Chebyshev's Inequality :

- Essentially, chebyshev's inequality asserts that nearly all the values of a prob. distribution stay close to the mean value and that the std. deviation serve as a good measure of the dispersion or the spread of the distribution about the mean.

- It is useful in determining probabilities of events centered about the mean, when the actual prob. distribution is not available and when we know only the mean and Vari. of the distribution.

Thm: Let X be a r.v. with mean μ and finite variance σ^2. Then for any real no. $k > 0$

$$P\left[|X-\mu| \geq k\sigma\right] \leq \frac{1}{k^2}$$

Corollary: Let X be a r.v. with mean μ and finite variance σ^2. Then for any real no. $k > 0$

$$P\left[|X-\mu| < k\sigma\right] \geq 1 - \frac{1}{k^2}$$

Corollary: Let X be a r.v. with mean μ and finite variance σ^2. Then for any real no $c > 0$,

$$P\left[|X-\mu| \geq c\right] \leq \frac{\sigma^2}{c^2}$$

$$\& \quad P\left[|X-\mu| < c\right] \geq 1 - \frac{\sigma^2}{c^2}$$

Ex:1 Use chebyshev's inequality to S.T., if x is the no. scored in a throw of a fair die,

$$P\{|X-3.5| > 2.5\} < 0.47.$$

$:\rightarrow$

X :	1	2	3	4	5	6
$f(x)$:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\mu = \frac{7}{2} = 3.5$$

$E(x) = \mu = \sum x\, p(x) = 7/2$

$$\& \text{ Vari} = \sigma^2 = \frac{35}{12} \qquad Var(x) = E(x^2) - (E(x))^2$$

$$= 2.9167$$

By chebyshev's inequality, we have

$$P\{|X-\mu| \geq C\} \leq \frac{\sigma^2}{C^2}$$

Taking $C = 2.5$, we have

$$P\{|X-3.5| > 2.5\} < \frac{2.9167}{6.25}$$

$$= 0.4667$$

$$\approx 0.47$$

Ex:2 A r.v. X has the density fu^n e^{-x}, $x \geq 0$. S.T chebyshev's inequality gives $P[|x-1| > 2] < \frac{1}{4}$ and show that actual prob. is e^{-3}.

$:\rightarrow$

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^{\infty} x e^{-x}\, dx = 1$$

$$E(x^2) = \int_0^{\infty} x^2 e^{-x}\, dx = 2$$

$$\therefore var(x) = 2 - (1)^2 = 1$$

By chebyshev's Inequality,

$$P[|X-\mu| > c] < \frac{\sigma^2}{c^2}$$

Taking $c = 2$

$$P[|X-1| > 2] < \frac{1}{4}$$

The actual prob. is

$$P(|X-1|>2) = 1-P\{|X-1|\le 2\}$$
$$= 1-P\{-2\le X-1\le 2\}$$
$$= \cancel{1-P(-1\le X\le 3)}$$
$$= 1-\int_0^3 e^{-x}\,dx$$
$$= e^{-3}.$$

Thus, the actual prob is e^{-3}, while bound given by chebyshev's inequality is $\frac{1}{4}$.

Ex. A r.v X has a mean $\mu=8$, a variance $\sigma^2=9$, and an unknown prob. distri. Find (i) $P(-4<X<20)$ (ii) $P(|X-8|\ge 6)$

:→ By chebyshev's inequality, we have

$$P[|X-\mu|<c] \ge 1-\frac{\sigma^2}{c^2}$$

Thus,
(i) $$P(-4<X<20) = P(|X-8|<12)$$
$$\ge 1-\frac{9}{12^2} = 1-\frac{1}{16}$$
$$= \frac{15}{16} = 0.9375$$

(ii) $$P[|X-\mu|\ge c] \le \frac{\sigma^2}{c^2}$$
$$\therefore P[|X-8|\ge 6] \le \frac{9}{6^2} = \frac{1}{4} = 0.25$$

Ex: Suppose that it is known that the no. of items produced in a factory during a week is a r.v with mean 50. If the variance of a week's production is known to equal 25, then what can be said about the productivity that will be betn 40 & 60?

