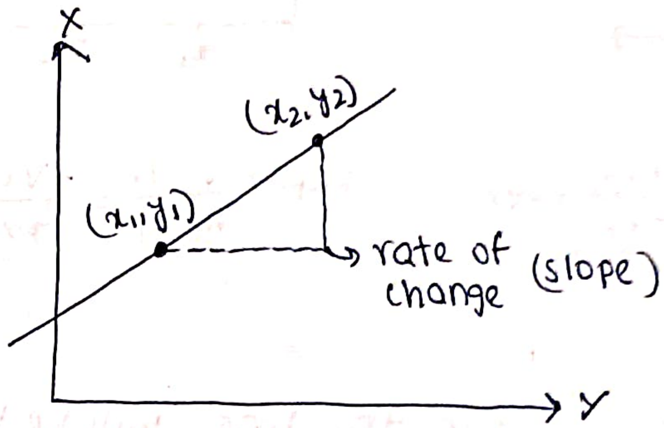
Slope ⇒ Derivative as a concept

A slope of a line is a measure of how steep the line is and represents the rate of change of one variable with respect to another. In the context of two-dimensional cartesian co-ordinate system, the slope indicates the ratio of vertical change to the horizontal change between two points on a line.

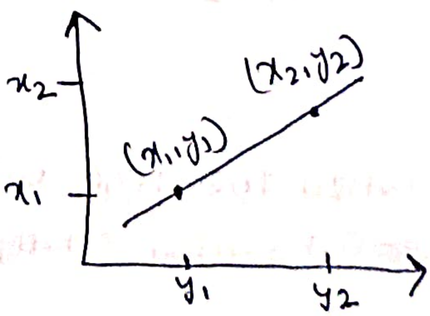


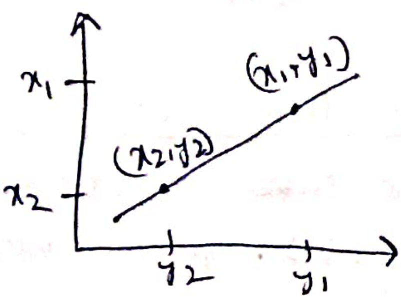
$$\boxed{\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{Rise (vertical)}}{\text{Run (Horizontal)}}}$$

Interpretation of slope :-

(i) Positive slope :-

The slope is positive when the line rises as it moves from left to right. The larger the slope, the steeper the line



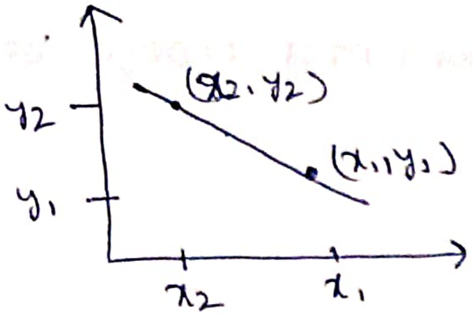


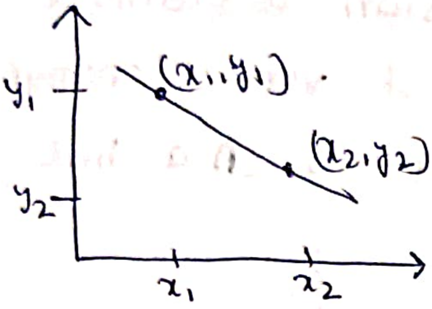
$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{+ve}{+ve} = +ve$$

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-ve}{-ve} = +ve$$

(ii) <u>Negative slope:-</u>

The slope will be negative when the line falls as it moves from left to right. The more negative the slope is, the steeper the line in the downward direction.



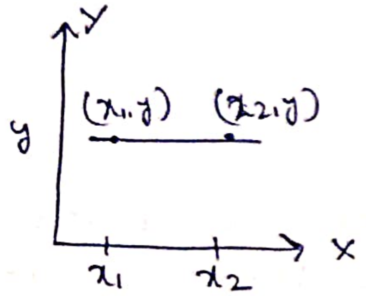


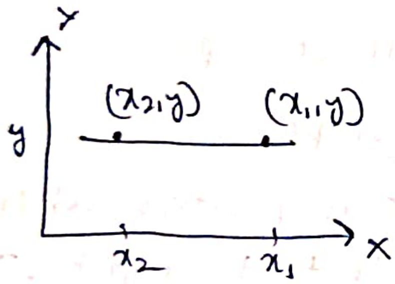
$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{+ve}{-ve} = -ve$$

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-ve}{+ve} = -ve$$

(iii) <u>Zero slope:-</u>

The slope will be zero when the line will be horizontal, This is because of no-vertical change as the line moves from left to right.



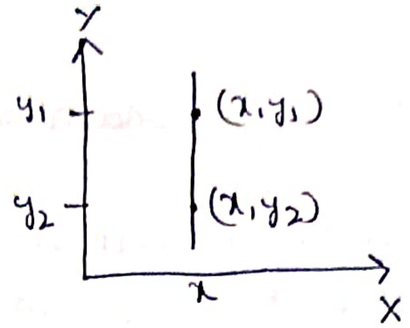


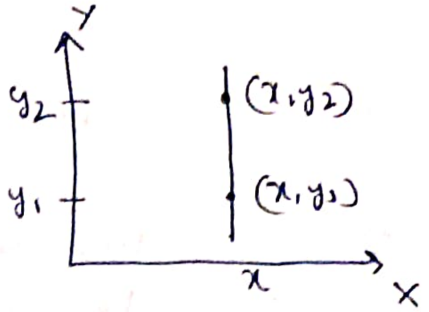
$$\text{Slope} = \frac{y - y}{x_2 - x_1} = 0$$

$$\text{Slope} = \frac{y - y}{x_1 - x_2} = 0$$

(iii) <u>undefined slope:-</u>

The slope will be undefined when the line will be vertical. This is because of no-horizontal change in the line.



