Solutions to 'Some Training Geometry Problems for Vietnam IMO Team 2024' – Tran Quang Hung

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Part I

Problems

Problem 1. *Let ABC be a triangle with centroid G and symmedian point L. Let D be the symmedian point of triangle GBC. Construct parallelogram $GDLX$. Similarly, we have the points D, E, Y, Z. Prove that lines AX, BY, CZ are concurrent.*

Problem 2. *Let ABC be a triangle with two isogonal conjugate points. Let I and J be the orthocenters of triangles QAC and QAB respectively. Let K be the second intersection point of (PAB) and BC. Let L be the intersection of AK and IJ. Prove that $PL \perp AC$.*

Problem 3. *Let P and Q be two isogonal conjugate points with respect to the triangle ABC. Let K and L be the circumcenters of triangles BPQ and CPQ respectively. Prove that the second intersection of the circles (ABK) and (ACL) lies on line KL.*

Problem 4. *Let P and Q be two isogonal conjuhate points with respect to a triangle ABC. Let D be the reflection of Q in line BC. Let K and L be the orthocenters of triangles BPQ and CPQ respectively. Prove the the circles (DBK), (DCL), and (PKL) are concurrent.*

Problem 5. *Consider triangle ABC inscribed in circle ω, with circumcevian triangle DEF of the incenter I. The Simson lines of points B and C with respect to triangle DEF intersect at point P. Let J denote the midpoint of segment AI. Additionally, let K and L represent the projections of point J onto lines DE and DF respectively. Prove that the midpoint of segment AP is equidistant from points K and L.*

Problem 6. *Consider triangle ABC inscribed in circle ω, with incenter I. A-mixtilinear incircle touches ω at D. Define similarly E and F. Prove that*

$$ID^2 + IE^2 + IF^2 = 4(R^2 - r^2)$$

iff triangle ABC is equilateral where R and r are circumradius and inradius of ABC.

Problem 7. *Let ABC be a triangle with altitudes AD, BE, CF. D-mixtilinear excircle of triangle DEF touches (DEF) at X. Define similarly Y and Z. Prove that*

$$\frac{DX^2 + EY^2 + FZ^2}{AX^2 + BY^2 + CZ^2} = \frac{144S^2}{(a^2 + b^2 + c^2)^2} + 1$$

where S is the area of ABC and $BC = a, CA = b, AB = c$.

Problem 8. *Let ABC be a triangle with altitudes AD, BE, and CF. The A-median of $\triangle ABC$ intersects the circumcircle of $\triangle DEF$ at X, such that X is not the midpoint of BC. Similarly, define points Y and Z. Prove that*

$$\frac{DX^2 + EY^2 + FZ^2}{AX^2 + BY^2 + CZ^2} = 4 - \frac{12d^2}{9\delta^2 + 4d^2},$$

where d is the distance between the orthocenter and the circumcenter, and δ is the distance between the two Fermat points of $\triangle ABC$.

Problem 9. *Given a triangle ABC. Construct an inscribed square $MNPQ$ within the triangle with M and Q on side BC. The incircle of triangle ANP touches NP at X. The external common tangent of the incircles of triangles ANP and BMN intersects at S. The external common tangent of the incircles of triangles ANP and CPQ intersects at T. MS intersects TQ at Y. Prove that XY bisects MQ.*

Problem 10. *Let the triangle ABC with P, Q as two isogonal conjugate points. E, F are the projections of P onto CA, AB respectively. M, N are symmetrical of Q through the midpoints of BE, CF respectively. MN meets EF at R. Prove that $PR \parallel BC$.*

Problem 11. *Let A be a point lying on the segment BC. Draw circle ω which is tangent to three circles diameters AB, AC and BC. Draw circle Ω passing through B and C and is tangent to ω. Ω meets the circles diameters AB and AC at N and M respectively. ω touches the circles diameters AB and AC at F and E respectively. Let P be the intersection of EM and FN. Prove that $PA \perp BC$.*

Problem 12. *Let $ABCD$ be a parallelogram with point P inside it. The incircle of $\triangle PAB$ touches AB at X. A circle that touches PB and CB also touches CD at Y. Another circle touches PA and DA and also touches DC at Z. Let W be the midpoint of YZ, and let O be the center of parallelogram $ABCD$. Prove that $OW \parallel PX$.*

Problem 13. *Let $MNPQ$ be a parallelogram inscribed in triangle ABC with points M and N lying on sides AB and AC, respectively, and points P and Q lying on side BC. The external common tangent of the incircles of triangles AMN and BMQ intersects the external common tangent of the incircles of triangles AMN and CNP at point J. The incircle of triangle AMN touches MN at point R. Prove that the line RJ passes through the center of parallelogram $MNPQ$.*

Problem 14. *Given triangle ABC with incircle (I) touching BC, CA, AB at D, E, F respectively. AX, BY, CZ are altitudes. P, Q, R are the reflections of X, Y, Z in the lines EF, FD, DE respectively. Prove that the inverse of de Longchamps point of triangle PQR through its circumcircle lies on the Euler line of ABC.*

Problem 15. *Given triangle ABC inscribed in (O, R), circumscribed about (I, r), centroid G, and Lemoine point L. Let P satisfy equation $3R\vec{GP} + r\vec{OI} = \vec{0}$. Prove that P lies on line IL.*

Problem 16. *The bicentric quadrilateral $ABCD$ satisfies the condition*

$$\frac{1}{S^2 + rx^3} + \frac{1}{S^2 + ry^3} + \frac{1}{S^2 + rz^3} + \frac{1}{S^2 + rt^3} = \frac{16}{15S^2}$$

where S is the area of the quadrilateral, r is the radius of the inscribed circle, and x, y, z, t are the lengths of the tangent segments drawn from A, B, C, D to the inscribed circle of the quadrilateral. Prove that $ABCD$ is a square.

Problem 17. *Given a triangle ABC inscribed in a circle ω. Choose a point P on ω such that AP is parallel to BC. The A-mixtilinear incircle of ABC is Ω, which is tangent to ω at D. The two tangents from P to Ω intersect AD at M and N, with M lying between A and D. Prove that two lines NB and CM intersect on ω.*

Problem 18. *Given A lying on the segment BC, construct circles with diameters BC, AB, and AC. A circle (K) is tangent to these circles. Let Y and Z be the midpoints of arcs AB and AC respectively, not containing K on the same side as BC. (K) is tangent to the circle with diameter BC at D. Choose E on the circle with diameter BC such that $DE \parallel BC$. Prove that the reflection of K over BC lies on the circle (EYZ).*

Problem 19. *Given a triangle ABC inscribed in circle (O) with diameter AD and altitude AK. The tangents to (O) at B and C intersect at T. The perpendicular bisector of OT intersects AD at P. Line TP intersects KO at S. Let J be the foot of the perpendicular from O to AT. Choose L on OT such that $JL \parallel OA$. Let Q be the intersection of LD and OJ. Prove that SQ bisects OA.*

Part II

Solution

Problem 1. Let ABC be a triangle with centroid G and symmedian point L. Let D be the symmedian point of triangle GBC. Construct parallelogram $GDLX$. Similarly, we have the points D, E, Y, Z. Prove that lines AX, BY, CZ are concurrent.

Proof. First, we will state a following lemma.

Lemma 1.1 *Given triangle ABC and L is the Lemoine point of the triangle. Then*

$$\sum \left(a^2 \times \overrightarrow{LA}\right) = \overrightarrow{0}.$$

Where a, b, c are the lengths of BC, CA, AB, respectively.

Proof. We have the familiar lemma: For any arbitary point P lies inside the triangle ABC, then

$$\sum \left(S_{\Delta PBC} \times \overrightarrow{PA}\right) = \overrightarrow{0}$$

Apllying the lemma for the *Lemoine* point, we get $\sum \left(S_{\Delta LBC} \times \overrightarrow{LA}\right) = \overrightarrow{0}$.

Let AL intersect BC at A'. Then, $\dfrac{S_{\Delta LAB}}{S_{\Delta LAC}} = \dfrac{A'B}{A'C} = \dfrac{AB^2}{AC^2}$. Similarly, we get $\dfrac{S_{\Delta LAB}}{S_{\Delta LBC}} = \dfrac{AB^2}{BC^2}$, $\dfrac{S_{\Delta LAC}}{S_{\Delta LBC}} = \dfrac{AC^2}{BC^2}$ nên $\dfrac{S_{\Delta LAB}}{AB^2} = \dfrac{S_{\Delta LAC}}{AC^2} = \dfrac{S_{\Delta LBC}}{BC^2}$, therefore $\sum \left(a^2 \times \overrightarrow{LA}\right) = \overrightarrow{0}$

Hence, the lemma is proved.

□

Back to the main problem, we have

$$a^2 \times \overrightarrow{LA} + b^2 \times \overrightarrow{LB} + c^2 \times \overrightarrow{LC} = \overrightarrow{0}$$

Equivalent to

$$(a^2 + b^2 + c^2) \times \overrightarrow{LA} + b^2 \times \overrightarrow{AB} + c^2 \times \overrightarrow{AC} = \overrightarrow{0}$$

$$\Leftrightarrow \overrightarrow{LA} = \frac{-b^2}{a^2 + b^2 + c^2} \times \overrightarrow{AB} + \frac{-c^2}{a^2 + b^2 + c^2} \times \overrightarrow{AC}$$

Similarly, we get $\overrightarrow{DG} = \dfrac{-GC^2}{GB^2 + GC^2 + BC^2} \times \overrightarrow{GB} + \dfrac{-GB^2}{GB^2 + GC^2 + BC^2} \times \overrightarrow{GC}$ (1)

Let M_A, M_B, M_C be the lengths of the median from A, B, C of triangle ABC, respectively. By *Apollonius's median theorem*, we get

$$GB^2 = \frac{4}{9}\, M_B^2 = \frac{2a^2 + 2c^2 - b^2}{9},\ GC^2 = \frac{2a^2 + 2b^2 - c^2}{9}.$$

It's clearly that $\overrightarrow{GB} = \dfrac{1}{3}(2\overrightarrow{AB} - \overrightarrow{AC})$, and $\overrightarrow{GC} = \dfrac{1}{3}(2\overrightarrow{AC} - \overrightarrow{AB})$

Combine with (1), we get

$$\overrightarrow{GD} = \frac{GC^2}{GB^2 + GC^2 + BC^2} \times \frac{1}{3}(2\overrightarrow{AB} - \overrightarrow{AC}) + \frac{GB^2}{GB^2 + GC^2 + BC^2} \times \frac{1}{3}(2\overrightarrow{AC} - \overrightarrow{AB})$$

$\Leftrightarrow \overrightarrow{GD} = \dfrac{1}{3}\overrightarrow{AC}\ (\dfrac{2GB^2}{GB^2 + GC^2 + BC^2} - \dfrac{GC^2}{GB^2 + GC^2 + BC^2}) + \dfrac{1}{3}\overrightarrow{AB}$
$(\dfrac{2GC^2}{GB^2 + GC^2 + BC^2} - \dfrac{GB^2}{GB^2 + GC^2 + BC^2})$

$\Leftrightarrow \overrightarrow{GD} = \dfrac{1}{3}\overrightarrow{AC}\ \dfrac{2GB^2 - GC^2}{GB^2 + GC^2 + BC^2} + \dfrac{1}{3}\overrightarrow{AB}\ \dfrac{2GC^2 - GB^2}{GB^2 + GC^2 + BC^2}$

Equivalant to

$$\overrightarrow{GD} = \frac{2a^2 + 5c^2 - 4b^2}{3(13a^2 + b^2 + c^2)}\overrightarrow{AC} + \frac{2a^2 + 5b^2 - 4c^2}{3(13a^2 + b^2 + c^2)}\overrightarrow{AB}$$

Because $GDLX$ is a parallelogram, we have $\overrightarrow{XL} = \overrightarrow{GD}$.

Therefore,

$$\overrightarrow{AX} = \overrightarrow{AL} + \overrightarrow{LX} = \overrightarrow{AL} - \overrightarrow{GD}$$

$$\Leftrightarrow \overrightarrow{AX} = (\frac{c^2}{a^2 + b^2 + c^2} - \frac{2a^2 + 5c^2 - 4b^2}{3(13a^2 + b^2 + c^2)})\overrightarrow{AC} + (\frac{b^2}{a^2 + b^2 + c^2} - \frac{2a^2 + 5b^2 - 4c^2}{3(13a^2 + b^2 + c^2)})\overrightarrow{AB}$$

$$\Leftrightarrow \overrightarrow{AX} = \frac{2((15a^2c^2 + 3b^4) - (a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2))}{3(a^2 + b^2 + c^2)(13a^2 + b^2 + c^2)}\overrightarrow{AC} + \frac{2((15a^2b^2 + 3c^4) - (a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2))}{3(a^2 + b^2 + c^2)(13a^2 + b^2 + c^2)}\overrightarrow{AB}$$

Let AX cut BC at A'. It implies that

$$\frac{2((15a^2c^2 + 3b^4) - (a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2))}{3(a^2 + b^2 + c^2)(13a^2 + b^2 + c^2)}\overrightarrow{A'C} + \frac{2((15a^2b^2 + 3c^4) - (a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2))}{3(a^2 + b^2 + c^2)(13a^2 + b^2 + c^2)}\overrightarrow{A'B} = \overrightarrow{0}$$

Therefore,

$$\Leftrightarrow \frac{\overline{A'B}}{\overline{A'C}} = -\frac{2((15a^2c^2 + 3b^4) - (a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2))}{2((15a^2b^2 + 3c^4) - (a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2))}.$$

Similarly, let BY cut AC at B', CZ cut AB at C' then we get $\dfrac{\overline{A'B}}{\overline{A'C}} \cdot \dfrac{\overline{B'C}}{\overline{B'A}} \cdot \dfrac{\overline{C'A}}{\overline{C'B}} = -1$, which implies that AX, BY, CZ are concurrent by $Ceva's$ theorem.

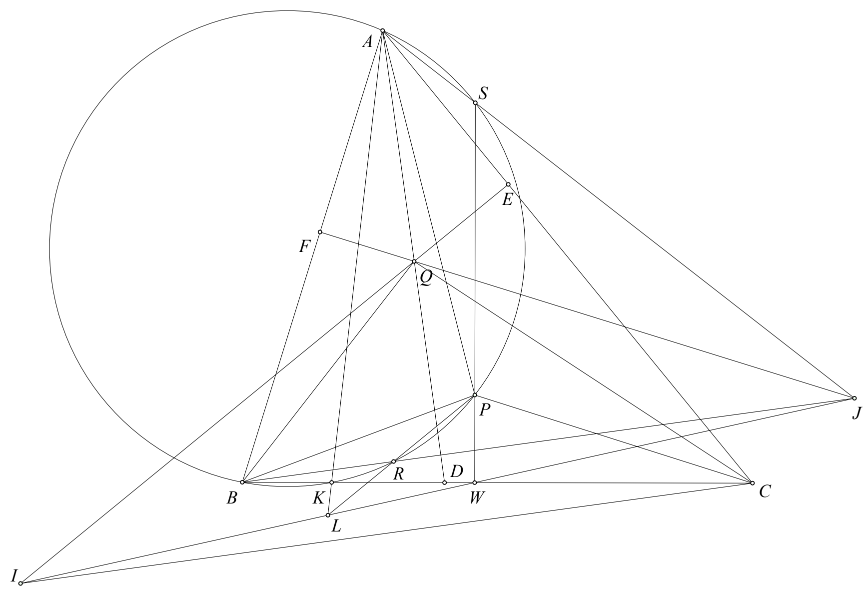
Hence, the problem is proved.

$\square$

Problem 2. Let ABC be a triangle with two isogonal conjugate points. Let I and J be the orthocenters of triangles QAC and QAB respectively. Let K be the second intersection point of (PAB) and BC. Let L be the intersection of AK and IJ. Prove that $PL \perp AC$.

Solution(Gia Bach). First, we will prove the following claim.

