CH: System Of Linear Equation.

* Matrix: A matrix is a rectangular arrangement of numbers enclosed in a bracket.

→ Eqⁿ

↓

system of linear eqⁿ

↓

Augmented matrix

↓ Gauss-elimination → Row-echlon

↓ Gauss-Jorden → Reduced row-echlon

→ Solⁿs:

1) Unique solⁿ
2) Infinite solⁿ
3) No solⁿ

Row-echlon:

→ Leading element must be in order starting from the first column and should be in order like. 1st leading, 2nd ...

× $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ × $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ✓ $\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\checkmark \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \times \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \checkmark \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

→ <u>Reduced-Row echlon:</u>

$$\checkmark \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \checkmark \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \checkmark \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Ex: Solve the system of linear eq^n using gauss elimination method i.e.

$$x - 2y - z = 0$$
$$2y - 8z = 8$$
$$-4x + 5y + 9z = -9$$

Sol^n

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

The augmented matrix: $[A/b]$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \quad R_{13}(4)$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \quad R_2(1/2)$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \quad R_{23}(3)$$

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$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Using Back Substitution

$x - 2y + z = 0$

$y - 4z = 4$

$\boxed{z = 3}$

$\boxed{y = 16}$ $\boxed{x = 29}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 29 \\ 16 \\ 3 \end{bmatrix}$$

System has unique soln.

4/10/19
Friday.

Q2) $\frac{-1}{x} + \frac{3}{y} + \frac{4}{z} = 30.$

$\frac{3}{x} + \frac{2}{y} - \frac{1}{z} = 9$

$\frac{2}{x} \cancel{+} \frac{-1}{y} + \frac{2}{z} = 10$

Soln. Let $\frac{1}{x} = X, \quad \frac{1}{y} = Y, \quad \frac{1}{z} = Z.$

$\therefore -X + 3Y + 4Z = 30$

$3X + 2Y - Z = 9$

$2X - Y + 2Z = 10$

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The augmented matrix:

$$\left[\begin{array}{ccc|c} -1 & 3 & 4 & 30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{array}\right] \quad R_1(-1)$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{array}\right] \quad \begin{array}{l} R_{12}(-3) \\ R_{13}(-2) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 11 & 11 & 99 \\ 0 & 5 & 10 & 70 \end{array}\right] \quad R_2\left(\frac{1}{11}\right)$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 5 & 10 & 70 \end{array}\right] \quad R_{23}(-5)$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 5 & 25 \end{array}\right] \quad R_3\left(\frac{1}{5}\right)$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Back-sub.

$z = 5 \Rightarrow z = 1/5$

$y = 4 \Rightarrow y = 1/4$

$x = -30 + 20 + 12$

$x = 2 \Rightarrow x = 1/2$

Sol^n: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/4 \\ 1/5 \end{bmatrix}$

Q.3) $2x_1 + 2x_2 + 2x_3 = 0$

$-2x_1 + 5x_2 + 2x_3 = 1$

$8x_1 + x_2 + 4x_3 = -1$

Soln

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right] R_1\left(\frac{1}{2}\right)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right] \begin{array}{l} R_{12}(2) \\ R_{13}(-8) \end{array}$$

$$\cancel{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right]} \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array}\right] R_{23}(1)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] R_2\left(\frac{1}{3}\right)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right] R_{21}(-1)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3/7 & -1/7 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

By Back-subst.

$x_2 = 1/7 - 4/7\, x_3$

$x_1 = -1/7 - 3/7\, x_3$

let $x_3 = t$, $t \in \mathbb{R}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/7 - 3/7\,t \\ 1/7 - 4/7\,t \\ t \end{bmatrix}$$

$\therefore$ System has infinite soln.

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- Homogeneous system of linear eq^n: The system of linear eq^n which has b matrix i.e $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is called homogeneous system of linear equation.

System of eq^n:

Non-homogeneous

Types of sol^n

- unique
  $\left[\begin{array}{ccc|c} 0 & 0 & a & b \end{array}\right] \Rightarrow$ If $a \neq 0$ always unique sol^n
- Infinite
  $\left[\begin{array}{ccc|c} 0 & 0 & a & b \end{array}\right] \Rightarrow$ if $a = 0 ; b = 0$ always infinite sol^n
- no sol^n $\left[\begin{array}{ccc|c} 0 & 0 & a & b \end{array}\right]$ if $a = 0, b \neq 0$ $\Rightarrow$ no sol^n

Homogeneous

Types of sol^n

- unique sol^n $\Rightarrow$ $\left[\begin{array}{ccc|c} \cdot & & & \\ 0 & 0 & a & 0 \end{array}\right]$ If $a \neq 0$ we get $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \rightarrow$ trivial sol^n
- Infinite sol^n $\left[\begin{array}{ccc|c} 0 & 0 & a & 0 \end{array}\right]$
  if $a = 0$, Infinite sol^n.

