Problem 8

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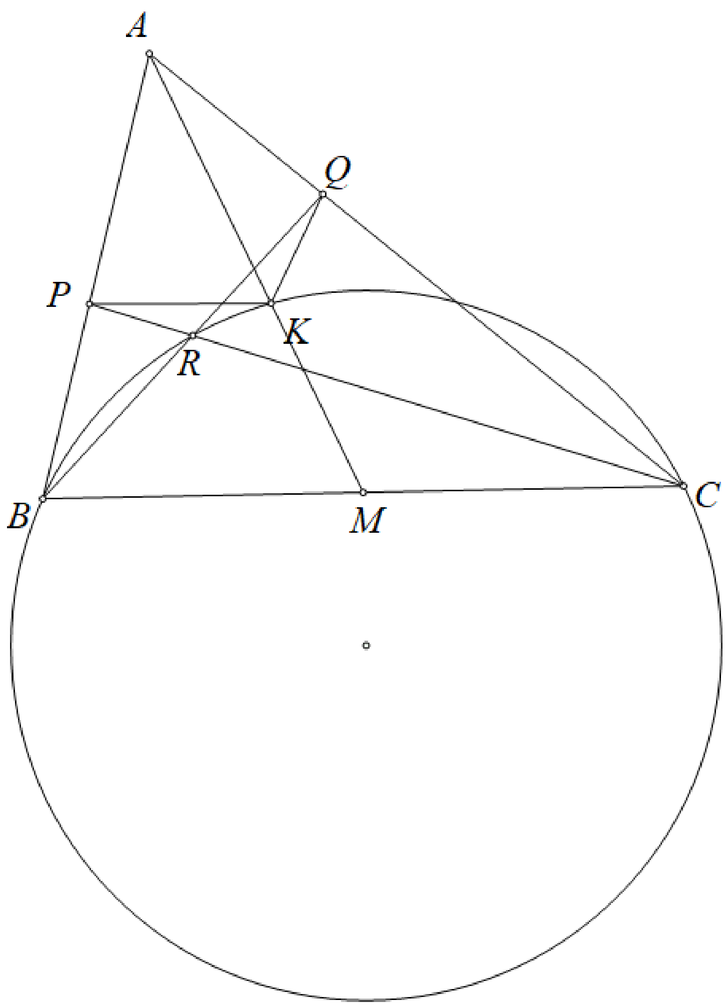
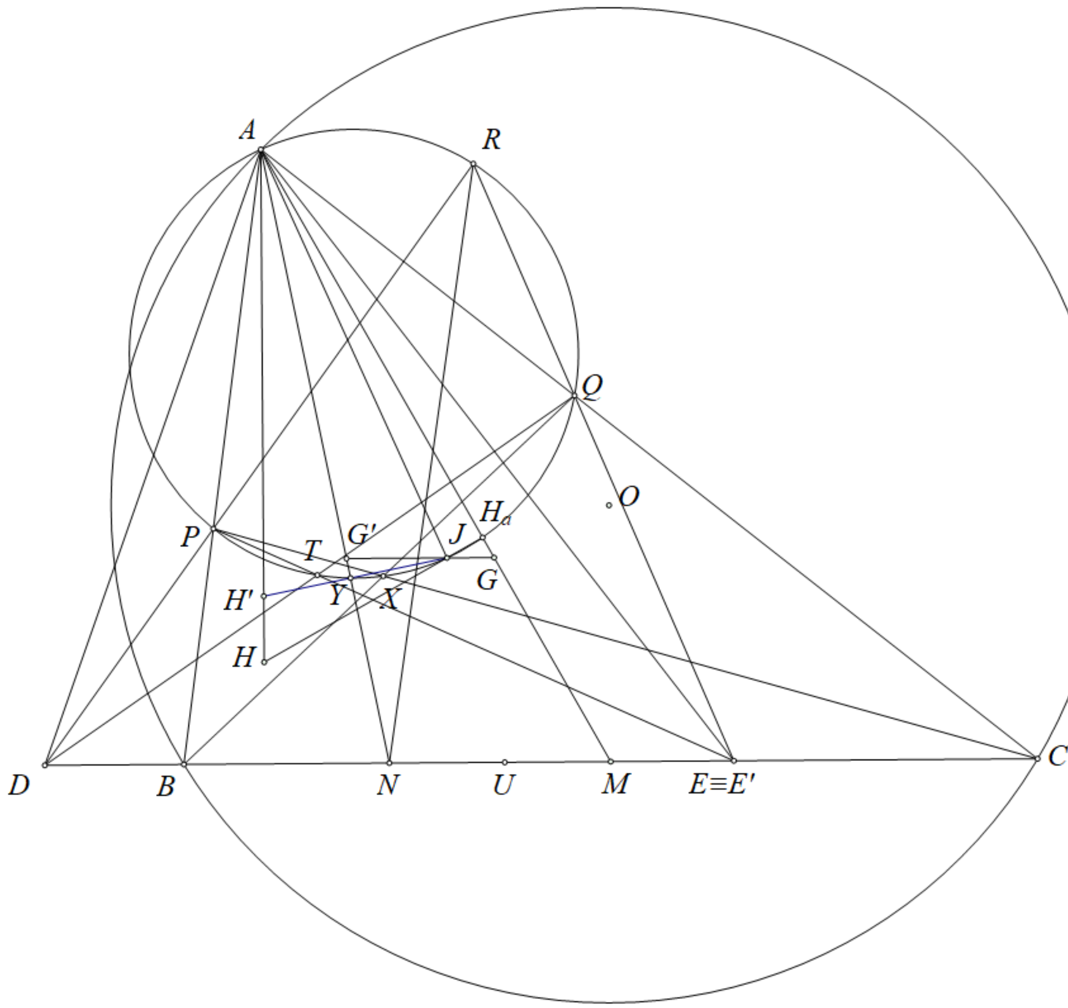
Lemma: triangle ABC with P, Q on AB, AC respectively. Let BQ cut CP at R. Then R lies on (APQ) if and only if $P(M, (APQ)) = MB^2$.



