Probability Distributions: (Standard) (Parametric)

:: Discrete Distribution:

1 Bernoulli Distribution : [Bernouli Trials]

The Bernouli distribution is named after swiss mathematicians James Bernoulli (1654-1705). This distribution is associated with the Bernoulli trial

- Bernoulli trial is an experiment with two outcome which are random, named as success and failure
- Prob. success is denoted by p and prob. of failure is denoted by q.
- $p+q=1$

Example: Many real life instances, which can be viewed as Bernoulli trial

1. A coin is tossed. It has two outcomes, H & T. $p=\frac{1}{2}$ & $q=\frac{1}{2}$

2. A fair die is tossed. consider getting no six as a success and any other no. as failure $p=\frac{1}{6}$ & $q=\frac{5}{6}$.

3. A card is drawn from a pack of cards. consider getting a card of spade as a success and a card of any other suit as a failure, then $p=\frac{1}{4}$ & $q=\frac{3}{4}$.

4. Two person A and B, are contesting in an electio A public opinion survey is conducted. A voter is selected at random, and he (or she) is asked whether he(orshe) would vote for the candidate A.

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Note: Bernoulli trials feature questions with "Yes" or "No" type of answers. "Yes" corresponds to a success & "no" corresponds to a failure.

Def^n: Bernoulli Distribution:

A r.v X is said to have a Bernoulli distri. if it can take only two values 0 and 1, with 1 being consider as a success and 0 as a failure, with Prob. p & q resp.

$$\text{ie } f(x;p) = P(X=x) = \begin{cases} p & \text{if } x=1 \\ q=1-p & \text{if } x=0 \end{cases}$$

Mean : $E(X) = p$

Var(X) : $Var(X) = pq$

$$\text{Mean} = \mu = E(X) = \sum x \cdot P(X=x)$$
$$= 0 \times f(0) + 1 \times f(1) = 0 + p = p$$

$$E(x^2) = \sum x^2 P(X=x)$$
$$= 0^2 \times f(0) + 1^2 \times f(1) = p$$

$$\therefore var(X) = E(x^2) - (E(x))^2$$
$$= p - p^2 = p(1-p) = pq$$

Binomial Distribution :

It was discovered by James Bernoulli in 1700.

It is a discrete probability distribution of the no. of successes in a sequence of n independent bernoulli trials with a ~~cr~~ constant prob. of success p in all the trials.

For eg : Consider a sequence of 3 Bernoulli trials with const. prob. of success p in each Trial

Then

$$P(X=0) = P(\text{All failure})$$
$$= q \times q \times q = q^3$$
$$P(X=1) = P(2 \text{ failure and } 1 \text{ success})$$
$$= P(SFF) + P(FSF) + P(FFS)$$
$$= pq^2 + pq^2 + pq^2$$
$$= 3pq^2$$
$$P(X=2) = P(1 \text{ failure and } 2 \text{ success})$$
$$= p^2q + p^2q + p^2q$$
$$= 3p^2q$$
$$P(X=3) = P(3 \text{ success}) = P(SSS)$$
$$= p \times p \times p = p^3$$

∴ For the sequence of n Indept Bernoulli trials we have

$$P(X=x) = P(x \text{ success and } (n-x) \text{ failure})$$

The x successes in n trials can occur in nC_x ways

The prob. for each of these ways is $p^x q^{n-x}$

$$\therefore P(X=x) = nC_x \, p^x q^{n-x}, \quad x = 0,1,2,\ldots,n$$

Def^n : A r. v. X is said to follow binomial distri. denoted by $B(n,p)$ if it assumes only non negative values and its PMF is given by

$$P(x) = P(X=x) = \begin{cases} nC_x \, p^x q^{n-x}, & x = 0,1,2,\ldots,n \\ 0, & \text{otherwise} \end{cases}$$

$\boxed{q = 1-p}$

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Binomial Frequency Distribution:

