Problem 5

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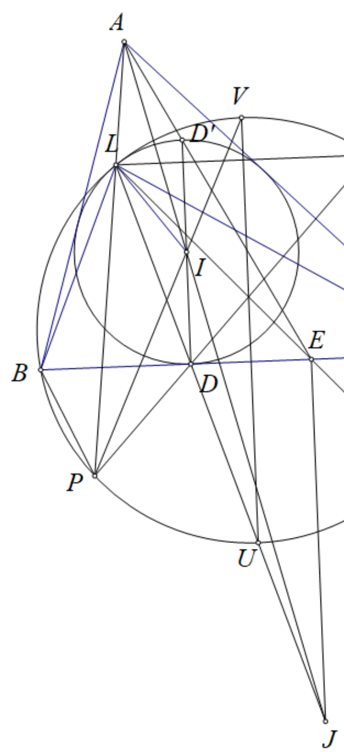
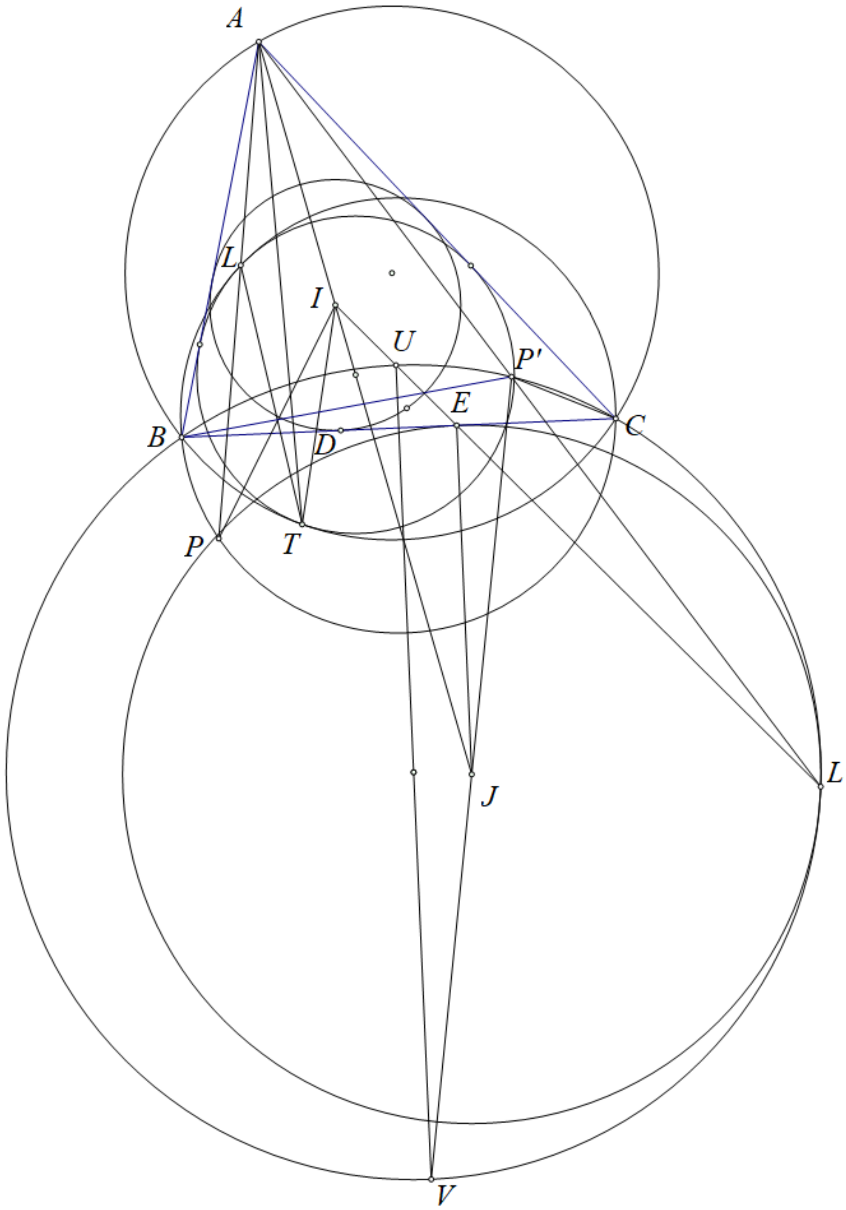
Lemma: Let ABC be a triangle with incenter (I), a circle pass through B, C tangents to (I) at L, AL cut (BLC) at P. Let (I) touch BC at D; U, V be the midpoint of arc BC not contain L and contain L respectively.
Then: $LIDP$ are cyclic, L, D, U and P, I, V are collinear.



Proof: Let J be the $A - excenter$ of triangle ABC then it is well known that L, D, J are collinear and LJ is the bisector of BLC hence LD pass through U which is the midpoint of arc BC that doesn't contain L.
Let (J) touch BC at E, construct diameter DD' of (I) it is well known that A, D', E are collinear therefore $-1 = D(IJ, AE) = D(D'L, AE) = L(D'D, AE)$ and since $\angle DLD' = 90°$
we get LD is the angle bisector of $\angle ALE$ and since LD is also the bisector of BLC we get LP, LE are isogonal wrt BLC.
Hence let LE cut (BLC) at Q' then $PQ \parallel BC$ hence P, D and Q, E are reflections through the perpendicular bisector of BC.
Therefore $\angle LPC = \angle LQC = \angle EQC = \angle BPD$ therefore PD, PL are isogonal wrt $\angle BLC$ then let PD cut (BLC) at L' then $LL' \parallel BC$.
Let VI cut (BLC) at P', since $VU \parallel ID$ we get $LIDP'$ are cyclic by Reim.
Since $IL = ID$ we get PI is the bisector of $LP'D$ and also of $BP'C$ hence $P'L, P'D$ are isogonal wrt $BP'C$ hence L', D, P' are collinear therefore $P' \equiv P$ therefore P, I, V are collinear $LIDP$ is cyclic.

Back to the main problem,



Let J be the $A-excenter$ of triangle ABC, the circle pass through B, C tangents to (J) at L', AL' cut $(BL'C)$ at P',
Let U, V be the midpoint of arc BC not contain L', contain L' respectively.
Perform extraversion for the lemma above we get $JEP'L'$ is cyclic, P, J, V are collinear, I, U, E, L' are collinear.
$a)$Consider an inversion about a circle at A with radius $\sqrt{AB \cdot AC}$, followed by a reflection across the bisector of $\angle BAC$. It swaps $T \mapsto E, I \mapsto J$, $(K) \mapsto (J)$ hence $L \mapsto L', P \mapsto P'$ and since $JEP'L'$ is cyclic we get $LITP$ is cyclic.

$b)$The problem is equivalent to proving PI be the bisector of BPC. From the inversion above we get L, P' are isogonal conjugate wrt ABC,
$\triangle API \sim \triangle AJP'$ hence $\angle BLC - \angle BAC = \angle LBA + \angle LCA = \angle P'BC + \angle P'CB = \angle BL'C$
hence $\angle BPC = 180^\circ - \angle BL'C - \angle BAC(1)$.

We have $\angle BPI = \angle APB + \angle API = \angle ACP' + \angle AJP' = \angle JP'C - \angle JAC = \angle CP'V - \angle BAC/2 = 90^\circ - \angle BL'C/2 - \angle BAC/2$.
combine with (1) we get $\angle BPC = 2 \cdot \angle BPI$ hence PI is the angle bisector of BPC.