Claim 2.1 Let W be the projection of P onto BC. Then I, W, J are collinear. □



Proof. Let IQ, JQ cut AC and AB at E, F, respectively.

Since PW is perpendicular to BC, we get $\dfrac{BW}{CW} = \dfrac{BP}{CP} \cdot \dfrac{sin\angle BPW}{sin\angle CPW}$

$$\iff \frac{BW}{CW} = \frac{BP}{CP} \cdot \frac{sin\angle BQF}{sin\angle CQE}$$

$$\iff \frac{BW}{CW} = \frac{BP}{CP} \cdot \frac{BF}{BQ} \cdot \frac{CQ}{CE} = \frac{BP}{CP} \cdot \frac{CQ}{BQ} \cdot \frac{BF}{CE}$$

Let AQ intersect BC at D. Then we have

$$\frac{AB}{AC} \cdot \frac{DC}{DB} = \frac{AB}{AC} \cdot \frac{S_{\Delta ACQ}}{S_{\Delta ABQ}}$$

Equivalant to

$$\frac{AB}{AC} \cdot \frac{DC}{DB} = \frac{AB}{AC} \cdot \frac{AC.CQsin\angle ACQ}{AB.BQsin\angle ABQ}$$

$$\Leftrightarrow \frac{AB}{AC} \cdot \frac{DC}{DB} = \frac{CQsin\angle ACQ}{BQsin\angle ABQ}$$

$$\Leftrightarrow \frac{AB}{AC} \cdot \frac{DC}{DB} = \frac{CQ}{BQ} \cdot \frac{sin\angle PCB}{sin\angle PBC}$$

$$\Leftrightarrow \frac{AB}{AC} \cdot \frac{DC}{DB} = \frac{CQ}{BQ} \cdot \frac{BP}{CP}$$

Therefore,

$$\Longleftrightarrow \frac{BW}{CW} = \frac{AB}{AC} \cdot \frac{DC}{DB} \cdot \frac{BF}{CE}$$

$$\Longleftrightarrow \frac{BW}{CW} = \frac{sin\angle CAD}{sin\angle BAD} \cdot \frac{BF}{CE}$$

$$\Longleftrightarrow \frac{BW}{CW} = \frac{sin\angle CIE}{sin\angle BJF} \cdot \frac{BF}{CE} = \frac{BJ}{CI}$$

Therefore, I, W, J are collinear.

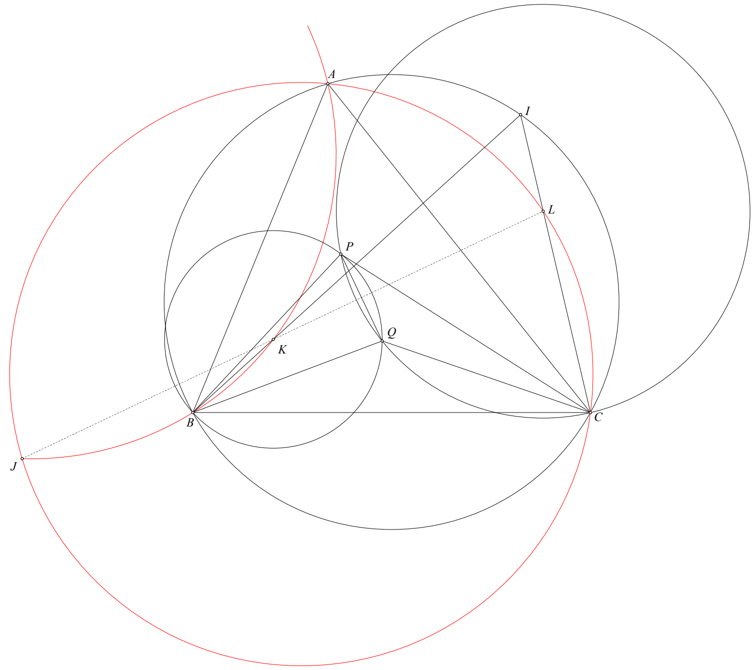
Back to the main problem, let PW intersect AJ at S. Then, $\angle ASP = 180^\circ - \angle PSJ = 180^\circ - \angle QBC = 180^\circ - \angle ABP$, therefore A, B, P, S are concyclic. Let LP cut again (ABP) at R. Applying *Pascal* theorem for $\begin{pmatrix} S & R & K \\ B & A & P \end{pmatrix}$ We get B, R, J are collinear, since L, W, J are collinear.

Therefore, $\angle APR = 180^\circ - \angle ABR = 180^\circ - \angle ABJ = \angle AQJ = 90^\circ + \angle QAB = 90^\circ + \angle PAC$, which implies that PR is perpendicular to AC.

Hence, the problem is proved.

$\square$

Problem 3. Let P and Q be two isogonal conjugate points with respect to the triangle ABC. Let K and L be the circumcenters of triangles BPQ and CPQ respectively. Prove that the second intersection of the circles (ABK) and (ACL) lies on line KL.



Proof. This is the solution for the problem in one circumstance of the configuration. The other forms of configurations is solved similarly.

We need to prove that BK and CL intersect at I, which lies on (ABC).

Since K and L are the centers of (BPQ) and (CPQ), respectively, we have:

$$\angle KBQ = 90^\circ - \angle BPQ, \quad \angle LCQ = 90^\circ - \angle CPQ$$

Thus:

$$\angle BIC = 180^\circ - \angle KBC - \angle LCB$$

$$\iff \angle BIC = 180^\circ - \angle KBQ - \angle QBC - \angle LCQ - \angle QCB$$

$$\iff \angle BIC = \angle BPQ + \angle CPQ - \angle QBC - \angle QCB$$

$$= \angle BPC - 180^\circ + \angle BQC = \angle A$$

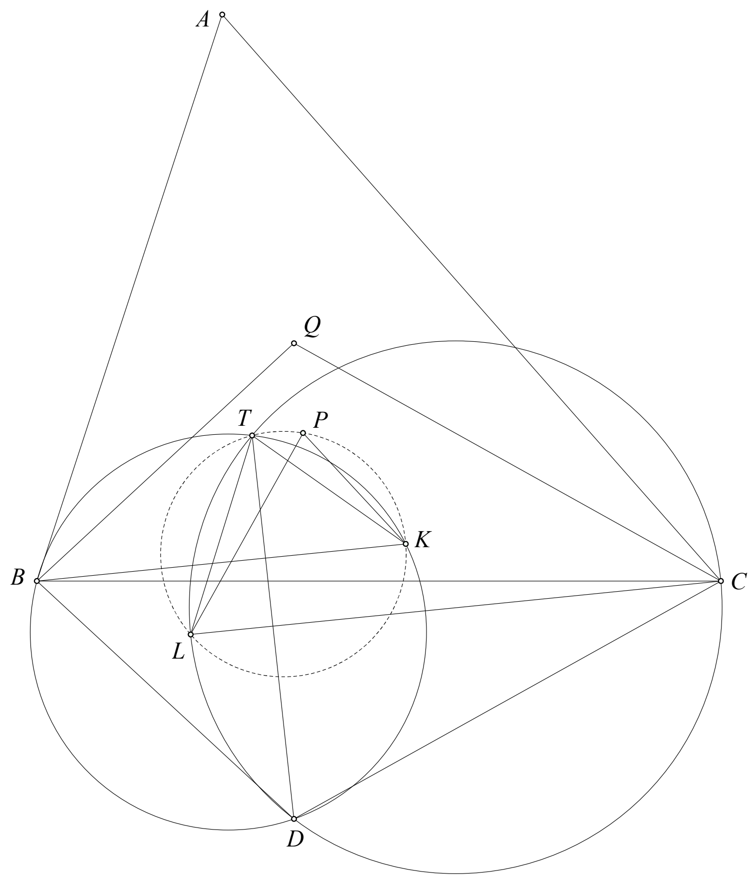
Therefore, A, B, C, I are concyclic. Let J be the second intersection of (ABK) and (ACL).Then we have

$$\angle AJK = \angle ABK = \angle ACL = \angle AJL$$

Furthermore, AI is not the angle bisector of $\angle KJL$. Therefore, J, K, L are collinear.

□

Problem 4. Let P and Q be two isogonal conjuhate points with respect to a triangle ABC. Let D be the reflection of Q in line BC. Let K and L be the orthocenters of triangles BPQ and CPQ respectively. Prove the the circles (DBK), (DCL), and (PKL) are concurrent.



Proof. This is the solution for the problem in one circumstance of the configuration. The other forms of configurations is solved similarly. Let T be the second intersection point of the two circles (DBK) and (DCL). We have the following angle chasing:

$$\angle LTK = \angle LTD + \angle KTD$$

$$\Leftrightarrow \angle LTK = \angle LCD + \angle KBD$$

$$\Leftrightarrow \angle LTK = \angle QBC + \angle KBC + \angle QCB - \angle LCB$$

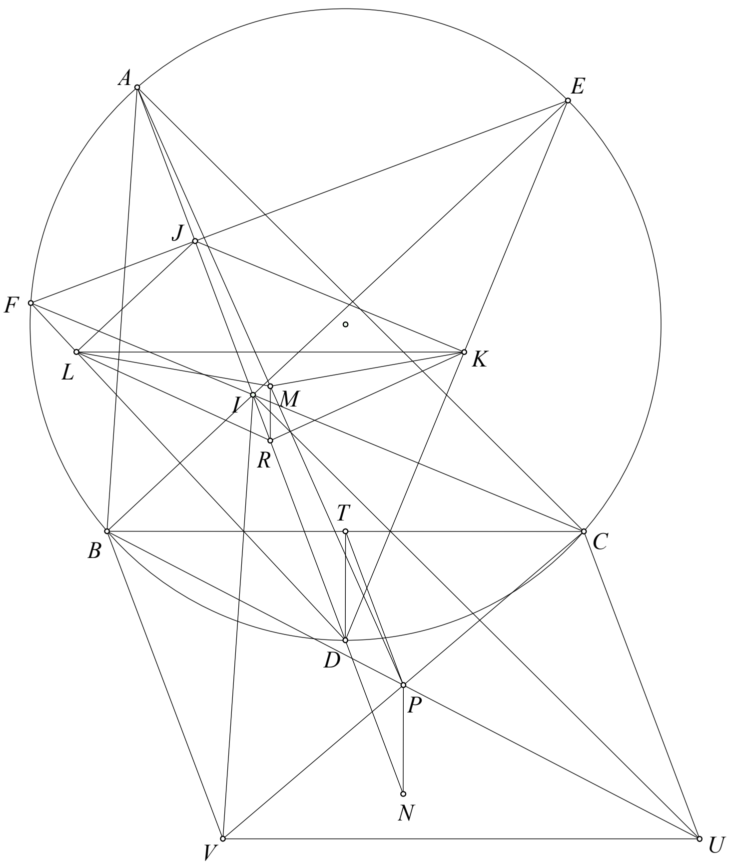
And since $BK \parallel CL$, $PK \perp BQ$, $PL \perp CQ$, therefore

$$\angle QBC + \angle KBC + \angle QCB - \angle LCB = 180^\circ - \angle BQC = \angle LPK.$$

Hence, the problem is proved.

□

Problem 5. Consider triangle ABC inscribed in circle ω, with circumcevian triangle DEF of the incenter I. The Simson lines of points B and C with respect to triangle DEF intersect at point P. Let J denote the midpoint of segment AI. Additionally, let K and L represent the projections of point J onto lines DE and DF respectively. Prove that the midpoint of segment AP is equidistant from points K and L.



Solution(Gia Bach).

Contruct a parallelogram $BCUV$ and P is the center of the parallelogram. So we get P is the midpoint of BU and CV. Because P lies on the *Simson* line of B and C with respect to triangle DEF, therefore IU, IV is the *Steiner* line of B, C with respect to triangle DEF, respectively.

By simple angle chasing, we get the *Simson* line of B and C with respect to triangle DEF parallel to AC, AB, respectively, therefore IU, IV is parallel to AC, AB, respectively. Construct point N on AI that $\overline{AJ} = \overline{DN}$. Let M, R, T be the midpoint of AP, JD, BC, respectively. Since $BCUV$ is a parallelogram, we get $AIVB$, $AIUC$ are also the parallelograms. Therefore, we get $AI = BV$, so $\frac{1}{2}AI = \frac{1}{2}BV \Leftrightarrow AJ = TP$, which implies that $DN = TP$. Combine with $DN \parallel TP$, we get $TDNP$ is a parallelogram.

Therefore, PN is perpendicular to BC, that makes MR is perpendicular to BC. By simple angle chasing, we get KL is parallel to BC. And since $RK = RL$, we get $MK = ML$.

Hence, the problem is proved.

□

Problem 6. Consider triangle ABC inscribed in circle ω, with incenter I. A-mixtilinear incircle touches ω at D. Define similarly E and F. Prove that

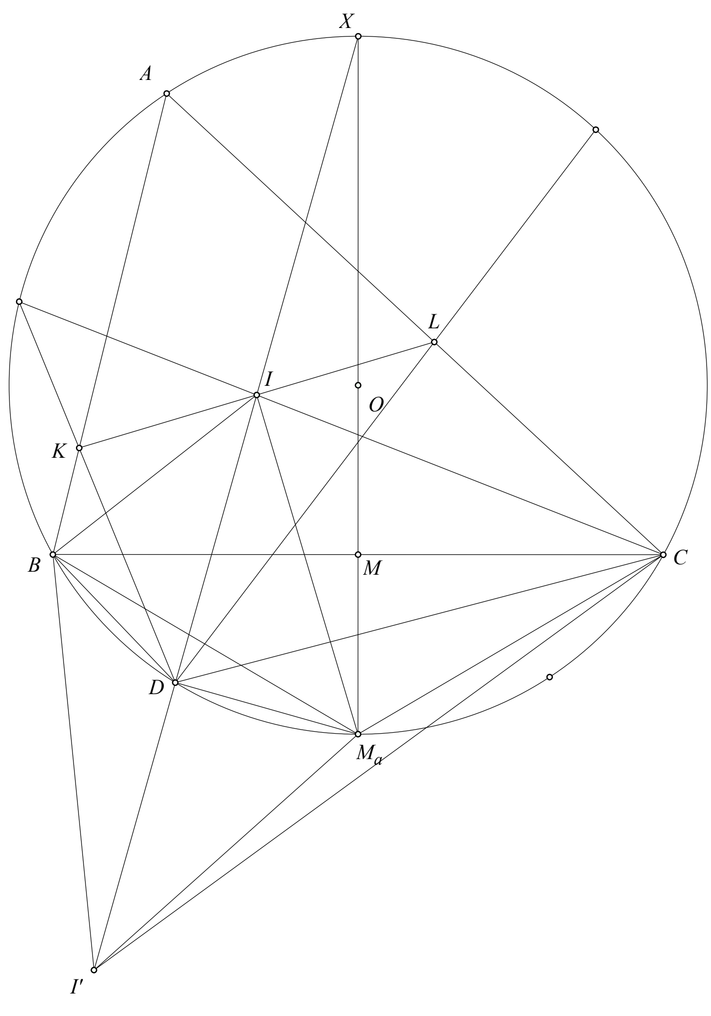
$$ID^2 + IE^2 + IF^2 = 4(R^2 - r^2)$$

iff triangle ABC is equilateral where R and r are circumradius and inradius of ABC.

Solution (Gia Bach).

It's trivial if the triangle is equilateral. We will prove that the formula come true if only the triangle is equilateral. WLOG, we assume that $\angle B$ and $\angle C$ are acuted

We will prove the following property: Let X be the midpoint of the larger arc BC. Then X, I, D are collinear.



Let K and L be the points of tangency of (A-mixtilinear) with AB and AC, respectively. We get DK and DL pass through the midpoints of the arcs AB and AC. Consequently, $\angle BDK = \frac{1}{2}\angle C = \angle BIK$, so $BKID$ is cyclic. Similarly, $CLID$ is cyclic. Therefore, $\angle BDI = \angle CDI = 90° - \frac{1}{2}\angle A$, which implies that DI bisects $\angle BDC$. As a result, X, I, D are collinear.

