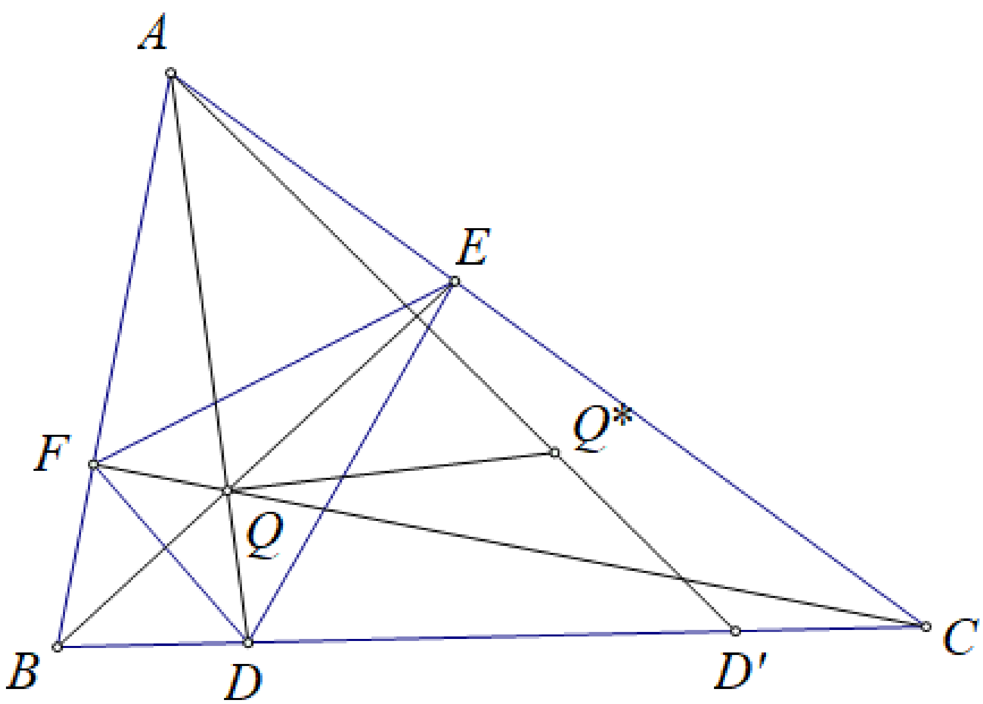
Vector Illustrative Problem

Ha Vu Anh

Lemma. Given a triangle ABC with cevian triangle DEF of an arbitrary point Q with respect to ABC. Then:

$$\overrightarrow{u} = \overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF}$$

is parallel to $\overrightarrow{QQ^*}$, where Q^* is the isotomic conjugate of Q with respect to triangle ABC.



Let $Q(a, b, c)$, then

$$\frac{DB}{DC} = \frac{b}{c}.$$

Let AQ^* intersect BC at D', then

$$\frac{D'B}{D'C} = \frac{c}{b},$$

hence $Q^*(1/a, 1/b, 1/c)$.

We have

$$\begin{aligned}\overrightarrow{u} &= \overrightarrow{QD} + \overrightarrow{QE} + \overrightarrow{QF} - (\overrightarrow{QA} + \overrightarrow{QB} + \overrightarrow{QC}) \\ &= \sum \left(\frac{b}{b+c}\overrightarrow{QB} + \frac{c}{b+c}\overrightarrow{QC} \right) - \sum \overrightarrow{QA} \\ &= \sum \left(\frac{a}{a+b} + \frac{a}{a+c} - 1 \right) \overrightarrow{QA}\end{aligned}$$

$$= \frac{1}{(a+b)(b+c)(c+a)} \sum (a^2 - bc)(b+c)\overrightarrow{QA} \quad (1)$$

