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For almost half a century, since the seminal work of Merton [1973] and Ross [1976], academics in finance have been writing about multi-factor models of asset returns. There have been hundreds of scholarly publications and over 300 candidates suggested as possible factors (Harvey, Liu, and Zhu [2016]). More recently, financial institutions have fully embraced the basic idea, coining terms such as *smart beta* to make multidimensional investing sound more sophisticated to clients.

There is little doubt that common pervasive factors are indeed the main drivers of returns for well-diversified portfolios. We know this must be true because diversification does not eliminate all volatility—perhaps only half of it for the average large fund. Moreover, there must be several factors because well-diversified portfolios in different asset classes or countries are not typically highly correlated (as they would be if there were only a single common pervasive factor).

One can go even further and assert the following fundamental truth: Diversification is factor investing. Almost 100% of the returns to a well-diversified portfolio are dictated by its sensitivities (often called *betas*) to common pervasive factors. True, many investors and investment managers do not know exactly which factors are driving their portfolios, but everyone should know by now that some factors are doing so.

Because the multifactor paradigm has become so prevalent among both academics and practitioners, there is a growing need for tools to manage multidimensional risks, which is essentially the difficult task of identifying the underlying pervasive factors and managing every diversified portfolio's sensitivities to them. This article is devoted to one particular highly useful tool, the mimicking portfolio.

A *mimicking portfolio* is a tradable fund engineered to closely copy the factor sensitivities of an individual asset, a fund, or a nontradable variable, such as a macroeconomic quantity. To be useful, the mimicking portfolio must have two attributes: (1) It is composed only of liquid and easily tradable assets, and (2) it retains only a small amount of return volatility that is not explained by the pervasive factors.

Mimicking portfolios have many potential uses, including (though not limited to)

- Evaluating active manager performance
- Substituting for a desired investment in illiquid assets
- Determining the true potential for improved diversification
- Understanding the sources of past return volatility
- Predicting the likely level of future return volatility

Our article is preceded by and closely related to work by Lamont [2001] and Cooper and Priestley [2011]. Lamont presented a method of constructing a mimicking portfolio to match various macroeconomic variables, such as gross domestic product and unemployment. Cooper and Priestley focused on the asset pricing risk factors first suggested by Chen, Roll, and Ross [1986], which in some cases were also macroeconomic variables and in other cases were higher frequency market observations intended to capture aggregate sentiment.[1] We intend to buttress this work by developing methods for constructing mimicking portfolios to a general set of factors and providing evidence for their robustness over time in the specific case of global market factors.

The next section provides the mathematical underpinning of mimicking portfolio analysis and gives a general closed form for their construction. We then provide an empirical implementation and numerical examples before concluding and offering suggestions for further research.

MIMICKING PORTFOLIO ALGEBRA

We begin by assuming that the underlying pervasive risk factors are known or else can be spanned by a fixed and equal number of observable indexes (or funds). The basic multiple-factor paradigm stipulates that the return on any asset, say stock j observed over an interval ending at time t, can be written as a linear function:

$$\tilde{R}_{j,t} = \alpha_j + \beta_{j,1}\tilde{f}_{1,t} + \beta_{j,2}\tilde{f}_{2,t} + \cdots + \beta_{j,K}\tilde{f}_{K,t} + \tilde{\varepsilon}_{j,t} \qquad (1)$$

where the fs denote the pervasive factors and the βs are factor sensitivities; α_j is an intercept constant, and $\tilde{\varepsilon}_{j,t}$ represents a residual, or diversifiable, risk that is unrelated to the factors and is also uncorrelated across individual assets. Throughout the duration of this article, we will interchangeably refer to the volatility of this term as the *idiosyncratic volatility* or the *residual volatility*. The time interval itself can be any length desired (e.g., daily, weekly, monthly).

Identifying the number of pervasive factors, K, is ultimately empirical; for now, we simply assume that we know it. Theoretically, K could be almost any number greater than one, but the fervent hope is that it is not too large and that Equation (1) is parsimonious. Because K is theoretically unspecified, the model in Equation (1) does not really assume very much. Presumably, just about any asset's return could be written as a function of K factors plus some residual volatility.

The factor sensitivities and the intercept cannot be just any numbers imaginable. To prevent arbitrage, Ross [1976] showed that the mean return for stock j (the expected value of the variable on the left side of Equation (1)) had to be the same linear function of the βs for every asset. This is the essence of Ross' arbitrage pricing theory (APT), which is widely used today as the underlying scientific foundation of multifactor investing.[2]

Mimicking portfolios can be a useful tool without knowing for sure whether Ross' no arbitrage result is fully in effect. We can employ the linear model in Equation (1) without worrying about possible equilibrium conditions for the βs.

A well-diversified portfolio has returns that are driven mainly by the nondiversifiable factors in Equation (1); its remaining residual risk is minimal because residuals in Equation (1) are uncorrelated across the assets within the portfolio and are consequently diversified away. Indeed, a fully diversified portfolio p's returns would be

$$\tilde{R}_{p,t} = \alpha_p + \beta_{p,1}\tilde{f}_{1,t} + \beta_{p,2}\tilde{f}_{2,t} + \cdots + \beta_{p,K}\tilde{f}_{K,t}. \qquad (2)$$

Note that the pervasive factors (fs) are exactly the same in Equation (2) as in Equation (1), but the factor sensitivities (βs) and intercept (α) are different; in Equation (2), they are weight averaged over the individual assets in the entire portfolio. There is no longer any residual volatility (ε), which algebraically portrays the fact that return variability of a completely diversified portfolio is determined entirely by the factors (the fs) acting through the sensitivities (the βs.)

The sensitivities themselves are a matter of financial engineering. By astute weighting of the individual assets in portfolio p, the βs can be determined in advance. Indeed, if short positions or futures are available, each of the K βs in Equation (2) can be set to virtually any level desired. Once the portfolio composition (individual stock weightings) are selected, the βs are simply weighted averages, with the same weights. The portfolio's return over the subsequent period ending at t is fully dictated by the predetermined βs (plus α_p) and whatever factor values turn out to prevail at t.

SOME USES OF MIMICKING PORTFOLIOS

Active management has just two ways of producing superior performance, selection and timing. In either case, a mimicking portfolio such as the perfectly diversified one depicted in Equation (2) can be used as a benchmark. If the manager is a stock picker, the βs of the benchmark are constructed to identically match those of the managed portfolio. Return performance is simply the managed portfolio's ex post intercept less the benchmark's intercept.

In contrast, a timing manager is attempting to predict the ex post realization of one or more factors, which implies that one or more βs are changed accordingly from period to period; that is, a β will increase in a period when its corresponding factor is predicted to be large and positive, and vice versa. In this case, the mimicking benchmark would have βs fixed as constant time series averages of the βs of the managed portfolio. Again, the ex post difference in intercepts measures performance.

Another important application of mimicking portfolios is in estimating the benefits of diversification. As discussed by Pukthuanthong-Le and Roll [2009] and Roll [2013] more forcefully, the simple correlation between already diversified portfolios is a completely inadequate metric for assessing whether combining the two would help reduce risk. The reason is simple: a mimicking portfolio for, say, portfolio B can be constructed by reweighting the individual assets already in, say, portfolio A. If that mimicking portfolio of A assets, denoted A*, has minimal residual risk, there is little benefit from combining A and B, even if A and B are only weakly correlated to start.

For example, Roll [2013] showed that the global minimum-variance portfolio that can be obtained by combining A*, the B-mimicking portfolio from A assets, with the original portfolio B has a weight in the original portfolio B equal to

$$w = Var(\varepsilon_{A*})/[Var(\varepsilon_{A*}) + Var(\varepsilon_B)]$$

where $Var(\varepsilon_{A*})$ is the residual variance of A*, the B-mimicking portfolio of A assets constructed to have B's set of βs; and $Var(\varepsilon_B)$ is the residual variance of the original portfolio B. In other words, when the mimicking portfolio has zero residual volatility, there is no diversification benefit at all. Notice that original correlation between A and B plays no role in this calculation.

