Solution for a hard concurrency problem involving Lemoine point

Ha Vu Anh

7/1/2023

1 Problem Statement

Problem.(Source: Mr. Le Xuan Hoang) Let triangle $\triangle ABC$ be inscribed in (O). M, N, P are respectively the midpoints of arcs BC, CA, AB not containing A, B, C. AO intersects NP at X. Define Y, Z similarly.

a) Prove that MX, NY, PZ are concurrent at K.

b) Prove that OK passes through the Lemoine point of $\triangle ABC$.

2 Solution

Lemma 1: In triangle $\triangle ABC$ with orthocenter H, let H' be the isotomic conjugate of H with respect to ABC, and let N_a, G_e be the Nagel point and Gergonne point of $\triangle ABC$, respectively.

1.1: Prove that $\overline{H', N_a, G_e}$ (Figure 1.1).

Let BH, CH intersect AC, AB at X, Y; BH', CH' intersect AC, AB at X', Y'; BN_a, CN_a intersect AC, AB at E', F', respectively; and let (I) be the incircle of ABC, tangent to AC, AB at E, F.

We need to prove that $\overline{H', N_a, G_e} \Leftrightarrow B(C, N_a, H', G_e) = C(B, N_a, H', G_e)$

$\Leftrightarrow (C, E', X', E) = (B, F', Y', F) \Leftrightarrow (A, E, X, E') = (A, F, Y, F')$.

Let J_B be the excenter opposite B of $\triangle ABC$. We have $E(J_B I, XA) = -1$. Since $IE \parallel BX$, line EJ_B bisects BX at U. Let BH intersect AJ_B at V. Then

$$(A, E, X, E') = J_B(A, E, X, E') = J_B(V, U, X, E') = \frac{XV}{XU} = \frac{XV}{XA} \cdot \frac{XA}{XU} = 2\cot\frac{A}{2} \cdot \cot A.$$

By a similar argument, we obtain

$$(A, E, X, E') = (A, F, Y, F') = 2\cot\frac{A}{2} \cdot \cot A. \quad \text{(as desired)}$$

Hence the lemma is proved.

