ECE521 Assignment 4

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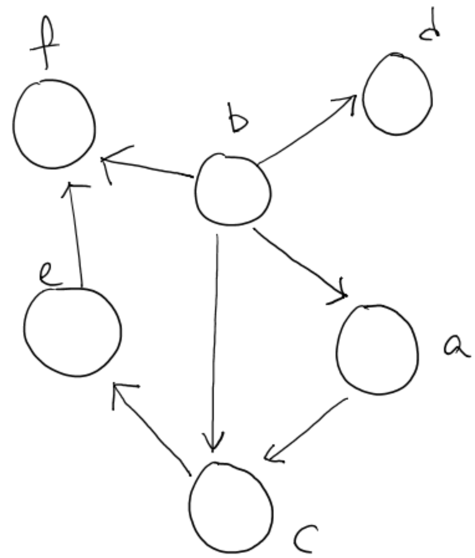
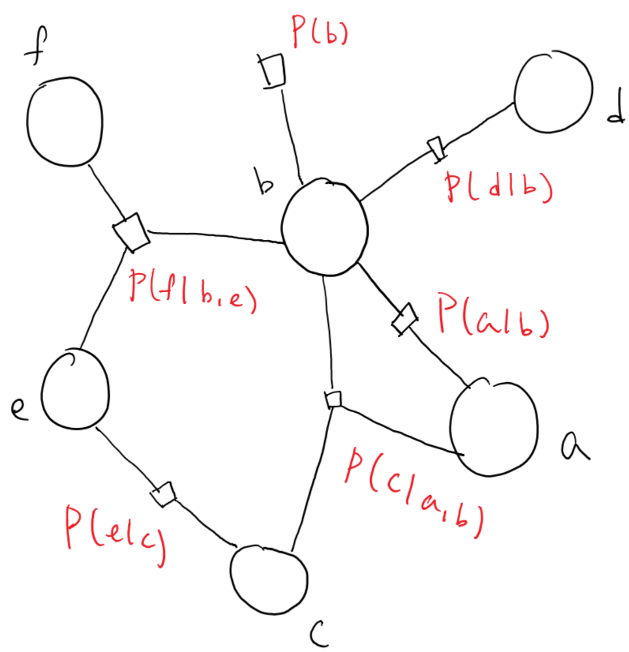
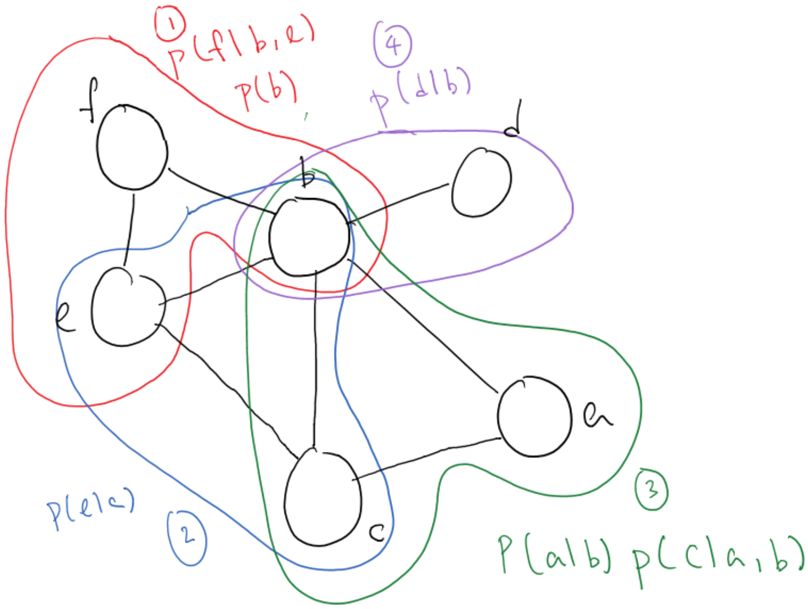
1.1.1



