Regression

Lesson 6 with *Ian Carroll*

Lesson Objectives

- Learn several functions and packages for statistical modeling
- Understand the “formula” part of model specification
- Introduce increasingly complex, still “linear”, regression models

Lesson Non-objectives

Pay no attention to these very important topics.

- Experimental/sampling design
- Model validation
- Hypothesis tests
- Model comparison

Specific Achievements

- Write a `lm` formula with an interaction term
- Use a non-gaussian family in the `glm` function
- Add a “random effect” to a `lmer` formula

Dataset

The dataset you will plot is an example of Public Use Microdata Sample (PUMS) produced by the US Census Beaurea.

worksheet-6.R

```
library(readr)
library(dplyr)

person <- read_csv(
  file = 'data/census_pums/sample.csv',
  col_types = cols_only(
    AGEP = 'i',  # Age
    WAGP = 'd',  # Wages or salary income past 12 months
    SCHL = 'i',  # Educational attainment
    SEX = 'f',   # Sex
    OCCP = 'f',  # Occupation recode based on 2010 OCC codes
    WKHP = 'i')) # Usual hours worked per week past 12 months
```

This CSV file contains individuals’ anonymized responses to the 5 Year American Community Survey (ACS) completed in 2017. There are over a hundred variables giving individual level data on household members income, education, employment, ethnicity, and much more. The technical documentation provided the data definitions quoted above in comments.

Collapse the education attainment levels for this lesson, and remove non wage-earners as well as the “top coded” individuals whose income is annonymized.

worksheet-6.R

```
person <- within(person, {
  SCHL <- factor(SCHL)
  levels(SCHL) <- list(
    'Incomplete' = c(1:15),
    'High School' = 16,
    'College Credit' = 17:20,
    'Bachelor\'s' = 21,
```

```
    'Master\'s' = 22:23,
    'Doctorate' = 24)}) %>%
  filter(
    WAGP > 0,
    WAGP < max(WAGP, na.rm = TRUE))
```

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Formula Notation

The basic function for fitting a regression in R is the `lm()` function, standing for linear model.

worksheet-6.R

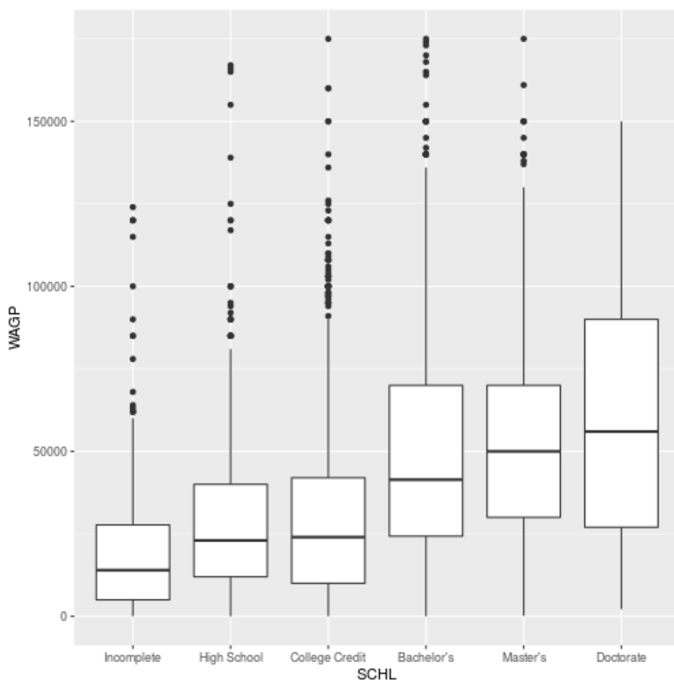
```
fit <- lm(
  formula = WAGP ~ SCHL,
  data = person)
```

The `lm()` function takes a formula argument and a data argument, and computes the best fitting linear model (i.e. determines coefficients that minimize the sum of squared residuals.

Data visualizations can help confirm intuition, but only with very simple models, typically with just one or two “fixed effects”.