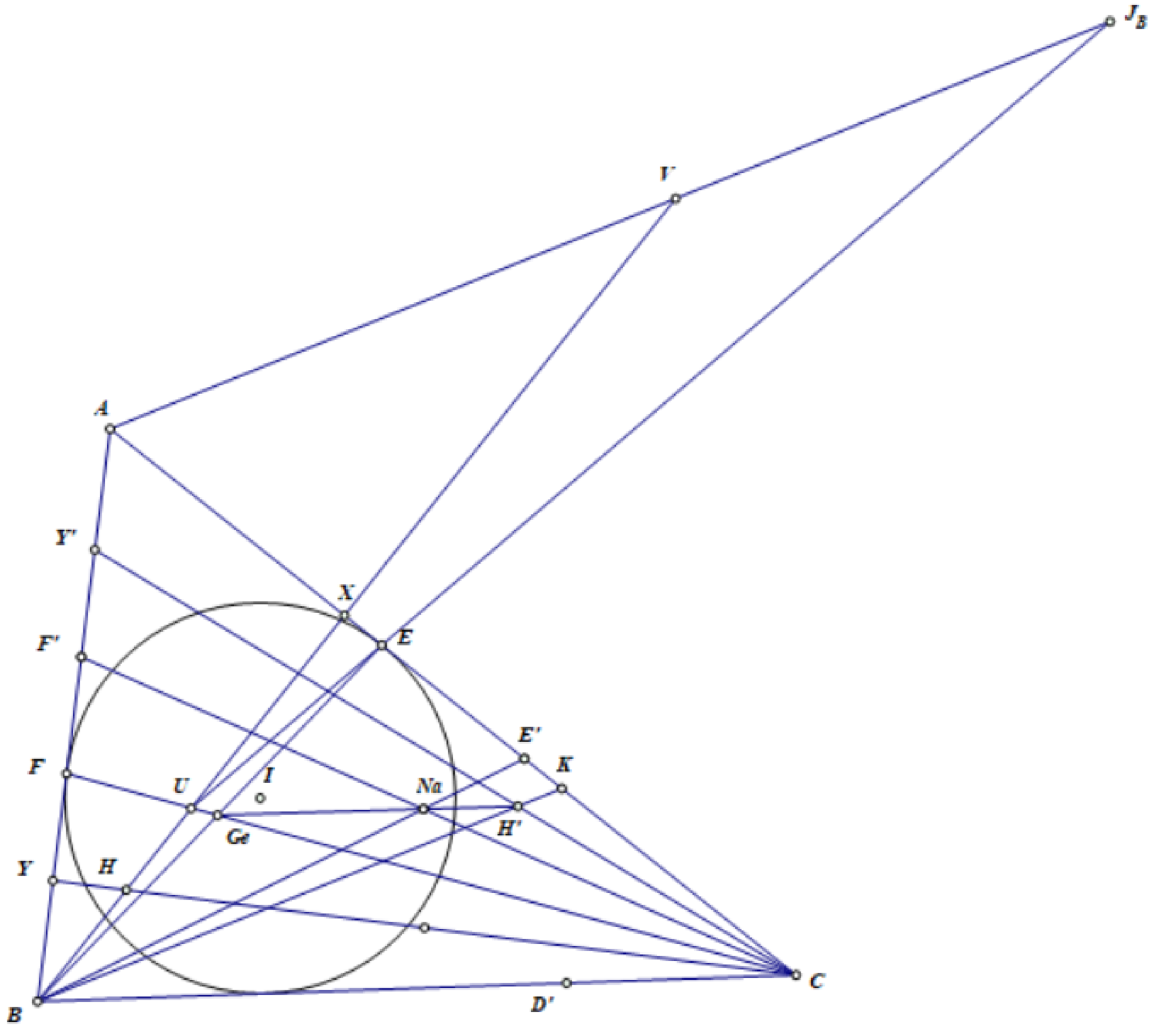

Figure 1: Figure 1.1

1.2: Let J_A, J_B, J_C be the excenters opposite to A, B, C of triangle ABC.
Find the ratio $\dfrac{H'N_a}{H'G_e}$ in terms of triangle $J_AJ_BJ_C$ (Figure 1.2).
Let AG_e, AN_a, AH', AH intersect BC at D, D', L, X, respectively. Let BG_e, BN_a intersect AC at E, E', respectively. J_AD bisects AX at N (as proved above). Denote a, b, c as the side lengths of triangle $J_AJ_BJ_C$.
We have $\dfrac{DX}{D'X} = \dfrac{IA}{AJ_A}$, $\dfrac{J_AD'}{XN} = \dfrac{2J_AD'}{XA}$, and $\dfrac{N_aA}{N_aD'} = \dfrac{E'A}{E'C} \cdot \dfrac{BC}{BD'} = \dfrac{BC}{E'C} = \dfrac{2a}{b+c-a}$
$\dfrac{AN_a}{AD'} = \dfrac{2a}{a+b+c} = \dfrac{2S}{(a+b+c)h_A} = \dfrac{2r}{h_A} = \dfrac{ID}{AN} = \dfrac{J_AI}{J_AA}$.
Since G_e is the Lemoine point of triangle DEF, let L be the Lemoine point of $\triangle ABC$, and let T be the intersection of the tangents to (ABC) at B and C. Then $\triangle ABC \cup L \sim \triangle DEF \cup G_e$, hence $\dfrac{AG_e}{AD} = \dfrac{SL}{SJ_A}$.
We have $\dfrac{\sin \angle H'AN_a}{\sin \angle H'AG_e} = \dfrac{\sin \angle LAD'}{\sin \angle LAD} \iff \dfrac{H'N_a}{H'G_e} \cdot \dfrac{AG_e}{AN_a} = \dfrac{AD}{AD'} \cdot \dfrac{LD'}{LD} \iff \dfrac{H'N_a}{H'G_e} = \dfrac{XD}{XD'} \cdot$
$\dfrac{AD}{AG_e} \cdot \dfrac{AN_a}{AD'} = \dfrac{AI}{AJ_A} \cdot \dfrac{IJ_A}{AJ_A} \cdot \dfrac{SJ_A}{SL} = \dfrac{IJ_A \cdot IA}{J_AA^2} \cdot \dfrac{SJ_A}{SL}$.