:→ $\mu=50$, $\sigma^2=25$, $P(40\le X\le 60) = P(|X-\mu|\le 20)$

$$P(|X-\mu|\le c) \ge 1-\frac{\sigma^2}{c^2}$$
$$P(|X-\mu|\le 20) \ge 1-\frac{25}{20^2} = 0.9375$$

Ex: If x is a r.v of the no. of heads obtained in tossing three coins, P.T $P(|x-3/2| \geq 2) \leq 3/16$

:→ $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

so, prob distri. is

x :	0	1	2	3
$P(x)$:	$1/8$	$3/8$	$3/8$	$1/8$

$\therefore \mu = E(x) = 3/2$

& $var(x) = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4} = \sigma^2$

By chebyshev's Inequality,

$$P(|x-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Taking $k = \frac{c}{\sigma}$ or $k\sigma = c$

$$P(|x-\mu| \geq c) \leq \frac{\sigma^2}{c^2}$$

Here $\mu = 3/2$, taking $c = 2$

$$\therefore P(|x - 3/2| \geq 2) \leq \frac{3}{4} \cdot \frac{1}{2^2} = \frac{3}{16}$$

Ex: If x is the no. scored in a draw of a fair die, S.T $P(|x-3.5| > 2.5) < 7/15$.

:→ we have, $\mu = E(x) = \frac{1}{6}(1+2+3+4+5+6) = 3.5$

$$E(x^2) = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2) = \frac{91}{6}$$

$$\therefore Var(x) = \sigma^2 = E(x^2) - (E(x))^2$$

$$= \frac{91}{6} - (3.5)^2 = \frac{35}{12}$$

By chebyshev's inequality

$$P(|x-\mu| > c) < \frac{\sigma^2}{c^2}$$

$$P(|x-\mu| > 2.5) < \frac{35}{12} \times \frac{1}{(2.5)^2} = \frac{7}{15}$$

5. If the joint pdf of (X, Y) is given by

$$f(x, y) = 2, \quad 0 \le x < y \le 1,$$

find the conditional mean and conditional variance of X given that $Y = y$.

$$\left[\textbf{Ans.: } \frac{y}{2}, \frac{y^2}{12}\right]$$

6. If the joint pdf of (X, Y) is given by

$$f(x, y) = 21\, x^2 y^3\,,\ 0 \le x < y \le 1$$

find the conditional mean and conditional variance of X, given that $Y = y,\ 0 < y < 1$.

$$\left[\textbf{Ans.: } \frac{3y}{4}, \frac{3y^2}{80}\right]$$

7. If the joint pdf of (X, Y) is given by

$$f(x, y) = 3xy\,(x + y)\,,\ 0 < x \le y \le 1,$$

verify that $E\{E(Y/X)\} = E(Y) = \dfrac{17}{24}$.

3.9 BOUNDS ON PROBABILITIES

If the probability distribution of a random variable is known $E(X)$ and $\mathrm{Var}(X)$ can be computed. Conversely, if $E(X)$ and $\mathrm{Var}(X)$ are known, probability distribution of X cannot be constructed and quantities such as $P\{|X - E(X)| \le k\}$ can not be evaluated. Several approximation techniques have been developed to yield upper and /or lower bounds to such probabilities. The most important of such techniques is Chebyshev's inequality.