We also have

$$\sum (a^2 - bc)(b+c)\overrightarrow{QA} = a(ab+bc+ca)\overrightarrow{QA} - bc(a+b+c)\overrightarrow{QA}$$

$$= (ab+bc+ca)\sum a\overrightarrow{QA} - (a+b+c)\sum bc\overrightarrow{QA}$$

$$= -(a+b+c)\sum bc\overrightarrow{QA} \quad (2)$$

From (1) and (2), we deduce

$$\overrightarrow{u} \parallel \sum bc\overrightarrow{QA}.$$

Moreover,

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\overrightarrow{QQ^*} = \sum \frac{1}{a}\overrightarrow{AQ^*} + \sum \frac{1}{a}\overrightarrow{QA}$$

$$= \frac{1}{abc}\sum bc\overrightarrow{QA}.$$

Hence,

$$\overrightarrow{QQ^*} \parallel \sum bc\overrightarrow{QA} \parallel \overrightarrow{u}.$$

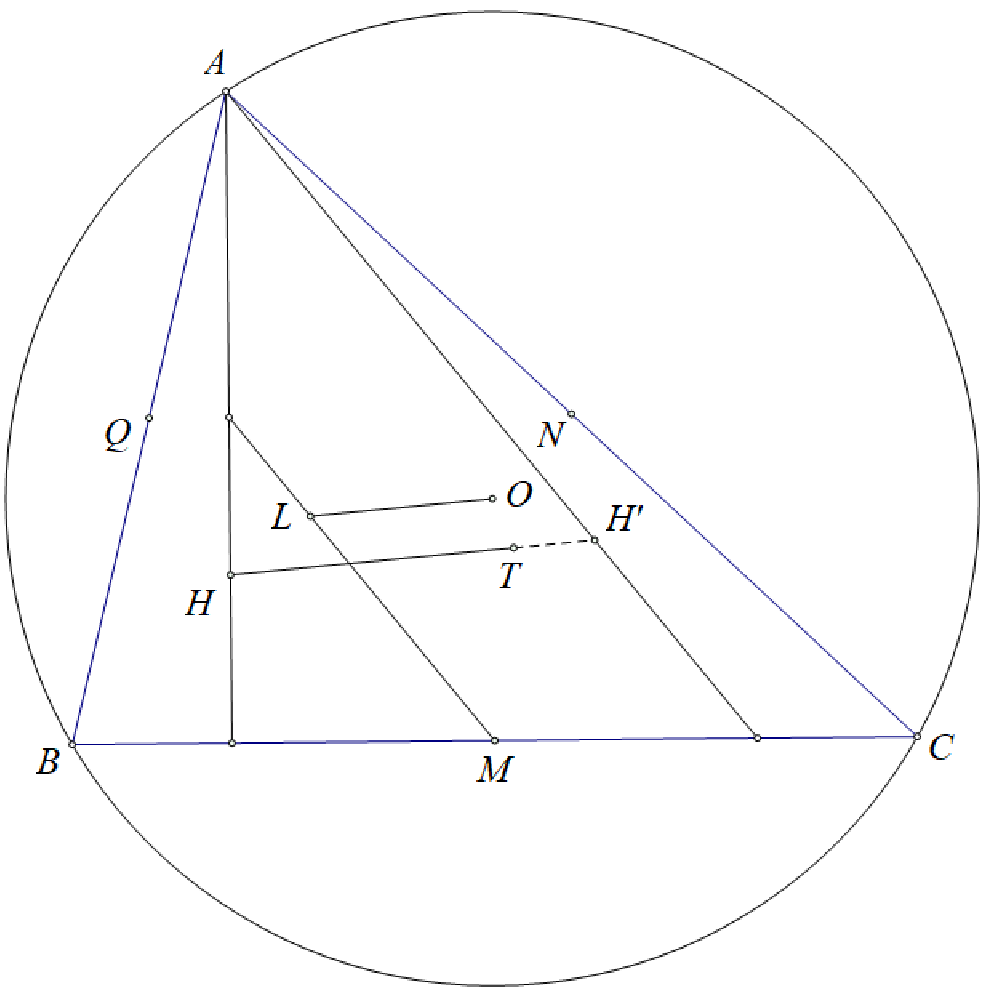
Thus, the lemma is proved.

