5 Multiple Integral

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1) evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

$$= \int_{y=0}^{1} \frac{1}{\sqrt{1-y^2}} \left[\int_{x=0}^{1} \frac{dx}{\sqrt{(1-x^2)}} \right] dy.$$

$$= \int_{y=0}^{1} \frac{1}{\sqrt{1-y^2}} \left[\sin^{-1} x \right]_0^1 dy$$

$$= \int_{y=0}^{1} \frac{1}{\sqrt{1-y^2}} \left[\sin^{-1} 1 - \sin^{-1} 0 \right] dy$$

$$= \int_{y=0}^{1} \frac{1}{\sqrt{1-y^2}} \left[\frac{\pi}{2} \right] dy$$

$$= \frac{\pi}{2} \int_{y=0}^{1} \frac{1}{\sqrt{1-y^2}} dy.$$

$$= \frac{\pi}{2} \left[\sin^{-1} y \right]_0^1$$

$$= \frac{\pi}{2} \left[\sin^{-1} 1 - \sin^{-1} 0 \right]$$

$$= \frac{\pi}{2} \left[\frac{\pi}{2} \right]$$

$$= \frac{\pi}{4}.$$

3) Evaluate $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{\frac{y}{\sqrt{x}}} \, dy \, dx.$

$$= \int_{x=1}^{4} \int_{y=0}^{\sqrt{x}} \frac{3}{2} e^{\frac{y}{\sqrt{x}}} \, dy \, dx.$$

$$= \frac{3}{2} \int_{x=1}^{4} \left[\int_{y=0}^{\sqrt{x}} e^{\frac{y}{\sqrt{x}}} \, dy \right] dx.$$

$$= \frac{3}{2} \int_{x=1}^{4} \left[\frac{e^{y/\sqrt{x}}}{1/\sqrt{x}} \right]_0^{\sqrt{x}} dx$$

$$= \frac{3}{2} \int_{x=1}^{4} \left[\sqrt{x} \, e^{y/\sqrt{x}} \right]_0^{\sqrt{x}} dx$$

$$= \frac{3}{2} \int_{x=1}^{4} \left[\sqrt{x} \, (e^{1} - e^{0}) \right] dx.$$

$$= \frac{3}{2} (e - 1) \int_{x=1}^{4} [\sqrt{x}] \, dx$$

$$= \frac{3}{2} (e - 1) \left[\frac{x^{3/2}}{3/2} \right]_1^4$$

$$= \frac{3}{2} (e - 1) \times \frac{2}{3} \cdot \left[4^{3/2} - 1^{3/2} \right]$$

$$= (e - 1) \cdot [8 - 1]$$

$$= 7 (e - 1)$$

8) Evaluate $\int_0^1 \int_0^x \int_0^{\sqrt{x+y}} z \, dz \, dy \, dx$.

$$= \int_{x=0}^{1} \int_{y=0}^{x} \left[\int_{z=0}^{\sqrt{x+y}} z \, dz \right] dy \, dx.$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} \left[\frac{z^2}{2} \right]_0^{\sqrt{x+y}} dy \, dx.$$

$$= \frac{1}{2} \int_{x=0}^{1} \left[\int_{y=0}^{x} (x+y) \, dy \right] dx.$$

$$= \frac{1}{2} \int_{x=0}^{1} \left[xy + \frac{y^2}{2} \right]_0^x dx.$$

$$= \frac{1}{2} \int_{x=0}^{1} \left(x^2 + \frac{x^2}{2} \right) dx.$$

$$= \frac{1}{2} \int_{x=0}^{1} \frac{3}{2} x^2 \, dx.$$

$$= \frac{3}{4} \int_0^1 x^2 \, dx$$

$$= \frac{3}{4} \left[\frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{4} \left[1 - 0 \right]$$

$$= \frac{1}{4}.$$

6) Evaluate the triple integral

$$= \int_0^1 \int_0^{\pi} \int_0^{\pi} y \sin z \, dx \, dy \, dz.$$

$$= \int_{z=0}^{1} \int_{x=0}^{\pi} \sin z \left[\int_{y=0}^{\pi} y \, dy \right] dx \, dz.$$

$$= \int_{z=0}^{1} \int_{x=0}^{\pi} \sin z \left[\frac{y^2}{2} \right]_0^{\pi} dx \, dz.$$

$$= \int_{z=0}^{1} \int_{x=0}^{\pi} \sin z \left[\frac{\pi^2}{2} \right] dx \, dz.$$

$$= \frac{\pi^2}{2} \int_{z=0}^{1} \sin z \left[\int_{x=0}^{\pi} 1 \cdot dx \right] dz.$$

$$= \frac{\pi^2}{2} \int_{z=0}^{1} \sin z \left[x \right]_0^{\pi} dz.$$

$$= \frac{\pi^2}{2} \int_{z=0}^{1} \sin z \left[\pi \right] dz.$$

$$= \frac{\pi^3}{2} \int_{z=0}^{1} \sin z \, dz.$$

$$= \frac{\pi^3}{2}\left[-\cos z\right]_0^1$$

$$= \frac{\pi^3}{2}\left[-\cos 1 + \cos 0\right]$$

$$= \frac{\pi^3}{2}\left[-\cos 1 + 1\right]$$

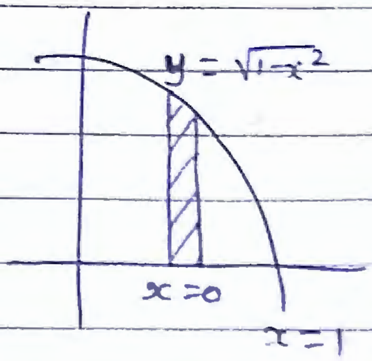
$$= \frac{\pi^3}{2}\left[1 - \cos 1\right]$$

26) $$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\,dy\,dx.$$

Let $x = r\cos\theta$, $y = r\sin\theta$

$dy\,dx = r\,dr\,d\theta$

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} e^{-r^2}\, r\,dr\,d\theta$$



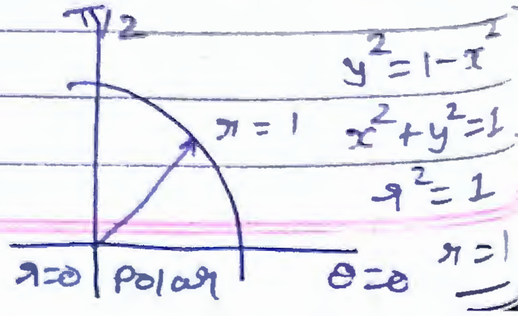
Cartesian.

