

# 6. Standard Error Models

## 6.1. Static Error Model

$$\varepsilon(t) = \tilde{\theta}^T(t)\omega(t), \quad (6.1)$$

where  $\varepsilon(t)$  is the output,  $\tilde{\theta}(t) = \theta - \hat{\theta}(t) \in \mathbb{C}^m$  is the vector of parametric errors,  $\omega(t) \in \mathbb{C}^m$  is the vector of measurable functions (regressor).

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Time derivative:

$$\dot{V} = \frac{1}{\gamma} \tilde{\theta}^T \dot{\tilde{\theta}} = -\frac{1}{\gamma} \tilde{\theta}^T \dot{\hat{\theta}} = ?$$

$$\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$$

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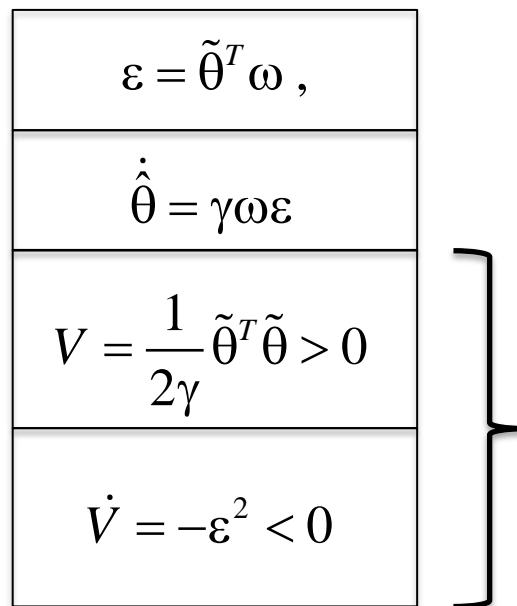
## Summary and Discussion

Error Model

Adaptation Algorithm

Lyapunov function

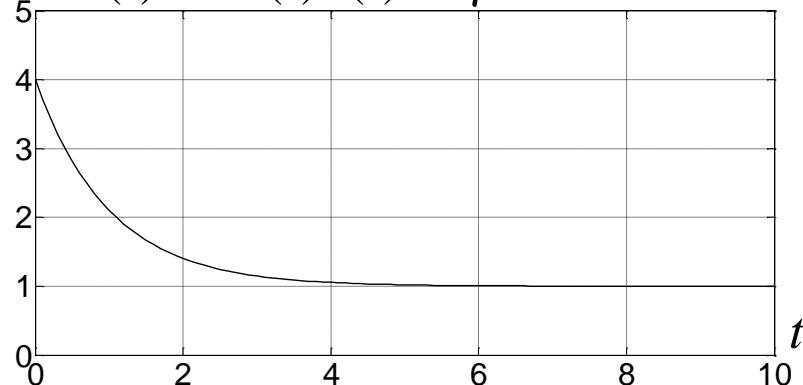
Its time derivative



What it means?

## 6.1. Static Error Model

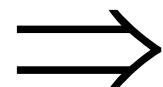
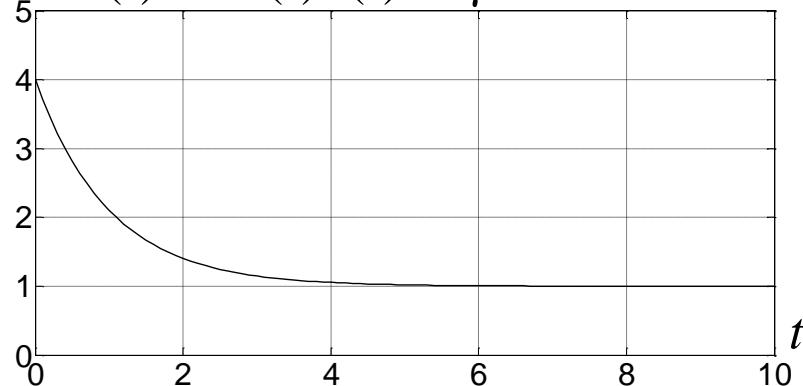
1.  $V(t) = \tilde{\theta}^T(t)\tilde{\theta}(t) / 2\gamma$



$\varepsilon \rightarrow 0$  *asymptotically fast*  
 $\|\tilde{\theta}\|^2$  *is monotonically*  
*decreasing function*

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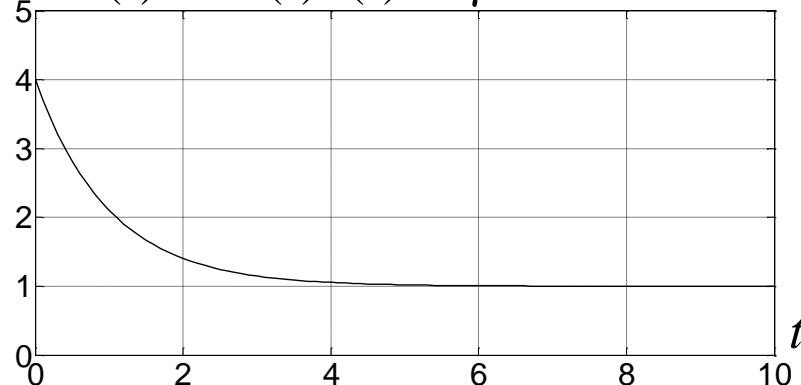
$\|\tilde{\theta}(t)\|^2$  is a monotonically  
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2.

Does  $\tilde{\theta}(t)$  always tend to zero  
or does the system has identification properties?

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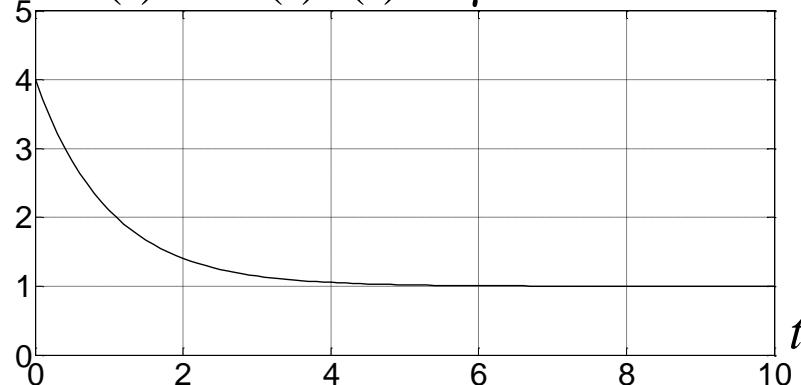
### 2. Example 6.1

For a given error model  $\varepsilon = \tilde{\theta}_1\omega_1 + \tilde{\theta}_2\omega_2$  there are the following scenarios:

- a)  $\omega_1 = 1, \omega_2 = 2$  and  $\tilde{\theta}_1 \rightarrow 2, \tilde{\theta}_2 \rightarrow -1$

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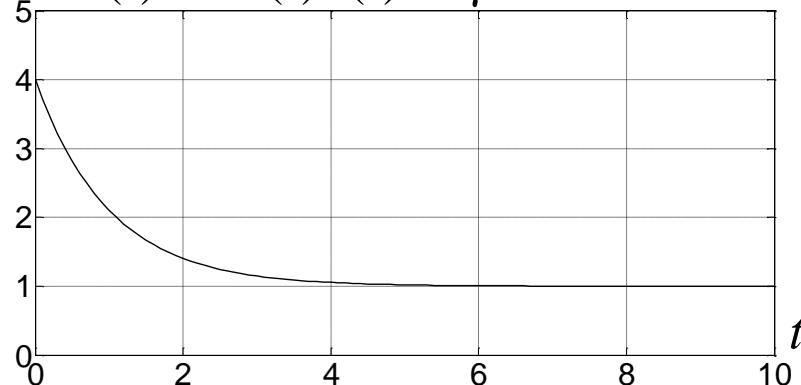
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**How many options for convergence?**

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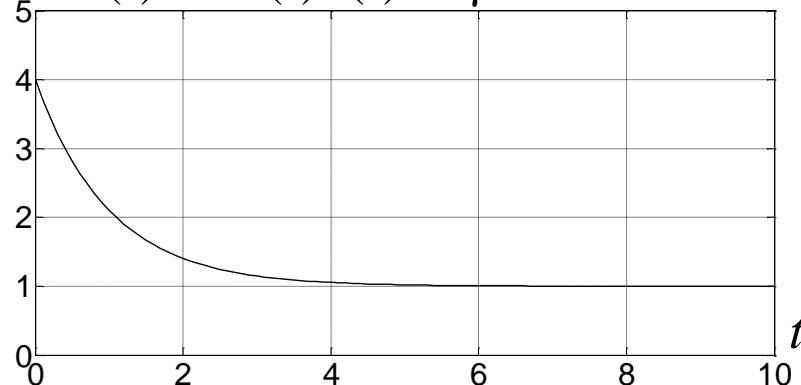
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- d)  $\omega_1 = \sin t, \omega_2 = 2 \sin 2t$  and  $\tilde{\theta}_1 \rightarrow ?, \tilde{\theta}_2 \rightarrow ?$

How many options for convergence?

## 6.1. Static Error Model

### Example 6.2

For the error model  $\varepsilon = \tilde{\theta}_1\omega_1 + \tilde{\theta}_2\omega_2 + \tilde{\theta}_3\omega_3$  there are the following scenarios:

- a)  $\omega_1 = \sin t, \omega_2 = 2 \sin t, \omega_3 = 3 \sin t$  and  $\tilde{\theta}_{1,2,3} \rightarrow ?$
- b)  $\omega_1 = \sin t, \omega_2 = 2 \sin t, \omega_3 = 3 \sin 2t$  and  $\tilde{\theta}_{1,2,3} \rightarrow ?$
- c)  $\omega_1 = \sin t, \omega_2 = 2 \sin 3t, \omega_3 = 3 \sin 2t$  and  $\tilde{\theta}_{1,2,3} \rightarrow ?$
- d)  $\omega_1 = \sin(t), \omega_2 = 2 \sin(t + \pi), \omega_3 = 3 \sin(t + \pi/2)$  and  $\tilde{\theta}_{1,2,3} \rightarrow ?$
- e)  $\omega_1 = \sin(2t), \omega_2 = 2 \sin(t + \pi), \omega_3 = 3 \sin(t + \pi/2)$  and  $\tilde{\theta}_{1,2,3} \rightarrow ?$

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*Vector  $\omega \in \mathbb{C}^m$  has to contain at least  $m/2$  different harmonics to provide identification properties*

## 6.1. Static Error Model

### Summary

#### Properties of the closed-loop system:

1. If  $\omega, \dot{\omega}$  are bounded, all the signals in the system are bounded;
2. Error  $\varepsilon(t)$  approaches zero asymptotically;
3. The function  $\|\tilde{\theta}(t)\|^2$  is nonincreasing;
4.  $\|\tilde{\theta}(t)\|^2$  approaches zero exponentially if  $\omega$  contains at least  $m/2$  harmonics and consists of linearly independent elements;

This property can be reformulated in terms of **Persistent Excitation**

#### Condition:

$$\int_t^{t+T} \omega(\tau) \omega^T(\tau) d\tau \geq \alpha I \quad (6.3)$$

for some positive  $\alpha, T$ .

## 6.1. Static Error Model

**Example 6.3. The problem of identification reduced to Static Error Model**

### Problem statement

Let a plant be described by

$$\ddot{y} + a_1 \dot{y} + a_0 y = b_0 u \quad (6.4)$$

with unknown parameters  $a_0, a_1, b_0$  and measurable input  $u$  and output  $y$ .

The objective is to design such estimates  $\hat{a}_0, \hat{a}_1, \hat{b}_0$  that

$$\lim_{t \rightarrow \infty} (a_0 - \hat{a}_0(t)) = \lim_{t \rightarrow \infty} (a_1 - \hat{a}_1(t)) = \lim_{t \rightarrow \infty} (b_0 - \hat{b}_0(t)) = 0. \quad (6.5)$$

## 6.1. Static Error Model

### Solution

Main idea of solution is to reduce the problem to the error model.

Then to get the adaptation algorithm generating the estimates.



## 6.1. Static Error Model

### Solution

1. Apply transfer function

$$H(s) = \frac{1}{K(s)} = \frac{1}{s^2 + k_1 s + k_0}$$

with Hurwitz polynomial  $K(s) = s^2 + k_1 s + k_0$  to the plant (6.4) assuming initial conditions  $y(0), \dot{y}(0)$  equaled to zero:

$$\ddot{y} + a_1 \dot{y} + a_0 y = b_0 u$$

$$\Downarrow H(s)[\square]$$

$$\frac{s^2}{K(s)}[y] + a_1 \frac{s}{K(s)}[y] + a_0 \frac{1}{K(s)}[y] = b_0 \frac{1}{K(s)}[u]$$

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 ~~$+k_1 s + k_0$~~

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## 6.1. Static Error Model

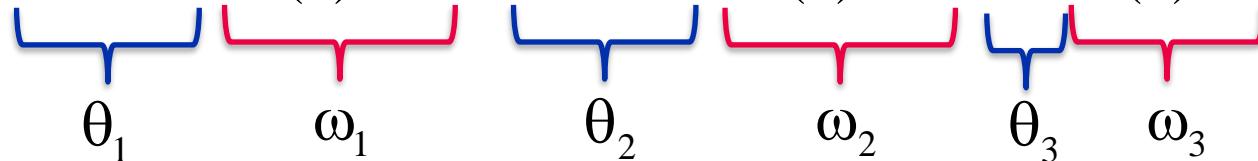
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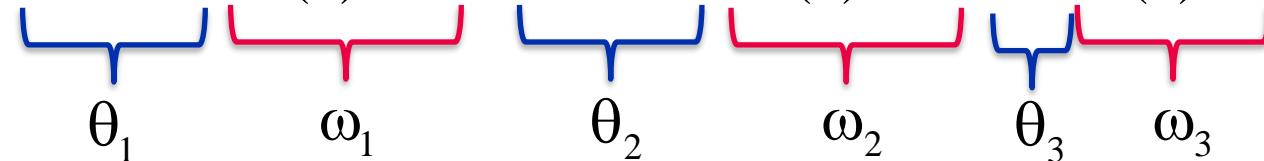
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Parameterized plant

$$y = \theta^T \omega \quad (6.6)$$

$$\theta = \text{col}(\theta_1, \theta_2, \theta_3), \quad \omega = \text{col}(\omega_1, \omega_2, \omega_3)$$

## 6.1. Static Error Model

### Solution

2. Design of error

$$\varepsilon = y - \hat{\theta}^T \omega \quad \text{Why this?} \quad (6.7)$$

where  $\hat{\theta}$  is the estimate of  $\theta$ .

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$$\downarrow \qquad \qquad \qquad y = \theta^T \omega$$

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$\tilde{\theta} = \theta - \hat{\theta}$  is parametric error vector.

