Problem 14

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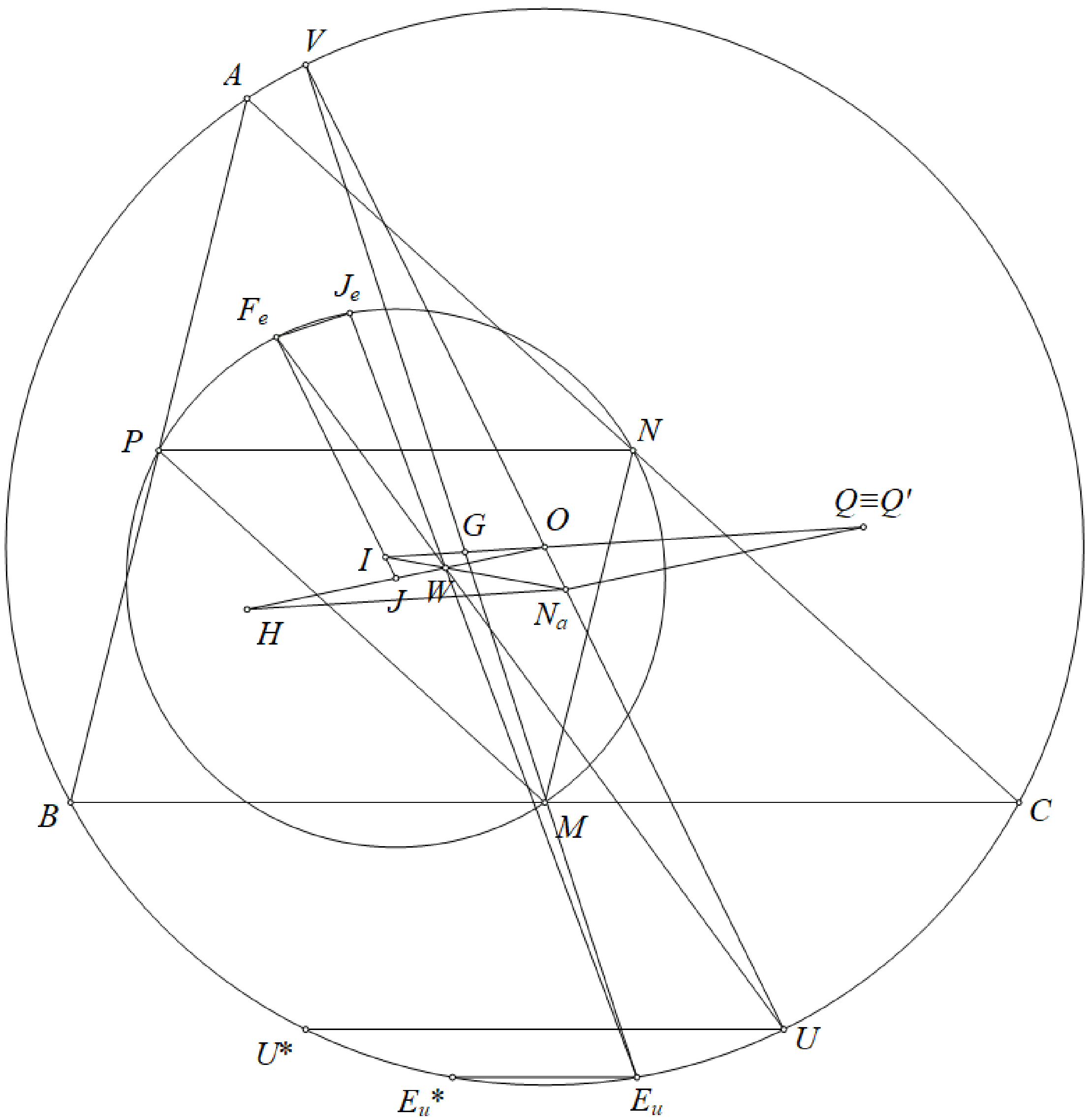
Let M, N, Q be the midpoints of BC, CA, AB, and W the centroid of triangle ABC.



The homothety centered at W with ratio 2 maps triangle MNQ to ABC, and (MNQ) to (O), so J_e is mapped to E_u (since J_e is the Euler reflection point of MNQ, and the Euler circle center J is mapped to O).
The Nagel point I of MNQ is mapped to the Nagel point N_a of ABC, hence F_e, the intersection of IJ with the Euler circle, is mapped to the intersection of ON_a with the smaller arc BC of (O), denoted by U.

Since F_e is also the anti-Steiner point of OI with respect to MNQ, U is the anti-Steiner point of HN_a with respect to ABC.

Let R, r be the circumradius and inradius respectively.
We have $IJ = JF_e - IF_e = \dfrac{R(ABC)}{2} - r(ABC)$, hence $ON_a = R(ABC) - 2r(ABC)$.
Let ON_a intersect (O) again at V distinct from U.
Let Q be a point on the ray opposite to OI such that $OQ = 2OI$. Then $OV \cdot ON_a = R(R-2r) = OI^2 = OG \cdot OQ$, so VGN_aQ is cyclic.
Thus $\angle N_aVG = \angle N_aQO$, and since $HN_a \parallel 2OI = OQ$, the quadrilateral HN_aQO is a parallelogram, hence $\angle N_aQO = \angle N_aHO$.

Draw through U and E_u the lines UU^* and $E_uE_u^*$ parallel to BC such that U^* and E_u^* lie on (O).
Then $AU^* \perp HN_a$ and $AE_u^* \perp OH$. Therefore, $\angle N_aHO = \angle U^*AE_u^* = \angle UAE_u$, so $\angle E_uVU = \angle E_uAU = \angle GVU$.
Hence V, G, E_u are collinear, implying $E_uG \perp E_uU$.
Since the homothety centered at W with ratio 2 maps F_eJ_e to E_uU, we conclude that $E_uG \perp F_eJ_e$, as desired.
Hence, the problem is proven.