Problem 1

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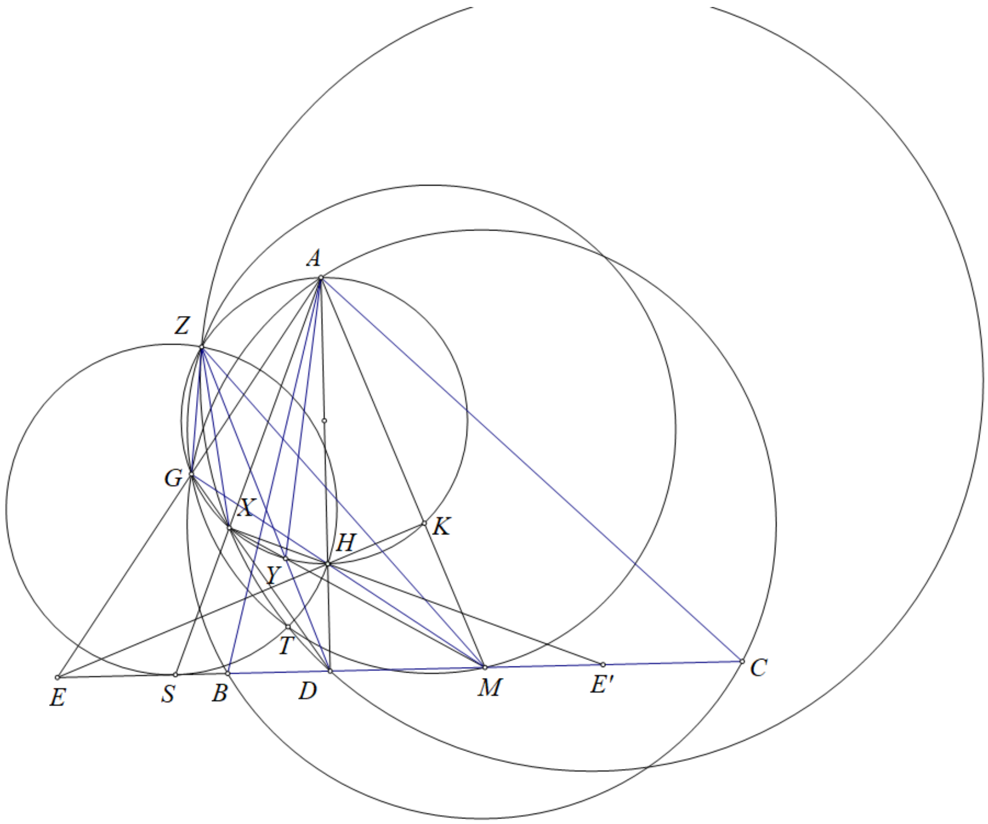
Claim: Let AX cut BC at S then (ZHS) is tangent to BC.



Proof: Let AM cut (AH) at K then $\angle DAM = \angle DGM = \angle XAH$ therefore $HX = HK$ therefore X and K are reflections through AD therefore $DM = DS$. It is well known that HK, AG, BC concurrent at a point E and $DH \cdot DA = DM \cdot DE = DS \cdot DE$, $MB^2 = MD \cdot ME$.

Consider an inversion about a circle at M with radius MB. It sends $Y \mapsto X$, $A \mapsto K$, $E \mapsto D$ therefore it sends $(YAE) \mapsto (XKD)$. Since $DX = DK$ and $DE \parallel XK$ we get (XKD) is tangent to BC at D therefore (YAE) is tangent to BC at E.

Consider an inversion about a circle at D with radius $\sqrt{DH \cdot DA}$. It sends $Z \mapsto Y$, $H \mapsto A$, $S \mapsto E$ therefore it send $(YAE) \mapsto (ZHS)$ and since (YAE) is tangent to BC at E we get (ZHS) is tangent to BC at S.

Hence the claim is proved.

Let (w) be the circle that pass through Z, H and tangents to BC. By the claim above we get (w) tangent to BC at S.