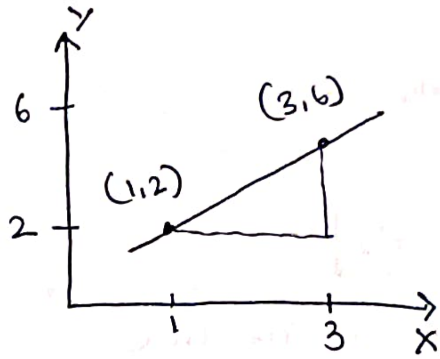


slope $= \dfrac{y_2 - y_1}{x - x_1} =$ undefined

slope $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{+ve}{0} =$ undefined

Ⓐ Examples:-

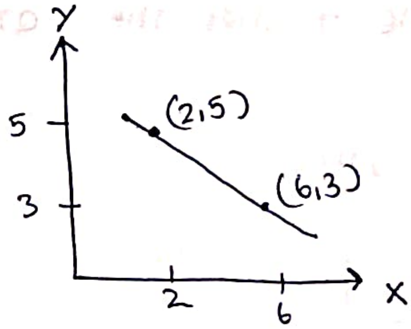
(i)



slope $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{6-2}{3-1} = \dfrac{4}{2} = 2$

This means that, for every one unit we move from left to right, the line moves two units vertically up.

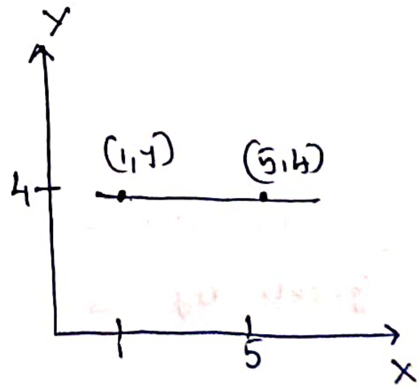
(ii)



slope $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3-5}{6-2} = \dfrac{-2}{4} = \dfrac{-1}{2}$

This means that, for every two units we move horizontally from left to right, the line moves one unit vertically down.

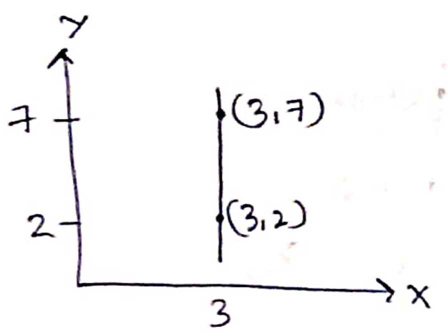
(iii)



slope $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4-4}{5-1} = 0$

Even if we move any number of units from left to right, the line doesn't change/move vertically. X is not related to Y.

(iv)

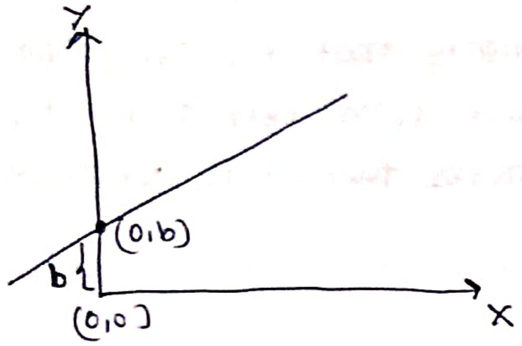


$$\text{slope} = \frac{y_2-y_1}{x_2-x_1} = \frac{7-2}{3-3} = \frac{5}{0} = \text{undefined}$$

This happens when line is vertical because irrespective of changes in line vertically, it doesn't change horizontally.

Slope in an equation of line:-

→ If line doesn't pass through origin,

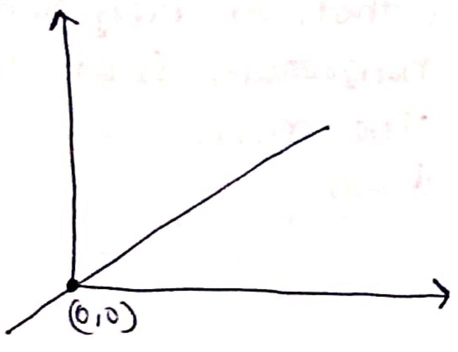


$\boxed{y = mx + b}$

m ⇒ slope of the line

b ⇒ y-intercept, where the line touches the y-axis

→ If the line passes through the origin,



$\boxed{y = mx}$

m ⇒ slope of the line

Example of interpreting equation of the line:-

suppose, there is a line equation given by

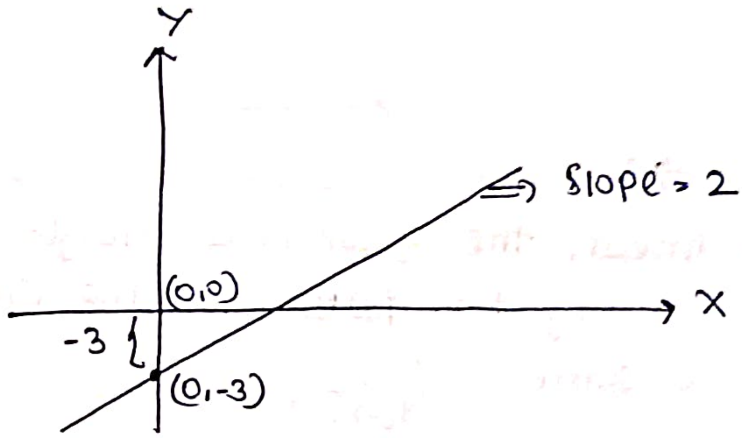
$y = 2x - 3$

comparing it with equation of line, $y = mx + b$

$m = +2$. and $b = -3$

$m = +2 \Rightarrow$ The slope is positive and hence the line rises vertically up as we move horizontally from left to right.

