Problem 13

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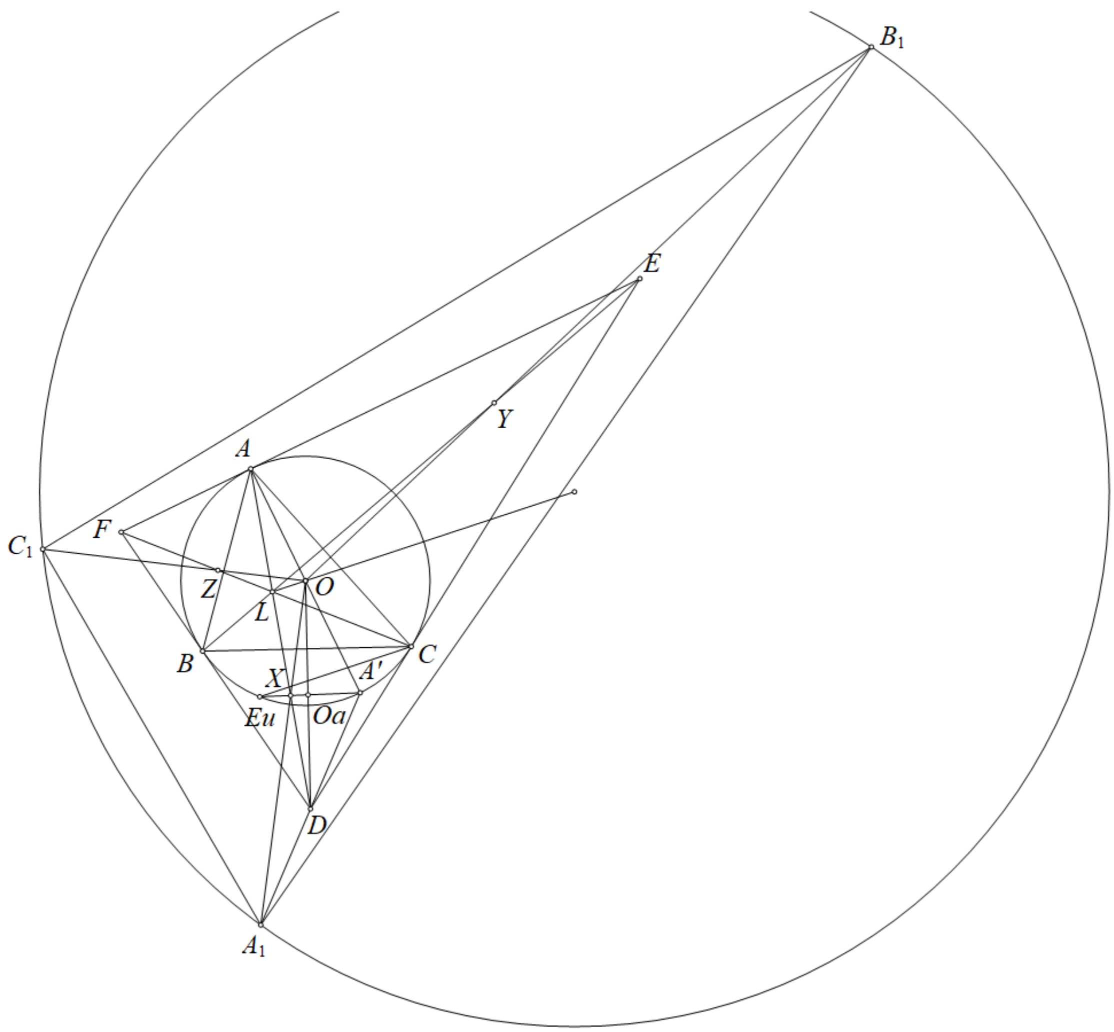
Let DEF be the triangle formed by the tangents at A, B, C of (O), and L the Lemoine point of triangle ABC.

