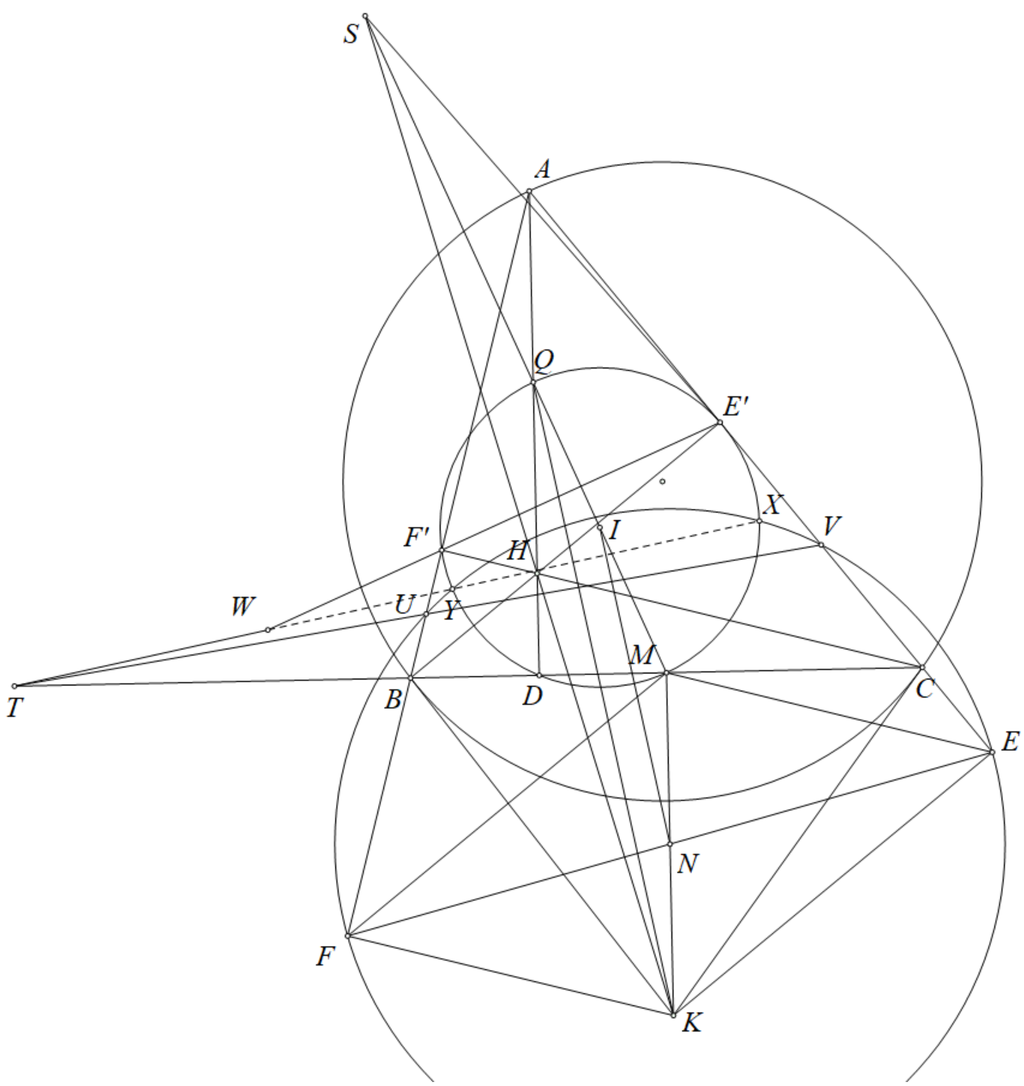
Inversion Illustrative Problem

Ha Vu Anh

Solution. By simple angle chasing: Let M be the midpoint of BC, then M is the orthocenter of $\triangle AEF$. Therefore, the midpoint N of EF is also the midpoint of MK. Let U, V be the projections of M on AB, AC, and let UV cut BC at T. Since A, M, U, D, V lie on the circle with diameter AM, we easily get that T lies on the radical axis of (EF) and the Euler circle.