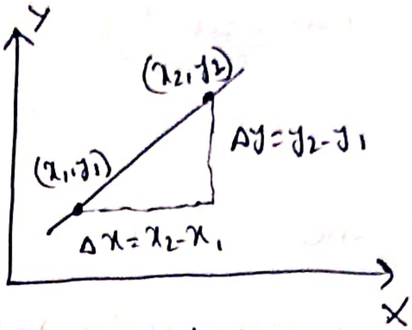
$b = -3 \Rightarrow$ The line touches the Y-axis at $y = -3$ i.e the line intercepts Y-axis at $y = -3$



In this way, we can obtain the insights like steepness of the line and Y-intercept by looking at equation of a straight line.

Derivative

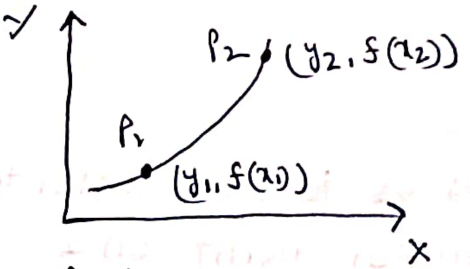
⇒ For linear graphs,
The equation of the straight line will be in the form $y = mx + b$



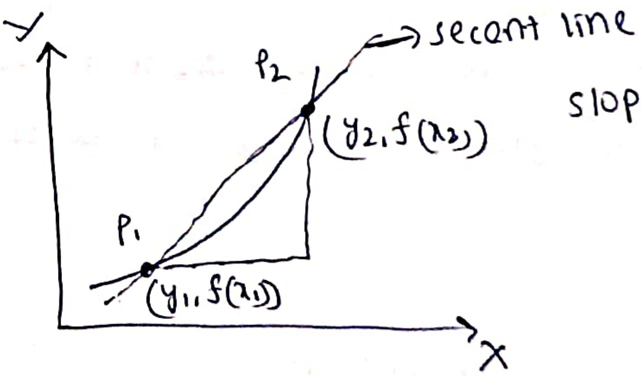
$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope between any two points on the straight line remains the same.

⇒ But for non-linear graphs,
The equation of the curve will be in the form $y = f(x)$

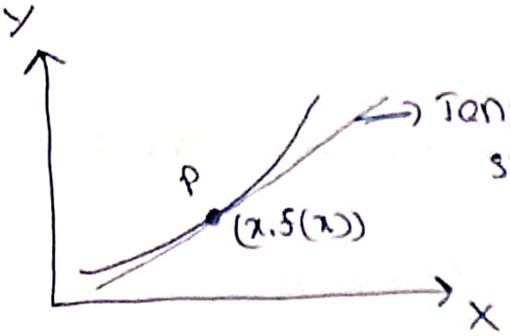


To find average slope between two points on the curve, we draw a secant line touching both the points and then derive the slope of the line



$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

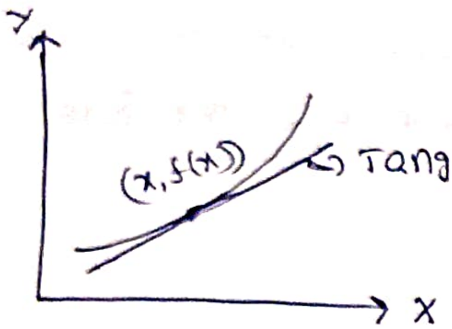
In this way, by using a secant line, we can obtain the AVERAGE slope between any two points on a non-linear curve. But, to obtain istantaneous rate of change (slope at a single point on non-linear curve), we draw a tangent line that touches exactly that single point on curve.



slope $= \frac{\partial y}{\partial x} = \frac{\text{Derivative of } y}{\text{Derivative of } x}$

Understanding importance of tangent and secant:-

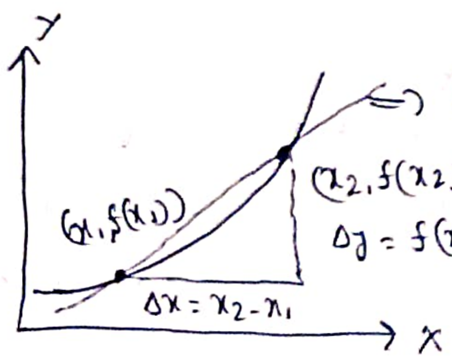
→ Tangent line touches the curve at only one specific point, without cutting through the curve.



Slope of this tangent line allows us to find instanteneous rate of change / slope at a particular point on a non-linear curve.

$$\boxed{\text{Slope} = \frac{\partial y}{\partial x} = \frac{\partial f(x)}{\partial x}}$$ } INSTANTANEOUS slope

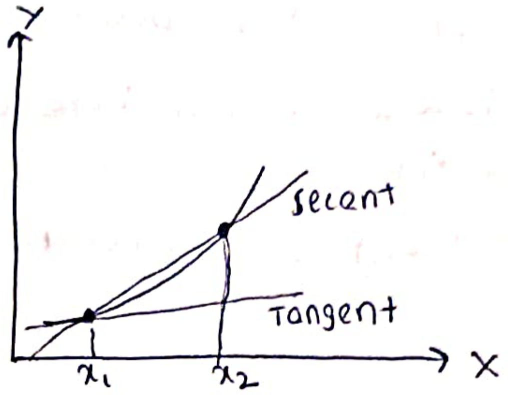
→ Secant line intersects the curve at two distinct points, effectively "cutting" through the curve at both of those points.



