Ch-2

Laplace Theorem.

* Laplace Transform :-

- Defination :- Laplace transform of any function $f(t)$ is defined as

$$\boxed{\mathcal{L} f(t) = \int_0^{\infty} e^{-st} f(t) dt} \quad \text{where } (t \geq 0)$$

(s is any parameter)

$$\boxed{\begin{array}{l} e^{\infty} = \infty \\ e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0 \end{array}}$$

(1) Prove $\mathcal{L}\, 1 = \frac{1}{s}$.

$$\mathcal{L} f(t) = \int_0^{\infty} e^{-st} f(t)\, dt$$

Let $f(t) = 1$

$$\mathcal{L}\, 1 = \int_0^{\infty} e^{-st}\, 1\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \frac{1}{-s}\left[e^{-st}\right]_0^{\infty}$$

$$= -\frac{1}{s}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{-1}{s}[0-1]$$

$$= \frac{1}{s}$$

$$\boxed{\mathcal{L}1 = \frac{1}{s}}$$

② Prove $\mathcal{L}e^{at} = \frac{1}{s-a}$

$$\mathcal{L}f(t) = \int_0^\infty e^{-st} f(t)\,dt$$

Let $f(t) = e^{at}$

$$\mathcal{L}e^{at} = \int_0^\infty e^{-st} e^{at}\,dt$$

$$= \int_0^\infty e^{(a-s)t}\,dt$$

$$= \left[\frac{e^{(a-s)t}}{(a-s)}\right]_0^\infty$$

$$= \frac{1}{(a-s)}\left[e^{(a-s)t}\right]_0^\infty$$

$$= \frac{1}{a-s}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{1}{a-s}\left[0 - 1\right]$$

$$= \frac{-1}{(a-s)}$$

$$\boxed{\mathcal{L}\, e^{at} = \frac{1}{(s-a)}}$$

(3) Prove $\mathcal{L}\, e^{-at} = \frac{1}{s+a}$

$$\mathcal{L}\, f(t) = \int_0^\infty e^{-st} f(t)\, dt$$

Let $f(t) = e^{-at}$

$$\mathcal{L}\, e^{-at} = \int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{-1}{(s+a)}\left[e^{-(s+a)t}\right]_0^\infty$$

$$= \frac{-1}{a+s}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{-1}{a+s}\left[0-1\right]$$

$$\boxed{\mathcal{L}\, e^{-at} = \frac{1}{a+s}}$$

(4) Prove $\mathcal{L} \cos at = \dfrac{s}{s^2+a^2}$.

$$\mathcal{L} f(t) = \int_0^\infty e^{-st} f(t)\, dt$$

Let $f(t) = \cos at$

$$\mathcal{L} \cos at = \int_0^\infty e^{-st} \cos at \, dt$$

$$a = -s \quad , \quad b = a$$

$$= \left[\frac{e^{-st}}{(-s)^2+a^2}(-s \cos at + a \sin at)\right]_0^\infty$$

$$= \frac{1}{s^2+a^2}\left[e^{-st}(-s \cos at + a \sin at)\right]_0^\infty$$

$$= \frac{1}{s^2+a^2}\left[0-1(-s \cos 0 + a \sin 0)\right]$$

abbb Formula :-

$$\int e^{at} \cos bt \, dt = \frac{e^{at}}{a^2+b^2}(a \cos bt + b \sin bt)$$

$$\int e^{at} \sin bt \, dt = \frac{e^{at}}{a^2+b^2}(a \sin bt - b \cos bt)$$

$$= \frac{1}{s^2+a^2}(-(-s))$$

$$\boxed{\mathcal{L} \cos at = \frac{s}{s^2+a^2}}$$

(5) Prove $\mathcal{L} \sin at = \frac{a}{s^2+a^2}$.

$$\mathcal{L} f(t) = \int_0^\infty e^{-st} f(t) \, dt$$

Let $f(t) = \sin at$

$$\mathcal{L} \sin at = \int_0^\infty e^{-st} \sin at \, dt$$

$a = -s, \; b = a$

$$= \left[\frac{e^{-st}}{(-s)^2+(a)^2}(-s \sin bt - a \cos bt)\right]_0^\infty$$

$$= \frac{1}{s^2+a^2}\left[e^{-st}(-s \sin bt - a \cos bt)\right]_0^\infty$$

$$= \frac{1}{s^2+a^2}\left[0 - 1(-s(0) - a(1))\right]$$

$= \frac{1}{s^2+a^2} \quad [-1(-a)]$

$$\boxed{\mathcal{L}\sin at = \frac{a}{s^2+a^2}}$$

(6) Prove $\mathcal{L}\cosh at = \frac{s}{s^2-a^2}$

Formulae:-

$$\sinh ax = \frac{e^{ax} - e^{-ax}}{2}$$

$$\cosh ax = \frac{e^{ax} + e^{-ax}}{2}$$

Function → Real (Trigo), Complex (Hyperbolic)

$\mathcal{L}f(t) = \int_0^\infty e^{-st} f(t)\, dt$

Let $f(t) = \cosh at$

$\mathcal{L}\cosh at = \int_0^\infty e^{-st} \cosh at \, dt$

$= \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right) dt$

$= \frac{1}{2}\int_0^\infty \left(e^{at-st} + e^{-at-st}\right) dt$

$= \frac{1}{2}\int_0^\infty \left[e^{(a-s)t} + e^{-(a+s)t}\right] dt$

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$$= \frac{1}{2}\left[\frac{e^{(a-s)t}}{(a-s)} + \frac{e^{-(a+s)t}}{-(a+s)}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{(a-s)} - \frac{e^{(a+s)t}}{a+s}\right]_0^{\infty}$$

$$= \frac{1}{2}\left\{0 - \left[\frac{1}{a-s} - \frac{1}{a+s}\right]\right\}$$

$$= \frac{1}{2}\left[\frac{(a+s)(a-s)}{(a-s)(a+s)}\right]$$

$$= \frac{-1}{2}\left[\frac{2s}{a^2-s^2}\right]$$

$$\boxed{\frac{s}{s^2-a^2} = \mathcal{L}\cosh at}$$

(7) Prove $\mathcal{L}\sinh at = \frac{a}{s^2-a^2}$

$$\mathcal{L}f(t) = \int_0^{\infty} e^{-st} f(t)\,dt$$

Let $f(t) = \sinh at$

$$\mathcal{L}\sinh at = \int_0^{\infty} e^{-st}\sinh at\,dt$$

$$= \int_0^{\infty} e^{-st}\left(\frac{e^{at} - e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\int_0^\infty \left[e^{at} e^{-st} - e^{-at} e^{-st}\right] dt$$

$$= \frac{1}{2}\int_0^\infty e^{(a-s)t} - e^{-(a+s)t}\, dt$$

$$= \frac{1}{2}\left[\frac{e^{(a-s)t}}{(a-s)} - \frac{e^{-(a+s)t}}{-(a+s)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{(a-s)} + \frac{e^{-(a+s)t}}{(a+s)}\right]_0^\infty$$

$$= \frac{1}{2}\left\{0 - \left[\frac{1}{a-s} + \frac{1}{a+s}\right]\right\}$$

$$= \frac{-1}{2}\left[\frac{(a+s)+(a-s)}{(a-s)(a+s)}\right]$$

$$= \frac{-1}{2}\,\frac{2a}{(a^2-s^2)}$$

$$\boxed{\mathcal{L}\sinh at = \frac{a}{s^2-a^2}}$$

(8) $\mathcal{L}\, t^n = \frac{\Gamma n+1}{s^{n+1}}$ [if n is the non-Integer]

$= \frac{n!}{s^{n+1}}$ [If n is an the Integer]

$$\mathcal{L} f(t) = \int_0^{\infty} e^{-st} \cdot f(t)\, dt$$

$$f(t) = t^n$$

$$\mathcal{L} t^n = \int_{t=0}^{\infty} e^{-st} t^n \, dt$$

Let $st = u$ $\quad t = \frac{u}{s}$ $\quad dt = \frac{du}{s}$

$t \to 0$, $u \to 0$

$t \to \infty$, $u \to \infty$

$$\int_{u=0}^{\infty} e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$\int_{u=0}^{\infty} e^{-u} \frac{u^n}{s^n} \frac{du}{s}$$

$$\frac{1}{s^{n+1}} \int_{u=0}^{\infty} e^{-u} u^n \, du$$

$$\boxed{\begin{aligned} \mathcal{L} t^n &= \frac{\Gamma n+1}{s^{n+1}} \quad \text{(if } n \text{ is non Integer)} \\ &= \frac{n!}{s^{n+1}} \quad \text{(if } n \text{ is +ve Integer)} \end{aligned}}$$