3.10 CHEBYSHEV'S INEQUALITY

If X is a random variable with mean μ and variance σ^2, then for any positive umber k,

$$P\{|X - \mu| \ge k\sigma\} \le \frac{1}{k^2}$$

or

$$P\{|X - \mu| < k\sigma\} \ge 1 - \frac{1}{k^2}$$

Proof

Let X be a continuous random variable.

$$\sigma^2 = E[X - E(X)]^2$$

$$= E[X - \mu]^2 \qquad [\because\ \mu = E(X)]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx \quad \text{where } f(x) \text{ is pdf of } X.$$

$$= \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)dx$$

$$\geq \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)dx \qquad \text{...(1)}$$

We know that $x \leq \mu - k\sigma$ and $x \geq \mu + k\sigma$

$\therefore$ $|x-\mu| \geq k\sigma$

Substituting in Eq. (1),

$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} k^2\sigma^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} k^2\sigma^2 f(x)dx$$

$$= k^2\sigma^2\left[\int_{-\infty}^{\mu-k\sigma} f(x)dx + \int_{\mu+k\sigma}^{\infty} f(x)dx\right]$$

$$= k^2\sigma^2[P(X \leq \mu - k\sigma) + P(X \geq \mu + k\sigma)]$$

$$= k^2\sigma^2[P(X-\mu \leq -k\sigma) + P(X-\mu \geq k\sigma)]$$

$$= k^2\sigma^2 P\{|X-\mu| \geq k\sigma\}$$

$$P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

$$\because P\{|X-\mu| \geq k\sigma\} + P\{|X-\mu| < k\sigma\} = 1$$

$$P\{|X-\mu| < k\sigma\} = 1 - P\{|X-\mu| \geq k\sigma\}$$

$$\geq 1 - \frac{1}{k^2}$$

Note

1. If $k\sigma = c > 0$

$$P\{|X-\mu| \geq c\} \leq \frac{\sigma^2}{c^2}$$

and $$P\{|X-\mu| < c\} \geq 1 - \frac{\sigma^2}{c^2}$$

2. To find the lower bound of probabilities following form of Chebyshev's inequality is used:

$$P\{|X-\mu| < k\sigma\} \geq 1 - \frac{1}{k^2}$$

or $$P\{|X-\mu|<c\}\geq 1-\frac{\sigma^2}{c^2}$$

3. To find the upper bound of probabilities following form of Chebyshev's inequality is used;

$$P\{|X-\mu|\geq k\sigma\}\leq\frac{1}{k^2}$$

or $$P\{|X-\mu|\geq c\}\leq\frac{\sigma^2}{c^2}$$

Example 1

A random variable X has a mean $\mu = 12$ and a variance $\sigma^2 = 9$ and unknown probability distribution. Find $P(6 < X < 18)$.

Solution

$$\mu = 12, \quad \sigma^2 = 9$$
$$\sigma = 3$$

By Chebyshev's inequality,

$$P\{|X-\mu|<k\sigma\}\geq 1-\frac{1}{k^2}$$
$$P\{-k\sigma<X-\mu<k\sigma\}\geq 1-\frac{1}{k^2}$$
$$P\{\mu-k\sigma<X<\mu+k\sigma\}\geq 1-\frac{1}{k^2}$$
$$P\{12-3k<X<12+3k\}\geq 1-\frac{1}{k^2}$$

Comparing with $P(6 < X < 18)$,

$$12-3k=6$$
$$12+3k=18$$
$$\therefore \quad k=2$$

$$P\{6<X<18\}\geq 1-\frac{1}{4}$$
$$P\{6<X<18\}\geq\frac{3}{4}$$

Example 2

A random variable X has a mean 10 and a variance 4 and unknown probability distribution. Find the value of c such that $P\{|X-10| \geq c\} \leq 0.04$.

Solution

$$\mu = 10,\ \sigma^2 = 4$$
$$\sigma = 2$$

By Chebyshev's inequality,

$$P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

Comparing with $P\{|X-10| \geq c\} \leq 0.04$,

$$\frac{1}{k^2} = 0.04$$
$$k = 5$$

and
$$k\sigma = c$$
$$c = 5(2) = 10$$

Example 3

A random variable X has pdf $f(x) = e^{-x}, x \geq 0$. Use Chebyshev's inequality to show that $P\{|X-1| > 2\} \leq \frac{1}{4}$ and also, show that the actual probability is given by e^{-3}.