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} e^{-r^2}\, r\,dr\,d\theta$$

Let $r^2 = t$

$2r\,dr = dt$

$r\,dr = \frac{1}{2}\,dt.$



$$\boxed{\begin{array}{llll} r \Rightarrow 0 & \Rightarrow & t \to 0 \\ r \to 1 & \Rightarrow & t \to 1 \end{array}}$$

$$\int_{\theta=0}^{\pi/2} \left[\int_{t=0}^{1} e^{-t} \frac{1}{2} \, dt \right] d\theta$$

$$\frac{1}{2} \int_{\theta=0}^{\pi/2} \left[-e^{-t} \right]_0^1 d\theta$$

$$= -\frac{1}{2} \int_{\theta=0}^{\pi/2} \left[e^{-1} - 1 \right] d\theta .$$

$$= -\frac{1}{2} (e^{-1} - 1) \int_0^{\pi/2} d\theta$$

$$= \frac{1}{2} (1 - e^{-1}) \left[\theta \right]_0^{\pi/2}$$

$$= \frac{1}{2} \left(1 - \frac{1}{e}\right) \left(\pi/_2\right)$$

$$= \frac{\pi}{4} \left(1 - \frac{1}{e}\right)$$

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4) $$I = \int_0^{\pi} \int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= \int_0^{\pi} \left[\frac{r^2}{2} \right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2} \int_0^{\pi} a^2 . (1+\cos\theta)^2 \, d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} (1+\cos\theta)^2 \, d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} (2\cos^2 \theta/2)^2 \, d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} (4\cos^4 \theta/2) \, d\theta$$

$$= 2a^2 \left[\frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} \right]$$

$$= \frac{3\pi a^2}{4}$$

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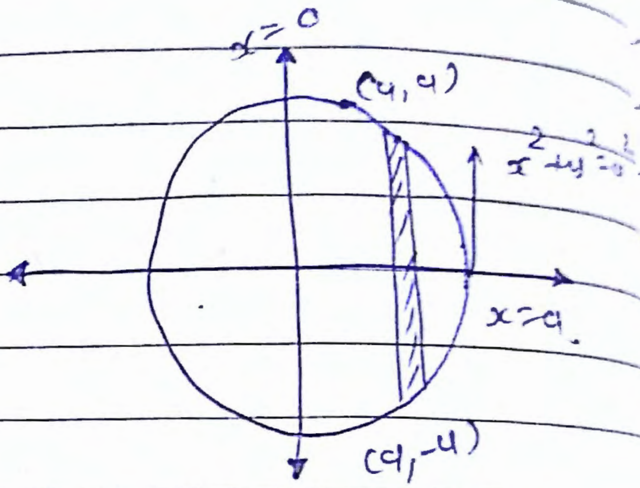
10) Evaluate $\iint_R dy\,dx$ where R is the region bounded by $x=0$, $x=a$ and the circle with center $(0,0)$ and radius a.



→ $I = \iint_R dy\,dx$

$x = 0, \quad x = a$

The limits are

$0 \leq x \leq a$

$0 \leq y \leq \sqrt{a^2-x^2}$

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx.$$

$$= \int_0^a [y]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left[\sqrt{a^2-x^2}\right] dx$$

$$= \int_0^a \left[\sqrt{a^2-x^2}\right. dx.$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]_0^a$$

$$= \left\{ 0 + \frac{a^2}{2}\frac{\pi}{2} \right\}$$

$$= \frac{a^2\pi}{4}$$

So for the whole covered region

$$= \cancel{2} \times \frac{a^2\pi}{\cancel{4}\,2}$$

$$= \frac{a^2\pi}{2}$$

9) Evaluate the integral $\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2\cos^2\theta + z^2)$ ~~$r\,d\theta\,dr\,dz$~~

$$\rightarrow I = \int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^3\cos^2\theta + rz^2)\,d\theta\,dr\,dz.$$

$$= \int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} \left(r^3\left(\frac{1+\cos 2\theta}{2}\right) + rz^2\right) d\theta\,dr\,dz$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} r^3 + r^3\cos 2\theta + 2rz^2\,d\theta\,dr\,dz$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{2}} \left[r^3\theta + \frac{r^3 \sin 2\theta}{2} + 2r\theta z^2 \right]_0^{2\pi} dr\,dz$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{2}} \left[2\pi r^3 + 2\pi \cdot 2rz^2 \right] dr\,dz$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{2}} 2\pi r^3 + 4\pi rz^2 \, dr\,dz$$

$$= \frac{1}{2}\int_0^1 \left[\frac{2\pi r z^2}{2} + \frac{2\pi z^3}{3} \right]_0 dz$$

$$= \frac{1}{2}\left[\frac{\pi}{2}\frac{z^3}{3} + \frac{2\pi z^4}{4} \right]_0^1$$

$$= \frac{1}{2}\left[\frac{\pi}{6} + \frac{2\pi}{4} \right]$$

$$= \frac{1}{2}\left[\frac{2\pi + 6\pi}{12} \right]$$

$$= \frac{8\pi}{24}$$

$$= \frac{\pi}{3}.$$

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3 Fourier Series

1) find the fourier series for $f(x) = x^2$ where $-\pi \le x \le \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx .$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx .$$

$$= \frac{2}{\pi} \left[\int_0^{\pi} x^2\, dx . \right]$$

$$= \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2\pi^3}{3\pi}$$

$$a_0 = \frac{2\pi^2}{3}$$

$$\rightarrow a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi} \left[x^2 \frac{\sin nx}{n} - 2x \left(-\frac{\cos nx}{n^2} \right) + x \left(-\frac{\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{2\pi (-1)^n}{n} \right]$$

$$a_n = \frac{4 (-1)^n}{n}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx .$$