Q) $\mathcal{L} t^3 = \frac{3!}{s^{3+1}}$

$$\mathcal{L}\, t^{3/2} = \frac{\Gamma 3/2+1}{s^{(3/2+1)}} = \frac{\Gamma 5/2}{s^{5/2}}$$

$$= \frac{\frac{3}{2}\,\frac{1}{2}\,\Gamma\frac{1}{2}}{s^{(5/2)}} = \frac{\frac{3}{4}\,\sqrt{\pi}}{s^{5/2}}$$

Type Q. Find out Laplace theorem of

(i) $f(t) = t^3 + e^{-3t} + t^{3/2}$

Solⁿ:-

$$\mathcal{L} f(t) = \mathcal{L} t^3 + \mathcal{L} e^{-3t} + \mathcal{L} t^{3/2}$$

$$f(s) = \frac{3!}{s^{3+1}} + \frac{1}{s+3} + \frac{\Gamma 5/2}{s^{3/2+1}}$$

$$= \frac{6}{s^4} + \frac{1}{s+3} + \frac{\Gamma 5/2}{s^{3/2+1}}$$

$$\boxed{f(s) = \frac{6}{s^4} + \frac{1}{s+3} + \frac{3}{4}\frac{\sqrt{\pi}}{s^{5/2}}}$$

(ii) $f(t) = (2t-1)^2$

Solⁿ:-

$$f(t) = 4t^2 - 4t + 1$$

$$\mathcal{L} f(t) = 4\mathcal{L} t^2 + \mathcal{L} 1 - 4\mathcal{L} t^1$$

$$= 4\left(\frac{2!}{s^{2+1}}\right) + \frac{1}{s} - 4\left(\frac{1}{s^{1+1}}\right)$$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$$\boxed{f(s) = \frac{8}{s^3} + \frac{1}{s} - \frac{4}{s^2}}$$

(iii) $f(t) = A + Bt^{1/2} + Ct^{-1/2}$

$$\mathcal{L} f(t) = A\,\mathcal{L}1 + B\,\mathcal{L}t^{1/2} + C\,\mathcal{L}t^{-1/2}$$

$$= A\frac{1}{s} + B\frac{\Gamma(1/2+1)}{s^{1/2+1}} + C\frac{\Gamma(-1/2+1)}{s^{-1/2+1}}$$

$$= \frac{A}{s} + \frac{B\,\Gamma 3/2}{s^{3/2}} + \frac{C\,\Gamma 1/2}{s^{1/2}}$$

$$= \frac{A}{s} + \frac{B\,\frac{1}{2}\Gamma 1/2}{s^{3/2}} + \frac{C\sqrt{\pi}}{s^{1/2}}$$

$$\boxed{f(s) = \frac{A}{s} + \frac{B\,\frac{1}{2}\sqrt{\pi}}{s^{3/2}} + \frac{C\sqrt{\pi}}{s^{1/2}}}$$

Type-I

Q(i) $f(t) = \sin(at+b)$

$= \sin at\cos b + \cos at\sin b$

$$\mathcal{L} f(t) = (\mathcal{L}\sin at)\cos b + (\mathcal{L}\cos at)\sin b$$

$$= \frac{a}{s^2+a^2}\cos b + \frac{s}{s^2+a^2}\sin b$$

$$\boxed{\mathcal{L} f(t) = \frac{a\cos b + s\sin b}{s^2+a^2}}$$

(ii) $f(t) = \cos(2t+3)$

$= \cos 2t \cos 3 - \sin 2t \cdot \sin 3$

$\mathcal{L} f(t) = (\mathcal{L}\cos 2t)\cos 3 - (\mathcal{L}\sin 2t)\sin 3$

$= \left(\frac{s}{s^2+4}\right)\cos 3 - \left(\frac{2}{s^2+4}\right)\sin 3$

$$\boxed{\mathcal{L} f(t) = \frac{s\cos 3 - 2\sin 3}{s^2+4}}$$

Note: Formula to use

$$\boxed{\begin{aligned} \mathcal{L}\cos at &= \frac{s}{s^2+a^2} \\ \mathcal{L}\sin at &= \frac{a}{s^2+a^2} \end{aligned}}$$

Type II:

Imp Formulae

$2SC = S+S$

$2CS = S-S$

$2CC = C+C$

$2SS = C-C$

Q.1. $f(t) = \sin 2t \cdot \cos t$

$= \frac{1}{2}[2\sin 2t \cdot \cos t]$

$= \frac{1}{2}[\sin(2t+t) + \sin(2t-t)]$

[Note:- Add angle first then after + sign subtract angle]

$f(t) = \frac{1}{2}[\sin 3t + \sin t]$

$$\mathcal{L} f(t) = \frac{1}{2}\left[\mathcal{L} \sin 3t + \mathcal{L} \sin t\right]$$

$$= \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{(s^2+1)3 + (s^2+9)}{(s^2+9)(s^2+1)}\right]$$

$$= \frac{1}{2}\left[\frac{3s^2+3+s^2+9}{(s^2+9)(s^2+1)}\right]$$

$$= \frac{1}{2}\left[\frac{4s^2+12}{(s^2+9)(s^2+1)}\right]$$

$$= \frac{4}{2}\left[\frac{(s^2+3)}{(s^2+9)(s^2+1)}\right]$$

$$\boxed{\mathcal{L} f(t) = \frac{2(s^2+3)}{(s^2+9)(s^2+1)}}$$

$$f(t) = \cos 4t \cos 2t.$$

$$= \frac{1}{2}\left[2 \cos 4t \cos 2t\right]$$

$$= \frac{1}{2}\left[\cos 6t + \cos 2t\right]$$

$$\mathcal{L} f(t) = \frac{1}{2}\left[\mathcal{L} \cos 6t + \mathcal{L} \cos 2t\right]$$

$$\mathcal{L} f(t) = \frac{1}{2}\left[\mathcal{L} \cos 6t + \mathcal{L} \cos 2t\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+36} + \frac{s}{s^2+4}\right]$$

$$= \frac{s}{2}\left[\frac{(s^2+4)+(s^2+36)}{(s^2+36)(s^2+4)}\right]$$

$$= \frac{s}{2}\left[\frac{2s^2+40}{(s^2+4)(s^2+36)}\right]$$

$$= \frac{2s}{2}\left[\frac{s^2+20}{(s^2+4)(s^2+36)}\right]$$

$$= s\left[\frac{s^2+20}{(s^2+4)(s^2+36)}\right]$$

Type II :- Power more than 1.

Formulas:-

$$\boxed{\begin{aligned} \sin^2\theta &= \frac{1-\cos 2\theta}{2} \\ \cos^2\theta &= \frac{1+\cos 2\theta}{2} \end{aligned}}$$

Q. (i) $f(t) = \sin^2 2t$

$$= \frac{1-\cos 4t}{2}$$

$$= \frac{1}{2}\left(1-\cos 4t\right)$$

$$= \frac{1}{2}\left[\mathcal{L}1 - \mathcal{L}\cos 4t\right]$$

$$\mathcal{L}f(t) = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

$$= \frac{1}{2}\left[\frac{(s^2+16)-s^2}{s(s^2+16}\right]$$

$$\boxed{\mathcal{L}f(t) = \frac{8}{s(s^2+16)}}$$

Q.(ii) $f(t) = \cos^2 3t$.

$$= \frac{1+\cos 6t}{2}$$

$$= \frac{1}{2}(1+\cos 6t)$$

$$\mathcal{L}f(t) = \frac{1}{2}[\mathcal{L}1 + \mathcal{L}\cos 6t]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+36+s^2}{s(s^2+36)}\right]$$

$$= \frac{2}{2}\left[\frac{s^2+18}{s(s^2+36)}\right]$$

$$\boxed{\mathcal{L}f(t) = \frac{s^2+18}{s(s^2+36)}}$$

$$\sin^3\theta = \frac{1}{4}[3\sin\theta - \sin 3\theta]$$

$$\cos^3\theta = \frac{1}{4}[3\cos\theta - \cos 3\theta]$$

Type: III

Q(1) $f(t) = \sin^3 2t$.

$$f(t) = \frac{1}{4}[3\sin 2t - \sin 6t]$$

$$\mathcal{L} f(t) = \frac{1}{4}[3\mathcal{L}\sin 2t - \mathcal{L}\sin 6t]$$

$$= \frac{1}{4}\left[3\left(\frac{2}{s^2+4}\right) - \left(\frac{6}{s^2+36}\right)\right]$$

$$= \frac{1}{4}\left[\frac{6}{(s^2+4)} - \frac{6}{(s^2+36)}\right]$$

$$= \frac{6}{4}\left[\frac{1}{(s^2+4)} - \frac{1}{(s^2+36)}\right]$$

$$= \frac{3}{2}\left[\frac{(s^2+36) - (s^2+4)}{(s^2+4)(s^2+36)}\right]$$

$$\boxed{\mathcal{L} f(t) = \frac{48}{(s^2+4)(s^2+36)}}$$

Q.2. $f(t) = \cos^3 3t$.