Console

```
> library(ggplot2)
>
> ggplot(person,
+   aes(x = SCHL, y = WAGP)) +
+   geom_boxplot()
```



Function, Formula, and Data

The developers of R created an approach to statistical analysis that is both concise and flexible from the users perspective, while remaining precise and well-specified for a particular fitting procedure.

The researcher contemplating a new regression analysis faces three initial challenges:

1. Choose the function that implements a suitable fitting procedure.
2. Write down the regression model using a formula expression combining variable names with algebra-like operators.

3. Build a data frame with columns matching these variables that have suitable data types for the chosen fitting procedure.

The formula expressions are usually very concise, in part because the function chosen to implement the fitting procedure extracts much necessary information from the provided data frame. For example, whether a variable is a factor or numeric is not specified in the formula because it is specified in the data.

Formula Mini-language(s)

- "base R" function `lm()`
- "base R" function `glm()`
- lme4 function `lmer()`
- lme4 function `glmer()`

Across different packages in R, the formula "mini-language" has evolved to allow complex model specification. Each package adds syntax to the formula or new arguments to the fitting function. The `lm` function admits only the simplest kinds of expressions, while the lme4 packagehas evolved this "mini-language" to allow specification of hierarchichal models.

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Linear Models

The formula requires a response variable left of a `~` and any number of predictors to its right.

Formula	Description
`y ~ a`	constant and one predictor
`y ~ a + b`	constant and two predictors
`y ~ a + b + a:b`	constant and two predictors and their interaction

The constant (i.e. intercept) can be explicitly removed, although this is rarely needed in practice. Assume an intercept is fitted unless its absence is explicitly noted.

Formula	Description
`y ~ -1 + a`	one predictor with no constant

More or higher-order interactions are included with a shorthand that is sort of consistent with the expression's algebraic meaning.

Formula	Description
`y ~ a*b`	all combinations of two variables
`y ~ (a + b)^2`	equivalent to `y ~ a*b`
`y ~ (a + b + ... )^n`	all combinations of all predictors up to order `n`

Data transformation functions are allowed within the formula expression.

Formula	Description
`y ~ log(a)`	log transformed variable
`y ~ sin(a)`	periodic function of a variable
`y ~ I(a*b)`	product of variables, protected by `I` from expansion

The `WAGP` variable is always positive and includes some values that are many orders of magnitude larger than the mean.

worksheet-6.R

```
fit <- lm(
  log(WAGP) ~ SCHL,
  person)
```

Console

```
> summary(fit)
```

```

Call:
lm(formula = log(WAGP) ~ SCHL, data = person)

Residuals:
    Min      1Q  Median      3Q     Max 
-8.4081 -0.4712  0.2427  0.8023  2.5084 

Coefficients:
                   Estimate Std. Error t value Pr(>|t|)    
(Intercept)         9.21967    0.05862 157.284  < 2e-16 ***
SCHLHigh School     0.57470    0.07011   8.197 3.23e-16 ***
SCHLCollege Credit  0.58083    0.06530   8.895  < 2e-16 ***
```

Metadata Matters

For the predictors in a linear model, there is a big difference between factors and numbers.

worksheet-6.R

```
fit <- lm(
  log(WAGP) ~ AGEP,
  person)
```

Console

```
> summary(fit)
```

```

Call:
lm(formula = log(WAGP) ~ AGEP, data = person)

Residuals:
    Min      1Q  Median      3Q     Max 
-8.8094 -0.5513  0.2479  0.8093  2.1933 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 8.842905   0.055483  159.38   <2e-16 ***
AGEP        0.025525   0.001226   20.81   <2e-16 ***
---
```

The difference between 1 and 5 model degrees of freedom between the last two summaries—with one fixed effect each—arises because `SCHL` is a a factor while `AGEP` is numeric. In the first case you get multiple intercepts, while in the second case you get a slope.

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Generalized Linear Models

The `lm` function treats the response variable as numeric—the `glm` function lifts this restriction and others. Not through the `formula` syntax, which is the same for calls to `lm` and `glm`, but through addition of the `family` argument.