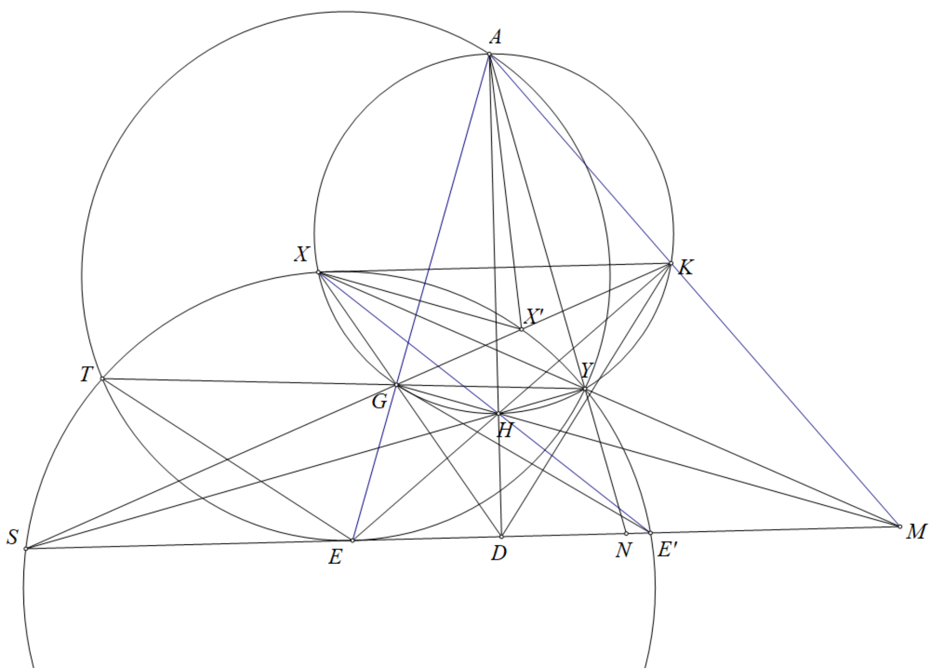
The problems will be equivalent to (w) pass through T or proving 3 cirlces $(ZHS), (ZXD)$ and (ZGM) are coaxial.$(*)$

Let E' be the reflection of E through D we get X, H, E' are collinear. Consider an inversion about a circle at D with radius $\sqrt{DH \cdot DA}$.

It sends $Z \mapsto Y$, $H \mapsto A$, $S \mapsto E$, $X \mapsto G$, $M \mapsto E'$ therefore it sends $(ZHS) \mapsto (YAE)$, $(ZXD) \mapsto GY$, $(ZGM) \mapsto (YXE')$.

Therefore $(*)$ is equivalent to proving GY is the radical axis of (YAE) and $(YXE')(**)$.

Proof: Let GY cut (YAE) at T, we will prove T lies on (YXE').



Since $DX \cdot DG = DH \cdot DA = DE \cdot DM = DE' \cdot DM$ we get $XGE'M$ are cyclic therefore $\angle GE'D = \angle GXM = \angle YAG$.

Let N be the midpoint of ME, we get $EG \cdot EA = ED \cdot EM = EE' \cdot EN$ therefore we get $AGNE'$ are cyclic therefore $\angle GAN = \angle GE'E = \angle YAG$ therefore A, Y, N are collinear.

Therefore GK, EM, HY are concurrent at a point S. Let X' be the reflection of X through AE since GK, GX are reflections through AB we get X' lies on GK. Since $XGE'M$ are cyclic we get $\angle XE'S = \angle DGM = \angle XYH$ therefore S lies on (XYE').

Since $\angle XX'S = 90^\circ - \angle AGK = \angle HE'S$ we get X' lies on (XYE') and since X' is the reflection of X through AE we get the center of (XYE') lies on AE.

Since $\angle GE'E = \angle YAG$, $\angle GEE' = \angle AKG = \angle AYG$ we get $\triangle AYG \sim \triangle E'EG$ therefore $\dfrac{EE'}{AY} = \dfrac{GE}{GY} = \dfrac{ET}{AY}$ therefore $EE' = ET$ combine with $\angle AET = \angle AYG = \angle AKG = \angle AEM$.

we get T is the reflection of E' through AE combine with the center of (XYE') lies on AE we get T lies on (XYE').

Therefore $(**)$ is true hence the problem is proven.