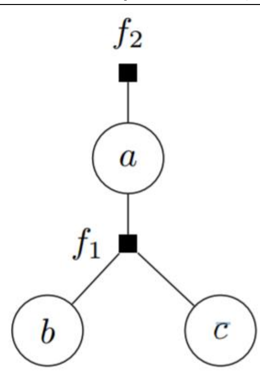
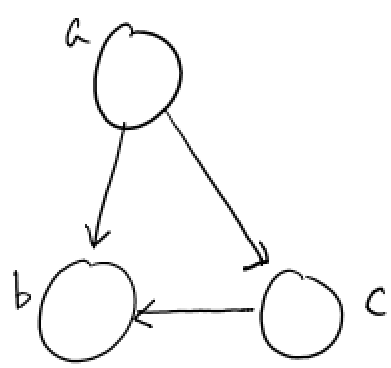
1.1.2



1.1.3



1.2.1.1a)

	Factor Graph (given in question)	Bayesian Network
Graph		
$a \perp b?$	False	False
$b \perp c?$	False	False
$a \perp c?$	False	False
$a \perp b \mid c?$	False	False
$b \perp c \mid a?$	False	False
$a \perp c \mid b?$	False	False

Conditional Probabilities in terms of factors using sum-product algorithm

$$p(b|a) = \frac{p(a,b)}{p(a)} = \frac{\sum_c f_1(a,b,c)f_2(a)}{\sum_{b,c} f_1(a,b,c)f_2(a)} = \frac{f_1(a,b)}{f_1(a)}$$

$$p(c|a) = \frac{p(a,c)}{p(a)} = \frac{\sum_b f_1(a,b,c)f_2(a)}{\sum_{b,c} f_1(a,b,c)f_2(a)} = \frac{f_1(a,c)}{f_1(a)}$$

$$p(b|c) = \frac{p(b,c)}{p(c)} = \frac{\sum_a f_1(a,b,c)f_2(a)}{\sum_{a,b} f_1(a,b,c)f_2(a)}$$

$$p(a|b,c) = \frac{p(a,b,c)}{p(b,c)} = \frac{f_1(a,b,c)f_2(a)}{\sum_a f_1(a,b,c)f_2(a)}$$

$$p(c|a,b) = \frac{p(a,b,c)}{p(a,b)} = \frac{f_1(a,b,c)f_2(a)}{\sum_c f_1(a,b,c)f_2(a)} = \frac{f_1(a,b,c)}{f_1(a,b)}$$

$$p(b|a,c) = \frac{p(a,b,c)}{p(a,c)} = \frac{f_1(a,b,c)f_2(a)}{\sum_b f_1(a,b,c)f_2(a)} = \frac{f_1(a,b,c)}{f_1(a,c)}$$

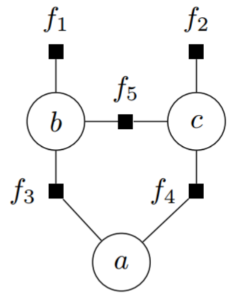
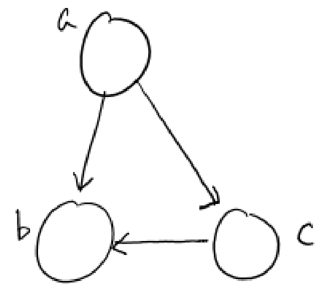
1.2.1.1b)

NOTE: A Bayesian Network representation of the factor graph below is impossible if

$$\nexists\ i,j \in \{1,2,5\}\ st\ f_i f_j = f_i$$

This is because the factors f_1 and f_2 indicates that b and c have no parent but f_5 indicates there's conditional dependence between the two. Thus either b is the parent or c is the parent, which is a contradiction.

However, if we relax our assumption, that is, assume $\exists\ i,j \in \{1,2,5\}\ st\ f_i f_j = f_i$, Then the simplified factor graph can be converted to the BN as follows.