Q.1)
$$x_3 + x_4 + x_5 = 0$$
$$-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$$
$$x_1 + x_2 - 2x_3 - x_5 = 0$$
$$2x_1 + 2x_2 - x_3 + x_5 = 0$$

Sol^n: Augmented matrix: $\left[\begin{array}{ccccc|c} 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{array}\right]$ R_{13} or $R_1 \rightleftharpoons R_3$

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$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_{12}(+1) \\ R_{14}(-2) \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \end{array}\right] \quad R_{23}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \end{array}\right] \quad R_{24}(-3)$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{array}\right] \quad R_{34}(-1)$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_4 = 0$

$x_3 = -x_5$

$x_1 + x_2 - 2x_3 - x_5 = 0$

$x_1 = -x_2 - 2x_5 + x_5$

$x_1 = -x_2 - x_5$

let $x_5 = t$

$x_2 = s$

Sol^n $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{bmatrix}$

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If number of variable is more than number of eq^n, we always get infinite sol^n.

Q.2) $x + 2y - 3z = 4$
$3x - y + 5z = 2$
$4x + y + (a^2 - 14)z = a + 2.$

sol^n Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right] \quad R_{12}(-3),\ R_{13}(-4)$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right] \quad R_2\left(-1/7\right)$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & 2 & 10/7 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right] \quad R_{23}(7)$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$$

(i) Now for unique sol^n
$a^2 - 16 \neq 0$
$a^2 \neq 16 \Rightarrow a \neq \pm 4$

(ii) for infinite sol^n.
$a^2 - 16 = 0$ & $a - 4 = 0$
$\Rightarrow a^2 = 16$ & $a = 4.$

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if $a=4$, we get infinite sol^n.

(iii) for no sol^n.

$a^2-16=0$ & $a-4\neq 0$
$\Rightarrow a=\pm 4$ & $a\neq 4$

$\Rightarrow$ i.e. $a=-4$, system has no sol^n.

Q.) $(\lambda-3)x + y = 0$
$x + (\lambda-3)y = 0$

Sol^n $\left[\begin{array}{cc|c} \lambda-3 & 1 & 0 \\ 1 & \lambda-3 & 0 \end{array}\right]$ R_{12}

$\left[\begin{array}{cc|c} 1 & \lambda-3 & 0 \\ \lambda-3 & 1 & 0 \end{array}\right]$ $R_{12}(-(\lambda-3))$

$\left[\begin{array}{cc|c} 1 & \lambda-3 & 0 \\ 0 & 1-(\lambda-3)^2 & 0 \end{array}\right]$

we get non-trivial sol^n
if $1-(\lambda-3)^2=0$

$1=(\lambda-3)^2$

$\lambda-3=\pm 1$

$\Rightarrow \lambda=4, \lambda=2$

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* Inverse of a Matrix by row-operation:

$[A | I]$

$\downarrow$ Row-operation

$[I | A^{-1}]$

Q) Find the inverse of given matrix if possible using gauss method.

(1) $\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$

Soln. $\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$ $R_{12}(-2)$, $R_{13}(1)$

$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$ $R_{23}(1)$

$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$ $\therefore$ A matrix is not invertible.

(2) $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$

Soln $\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$ $R_1\left(\frac{1}{2}\right)$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] R_{12}(-2)$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] R_{13}(-2)$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] R_{23}(-1)$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] R_{31}(-3)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

A^{-1}

* Rank of a matrix:

→ The number of non-zero rows in a non-zero rows in a row-echlon form of a matrix is called the Rank of a matrix. and denoted by Rank(A) or $\rho(A)$

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1) $\begin{bmatrix} 1 & 5 & 3 & -2 \\ 2 & 0 & 4 & 1 \\ 4 & 8 & 9 & -1 \end{bmatrix}$

Solⁿ $\begin{bmatrix} 1 & 5 & 3 & -2 \\ 2 & 0 & 4 & 1 \\ 4 & 8 & 9 & -1 \end{bmatrix}$ $R_{12}(-2)$

$\begin{bmatrix} 1 & 5 & 3 & -2 \\ 0 & -10 & -2 & 5 \\ 4 & 8 & 9 & -1 \end{bmatrix}$ $R_{13}(-4)$

$\begin{bmatrix} 1 & 5 & 3 & -2 \\ 0 & -10 & -2 & 5 \\ 0 & -12 & -3 & 7 \end{bmatrix}$ $R_2\left(\frac{-1}{10}\right)$

$\begin{bmatrix} 1 & 5 & 3 & -2 \\ 0 & 1 & 1/5 & -1/2 \\ 0 & -12 & -3 & 7 \end{bmatrix}$ $R_{23}(12)$

$\begin{bmatrix} 1 & 5 & 3 & -2 \\ 0 & 1 & 1/5 & -1/2 \\ 0 & 0 & -3/5 & 1 \end{bmatrix}$

$\frac{12}{5} - 3$

$= \frac{12 - 15}{5}$

$= \frac{-3}{5}$

EIGEN VALUE AND EIGEN VECTOR.