Suppose that n trials consistitute an experiment and if this experiment is repeated N times, the frequency funⁿ of the binomial distribution is given by

$$N.P(x) = N \times {}^nC_x \, p^x q^{n-x} , \quad x = 0,1,2,----,n$$

Properties:

A binomial experiment must possess the following

1. Each trial results in two mutually disjoint outcomes, termed as success and failure
2. The trials must be independent of each other
3. All trials have same const. prob. of success
4. The no. of trials n is finite

Note: Let p = prob. of success

$P(\text{all success}) = p^n$

$P(\text{all failure}) = q^n$

$P(\text{at least one success}) = 1-q^n$

$P(\text{at least one failure}) = 1-p^n$

Mean of Binomial Distribution:

$$\mu_1 = \text{Mean} = E(x) = \sum_x x P(x)$$

$$= \sum_{x=0}^{n} x . {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x . \frac{n!}{(n-x)! \, x!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x . \frac{n(n-1)!}{(n-x)! \, x(x-1)!} p^x q^{n-x}$$

$$\therefore \quad E(x) = \sum_{x=1}^{n} \frac{n(n-1)!}{[(n-1)-(x-1)]!\,(x-1)!} p^{x-1} \cdot p \cdot q^{[(n-1)-(x-1)]}$$

$$= np \sum_{x=1}^{\infty} {}^{n-1}C_{x-1}\, p^{x-1} q^{[(n-1)-(x-1)]}$$

$$= np(q+p)^{n-1}$$

$$= np \qquad (\because q+p=1)$$

Variance of Binomial Distribution:

$$Var(x) = E(x^2) - [E(x)]^2$$

where

$$E(x^2) = E[x(x-1)+x]$$

$$= E \sum_{x=0}^{n} [x(x-1)+x]\, P(x)$$

$$= \sum_{x=0}^{n} x(x-1)P(x) + \sum_{x=0}^{n} x P(x)$$

$$= \sum_{x=0}^{n} x(x-1)\, {}^{n}C_x\, p^x q^{n-x} + E(x)$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{(n-x)!\,x(x-1)(x-2)!} p^x q^{n-x} + E(x)$$

$$= \sum_{x=2}^{n} \frac{n(n-1)(n-2)!}{[(n-2)-(x-2)]!\,(x-2)!} p^{x-2} \cdot p^2\, q^{[(n-2)-(x-2)]} + E(x)$$

$$= n(n-1)p^2 \sum_{x=2}^{\infty} {}^{n-2}C_{x-2}\, p^{x-2} q^{[(n-2)-(x-2)]} + E(x)$$

$$= n(n-1)p^2 (q+p)^{n-2} + E(x)$$

$$= n(n-1)p^2 + np$$

$$= n^2p^2 - np^2 + np$$

$\mu_2 = \therefore Var(x) = n^2p^2 - np^2 + np - (np)^2$

$= np - np^2$

$= np(1-p) = npq$

Note: - $\mu_3 = npq(q-p)$

- $\mu_4 = npq(1-6pq+3npq)$
- Mean of the binomial distri. is always greater than its variance

Ex:1 The mean and variance of a binomial distribution are 4 and 3 resp. Find $P(x=0)$, $P(x=1)$ and $P(X \geq 2)$

:→

Mean $= np$ $\quad\quad$ $var(x) = npq$

ie $4 = np$ $\quad\quad$ ie $3 = npq$

$\therefore 3 = (4)q \quad \therefore q = \frac{3}{4}$

$\therefore p = 1-q$

$= \frac{1}{4}$

Mean $= np$

$\therefore 4 = n(\frac{1}{4}) \quad \therefore n = 16$

$P(x=x) = nC_x \, p^x \, q^{n-x}$

$\therefore P(x=0) = 16C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{16-0} = \left(\frac{3}{4}\right)^{16} = 0.01$

$P(x=1) = 16C_1 \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{15} = 16 \times \frac{1}{4} \times \left(\frac{3}{4}\right)^{15} = 0.053$

$P(X \geq 2) = 1 - P(X < 2) = 1 - [P(x=0) + P(x=1)]$

$= 1 - [0.01 + 0.053]$

$= 0.93$

Ex: 2 The mean and std. deviation of a binomial distribution are 6 and 2 resp. Determine the distribution