$$f(t) = \frac{1}{4}[3\cos 3t + \cos 9t]$$

$$\mathcal{L} f(t) = \frac{1}{4}[3\mathcal{L}\cos 3t + \mathcal{L}\cos 9t]$$

$$f(S) = \frac{1}{4}\left[3\cdot\frac{S}{S^2+9} + \frac{S}{S^2+81}\right]$$

$$= \frac{S}{4}\left[\frac{3S^2+243+S^2+9}{(S^2+9)(S^2+81)}\right]$$

$$= \frac{S}{4}\left[\frac{S^2+63}{(S^2+9)(S^2+81)}\right]$$

* First Shifting Theorem :-

If $\mathcal{L} f(x) = f(S)$ then

$$\mathcal{L} e^{at} f(x) = f(S-a) \quad \& \quad \mathcal{L} e^{-at} f(x) = f(S+a)$$

Q.1. $f(t) = e^{-3t} t^{3/2}$

$$\mathcal{L} t^{3/2} = \frac{\Gamma 3/2+1}{S^{3/2+1}} = \frac{\Gamma 5/2}{S^{5/2}} = \frac{3/4\sqrt{\pi}}{S^{5/2}}$$

$$\mathcal{L} e^{-3t} t^{3/2} = \frac{3/4\sqrt{\pi}}{(S+3)^{5/2}}$$

Q.2. $f(t) = e^{t}(t+1)^2$

$$(t+1)^2 = t^2+2t+1$$

$$\mathcal{L}(t+1)^2 = \frac{2!}{S^3} + 2\left(\frac{1!}{S^2}\right) + \frac{1}{S}$$

$$= \frac{2}{S^3} + \frac{2}{S^2} + \frac{1}{S}$$

* Laplace by definition :-

$$\mathcal{L} f(t) = \int_0^\infty e^{-st} f(t)\, dt .$$

Q. $f(t) = 0 \quad ; \quad 0 < t < 2$
$\quad = (t-2)^2 \quad ; \quad t \leq 2$

$$\mathcal{L} f(t) = \int_0^2 0\, dt + \int_2^\infty e^{-st} (t-2)^2 dt$$

$$= \left[(t-2)^2 \left(\frac{e^{-st}}{s}\right)\right]_2^\infty - \left[2(t-2)\left(\frac{e^{-st}}{s^2}\right)\right]_2^\infty$$

$$+ \left[2\left(\frac{e^{-st}}{-s^3}\right)\right]_2^\infty$$

$$= \left\{\frac{-1}{s}(0-0) - 2(0-0) - \frac{2}{s^3}(0-e^{-2s})\right\}$$

$$= \frac{2}{s^3} e^{-2s} .$$

Q. $f(t) = 0 \qquad 0 < t < \pi$
$\quad = \sin t \qquad t \geq \pi$

$$\mathcal{L} f(t) = \int_0^\pi 0\, dt + \int_\pi^\infty e^{-st} \sin t\, dt$$

$$abbb = \frac{e^{at}}{a^2+b^2} (a \sin bt - b \cos bt)$$

$$= \int_{\pi}^{\infty} e^{-st} \sin t \, dt$$

$$= \left[\frac{e^{-st}}{s^2+1} (-s \sin t - \cos t) \right]_{\pi}^{\infty}$$

$$= \frac{1}{s^2+1} \left[0 - e^{-s\pi} (0 - \cos \pi) \right]$$

$$\boxed{= \frac{e^{-s\pi}}{s^2+1}}$$

Q. $f(t) = 0 \quad , \quad 0 < t < \pi$

$= \cos t \quad \quad t \geq \pi$

$$abbb = \frac{e^{at}}{a^2+b^2} [a \cos bt + b \sin bt]$$

$$= \int_{\pi}^{\infty} e^{-st} \cos t \, dt$$

$$= \left[\frac{e^{-st}}{s^2+1} (-s \cos t + \sin t) \right]_{\pi}^{\infty}$$

$$= \frac{1}{s^2+1} \left[0 - e^{-s\pi} (-s \cos \pi + \sin \pi) \right]$$

$$= \frac{1}{s^2+1} \left[0 - e^{-s\pi} [-s(-1) + 0] \right]$$

$$= \frac{1}{s^2+1} \left[-s e^{-s\pi} \right] = \frac{-s e^{-s\pi}}{s^2+1}.$$

* Differentiation Method:-

- If $\mathcal{L} f(t) = f(s)$ then

$$\mathcal{L}\, t^n f(t) = (-1)^n \frac{d^n}{ds^n}\left[f(s)\right]$$

$$\mathcal{L}\, t^3 e^{at} = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{(s-a)}\right)$$

$$\mathcal{L}\, t^2 \sin t = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right)$$

$$y = \frac{-1}{s^2}$$
$$= (-s^{-2})$$
$$= -2\, s^{-2-1}$$
$$= -2\, s^{-3}$$
$$= -\frac{2}{s^3}$$

Differentiations:-

$y = \frac{1}{y}$ $\qquad y_3 = \frac{-6}{s^4}$

$y_1 = \frac{-1}{s^2}$ $\qquad y_4 = \frac{+24}{y^5}$

$y_2 = +\frac{2}{s^3}$

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Q. Find out Laplace theorem of

(i) $f(t) = t^4 e^{at}$

Soln:- $\mathcal{L}\, t^4 e^{at} = (-1)^4 \frac{d^4}{ds^4}\left(\frac{1}{s-a}\right)$

$= \frac{d^3}{ds^3}\left(\frac{-1}{(s-a)^2}\right)$

$= \frac{d^2}{ds^2}\left(\frac{2}{(s-a)^3}\right)$

$= \frac{d}{ds}\left(\frac{-6}{(s-a)^4}\right)$

$= \frac{24}{(s-a)^5}$

(ii) $f(t) = t^2 \sin \omega t$

$\mathcal{L}\, t^2 \sin \omega t = (-1)^2 \frac{d^2}{ds^2}\left(\frac{\omega}{s^2+\omega^2}\right)$

$= \omega \frac{d^2}{ds^2}\left(\frac{1}{s^2+\omega^2}\right)$

$= \omega \frac{d}{ds}\left[\frac{-1}{(s^2+\omega^2)^2}\right] 2s$

$= -2\omega \frac{d}{ds}\left[\frac{s}{(s^2+\omega^2)^2}\right]$

$$= -2\omega \left\{ \frac{(s^2+\omega^2)^2 (1) - s\,[2(2s)(s^2+\omega^2)]}{(s^2+\omega^2)^4} \right\}$$

$$= -2\omega\,(s^2+\omega^2) \left\{ \frac{(s^2+\omega^2) - 4s^2}{(s^2+\omega^2)^4} \right\}$$

$$= -2\omega \left[\frac{\omega^2 - 4s^2}{(s^2+\omega^2)^3} \right]$$

$$= 2\omega \left[\frac{3s^2 - \omega^2}{(s^2+\omega^2)^3} \right]$$

(iii) $f(t) = t^2 \cos t.$

$$\mathcal{L}\, t^2 \cos t = (-1)^2 \frac{d s^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{d}{ds}\left[\frac{(s^2+1)\cdot 1 - 2s\cdot s}{(s^2+1)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+1) - 2s^2}{(s^2+1)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{1-s^2}{(s^2+1)^2}\right]$$

$$= \left[\frac{(s^2+1)^2(-2s) - \{(-s^2)(2)(2s)(s^2+1)\}}{(s^2+1)^4}\right]$$

$$= (s^2+1)\left[-2s(s^2+1) - 4s(1-s^2)\right]$$

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(iv) $f(t) = t e^{-2t} \sin 3t$

$$\mathcal{L} \sin 3t = \left(\frac{3}{s^2+9}\right)$$

$$\mathcal{L} e^{-2t} = \left(\frac{3}{(s+2)^2+9}\right)$$

$$f(t) = e^{-2t} \sin 3t$$

$$\mathcal{L}\, t f(t) = \mathcal{L}\, t e^{-2t} \sin 3t$$

$$= (-1) \frac{d}{ds}\left(\frac{3}{(s+2)^2+9}\right)$$

$$= -3 \frac{d}{ds}\left[\frac{1}{(s+2)^2+9}\right]$$

$$= -3 \left[\frac{-12(s+2)}{\{(s+2)^2+9\}^2}\right]$$

$$= -3 \left[\frac{-12(s+2)}{\{(s+2)^2+9\}^2}\right]$$

$$= \frac{6(s+2)}{[(s+2)^2+9]^2}$$

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(v) $f(t) = t^2 \sinh t$

$n = 2$

$$\mathcal{L}\, t^2 \sinh t = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s^2-1}\right)$$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s^2-1}\right)$$