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$$\begin{array}{c} \downarrow \\ \varepsilon = \tilde{\theta}^T \omega, \end{array}$$

#### 3. Adaptation algorithm design.

*Adaptation algorithm*

$$\dot{\hat{\theta}} = \gamma \omega \varepsilon \quad (6.8)$$



## 6.1. Static Error Model

### Solution Summary

*Error*

$$\varepsilon = y - \hat{\theta}^T \omega = y - (k_1 - \hat{a}_1) \frac{s}{K(s)} [y] - (k_0 - \hat{a}_0) \frac{1}{K(s)} [y] - \hat{b}_0 \frac{1}{K(s)} [u]$$

*Adaptation Algorithms*

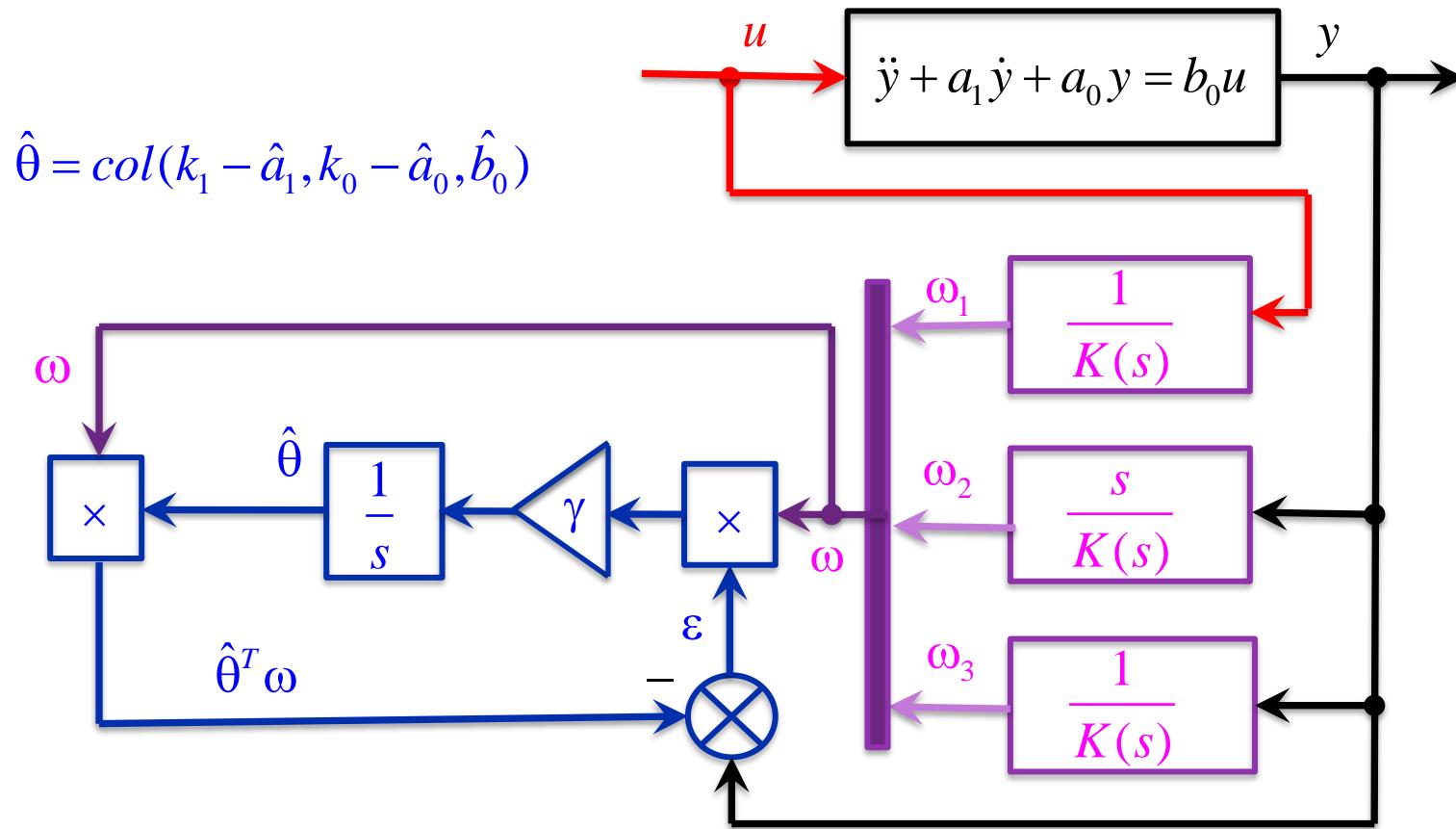
$$\dot{\hat{\theta}} = \gamma \omega \varepsilon$$

The diagram shows three arrows originating from the error term  $\varepsilon$  in the equation above and pointing to the corresponding adaptation equations below.

$$\dot{\hat{a}}_0 = -\gamma \frac{1}{K(s)} [y] \varepsilon$$
$$\dot{\hat{a}}_1 = -\gamma \frac{s}{K(s)} [y] \varepsilon$$
$$\dot{\hat{b}}_0 = \gamma \frac{1}{K(s)} [u] \varepsilon$$

## 6.1. Static Error Model

### General Scheme



## 6.1. Static Error Model

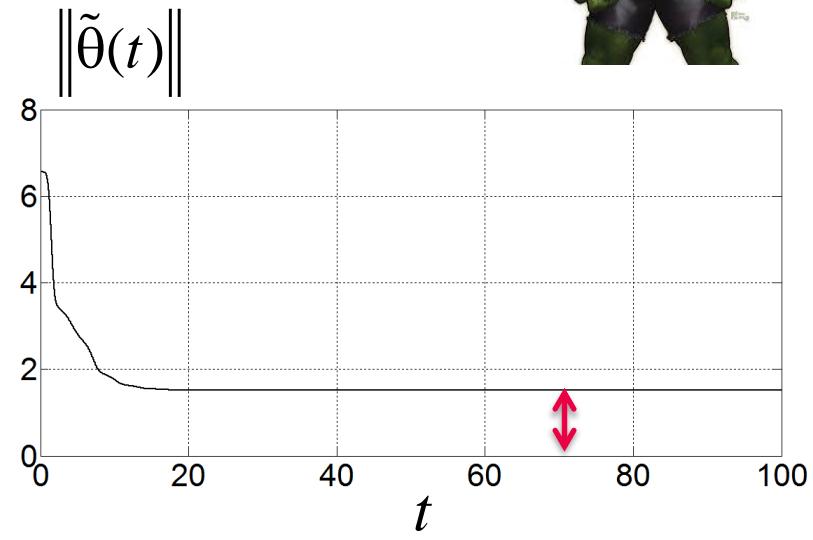
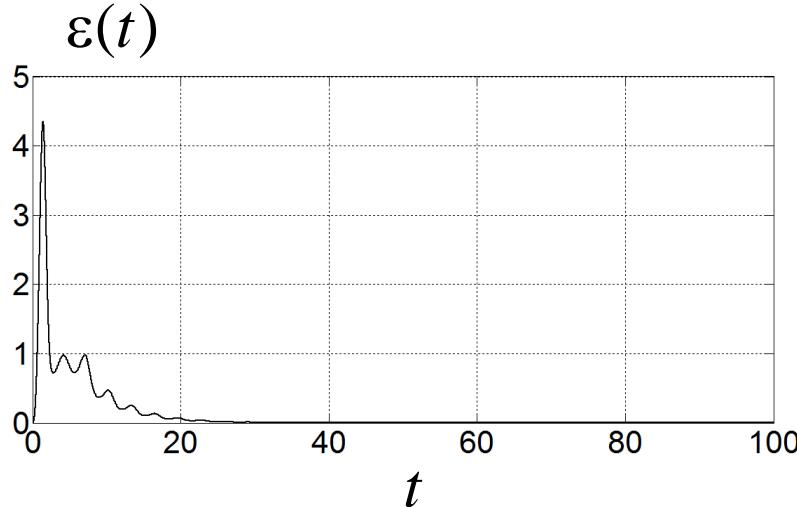
### Simulation results

*Plant*  $\ddot{y} + 2\dot{y} + y = 3u$

*Filters polynomial*  $K(s) = s^2 + 5s + 6$        $\theta = \text{col}(3, 5, 3)$

*Adaptation gain*  $\gamma = 1$

$$u(t) = 10 \sin t$$



## 6.1. Static Error Model

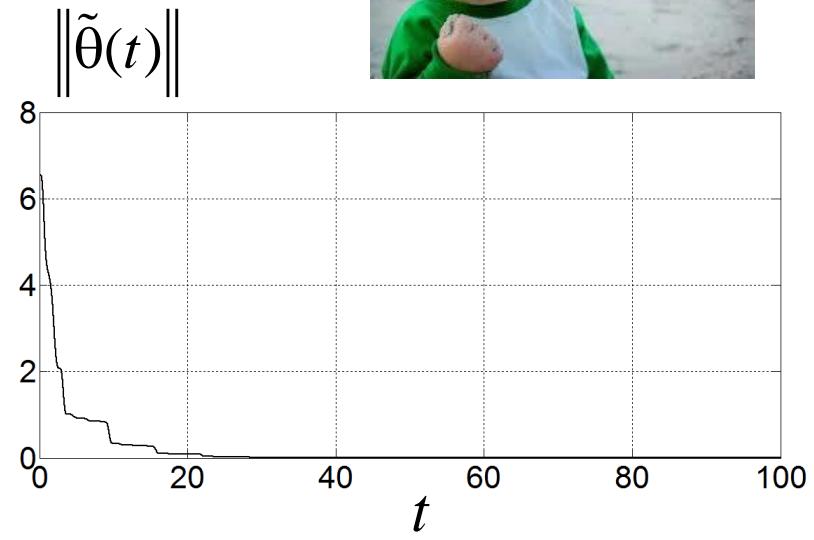
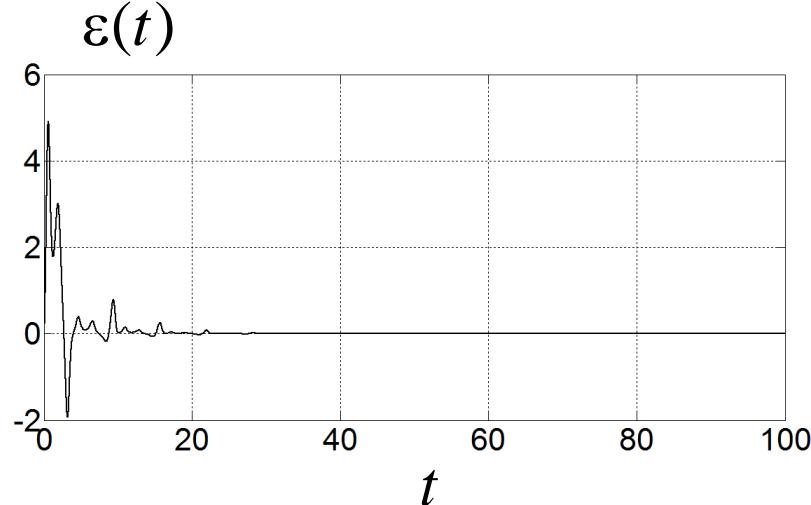
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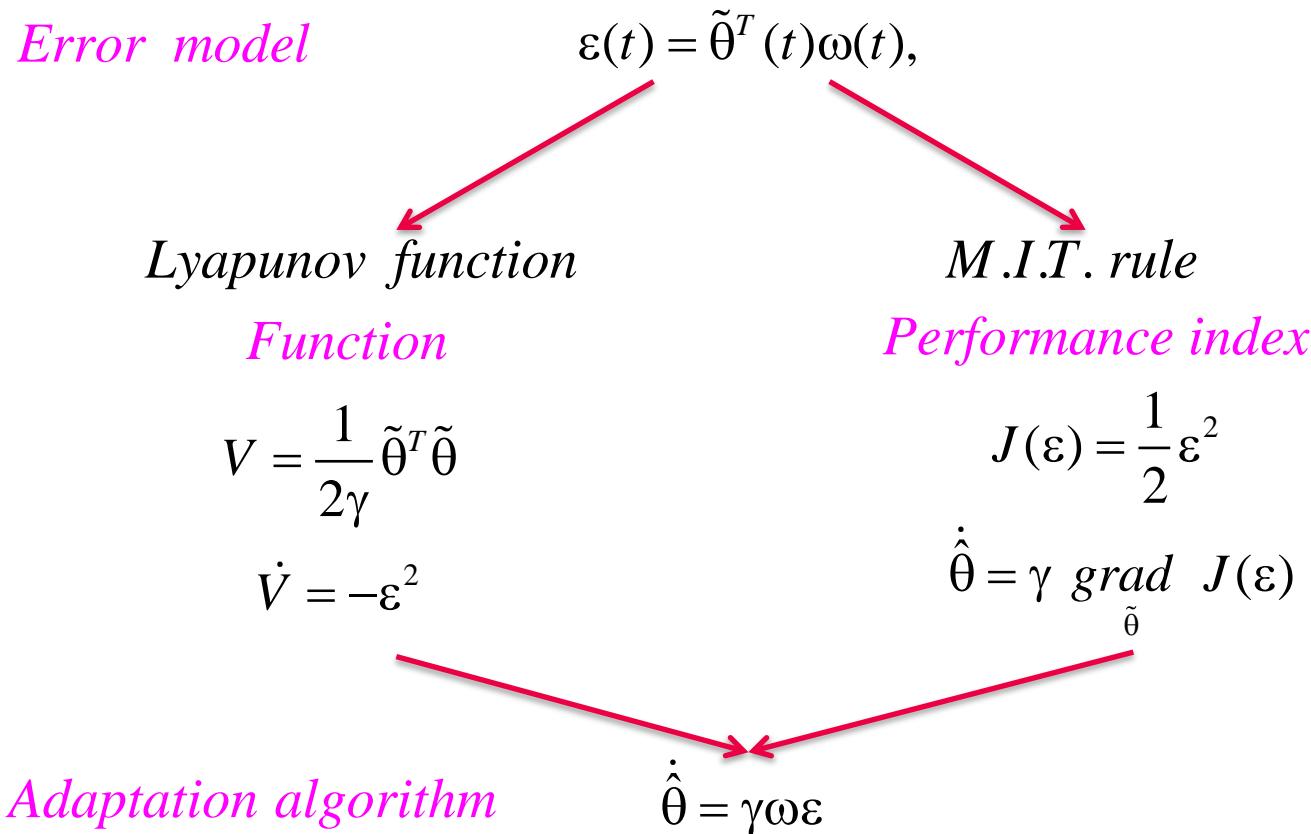
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$$u(t) = 10\sin t + 20\cos 2t$$



## 6.1. Static Error Model

### M.I.T. rule as an alternative methodology to Lyapunov functions



## 6.2. Dynamic error model with measurable state

$$\begin{aligned}\dot{e}(t) &= Ae(t) + b\tilde{\theta}^T(t)\omega(t), \\ \varepsilon(t) &= Ce(t)\end{aligned}\tag{6.9}$$

where  $e \in \mathbb{R}^n$  is the state,  $\varepsilon$  is the output,  $\tilde{\theta} \in \mathbb{R}^m$  is the vector of parametric errors,  $\omega \in \mathbb{R}^m$  is the vector of measurable functions (regressor).