Sol →

$$\text{mean} = np = 6$$

$$s.d = 2 \Rightarrow Var(x) = (s.d)^2 = 4 = npq$$

$$\therefore p = \frac{1}{3}, \quad q = \frac{2}{3}, \quad n = 18$$

The binomial distribution is

$$P(x) = P(X=x) = nC_x p^x q^{n-x}$$

$$= 18C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{n-x}, \quad x = 0, 1, \ldots, 18$$

∴ **Moment Generating fun of Binomial Distribution:**

$$M_X(t) = E(e^{tX})$$

$$= \sum_{x=0}^{n} e^{tx} \, nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x}$$

$$= q^n + nC_1 (pe^t)^1 q^{n-1} + nC_2 (pe^t)^2 q^{n-2} + \cdots + (pe^t)^n$$

$$= (q + pe^t)^n$$

Mean of M.G fun

$$E(X) = \mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt} (q + pe^t)^n\right]_{t=0}$$

$$= \left[n(q+pe^t)^{n-1} \cdot pe^t\right]_{t=0}$$

$$= np(p+q)^{n-1}$$

$$= np$$

$$E(x^2) = \mu_2' = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(n(q+pe^t)^{n-1} pe^t\right)\right]_{t=0}$$

$$= \left[npe^t(q+pe^t)^{n-1} + n(n-1)(q+pe^t)^{n-2}(pe^t)^2\right]_{t=0}$$

$$= np(q+p)^{n-1} + n(n-1)p^2(q+p)^{n-2}$$

$$= np + n(n-1)p^2$$

$$= np + (n^2-n)p^2$$

$$Var(x) = E(x^2) - (E(x))^2$$

$$= np + n^2p^2 - np^2 - (np)^2$$

$$= np - np^2$$

$$= np(1-p) = npq$$

Ex For a r.v X, $M_X(t) = \frac{1}{81}(e^t+2)^4$. Find $P(X \le 2)$

:→ For a binomial r.v Y with parameters n and p the moment generating funⁿ of x is given by

$$M_Y(t) = (q+pe^t)^n$$

Given that

$$M_X(t) = \frac{1}{81}(e^t+2)^4$$

$$= \left(\frac{2}{3} + \frac{1}{3}e^t\right)^4$$

By the uniquness thm of M.G.F $M_X(t) = M_Y(t)$ $\Leftrightarrow$ $F_X = F_Y$

$$\therefore M_X(t) = M_Y(t)$$

$$= (q+pe^t)^n$$

$$= \left(\frac{1}{3}e^t + \frac{2}{3}\right)^4$$

$$\therefore p = \frac{1}{3}, \; q = \frac{2}{3}, \; n = 4$$

$$P(X=x) = 4C_x \, p^x \, q^{4-x}$$
$$= 4C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}, \quad x = 0,1,2,3,4$$

$$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$$
$$= q^4 + 4pq^3 + 6p^2q^2$$
$$= \frac{16}{81} + \frac{32}{81} + \frac{8}{27} = \frac{8}{9}$$

Ex: A box contains 100 cellphones, 20 of which are defective 10 cellphones are selected for inspection. Find the probability that (i) at least one is defective
(ii) at the most three are defective
(iii) all the ten are defective
(iv) none of the ten is defective

:→ X — no. of defective cellphone
$n = 10$
$p = \frac{20}{100} = \frac{1}{5}$ $\quad \therefore q = 1 - \frac{1}{5} = \frac{4}{5}$

$$P(X=x) = 10C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{10-x}, \quad x = 0,1,2,\ldots,10$$
$$= 10C_x \frac{4^{10-x}}{5^{10}}$$

(i) at least one cellphone defective

$$P(X \ge 1) = 1 - P(X=0)$$
$$= 1 - \left(\frac{4}{5}\right)^{10} = 0.8926$$

(ii) at the most three cellphones are defective

$$P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$
$$= \frac{1}{5^{10}}\left[4^{10} + (10 \times 4^9) + (45 \times 4^8) + (120 \times 4^7)\right]$$
$$= 0.8791$$

