

7. Schemes #1, #2 of LTI Plants Parameterization

7.1. Scheme #1. Output parameterization

Plant

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0y = b_m u^{(m)} + \dots + b_0u, \quad (7.1)$$

where $a_i, b_j, i = \overline{0, n-1}, j = \overline{0, m}$ are the constant parameters.

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Apply transfer function

$$H(s) = \frac{1}{K(s)} = \frac{1}{s^n + k_{n-1}s^{n-1} + \dots + k_0}$$

with a Hurwitz polynomial $K(s) = s^n + k_{n-1}s + \dots + k_0$ to (7.1) assuming initial conditions $y(0), \dots, y^{(n-1)}(0)$ zero .

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with a Hurwitz polynomial $K(s) = s^n + k_{n-1}s + \dots + k_0$ to (7.1) assuming initial conditions $y(0), \dots, y^{(n-1)}(0)$ zero . Using *Example 6.3* we have

7.1. Scheme #1. Output parameterization

Parameterization

$$y = (k_{n-1} - a_{n-1}) \frac{s^{n-1}}{K(s)} [y] + \dots + (k_1 - a_1) \frac{s}{K(s)} [y] + (k_0 - a_0) \frac{1}{K(s)} [y] +$$

$$b_m \frac{s^m}{K(s)} [u] + \dots + b_1 \frac{s}{K(s)} [u] + b_0 \frac{1}{K(s)} [u],$$

7.1. Scheme #1. Output parameterization

Parameterization

$$y = \left(k_{n-1} - a_{n-1} \right) \frac{s^{n-1}}{K(s)} [y] + \dots + \left(k_1 - a_1 \right) \frac{s}{K(s)} [y] + \left(k_0 - a_0 \right) \frac{1}{K(s)} [y] +$$
$$b_m \frac{s^m}{K(s)} [u] + \dots + b_1 \frac{s}{K(s)} [u] + b_0 \frac{1}{K(s)} [u],$$

$$y = \theta^T \omega, \quad (7.2)$$

where $\theta = \text{col}(\theta_1, \theta_2, \dots, \theta_{n+m+1}) \in \square^{n+m+1}$,

$$\omega = \text{col}(\xi_1, \xi_2, \dots, \xi_n, v_1, v_2, \dots, v_{m+1}) \in \square^{n+m+1}.$$

7.1. Scheme #1. Output parameterization

Parameterization

$$\begin{aligned}
 y = & \underbrace{\left(k_{n-1} - a_{n-1} \right) \frac{s^{n-1}}{K(s)} [y]}_{\theta_n} + \dots + \underbrace{\left(k_1 - a_1 \right) \frac{s}{K(s)} [y]}_{\theta_2} + \underbrace{\left(k_0 - a_0 \right) \frac{1}{K(s)} [y]}_{\theta_1} + \\
 & \underbrace{b_m \frac{s^m}{K(s)} [u]}_{\theta_{n+m+1} v_{m+1}} + \dots + \underbrace{b_1 \frac{s}{K(s)} [u]}_{\theta_{n+2} v_2} + b_0 \frac{1}{K(s)} [u],
 \end{aligned}$$

$$\left\{
 \begin{array}{l}
 \dot{\xi}_1 = \xi_2 \\
 \dot{\xi}_2 = \xi_3 \\
 \dots \\
 \dot{\xi}_{n-1} = \xi_n \\
 \dot{\xi}_n = -k_0 \xi_1 - k_1 \xi_2 - \dots - k_{n-1} \xi_n + y
 \end{array}
 \right. \quad (7.3) \quad \left\{
 \begin{array}{l}
 \dot{v}_1 = v_2 \\
 \dot{v}_2 = v_3 \\
 \dots \\
 \dot{v}_{n-1} = v_n \\
 \dot{v}_n = -k_0 v_1 - k_1 v_2 - \dots - k_{n-1} v_n + u
 \end{array}
 \right. \quad (7.4)$$

7.1. Scheme #1. Output parameterization

Parameterization

$$y = \underbrace{\left(k_{n-1} - a_{n-1} \right) \frac{s^{n-1}}{K(s)} [y]}_{\theta_n} + \dots + \underbrace{\left(k_1 - a_1 \right) \frac{s}{K(s)} [y]}_{\theta_2} + \underbrace{\left(k_0 - a_0 \right) \frac{1}{K(s)} [y]}_{\theta_1} +$$

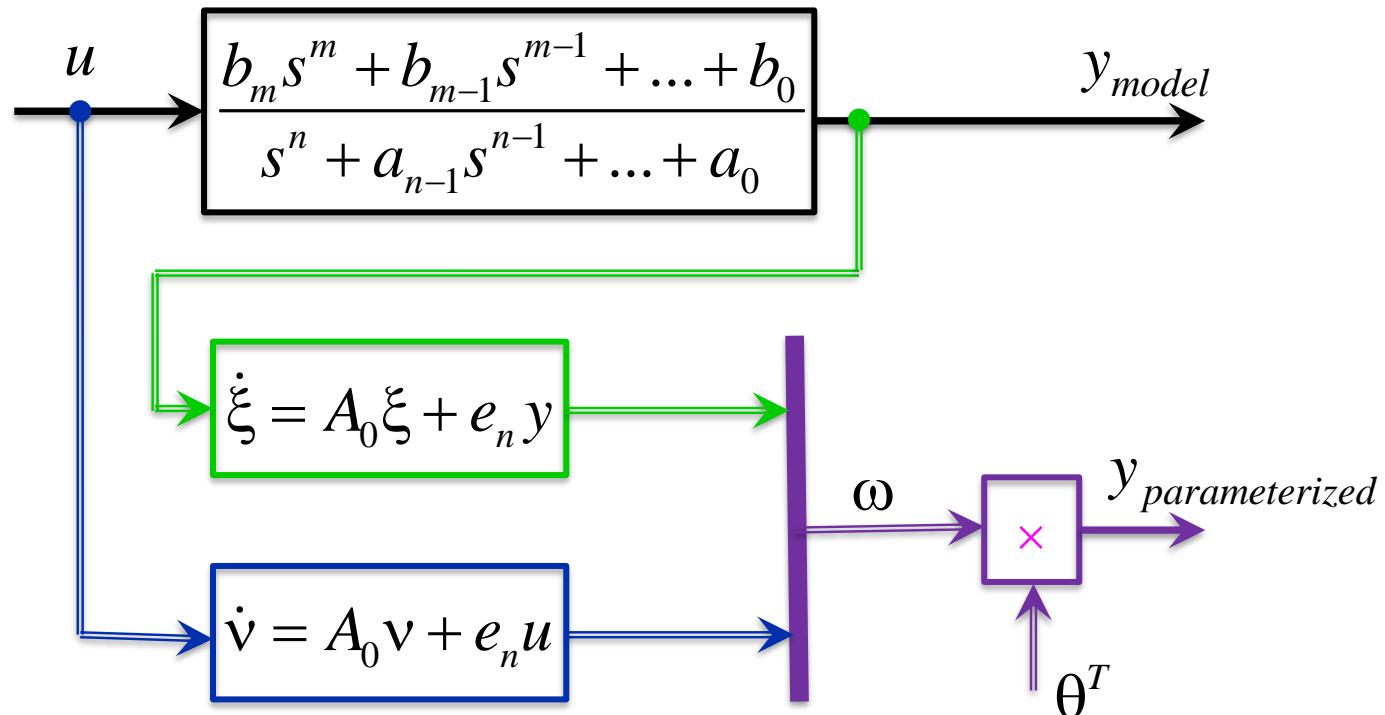
$$b_m \frac{s^m}{K(s)} [u] + \dots + b_1 \frac{s}{K(s)} [u] + b_0 \frac{1}{K(s)} [u],$$

$$\underbrace{\theta_{n+m+1} v_{m+1}}_{\theta_{n+2} v_2} \quad \underbrace{\theta_{n+1} v_1}_{\theta_{n+2} v_2}$$

$$\begin{cases} \dot{\xi} = A_0 \xi + e_n y, \\ \dot{v} = A_0 v + e_n u \end{cases} \quad (7.3)$$

$$(7.4) \quad A_0 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -k_0 & -k_1 & -k_2 & \dots & -k_{n-1} \end{bmatrix}, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

7.1. Scheme #1. Output parameterization



$$y_{parameterized} = \theta^T \omega$$

7.2. Scheme #2. State parameterization

Plant is given in canonical form

$$\begin{cases} \dot{x} = Ax + bu, \\ y = c^T x, \end{cases} \quad (7.5)$$

where

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$a_i, b_j, i = \overline{0, n-1}, j = \overline{0, m}$ are the constant parameters.

7.2. Scheme #2. State parameterization

Apply transfer matrix

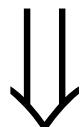
$$\Phi(s) = (I_{n \times n} s - A_0^*)^{-1} \quad (7.6)$$

with a Hurwitz matrix

$$A_0^* = \begin{bmatrix} -k_{n-1} & 1 & 0 & \cdots & 0 \\ -k_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_1 & 0 & 0 & \cdots & 1 \\ -k_0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

to (7.5) assuming initial conditions $x(0)$ zero:

$$(I_{n \times n} s - A_0^*)^{-1} [\dot{x}] = (I_{n \times n} s - A_0^*)^{-1} A[x] + (I_{n \times n} s - A_0^*)^{-1} b[u]$$



7.2. Scheme #2. State parameterization

$$\left(I_{n \times n} s - A_0^* \right)^{-1} \cdot s [x] = \left(I_{n \times n} s - A_0^* \right)^{-1} A [x] + \left(I_{n \times n} s - A_0^* \right)^{-1} b [u]$$

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$$\left(I_{n \times n} s - A_0^* \right)^{-1} \cdot s [x] = \left(I_{n \times n} s - A_0^* \right)^{-1} A [x] + \left(I_{n \times n} s - A_0^* \right)^{-1} b [u]$$

~~A_0^*~~

$$\downarrow$$
$$\left(I_{n \times n} s - A_0^* \right)^{-1} \left(I_{n \times n} s \pm A_0^* \right) [x] = \left(I_{n \times n} s - A_0^* \right)^{-1} A [x] + \left(I_{n \times n} s - A_0^* \right)^{-1} b [u]$$

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~~A_0^*~~



$$\left(I_{n \times n} s - A_0^* \right)^{-1} \left(I_{n \times n} s \pm A_0^* \right) [x] = \left(I_{n \times n} s - A_0^* \right)^{-1} A [x] + \left(I_{n \times n} s - A_0^* \right)^{-1} b [u]$$



$$x = \left(I_{n \times n} s - A_0^* \right)^{-1} \left(A - A_0^* \right) [x] + \left(I_{n \times n} s - A_0^* \right)^{-1} b [u]$$

7.2. Scheme #2. State parameterization

$$\begin{aligned}
 & \left(I_{n \times n} s - A_0^* \right)^{-1} \cdot s [x] = \left(I_{n \times n} s - A_0^* \right)^{-1} A [x] + \left(I_{n \times n} s - A_0^* \right)^{-1} b [u] \\
 & \quad \downarrow \\
 & \left(I_{n \times n} s - A_0^* \right)^{-1} \left(I_{n \times n} s \pm A_0^* \right) [x] = \left(I_{n \times n} s - A_0^* \right)^{-1} A [x] + \left(I_{n \times n} s - A_0^* \right)^{-1} b [u] \\
 & \quad \downarrow \\
 & x = \left(I_{n \times n} s - A_0^* \right)^{-1} \underbrace{\left(A - A_0^* \right)}_{\text{blue bracket}} [x] + \left(I_{n \times n} s - A_0^* \right)^{-1} b [u]
 \end{aligned}$$

$$\begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} -k_{n-1} & 1 & 0 & \cdots & 0 \\ -k_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_1 & 0 & 0 & \cdots & 1 \\ -k_0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} k_{n-1} - a_{n-1} \\ \vdots \\ k_1 - a_1 \\ k_0 - a_0 \end{bmatrix} [1 \ 0 \ \cdots \ 0]$$

7.2. Scheme #2. State parameterization

$$\begin{aligned}
 & \left(I_{n \times n} s - A_0^* \right)^{-1} \cdot s \begin{bmatrix} x \end{bmatrix} = \left(I_{n \times n} s - A_0^* \right)^{-1} A \begin{bmatrix} x \end{bmatrix} + \left(I_{n \times n} s - A_0^* \right)^{-1} b \begin{bmatrix} u \end{bmatrix} \\
 & \quad \downarrow \\
 & \left(I_{n \times n} s - A_0^* \right)^{-1} \left(I_{n \times n} s \pm A_0^* \right) \begin{bmatrix} x \end{bmatrix} = \left(I_{n \times n} s - A_0^* \right)^{-1} A \begin{bmatrix} x \end{bmatrix} + \left(I_{n \times n} s - A_0^* \right)^{-1} b \begin{bmatrix} u \end{bmatrix} \\
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 & x = \left(I_{n \times n} s - A_0^* \right)^{-1} \underbrace{\left(A - A_0^* \right)}_{\text{blue bracket}} \begin{bmatrix} x \end{bmatrix} + \left(I_{n \times n} s - A_0^* \right)^{-1} b \begin{bmatrix} u \end{bmatrix}
 \end{aligned}$$

$$\begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} -k_{n-1} & 1 & 0 & \cdots & 0 \\ -k_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_1 & 0 & 0 & \cdots & 1 \\ -k_0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} k_{n-1} - a_{n-1} \\ \vdots \\ k_1 - a_1 \\ k_0 - a_0 \end{bmatrix} c^T$$

7.2. Scheme #2. State parameterization

$$\left(I_{n \times n} s - A_0^* \right)^{-1} \cdot s [x] = \left(I_{n \times n} s - A_0^* \right)^{-1} A [x] + \left(I_{n \times n} s - A_0^* \right)^{-1} b [u]$$

~~A_0^*~~



$$\left(I_{n \times n} s - A_0^* \right)^{-1} \left(I_{n \times n} s \pm A_0^* \right) [x] = \left(I_{n \times n} s - A_0^* \right)^{-1} A [x] + \left(I_{n \times n} s - A_0^* \right)^{-1} b [u]$$



$$x = \left(I_{n \times n} s - A_0^* \right)^{-1} \begin{bmatrix} k_{n-1} - a_{n-1} \\ \vdots \\ k_1 - a_1 \\ k_0 - a_0 \end{bmatrix} c^T [x] + \left(I_{n \times n} s - A_0^* \right)^{-1} b [u]$$

7.2. Scheme #2. State parameterization

$$\left(I_{n \times n} s - A_0^* \right)^{-1} \cdot s [x] = \left(I_{n \times n} s - A_0^* \right)^{-1} A [x] + \left(I_{n \times n} s - A_0^* \right)^{-1} b [u]$$

~~A_0^*~~



$$\left(I_{n \times n} s - A_0^* \right)^{-1} \left(I_{n \times n} s \pm A_0^* \right) [x] = \left(I_{n \times n} s - A_0^* \right)^{-1} A [x] + \left(I_{n \times n} s - A_0^* \right)^{-1} b [u]$$



$$x = \left(I_{n \times n} s - A_0^* \right)^{-1} \begin{bmatrix} k_{n-1} - a_{n-1} \\ \vdots \\ k_1 - a_1 \\ k_0 - a_0 \end{bmatrix} [y] + \left(I_{n \times n} s - A_0^* \right)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix} [u]$$

7.2. Scheme #2. State parameterization

$$\begin{bmatrix} k_{n-1} - a_{n-1} \\ \vdots \\ k_1 - a_1 \\ k_0 - a_0 \end{bmatrix} = \sum_{i=0}^{n-1} (k_i - a_i) e_{n-i},$$

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix} = \sum_{j=0}^m b_j e_{m+1-j},$$

where $e_i = \text{col}(0, \dots, 0, \underset{i}{1}, 0, \dots, 0)$.