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**Remark 6.2.** The model is widely used in the problems of state adaptive control (see example below).

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Lyapunov function?

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**Remark 6.2.** The model is widely used in the problems of state adaptive control (see example below).

$$V = \frac{1}{2}e^T Pe + \frac{1}{2\gamma} \tilde{\theta}^T \tilde{\theta} \tag{6.10}$$

with a positive gain  $\gamma$  and positively defined symmetric matrix  $P = P^T \succ 0$  defined later.

## 6.2. Dynamic error model with measurable state

Time derivative:

$$\dot{e} = Ae + b\tilde{\theta}^T \omega$$
$$\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$$
$$\dot{V} = \frac{1}{2}\dot{e}^T Pe + \frac{1}{2}e^T P\dot{e} + \frac{1}{\gamma}\tilde{\theta}^T \dot{\tilde{\theta}}$$

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$$\dot{e} = Ae + b\tilde{\theta}^T \omega$$
$$\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$$
$$\dot{V} = \frac{1}{2}\dot{e}^T Pe + \frac{1}{2}e^T P\dot{e} + \frac{1}{\gamma}\tilde{\theta}^T \dot{\tilde{\theta}} = \frac{1}{2}(Ae + b\tilde{\theta}^T \omega)^T Pe + \frac{1}{2}e^T P(Ae + b\tilde{\theta}^T \omega) - \frac{1}{\gamma}\tilde{\theta}^T \dot{\hat{\theta}} =$$

## 6.2. Dynamic error model with measurable state

Time derivative:

$$\dot{e} = Ae + b\tilde{\theta}^T \omega$$
$$\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$$
$$\dot{V} = \frac{1}{2}\dot{e}^T Pe + \frac{1}{2}e^T P\dot{e} + \frac{1}{\gamma}\tilde{\theta}^T \dot{\tilde{\theta}} = \frac{1}{2}(Ae + b\tilde{\theta}^T \omega)^T Pe + \frac{1}{2}e^T P(Ae + b\tilde{\theta}^T \omega) - \frac{1}{\gamma}\tilde{\theta}^T \dot{\hat{\theta}} =$$
$$\frac{1}{2}e^T A^T Pe + \frac{1}{2}e^T PAe + b^T \tilde{\theta}^T \omega Pe - \frac{1}{\gamma}\tilde{\theta}^T \dot{\hat{\theta}}$$

## 6.2. Dynamic error model with measurable state

Time derivative:

$$\dot{e} = Ae + b\tilde{\theta}^T \omega$$

$$\dot{\tilde{\theta}} = -\dot{\tilde{\theta}}$$

$$\dot{V} = \frac{1}{2}\dot{e}^T Pe + \frac{1}{2}e^T P\dot{e} + \frac{1}{\gamma}\tilde{\theta}^T \dot{\tilde{\theta}} = \frac{1}{2}(Ae + b\tilde{\theta}^T \omega)^T Pe + \frac{1}{2}e^T P(Ae + b\tilde{\theta}^T \omega) - \frac{1}{\gamma}\tilde{\theta}^T \dot{\tilde{\theta}} =$$

$$\frac{1}{2}e^T A^T Pe + \frac{1}{2}e^T PAe + b^T \tilde{\theta}^T \omega Pe - \frac{1}{\gamma}\tilde{\theta}^T \dot{\tilde{\theta}} = \frac{1}{2}e^T (A^T P + PA)e + \tilde{\theta}^T \omega b^T Pe - \frac{1}{\gamma}\tilde{\theta}^T \dot{\tilde{\theta}}$$

Since matrix  $A$  is Hurwitz, it is related to the matrix  $P$   
via Lyapunov equation  $A^T P + PA = -Q$  with  $Q = Q^T \succ 0$

$$\dot{V} = -\frac{1}{2}e^T Qe + \tilde{\theta}^T \omega b^T Pe - \frac{1}{\gamma}\tilde{\theta}^T \dot{\tilde{\theta}}$$

## 6.2. Dynamic error model with measurable state

Time derivative:

$$\dot{e} = Ae + b\tilde{\theta}^T \omega$$

$$\dot{\tilde{\theta}} = -\dot{\tilde{\theta}}$$

$$\dot{V} = \frac{1}{2}\dot{e}^T Pe + \frac{1}{2}e^T P\dot{e} + \frac{1}{\gamma}\tilde{\theta}^T \dot{\tilde{\theta}} = \frac{1}{2}(Ae + b\tilde{\theta}^T \omega)^T Pe + \frac{1}{2}e^T P(Ae + b\tilde{\theta}^T \omega) - \frac{1}{\gamma}\tilde{\theta}^T \dot{\tilde{\theta}} =$$

$$\frac{1}{2}e^T A^T Pe + \frac{1}{2}e^T PAe + b^T \tilde{\theta}^T \omega Pe - \frac{1}{\gamma}\tilde{\theta}^T \dot{\tilde{\theta}} = \frac{1}{2}e^T (A^T P + PA)e + \tilde{\theta}^T \omega b^T Pe - \frac{1}{\gamma}\tilde{\theta}^T \dot{\tilde{\theta}}$$

Since matrix  $A$  is Hurwitz, it is related to the matrix  $P$   
via Lyapunov equation  $A^T P + PA = -Q$  with  $Q = Q^T \succ 0$

$$\dot{V} = -\frac{1}{2}e^T Qe + \tilde{\theta}^T \omega b^T Pe - \frac{1}{\gamma}\tilde{\theta}^T \dot{\tilde{\theta}}$$

Adaptation algorithm?

## 6.2. Dynamic error model with measurable state

$$\dot{V} = -\frac{1}{2}e^T Q e + \tilde{\theta}^T \omega b^T P e - \frac{1}{\gamma} \tilde{\theta}^T \dot{\tilde{\theta}}$$

If  $\dot{\tilde{\theta}} = \gamma \omega b^T P e$  ,

$$\dot{V} = -\frac{1}{2}e^T Q e < 0 \quad (6.11)$$

## 6.2. Dynamic error model with measurable state

### Summary and Discussion

*Error Model*

$$\dot{e} = Ae + b\tilde{\theta}^T \omega$$

*Adaptation Algorithm*

$$\dot{\tilde{\theta}} = \gamma \omega b^T P e$$

*Lyapunov function*

$$V = \frac{1}{2} e^T P e + \frac{1}{2\gamma} \tilde{\theta}^T \tilde{\theta}$$

*Its time derivative*

$$\dot{V} = -\frac{1}{2} e^T Q e < 0$$

What it means?

## 6.2. Dynamic error model with measurable state

### Summary

#### Properties of the closed-loop system:

1. If  $\omega$  is bounded, all the signals in the system are bounded;
2. Error  $\|e(t)\|$  approaches zero asymptotically;
3. The function  $V(t)$  is nonincreasing;
4.  $\|\tilde{\theta}(t)\|^2$  approaches zero asymptotically if  $\omega$  contains at least  $m/2$  harmonics and consists of linearly independent elements;

This property can be reformulated in terms of **Persistent Excitation**

#### Condition:

$$\int_t^{t+T} \omega(\tau) \omega^T(\tau) d\tau \geq \alpha I$$

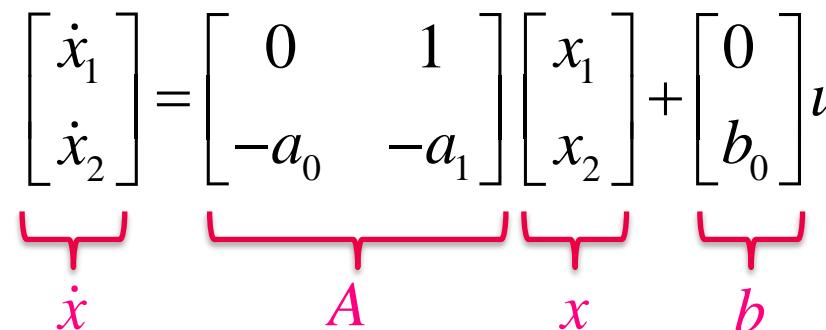
for some positive  $\alpha, T$ .

## 6.2. Dynamic error model with measurable state

**Example 6.4. The problem of state adaptive control**

### Problem statement

Let a plant be described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b_0 \end{bmatrix} u \quad (6.12)$$


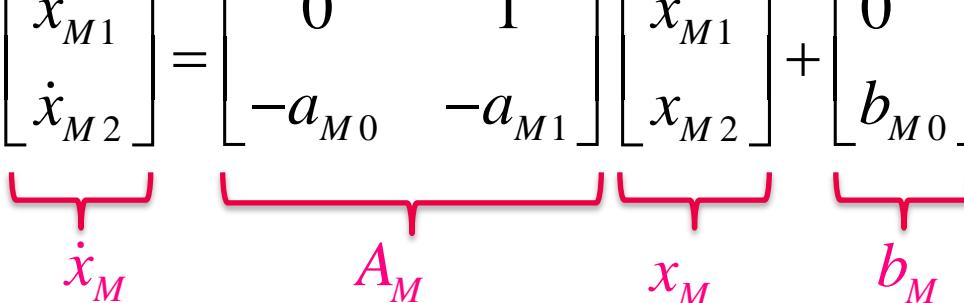
with **unknown** parameters  $a_0, a_1$ , known  $b_0$  and measurable state input  $u$  and output  $y$ .

The objective is to design a control  $u$  such that

$$\lim_{t \rightarrow \infty} \|x_M(t) - x(t)\| = 0 \quad (6.13)$$

## 6.2. Dynamic error model with measurable state

$x_M$  is the state of reference model

$$\begin{bmatrix} \dot{x}_{M1} \\ \dot{x}_{M2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_{M0} & -a_{M1} \end{bmatrix} \begin{bmatrix} x_{M1} \\ x_{M2} \end{bmatrix} + \begin{bmatrix} 0 \\ b_{M0} \end{bmatrix} g \quad (6.14)$$


with parameters  $a_{M0}, a_{M1}, b_{M0}$  responsible for transient performance of the closed-loop system and reference signal  $g$ .

## 6.2. Dynamic error model with measurable state

$x_M$  is the state of reference model

$$\begin{bmatrix} \dot{x}_{M1} \\ \dot{x}_{M2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_{M0} & -a_{M1} \end{bmatrix} \begin{bmatrix} x_{M1} \\ x_{M2} \end{bmatrix} + \begin{bmatrix} 0 \\ b_{M0} \end{bmatrix} g \quad (6.14)$$

$\underbrace{\dot{x}_M}_{\dot{x}_{M1}, \dot{x}_{M2}}$      $\underbrace{A_M}_{0, 1, -a_{M0}, -a_{M1}}$      $\underbrace{x_M}_{x_{M1}, x_{M2}}$      $\underbrace{b_M}_{0, b_{M0}}$

with parameters  $a_{M0}, a_{M1}, b_{M0}$  responsible for transient performance of the closed-loop system and reference signal  $g$ .

Main idea of solution is to reduce the problem to the error model.

Then to get the adaptation algorithm.



## 6.2. Dynamic error model with measurable state

### Solution

1. Let the parameters  $a_0, a_1$  be known.

Form the error signal  $e = x_M - x$  and take its derivative in view of the plant and reference model equations:

$$\dot{e} = \dot{x}_M - \dot{x} = A_M x_M + b_M g - Ax - bu$$

## 6.2. Dynamic error model with measurable state

### Solution

1. Let the parameters  $a_0, a_1$  be known.

Form the error signal  $e = x_M - x$  and take its derivative in view of the plant and reference model equations:

$$\dot{e} = \dot{x}_M - \dot{x} = A_M x_M + b_M g - Ax - bu$$

Let  $\dot{e} \square A_M e$  ( $e(t) = \exp(A_M t)e(0) \rightarrow 0$  exponentially fast).

Then

$$A_M x_M + b_M g - Ax - bu \square A_M e$$

$$A_M x_M + b_M g - Ax - bu \square A_M x_M - A_M x$$

## 6.2. Dynamic error model with measurable state

### Solution

1. Let the parameters  $a_0, a_1$  be known.

Form the error signal  $e = x_M - x$  and take its derivative view of the plant and reference model equations:

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Let  $\dot{e} \square A_M e$  ( $e(t) = \exp(A_M t)e(0) \rightarrow 0$  exponentially fast).

