Problem 4

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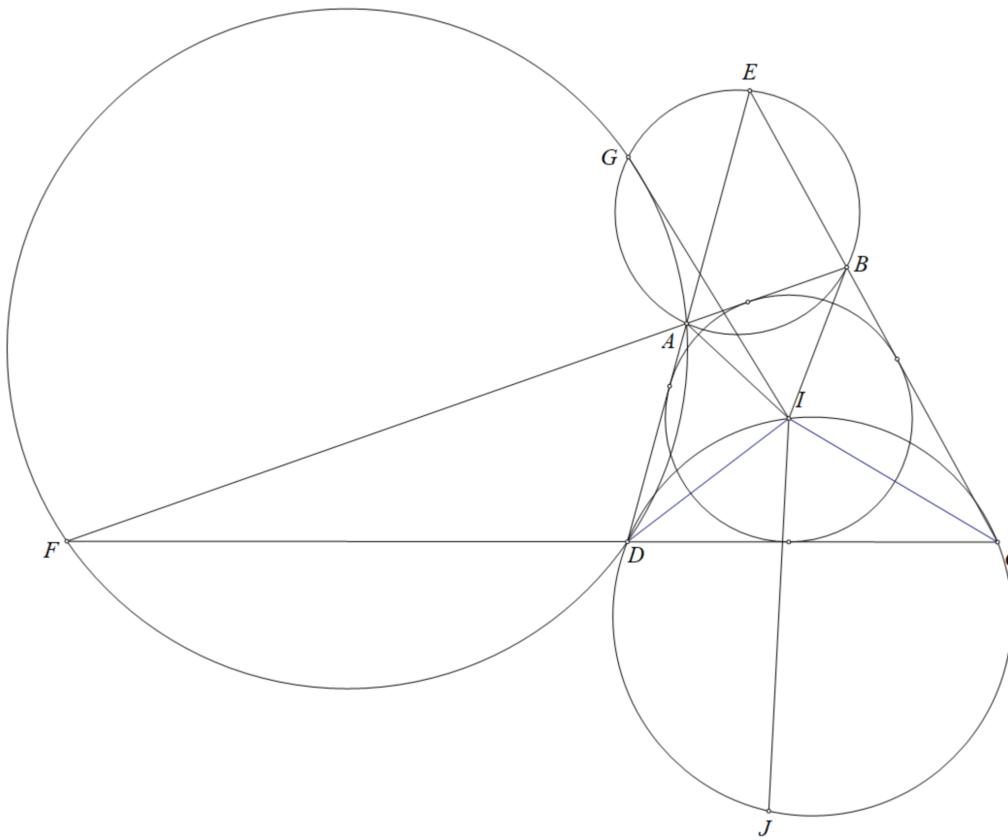
Lemma: Given tangential quadrilateral $ABCD$ with (I) being its incircle.
Let AD cut BC at E, AB cut CD at F, let G be the Miquel point of the complete quadrilateral $ABCD.EF$ then GI is the angle bisector of EGF.
Proof: Construct point J such that $\triangle GAD \sim \triangle GBC \sim \triangle GIJ$.