7.2. Scheme #2. State parameterization

$$\begin{bmatrix} k_{n-1} - a_{n-1} \\ \vdots \\ k_1 - a_1 \\ k_0 - a_0 \end{bmatrix} = \sum_{i=0}^{n-1} (k_i - a_i) e_{n-i},$$

where $e_i = \text{col}(0, \dots, 0, \underset{i}{1}, 0, \dots, 0)$.

$$x = (I_{n \times n} s - A_0^*)^{-1} \begin{bmatrix} k_{n-1} - a_{n-1} \\ \vdots \\ k_1 - a_1 \\ k_0 - a_0 \end{bmatrix} [y] + (I_{n \times n} s - A_0^*)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix} [u]$$

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$$\begin{bmatrix} k_{n-1} - a_{n-1} \\ \vdots \\ k_1 - a_1 \\ k_0 - a_0 \end{bmatrix} = \sum_{i=0}^{n-1} (k_i - a_i) e_{n-i}, \quad \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix} = \sum_{j=0}^m b_j e_{m+1-j},$$

where $e_i = \text{col}(0, \dots, 0, \underset{i}{1}, 0, \dots, 0)$.

$$x = \sum_{i=0}^{n-1} (k_i - a_i) (I_{n \times n} s - A_0^*)^{-1} e_{n-i}[y] + \sum_{j=0}^m b_j (I_{n \times n} s - A_0^*)^{-1} e_{m+1-j}[u]$$

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$$\begin{bmatrix} k_{n-1} - a_{n-1} \\ \vdots \\ k_1 - a_1 \\ k_0 - a_0 \end{bmatrix} = \sum_{i=0}^{n-1} (k_i - a_i) e_{n-i}, \quad \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix} = \sum_{j=0}^m b_j e_{m+1-j},$$

where $e_i = \text{col}(0, \dots, 0, \underset{i}{1}, 0, \dots, 0)$.

$$x = \sum_{i=0}^{n-1} (k_i - a_i) (I_{n \times n} s - A_0^*)^{-1} e_{n-i}[y] + \sum_{j=0}^m b_j (I_{n \times n} s - A_0^*)^{-1} e_{m+1-j}[u]$$

$$x = \sum_{i=0}^{n-1} \theta_{i+1} (I_{n \times n} s - A_0^*)^{-1} e_{n-i}[y] + \sum_{j=0}^m \theta_{j+1+n} (I_{n \times n} s - A_0^*)^{-1} e_{m+1-j}[u] \quad (7.7)$$

$$\theta = \text{col}(k_0 - a_0, k_1 - a_1, \dots, k_{n-1} - a_{n-1}, b_0, b_1, \dots, b_m) \in \square^{n+m+1}$$

7.2. Scheme #2. State parameterization

$$x = \sum_{i=0}^{n-1} \theta_{i+1} \left(I_{n \times n} s - A_0^* \right)^{-1} e_{n-i} [y] + \sum_{j=0}^m \theta_{j+1+n} \left(I_{n \times n} s - A_0^* \right)^{-1} e_{m+1-j} [u]$$

Parameterization with sets of filters

$$x = \sum_{i=1}^n \theta_i \xi_i^* + \sum_{j=1}^{m+1} \theta_{j+n} \nu_j^*, \quad (7.8)$$

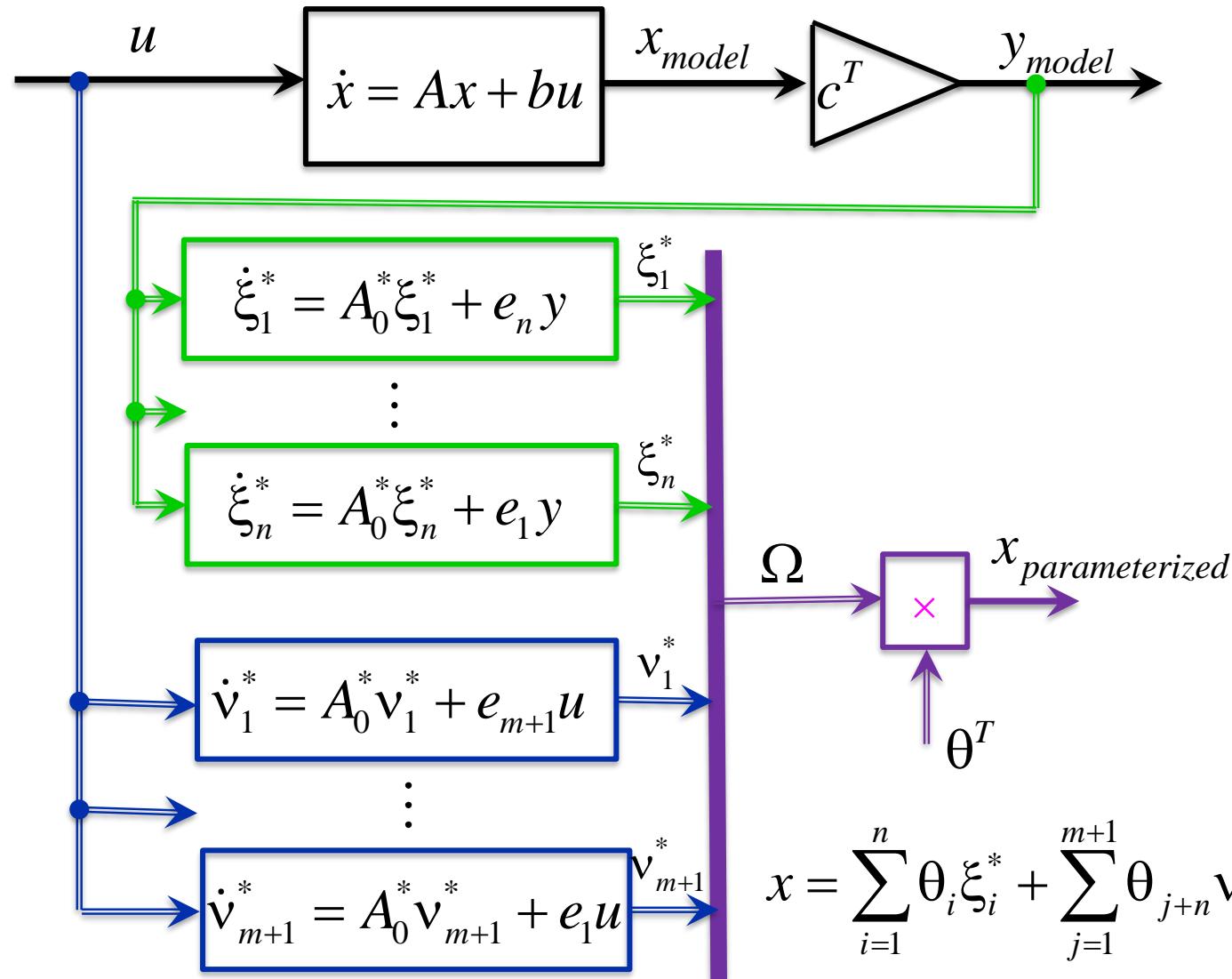
where $e_i = \text{col}(0, \dots, 0, \underset{i}{1}, 0, \dots, 0)$,

$$\begin{cases} \dot{\xi}_i^* = A_0^* \xi_i^* + e_{n+1-i} y, & i = \overline{1, n}, \\ \dot{\nu}_j^* = A_0^* \nu_j^* + e_{m+2-j} u, & j = \overline{1, m+1}, \end{cases} \quad (7.9) \quad (7.10)$$

$$A_0^* = \begin{bmatrix} -k_{n-1} & 1 & 0 & \cdots & 0 \\ -k_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_1 & 0 & 0 & \cdots & 1 \\ -k_0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\theta = \text{col}(k_0 - a_0, k_1 - a_1, \dots, k_{n-1} - a_{n-1}, b_0, b_1, \dots, b_m) \in \square^{n+m+1}$$

7.2. Scheme #2. State parameterization



8. Adaptive observer design

Problem statement

Plant

$$\begin{cases} \dot{x} = Ax + bu, \\ y = c^T x, \end{cases} \quad (8.1)$$

where $x \in \mathbb{C}^n$ is the unmeasurable state vector of known dimension, $u, y \in \mathbb{C}$ are the measurable input and output, respectively,

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$a_i, b_j, i = \overline{0, n-1}, j = \overline{0, m}$ are the unknown constant parameters.

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Problem statement

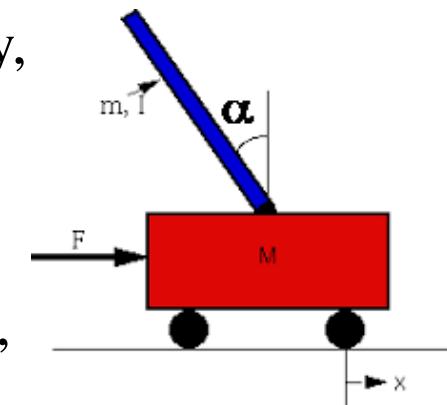
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$$\begin{cases} \dot{x} = Ax + bu, \\ y = c^T x, \end{cases} \quad (8.1)$$

where $x \in \mathbb{R}^n$ is the unmeasurable state vector of known dimension,

$u, y \in \mathbb{R}$ are the measurable input and output, respectively,

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$



$a_i, b_j, i = \overline{0, n-1}, j = \overline{0, m}$ are the unknown constant parameters.

Problem statement

Assumption 8.1. Matching conditions (general case): There exists such a Hurwitz matrix A_0^* and a vector of unknown parameters θ that

$$A_0^* = A + \theta c^T. \quad (8.2)$$

Objective to design and estimator that generates the vector $\hat{x}(t)$ satisfying limiting equality

$$\lim_{t \rightarrow \infty} \|x(t) - \hat{x}(t)\| = 0. \quad (8.3)$$

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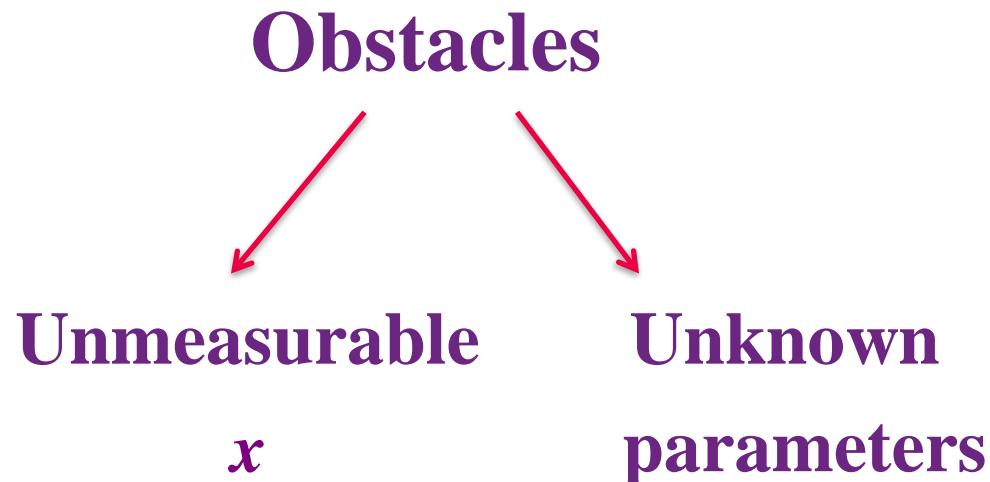
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Assumption 8.3. The matrix A is Hurwitz. The pair (A, c) is observable.

Solution



Solution

1. The obstacle of unmeasurable state is resolved by using parameterization

*Scheme #2 of
parameterization*

$$x = \sum_{i=1}^n \theta_i \xi_i^* + \sum_{j=1}^{m+1} \theta_{j+n} v_j^*,$$

where $\theta = \text{col}(k_0 - a_0, k_1 - a_1, \dots, k_{n-1} - a_{n-1}, b_0, b_1, \dots, b_m) \in \mathbb{C}^{n+m+1}$,

$$\begin{cases} \dot{\xi}_i^* = A_0^* \xi_i^* + e_{n+1-i} y, & i = \overline{1, n}, \\ \dot{v}_j^* = A_0^* v_j^* + e_{m+2-j} u, & j = \overline{1, m+1}, \end{cases} \quad (8.5)$$

$$\begin{cases} \dot{\xi}_i^* = A_0^* \xi_i^* + e_{n+1-i} y, & i = \overline{1, n}, \\ \dot{v}_j^* = A_0^* v_j^* + e_{m+2-j} u, & j = \overline{1, m+1}, \end{cases} \quad (8.6)$$

$$e_i = \text{col}(0, \dots, 0, \underset{i}{1}, 0, \dots, 0),$$

$$A_0^* = \begin{bmatrix} -k_{n-1} & 1 & 0 & \cdots & 0 \\ -k_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_1 & 0 & 0 & \cdots & 1 \\ -k_0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

Solution

and replacement of the vector θ by its estimate $\hat{\theta}$:

$$\hat{x} = \sum_{i=1}^n \hat{\theta}_i \xi_i^* + \sum_{j=1}^{m+1} \hat{\theta}_{j+n} v_j^*, \quad (8.4)$$

where

$$\begin{cases} \dot{\xi}_i^* = A_0^* \xi_i^* + e_{n+1-i} y, & i = \overline{1, n}, \\ \dot{v}_j^* = A_0^* v_j^* + e_{m+2-j} u, & j = \overline{1, m+1}, \end{cases} \quad (8.5)$$

$$\begin{cases} \dot{\xi}_i^* = A_0^* \xi_i^* + e_{n+1-i} y, & i = \overline{1, n}, \\ \dot{v}_j^* = A_0^* v_j^* + e_{m+2-j} u, & j = \overline{1, m+1}, \end{cases} \quad (8.6)$$

$$e_i = \text{col} \left(0, \dots, 0, \underset{i}{1}, 0, \dots, 0 \right),$$

$$A_0^* = \begin{bmatrix} -k_{n-1} & 1 & 0 & \cdots & 0 \\ -k_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_1 & 0 & 0 & \cdots & 1 \\ -k_0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

Solution

2. The obstacle of unknown parameters is resolved using parameterization

*Scheme #1 of
parameterization*

$$y = \theta^T \omega,$$

where $\omega = \text{col}(\xi_1, \xi_2, \dots, \xi_n, v_1, v_2, \dots, v_{m+1}) \in \mathbb{R}^{n+m+1}$,

$$\begin{cases} \dot{\xi} = A_0 \xi + e_n y, \\ \dot{v} = A_0 v + e_n u, \end{cases} \quad (8.7)$$

$$(8.8)$$

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -k_0 & -k_1 & -k_2 & \cdots & -k_{n-1} \end{bmatrix}, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Solution

2. The obstacle of unknown parameters is resolved using parameterization

*Scheme #1 of
parameterization*

$$y = \theta^T \omega, \quad (8.9)$$

where $\omega = \text{col}(\xi_1, \xi_2, \dots, \xi_n, v_1, v_2, \dots, v_{m+1}) \in \mathbb{C}^{n+m+1}$.

