Science Atlantic Math Competition 2020 Solutions

Problem 1
Write $\tan(x) + \cot(2x)$ in the form $bf(cx)$, where b and c are real numbers, and f is a standard trigonometric function.

Solution

$$\begin{aligned}
\tan(x) + \cot(2x) &= \frac{\sin(x)}{\cos(x)} + \frac{\cos(2x)}{\sin(2x)} \\
&= \frac{\sin(x)}{\cos(x)} + \frac{\cos^2(x) - \sin^2(x)}{2\sin(x)\cos(x)} \\
&= \frac{2\sin^2(x) + \cos^2(x) - \sin^2(x)}{2\sin(x)\cos(x)} \\
&= \frac{\sin^2(x) + \cos^2(x)}{2\sin(x)\cos(x)} \\
&= \frac{1}{\sin(2x)} = \csc(2x)
\end{aligned}$$

Problem 2
A bug located at $(2,0,0)$ in $\mathbb{R}^3$ wants to get to $(-2,0,0)$ but is impeded by an impenetrable sphere of radius 1 centred at the origin. Describe and give the length of the shortest path available to this bug.

Solution
The projection of a path onto a plane is no longer than the path, so with no loss of generality, model the problem as the shortest path from $(2,0)$ to $(-2,0)$ which does not pass through the unit circle. If a string is stretched from $(2,0)$ to $(-2,0)$, its path consists of two line segments from $(-2,0)$ and $(2,0)$ meeting the unit circle at the points of tangency and the arc between those points. The points of tangency lie at $(-1/2,\sqrt{3}/2)$ and $(1/2,\sqrt{3}/2)$, or at angles of $2\pi/3$ and $\pi/3$ on the unit circle, respectively. The distances from $(-2,0)$ to $(-1/2,\sqrt{3}/2)$ and from $(2,0)$ to $(1/2,\sqrt{3}/2)$ are both $\sqrt{3}$, and the length of the arc between $\pi/3$ and $2\pi/3$ is $\pi/3$, so the length of the path is $2\sqrt{3}+\pi/3$.

Problem 3
A tetrahedron has vertices $ABCD$. Suppose that the plane γ bisects the (internal) dihedral angle along edge AB and meets edge CD at G. Prove that

$$\frac{|ABC|}{|ABD|} = \frac{|CG|}{|GD|}$$

Solution

Problem 4

A sequence of natural numbers is *eccentric* if no term can be written as a sum of terms that come before it (repetition is allowed). For instance, the finite sequence $12, 3, 13, 17, 2$ is eccentric, but $11, 3, 13, 17, 2$ is not because $11 + 3 + 3 = 17$. Does there exist an infinite eccentric sequence?

Solution

Let $(a_n)_{n=0}^{\infty}$ be a sequence of natural numbers. If any terms are repeated, the sequence is not eccentric (because a single term is considered a sum).

Suppose no terms are repeated. By the pigeonhole principle, there exists b such that $0 \leq b < a_0$ and infinitely many $a_n \equiv b \pmod{a_0}$. Then there exists a subsequence $(a_{n_i})_{i=0}^{\infty}$ of (a_n) such that each $a_{n_i} \equiv b \pmod{a_0}$. Then there exists j such that $a_{n_0} = ja_0 + b$. Since (a_n) has no repetitions, there exists $k \leq j + 1$ such that $a_{n_k} > a_{n_0}$, implying that there exists $\ell > j$ such that $a_{n_i} = \ell a_0 + b$. Then $a_{n_i} = a_{n_0} + (\ell - j)a_0$, so (a_n) is not eccentric.

Therefore no infinite eccentric sequences exist.

Problem 5

Define a *factorial M-partition* of N to be a set $\{a_1, a_2, \cdots, a_M\}$ of positive integers such that $N = \sum_{i=1}^{M} a_i!$. Furthemore, define a factorial M-partition to be *proper* if the a_i are all unequal. Show that if for some M, N there is a proper factorial M-partition of N, then every other proper factorial M-partition of N is a permutation of it.

Solution

First we show that $1! + \cdots + (n-1)! < n!$ by induction. Indeed, $1! < 2!$ and if $n > 1$ and $1 + \cdots + (n-1)! < n!$, then $1 + \cdots n! < 2n! < (n+1)n! < (n+1)!$.

Now we show the main result by induction on M. For $M = 1$, $\{a!\} = \{b!\} \iff a = b$. Now suppose for a given M that for every N with a proper factorial M-partition, N admits only one such partition up to ordering. Let N be a number with a proper factorial $(M+1)$-partition. Let n be the unique natural number such that $n! \leq N < (n+1)!$. Since $1! + \cdots + (n-1)! < n! \leq N$, at least one element of the partition is $\geq n$. On the other hand, all terms of the partition are $< (n+1)$ so the largest element is n. Now by supposition, $N - n$ has a unique ordered proper factorial M-partition, so appending n to this gives a unique ordered proper factorial $(M+1)$-partition of N.

