Linear Topological Spaces

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Outline

Introduction

- Linear topological spaces bridge algebra and topology, enabling the study of continuity and convergence in vector spaces.
- These spaces, such as Banach and Hilbert spaces, provide tools for analyzing limits, sequences, and functional dynamics.
- Applications span many areas, including partial differential equations.

Definition of a Vector Space

Definition

Let V be a **vector space** over a field F if it is equipped with two operations: vector addition and scalar multiplication, such that the following axioms hold for all $u, v, w \in V$ and scalars $a, b \in F$:

1. $(u + v) + w = u + (v + w)$
2. $u + v = v + u$
3. There exists a vector $0 \in V$ such that $v + 0 = v \quad \text{for all} \quad v \in V$
4. For every $v \in V$, there exists a vector $-v \in V$ such that $v + (-v) = 0$
5. $a(u + v) = au + av$
6. $(a + b)v = av + bv$
7. $(ab)v = a(bv)$
8. $1v = v \quad \text{for all} \quad v \in V$

For $x, y \in V$ and $\alpha \in F$, $x + y$ is the **sum**, and αy is the **product**.

Example: Space of Continuous Functions

Remark

Consider the space $C([a,b])$ of continuous functions on $[a,b]$. The vector addition and scalar multiplication are defined as follows:

$$(f+g)(x) = f(x) + g(x), \quad (af)(x) = af(x),$$

for $f, g \in C([a,b])$ and $a \in \mathbb{R}$.

1. **Closure under addition:** If $f, g \in C([a,b])$, then $(f+g)(x) = f(x) + g(x)$ is continuous on $[a,b]$. Since the sum of two continuous functions is continuous, $f + g \in C([a,b])$.
2. **Closure under scalar multiplication:** If $f \in C([a,b])$ and $a \in \mathbb{R}$, then $(af)(x) = af(x)$ is continuous on $[a,b]$. Since the product of a scalar and a continuous function remains continuous, $af \in C([a,b])$.

These properties confirm that $C([a,b])$ satisfies the vector space axioms.

Linear Transformation

Definition

A function T is called a *linear transformation* from the linear space X to a linear space Y over a scalar field $\mathbb{F}$ if it satisfies:

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y),$$

where $\alpha, \beta \in \mathbb{F}$ and $x, y \in X$.

Example: Linear Transformation in $\mathbb{R}^2$

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation defined by the matrix:

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x \\ 3y \end{pmatrix}.$$

In this case, for any vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$, the transformation T scales the x-coordinate by 2 and the y-coordinate by 3.

- $T(av_1 + bv_2) = T\left(a\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + b\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = T\left(\begin{pmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \end{pmatrix}\right)$

$$= a\begin{pmatrix} 2x_1 \\ 3y_1 \end{pmatrix} + b\begin{pmatrix} 2x_2 \\ 3y_2 \end{pmatrix} = aT(v_1) + bT(v_2).$$

Therefore, T is a linear transformation

Linearly Independent

Definition

A subset $E = \{x_1, x_2, \dots, x_n\}$ of a linear space X is said to be *linearly independent* if:

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0 \quad \text{if and only if} \quad \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.$$

Definition of a Basis

Definition

A *basis* β of a vector space V is a linearly independent subset of V that generates V. We say the vectors of β form a basis for V.

Example

The set $\beta = \{1, x, x^2, x^3, \dots\}$ is a basis in $P(F)$, the space of all polynomials with coefficients in F and degree at most n, because their linear combinations can form any element $f(x)$ in $P(F)$.

Dimension of a Vector Space

Definition

A vector space is called *finite-dimensional* if it has a basis consisting of a finite number of vectors. Every basis of V consists of exactly n vectors, where n is the *dimension* of V. Denote: $n = \dim(V)$.
If a vector space is not finite-dimensional, it is *infinite-dimensional*.

Example

Consider the vector space $P_n(F)$. A basis for $P_n(F)$ is given by:

$$\{1, x, x^2, \ldots, x^n\}.$$

Since the basis consists of $n+1$ elements, the dimension of $P_n(F)$ is:

$$\dim(P_n(F)) = n + 1.$$

Definition of a Topological Space

Definition

Let X be a set. A *topology* on X is a collection τ of subsets of X satisfying the following conditions:

- The total set X and the empty set $\emptyset$ are elements of τ;
- If $\{U_\gamma\}_{\gamma\in\Gamma}$ is a (possibly infinite) family of elements of τ, then $\cup_{\gamma\in\Gamma} U_\gamma \in \tau$;
- If $\{U_1, \ldots, U_n\} \subseteq \tau$ is a finite family of elements of τ, then $\bigcap_{i=1}^n U_i \in \tau$.