Introduce the error of identification (see Example 6.3)

$$e = y - \hat{\theta}^T \omega, \quad (8.10)$$

which after replacement of (8.9) gives the static error model
(see Lecture 6.1)

$$e = \tilde{\theta}^T \omega \quad (8.11)$$

with the vector of parametric errors $\tilde{\theta} = \theta - \hat{\theta}$.

Solution

The static error model motivates design of adaptation algorithm

$$\dot{\hat{\theta}} = \gamma \omega e \quad (8.12)$$

with a positive gain γ .

Solution Summary

Adjustable parameterized model of the plant

$$\hat{x} = \sum_{i=1}^n \hat{\theta}_i \xi_i^* + \sum_{j=1}^{m+1} \hat{\theta}_{j+n} v_j^* \quad (8.4)$$

Filters

$$\begin{cases} \dot{\xi}_i^* = A_0^* \xi_i^* + e_{n+1-i} y, & i = \overline{1, n}, \\ \dot{v}_j^* = A_0^* v_j^* + e_{m+2-j} u, & j = \overline{1, m+1} \end{cases} \quad (8.5)$$

$$\begin{cases} \dot{\xi}_i^* = A_0^* \xi_i^* + e_{n+1-i} y, & i = \overline{1, n}, \\ \dot{v}_j^* = A_0^* v_j^* + e_{m+2-j} u, & j = \overline{1, m+1} \end{cases} \quad (8.6)$$

Identification error

$$e = y - \hat{\theta}^T \omega, \quad (8.10)$$

$$\omega = \text{col}(\xi_1, \xi_2, \dots, \xi_n, v_1, v_2, \dots, v_{m+1})$$

Filters

$$\begin{cases} \dot{\xi} = A_0 \xi + e_n y, \\ \dot{v} = A_0 v + e_n u, \end{cases} \quad (8.7)$$

$$\begin{cases} \dot{\xi} = A_0 \xi + e_n y, \\ \dot{v} = A_0 v + e_n u, \end{cases} \quad (8.8)$$

Adaptation algorithm

$$\dot{\hat{\theta}} = \gamma \omega e \quad (8.12)$$

Solution Summary

Adjustable parameterized model of the plant

$$\hat{x} = \sum_{i=1}^n \hat{\theta}_i \xi_i^* + \sum_{j=1}^{m+1} \hat{\theta}_{j+n} v_j^* \quad (8.4)$$

Filters

$$\begin{cases} \dot{\xi}_i^* = A_0^* \xi_i^* + e_{n+1-i} y, & i = \overline{1, n}, \\ \dot{v}_j^* = A_0^* v_j^* + e_{m+2-j} u, & j = \overline{1, m+1} \end{cases} \quad (8.5)$$

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Identification error

$$e = y - \hat{\theta}^T \omega, \quad (8.10)$$

$$\omega = \text{col}(\xi_1, \xi_2, \dots, \xi_n, v_1, v_2, \dots, v_{m+1})$$

Filters

$$\begin{cases} \dot{\xi} = A_0 \xi + e_n y, \\ \dot{v} = A_0 v + e_n u, \end{cases} \quad (8.7)$$

Adaptation algorithm

$$\dot{\hat{\theta}} = \gamma \omega e \quad (8.12)$$

Homework
Simplify
the Structure

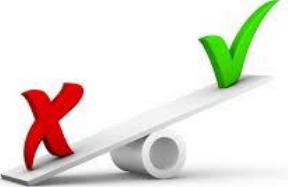
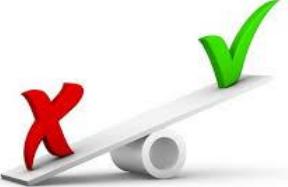
Solution Summary

Properties

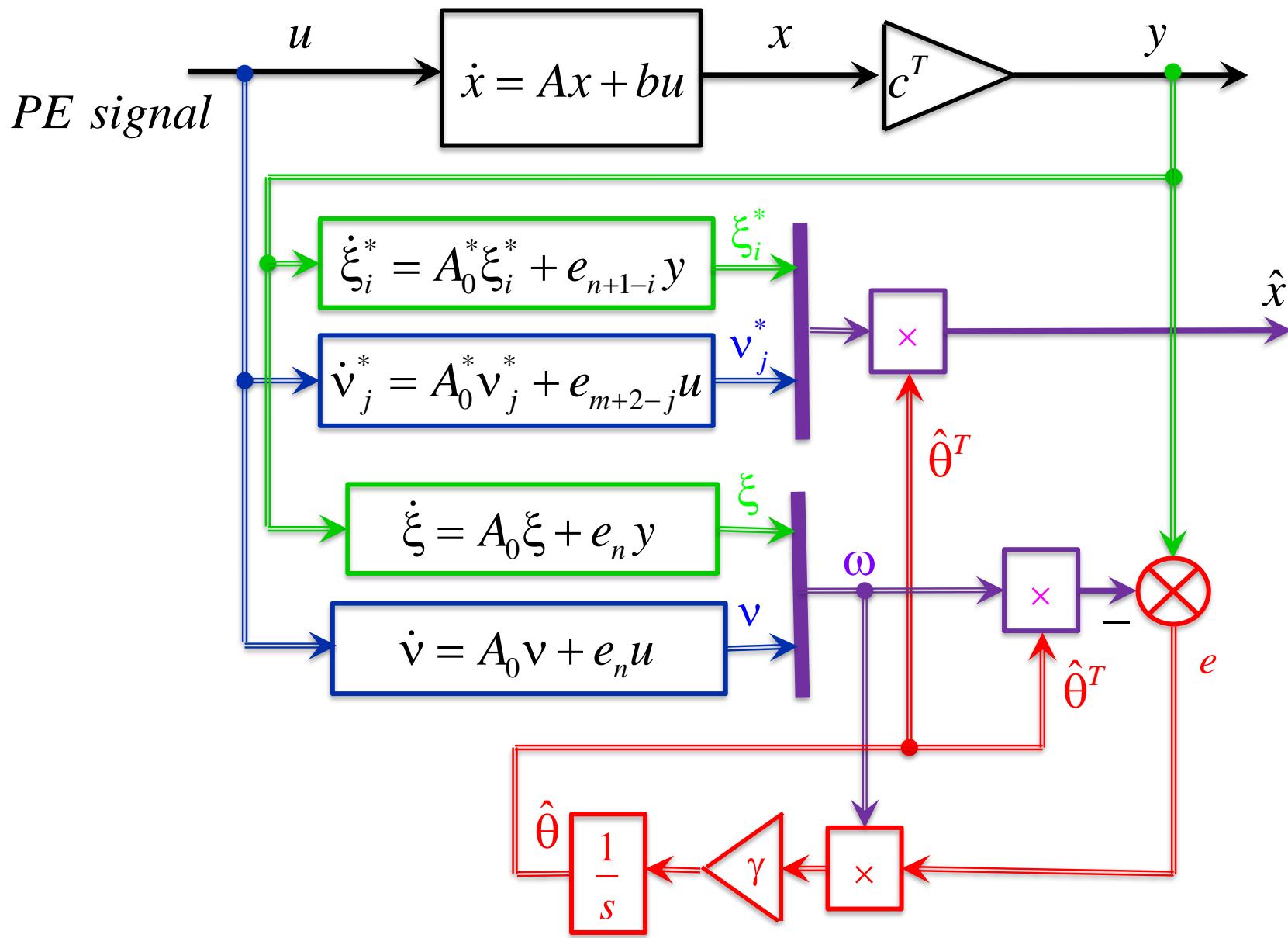
1. If u is bounded, all the signals in the system are bounded; 
2. If the signal u contains at least $(n+m+1)/2$ harmonics, then the norms $\|x(t) - \hat{x}(t)\|$, $\|\tilde{\theta}(t)\|$ approach zero asymptotically;
3. The function $\|\tilde{\theta}(t)\|$ is nonincreasing;
4. If the signal u contains at least $(n+m+1)/2$ harmonics, there exists an optimal γ , for which the rate of convergence of $\|\tilde{\theta}(t)\|$ is maximum.

Solution Summary

Properties

1. If u is bounded, all the signals in the system are bounded 
2. If the signal u contains at least $(n+m+1)/2$ harmonics, then the norms $\|x(t) - \hat{x}(t)\|$, $\|\tilde{\theta}(t)\|$ approach zero asymptotically; 
3. The function $\|\tilde{\theta}(t)\|$ is nonincreasing;
4. If the signal u contains at least $(n+m+1)/2$ harmonics, there exists an optimal γ , for which the rate of convergence of $\|\tilde{\theta}(t)\|$ is maximum.

Significant restriction of practical implementation



Simulation results

Plant

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 \\ -a_0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_0 \end{bmatrix} u$$

Unknown parameters

$$a_0 = -1, \quad a_1 = -2, \\ b_0 = 3, \quad b_1 = 4$$

Filters

$$\begin{bmatrix} \dot{\xi}_{11}^* \\ \dot{\xi}_{12}^* \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} \xi_{11}^* \\ \xi_{12}^* \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y, \quad \begin{bmatrix} \dot{v}_{11}^* \\ \dot{v}_{12}^* \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} v_{11}^* \\ v_{12}^* \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

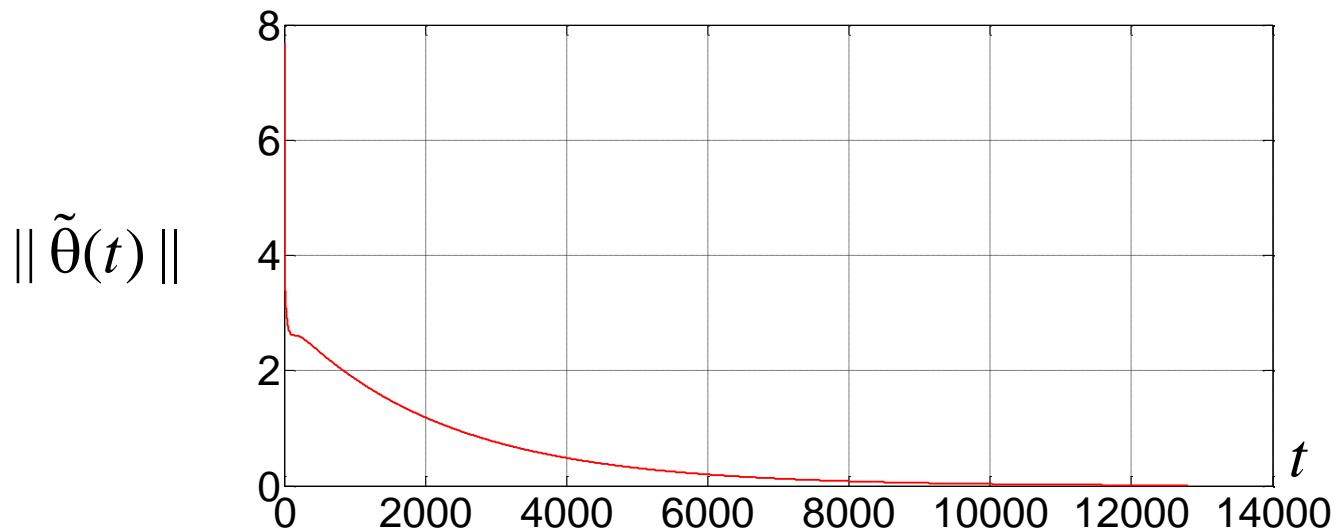
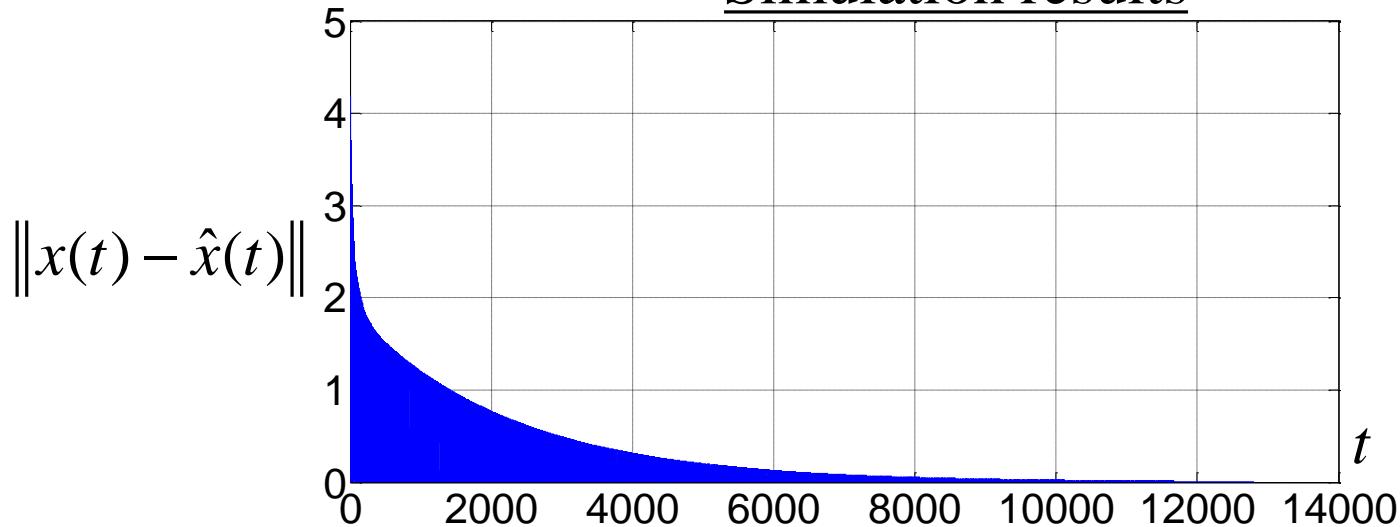
$$\begin{bmatrix} \dot{\xi}_{21}^* \\ \dot{\xi}_{22}^* \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} \xi_{21}^* \\ \xi_{22}^* \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} y, \quad \begin{bmatrix} \dot{v}_{21}^* \\ \dot{v}_{22}^* \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} v_{21}^* \\ v_{22}^* \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u,$$

