Problem 5

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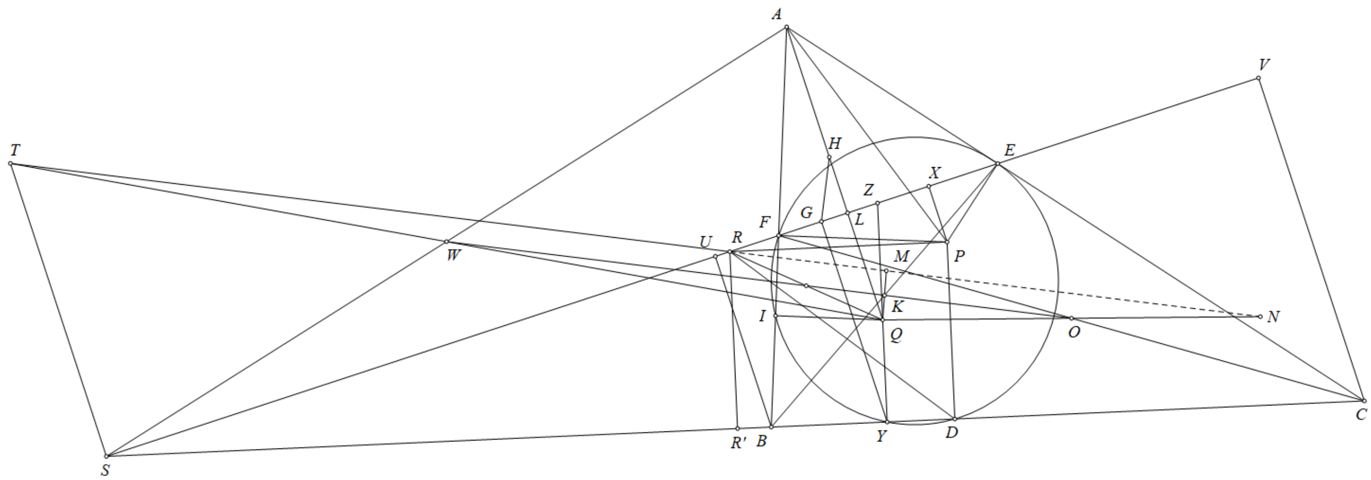

Redefine R as the intersection of the line through P parallel to BC and EF, we will prove M, N, R are collinear.

Let EF cut BC at S, AQ cut EF at L, W, K, O be the midpoint of AS, BE, CF we get W, K, O are collinear.

Therefore, let T be the reflection of Q across W we get T, M, N are collinear and $ST = AQ$, $ST \perp EF$(since $AQ \perp EF$).

Let U, V be the projection of B, C on EF; D, Y be the projection of P, Q on BC we get D, E, F, Y are concyclic therefore $\angle YFG = \angle EDC = \angle EPC = \angle CQY$ therefore triangles YGF and CYQ are similiar.

Similiarly triangles YGE and BYQ are similiar therefore $\dfrac{GF}{GE} = \dfrac{YB}{YC} = \dfrac{GU}{GV}$ therefore $GU \cdot GE = GV \cdot GF$ therefore G lies on radical axis of (BE) and (CF).

Let H be the orthocenter of triangle AEF we get H lies on radical axis of (BE) and (CF) therefore HG is the radical axis of (BE) and (CF) and so: $HG \perp KO$ therefore $HG \perp MN$ (1).

We will prove triangles TSR and GLH are similiar and since $\angle TSR = \angle GLH = 90^\circ$, this is equivalent to proving: $\dfrac{ST}{SR} = \dfrac{LG}{LH}$ $(*)$.

Let QY cut EF at Z, R' be the projection of R on BC we get $RR' = PD$, triangles SRR' and QZL are similiar therefore $\dfrac{SR}{RR'} = \dfrac{ZQ}{ZL} = \dfrac{QY}{GL} \rightarrow \dfrac{1}{SR} = \dfrac{GL}{QY \cdot PD}$ therefore $(*)$ is equivalent to:
$\dfrac{GL \cdot ST}{QY \cdot PD} = \dfrac{LG}{LH} \leftrightarrow \dfrac{QA}{QY} = \dfrac{ST}{QY} = \dfrac{PD}{LH}(**)$.

Let X be the projection of P on EF, since $EHFP$ is a parallelogram we get $PX = LH$.

Let I be the projection of Q on AB, by simple angle chasing, we get: $\triangle PXF \sim \triangle QIA$, $\triangle PFD \sim \triangle QYI$ therefore:

$\dfrac{QA}{QY} = \dfrac{QA}{QI} \cdot \dfrac{QI}{QY} = \dfrac{PF}{PX} \cdot \dfrac{PD}{PF} = \dfrac{PD}{PX} = \dfrac{PD}{LH}$ and so $(**)$ is true therefore $(*)$ is true.

Therefore we get triangles TSR and GLH are similiar and since $TS \perp GL, SR \perp LH$ we get $TR \perp GH$ combine with $(1) : HG \perp MN$ we get $TR \parallel MN$ and since T, M, N are collinear we get R, M, N are collinear.
Hence the problem is proved.