Then

$$A_M x_M + b_M g - Ax - bu \square A_M e$$

$$\cancel{A_M x_M + b_M g - Ax - bu} \square \cancel{A_M x_M} - A_M x$$

$$bu = (A_M - A)x + b_M g$$

## 6.2. Dynamic error model with measurable state

Solution

$$bu = (A_M - A)x + b_M g$$



$$\begin{bmatrix} 0 \\ b_0 \end{bmatrix} u = \left( \begin{bmatrix} 0 & 1 \\ -a_{M0} & -a_{M1} \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \right) x + \begin{bmatrix} 0 \\ b_{M0} \end{bmatrix} g$$

## 6.2. Dynamic error model with measurable state

Solution

$$bu = (A_M - A)x + b_M g$$



$$\begin{bmatrix} 0 \\ b_0 \end{bmatrix} u = \left( \begin{bmatrix} 0 & 1 \\ -a_{M0} & -a_{M1} \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \right) x + \begin{bmatrix} 0 \\ b_{M0} \end{bmatrix} g$$



$$u = \frac{1}{b_0} \left[ (a_0 - a_{M0})x_1 + (a_1 - a_{M1})x_2 + b_{M0}g \right]$$

## 6.2. Dynamic error model with measurable state

Solution

$$bu = (A_M - A)x + b_M g$$



$$\begin{bmatrix} 0 \\ b_0 \end{bmatrix} u = \left( \begin{bmatrix} 0 & 1 \\ -a_{M0} & -a_{M1} \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \right) x + \begin{bmatrix} 0 \\ b_{M0} \end{bmatrix} g$$



$$u = \frac{1}{b_0} \left[ \underbrace{(a_0 - a_{M0})x_1}_{\theta_1} + \underbrace{(a_1 - a_{M1})x_2}_{\theta_2} + b_{M0}g \right]$$



$$u = \frac{1}{b_0} [\theta^T x + b_{M0} g]$$

*Nonadaptive control*

(6.15)

## 6.2. Dynamic error model with measurable state

### Solution

2. Let the parameters  $a_0, a_1$  be unknown. Control

$$u = \frac{1}{b_0} [\theta^T x + b_{M0} g]$$

is not implementable. Substitute estimate  $\hat{\theta}$  for  $\theta$  and obtain the implementable adjustable control:

*Adjustable control*       $u = \frac{1}{b_0} [\hat{\theta}^T x + b_{M0} g] \quad (6.16)$

## 6.2. Dynamic error model with measurable state

### Solution

2. Let the parameters  $a_0, a_1$  be unknown. Control

$$u = \frac{1}{b_0} [\theta^T x + b_{M0} g]$$

is not implementable. Substitute estimate  $\hat{\theta}$  for  $\theta$  and obtain the implementable adjustable control:

*Adjustable control*       $u = \frac{1}{b_0} [\hat{\theta}^T x + b_{M0} g] \quad (6.16)$

Replace (6.16) in the plant equation  $\dot{x} = Ax + bu$  :

$$\dot{x} = A x + b \frac{1}{b_0} [\hat{\theta}^T x + b_{M0} g]$$

## 6.2. Dynamic error model with measurable state

### Solution

Evaluate time derivative of error:

$$\dot{e} = \dot{x}_M - \dot{x} = A_M x_M + b_M g - Ax - b \frac{1}{b_0} \left[ \hat{\theta}^T x + b_{M0} g \right]$$

## 6.2. Dynamic error model with measurable state

### Solution

Evaluate time derivative of error:

$$\begin{aligned}\dot{e} &= \dot{x}_M - \dot{x} = A_M x_M + b_M g - Ax - b \frac{1}{b_0} [\hat{\theta}^T x + b_{M0} g] \pm A_M x \\ &\quad \downarrow \\ \dot{e} &= \textcolor{blue}{A}_M e + b_M g + (\textcolor{blue}{A}_M - A)x - b \frac{1}{b_0} [\hat{\theta}^T x + b_{M0} g]\end{aligned}$$

## 6.2. Dynamic error model with measurable state

### Solution

Evaluate time derivative of error:

$$\dot{e} = \dot{x}_M - \dot{x} = A_M x_M + b_M g - Ax - b \frac{1}{b_0} [\hat{\theta}^T x + b_{M0} g] \pm A_M x$$

↓

$$\dot{e} = \cancel{A_M e} + \cancel{b_M g} + (\cancel{A_M} - A)x - b \frac{1}{b_0} [\hat{\theta}^T x + \cancel{b_{M0} g}]$$

↓

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_{M0} & -a_{M1} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} +$$

$$\left( \begin{bmatrix} 0 & 1 \\ -a_{M0} & -a_{M1} \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ b_0 \end{bmatrix} \frac{1}{b_0} \hat{\theta}^T x$$

## 6.2. Dynamic error model with measurable state

### Solution

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_{M0} & -a_{M1} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ (a_0 - a_{M0}) + (a_1 - a_{M1}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{\theta}^T x$$

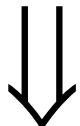
$\theta_1$                      $\theta_2$                      $k$

## 6.2. Dynamic error model with measurable state

### Solution

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_{M0} & -a_{M1} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ (a_0 - a_{M0}) + (a_1 - a_{M1}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{\theta}^T x$$

$\theta_1$                      $\theta_2$                      $k$



$$\dot{e} = A_M e + k\theta^T x - k\hat{\theta}^T x$$

## 6.2. Dynamic error model with measurable state

### Solution

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_{M0} & -a_{M1} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ (a_0 - a_{M0}) + (a_1 - a_{M1}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{\theta}^T x$$

$\theta_1$        $\theta_2$        $k$

↓

$$\dot{e} = A_M e + k\theta^T x - k\hat{\theta}^T x$$

↓

$$\dot{e} = A_M e + k\tilde{\theta}^T x$$



(6.17)

with parametric error  $\tilde{\theta} = \theta - \hat{\theta}$ .

## 6.2. Dynamic error model with measurable state

### Solution

*Error model*

$$\dot{e} = A_M e + k \tilde{\theta}^T x$$



*Adaptation Algorithm*

$$\dot{\hat{\theta}} = \gamma x k^T P e \quad (6.18)$$

where  $\gamma$  is a positive gain,  $P = P^T \succ 0$  is the solution of the Lyapunov equation

$$A_M^T P + P A_M = -Q \quad (6.19)$$

with preliminary selected  $Q = Q^T \succ 0$ .

## 6.2. Dynamic error model with measurable state

### Solution Summary

*Adjustable control*

$$u = \frac{1}{b_0} \left[ \hat{\theta}^T x + b_{M0} g \right] \quad (6.16)$$

*Adaptation Algorithm*

$$\dot{\hat{\theta}} = \gamma x k^T P e \quad (6.18)$$

*Error*

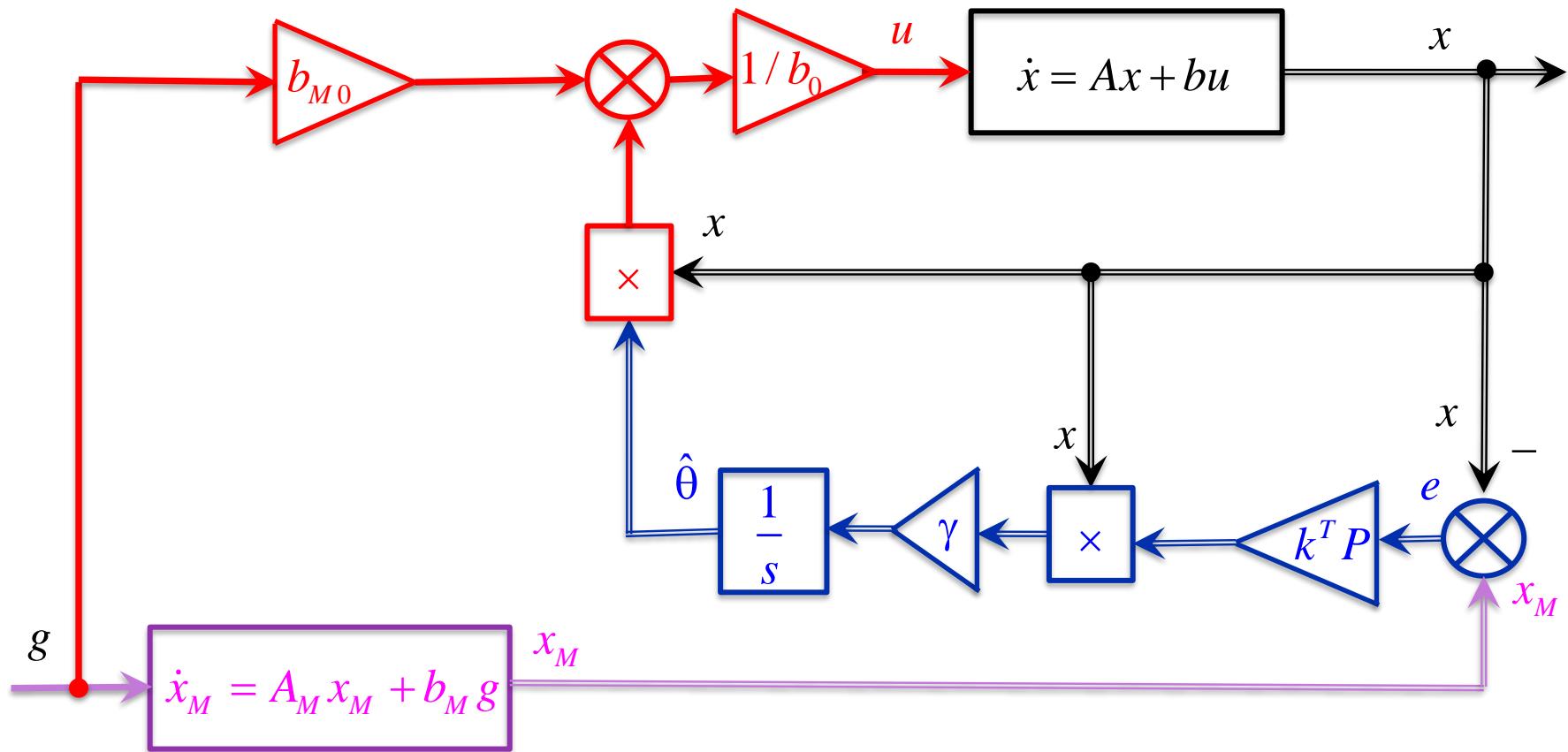
$$e = x_M - x$$

*Lyapunov equation*

$$A_M^T P + P A_M = -Q \quad (6.19)$$

## 6.2. Dynamic error model with measurable state

General Scheme



## 6.2. Dynamic error model with measurable state

### Simulation results

*Plant (unstable)*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

*Unknown parameters*

$$a_0 = 1, \quad a_1 = -2$$

*Reference model*

$$\begin{bmatrix} \dot{x}_{M1} \\ \dot{x}_{M2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_{M1} \\ x_{M2} \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \end{bmatrix} g$$

*Adaptation gain*

$$\gamma = 100$$

*Matrix  $P$*

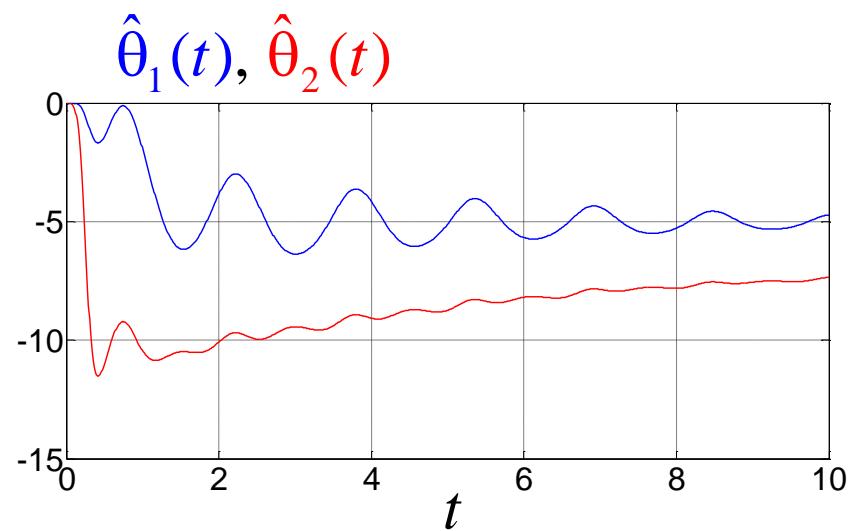
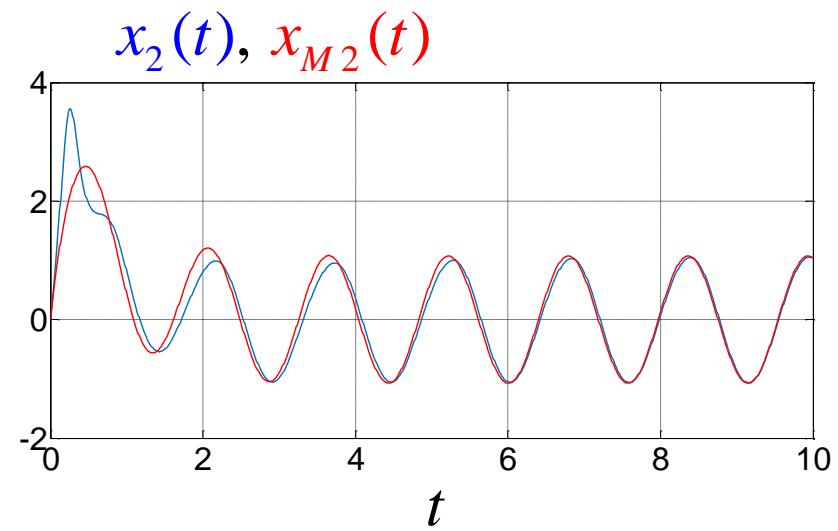
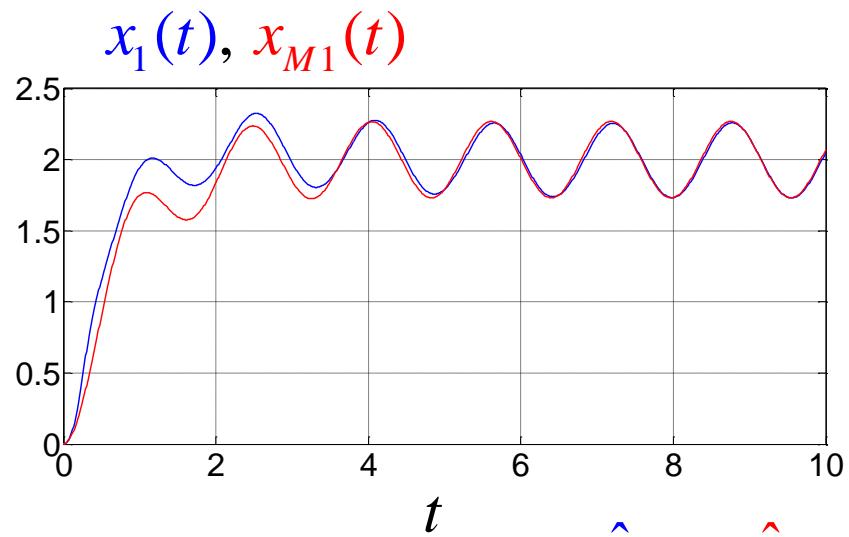
$$P = \begin{bmatrix} 1.1167 & 0.0833 \\ 0.0833 & 0.1167 \end{bmatrix}$$