Let I' be the reflection of I across D. Let M_a be the midpoint of the arc BC that not contain A. Since $\angle M_aDI = 90^\circ$, we have $M_aI = M_aI'$. Additionally, $M_aI = M_aB = M_aC$, so $BICI'$ is cyclic. Moreover, since XB and XC are tangents to (BIC), and X, I, I' are collinear, the quadrilateral $BICI'$ is harmonic. This implies $XI \cdot XI' = XB^2$, which simplifies to:

$$XI(XI + 2ID) = XB^2.$$

Using Euler's formula, $2XI \cdot ID = 2(R^2 - OI^2) = 4rR$, we get:

$$XI^2 = XB^2 - 4rR.$$

Therefore,

$$ID^2 = \frac{4R^2 \cdot r^2}{XI^2} = \frac{4R^2 \cdot r^2}{XB^2 - 4rR}.$$

Let M be the midpoint of BC. Then $XB^2 = XM \cdot XMa = XM \cdot 2R$, so:

$$ID^2 = \frac{4R^2 \cdot r^2}{2R(XM - 2r)} = \frac{2R \cdot r^2}{XM - 2r}.$$

Let Y and Z be the midpoints of the larger arcs CA and AB, and let N and P be the midpoints of AC and AB, respectively. We aim to prove:

$$Rr^2\left(\frac{1}{XM - 2r} + \frac{1}{YN - 2r} + \frac{1}{ZP - 2r}\right) \leq 2R^2 - 2r^2,$$

or equivalently:

$$r^2\left(\frac{R}{XM - 2r} + \frac{R}{YN - 2r} + \frac{R}{ZP - 2r} + 2\right) \leq 2R^2.$$

Let $A = \dfrac{R}{XM - 2r} + \dfrac{R}{YN - 2r} + \dfrac{R}{ZP - 2r} + 2$

Case 1: Triangle $\triangle ABC$ is acuted

Then, we have

$$\frac{1}{XM - 2r} = \frac{1}{R - 2r + OM} = \frac{1}{R + R\cos A - 2r}.$$

Case 2: Obtuse triangle $\triangle ABC$ (considering $\angle A$ is obtused)

In this case:

$$\frac{1}{XM - 2r} = \frac{1}{R - 2r - OM},$$

where $OM = R\cos(180^\circ - A) = -R\cos A$. Hence:

$$\frac{1}{R - 2r - OM} = \frac{1}{R - 2r + R\cos A},$$

which is identical to the case of an acute triangle.

Thus, the two cases are fundamentally the same, and we proceed by considering the case of an acute triangle.

Set $BC = a, CA = b, AB = c$. Using the area formula and the $Heron's$ area formula, we have:

$$R = \frac{abc}{4S}, \quad r = \frac{S}{p}, \quad S^2 = p(p-a)(p-b)(p-c)$$

where p is the semiperimeter. Consequently:

$$\frac{r}{R} = \frac{4S^2}{pabc}.$$

$$\frac{r}{R} = \frac{4p(p-a)(p-b)(p-c)}{pabc} = \frac{\prod(a+b-c)}{2abc}$$

Then,

$$A = \sum \frac{R}{R + R\cos A - R\frac{\prod(b+c-a)}{abc}} + 2$$

$$\Leftrightarrow A = \sum \frac{1}{1 + \frac{b^2+c^2-a^2}{2bc} - \frac{\prod(b+c-a)}{abc}} + 2$$

,by the law of cosine.

The formula equivalant to

$$A = \sum \frac{2abc}{2(abc - \prod(a+b-c)) + a(b^2+c^2-a^2)} + 2$$

We will prove the following inequality

$$abc \sum \frac{1}{2(abc - \prod(a+b-c)) + a(b^2+c^2-a^2)} + 1 \leq \frac{4(abc)^2}{\prod(a+b-c)^2}$$

Since a, b, c are the lengths in one triangle, we can substitute $a = x+y$, $b = y+z$, $c = z+x$, with x, y, z are real positive numbers. Then the inequality turns into

$$(x+y)(y+z)(z+x) \sum \frac{1}{2((x+y)(y+z)(z+x) - 8xyz + (y+z)(x^2+xy+xz-yz))} + 1 \leq \frac{4((x+y)(y+z)(z+x))^2}{64x^2y^2z^2}$$

$$\Leftrightarrow (x+y)(y+z)(z+x) \sum \frac{1}{\sum x^2(y+z) - 6xyz + x^2(y+z) + x(y^2+z^2) + 2xyz - yz(y+z)} + 2 \leq \frac{((x+y)(y+z)(z+x))^2}{8x^2y^2z^2}$$

$$\Leftrightarrow (x+y)(y+z)(z+x) \sum \frac{1}{2x(x(y+z) + (y-z)^2)} + 2 \leq \frac{((x+y)(y+z)(z+x))^2}{8x^2y^2z^2}$$

Then we need to prove that

$$(x+y)(y+z)(z+x) \sum \frac{1}{2x^2(y+z)} + 2 \leq \frac{((x+y)(y+z)(z+x))^2}{8x^2y^2z^2}$$

$$\Leftrightarrow \sum \frac{(x+y)(x+z)}{2x^2} + 2 \leq \frac{((x+y)(y+z)(z+x))^2}{8x^2y^2z^2}$$

$$\Leftrightarrow 16x^2y^2z^2 + 4\sum y^2z^2(x+y)(x+z) \leq ((x+y)(y+z)(z+x))^2$$

$$LHS \leq \sum yz(y+z)^2(x+y)(x+z) + 2xyz(x+y)(y+z)(z+x)$$

$$\Leftrightarrow LHS \leq (x+y)(y+z)(z+x)(\sum yz(y+z) + 2xyz) = ((x+y)(y+z)(z+x))^2 = RHS$$

Because the $LHS = RHS$ from the formula of the problem, therefore all the equal sign must occur to satisfy the inequality, it implies that $x = y = z$, so $BC = CA = AB$, and the triangle ABC is equilateral.

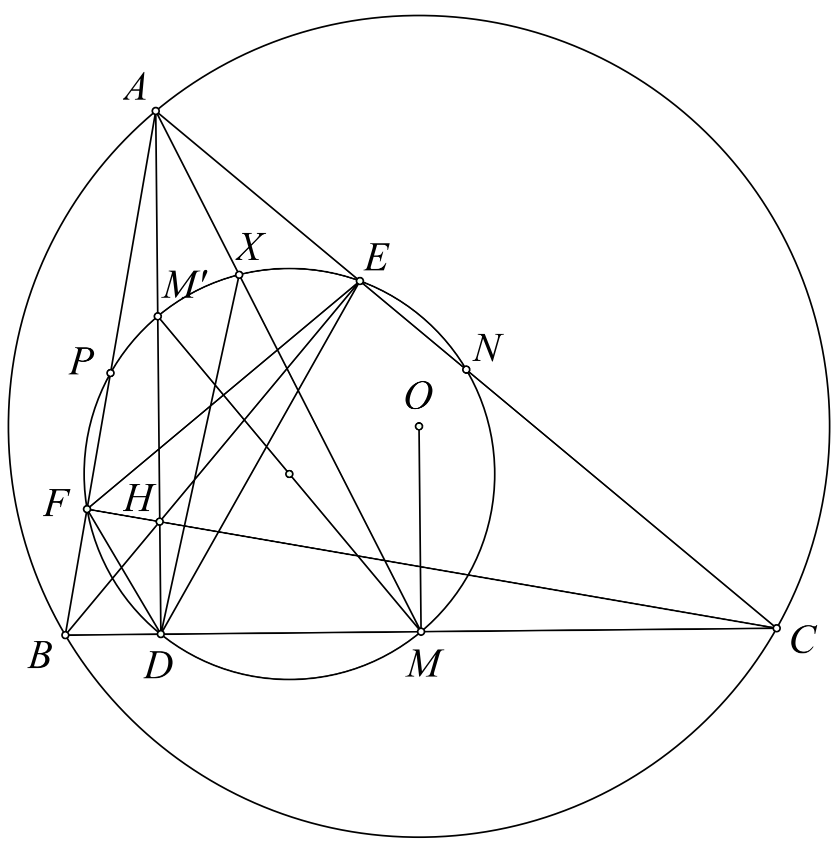
Hence, the problem is proved.

$\square$

Problem 7. Let ABC be a triangle with altitudes AD, BE, CF. D-mixtilinear excircle of triangle DEF touches (DEF) at X. Define similarly Y and Z. Prove that

$$\frac{DX^2 + EY^2 + FZ^2}{AX^2 + BY^2 + CZ^2} = \frac{144S^2}{(a^2 + b^2 + c^2)^2} + 1$$

where S is the area of ABC and $BC = a, CA = b, AB = c$.



Solution (Vu Anh). Let M, N, P be the midpoints of BC, CA, AB, and O be the circumcenter of $\triangle ABC$. According to the proof in problem 6, similarly we have X lies on the D-excenter of triangle DEF and the midpoint of arc EDF of $\odot(DEF)$, which is AM. Therefore, let X be the intersection of $\odot(DEF)$ and AM. Then,

$$AX \cdot AM = \frac{AF \cdot AB}{2} = \frac{\cos A \cdot bc}{2} = \frac{b^2+c^2-a^2}{4}, \quad \text{hence } AX = \frac{b^2+c^2-a^2}{4AM}.$$

Also,
$AX = \dfrac{AH \cdot AD}{2AM} = \dfrac{OM \cdot AD}{AM}.$
Let MM' be the diameter of $\odot(DEF)$. Then M' lies on AD, and $MM' = 2R$. Since $\dfrac{DX}{MM'} = \dfrac{AD}{AM}$, we get

$$DX = \frac{AD \cdot 2R}{AM} = \frac{bc}{AM}.$$

We have

$$DX^2 - AX^2 = \frac{AD^2 \cdot R^2}{AM^2} - \frac{AD^2 \cdot OM^2}{AM^2} = \frac{AD^2}{AM^2} \cdot (OB^2 - OM^2) = \left(\frac{AD \cdot BM}{AM}\right)^2 = \frac{S^2}{AM^2}.$$

Proving the problem is equivalent to

$$\frac{144S^2}{(a^2+b^2+c^2)^2} = \text{LHS}-1 = \frac{(DX^2-AX^2)+(EY^2-BY^2)+(FZ^2-CZ^2)}{AX^2+BY^2+CZ^2}. = \frac{S^2 \cdot \left(\frac{1}{AM^2}+\frac{1}{BN^2}+\frac{1}{CP^2}\right)}{AX^2+BY^2+CZ^2}$$

This simplifies to

$$\frac{144}{(a^2+b^2+c^2)^2} = \left(\frac{1}{AM^2}+\frac{1}{BN^2}+\frac{1}{CP^2}\right) \cdot \frac{1}{AX^2+BY^2+CZ^2}.$$

Rewriting,

$$\frac{(a^2+b^2+c^2)^2}{144} = \frac{AX^2+BY^2+CZ^2}{\frac{1}{AM^2}+\frac{1}{BN^2}+\frac{1}{CP^2}} = \frac{\sum \frac{(b^2+c^2-a^2)^2}{16AM^2}}{\sum \frac{4}{2b^2+2c^2-a^2}}.$$

Which is equivalent to,

$$\frac{(a^2+b^2+c^2)^2}{9} = \frac{\sum \frac{(a^2+b^2+c^2)^2-4a^2(b^2+c^2)}{2b^2+2c^2-a^2}}{\sum \frac{1}{2b^2+2c^2-a^2}}.$$

Implying:

$$\frac{2(a^2+b^2+c^2)^2}{9} = \frac{\sum \frac{a^2(b^2+c^2)}{2b^2+2c^2-a^2}}{\sum \frac{1}{2b^2+2c^2-a^2}}.$$

$$= \frac{\sum a^2(b^2+c^2)(2a^2+2b^2-c^2)(2a^2+2c^2-b^2)}{\sum (2a^2+2b^2-c^2)(2a^2+2c^2-b^2)}.(*)$$

We have:

$$\sum(2a^2+2b^2-c^2)(2a^2+2c^2-b^2)=9(a^2b^2+b^2c^2+c^2a^2).$$

Hence, $(*)$ is equivalent to:

$$2(a^2+b^2+c^2)^2(a^2b^2+b^2c^2+c^2a^2)=\sum a^2(b^2+c^2)(2a^2+2b^2-c^2)(2a^2+2c^2-b^2).$$
$$=(a^2b^2+b^2c^2+c^2a^2)(\sum(2a^2+2b^2-c^2)(2a^2+2c^2-b^2))-\sum(b^2c^2(2a^2+2b^2-c^2)(2a^2+2c^2-b^2)).$$

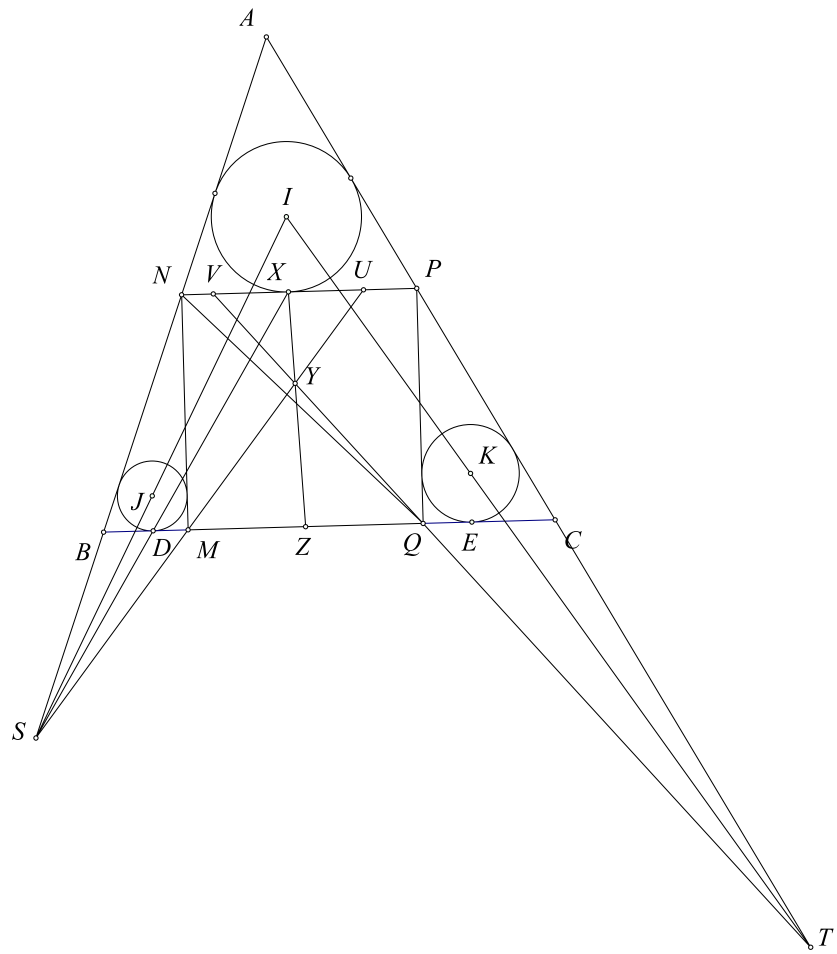
$$=9(a^2b^2+b^2c^2+c^2a^2)^2-\sum(4a^4b^2c^2+2a^2b^4c^2+2a^2b^2c^4+5b^4c^4-2b^2c^2(b^4+c^4+a^4)+2a^4b^2c^2).$$

$$\begin{aligned}&\iff (a^2b^2+b^2c^2+c^2a^2)(5a^2b^2+5b^2c^2+5c^2a^2-2a^4-2b^4-2c^4)\\&\quad=\sum\left(6a^4b^2c^2+2a^2b^4c^2+2a^2b^2c^4+5b^4c^4-2b^2c^2(b^4+c^4+a^4)\right)\\&\quad=10a^2b^2c^2(a^2+b^2+c^2)+5(a^4b^4+b^4c^4+c^4a^4)-2(b^4+c^4+a^4)(a^2b^2+b^2c^2+c^2a^2).\end{aligned}$$
$$\iff 5(a^2b^2+b^2c^2+c^2a^2)^2=10a^2b^2c^2(a^2+b^2+c^2)+5(a^4b^4+b^4c^4+c^4a^4),$$

which is true. Hence, $(*)$ is correct, and the problem is proved.