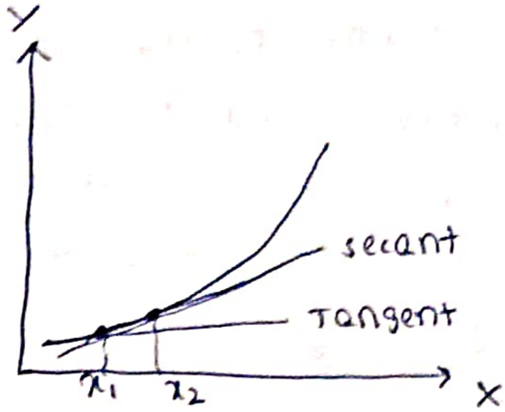
$$\boxed{\text{slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$ } AVERAGE slope

The slope of the secant line allows us to findout the average slope between two distinct points on the non-linear curve.

Ⓐ Tangent secant theorem:-



when x_2 is distant from x_1, the slopes of tangent and secant are completely distant / have much difference



As x_2 approaches closer to x_1, the slope of secant (average slope) converges to exact slope at a single point (tangent / instantaneous slope)

As x_2 approaches x_1, geometrically, the secant line "leans in" to become tangent line, capturing the behavior of the curve at the single point.

Even though, the tangent is only concerned with the instantaneous rate at one specific point, its mathematical definition fundamentally depends on comparing the function at two points and letting the gap between them shrink infinitely small using limits.

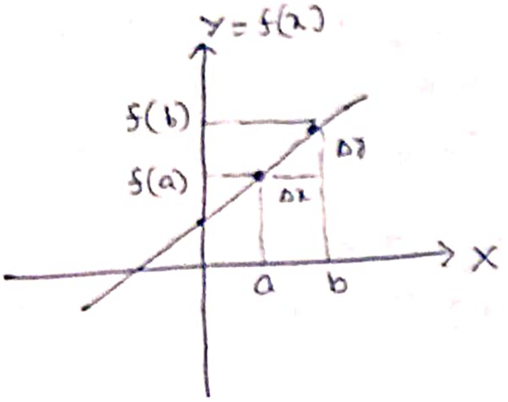
$$\boxed{\text{Instantaneous slope} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

Mathematical Notation of derivative with limits

⇒ The derivative is a fundamental concept in calculus that represents the rate at which a function is changing at any given point. Derivative is essentially the slope of the tangent line to the function's graph at that point.

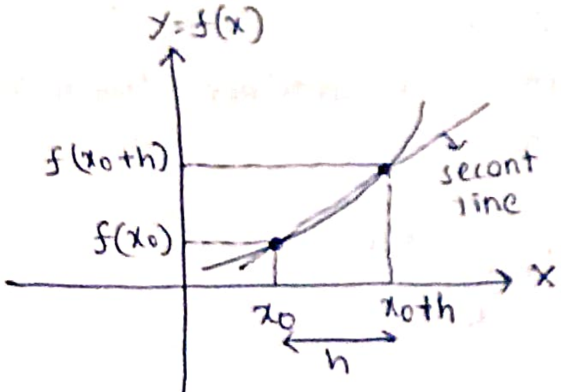
⇒ Derivative is used to understand how a function behaves as its input changes, and is key tool for analyzing the dynamics of systems in mathematics, physics, economics and engineering.

(a) For a linear function / straight line:-



$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a}$$

(a) For a curve / non-linear function:-



The slope of secant line will give the average rate of change

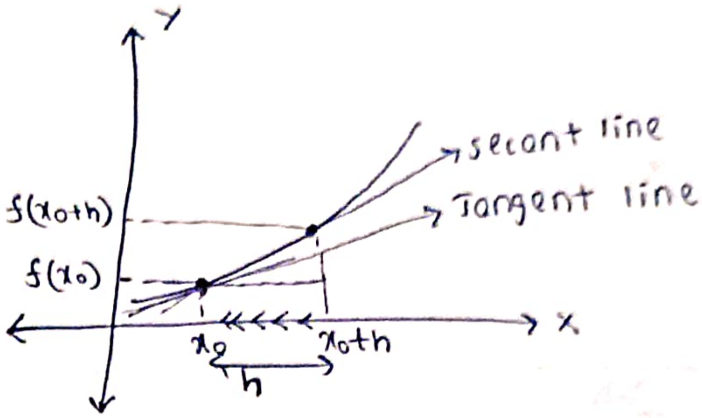
$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{f(x_0+h) - f(x_0)}{(x_0+h) - (x_0)}$$

$$= \frac{f(x_0+h) - f(x_0)}{h}$$

Now, let us use tangent to get the instantaneous slope.

The instantaneous slope will be:-

As per the tangent secant theorem.
As we bring x_0+h closer to x_0, the secant line drawn between the two points converge closer to the tangent line.



So, to bring x_0+h closer to x_0, we must reduce h closer to 0. Then, the secant and tangent lines will have similar slope. To do this, we use limits.

$$\boxed{\text{Instantaneous slope} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

It is also called as derivative of $f(x)$. denoted by $f'(x)$

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

So, as $h \to 0$, x_0+h approaches x_0 and hence secant line will be slowly converged into a tangent line. So, we can obtain the instantaneous slope using this approach.

Mathematical Notation of derivative with limits

⇒ The derivative is a fundamental concept in calculus that represents the rate at which a function is changing at any given point. Derivative is essentially the slope of the tangent line to the function's graph at that point.