Solution

$$f(x) = e^{-x}$$

The random variable X follows exponential distribution with parameter $\lambda = 1$.

$$E(X) = \mu = \frac{1}{\lambda} = 1$$
$$\text{Var}(X) = \sigma^2 = \frac{1}{\lambda^2} = 1$$

By Chebyshev's inequality,

$$P\{|X-\mu| > k\sigma\} \leq \frac{1}{k^2}$$

Comparing with $P\{|X-\mu|>2\}$,

$$k\sigma = 2$$
$$k(1) = 2$$
$$k = 2$$

$$\therefore \qquad P\{|X-1|>2\} \le \frac{1}{4}$$

The actual probability is given by

$$\begin{aligned} P\{|X-1|>2\} &= 1-P\{|X-1|\le 2\} \\ &= 1-P\{-1<X\le 3\} \\ &= 1-P\{0<X\le 3\} \\ &= 1-\int_0^3 e^{-x}dx \\ &= 1-\left|e^{-x}\right|_0^3 \\ &= 1-e^{-3} \end{aligned}$$

Example 4

A random variable X is exponentially distributed with parameter 1. *Use Chebyshev's inequality to show that* $P\{-1\le X\le 3\}\ge \frac{3}{4}$. *Find the actual probability also.*

Solution

For an exponential distribution with parameter $\lambda = 1$,

$$E(X) = \mu = \frac{1}{\lambda} = 1$$

$$\text{Var}(X) = \sigma^2 = \frac{1}{\lambda^2} = 1$$

$$\sigma = 1$$

By Chebyshev's inequality,

$$P\{|X-\mu|<k\sigma\} \ge 1-\frac{1}{k^2}$$

$$P\{-k\sigma < X-\mu < k\sigma\} \ge 1-\frac{1}{k^2}$$

$$P\{\mu - k\sigma < X < \mu + k\sigma\} \geq 1 - \frac{1}{k^2}$$

$$P\{1 - k < X < 1 + k\} \geq 1 - \frac{1}{k^2}$$

Comparing with $P\{-1 \leq X \leq 3\} \geq \frac{3}{4}$,

$$1 - k = -1$$

$$k = 2$$

$$\therefore \quad P\{-1 \leq X \leq 3\} \geq 1 - \frac{1}{4}$$

$$\geq \frac{3}{4}$$

The actual probability is given by

$$P\{-1 \leq X \leq 3\} = P\{0 \leq X \leq 3\} \quad [\because x > 0 \text{ for exponential distribution}]$$

$$= \int_0^3 f(x)dx$$

$$= \int_0^3 e^{-x}dx$$

$$= \left| -e^{-x} \right|_0^3$$

$$= -e^{-3} + e^0$$

$$= 1 - e^{-3}$$

$$= 0.9502$$

Example 5

A fair dice is tossed 120 times. Use Chebyshev's inequality to find a lower bound for the probability of getting 80 to 120 sixes.

Solution

Let X be the random variable which denotes number of sixes obtained when a fair dice is tossed by 720 times.

$$n = 720$$

Probability of getting 6 in single toss

$$p = \frac{1}{6}$$

$$q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

X follows a binomial distribution.

$$\mu = np-(720)\left(\frac{1}{6}\right)=120$$

$$\sigma^2 = npq = (720)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)=100$$

$$\sigma = 10$$

By Chebyshev's inequality,

$$P\{|X-\mu| < k\sigma\} \ge 1-\frac{1}{k^2}$$

$$P\{-k\sigma < X-\mu < k\sigma\} \ge 1-\frac{1}{k^2}$$

$$P\{\mu-k\sigma < X < \mu+k\sigma\} \ge 1-\frac{1}{k^2}$$

$$P\{120-10k < X < 120+10k\} \ge 1-\frac{1}{k^2}$$

Comparing with $P\{80 < X < 120\}$,

$$120-10k=80$$

$$k=4$$

$$P\{80 < X < 120\} \ge 1-\frac{1}{4^2}$$

$$P\{80 < X < 120\} \ge \frac{15}{16}$$

Hence, the lower bound for probability $= \dfrac{15}{16}$

Example 6

Two dice are thrown once. If X is the sum of the numbers sharing up, prove that $P\{|X-7| \ge 3\} \le \dfrac{35}{34}$. *Compare this value with the exact probability.*