A *topological space* is a pair (X, τ) where τ is a topology on X.

Example: Topological Space (Finite Case)

Example

Let (X, τ) be a topological space where $X = \{1, 2, 3, 4\}$ and

$$\tau = \{\emptyset, X, \{1, 2\}, \{2\}, \{1, 2, 3\}, \{2, 3, 4\}, \{3\}\}.$$

To verify that τ forms a topology:

1. **Empty set and total set**: Clearly, $\emptyset \in \tau$ and $X \in \tau$, satisfying the first condition of a topology.
2. **Closure under unions**: For example, $\{1, 2\} \cup \{2, 3, 4\} = X \in \tau$, showing closure under unions.
3. **Closure under intersections**: For instance, $\{1, 2, 3\} \cap \{2, 3, 4\} = \{2, 3\} \in \tau$. Other intersections also result in elements of τ or $\emptyset$.

Since τ satisfies all conditions, (X, τ) is a topological space.

Definition of Openness

Definition

A set is called **open** in a topological space if, for each point in the set, there exists a neighborhood of that point entirely contained within the set. Formally, a set U is open in a topological space (X, τ) if $U \in \tau$, where τ is the topology on X.

Example (Discrete Topology)

Consider the **discrete topology** on $X = \{a, b, c\}$, where every subset of X is open. For instance, the set $\{a\}$ is open because all subsets of X are open by definition.

Definition of Closedness

Definition

A set is called **closed** in a topological space if its complement (with respect to the entire space) is open. That is, a set A is closed in (X, τ) if $X \setminus A \in \tau$.

Example (Finite Complement Topology)

In the **finite complement topology** on $X = \mathbb{Z}$, open sets are those with finite complements. The set $A = \mathbb{Z} \setminus \{1, 2, 3\}$ is closed because its complement $\{1, 2, 3\}$ is finite and therefore open.

Definition of Continuity

Definition

A function $f : X \to Y$ is *continuous* if small changes in the input result in small changes in the output.
In the context of metric spaces, f is continuous if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_1, x_2 \in X$:

$$d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \epsilon.$$

This ensures f has no jumps or breaks.

Example: Continuity of $f(x) = 2x$

Example

Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$, where $\mathbb{R}$ has the standard topology.
Let $V = (a, b)$ be an open interval in $\mathbb{R}$. The preimage of V under f is:

$$f^{-1}(V) = \{x \in \mathbb{R} \mid f(x) = 2x \in (a, b)\} = \left(\frac{a}{2}, \frac{b}{2}\right).$$

Since $\left(\frac{a}{2}, \frac{b}{2}\right)$ is an open interval in $\mathbb{R}$, $f^{-1}(V)$ is open in $\mathbb{R}$. Therefore, $f(x) = 2x$ is continuous.

Definition of a Metric Space

Definition

A metric space is a set X together with a function $\rho : X \times X \to \mathbb{R}$ (called the metric) which satisfies the following properties for all $x, y, z \in X$:

- **Positive definiteness:** $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$.
- **Symmetry:** $\rho(x, y) = \rho(y, x)$.
- **Triangle inequality:** $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

Note: $\rho(x, y)$ is finite for all $x, y \in X$.

Example

Consider $C([a, b])$ with the supremum metric:

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|, \quad f, g \in C([a, b]).$$

Example: Metric on $C([a, b])$ (Cont.)

Example

- **Positive definiteness:** $d(f, g) \geq 0$, and $d(f, g) = 0$ if and only if $f(x) = g(x)$ for all $x \in [a, b]$.
- **Symmetry:** $|f(x) - g(x)| = |g(x) - f(x)|$ implies $d(f, g) = d(g, f)$.
- **Triangle inequality:** For all $x \in [a, b]$,

$$|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|,$$

and taking the supremum yields

$$d(f, h) \leq d(f, g) + d(g, h).$$

Hence, $(C([a, b]), d)$ is a metric space.