GLM Families

The `family` argument determines the family of probability distributions in which the response variable belongs. A key difference between families is the data type and range.

Family	Data Type	(*default*) link functions
`gaussian`	`double`	*identity*, log, inverse
`binomial`	`boolean`	*logit*, probit, cauchit, log, cloglog

Family	Data Type	(*default*) link functions
`poisson`	`integer`	*log*, identity, sqrt
`Gamma`	`double`	*inverse*, identity, log
`inverse.gaussian`	`double`	*"1/mu^2"*, inverse, identity, log

The previous model fit with `lm` is (almost) identical to a model fit using `glm` with the Gaussian family and identity link.

worksheet-6.R

```
fit <- glm(log(WAGP) ~ SCHL,
    family = gaussian,
    data = person)
```

Console

```
> summary(fit)
```

```

Call:
glm(formula = log(WAGP) ~ SCHL, family = gaussian, data = person)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-8.4081  -0.4712   0.2427   0.8023   2.5084

Coefficients:
                   Estimate Std. Error t value Pr(>|t|)
(Intercept)         9.21967    0.05862 157.284  < 2e-16 ***
SCHLHigh School     0.57470    0.07011   8.197 3.23e-16 ***
SCHLCollege Credit  0.58083    0.06530   8.895  < 2e-16 ***
```

Question

Should the coeficient estimates between this Gaussian `glm()` and the previous `lm()` be different?

Answer

It's possible. The `lm()` function uses a different (less general) fitting procedure than the `glm()` function, which uses IWLS. But with 64 bits used to store very precise numbers, it's rare to encounter an noticeable difference in the optimum.

Logistic Regression

Calling `glm` with `family = binomial` using the default "logit" link performs logistic regression.

worksheet-6.R

```
fit <- glm(SEX ~ WAGP,
  family = binomial,
  data = person)
```

Interpretting the coefficients in the summary is non-obvious, but always works the same way. In the `person` table, where men are coded as `1`, the `levels` function shows that `2`, or women, are the first level. When using the binomial family, the first level of the factor is consider 0 or "failure", and all remaining levels (typically just one) are considered 1 or "success". The predicted value, the linear combination of variables and coeficients, is the log odds of "success".

The positive coefficient on `WAGP` implies that a higher wage tilts the prediction towards men.

Console

```
> summary(fit)
```

```

Call:
glm(formula = SEX ~ WAGP, family = binomial, data = person)

Deviance Residuals:
```

```
    Min       1Q   Median       3Q      Max
-2.1479  -1.1225   0.6515   1.1673   1.3764

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept) -4.567e-01  4.906e-02  -9.308   <2e-16 ***
WAGP         1.519e-05  1.166e-06  13.028   <2e-16 ***
```

The always popular R^2 indicator for goodness-of-fit is absent from the summary of a `glm` result, as is the defult F-Test of the model's significance.

The developers of `glm`, detecting an increase in user sophistication, are leaving more of the model assessment up to you. The "null deviance" and "residual deviance" provide most of the information we need. The `?anova.glm` function allows us to apply the F-test on the deviance change, although there is a better alternative.