*Reference*

$$g(t) = \sin 4t + 2$$

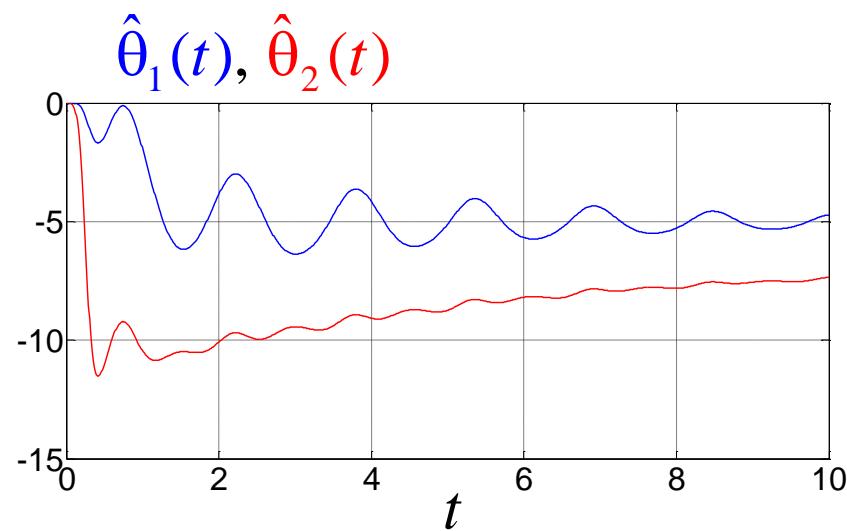
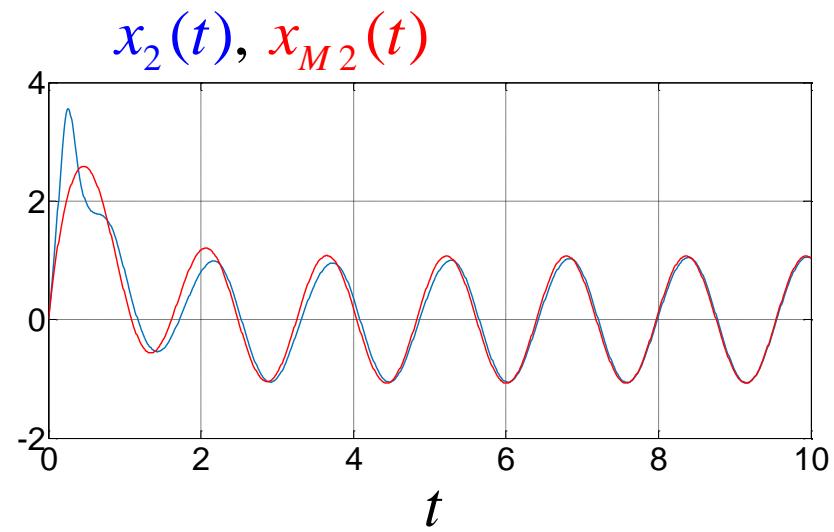
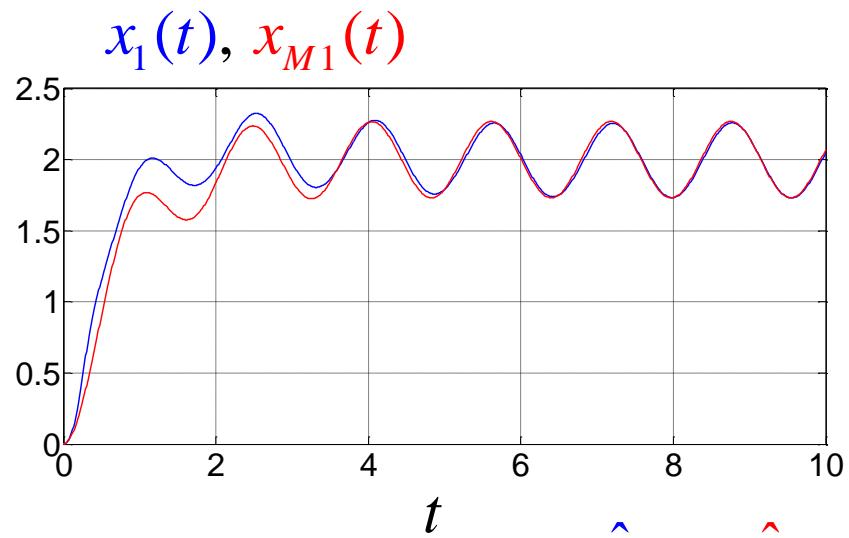
## 6.2. Dynamic error model with measurable state

### Simulation results



## 6.2. Dynamic error model with measurable state

### Simulation results



*Does AA provide  
identification  
properties?*

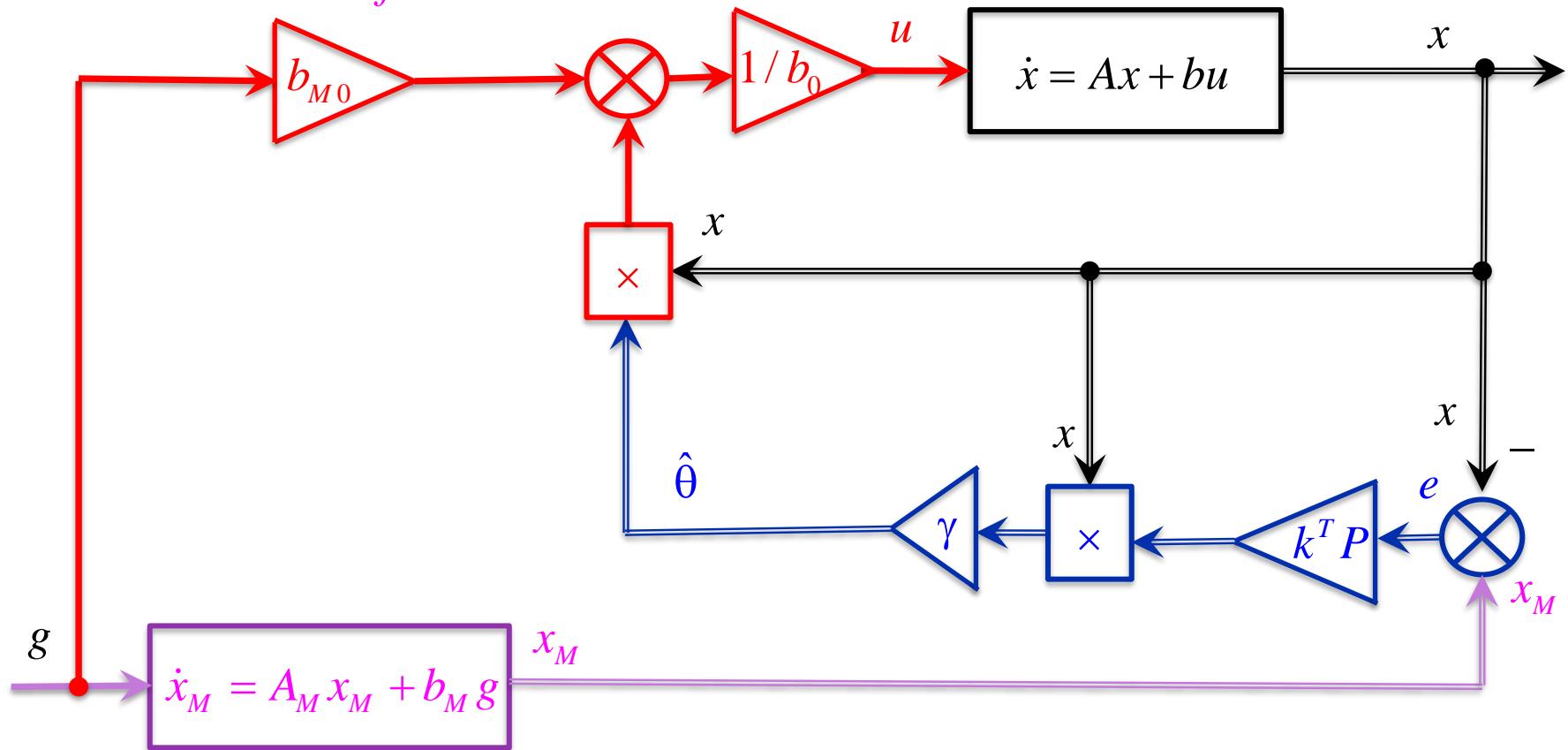
## 6.2. Dynamic error model with measurable state

### Robust modifications of adaptation algorithm

*Modification*

*with nonlinear feedback*

$$\hat{\theta} = \gamma \omega b^T P e$$



## 6.2. Dynamic error model with measurable state

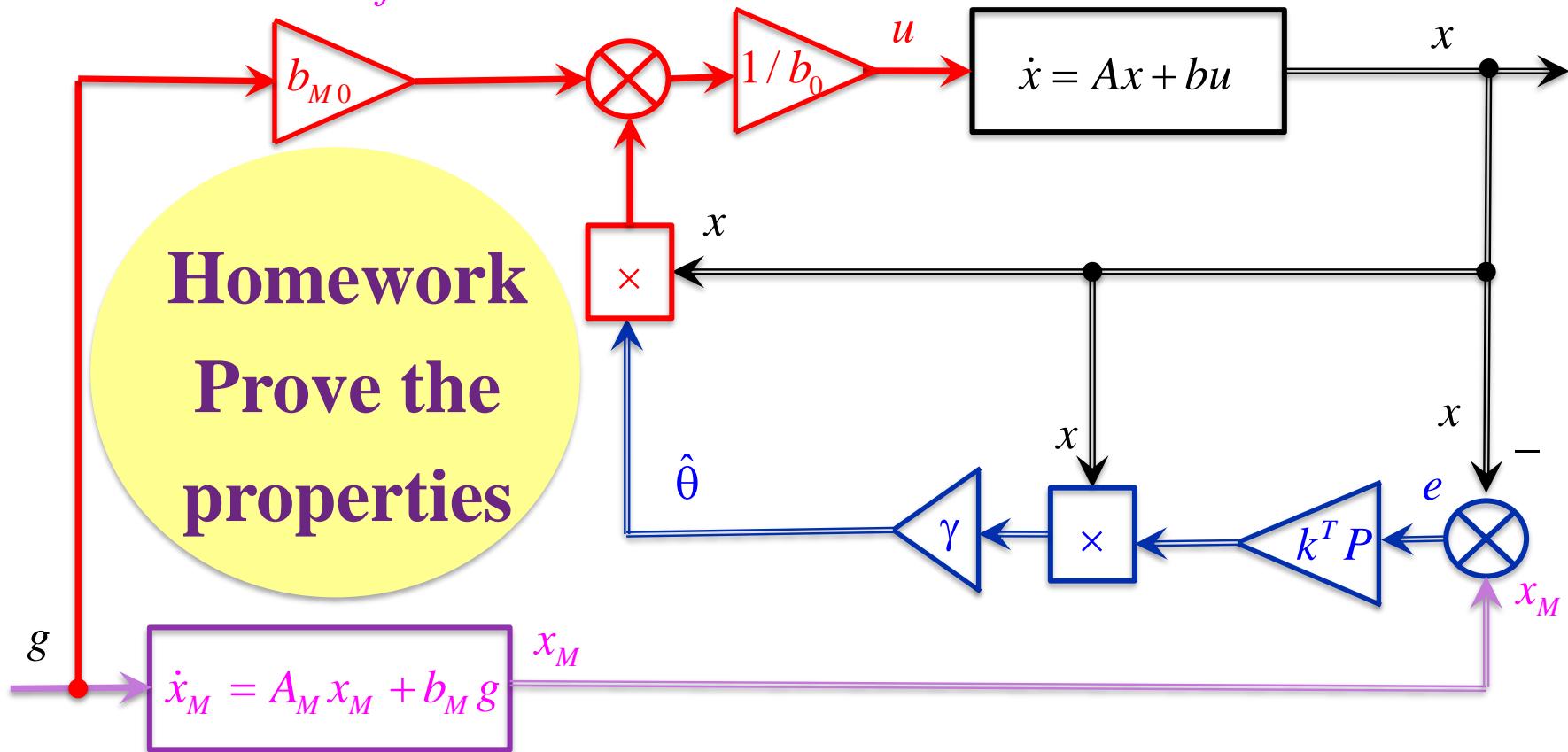
### Robust modifications of adaptation algorithm

