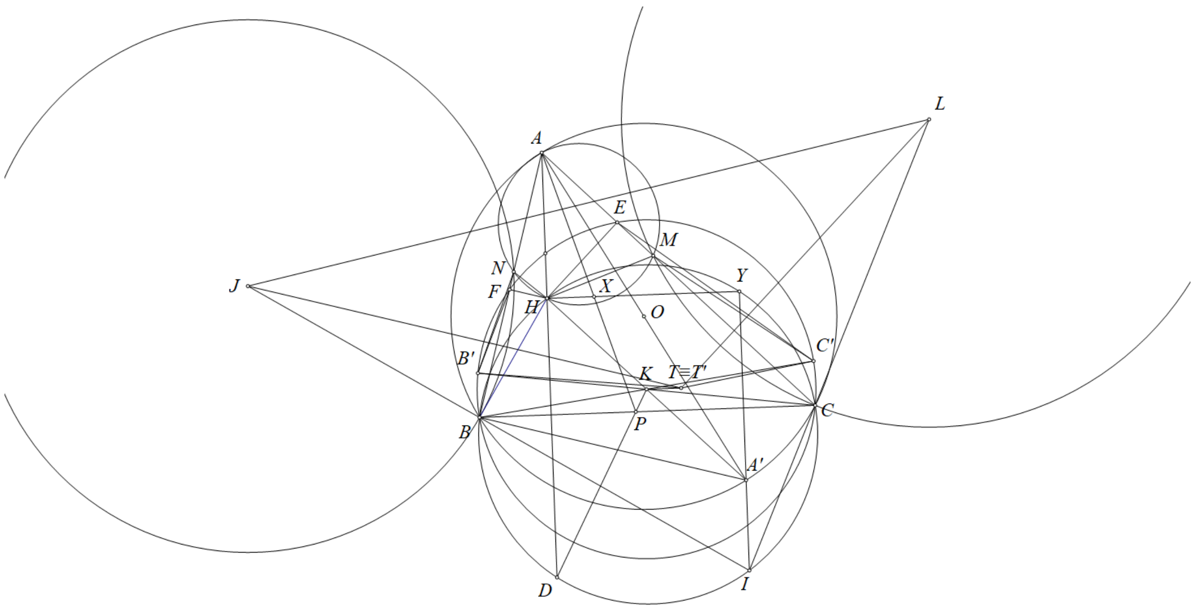
$$\frac{W^*D}{W^*A'} = \frac{UV}{BC} = \left(\frac{NA}{NR} \cdot \frac{AH'}{AH} - 1\right) \cdot \frac{RN}{RA} = \left(\frac{x + y/2}{z - y/2} \cdot \frac{x + 2z}{x} - 1\right) \cdot \frac{z - y/2}{x + z}$$

$$= \frac{(2(x^2 + xy + yz + xz)}{2z - y} \cdot \frac{z - y/2}{x + z} = \frac{2(x + y)(x + z)}{2x(x + z)} = \frac{x + y}{x} = \frac{AD}{AH}$$

Combine with the claim $(*)$ we get $\dfrac{W^*D}{W^*A'} = \dfrac{WD}{WA'} = \dfrac{AD}{AH}$ therefore $W^* \equiv W$ and since W^* lies on the radical axis of (X, XB), (Y, YC) we got W lies on the radical axis of (X, XB), (Y, YC).

Hence the lemma is proved.

Back to the main problem,



Let T be the intersection of the line from J perpendicular to AB and the line from L perpendicular AC, we will prove $KT \parallel BC$.

Construct diameter BB', CC' of (K). Since $AH \perp BC$ we get that $\dfrac{FB'}{EC'} = \dfrac{FB'}{BB'} \cdot \dfrac{CC'}{EC'} = \dfrac{sin\angle B'BF}{sin\angle B'FB} \cdot \dfrac{sin\angle C'EC}{sin\angle C'CE} = \dfrac{sin\angle BAH}{sin\angle KCB} \cdot \dfrac{sin\angle KBC}{sin\angle CAH} = \dfrac{sin\angle HEF}{sin\angle HFE} = \dfrac{HF}{HE}$.

Combine with $\angle HFB' = \angle HEC'$ we get $\triangle HFB' \sim \triangle HEC'$. Combine this with $\triangle HFN \sim \triangle HEM$ we get $\triangle NFB' \sim \triangle EMC'$.

Let T' be the center of (BNB') we get $KT' \parallel BC$ therefore $\angle B'T'K = \angle B'NB$, $\angle BKT' = 180^\circ - \angle B'FB$ $= \angle B'FN$ therefore $\triangle B'KT' \sim \triangle B'FN \sim \triangle C'ME$ therefore $\dfrac{KC'}{KT'} = \dfrac{KB'}{KT'} = \dfrac{FB'}{FN} = \dfrac{MC'}{ME}$.

combine with $\angle C'KT' = \angle C'EM$ we get $\triangle C'T'K \sim \triangle C'ME$ therefore $180^\circ - \angle C'T'K = \angle C'MC$ combine with $T'C' = T'C$ we get T' is the center of (CMC').

Therefore T' lies on the perpendicular bisector of BN and CM therefore $T' \equiv T$. Since $KT' \parallel BC$ we get $KT \parallel BC$.

Let AA' be the diameter of (ABC), It is well known that H, K, A' are collinear, and since $AH \perp BC$ we get K is the midpoint of HA'. Let HI be the diameter of (BHC) we get I, B, J and I, C, L are collinear. Let IA' cut (BHC) at Y then $HY \parallel BC$ therefore X lies on HY

Applying the lemma for triangle IBC with A' be an arbitrary point that lies on the line from I perpendicular to BC, T be an arbitrary point that lies on the line from K parallel to BC we get that X lies on the radical axis of (J, JB) and (L, LC) .