Science Atlantic Math Competition 2021 Solutions

Problem 1

Determine, with proof, all real numbers m such that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(f(x)) = 1 + mx$ for all $x \in \mathbb{R}$.

Solution

Problem 2
Find, with proof, all real solutions to

$$x2^{1/x} + \frac{1}{x}2^{x} = 4.$$

Solution

Problem 3
For what integer values of a is $n(n+5)(2n+a)$ divisible by 6 for every natural number n?

Solution

Problem 4
Let ω be a plane in 3-space which passes through a vertex A of a unit cube. For the vertices B_1, B_2, B_3 of the cube which are adjacent to A, let F_1, F_2, F_3 in turn be the feet of the perpendiculars dropped to ω. Determine the value of

$$(AF_1)^2 + (AF_2)^2 + (AF_3)^2.$$

Solution

Problem 5

Recall that the *population standard deviation* is defined to be

$$\sigma := \sqrt{\frac{\sum_{i=1}^{n}(\mu - x_i)^2}{n}},$$

where μ is the mean of the x_i. The polynomial

$$a_0 + a_1 x + \cdots + a_{2022} x^{2022}$$

has 2022 roots $\{x_i : 1 \leq i \leq 2022\}$ (not necessarily distinct). Find μ and σ for these 2022 values.

Solution

Problem 6

Find all triples (x, y, z) of non-negative real numbers satisfying the system of equations

$$\sqrt[3]{x} - \sqrt[3]{y} - \sqrt[3]{z} = 16,$$
$$\sqrt[4]{x} - \sqrt[4]{y} - \sqrt[4]{z} = 8,$$
$$\sqrt[6]{x} - \sqrt[6]{y} - \sqrt[6]{z} = 4.$$

Solution

Problem 7
Give a closed-form expression for the function

$$f(x) = \lim_{n\to\infty} \lim_{m\to\infty} \cos^m(n!\pi x),$$

where m, n are natural numbers.

Solution

Problem 8

Suppose $p(x, y)$ is a polynomial in two variables and with real coefficients such that

(1) $p(x, y) = p(y, x)$, and
(2) $x - y$ is a factor of $p(x, y)$.

Prove that $(x - y)^2$ is a factor of $p(x, y)$.

Solution