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y, \quad \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Adaptation gain

$$\gamma = 1000$$

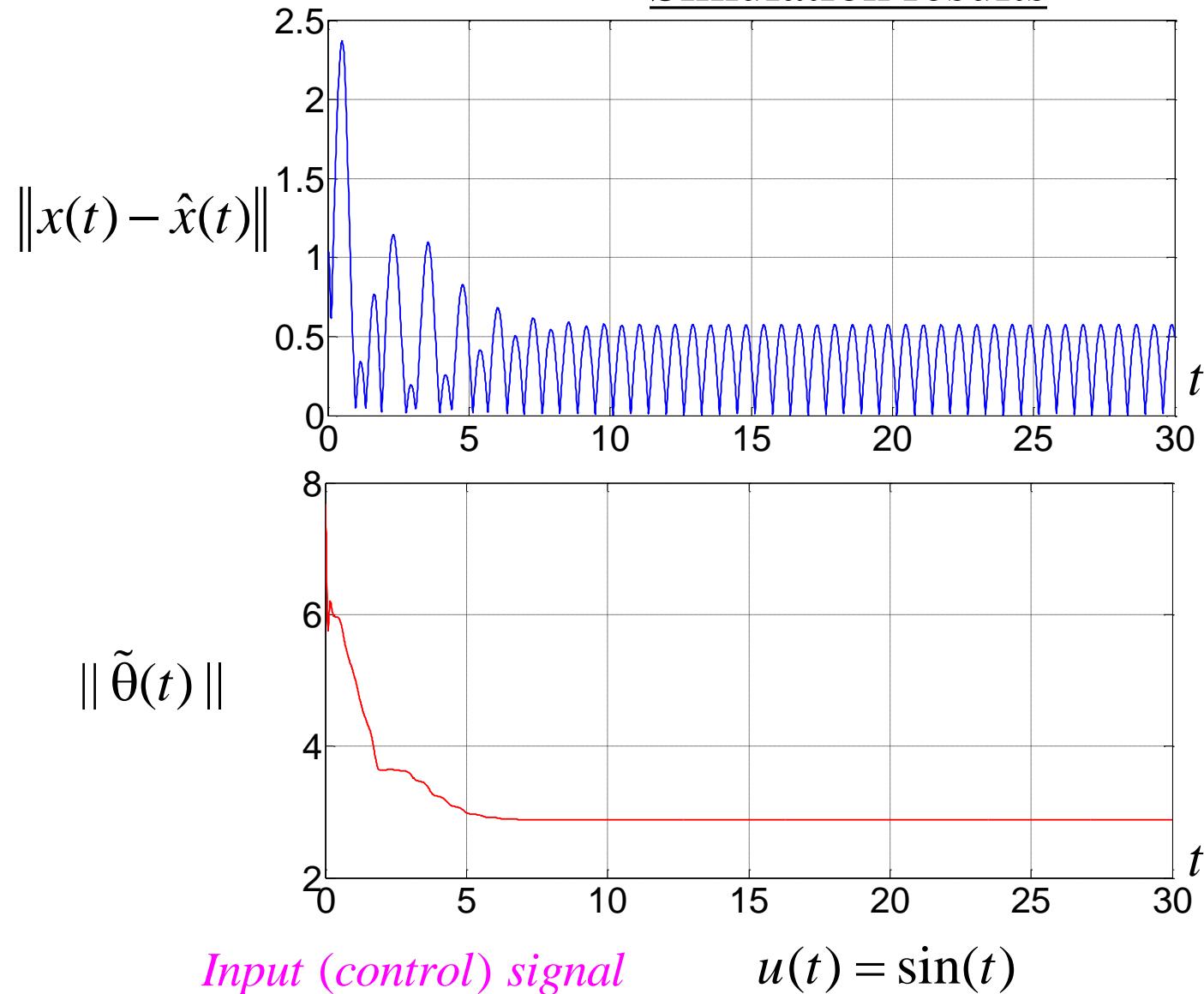
Simulation results



Input (control) signal

$$u(t) = \sin(t) + \cos(5t)$$

Simulation results



9. Scheme#3 of LTI plants parameterization

Application of adaptive observers in the output control can deteriorate the closed-loop system performance due to the dependence of identification process from PE condition.

9. Scheme#3 of LTI plants parameterization

Application of adaptive observers in the output control can deteriorate the closed-loop system performance due to the dependence of identification process from PE condition.

In was observed in ([K. Åstrom and B. Wittenmark, On Self-tuning Regulators, Automatica, Vol. 9, pp. 185-199, 1973.](#)) that it is not necessary to identify the plant parameters, but instead it is sufficient to directly tune the controller parameters and overcome the problem of PE condition.

9. Scheme#3 of LTI plants parameterization

Plant

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0y = b_m u^{(m)} + \dots + b_0u, \quad (9.1)$$

where $a_i, b_j, i = \overline{0, n-1}, j = \overline{0, m}$ are the constant parameters.

Transfer function representation

$$y(t) = \frac{b_m s^m + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} [u(t)] = \frac{b(s)}{a(s)} [u(t)] \quad (9.2)$$

with coprime polynomials $a(s), b(s)$.

Assume initial conditions $y(0), \dots, y^{(n-1)}(0)$ zero.

9. Scheme#3 of LTI plants parameterization

Lemma (R. V. Monopoli, 1974, direct control parameterization lemma)

There exists a constant vector $\theta \in \mathbb{C}^{2n-1}$ such that

$$y(t) = \frac{1}{\delta_M(s)} [\theta^T \omega(t) + b_m u(t)], \quad (9.3)$$

where $\omega = \text{col}(y, v_1, v_2) \in \mathbb{C}^{2n-1}$ is the regressor vector,

$\delta_M(s)$ is an arbitrary Hurwitz polynomial of degree

$(n-m)$ (the same as the plant relative degree),

$v_1, v_2 \in \mathbb{C}^{n-1}$ are the vectors generated by the stable filters

$$\begin{cases} \dot{v}_1 = \Lambda v_1 + e_{n-1} y, \\ \dot{v}_2 = \Lambda v_2 + e_{n-1} u, \end{cases} \quad (9.4)$$

$$(9.5) \quad \Lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\gamma_0 & -\gamma_1 & -\gamma_2 & \cdots & -\gamma_{n-2} \end{bmatrix}, \quad e_{n-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

10. Model Reference Adaptive Control (MRAC) Schemes

Problem statement

Plant

$$y(t) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_0} [u(t)] = \frac{b(s)}{a(s)} [u(t)] \quad (10.1)$$

with **unknown** parameters a_i, b_j , $i = \overline{0, n-1}$, $j = \overline{0, m}$, unmeasurable state , the measurable input u and output y , the known order n and the relative degree $\rho = n - m$.

10. Model Reference Adaptive Control (MRAC) Schemes

Problem statement

Plant

$$y(t) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_0} [u(t)] = \frac{b(s)}{a(s)} [u(t)] \quad (10.1)$$

with **unknown** parameters a_i, b_j , $i = \overline{0, n-1}$, $j = \overline{0, m}$, unmeasurable state, the measurable input u and output y , the known order n and the relative degree $\rho = n - m$.

The objective is to design a control $u(t)$ such that

$$\lim_{t \rightarrow \infty} \|y_M(t) - y(t)\| = 0, \quad (10.2)$$

where y_M is the output of reference model with PWC reference signal g :

$$y_M^\rho + a_{M,n-1} y_M^{\rho-1} + \dots + a_{M,0} y_M = a_{M,0} g \quad (10.3)$$

10. Model Reference Adaptive Control (MRAC) Schemes

Problem statement

Plant

$$y(t) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_0} [u(t)] = \frac{b(s)}{a(s)} [u(t)] \quad (10.1)$$

with **unknown** parameters a_i, b_j , $i = \overline{0, n-1}$, $j = \overline{0, m}$, unmeasurable state, the measurable input u and output y , the known order n and the relative degree $\rho = n - m$.

The objective is to design a control $u(t)$ such that

$$\lim_{t \rightarrow \infty} \|y_M(t) - y(t)\| = 0, \quad (10.2)$$

where y_M is the output of reference model with PWC reference signal g :

$$y_M(t) = \frac{a_{M0}}{s^\rho + a_{M\rho-1}s^{\rho-1} + a_{M\rho-2}s^{\rho-2} + \dots + a_{M0}} [g(t)] = \frac{a_{M0}}{\delta_M(s)} [g(t)] \quad (10.3)$$

Problem statement

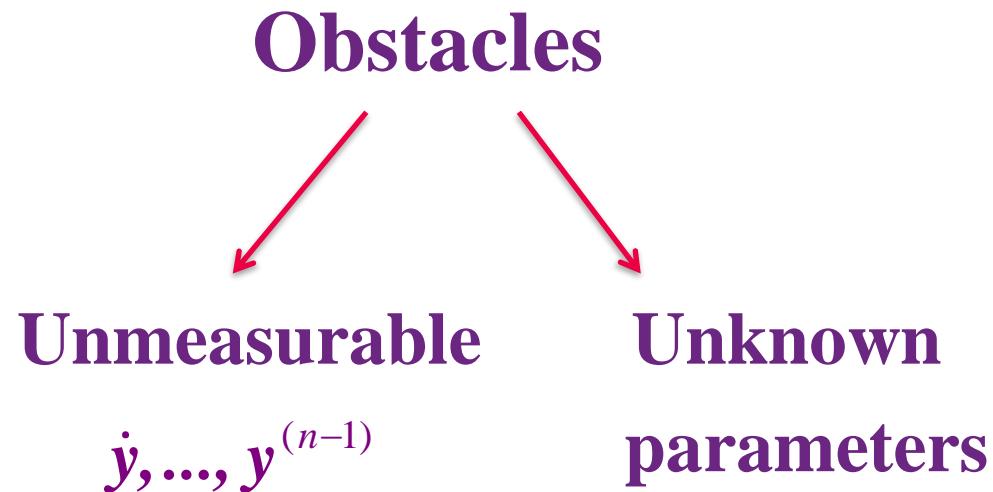
Assumption 10.1. Plant is controllable ;

Assumption 10.2. Plant is minimum-phase, i.e. The polynomial

$$b(s) = b_m s^m + \dots + b_1 s + b_0 \text{ is Hurwitz;}$$

Assumption 10.3. b_m is known (for the sake of simplicity and without loss of generality).

Solution



Solution

1. Obstacle of unmeasurable state.

Introduce the output tracking error

$$\varepsilon = y_M - y \quad (10.4)$$

Solution

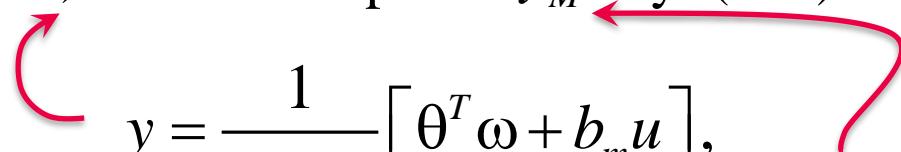
1. Obstacle of unmeasurable state.

Introduce the output tracking error

$$\varepsilon = y_M - y \quad (10.4)$$

and using *direct control parameterization lemma* we replace y by

(9.3) and then replace y_M by (10.3):

$$y = \frac{1}{\delta_M(s)} [\theta^T \omega + b_m u], \quad y_M = \frac{a_{M0}}{\delta_M(s)} [g]$$


Solution

1. Obstacle of unmeasurable state.

Introduce the output tracking error

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Solution

1. Obstacle of unmeasurable state.

Introduce the output tracking error

$$\varepsilon = y_M - y \quad (10.4)$$

and using *direct control parameterization lemma* we replace y by

(9.3) and then replace y_M by (10.3):

$$\begin{aligned} \varepsilon &= \frac{a_{M0}}{\delta_M(s)}[g] - \frac{1}{\delta_M(s)}[\theta^T \omega + b_m u], \\ &\quad \downarrow \\ \varepsilon &= \frac{1}{\delta_M(s)}[a_{M0}g - \theta^T \omega - b_m u] \end{aligned} \quad (10.5)$$

**See also Example 6.6. in page 6.50 (second order system)
with similar notations**

Solution

2. Obstacle of unknown parameters.

Error equation

$$\varepsilon = \frac{1}{\delta_M(s)} [a_{M0}g - \theta^T \omega - b_m u]$$

Certainty equivalence principle

Adjustable control

$$u = \frac{1}{b_m} (a_{M0}g - \hat{\theta}^T \omega)$$



Error model

$$\varepsilon = -\frac{1}{\delta_M(s)} [\tilde{\theta}^T \omega]$$

R. V. Monopoli, 1974,

A. Feuer and A.S. Morse, 1978

$$u = \frac{1}{b_m} \delta_M(s) \left[\frac{a_{M0}}{\delta_M(s)} [g] - \hat{\theta}^T \bar{\omega} \right]$$



$$\varepsilon = -\tilde{\theta}^T \bar{\omega},$$

$$\bar{\omega} = \frac{1}{\delta_M(s)} [\omega]$$

A. S. Morse, 1994

Augmented Error Solution

2. Obstacle of unknown parameters.

Based on the error equation

$$\varepsilon = \frac{1}{\delta_M(s)} [a_{M0}g - \theta^T \omega - b_m u] \quad (10.5)$$

and the certainty equivalence principle we design adjustable control

$$u = \frac{1}{b_m} (a_{M0}g - \hat{\theta}^T \omega) \quad (10.6)$$

that after replacement to (10.5) gives the **dynamic** error model

(see Lecture 6, Pages 6.42-6.57)

$$\varepsilon = -\frac{1}{\delta_M(s)} [\tilde{\theta}^T \omega]. \quad (10.7)$$

Augmented Error Solution

2. Obstacle of unknown parameters.

Model (10.7) motivates the design of *augmented error* adaptation algorithm (see Lecture 6, Pages 6.42-6.57)

$$\dot{\hat{\theta}} = -\gamma \bar{\omega} \hat{\varepsilon}, \quad (10.8)$$

$$\hat{\varepsilon} = \varepsilon + \hat{\theta}^T \bar{\omega} - \frac{1}{\delta_M(s)} [\hat{\theta}^T \omega], = -\tilde{\theta}^T \bar{\omega} \quad (10.9)$$

$$\bar{\omega} = \frac{1}{\delta_M(s)} [\omega],$$

where γ is the normalized adaptation gain defined as

$$\gamma(t) = \frac{\gamma_0}{1 + \bar{\omega}^T \bar{\omega}}, \quad \gamma_0 = \text{const} > 0. \quad (10.10)$$

Augmented Error Solution Summary

Adjustable control

$$u = \frac{1}{b_m} \left(a_M g - \hat{\theta}^T \omega \right) \quad (10.6)$$

Regressor

$$\omega = \text{col}(y, v_1, v_2)$$

Filters

$$\begin{cases} \dot{v}_1 = \Lambda v_1 + e_{n-1} y, \\ \dot{v}_2 = \Lambda v_2 + e_{n-1} u \end{cases} \quad (9.4)$$

$$\begin{cases} \dot{v}_1 = \Lambda v_1 + e_{n-1} y, \\ \dot{v}_2 = \Lambda v_2 + e_{n-1} u \end{cases} \quad (9.5)$$

Adaptation Algorithm

$$\dot{\hat{\theta}} = -\gamma \bar{\omega} \hat{\varepsilon} \quad (10.8)$$

Augmented error

$$\hat{\varepsilon} = \varepsilon + \hat{\theta}^T \bar{\omega} - \frac{1}{\delta_M(s)} [\hat{\theta}^T \omega] \quad (10.9)$$

Error

$$\varepsilon = y_M - y$$

Filtered regressor

$$\bar{\omega} = \frac{1}{\delta_M(s)} [\omega]$$

Normalized adaptation

gain

$$\gamma(t) = \frac{\gamma_0}{1 + \bar{\omega}^T(t) \bar{\omega}(t)}, \quad \gamma_0 = \text{const} > 0. \quad (10.10)$$

Augmented Error Solution Summary

Adjustable control

$$u = \frac{1}{b_m} \left(a_{M0} g - \hat{\theta}^T \omega \right) \quad (10.6)$$

Regressor

$$\omega = \text{col}(y, v_1, v_2)$$

Filters

$$\begin{cases} \dot{v}_1 = \Lambda v_1 + e_{n-1} y, \\ \dot{v}_2 = \Lambda v_2 + e_{n-1} u \end{cases} \quad (9.4)$$