Solution

Let X_1 and X_2 be the random variables which denote the outcomes of first and second dice.

$$E(X_1)=E(X_2)=\frac{1}{6}(1+2+3+4+5+6)=\frac{7}{2}$$

$$E(X)=E(X_1)+E(X_2)=\mu=\frac{7}{2}+\frac{7}{2}=7$$

$$E(X_1^2) = E(X_2^2) = \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = \frac{91}{6}$$

$$\text{Var}(X_1) = \text{Var}(X_2) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

$$\text{Var}(X) = \text{Var}(X_1 + X_2) = (1)^2\,\text{Var}(X_1) + (1)^2\,\text{Var}(X_2)$$

$$\sigma^2 = \frac{35}{12} + \frac{35}{12} = \frac{35}{6}$$

$$\sigma = \sqrt{\frac{35}{6}}$$

By Chebyshev's inequality,

$$P\{|X - \mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

Comparing with $P\{|X-7| \geq 3\}$,

$$\mu = 7$$

$$k\sigma = 3$$

$$k\sqrt{\frac{35}{6}} = 3$$

$$k = 3\sqrt{\frac{6}{35}}$$

$$\therefore \quad P\{|X-7| \geq 3\} \leq \frac{1}{\left(3\sqrt{\frac{6}{35}}\right)^2}$$

$$\leq \frac{35}{54}$$

Actual probability is given by

$$P\{|X-7| \geq 3\} = P\{X = 1, 2, 3, 4, 10, 11, 12\}$$

$$= \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36}$$

$$= \frac{4}{9}$$

Example 7

Use Chebyshev's inequality to find how many times a fair coin must be tossed in order that probability that the ratio of the number of heads

to the number of tosses will the between 0.45 and 0.55 will be at le
0.95.

Solution

Let X be the random variable which denotes the number of heads obtained when a fa
coin is tossed n times.

$$p = q = \frac{1}{2}$$

X follows a binomial distribution.

$$\text{Mean} = np \quad \text{and} \quad \text{Var}(X) = npq$$

Mean of required ratio $\frac{x}{n} = E\left(\frac{1}{n}X\right) = \frac{1}{n}E(X)$

$$= \frac{1}{n}np = p = \frac{1}{2}$$

$$\therefore \qquad \mu = \frac{1}{2}$$

$$\text{Var}\left(\frac{X}{n}\right) = \left(\frac{1}{n}\right)^2 \text{Var}(X) = \frac{1}{n^2}npq = \frac{pq}{n}$$

$$\sigma = \sqrt{\frac{pq}{n}} = \sqrt{\frac{\frac{1}{2}\cdot\frac{1}{2}}{n}} = \frac{1}{2\sqrt{n}}$$

By Chebyshev's inequality,

$$P\left\{\left|\frac{X}{n} - \mu\right| < k\sigma\right\} \geq 1 - \frac{1}{k^2}$$

$$P\left\{-k\sigma < \frac{X}{n} - \mu < k\sigma\right\} \geq 1 - \frac{1}{k^2}$$

$$P\left\{\mu - k\sigma < \frac{X}{n} < \mu + k\sigma\right\} \geq 1 - \frac{1}{k^2}$$

But $P\left\{0.45 < \frac{X}{n} < 0.55\right\} \geq 0.95$

$$1 - \frac{1}{k^2} = 0.95$$

$$\frac{1}{k^2} = 0.05$$

$$k = \sqrt{20}$$

$$\mu - k\sigma = 0.45$$

$$0.5 - \left(\frac{1}{2\sqrt{n}}\right) = 0.45$$

$$n = 2000$$

Hence, the fair coin must be tossed 2000 times.

