Science Atlantic Math Competition 2021 Solutions

Problem 1
True or false? A 5-5-6 triangle has the same area as a 5-5-8 triangle.

Solution

Problem 2
True or false? $31^{11} < 17^{14}$

Solution

Problem 3
True or false? $\log_1 0550 < 2.5$.

Solution

Problem 4
True or false? There are exactly 10! seconds in 6 weeks.

Solution

Problem 5
True or false? There is a permutation of $\{0, \ldots, 9\}$ such that the sum of any three consecutive numbers is < 15.

Solution

Problem 6
True or false? The area in the first quadrant enclosed by the graphs of $y = x^{2023}$ and $x^{1/2023}$ is > 0.999.

Solution

Problem 7
Alice and Bob play a game in which Alice tosses 2023 fair coins and Bob tosses 2022 fair coins. If Alice obtains a greater number of heads, she wins the game; otherwise Bob wins. True or false? Bob has a greater probability of winning the game than does Alice.

Solution

Problem 8
A farmer owns a triangular field, as shown below (to be added later). The farmer figures that 5 sheep can graze in the west field, 10 in the south field, and 8 in the east field. Assume that all sheep eat the same amount of grass. True or false? At least 21 sheep can graze in the north field.

Solution

Problem 9
Suppose that

$$\tan(x) + \cot(x) + \sec(x) + \csc(x) = 2023.$$

True or false? $\sin(x) + \cos(x) \geq \frac{2}{\sqrt{3}}$.

Solution

Problem 10
True or false? When a prime number is divided by 60, the remainder is either 1 or a prime number.

Solution
109 is prime, but the remainder upon division by 60 is $49 = 7^2$.

Problem 11
Two people plan to meet at a given place between noon and 1PM under the agreement that neither person will wait longer than 12 minutes for the other. Assume that each person chooses their time of arrival at random. True or false? The probability that a meeting takes place is $\geq 1/3$.

Solution

Problem 12
A road crew can build a section of highway in 3 days with their present supply of machines. With 3 more machines, however, they could do the job in only 2 days. True or false? If the road crew had only one machine, it would take more than two weeks to build the highway?

Solution
The number of machines is inversely proportional to time. Let m be the current number of machines; then $m + 3 = 3m/2$ and $m = 6$. If the crew had only one machine, it would take $3 \cdot 6 = 18$ days, which is more than two weeks.

Problem 13
The numbers $1, \ldots, 6$ are assigned to the six faces of a cube, one number to each face. For each of the eight vertices of the cube, compute the product of the numbers assigned to the three faces incident with that vertex, and then let S be the sum of the eight products. True or false? Regardless of how the numbers are assigned to the faces, $S < 321$.

Solution

Problem 14
Let A and B be 2×2 matrices with real entries. True or false? If $AB = 0$ then $BA = 0$.

Solution

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

but

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Problem 15
True or false? There are at least 280 positive integer divisors of 20^{23} which are perfect squares.

Solution

Problem 16
True or false? It is possible to partition the set $\{1, \ldots, 15\}$ into 5 disjoint 3-element subsets $S_k = \{a_k, b_k, c_k\}$ such that $a_k + b_k = c_k$ for all $k \in \{1, \ldots, 5\}$.

Solution

Problem 17
Let $f(x) = \ln(\sqrt{1 + e^{\cos(x)}})$. True or false? $f(0)f'(0)f''(0) = -1$.

Solution

Problem 18
True or false? The value of $\sqrt{9 + \sqrt{9 + \sqrt{9 + \sqrt{9 + \ldots}}}}$ is less than 3.5.

Solution

Problem 19
A long round wire has a cross-section with a radius of 1 millimetre. The wire is tightly wound into a spherical shape, resulting in a ball of radius 20 centimetres. True or false? When unravelled, the wire will span (at least) the entire length of the Confederation Bridge (12.9 kilometres).

Solution

Problem 20
The golden ratio $\phi = (1 + \sqrt{5})/2$ has the remarkable property that its reciprocal is equal to its fractional part. True or false? The golden ratio is the only positive real number with this property.

Solution

Problem 21
A finite set S of real numbers has the following properties:

- $1 \notin S$ and $2001 \notin S$,
- The mean of $S \cup \{1\}$ is 13 less than the mean of S, and
- The mean of $S \cup \{2001\}$ is 27 more than the mean of S.

True or false? There are at least 50 numbers in S.

Solution

Problem 22
True or false? The equation $x^3 = 3x^2 + 1$ has exactly three real solutions.

Solution

Problem 23
True or false?

$$\frac{1}{2} \times \frac{3}{4} \times \cdots \times \frac{99}{100} < \frac{1}{10}.$$

Solution

Problem 24
Fifteen points $P_1, \ldots, P_{15}$ are equally spaced around a circle. True or false? The size of the angle $P_1P_3P_7$ is less than 107 degrees.

Solution

Problem 25
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function having the property that $f(x) \cdot f(f(x)) = 1$ for all real numbers x. Moreover, suppose that $f(4046) = 4045$. True or false? $f(2023) < 2023$.

Solution

Problem 26
Suppose that $(1 + x + x^2)^{2023} = a_0 + \cdots + a_{4046}x^{4046}$. True or false? The integer $a_1 + a_3 + a_5 + \cdots + a_{4045}$ is odd.

Solution

Problem 27
The integer 99 was multiplied by an integer to obtain the 7-digit integer $62ab427$, but the digits a, b were illegible. True or false? The only possible values for the illegible digits are $a = 2$ and $b = 4$.

Solution

Problem 28
True or false?

$$\int_0^{\sqrt{\pi/6}} \sin(x^2)\ dx + \int_{-\sqrt{\pi/6}}^{\sqrt{\pi/6}} x^2 \cos(x^2)\ dx < \frac{1}{2\sqrt{2}}.$$

Solution

Problem 29
Alice drives a truck that gets 10 kilometres per litre. Bob drives a small car that gets 40 kilometres per litre. Each week, Alice and Bob drive the same distance. Alice decides to trade her truck for a new one that gets 15 kilometres per litre. Bob now wants to trade his car for one which gets x kilometres per litre so that his fuel savings over a week will match those of Alice. True or false? $x \leq 80$.

Solution

Problem 30
Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined as follows: $f(x) = \min\{2x + 2, -\frac{3}{4}x + 7\}$. True or false? The maximum value of $f(x)$ is at least 5.

Solution