$$= \frac{d}{ds}\left(\frac{-2s}{(s^2-1)^2}\right)$$

$$= -2\left\{\frac{d}{ds}\left(\frac{s}{(s^2-1)^2}\right)\right.$$

$$= -2\left[\frac{(s^2-1)^2(1) - s(2s)(2)(s^2-1)}{(s^2-1)^4}\right]$$

$$= -2\left[(s^2-1)\left(\frac{(s^2-1) - 4s^2}{(s^2-1)^4}\right)\right]$$

$$= -2\left[\frac{-1-3s^2}{(s^2-1)^3}\right]$$

$$\boxed{\mathcal{L}\, f(t) = \frac{2-6s^2}{(s^2-1)^3}}$$

* **Integration Method :-**

If $\mathcal{L} f(t) = f(s)$ then

$$\mathcal{L} \frac{f(t)}{t} = \int_{s}^{\infty} f(s)\, ds$$

Eg $$\mathcal{L} \frac{\sin t}{t} = \int_{s}^{\infty} \frac{1}{s^2+1}\, ds$$

* **Imp formulas :-**

→ $$\int \frac{1}{s^2+a^2}\, ds = \frac{1}{a} \tan^{-1}\left(\frac{s}{a}\right)$$

→ $$\int \frac{1}{s^2+a^2}\, ds = \frac{1}{2} \int \frac{2s}{s^2+a^2}\, ds$$

$$= \frac{1}{2} \log(s^2+a^2)$$

→ $$\int \frac{1}{s+a}\, ds = \log(s+a)$$

→ $$\int \frac{1}{s^2+a^2}\, ds = \frac{1}{2a} \log \left|\frac{s-a}{s+a}\right|$$

→ $\cot^{-1}$

$\boxed{\text{Ans, } \cot^{-1}, \log s}$

* $\tan \infty = \frac{\pi}{2}$
* $\tan^{-1} x + \cot^{-1} x = \pi/2$
* $\cot^{-1} x = \pi/2 - \tan^{-1} x$

Q. $\mathcal{L}\dfrac{\sin\omega t}{t}$.

$$= \int_s^{\omega}\left(\frac{\omega}{s^2+\omega^2}\right) ds.$$

$$= \omega \int_s^{\infty} \frac{1}{s^2+\omega^2}\, ds.$$

$$= \omega\left[\frac{1}{\omega}\tan^{-1}\left(\frac{s}{\omega}\right)\right]_s^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{\omega}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{\omega}\right)$$

$$= \cot^{-1}\left(\frac{s}{\omega}\right).$$

Note:- If Qs in trigo ans is $\cot^{-1}$
" Qs in other form ans : is log.

Q. $f(t) = \dfrac{e^{-at} - e^{-bt}}{t}$

Solⁿ :-

$$\mathcal{L}\, f(t) = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b} \right) ds.$$

$$= \left[\log(s+a) - \log(s+b) \right]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{s+b}\right) \right]_s^\infty$$

$$= \log \left[\frac{s\,(1+a/s)}{s\,(1+b/s)} \right]_s^\infty$$

$$= \log \left[\frac{1+a/s}{1+b/s} \right]_s^\infty$$

$$= \log 1 - \log\left(\frac{s+a}{s+b}\right)$$

$$= 0 - \log\left(\frac{s+a}{s+b}\right)$$

$$= \log\left(\frac{s+b}{s+a}\right).$$

Q. $f(t) = \frac{\cos 2t - \cos 3t}{t}$

$$\mathcal{L} f(t) = \int_s^{\infty} \left(\frac{s}{s^2 + 2^2} - \frac{s}{s^2 + 3^2} \right) ds$$

$$= \left[\frac{1}{2} \log (s^2 + 2^2) - \frac{1}{2} \log (s^2 + 3^2) \right]_s^{\infty}$$

$$= \frac{1}{2} \left[\log (s^2 + 2^2) - \log (s^2 + 3^2) \right]_s^{\infty}$$

$$= \frac{1}{2} \left[\log \left(\frac{s^2 + 4}{s^2 + 9} \right) \right]_s^{\infty}$$

$$= \frac{1}{2} \left[\log 1 - \log \left(\frac{s^2 + 4}{s^2 + 9} \right) \right]$$

$$= \frac{1}{2} \left[0 + \log \left(\frac{s^2 + 9}{s^2 + 4} \right) \right]$$

$$= \frac{1}{2} \log \left(\frac{s^2 + 9}{s^2 + 4} \right)$$

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Q. $\mathcal{L}\, \frac{e^{-t}\sin t}{t}$

$$\mathcal{L}\, f(t) = \int_s^\infty \frac{1}{(s+1)^2+1}\, ds$$

$$= \left[\tan^{-1}(s+1)\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}(s+1)$$

$$= \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1).$$

Q. $\mathcal{L}\, \frac{1-e^{t}}{t}$

$$\mathcal{L}\, f(t) = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds.$$

$$= \left[\log s - (\log s-1)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{(s-1)}\right)\right]_s^\infty$$

$$= \log 1 - \log\left(\frac{s}{s-1}\right)$$

$$= \cancel{\log} 0 + \log\left(\frac{s-1}{s}\right) = \boxed{\log\left(\frac{S-1}{s}\right)}$$

Q. $f(t) = \frac{1-\cos t}{t}$

$$\mathcal{L}\, f(t) = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds.$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{1}{2}\frac{2s}{s^2+1}\right] ds.$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1^2).\right]_s^\infty$$

$$= \left[\log s - \log\sqrt{s^2+1^2}\right]_s^\infty$$

$$= \log\left[\frac{s}{\sqrt{s^2+1^2}}\right]_s^\infty$$

$$= \log 1 - \log\left(\frac{s}{\sqrt{s^2+1^2}}\right)$$

$$= 0 + \log\frac{\sqrt{s^2+1^2}}{s}$$

$$= \boxed{\log\frac{\sqrt{s^2+1^2}}{s}}$$

Q. $f(t) = \frac{1-\cos t}{t^2}$

Solⁿ: $\mathcal{L}\, f(t) = \int_s^\infty \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1} \right) ds\, ds$

$= \int_s^\infty \log\left(\frac{\sqrt{s^2+1}}{s} \right) ds.$

$= \int_s^\infty \underset{v}{1} \; \underset{u}{\log\left(\frac{\sqrt{s^2+1}}{s} \right)} ds.$

Note:- $\int \underset{v}{1} . \underset{u}{\log s}\, ds =$

$= (\log s) \int \log 1 - \int \frac{1}{s} . s\, ds.$

$= s \log s - \int ds.$

$= s \log s - s.$

ifferentiation of u.

$$u = \log \frac{\sqrt{s^2+1}}{s}$$

$$= \log \sqrt{s^2+1} - \log s.$$

$$u = \frac{1}{2} \log (s^2+1) - \log s.$$

$$\frac{du}{ds} = \frac{1}{2}\left(\frac{2s}{s^2+1}\right) - \frac{1}{s}$$

$$= \frac{s^2 - (s^2+1)}{s(s^2+1)} = \frac{-1}{s(s^2+1)}$$

$$= \int_s^\infty 1 \cdot \log \frac{\sqrt{s^2+1}}{s} \, ds.$$

$$= \left[\log\left(\frac{\sqrt{s^2+1}}{s}\right) \cdot s\right]_s^\infty - \int_s^\infty -\frac{1 \cdot s}{s(s^2+1)} \, ds$$

$$= \left[\log\left(\frac{\sqrt{s^2+1}}{s}\right) \cdot s\right]_s^\infty - [\tan^{-1} s]_s^\infty$$

$$0 - \log\left(\frac{\sqrt{s^2+1}}{s}\right) \cdot s + \cot^{-1} s.$$

$$= \boxed{\cot^{-1} s - s \log \frac{\sqrt{s^2+1}}{s}}.$$

* Laplace of Integration :-

$$\mathcal{L} \int_0^t f(t)\, dt$$

Steps

1.* Find out Laplace of $f(t)$

2.* Divide this laplace by 's'.