Definition: Open and Closed Balls

Definition

Let $a \in X$ and $r > 0$:

- The *open ball* with center a and radius r is:

$$B_r(a) := \{x \in X : \rho(x, a) < r\}.$$

- The *closed ball* with center a and radius r is:

$$\overline{B}_r(a) := \{x \in X : \rho(x, a) \leq r\}.$$

Example

In the space ℓ^2 of square-summable sequences:

$$\ell^2 = \left\{ x = (x_n)_{n \in \mathbb{N}} : \sum_{n=1}^{\infty} x_n^2 < \infty \right\},$$

Example: Open and Closed Balls in ℓ^2

Example

With metric:

$$\rho(x, y) = \left(\sum_{n=1}^{\infty} (x_n - y_n)^2 \right)^{\frac{1}{2}},$$

The open ball around 0 is:

$$B_r(0) = \left\{ x \in \ell^2 : \left(\sum_{n=1}^{\infty} x_n^2 \right)^{\frac{1}{2}} < r \right\}.$$

Similarly, the closed ball is:

$$\overline{B}_r(0) = \left\{ x \in \ell^2 : \left(\sum_{n=1}^{\infty} x_n^2 \right)^{\frac{1}{2}} \leq r \right\}.$$

Definition: Open and Closed Sets

Definition

- A set $V \subseteq X$ is **open** if for every $x \in V$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq V$.
- A set $E \subseteq X$ is **closed** if its complement $E^c = X \setminus E$ is open. This implies E contains all its limit points.

Remark

Every open ball is open, and every closed ball is closed in a metric space.

Definition of Boundedness

Definition

Let (X, d) be a metric space. A subset $A \subseteq X$ is **bounded** if there exists a real number $M > 0$ such that for all $x \in A$ and for some fixed point $x_0 \in X$:

$$d(x, x_0) \leq M.$$

This means A is contained within a ball of radius M around x_0.

Example

Consider $(\mathbb{R}^2, d)$ with the Euclidean metric. Let A be the closed disk of radius 1 centered at the origin:

$$A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}.$$

This set A is bounded because the distance between any two points in A is at most 2. Thus, A is bounded in $\mathbb{R}^2$ with $M = 2$.

Completeness: Cauchy Sequences and Cauchy Nets

Definition

A sequence $\{x_n\}$ in a metric space (X, d) is a **Cauchy sequence** if, for every $\epsilon > 0$, there exists an integer N such that for all $m, n \geq N$:

$$d(x_m, x_n) < \epsilon.$$

Definition

A net $\{x_\alpha\}$ in a topological space X is a **Cauchy net** if, for every neighborhood U of the diagonal $\Delta = \{(x, x) : x \in X\}$ in $X \times X$, there exists $\alpha_0 \in A$ such that for all $\alpha, \beta \geq \alpha_0$:

$$(x_\alpha, x_\beta) \in U.$$

Definition of Completeness

Definition

A topological space X is **complete** if every Cauchy net in X converges to a point in X. That is, for every Cauchy net $\{x_\alpha\}$, there exists $x \in X$ such that $x_\alpha \to x$.

Example

In $\mathbb{R}$ with the standard topology, every Cauchy net converges to a real number. Thus, $\mathbb{R}$ is complete.

Definition of Linear Topological Space

Definition

A linear space X over a field $\mathbb{F}$ is a **linear topological space** if there exist topologies ρ_1 on X and ρ_2 on $\mathbb{F}$ such that the following functions are continuous:

1. $X \times X \to X$, $(x, y) \mapsto x + y$,
2. $\mathbb{F} \times X \to X$, $(\alpha, x) \mapsto \alpha x$.

Example: Banach Space

Definition

A **Banach space** is a vector space B over $\mathbb{R}$ or $\mathbb{C}$, equipped with a norm $\|\cdot\|$, that is complete with respect to this norm. This means every Cauchy sequence in B converges to a point in B.

Example

The space $C([a, b])$ with the supremum norm:

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|,$$

is a Banach space. Any Cauchy sequence of functions in $C([a, b])$ converges uniformly to a function in $C([a, b])$.

Definition of Functionals and Linear Functionals

Definition

A **functional** is a mapping from a space of functions into the real numbers $\mathbb{R}$ (or another field). It takes a function as input and outputs a scalar.

Definition

Let V be a vector space over a field F. A function $f : V \to F$ is a **linear functional** if it satisfies:

- **Additivity:** $f(u + v) = f(u) + f(v)$ for all $u, v \in V$,
- **Homogeneity:** $f(a \cdot v) = a \cdot f(v)$ for all $a \in F$ and $v \in V$.