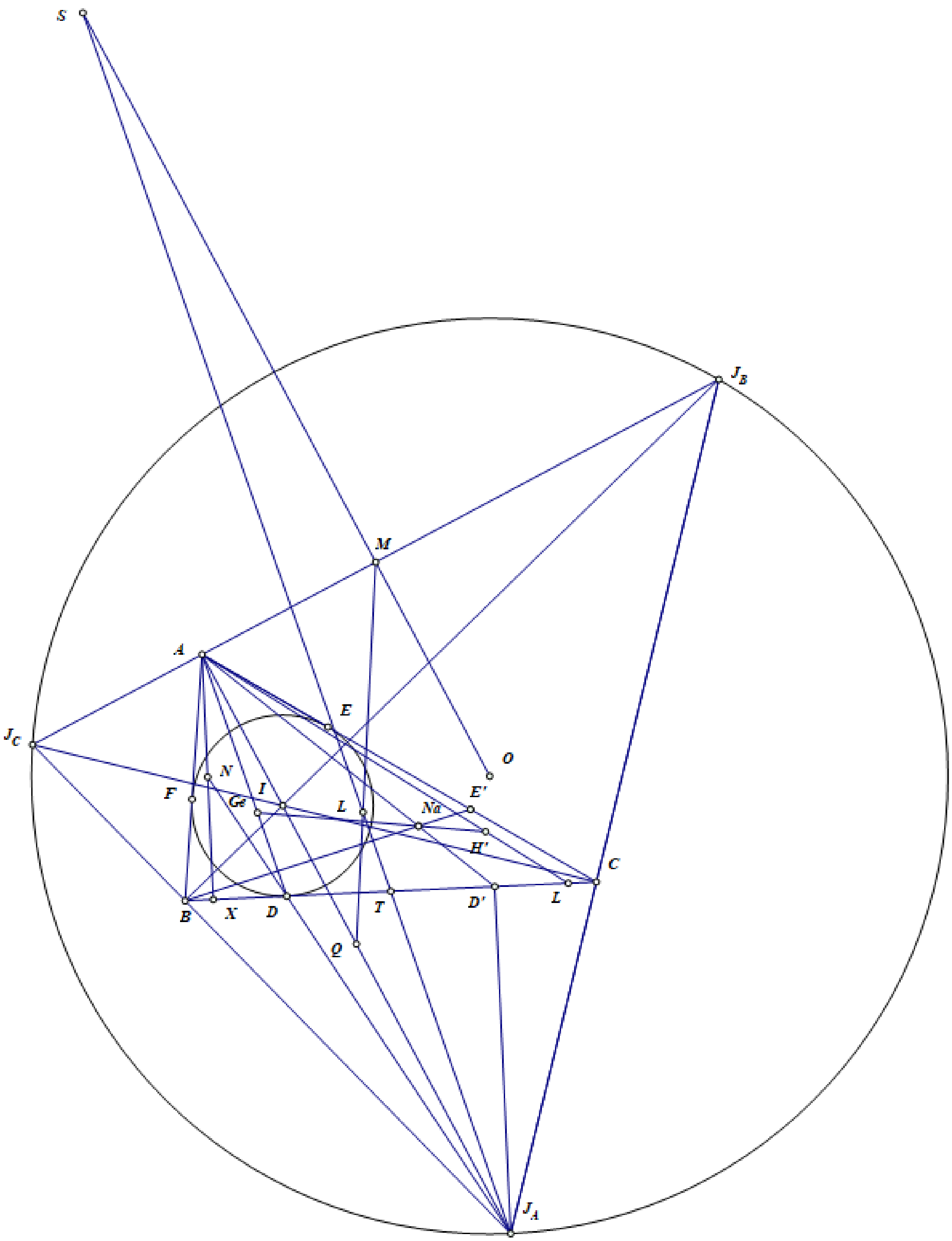

Figure 2: Figure 1.2

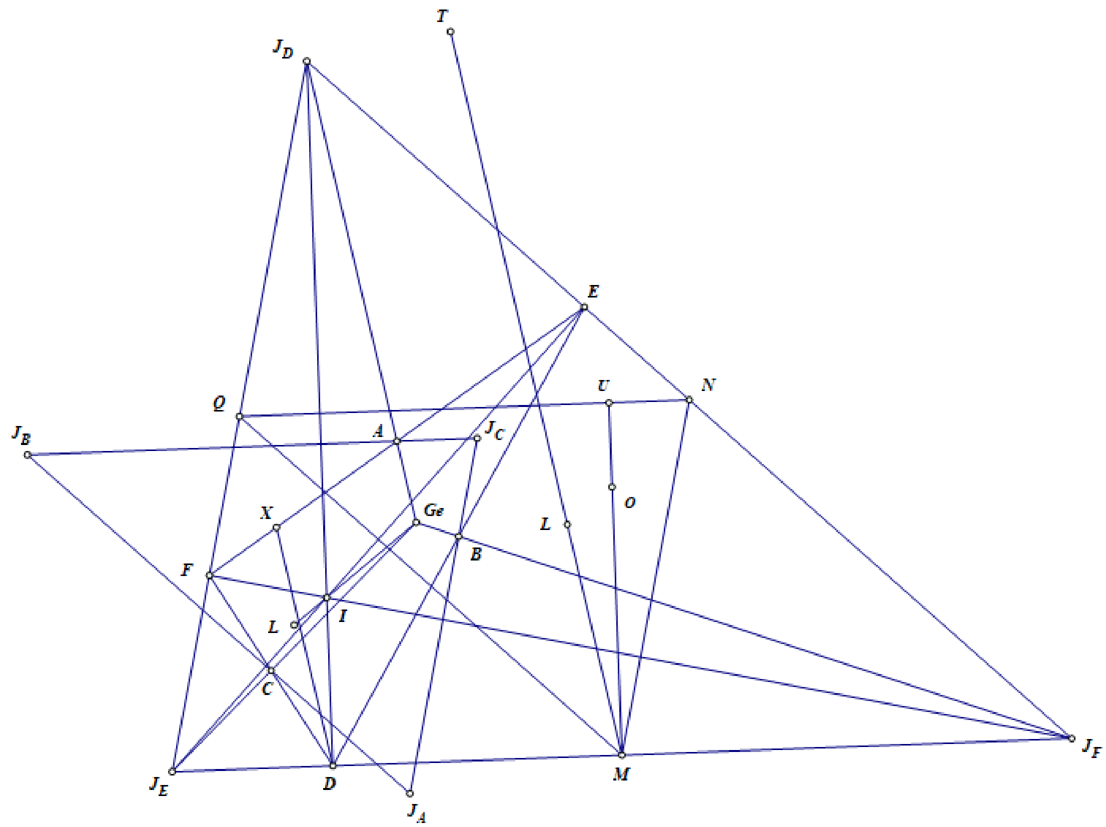

Figure 3: Figure 1.3

1.3 (Figure 1.3) Construct $\triangle DEF$ as the anti-medial triangle of $\triangle ABC$. Let J_D, J_E, J_F be the excenters opposite D, E, F of triangle DEF. Let I be the orthocenter of $\triangle J_D J_E J_F$, which is also the incenter of $\triangle DEF$.
Let (I) be tangent to EF at X. Then AI bisects DX, and since $A(J_D I, XA) = -1$, we have $DX \parallel AJ_D$.
Let G_e' be the Gergonne point of $\triangle DEF$. Because triangles ABC and DEF have corresponding sides parallel, we get $AG_e' \parallel DG_e$, implying that AJ_D passes through G_e. Similarly, G_e is the intersection of AJ_D, BJ_E, CJ_F.
Since AJ_D is the symmedian from vertex A of $\triangle ABC$, G_e is the Lemoine point of $\triangle J_D J_E J_F$.
Let L be the Lemoine point of $\triangle DEF$. Then L is the isotomic conjugate of the orthocenter of $\triangle ABC$, and since I is the Nagel point of $\triangle ABC$, by Lemma 1.1 we have $\overline{I, G_e, L}$.
Let MNQ be the medial triangle of $\triangle J_D J_E J_F$, and let O be the orthocenter of $\triangle MNQ$. Let MO intersect NQ at U. Let L be the Lemoine point of $\triangle MNQ$, and let T be the intersection of the tangents to (MNQ) at N and Q. Since $\triangle J_A J_B J_C \sim \triangle MNQ$, applying Lemma 1.2 gives

$$\frac{LI}{LG_e} = \frac{OM \cdot OU}{MU^2} \cdot \frac{TL}{TM}.$$

Lemma 2: $\triangle ABC$ is inscribed in (O) with orthocenter H. Let H' and O' be the isotomic conjugates of H and O with respect to $\triangle ABC$.

2.1: $\overline{H, H', O'}$ (Figure 2.1)

Let XYZ be the anti-medial triangle of $\triangle ABC$. Draw the altitudes BE and CF of (O), which intersect at the orthocenter J of $\triangle ABC$. Since CH' and BH' bisect ZF and YB at

