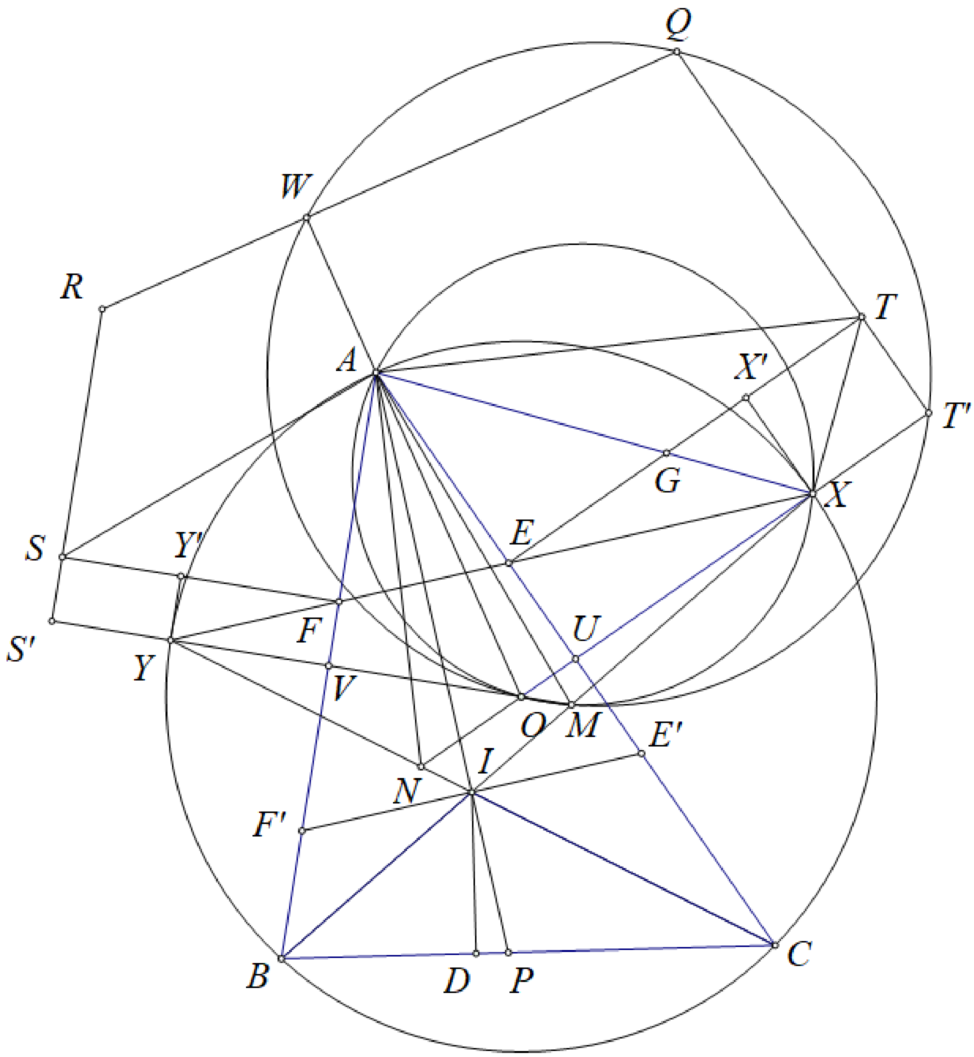
Problem 3

Ha Vu Anh



Let U, V be the midpoint of AC, AB respectively; BI, CI cut (O) at X, Y respectively we get X, Y lies on EF.

Let S', T' be the projection of S, T on OY, OX respectively; X', Y' be the projection of X, Y on ET, FS respectively.

Let E', F' be the reflection of A through E, F we get E', F' lies on the line from I perpendicular to AI.

Claim: T' lies on the opposite ray of $XO(1)$

Since $\angle NAC = \angle ICA < 90^\circ$ and T lies on the line from A perpendicular to AN and the line from E perpendicular to AC we get T lies on the half plane of BC that contain X.

Since $\angle AIC = 90^\circ + \angle ABC/2 > 90^\circ$ and $\angle AIE' = 90^\circ$ we get E' lies on segment AC therefore $AE' < AC$ therefore $AE < AU$ and $ET \parallel UX$ therefore let AX cut ET at G then G lies on segment AX therefore $\angle XET < \angle GEU = 90^\circ$.

Since $\angle AXE = \angle AXY = \angle ACI = \angle NAC = \angle ATE$ we get $AEXT$ is cyclic therefore $\angle XTE = \angle XAC = \angle IBC$ therefore $\angle XTE < 90^\circ$
combine with $\angle XET < 90^\circ$ we get the projection of X on ET which is X' lies on segment ET therefore X lies on segment UT' therefore (1) is true.

Let W be the projection of Q on AO we get $OWT'M$ cyclic in a circle with diameter OQ.
Combine with $\angle AMX = \angle ABC = \angle AOX$ we get $OAXM$ is cyclic therefore $\triangle MAW \sim \triangle MXT'$ and since T' lies on the opposite ray of XO we get W lies on the opposite ray of AO.

We also get $\dfrac{AW}{AM} = \dfrac{XT'}{XM}$ combine with $\triangle MAX \sim \triangle BAC$ therefore $AW = \dfrac{X'T \cdot AM}{XM} = \dfrac{X'T \cdot BA}{BC}$.

Similiarly let W' be the projection of R on AO we get W' lies on the opposite ray of AO and $AW' = \dfrac{SY' \cdot AC}{BC}$.

We will prove $W' \equiv W$ which is equivalent to $AW' = AW$ since W, W' both lies on the opposite ray of AO. Therefore we need to prove $\dfrac{X'T \cdot BA}{BC} = \dfrac{SY' \cdot AC}{BC}$ which is equivalent to $\dfrac{TX'}{SY'} = \dfrac{AC}{AB}(*)$.

Let D be the projection of I on BC, AI cut BC at G. From above we have proved $AEXT$ is cyclic therefore $\angle XTE = \angle XAE = \angle IBC$ therefore $\triangle XX'T \sim \triangle IDB$ therefoe $\dfrac{DI}{DB} = \dfrac{XX'}{TX'} = \dfrac{UE}{TX'} = \dfrac{CE'}{2TX'}$

Similiarly $\dfrac{DC}{DI} = \dfrac{2SY'}{BF'}$ therefore $\dfrac{TX'}{SY'} = \dfrac{DB}{DC} \cdot \dfrac{CE'}{BF'}$.

Since ID, IP are isogonal in $\angle BIC$ we get $\dfrac{IB^2}{IC^2} = \dfrac{PB}{PC} \cdot \dfrac{DB}{DC}$.

Since $\triangle BF'I \sim \triangle BIC$ we get $BF' = \dfrac{BI^2}{BC}$ and similiarly $CE' = \dfrac{CI^2}{BC}$ therefore $\dfrac{CE'}{BF'} = \dfrac{IC^2}{IB^2} = \dfrac{DC}{DB} \cdot \dfrac{PC}{PB}$ therefore $\dfrac{TX'}{SY'} = \dfrac{DB}{DC} \cdot \dfrac{DC}{DB} \cdot \dfrac{PC}{PB} = \dfrac{PC}{PB} = \dfrac{AC}{AB}$ therefore $(*)$ is true.

Therefore $W' \equiv W$ therefore $AO \perp RQ$ hence the problem is proved.