	Factor Graph (given in question)	Bayesian Network
Graph		
$a \perp b?$	False	False
$b \perp c?$	False	False
$a \perp c?$	False	False
$a \perp b \mid c?$	False	False
$b \perp c \mid a?$	False	False
$a \perp c \mid b?$	False	False

The conditional Probabilities in terms of factors are computed as follows.

$$p(b|a) = \frac{p(a,b)}{p(a)} = \frac{\sum_c f_1(b)f_2(c)f_3(a,b)f_4(a,c)f_5(b,c)}{\sum_{b,c} f_1(b)f_2(c)f_3(a,b)f_4(a,c)f_5(b,c)}$$

$$p(c|a) = \frac{p(a,c)}{p(a)} = \frac{\sum_b f_1(b)f_2(c)f_3(a,b)f_4(a,c)f_5(b,c)}{\sum_{b,c} f_1(b)f_2(c)f_3(a,b)f_4(a,c)f_5(b,c)}$$

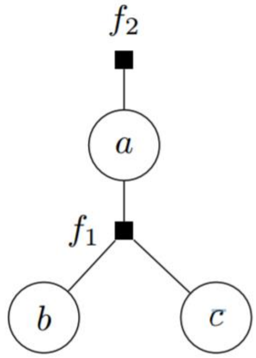
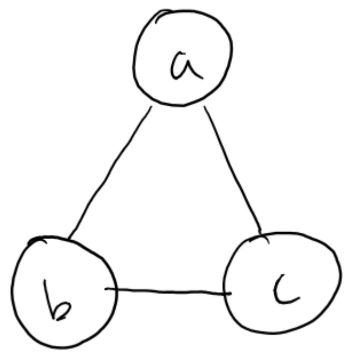
$$p(b|c) = \frac{p(b,c)}{p(c)} = \frac{\sum_a f_1(b)f_3(a,b)f_4(a,c)f_5(b,c)}{\sum_{a,b} f_1(b)f_3(a,b)f_4(a,c)f_5(b,c)}$$

$$p(a|b,c) = \frac{p(a,b,c)}{p(b,c)} = \frac{f_3(a,b)f_4(a,c)}{\sum_a f_3(a,b)f_4(a,c)}$$

$$p(b|a,c) = \frac{p(a,b,c)}{p(a,c)} = \frac{f_1(b)f_3(a,b)f_5(b,c)}{\sum_b f_1(b)f_3(a,b)f_5(b,c)}$$

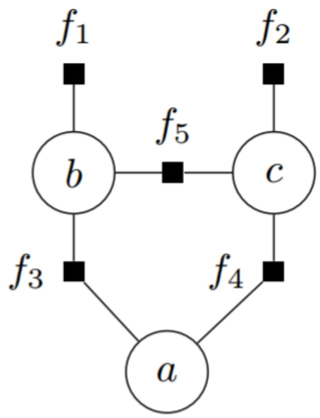
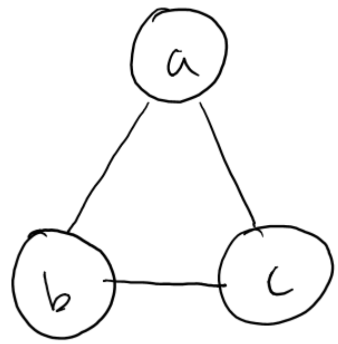
$$p(c|a,b) = \frac{p(a,b,c)}{p(a,b)} = \frac{f_2(c)f_4(a,c)f_5(b,c)}{\sum_c f_2(c)f_4(a,c)f_5(b,c)}$$

1.2.1.2a)

<table>
<tr><th></th><th>Factor Graph
(given in question)</th><th>Markov Random field</th></tr>
<tr><td>Graph</td><td>
</td><td>
</td></tr>
<tr><td>$a \perp b$?</td><td>False</td><td>False</td></tr>
<tr><td>$b \perp c$?</td><td>False</td><td>False</td></tr>
<tr><td>$a \perp c$?</td><td>False</td><td>False</td></tr>
<tr><td>$a \perp b \mid c$?</td><td>False</td><td>False</td></tr>
<tr><td>$b \perp c \mid a$?</td><td>False</td><td>False</td></tr>
<tr><td>$a \perp c \mid b$?</td><td>False</td><td>False</td></tr>
</table>