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(iii) All ten cellphones are defective

$$P(X=10) = 1 \times \left(\frac{1}{5}\right)^{10} = 1.0240 \times 10^{-7}$$

(iv) None of the ten is defective is

$$P(X=0) = 1 \times \left(\frac{4}{5}\right)^{10} = 0.1074$$

Ex: It is known that 5% of the books bound at a certain bindery have defective bindings. Find the prob. that 2 of 100 books bound by this bindery will have defective bindings.

:→ X — no. of books with defective bindings.

$n = 100$

$p = \frac{5}{100} = \frac{1}{20}$

$$P(X=x) = {}^{100}C_x \left(\frac{1}{20}\right)^x \left(\frac{19}{20}\right)^{100-x} \quad ; \ x = 0,1,2,\ldots,100$$

$$\therefore P(X=2) = {}^{100}C_2 \left(\frac{1}{20}\right)^2 \left(\frac{19}{20}\right)^{98} = 0.0812$$

Ex: Six dice are thrown 729 times. How many times do you expect at least 3 dice to show 5 or 6?

:→ Success — Event getting a 5 or 6

$\therefore p = \frac{2}{6} = \frac{1}{3} \quad \therefore q = \frac{2}{3}$

When 6 dice are thrown, the no. of successes X, follows a binomial distribution with

$n = 6$

$p = \frac{1}{3}$

$$P(X=x) = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x} \quad x = 0,1,\ldots,6$$

$$P(X \geq 3) = 1 - P(X < 3) = 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - \frac{1}{3^6}\left[2^6 + (6 \times 2^5) + (10 \times 2^4)\right] = \frac{233}{729}$$

Thus, when 6 dice are thrown 729 times, the no. of times we expect at least 3 dice to show 5 or 6 is

$$N \times P(X \geq 3) = 729 \times \frac{233}{729} = 233$$

Ex: In a certain ~~town~~ town 20% samples of the population are literate. Assume that 200 investigators each take sample of 10 individuals to see whether they are literate. How many investigators would you expect to report that 3 people or less are literate in the samples?

$\rightarrow$ Given:

$$P(\text{an individuals is literate}) = \frac{20}{100} = 0.2$$

ie $p = 0.2$

$q = 1 - p = 0.8$

$n = 10$

X – no. of literate

$$P(X = x) = nC_x\, p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

$$= 10C_x (0.2)^x (0.8)^{10-x}$$

P(Investigator reporting 3 or less as literate)

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= 10C_0 (0.2)^0 (0.8)^{10} + 10C_1 (0.2)^1 (0.8)^9 + 10C_2 (0.2)^2 (0.8)^8 + 10C_3 (0.2)^3 (0.8)^7$$

$$= (0.8)^7 [(0.8)^3 + 10(0.2)(0.8)^2 + 45(0.2)^2(0.8) + 120(0.2)^3]$$

$$= 0.2097 (0.512 + 1.28 + 1.44 + 0.96)$$

$$= 0.8790$$

The No. of investigators reporting 3 or less as literate out of 200 $= 200 \times 0.8790 = 175.82 = 176.$

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Ex: A variate takes values 0, 1, 2, ..., n with frequencies proportional to the binomial coefficients $1, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$. Find the mean and the variance of the distribution and show that the variance is half of the mean.

:→ The total frequency $= 1 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n$

$= (1+1)^n = 2^n$

Given: The variate takes values 0, 1, 2, ..., n with frequencies proportional to the binomial coeffi. $1, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$

∴ The probabilities for 0, 1, 2, ..., n values of the variates are $\frac{1}{2^n}, \frac{{}^nC_1}{2^n}, \frac{{}^nC_2}{2^n}, \ldots, \frac{{}^nC_n}{2^n}$ resp.