*Modification*

*with nonlinear feedback*

$$\hat{\theta} = \gamma \omega b^T P e$$

**Homework**  
Prove the  
properties

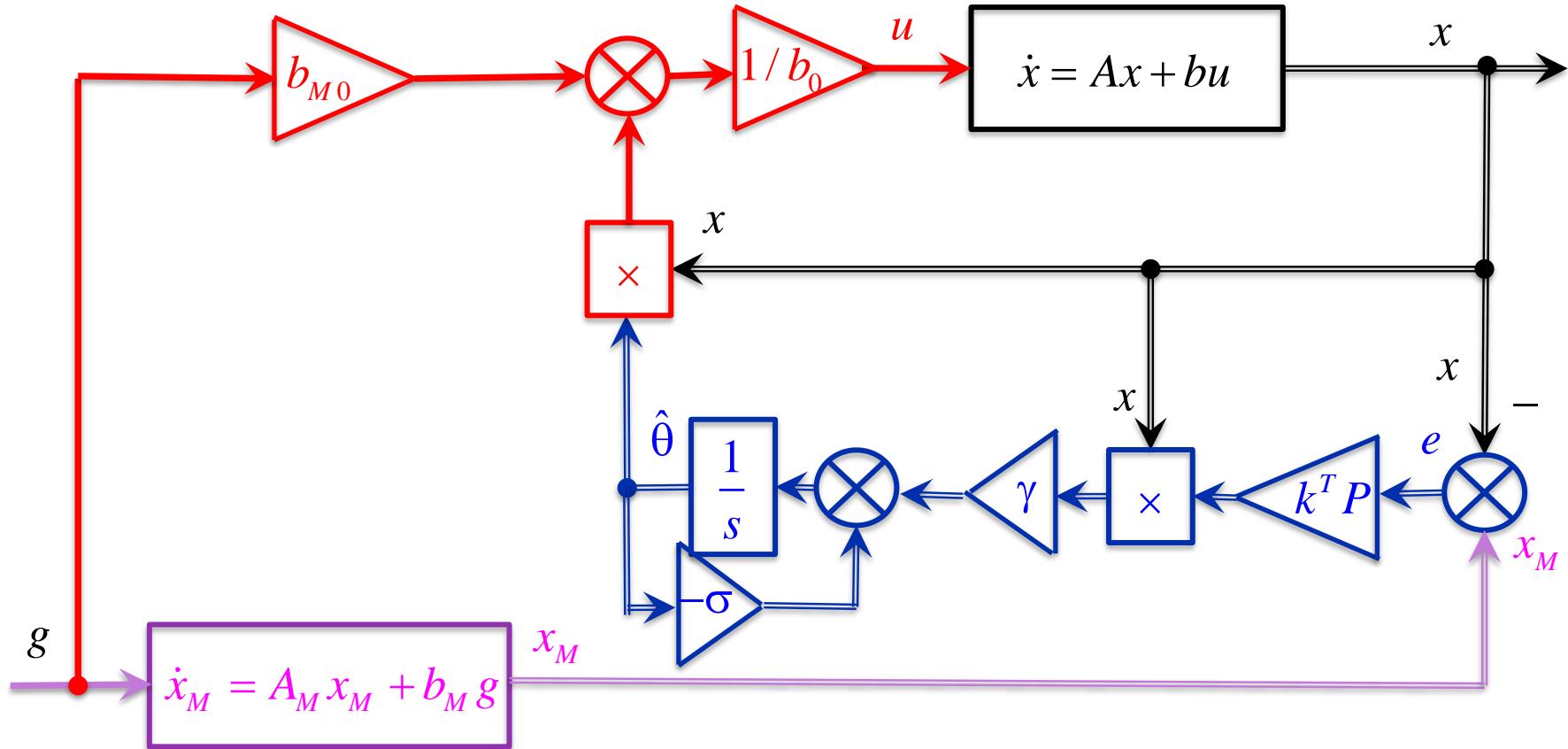


## 6.2. Dynamic error model with measurable state

### Robust modifications of adaptation algorithm

*$\sigma$  - Modification*

$$\dot{\hat{\theta}} = -\sigma \hat{\theta} + \gamma \omega b^T P e$$

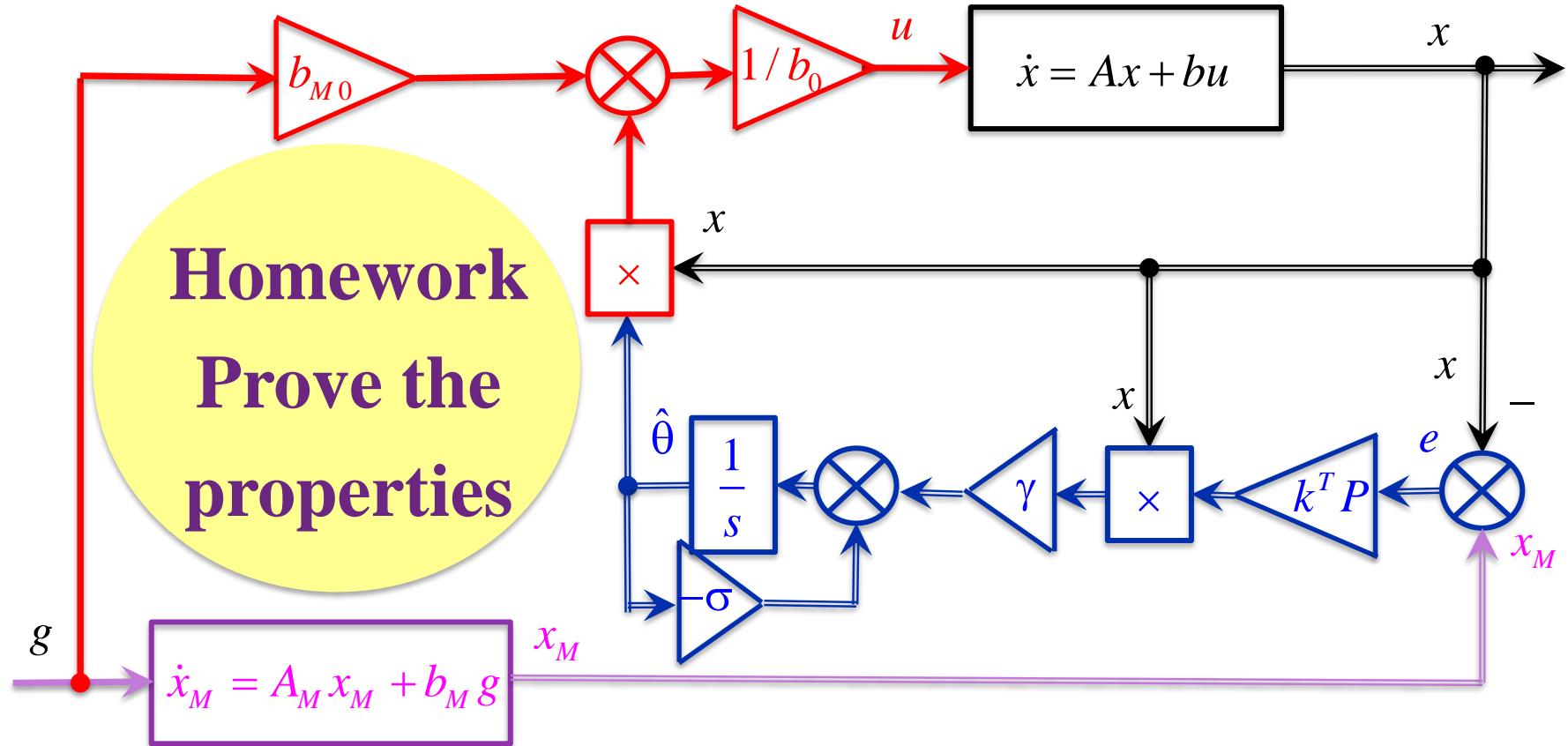


## 6.2. Dynamic error model with measurable state

### Robust modifications of adaptation algorithm

*σ - Modification*

$$\dot{\hat{\theta}} = -\sigma \hat{\theta} + \gamma \omega b^T P e$$



## 6.3. Dynamic error model with measurable output

$$\begin{aligned}\dot{e}(t) &= Ae(t) + b\tilde{\theta}^T(t)\omega(t), \\ \varepsilon(t) &= c^T e(t)\end{aligned}\tag{6.20a}$$

where  $e \in \mathbb{C}^n$  is the unmeasurable state  $\varepsilon$  is the output,  $\tilde{\theta} \in \mathbb{C}^m$  is the vector of parametric errors,  $\omega \in \mathbb{C}^m$  is the vector of measurable functions (regressor).

**Remark 6.3.** Since vector is not measurable, the model (6.20a) can be presented in the “Input-Output” form

$$\varepsilon(t) = W(s) \left[ \tilde{\theta}^T(t)\omega(t) \right] \tag{6.20b}$$

with transfer function  $W(s) = c^T (Is - A)^{-1} b$ .

## 6.3. Dynamic error model with measurable output

*Remark 6.4.* The model is widely used in the problems of output adaptive control (see example below).

The problem is to design an adaptation algorithm/algorithms based on (6.20)

## 6.3. Dynamic error model with measurable output

Solution #1

Can we just apply adaptation algorithm

$$\dot{\hat{\theta}} = \gamma \omega \varepsilon$$

used for static error model?

## 6.3. Dynamic error model with measurable output

Solution #1

Can we just apply adaptation algorithm

$$\dot{\hat{\theta}} = \gamma \omega \varepsilon$$

used for static error model?

IF YES, WHEN???

## 6.3. Dynamic error model with measurable output

### Solution #1

$$\dot{e} = Ae + b\tilde{\theta}^T \omega,$$

$$\varepsilon = c^T e$$

$$\downarrow$$
$$\dot{\hat{\theta}} = \gamma \omega b^T P e$$

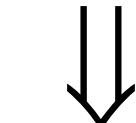
" $e$ " unmeasurable

## 6.3. Dynamic error model with measurable output

### Solution #1

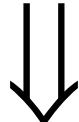
$$\dot{e} = Ae + b\tilde{\theta}^T \omega,$$

$$\varepsilon = c^T e$$



$$\dot{\hat{\theta}} = \gamma \omega b^T P e$$

" $e$ " unmeasurable



If  $b^T P = c^T$ , adaptation algorithm becomes implementable since

$$\dot{\hat{\theta}} = \gamma \omega b^T P e = \gamma \omega \varepsilon \quad (6.21)$$

## 6.3. Dynamic error model with measurable output

### Solution #1

**Lemma (Yakubovich-Kalman-Popov):**

Matrix  $P = P^T \succ 0$  satisfies both Lyapunov equation

$$A^T P + PA = -Q$$

and equation

$$b^T P = c^T$$

simultaneously iff transfer function

$$W(s) = c^T (Is - A)^{-1} b.$$

is Strictly Positive Real (SPR).

## 6.3. Dynamic error model with measurable output

### Solution #1

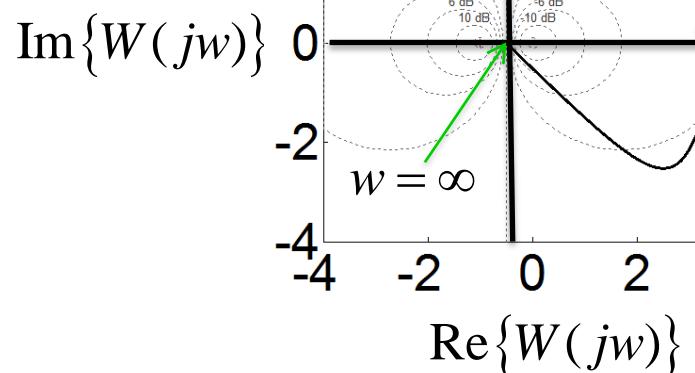
**Definition 6.1. Transfer function**  $W(s) = c^T (Is - A)^{-1} b$  **is SPR if**

1. it is stable, i.e. polynomial of its denominator is Hurwitz (has all the roots in the left half plane of root locus);
2. Nyquist plot is placed in the right half plane of the diagram.

$$\operatorname{Re}\{W(jw)\} > 0, \quad \forall w \in [0, \infty).$$

3. the limit equality hold

$$\lim_{w \rightarrow \infty} w^2 \operatorname{Re}\{W(jw)\} > 0$$



## 6.3. Dynamic error model with measurable output

**Example 6.5.** SPR transfer function of first order block

$$W(s) = \frac{K}{Ts + 1}$$

with some positive constant parameters  $K$  and  $T$ .

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### Verification

1. Frequency transfer function

$$W(jw) = \frac{K}{Tjw + 1} = \frac{K(-Tjw + 1)}{(Tjw + 1)(-Tjw + 1)} = \frac{K}{\underbrace{T^2 w^2 + 1}_{\text{Re}\{W(jw)\}}} - j \frac{KTw}{\underbrace{T^2 w^2 + 1}_{-\text{Im}\{W(jw)\}}}$$

## 6.3. Dynamic error model with measurable output

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2. The first condition:  $Ts + 1 = 0 \Rightarrow s_1 = -1/T \Rightarrow W(s)$  is Hurwitz



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2. The first condition:  $Ts + 1 = 0 \Rightarrow s_1 = -1/T \Rightarrow W(s)$  is Hurwitz
3. The second condition:

$$\operatorname{Re}\{W(jw)\} = \frac{K}{T^2 w^2 + 1} > 0, \quad \forall w \in [0, \infty).$$



## 6.3. Dynamic error model with measurable output

4. The third condition:

$$\lim_{w \rightarrow \infty} w^2 \operatorname{Re}\{W(jw)\} = \lim_{w \rightarrow \infty} \frac{Kw^2}{T^2 w^2 + 1} = \frac{K}{T^2} > 0.$$



## 6.3. Dynamic error model with measurable output

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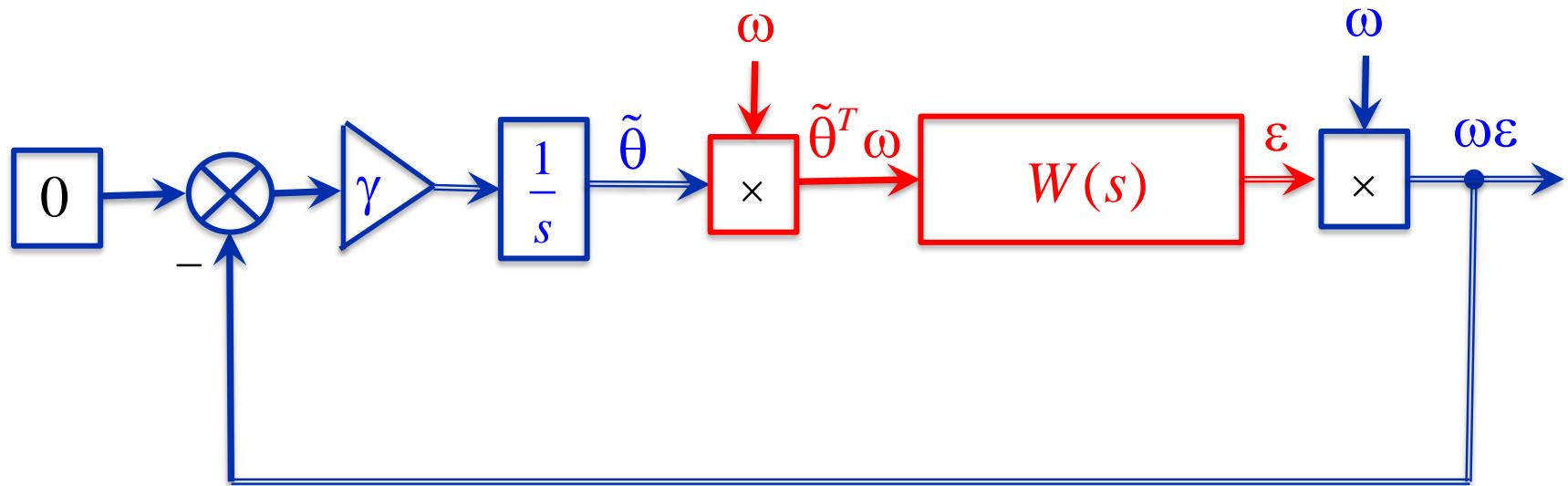
$$\lim_{w \rightarrow \infty} w^2 \operatorname{Re}\{W(jw)\} = \lim_{w \rightarrow \infty} \frac{Kw^2}{T^2 w^2 + 1} = \frac{K}{T^2} > 0.$$



**SPR transfer function is a function with  
property of the first order block,  
i.e. relative degree less than 2 (0 or 1)**

## 6.3. Dynamic error model with measurable output

*Remark 6.5. One-syllable words about adaptation algorithm and SPR transfer functions*



Error model

$$\varepsilon(t) = W(s) [\tilde{\theta}^T(t) \omega(t)]$$

Adaptation algorithm

$$\dot{\tilde{\theta}}(t) = -\dot{\hat{\theta}}(t) = -\gamma \omega(t) \varepsilon(t)$$

## 6.3. Dynamic error model with measurable output

### Solution #1

### Summary and Discussion

*Error Model*

$$\varepsilon = W(s) \begin{bmatrix} \tilde{\theta}^T \omega \end{bmatrix},$$

*Adaptation Algorithm*

$$\dot{\hat{\theta}} = \gamma \omega \varepsilon,$$

where  $W(s)$  is an SPR transfer function.

## 6.3. Dynamic error model with measurable output

### Solution #1

### Summary and Discussion

*Error Model*

$$\varepsilon = W(s) \begin{bmatrix} \tilde{\theta}^T \omega \end{bmatrix},$$

*Adaptation Algorithm*

$$\dot{\hat{\theta}} = \gamma \omega \varepsilon,$$

where  $W(s)$  is an SPR transfer function.

