Problem 4

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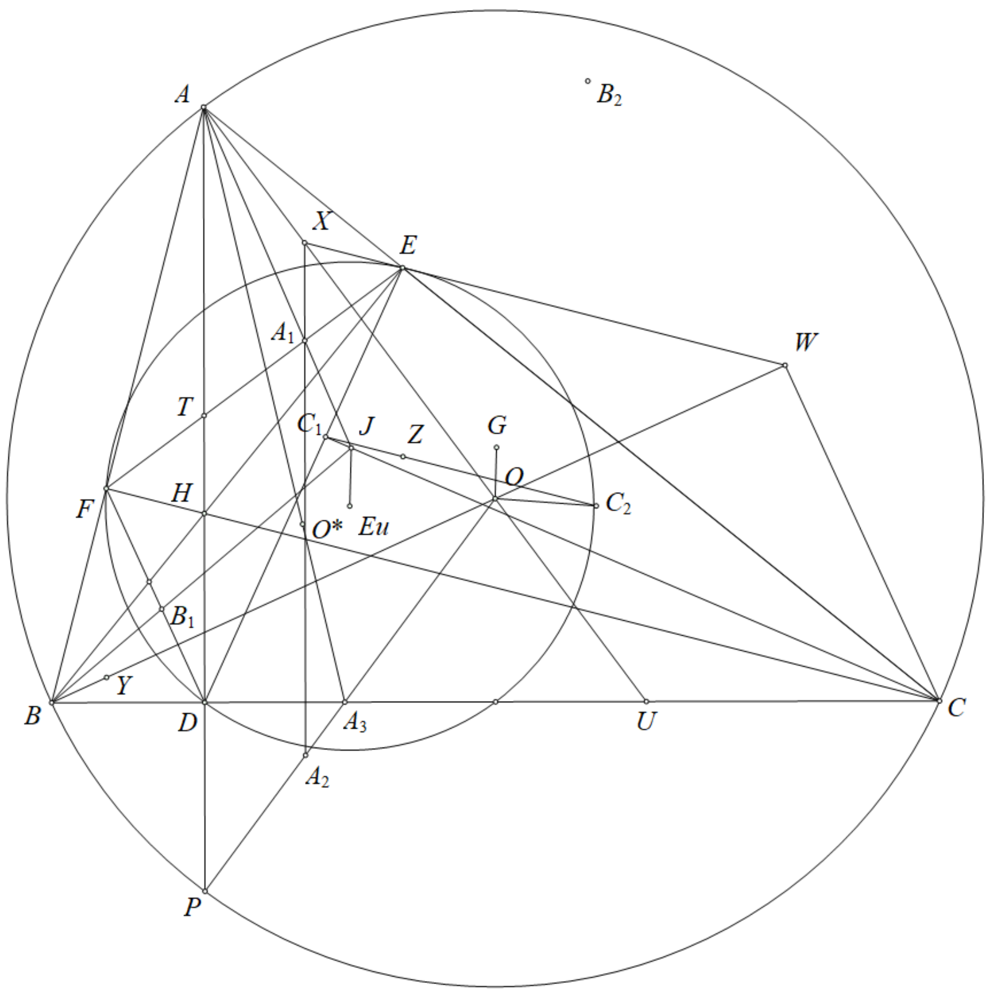
Since $XE \parallel FH$ and $XF \parallel HE$, the quadrilateral $XFHE$ is a parallelogram.
Let AD intersect EF at T, then $FA_3 = EA_1$.
Let AO and OP meet at U and A_3 respectively. By simple angle chasing, we find that A_3 and U are reflections with respect to the midpoint of BC.
Hence, $\triangle ABCU(A_1, T) \sim \triangle AEFU(A_3, U)$, and therefore AA_3 and AA_1 are isogonal with respect to $\angle BAC$.



Define B_3 and C_3 similarly. Then AA_3, BB_3, CC_3 are concurrent at the isotomic conjugate of O with respect to $\triangle ABC$, denoted as O^*.
Hence, AA_1, BB_1, CC_1 are concurrent at J, which is the isogonal conjugate of O^* with respect to $\triangle ABC$.