CONSTRUCTION METHOD FOR MIMICKING PORTFOLIOS

A mimicking portfolio matches the exposures of a target underlying asset, which may itself be an individual asset or a portfolio, while minimizing residual volatility. Let N denote the number of assets used in forming the mimicking portfolio, and let T denote the number of return observations available for all relevant time series.

For the target asset j, Equation (1) is fit using ordinary least squares (OLS), which results in estimates of all its parameters; the K estimates of the βs are $\hat{\beta}_{j,1},\ldots,\hat{\beta}_{j,K}$, the intercept estimate is $\hat{\alpha}_j$ and the OLS residual at time t is $\hat{\varepsilon}_{j,t} = \tilde{R}_{j,t} - [\hat{\alpha}_j + \hat{\beta}_{j,1}\tilde{f}_{1,t} + \hat{\beta}_{j,2}\tilde{f}_{2,t} + \cdots + \hat{\beta}_{j,K}\tilde{f}_{K,t}]$. The unbiased OLS residual variance is given by $\hat{\sigma}_j^2 = \sum_{t=1}^{T}\hat{\varepsilon}_{j,t}^2/(T-K-1)$.[3]

Then, for each of the N assets preselected to be included in the mimicking portfolio, the same OLS estimation is conducted, not necessarily over the same time period, although the form of the estimates is identical to that of the target asset j just presented. Under the utmost generality, dependence is allowed among these N residuals, and the covariance between the residuals of asset i and m is denoted $\hat{\sigma}_{i,m}$. Note that the sample residual variance for asset i is simply $\hat{\sigma}_{i,i}$.

A mimicking portfolio is a weighted average of the N assets, with weights given by the column vector $\mathbf{w} = (w_1, \ldots, w_N)'$.[4] Formally, the optimization problem whose solution gives the mimicking portfolio weights is

$$Min_{\mathbf{w}} \sum_{i=1}^{N}\sum_{m=1}^{N} w_i w_m \hat{\sigma}_{i,m} \quad \text{subject to} \quad \sum_{i=1}^{N} w_i = 1 \quad \text{and} \quad \sum_{i=1}^{N} w_i \hat{\beta}_{i,k} = \hat{\beta}_{j,k}\,\forall k.$$

This is a constrained multivariate optimization which can be solved using Lagrangian multipliers. The associated Lagrangian is

$$\mathcal{L} = \sum_{i=1}^{N} w_i^2\hat{\sigma}_i^2 + \sum_{i<m} 2w_i w_m \hat{\sigma}_{i,m} + \xi_1\left[-\hat{\beta}_{j,1} + \sum_{i=1}^{N} w_i\,\hat{\beta}_{i,1}\right] + \cdots + \xi_K\left[-\hat{\beta}_{j,K} + \sum_{i=1}^{N} w_i\hat{\beta}_{i,K}\right] + \xi_{K+1}\left[-1 + \sum_{i=1}^{N} w_i\right]$$

$$\Rightarrow \mathcal{L} = \sum_{i=1}^{N} w_i^2 \hat{\sigma}_i^2 + \sum_{i<m} 2 w_i w_m \hat{\sigma}_{i,m}$$
$$+ \sum_{m=1}^{K} \left[\xi_m \left(-\hat{\beta}_{j,m} + \sum_{i=1}^{N} w_i \hat{\beta}_{i,m} \right) \right] + \xi_{K+1} \left(-1 + \sum_{i=1}^{N} w_i \right)$$

This system can be solved exactly from first-order conditions, taking the partial derivatives of the above Lagrangian first with respect to each w_i and then with respect to the ξ_i. The partial derivative of Equation (1) with respect to w_i is

$$\frac{\partial \mathcal{L}}{\partial w_i} = \sum_{m=1}^{N} 2 w_m \hat{\sigma}_{i,m} + \sum_{m=1}^{K} \xi_m \hat{\beta}_{i,m} + \xi_{K+1} = 0 \quad for\ i = 1, 2, \ldots, N \tag{1}$$

There are N equations of this form, each one linear in $\mathbf{w}$, so the entire system can be expressed compactly in matrix notation. Define $\boldsymbol{\Sigma}$ to be the $N \times N$ variance–covariance matrix of the regression residuals, and define

$$\mathbf{b} = \begin{bmatrix} \hat{\beta}_{1,1} & \hat{\beta}_{2,1} & \cdots & \hat{\beta}_{N,1} \\ \hat{\beta}_{1,2} & \hat{\beta}_{2,2} & \cdots & \hat{\beta}_{N,2} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\beta}_{1,K} & \hat{\beta}_{2,K} & \cdots & \hat{\beta}_{N,K} \end{bmatrix}$$

$$\xi = (\xi_1, \xi_2, \ldots, \xi_{K+1})'$$

$$\mathbf{c} = (\hat{\beta}_{j,1}, \hat{\beta}_{j,2}, \ldots, \hat{\beta}_{j,K}, 1)'$$

$$\mathbf{A} = [\mathbf{b}' 1]$$

The system of Equations (1) can thus be expressed as

$$2\boldsymbol{\Sigma}\mathbf{w} = -\mathbf{A}\xi \rightarrow \mathbf{w} = -\frac{1}{2}\Sigma^{-1}\mathbf{A}\xi \tag{2}$$

Note that, for Equation (2) to be valid, the covariance matrix $\boldsymbol{\Sigma}$ must be nonsingular, which will not be the case when there are fewer sample dates than assets. This problem can be overcome in at least two ways. First, given a sample of size T, the number of assets N used in constructing a mimicking portfolio can be made strictly less than T and, for efficiency, considerably less than T. This would seem to be an easy enough condition unless T is very small. Alternatively, the problem can be finessed by making a particular reasonable assumption that results in the alternative procedure outlined in the next section.

We proceed by taking the partials of Equation (1) with respect to each ξ_j for $j = 1, 2, \ldots, K$ to produce K additional equations, each of the form

$$\frac{\partial \mathcal{L}}{\partial \xi_i} = \sum_{m=1}^{N} w_m \hat{\beta}_{m,i} - \hat{\beta}_{j,i} = 0 \tag{3}$$

Taking the partial with respect to ξ_{K+1} gives the normalization condition on the system as

$$\sum_{i=1}^{N} w_i - 1 = 0 \tag{4}$$

Note that the system of $K+1$ equations formed by Equations (3) and (4) is $\mathbf{A'w} = \mathbf{c}$. Inserting the expression for $\mathbf{w}$ from Equation (2) into this gives

$$\mathbf{A}'\sum^{-1}\mathbf{A}\xi = -2\mathbf{c} \tag{5}$$

Note that $\mathbf{A}'\Sigma^{-1}\mathbf{A}$ is a square $(K + 1) \times (K + 1)$ matrix that is assumed to be invertible. However, $\mathbf{A}$ is not square, so $(\mathbf{A}'\Sigma^{-1}\mathbf{A})^{-1} \neq \mathbf{A}^{-1}\Sigma(\mathbf{A}')^{-1}$. With this in mind, we can write that

$$\xi = -2\left(\mathbf{A}'\sum^{-1}\mathbf{A}\right)^{-1}\mathbf{c} \tag{6}$$

Inserting Equation (6) into Equation (2) provides the solution

$$\mathbf{w} = -\frac{1}{2}\sum^{-1}\mathbf{A}\xi = \sum^{-1}\mathbf{A}\left(\mathbf{A}'\sum^{-1}\mathbf{A}\right)^{-1}\mathbf{c} \tag{7}$$

ASSUMING ORTHOGONAL RESIDUALS

The solution just given requires nonsingularity of the covariance matrix Σ and the matrix $\mathbf{A}'\Sigma^{-1}\mathbf{A}$. If the residuals are assumed to be mutually orthogonal, these conditions are assured. With many factor models, mutual orthogonality should be approximately valid because the factors subsume most of the commonality. The assumption of orthogonality also obviates any worry about nonstationarity in residual correlations; they are always zero, by assumption.

