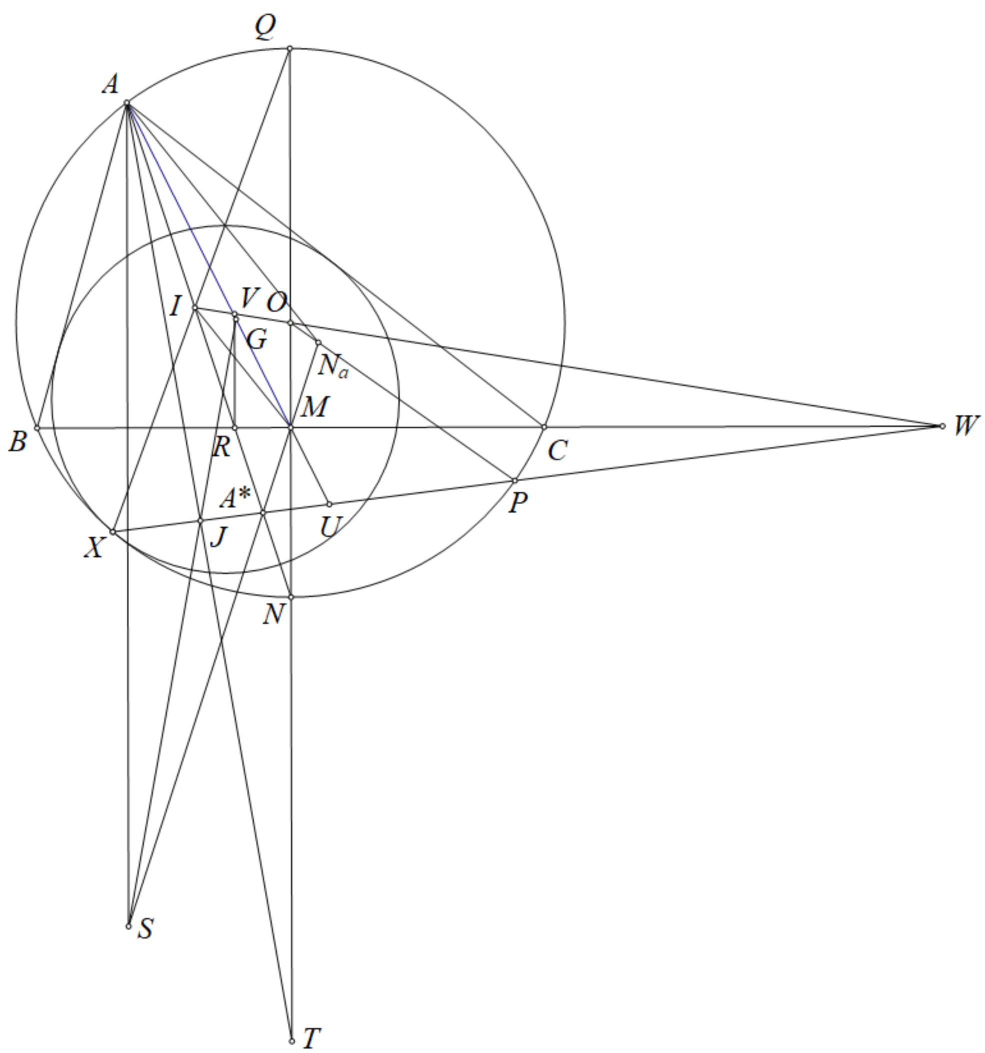
Problem 11

Ha Vu Anh



Construct $A_1B_1C_1$ as the anti-medial triangle of triangle ABC
We get P as the Feuerbach point of triangle $A_1B_1C_1$
By Fontené's theorem, P is the anti-Steiner point of HNa with respect to triangle ABC, where H is the orthocenter of ABC
Since $HNa \parallel OI$, the isogonal conjugate of AP in angle BAC is perpendicular to OI
Moreover, the familiar result: OI intersects BC at W, A^* is the reflection of A across I, then X, A^*, W are collinear and OI is tangent to (IXA^*)
Hence $\angle QXA^* = \angle OIN$, and after angle chasing we get X, A^*, P collinear
Since $\overrightarrow{ANa} = 2 \cdot \overrightarrow{IM}$, MNa passes through A^*

Redefine S as the intersection of XP with the A-median of triangle ABC
Proving the problem is equivalent to proving that S, G, J collinear $\iff A^*(JG, SA) = A(JG, SA^*) \iff A^*(UG, MA) = A(TM, SN)$
(T is the intersection of the tangents at B and C of (O))

$$\iff \frac{2UM}{UA} = \frac{NM}{NT} = \frac{BM}{BT} = \frac{OM}{OB} \quad (*)$$

Let AI intersect BC at R, and the line through R perpendicular to BC intersects OI at V
Applying Menelaus' theorem to triangle ARM with three collinear points A^*, U, W, we get

$$\frac{UM}{UA} = \frac{WM}{WR} \cdot \frac{A^*R}{A^*A}$$
$$= \frac{OM}{VR} \cdot \frac{A^*R}{2AI}$$
$$= \frac{OM}{ON} \cdot \frac{ON}{VR} \cdot \frac{A^*R}{2AI}$$
$$= \frac{OM}{ON} \cdot \frac{IN}{IR} \cdot \frac{A^*R}{2AI}$$

Hence $(*) \iff \dfrac{OM}{ON} \cdot \dfrac{IN}{IR} \cdot \dfrac{A^*R}{AI} = \dfrac{OM}{OB}$

$$\iff \frac{RA^*}{RI} = \frac{IA}{IN}$$
$$\iff \frac{IA^*}{IR} = \frac{NI}{NA}$$
$$\iff \frac{IA}{IR} = \frac{NB}{NA} = \frac{BA}{BR} \text{ (which is true).}$$

Hence $(*)$ is true which is, as stated above, equivalent to S, G, J being collinear, as desired.
Therefore, the problem is proven