Example 8

If X is the number on a dice when it is thrown, prove that $P\{|X-\mu| \geq 2.5\} \leq 0.47$, where μ is the mean.

Solution

Let x be the random variable which denotes the number on a dice. The probability function is

X	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = \mu = \sum xp(x)$$

$$= 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right)$$

$$= \frac{7}{2}$$

$$\text{Var}(X) = \sigma^2 = \sum x^2 p(x) - \mu^2$$

$$= 1\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 9\left(\frac{1}{6}\right) + 16\left(\frac{1}{6}\right) + 25\left(\frac{1}{6}\right) + 36\left(\frac{1}{6}\right) - \left(\frac{7}{2}\right)^2$$

$$= 2.9167$$

$$\sigma = 1.707$$

By Chebyshev's inequality,

$$P\{|X-\mu| > k\sigma\} < \frac{1}{k^2}$$

Comparing with $P\{|X-\mu| > 2.5\}$,

$$k\sigma = 2.5$$

$$k(1.707) = 2.5$$

$$k = 1.46$$

$$\therefore P\{|X-\mu|>2.5\}<\frac{1}{(1.46)^2}$$

$$P\{\{X-\mu\}>2.5\}<0.47$$

Example 9

The number of planes landing at an airport in a 30 minutes interv obeys the Poisson law with mean 25. Use Chebyshev's inequality to fi the least chance that the number of planes landing within a given 3 minutes interval will be between 15 and 25.

Solution

Let x be a random variable which denotes the number of planes landing at an airport. For Poisson distribution,

$$E(X)=\mu=25$$

$$\text{Var}(X)=\sigma^2=\mu=25$$

$$\sigma=5$$

By Chebyshev's inequality,

$$P\{|X-\mu|<k\sigma\}\geq 1-\frac{1}{k^2}$$

$$P\{-k\sigma<X-\mu<k\sigma\}\geq 1-\frac{1}{k^2}$$

$$P\{\mu-k\sigma<X<\mu+k\sigma\}\geq 1-\frac{1}{k^2}$$

$$P\{25-5k<X<25+5k\}\geq 1-\frac{1}{k^2}$$

Comparing with $P\{15<X<25\}$,

$$25-5k=15 \text{ and } 25+5k=25$$

$$k=2$$

$$\therefore P\{15<X<25\}\geq 1-\frac{1}{(2)^2}$$

$$\geq \frac{3}{4}$$

03. A machine produces on the average of 10% defectives. Find the probability that in a sample of 10 tools chosen at random, exactly two will be defective.

04. The continuous random variable X has a standard normal distribution. Calculate the probability that (a) $0 < X < 1$, (b) $-1 < X < 1$, (c) $-0.5 \le X \le 2$.

05. If the average height of a certain type of corn is 12 *cm* with standard deviation of 1.8 *cm*. What percentage of these corns exceeds 14 *cms* in height, assuming that the heights are normally distributed ?

06. The following table gives the probabilities that a certain computer will malfunction 0, 1, 2, 3, 4, 5 or 6 times on any one day.

No. of malfunctions(x)	:	0	1	2	3	4	5	6
Probabilities $f(x)$	:	0.17	0.29	0.27	0.16	0.07	0.03	0.01

Find the mean and standard deviation of this probability distribution.

[GTU, May 2017]

07. The breaking strength $X(kg)$ of a certain type of plastic block is normally distributed with a mean of 1250 kg and a standard deviation of 55 kg. What is the maximum load such that we can expect no more than 5% of the block to break?

[GTU, May 2017]

08. Find the expectation for the following discrete probability distribution.

x	:	10	14	18	25	35
$p(x)$	:	0.125	0.225	0.325	0.200	0.125

[GTU, May 2017 - Comp.]