$f(x) = x^2$ so, $f(x)$ is even

$f(-x) = x^2$ $\sin nx$ is odd.

so,

$b_n = 0$

$$f(x) = \frac{2\pi^3}{6} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n .$$

3) obtain the fourier series to represent the function $f(x) = \frac{1}{4}(\pi - x^2)$, $0 < x < 2\pi$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \, dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \, dx$$

$$= \frac{1}{4\pi}\left[-\frac{(\pi - x)^3}{3}\right]_0^{2\pi}$$

$$= \frac{-1}{4\pi}\left[-\frac{\pi^3}{3} - \frac{\pi^3}{3}\right]$$

$$a_0 = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx.$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx \, dx.$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(\frac{-\cos nx}{n^2}\right) + (-1)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{2(-\pi)(-1)}{n^2} + \frac{2(\pi)}{n^2}\right]$$

$$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right]$$

$$= \frac{1}{\cancel{4\pi}}\left[\frac{\cancel{4\pi}}{n^2}\right]$$

$$= \frac{1}{n^2}$$

$$a_n = \frac{1}{n^2}$$

$$b_n = \frac{1}{4}\int_0^{2\pi} (\pi-x)^2 \sin nx \, dx$$

$$= \frac{1}{4}\left[(\pi-x)^2 \frac{(-\cos nx)}{n} - 2(\pi-x)(-1)\frac{(-\sin nx)}{n^2} - 2(-1)\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{4}\left[\frac{-\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right]$$

$$b_n = 0$$

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$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.$$

4) find fourier series of $f(x) = 1 + \frac{2x}{\pi}$; $-\pi \le x \le 0$

$= 1 - \frac{2x}{\pi}$; $0 \le x \le \pi$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 1 + \frac{2x}{\pi}\, dx + \int_{0}^{\pi} 1 - \frac{2x}{\pi}\, dx\right]$$

$$= \frac{1}{\pi}\left[\left[x + \frac{x^2}{\pi}\right]_{-\pi}^{0} + \left[x - \frac{x^2}{\pi}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[[-\pi + \pi] + [\pi - \pi]\right]$$

$a_0 = 0$

$$a_n = \frac{1}{\pi}\left[\int_{\pi}^{0} \left(1 + \frac{2x}{\pi}\right) \cos nx\, dx + \int_{0}^{\pi} \left(1 - \frac{2x}{\pi}\right) \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(1+\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(\frac{2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{1}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) + \left(\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{2}{\pi}\cdot\frac{1}{n^2} - \frac{2}{\pi}\frac{(-1)^n}{n^2}\right] + \frac{1}{\pi}\left[\frac{-2}{\pi}\frac{(-1)^n}{n^2} + \frac{2}{\pi}\frac{1}{n^2}\right]$$

$$= \frac{2}{n^2\pi^2} - \frac{2(-1)^n}{\pi^2 n^2} - \frac{2(-1)^n}{\pi^2 n^2} + \frac{2}{\pi^2 n^2}$$

$$= \frac{4}{n^2\pi^2} - \frac{4(-1)^n}{\pi^2 n^2} = \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]$$

$$\rightarrow b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(1+\frac{2x}{\pi}\right)\sin nx\, dx + \int_{0}^{\pi}\left(1-\frac{2x}{\pi}\right)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(1+\frac{2x}{\pi}\right)\left(\frac{-\cos nx}{n}\right) - \left(\frac{2}{\pi}\right)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{1}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{-\cos nx}{n}\right) + \left(\frac{2}{\pi}\right)\left(\frac{-\sin nx}{n^2}\right)\right.$$

$$= \frac{1}{\pi}\left[-\frac{1}{n} + \frac{(-1)^n}{n}\right] + \frac{1}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n}\right]$$

$$= \frac{-1}{n\pi} + \frac{(-1)^n}{n\pi} + \frac{1}{n\pi} - \frac{(-1)^n}{n\pi}$$

$$= 0$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left[1 - (-1)^n\right] \cos n\pi$$

$$= \frac{4}{\pi^2}\left[2 + \frac{2}{3^2} + \frac{2}{5^2} + \cdots\right]$$

$$f(x) = \frac{8}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$x = 0$

$f(x) = 1$

$$1 = \frac{8}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

5) Expand f(x) in fourier series in the interval $(0, 2\pi)$ if $f(x) = -\pi \;;\; -\pi < x < 0$

$= x \;;\; 0 < x < \pi$

and hence show that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} -\pi\, dx + \int_{0}^{\pi} x\, dx \right]$$

$$= \frac{1}{\pi} \left[-\pi [x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[-\pi^2 + \frac{\pi^2}{2} \right]$$

$$a_0 = \frac{-\pi}{2}$$

$$a_n = \frac{1}{\pi} \left[\int_{-\pi}^{0} -\pi \cos nx\, dx + \int_{0}^{\pi} x \cos nx\, dx \right]$$

$$= \frac{1}{\pi} \left[-\pi \left(\frac{\sin nx}{n} \right) \right]_{-\pi}^{0} + \frac{1}{\pi} \left[x \left(\frac{\sin nx}{n} \right) - \left(\frac{-\cos nx}{n} \right) \right]_0^{\pi}$$

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$$= 0 + \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - 1\right]$$

$$= \frac{(-1)^n - 1}{n^2 \pi}$$

if $n = \text{odd}$ $\quad a_n = \frac{-2}{n^2\pi^2}$ $\quad n = \text{even}$ $\quad a_n = 0$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx\, dx + \int_{0}^{\pi} x \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{\pi(-1)^n}{n}\right] + \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n}\right]$$

$$= \frac{1}{n} - \frac{(-1)^n}{n} - \frac{(-1)^n}{n}$$

$$= \frac{1}{n}\left(1 - 2(-1)^n\right)$$

$$f(x) = \frac{-\pi}{4} + \sum_{n=\text{odd}}^{\infty} \frac{-2}{n^2\pi} \cos nx + \sum_{n=1}^{\infty} \frac{1}{n}\left(1 - 2(-1)^n\right) \sin nx.$$