Lemma. Let $I(x, y, z)$ be the barycentric coordinates of a point I with respect to $\triangle ABC$, and let a, b, c be the side lengths opposite A, B, C, respectively.

Then, if I' and I^* are the isogonal and isotomic conjugates of I with respect to $\triangle ABC$, their barycentric coordinates are $I^*\left(\dfrac{1}{x},\dfrac{1}{y},\dfrac{1}{z}\right)$ and $I'\left(\dfrac{a^2}{x},\dfrac{b^2}{y},\dfrac{c^2}{z}\right)$, respectively.

Back to the main problem, let $O(x,y,z)$ be the barycentric coordinates of O with respect to $\triangle ABC$.
Applying the lemma to O, we get $O^*\left(\dfrac{1}{x},\dfrac{1}{y},\dfrac{1}{z}\right)$.
Applying the lemma again to O^*, we get $J(a^2x,b^2y,c^2z)$.

Hence, $a^2x\,\overrightarrow{JA}+b^2y\,\overrightarrow{JB}+c^2z\,\overrightarrow{JC}=\overrightarrow{0}$.
Since $\overrightarrow{E_uJ}=\overrightarrow{AJ}+\overrightarrow{E_uA}$, we have $a^2x\,\overrightarrow{E_uJ}=a^2x\,\overrightarrow{E_uA}+a^2x\,\overrightarrow{AJ}$.
Therefore, $(a^2x+b^2y+c^2z)\,\overrightarrow{E_uJ}=\sum a^2x\,\overrightarrow{E_uA}+\sum a^2x\,\overrightarrow{AJ}=\sum a^2x\,\overrightarrow{E_uA}$. $\quad(*)$
It is well known that

$$(x,y,z)=\big(a^2(b^2+c^2-a^2),\ b^2(a^2+c^2-b^2),\ c^2(a^2+b^2-c^2)\big).$$

Hence, $(*)$ is equivalent to

$$(a^2x+b^2y+c^2z)\cdot\overrightarrow{E_uJ}=\sum a^4(b^2+c^2-a^2)\cdot\overrightarrow{E_uA}.$$

By simple angle chasing, we have $OA_2=OX$ and $OA_2\parallel DE_u$.
Let W be the projection of C on BO. We get that B,W,E,C lie on the circle with diameter BC, therefore $\angle WEC=\angle OBC=\angle AEX$, hence W,E,X are collinear.
Thus, $\angle OXW=\angle OXE=\angle AFE=\angle ACB=\angle EWO=\angle XWO$.

Hence

$$OA_2=OX=OW=\cos(\angle WOC)\cdot OC=-\cos(\angle BOC)\cdot R.$$

Therefore,

$$\overrightarrow{OA_2}=\frac{R}{E_uD}\cdot\big(-\cos(\angle BOC)\big)\cdot\overrightarrow{E_uD}=2\cdot\big(-\cos(\angle BOC)\big)\cdot\overrightarrow{E_uD}.$$

We will denote the angles of triangle ABC as $\angle A,\angle B,\angle C$.
We have

$$\cos(\angle BOC)=\cos(2A)=2\cos^2A-1=2\left(\frac{b^2+c^2-a^2}{2bc}\right)^2-1,$$

so

$$\cos(\angle BOC)=\frac{(b^2+c^2-a^2)^2-2b^2c^2}{2b^2c^2}=\frac{a^4+b^4+c^4-2a^2b^2-2a^2c^2}{2b^2c^2}.$$