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Adaptation Algorithm

$$\dot{\hat{\theta}} = -\gamma \bar{\omega} \hat{\varepsilon} \quad (10.8)$$

Augmented error

$$\hat{\varepsilon} = \varepsilon + \hat{\theta}^T \bar{\omega} - \frac{1}{\delta_M(s)} [\hat{\theta}^T \omega] \quad (10.9)$$

Error

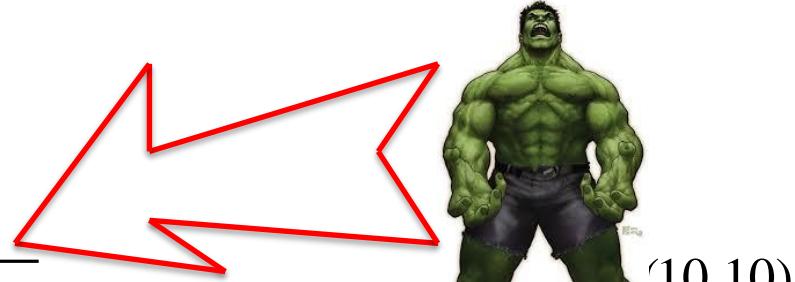
$$\varepsilon = y_M - y$$

Filtered regressor

$$\bar{\omega} = \frac{1}{\delta_M(s)} [\omega]$$

Normalized adaptation gain

$$\gamma(t) = \frac{\gamma_0}{1 + \bar{\omega}^T(t) \bar{\omega}(t)} \quad (10.10)$$



Augmented Error Solution Summary

Apply the Swapping Lemma

Adaptation Algorithm $\dot{\hat{\theta}} = -\gamma \bar{\omega} \hat{\varepsilon}$ (10.8)

Augmented error $\hat{\varepsilon} = \varepsilon + \hat{\theta}^T \bar{\omega} - \frac{1}{\delta_M(s)} [\hat{\theta}^T \omega]$ (10.9)

Filtered regressor $\bar{\omega} = \frac{1}{\delta_M(s)} [\omega]$

Normalized adaptation gain $\gamma(t) = \frac{\gamma_0}{1 + \bar{\omega}^T(t) \bar{\omega}(t)}$ (10.10)

Augmented Error Solution Summary

Apply the Swapping Lemma

Adaptation Algorithm $\dot{\hat{\theta}} = -\gamma \bar{\omega} \hat{\varepsilon}$ (10.8)

Augmented error $\hat{\varepsilon} = \varepsilon + W_c(s) \left[W_b(s) [\omega^T] \dot{\hat{\theta}} \right]$ (10.9)

$$W_C(s) = c^T (Is - A)^{-1}, \quad W_b(s) = (Is - A)^{-1} b$$

Filtered regressor $\bar{\omega} = \frac{1}{\delta_M(s)} [\omega] = c^T (Is - A)^{-1} b$

Normalized adaptation gain $\gamma(t) = \frac{\gamma_0}{1 + \bar{\omega}^T(t) \bar{\omega}(t)}$ (10.10)

Augmented Error Solution Summary

Apply the Swapping Lemma

Adaptation Algorithm

$$\dot{\hat{\theta}} = -\gamma \bar{\omega} \hat{\varepsilon} \quad (10.8)$$

Augmented error

$$\hat{\varepsilon} = \varepsilon + W_c(s) \left[W_b(s) [\omega^T] \dot{\hat{\theta}} \right] \quad (10.9)$$

$$W_C(s) = c^T (Is - A)^{-1}, \quad W_b(s) = (Is - A)^{-1} b$$

Filtered regressor

$$\bar{\omega} = \frac{1}{\delta_M(s)} [\omega]$$

Normalized adaptation

$$gain \quad \gamma(t) = \frac{\gamma_0}{1 + \bar{\omega}^T(t) \bar{\omega}(t)} \quad (10.10)$$

Augmented Error Solution Summary

Augmented error $\hat{\varepsilon} = \varepsilon - W_c(s) \left[W_b(s) \left[\omega^T \right] \frac{\gamma_0 \bar{\omega} \hat{\varepsilon}}{1 + \bar{\omega}^T \bar{\omega}} \right]$ (10.9)

$$W_C(s) = c^T (Is - A)^{-1}, \quad W_b(s) = (Is - A)^{-1} b$$

Filtered regressor $\bar{\omega} = \frac{1}{\delta_M(s)} [\omega]$

Augmented Error Solution Summary

Augmented error $\hat{\varepsilon} = \varepsilon - \gamma_0 W_c(s) \left[W_b(s) \begin{bmatrix} \omega^T \\ \frac{\bar{\omega}\hat{\varepsilon}}{1 + \bar{\omega}^T \bar{\omega}} \end{bmatrix} \right]$ (10.9)

Filtered regressor $W_C(s) = c^T (Is - A)^{-1}, \quad W_b(s) = (Is - A)^{-1} b$

Augmented error **Goes to zero**

Filtered regressor **Bounded**

The diagram consists of two red arrows. One arrow points from the text "Goes to zero" to the term $\frac{\bar{\omega}\hat{\varepsilon}}{1 + \bar{\omega}^T \bar{\omega}}$ in the augmented error equation. The other arrow points from the text "Bounded" to the term $W_b(s)$ in the same equation.

Augmented Error Solution Summary

Properties of the closed-loop system (See Lecture 6.3):

1. All the signals in the system are bounded;
2. The error $\hat{\varepsilon}(t)$ approaches zero asymptotically;
3. The norm $\|\tilde{\theta}(t)\|$ is nonincreasing;
4. The $\|\tilde{\theta}(t)\|$ approaches zero asymptotically if ω satisfies the Persistent Excitation condition;
5. The error $\varepsilon(t)$ approaches zero asymptotically.

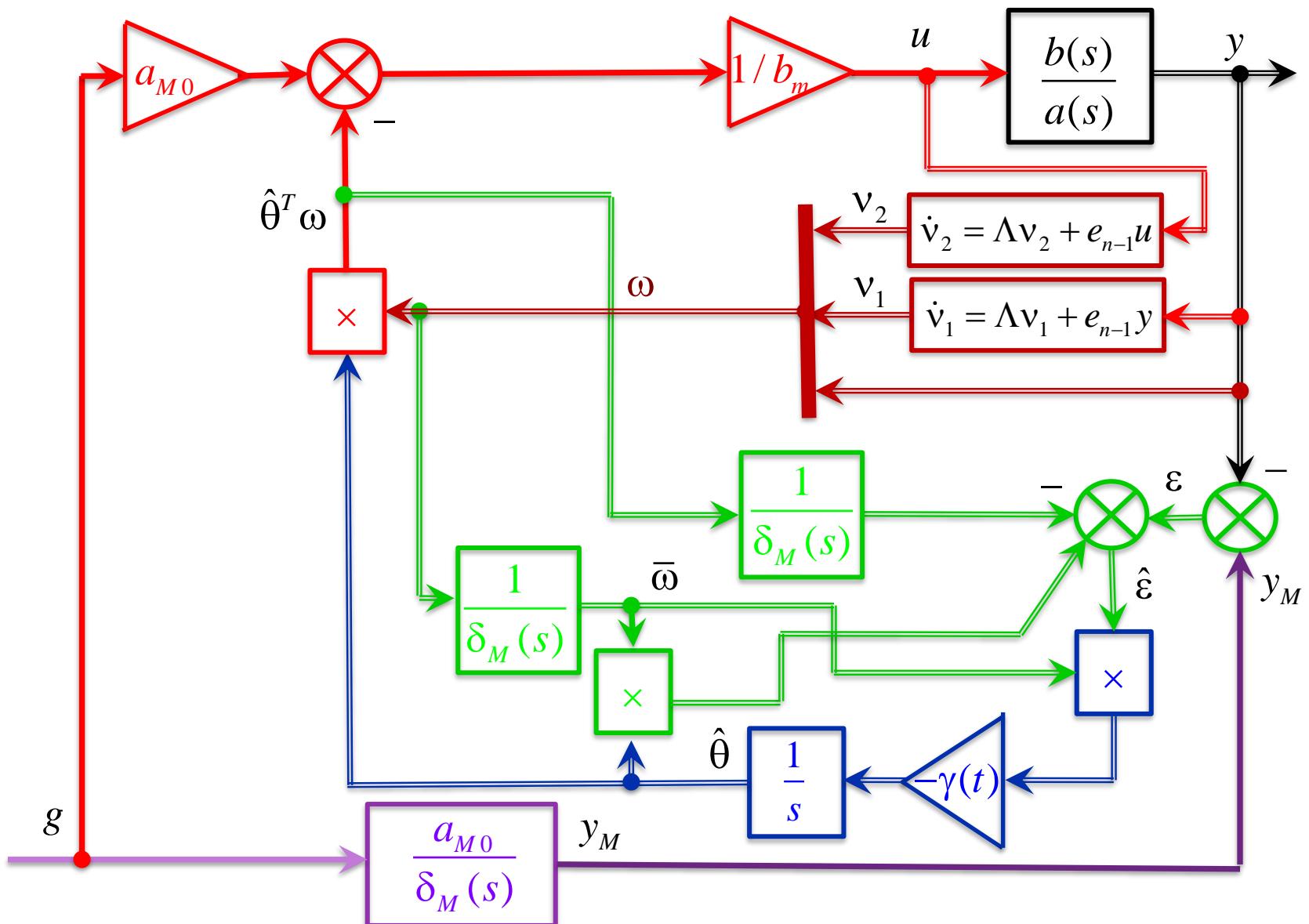
Augmented Error Solution Summary

Properties of the closed-loop system (See Lecture 6.3):

1. All the signals in the system are bounded;
2. The error $\hat{\varepsilon}(t)$ approaches zero asymptotically;
3. The norm $\|\tilde{\theta}(t)\|$ is nonincreasing;
4. The $\|\tilde{\theta}(t)\|$ approaches zero asymptotically if ω satisfies the Persistent Excitation condition;
5. The error $\varepsilon(t)$ approaches zero asymptotically;
6. Rate of parametric convergence is bounded due to normalization factor

$$\dot{\tilde{\theta}} = -\gamma_0 \frac{\bar{\omega} \bar{\omega}^T}{1 + \bar{\omega}^T \bar{\omega}} \tilde{\theta}$$





Simulation results

See Lecture 6, pages 6.55-6.57 (second order system)

High Order Tuner Solution

2. Obstacle of unknown parameters.

Based on the error equation

$$\varepsilon = \frac{1}{\delta_M(s)} [a_{M0}g - \theta^T \omega - b_m u] \quad (10.11)$$

and the certainty equivalence principle we design adjustable control

$$u = \frac{1}{b_m} \delta_M(s) \left[\frac{a_{M0}}{\delta_M(s)} [g] - \hat{\theta}^T \bar{\omega} \right] \quad (10.12)$$

that after replacement to (10.11) gives the **static** error model

(see Lecture 6, Pages 6.1-6.14)

$$\varepsilon = -\tilde{\theta}^T \bar{\omega} \quad (10.13)$$

with filtered regressor $\bar{\omega} = 1/\delta_M(s)[\omega]$.

Two problems arise immediately



**Control depends on the
high order time derivatives
 $\dot{\hat{\theta}}, \dots, \dot{\hat{\theta}}^{(p-1)}$ hidden in the term**

$$\delta_M(s) \left[\hat{\theta}^T \bar{\omega} \right]$$

$$u = \frac{1}{b_m} \delta_M(s) \left[\frac{a_{M0}}{\delta_M(s)} [g] - \hat{\theta}^T \bar{\omega} \right]$$



Two problems arise immediately

Control depends on the
high order time derivatives
 $\dot{\hat{\theta}}, \dots, \dot{\hat{\theta}}^{(p-1)}$ hidden in the term

$$\delta_M(s) [\hat{\theta}^T \bar{\omega}]$$

$$u = \frac{1}{b_m} \delta_M(s) \left[\frac{a_{M0}}{\delta_M(s)} [g] - \hat{\theta}^T \bar{\omega} \right]$$



Adaptation algorithm

$$\dot{\hat{\theta}} = -\gamma \bar{\omega} \varepsilon$$

based on the static error
model $\varepsilon = -\tilde{\theta}^T \bar{\omega}$ is unable
to generate $\dot{\hat{\theta}}, \dots, \dot{\hat{\theta}}^{(p-1)}$ for
control



High Order Tuner Solution

2. Obstacle of unknown parameters.

The High Order Tuner **(A. S. Morse, 1994)**

$$\begin{cases} \dot{\hat{\psi}}_i = -\bar{\omega}_i \varepsilon, \\ \dot{\eta}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) (\bar{A} \eta_i + \bar{b} \hat{\psi}_i), \\ \hat{\theta}_i = \bar{c}^T \eta_i, \end{cases} \quad \begin{array}{l} (10.14) \\ (10.15) \\ (10.16) \end{array}$$

where $\bar{\omega}_i, \hat{\psi}_i, \hat{\theta}_i$ $i = \overline{1, 2n-1}$ are the elements of vectors $\bar{\omega}, \hat{\psi}, \hat{\theta}$, respectively,

$$\mu \geq \frac{(2n-1)}{2} \left\| \bar{c} - P \bar{A}^{-1} \bar{b} \right\|^2$$

is a constant gain, $P = P^T \succ 0$ is the solution of Lyapunov equation $\bar{A}^T P + P \bar{A} = -2I_{p-1 \times p-1}$, $(\bar{c}^T, \bar{A}, \bar{b})$ is the matrices triple being the minimal realization of stable transfer function $\alpha(0)/\alpha(s)$.