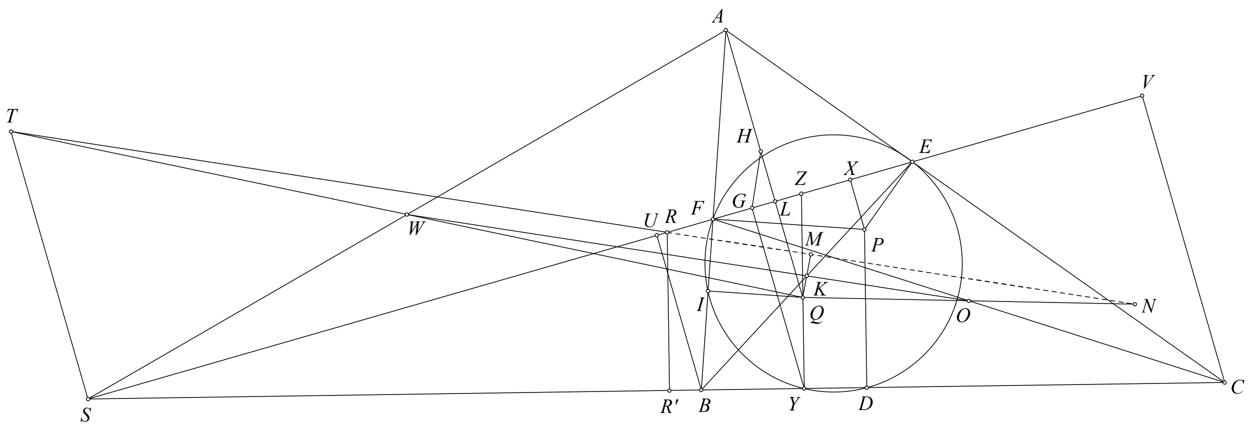
$\square$

Problem 9. Given a triangle ABC. Construct an inscribed square $MNPQ$ within the triangle with M and Q on side BC. The incircle of triangle ANP touches NP at X. The external common tangent of the incircles of triangles ANP and BMN intersects at S. The external common tangent of the incircles of triangles ANP and CPQ intersects at T. MS intersects TQ at Y. Prove that XY bisects MQ.



Solution(Vu Anh). Let SM, QT cut NP at U, V respectively; let I, J, K be the incenter of triangles ANP, BMN, CPQ respectively, incircles of triangle BMN, CPQ touch BC at D, E respectively.
Since $\angle JMD = 45°$, we get $DJ = DM$. We have that S is the exsimilicenter of (J) and (I) therefore the homothety center S sends (J) to (I) also sends $D \mapsto X, M \mapsto U$. This implies that $\triangle JDM \sim \triangle IXU$, and since $DJ = DM$, we get $XU = XI$.
Similiarly we get $XV = XU = XI$ therefore XY bisects MQ.
Hence the problem is proved. $\square$

Problem 10. Let the triangle ABC with P, Q as two isogonal conjugate points. E, F are the projections of P onto CA, AB respectively. M, N are symmetrical of Q through the midpoints of BE, CF respectively. MN meets EF at R. Prove that $PR \parallel BC$.



Solution(Vu Anh). Redefine R as the intersection of the line through P parallel to BC and EF, we will prove M, N, R are collinear. Let EF cut BC at S, AQ cut EF at L; W, K, O be the midpoint of AS, BE, CF we get W, K, O are collinear.

Therefore, let T be the reflection of Q across W we get T, M, N are collinear and $ST = AQ$, $ST \perp EF$(since $AQ \perp EF$). Therefore, let T be the reflection of Q across W we get T, M, N are collinear and $ST = AQ$, $ST \perp EF$(since $AQ \perp EF$).

Let U, V be the projection of B, C on EF; D, Y be the projection of P, Q on BC we get D, E, F, Y are concyclic therefore $\angle YFG = \angle EDC = \angle EPC = \angle CQY$ therefore triangles YGF and CYQ are similiar. Similiarly triangles YGE and BYQ are similiar therefore $\dfrac{GF}{GE} = \dfrac{YB}{YC} = \dfrac{GU}{GV}$ therefore $GU \cdot GE = GV \cdot GF$ therefore G lies on radical axis of (BE) and (CF).

Let H be the orthocenter of triangle AEF, then we get H lies on radical axis of (BE) and (CF) therefore HG is the radical axis of (BE) and (CF) and so: $HG \perp KO$ therefore $HG \perp MN$ (1). We will prove triangles TSR and GLH are similiar and since $\angle TSR = \angle GLH = 90°$, this is equivalent to proving: $\dfrac{ST}{SR} = \dfrac{LG}{LH}$ $(*)$.

Let QY cut EF at Z, R' be the projection of R on BC we get $RR' = PD$, triangles SRR' and QZL are similiar therefore $\dfrac{SR}{RR'} = \dfrac{ZQ}{ZL} = \dfrac{QY}{GL}$, hence $\dfrac{1}{SR} = \dfrac{GL}{QY \cdot PD}$ therefore $(*)$ is equivalent to:

$$\frac{GL \cdot ST}{QY \cdot PD} = \frac{LG}{LH} \leftrightarrow \frac{QA}{QY} = \frac{ST}{QY} = \frac{PD}{LH}(**).$$

Let X be the projection of P on EF, since $EHFP$ is a parallelogram we get $PX = LH$. Let I be the projection of Q on AB, by simple angle chasing, we get: $\triangle PXF \sim \triangle QIA$, $\triangle PFD \sim \triangle QYI$ therefore

$$\frac{QA}{QY} = \frac{QA}{QI} \cdot \frac{QI}{QY} = \frac{PF}{PX} \cdot \frac{PD}{PF} = \frac{PD}{PX} = \frac{PD}{LH}$$

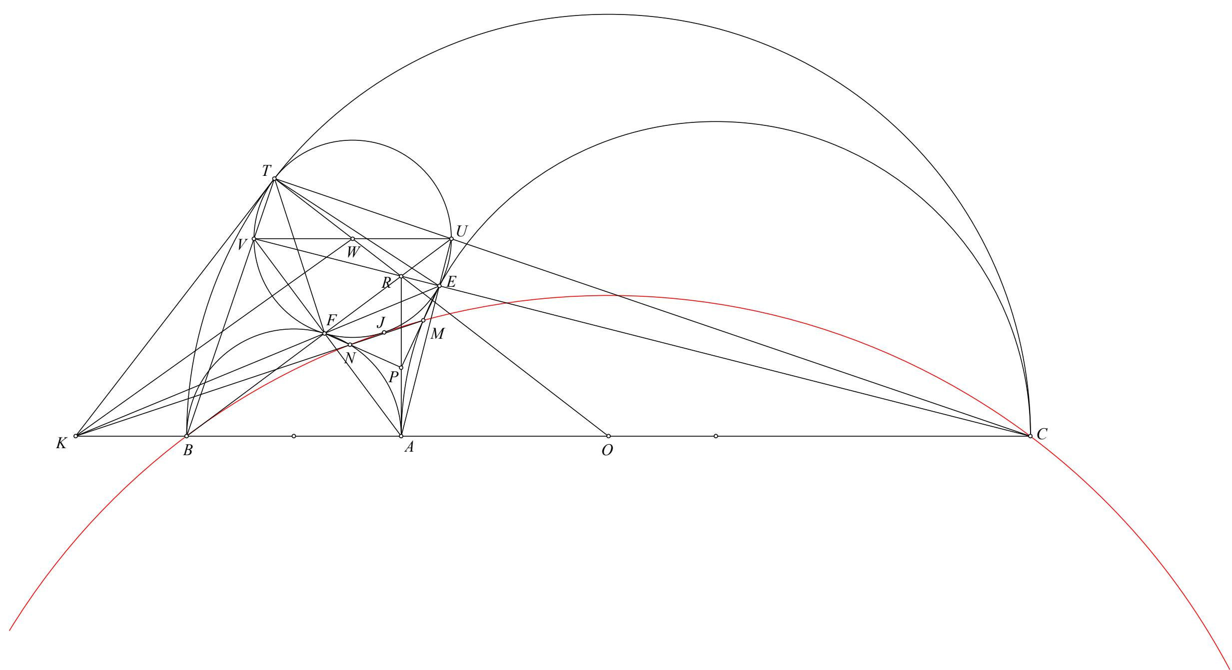
,and so $(**)$ is true, therefore $(*)$ is true.

Therefore we get triangles TSR and GLH are similiar and since $TS \perp GL, SR \perp LH$ we get $TR \perp GH$ combine with $(1) : HG \perp MN$ we get $TR \parallel MN$ and since T, M, N are collinear we get R, M, N are collinear.

Hence the problem is proved.

□

Problem 11. Let A be a point lying on the segment BC. Draw circle ω which is tangent to three circles diameters AB, AC and BC. Draw circle Ω passing through B and C and is tangent to ω. Ω meets the circles diameters AB and AC at N and M respectively. ω touches the circles diameters AB and AC at F and E respectively. Let P be the intersection of EM and FN. Prove that $PA \perp BC$.



Solution(Gia Bach).

We have a small lemma.

Lemma. *Given 2 circle ω_1 and ω_2 with the center O_1, O_2 and K is the exsimilicenter of the two circles. A line pass though K intersect 2 circles at B, C, respectively, so that O_1B is not parallel to O_2C. A circle pass though B, C, intersect the ω_1 and ω_2 at P, Q. Then P, Q, K are collinear.*

This lemma is very familiar and not hard to be proved, therefore I will leave as the small challenge for the readers.

Back to the main problem. Let O be the midpoint of BC, K be the exsimilicenter of 2 circle (AB), (AC). Let T lies on (BC) that KT is tangent to (BC). Draw a line from A that perpendicular to BC, cut OT at R. Define RC, RB intersect (AC), (AB) at E, F, respectively. Then we will prove that (TEF) is tangent to (AB), (AC), (BC).

Beacuse K is the exsimilicenter of 2 circle (AB), (AC), then we get $\dfrac{KB}{KA} = \dfrac{KA}{KC} = \dfrac{AB}{AC} \iff \dfrac{KB}{KA} \cdot \dfrac{KA}{KC} = \dfrac{AB}{AC}^2 \iff \dfrac{KB}{KC} = \dfrac{AB}{AC}^2 \iff \dfrac{TB}{TC}^2 = \dfrac{AB}{AC}^2 \iff \dfrac{TB}{TC} = \dfrac{AB}{AC}$, therefore TA is the angle bisector of $\angle BAC$, and from that we can easily have $RA = RT$.

We have $RA^2 = \overline{RE} \cdot \overline{RC} = \overline{RB} \cdot \overline{RF} \iff RT^2 = \overline{RE} \cdot \overline{RC} = \overline{RB} \cdot \overline{RF}$, therefore $\angle RET = \angle RTC = \angle TCB \iff \angle TEA + \angle TBA = 180^o$. So we get T, E, A, B are concyclic. Also, we get T, F, A, C are concyclic

Let AE cut CT at U, CE cut BT at V. Because $TVEU$ is the cyclic quadrilateral, we get $\angle TVU = \angle TEU = \angle TBA$, therefore UV is parallel to BC. We have $\angle FTU = \angle FAB = \angle FRA = \angle FEA$, so T, F, E, U are concyclic. Therefore, T, V, F, E, U are concyclic, and we can easily prove that B, R, U and A, F, V are collinear. Combine with UV is parallel to BC, we get (TEF) is tangent to (AB), (AC)

Let W be the midpoint of UV. Then W is the center of (TEF). Since UV is parallel to BC, it's clearly that W, R, O are collinear, therefore (TEF) is tangent to (BC).

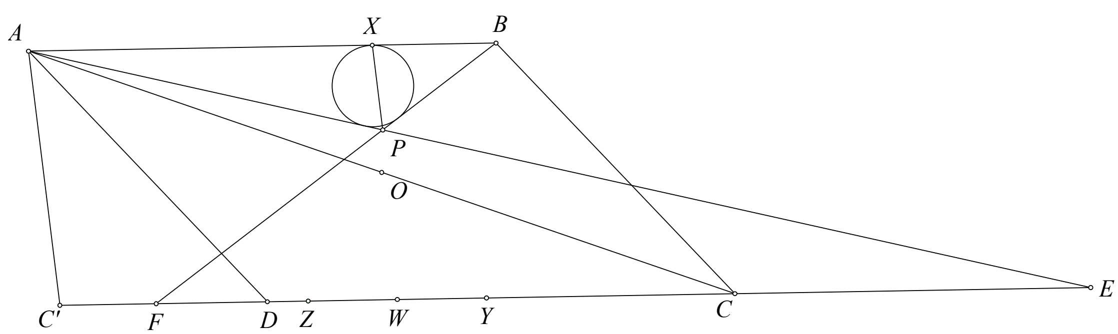
Applying the lemma for the cyclic quadrilateral $BFEC$ and $BNMC$, and K is the exsimilicenter of 2 circle (AB), (AC), then E, F, K and M, N, K are collinear, and $\overline{KB} \cdot \overline{KC} = \overline{KE} \cdot \overline{KF} = \overline{KM} \cdot \overline{KN}$, therefore E, F, N, M are concyclic. So we get P lies on the radical axis of (AB), (AC), therefore PA is perpendicular to BC.

Hence the problem is proved.

Comment. The problem also works with any circles (Ω) that pass though B and C.

□

Problem 12. Let $ABCD$ be a parallelogram with point P inside it. The incircle of $\triangle PAB$ touches AB at X. A circle that touches PB and CB also touches CD at Y. Another circle touches PA and DA and also touches DC at Z. Let W be the midpoint of YZ, and let O be the center of parallelogram $ABCD$. Prove that $OW \parallel PX$.



Solution (Vu Anh). Let PA and PB intersect CD at E and F, respectively then the incircles of $\triangle ADE$ and $\triangle BCF$ are tangent to CD at points Z and Y, respectively. Thus, Y lies on ray CD, Z lies on ray DC. Let C' be the reflection of C through W. By construction, we have $OW \parallel AC'$. To derive this, we calculate EC':

$$EC' = CC' + EF - FC = CZ + CY + EF - FC = DC - DZ - FY + EF.$$

Therefore:

$$EC' = DC + EF - \frac{DA + DE - AE}{2} - \frac{FC + FB - BC}{2}.$$

Simplifying further:

$$EC' = DC + EF - \frac{DE + CF}{2} + \frac{AE - BF}{2} = \frac{DC + EF}{2} + \frac{AE - BF}{2}.$$

By Thales, we have:

$$\frac{AP}{AE} = \frac{BP}{BF} = \frac{AB}{AB + EF} = k.$$

Therefore:

$$EC' = \frac{AB}{2k} + \frac{AP}{2k} - \frac{BP}{2k} = \frac{1}{k}\left(\frac{AB + AP - BP}{2}\right) = \frac{1}{k} \cdot AX.$$

$$\text{Hence} \quad \frac{AX}{EC'} = k = \frac{AP}{AE}, \angle XAP = \angle AEC'.$$

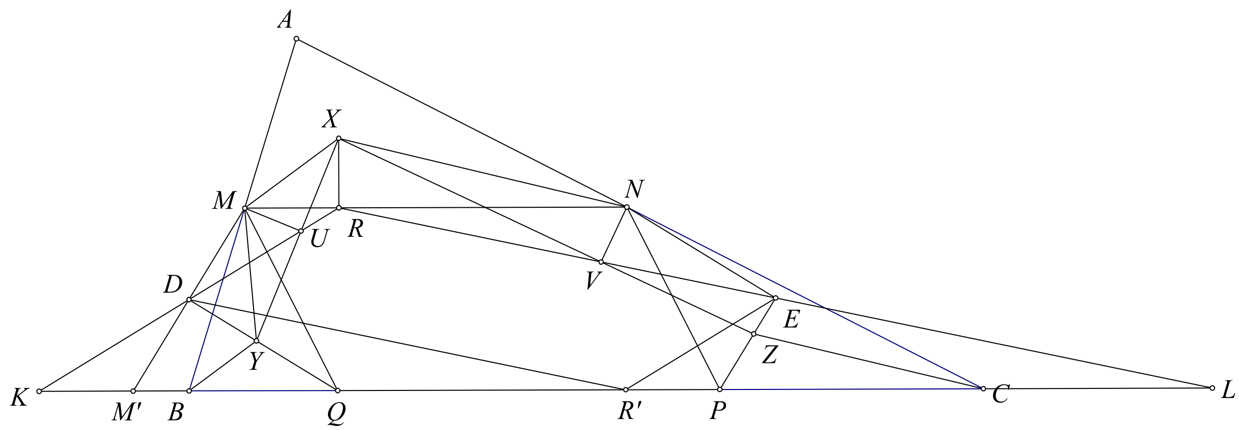
This implies that $\triangle APX$ and $\triangle EAC'$ are similar. Consequently, $AC' \parallel PX$, which implies that $OW \parallel PX$, as desired.