$$= \frac{-\pi}{4} + \frac{2}{\pi}\left[1 \cos x + \frac{1}{3^2} \cos 3x + \cdots\right]$$

$$3 \sin x - \frac{1}{2} \sin 2x \cdots$$

$x = 0$

$$-\frac{\pi}{2} = -\frac{\pi}{4} + \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$f(x) = \frac{\pi}{2}$

$$-\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

6) Find the F.S of $f(x) = 2x - x^2$ in the Interval of $(0, 3)$ Hence, deduce that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{2} + b_n \sin\frac{n\pi x}{l}\right]$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_0 = \frac{2}{3}\int_0^3 (2x - x^2)\,dx$$

$$= \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3$$

$$= 0$$

$$a_n = \frac{2}{3}\int_0^3 (2x - x^2)\cos\left(\frac{2n\pi x}{3}\right) dx.$$

$$= \frac{2}{3}\left[\frac{(2x-x^2)}{\frac{2n\pi}{3}} \sin\left(\frac{2n\pi x}{3}\right) - \frac{(2-2x)}{\frac{4n^2\pi^2}{9}}\left(-\cos\frac{2n\pi x}{3}\right)\right.$$

$$\left. + \frac{(-2)}{\frac{8n^3\pi^3}{27}}\left(-\sin\frac{2\pi n x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{-9}{n^2\pi^2} - \frac{9}{2n^2\pi^2}\right] = \frac{-9}{n^2\pi^2}$$

$$a_n = \frac{-9}{n^2\pi^2}$$

$$\underline{\underline{b_n}} = \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}\, dx$$

$$= \frac{2}{3}\left[\frac{3(2x-x^2)}{2n\pi}\left(-\cos\frac{2n\pi x}{3}\right) + \frac{9(2-2x)}{4n^2\pi^2}\sin\frac{2n\pi}{3}\right.$$

$$\left. - \frac{(54)}{8n^3\pi^3}\cos\frac{2n\pi x}{3}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{9}{2n\pi} - \frac{54}{8n^3\pi^3} + \frac{54}{8n^3\pi^3}\right]$$

$$= \frac{3}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{(-9)}{n^2\pi^2} \cos \frac{2n\pi x}{3} +$$

$$\sum_{n=1}^{\infty} \frac{3}{n\pi} \sin\left(\frac{2n\pi x}{3}\right)$$

$x = 0$

$f(x) = 0$

$$0 = 9\left[\frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2 \cdot \pi^2} + \cdots\right].$$

8) Find the half range sine series of $f(x) = x^3$ in $0 \le x \le \pi$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \, dx.$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cancel{x^3 \sin nx\,dx}$$

$$= \frac{2}{\pi}\left[x^3\frac{(-\cos nx)}{n} - 3x^2\frac{(-\sin nx)}{n^2} + 6x\frac{(\cos nx)}{n^3} - \frac{6(\sin nx)}{n^4}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3}{n}(-1)^n + \frac{6\pi}{n^3}(-1)^n\right]$$

$$= \frac{2}{n^3}(-1)^n\left[6 - n^2\pi^2\right]$$

$$f(x) = 2\sum_{n=1}^{\infty}\frac{1}{n^3}(-1)^n\left[6 - n^2\pi^2\right]$$

9) Obtain Half Range cosine series of ~~two~~ $f(x) = \sin x$ in the interval $0 < x < \pi$ Hence deduce that $\frac{\pi^2}{12} - \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \sin x \, dx.$$

$$= -\frac{2}{\pi} [\cos x]_0^{\pi}$$

$$= -\frac{2}{\pi} [-1 - 1] = \frac{4}{\pi}$$

$$a_0 = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} 2 \sin x \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} (\sin (1+n) x + \sin (1-n) x) dx$$

$$= \frac{1}{\pi} \left[\frac{(-\cos (1+n) n)}{1+n} + \frac{(-\cos (1-n) x}{(1-n)} \right]_0^{\pi}$$

$$= \frac{-1}{\pi}\left[\frac{(-1)^{n+1}}{1+n} + \frac{(-1)^{1-n}}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{n}+1}{1+n} + \frac{(-1)^{n}+1}{1-n}\right]$$

$$= \frac{(-1)^{n}+1}{\pi}\left[\frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{(-1)^{n}+1}{\pi} \cdot \frac{2}{1-n^2}$$

$$= \frac{2}{\pi(n^2-1)}\left[-(-1)^{n+1}+1\right]$$

$$f(x) = \frac{4}{2\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi(n^2-1)}\left[1-(-1)^{n+1}\right]\cos nx$$

$x = 0$

$f(x) = 0$

2 Infinite Sequences & Series

4) Determine whether following series converge or diverge. find sum of series $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \cdots$

$a = 1 \qquad ar = -\frac{1}{2} \qquad r = -\frac{1}{2}$

$|r| < 1$

So this series is convergent.

Sum of this series $= \frac{a}{1-r}$

$= \frac{1}{1+\frac{1}{2}}$

$= \frac{2}{3}.$

5) Test convergence of $\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{1}{5^n} \right)$

$U_n = \frac{1}{2^n} + \frac{1}{5^n}$

$= \left(\frac{1}{2}\right)^n + \left(\frac{1}{5}\right)^n$

Both are convergent.

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so the series is convergent.

$$\text{sum} = \frac{1/2}{1 - 1/2} + \frac{1/5}{1 - 1/5}$$

$$= 1 + \frac{1}{4}$$

$$= \frac{5}{4}.$$

6) Test the convergence of $\sum_{n=1}^{\infty} \frac{4^n + 1}{5^n}$

$$U_n = \frac{4^n + 1}{5^n} = \left(\frac{4}{5}\right)^n + \left(\frac{1}{5}\right)^n$$

Both are convergent.

So the series is convergent.

$$\text{Sum} = \frac{4/5}{1 - 4/5} + \frac{1/5}{1 - 1/5}$$

$$= 4 + \frac{1}{4}$$

$$= \frac{17}{4}.$$

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7) Test the series (by integral test.)

1) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$

$= \lim_{b \to \infty} \int_2^b \frac{1/x}{\log x} \, dx.$

$= \lim_{b \to \infty} \left[\log |\log x| \right]_2^b$

$= \lim_{b \to \infty} \left[\log |\log b| - \log |\log 2| \right]$

$= \infty$

So the series is divergent.