Back to the problem.
Let MNQ be the median triangle of ABC, then ML bisects the altitude. Similarly, L is the isotomic conjugate of O with respect to MNQ. Since O is the orthocenter of ABC, we have

$$OL \parallel HH^*,$$

where H^* is the isotomic conjugate of H with respect to ABC.

We prove that H, T, H^* are collinear.



We have

$$\overrightarrow{HH^*} \parallel \overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF},$$

with AD, BE, CF being the altitudes. Therefore, we will prove

$$\overrightarrow{HT} \parallel \overrightarrow{u} = \overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF}.$$

Next,

$$L(a^2, b^2, c^2) \Rightarrow T(1/a^2, 1/b^2, 1/c^2),$$
$$\overrightarrow{HT} = \overrightarrow{HA} + \overrightarrow{AT}.$$

Hence,

$$\overrightarrow{HT}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=\sum\frac{\overrightarrow{AT}}{a^2}+\sum\frac{\overrightarrow{HA}}{a^2}=\sum\frac{\overrightarrow{HA}}{a^2}.$$

$$=\sum\frac{\cos B\cos C\cdot bc}{(a^2+c^2-b^2)(a^2+b^2-c^2)}\overrightarrow{HA}.$$

$$=\frac{1}{(a^2+c^2-b^2)(a^2+b^2-c^2)(b^2+c^2-a^2)}\sum\cos B\cos C\cdot bc\cdot(b^2+c^2-a^2)\overrightarrow{HA}.$$

We know

$$\frac{AH}{R}=\frac{2OM}{R}=2\cos A,\quad \frac{AD}{AH}=\frac{S}{a\cdot AH}=\frac{S}{2aR\cos A}.$$

Hence,

$$\sum\overrightarrow{AD}=\frac{S}{2R}\sum\frac{\overrightarrow{AH}}{a\cos A}=\frac{S}{R\cos A\cos B\cos C\cdot abc}\sum\cos B\cos C\cdot bc\overrightarrow{AH}.$$

$$=\frac{S}{R}(a^2+b^2+c^2)\cos A\cos B\cos C\cdot abc\sum\cos B\cos C\cdot bc\cdot(a^2+b^2+c^2)\overrightarrow{AH}.$$

We have

$$H(\cos B\cos C,\cos A\cos C,\cos B\cos A)\Rightarrow\sum\cos B\cos C\overrightarrow{AH}=0.$$

$$\sum\cos B\cos C\cdot 2abc(a+b+c)\overrightarrow{HA}=0.$$

Let

$$x=\frac{1}{(a^2+c^2-b^2)(a^2+b^2-c^2)(b^2+c^2-a^2)},\quad y=\frac{S}{R}(a^2+b^2+c^2)\cos A\cos B\cos C\cdot abc,\quad z=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

Then

$$\overrightarrow{HT}\cdot z=x\sum\cos B\cos C\cdot bc\cdot(b^2+c^2-a^2)\overrightarrow{HA},$$

$$\overrightarrow{HT}=\frac{x}{z}\sum\cos B\cos C\cdot bc\cdot(b^2+c^2-a^2)\overrightarrow{HA}.$$

$$=\frac{x}{z}\left(\sum\cos B\cos C\cdot bc\cdot(b^2+c^2-a^2)\overrightarrow{HA}+\sum\cos B\cos C\cdot 2abc(a+b+c)\overrightarrow{HA}\right)$$

$$=\frac{x}{z}\sum\cos B\cos C\cdot bc\cdot(b^2+c^2+a^2).$$

$$\overrightarrow{u}=y\sum\cos B\cos C\cdot bc\overrightarrow{AH}.$$

$$\overrightarrow{u}=\frac{y\cdot z}{x}(a^2+b^2+c^2)\overrightarrow{HT}.$$

Hence,

$$\overrightarrow{HT}\parallel\overrightarrow{u}.$$

Thus,

$$\overrightarrow{HT}\parallel\overrightarrow{HH^*},$$

as desired.