Hence, $\boldsymbol{\Sigma}^{-1}$ becomes diagonal:

$$\begin{bmatrix} 1/\hat{\sigma}_1^2 & 0 & \cdots & 0 \\ 0 & 1/\hat{\sigma}_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\hat{\sigma}_N^2 \end{bmatrix}$$

Let $\mathbf{C} = \sqrt{\Sigma^{-1}}\mathbf{A}$ so that Equation (7) can be written

$$\mathbf{w} = \sum^{-1}\mathbf{A}(\mathbf{C}'\mathbf{C})^{-1}\mathbf{c} \tag{8}$$

Now, the only matrix inverse required is $(\mathbf{C}'\mathbf{C})^{-1}$ (because $\boldsymbol{\Sigma}^{-1}$ is already given explicitly earlier), and $\mathbf{C}$ is an $N \times (K + 1)$ matrix in which we have that $N \gg K$ because the set of factors is relatively parsimonious compared to the number of stocks, N. Thus, there should be no issues with singularity in this case, and Equation (8) should always be computationally feasible.

In the remainder of the article, we assume orthogonal residuals. Even with this simplification, our method brings significant reductions in the idiosyncratic volatility of mimicking portfolios.

EMPIRICAL APPLICATION: GLOBAL FACTORS

Factor Selection

To provide an empirical demonstration of our method, we now move away from the full generality of mimicking portfolios for any set of factors to focus on some specific global factors. In accordance with previous literature (Pukthuanthong-Le and Roll [2009] and many others), we assume the existence of global factors that drive asset returns across well-integrated international markets. The covariance matrix of country-specific index returns for equities, fixed income, and real estate can be employed to derive orthogonal principal components directly as factors (Cotter, Gabriel, and Roll [2016]). However, we adopt a slightly different approach designed to proxy for the same underlying global market factors, which has the advantage of being both economically intuitive and easily observable.

Specifically, the vector space spanned by orthogonal factors obtained from a principal components analysis can be also be spanned by a number of exchange-traded funds (ETFs), provided that they are sufficiently heterogeneous. Hence, we consider a large number of ETFs exposed to major asset classes in broad geopolitical regions (United States, Eurozone, Asia, and Emerging Markets) as proxy factors. However, many existing ETFs may be redundant in the sense that their addition to the set of proxy factors does not change the cumulative vector space spanned. To account for this fact, and to make our model more parsimonious, we compute the correlation matrix of candidate ETFs and retain only one from each pair whose correlation is higher than 0.7. The resulting eight ETFs, listed in Exhibit 1, Panel A, are used in the remainder of this study as proxies for global market factors.

ETFs are a relatively recent innovation with various inception dates. Hence, our sample period begins when a sufficient number of heterogeneous ETFs becomes available, which in our case falls on January 30, 2009. We then continue the sample through December 30, 2016. Panel B of Exhibit 1 provides summary statistics for the daily returns for the eight chosen ETFs over this sample period. Panel C reports the correlation matrix for their daily returns.

It is important to note that although ETF proxy factors likely span a vector space similar to that of the various factors considered in the previous literature, they are not mutually orthogonal, which implies that exposures to individual factors need not correspond to any factors explored in the earlier literature or have any particular macroeconomic interpretation. Nonetheless, the magnitude of a given asset's loadings over all of our ETF-based factors collectively should capture most of the relevant underlying information.

We define a mimicking portfolio for a given target asset as a portfolio with identical exposures to the factor-proxy ETFs, along with minimal remaining idiosyncratic volatility among all possible such portfolios. The idiosyncratic volatility of the target asset, in the case of global market factors, can be regarded as encapsulating both specific isolated events about that asset and the local or regional volatility. Consequently, the derived mimicking portfolio essentially presents the same exposure profile to global factors as the asset being mimicked, while reducing (and attempting to eliminate) the local risk involved. Such a mimicking portfolio should be useful for hedging away risk caused by global market movements.

Exhibit 1

ETF Proxies for Global Factors

Panel A: Identities of Factor-Proxy ETFs

Name	Ticker	Region	Asset Class
iShares Core U.S. Aggregate Bond ETF	AGG	USA	Fixed-Income
iShares U.S. Real Estate ETF	IYR	USA	Real-Estate
iShares International Treasury Bond	IGOV	Eurozone	Fixed-Income
iShares Europe Developed Real Estate ETF	IFEU	Eurozone	Real-Estate
iShares J.P. Morgan USD Emerging Markets Bond ETF	EMB	Asia ex Japan	Fixed-Income
iShares MSCI Japan ETF	EWJ	Japan	Equities
iShares MSCI Hong Kong ETF	EWH	Hong Kong	Equities
iShares S&P GSCI Commodity-Indexed Trust	GSG	International	Commodities

Panel B: Factor-Proxy ETF Daily Return Summary Statistics

	AGG	IYR	IGOV	IFEU	EMB	EWJ	EWH	GSG
mean	0.0149	0.0741	0.0053	0.0559	0.0322	0.0311	0.0543	–0.0166
std	0.2316	1.6921	0.5592	1.5538	0.5084	1.2196	1.2991	1.3858
min	–1.244	–10.7043	–2.1629	–12.1189	–4.4433	–7.0305	–6.1665	–7.0648
25%	–0.119	–0.5666	–0.3355	–0.7548	–0.2005	–0.622	–0.6137	–0.8043
50%	0.0267	0.0976	0.0194	0.107	0.044	0.081	0.0519	0
75%	0.1578	0.7305	0.3493	0.8686	0.2703	0.7293	0.7394	0.7428
max	1.5891	14.9897	2.6738	7.3864	3.5025	7.8227	8.7845	5.5844

Panel C: Factor-Proxy ETF Daily Return Correlations

	IYR	IGOV	IFEU	EMB	EWJ	EWH	GSG
AGG	–0.078	0.31	–0.124	0.195	–0.151	–0.196	–0.191
IYR		0.203	0.539	0.299	0.545	0.586	0.38
IGOV			0.358	0.251	0.181	0.127	0.218
IFEU				0.391	0.558	0.566	0.444
EMB					0.281	0.348	0.262
EWJ						0.632	0.384
EWH							0.446

Panel D: Case Study Correlation Matrix

	LQD	RWX
AGG	0.785	–0.151
IYR	0.136	0.704
IGOV	0.333	0.329
IFEU	0.065	0.783
EMB	0.306	0.418
EWJ	0.031	0.742
EWH	0.016	0.752
GSG	–0.056	0.51