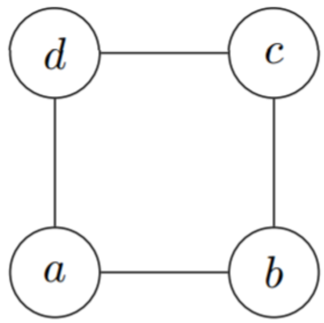
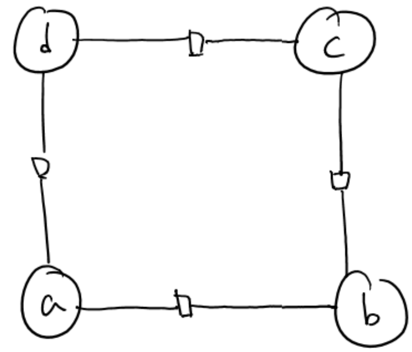
$$\psi(a,b) = f_1(a,b)f_2(a)$$
$$\psi(a,c) = f_1(a,c)f_2(a)$$
$$\psi(b,c) = f_1(b,c)$$
$$\psi(a,b,c) = f_1(a,b,c)f_2(a)$$

1.2.1.2b)

	Factor Graph (given in question)	Markov Random field
Graph		
$a \perp b$?	False	False
$b \perp c$?	False	False
$a \perp c$?	False	False
$a \perp b\|c$?	False	False
$b \perp c\|a$?	False	False
$a \perp c\|b$?	False	False

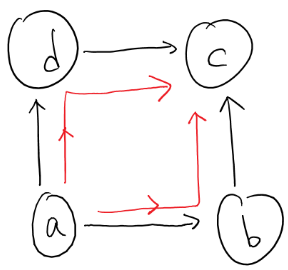
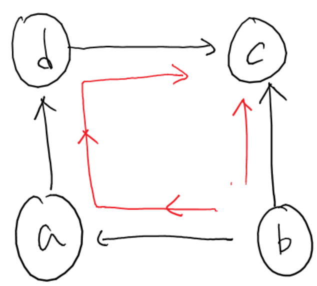
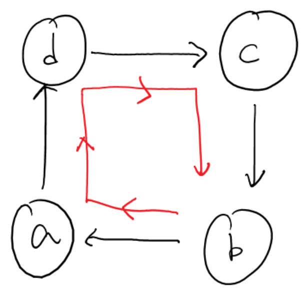
$$\psi(a,b) = f_3(a,b)f_1(b)f_5(b)f_4(a)$$
$$\psi(a,c) = f_4(a,c)f_5(c)f_2(c)f_3(a)$$
$$\psi(b,c) = f_5(b,c)f_1(b)f_3(b)f_2(c)f_4(c)$$
$$\psi(a,b,c) = f_1(b)f_2(c)f_3(a,b)f_4(a,c)f_5(b,c)$$

1.2.2.1

	Markov Random field (given in question)	Factor graph
Graph		
$a \perp b \mid c?$	False	False
$a \perp b \mid d?$	False	False
$a \perp b \mid c, d?$	False	False
$b \perp c \mid a?$	False	False
$b \perp c \mid d?$	False	False
$b \perp c \mid a, d?$	False	False
$a \perp c \mid b?$	False	False
$a \perp c \mid d?$	False	False
$a \perp c \mid b, d?$	True	True
$a \perp d \mid b?$	False	False
$a \perp d \mid c?$	False	False
$a \perp d \mid b, c?$	False	False
$b \perp d \mid a?$	False	False
$b \perp d \mid c?$	False	False
$b \perp d \mid a, c?$	True	True
$c \perp d \mid a?$	False	False
$c \perp d \mid b?$	False	False
$c \perp d \mid a, b?$	False	False

1.2.2.2

No. First given that $a \perp c \mid b, d$ and $b \perp d \mid a, c$ are true for the given MRF, there cannot be any connections between the diagonal vertices in BN. Thus we look at the BNs without diagonal connection$\{e_{bd}, e_{ac}\}$, with only 4 edges connecting the adjacent vertices $\{e_{ad}, e_{dc}, e_{cb}, e_{ba}\}$ Moreover, the 2^4 combinations made by the 4 edges in the graph can be boiled down to 3 by Symmetry, as follows. It is obvious that adjacent nodes are always conditionally dependent. We investigate conditional independence between diagonal vertices.