2) $\sum_{n=3}^{\infty} \frac{1}{n \log n \sqrt{(\log n)^2 - 1}}$

$= \lim_{b \to \infty} \int_3^b \frac{dx}{x \log x \sqrt{(\log x)^2 - 1}}$

$\log x = t$

$1/x \, dx = \cancel{dx}.\ dt.$

$x \to 3 \Rightarrow t \to \log 3$

$x \to b \Rightarrow t \to \log b$

$$= \lim_{b \to \infty} \int_{\log 3}^{\log b} \frac{dt}{\log t \sqrt{t^2 - 1}}$$

$$= \lim_{b \to \infty} \left[\sec^{-1} t \right]_{\log 3}^{\log b}$$

$$= \lim_{b \to \infty} \left[\sec^{-1}(\log b) - \sec^{-1}(\log 3) \right]$$

$$= \sec^{-1} \infty - \sec^{-1}(\log 3)$$

$$= \frac{\pi}{2} - \sec^{-1}(\log 3)$$

$=$ finite.

$\rightarrow$ The series is convergent.

8) Test the series (by comparison test.)

1) $$\sum_{n=1}^{\infty} \frac{(2n^2 - 1)^{1/3}}{(3n^3 + 2n + 5)^{1/4}}.$$

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$$U_n = \frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}}$$

$$= \frac{n^{2/3}\,(2 - 1/n^2)^{1/3}}{n^{3/4}\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{1/4}}$$

$$= \frac{2 - 1/n^2}{\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{1/4}}$$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{(2-1/n^2)^{1/3}}{\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)}$$

$$\Sigma v_n = \frac{1}{n^p} = \frac{1}{n^{1/12}} \qquad \boxed{p = \frac{1}{12}}$$

2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2^2+3^2+4^2+\cdots}$

$$U_n = \frac{6}{n(n+1)(2n+1)}$$

$$= \frac{6}{n^3(1+\frac{1}{n})(2+\frac{1}{n})}$$

$$v_n = \frac{1}{n^3}$$

Here $p = 3 > 1$

By p series test v_n is convergent.

Now, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{6}{(1+\frac{1}{n})(2+1/n)}$

$$= \frac{6}{2} = \boxed{3}$$

By comparison test given series is convergent

3) $\sum_{n=1}^{\infty} \sqrt[3]{n^3+1} - n$

$$u_n = \sqrt[3]{n^3+1} - n$$

$$= \sqrt[3]{n^3+1} - n \times \frac{\sqrt[3]{n^3+1} + n}{\sqrt[3]{n^3+1} + n}$$

$$= \frac{(n^3+1)^{2/3} - n^2}{(n^3+1)^{1/3} + n}$$

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$$= \frac{n^2\left(1+1/n^3\right)^{2/3} - n^2}{n\left(1+\frac{1}{n^3}\right)^{1/3} + n}$$

$$= \frac{n\left(1+1/n^3\right)^{2/3} - n^2}{\left(1+1/n^3\right)^{1/3} + n}$$

$$= \frac{1}{n^{-1}} \frac{\left(1+1/n^3\right)^{2/3} - 1^2}{\left(1+1/n^3\right)^{1/3} + 1}$$

$$v_n = \frac{1}{n^{-1}}$$

$$p = -1 < 1$$

v_n is divergent by p series test.

Now,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\left(1+1/n^3\right)^{2/3} - 1}{\left(1+1/n^3\right)^{1/3} + 1}$$

$$= \frac{0}{2}$$

$$= 0$$

$\therefore u_n$ is divergent by comparison test.

5) $\sum_{n=1}^{\infty} \frac{n+1}{n^3 - 3n + 2}$.

$$U_n = \frac{n+1}{n^3 - 3n + 2}.$$

$$= \frac{n(1 + 1/n)}{n^3\left(1 - \frac{3}{n^2} + \frac{2}{n^3}\right)}$$

$$= \frac{(1 + 1/n)}{n^2\left(1 - \frac{3}{n^2} + \frac{2}{n^3}\right)}$$

$\lim_{n \to \infty} \frac{U_n}{V_n} = 1$ (finite and non zero).

$\Sigma V_n = \frac{1}{n^p} = \frac{1}{n^2}$ $\qquad p = 2 > 1$

V_n is convergent.

so u_n is also convergent.

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9) $\sum_{n=1}^{\infty} \frac{n^3+1}{2^n+2}$

$$u_n = \frac{n^3+1}{2^n+2}$$

$$u_{n+1} = \frac{(n+1)^3+1}{2^{(n+1)}+2}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^3+1}{2^{(n+1)}+2} \times \frac{2^n+2}{n^3+2}$$

$$= \frac{n^3\left[(1+1/n)^3+1/n^3\right] 2^n\left[1+\frac{2}{2^n}\right]}{n^3\left[1+\frac{2}{n^3}\right] 2^n\left[2+2/2^n\right]}$$

$$\lim \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1 = \text{convergent.}$$

2) $x - \frac{x^3}{3} + \frac{x^5}{5}$

$$U_n = \frac{(-1)^{n+1} x^{2n-1}}{2n-1}$$

$$U_{n+1} = (-1)^{n+2} \frac{x^{2n+1}}{2n+1}$$

$$\frac{U_n}{U_{n+1}} = (-1)^{n+1} \frac{x^{2n-1}}{2n-1} \times \frac{2n+1}{(-1)^{n+2} x^{2n+1}}$$

$$= -\frac{\cancel{x^{2n}}}{x} \times \frac{1}{\cancel{x^{2n}} \cdot x} \frac{(2+1/n)}{(2-1/n)}$$

$$= \frac{1}{x^2}$$

Given series is divergent if $1/x^2 < 1 \Rightarrow x > 1$

given series is convergent if $\frac{1}{x^2} > 1 \Rightarrow x < 1$

10) 11 $\sum_{n=1}^{\infty} \frac{2^{3n}}{3^{2n}}$

$$U_n = \frac{8^n}{9^n}$$

$$\sqrt[n]{u_n} = \frac{8}{9}$$

$$\lim_{n \to \infty} \sqrt[n]{u_n} = \frac{8}{9} \; ; \; (\text{finite}.$$

So the Series is convergent.