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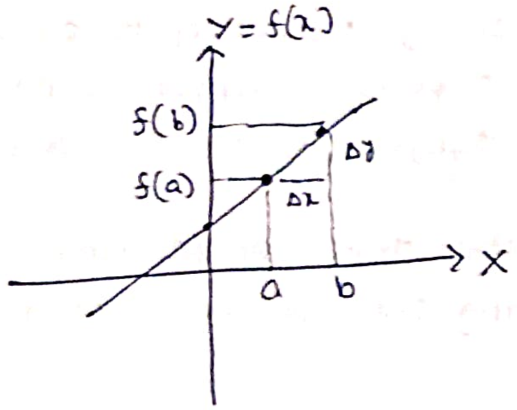
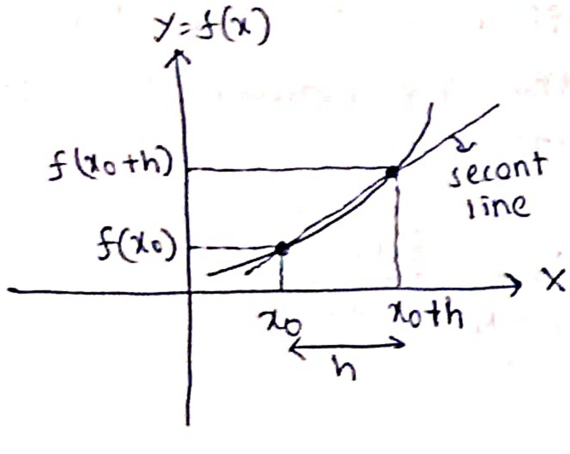
Ⓐ For a linear function / straight line:-



$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a}$$

Ⓑ For a curve / non-linear function:-



The slope of secant line will give the average rate of change

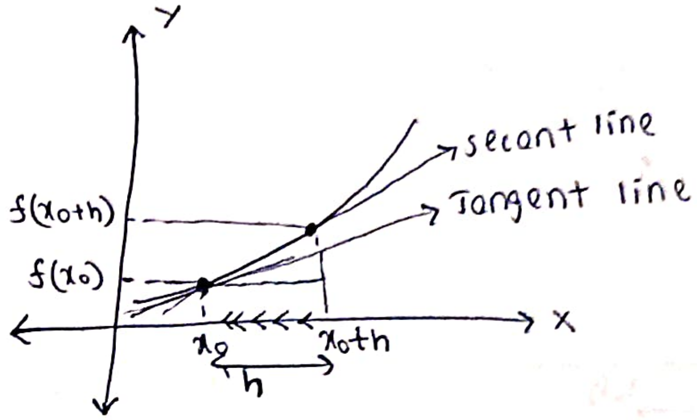
$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{f(x_0+h) - f(x_0)}{(x_0+h) - (x_0)}$$

$$= \frac{f(x_0+h) - f(x_0)}{h}$$

Now, let us use tangent to get the instantaneous slope.

The instantaneous slope will be:-

As per the tangent secant theorem,
As we bring x_0+h closer to x_0, the secant line drawn between the two points converge closer to the tangent line.



So, to bring x_0+h closer to x_0, we must reduce h closer to 0. Then, the secant and tangent lines will have similar slope. To do this, we use limits.

$$\boxed{\text{Instantaneous slope} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

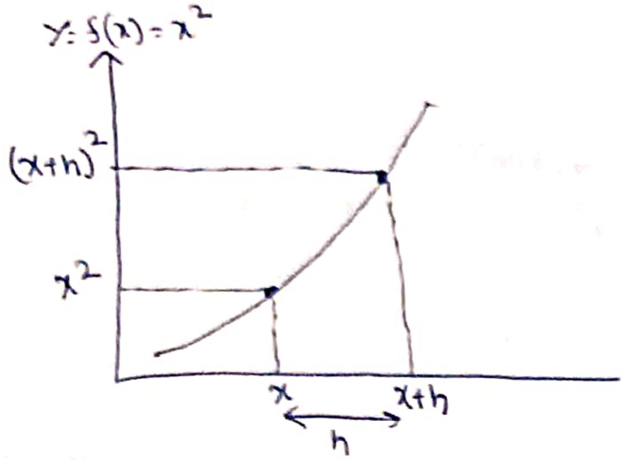
It is also called as derivative of $f(x)$. denoted by $f'(x)$

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

So, as $h \to 0$, x_0+h approaches x_0 and hence secant line will be slowly converged into a tangent line. So, we can obtain the instantaneous slope using this approach.

Finding Derivative at a Point with Examples

Let us take a curve with non-linear function $f(x) = x^2$



$$\text{Instantaneous slope} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - (x)}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + h^2 + 2hx - x^2}{h}$$

$$= \lim_{h \to 0} \frac{h(h+2x)}{h} = \lim_{h \to 0} h + 2x$$

$$= 2x$$

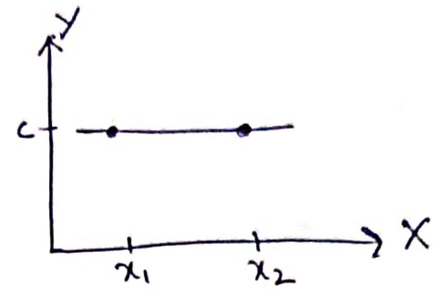
So, mathematically, the derivative of $f(x) = x^2$ is $2x$. It is mathematically represented as

$$\text{Instantaneous slope} = \frac{\partial y}{\partial x} = \frac{\partial f(x)}{\partial x} = \frac{\partial}{\partial x}(x^2) = 2x = f'(x)$$