High Order Tuner Solution

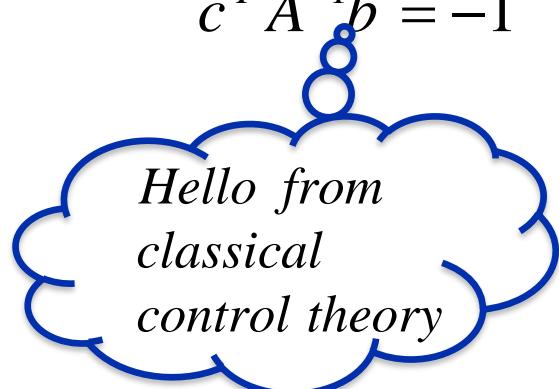
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The High Order Tuner **(A. S. Morse, 1994)**

$$\begin{cases} \dot{\hat{\psi}}_i = -\bar{\omega}_i \varepsilon, \\ \dot{\eta}_i = (1 + \mu \bar{\omega}^T \bar{\omega})(\bar{A}\eta_i + \bar{b}\hat{\psi}_i), \\ \hat{\theta}_i = \bar{c}^T \eta_i, \end{cases} \quad \begin{array}{l} (10.14) \\ (10.15) \\ (10.16) \end{array}$$

Since $\bar{c}^T \bar{A}^{i-1} \bar{b} = 0$, $i = 1, \rho - 2$, we obtain for the case $\rho = 2$:

$$\begin{aligned} \bar{c}^T \bar{A}^{-1} \bar{b} &= -1 & \dot{\hat{\theta}}_i &= \bar{c}^T \dot{\eta}_i = \bar{c}^T (1 + \mu \bar{\omega}^T \bar{\omega})(\bar{A}\eta_i + \bar{b}\hat{\psi}_i) \\ &\text{8} & \ddot{\hat{\theta}}_i &= 2\bar{c}^T \mu \bar{\omega}^T \dot{\bar{\omega}}(\bar{A}\eta_i + \bar{b}\hat{\psi}_i) + \\ &&& \bar{c}^T (1 + \mu \bar{\omega}^T \bar{\omega})(\bar{A}\dot{\eta}_i + \bar{b}\dot{\hat{\psi}}_i) = \\ &&& 2\bar{c}^T \mu \bar{\omega}^T \dot{\bar{\omega}}(\bar{A}\eta_i + \bar{b}\hat{\psi}_i) + \\ &&& (1 + \mu \bar{\omega}^T \bar{\omega})^2 (\bar{c}^T \bar{A}^2 \eta_i + \bar{c}^T \bar{A} \bar{b} \hat{\psi}_i) - (1 + \mu \bar{\omega}^T \bar{\omega}) \bar{c}^T \bar{b} \bar{\omega}_i \varepsilon \end{aligned}$$



High Order Tuner Solution

Remark 10.1. One-syllable words about high order tuner

Introduce auxiliary signals

$$z_i = \eta_i + \bar{A}^{-1} \bar{b} \hat{\psi}_i \quad (10.17)$$

High Order Tuner Solution

Remark 10.1. One-syllable words about high order tuner

Introduce auxiliary signals

$$z_i = \eta_i + \bar{A}^{-1} \bar{b} \hat{\psi}_i \quad (10.17)$$

and calculate their time derivatives:

$$\dot{z}_i = \dot{\eta}_i + \bar{A}^{-1} \bar{b} \dot{\hat{\psi}}_i \quad (10.15) \quad \dot{\eta}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) (\bar{A} \eta_i + \bar{b} \hat{\psi}_i) \quad (10.14)$$

High Order Tuner Solution

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High Order Tuner Solution

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$$z_i = \eta_i + \bar{A}^{-1} \bar{b} \hat{\psi}_i \quad (10.17)$$

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$$\dot{z}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon. \quad (10.18)$$

High Order Tuner Solution

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$$\dot{z}_i = (1 + \mu\bar{\omega}^T\bar{\omega})\bar{A}z_i - \bar{A}^{-1}\bar{b}\bar{\omega}_i\varepsilon. \quad (10.18)$$

Multiply (10.17) by \bar{c}^T :

$$\bar{c}^T z_i = \underbrace{\bar{c}^T \eta_i}_{\text{red wavy arrow}} + \underbrace{\bar{c}^T \bar{A}^{-1}\bar{b}\hat{\psi}_i}_{\text{blue bracket}}$$

$$\hat{\theta}_i = \bar{c}^T \eta_i \quad (10.16)$$

$$\bar{c}^T \bar{A}^{-1}\bar{b} = -\alpha(0) / \alpha(s) \Big|_{s=0} = -1$$


High Order Tuner Solution

Remark 10.1. One-syllable words about high order tuner

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High Order Tuner Solution

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Multiply (10.17) by \bar{c}^T :

$$\begin{aligned} \bar{c}^T z_i &= \hat{\theta}_i - \hat{\psi}_i \\ \tilde{\theta}_i &= \theta_i - \hat{\theta}_i \qquad \qquad \qquad \tilde{\psi}_i = \theta_i - \hat{\psi}_i \end{aligned}$$


High Order Tuner Solution

Remark 10.1. One-syllable words about high order tuner

Introduce auxiliary signals

$$z_i = \eta_i + \bar{A}^{-1} \bar{b} \hat{\psi}_i \quad (10.17)$$

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Multiply (10.17) by \bar{c}^T :

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High Order Tuner Solution

Remark 10.1. One-syllable words about high order tuner

Introduce auxiliary signals

$$z_i = \eta_i + \bar{A}^{-1} \bar{b} \hat{\psi}_i \quad (10.17)$$

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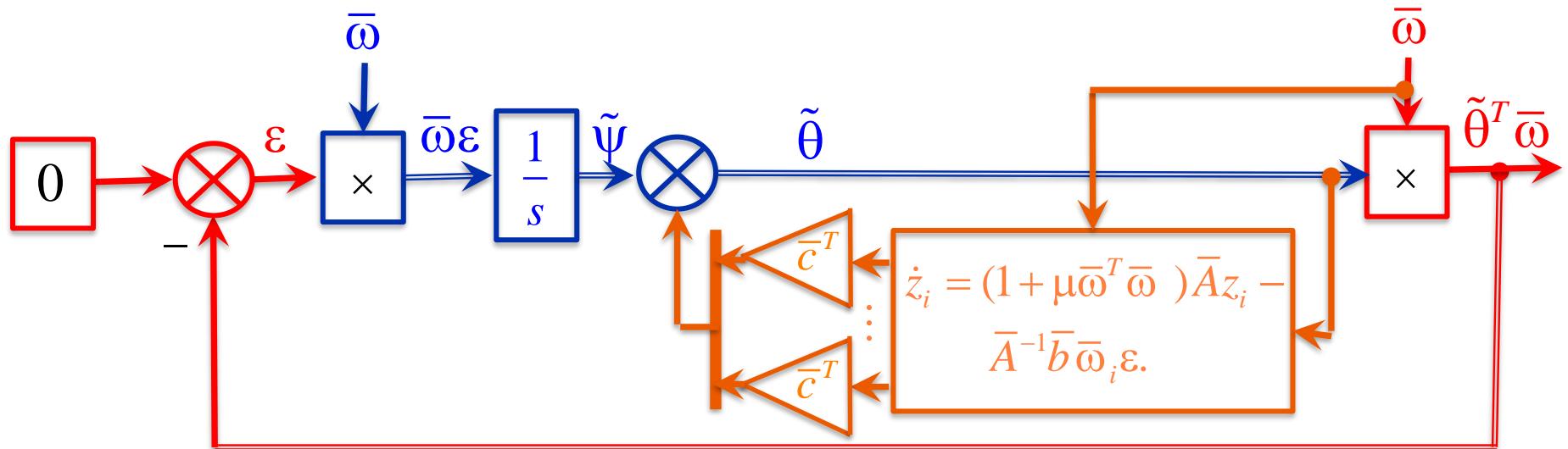
$$\dot{z}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon. \quad (10.18)$$

Multiply (10.17) by \bar{c}^T :

$$\tilde{\theta}_i = \tilde{\psi}_i - \bar{c}^T z_i \quad (10.19)$$

High Order Tuner Solution

Remark 10.1. One-syllable words about high order tuner



Error model

$$\varepsilon = -\tilde{\theta}^T \bar{\omega} \quad (10.13)$$

Auxiliary dynamics

$$\dot{z}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \quad (10.18)$$

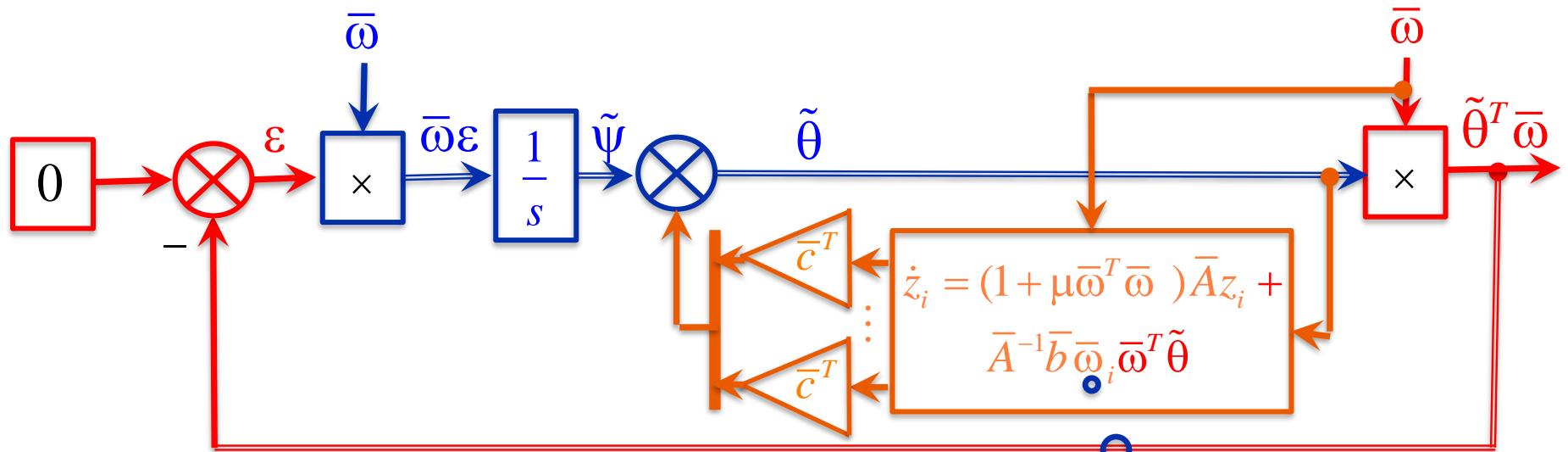
Parametric error model

$$\tilde{\theta}_i = \tilde{\psi}_i - \bar{c}^T z_i \quad (10.19)$$

$$\dot{\tilde{\psi}}_i = \bar{\omega}_i \varepsilon \quad (10.14)$$

High Order Tuner Solution

Remark 10.1. One-syllable words about high order tuner



Error model

$$\varepsilon = -\tilde{\theta}^T \bar{\omega} \quad (10.13)$$

Auxiliary dynamics

$$\dot{z}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \bar{\omega}^T \tilde{\theta} \quad (10.18)$$

Parametric error model

$$\begin{aligned} \tilde{\theta}_i &= \tilde{\Psi}_i - \bar{c}^T z_i \\ \dot{\tilde{\Psi}}_i &= \bar{\omega}_i \varepsilon \end{aligned} \quad (10.19)$$

Filters with expanding cutoff frequency

Lyapunov function?

Error model

$$\varepsilon = -\tilde{\theta}^T \bar{\omega} \quad (10.13)$$

Auxiliary dynamics

$$\dot{z}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \quad (10.18)$$

Parametric error
model

$$\tilde{\theta}_i = \tilde{\Psi}_i - \bar{c}^T z_i \quad (10.19)$$

$$\dot{\tilde{\Psi}}_i = \bar{\omega}_i \varepsilon \quad (10.14)$$

$$V = \frac{1}{2} \sum_{i=1}^{2n-1} z_i^T P z_i + \frac{1}{2} \sum_{i=1}^{2n-1} \tilde{\Psi}_i^2 \quad (10.20)$$

Error model

$$\varepsilon = -\tilde{\theta}^T \bar{\omega} \quad (10.13)$$

Auxiliary dynamics

$$\dot{z}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \quad (10.18)$$

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$$\dot{V} = \frac{1}{2} \sum_{i=1}^{2n-1} \dot{z}_i^T P z_i + \frac{1}{2} \sum_{i=1}^{2n-1} z_i^T P \dot{z}_i + \sum_{i=1}^{2n-1} \tilde{\psi}_i \dot{\tilde{\psi}}_i$$

Error model

Auxiliary dynamics

Parametric error
model

$$\varepsilon = -\tilde{\theta}^T \bar{\omega} \quad (10.13)$$

$$\dot{z}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \quad (10.18)$$

$$\dot{\tilde{\theta}}_i = \tilde{\psi}_i - \bar{c}^T z_i \quad (10.19)$$

$$\dot{\tilde{\psi}}_i = \bar{\omega}_i \varepsilon \quad (10.14)$$

$$V = \frac{1}{2} \sum_{i=1}^{2n-1} z_i^T P z_i + \frac{1}{2} \sum_{i=1}^{2n-1} \tilde{\Psi}_i^2 \quad (10.20)$$

$$\begin{aligned} \dot{V} &= \frac{1}{2} \sum_{i=1}^{2n-1} \dot{z}_i^T P z_i + \frac{1}{2} \sum_{i=1}^{2n-1} z_i^T P \dot{z}_i + \sum_{i=1}^{2n-1} \tilde{\Psi}_i \dot{\tilde{\Psi}}_i = \\ &\frac{1}{2} \sum_{i=1}^{2n-1} \left((1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \right)^T P z_i + \frac{1}{2} \sum_{i=1}^{2n-1} z_i^T P \left((1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \right) + \\ &\sum_{i=1}^{2n-1} \tilde{\Psi}_i \bar{\omega}_i \varepsilon \end{aligned}$$

Error model

$$\varepsilon = -\tilde{\theta}^T \bar{\omega} \quad (10.13)$$

Auxiliary dynamics

$$\dot{z}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \quad (10.18)$$

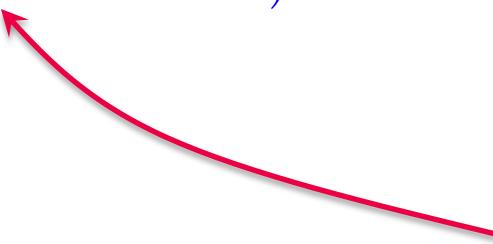
Parametric error model

$$\tilde{\theta}_i = \tilde{\Psi}_i - \bar{c}^T z_i \quad (10.19)$$

$$\dot{\tilde{\Psi}}_i = \bar{\omega}_i \varepsilon \quad (10.14)$$

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Error model

$$\varepsilon = -\tilde{\theta}^T \bar{\omega} \quad (10.13)$$

Auxiliary dynamics

$$\dot{z}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \quad (10.18)$$

Parametric error model

$$\tilde{\theta}_i = \tilde{\Psi}_i - \bar{c}^T z_i \quad (10.19)$$

$$\dot{\tilde{\Psi}}_i = \bar{\omega}_i \varepsilon \quad (10.14)$$

$$V = \frac{1}{2} \sum_{i=1}^{2n-1} z_i^T P z_i + \frac{1}{2} \sum_{i=1}^{2n-1} \tilde{\psi}_i^2 \quad (10.20)$$

$$\dot{V} = \frac{1}{2} \sum_{i=1}^{2n-1} \dot{z}_i^T P z_i + \frac{1}{2} \sum_{i=1}^{2n-1} z_i^T P \dot{z}_i + \sum_{i=1}^{2n-1} \tilde{\psi}_i \dot{\tilde{\psi}}_i =$$

$$\frac{1}{2} \sum_{i=1}^{2n-1} \left((1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \right)^T P z_i + \frac{1}{2} \sum_{i=1}^{2n-1} z_i^T P \left((1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \right) +$$

$$\sum_{i=1}^{2n-1} \left(\tilde{\theta}_i \bar{\omega}_i \varepsilon + \bar{c}^T z_i \bar{\omega}_i \varepsilon \right) = (1 + \mu \bar{\omega}^T \bar{\omega}) \frac{1}{2} \sum_{i=1}^{2n-1} z_i^T \underbrace{\left(\bar{A}^T P + P \bar{A} \right)}_{-2I} z_i -$$

$$\frac{1}{2} \sum_{i=1}^{2n-1} \left[\left(\bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \right)^T P z_i + z_i^T P \left(\bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \right) \right] - \varepsilon^2 + \sum_{i=1}^{2n-1} \bar{c}^T z_i \bar{\omega}_i \varepsilon$$