Therefore the result holds for all M, N by induction.

Problem 6
Show for integers ≥ 1 that

$$\frac{d^n}{dx^n}(1+x)^{n-1}e^{\frac{x}{x+1}} = \frac{e^{\frac{x}{x+1}}}{(1+x)^{n+1}}$$

Solution
Let $y = 1 + x$ and note that $(1+x)^{n-1}e^{\frac{x}{x+1}} = y^{n-1}e^{1-\frac{1}{y}}$. For each m, n let $f_n^{(m)} = \dfrac{d^m}{dx^m}y^{n-1}e^{1-\frac{1}{y}}$.

First we show by induction on m that $f_{(n+1)}^{(m)} = mf_n^{(m-1)} + yf_n^{(m)}$. For the base case $m = 1$, $f_{(n+1)}^{(1)} = \dfrac{d}{dx}yf_n = f_n + yf_n^{(1)}$. For the step case, if $f_{(n+1)}^{(m)} = mf_n^{(m-1)} + yf_n^{(m)}$, then $f_{(n+1)}^{(m+1)} = mf_n^{(m)} + f_n^{(m)} + yf_n^{(m+1)} = (m+1)f_n^{(m)} + yf_n^{(m+1)}$.

Now we show by induction on n that $f_n^{(n)} = y^{-(n+1)}e^{1-\frac{1}{y}}$ as desired. The base case $n = 0$ is obvious. For the step case, $f_{(n+1)}^{(n)} = nf_n^{(n-1)} + yf_n^{(n)}$ and so $f_{(n+1)}^{(n+1)} = nf_n^{(n)} + f_n^{(n)} + yf_n^{(n+1)} = (n+1)f_n^{(n)} + yf_n^{(n+1)}$. By supposition, $f_n^{(n)} = y^{-(n+1)}e^{1-\frac{1}{y}}$ and so $f_n^{(n+1)} = (-(n+1)y^{-(n+2)} + y^{-(n+3)})e^{1-\frac{1}{y}}$. Substituting, we have $f_{n+1}^{(n+1)} = y^{-(n+2)}e^{1-\frac{1}{y}}$ as desired.

Problem 7

Assess the convergence of

$$\sum_{n=0}^{\infty} \arctan\left(\frac{2n-1}{n^4-2n^3+n^2+1}\right)$$

If convergent, give the value of the series.

Solution

By the rule

$$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$$

we have that

$$\arctan\left(\frac{2n-1}{n^4-2n^3+n^2+1}\right) = \arctan\left(\frac{n^2-(n-1)^2}{1+n^2(n-1)^2}\right) = \arctan(n^2)-\arctan((n-1)^2)$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty} \arctan\left(\frac{2n-1}{n^4-2n^3+n^2+1}\right) &= \sum_{n=0}^{\infty} \arctan(n^2) - \arctan((n-1)^2) \\ &= -\arctan((-1)^2) + \lim_{n\to\infty} \arctan(n^2) \\ &= -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}\end{aligned}$$

Problem 8

The symmetric $n \times n$ matrix A_n has ij-th entry

$$\begin{cases} (3-(-1)^i)/2 & i=j \\ -1 & i=j\pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine $\det(A_n)$ for all positive integers n.

Solution

Let $e_i = (3-(-1)^i)/2$. An easy calculation shows that $A_1 = [2]$ and $A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ have determinants 2 and 1 respectively. In the general case, A_n has A_{n-2} as the top left $(n-2)\times(n-2)$ sub-block and A_{n-1} as the top left $(n-1)\times(n-1)$ sub-block.

$$\begin{bmatrix} & & & \cdots & \cdots \\ & A_{n-2} & & \cdots & \cdots \\ & & & -1 & \cdots \\ \cdots & \cdots & -1 & e_{n-1} & -1 \\ \cdots & \cdots & \cdots & -1 & e_n \end{bmatrix}$$

Then $\det(A_n) = e_n \det(A_{n-1} - \det(A_{n-2})$ by evaluating first along the n-th row, then the n-th column of the resulting minors. This recurrence yields $\det(A_3) = 0$, $\det(A_4) = -1$, $\det(A_5) = -2$, $\det(A_6) = -1$, $\det(A_7) = 0$, $\det(A_8) = 1$, $\det(A_9) = 2$, and $\det(A_{10}) = 1$. Since $\det(A_9) = \det(A_1)$ and $\det(A_{10}) = \det(A_2)$, the sequence of determinants has period 8 and follows this pattern.