The above are the terms of a binomial distribution $\left(\frac{1}{2} + \frac{1}{2}\right)^n$ where $p = \frac{1}{2}$ and $q = \frac{1}{2}$

$$\text{Mean} = np = \frac{n}{2}$$

$$\text{Vari} = npq = n \times \frac{1}{2} \times \frac{1}{2} = \frac{n}{4} = \frac{\text{mean}}{2}$$

Ex: An irregular six-faced die is thrown and the expectation that in 10 throws it will give 5 even nos. is twice the expectation that it will give 4 even nos. How many times in 10000 sets of 10 throws each, would you expect it to give no even no.?

:→ Let the r.v. X denote the no. of even nos

$$P(\text{getting } x \text{ even no}) = P(X = x)$$

$$= {}^nC_x \, p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

Given: n = 10

$$\therefore P(X = x) = {}^{10}C_x \, p^x q^{10-x}, \quad x = 0, 1, 2, \ldots, 10.$$

$\therefore P(\text{getting 5 even no.}) = 2P(\text{getting 4 even no.})$

$$P(X=5) = 2P(X=4)$$

$$\text{i.e. } 10C_5\, p^5 q^5 = 2\,(10C_4\, p^4 q^6)$$

$$\text{i.e. } \frac{p}{5} = \frac{q}{3} \Rightarrow 3p = 5q$$

$$\Rightarrow 3p = 5(1-p)$$

$$\Rightarrow 8p = 5$$

$$\therefore p = \frac{5}{8}$$

$$\therefore q = \frac{3}{8}$$

$\therefore P(\text{getting } x \text{ even no.}) = 10C_x \left(\frac{5}{8}\right)^x \left(\frac{3}{8}\right)^{10-x} \quad x = 0,1,\dots,n$

$\therefore$ The required no. of times that in 10000 sets of 10 throws each we get no even nos

$$= 10000 \times P(X=0)$$

$$= 10000 \times 10C_0 \left(\frac{5}{8}\right)^0 \left(\frac{3}{8}\right)^{10} = 1 \text{ (appr.)}$$

Ex: If at least 1 child in a family with 2 children is a boy, what is the prob. that both children are boys?

$\to$

$$P(\text{child is a boy}) = \frac{1}{2} = p \quad \therefore q = \frac{1}{2}$$

$$n = 2$$

X - no. of boys

$$P(X=x) = 2C_x \left(\frac{1}{2}\right)^2, \quad x = 0,1,2.$$

$$P(\text{at least 1 boy}) \quad P(X = 1 \text{ or } 2) = 2C_1\left(\frac{1}{2}\right)^2 + 2C_2\left(\frac{1}{2}\right)^2 = \frac{3}{4}$$

$P(\text{both boys} \mid \text{at least 1 boy})$

$$P(X=2 \mid X=1 \text{ or } 2) = \frac{P(X=2 \cap X=1 \text{ or } 2)}{P(X=1 \text{ or } 2)}$$

$$= \frac{P(X=2)}{P(X=1)+P(X=2)} = \frac{2C_2\left(\frac{1}{2}\right)^2}{\frac{3}{4}} = \frac{1}{3}$$

Ex: If the probability of success is 1/100, how many trials are necessary in order that probability of at least one success is greater than 1/2 ?

→ X — no. of successes

Given: $p = \frac{1}{100}$ $\therefore q = \frac{99}{100}$

$$\therefore P(X=x) = nC_x \left(\frac{1}{100}\right)^x \left(\frac{99}{100}\right)^{n-x}, \quad x = 0,1,2,\ldots,n$$

$$P(X=0) = (0.99)^n = q^n$$

P(hitting at least once)

$$P(X\geq 1) = 1 - P(X<1)$$
$$= 1 - P(X=0)$$
$$= 1 - q^n$$

$P(X\geq 1) > \frac{1}{2}$ $\therefore 1 - q^n > \frac{1}{2}$

ie $1 - (0.99)^n > \frac{1}{2}$

ie $1 - \frac{1}{2} > (0.99)^n$

ie $\frac{1}{2} > (0.99)^n$

$\Rightarrow \log(0.5) > n \log(0.99)$

$\Rightarrow \frac{0.3010}{0.0044} < n$

$\Rightarrow 68.4 < n$

$\Rightarrow \boxed{n = 69}$

Ex: Ten coins are tossed 1024 times and the following frequencies are observed. Compare these frequencies with the expected frequencies:

No. of heads :	0	1	2	3	4	5	6	7	8	9	10
Frequencies :	2	10	38	106	188	257	226	128	59	7	3

$\therefore \rightarrow$ Given $n = 10$

$N = \Sigma f = 1024$

P = Prob. of getting a head in one toss

$\therefore P = \frac{1}{2}$

$\& \ q = 1/2$

The expected frequency of x heads

$= 1024 \times 10C_x \left(\frac{1}{2}\right)^{10-x} \left(\frac{1}{2}\right)^{x}, \quad x = 0, 1, 2, --, 10$

Hence

Observed fre. :	2	10	38	106	188	257	226	128	59	7	3
Expected fre. :	1	10	45	120	210	252	210	120	45	10	1

Ex: A biased coin is tossed 8 times and the no. of heads are noted. The experiment is repeated 100 times and the following frequency distribution is obtained.

x	0	1	2	3	4	5	6	7	8
f	2	7	13	15	25	16	11	8	3

Fit a binomial distribution to these data Calculate the theoretical frequencies.

$\therefore \rightarrow$

$$\text{Mean} = \mu = \frac{\Sigma fx}{\Sigma f} = \frac{404}{100} = 4.04$$

Thus $P = \frac{\mu}{n} = \frac{4.04}{8} = 0.505$ (Mean $= np$)

$\therefore q = 1 - P = 0.495$

Since $N = 100$, the theoretical frequencies is given by

$$N f(x) = N \cdot {}^{n}C_x \, P^{x} q^{n-x} = 100 \cdot {}^{8}C_x (0.505)^{x} (0.495)^{8-x}$$

$x = 0, 1, 2, ---, 8$

$Nf(0) = 100 \times (0.495)^8 = 0.3604$

$Nf(1) = 100 \times 8C_1 (0.505)(0.495)^7 = 2.9414$ and so on upto $Nf(8)$

No of heads	observed freq	Theo. fre
0	2	0.3604
1	7	2.9414
2	13	10.5030
3	15	21.4303
4	25	27.3290
5	16	22.3049
6	11	11.3777
7	8	3.3164
8	3	0.4229

Generalization of Bernoulli Theorem / Multinomial Distribution:

If $A_0, A_1, A_2, \ldots, A_n$ are exhaustive and mutually exclusive events associated with a random experiment such that $P(A_i) = P_i$ where $P_1 + P_2 + \cdots + P_k = 1$ and if the experiment is repeated n times, then the probability that A_1 occurs r_1 times, A_2 occurs r_2 times, ----, A_k occurs r_k times such that $r_1 + r_2 + \cdots + r_k = n$ is given by

$$P_n(r_1, r_2, \ldots, r_k) = \frac{n!}{r_1!\, r_2! \cdots r_k!} P_1^{r_1} P_2^{r_2} \cdots P_k^{r_k}$$

eg A fair die is rolled 5 times. Find the prob. that 1 shown twice, and 6 shown once.

→ Given $r_1 = 5$

A_1 = Getting 1

A_2 = " 3

A_3 = " 6

$P(A_1) = \frac{1}{6} = P_1$ $P(A_2) = \frac{1}{6} = P_2$ $P(A_3) = \frac{1}{6} = P_3$

$r_1 = 2$, $r_2 = 2$, $r_3 = 1$

$\therefore r_1 + r_2 + r_3 = 5 = n$

Using the multinomial distribution, the required pro.

