Metric Spaces

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Overview

Why is Metric Space Important?

- Introduced in 1906 by Maurice Frechet, and further developed by Felix Hausdorff.
- Generalize the notion of distance.
- Their properties are widely used such as in geometry or limit problems.

Definition of a Metric Space

Definition

A metric space is a set X together with a function $p : X \times X \longrightarrow \mathbb{R}$ (called the metric of X) which satisfies the following properties for all $x, y, z \in X$:

POSITIVE DEFINITE: $\rho(x, y) \geq 0$ with $\rho(x, y) = 0$ if and only if $x = y$,
SYMMETRIC: $\rho(x, y) = \rho(y, x)$,
TRIANGLE INEQUALITY: $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

[Notice that by definition, $\rho(x, y)$ is finite valued for all $x, y \in X$.]

Example 1 - Discrete

Example

The set S is a metric space that is defined by $\rho : X \times X \longrightarrow \mathbb{R}$ such that

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases} \tag{1}$$

Example 1 (cont.)

Proof.

Positive definite: For $x, y \in S$, it is always true that $\rho(x, y) \geq 0$ where $\rho(x, y) = 0 \iff x = y$.

Symmetric: We observe that 0 and 1 are the only solutions of $\rho(x, y)$ such that the position of x and y are not important. Therefore, $\rho(x, y) = \rho(y, x)$.

Triangle inequality: Let $z \in S$. Since $max\{\rho(x, y)\}_{x,y \in S} = 1$. We have the following cases:

Case $x = y = z$: Then $\rho(x, z) + \rho(z, y) = 0$. And having $\rho(x, y) = 0$.

So $\rho(x, z) + \rho(z, y) = \rho(x, y)$.

Case $x \neq y \neq z$: Then $\rho(x, y) = 1 < \rho(x, z) + \rho(z, y) = 1 + 1 = 2$

Case one is difference: Assume x = y and $x \neq z$.

Then, $\rho(x, y) = 0 < \rho(x, z) + \rho(z, y) = 2$

Assume $x \neq y$ and, WLOG $x = z$.

Then, $\rho(x, y) = \rho(x, z) + \rho(z, y) = 1$

Therefore, $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$. □

Example 2 - Continuous

Example

$\mathbb{R}^2$ is a metric space with the function $\rho(x, y) = |x - y|$ for $x, y \in \mathbb{R}$

Example 2 (cont.)

Proof.

Positive definite: For $x, y \in \mathbb{R}$, it is always true that $\rho(x,y) = |x-y| \geq 0$ where $\rho(x,y) = 0 \iff x = y$.
Symmetric: By definition of absolute value, we have:

$$\begin{aligned}\rho(x,y) &= |x-y| \\ &= |-(x-y)| \\ &= |y-x| \\ &= \rho(y,x)\end{aligned}$$

Triangle inequality: Let $z \in \mathbb{R}$. We have:

$$\begin{aligned}\rho(x,z) + \rho(z,y) &= |x-z| + |z-y| \\ &\geq |x-z+z-y| = |x-y| = \rho(x,y)\end{aligned}$$

Therefore, $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$. $\square$

Openness and Closeness

Definition

Let $a \in X$ and $r > 0$. The *open ball* (in X) with *center* a and *radius* r is the set

$$B_r(a) := \{x \in X : \rho(x, a) < r\},$$

and the *closed ball* (in X) with *center* a and *radius* r is the set

$$\{x \in X : \rho(x, a) \leq r\}.$$

Definition

i) A set $V \subseteq X$ is said to be *open* if and only if for every $x \in V$ there is an $\epsilon > 0$ such that the open ball $B_\epsilon(x)$ is contained in V.
ii) A set $E \subseteq X$ is said to be *closed* if and only if $E^c := X \backslash E$ is open.

Proposition

Every open ball is open, and every closed ball is closed.

Proof.

Let $B_r(a)$ be an open ball, $x \in B_r(a)$, and $\epsilon = r - \rho(x, a)$.
For $y \in B_\epsilon(x)$, by the Triangle Inequality and chosen ϵ, we have:

$$\begin{aligned} \rho(y, a) &\leq \rho(y, x) + \rho(x, a) \\ &< \epsilon + \rho(x, a) \\ &= r \end{aligned}$$

Thus, $y \in B_r(a)$. In other words, $B_\epsilon(x) \subseteq B_r(a)$.
By the definition of openness and closeness, $B_r(a)$ is open. Hence, every open ball is open.
Similarly, the set $\{x \in X : \rho(x, a) > r\}$ is open, so its complement $\{x \in X : \rho(x, a) \leq r\}$ closed. Hence, every closed ball is closed. $\square$

Theorem

Let X be a metric space.
i) If $\{V_\alpha\}_{\alpha \in A}$ is any collection of open sets in X, then
$$\bigcup_{\alpha \in A} V_\alpha$$
is open.
ii) If $\{V_k := 1, 2, ..., n\}$ is a finite collection of open sets in X, then
$$\bigcap_{k=1}^{n} V_k := \bigcap_{k \in \{1,2,..,n\}} V_k$$
is open.