Power rule in derivatives

$\Rightarrow$ Derivative of a constant is 0

$f(x) = c \Rightarrow f'(x) = 0$



$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{c - c}{x_2 - x_1} = 0$$

$\Rightarrow$ As per the power rule of derivative for polynomial expressions, the derivative of x^n is $n \cdot x^{n-1}$, where $n \neq 0$

$$\boxed{f(x) = x^n \Rightarrow f'(x) = n \cdot x^{n-1}}$$

Examples :-

(i) $f(x) = x^3 \Rightarrow f'(x) = 3 \cdot x^{3-1} = 3x^2$

(ii) $f(x) = 3x^2 \Rightarrow f'(x) = \frac{\partial}{\partial x}(3x^2) = 3 \cdot \frac{\partial}{\partial x}(x^2) = 3 \cdot (2x^{2-1}) = 6x$

(iii) $f(x) = 1/x \Rightarrow f'(x) = \frac{\partial}{\partial x}(x^{-1}) = -1 \cdot x^{-1-1} = -1 \cdot x^{-2} = \frac{-1}{x^2}$

Derivative rules :- constant, sum, difference and constant multiplication

$\Rightarrow$ Derivative of scaler multiplication with the function

$$y = c \cdot f(x) \Rightarrow \frac{\partial y}{\partial x} = \frac{\partial}{\partial x} c \cdot f(x) = c \cdot \frac{\partial}{\partial x} f(x) = c \cdot f'(x)$$

So, if $y = c \cdot f(x)$, then $y' = c \cdot f'(x)$

Example :-

$$y = 3x^4 \Rightarrow \frac{\partial y}{\partial x} = \frac{\partial}{\partial x}(3x^4) = 3 \cdot \frac{\partial}{\partial x}(x^4)$$

$$= 3\,(4 \cdot x^{4-1}) = 12x^3$$

$y = 3x^4 \Rightarrow y' = 12x^3$

⇒ Derivative for sum of two functions

suppose, $y = f(x) + g(x)$, then the derivative will be

$$y = f(x) + g(x)$$

$$\Rightarrow \frac{\partial y}{\partial x} = \frac{\partial}{\partial x}(f(x) + g(x))$$

$$\Rightarrow y' = \frac{\partial}{\partial x}(f(x)) + \frac{\partial}{\partial x}(g(x))$$

$$\boxed{y' = f'(x) + g'(x)}$$

Example:-

$$y = x^3 + 3x^2$$

$$\Rightarrow \frac{\partial y}{\partial x} = \frac{\partial}{\partial x}(x^3 + 3x^2)$$

$$\Rightarrow y' = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(3x^2)$$

$$\Rightarrow y' = 3 \cdot x^{3-1} + 3 \cdot 2 \cdot x^{2-1}$$

$$y' = 3x^2 + 6x$$

⇒ Derivative for multiplication of two functions:-

suppose $y = f(x) \cdot g(x)$, then

$$\Rightarrow \frac{\partial y}{\partial x} = y' = \frac{\partial}{\partial x}(f(x) \cdot g(x))$$

$$\Rightarrow y' = f(x) \cdot \frac{\partial}{\partial x} g(x) + g(x) \cdot \frac{\partial}{\partial x} f(x)$$

$$\boxed{y' = f(x) \cdot g'(x) + g(x) \cdot f'(x)}$$

Example:-

$$y = (3x)(x^2)$$

$$\frac{\partial y}{\partial x} = y' = \frac{\partial}{\partial x}(3x)(x)^2$$

$$\Rightarrow y' = 3x \frac{\partial}{\partial x}(x^2) + x^2 \cdot \frac{\partial}{\partial x}(3x)$$

$\Rightarrow y' = 3x\,(2 \cdot x^{2-1}) + 3x^2(1 \cdot x^{1-0})$

$\Rightarrow y' = 6x^2 + 3x^2$

$y' = 9x^2$

we can confirm by multiplication initially that,

$y = (3x)\,(x^2) = 3x^3$

$\frac{\partial y}{\partial x} = y' = 3\,\frac{\partial}{\partial x}x^3 = 3 \cdot 3 \cdot x^{3-1} = 9x^2$

So,

If $y = f(x) + g(x) \Rightarrow y' = f'(x) + g'(x)$

If $y = f(x) - g(x) \Rightarrow y' = f'(x) - g'(x)$

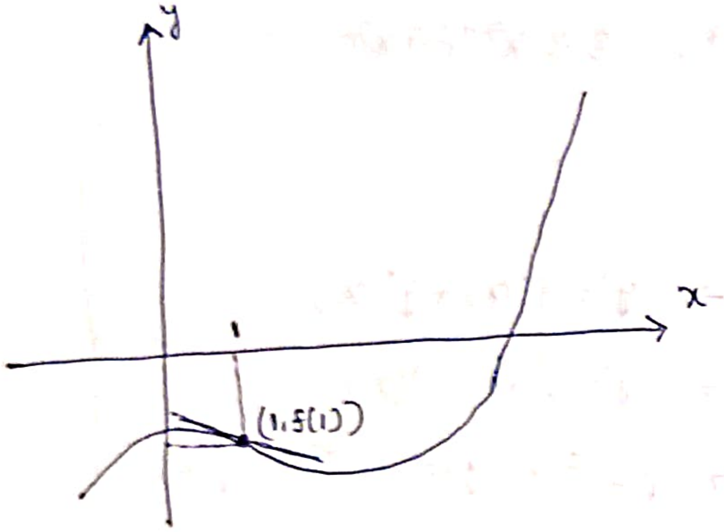
If $y = f(x) \cdot g(x) \Rightarrow y' = f(x) \cdot g'(x) + f'(x) \cdot g(x)$

Tangent of polynomials

let us take a polynomial function,

$y = f(x) = x^3 - 6x^2 + x - 7$

It's curve looks like :-



At $x=1 \Rightarrow y = f(x) = f(1) = (1)^3 - 6(1)^2 + (1) - 7 = -11$

so, when we try to find slope at $(1, f(1)) \Rightarrow (1, -11)$, we will obtain its value using derivative at that point.

$$\text{slope} = f'(x) = \frac{\partial}{\partial x} f(x) = \frac{\partial}{\partial x}(x^3 - 6x^2 + x - 7)$$

$$= \frac{\partial}{\partial x}(x^3) - 6\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial x}(7)$$

$$\boxed{f'(x) = 3x^2 - 12x + 1}$$

At $x=1 \Rightarrow f'(1) = 3(1) - 12(1) + 1 = -8$

so, the slope of line at $x=1$ is -8.