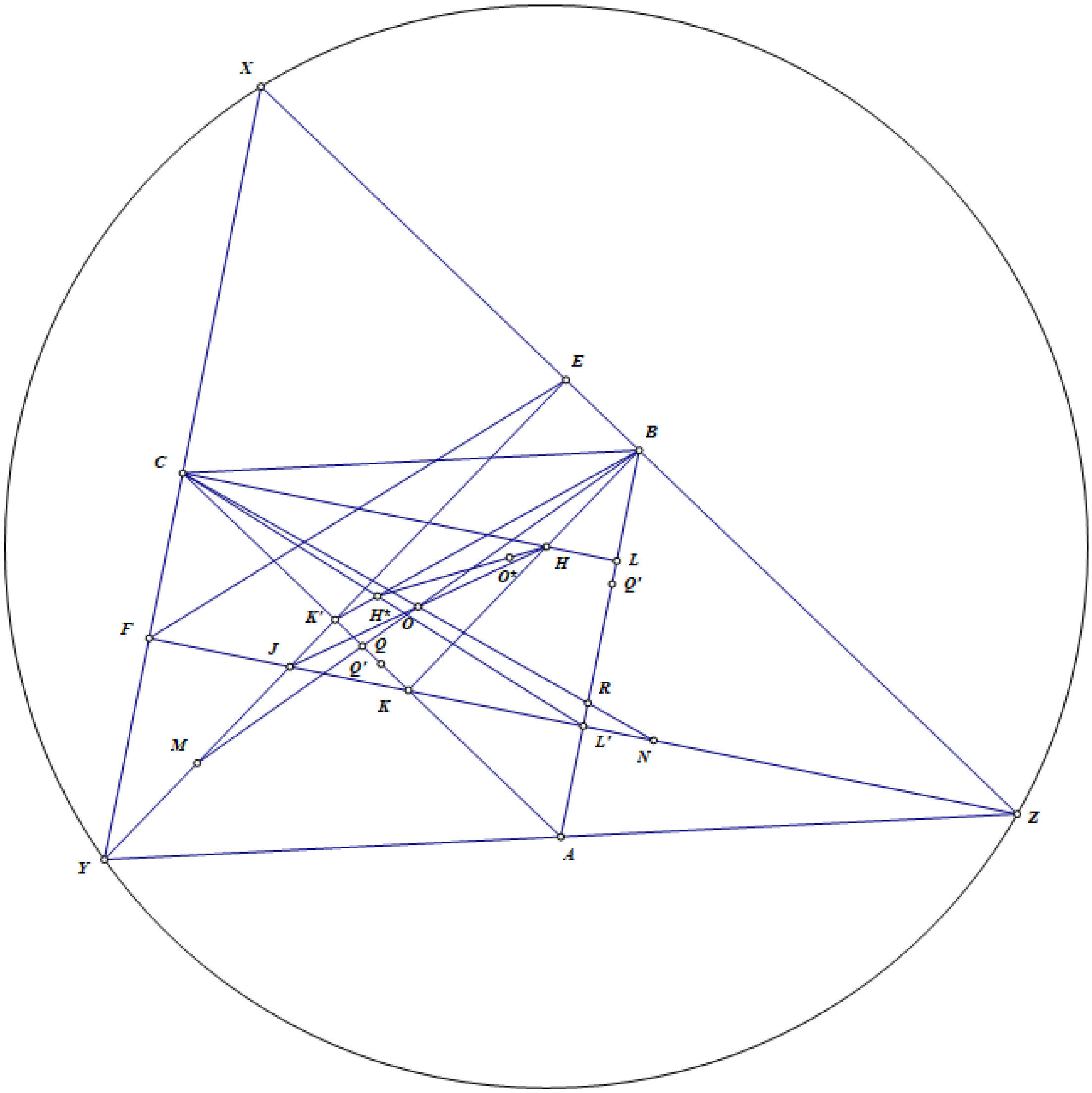

Figure 4: Figure 2.1

L' and K' respectively, H' is the Lemoine point of $\triangle ABC$.

Let BH, CH intersect AC, AB at K, L; BO', CO' intersect AC, AB at Q', R'; and BO, CO intersect AC, AB at Q, R.

We need to prove that $\overline{H', H, O'} \iff B(C, H, O', H') = C(B, H, O', H') \iff (C, K, Q', K') = (B, L, R', L') \iff (A, K', Q, K) = (A, L', R, L)$.

Since O is the Euler center of $\triangle XYZ$, CO and BO bisect BJ and CJ at M and N respectively. Hence,

$$B(A, K', Q, K) = B(BA \cap YE, K', M, K) = \frac{M(BA \cap YE)}{MK'} = A(B, K', M, N)$$

(because $AN \parallel YE$, $AM \parallel ZF$),

$$= A(L', K, M, N) = \frac{NK}{NL'} = C(K, L', N, L) = C(A, L', R, L),$$

which completes the proof. (Q.E.D.)