Q. Find out Laplace transform of $\mathcal{L} \int_0^t \sin t\, dt$

Soln:-

Here $f(t) = \sin t$

$\mathcal{L} f(t) = \mathcal{L} \sin t$

$f(s) = \dfrac{1}{s^2+1}$

Now $\mathcal{L} \int_0^t \sin t\, dt = \dfrac{1}{s}\left(\dfrac{1}{s^2+1}\right)$.

Q. Find out Laplace of $\mathcal{L} \int_0^t e^t \cos t\, dt$.

$f(t) = e^t \cos t$

$\mathcal{L} f(t) = \dfrac{s-1}{(s-1)^2+1}$

Now $\mathcal{L} \int_0^t e^t \cos t\, dt = \dfrac{1}{s}\left[\dfrac{s-1}{(s-1)^2+1}\right]$

Q. Find out Laplace transform of $L\int_0^t t\sin t\,dt$.

$$L\, t\sin t = (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= (-1)\left(-\frac{1\,(2s)}{(s^2+1)^2}\right)$$

$$= \frac{2s}{(s^2+1)^2}$$

Now $L\int_0^t t\sin t\,dt = \frac{1}{s}\left[\frac{2s}{(s^2+1)^2}\right]$

$$= \frac{2}{(s^2+1)^2}$$

Q. Find out Laplace of $L\int_0^t \frac{\sin t}{t}$.

$f(t) = \frac{\sin t}{t}$

$$L f(t) = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \left[\tan^{-1}s\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}s.$$

$$= \pi/2 - \tan^{-1}s. = \cot^{-1}s.$$

Now;

$$\mathcal{L}\int_0^t \frac{\sin t}{t}\,dt = \frac{\cot^{-1} s}{s}.$$

Note:- $\int_0^t \int_0^t \int_0^t \frac{\sin t}{t}\,dt\,dt\,dt.$

As integration rises power for ans $\frac{\cot^{-1} s}{s}$ increases i.e. $\frac{\cot^{-1} s}{s^3}$ for triple integration.

Q. Find out Laplace of $\mathcal{L}\int_0^t\int_0^t \sin at\,dt\,dt.$

Soln:- $f(t) = \sin at$

$$\mathcal{L}\sin at = \frac{a}{s^2+a^2}$$

Now

$$\mathcal{L}\int_0^t\int_0^t \sin at\,dt = \frac{1}{s^2}\left(\frac{a}{s^2+a^2}\right).$$

Q. $\mathcal{L}\, e^{t} \int_0^t \cos t \; dt$.

$$\mathcal{L} \cos t = \frac{s}{s^2+1}$$

$$\mathcal{L} \int_0^t \cos t \; dt = \frac{1}{s}\left(\frac{s}{s^2+1}\right)$$

$$= \left(\frac{1}{s^2+1}\right)$$

Now $\mathcal{L}\, e^{t} \int_0^t \cos t \; dt = \dfrac{1}{(s-1)^2+1}$.

Q. Find out laplace of $\mathcal{L}\, t \int_0^t \cosh at \; dt$.

$$\mathcal{L} \cosh at = \frac{s}{s^2-a^2}$$

$$\mathcal{L} \int_0^t \cosh at = \frac{1}{s}\left(\frac{s}{s^2-a^2}\right)$$

$$= \left(\frac{1}{s^2-a^2}\right)$$

Now $\mathcal{L}\, t \int_0^t \cosh at = (-1)\dfrac{d}{ds}\left(\dfrac{1}{s^2-a^2}\right)$

$$= -\left[\frac{-1\,(2s)}{(s^2-a^2)^2}\right]$$

$$= \frac{2s}{(s^2-a^2)^2} \quad //$$

Q $\mathcal{L}\, e^{-3t} \frac{1}{t} \int_0^t \cos t \, dt$

$\mathcal{L} \cos t = \frac{s}{s^2+1}$

$\mathcal{L} \int_0^t \cos t \, dt = \frac{1}{s} \left(\frac{s}{s^2+1} \right)$

$= \frac{1}{s^2+1}$

$\mathcal{L} \frac{1}{t} \int_0^t \cos t \, dt = \int_s^\infty \frac{1}{s^2+1} \, ds$

$= \left[\tan^{-1} s \right]_s^\infty$

$= \tan^{-1}\infty - \tan^{-1} s$

$= \frac{\pi}{2} - \tan^{-1} s$

$= \cot^{-1} s$.

Now,

$\mathcal{L}\, e^{-3t} \frac{1}{t} \int_0^t \cos t \, dt = \cot^{-1}(s+3)$

* Heaviside's Unit Step Function :-

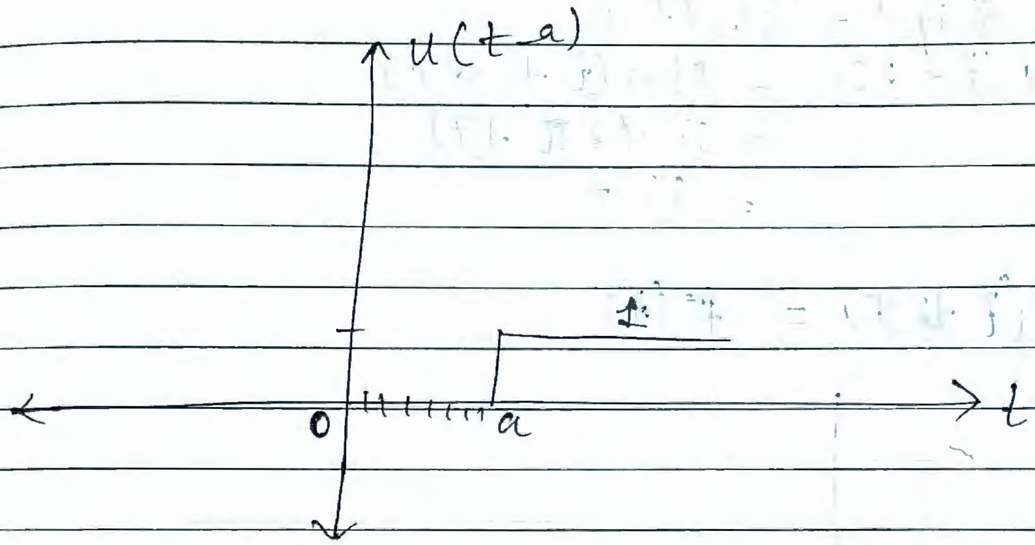
$H(t-1)$.

$f(x) = x\,L$ $u(t-1)$.

unit step function is defined as.

$u(t-a) = 0 \quad 0 < t < a$

$= 1 \quad t \geq a$.



$$Lf(t) = \int_0^{\infty} e^{-st} f(t)\,dt$$

$$L\,u(t-a) = \int_0^{\infty} e^{-st} u(t-a)\,dt$$

$$= \int_0^{a} e^{-st}\, 0\,dt + \int_a^{\infty} e^{-st}\,dt.$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^{\infty}$$

$$= \frac{-1}{s}\left[0 - e^{-as}\right]$$

$$\boxed{L\,u(t-a) = \frac{e^{-as}}{s}}$$

* Laplace of periodic function :-

Any function $f(t)$ is said to be periodic of period 'p' if

$$f(t+p) = f(t)$$

$$f(t) = \sin t$$

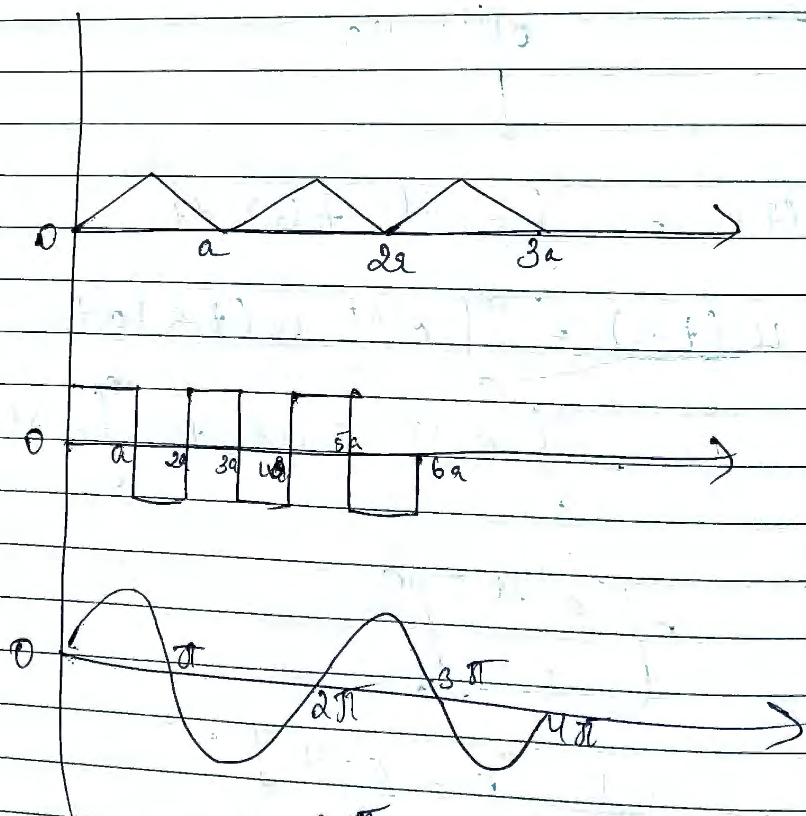
$$f(t+p) = \sin(t+p)$$

$$f(t+2\pi) = \sin(t+2\pi)$$

$$= \sin(2\pi + t)$$

$$= \sin t$$

$$f(t+2\pi) = f(t)$$



$p = 2\pi$

$$\mathcal{L} f(t) = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$$

$$\mathcal{L} f(t) = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$$

$$\mathcal{L} f(t) = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-as}} \int_0^a \underset{u}{t}\ \underset{v}{e^{-st}}\, dt$$