3 Chebyshev's Inequality

e know that the standard deviation σ gives us idea about the variability of the servations about the mean, and thus it controls the concentration of probability in neighbourhood of the mean. For smaller values of σ, there is a high probability of ting values close to the mean. The Chebyshev's inequality, in general, gives us nds on probability that how far a random variable X is deviated when both mean d variance σ^2 of the distribution are known. The inequality is also helpful when probability distribution (either discrete or continuous) of X is not known.

Moreover, the bounds given by the inequality is universal; that is, it is the for all random variables X with a given μ and σ^2, with the drawbacks that the ds are not sharp in general. If there is more information about the distribution of en it might be possible to get a better bounds as compared to Chebyshev's ality (refer Example 6.11). The only restriction with this inequality is that X d have finite σ^2.

Theorem 6.1 (Chebyshev's Inequality) If X is a random variable with mean μ and finite variance σ^2, then for every $k > 0$,

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2} \text{ or } P(|X-\mu| < k\sigma) \geq 1-\frac{1}{k^2} \quad \text{...(6.12)}$$

A convenient form Chebyshev's inequality can be obtained by taking $k\sigma = \rho$ and $\rho > 0$. Therefore, (6.12) becomes

$$P(|X-\mu| \geq \rho) \leq \frac{\sigma^2}{\rho^2} \text{ or } P(|X-\mu| < \rho) \geq 1-\frac{\sigma^2}{\rho^2} \quad \text{...(6.13)}$$

Example 6.11

Suppose that a random variable X is $N(\mu, \sigma^2)$. Compute $P(|X-\mu| \geq 2\sigma)$.

Solution

Here, X is normally distributed with $E(X) = \mu$ and $V(X) = \sigma^2$.

Now,

$$P(|X-\mu| \geq 2\sigma) = P\left(\left|\frac{X-\mu}{\sigma}\right| \geq 2\right)$$

$$= P(|Z| \geq 2)$$

$$\approx 0.0456.$$

(Using Table I in Appendix *A*) ...(i)

By direct application of (6.13), we have

$$P(|X-\mu| \geq 2\sigma) \leq \frac{\sigma^2}{(2\sigma)^2} = \frac{1}{4} = 0.25,$$

which is substantially large compared to the more exact value 0.0456 obtained in (i).

This example justifies the statement of the Introduction that the bounds obtained by Chebyshev's inequality may not be sharp in general. But a better bounds can be obtained if we know exact distribution of X.

Example 6.12

The number of customers who visit a car dealer's showroom on sunday morning is a random variable with mean 18 and standard deviation 2.5. What is the probability that on sunday morning the customer's will be between 8 to 28.

Solution

Given that

$$\mu = E(X) = 18 \text{ and } \sigma^2 = \text{var}(X) = (2.5)^2 = 6.25.$$

Using (6.13), the required probability is

$$P(|X - 18| < 10) \geq 1 - \frac{6.25}{100} \qquad \text{(Using } \rho = 10\text{)}$$

$$\Rightarrow P(8 < X < 28) \geq 1 - \frac{1}{16}$$

$$\Rightarrow P(8 < X < 28) \geq \frac{15}{16}.$$

Answer

Example 6.13

Determine the smallest value of k in the Chebyshev's inequality for which the probability is at least 0.95.

Solution

Using (6.12),

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}.$$

Given that

$$0.95 \geq 1 - \frac{1}{k^2} \Rightarrow \frac{1}{k^2} \geq 1 - 0.95$$

$$\Rightarrow k^2 \leq \frac{1}{0.05}$$

$$\Rightarrow k^2 = \sqrt{20}.$$

Answer

Exercises 6.2

1. Suppose X is a random variable such that $E(X) = 3$ and $E(X^2) = 13$. Calculate a lower bound for the probability that X lies between -2 and 8 using Chebyshev's inequality.
2. The number of items cleared by an assembly line during a week is a random variable with mean 50 and variance 25. What can be said about the probability that this week's clearance will be between 40 to 60 ?