Error model

$$\varepsilon = -\tilde{\theta}^T \bar{\omega} \quad (10.13)$$

Auxiliary dynamics

$$\dot{z}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \quad (10.18)$$

Parametric error model

$$\tilde{\theta}_i = \tilde{\psi}_i - \bar{c}^T z_i \quad (10.19)$$

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$$V = \frac{1}{2} \sum_{i=1}^{2n-1} z_i^T P z_i + \frac{1}{2} \sum_{i=1}^{2n-1} \tilde{\Psi}_i^2 \quad (10.20)$$

$$\dot{V} = -(1 + \mu \bar{\omega}^T \bar{\omega}) \sum_{i=1}^{2n-1} z_i^T z_i - \sum_{i=1}^{2n-1} [z_i^T P \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon] - \varepsilon^2 + \sum_{i=1}^{2n-1} \bar{c}^T z_i \bar{\omega}_i \varepsilon$$

Error model

$$\varepsilon = -\tilde{\theta}^T \bar{\omega} \quad (10.13)$$

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$$\dot{z}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \quad (10.18)$$

Parametric error model

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$$V = \frac{1}{2} \sum_{i=1}^{2n-1} z_i^T P z_i + \frac{1}{2} \sum_{i=1}^{2n-1} \tilde{\Psi}_i^2 \quad (10.20)$$

$$\begin{aligned} \dot{V} = & -(1 + \mu \bar{\omega}^T \bar{\omega}) \sum_{i=1}^{2n-1} z_i^T z_i - \sum_{i=1}^{2n-1} \left[z_i^T P \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \right] - \varepsilon^2 + \sum_{i=1}^{2n-1} \bar{c}^T z_i \bar{\omega}_i \varepsilon \leq \\ & - \sum_{i=1}^{2n-1} z_i^T z_i - \mu \sum_{i=1}^{2n-1} \bar{\omega}_i^2 \|z_i\|^2 + \sum_{i=1}^{2n-1} \|z_i\| \|\bar{c} - P \bar{A}^{-1} \bar{b}\| \|\bar{\omega}_i\| |\varepsilon| - \frac{\varepsilon^2}{2} - \frac{\varepsilon^2}{2} \end{aligned}$$

Error model

$$\varepsilon = -\tilde{\theta}^T \bar{\omega} \quad (10.13)$$

Auxiliary dynamics

$$\dot{z}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \quad (10.18)$$

Parametric error model

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$$\dot{\tilde{\Psi}}_i = \bar{\omega}_i \varepsilon \quad (10.14)$$

$$V = \frac{1}{2} \sum_{i=1}^{2n-1} z_i^T P z_i + \frac{1}{2} \sum_{i=1}^{2n-1} \tilde{\Psi}_i^2 \quad (10.20)$$

$$\begin{aligned} \dot{V} &= -(1 + \mu \bar{\omega}^T \bar{\omega}) \sum_{i=1}^{2n-1} z_i^T z_i - \sum_{i=1}^{2n-1} \left[z_i^T P \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \right] - \varepsilon^2 + \sum_{i=1}^{2n-1} \bar{c}^T z_i \bar{\omega}_i \varepsilon \leq \\ &\leq -\sum_{i=1}^{2n-1} z_i^T z_i - \mu \sum_{i=1}^{2n-1} \bar{\omega}_i^2 \|z_i\|^2 + \sum_{i=1}^{2n-1} \|z_i\| \|\bar{c} - P \bar{A}^{-1} \bar{b}\| \|\bar{\omega}_i\| |\varepsilon| - \frac{\varepsilon^2}{2} - \frac{\varepsilon^2}{2} = \\ &= -\sum_{i=1}^{2n-1} z_i^T z_i - \frac{\varepsilon^2}{2} - \sum_{i=1}^{2n-1} \left(\mu \bar{\omega}_i^2 \|z_i\|^2 - \|\bar{c} - P \bar{A}^{-1} \bar{b}\| \|z_i\| \|\bar{\omega}_i\| |\varepsilon| + \frac{\varepsilon^2}{2(2n-1)} \right) \end{aligned}$$

Error model

$$\varepsilon = -\tilde{\theta}^T \bar{\omega} \quad (10.13)$$

Auxiliary dynamics

$$\dot{z}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) \bar{A} z_i - \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \quad (10.18)$$

Parametric error model

$$\tilde{\theta}_i = \tilde{\Psi}_i - \bar{c}^T z_i \quad (10.19)$$

$$\dot{\tilde{\Psi}}_i = \bar{\omega}_i \varepsilon \quad (10.14)$$

$$V = \frac{1}{2} \sum_{i=1}^{2n-1} z_i^T P z_i + \frac{1}{2} \sum_{i=1}^{2n-1} \tilde{\Psi}_i^2 \quad (10.20)$$

$$\begin{aligned} \dot{V} &= -(1 + \mu \bar{\omega}^T \bar{\omega}) \sum_{i=1}^{2n-1} z_i^T z_i - \sum_{i=1}^{2n-1} \left[z_i^T P \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \right] - \varepsilon^2 + \sum_{i=1}^{2n-1} \bar{c}^T z_i \bar{\omega}_i \varepsilon \leq \\ &\leq - \sum_{i=1}^{2n-1} z_i^T z_i - \mu \sum_{i=1}^{2n-1} \bar{\omega}_i^2 \|z_i\|^2 + \sum_{i=1}^{2n-1} \|z_i\| \|\bar{c} - P \bar{A}^{-1} \bar{b}\| \|\bar{\omega}_i\| |\varepsilon| - \frac{\varepsilon^2}{2} - \frac{\varepsilon^2}{2} = \\ &= - \sum_{i=1}^{2n-1} z_i^T z_i - \frac{\varepsilon^2}{2} - \sum_{i=1}^{2n-1} \left(\mu \bar{\omega}_i^2 \|z_i\|^2 - \|\bar{c} - P \bar{A}^{-1} \bar{b}\| \|z_i\| \|\bar{\omega}_i\| |\varepsilon| + \frac{\varepsilon^2}{2(2n-1)} \right) \end{aligned}$$

What about parameter μ ?

$$V = \frac{1}{2} \sum_{i=1}^{2n-1} z_i^T P z_i + \frac{1}{2} \sum_{i=1}^{2n-1} \tilde{\Psi}_i^2 \quad (10.20)$$

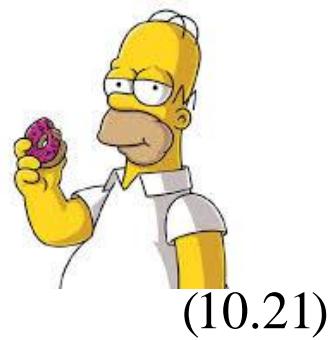
$$\begin{aligned} \dot{V} &= -(1 + \mu \bar{\omega}^T \bar{\omega}) \sum_{i=1}^{2n-1} z_i^T z_i - \sum_{i=1}^{2n-1} \left[z_i^T P \bar{A}^{-1} \bar{b} \bar{\omega}_i \varepsilon \right] - \varepsilon^2 + \sum_{i=1}^{2n-1} \bar{c}^T z_i \bar{\omega}_i \varepsilon \leq \\ &\leq -\sum_{i=1}^{2n-1} z_i^T z_i - \mu \sum_{i=1}^{2n-1} \bar{\omega}_i^2 \|z_i\|^2 + \sum_{i=1}^{2n-1} \|z_i\| \|\bar{c} - P \bar{A}^{-1} \bar{b}\| \|\bar{\omega}_i\| |\varepsilon| - \frac{\varepsilon^2}{2} - \frac{\varepsilon^2}{2} = \\ &= -\sum_{i=1}^{2n-1} z_i^T z_i - \frac{\varepsilon^2}{2} - \sum_{i=1}^{2n-1} \left(\mu \bar{\omega}_i^2 \|z_i\|^2 - \|\bar{c} - P \bar{A}^{-1} \bar{b}\| \|z_i\| \|\bar{\omega}_i\| |\varepsilon| + \frac{\varepsilon^2}{2(2n-1)} \right) \end{aligned}$$

If

$$\mu \geq \frac{(2n-1)}{2} \|\bar{c} - P \bar{A}^{-1} \bar{b}\|^2.$$

then

$$\dot{V} \leq -\sum_{i=1}^{2n-1} z_i^T z_i - \frac{\varepsilon^2}{2} < 0. \quad (10.21)$$



High Order Tuner Solution Summary

Adjustable control

$$u = \frac{1}{b_m} \delta_M(s) \left[\frac{a_{M0}}{\delta_M(s)} [g] - \hat{\theta}^T \bar{\omega} \right] \quad (10.12)$$

Regressor

$$\omega = \text{col}(y, v_1, v_2)$$

Filters

$$\begin{cases} \dot{v}_1 = \Lambda v_1 + e_{n-1} y, \\ \dot{v}_2 = \Lambda v_2 + e_{n-1} u \end{cases} \quad (9.4)$$

$$\begin{cases} \dot{v}_1 = \Lambda v_1 + e_{n-1} y, \\ \dot{v}_2 = \Lambda v_2 + e_{n-1} u \end{cases} \quad (9.5)$$

Adaptation Algorithm

$$\begin{cases} \dot{\hat{\psi}}_i = -\bar{\omega}_i \varepsilon, \\ \dot{\eta}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) (\bar{A} \eta_i + \bar{b} \hat{\psi}_i), \\ \hat{\theta}_i = \bar{c}^T \eta_i \end{cases} \quad (10.14)$$

$$\begin{cases} \dot{\hat{\psi}}_i = -\bar{\omega}_i \varepsilon, \\ \dot{\eta}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) (\bar{A} \eta_i + \bar{b} \hat{\psi}_i), \\ \hat{\theta}_i = \bar{c}^T \eta_i \end{cases} \quad (10.15)$$

$$\begin{cases} \dot{\hat{\psi}}_i = -\bar{\omega}_i \varepsilon, \\ \dot{\eta}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) (\bar{A} \eta_i + \bar{b} \hat{\psi}_i), \\ \hat{\theta}_i = \bar{c}^T \eta_i \end{cases} \quad (10.16)$$

Error

$$\varepsilon = y_M - y$$

Filtered regressor

$$\bar{\omega} = \frac{1}{\delta_M(s)} [\omega] \quad \text{Gain} \quad \mu \geq \frac{(2n-1)}{2} \|\bar{c} - P \bar{A}^{-1} \bar{b}\|^2$$

High Order Tuner Solution Summary

Adjustable control

$$u = \frac{1}{b_m} \delta_M(s) \left[\frac{a_{M0}}{\delta_M(s)} [g] - \hat{\theta}^T \bar{\omega} \right] \quad (10.12)$$

Regressor

$$\omega = \text{col}(y, v_1, v_2)$$

Filters

$$\begin{cases} \dot{v}_1 = \Lambda v_1 + e_{n-1} y, \\ \dot{v}_2 = \Lambda v_2 + e_{n-1} u \end{cases} \quad (9.4)$$

$$\begin{cases} \dot{v}_1 = \Lambda v_1 + e_{n-1} y, \\ \dot{v}_2 = \Lambda v_2 + e_{n-1} u \end{cases} \quad (9.5)$$

Adaptation Algorithm

High Order Time Derivatives ($\rho = 2$)

$$\begin{cases} \dot{\hat{\psi}}_i = -\bar{\omega}_i \varepsilon, \\ \dot{\eta}_i = (1 + \mu \bar{\omega}^T \bar{\omega}) (\bar{A} \eta_i + \bar{b} \hat{\psi}_i), \\ \dot{\hat{\theta}}_i = \bar{c}^T \eta_i \end{cases} \quad \begin{aligned} \dot{\hat{\theta}}_i &= \bar{c}^T \dot{\eta}_i = \bar{c}^T (1 + \mu \bar{\omega}^T \bar{\omega}) (\bar{A} \eta_i + \bar{b} \hat{\psi}_i) \\ \ddot{\hat{\theta}}_i &= 2 \bar{c}^T \mu \bar{\omega}^T \bar{\omega} (\bar{A} \eta_i + \bar{b} \hat{\psi}_i) + \\ &\quad (1 + \mu \bar{\omega}^T \bar{\omega})^2 \bar{c}^T (\bar{A}^2 \eta_i + \bar{A} \bar{b} \hat{\psi}_i - \bar{b} \bar{\omega}_i \varepsilon) \end{aligned}$$

Error

$$\varepsilon = y_M - y$$

Filtered regressor

$$\bar{\omega} = \frac{1}{\delta_M(s)} [\omega]$$

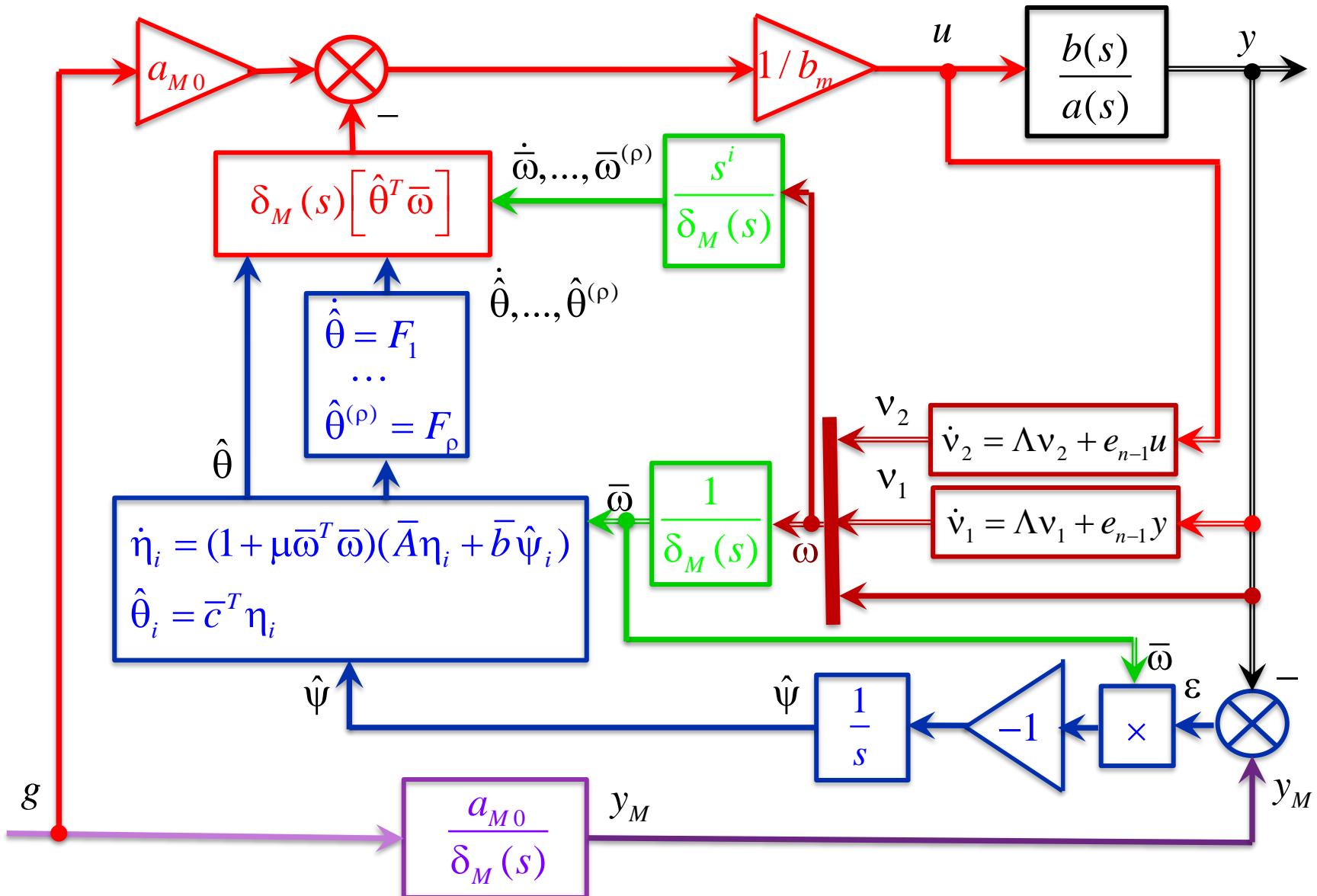
$$\text{Gain } \mu \geq \frac{(2n-1)}{2} \left\| \bar{c} - P \bar{A}^{-1} \bar{b} \right\|^2$$