2 “head to tail” arrows	3 “head to tail arrows”	4 head to tail arrows
		
$a \perp c \mid b$ FALSE	$a \perp c \mid b$ FALSE	NOT POSSIBLE SINCE BN CONTAINS CYCLE
$a \perp c \mid d$ FALSE	$a \perp c \mid d$ FALSE	
$b \perp d \mid c$ FALSE	$b \perp d \mid c$FALSE	
$b \perp d \mid a$ TRUE	$b \perp d \mid a$TRUE	

In both cases, there exists one conditional independence between the diagonal vertices.

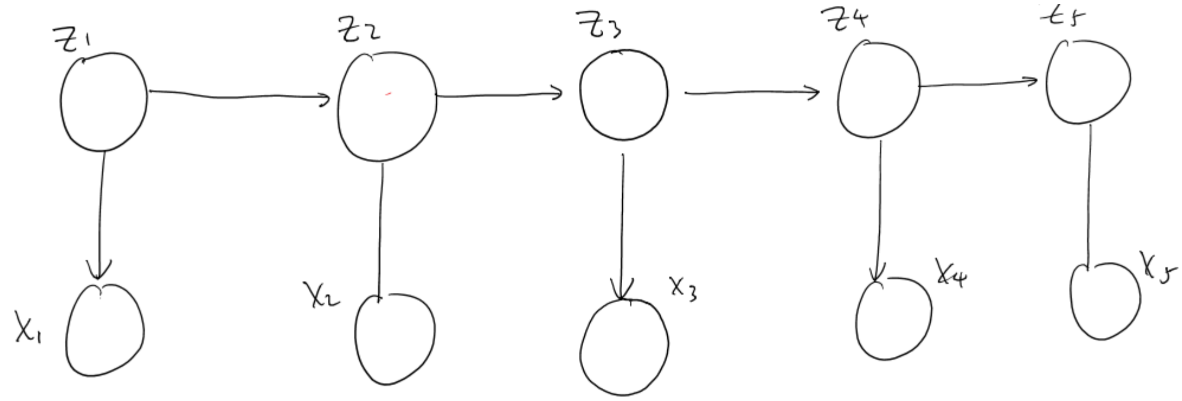
1.3.1

$$p(a,b,c,d,e,f) = p(a)p(b|a)p(c|b,e)p(d|b,c)p(e)p(f|b)$$

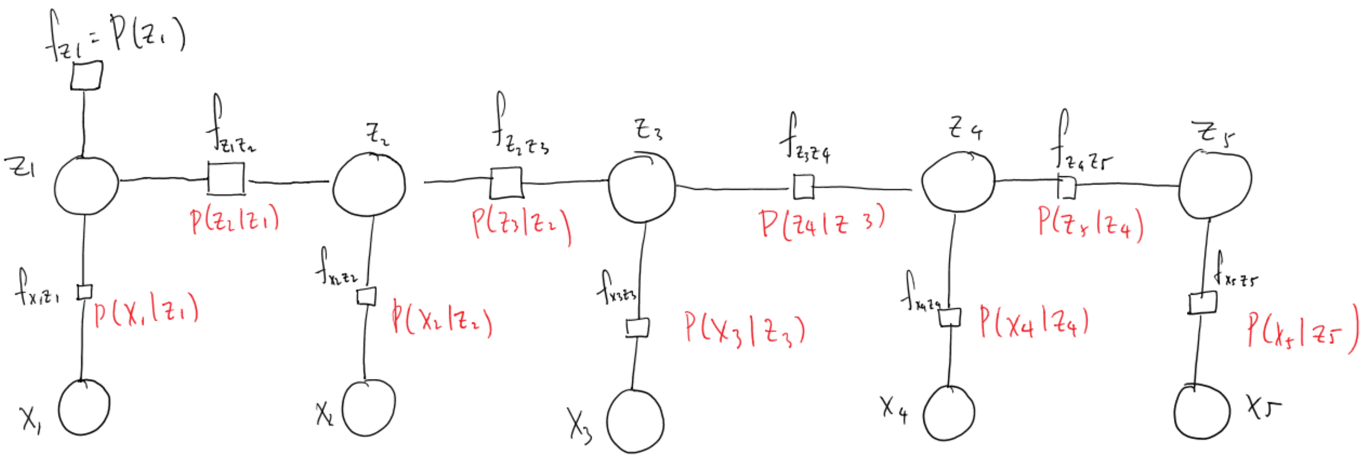
1.3.2

$a \perp c$	FALSE
$a \perp c \mid b$	TRUE
$e \perp b$	TRUE
$e \perp b \mid c$	FALSE
$a \perp e$	TRUE
$a \perp e \mid c$	FALSE

