# 2020 Science Atlantic Math Competition

Time: 3 hours.

Contestants are on their honor to work alone without help. Calculators, mathematical software, books, notes, etc are forbidden.

Please write the answer to each question on a separate sheet (or sheets) of paper, and do not refer to other answers, as your answers to the various questions will be graded separately. Do not put your name or university on the answer sheets. Show all work.

Answers should be photographed or scanned, and mailed to this address: sa_mscs_20@cs.smu.ca

IMPORTANT: The subject line MUST be:

SAMATH_[question number]_[ID]

So if you are competitor 007 and you have answered question 9, you would send it as an attachment to an email with subject

SAMATH_9_007

This is how we will route the answers to the judges. Submissions not following this may not be graded.

No answer mailed after 5:00 Atlantic time (5:30 in NL) will be graded.

Each of the eight questions carries equal weight.

QUESTIONS

There are eight questions, over two pages. Not all questions are of equal difficulty.

1. Write $\tan(x)+\cot(2z)$ in the form $bf(cx)$, where $b$ and $c$ are real numbers, and $f$ is a standard trigonometric function.

2. A bug located at $(2,0,0)$ in $\mathbb{R}^3$ wants to get to $(-2,0,0)$ but is impeded by an impenetrable sphere of radius 1 centred at the origin. Describe and give the length of a shortest path available to this bug.

3. A tetrahedron has vertices $ABCD$. Suppose that the plane $\gamma$ bisects the (internal) dihedral angle along edge $AB$ and meets edge $CD$ at $G$. Prove that
$$\frac{(ABC)}{(ABD)} = \frac{CG}{GD}.$$
Notation: $(ABC)$ denotes the area of triangle $ABC$.

4. A sequence of natural numbers is *eccentric* if no term can be written as a sum of terms that come before it (repetition is allowed.) For instance, the finite sequence 12, 3, 13, 17, 2 is eccentric, but 11,3,13,17,2 is not because 11+3+3=17. Does there exist an infinite eccentric sequence? Prove your result!

5. Define a *factorial M-partition* of N to be a set $\{a_1, a_2, \ldots, a_M\}$ of positive integers such that $N = \sum_{i=1}^{M} a_i!$ . We will call a factorial M-partition *proper* if the $a_i$ are all unequal. Show that if for some $M, N$ there is a proper factorial M-partition of N, then every other factorial M-partition of N is a permutation of it.

6. Show for integers $n \geq 1$ that

$$\frac{d^n}{dx^n}(1+x)^{n-1}e^{\frac{x}{x+1}} = \frac{e^{\frac{x}{x+1}}}{(1+x)^{n+1}}$$

7. Assess the convergence of

$$\sum_{n=0}^{\infty} \arctan\left(\frac{2n-1}{n^4-2n^3+n^2+1}\right)$$

If convergent, give the value of this series.

8. The symmetric $n \times n$ matrix $A_n$ has $ij^{\text{th}}$ entry

$$\begin{cases} (3-(-1)^i)/2, & \text{if } i=j; \\ -1, & \text{if } i=j\pm 1; \\ 0, & \text{otherwise.} \end{cases}$$

Determine $\det(A_n)$ for all positive integers $n$.