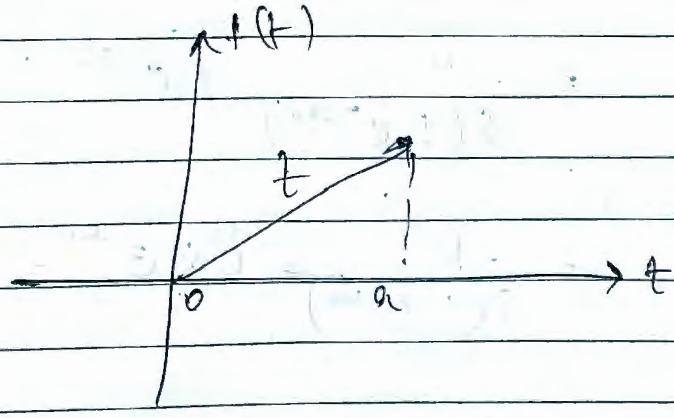
$$= \frac{1}{1-e^{-as}} \left\{ \left[t \frac{e^{-st}}{-s} \right]_0^a - 1 \left[\frac{e^{-st}}{s^2} \right]_0^a \right\}$$

$$= \frac{1}{1-e^{-as}} \left\{ -\frac{1}{s} \left(t e^{-st} \right)_0^a - \frac{1}{s^2} \left(e^{-st} \right)_0^a \right\}$$

$$= \frac{1}{1-e^{-as}} \left[\frac{-1}{s} \left(a e^{-as} - 0 \right) - \frac{1}{s^2} \left(e^{-as} - 1 \right) \right]$$

$$= \frac{1}{1-e^{-as}} \left[-\frac{a}{s} e^{-as} - \frac{1}{s^2} \left(e^{-as} - 1 \right) \right]$$

$f(t) = t$



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Q Find out Laplace transform of $f(t) = K \quad 0 < t < a$ where $f(t + 2a) = f(t)$

$= -K \quad a < t < 2a$

Sol^n: $\displaystyle \mathcal{L} f(t) = \frac{1}{1-e^{-Ps}} \int_0^P e^{-st} f(t)\, dt$

$$\mathcal{L} f(t) = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$\mathcal{L} f(t) = \frac{1}{1-e^{-2as}} \left[K\int_0^{a} e^{-st}\, dt - K \int_0^{2a} e^{-st}\, dt \right]$$

$$= \frac{K}{1-e^{-2as}} \left[\left(\frac{e^{-st}}{-s}\right)_0^a - \left(\frac{e^{-st}}{-s}\right)_0^{2a} \right]$$

$$= \frac{-K}{s(1-e^{-2as})} \left[(e^{-st})_0^a - (e^{-st})_a^{2a} \right]$$

$$= \frac{-K}{s(1-e^{-2as})} \left[(e^{-as} - e^{0}) - (e^{-2as} - e^{-as}) \right]$$

$$= \frac{-K}{s(1-e^{-2as})} \left[e^{-as} - 1 - e^{-2as} + e^{-as} \right]$$

$$= \frac{-K}{s(1-e^{-2as})} \left[2e^{-as} - 1 - e^{-2as} \right]$$

$$= \frac{-K}{s(1-e^{-2as})}\left[2e^{-as} - 1 - e^{-2as}\right]$$

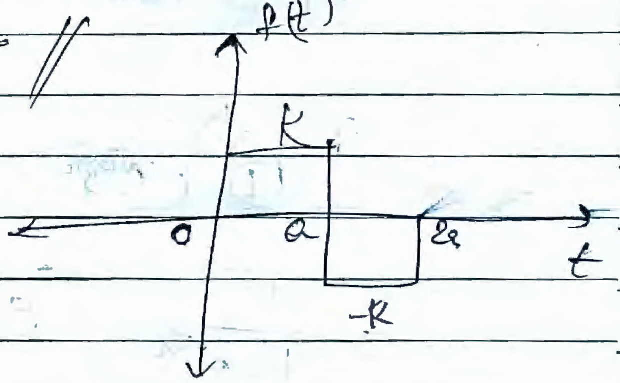
$$= \frac{K}{s(1-e^{-2as})}\left[1 + e^{-2as} - 2e^{-as}\right]$$

$$= \frac{K}{s(1-e^{-2as})}\left[1 - e^{-as}\right]^2$$

$$= \frac{K}{s}\,\frac{\left[1-e^{-as}\right]^2}{\left[1-(e^{-as})^2\right]}$$

$$= \frac{K}{s}\,\frac{(1-e^{-as})^2}{\left[(1-e^{-as})(1+e^{-as})\right]}$$

$$= \frac{K}{s}\left(\frac{(1-e^{-as})}{(1+e^{-as})}\right)$$



Q. Find out Laplace transform of full wave rectification ~~$f(t) = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)\,dt$~~

$f(t) = |\sin \omega t|$ $t \geq 0$.

Soln.

$$\mathcal{L} f(t) = \frac{1}{1-e^{-PS}} \int_0^P e^{-st} f(t)\, dt.$$

$$f(t) = \sin \omega t. \quad \left(0 < t < \frac{\pi}{\omega}\right)$$

$$\mathcal{L} f(t) = \frac{1}{1-e^{-\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin \omega t\, dt.$$

$$a = s, \quad b = \omega.$$

$$= \frac{1}{1-e^{-\pi s/\omega}} \left[\frac{e^{-st}}{(-s)^2 + \omega^2} (-s \sin \omega t - \omega \cos \omega t) \right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-\pi s/\omega}} \left[e^{-\pi s/\omega} (0 - \omega \cos \pi) - 1 (0 - \omega \cos 0) \right]$$

$$= \frac{1}{(1-e^{-\pi s/\omega})(s^2+\omega^2)} \left(\omega e^{-\pi s/\omega} + \omega \right)$$

$$= \boxed{\frac{\omega}{(s^2+\omega^2)} \frac{(1 + e^{-\pi s/\omega})}{(1 - e^{-\pi s/\omega})}}$$

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* Inverse Laplace :-

- $\mathcal{L}1 = \frac{1}{s} \Rightarrow \boxed{\mathcal{L}^{-1}\frac{1}{s} = 1}$

- $\mathcal{L}e^{at} = \frac{1}{s-a} \Rightarrow \boxed{\mathcal{L}^{-1}\frac{1}{s-a} = \cancel{\phi}\, e^{at}. \quad \mathcal{L}^{-1}\frac{1}{s+a} = e^{-at}}$ Power should be 1.