worksheet-6.R

```
anova(fit, update(fit, SEX ~ 1), test = 'Chisq')
```

```
Analysis of Deviance Table

Model 1: SEX ~ WAGP
Model 2: SEX ~ 1
  Resid. Df Resid. Dev Df Deviance  Pr(>Chi)
1      4244     5692.7
2      4245     5883.3 -1  -190.51 < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Observation Weights

Both the `lm()` and `glm()` function allow a vector of weights the same length as the response. Weights can be necessary for logistic regression, depending on the format of the data. The `binomial` family `glm()` works with three different response variable formats.

1. `factor` with only, or coerced to, two levels (binary)
2. `matrix` with two columns for the count of "successes" and "failures"
3. `numeric` proprtion of "successes" out of `weights` trials

Depending on your model, these three formats are not necesarilly interchangeable.

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Linear Mixed Models

The lme4 package expands the formula "mini-language" to allow descriptions of "random effects".

- normal predictors are "fixed effects"
- `(...|...)` expressions describe "random effects"

In the context of this package, variables added to the right of the `~` in the usual way are "fixed effects"—they consume a well-defined number of degrees of freedom. Variables added within `(...|...)` are "random effects".

Models with random effects should be understood as specifying multiple, overlapping probability statements about the observations.

$$\log(WAGP_i) \sim Normal(\mu_i, \sigma_0^2)$$
$$\mu_i = \beta_0 + \boldsymbol{\beta}_1[OCCP_i] + \beta_2 \log(SCHL_i)$$
$$\boldsymbol{\beta}_1 \sim Normal(0, \boldsymbol{\Sigma}_1)$$

The "random intercepts" and "random slopes" models are the two most common extensions to a formula with one variable.

Formula	Description
`y ~ (1 | b) + a`	random intercept for each level in `b`

Formula	Description
`y ~ a + (a | b)`	random intercept and slope w.r.t. `a` for each level in `b`

Random Intercept

The `lmer` function fits linear models with the lme4 extensions to the `lm` formula syntax.

worksheet-6.R

```
library(lme4)
fit <- lmer(
  log(WAGP) ~ (1|OCCP) + SCHL,
  data = person)
```

Console

```
> summary(fit)
```

```
Linear mixed model fit by REML ['lmerMod']
Formula: log(WAGP) ~ (1 | OCCP) + SCHL
   Data: person

REML criterion at convergence: 12766.4

Scaled residuals: 
    Min      1Q  Median      3Q     Max 
-8.6724 -0.3421  0.2034  0.6046  2.8378 

Random effects:
 Groups   Name        Variance Std.Dev.
 OCCP     (Intercept) 0.3945   0.6281  
```

The familiar assessment of model residuals is absent from the summary due to the lack of a widely accepted measure of null and residual deviance. The notions of model saturation, degrees of freedom, and independence of observations have all crossed onto thin ice.

In a `lm` or `glm` fit, each response is conditionally independent, given it's predictors and the model coefficients. Each observation corresponds to it's own probability statement. In a model with random effects, each response is no longer conditionally independent, given it's predictors and model coefficients.

Random Slope

Adding a numeric variable on the left of a grouping specified with `(...|...)` produces a "random slope" model. Here, separate coefficients for hours worked are allowed for each level of education.

worksheet-6.R

```
fit <- lmer(
  log(WAGP) ~ (WKHP | SCHL),
  data = person)
```

```
Warning in checkConv(attr(opt, "derivs"), opt$par, ctrl = control
$checkConv, : Model failed to converge with max|grad| = 0.207748 (tol =
0.002, component 1)
```

The warning that the procedure `failed to converge` is worth paying attention to. In this case, we can proceed by switching to the slower numerical optimizer that lme4 previously used by default.

worksheet-6.R

```
fit <- lmer(
  log(WAGP) ~ (WKHP | SCHL),
  data = person,
  control = lmerControl(optimizer = "bobyqa"))
```

Console

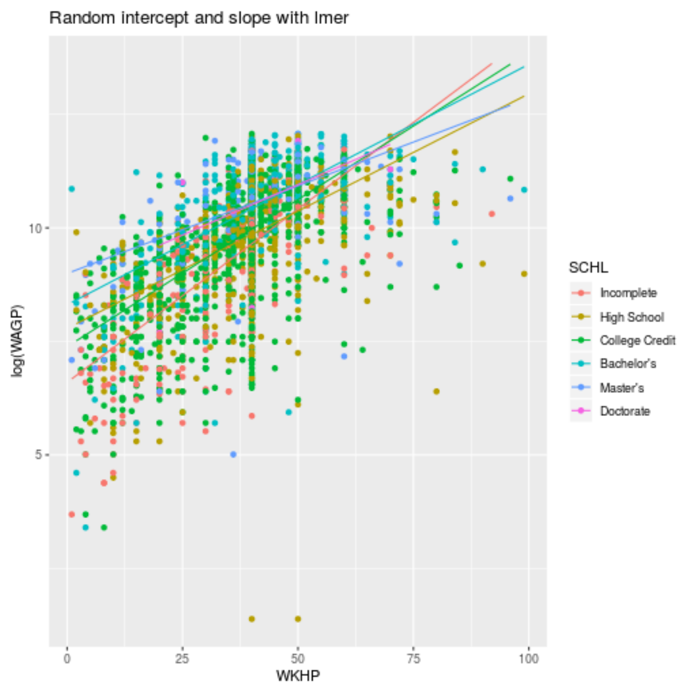
```
> summary(fit)
```