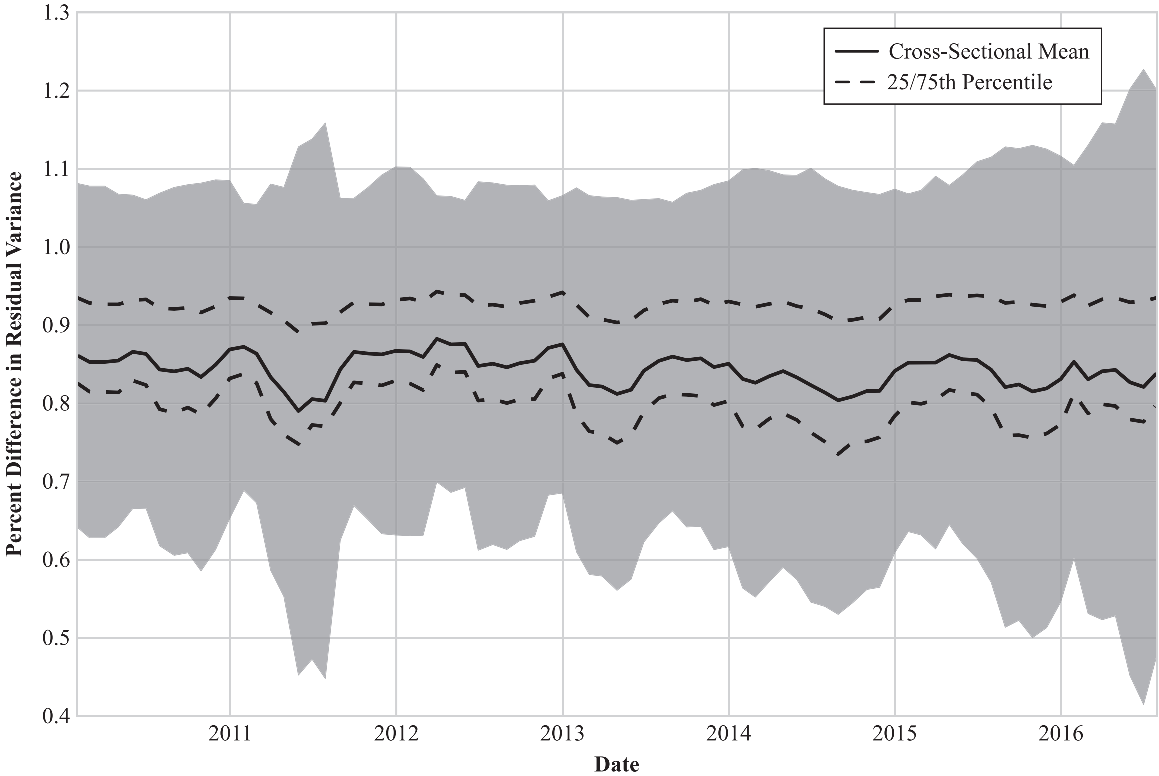
Notes: Eight ETFs, representing various asset classes and major geographic regions, serve as our proxies for global factors. Data for these ETFs span January 30, 2009 through December 30, 2016, or 1,995 trading days. Larger correlations in Panels C and D are more darkly colored.

Sample Description

To obtain a tradeable set of individual assets to use as targets and as constituents of mimicking portfolios, we download NYSE-listed stocks from CRSP[5] and remove those with incomplete data over the 2009–2016 period, yielding a universe of 1,634 assets. These stocks might be susceptible to a survivorship bias because they do not include stocks that are delisted before the end of the period or initial public offerings that go public during

Exhibit 2

The Relative Difference between Target and Mimicking Portfolio Idiosyncratic Variance



Notes: The idiosyncratic variance of 1,634 NYSE individual target stocks and the idiosyncratic variance of each target's mimicking portfolio are estimated over 100 days after each month-end rebalancing date. The relative difference is $1-\hat{\sigma}^2_{M_j}/\hat{\sigma}^2_j$ *for target j and mimicking portfolio* M_j*. The cross-sectional mean and the 25th and 75th percentiles of the cross-sectional distribution of these differences are plotted over the sample of rebalancing dates. The shading represents plus or minus two standard deviations from the cross-sectional mean. Note that no observations necessarily attain the values indicated by the shading; rather, it indicates the spread in observations.*

the period. However, because we are not concerned with average return performance per se, survivorship effects are not pertinent to this study and should not confound any of our conclusions.

Cross-Sectional Analysis

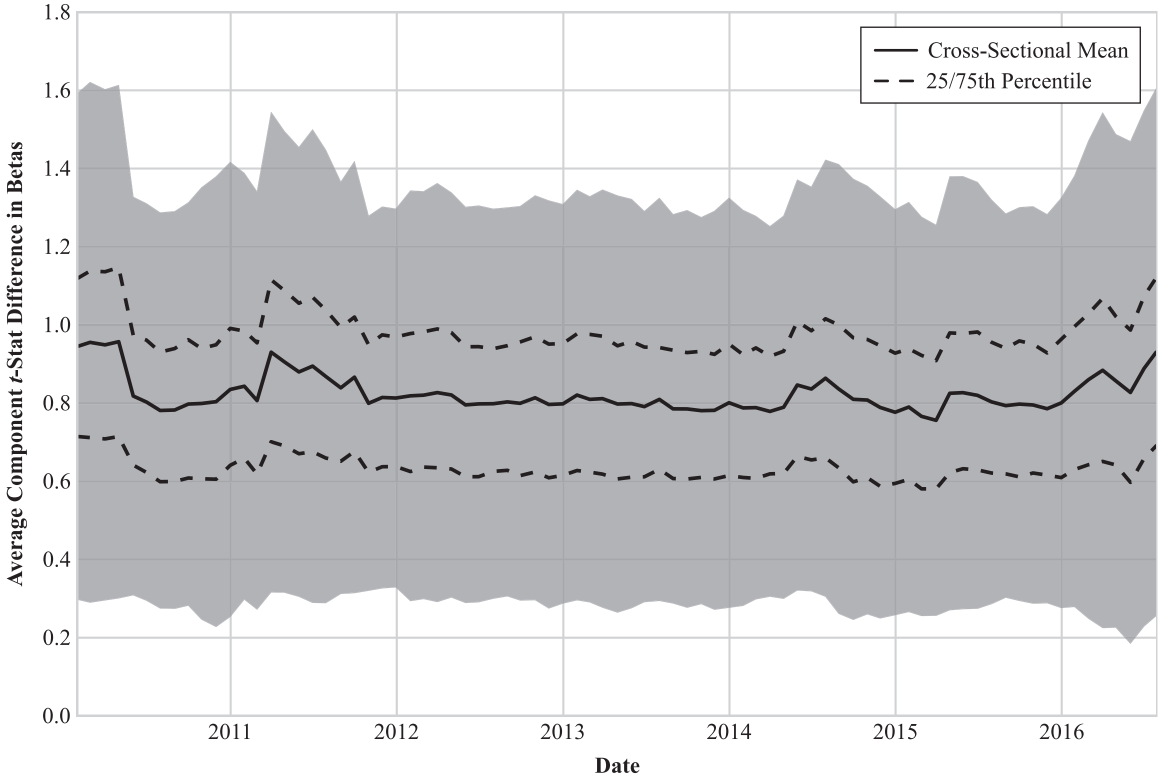
We now provide evidence that mimicking portfolios constructed as described, under the assumption of orthogonal residuals, perform as expected, and we report further on the robustness of their main characteristics.

We construct a mimicking portfolio for each of the 1,634 target assets on the last day of each calendar month,[6] provided that there were at least 250 trading days before that date. To do so, we first estimate the OLS parameters of the target asset using data from the 300 prior trading days (or up to 300 but not less than 250 for the initial dates when we have insufficient data). The estimated parameters then are inserted into Equation (8) to obtain mimicking portfolio weights, which we hold constant until the next rebalancing date, at which time the procedure is repeated. This yields a time series of returns for 1,634 mimicking portfolios, one for each target asset. Note that this time series excludes the initial period during which there are insufficient data (before January 29, 2010).