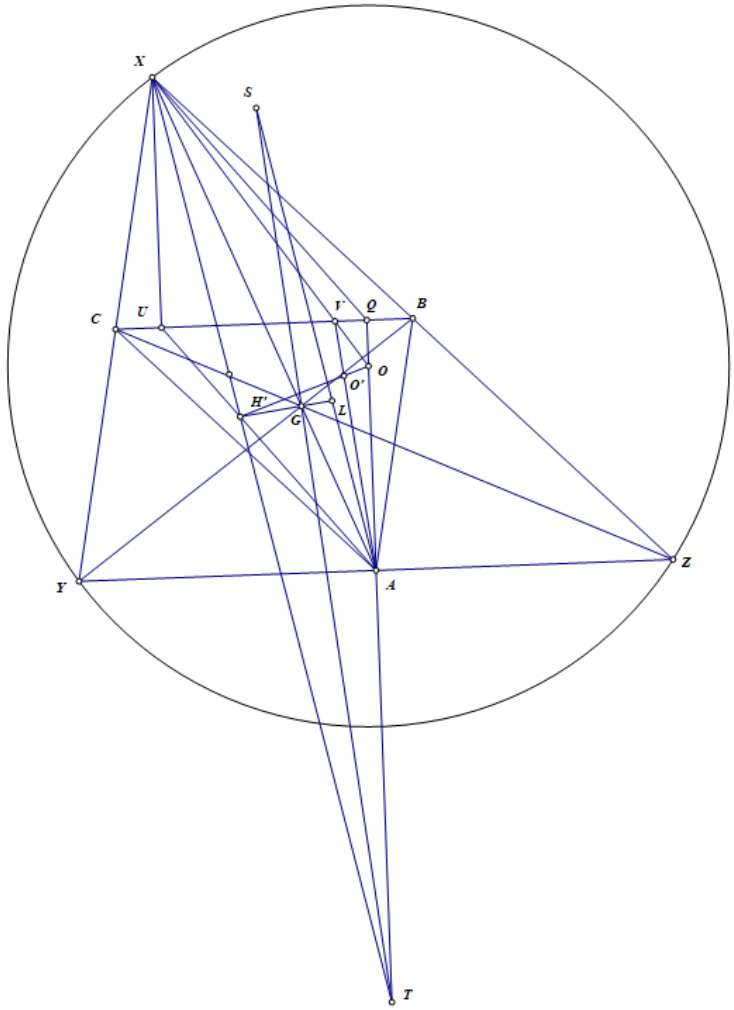

Figure 5: Figure 2.2

2.2: Compute the ratio $\dfrac{O'O}{O'H'}$ in terms of triangle XYZ. (Figure 2.2)

Let T be the intersection of the tangents to (XYZ) at Y and Z. Then $\overline{X, H', T}$ (since H' is the Lemoine point of $\triangle XYZ$, as proven above). Let XO intersect BC at V, giving $\overline{A, O', V}$. We have $\dfrac{VQ}{VU} = \dfrac{OQ}{AU} = \dfrac{QO}{QA}$.

Let S be the intersection of the tangents to (ABC) at B and C, and let L be the Lemoine point of $\triangle ABC$. Since the homothety centered at G with ratio $1/2$ transforms $\triangle ABC$ into $\triangle XYZ$, we have $\triangle ABC \cup \{L, S\} \sim \triangle XYZ \cup \{H', T\}$, thus $\dfrac{AH'}{AU} = \dfrac{TH'}{TX} = \dfrac{SL}{SA}$.

Moreover,

$$\frac{\sin \angle O'AO}{\sin \angle O'AH'} = \frac{\sin \angle VAQ}{\sin \angle VAU} \iff \frac{O'O}{O'H'} \cdot \frac{AH'}{AO} = \frac{VQ}{VU} \cdot \frac{AU}{AQ}$$

$$\iff \frac{O'O}{O'H'} = \frac{QO}{QA} \cdot \frac{AU}{AH'} \cdot \frac{AO}{AQ} = \frac{SA}{SL} \cdot \frac{OQ \cdot OA}{AQ^2}.$$

3. Returning to the main problem: (Figure 3)

a) MX, NY, PZ are concurrent at the isotomic conjugate of O with respect to $\triangle MNP$.

b) Let I be the incenter of $\triangle ABC$, and let DEF be the cevian triangle of I with respect to $\triangle MNP$. Then I is the orthocenter of $\triangle MNP$, with D, E, F being the feet of the altitudes. Let XYZ be the medial triangle of $\triangle MNP$, and let O' be the isotomic conjugate of the circumcenter of $\triangle XYZ$. Denote L, J, L' as the Lemoine points of $\triangle ABC$, $\triangle DEF$, and $\triangle MNP$, respectively.

By applying (1.3) and (2.2), we obtain $\overline{L, I, L'}$, $\overline{O, O', L'}$, and

$$\frac{JI}{JL'} = \frac{O'O}{O'L'}.$$

Since $\overrightarrow{KI} = 2 \cdot \overrightarrow{OO'}$, we have $KI \parallel OL'$ (1).

Also,

$$\frac{JI}{JL'} = \frac{KI/2}{OL' - KI/2} \iff \frac{LJ}{LL'} = \frac{KI/2}{OL'} \iff \frac{LI}{LL'} = \frac{KI}{OL'} \quad (2).$$

From (1) and (2), it follows that OK passes through L. (Q.E.D.)