```
Linear mixed model fit by REML ['lmerMod']
Formula: log(WAGP) ~ (WKHP | SCHL)
   Data: person
Control: lmerControl(optimizer = "bobyqa")

REML criterion at convergence: 11670.1

Scaled residuals:
    Min      1Q  Median      3Q     Max
-9.4561 -0.3967  0.1710  0.6409  2.7948

Random effects:
 Groups   Name        Variance  Std.Dev. Corr
```

The `predict` function extracts model predictions from a fitted model object (i.e. the output of `lm`, `glm`, `lmer`, and `gmler`), providing one easy way to visualize the effect of estimated coeficients.

worksheet-6.R

```
ggplot(person,
  aes(x = WKHP, y = log(WAGP), color = SCHL)) +
  geom_point() +
  geom_line(aes(y = predict(fit))) +
  labs(title = 'Random intercept and slope with lmer')
```



Generalized Mixed Models

The `glmer` function adds upon `lmer` the option to specify the same group of exponential family distributions for the residuals available to `glm`.

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A Little Advice

- Don't *start* with generalized linear mixed models! Work your way up through

1. `lm()`
2. `glm()`
3. `lmer()`
4. `glmer()`

- Take time to understand the `summary()`—the hazards of secondary data compel you! The vignettes for lme4 provide excellent documentation.

When stuck, ask for help on Cross Validated. If you are having trouble with lme4, who knows, Ben Bolker himself might provide the answer.

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Exercises

Exercise 1

Regress `WKHP` against `AGEP`, adding a second-order interaction to allow for possible curvature in the relationship. Plot the predicted values over the data to verify the coefficients indicate a downward quadratic relationship.

Exercise 2

Controlling for a person's educational attainment, fit a linear model that addresses the question of wage disparity between men and women in the U.S. workforce. What other predictor from the `person` data frame would increases the goodness-of-fit of the "control" model, before SEX is considered.

Exercise 3

Set up a generalized mixed effects model on whether a person attained an advanced degree (Master's or Doctorate). Include sex and age as fixed effects, and include a random intercept according to occupation.

Exercise 4

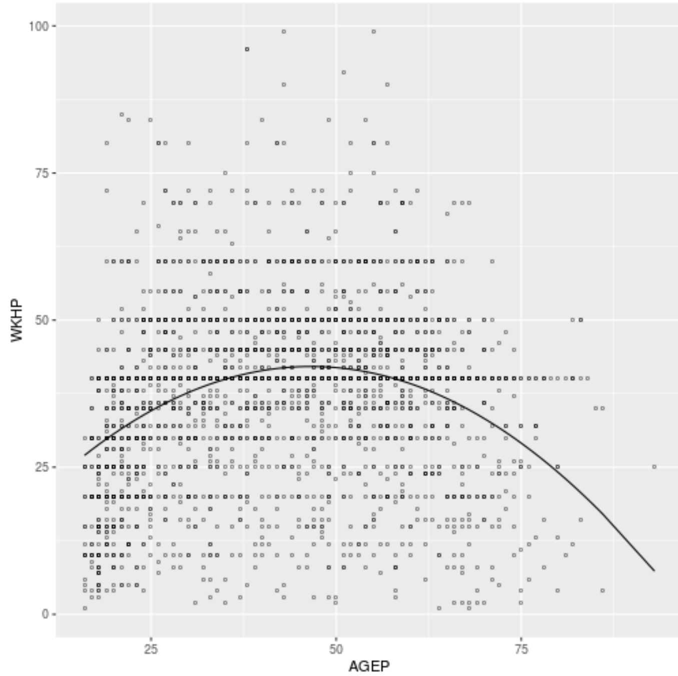
Write down the formula for a random intercepts model for earned wages with a fixed effect of sex and a random effect of educational attainment.

Solutions

Solution 1