$$P_n(r_1, r_2, r_3) = \frac{n!}{r_1!\, r_2!\, r_3!} P_1^{r_1} P_2^{r_2} P_3^{r_3}$$

$$= \frac{5!}{2!\,2!\,1!} \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)$$

$$= 0.0039$$

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Ex: If the moment generating funⁿ of a r.v X is $\left(\frac{2}{3}+\frac{1}{3}e^t\right)^9$ show that $P(\mu-2\sigma < X < \mu+2\sigma) = \sum_{x=1}^{5} {}^9C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$

Sol → Given: Moment generating funⁿ

$$M_X(t) = (q+pe^t)^n = \left(\frac{2}{3}+\frac{1}{3}e^t\right)^9$$

$\therefore$ $q = \frac{2}{3}$ & $p = \frac{1}{3}$ & $n = 9$

$\therefore$ X follows binomial distribution $\sim B(9, \frac{1}{3})$

$\therefore$ $E(X) = np = 9 \times \frac{1}{3} = 3 = \mu$

$Var(X) = npq = 9\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = 2 = \sigma^2$

$\therefore$ $\sigma = \sqrt{Var(X)} = \sqrt{2}$

$\mu \pm 2\sigma = 3 \pm 2\sqrt{2} = 0.2$ or 5.8

$\therefore$ $P(\mu-2\sigma < X < \mu+2\sigma) = P(0.2 < X < 5.8)$

$= P(1 \le X \le 5)$

$= \sum_{x=1}^{5} {}^nC_x p^x q^{n-x}$

$= \sum_{x=1}^{5} {}^9C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$

Ex: If the moment generating funⁿ of a r.v. X is of the form $(0.4e^t + 0.6)^8$, find the moment generating funⁿ of $3X+2$.

Sol → Here $q = 0.6$, $p = 0.4$, $n = 8$

$M_{3X+2}(t) = E\left[e^{(3X+2)t}\right] = E\left[e^{2t} \cdot e^{3Xt}\right]$

$= e^{2t} E(e^{3Xt})$

$= e^{2t} E\left[e^{X(3t)}\right]$

$= e^{2t}(0.6 + 0.4e^{3t})^8$

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Recurrence Formula for the Central Moments of the Binomial Distribution.

By def^n, the k^{th} order central moment μ_k is given by

$$\mu_k = E\{X - E(X)\}^k$$

$$= \sum_{x=0}^{n} (x-np)^k \, {}^nC_x p^x q^{n-x} \quad \text{--- (1)}$$

Diff^{ing} (1) w.r.t p, we get

$$\frac{d\mu_k}{dp} = \sum_{x=0}^{n} {}^nC_x \Big\{ -nk(x-np)^{k-1} p^x q^{n-x} + (x-np)^k \left[x p^{x-1} q^{n-x} + p^x (n-x) q^{n-x-1} (-1) \right] \Big\}$$

$$= -nk\mu_{k-1} + \sum_{x=0}^{n} {}^nC_x (x-np)^k p^{x-1} q^{n-x-1} \left[xq - (n-x)p \right]$$

$$= -nk\mu_{k-1} + \frac{1}{pq} \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} (x-np)^{k+1}$$

$$= -nk\mu_{k-1} + \frac{1}{pq}\mu_{k+1}$$

$$\therefore \mu_{k+1} = pq\left(\frac{d\mu_k}{dp} + nk\mu_{k-1}\right)$$

Moments of higher order

Put $k=1$ in above, $\mu_2 = pq\left(\frac{d\mu_1}{dp} + n\mu_0\right)$

$= npq \quad (\mu_0 = 1 \ \& \ \mu_1 = 0)$

$k=2$ $\quad \mu_3 = pq\left(\frac{d\mu_2}{dp} + 2n\mu_1\right)$

$= pq\frac{d}{dp}(np(1-p)) = npq(1-2p)$

$= npq(q-p)$

$k=3$, simplifying, $\mu_4 = npq[1+3pq(n-2)]$

Note: μ_2 = Variance

μ_3 = measure of skewness (β_1)

μ_4 = " " kurtosis (β_2)

Where $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$ $\quad \beta_2 = \dfrac{\mu_4}{\mu_2^2}$

Note: The mean of binomial distri. is always greater than variance.

n]}

p]