**SPR condition is quite restrictive and can narrow practical meaning of the problem**

## 6.3. Dynamic error model with measurable output

### Solution #2 Augmented error algorithm (Monopoli, 1974)

Consider error model

$$\varepsilon = W(s) \begin{bmatrix} \tilde{\theta}^T \omega \end{bmatrix}$$


and introduce augmentation signal

$$\hat{\varepsilon} = \varepsilon - \hat{\theta}^T W(s) [\omega] + W(s) \begin{bmatrix} \hat{\theta}^T \omega \end{bmatrix}. \quad (6.22)$$

## 6.3. Dynamic error model with measurable output

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Substitution of error model into (6.22) gives static error model !!!

$$\hat{\varepsilon} = \tilde{\theta}^T W(s) [\omega]. \quad (6.23)$$

## 6.3. Dynamic error model with measurable output

### Solution #2 Augmented error algorithm (Monopoli, 1974)

Consider error model

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Substitution of error model into (6.22) gives static error model !!!

$$\hat{\varepsilon} = \tilde{\theta}^T W(s) [\omega]. \quad (6.23)$$

Adaptation algorithm (see section 6.1. Static error model)

$$\dot{\hat{\theta}} = \gamma W(s) [\omega] \hat{\varepsilon}. \quad (6.24)$$

## 6.3. Dynamic error model with measurable output

Solution #2 Augmented error algorithm (Monopoli, 1974)

### Summary and Discussion

Error Model

$$\varepsilon = W(s) [\tilde{\theta}^T \omega],$$

Augmented error

$$\hat{\varepsilon} = \varepsilon - \hat{\theta}^T W(s) [\omega] + W(s) [\hat{\theta}^T \omega],$$

Adaptation Algorithm

$$\dot{\hat{\theta}} = \gamma W(s) [\omega] \hat{\varepsilon}.$$

The solution relaxes restriction on the class  
of transfer functions

## 6.3. Dynamic error model with measurable output

**Solution #2 Augmented error algorithm (Monopoli, 1974)**

### Summary and Discussion

*Error Model*

$$\varepsilon = W(s) [\tilde{\theta}^T \omega],$$

*Augmented error*

$$\hat{\varepsilon} = \varepsilon - \hat{\theta}^T W(s) [\omega] + W(s) [\hat{\theta}^T \omega],$$

*Adaptation Algorithm*

$$\dot{\hat{\theta}} = \gamma W(s) [\omega] \hat{\varepsilon}.$$

Proved using *the Swapping lemma*:

$$W(s) [\hat{\theta}^T \omega] = \hat{\theta}^T W(s) [\omega] - W_C(s) [W_b(s) [\omega^T] \dot{\hat{\theta}}]$$

where  $W_C(s) = c^T (Is - A)^{-1}$ ,  $W_b(s) = (Is - A)^{-1} b$  are the transfer matrices.

## 6.3. Dynamic error model with measurable output

### Solution #2 Augmented error algorithm (Monopoli, 1974)

### Summary and Discussion

*Error Model*

$$\varepsilon = W(s) [\tilde{\theta}^T \omega],$$

*Augmented error*

$$\hat{\varepsilon} = \varepsilon - \hat{\theta}^T W(s) [\omega] + W(s) [\hat{\theta}^T \omega],$$

*Adaptation Algorithm*

$$\dot{\hat{\theta}} = \gamma W(s) [\omega] \hat{\varepsilon}.$$

Augmented error simplified using the *Swapping lemma*:

$$\hat{\varepsilon} = \varepsilon - W_C(s) [W_b(s) [\omega^T] \dot{\hat{\theta}}],$$

where  $W_C(s) = c^T (Is - A)^{-1}$ ,  $W_b(s) = (Is - A)^{-1} b$  are the transfer matrices.

## 6.3. Dynamic error model with measurable output

### Solution #2 Augmented error algorithm (Monopoli, 1974)

#### Summary and Discussion

*Error Model*

$$\varepsilon = W(s) [\tilde{\theta}^T \omega],$$

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$$\hat{\varepsilon} = \varepsilon - \hat{\theta}^T W(s) [\omega] + W(s) [\hat{\theta}^T \omega],$$

*Adaptation Algorithm*

$$\dot{\hat{\theta}} = \gamma W(s) [\omega] \hat{\varepsilon}.$$

Augmented error simplified using the *Swapping lemma*:

$$\hat{\varepsilon} = \varepsilon - \gamma W_C(s) [W_b(s) [\omega^T] W(s) [\omega]] \hat{\varepsilon},$$

where  $W_C(s) = c^T (Is - A)^{-1}$ ,  $W_b(s) = (Is - A)^{-1} b$  are the transfer matrices.

## 6.3. Dynamic error model with measurable output

**Solution #2 Augmented error algorithm (Monopoli, 1974)**

### Summary

**Properties of the closed-loop system:**

1. If  $\omega$  is bounded, all the signals in the system are bounded;
2. Error  $\hat{\varepsilon}(t)$  approaches zero asymptotically;
3. The norm  $\|\tilde{\theta}(t)\|$  is nonincreasing;
4. The norm  $\|\tilde{\theta}(t)\|$  approaches zero asymptotically, if  $\omega$  satisfies the Persistent Excitation condition;
- 5.

$$\hat{\varepsilon} = \varepsilon - \gamma W_C(s) \left[ W_b(s) [\omega^T] W(s) [\omega] \hat{\varepsilon} \right],$$

## 6.3. Dynamic error model with measurable output

Solution #2 Augmented error algorithm (Monopoli, 1974)

### Summary

Properties of the closed-loop system:

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4. The norm  $\|\tilde{\theta}(t)\|$  approaches zero asymptotically, if  $\omega$  satisfies the Persistent Excitation condition;
- 5.

$$\hat{\varepsilon} = \varepsilon - \gamma W_C(s) [W_b(s) [\omega^T] W(s) [\omega] \hat{\varepsilon}],$$

Does  $\varepsilon$   
go to zero  
???

## 6.3. Dynamic error model with measurable output

**Solution #2 Augmented error algorithm (Monopoli, 1974)**

### Summary

**Properties of the closed-loop system:**

1. If  $\omega$  is bounded, all the signals in the system are bounded;
2. Error  $\hat{\varepsilon}(t)$  approaches zero asymptotically;
3. The norm  $\|\tilde{\theta}(t)\|$  is nonincreasing;
4. The norm  $\|\tilde{\theta}(t)\|$  approaches zero asymptotically, if  $\omega$  satisfies the Persistent Excitation condition;
5. If  $\omega$  is bounded, error  $\varepsilon(t)$  approaches zero asymptotically.

## 6.3. Dynamic error model with measurable output

**Example 6.6. The problem of output adaptive control**

### Problem statement

Let a plant be described by

$$\ddot{y} + a_1 \dot{y} + a_0 y = b_0 u \quad (6.25)$$

with **unknown** parameters  $a_0, a_1$ , known  $b_0$  and unmeasurable state  $\dot{y}$ , known input  $u$  and output  $y$ .

The objective is to design a control  $u$  such that

$$\lim_{t \rightarrow \infty} \|y_M(t) - y(t)\| = 0, \quad (6.26)$$

where  $y_M$  is the output of reference model

$$\ddot{y}_M + a_{M1} \dot{y}_M + a_{M0} y_M = b_{M0} g \quad (6.27)$$

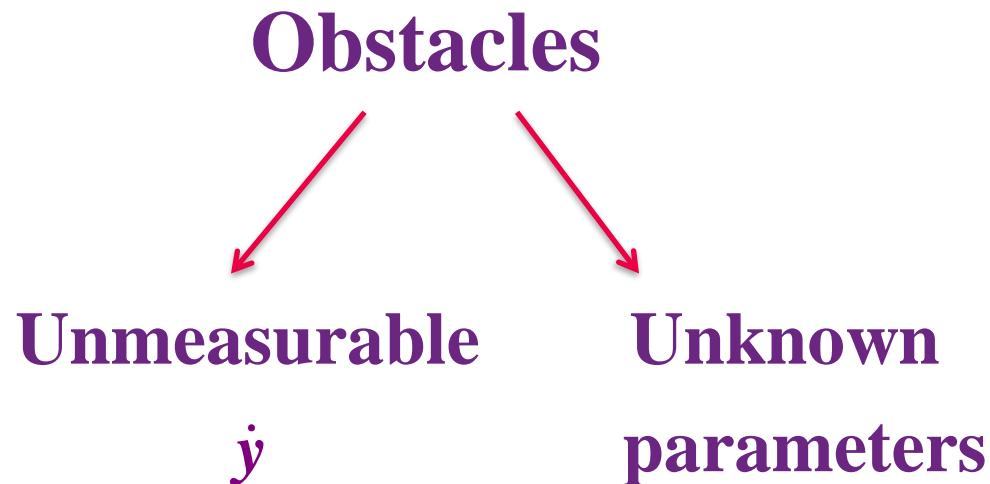
with the reference signal  $g$ .

## 6.3. Dynamic error model with measurable output

Solution

## 6.3. Dynamic error model with measurable output

### Solution



## 6.3. Dynamic error model with measurable output

### Solution

1. Obstacle of unmeasurable state.

Apply first order ( $n-1$ )th filter

$$\frac{1}{s+k}, \quad k > 0$$

to the plant equation:

$$\frac{1}{s+k} [\ddot{y} + a_1 \dot{y} + a_0 y] = b_0 \frac{1}{s+k} [u]$$

## 6.3. Dynamic error model with measurable output

### Solution

1. Obstacle of unmeasurable state.

Apply first order ( $n-1$ )th filter

$$\frac{1}{s+k}, \quad k > 0$$

to the plant equation:

$$\frac{1}{s+k} [\ddot{y} + a_1 \dot{y} + a_0 y] = b_0 \frac{1}{s+k} [u]$$

↓

$$\frac{s+k}{s+k} [\dot{y}] + a_1 \frac{s+k}{s+k} [y] + a_0 \frac{1}{s+k} [y] = b_0 \frac{1}{s+k} [u]$$

~~$\frac{s+k}{s+k}$~~

## 6.3. Dynamic error model with measurable output

### Solution

1. Obstacle of unmeasurable state.

Apply first order  $(n-1)$ th filter

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to the plant equation:

$$\frac{1}{s+k} [\ddot{y} + a_1 \dot{y} + a_0 y] = b_0 \frac{1}{s+k} [u]$$



$$\frac{s+k}{s+k} [\dot{y}] + a_1 \frac{s+k}{s+k} [y] + a_0 \frac{1}{s+k} [y] = b_0 \frac{1}{s+k} [u]$$

$$\dot{y} = (k - a_1) y + (a_1 k - k^2 - a_0) \frac{1}{s+k} [y] + b_0 \frac{1}{s+k} [u]$$

## 6.3. Dynamic error model with measurable output

### Solution

$$\dot{y} = (k - a_1)y + (a_1 k - k^2 - a_0) \frac{1}{s+k} [y] + b_0 \frac{1}{s+k} [u]$$
$$\downarrow$$
$$\dot{y} = \theta^{*T} \omega$$

The derivative  $\dot{y}$  is still not accessible, however presentable in the useful form of linear regression

## 6.3. Dynamic error model with measurable output

### Solution

2. Obstacle of unknown parameters.

Main idea of solution is to reduce the problem to the  
error model.

Then to get the adaptation algorithm.



## 6.3. Dynamic error model with measurable output

### Solution

2. Obstacle of unknown parameters.

Evaluate the **second** time derivative of error  $\varepsilon = y_M - y$  in view of plant  $\ddot{y} + a_1\dot{y} + a_0y = b_0u$

and reference model  $\ddot{y}_M + a_{M1}\dot{y}_M + a_{M0}y_M = b_{M0}g$  :

$$\ddot{\varepsilon} = \ddot{y}_M - \ddot{y} =$$

## 6.3. Dynamic error model with measurable output

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$$\ddot{\varepsilon} = \ddot{y}_M - \ddot{y} = -a_{M1}\dot{y}_M - a_{M0}y_M + b_{M0}g + a_1\dot{y} + a_0y - b_0u =$$

## 6.3. Dynamic error model with measurable output

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$$\ddot{\varepsilon} = -a_{M1}(\dot{y}_M \pm \dot{y}) - a_{M0}(y_M \pm y) + b_{M0}g + a_1\dot{y} + a_0y - b_0u =$$

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$$\ddot{\varepsilon} = -a_{M1}(\dot{y}_M \pm \dot{y}) - a_{M0}(y_M \pm y) + b_{M0}g + a_1\dot{y} + a_0y - b_0u =$$

$$-a_{M1}\dot{\varepsilon} - a_{M1}\dot{y} - a_{M0}\varepsilon - a_{M0}y + b_{M0}g + a_1\dot{y} + a_0y - b_0u$$

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$$\ddot{\varepsilon} = \ddot{y}_M - \ddot{y} = -a_{M1} \dot{y}_M - a_{M0} y_M + b_{M0} g + a_1 \dot{y} + a_0 y - b_0 u$$

$$\begin{aligned}\ddot{\varepsilon} &= -a_{M1} (\dot{y}_M \pm \dot{y}) - a_{M0} (y_M \pm y) + b_{M0} g + a_1 \dot{y} + a_0 y - b_0 u = \\ &= -a_{M1} \dot{\varepsilon} - a_{M1} \dot{y} - a_{M0} \varepsilon - a_{M0} y + b_{M0} g + a_1 \dot{y} + a_0 y - b_0 u\end{aligned}$$

## 6.3. Dynamic error model with measurable output

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$$\ddot{\varepsilon} = \ddot{y}_M - \ddot{y} = -a_{M1} \dot{y}_M - a_{M0} y_M + b_{M0} g + a_1 \dot{y} + a_0 y - b_0 u$$

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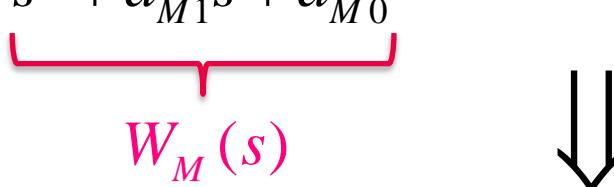


$$\varepsilon = \frac{1}{s^2 + a_{M1}s + a_{M0}} [-a_{M1} \dot{y} - a_{M0} y + b_{M0} g + a_1 \dot{y} + a_0 y - b_0 u]$$

## 6.3. Dynamic error model with measurable output

Solution

$$\varepsilon = \frac{1}{s^2 + a_{M1}s + a_{M0}} [-a_{M1}\dot{y} - a_{M0}y + b_{M0}g + a_1\dot{y} + a_0y - b_0u]$$



$$\varepsilon = W_M(s) [(a_1 - a_{M1})\dot{y} + (a_0 - a_{M0})y + b_{M0}g - b_0u]$$

## 6.3. Dynamic error model with measurable output

Solution

$$\varepsilon = \frac{1}{s^2 + a_{M1}s + a_{M0}} [-a_{M1}\dot{y} - a_{M0}y + b_{M0}g + a_1\dot{y} + a_0y - b_0u]$$

$\underbrace{s^2 + a_{M1}s + a_{M0}}_{W_M(s)}$

↓

$$\varepsilon = W_M(s) [(a_1 - a_{M1})\dot{y} + (a_0 - a_{M0})y + b_{M0}g - b_0u]$$


$$\varepsilon = W_M(s) [(a_1 - a_{M1})\dot{y} + (a_0 - a_{M0})y + b_{M0}g - b_0u]$$



$$\dot{y} = \theta^{*T} \omega$$

$$\varepsilon = W_M(s) [\theta^T \omega + b_{M0}g - b_0u]$$

## 6.3. Dynamic error model with measurable output

### Solution

$$\varepsilon = W_M(s) \left[ \theta^T \omega + b_{M0}g - b_0u \right]$$

where  $\omega = \text{col} \left( y, \frac{1}{s+k} [y], \frac{1}{s+k} [u] \right)$

$$\theta = \text{col} \left( (a_1 - a_{M1})\theta_1^* + (a_0 - a_{M0}), (a_1 - a_{M1})\theta_2^*, (a_1 - a_{M1})\theta_3^* \right)$$