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2) $$\sum_{n=1}^{\infty} \frac{[(n+1)x]^{n}}{(n)^{n+1}}$$

$$u_n = \frac{[(n+1)x]^{n}}{n^{n+1}}$$

$$\sqrt[n]{u_n} = \frac{(n+1)x}{n^{1+1/n}}$$

$$= \frac{\cancel{n}(1+1/n)x}{n(n+n^{1/n})\cancel{x} \cdot n^{1/n}}$$

$$\lim_{n\to\infty} \sqrt[n]{u_n} = x.$$

if $0 \leq x < 1$ series is convergent

$x > 1$ series is divergent.

11) I) $$\sum_{n=1}^{\infty} \frac{1}{1^2+2^2+3^2+\cdots n^2}.$$

$$u_n = \frac{1}{1^2+2^2+3^2+\cdots+n^2}$$

$$u_{n+1} = \frac{1}{1^2+2^2+\cdots n^2+(n+1)^2}$$

$$\frac{U_{n+1}}{u_n} = \frac{1^2+2^2+\dots+n^2}{1^2+2^2+\dots+n^2+(n+1)^2}$$

$$\frac{U_{n+1}}{u_n} = \frac{-(n+1)^2}{1^2+2^2+\dots+(n+1)^2}$$

$$l = \lim_{n\to\infty} n\left(\frac{U_{n+1}}{U_n} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{-n^2(1+1/n)^2}{n^2\left(1/n^2+2/n^2+\dots+(1+\frac{1}{n})^2\right)}\right)$$

$= \infty > 1 \Rightarrow$ Convergent.

(2) $\frac{2}{7} + \frac{2\cdot 5}{7\cdot 10} + \frac{2\cdot 5\cdot 8}{7\cdot 10\cdot 13} + \dots$

$$U_n = \frac{3n-1}{3n+4}$$

$$U_{n+1} = \frac{3(n+1)-1}{3(n+1)+4} = \frac{3n+2}{3n+7}.$$

$$\frac{U_{n+1}}{U_n} = \frac{3n+2}{3n+9} \times \frac{3n+4}{3n-1} = \frac{9n^2+12n+6n+8}{9n^2-3n+21n-7}$$

$$\frac{U_{n+1}}{U_n} - 1 = \frac{9n^2+12n+6n+8-9n^2+3n-21n+7}{9n^2-3n+21n-7}$$

$$l = \lim_{n\to\infty} n\left(\frac{U_{n+1}}{U_n} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{15}{9n^2-15n-7}\right)$$

$$= \lim_{n\to\infty} n\left(\frac{5}{3n+2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{5}{3+2/n}\right) = \frac{5}{3} > 1 \quad \therefore l > 1$$

$\Downarrow$

Convergent.

12) Test the series by Leibnitz test.

$$5 - \frac{10}{3} + \frac{20}{4} - \frac{40}{27} + \cdots$$

$$5\left(\frac{2}{3}\right) - 5\left(\frac{2}{3}\right)^1 (-1)^3 + 5\left(\frac{2}{3}\right)^2 (-1)^4 - 5\left(\frac{2}{3}\right)^3 \cdots$$

$$\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1} (-1)^{n+1}$$

$$U_n = 5\left(\frac{2}{3}\right)^{n-1}$$

$$U_{n+1} = 5\left(\frac{2}{3}\right)^{n}$$

$\therefore U_n > U_{n+1}$

$\therefore$ series is convergent.

13) check the convergence of the series

$$\frac{1}{1.2} - \frac{1}{3.4} + \frac{1}{5.6} - \frac{1}{7.8} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n)}$$

$$U_n = \frac{1}{(2n-1)(2n)}$$

$$U_{n+1} = \frac{1}{(2n+1)(2n+2)}$$

$\therefore U_n > U_{n+1}$

By leibnits test series is convergent.

14) Check for absolute or conditional convergence

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n} + \sqrt{n+1}}$$

$$U_n = \frac{(-1)^n}{\sqrt{n} + \sqrt{n+1}}$$

$$\sum_{n=1}^{\infty} |U_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n} + \sqrt{n+1}} \right|$$

$$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n)^{1/2} + \left(1 + (1+1/n)^{1/2}\right)}$$

$$V_n = \frac{1}{n^2}$$

$$P = \frac{1}{2} < 1$$

There for it is not absolutely convergent

$$\frac{1}{1+\sqrt{2}} > \frac{1}{\sqrt{2}+\sqrt{3}} > \frac{1}{\sqrt{3}+2} > \ldots$$

therfor it is conditionally convergent.

15) For what values of x the series converges absolutely or conditionally.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (3x-1)^n}{n^2}$$

$$U_n = \frac{(-1)^{n-1} (3x-1)^n}{n^2}$$

$$U_{n+1} = \frac{(-1)^{n} (3x-1)^{n+1}}{(n+1)^2}$$

$$= \lim_{n\to\infty} \left| \frac{U_{n+1}}{U_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n (3x-1)^{n+1} \times n^2}{(-1)^n (3x-1)^n \times (n+1)^2} \right|$$

$$= \lim_{n\to\infty} \frac{(3x-1)^{n+1} \times n^2}{n^2 \left(1+\frac{1}{n}\right)(3x-1)^n}$$

$$= \lim_{n\to\infty} \frac{(3x-1)}{\left(1+\frac{1}{n}\right)^2}$$

$$= 3x-1$$

$\therefore$ $3x-1 < 1$ Convergent.