- $\mathcal{L}\cos at = \frac{s}{s^2+a^2}$

$\therefore \cos at = \mathcal{L}^{-1}\, \cancel{\beta}\, \frac{s}{s^2+a^2}$

- $\mathcal{L}^{-1}\frac{s}{s^2+a^2} = \cos at$

- $\mathcal{L}^{-1}\frac{s}{s^2-a^2} = \cosh at$

- $\mathcal{L}\sin at = \frac{a}{s^2+a^2}$

$\therefore \sin at = a\,\mathcal{L}^{-1}\frac{1}{s^2+a^2}$

$\therefore \frac{\sin at}{a} = \mathcal{L}^{-1}\frac{1}{s^2+a^2}$

$$\rightarrow L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at$$

$$\rightarrow L^{-1}\frac{1}{s^2-a^2} = \frac{1}{a}\sinh at$$

$$\rightarrow L\, t^n = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\rightarrow t^n = \Gamma(n+1)\, L^{-1}\frac{1}{s^{n+1}}$$

$$\rightarrow \frac{t^n}{\Gamma(n+1)} = L^{-1}\frac{1}{s^{n+1}}$$

$$n = n-1$$

$$\rightarrow L^{-1}\frac{1}{s^n} = \frac{t^{n-1}}{\Gamma(n)} \quad (\text{if } n \text{ is non-Integer})$$

$$= \frac{t^{n-1}}{(n-1)!} \quad (\text{if } n \text{ is +ve Integer})$$

* Inverse Laplace with 1st shifting :-

$$L^{-1}\frac{s-a}{(s-a)^2+b^2} = e^{at}\cos bt$$

→ $$L^{-1}\frac{s-a}{(s-a)^2-b^2} = e^{at}\cosh bt.$$

→ $$L^{-1}\frac{1}{(s-a)^2+b^2} = e^{at}\frac{1}{b}\sin bt.$$

→ $$L^{-1}\frac{1}{(s-a)^2-b^2} = e^{at}\frac{1}{b}\sin bt.$$

$$L^{-1}\frac{1}{(s-a)^n} = e^{at}\frac{t^{n-1}}{(n-1)!}$$

$$= \frac{t^{n-1}}{\lceil n}$$

Type 1: Find out inverse Laplace.

(i) $f(s) = \frac{s^3 - 3s^2 + 2}{s^4}$

$$= \frac{s^3}{s^4} - \frac{3s^2}{s^4} + \frac{2}{s^4}$$

$$= \frac{1}{s} - \frac{3}{s^2} + \frac{2}{s^4}$$

$$L^{-1} f(s) = L^{-1}\frac{1}{s} - 3L^{-1}\frac{1}{s^2} + 2L^{-1}\frac{1}{s^4}$$

$$= 1 - 3\frac{t^1}{1!} + 2\frac{t^3}{3!}$$

$$\boxed{f(t) = 1 - 3t + \frac{t^3}{3}}$$

(ii)

(iii)

(ii) $f(s) = \frac{(2s^2-1)^2}{s^5}$

$= \frac{4s^4+1-4s^2}{s^5}$

$= \frac{4}{s} + \frac{1}{s^5} - \frac{4}{s^3}$

$\mathcal{L}^{-1} f(s) = 4\mathcal{L}^{-1}\frac{1}{s} + \mathcal{L}^{-1}\frac{1}{s^5} - 4\mathcal{L}^{-1}\frac{1}{s^3}$

$= 4(1) + \frac{t^4}{4!} - 4\frac{t^2}{2!}$

$$\boxed{f(t) = 4 + \frac{t^4}{24} - 2t^2}$$

(iii) $f(s) = \frac{s}{(s+1)^5}$

$= \frac{(s+1)-1}{(s+1)^5}$

$= \frac{1}{(s+1)^4} - \frac{1}{(s+1)^5}$

$\mathcal{L}^{-1} f(s) = \mathcal{L}^{-1}\frac{1}{(s+1)^4} - \mathcal{L}^{-1}\frac{1}{(s+1)^5}$

$$\boxed{L^{-1} f(s) = e^{-t}\frac{t^3}{3!} - e^{-t}\frac{t^4}{4!}}$$

(iv) $f(s) = \frac{s+3}{(2s+1)^4}$

$= \frac{s+3}{2^4\,(2s}$

~~Inverse~~

→ Inverse Laplace.

$$\boxed{f(s) = \frac{g(s)}{h(s)}}$$

If h(s) factorise ↙ ↘ If h(s) does not factorise

Partial fraction

① If factors in deno. are distinct & linear. $\frac{1}{(s+2)^1(s+6)^1}$

② Outside & inside power. $\frac{1}{(s+2)^1(s+6)^2}$

③ Out. & inside power ones. $\frac{1}{(s+2)(s^2+6^2)}$

Method-2

① Equation in deno. is quadratic.

②

Partial fraction. Note Power is 1.

Type-I If factors in dinominator are distinct & linear.

Note :- Ans in Type I

$$\mathcal{L}^{-1}\frac{1}{s+a} = e^{-at}$$

$$\mathcal{L}^{-1}\frac{1}{s-a} = e^{at}$$

always in exponent.

Q $$f(s) = \frac{s}{s^2+3s+2}$$

$$= \frac{s}{(s+1)^1(s+2)^1}$$

Let $$\frac{s}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} \longrightarrow ①$$

$$= \frac{A(s+2) + B(s+1)}{(s+1)(s+2)}$$

$$s = A(s+2) + B(s+1).$$

Inverse Laplace.

$$\mathcal{L}^{-1} f(s) = -\mathcal{L}^{-1} \frac{1}{(s+1)} + 2\,\mathcal{L}^{-1} \frac{1}{(s+2)}$$

$$\boxed{f(t) = -e^{-t} + 2e^{-2t}}$$

A	B	C
$s = -1$	$s = -2$	$s = -3$
$3(-1)^2 + 2 = (1)(2)A$	$3(-2)^2 + 2 = (-1)(1)B$	$3(-3)^2 + 2 = (-2)(-1)C$
$5 = 2A$	$3(4) + 2 = -B$	$3(9) + 2 = 2C$
$\boxed{A = \frac{5}{2}}$	$12 + 2 = -B$	$\boxed{\frac{29}{2} = C}$
	$\boxed{B = -14}$	

Put values of A, B & C in (1)

$$L^{-1} f(s) = \frac{5}{2}\,\frac{1}{s+1} - 14\,\frac{1}{s+2} + \frac{29}{2}\,\frac{1}{s+3}$$

Inverse Laplace

$$L^{-1} f(s) = \frac{5}{2}\,L^{-1}\frac{1}{s+1} - 4\,L^{-1}\frac{1}{s+2} + \frac{29}{2}\,L^{-1}\frac{1}{s+3}$$

$$\boxed{f(t) = \frac{5}{2}\,e^{-t} - 4\,e^{-2t} + \frac{29}{2}\,e^{-3t}}$$

Q $f(s) = \frac{1}{(s+\sqrt{2})(s-\sqrt{3})}$ [GTU Q]

Sol^n

Let $\frac{1}{(s+\sqrt{2})(s-\sqrt{3})} = \frac{A}{(s+\sqrt{2})} + \frac{B}{(s-\sqrt{3})}$

$$\frac{1}{(s+\sqrt{2})(s-\sqrt{3})} = \frac{(s-\sqrt{3})A + (s+\sqrt{2})B}{(s+\sqrt{2})(s-\sqrt{3})}$$

$$1 = (s-\sqrt{3})A + (s+\sqrt{2})B \longrightarrow (1)$$

(A)

$s = -\sqrt{2}$

$1 = (-\sqrt{2}-\sqrt{3})A$

$1 = -(\sqrt{2}+\sqrt{3})A$

$$\boxed{A = \frac{-1}{\sqrt{2}+\sqrt{3}}}$$

(B)

$s = -\sqrt{3}$

$1 = -\sqrt{3} - \sqrt{2}B$

$1 = -(\sqrt{3}+\sqrt{2})B$

$$\boxed{B = \frac{-1}{\sqrt{3}+\sqrt{2}}}$$

Put values of A & B in (1).

$$f(s) = (s-\sqrt{3})\left[\frac{-1}{\sqrt{2}+\sqrt{3}}\right] + (s+\sqrt{2})\left[\frac{-1}{\sqrt{3}+\sqrt{2}}\right]$$

Inverse Laplace of $f(s)$ is

$$f(t) = \frac{1}{\sqrt{2}+\sqrt{3}}\left[\mathcal{L}^{-1}\frac{-1}{s+\sqrt{2}} + \mathcal{L}^{-1}\frac{1}{s-\sqrt{3}}\right]$$

$$f(t) = \frac{1}{\sqrt{2}+\sqrt{3}}\left[-e^{-\sqrt{2}t} + e^{\sqrt{3}t}\right]$$

Q $f(s) = \dfrac{s}{(s-1)(s-3)(s-4)}$

Solⁿ:- Let $\dfrac{s}{(s-1)(s-3)(s-4)} = \dfrac{A}{(s-1)} + \dfrac{B}{(s-3)} + \dfrac{C}{(s-4)}$ $\to$ (1).

$$\frac{s}{(s-1)(s-3)(s-4)} = \frac{(s-3)(s-4)A + B(s-1)(s-4) + C(s-1)(s-3)}{(s-1)(s-3)(s-4)}$$

$s = (s-3)(s-4)A + B(s-1)(s-4) + C(s-1)(s-3)$ $\longrightarrow$ (1)