3.1.1



3.1.2



3.2.1

$$\mu_{z_4 \to f_{z_3 z_4}}(z_4) = \prod_{f_i \in Ne(z_4)} \mu_{f_i \to z_4}(z_4)$$

$$\mu_{z_4 \to f_{z_3 z_4}}(z_4) = \mu_{f_{z_4 z_5} \to z_4}(z_4)\mu_{f_{x_4 z_4} \to z_4}(z_4)$$

3.3.1 NOTE: · denotes element-wise product

$$\boldsymbol{f}_{\boldsymbol{x_1 z_1}}(x_1, z_1) = \begin{pmatrix} f(x_1 = 1, z_1 = 1) & f(x_1 = 2, z_1 = 1) & \cdots & f(x_1 = M, z_1 = 1) \\ f(x_1 = 1, z_1 = 2) & f(x_1 = 2, z_1 = 2) & \cdots & f(x_1 = M, z_1 = 2) \\ \vdots & \vdots & \ddots & \vdots \\ f(x_1 = 1, z_1 = K) & f(x_1 = 2, z_1 = K) & \cdots & f(x_1 = M, z_1 = K) \end{pmatrix}$$

Where

$$\boldsymbol{f}_{\boldsymbol{x_1 z_1}}(x_1 = i, z_1 = j) = P(x_1 = i | z_1 = j)$$

$$W = \begin{pmatrix} P(x_1 = 1 | z_1 = 1) & P(x_1 = 1 | z_1 = 2) & \cdots & P(x_1 = 1 | z_1 = K) \\ P(x_1 = 2 | z_1 = 1) & P(x_1 = 2 | z_1 = 2) & \cdots & P(x_1 = 2 | z_1 = K) \\ \vdots & \vdots & \ddots & \vdots \\ P(x_1 = M | z_1 = 1) & P(x_1 = M | z_1 = 2) & \cdots & P(x_1 = M | z_1 = K) \end{pmatrix}$$

Moreover

$$\boldsymbol{f}_{\boldsymbol{z_1 z_2}}(z_1, z_2) = \begin{pmatrix} f(z_1 = 1, z_2 = 1) & f(z_1 = 2, z_2 = 1) & \cdots & f(z_1 = K, z_2 = 1) \\ f(z_1 = 1, z_2 = 2) & f(z_1 = 2, z_2 = 2) & \cdots & f(z_1 = K, z_2 = 2) \\ \vdots & \vdots & \ddots & \vdots \\ f(z_1 = 1, z_2 = K) & f(z_1 = 2, z_2 = K) & \cdots & f(z_1 = K, z_2 = K) \end{pmatrix}$$

$$\boldsymbol{f}_{\boldsymbol{z_1 z_2}}(z_1 = i, z_2 = j) = P(z_2 = i | z_1 = j)$$

$$T = \begin{pmatrix} P(z_2 = 1 | z_1 = 1) & P(z_2 = 1 | z_1 = 2) & \cdots & P(z_2 = 1 | z_1 = K) \\ P(z_2 = 2 | z_1 = 1) & f(z_1 = 2, z_2 = 2) & \cdots & f(z_1 = 2, z_2 = K) \\ \vdots & \vdots & \ddots & \vdots \\ f(z_1 = K, z_2 = 1) & f(z_1 = K, z_2 = 2) & \cdots & f(z_1 = K, z_2 = K) \end{pmatrix}$$