* Now let us obtain the equation of tangent at that point:-

The equation of straight line can be represented as

$$y = mx + c$$

we have already obtained the slope as -8, so the equation of tangent will be

$$\boxed{y = -8x + c}$$

since the tangent line touches the point on curve at $x=1$, the co-ordinates $(x,y) \Rightarrow (x, f(x)) \Rightarrow (1, f(1)) \Rightarrow (1,-11)$ can be substituted in the equation to obtain y-intercept value

$$y = -8x + c$$

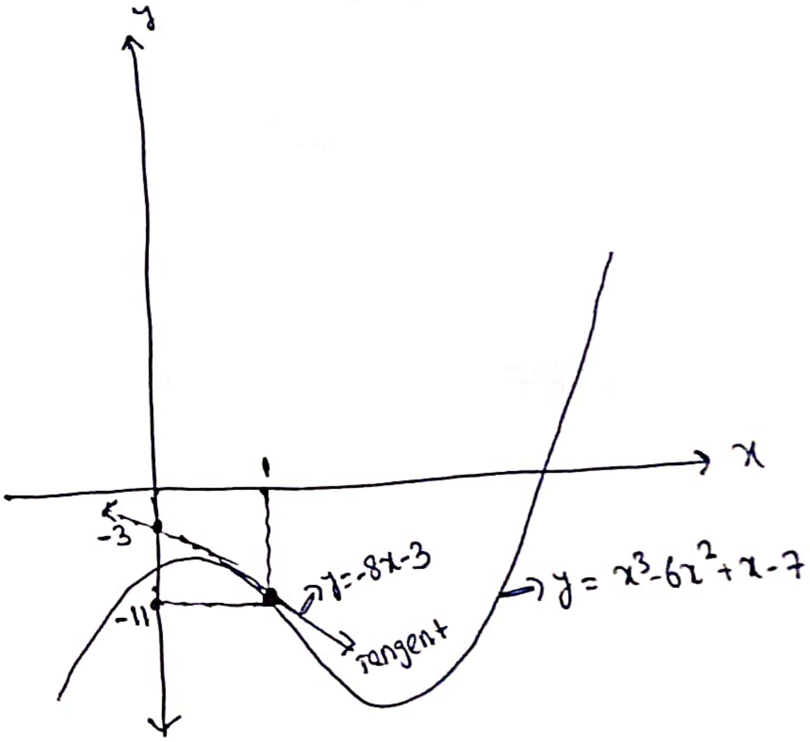
$$\Rightarrow c = y + 8x = -11 + 8(1) = -3$$

$$\boxed{c = -3}$$

So, the equation of tangent with slope$(m) = -8$ and y-intercept$(c) = -3$ will be

$$y = mx + c \Rightarrow \boxed{y = -8x - 3}$$

The representative curve looks like :-



This concept is very useful when we will be dealing with data that is curved, for prediction tasks. we need to find slope using derivative and value of y-intercept that satisfies the equation at that point to obtain predictions. This will help for optimization.

① Derivative of trigonometric and logarithmic functions:-

$\Rightarrow f(x) = \sin x \Rightarrow f'(x) = \frac{\partial}{\partial x}(\sin x) = \cos x$

$\Rightarrow f(x) = \cos x \Rightarrow f'(x) = \frac{\partial}{\partial x}(\cos x) = -\sin x$

$\Rightarrow f(x) = \sec x \Rightarrow f'(x) = \frac{\partial}{\partial x}(\sec x) = \sec x . \tan x$

$\Rightarrow f(x) = \operatorname{cosec} x \Rightarrow f'(x) = \frac{\partial}{\partial x}(\operatorname{cosec} x) = -\operatorname{cosec} x . \cot x$

$\Rightarrow f(x) = \tan x \Rightarrow f'(x) = \frac{\partial}{\partial x}(\tan x) = \sec^2 x$

$\Rightarrow f(x) = \cot x \Rightarrow f'(x) = \frac{\partial}{\partial x}(\cot x) = -\operatorname{cosec}^2 x$

For logarithmic functions:-

$\Rightarrow f(x) = \ln(x) \Rightarrow f'(x) = \frac{\partial}{\partial x}(\ln x) = \frac{1}{x}$

For exponential functions:-

$\Rightarrow f(x) = e^x \Rightarrow f'(x) = e^x$

Power rule:-

$\Rightarrow f(x) = x^n \Rightarrow f'(x) = \frac{\partial}{\partial x}(x^n) \Rightarrow f'(x) = n . x^{n-1}$

Product rule in derivatives

For example, we have product of two functions,

$y = f(x) \cdot g(x)$

Then its derivative can be obtained by,

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}(f(x) \cdot g(x))$$

$$= f(x) \cdot \frac{\partial}{\partial x} g(x) + g(x) \cdot \frac{\partial}{\partial x} f(x)$$

$$= f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

So, $$\boxed{\frac{\partial}{\partial x}(f(x) \cdot g(x)) = f(x) \cdot g'(x) + g(x) \cdot f'(x)}$$

$\Rightarrow$ An example for the same is,

$y = x^2 \cos x$.