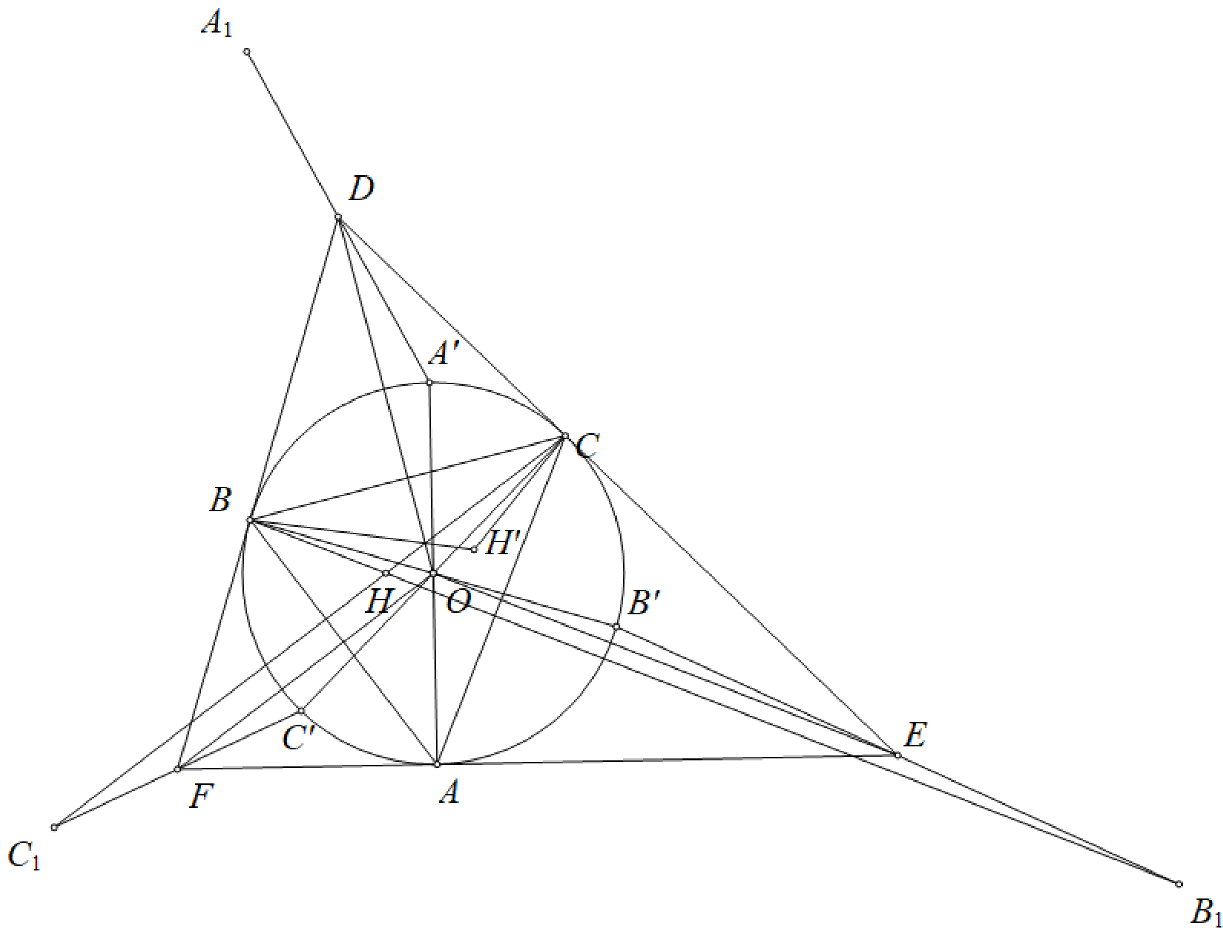


Draw the diameter AA' of (O), then the familiar result G, O_a, A' and the midpoint of DO are collinear, giving $\dfrac{XD}{XA} = \dfrac{1}{2}$.

So, let OX intersect DA' at A_1, then D is the midpoint of $A'A_1$ and $\dfrac{OX}{OA_1} = \dfrac{1}{3}$.

