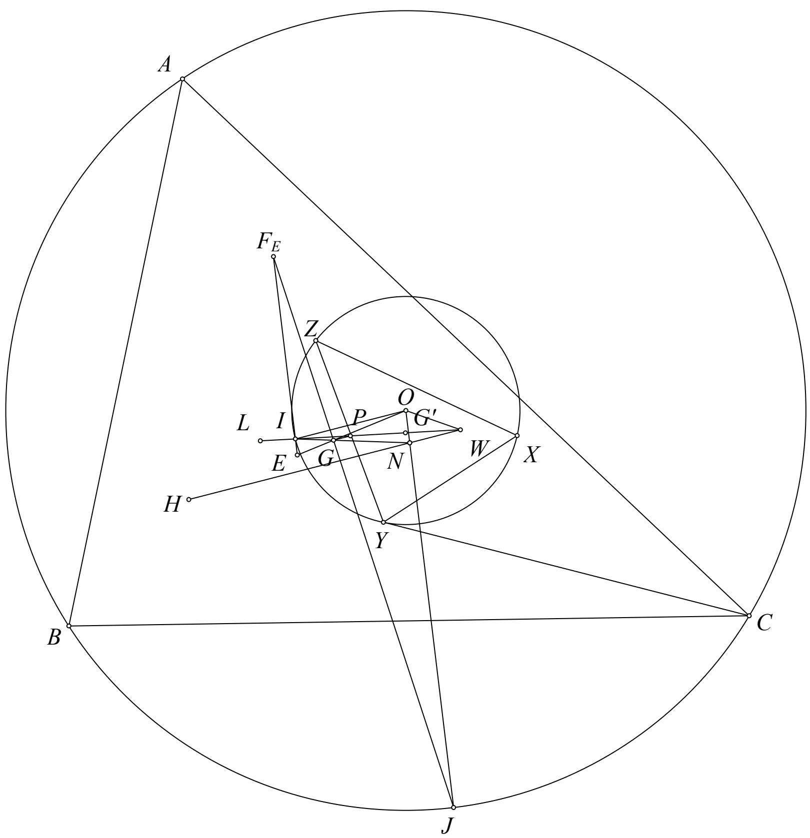
Problem 1

Ha Vu Anh

Let X, Y, Z be the reflection of I through the perpendicular bisector of BC, CA, AB respectively we get X, Y, Z, I lies on a circle with center O. Let H, N, E, F_E be the orthocenter, Nagel point, Nine-point-center and Feuerbach point of ABC respectively, W, G' be the Feuerbach point, centroid of the triangle XYZ.



Claim: *I, P, W, G' are collinear.*

Consider homothety center G with scaling factor -2, it sends $E \mapsto O, I \mapsto N, F_E \mapsto J$ then we have $OJ = 2 \cdot EF_E = R$ therefore J lies on (O). We also have:

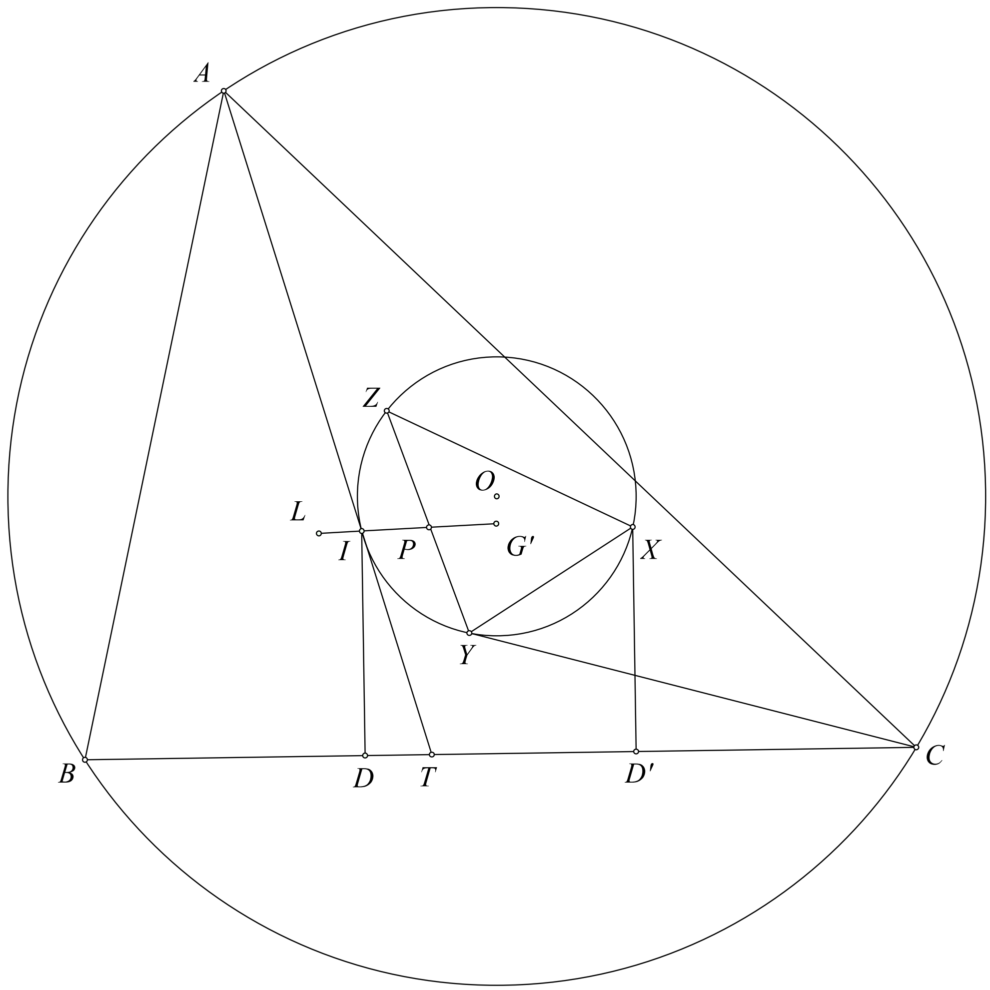
$ON \cdot OJ = 4 \cdot EI \cdot EF_E = 4 \cdot (EF_E - IF_E) \cdot EF_E = 4 \cdot \left(\dfrac{R}{2} - r\right) \cdot \dfrac{R}{2} = R \cdot (R - 2r) = OI^2.$

It is well known that F_E is the Anti-Steiner point of medial triangle of ABC wrt OI therefore J is the Anti-Steiner point of ABC wrt HN, therefore AJ and the line through A perpendicular to HN, which parallel to OI, are isogonal wrt $\angle BAC$. Simple angle chasing yields

$\angle JBC = \angle JAC = 90^\circ - \angle OIZ = \angle IYZ$ therefore $\triangle IYZ \sim \triangle JBC$.

We have $\angle XYZ = \angle XIZ = \angle ABC$; similiarly, we get $\triangle XYZ \sim \triangle ABC$. Denote N' as the incenter of XYZ we get $\triangle XYZ \cup (O, I, N') \sim \triangle ABC \cup (O, J, I)$ therefore $\triangle OIN' \sim \triangle OJI$ therefore N' lies on OJ and $OI^2 = ON \cdot OJ$ which combine with (1) implies that $N' \equiv N$ hence N is the incenter of XYZ.

Therefore we get $\triangle XYZ \cup (O, N, W) \sim \triangle ABC \cup (O, I, F_E)$ and so $\angle ONW = \angle OIF_E$. Combine this with the fact that $IF_E \parallel ON$ we get $NW \parallel OI$. Also, we have $\dfrac{NW}{IF_E} = \dfrac{OI}{OJ}$ and $\dfrac{GP}{OI} = \dfrac{r}{3R}$ therefore $\dfrac{GP}{NW} = \dfrac{GP}{OI} \cdot \dfrac{OI}{NW} = \dfrac{r}{3R} \cdot \dfrac{R}{r} = \dfrac{1}{3} = \dfrac{IG}{IN}$.
Combine this with the fact that $GP \parallel OI \parallel NW$ we get P lies on IW. Also, since $\triangle XYZ \cup (W, G', I) \sim \triangle ABC \cup (F_E, G, J)$ and F_E, G, J are collinear we get W, G', I are collinear therefore I, P, G', W are collinear.