Even though they are formed under the simplified assumption of orthogonal residuals, the resulting mimicking portfolios have much smaller idiosyncratic volatility than the original assets. To see this, we estimate the residual variance of each target asset j and its associated mimicking portfolio M_j over the 100 days subsequent to each rebalancing date and then record

EXHIBIT 3

Exposure to Factors



Notes: We use the 100 days subsequent to each rebalancing date to estimate the factor betas for every target asset and its associated mimicking portfolio and estimate the standard errors of the target betas. The resulting normalized estimated beta difference between target j and its mimicking portfolio M_j is $|\hat{\beta}_{j,k} - \hat{\beta}_{M_{j,k}}|/\hat{\sigma}_{\hat{\beta}_{j,k}}$ where $\hat{\sigma}_{\hat{\beta}_{j,k}}$ is the OLS standard error for target j's beta on the kth factor The solid line is the cross-sectional mean difference, and the two dashed lines are the 25th and 75th percentiles of the cross-sectional distribution of differences. The shading represents plus or minus two standard deviations from the cross-sectional mean. Note that no observations necessarily attain the values indicated by the shading; rather, it indicates the spread in observations.

their relative difference, $1-\hat{\sigma}^2_{M_j}/\hat{\sigma}^2_j$. Exhibit 2 depicts the cross-sectional distribution of this difference; the cross-sectional mean is solid, the 25th and 75th percentiles are dashed, and two standard deviations above and below the mean are shaded.[7]

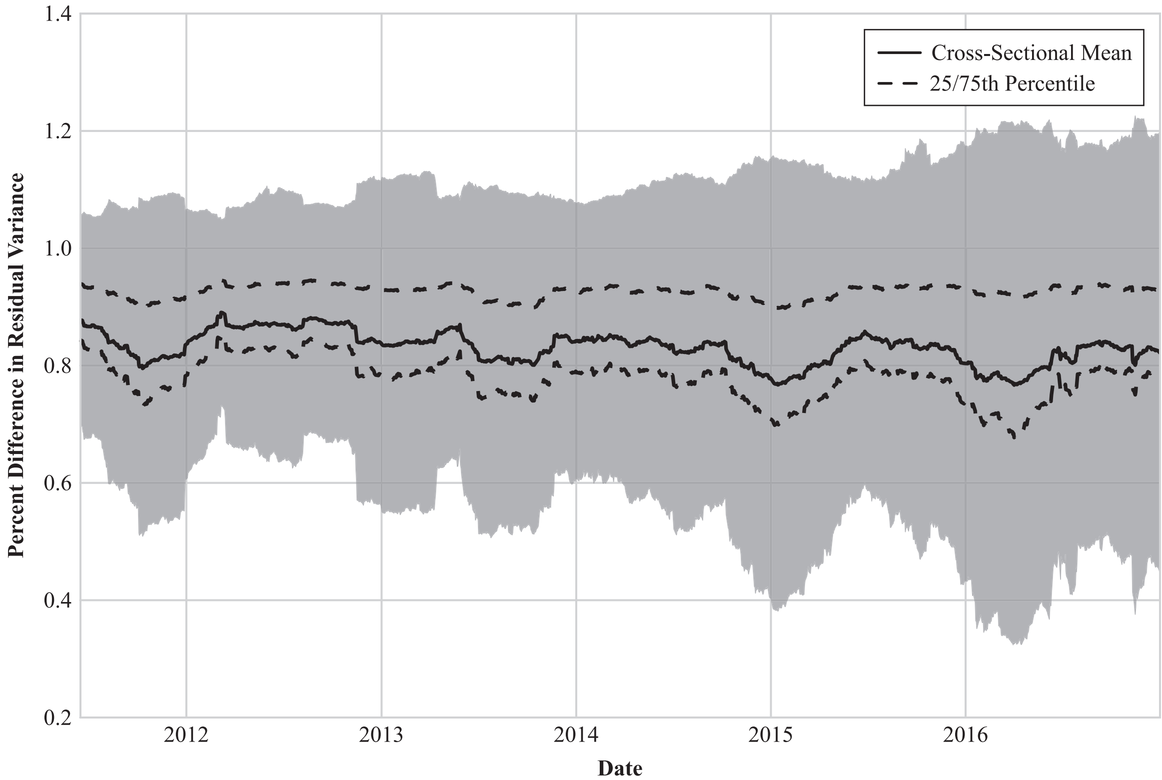
As would be expected given that the objective of portfolio formation is minimization of variance, mimicking portfolios do provide roughly an 80% reduction on average. This is accomplished, by the way, despite having assumed, perhaps wrongly, that individual stock residuals are orthogonal to one another. Note that each target asset can be held within its own mimicking portfolio, but its own weight is likely to be very small because of diversification.

Moreover, mimicking portfolios match the betas of their underlying target assets reasonably well, even with rather slow (monthly) rebalancing. Of course, the betas match exactly on the rebalancing date itself, but we look further in the future and examine how they fare over the 100 days immediately afterward. To facilitate this examination, we recompute OLS regressions for each target and its mimicking portfolio over the 100 postrebalancing days. The familiar OLS standard error of beta is $\hat{\sigma}_{\hat{\beta}_{j,k}}$ for target j's beta on the kth factor. The normalized beta difference between target j and its mimicking portfolio M_j is thus $|\hat{\beta}_{j,k} - \hat{\beta}_{M_j,k}|/\hat{\sigma}_{\hat{\beta}_{j,k}}$.

We record the mean across eight factors of this difference and report the cross-sectional distribution in Exhibit 3. As in Exhibit 2, the cross-sectional mean is solid, and the 25th and 75th percentiles are dashed with shading indicating plus or minus two standard deviations

Exhibit 4

Idiosyncratic Volatility Reductions under Stale Information



Notes: The idiosyncratic variance of 1,634 NYSE individual target stocks and the idiosyncratic variance of each target's stale mimicking portfolio are estimated over 100 days before each date after construction. The stale mimicking portfolios have their weights computed exactly once and then held constant for the remainder of the period. The relative difference is $1-\hat{\sigma}^2_{M_j}/\hat{\sigma}^2_j$ *for target j and mimicking portfolio* M_j*. The cross-sectional mean and the 25th and 75th percentiles of the cross-sectional distribution of these differences are plotted over the sample of rebalancing dates. The shading represents plus or minus two standard deviations from the cross-sectional mean. Note that no observations necessarily attain the values indicated by the shading; rather, it indicates the spread in observations.*

around the mean. The mean is less than one, and 50% of the values lie within plus or minus 0.2 target standard errors of one. Hence, in the majority of cases, mimicking portfolios are tracking their targets quite well over the 100 days after portfolio formation.