let us represent $f(x) = x^2$ and $g(x) = \cos x$

$$y = f(x) \cdot g(x) \Rightarrow \frac{\partial y}{\partial x} = g(x) \cdot \frac{\partial}{\partial x} f(x) + f(x) \cdot \frac{\partial}{\partial x} g(x)$$

$$\Rightarrow \frac{\partial y}{\partial x} = \cos x \cdot \frac{\partial}{\partial x}(x^2) + x^2 \cdot \frac{\partial}{\partial x}(\cos x)$$

$$\Rightarrow \frac{\partial y}{\partial x} = \cos x \cdot (2x) + x^2 \cdot (-\sin x)$$

$$\boxed{\frac{\partial y}{\partial x} = 2x \cdot \cos x - x^2 \sin x}$$

$\Rightarrow$ Another example can be

$y = \sin x \cos x$

Let us represent $f(x) = \sin x$ and $g(x) = \cos x$

$$y = f(x) \cdot g(x) \Rightarrow \frac{\partial y}{\partial x} = f(x) \cdot \frac{\partial}{\partial x} g(x) + g(x) \cdot \frac{\partial}{\partial x} f(x)$$

$$\Rightarrow \frac{\partial y}{\partial x} = \sin x \cdot \frac{\partial}{\partial x}(\cos x) + \cos x \cdot \frac{\partial}{\partial x}(\sin x)$$

$$\Rightarrow \frac{\partial y}{\partial x} = \sin x(-\sin x) + \cos x \cdot (\cos x)$$

$$\Rightarrow \frac{\partial y}{\partial x} = -\sin^2 x + \cos^2 x$$

$$\boxed{\frac{\partial y}{\partial x} = \cos^2 x - \sin^2 x}$$

In this way, we can represent complex functions as product of two functions and then use product rule to obtain derivative.

Chain rule of derivative

The chain rule is the fundamental theorem in calculus that is used to find the derivative of a composite function. when a function is composed of other functions, the chain rule allows us to differentiate it with respect to the innermost variable

Formal definition:-

If $y = f(g(x))$, we can represent it as

$y = f(u)$, where $u = g(x)$

Then derivative of y as per chain rule of derivatives is

$$y = f(u) \Rightarrow \boxed{\frac{\partial y}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x}}$$

In the simpler terms, this can be expressed as

$$\boxed{\frac{\partial y}{\partial x} = f'(g(x)) \cdot g'(x)}$$

This means that to differentiate a composite function, you first differentiate the outer function with respect to the inner function and then multiply by the derivative of the inner function.

Examples of chain rule of derivatives:-

$\Rightarrow$ $y = (3x^2 + 2x + 1)^5$

Let $u = 3x^2 + 2x + 1$, then $y = (3x^2+2x+1)^5 = u^5$

The outer function is u^5

The inner function is $u = 3x^2 + 2x + 1$

Differentiating the outer function, $f(u) = u^5 \Rightarrow \frac{\partial f}{\partial u} = 5 \cdot u^4$

Differentiating the inner function, $u = 3x^2 + 2x + 1$

$$\Rightarrow \frac{\partial u}{\partial x} = 6x + 2$$

Derivative of y is,

$$\boxed{\frac{\partial y}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} = (5u^4)(6x+2) = 5(6x+2)(3x^2+2x+1)^4}$$

→ With respect to trigonometric function

$y = \sin(4x^3 + x)$

Let $u = 4x^3 + x$, then $y = f(u) = \sin u$

Outer function ⇒ $f(u) = \sin u$

Inner function ⇒ $u = 4x^3 + x$

Differentiating the outer function:-

$f(u) = \sin u \Rightarrow \frac{\partial f}{\partial u} = \cos u$

Differentiating the inner function:-

$u = 4x^3 + x \Rightarrow \frac{\partial u}{\partial x} = 12x^2 + 1$

Now, the derivative of y will be

$$\boxed{\frac{\partial y}{\partial x} = \frac{\partial f}{\partial u} \times \frac{\partial u}{\partial x} = (\cos u)(12x^2+1) = \cos(4x^3+x)(12x^2+1)}$$

② Applications of chain rule in data science:-

(i) Back propogation in neural networks

(ii) Gradient descent optimization

(iii) Feature engineering

(iv) Regularization techniques

Composite of three or more functions

⇒ Example of such problem is

$y = \sqrt{\sin 3x}$

It has three composite functions:-

Outer function ⇒ $y = f(u) = \sqrt{u} = u^{1/2}$, where $u = \sin 3x$

Middle function ⇒ $u = \sin 3x$, where if $v = 3x$, then $u = \sin v$

Inner function ⇒ $v = 3x$

Differentiating the outer function:-

$$\frac{\partial f}{\partial u} = \frac{\partial}{\partial u}(u^{1/2}) = \frac{1}{2}u^{-1/2}$$

Differentiating the middle function:-

$$\frac{\partial u}{\partial v} = \frac{\partial}{\partial v}(\sin v) = \cos v$$

Differentiating the inner function:-

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(3x) = 3$$

Now, as per chain rule of derivative,

$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial u} \times \frac{\partial u}{\partial v} \times \frac{\partial v}{\partial x} = \left(\frac{1}{2}u^{-1/2}\right)(\cos v)(3)$$

$$= \frac{3}{2\sqrt{u}} \times \cos v = \frac{3\cos 3x}{2\sqrt{\sin 3x}}$$

$$\boxed{\frac{\partial y}{\partial x} = \frac{3\cos 3x}{2\sqrt{\sin 3x}}}$$

The chain rule is the powerful tool in calculus for differentiating composite functions. By breaking down the differentiation process into manageable steps, it allows us to compute derivatives of complex expressions efficiently.