Definition of Dual Space

Definition

The **dual space** of a linear topological space X, denoted as X^*, is the set of all continuous linear functionals from X to $\mathbb{R}$:

$$X^* = \{f \mid f : X \to \mathbb{R} \text{ is linear and continuous}\}.$$

Example: Dual Space of $C([0,1])$

Example

Consider the space $C([0,1])$:

- **Space:** $C([0,1])$ is infinite-dimensional because there are infinitely many linearly independent continuous functions on $[0,1]$.
- **Dual Space:** The dual space $C([0,1])^*$ consists of all continuous linear functionals $f : C([0,1]) \to \mathbb{R}$.

Example of a Functional: The Riemann integral defines a functional in $C([0,1])^*$:

$$f(g) = \int_0^1 g(x)\,dx, \quad g \in C([0,1]).$$

This functional is linear and continuous, mapping a function $g(x)$ to a real number.

Definition of Weak Topology

Definition

Let X be a topological vector space, and let X^* be its **dual space**. The **weak topology** on X is the coarsest topology such that every functional $f \in X^*$ is continuous.
A net $\{x_\alpha\} \subseteq X$ converges to $x \in X$ in the weak topology if and only if:

$$f(x_\alpha) \to f(x) \quad \text{for all } f \in X^*.$$

This means convergence is determined solely by the functionals, not by norms or distances.

Example: Weak Topology on $C([a, b])$

Example

Consider $X = C([a, b])$ with the dual space $X^* = \mathcal{M}([a, b])$, the space of signed measures.

Weak Convergence: A sequence $\{f_n\} \subset C([a, b])$ converges weakly to $f \in C([a, b])$ if:

$$\int_a^b f_n(x)\, d\mu \to \int_a^b f(x)\, d\mu, \quad \text{for all } \mu \in \mathcal{M}([a, b]).$$

This means convergence is determined by the action of μ, not the norm of $f_n - f$.

Convex Set and Local Convexity

Definition (Convex Set)

A set C in a vector space is **convex** if for any $x, y \in C$ and $t \in [0, 1]$:

$$tx + (1 - t)y \in C.$$

Definition (Local Convexity)

A topological space is **locally convex** if every point has a neighborhood basis consisting of convex sets.

Example

In $\mathbb{R}^2$ with the standard topology, the unit disk

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

is convex. Open disks around any point in $\mathbb{R}^2$ are also convex, showing that $\mathbb{R}^2$ is locally convex.

Seminorm

Definition (Seminorm)

A **seminorm** $p : V \to \mathbb{R}$ on a vector space V satisfies:

- **Non-negativity:** $p(x) \geq 0$ for all $x \in V$,
- **Homogeneity:** $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{R}, x \in V$,
- **Subadditivity:** $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$.

Unlike a norm, $p(x) = 0$ is possible for $x \neq 0$.

Example of a Seminorm

Example

Let $V = C([a, b])$. Define:

$$p(f) = \left| \frac{1}{b-a} \int_a^b f(x)\, dx \right|.$$

Properties:

- **Non-negativity:** $p(f) \geq 0$ since absolute value is non-negative.
- **Homogeneity:** For $\alpha \in \mathbb{R}$,

$$p(\alpha f) = |\alpha| p(f).$$

- **Subadditivity:**

$$p(f+g) \leq p(f) + p(g).$$

Note: $p(f) = 0$ for some non-zero $f(x)$, such as $f(x) = x - \frac{a+b}{2}$, so p is not a norm.

Open Graph Theorem

Theorem (Open Graph Theorem)

Let X and Y be Banach spaces, and let $T : X \to Y$ be a linear operator. Suppose the graph of T, defined as:

$$G(T) = \{(x, T(x)) \mid x \in X\} \subseteq X \times Y,$$

is an open subset of $X \times Y$ in the product topology (induced by the norms of X and Y). Then T is continuous.

Assume T is not continuous.

- If T is not continuous, there exists a sequence $\{x_n\} \subset X$ such that:

$$x_n \to 0 \quad \text{in } X, \quad \text{but } T(x_n) \not\to 0 \quad \text{in } Y.$$

- Specifically, there exists $\epsilon > 0$ such that:

$$\|T(x_n)\|_Y \geq \epsilon \quad \text{for all } n.$$

Proof of the Open Graph Theorem (Step 2)

Openness of the Graph $G(T)$:

- The openness of $G(T)$ implies that there exists a neighborhood $U \subset X \times Y$ around any point $(0, 0) \in G(T)$.
- For sufficiently large n, the points $(x_n, T(x_n))$ belong to this neighborhood.