□

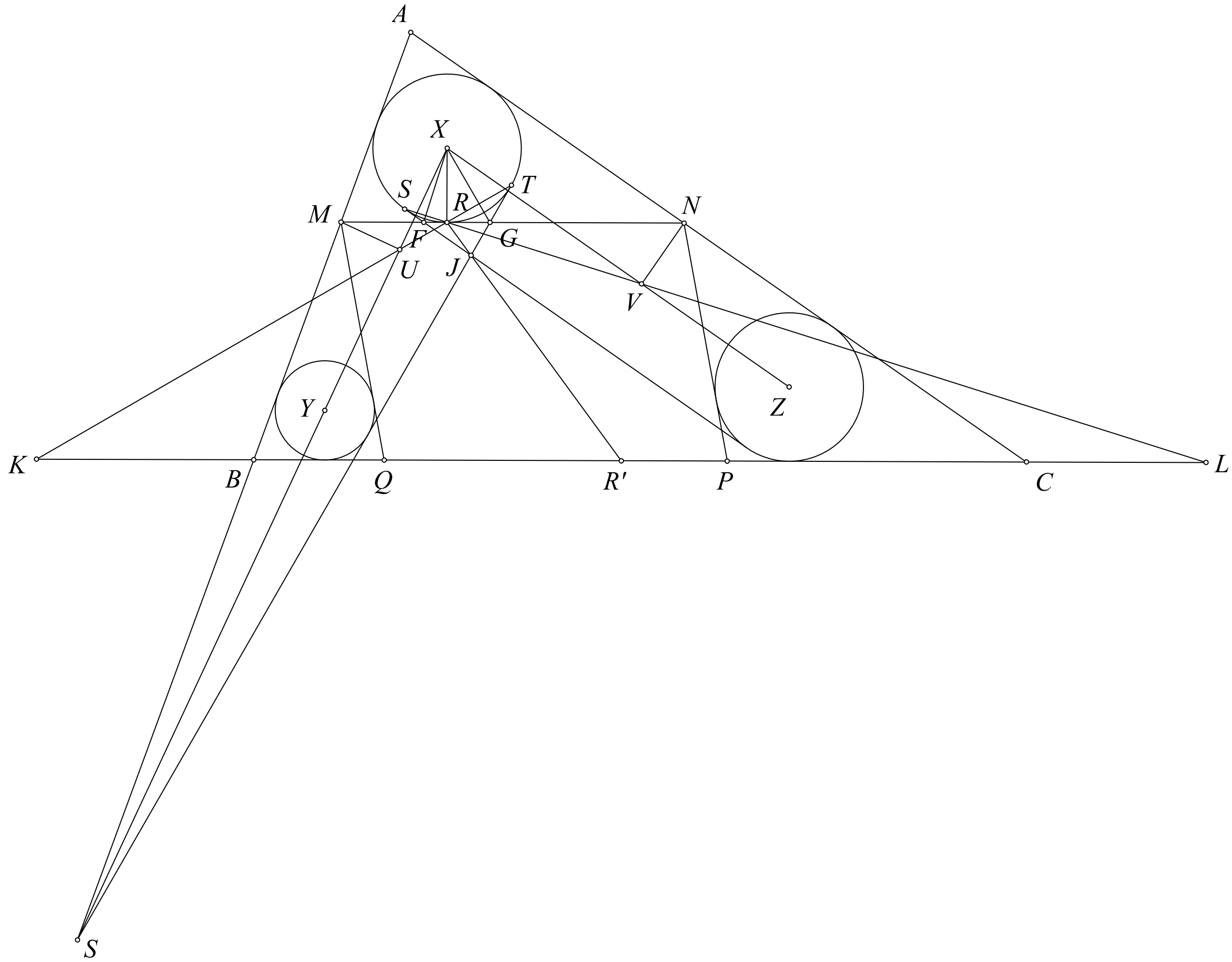
Problem 13. Let $MNPQ$ be a parallelogram inscribed in triangle ABC with points M and N lying on sides AB and AC, respectively, and points P and Q lying on side BC. The external common tangent of the incircles of triangles AMN and BMQ intersects the external common tangent of the incircles of triangles AMN and CNP at point J. The incircle of triangle AMN touches MN at point R. Prove that the line RJ passes through the center of parallelogram $MNPQ$.

Solution(Vu Anh & Gia Bach). Denote X, Y, Z as the incenter of AMN, BMQ and CNP. Let U, V as the projection of M, N on XY, XZ respectively, RU, RV cut BC at K, L respectively. Let R' be the point on segment PQ such that $MR = PR'$.

Claim 13.1. *R' is the midpoint of KL.*



Proof. (Gia Bach). Let QY cut RK at D and PZ cut RL at E. Since $MXRU$ is cyclic, we have: $\angle MUD = \angle MXR = 90° - \angle AMN/2 = 90° - \angle ABC/2 = \angle MYD$ therefore $MDYU$ is cyclic therefore $\angle MDY = 90°$. Let MD cut BC at M' since QD is the angle bisector of MQB we get D is the midpoint MM' therefore D is the midpoint of RK.
Similiarly, we get E is the midpoint of RL and $\angle NEP = 90°$. By simple angle chasing we get $QY \perp PZ$ therefore both NE and QY are perpendicular to PZ which yield $NE \parallel QD$. Similiarly, $MD \parallel PE$ therefore we get $\triangle MDQ = \triangle PEN$ therefore $DQ = NE$. Consider two triangles DQR' and ENR we have $NR \parallel QR', DQ \parallel NE, RN = QR', DQ = NE$ therefore we get $\triangle DQR' = \triangle ENR$. This implies that $DR' \parallel RL$ and since D the midpoint of RK we get R' is the midpoint of KL. □

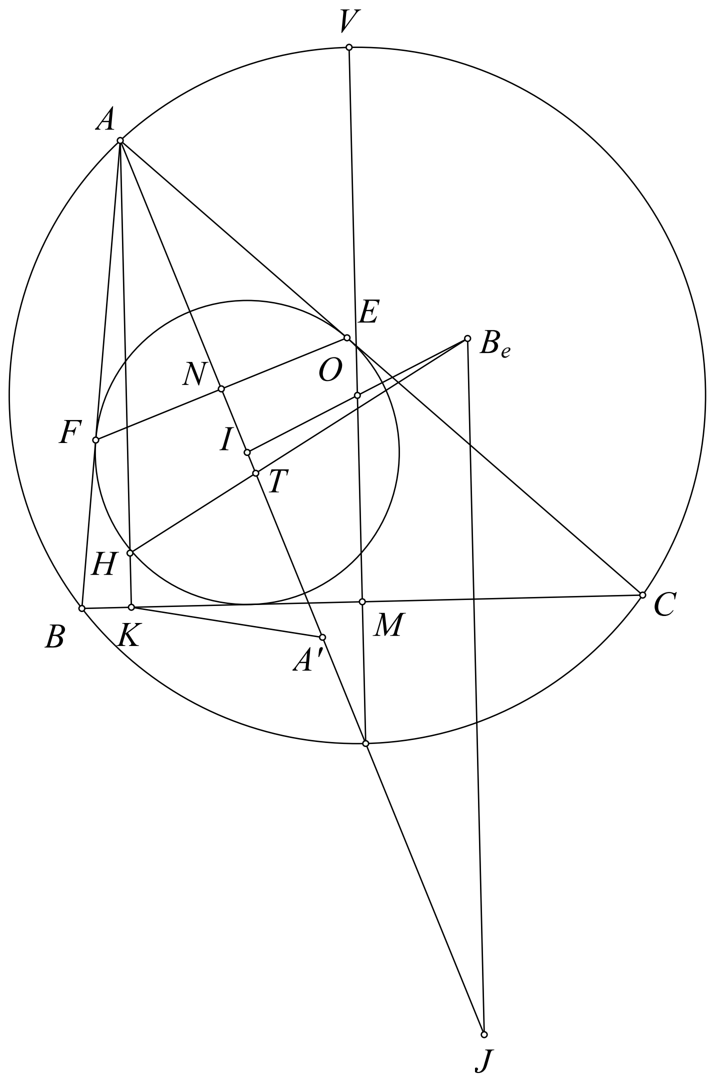


Back to the main problem, Let the external common tangents of the incircles of triangles AMN and BMQ; AMN and CNP cut MN at G, F respectively and touch the incircle of AMN at T, S respectively. We have: JG and AB are reflections through XY therefore XY, GJ, AB concurrent as S. Since X is the $S-excenter$ of triangle SMG we get XY pass through the circumcenter of (XMG) therefore $\angle RXG = \angle UXM = \angle URM$ therefore $RK \perp XG$ therefore $RK \parallel RT$ therefore R, T, K are collinear and similiarly we get R, S, L are collinear.
Using the claim above we get R' is the midpoint of KL. Therefore $R(R'M, TS) = R(R'M, KL) = -1 = R(JM, TS)$ therefore RJ pass through R' and since RR' pass through the center of $MNPQ$ we get RJ pass through the center of $MNPQ$.
Hence the problem is proved.

□

Problem 14. Given triangle ABC with incircle (I) touching BC, CA, AB at D, E, F respectively. AX, BY, CZ are altitudes. P, Q, R are the reflections of X, Y, Z in the lines EF, FD, DE respectively. Prove that the inverse of de Longchamps point of triangle PQR through its circumcircle lies on the Euler line of ABC.

Solution(Gia Bach & Vu Anh).



First, we will prove the following lemma.

Lemma 14.1. *Given a triangle ABC inscribed in circle (O) with the altitude AK. Let I, H, B_e be the incenter, orthocenter and the Bevan point of the triangle ABC, respectively. (I) touches AC, AB at E, F, respectively. Let A' be the symmetric point of A with respect to EF. Then, $\angle AHB_e = \angle AA'K$.*

Proof (Vu Anh). Let N be the midpoint of EF, HB_e intersect AI at T, V being the midpoint of arc BAC of (O). Since we need to prove $\angle AHB_e = \angle AA'K$, it is suffices to show that the quadrilateral $HKA'T$ is cyclic which is equivalent to: $AH \cdot AK = AT \cdot AA' = 2AT \cdot AN$, which is equivalent to $\dfrac{AT}{OM} = \dfrac{AK}{AN}.(*)$
Denote the center of the $A-excircle$ of ABC as J then $JB_e \perp BC$ and $JB_e = 2R$ therefore $\dfrac{AT}{TJ} = \dfrac{AH}{2R}$.
From this, it follows: $\dfrac{AT}{AJ} = \dfrac{OM}{OM+R}$. and so $\dfrac{AT}{OM} = \dfrac{AJ}{OM+R} = \dfrac{AJ}{MV}(1)$.

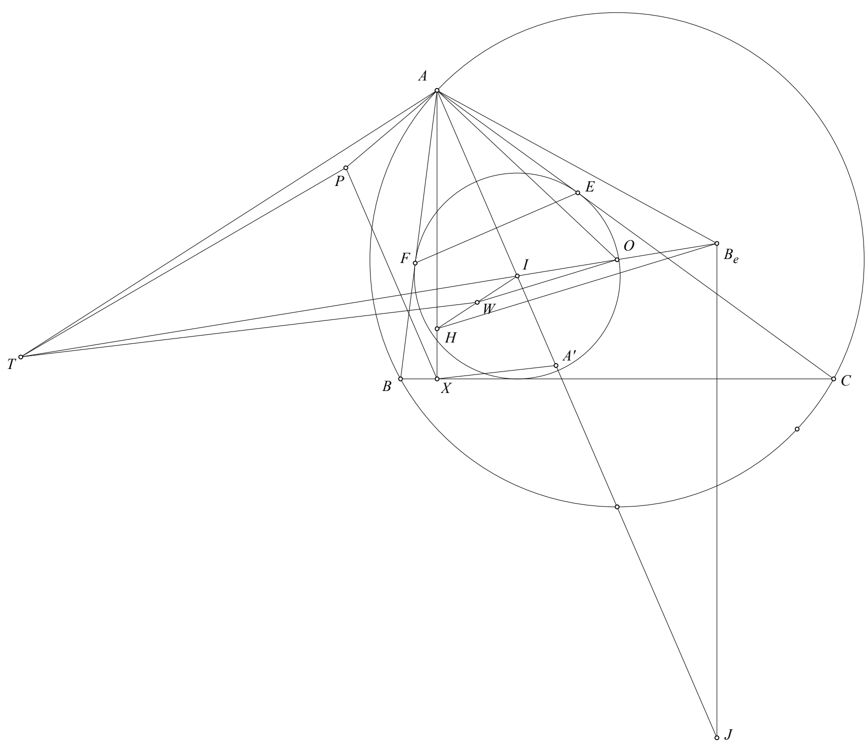
Since $\triangle VBC \sim \triangle AFE$, it follows that: $\dfrac{AN}{MV} = \dfrac{AI}{2R}$.. Combine with the fact that because $AI \cdot AJ = AB \cdot AC = AK \cdot 2R \Leftrightarrow \dfrac{AI}{2R} = \dfrac{AK}{AJ}$, we get $\dfrac{AN}{MV} = \dfrac{AI}{2R} = \dfrac{AK}{AJ}$ therefore $\dfrac{AJ}{MV} = \dfrac{AK}{AN}(2)$. From $(1), (2)$ we get $\dfrac{AK}{AN} = \dfrac{AJ}{MV} = \dfrac{AT}{OM}$ hence $(*)$ is proved and so is the claim. □

Back to the main problem, we have the following claim:

Claim 14.1. *Let O and H be the center and orthocenter of triangle ABC, respectively, W be the midpoint of IH, and T be the point satisfies that*

$$\overline{OI} \cdot \overline{OT} = OA^2$$

Then $\Delta TPA \sim \Delta TWO$.