Also, since $\overrightarrow{EuD}=\dfrac{CD}{CB}\cdot\overrightarrow{EuB}+\dfrac{BD}{BC}\cdot\overrightarrow{EuC}$
$=\dfrac{\cos C\cdot b}{a}\cdot\overrightarrow{E_uB}+\dfrac{\cos B\cdot c}{a}\cdot\overrightarrow{E_uC}$
We get

$$\overrightarrow{E_uD}=\frac{CD}{CB}\cdot\overrightarrow{E_uB}+\frac{BD}{BC}\cdot\overrightarrow{E_uC}=\frac{\cos C\cdot b}{a}\cdot\overrightarrow{E_uB}+\frac{\cos B\cdot c}{a}\cdot\overrightarrow{E_uC}.$$

From (1) and (3), we get We have

$$3\overrightarrow{OG}=\overrightarrow{OA_2}+\overrightarrow{OB_2}+\overrightarrow{OC_2}=\sum\overrightarrow{OA_2}.$$

$$3\overrightarrow{OG}=-2\cdot\sum\left(\frac{a^4+b^4+c^4-2a^2b^2-2a^2c^2}{2b^2c^2}\right)\left(\frac{\cos C\cdot b}{a}\overrightarrow{E_uB}+\frac{\cos B\cdot c}{a}\overrightarrow{E_uC}\right).$$

Expanding,

$$3\overrightarrow{OG}=-2\left[\sum\left(\frac{a^4+b^4+c^4-2a^2b^2-2a^2c^2}{2b^2c^2}\cdot\frac{\cos C\cdot b}{a}\right)\overrightarrow{E_uB}+\sum\left(\frac{a^4+b^4+c^4-2a^2b^2-2a^2c^2}{2b^2c^2}\cdot\frac{\cos B\cdot c}{a}\right)\overrightarrow{E_uC}\right].$$

Since

$$\frac{\cos C \cdot b}{a} = \frac{(a^2+b^2-c^2)b}{2aba} = \frac{a^2+b^2-c^2}{2a^2},$$

we get

$$3\overrightarrow{OG} = -2\sum\left(\frac{(a^4+b^4+c^4-2a^2b^2-2a^2c^2)(a^2+b^2-c^2)}{4a^2b^2c^2}\right)\overrightarrow{E_uB}$$
$$-2\sum\left(\frac{(a^4+b^4+c^4-2a^2b^2-2a^2c^2)(a^2+c^2-b^2)}{4a^2b^2c^2}\right)\overrightarrow{E_uC}.$$

Hence,

$$3\overrightarrow{OG} = -2\sum\left(\frac{(a^4+b^4+c^4-2b^2a^2-2b^2c^2)(a^2+b^2-c^2)+(a^4+b^4+c^4-2c^2a^2-2c^2b^2)(c^2+a^2-b^2)}{4a^2b^2c^2}\right)\overrightarrow{E_uA}.$$

Simplifying,

$$3\overrightarrow{OG} = \frac{-2}{4a^2b^2c^2}\sum\left[(a^4+b^4+c^4-2b^2a^2-2b^2c^2)(a^2+b^2-c^2)+(a^4+b^4+c^4-2c^2a^2-2c^2b^2)(c^2+a^2-b^2)\right]\overrightarrow{E_uA}.$$

Therefore,

$$3\overrightarrow{OG} = \frac{-1}{2a^2b^2c^2}\sum\left(2a^4(a^2-b^2-c^2)\right)\overrightarrow{E_uA} = \frac{1}{a^2b^2c^2}\sum\left(a^4(b^2+c^2-a^2)\right)\overrightarrow{E_uA}.$$

Considering (1) and (4), let

$$m = a^2x+b^2y+c^2z, \quad n = \frac{1}{a^2b^2c^2}, \quad \overrightarrow{W} = \sum\left(a^4(b^2+c^2-a^2)\right)\overrightarrow{E_uA}.$$

Then (1) is equivalent to

$$m\overrightarrow{E_uJ} = \overrightarrow{W},$$

and (4) is equivalent to

$$3\overrightarrow{OG} = n\overrightarrow{W} = mn\overrightarrow{E_uJ}.$$

Since m, n are real numbers, we get $\overrightarrow{OG} \parallel \overrightarrow{E_uJ}$ as desired.
Hence, the problem is proven