Definition: The roots of the determinant characteristic eqⁿ $|A-\lambda I| = 0$ is said to be a eigen value.

Eigen vector: The solution space of $[A-\lambda I]x = 0$ is said to be a eigen vector of the matrix A.

Q Find the eigen value and eigen vector of matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Solⁿ.

$$[A-\lambda I] = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 - 4 = 0$$
$$1 - 2\lambda + \lambda^2 - 4 = 0$$
$$\lambda^2 - 3\lambda + \lambda - 3 = 0$$
$$\lambda(\lambda-3) + 1(\lambda-3) = 0$$
$$\lambda = 3, -1$$

$\Rightarrow$ Another method : For any 2x2 matrix, general formula:

$$\boxed{\lambda^2 - \underbrace{trace(A)}_{\text{addⁿ of diagonal elements.}} \cdot \lambda + det(A) = 0}$$

$\rightarrow$ $[A-\lambda I]x = 0$

$$\begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$R_2 \to R_2 + R_1$, $R_1 \to R_1/2$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_2 = 0$

Suppose $x_2 = t$

$x_1 = t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ ← eigenvector

15/11/19
friday.

* Caley - Hamilton Theorem:

→ Every square matrix satisfies its own characteristic equation.

→ $|A - \lambda I| = (-1)^n \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_{n-1}\lambda + a_n = 0$
is characteristic equation of matrix of order n.

$(-1)^n A^n + a_1 A^{n-1} \dots + a_{n-1} A + a_n I = 0$

Cor 1: Inverse of non-singular matrix:

$$A^{-1} = \frac{-1}{a_n} \left[(-1)^n A^{n-1} + a_1 A^{n-2} + \dots + a_{n-2} A + a_{n-1} I \right]$$

Cor 2: To find higher order:

$$A^n = \frac{-1}{(-1)^n} \left[a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_{n-1} A + a_n I \right]$$

Q.1) Find A^2, A^3, A^{-1}, A^{-2} using Caley - Hamilton theorem

if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Soln: $|A - \lambda I| = 0$

$= \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = 0$

$(1-\lambda)(4-\lambda) - 6 = 0$

$4 - 5\lambda + \lambda^2 - 6 = 0$

$\lambda^2 - 5\lambda - 2 = 0$

$\therefore A^2 - 5A - 2 = 0$

$\therefore A^2 = 5A + 2I$

$= 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \cancel{5 \; 10 \; 15} \quad \cancel{5(4-6)+2}$

$\cancel{= 5(-2)+2}$

$\cancel{= -10+2}$

$\cancel{= -8}$

$\begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$

→ $A^3 = A^2 \times A = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 7\times1 + 10\times3 & 7\times2 + 10\times4 \\ 15\times1 + 22\times3 & 15\times2 + 22\times4 \end{bmatrix}$

$= \begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix}$

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→ For A^{-1}

$$A^{-1}(A^2-5A-2I)=0$$
$$A-5I-2A^{-1}=0$$
$$A^{-1}=\left(\frac{5I-A}{2}\right)$$
$$2A^{-1}=A-5I$$
$$=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}-\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}$$
$$=\begin{bmatrix}-4 & 2\\ 3 & -1\end{bmatrix} \qquad \therefore A^{-1}=\begin{bmatrix}-2 & 1\\ +3/2 & -1/2\end{bmatrix}.$$

→ for A^{-2}

$$A^{-2}=A^{-1}\times A^{-1}$$
$$=\begin{bmatrix}-2 & 1\\ +3/2 & -1/2\end{bmatrix}\times\begin{bmatrix}-2 & 1\\ 3/2 & -1/2\end{bmatrix}.$$

or $A^{-2}(A^2-5A-2I)=0A^{-2}$

$$I-5A^{-1}-2A^{-2}=0$$
$$2A^{-2}=I-5A^{-1}$$
$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\frac{5}{2}\begin{bmatrix}-4 & 2\\ 3 & -1\end{bmatrix}$$
$$A^{-2}=\frac{1}{4}\begin{bmatrix}22 & -10\\ -15 & 7\end{bmatrix}$$

Q.2) Verify Caley-Hamilton theorem for $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$

Hence find A^{-1}.

