Problem 2

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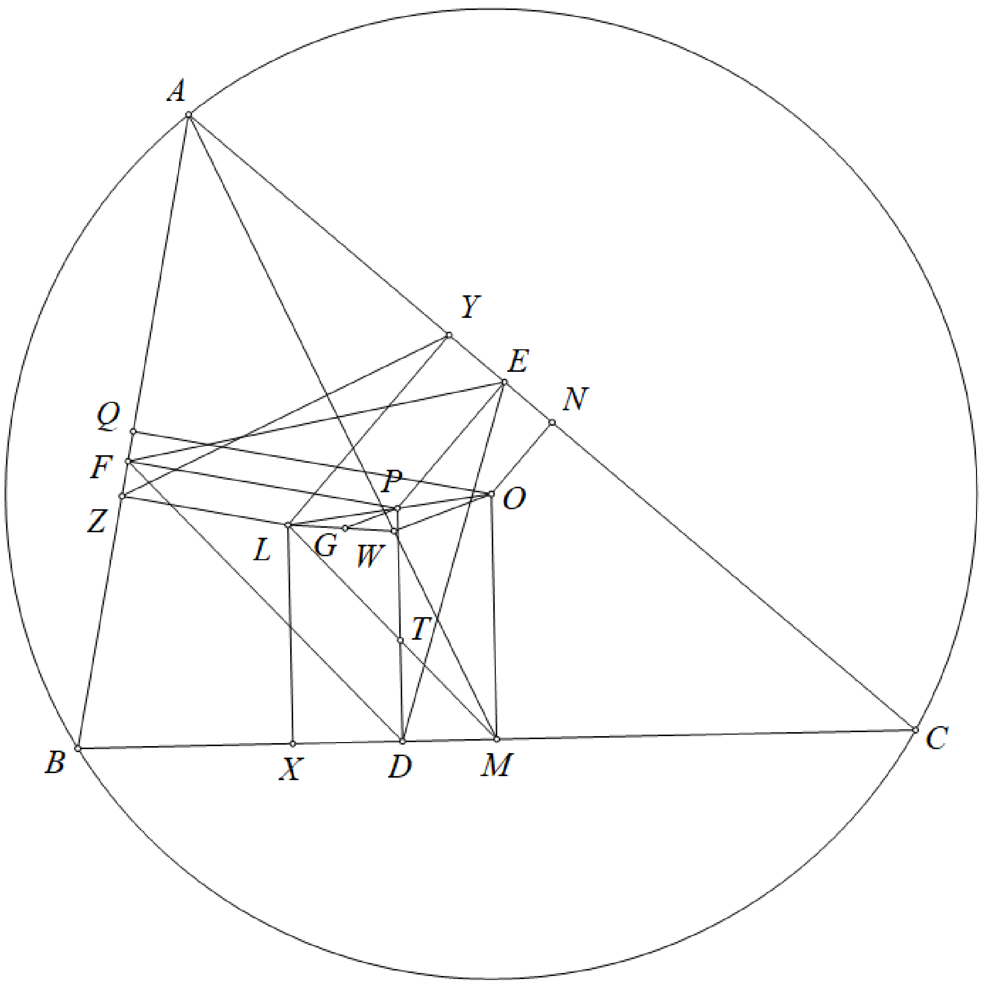
Let W be the centroid of triangle ABC, XYZ be the pedal triangle of L wrt triangle ABC, MNQ be the pedal triangle of O wrt triangle ABC.
Claim: L is the centroid of triangle XYZ.



Proof: By simple angle chasing, we get $AM \perp YZ$, let Ax be a ray being parallel to BC, Ly be a ray parallel to YZ we get
$(Ly, LX, LY, LZ) = (AM, Ax, AC, AB) = -1$, hence LX bisects YZ. Similarly, we get that L is the centroid of the triangle XYZ, as desired.
Back to the main problem,
From the claim above, $LX + \overrightarrow{LY} + \overrightarrow{LZ} = \overrightarrow{0}$ Let LM cut PD at T, let $\frac{LP}{LO} = k$, we have:
$\frac{PT}{OM} = \frac{LP}{LO} = k; \frac{DT}{LX} = \frac{MT}{ML} = \frac{OP}{OL} = 1 - k$

Hence $PD = PT + DT = k \cdot OM + (1-k) \cdot LX$, and since $LX \parallel PD \parallel OM$:
$\overrightarrow{PD} = k \cdot \overrightarrow{OM} + (1-k) \cdot \overrightarrow{LX}$.
Similarly, we get
$3 \cdot \overrightarrow{PG} = \overrightarrow{PD} + \overrightarrow{PE} + \overrightarrow{PF}$
$= k \cdot \left(\overrightarrow{OM} + \overrightarrow{ON} + \overrightarrow{OQ}\right) + (1-k) \cdot \left(\overrightarrow{LX} + \overrightarrow{LY} + \overrightarrow{LZ}\right) = 3k \cdot \overrightarrow{OW}$.
Hence $PG \parallel OW$, which is the Euler line of ABC, or the problem is proven.

Remark: By similar approach, we can prove the generalized problem:
Let X, Y be 2 arbitrary points such that L, X, Y are collinear, then the line connecting X with the centroid of its pedal triangle WRT triangle ABC, and the line connecting Y with the centroid of its pedal triangle WRT triangle ABC, are parallel.