```
fit <- lm(
  WKHP ~ AGEP + I(AGEP^2),
  person)

ggplot(person,
  aes(x = AGEP, y = WKHP)) +
  geom_point(shape = 'o') +
  geom_line(aes(y = predict(fit)))
```



Solution 2

```
fit <- lm(WAGP ~ SCHL, person)
summary(fit)
```

```
Call:
lm(formula = WAGP ~ SCHL, data = person)

Residuals:
   Min     1Q Median     3Q    Max 
-61303 -18827  -5827  12325 145173 

Coefficients:
                    Estimate Std. Error t value Pr(>|t|)    
(Intercept)            19614       1363  14.391  < 2e-16 ***
SCHLHigh School         8062       1630   4.945 7.89e-07 ***
SCHLCollege Credit     10214       1518   6.727 1.96e-11 ***
```

The “baseline” model includes educational attainment, when compared to a model with sex as a second predictor, there is a strong indication of signifigance.

Solution 2

```
anova(fit, update(fit, WAGP ~ SCHL + SEX))
```

```
Analysis of Variance Table

Model 1: WAGP ~ SCHL
Model 2: WAGP ~ SCHL + SEX
  Res.Df        RSS Df  Sum of Sq      F    Pr(>F)    
1   4240 3.2450e+12                                   
2   4239 3.0469e+12  1 1.9806e+11 275.55 < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Adding WKHP to the baseline increases the R^2 to `.32` from `.14` with just `SCHL`. There is no way to include the possibility that hours worked also dependends on sex, giving a direct and indirect path to the gender wage gap, in a linear model.

Solution 2

```
summary(update(fit, WAGP ~ SCHL + WKHP))
```

```

Call:
lm(formula = WAGP ~ SCHL + WKHP, data = person)

Residuals:
   Min     1Q Median     3Q    Max 
-82762 -14392  -3919   9306 142326 

Coefficients:
                    Estimate Std. Error t value Pr(>|t|)    
(Intercept)        -16510.68    1627.64 -10.144  < 2e-16 ***
SCHLHigh School      3230.58    1458.67   2.215   0.0268 *  
SCHLCollege Credit   7146.98    1354.98   5.275 1.40e-07 ***
```

Solution 3

```
df <- person
levels(df$SCHL) <- c(0, 0, 0, 0, 1, 1)
fit <- glmer(
  SCHL ~ (1 | OCCP) + AGEP,
  family = binomial,
  data = df)
anova(fit, update(fit, . ~ . + SEX), test = 'Chisq')
```

```
Data: df
Models:
fit: SCHL ~ (1 | OCCP) + AGEP
update(fit, . ~ . + SEX): SCHL ~ (1 | OCCP) + AGEP + SEX
                         Df    AIC    BIC  logLik deviance  Chisq Chi Df
fit                       3 1996.5 2015.6 -995.27   1990.5              
update(fit, . ~ . + SEX)  4 1997.6 2023.0 -994.79   1989.6 0.9572      1
                         Pr(>Chisq)
fit                                
update(fit, . ~ . + SEX)     0.3279
```

Solution 4

```
WAGP ~ (1 | SCHL) + SEX
```

```
WAGP ~ (1 | SCHL) + SEX
```

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If you need to catch-up before a section of code will work, just squish it's to copy code above it into your clipboard. Then paste into your interpreter's console, run, and you'll be ready to start in on that section. Code copied by both and will also appear below, where you can edit first, and then copy, paste, and run again.

```
# Nothing here yet!
```