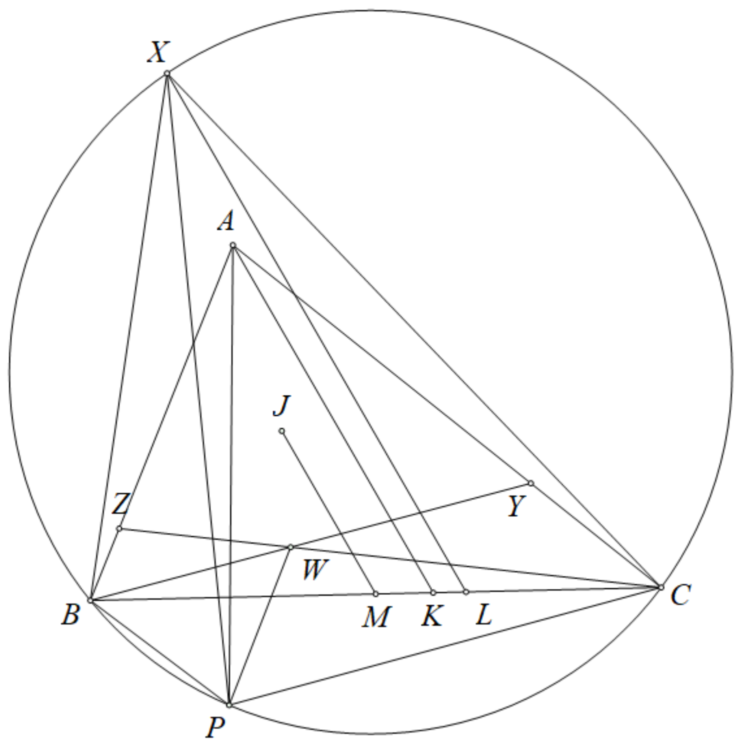


Then, we have $\triangle GAB \cup (I) \sim \triangle GDC \cup (J)$ therefore $\triangle AIB \sim \triangle DJC$.
Therefore $\angle DJC = \angle AIB = 180° - \angle DIC$ hence $DICJ$ is cyclic.
By simple angle chasing, we obtain that $\triangle BIC$ is similar to $\triangle IDJ$, hence

$$\frac{DJ}{IC} = \frac{IJ}{BC} = \frac{GJ}{GC}.$$

We have $\angle GJI = \angle GCB$ and $\angle IJD = \angle BCI$, so $\angle GJD = \angle GIC$, therefore $\triangle GJD$ is similar to the angle at GIC.
Consequently, $\triangle GDI$ is similar to $\triangle GJC$, which is similar to $\triangle GIB$, so GI is the internal bisector of $\angle BGD$.
Hence GI is the angle bisector of $\angle EGF$, as desired.

Back to the main problem.



Construct point X such that J is the incenter of triangle XBC, let the internal $X-mixitilinear$ of triangle XBC tangents to (XBC) at P'.
Then it is a common result that $P'J$ is the angle bisector of $\angle BP'C$ hence

$$\angle BJC = \frac{180 - \angle BAC}{2} = \frac{\angle BP'C}{2} = \angle JP'C \text{ hence } \angle BJP' = \angle JCP'.$$

Therefore $\triangle P'BJ \sim \triangle P'JC$ by (angle-angle) hence $P' \equiv P$ or P is the touchpoint of $(X-mixitilinear)$ and (XBC).
Let Y, Z be 2 points on AC, AB such that BJ, CJ is the angle bisector of $\angle ABY, \angle ACZ$ respectively.
Since AJ is the angle bisector of BAC, we get that J is the incenter of $\triangle ABY$ and ACZ and BY, CZ.
Let BY cut CZ at W, consider quadrilateral $AYWZ$ then we have AJ, ZJ, YJ is the angle bisector of $A, \angle Z, \angle Y$ respectively hence J is the incenter of quadrilateral $AYWZ$.
Let P^* be the miquel point of the complete quadrilateral $AYWZ.BC$ then applying the lemma above, we get P^*J is the angle bisector of $\angle BP^*C$
Furthermore, we have $\angle BP^*C = \angle BP^*W + \angle CP^*W = \angle AZW + \angle AYW = 360° - \angle BAC - \angle BWC = 180 - \angle BXC$(since A, W are isogonal conjugates $W.R.T$ triangle XBC).
Hence P^* lies on (XBC) and since P^*J is also the angle bisector of $\angle BP^*C$, we get P^* is the touchpoint of $(X-mixitilinear)$ and (XBC) hence $P^* \equiv P$
Therefore, P is the miquel point of the complete quadrilateral $AYWZ.BC$ hence P lies on $(ABY), (ACZ), (WBZ), (WCY)$
Let L be the touchpoint of $X-excenter$ and BC then it is well known that XP, XL are reflections through XJ and $JM \parallel XL$, since J is the incenter of triangle XBC.
Hence, the problem is equivalent to proving that $AK \parallel XL$.
We have that $\angle XLC = \angle XBP = \angle XBA + \angle ABP = \angle YBC + \angle PBC + \angle ABC = \angle PBY + \angle ABC = \angle PAY + \angle ABC = \angle BAC - \angle PAB + \angle ABC = \angle BAC + \angle ABC - \angle KAC = 180° - \angle ACB - \angle KAC = \angle AKC$.
Hence $AK \parallel XL \parallel JM$, as desired.
Hence the problem is proved.