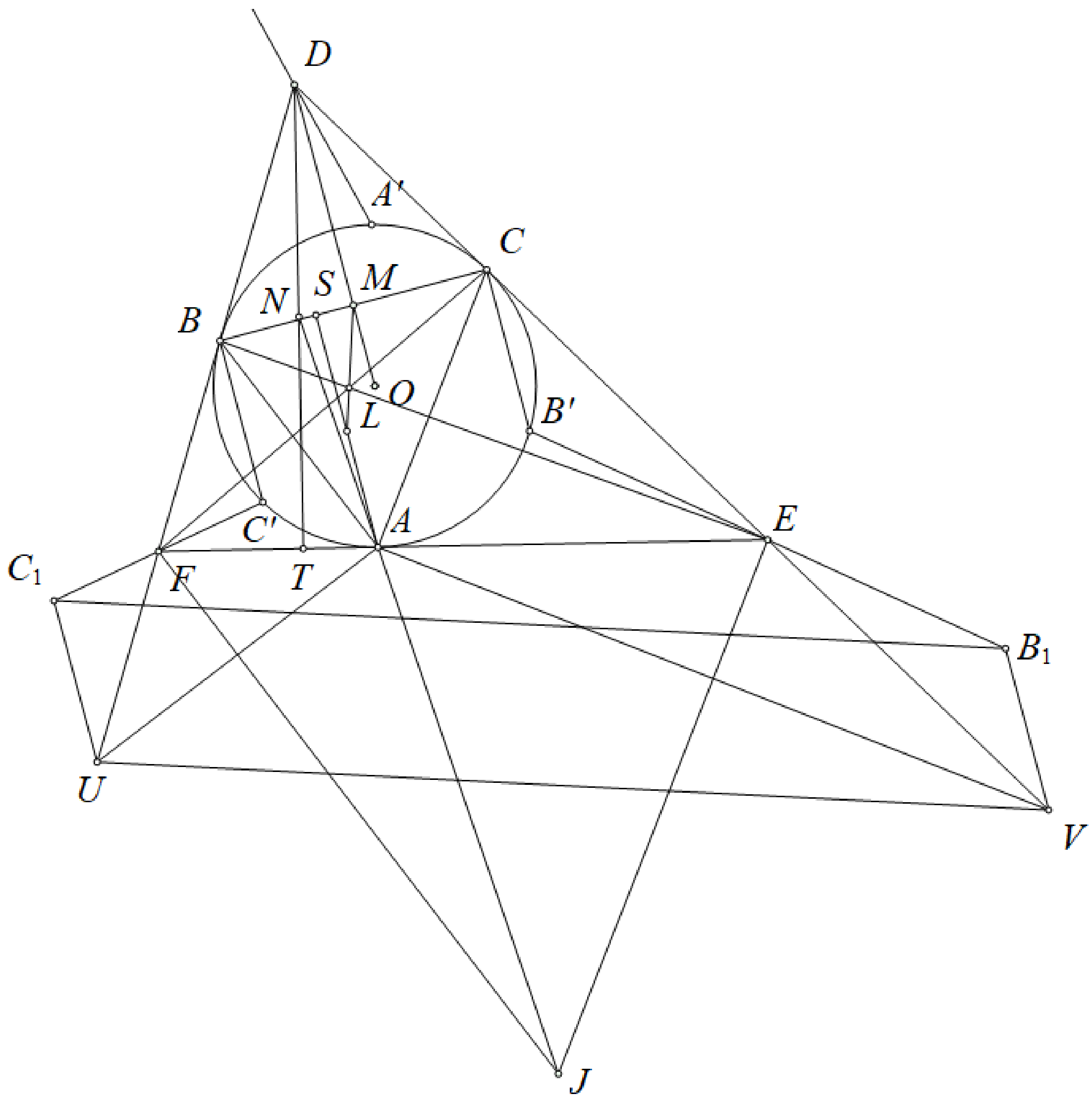
Similarly define B_1, C_1, we get $A_1B_1C_1$ as the image of XYZ under the homothety with center O and ratio 3, hence we need to prove that that the center of $(A_1B_1C_1)$ lies on OL.

+) Result 1: DO passes through the center of (HB_1C_1) where H is the orthocenter of ABC.