High Order Tuner Solution Summary

Properties of the closed-loop system:

1. All the signals in the system are bounded;
2. The norm $\|z(t)\|$ approaches zero asymptotically;
3. The error $\varepsilon(t)$ approaches zero asymptotically;
4. The $\|\tilde{\theta}(t)\|$ approaches zero asymptotically if ω satisfies the Persistent Excitation condition **(R. Ortega, 1995)** ;
5. The adaptation algorithm does not have normalization factor restricting the transient performance of the adaptation law

$$\begin{cases} \dot{\hat{\psi}}_i = -\bar{\omega}_i \varepsilon, \\ \dot{\eta}_i = (1 + \mu \bar{\omega}^T \bar{\omega})(\bar{A}\eta_i + \bar{b}\hat{\psi}_i), \\ \hat{\theta}_i = \bar{c}^T \eta_i. \end{cases}$$



Simulation results (conditions from Lecture 6.3)

Plant

$$\ddot{y} + a_1 \dot{y} + a_0 y = u$$

Unknown parameters

$$a_0 = 1, \quad a_1 = 2$$

Reference model

$$\ddot{y}_M + 5\dot{y}_M + 6y_M = 6g$$

$$W_M(s) = \frac{6}{s^2 + 5s + 6} = \frac{6}{\delta_M(s)}$$

Regressor

$$\omega = \text{col} \left(y, \frac{1}{s+8}[y], \frac{1}{s+8}[u] \right)$$

Filtered regressor

$$\bar{\omega} = \frac{1}{s^2 + 5s + 6}[\omega] = \frac{1}{\delta_M(s)}[\omega]$$

Augmented Error Solution. Simulation results

Augmented error $\hat{\varepsilon} = \varepsilon + \hat{\theta}^T \bar{\omega} - \frac{1}{s^2 + 5s + 6} [\hat{\theta}^T \omega]$

Error $\varepsilon = y_M - y$

Adjustable control $u = -\hat{\theta}^T \omega + 6g$

Adaptation algorithm $\dot{\hat{\theta}} = -\frac{100}{1 + \bar{\omega}^T(t) \bar{\omega}(t)} \bar{\omega} \hat{\varepsilon}$

Gain $\gamma = 100$

High Order Tuner Solution. Simulation results

Adjustable control

$$u = -\ddot{\hat{\theta}}^T \bar{\omega} - 2\dot{\hat{\theta}}^T \dot{\bar{\omega}} - \hat{\theta}^T \ddot{\bar{\omega}} - 5\dot{\hat{\theta}}^T \bar{\omega} - 5\hat{\theta}^T \dot{\bar{\omega}} - 6\hat{\theta}^T \bar{\omega} + 6g$$

Adaptation algorithm

$$\begin{cases} \dot{\hat{\psi}}_i = -\bar{\omega}_i \varepsilon, & i = 1, 2, 3, \\ \dot{\eta}_i = (1 + \mu \bar{\omega}^T \bar{\omega})(-5\eta_i + 5\hat{\psi}_i), \\ \dot{\hat{\theta}}_i = \eta_i \end{cases}$$

High order time derivatives

$$\begin{aligned} \dot{\hat{\theta}}_i &= (1 + \mu \bar{\omega}^T \bar{\omega})(-5\eta_i + 5\hat{\psi}_i) \\ \ddot{\hat{\theta}}_i &= 2\mu \bar{\omega}^T \dot{\bar{\omega}}(-5\eta_i + 5\hat{\psi}_i) + \\ &\quad (1 + \mu \bar{\omega}^T \bar{\omega})^2 \bar{c}^T (25\eta_i - 25\hat{\psi}_i - 5\bar{\omega}_i \varepsilon) \end{aligned}$$

*Filtered regressor
and its derivatives*

$$\bar{\omega} = \frac{1}{s^2 + 5s + 6} [\omega],$$

$$\dot{\bar{\omega}} = \frac{s}{s^2 + 5s + 6} [\omega], \quad \ddot{\bar{\omega}} = \frac{s^2}{s^2 + 5s + 6} [\omega]$$

Error

$$\varepsilon = y_M - y$$

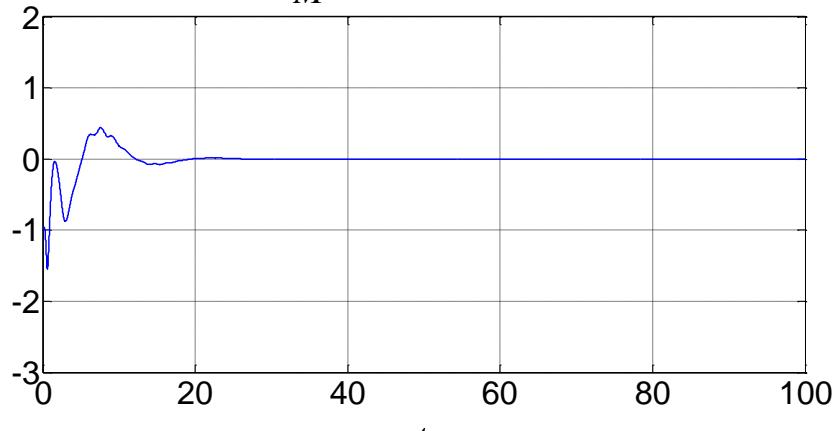
Gain

$$\mu = 54$$

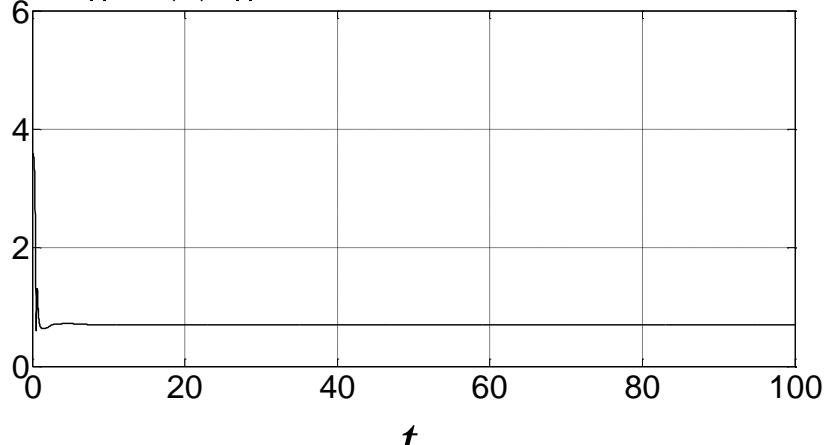
Simulation results

$$g(t) = 15\sin(2t) \text{ (NOT persistently exciting)}$$

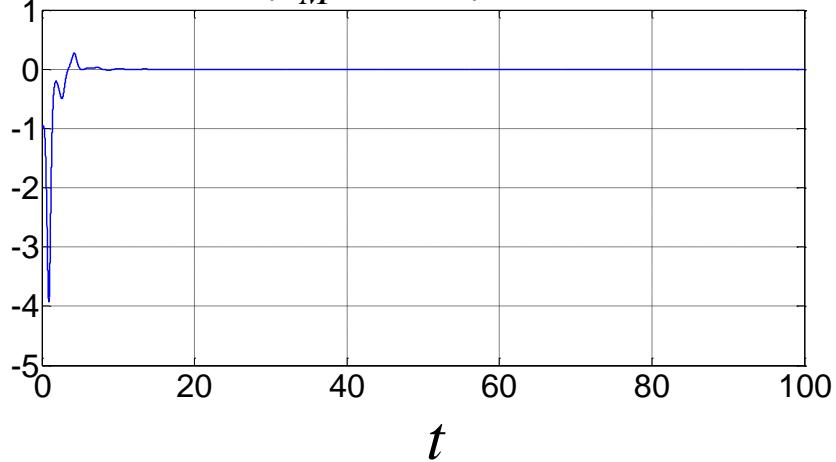
$$\varepsilon(t) = y_M(t) - y(t)$$



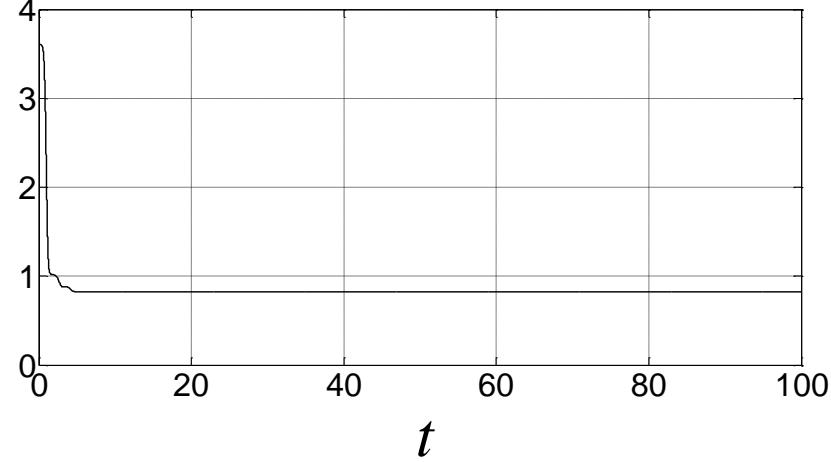
$$\|\tilde{\theta}(t)\|$$



$$\varepsilon(t) = y_M(t) - y(t)$$



$$\|\tilde{\theta}(t)\|$$



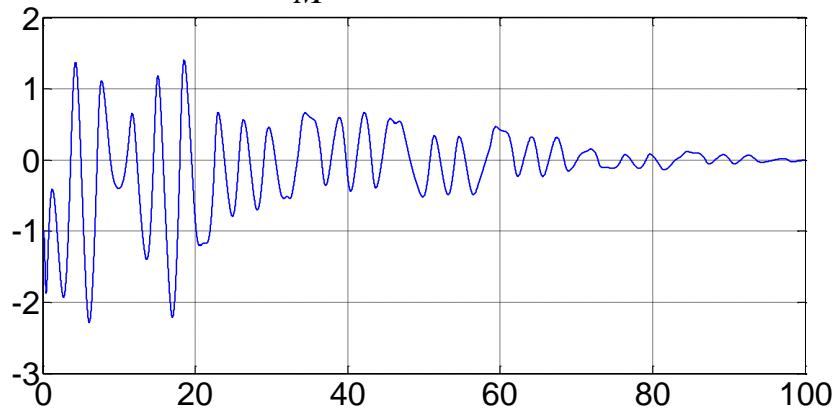
AE

HOT

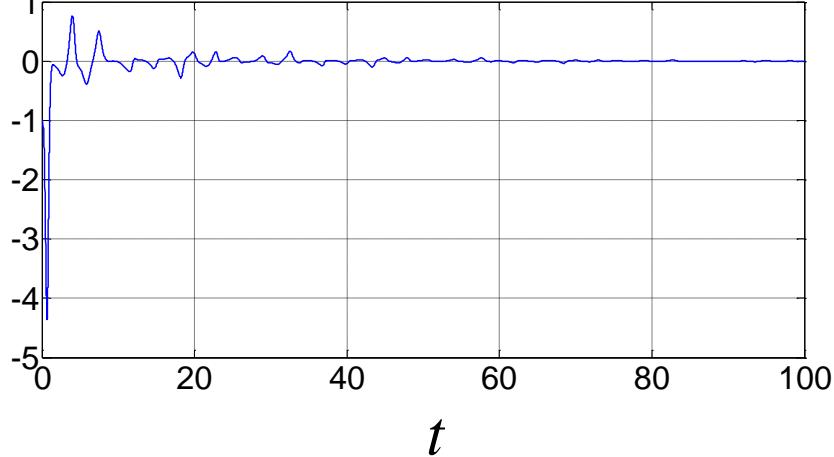
Simulation results

$$g(t) = 15\sin(2t) + 10\cos(0.25t) \text{ (persistently exciting)}$$

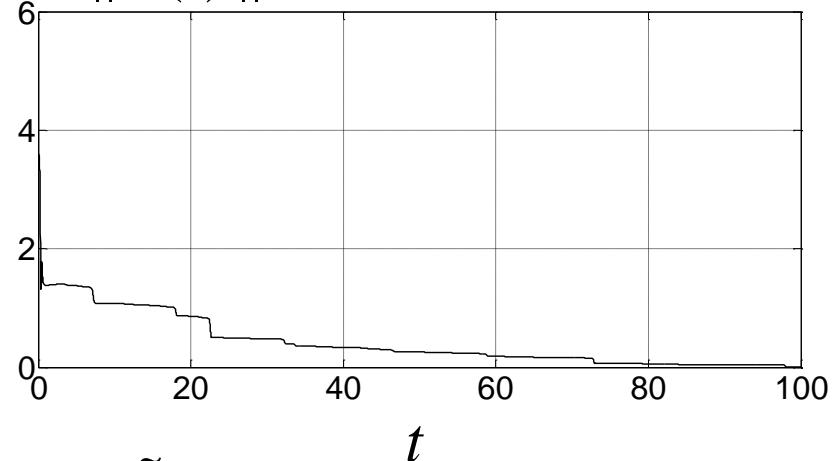
$$\varepsilon(t) = y_M(t) - y(t)$$



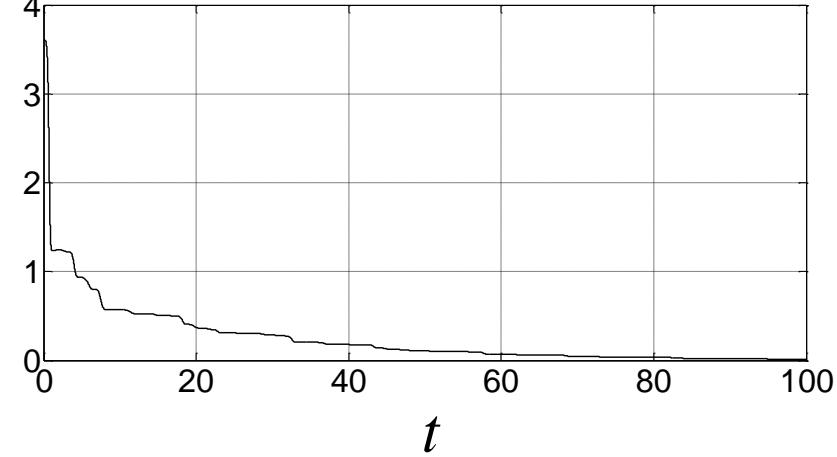
$$\varepsilon(t) = y_M(t) - y(t)$$



$$\|\tilde{\theta}(t)\|$$



$$\|\tilde{\theta}(t)\|$$



AE

HOT