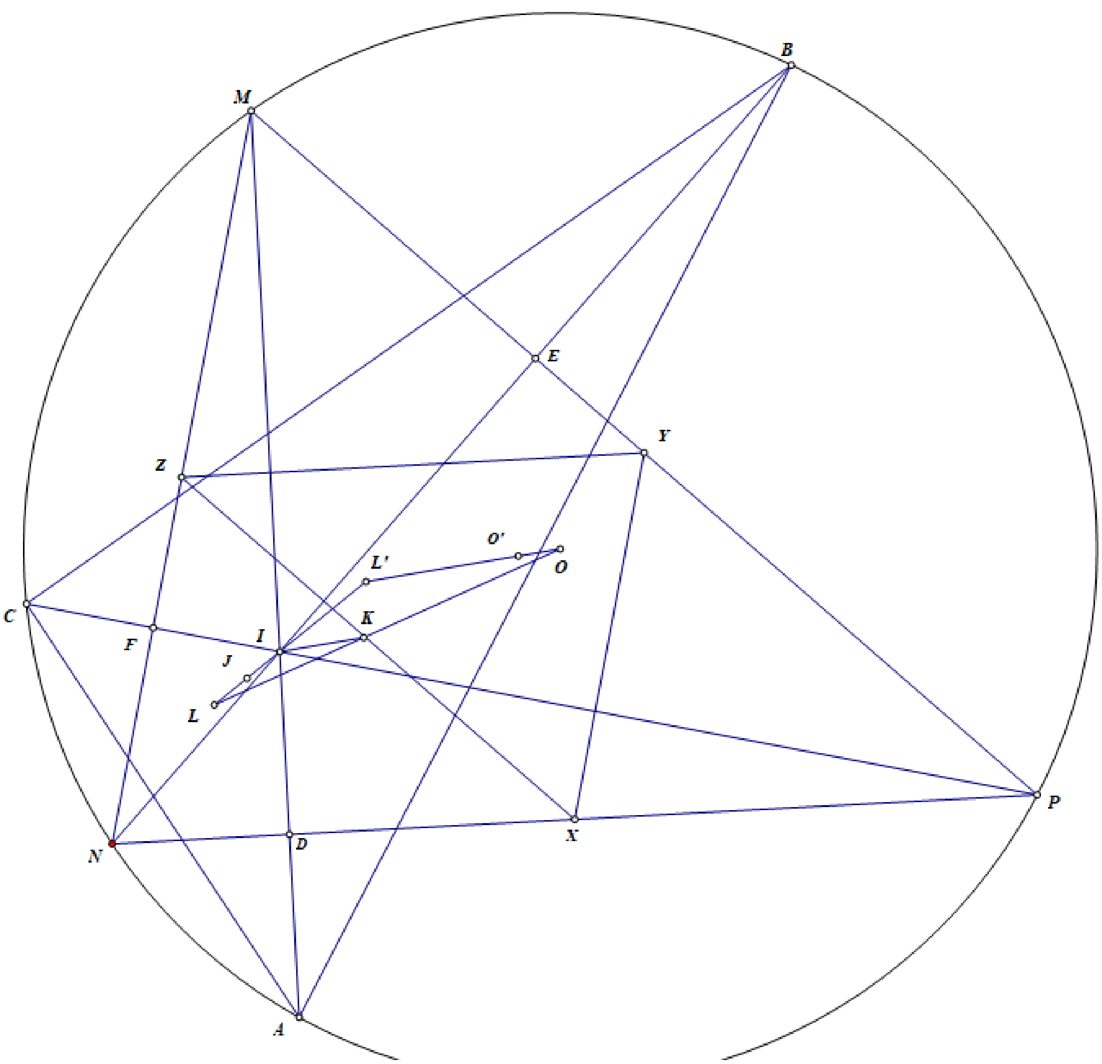

Figure 6: Figure 3