$x < 2/3 \Rightarrow$ Convergent.

therfor series is convergent in $(0, 2/3)$,

16) find ROC & IOC for series

(i) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x)^{2n-1}}{2n-1}$$

2) $$\sum_{n=1}^{\infty} \frac{(2x+3)^{2n}}{n!}$$

$$U_{n+1} = \frac{(2x+3)^{2n+1}}{n!}$$

$$U_{n+1} = \frac{(2x+3)^{(2n+3)}}{(n+1)!}$$

$$\frac{U_n}{U_{n+1}} = \frac{(2x+3)^{2n+1}}{n!} \times \frac{(n+1)\, n!}{(2x+3)^{2n+3}}$$

$$= \frac{(2x+3)(n+1)}{(2x+3)^3}$$

$$= \frac{(n+1)}{(2x+3)^2}$$

$$\left|\frac{U_n}{U_{n+1}}\right| = \left|\frac{(n+1)}{(2x+3)^2}\right|$$

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$$= \left| \frac{1}{x^2} \frac{(n+1)}{(2+3/x)^2} \right|$$

$$\sum_{n=1}^{\infty} \left| \frac{U_n}{U_{n+1}} \right| = \sum_{n=1}^{\infty} \left| \frac{n(p+1/n)}{(2+3/x)^2} \right|$$

$$= \infty$$

for all x it's convergent $r = 8$

@ ~~Type~~ Improper Integral.

1) check the convergence of $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Ans $\int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx$

$= \lim_{a \to -\infty} \int_{a}^{0} \frac{1}{1+x^2} dx + \lim_{b \to \infty} \int_{0}^{b} \frac{1}{1+x^2} dx.$

$= \lim_{a \to -\infty} [\tan^{-1} x]_{a}^{0} + \lim_{b \to \infty} [\tan^{-1} x]_{0}^{b}$

$= \lim_{a \to -\infty} [\tan^{-1} 0 - \tan^{-1} a] + \lim_{b \to \infty} [\tan^{-1} b - \tan^{-1} 0$

$= 0 - \tan^{-1}(-\infty) + \tan^{-1}(\infty) - 0$

$= \pi - \frac{\pi}{2} - \frac{\pi}{2}$

$= \pi > 0 \Rightarrow$ convergent.

2) Evaluate the improper integral $\int_5^\infty \frac{5x}{(x^2+1)^3}$

$$\rightarrow \lim_{c \to \infty} \int_5^c \frac{5x}{(x^2+1)^3} \, dx$$

$$= \lim_{c \to \infty} \int_5^c \frac{2x}{\frac{2}{5}(x^2+1)} \, dx.$$

$$= \lim_{c \to \infty} \frac{5}{2} \int_5^c \frac{2x}{(x^2+1)^3} \, dx.$$

$$\lim_{c \to \infty} \frac{5}{2} \left[\frac{(x^2+1)^{-2}}{-2} \right]_5^c$$

$$= \lim_{c \to \infty} \frac{5}{2} \left[\frac{(c^2+1)^{-2}}{-2} - \frac{(25+1)^{-2}}{-2} \right]$$

$$= \lim_{c \to \infty} -\frac{5}{4} (c^2+1)^{-2} + \frac{5}{4}(26)^{-2}$$

$$= -\frac{5}{4} \frac{1}{(\infty+1)^2} + \frac{5}{4}(26)^{-2}$$

$$= 0 + \frac{5}{4(26)^2}$$

$$= \frac{5}{2704} > 0 \Rightarrow \text{Convergent.}$$

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3) Evaluate the following improper integral.

$$\int_0^{\infty} \frac{dx}{(1+x^2)(1+\tan^{-1}x)}$$

Ans: $= \int_0^{\pi/2} \frac{dx}{(1+x^2)(1+\tan^{-1}x} + \int_{\pi/2}^{\infty} \frac{dx}{(1+x^2)(1+\tan^{-1}x)}$

$= \lim_{a\to 0} \int_a^{\pi/2} \frac{1}{(1+x^2)(1+\tan^{-1}x)} dx + \lim_{b\to\infty} \int_{\pi/2}^{\infty} \frac{1}{(1+x^2)(1+\tan^{-1}}$

$= \lim_{a\to 0} \int_a^{\pi/2} \frac{1}{(1+x^2)(1+\tan^{-1}x)} + \lim_{b\to\infty} \int_{\pi/2}^{b} \frac{1}{(1+x^2)(1+\tan^{-1}x)}$

$= \lim_{a\to 0} \int_a^{\pi/2} \frac{1/1+x^2}{1+\tan^{-1}x} + \lim_{b\to\infty} \int_{\pi/2}^{b} \frac{1/1+x^2}{1+\tan^{-1}x}.$

$= \lim_{a\to 0} \left[\log|1+\tan^{-1}x\right]_a^{\pi/2} + \lim_{b\to\infty} \left[\log|1+\tan^{-1}x\right]_{\pi/2}^{b}$

$= \lim_{a\to 0} \left[\log|1+\tan^{-1}\pi/2| - \log|1+\tan^{-1}a|\right]$

$+ \lim_{b\to\infty} \left[\log|1+\tan^{-1}\infty|^{\infty} - \log|1+\tan^{-1}\pi/2|\right]$

$= \log|1+\infty| - \log 1 + \log|1+\tan^{-1}\infty|$
$- \log 1 + \infty$

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$$= \log(1 + \tan^{-1}\infty)$$

$$= \log\left(1 + \frac{\pi}{2}\right) > 0 \implies \text{convergent.}$$

4) Is $\displaystyle\int_1^{\infty} \frac{dx}{x^{\sqrt{2}}}$ convergent ?

$$\rightarrow \lim_{c \to \infty} \int_1^{c} \frac{1}{x^{\sqrt{2}}} \, dx$$

$$= \lim_{c \to \infty} \left[\frac{x^{-\sqrt{2}+1}}{-\sqrt{2}+1} \right]_1^{c}$$

$$= \lim_{c \to \infty} \left[\frac{c^{-\sqrt{2}+1}}{-\sqrt{2}+1} - \frac{1^{-\sqrt{2}+1}}{-\sqrt{2}+1} \right]$$

$$= \infty - \frac{1}{-\sqrt{2}+1} = \frac{-1}{1-\sqrt{2}} = \frac{1}{1-\sqrt{2}} > 0$$

$\Downarrow$

convergent.

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Beta and gamma function

1) State the relation between beta and gamma function.