Contradiction:

- Since $x_n \to 0$ in X, $(x_n, T(x_n)) \to (0, 0)$ in $X \times Y$.
- By the openness of $G(T)$, $T(x_n)$ must approach 0 in Y, contradicting $\|T(x_n)\|_Y \geq \epsilon$.

Proof of the Open Graph Theorem (Conclusion)

Conclude Continuity:

- The assumption that T is not continuous leads to a contradiction.
- Therefore, T must be continuous.

Summary:

- The openness of the graph $G(T)$ guarantees that small perturbations in x lead to predictable changes in $T(x)$.
- This ensures the operator T is stable and well-behaved under small perturbations.

Some Useful Theorems

- **Sololev Embedding**
  Let $H^k(\Omega)$ be a Sobolev space on a domain $\Omega \subset \mathbb{R}^n$. If $k > \frac{n}{2}$, then:

$$H^k(\Omega) \subset C^0(\Omega) \cap L^\infty(\Omega).$$

- **Elliptic Regularity**
  Consider the PDE:

$$-\Delta u + \lambda u = g, \quad \lambda \notin \sigma_p(-\Delta), \quad g \in L^2(\Omega).$$

  Then $u \in H^2(\Omega) \cap H^1_0(\Omega)$.
- **Contraction Mapping**
  Let (X, d) be a complete metric space. If $T : X \to X$ satisfies:

$$d(T(x), T(y)) \leq c\, d(x, y), \quad 0 \leq c < 1,$$

  then T has a unique fixed point $x^* \in X$.

Non-linear Elliptic PDE: Problem Setup

Problem:

$$\begin{cases} \Delta u + \lambda u + u^2 = g, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^3$: Open, bounded, smooth boundary.
- $\lambda \in \mathbb{R}$: Constant parameter.
- $g \in L^2(\Omega)$: Given function.

Goal: Find u satisfying this equation.

Functional Framework

Function Space: Sobolev space $X = H^2(\Omega) \cap H_0^1(\Omega)$, equipped with the H^2-norm.

Why Sobolev Spaces?

- Ensure $\Delta u + \lambda u \in L^2(\Omega)$.
- Incorporate boundary condition $u = 0$ on $\partial\Omega$ (trace sense).
- Provide the required regularity for u^2 to lie in $L^2(\Omega)$.

Linear Auxiliary Problem

Step 1: Reformulate Non-linear Term:

$$h(u) = g - u^2 \quad \text{where } h(u) \in L^2(\Omega).$$

Step 2: Solve Linear Problem:

$$\begin{cases} \Delta w + \lambda w = h(u), & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

Result: Using **Elliptic Regularity Theory**, if $\lambda \notin \sigma_p(-\Delta)$, the solution w satisfies:

$$w \in H^2(\Omega) \cap H_0^1(\Omega).$$

Define Operator: $A(u) = w$, mapping $X \to X$.

Fixed Point and Contraction Mapping

Fixed Point:

- $A(u) = u$ implies u solves the original PDE:

$$\Delta u + \lambda u + u^2 = g, \quad u = 0 \text{ on } \partial\Omega.$$

Contraction Property:

- For $u \in B_X(0, r) \subset X$:

$$\|A(u)\|_{H^2(\Omega)} \leq C(\|g\|_{L^2(\Omega)} + \|u\|^2_{H^2(\Omega)}).$$

- For small $\|g\|_{L^2(\Omega)}$ and r, A satisfies:

$$\|A(u) - A(v)\|_{H^2(\Omega)} \leq c\|u - v\|_{H^2(\Omega)}, \quad 0 \leq c < 1.$$

Conclusion: Existence of Solution

Using the Contraction Mapping Theorem:

- A has a unique fixed point $u_0 \in B_X(0, r)$.
- u_0 is the unique weak solution to the PDE in $H^2(\Omega) \cap H_0^1(\Omega)$.

Conclusion

- Explored linear topological spaces, from vector spaces to advanced concepts like weak and weak* topologies, and duality.
- Highlighted Banach and Hilbert spaces as frameworks for analyzing continuous mappings, linear operators, and PDE solutions.
- Future research into fixed-point theorems and optimization.