Proof.

i) Let $x \in \bigcup_{\alpha \in A} V_\alpha$. Then, $x \in V_\alpha$ which is open for some $\alpha \in A$. Thus, there is an $r > 0$ such that $B_r(x) \subseteq \bigcup_{\alpha \in A} V_\alpha$.
By the definition of openness and closeness, $\bigcup_{\alpha \in A} V_\alpha$ is open.
ii) Let $x \in \bigcap_{k=1}^{n} V_k$. Then $x \in V_k$ which is open for some $k = 1, 2, ..., n$. Thus, there are some numbers $r_k > 0$ such that $B_{r_k}(x) \subseteq V_k$. For any $r \in r_k$, $r > 0$ and $B_r(x) \subseteq V_k$.
By the definition of openness and closeness, $\bigcap_{k=1}^{n} V_k$ is open. □

Compactness

Definition

Let $\nu = \{V_\alpha\}_{\alpha \in A}$ be a collection of subsets of a metric space X and suppose that E is a subset of X.

i) ν is said to *cover* E (or be a *covering* of E) if and only if

$$E \subseteq \bigcup_{\alpha \in A} V_\alpha$$

ii) ν is said to be an *open covering* of E if and only if ν covers E and each V_α is open.

iii) Let ν be a covering of E. ν is said to have a *finite* (respectively, *countable*) *subcovering* if and only if there is a finite (respectively, countable) subset A_0 of A such that $\{V_\alpha\}_{\alpha \in A_0}$ covers E.

Example

$\nu = \{(\frac{1}{k+1}, \frac{k}{k+1})\}_{k \in \mathbb{N}}$ is an open covering of (0,1).

ν is a covering of $(0,1)$ since $(0,1) \subseteq \bigcup_{k \in \mathbb{N}} (\frac{1}{k+1}, \frac{k}{k+1})$

And since $(\frac{1}{k+1}, \frac{k}{k+1})$ is open for every $k \in \mathbb{N}$, ν is an open covering.

Compact Sets

Definition

A subset H of a metric space X is said to be *compact* if and only if every open subcovering of H has a finite subcover.

Proposition

A compact set is always closed.

Theorem

Let H be a subset of a metric space X. If H is compact, then H is closed and bounded.

Proof.

Let H be a compact subset of a metric space X. By Proposition above, we know that a compact set is always closed, so H is closed.
Take $b \in X$ and note $\{B_n(b) : n \in N\}$ covers X.
As given that H is compact, we have:

$$H \subset \bigcup_{n=1}^{N} B_n(b) \text{ for some } N \in \mathbb{N}.$$

Hence, $H \subset B_N(b)$ or H is bounded.
Therefore, H is closed and bounded. $\square$

Connectedness

Definition

Let X be a metric space.
i) A pair of nonempty open sets U, V in X is said to *separate* X if and only if $X = U \cup V$ and $U \cap V = \emptyset$.
ii) X is said to be *connected* if and only if X cannot be separated by any pair of open sets U, V.

Example

$\mathbb{R}$ under discrete metric is separate because $(-\infty, 0]$ and $(0, \infty)$ are both open subsets of $\mathbb{R}$ such that $\mathbb{R} = (-\infty, 0] \cup (0, \infty)$ and $(-\infty, 0] \cap (0, \infty) = \emptyset$.
$\mathbb{R}$ under a continuous metric such as $|x - y|$ is connected because the metric itself is continuous. So, S is not separated by any pairs of open sets U, V.

Definition of a Topological Space

Definition

Let X be a set. A *topology* on X is a collection τ of subsets of X satisfying the following conditions

(T1) the total set, X, and the empty set, $\emptyset$, are elements of τ;

(T2) if $\{U_\gamma\}_{\gamma\in\Gamma}$ is a (possibly infinite) family of elements of τ, then
$$\bigcup_{\gamma\in\Gamma} U_\gamma \in \tau;$$

(T3) if $\{U_1, ..., U_n\} \subseteq \tau$ is a finite family of elements of τ, then
$$\bigcap_{i=1}^{n} U_i \in \tau.$$

A *topological space* is a pair (X, τ) where τ is a topology on X.

Example 1

Example

(X, τ) is a topological space where $X = \{1, 2, 3\}$ and
$\tau = \{\emptyset, X, \{2\}, \{1, 2\}, \{2, 3\}\}$

Proof.

(T1): As we observe, $X, \emptyset \in \tau$.
(T2): From the given τ, we have: $\bigcup_{\gamma=1}^{5} U_\gamma \in \tau$. That is the union of families of elements of τ is also in τ.
(T3): Since any intersection of families of elements in τ is in τ, we conclude that $\bigcap_{i=1}^{5} U_i \in \tau$.
Thus, τ is a topology. Hence, (X, τ) is a topological space. $\square$

Example 2

Example

Let's consider the same set X with a different τ, say τ_0.
We have $X = \{1, 2, 3\}$ and $\tau_0 = \{\emptyset, X, \{2\}, \{3\}\}$
(T1): We can see that $\emptyset, X \in \tau_0$.
(T2): $\{2\} \cup \{3\} = \{2, 3\} \notin \tau_0$.
Thus, τ_0 is not a topology.
Hence, the pair (X, τ_0) is not a topological space.

Every Metric Space is a Topological Space

Proof.

Recall that the collection of all open sets of a metric space is closed under union and intersection.

For a metric space (X, ρ), let τ be the collection of all open subsets of X which respect to metric ρ. Thus, τ is a topology on X.

Hence, the pair (X, τ) is a topological space. $\square$

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