Let E', F' be the projections of B, C on AC, AB, and let Q be the midpoint of AH. Let I be the midpoint of MQ. We get that I is the center of the Euler circle. Let KH cut MQ at S. We can easily prove that

$$\frac{SQ}{SM} = \frac{HQ}{MK} = \frac{E'Q^2}{E'M^2}.$$

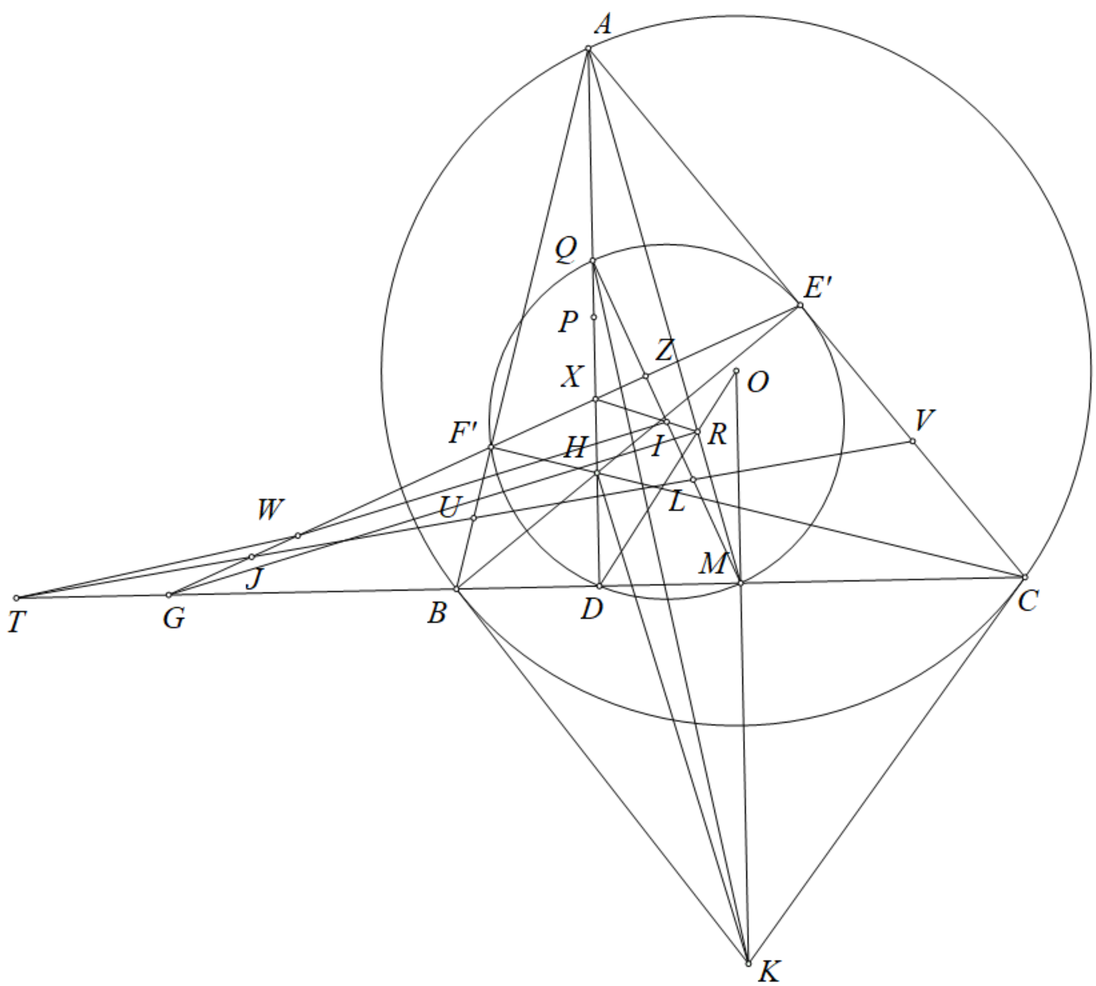
Therefore, SE' is the tangent at E' of (I), so the polar of $E'F'$ with respect to (I) lies on KH. Therefore, the polar of KH with respect to (I) lies on $E'F'$. We call it W.