Solⁿ: $\begin{vmatrix}2-\lambda & -1 & 1\\ -1 & 2-\lambda & -1\\ 1 & -1 & 2-\lambda\end{vmatrix}=0$

$$(2-\lambda)\left[(2-\lambda)^2-1\right]+1\left[-2+\lambda+1\right]+1\left[1-2+\lambda\right]$$

$$\therefore 2\lambda^2-8\lambda+6-\lambda^3+4\lambda^2-3\lambda-2+2\lambda$$

$$\therefore \lambda^3+2\lambda^2+7\lambda=0$$

$$\rightarrow A^3-6A^2+9A-4=0$$

$$A^2=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}$$

$$A^3=\begin{bmatrix}22 & -22 & -21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}$$

$$A^{-1}\left[A^3-6A^2+9A-4\right]=0$$

$$4A=A^2-6A+9I=0$$

$$A^{-1}=\frac{1}{4}\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$$

Q.3) Find characteristic eq^n of

$$A=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$$, Hence prove

$$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I=\begin{bmatrix}8 & 5 & 5\\ 0 & 3 & 0\\ 5 & 5 & 8\end{bmatrix}$$

A

Sol^n Characteristic eq^n : $|A - \lambda I| = 0$

$$\begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\therefore \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(1-\lambda)(2-\lambda) - (1)\times 0\right] - 1(0-0) + 1(0-(1-\lambda))$$

$$\therefore (2-\lambda)\left[2-\lambda-2\lambda+\lambda^2 - 1\times 0\right] + 0 - 1 + \lambda$$

$$\therefore (2-\lambda)\left[\lambda^2 - 3\lambda + 2\right] - 1 + \lambda$$

$$\therefore 2\lambda^2 - 6\lambda + 2 - \lambda^3 + 3\lambda^2 - \lambda - 1 + \lambda$$

$$\therefore -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$$

$$\therefore \quad \lambda^3 - 5\lambda^2 + 7\lambda - 3I = 0$$

By Caley-Hamilton theorem:

$$A^3 - 5A^2 + 7A - 3I = 0$$

$$\therefore A^5\left[A^3 - 5A^2 + 7A - 3I\right] + 8A\left[A^3 - 5A^2 + 7A - 3I\right] + (A^2 + A + I)$$

$A(A^3 - 5A^2$

$$\therefore A^5(0) + A(0) + A^2 + A + I$$

* Diagonalization:

→ Let A be a square matrix of order N. The matrix A is said to be diagonalizable if there exist an invertible matrix P such that $P^{-1}AP = D$ where D is the diagonal matrix.

→ **Imp. Note:**

1) A square matrix of order N is diagonalizable if the matrix A has exactly N linearly independent eigen vectors.

2) If all the eigen values of a square matrix A a distinct then A is always diagonalizable

3) Two similar matrices have the same determinant

Q) Determine the diagonal matrix from $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ is A is diagonalizable

Ans: The characteristic eqⁿ: $\begin{vmatrix} 1-\lambda & 0 \\ -1 & 2-\lambda \end{vmatrix} = 0$

$(1-\lambda)(2-\lambda) - 0 = 0$

$\lambda = 1, \ \lambda = 2$

To find eigen vectors:

(i) for $\lambda = 1$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$-x_1 + x_2 = 0$

$x_1 = x_2$

let $x_2 = t$

$\Rightarrow x_1 = t$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(ii) for $\lambda = 2$

$$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 0$, $x_2 = s$ (any variable) (as 2^{nd} row is zero)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ s \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \therefore P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$\therefore$ A is diagonalizable

$\therefore \cancel{A^{-1}RA}\ P^{-1}AP = D$

$$\therefore P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

3) Check : $A = \begin{bmatrix} 2 & 2 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$\begin{vmatrix} 2-\lambda & 2 & -2 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$(2-\lambda)(3-\lambda)^2 - 2(0-0) \cancel{+} -2(0-0) = 0$

$(2-\lambda)(9+\lambda^2-6\lambda) = 0$ $\quad \therefore (\lambda^2-6\lambda+9) = \lambda^2-3\lambda-3\lambda+9 = \cdot$
$\lambda(\lambda-3)-3(\lambda$