## 6.3. Dynamic error model with measurable output

### Solution

$$\varepsilon = W_M(s) \left[ \theta^T \omega + b_{M0} g - b_0 u \right]$$

where  $\omega = \text{col} \left( y, \frac{1}{s+k} [y], \frac{1}{s+k} [u] \right)$

$$\begin{aligned} \theta = & \text{col} \left( (a_1 - a_{M1})(k - a_1) + a_0 - a_{M0}, \right. \\ & \left. (a_1 - a_{M1})(a_1 k - k^2 - a_0), a_1 b_0 - a_{M1} b_0 \right) \end{aligned}$$

## 6.3. Dynamic error model with measurable output

### Solution

$$\varepsilon = W_M(s) \left[ \theta^T \omega + b_{M0} g - b_0 u \right]$$

where  $\omega = \text{col} \left( y, \frac{1}{s+k} [y], \frac{1}{s+k} [u] \right)$

$$\theta = \text{col} \left( (a_1 - a_{M1})(k - a_1) + a_0 - a_{M0}, \right. \\ \left. (a_1 - a_{M1})(a_1 k - k^2 - a_0), a_1 b_0 - a_{M1} b_0 \right)$$

*Adjustable control*  $u = \frac{1}{b_0} \left[ \hat{\theta}^T \omega + b_{M0} g \right]$

(6.28)

## 6.3. Dynamic error model with measurable output

### Solution

$$\varepsilon = W_M(s) \left[ \theta^T \omega + b_{M0} g - b_0 u \right]$$

where  $\omega = \text{col} \left( y, \frac{1}{s+k} [y], \frac{1}{s+k} [u] \right)$

$$\theta = \text{col} \left( (a_1 - a_{M1})(k - a_1) + a_0 - a_{M0}, \right. \\ \left. (a_1 - a_{M1})(a_1 k - k^2 - a_0), a_1 b_0 - a_{M1} b_0 \right)$$

*Adjustable control*       $u = \frac{1}{b_0} \left[ \hat{\theta}^T \omega + b_{M0} g \right]$       (6.28)

*Error model*       $\varepsilon = W_M(s) \left[ \tilde{\theta}^T \omega \right]$       (6.29)

with parametric errors  $\tilde{\theta} = \theta - \hat{\theta}$ .



## 6.3. Dynamic error model with measurable output

### Solution

$$\varepsilon = W_M(s) \begin{bmatrix} \tilde{\theta}^T \omega \end{bmatrix},$$



where

*Augmented error*

$$\hat{\varepsilon} = \varepsilon - \hat{\theta}^T W_M(s) [\omega] + W_M(s) \begin{bmatrix} \hat{\theta}^T \omega \end{bmatrix} \quad (6.30)$$

*Adaptation Algorithm*

$$\dot{\hat{\theta}} = \gamma W_M(s) [\omega] \hat{\varepsilon} \quad (6.31)$$

## 6.3. Dynamic error model with measurable output

### Solution Summary

*Adjustable control*

$$u = \frac{1}{b_0} [\hat{\theta}^T \omega + b_{M0} g] \quad (6.28)$$

*Augmented error*

$$\hat{\varepsilon} = \varepsilon - \hat{\theta}^T W_M(s) [\omega] + W_M(s) [\hat{\theta}^T \omega] \quad (6.30)$$

*Adaptation Algorithm*

$$\dot{\hat{\theta}} = \gamma(t) W_M(s) [\omega] \hat{\varepsilon} \quad (6.31)$$

*Error*

$$\varepsilon = y_M - y$$

*Regressor with filters*

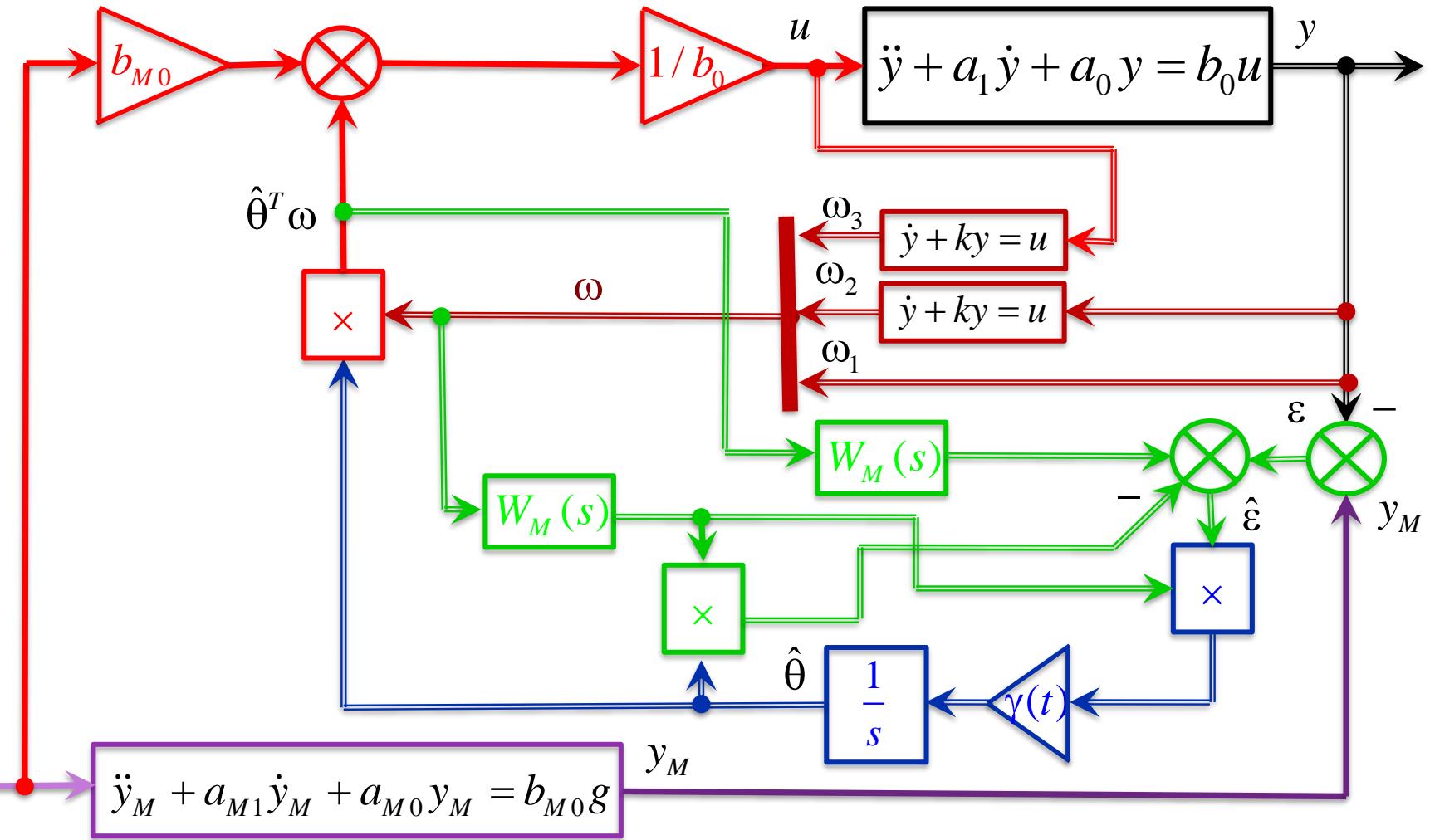
$$\omega = \text{col} \left( y, \frac{1}{s+k} [y], \frac{1}{s+k} [u] \right)$$

*Normalization ( $\omega$  is bounded?)*

$$\gamma(t) = \frac{\gamma_0}{1 + W_M(s) [\omega^T] W_M(s) [\omega]} \quad (6.32)$$

## 6.3. Dynamic error model with measurable output

General Scheme



## 6.3. Dynamic error model with measurable output

### Simulation results

*Plant*

$$\ddot{y} + a_1 \dot{y} + a_0 y = u$$

*Unknown parameters*

$$a_0 = 1, \quad a_1 = 2$$

*Reference model*

$$\ddot{y}_M + 5\dot{y}_M + 6y_M = 6g$$

*Adaptation gain*

$$\gamma(t) = \frac{1000}{1 + W_M(s)[\omega^T]W_M(s)[\omega]}$$

*Reference transfer function (with unity nominator)*

$$W_M(s) = \frac{1}{s^2 + 5s + 6}$$

*Reference*

$$g(t) = \sin 4t$$

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### Simulation results

*Regressor*

$$\omega = \text{col} \left( y, \frac{1}{s+8}[y], \frac{1}{s+8}[u] \right)$$

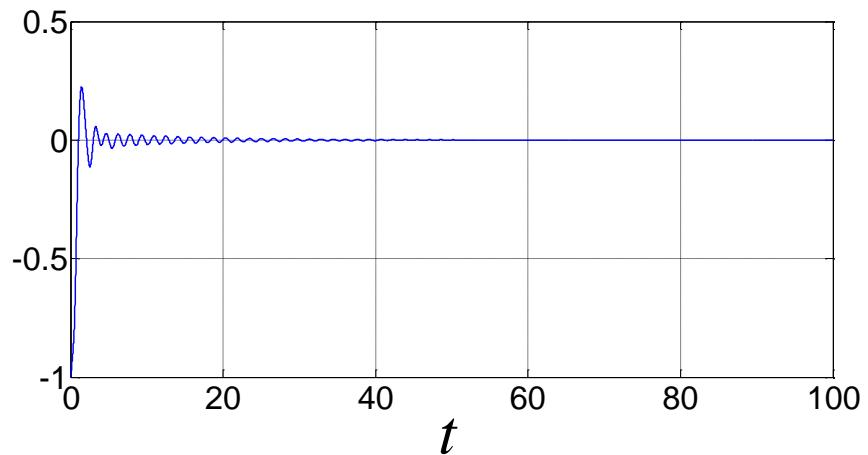
*Augmented error*     $\hat{\varepsilon} = \varepsilon - \hat{\theta}^T \frac{1}{s^2 + 5s + 6} [\omega] + \frac{1}{s^2 + 5s + 6} [\hat{\theta}^T \omega]$ ,

*Error*                   $\varepsilon = y_M - y$

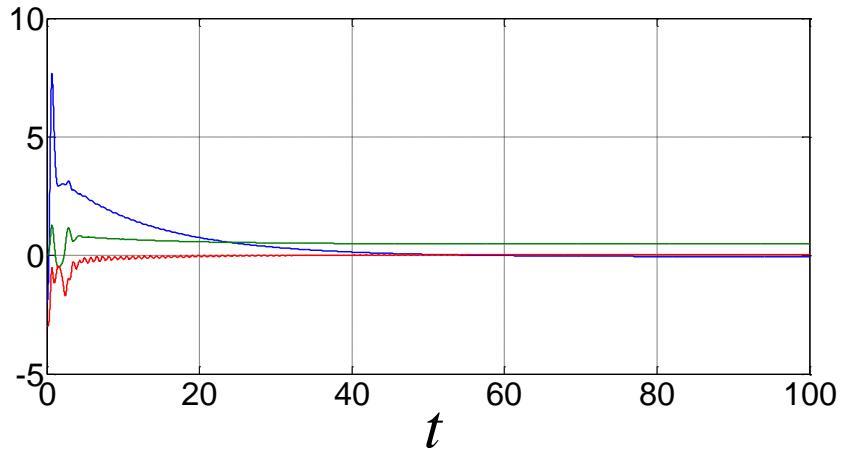
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### Simulation results

$$\varepsilon(t) = y_M(t) - y(t)$$



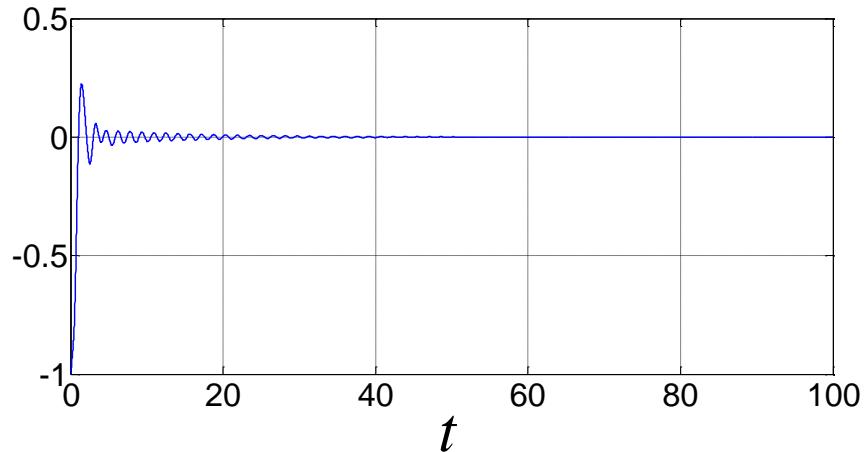
$$\tilde{\theta}_1(t), \tilde{\theta}_2(t), \tilde{\theta}_3(t)$$



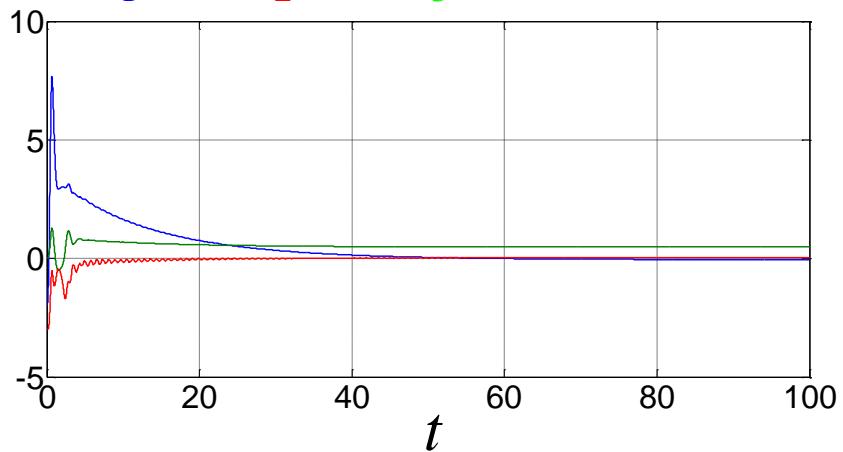
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### Simulation results

$$\varepsilon(t) = y_M(t) - y(t)$$



$$\tilde{\theta}_1(t), \tilde{\theta}_2(t), \tilde{\theta}_3(t)$$



*Does AA provide  
identification  
properties?*