Proving the problem is equivalent to proving that the pole of XY with respect to (I) lies on KH, which is equivalent to proving that the pole of KH with respect to (I), which is W, lies on XY, which is the radical axis of (EF) and (I). Since T lies on the radical axis of (EF) and (I), we need to prove $TW \perp NI$ and since $NI \parallel KQ$, we will prove $TW \perp KQ$.

Here, we have reduced the problem to proving that two lines are perpendicular, so we can consider handling it using ratio chasing.

Let $E'F'$ cut BC at G. Since $TG \perp MK$ and $GW \perp MQ$, to prove $TW \perp KQ$, we will prove $\triangle GWT \sim \triangle MQK$, or equivalently, we will prove

$$\frac{GW}{GT} = \frac{MQ}{MK} \quad (*).$$



Let Z be the midpoint of $E'F'$. Since $UZVM$ is a parallelogram, UV bisects MZ. Let UV cut $E'F'$ at J. Applying Menelaus' theorem to $\triangle ZGM$ with $\overline{T, J, L}$, we get

$$\frac{TG}{TM} = \frac{JG}{JZ}.$$

Since $UV \perp AK$, by angle chasing, we get $\triangle LZJ \sim \triangle MDA$. Let P be the midpoint of AD, then $\triangle MZJ \sim \triangle MDP$. Let AH cut $E'F'$ at X, we get $(AX, HD) = -1$. Then $\triangle MZG \sim \triangle MDQ$, therefore

$$\frac{DP}{DQ} = \frac{ZJ}{ZG}.$$

So,

$$\frac{TG}{TM} = \frac{JG}{JZ} = \frac{PQ}{PD} = \frac{PQ}{PA} \Longleftrightarrow \frac{GT}{GM} = \frac{QP}{QA} = \frac{HD}{2QA} \quad (1).$$

A well-known lemma states that: Let AM cut OD at R, then R lies on the radical axis of (BOC) and (I). Since we can easily prove G lies on the radical axis of these two circles, GR is perpendicular to the line connecting centers of these two circles that are the midpoint of OH, OK, which is parallel to HK, we get that $KH \perp GR$, and so $IW \parallel GR$. Let AH cut $E'F'$ at X, we get $(AX, HD) = -1$. Applying Desargues' theorem for $\triangle ODQ$ and $\triangle MAH$ with $OM \parallel AD \parallel HQ$, we get X, I, R are collinear. Therefore,

$$\frac{GW}{GX} = \frac{RI}{RX}.$$

Applying Menelaus' theorem for $\triangle QXI$ with $\overline{A, R, M}$, we get

$$\frac{RI}{RX} \cdot \frac{AX}{AQ} \cdot \frac{MQ}{MI} = 1.$$

This implies

$$\frac{GW}{GX} = \frac{RI}{RX} = \frac{AQ}{2AX} = \frac{DQ}{2DA} \quad (2).$$

From (1) and (2), combined, we get

$$\frac{GW}{GT} = \frac{GW}{GX} \cdot \frac{GM}{GT} \cdot \frac{GX}{GM} = \frac{DQ}{2DA} \cdot \frac{2QA}{HD} \cdot \frac{XD}{ZM}.$$

Therefore, from $(*)$, we need to prove

$$\frac{MQ}{MK} = \frac{GW}{GT} = \frac{QA \cdot QD}{DA \cdot DH} \cdot \frac{XD}{ZM} = \frac{QA \cdot QD}{DX \cdot DQ} \cdot \frac{XD}{ZM} = \frac{QA}{ZM}.$$

This is true since

$$\frac{QA}{QM} = \frac{QZ}{QA} = \frac{MZ}{MK}.$$

Hence, the problem is solved.