$18 + 2\lambda^2 - 12\lambda - 9\lambda - \lambda^3 + 6\lambda^2 = 0$

$-\lambda^3 + 8\lambda^2 - 21\lambda + 18 = 0$

$\therefore \lambda = 2, \quad \lambda = 3.$

→ for $\lambda = 3$

$$\left[\begin{array}{ccc|c} -1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-x_1 = -2x_2 + 2x_3.$
let $x_2 = t$, $x_3 = s$.
$x_1 = 2t - 2s.$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t-2s \\ t \\ s \end{bmatrix} \quad \therefore \Rightarrow t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

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Theorem :-

If the matrix A is diagonalizable & $P^{-1}AP = D$, then $A^K = P^{-1}D^K P$

$$P = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}AP = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A^5 = P^{-1} D^5 . P$$

$$= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 2^5 & 0 & 0 \\ 0 & 3^5 & 0 \\ 0 & 0 & 3^5 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

20/11/19 thursday

Q) Find a matrix P which diagonalize a matrix A, verify $P^{-1}AP = D$, where $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

Sol^n: eigen values:

The characteristic eq^n: $\lambda^2 - S_1\lambda + S_2 = 0$.

S_1 = sum of diagonal elements $= 7$

$S_2 = \det(A) = 10$

$\lambda^2 - 7\lambda + 10 = 0$

$\therefore$ Eigen values are 5 & $2 = \lambda$

$$\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix}$$

$= (4-\lambda)(3-\lambda) - 2 = 0$

$= 12 - 4\lambda - 3\lambda + \lambda^2 - 2 = 0$

$= \lambda^2 - 7\lambda + 10 = 0$

$\lambda^2 - 5\lambda - 2\lambda + 10 = 0$

$\lambda(\lambda - 5) - 2(\lambda - 5) = 0$

$\lambda = 5, 2.$

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To find eigen vectors:

(i) for $\lambda = 2$ $[A - \lambda I / 0] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right]$

$\Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\Rightarrow$ $2x_1 + x_2 = 0$

$x_2 = -2x_1$

let $x_1 = t$.

$\therefore x_2 = -2t$.

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\therefore V_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(ii) <u>for $\lambda = 5$</u>

$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\Rightarrow$ $-x_1 + x_2 = 0$

$x_1 = x_2$

$x_2 = t$.

$x_1 = t$.

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Note: Any matrix having lower or upper triangular matrix then its eigen values are the diagonal values itself

$$P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$\therefore$ A is diagonalizable

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \quad -(i) \qquad \therefore P^{-1} = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

LHS: $P^{-1}AP$

$$= \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2/3 & -2/3 \\ 10/3 & 5/3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

from (i) & (ii)

$PA^{-1}P = D$

q2) Show that $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalizable.

Sol^n → Eigen values are 1, 2, 2.

for $\lambda = 1$

$$A = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 5 & 1 & 0 \end{array}\right]$$

$$A = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -4 & 4 & 1 & 0 \end{array}\right] \Rightarrow x_1 + x_2 = 0$$

$-4x_1 + 4x_2 + x_3 = 0$

$\therefore x_1 = -x_2$

if $x_2 = t$, $x_1 = -t$.

$\therefore 4t + 4t + x_3 = 0$

$x_3 = -8t$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ -8 \end{bmatrix}$$

for $\lambda = 2$:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{bmatrix}$$

$-x_1 = 0$
$x_1 = 0.$
$-3x_1 + 5x_2 = 0.$ $\therefore 5x_2 = 0$, $x_2 = 0$
$x_3 = s.$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \Rightarrow s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ -8 & 1 \end{bmatrix}$$

$\therefore$ We can't get such square matrix P such that $P^{-1}AP = D$

$\therefore A$ is not diagonalizable.

Q) Find a matrix P that diagonalize matrix A

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$ & also find A^{10}

$$A = \begin{vmatrix} 1-\lambda & -1 \\ 0 & 2-\lambda \end{vmatrix}$$

$\therefore (1-\lambda)(2-\lambda) + 0 = 0.$
$\lambda = 1, 2.$

for $\lambda = 1$

$$A = \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$-x_2 = 0.$
$x_2 = 0.$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ 0 \end{bmatrix} \quad \therefore s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

<u>for $\lambda = 2$</u>

$A = \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$-x_1 - x_2 = 0$

$-x_1 = x_2 \qquad$ let $x_1 = t.$

$\therefore x_2 = -t.$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 0 \\ +1 \end{bmatrix}$

How to find P^{-1}?

$\therefore P = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \qquad \therefore P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$A^{10} = P^{-1} D^{10} P \qquad \Rightarrow D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$A^{10} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^{10} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ -1 & 1024 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$\underline{A^{10}} = \begin{bmatrix} 1 & 0 \\ 1023 & 1024 \end{bmatrix}$