Proof. Let A' be the symmetric point of A with respect to EF, and *Bevan* point B_e. Applying the lemma, we have $\angle AHB_e = \angle AA'X = \angle PAI$. We have

$$\angle TAP = \angle TAO - \angle OAI - \angle PAI$$

Equivalent to

$$\angle TAP = \angle OIA - \angle HAI - \angle AHB_e$$

So we get

$$\angle TAP = \angle HAI + \angle HB_eI + \angle AHB_e - \angle HAI - \angle AHB_e = \angle HB_eI = \angle TOW.$$

We will prove that $\dfrac{TA}{TO} = \dfrac{PA}{WO}$. That is equivalant to $\dfrac{AI}{AO} = \dfrac{2XA'}{HB_e}$

We have

$$\frac{2XA'}{HB_e} = \frac{2XA'}{HB_e} \cdot \frac{AX}{AB_e} \cdot \frac{AB_e}{AX} = 2\,\frac{AB_e}{HB_e} \cdot \frac{AX}{AB_e} \cdot \frac{XA'}{XA}$$

Equivalant to

$$\frac{2XA'}{HB_e} = 2\,\frac{sin\angle AHB_e}{sin\angle HAB_e} \cdot \frac{AX}{AB_e} \cdot \frac{sin\angle XAA'}{sin\angle XA'A} = 2\,\frac{AX}{AB_e} \cdot \frac{sin\angle XAA'}{sin\angle HAB_e}$$

Therefore

$$\frac{2XA'}{HB_e} = 2\,\frac{AX}{AB_e} \cdot \frac{sin\angle AJB_e}{sin\angle AB_eJ} = 2\,\frac{AX}{AB_e} \cdot \frac{AB_e}{AJ} = 2\,\frac{AX}{AJ}$$

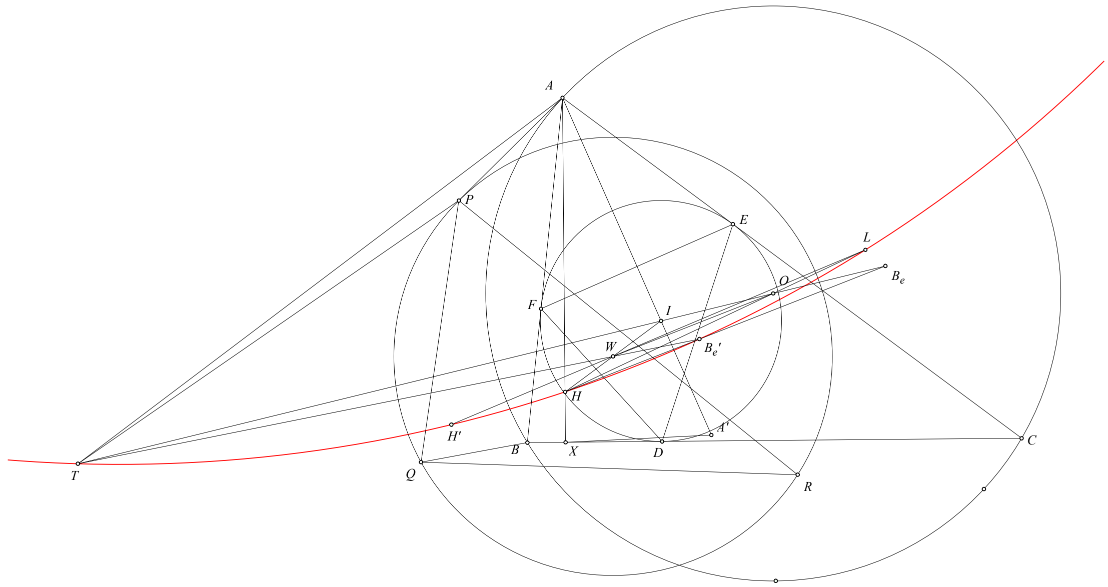
,since JB_e is perpendicular to BC.

So we get

$$\frac{2XA'}{HB_e} = 2\,\frac{AX}{AJ} = \frac{AI}{AO}$$

,because $2AO \cdot AX = AI \cdot AJ = AB \cdot AC$.

Therefore, $\Delta TPA \sim \Delta TWO$. Hence, the claim is proved. $\square$



Back to the main problem. Similarly, we have $\Delta TPA \sim \Delta TQB \sim \Delta TRC \sim \Delta TWO$, therefore the triangle ABC and PQR are similarity, with O and W are the center of the two triangles, respectively, and T is the center of a spiral similarity mapping triangle ABC to triangle PQR. Let H' be the orthocenter of triangle PQR, $H'W$ intersect HO at L. Then, it's clearly that T, H', H, L and T, W, O, L are concyclic.

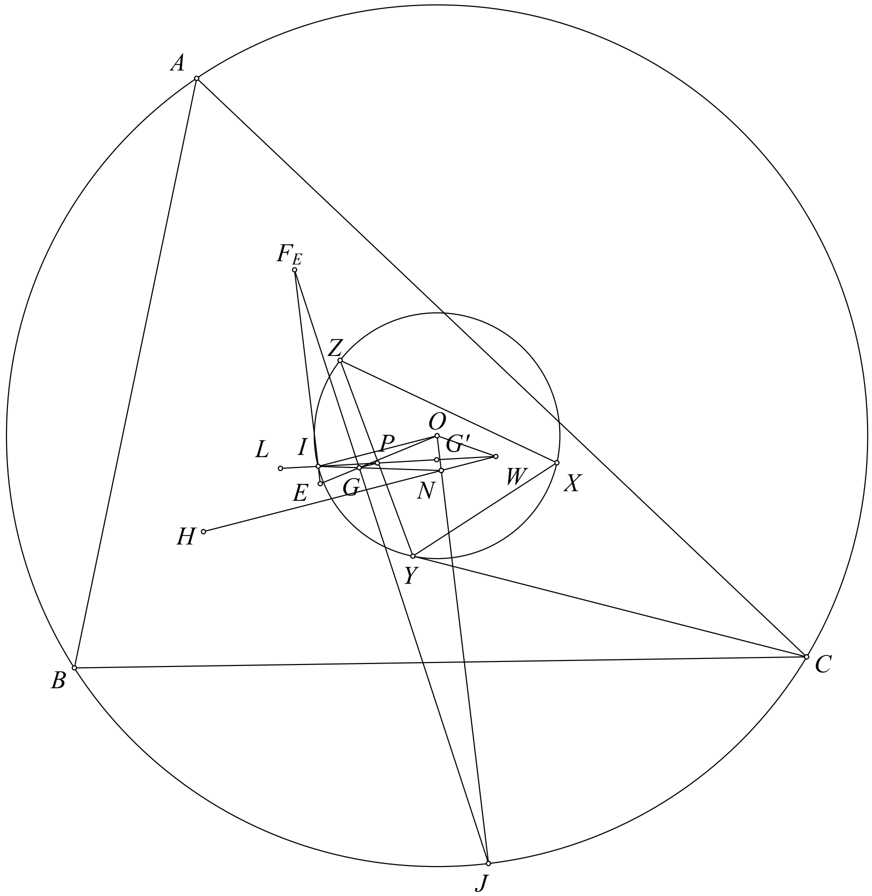
Let TW intersect HB_e at B_e'. Since OW is parallel to B_eB_e', we get that B_e' is the *Bevan* point of triangle PQR. Because OW is parallel to HB_e' and T, W, O, L are concyclic, we get that T, H, B_e', L are concyclic by *Reim's* theorem. Therefore, $\overline{WH'} \cdot \overline{WL} = \overline{WT} \cdot \overline{WB_e'} = R^2_{(PQR)}$, which implies that L is the inverse point of *deLongchamps* point of triangle PQR.

Hence, the problem is proved.

□

Problem 15. Given triangle ABC inscribed in (O, R), circumscribed about (I, r), centroid G, and Lemoine point L. Let P satisfy equation $3R\vec{GP} + r\vec{OI} = \vec{0}$. Prove that P lies on line IL.

Solution(Vu Anh). Let X, Y, Z be the reflection of I through the perpendicular bisector of BC, CA, AB respectively we get X, Y, Z, I lies on a circle with center O. Let H, N, E, F_E be the orthocenter, Nagel point, Nine-point-center and Feuerbach point of ABC respectively, W, G' be the Feuerbach point, centroid of the triangle XYZ.



Claim: *I, P, W, G' are collinear.*

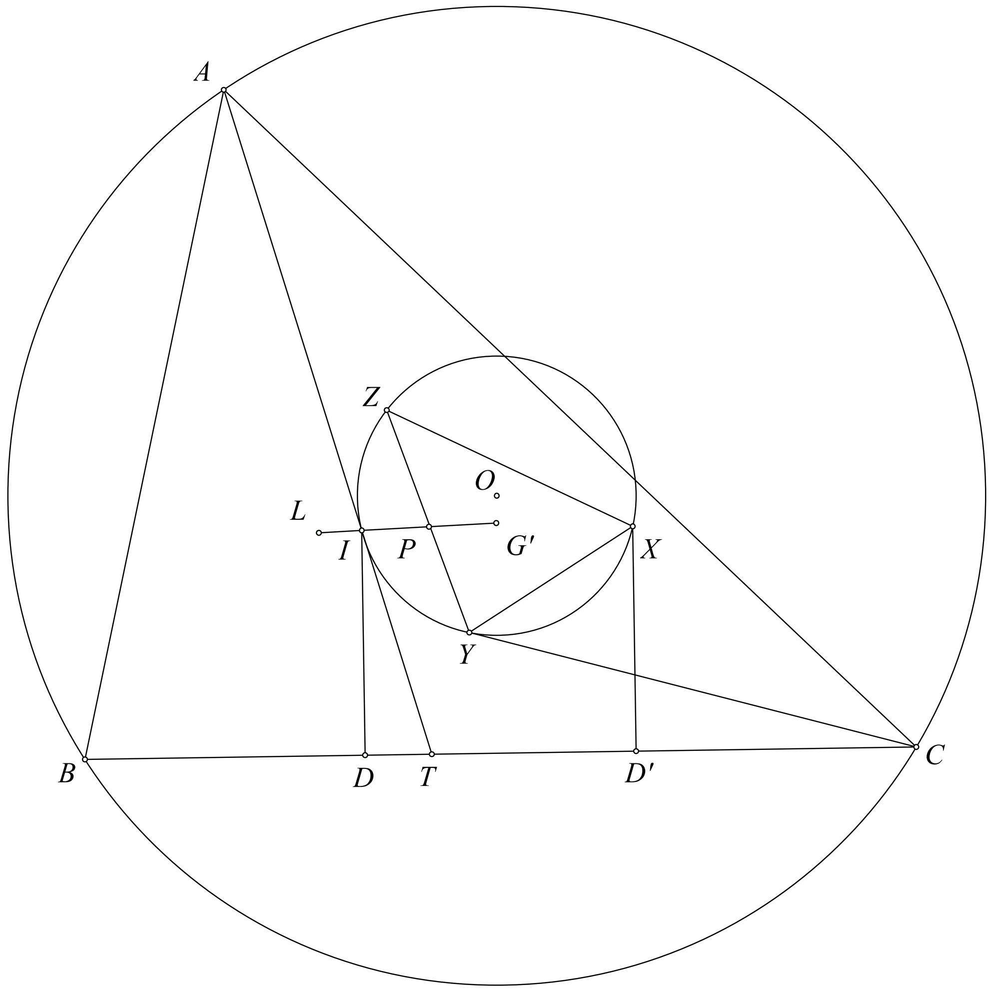
Consider homothety center G with scaling factor -2, it sends $E \mapsto O, I \mapsto N, F_E \mapsto J$ then we have $OJ = 2 \cdot EF_E = R$ therefore J lies on (O). We also have:

$ON \cdot OJ = 4 \cdot EI \cdot EF_E = 4 \cdot (EF_E - IF_E) \cdot EF_E = 4 \cdot \left(\frac{R}{2} - r\right) \cdot \frac{R}{2} = R \cdot (R - 2r) = OI^2.$

It is well known that F_E is the Anti-Steiner point of medial triangle of ABC wrt OI therefore J is the Anti-Steiner point of ABC wrt HN, therefore AJ and the line through A perpendicular to HN, which parallel to OI, are isogonal wrt $\angle BAC$. Simple angle chasing yields $\angle JBC = \angle JAC = 90^\circ - \angle OIZ = \angle IYZ$ therefore $\triangle IYZ \sim \triangle JBC$.

We have $\angle XYZ = \angle XIZ = \angle ABC$; similiarly, we get $\triangle XYZ \sim \triangle ABC$. Denote N' as the incenter of XYZ we get $\triangle XYZ \cup (O, I, N') \sim \triangle ABC \cup (O, J, I)$ therefore $\triangle OIN' \sim \triangle OJI$ therefore N' lies on OJ and $OI^2 = ON \cdot OJ$ which combine with (1) implies that $N' \equiv N$ hence N is the incenter of XYZ. Therefore we get $\triangle XYZ \cup (O, N, W) \sim \triangle ABC \cup (O, I, F_E)$ and so $\angle ONW = \angle OIF_E$. Combine this with the fact that $IF_E \parallel ON$ we get $NW \parallel OI$. Also, we have $\frac{NW}{IF_E} = \frac{OI}{OJ}$ and $\frac{GP}{OI} = \frac{r}{3R}$ therefore $\frac{GP}{NW} = \frac{GP}{OI} \cdot \frac{OI}{NW} = \frac{r}{3R} \cdot \frac{R}{r} = \frac{1}{3} = \frac{IG}{IN}$.

Combine this with the fact that $GP \parallel OI \parallel NW$ we get P lies on IW. Also, since $\triangle XYZ \cup (W, G', I) \sim \triangle ABC \cup (F_E, G, J)$ and F_E, G, J are collinear we get W, G', I are collinear therefore I, P, G', W are collinear.



Back to the main problem, Any case of triangle ABC being isosceles is trivial. Hence, WLOG, assume that $AB < BC < AC$.

since we need to prove P lies on IL, we will prove I, L, G' are collinear, which is equivalent to $\overrightarrow{LI} \parallel \overrightarrow{IG'}$.

Denote a, b, c as the length of BC, CA, AB respectively we get $c < a < b$ We have:

Let D, D' be the projection of I, X on BC respectively then $IX = DD' = DB - DC = \dfrac{b-c}{2}$

$\overrightarrow{IG'} = \overrightarrow{IX} + \overrightarrow{IY} + \overrightarrow{IZ} = \dfrac{b-c}{2a}\overrightarrow{BC} + \dfrac{a-c}{2b}\overrightarrow{AC} + \dfrac{b-a}{2a}\overrightarrow{BA}$ therefore

$2abc \cdot \overrightarrow{IG'} = (a^2c - c^2a) \cdot \overrightarrow{AC} + (b^2a - a^2b) \cdot \overrightarrow{BA} + (b^2c - c^2b) \cdot \overrightarrow{BC}(1)$

$(a^2+b^2+c^2) \cdot \overrightarrow{LI} = (a^2 \cdot \overrightarrow{LA} + b^2 \cdot \overrightarrow{LB} + c^2 \cdot \overrightarrow{LC}) + (a^2 \cdot \overrightarrow{AI} + b^2 \cdot \overrightarrow{BI} + c^2 \cdot \overrightarrow{CI}) = a^2 \cdot \overrightarrow{AI} + b^2 \cdot \overrightarrow{BI} + c^2 \cdot \overrightarrow{CI}$.

Let AI cut BC at T, we have $\dfrac{IA}{IT} = \dfrac{BA}{BT} = \dfrac{CA}{CT} = \dfrac{AB+AC}{BC} = \dfrac{b+c}{a}$ therefore $\dfrac{AI}{AT} = \dfrac{b+c}{a+b+c}$.

Therefore: $a^2 \cdot \overrightarrow{AI}$
$= \dfrac{a^2 \cdot (b+c)}{a+b+c} \cdot \overrightarrow{AT}$
$= \dfrac{a^2 \cdot (b+c)}{a+b+c} \cdot (\dfrac{BT}{BC} \cdot \overrightarrow{AC} + \dfrac{CT}{CB} \cdot \overrightarrow{AB})$
$= \dfrac{a^2 \cdot (b+c)}{a+b+c} \cdot (\dfrac{c}{b+c} \cdot \overrightarrow{AC} + \dfrac{b}{b+c} \cdot \overrightarrow{AB})$
$= \dfrac{1}{a+b+c} \cdot (a^2c \cdot \overrightarrow{AC} + a^2b \cdot \overrightarrow{AB})$. Similiarly we get $(a^2+b^2+c^2) \cdot \overrightarrow{LI} = a^2 \cdot \overrightarrow{AI} + b^2 \cdot \overrightarrow{BI} + c^2 \cdot \overrightarrow{CI} = \dfrac{1}{a+b+c} \cdot \Big((a^2c - c^2a) \cdot \overrightarrow{AC} + (b^2a - a^2b) \cdot \overrightarrow{BA} + (b^2c - c^2b) \cdot \overrightarrow{BC}\Big)$ (2).
From (1), (2) we get $(a^2+b^2+c^2) \cdot \overrightarrow{LI} = \dfrac{1}{a+b+c} \cdot 2abc \cdot \overrightarrow{IG'}$ therefore $\overrightarrow{LI} \parallel \overrightarrow{IG'}$, which imply I, L, G' are collinear, as desired.
Hence the problem is proved.