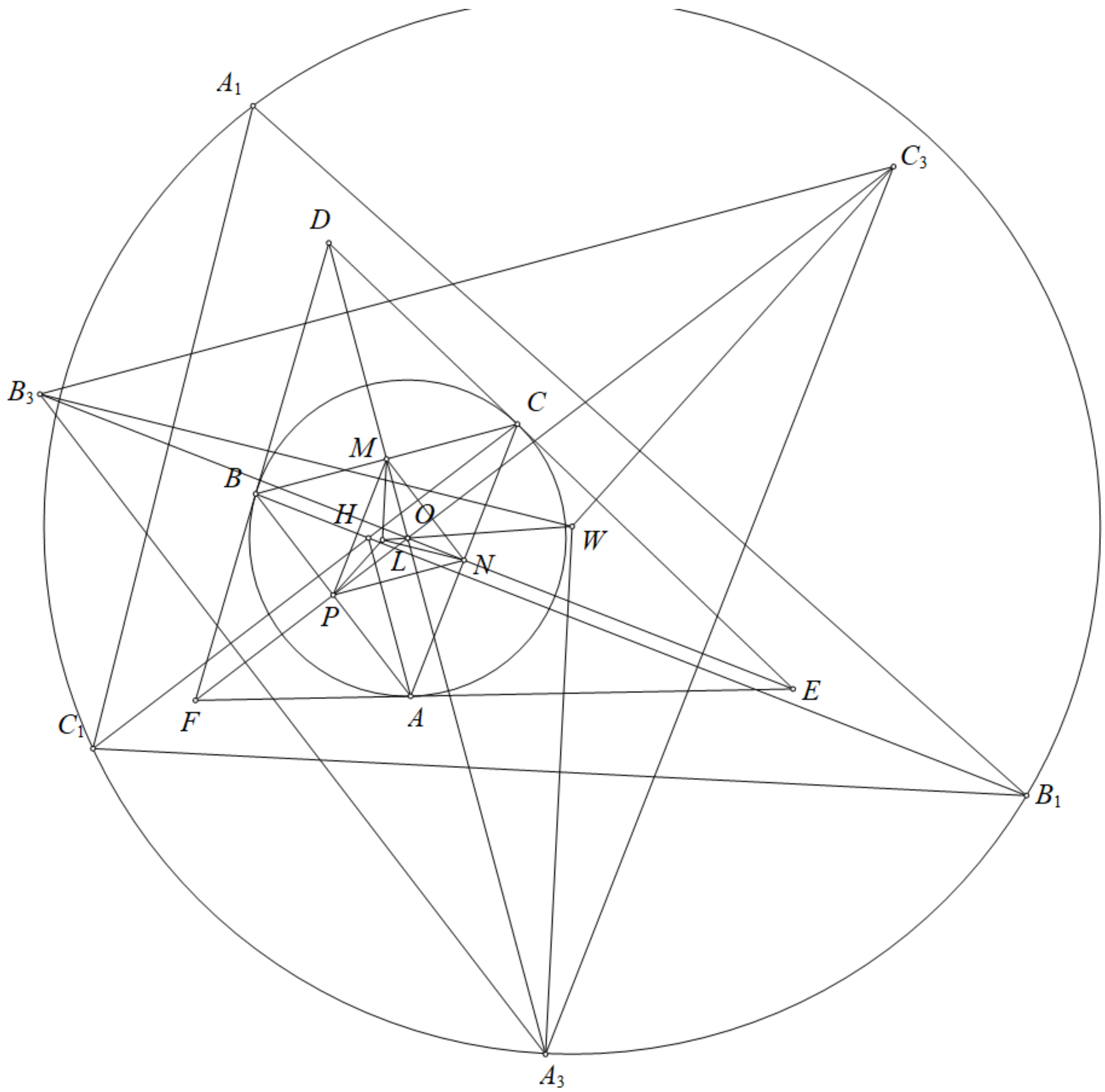
It is clear that B, H, B_1 and C, H, C_1 are collinear because OE is the midline of triangle $B'BB_1$.
Let H' be the reflection of H across DO, then $\dfrac{H'B}{H'C} = \dfrac{HB}{HC} = \dfrac{OE}{OF} = \dfrac{BB_1}{CC_1}$,
so $\triangle H'BB_1 \sim \triangle H'CC_1$ (SAS), hence $HH'B_1C_1$ is cyclic, and DO passes through the center of (HB_1C_1), denoted A_3.

+) Result 2: $ML \perp B_1C_1$ where M is the midpoint of BC.



Let V, U be reflections of C, B across E, F, then $\overrightarrow{B_1V} = \overrightarrow{CB'} = \overrightarrow{BC'} = \overrightarrow{C_1U}$ therefore $UV \parallel B_1C_1$.
Let T be the projection D on EF, J the $D-excircle$ in triangle DEF, then $JV = JA = JU$ and JA bisects DT at N.
Let S be the project of A on BC.
We also have $\angle SAM = \angle OMA = \angle DAO = \angle ADT$,
hence $\triangle DTA \sim \triangle ASM$, and ML bisects AS, so $\angle LMS = \angle NAT = \angle JAF$.
Since J is the center of (AUV),
$\angle AVU = 90° - \angle JAU = \angle NAB$ (because $\angle BAU = 90°$) $= \angle BAE - \angle NAT = 180° - \angle ACB - \angle LMC = \angle(ML, AC)$,
and $AC \perp AV$, thus $UV \perp ML$.

Back to the main problem,



Let M, N, P be the midpoints of BC, CA, AB, and A_3, B_3, C_3 the centers of $(HB_1C_1), (HA_1C_1), (HA_1B_1)$. By Result 1, we have $O, M, A_3; O, N, B_3; O, P, C_3$ collinear.
Since B_3C_3 is the perpendicular bisector of HB_1, we get $B_3C_3 \parallel NP$, and similarly O is the homothety center mapping $A_3B_3C_3$ to MNP.
Let W be the center of $(A_1B_1C_1)$, then $A_3W \perp B_1C_1$, and by Result 2, $A_3W \parallel ML$.
Similarly, $B_3W \parallel NL$, $C_3W \parallel PL$, and since O is the homothety center mapping $A_3B_3C_3$ to MNP, we conclude O, W, L are collinear, as desired.
Hence, the problem is proven.