Proof: Let K be the $A-Humpty$ point of triangle ABC
Assume that R lies on (APQ). We have $\angle BRC = \angle PRQ = 180^\circ - \angle BAC = \angle BKC$ hence K lies (BRC). Hence $\angle KRC = \angle KBC = \angle BAK$ hence $APRK$ lies on a circle hence R, P, A, Q, K lies on a circle. Assume that $P(M, (APQ)) = MB^2$ then $P(M, (APQ)) = MK \cdot MA$ hence K lies on (APQ).
Therefore, $\angle KPQ = \angle KAQ = \angle MAC = \angle KCB$ and similarly, $\angle KQP = \angle KBC$ hence $\triangle KQP \sim \triangle KBC$.
Therefore $\triangle KQB \sim \triangle KPC$ hence $\angle KQR = \angle KQB = \angle KPC = \angle KPR$ or $R, K.P, Q$ lie on a circle hence R, K, P, Q, A lies on a circle, as desire.

Back to the main problem,



Claim: (AJ) cut AB, AC at P, Q; TP, TQ cut BC at E, D respectively.
Then, the Euler line of $\triangle ADE$ is parallel to the Euler line of $\triangle ABC$.

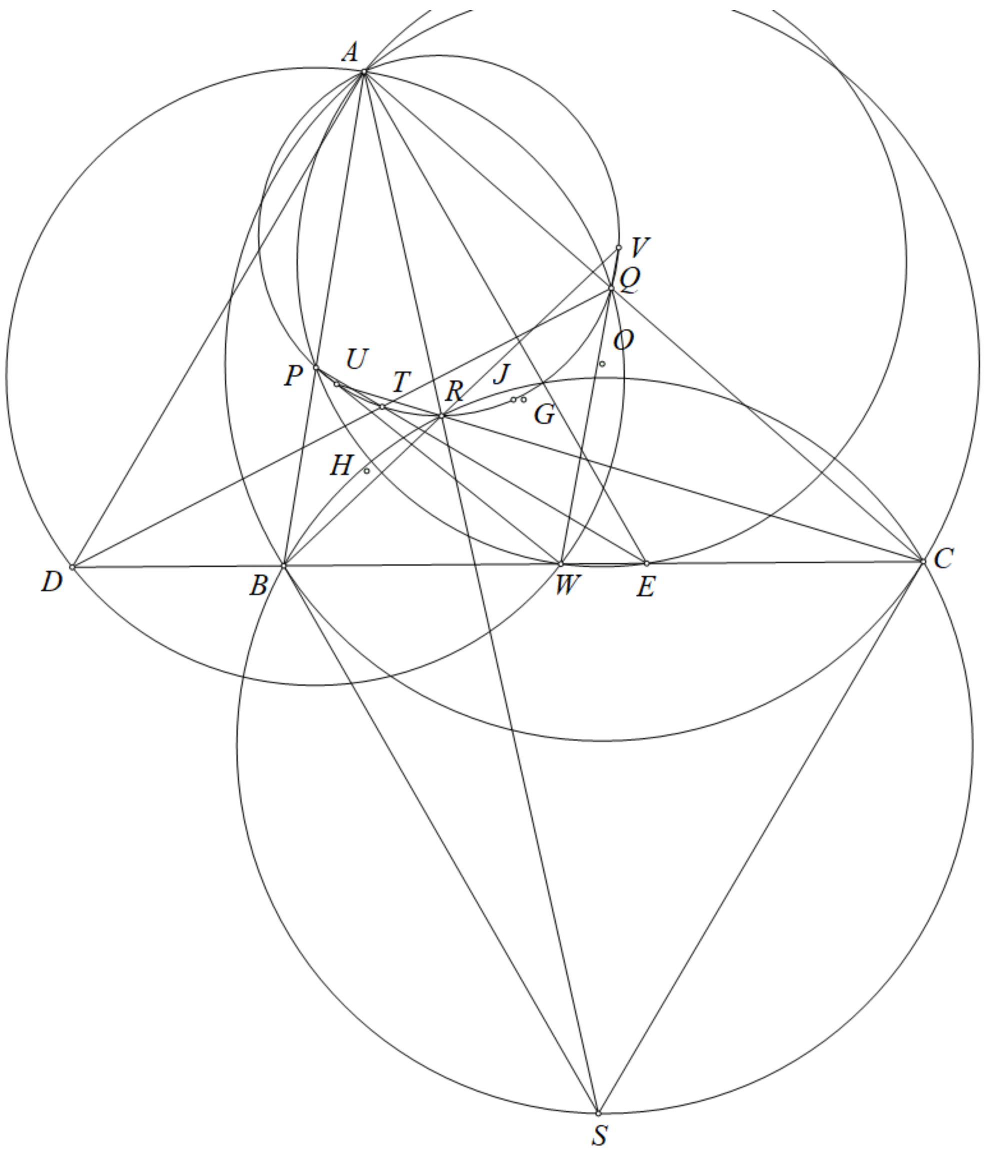
Proof: AG cut HJ, BC at H_a, M respectively then H_a is the $A-Humpty$ point of $\triangle ABC$ or $P(M,(APQ)) = MB^2$.
Therefore, let BQ cut CP at X, then applying the lemma above for triangle ABC with P, Q lies on AB, AC, we get X lies on (APQ).
Let U be the intersection of tangents of (AJ) at P and Q. Applying Pascal theorem for $\begin{pmatrix} P & A & Q \\ Q & X & P \end{pmatrix}$ we get B, C, U are collinear.
Let DP cut (AJ) at R, RQ cut PT at E'. Applying Pascal theorem for $\begin{pmatrix} P & R & Q \\ Q & T & P \end{pmatrix}$ we get D, U, E' are collinear.
Combine this with the fact that D, U lies on BC we get E' lies on BC, hence E' is the intersection of PT and BC or $E' \equiv E$.
Hence, DP, EQ intersects at R lies on (AJ).
Let N be the midpoint of DE. Applying the lemma above for triangle RDE with P, Q lies on RD, RE, DQ cut EP at T and T lies on (RPQ), we get $ND^2 = P(N,(RPQ)) = P(N,(AJ))$.