□

Problem 16. The bicentric quadrilateral $ABCD$ satisfies the condition

$$\frac{1}{S^2+rx^3} + \frac{1}{S^2+ry^3} + \frac{1}{S^2+rz^3} + \frac{1}{S^2+rt^3} = \frac{16}{15S^2}$$

where S is the area of the quadrilateral, r is the radius of the inscribed circle, and x, y, z, t are the lengths of the tangent segments drawn from A, B, C, D to the inscribed circle of the quadrilateral. Prove that $ABCD$ is a square.

About this problem, I think there is something wrong with the formula. If $ABCD$ is a square, then

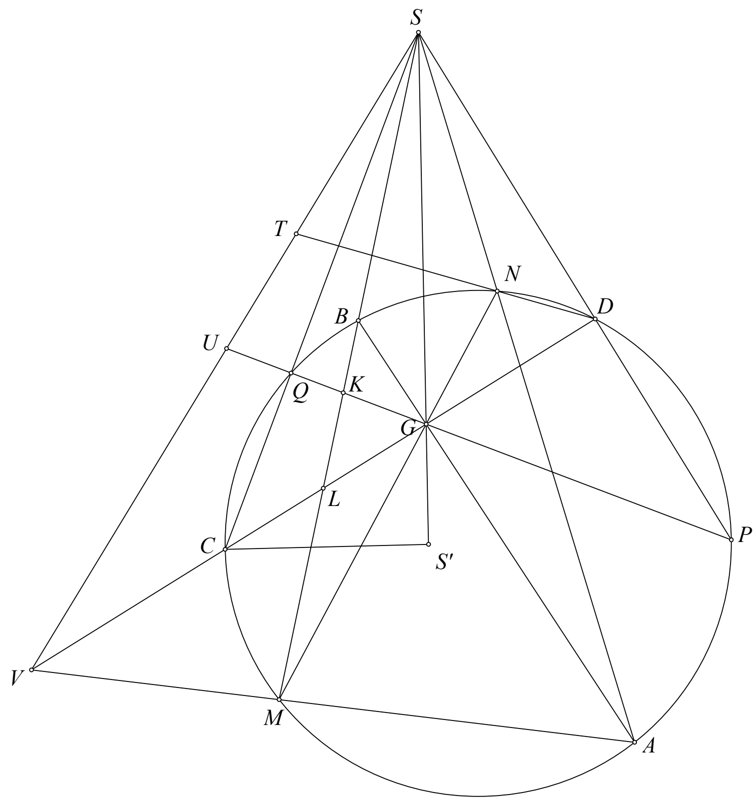
$$LHS = \frac{4}{S^2 + \frac{1}{16}S^2} = \frac{64}{17S^2},$$

Which is contradictory to the formula of the problem.

Problem 17. Given a triangle ABC inscribed in a circle ω. Choose a point P on ω such that AP is parallel to BC. The A-mixtilinear incircle of ABC is Ω, which is tangent to ω at D. The two tangents from P to Ω intersect AD at M and N, with M lying between A and D. Prove that two lines NB and CM intersect on ω.

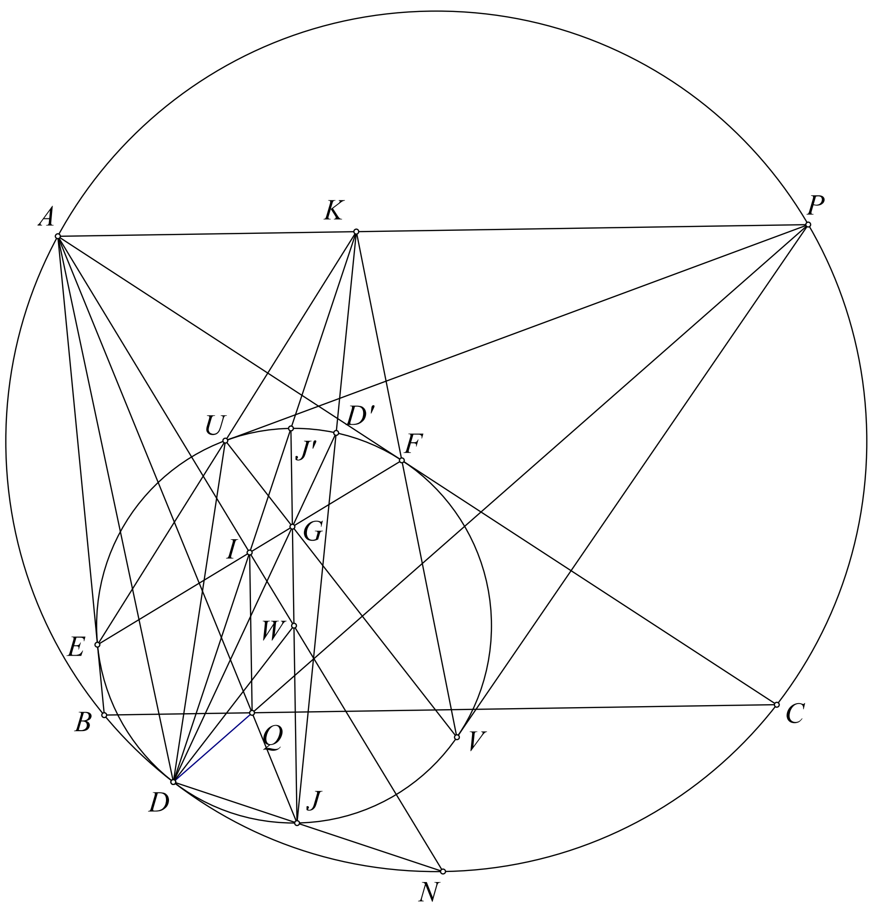
Solution(Vu Anh). First, we will prove the following lemma.
Lemma 17.1. *Given circle (O), Let B, C, D, N, P, Q be arbitrary points lies on (O) such that CQ, BM, PD are concurrent at a point S (Q near S than C, D near S than P, B near S than M. Let PQ cut CD at G, GB, GM cut (O) at A, N respectively. Prove that: $(AB, CD) = (MN, PQ)$.*

Proof. We will prove A, N, S are collinear. Let SG cut (CBG) at S' we get $\angle SS'C = \angle SQG = \angle SDC$ therefore $SDS'C$ is cyclic therefore $GS \cdot GS' = GC \cdot GD$ and $SG \cdot SS' = SQ \cdot SC = SB \cdot SM$ therefore $BGS'M$ is cyclic.
Consider an inversion about a circle with center G and radius $\sqrt{GC \cdot GD}$. It sends $S \mapsto S'$, $N \mapsto M$, $A \mapsto B$ and since $GBMS'$ is cyclic we get S, N, A are collinear.
Let BN cut PQ at U, CD cut AM at V, MQ cut ND at T, Applying Pascal theorem for $\begin{pmatrix} B & Q & D \\ P & N & M \end{pmatrix}$ we get S, T, U are collinear.
Also applying Pascal theorem for $\begin{pmatrix} A & D & Q \\ C & M & N \end{pmatrix}$ we get V, T, S are collinear combine with above we get



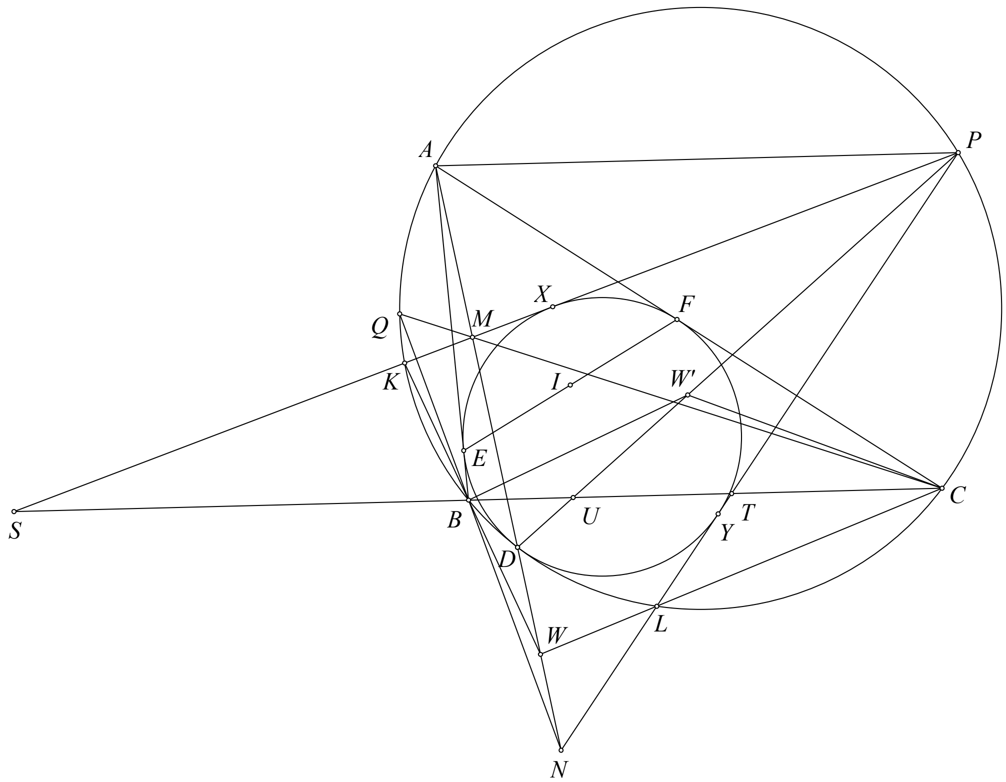
S, T, V, U is collinear.
Let SM cut PQ, CD at K, L. Since UV, CQ, KL, PD concurrent at a point S we get $(UK, QP) = (VL, CD)$.
Also since $B(UK, QP) = (NM, QP)$ and $M(VL, CD) = (AB, CD)$ we get $(NM, QP) = (AB, CD)$.
Hence the lemma is proved. $\square$

Back to the main problem, we will prove the following claims:
Claim 17.1. *Let (I) touch BC at U then $P(DA, NM) = \dfrac{UB}{UC}$.*


A
K
P
U
D'
F
J'
G
I
E
W
B
Q
C
V
D
J
N

Proof. Denote W as the center of (Ω), (Ω) touch AB, AC at E, F respectively. It is well known that P, D, U are collinear, the midpoint of EF is also the incenter of triangle ABC which we will denote as I, DI cut AP at K, EK, FK cut (Ω) at X, Y. We will prove PX and PY similiarly tangent to (Ω).
Let AD cut (Ω) at G, GK cut (Ω) at R, the line from W perpendicular to BC cut (Ω) at J', J respectively such that J' is nearer to A. We will prove P, R, J' are collinear.
Since D is the exsimilicenter of (Ω) and (O) we get DJ' pass through the midpoint of arc BAC of (O) therefore D, I, J' are collinear.
Let GI cut (Ω) at V, since A is the exsimilicenter of (I) and (Ω) we get AJ pass through U and AJ' pass through U' which is the reflection of U through I therefore AJ', AD are reflections through AI therefore G, J' are reflections through AI therefore $\angle AJ'I = \angle AGI = 180^\circ - \angle DGI = 180^\circ - \angle DJ'V$ therefore A, J', V are collinear.
Therefore $(J'K, J'x \parallel BC, J'A, J'R) = (DJ', VR) = G(DJ', VR) = (DJ', IK)(1)$.
Let AJ' cut (O) at Q since AD, AQ are reflections through AI, DI is the bisector of $\angle ADP$ we get $A(DJ', IK) = \dfrac{AD}{AQ} = \dfrac{DA}{DP} = \dfrac{KA}{KP} = (J'K, J'x \parallel BC, J'A, J'P)(2)$.
From $(1), (2)$ we get P, R, J' are collinear
Let ER cut DX at L. Applying Pascal theorem for $\begin{pmatrix} E & G & X \\ D & E & R \end{pmatrix}$ we get A, K, L are collinear.
Let P' be the intersection of RJ' and the tangent at X of (Ω). Also applying Pascal theorem for $\begin{pmatrix} R & X & D \\ X & J' & E \end{pmatrix}$ we get P', K, L are collinear therefore P' is the intersection of AP and RJ' therefore $P' \equiv P$ therefore PX tangent to (Ω) at X therefore P, X, M are collinear, P, Y, N are collinear therefore from the claim we will prove $P(DA, YX) = \dfrac{UB}{UC}(*)$.
Let XY cut EF at Z we get Z is the polar of AP wrt (Ω) therefore $WZ \perp BC$ therefore Z lies on JJ'.
Let DZ cut (Ω) at D'. We have
$P(DA, YX) = (Wx \perp PD, WZ, WY, WX) = D(DD', YX) = (DD', YX)(3)$
$\dfrac{UB}{UC} = A(UP, BC) = A(JP, BC) = (Wx \perp AJ, WZ, WE, WF) = J(JJ', EF)(4)$.
Applying the Lemma above for E, F, X, Y, D, J' be arbitrary points lies on (Ω) with EX, FY, DJ' concurrent at K, XY cut EF at Z, ZJ', ZD cut (Ω) at J, D' we get $(JJ', EF) = (DD', YX)$. Combine with $(3), (4)$ we get $P(DA, YX) = \dfrac{UB}{UC}$ therefore $(*)$ is proved.
Hence the claim is proved. □

Back to the main problem,



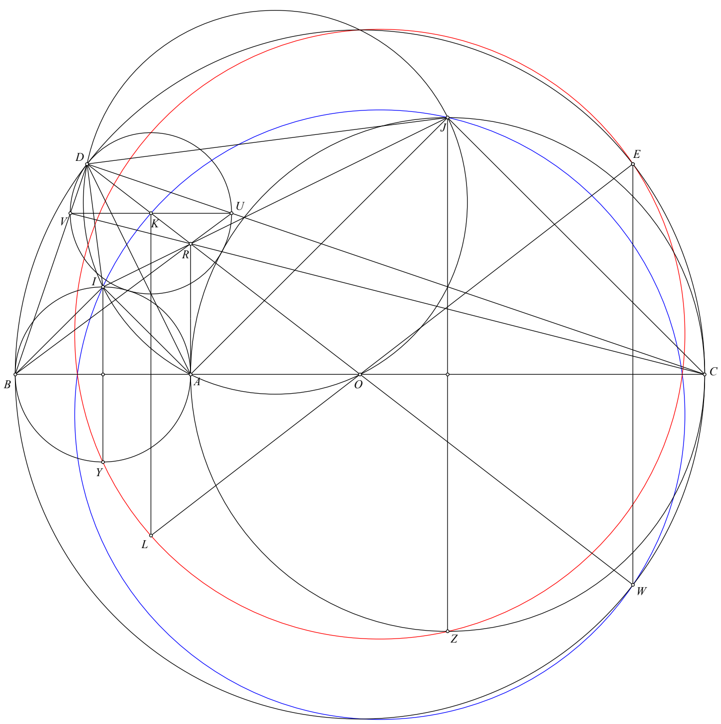
Let PM cut (O) at K, cut BC at S, PN cut (O) at L, cut BC at T.
From the claim we get $\dfrac{UB}{UC} = P(DA, NM) = P(UA, TS) = \dfrac{UT}{US}$ therefore $UB \cdot US = UC \cdot UT$ therefore U lies on the radical axis of (PBS) and (PCT) therefore let (PBS) cut (PCT) at $W' \neq P$ we get W' lies on PD.
Let BK cut CL at W we get $\angle WBD = \angle SPU = \angle W'BC$ and similiarly $\angle WCD = \angle W'CB$ therefore W, W' are isogonal conjugates wrt triangle BDC.
Therefore DW, DP are isogonal wrt $\angle BDC$ therefore D, W, A are collinear hence BK, CL, AD concurrent at W. We need to prove BN cut CM on (O) which is equivalent to $B(NADP) = C(MADP) \iff P(NADB) = P(MADC) \iff C(LA, DB) = B(KA, DC) \iff C(WA, DB) = B(WA, DC)$ which is true since W, A, D are collinear.
Hence the problem is proved.