Thus

$$\boldsymbol{f}_{\boldsymbol{x_1 z_1}}(x_1, z_1) = W^T$$

$$\mu_{f_{x_1 z_1} \to z_1}(z_1) = \sum_{x_1} f_{x_1 z_1}(x_1, z_1) \mu_{x_1 \to f_{x_1 z_1}}(x_1) = W^T x_1$$

$$\mu_{f_{z_1} \to z_1}(z_1) = \pi$$

$$\mu_{z_1 \to f_{z_1 z_2}}(z_1) = \mu_{f_{z_1} \to z_1}(z_1) \cdot \mu_{f_{x_1 z_1} \to z_1}(z_1) = \pi \cdot W^T x_1$$

$$\boldsymbol{f}_{\boldsymbol{x_2 z_2}}(x_2, z_2) = W^T$$

$$\mu_{f_{x_2 z_2} \to z_2}(z_2) = \sum_{x_2} f_{x_2 z_2}(x_2, z_2)\, \mu_{x_2 \to f_{x_2 z_2}}(x_2) = W^T x_2$$

$$\boldsymbol{f}_{\boldsymbol{z_1 z_2}}(z_1, z_2) = T$$

$$\mu_{f_{z_1 z_2} \to z_2}(z_2) = \sum_{z_1} f_{z_1 z_2}(z_1, z_2) \mu_{z_1 \to f_{z_1 z_2}}(z_1) = T(\pi \cdot W^T x_1)$$

$$\mu_{z_2 \to f_{z_2 z_3}}(z_2) = \mu_{f_{z_1 z_2} \to z_2}(z_2) \cdot \mu_{f_{x_2 z_2} \to z_2}(z_2) = (T(\pi \cdot W^T x_1)) \cdot (W^T x_2)$$

$$\boldsymbol{f}_{\boldsymbol{z_2 z_3}}(z_2, z_3) = T$$

$$\mu_{f_{z_2 z_3} \to z_3}(z_3) = \sum_{z_2} f_{z_2 z_3}(z_2, z_3)\, \mu_{z_2 \to f_{z_2 z_3}}(z_2) = T\big((T(\pi \cdot W^T x_1)) \cdot (W^T x_2)\big)$$

3.3.2 NOTE: · denotes element-wise product

$$\mu_{f_{x_5 z_5} \to z_5}(z_5) = \sum_{x_5} f_{x_5 z_5}(x_5, z_5)\, \mu_{x_5 \to f_{x_5 z_5}}(x_5) = W^T x_5$$

$$\mu_{z_5 \to f_{z_4 z_5}}(z_5) = \mu_{f_{x_5 z_5} \to z_5}(z_5) = W^T x_5$$

$$\mu_{f_{z_4 z_5} \to z_4}(z_4) = \sum_{z_5} f_{z_4 z_5}(z_4, z_5)\, \mu_{z_5 \to f_{z_4 z_5}}(z_5) = T^T W^T x_5$$

Notice, here we are summing $f_{z_i z_{i+1}}(z_i, z_{i+1})$ over z_{i+1} (NOT over z_i as in the previous question). This is equivalent to multiplying T^T not T

$$\mu_{f_{x_4 z_4} \to z_4}(z_4) = \sum_{x_4} f_{x_4 z_4}(x_4, z_4)\, \mu_{x_4 \to f_{x_4 z_4}}(x_4) = W^T x_4$$

$$\mu_{z_4 \to f_{z_3 z_4}}(z_4) = \mu_{f_{z_4 z_5} \to z_4}(z_4) \cdot \mu_{f_{x_4 z_4} \to z_4}(z_4) = (T^T W^T x_5) \cdot (W^T x_4)$$

$$\mu_{f_{z_3 z_4} \to z_3}(z_3) = \sum_{z_4} f_{z_3 z_4}(z_3, z_4)\, \mu_{z_4 \to f_{z_3 z_4}}(z_4) = T^T\big((T^T W^T x_5) \cdot (W^T x_4)\big)$$

$$\mu_{f_{x_3 z_3} \to z_3}(z_3) = \sum_{x_3} f_{x_3 z_3}(x_3, z_3)\, \mu_{x_3 \to f_{x_3 z_3}}(x_3) = W^T x_3$$

$$\mu_{z_3 \to f_{z_2 z_3}}(z_3) = \mu_{f_{z_3 z_4} \to z_3}(z_3) \cdot \mu_{f_{x_3 z_3} \to z_3}(z_3) = \Big(T^T\big((T^T W^T x_5) \cdot (W^T x_4)\big)\Big) \cdot (W^T x_3)$$