Problem 6

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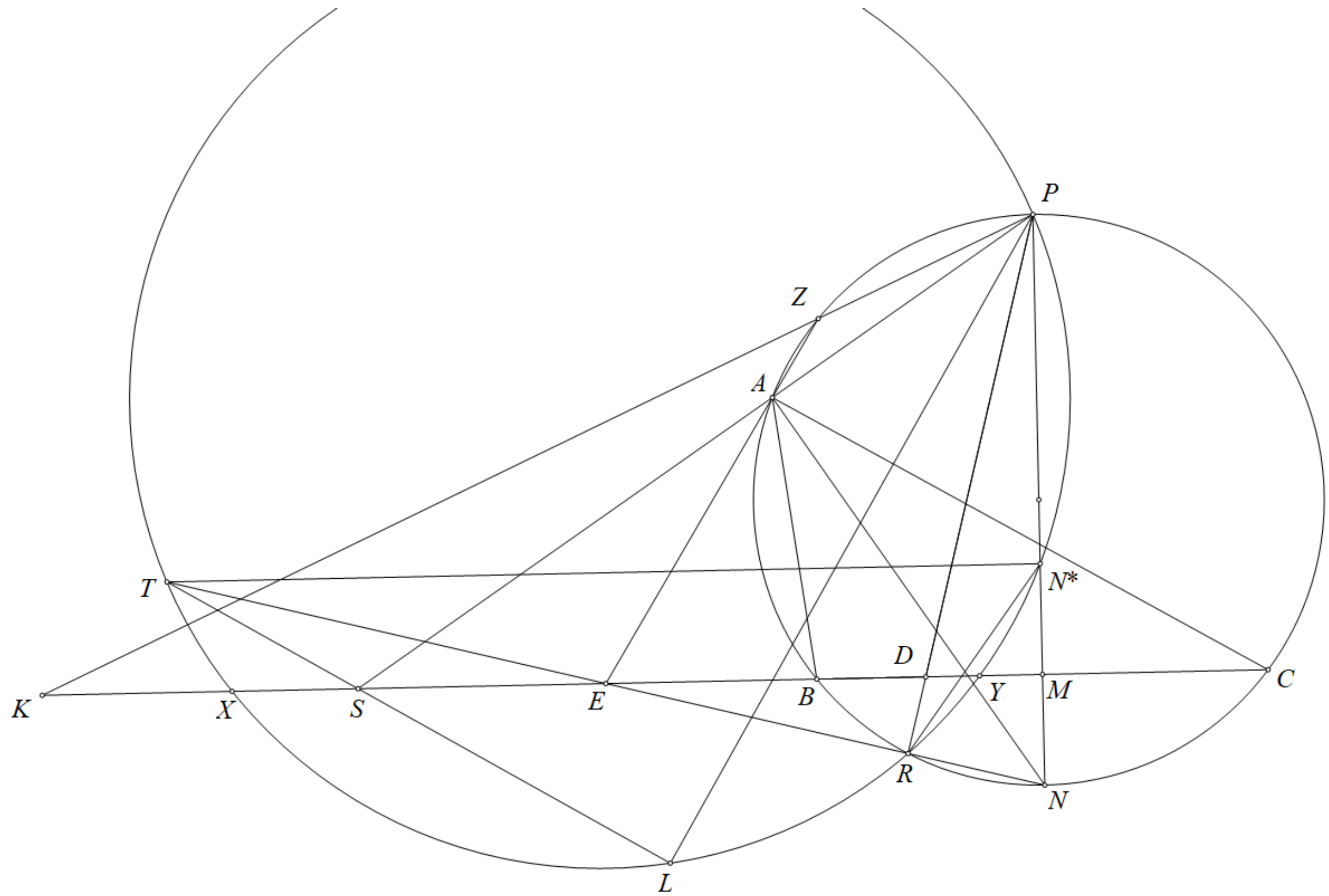

Let PD intersect (O) at U. Then RD is the angle bisector of $\angle BRC$, and since $(BC, DE) = -1$, we have that $\angle ERD$ is a right angle. Hence, E, R, N are collinear.
Let RN intersect (PN^*R) at T. Then AN^*T is a right angle, so $TN^* \parallel BC$, and E is the midpoint of TN.
We have $\angle ASM = \angle ANP = \angle AZK$, thus $AZKS$ is cyclic, and so

$$EA \cdot EZ = ES \cdot EK.$$

Let (PN^*R) intersect BC at X, Y. Then, we have

$$MB^2 = MN \cdot MP = MY \cdot MX \quad \Rightarrow \quad (XY, BC) = -1.$$

Since $TN^* \parallel XY$, the quadrilateral TN^*YX is an isosceles trapezoid,
and since E is the midpoint of TN, we also have $ET = EN^*$.

Because $E \in XY$, it follows that E is the midpoint of XY.
Therefore,

$$EX^2 = EY^2 = EB \cdot EC = EA \cdot EZ = ES \cdot EK = ES^2 + SE \cdot SK,$$

so

$$SE \cdot SK = EX^2 - ES^2 = SX \cdot SY \quad \text{(since } E \text{ is the midpoint of } XY\text{)} = P(S/(PN^*R)).$$

Let TS intersect (PN^*R) at L^*. Then $SL^* \cdot ST = SE \cdot SK$, which implies that T, E, L^*, K are concyclic, hence

$$\angle L^*PD = \angle L^*TR = \angle L^*KD \implies L \in (PDK).$$

Hence L^* is the intersection of (PN^*R) and (PDK), so L^* coincides with L.
Since $LT \perp LP$, it follows that $LS \perp LP$, as desired.
Therefore, the problem is proved.