$\square$

Problem 18. Given A lying on the segment BC, construct circles with diameters BC, AB, and AC. A circle (K) is tangent to these circles. Let Y and Z be the midpoints of arcs AB and AC respectively, not containing K on the same side as BC. (K) is tangent to the circle with diameter BC at D. Choose E on the circle with diameter BC such that $DE \parallel BC$. Prove that the reflection of K over BC lies on the circle (EYZ).

Solution(Gia Bach).

Let I, J, W be the reflection of Y, Z, E with respect to BC, respectively. Then we need to prove that I, K, J, W are concyclic. Point R lies on OD that $RD = RA$. Let BR, CR intersect DC, DB at U, V, respectively. According to the proof of **Problem 11**, we get AD is the angle bisector of $\angle BDC$, and K is the midpoint of UV.



Because $\angle ADB = \angle ADC = 45^o$, BIA and CJA are right isosceles triangles, therefore I, J is the center of (ADB) and (ADC), respectively. So we get IJ is the perpendicular bisector of AD, and I, R, J are collinear. By simple angle chasing, we get D, I, A, O are concyclic, and similarly with point J, we get D, I, A, O, J are concyclic. Therefore we get $\overline{RD} \cdot \overline{RO} = \overline{RI} \cdot \overline{RJ}$.

We have $\dfrac{DK}{DO} = \dfrac{UV}{BC} = \dfrac{RK}{RO}$, so $(DRKO) = -1$. Therefore, we get $\dfrac{KD}{KR} = \dfrac{OD}{OR} \iff \dfrac{DR}{KR} = \dfrac{RW}{OR} \iff RD \cdot RO = RK \cdot RW \iff \overline{RD} \cdot \overline{RO} = \overline{RK} \cdot \overline{RW}$. Combine with $\overline{RD} \cdot \overline{RO} = \overline{RI} \cdot \overline{RJ}$, we get $\overline{RI} \cdot \overline{RJ} = \overline{RK} \cdot \overline{RW}$. Therefore, I, K, J, W are concyclic.

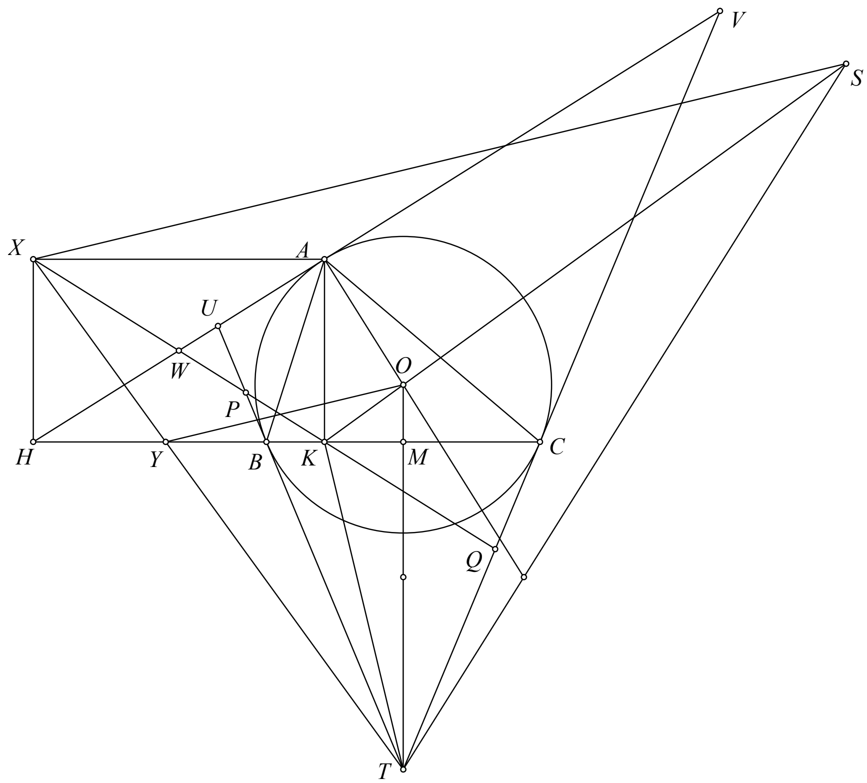
Hence, the problem is proved.

$\square$

Problem 19. Given a triangle ABC inscribed in circle (O) with diameter AD and altitude AK. The tangents to (O) at B and C intersect at T. The perpendicular bisector of OT intersects AD at P. Line TP intersects KO at S. Let J be the foot of the perpendicular from O to AT. Choose L on OT such that $JL \parallel OA$. Let Q be the intersection of LD and OJ. Prove that SQ bisects OA.

Solution(Gia Bach). First, we will prove the following lemmas.

Lemma 19.1. *Given a triangle ABC inscribed in circle (O) with the altitude AK. The tangents to (O) at B and C intersect at T, tangents to (O) at A intersect BC at H. Draw a rectangle $AHKX$. Construct point S which lies on ray OK, and ST, AO intersect at the point that lies in the perpendicular bisector of OT. Then, K is the orthocenter of triangle TSX.*



Proof. Let ray XK intersect AH, TB, TC at W, P, Q, respectively. Let ray AH intersect TB, TC at U, V.

Beacuse $AHKX$ is a rectangle, so $WA = WK \iff \angle WAK = \angle WKA \iff \angle WKB = \angle OAK$

We have simple angle chasing: $\angle UVQ = \angle AVC = 180^o$ - $\angle AOC = 180^o$ - $2\angle ABC = \angle KAB + \angle OAC =$ $\angle BAC$ - $\angle OAK = \angle TBC$ - $\angle WKB = \angle BPK = \angle WPU$, so U, P, Q, V is concyclic.

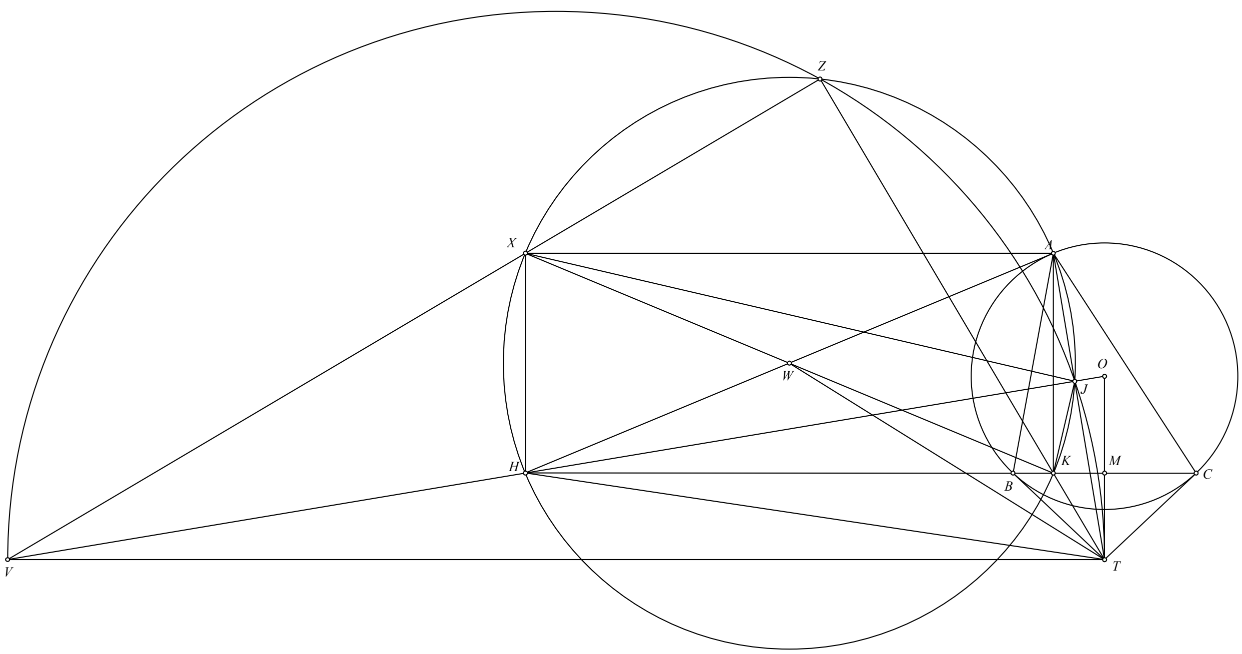
Therefore, we get $\overline{WP} \cdot \overline{WQ} = \overline{WU} \cdot \overline{WV}$. It's well-known that $T(BCAH)$ = -1; $\iff (UVAH) =$ $T(UVAH) = T(BCAH)$ = -1. Combine with W is the midpoint of AH, we get $WA^2 = \overline{WU} \cdot \overline{WV} \iff$ $WK^2 = \overline{WP} \cdot \overline{WQ}$. And since W is the midpoint of XK, we get $(PQXK)$ = -1 $\iff T(PQXK)$ = -1. Let TX intersect BC at Y. $\iff (BCYK)$ $T(BCYK) = T(PQXK)$ = -1.

Therefore, we get $MB^2 = \overline{MK} \cdot \overline{MY}$. $\iff -\overline{MO} \cdot \overline{MT} = \overline{MK} \cdot \overline{MY}$. Combine with YM is perpendicular to OT at M, we get K is the orthocenter of triangle TOY. Therefore OK is perpendicular to XT. $\iff$ $SK \perp XT$. It's easy to prove that XK is perpendiculr to ST by simple angle chasing, therefore K is the orthocenter of triangle TSX.

Hence, the lemma is proved.

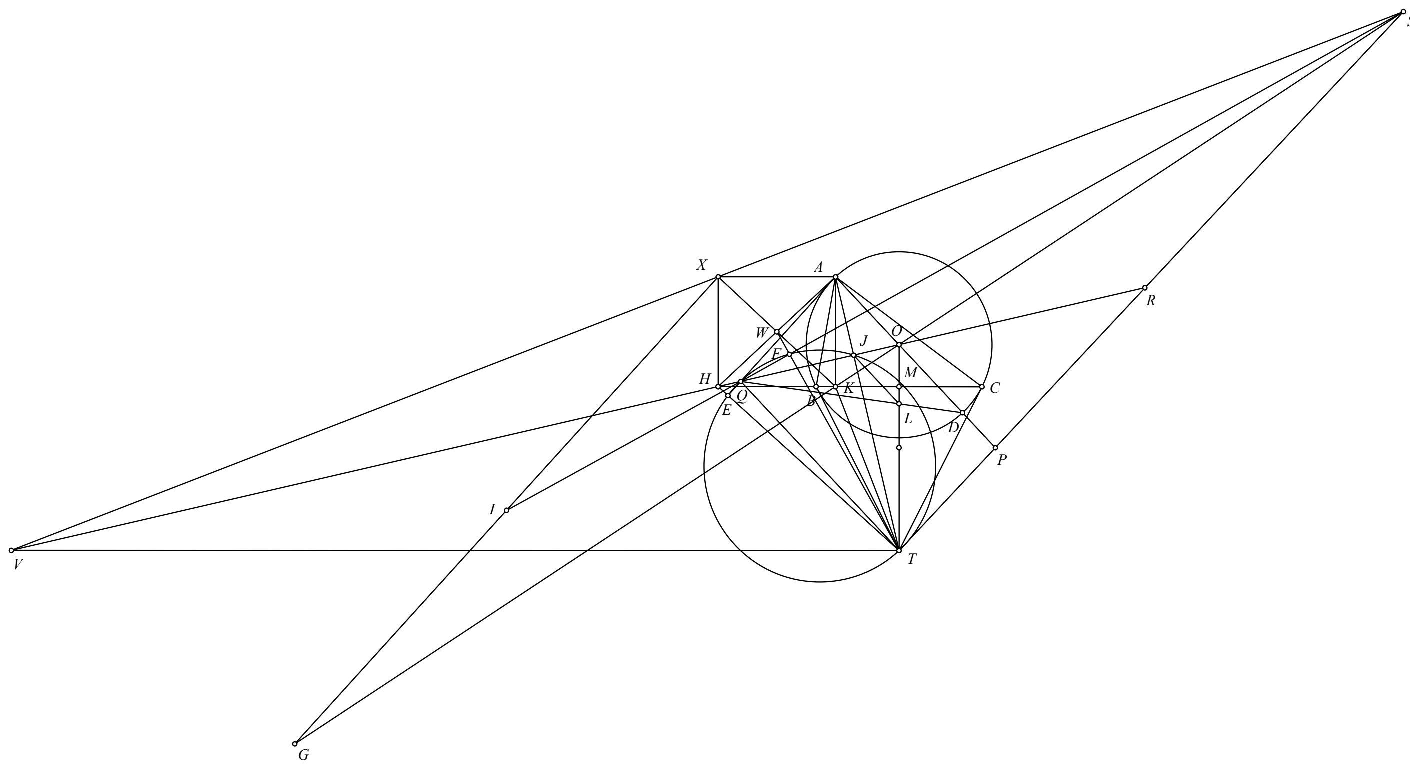
$\square$

Lemma 19.2. *Given a triangle ABC inscribed in circle (O) with the altitude AK. The tangents to (O) at B and C intersect at T, tangents to (O) at A intersect BC at H. Draw a rectangle $AHKX$. A line from X that perpendicular to TK intersect OH at V. Then, TV is parallel to BC.*



Proof. Let VX cut TK at Z, OH cut AT, ST at J, R, respectively. Since OH is perpendicular to AT, we get V, Z, J, T and $H; X; Z; J; K$ are concyclic. Therefore, we get HK is parallel to TV by *Reim*'s theorem. Hence, the lemma is proved. $\square$

Back to the main problem.



Let the tangents to (O) at A intersect BC at H. Construct a rectangle $AKHX$ with the center W. Let AQ, SQ cut TH, TW at E, F, respectively.

We have $\dfrac{QJ}{QO} = \dfrac{JL}{OD} = \dfrac{JL}{OA} = \dfrac{TJ}{TA}$, so $OA \parallel QT$, and QT is perpendicular to AH. Combine with $O; J; Q; H$ are collinear, we get Q is the orthocenter of triangle ATH, therefore $AE \perp TH$.

From X, draw a line that parallel to ST, intersect OK and SQ at G and I (**1**). Let SX intersect OH at V. Applying the lemma 1 for the problem, we get K is the orthocenter of triangle TSX $(*)$, therefore $TK \perp SX$. Combine with the lemma 2, we get TV is parallel to BC.

It's easy to prove that OT is the bisector of $\angle QTR$, and $TO \perp TV$, so $(OVQR) =$ -1. We have: $S(GXIR) = S(OVQR) = (OVQR) =$ -1. Combine with (1) we get I is the midpoint of GX. We have $SG \perp TX$ beacuse of $(*)$, therefore 2 triangles SXG and TKX have corresponding sides that perpendicular to each other. And TW, SI are two corresponding medians of two triangles, so $TW \perp SI$. Therefore, we get $E; Q; F; J; T$ lies on (QT).

Since W is the midpoint of AH, we get WE, WJ are the tangents of (QT). Therefore we get $(EJFT) =$ -1 $\iff Q(EJFT) =$ -1 $\iff Q(AOST) =$ -1. Combine with $AO \parallel ST$, we get QS pass though the midpoint of OA.

Hence, the problem is proved.

$\square$