Therefore, let NA cut (AJ) at Y then $ND^2 = NY \cdot NA$, hence Y is the $A-Humpty$ point of triangle ADE.

Therefore let H' be the orthocenter of $\triangle ADE$ then $H'Y \perp AN$ and since $YJ \perp AY$ we get H', Y, J being collinear.

Let G' be the centroid of $\triangle ADE$, then $\dfrac{AG'}{AN} = \dfrac{AG}{AM}$

, hence $GG' \parallel BC$. Therefore J also lies on GG' since $JG' \perp AH$.
Consider triangle $AH'J$ then $JG' \perp AH', AG' \perp H'J$ therefore G' is the orthocenter of $\triangle AH'J$.
Therefore, $H'G' \perp AJ$. Since HG is also perpendicular to AJ, we get that $H'G' \parallel HG$ or the Euler lines of $\triangle ADE$ and $\triangle ABC$ are parallel, as desired.
Hence the claim is proven.

Back to the main problem,



Let RB, RC cut (AJ) at V, U, let (AJ) cut AB, AC at P, Q.

Let PU cut QV at W, Applying Pascal theorem for
$\begin{pmatrix} P & R & Q \\ V & A & U \end{pmatrix}$ we get B, C, W are collinear.
Let $(AQW), (APW)$ cut BC at D, E respectively.
We have that $\angle AQD + \angle APE = \angle AWD + \angle AWE = 180^\circ$ hence QD cut PE at a point that lies on (AJ), which we will denote as T.
Applying the claim above for point T lies on (AJ), we get the Euler line of $\triangle ADE$ is parallel to the Euler line of $\triangle ABC (*)$.
We have $\angle SBC = \angle SRC = \angle ARU = 180^\circ - \angle APU = 180^\circ - \angle APW = \angle AEW$, hence $AE \parallel SB$.
Similarly, we get $AD \parallel SC$, hence $\triangle SBC$ and $\triangle ADE$ have their corresponding sides parallel.
Hence the Euler lines of $\triangle SBC$ and $\triangle ADE$ are parallel, combine with $(*)$ we get the Euler lines of $\triangle SBC$ and $\triangle ABC$ are parallel, as desired.
Hence the problem is proven.