Ans → $$\beta(m,n) = \frac{\sqrt{m}\,\sqrt{n}}{\sqrt{m}+\sqrt{n}}$$

$$\sqrt{m} = 2\int_0^\infty e^{-x^2} x^{2m-1}\,dx$$

$$\sqrt{m} = 2\int_0^\infty e^{-y^2} x^{2m-1}\,dx.$$

$$\sqrt{m}\,\sqrt{n} = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} x^{2n-1}\,dx\,dy.$$

Use polar co-ordinates.

$x = r\cos\theta$, $y = r\sin\theta$.

$$\sqrt{m}\,\sqrt{n} = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1}\, r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^\infty r^{2(m+n)-1} e^{-r^2} \cos^{2m-1}\theta\,\sin^{2n-1}\theta\,dr\,d\theta.$$

$$= \left[2\int_0^{\pi/2} \cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta\right]\left[2\int_0^\infty r^{2(m+n)-1} e^{-r^2}\,dr\right]$$

$$= \beta(m,n) \cdot 2 \int_0^\infty r^{2(m+n)-1} e^{-r^2} \, dr.$$

$$\sqrt{m}\sqrt{n} = \beta(m,n) \sqrt{m+n}$$

Thus $\boxed{\beta(m,n) = \dfrac{\sqrt{m}\sqrt{n}}{\sqrt{m+n}}}$

2) Evaluate $\sqrt{\tfrac{13}{2}}$

$$\sqrt{\tfrac{13}{2}} = \sqrt{\tfrac{11}{2}+1} = \cancel{\tfrac{11}{2} \cdot \tfrac{9}{2} \cdot \tfrac{7}{2} \sqrt{\tfrac{5}{2}+1}}$$

$$= \frac{11}{2} \sqrt{\tfrac{11}{2}}$$

$$= \frac{11}{2} \sqrt{\tfrac{9}{2}+1}$$

$$= \frac{11}{2} \cdot \frac{9}{2} \sqrt{\tfrac{9}{2}}$$

$$= \frac{11}{2} \cdot \frac{9}{2} \cdot \sqrt{\tfrac{7}{2}+1}$$

$$= \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \sqrt{\tfrac{7}{2}}$$

$$= \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \sqrt{\tfrac{5}{2}+1}$$

$$= \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \sqrt{\tfrac{5}{2}}$$

$$= \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \Gamma\left(\frac{3}{2}+1\right)$$

$$= \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2}\, \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2}\, \Gamma\left(\frac{1}{2}+1\right)$$

$$= \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\, \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}$$

$$= \frac{10395}{26} \sqrt{\pi}$$

3) find $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\Gamma n = \int_0^\infty e^{x}\, x^{n-1} \cdot dx.$$

$$= 2\int_0^\infty e^{-x^2}\, x^{2n-1}\, dx.$$

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-x^2} \cdot x^{2 \cdot \frac{1}{2} - 1} \cdot dx.$$

$$= 2\int_0^\infty e^{-x^2}\, x^{1-1}\, dx$$

$$= 2\int_0^\infty e^{-x^2}\, dx.$$

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$$\Gamma\tfrac{1}{2} \cdot \Gamma\tfrac{1}{2} = 2\int_0^\infty e^{-x^2} dx \cdot 2\int_0^\infty e^{-y^2} dy.$$

$$= 4\int_0^\infty \int_0^\infty e^{-x^2-y^2} dx\, dy.$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$$

changing to proper form.

$$x = r\cos\theta \qquad y = r\sin\theta.$$

$$dx\, dy = r\, d\theta\, dr$$

Unit. $x \to 0$ to $x \to \infty$

$y \to 0$ to $y \to \infty$

$r = 0$ to $r \to \infty$

$\theta = 0$ to $\theta \to \pi/2$

$$\Gamma\tfrac{1}{2} \cdot \Gamma\tfrac{1}{2} = 4\int_0^{\pi/2} \int_0^\infty e^{-r^2} d\theta\, dr.$$

$$= 4\int_0^{\pi/2} \int_0^\infty \left(-\tfrac{1}{2}\right) e^{-r^2} (-2r)\, dr\, d\theta$$

$$= 4\int_0^{\pi/2} \left(-\tfrac{1}{2}\right)\left[e^{-r^2}\right]_0^\infty d\theta$$

$$= \frac{4}{2} \int_0^{\pi/2} - d\theta$$

$$= -\frac{4}{2} \left[\theta \right]_0^{\pi/2}$$

$$= -2 \left[-\frac{\pi}{2} \right]$$

$$= \boxed{-\pi}$$

① $\Gamma -\frac{5}{2}$ $\longrightarrow$ $\frac{\Gamma n+1}{\Gamma n}$ = ~~$n\Gamma n$~~ $\frac{n\Gamma n}{\Gamma \frac{n+1}{n}}$

$$\Gamma \frac{-5}{2} = \Gamma \frac{-\frac{5}{2}+1}{-3/2}$$

$$= -\frac{2}{5} \Gamma \frac{-3}{2}$$

$$= \frac{-2}{5} \Gamma \frac{-3/2+1}{-3/2}$$

$$= \frac{4}{15} \Gamma \frac{-1}{2}$$

$$= \frac{4}{15} \Gamma \frac{-\frac{1}{2}+1}{-1/2}$$

$$= -\frac{8}{15} \Gamma \frac{1}{2}$$

(4)

$$-\frac{8}{15}\sqrt{\pi}$$

4)
$$\int_0^\infty e^{-h^2x^2}\,dx.$$

let $h^2x^2 = t$

$(hx)^2 = t$

$hx = t^{1/2}$

$h\,dx = \frac{1}{2}\frac{1}{\sqrt{t}}\,dt.$

$x \to 0$ to $t \to 0$

$x \to \infty$ to $t \to \infty$

$$= \int_0^\infty e^{-h^2x^2}\,dx = \int_0^\infty e^{-t}\,\frac{1}{2h}\cdot\frac{1}{\sqrt{t}}\,dt.$$

$$= \frac{1}{2h}\int_0^\infty e^{-t}\,t^{-1/2}\,dt.$$

$$= \frac{1}{2h}\int_0^\infty e^{-t}\cdot t^{1/2-1}\,dt.$$

$$= \frac{1}{2h}\Gamma\frac{1}{2}$$

$$\boxed{= \frac{1}{2h}\pi.}$$