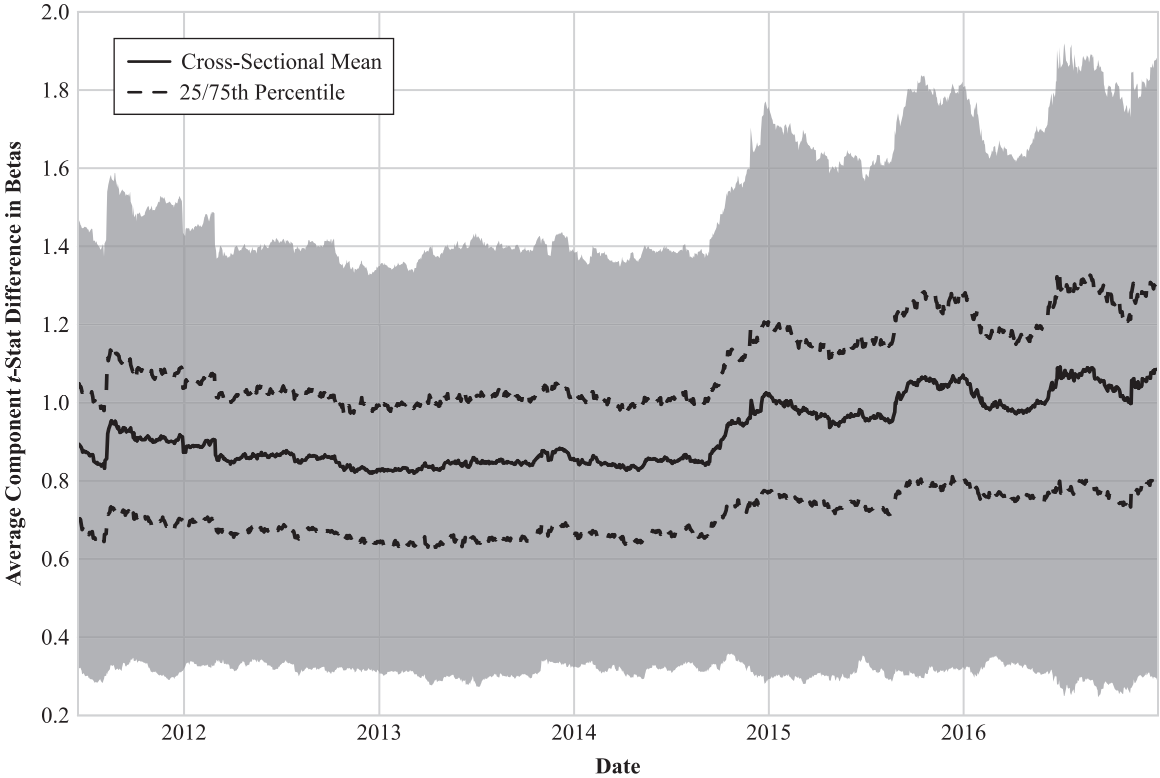
DECAY IN THE QUALITY OF MIMICKING INDUCED BY NONSTATIONARITIES

We now investigate how well mimicking persists without rebalancing. Because mimicking portfolios are constructed to mimic their target assets only at a specific point in time, nonstationarity in the assets used for portfolio construction and in the target assets themselves could very well bring decay in the quality of mimicking. We explore this effect with our global market factor model but note that the decay might be different for other models.

Specifically, we construct a mimicking portfolio for each asset in our stock universe on January 25, 2011 (the 500th date in our sample, chosen as an arbitrary date with sufficient return history). The necessary parameters for construction are estimated using the 300 trading days prior to this date, and the calculation is performed using the same assumption of orthogonal residuals employed in the cross-sectional analysis. However, because we are now interested in analyzing how quickly mimicking portfolios lose their desired properties in the absence of periodic updating, the original portfolio weights are fixed for the remainder of the sample period. Subsequent to January 25, 2011, we simply calculate the returns of these fixed-weight portfolios.

Exhibit 5

Exposures to Factors under Stale Information



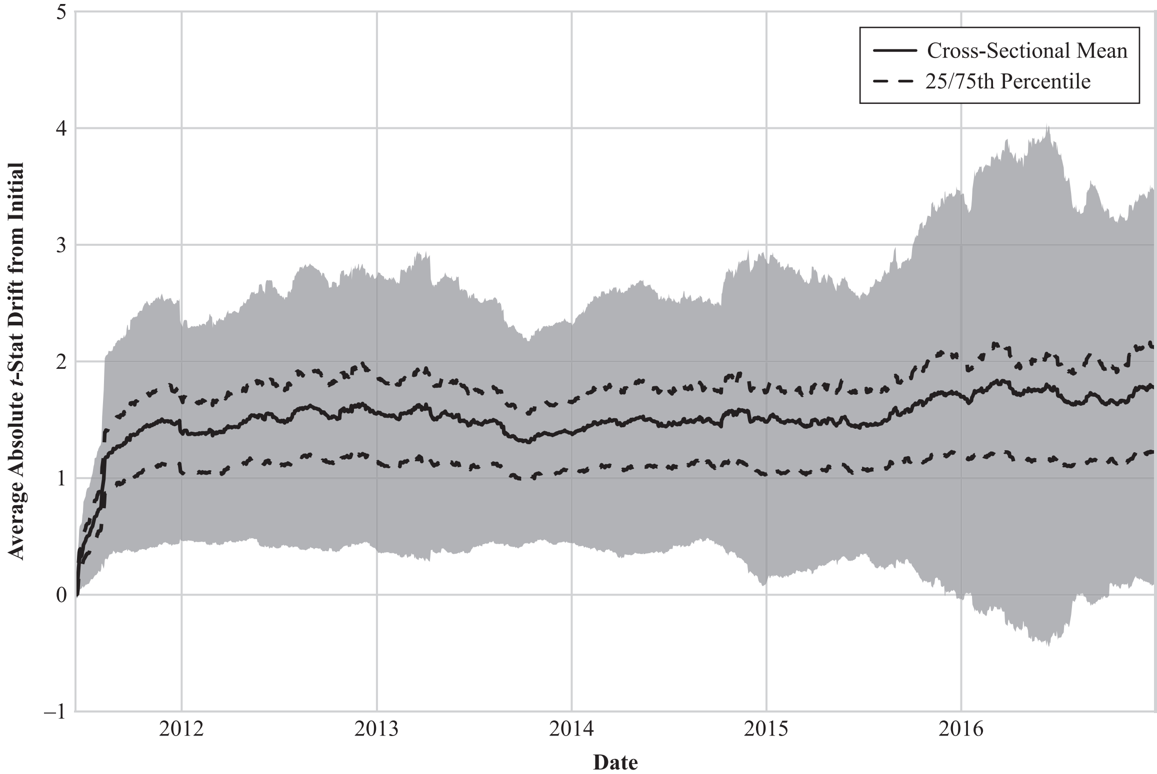
Notes: We use the 100 days prior to each date after construction to estimate the factor betas for every target asset and its associated stale mimicking portfolio and estimate the standard errors of the target betas. The stale mimicking portfolios have their weights computed exactly once and then held constant for the remainder of the period. The resulting normalized estimated beta difference between target j and its mimicking portfolio M_j *is* $|\hat{\beta}_{j,k} - \hat{\beta}_{M_j,k}| / \hat{\sigma}_{\hat{\beta}_{j,k}}$ *where* $\hat{\sigma}_{\hat{\beta}_{j,k}}$ *is the OLS standard error for target j's beta on the kth factor The solid line is the cross-sectional mean difference, and the two dashed lines are the 25th and 75th percentiles of the cross-sectional distribution of differences. The shading represents plus or minus two standard deviations from the cross-sectional mean. Note that no observations necessarily attain the values indicated by the shading; rather, it indicates the spread in observations.*

These portfolios could eventually become stale because they do not incorporate any new information after construction. To measure this quantitatively, we use the same metrics as before but now compare possibly stale mimicking portfolios (which are never rebalanced) to their underlying target assets. On each date subsequent to January 25, 2011, we use the 100 prior days to estimate the factor betas and idiosyncratic volatilities of the target asset and its stale mimicking portfolio, and then we compute their normalized spreads. This differs from our previous analysis in two ways. First, the mimicking portfolios are now possibly stale instead of being rebalanced monthly. Second, instead of using the 100 days after each rebalancing date for parameter estimation, we now look at every date subsequent to the construction date and use the 100 prior days for estimation.

Exhibit 4 reports on the spread in idiosyncratic volatility, again computed as a simple percentage difference. Interestingly, this figure reveals that even several years after portfolio construction, at which point in time one would almost surely expect the original information to be stale, mimicking portfolios offer significant reductions in idiosyncratic volatility, at least on average. However, the shading shows that the cross-sectional spread of the metric is widening over time, which suggests some decay for individual cases.