Back to the main problem, Any case of triangle ABC being isosceles is trivial. Hence, WLOG, assume that $AB < BC < AC$.
since we need to prove P lies on IL, we will prove I, L, G' are collinear, which is equivalent to $\overrightarrow{LI} \parallel \overrightarrow{IG'}$.
Denote a, b, c as the length of BC, CA, AB respectively we get $c < a < b$ We have:
Let D, D' be the projection of I, X on BC respectively then $IX = DD' = DB - DC = \dfrac{b-c}{2}$
$\overrightarrow{IG'} = \overrightarrow{IX} + \overrightarrow{IY} + \overrightarrow{IZ} = \dfrac{b-c}{2a}\overrightarrow{BC} + \dfrac{a-c}{2b}\overrightarrow{AC} + \dfrac{b-a}{2a}\overrightarrow{BA}$ therefore
$2abc \cdot \overrightarrow{IG'} = (a^2c - c^2a) \cdot \overrightarrow{AC} + (b^2a - a^2b) \cdot \overrightarrow{BA} + (b^2c - c^2b) \cdot \overrightarrow{BC}(1)$
$(a^2+b^2+c^2) \cdot \overrightarrow{LI} = (a^2 \cdot \overrightarrow{LA} + b^2 \cdot \overrightarrow{LB} + c^2 \cdot \overrightarrow{LC}) + (a^2 \cdot \overrightarrow{AI} + b^2 \cdot \overrightarrow{BI} + c^2 \cdot \overrightarrow{CI}) = a^2 \cdot \overrightarrow{AI} + b^2 \cdot \overrightarrow{BI} + c^2 \cdot \overrightarrow{CI}$.
Let AI cut BC at T, we have $\dfrac{IA}{IT} = \dfrac{BA}{BT} = \dfrac{CA}{CT} = \dfrac{AB+AC}{BC} = \dfrac{b+c}{a}$ therefore $\dfrac{AI}{AT} = \dfrac{b+c}{a+b+c}$.

Therefore: $a^2 \cdot \overrightarrow{AI}$
$= \dfrac{a^2 \cdot (b+c)}{a+b+c} \cdot \overrightarrow{AT}$
$= \dfrac{a^2 \cdot (b+c)}{a+b+c} \cdot (\dfrac{BT}{BC} \cdot \overrightarrow{AC} + \dfrac{CT}{CB} \cdot \overrightarrow{AB})$
$= \dfrac{a^2 \cdot (b+c)}{a+b+c} \cdot (\dfrac{c}{b+c} \cdot \overrightarrow{AC} + \dfrac{b}{b+c} \cdot \overrightarrow{AB})$
$= \dfrac{1}{a+b+c} \cdot (a^2c \cdot \overrightarrow{AC} + a^2b \cdot \overrightarrow{AB})$. Similiarly we get $(a^2+b^2+c^2) \cdot \overrightarrow{LI} = a^2 \cdot \overrightarrow{AI} + b^2 \cdot \overrightarrow{BI} + c^2 \cdot \overrightarrow{CI} = \dfrac{1}{a+b+c} \cdot \Big((a^2c - c^2a) \cdot \overrightarrow{AC} + (b^2a - a^2b) \cdot \overrightarrow{BA} + (b^2c - c^2b) \cdot \overrightarrow{BC}\Big)$ (2).
From $(1), (2)$ we get $(a^2+b^2+c^2) \cdot \overrightarrow{LI} = \dfrac{1}{a+b+c} \cdot 2abc \cdot \overrightarrow{IG'}$ therefore $\overrightarrow{LI} \parallel \overrightarrow{IG'}$, which imply I, L, G' are collinear, as desired.
Hence the problem is proved.