Exhibit 5 reports on the spread in factor betas, again using the average absolute t-statistic described in the previous cross-sectional analysis. This metric is also

EXHIBIT 6
Drift in Factor Exposures: Underlying Assets



Notes: On each date and for each target asset, we compute the absolute difference between the initial factor betas and the current factor betas, normalized with respect to the standard error in the initial factor beta estimates, and then take the mean of the resulting vector. The solid line is the cross-sectional mean, and the two dashed lines are the 25th and 75th percentiles of the cross-sectional distribution. The shading represents plus or minus two standard deviations from the cross-sectional mean. Note that no observations necessarily attain the values indicated by the shading; rather, it indicates the spread in observations.

stable, with the mean cross-sectional value remaining below two even five years after portfolio construction. There is a modest increase after about three years with little subsequent increase. However, the top of the shaded area shows a larger upswing, which is evidence that decay is more substantial in some stocks.

To verify the stability in target assets' betas, we compute them and their associated standard errors for every underlying asset on the construction date; then, on each subsequent date, we recalculate them and record the difference between the new betas and the original betas, normalized by the estimated standard errors from the initial date. The cross-sectional distribution of the resulting vector is plotted in Exhibit 6. The mean difference grows for about a year and is virtually stationary thereafter. This confirms that, on average, betas are quite stable. Again, however, individual stocks in the extremes depicted by the shading exhibit substantial nonstationarity, which is accentuated as time progresses.

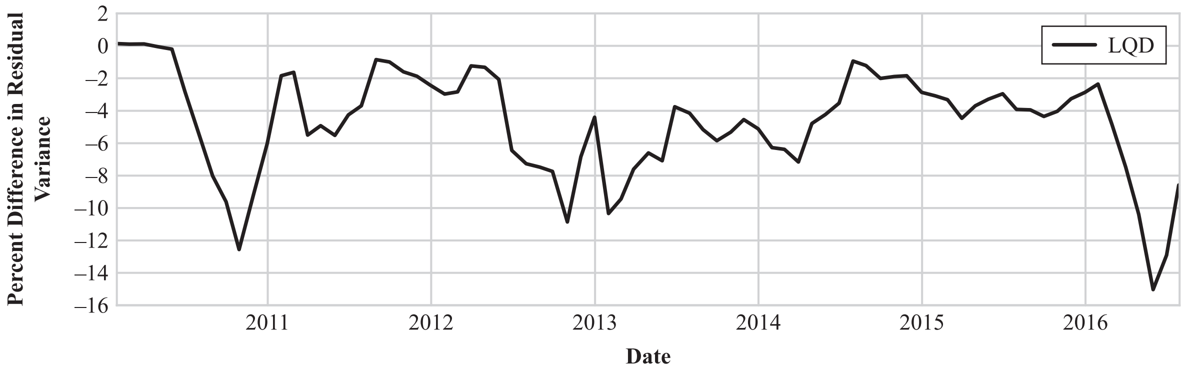
A CASE STUDY: MIMICKING PORTFOLIOS OF EQUITIES FOR NONEQUITY ASSETS

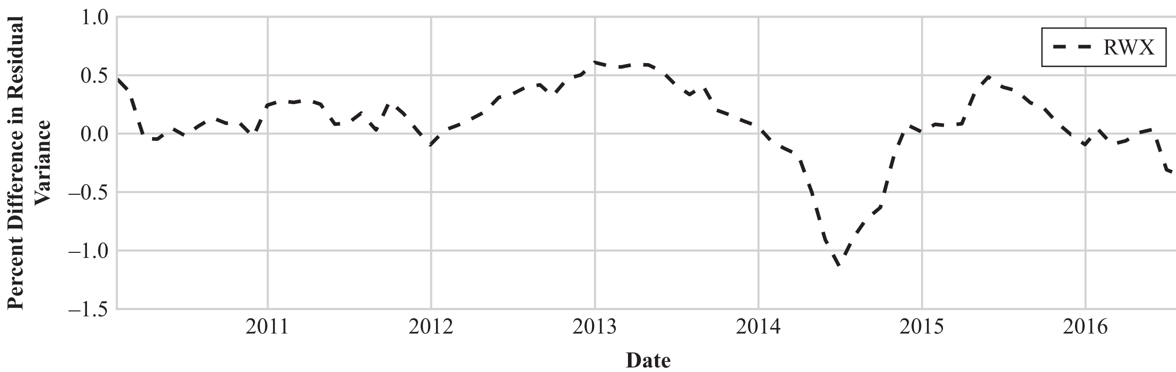
The analysis thus far shows that mimicking portfolios reduce idiosyncratic volatility relative to underlying single-name equities, while still providing closely matched exposures to market risk factors. We now examine whether portfolios composed only of equities can mimic nonequity portfolios.

As a demonstration, we consider as targets two ETFs not previously used: LQD, an investment-grade corporate bond ETF, and RWX, an international real-estate ETF. Note that both of these ETFs are already broad, diversified portfolios. We construct mimicking

Exhibit 7

Equity Mimicking Portfolios for Nonequity Targets: Idiosyncratic Volatility Comparison





Notes: The idiosyncratic variance of two target nonequity ETF targets and the idiosyncratic variance of each target's equity-only mimicking portfolio are estimated over 100 days after each month-end rebalancing date. The relative difference is $1-\hat{\sigma}^2_{M_j}/\hat{\sigma}^2_j$ *for target j and mimicking portfolio,* M_j *which is plotted for the two targets discussed in the text.*

portfolios for these two targets using the exact same procedure previously described. Exhibit 7 compares idiosyncratic volatilities, and Exhibit 8 depicts the differences in their factor exposures.

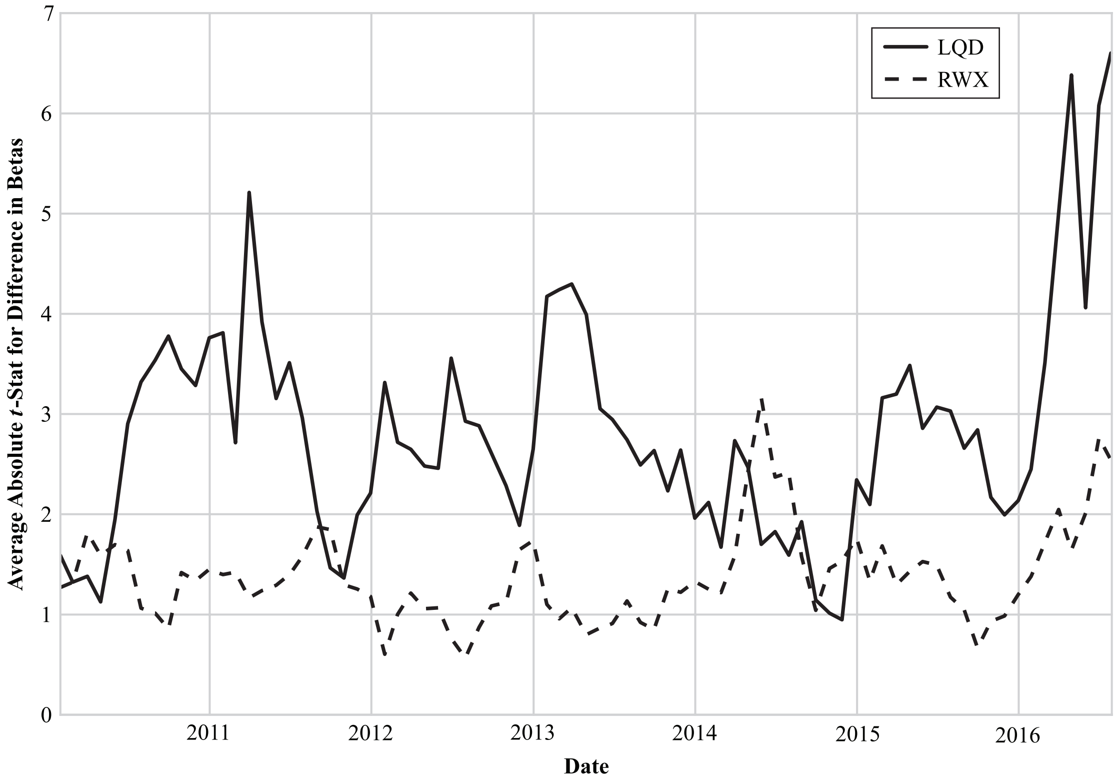
As shown in Exhibit 7, the idiosyncratic variance of the mimicking portfolio for LQD is often higher than that of LQD itself. This suggests that bond portfolios have very low idiosyncratic variance, and equity-only portfolios are unlikely to lower it further. In contrast, for the real estate ETF, RWX, there is some reduction in idiosyncratic variance except during a brief interval between 2014 and 2015.

Exhibit 8 demonstrates that the mimicking portfolio for RWX matches the exposures of its underlying assets fairly well, although we see significantly more deviation from the underlying asset for LQD than was observed in the cross-sectional results presented earlier.

The rather discouraging results in Exhibit 7, particularly for LQD, can be explained in part by the residual volatilities themselves, as plotted in Exhibit 9. Both RWX and LQD have small residual volatilities, but LQD's is much the smaller of the two, often only quarter to half that of RWX and indeed is sometimes close to zero. Any mimicking equity-only portfolio is unlikely to improve on this situation. Also, very low standard error of betas magnifies the difference shown in Exhibit 8. The one episode when the mimicking portfolio for RWX performs especially poorly, between 2014 and 2015, corresponds exactly to when the residual volatility of RWX drops by nearly 50%, providing further evidence for low idiosyncratic volatility causing the degradation in results.

These results could very well have been anticipated. Any extremely well-diversified portfolio such as LQD, whose idiosyncratic volatility is already very small, is not likely to be well mimicked by a portfolio constructed of assets in another class. In this particular

EXHIBIT 8
Case Study: Spread in Factor Betas



Note: The normalized difference in factor betas described in the text is plotted again but calculated for the two targets of our case study: LQD and RWX.

instance, the simple correlation is quite high between LQD and one of our proxy factors, AGG. AGG is a U.S. aggregate bond ETF—which is highly sensitive, of course, to bond prices, including the corporate bonds in LQD. Essentially, the resulting poor mimicking property cannot be ascribed to the method itself but rather to the choice of proxy factors: If one of those proxies is highly correlated with the target, the residual volatility will be minimal by construction. The method works much better when no proxy factor is too highly correlated with a target.

CONCLUSION

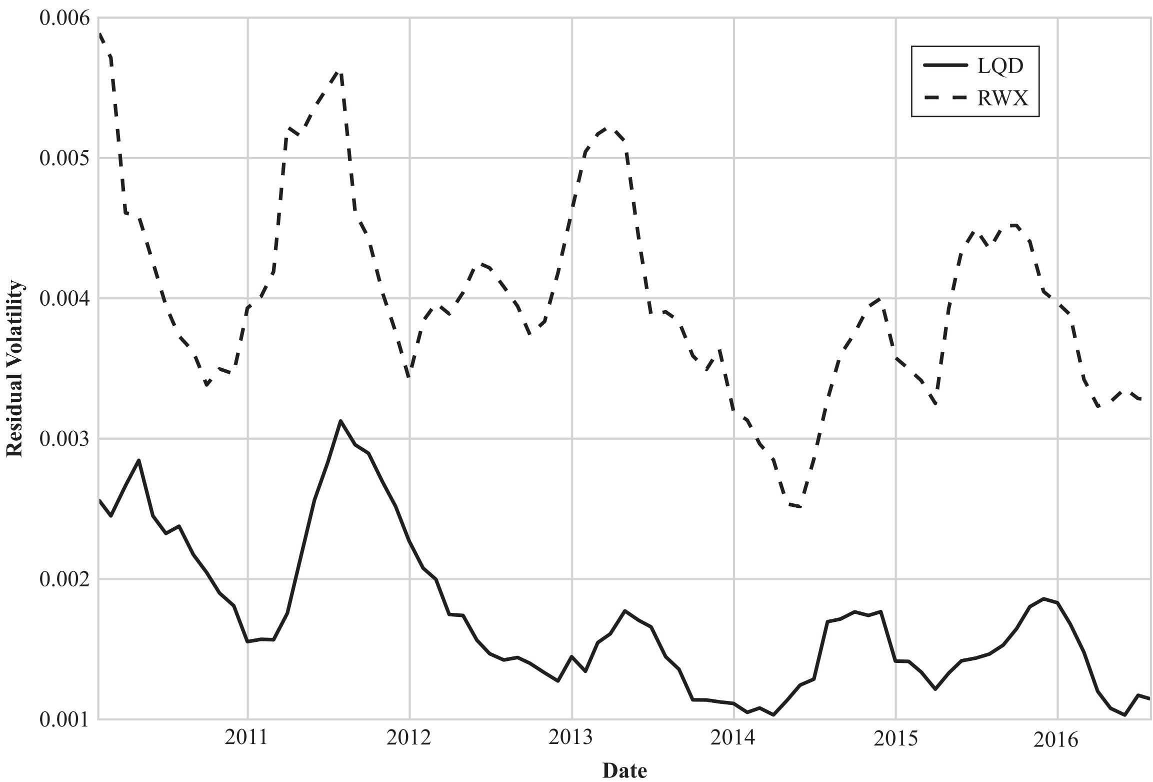
In the multifactor world of recent finance, mimicking portfolios have many uses, ranging from estimating the true potential for diversification to evaluating active manager performance. In this article, we propose a new method for constructing mimicking portfolios.

In a nutshell, our estimation guarantees that the factor sensitivities of the target asset being mimicked are exactly matched on the mimicking portfolio's construction date. This is achieved by a constrained optimization that minimizes residual volatility. To illustrate the application of our procedure, we employ a set of ETFs as proxies for global factors and construct mimicking portfolios for each NYSE stock. The mimicking portfolios have smaller residual volatilities than the individual stocks, as one would expect. Furthermore, the factor sensitivities (betas) match perfectly on the initial date.

We then study how well the mimicking portfolios retain their matching properties as individual stock betas evolve along with the betas of the constituents of the mimicking portfolios. Perhaps surprisingly, the matching is retained for several weeks, even months, after portfolio formation, but eventual rebalancing becomes essential at some point, depending on the underlying extent of nonstationarity in target assets. This implies that mimicking

Exhibit 9

Case Study: Idiosyncratic Volatilities of Underlying Assets



Note: We compute the idiosyncratic volatility of both underlying assets discussed in the case study using the 100 days subsequent to each rebalancing date again and plot them.

portfolios for diversified targets would likely retain their desirable properties even longer because such targets are inherently more stable than individual stocks.

We also study (briefly) the use of mimicking portfolios constructed from equities for matching assets in other classes such as fixed income. Understandably, the mimicking in such instances is less precise, which suggests that mimicking portfolios in various asset classes should be constructed with some assets within the same class.

ENDNOTES

[1]For example, Chen, Roll, and Ross used the change in corporate bond credit spreads as a proxy for investor confidence.

[2]Ross' no arbitrage condition is $E(R_j) = \lambda_0 + \lambda_1\beta_{j,1} + \cdots + \lambda_K\beta_{j,K}$ for all j, where the λs are the same for every j, λ_i is a risk premium associated with pervasive factor i, and λ_0 is a riskless rate of return.

[3]Henceforth, a statistical estimate will be indicated by ^ above the symbol.

[4]Vectors and matrixes are indicated by boldface.

[5]Center for Research in Security Prices. We use only NYSE stocks because they are typically more liquid.

[6]When this date falls on a nontrading day, we use the closest available trading day instead.

[7]Note that the data are asymmetric so that no value necessarily lies at the boundary of the shaded region; in particular, the ratio of variances is strictly positive, so there are no values in Exhibit 2 above 1.0, although the shading extends above that level.

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