A	B	C
$s=1$	$s=3$	$s=4$
$1 = (1-3)(1-4)A$	$3 = (3-1)(3-4)B$	$4 = (4-1)(4-3)C$
$1 = (-2)(-3)A$	$3 = (2)(-1)B$	$4 = (3)(1)C$
$\boxed{A = \frac{1}{6}}$	$3 = -2B$	$\boxed{C = \frac{4}{3}}$
	$\boxed{B = -\frac{3}{2}}$	

Put values of A, B, C in (1).

$$\cancel{f(s) = (S-3)(S-4)\frac{1}{6} + (S-1)(S-4)\frac{3}{2} + (S-1)(S-3)\,4}$$

$$f(s) = \frac{1}{6(S-1)} - \frac{2}{3(S-3)} + \frac{4}{3(S-4)}$$

Inverse Laplace

$$\mathcal{L}^{-1} f(s) = \frac{1}{6}\mathcal{L}^{-1}\frac{1}{(S-1)} - \frac{2}{3}\mathcal{L}^{-1}\frac{1}{(S-3)} + \frac{4}{3}\mathcal{L}^{-1}\frac{1}{(S-4)}$$

$$\boxed{\mathcal{L}^{-1} f(s) = \frac{1}{6}e^{-t} - \frac{2}{3}e^{-3t} + \frac{4}{3}e^{-4t}}$$

Type-2 If factors in denominator are repeated :- Power outside bracket & more than 1.

Note :- Answer should be

$$\mathcal{L}^{-1}\frac{1}{(S+a)^1} = e^{-at}$$

$$\mathcal{L}^{-1}\frac{1}{(S+a)^n} = e^{-at}\frac{t^{n-1}}{(n-1)!}$$

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$$)) = \frac{4S+5}{(S-1)^2(S+2)}$$

$$\frac{4S+5}{(S-1)^2(S+2)} = \frac{A}{(S-1)} + \frac{B}{(S-1)^2} + \frac{C}{(S+2)}$$

$$4S+5 = A(S-1)(S+2) + B(S-1)(S+2) + C(S-1)^2$$

B

$S = 1$

$9 = 3B$

$\frac{9}{3} = B$

$\Rightarrow \boxed{B = 3}$

C

$S = -2$

$4(-2) + 5 = (-3)^2 C$

$-8 + 5 = (-3)^2 C$

$-3 = 9C$

$\Rightarrow \boxed{C = \frac{-1}{3}}$

$S = 0$ in (V)

$5 = (-1)(2)A + 2B + C(-1)^2$

$5 = -2A + 2B + C$

$2A = 2B + C - 5$

$2A = 2(3) - \frac{1}{3} - 5$

$2A = 1 - \frac{1}{3}$

$2A = \frac{2}{3}$

$\boxed{A = \frac{1}{3}}$

$$f(s) = \frac{1}{3}\frac{1}{(S-1)} + 3\frac{1}{(S-1)^2} - \frac{1}{3}\frac{1}{(S+2)}$$

Inverse Laplace.

$$\boxed{f(t) = \frac{1}{3}e^{t} + 3e^{t}\frac{t!}{1!} + \frac{1}{3}e^{-2t}}$$

Type 8:- If factors in denominator are Quadratic :- Power inside is more than 1

Note:-

$$\boxed{\begin{array}{l} \text{Answer in.} \\ \mathcal{L}^{-1}\frac{1}{(s+a)^1} = e^{-at} \\ \mathcal{L}^{-1}\frac{s}{s^2+a^2} = \cos at. \\ \mathcal{L}^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at. \end{array}}$$

Q. $f(s) = \dfrac{s+29}{(s+4)(s^2+9)}$

Soln:-

Let $\dfrac{s+29}{(s+4)(s^2+9)} = \dfrac{A}{s+4} + \dfrac{Bs+C}{s^2+9}$

$$s+29 = A(s^2+9) + (Bs+C)(s+4)$$

A

S = -4

$25 = A(16+9)$

$25 = A\,25$

$\boxed{A = 1}$

C

S = 0

$29 = 9A + 4C$

$29 = 9 + 4C$

$20 = 4C$

$\boxed{C = 5}$

Put $s = 1$ in (1)

$30 = 10A + 5(B+C)$

$30 = 10 + 5(B+5)$

$20 = 5(B+5)$

$\frac{20}{5} = B+5$

$$L.S. = \frac{(m.s)^2}{4 \times F.S.}$$

Method 2

(1) If equation in deno. is Quadratic (2). (Set last term)

Q. $$f(s) = \frac{1}{s^2+s+1}$$

$$L.S. = \frac{(m.s)^2}{4 \times F.S.}$$

$$= \frac{s^2}{4s^2}$$

$$= \frac{1}{4}$$

$$f(s) = \frac{1}{s^2+s+1}$$

$$= \frac{1}{s^2+s+\frac{1}{4}+\frac{3}{4}}$$

$$= \frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

Inverse Laplace

$$L^{-1} f(s) = e^{-1/2 t} \frac{1}{\sqrt{3}/2} \sin\left(\frac{\sqrt{3}}{2} t\right)$$

$$\boxed{f(t) = \frac{2}{\sqrt{3}} e^{-1/2 t} \sin\left(\frac{\sqrt{3}}{2} t\right)}$$

Note: If there is s in numerator than shifting is done in denominator as well as numerator

Eg:- $\dfrac{(s+\frac{1}{2})-\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$

Q $f(s) = \dfrac{1}{s^2+4s+5}$

$= \dfrac{1}{s^2+4s+4+1}$

$= \dfrac{1}{(s+2)^2+(1)^2}$

Inverse Laplace

$$\boxed{f(t) = e^{-2t}\sin t}$$

Q $f(s) = \dfrac{s}{s^2+s+1}$

$= \dfrac{s}{(s^2+s+\frac{1}{4})+\frac{3}{4}}$

$= \left(s+\frac{1}{2}\right)-\frac{1}{2}$

$= \dfrac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} - \dfrac{1}{2}\,\dfrac{1}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$

$$\boxed{f(t) = e^{-\frac{1}{2}t}\left[\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\left(\frac{1}{\sqrt{3}/2}\sin\frac{\sqrt{3}}{2}t\right)\right]}$$

Q $f(s) = \frac{s}{s^2+4s+5}$

$= \frac{s}{s^2+4s+4+1}$

$= \frac{(s+2)-2}{(s+2)^2+(1)^2}$

$= \frac{(s+2)}{(s+2)^2+(1)^2} - \frac{2}{(s+2)^2+(1)^2}$

Invers Laplace.

$$\boxed{f(t) = e^{-2t}[\cos t - 2\sin t]}$$

→ Inverse Laplace by differentiation:-

$f(s) = \log\left(\frac{s+b}{s+a}\right)$

$f(s) = \log(s+b) - \log \cancel{\log}(s+a)$

Differentiate w.r.t. s

$\frac{d}{ds} f(s) = \frac{1}{s+b} - \frac{1}{s+a}$

$-\frac{d}{ds} f(s) = \frac{1}{s+b} - \frac{1}{s+a}$

$\rightarrow \mathcal{L}\, t^n f(t) = (-1)^n \frac{d^n}{ds^n} f(s)$

$\boxed{n=1}$

$\rightarrow \mathcal{L}^{-1}\, t\, f(t) = (-1) \frac{d}{ds} f(s)$

$\rightarrow t\, f(t) = \mathcal{L}^{-1} \frac{d}{ds} f(s)$

$$= \left(\frac{1}{s+a} - \frac{1}{s+b}\right)$$

$$\mathcal{L}^{-1} \frac{-d}{ds} f(s) = \mathcal{L}^{-1}\left(\frac{1}{s+a} - \frac{1}{s+b}\right)$$

$$\boxed{\{t\, f(t)\} = e^{-at} - e^{-bt}}$$

$$\boxed{\therefore f(t) = \frac{e^{-at} - e^{-bt}}{t}}$$

d. $f(s) = \tan^{-1}\left(\frac{2}{s}\right)$

$$\frac{d}{ds} f(s) = \frac{1}{\left(1 + \frac{4}{s^2}\right)} \left(\frac{-2}{s^2}\right)$$

$$= \frac{1}{\frac{s^2+4}{s^2}} \left(\frac{-2}{s^2}\right)$$

$$= \frac{s^2}{s^2+4} \cdot \frac{-2}{s^2}$$

$$\boxed{f'(s) = \frac{-2}{s^2+4}}$$

$$\mathcal{L}^{-1} F(s) = \frac{-1}{t} \mathcal{L}^{-1}\left[F'(s)\right]$$

$$= \frac{-1}{t} \mathcal{L}^{-1}\left[\frac{-2}{s^2+4}\right]$$

$$= \frac{\alpha}{t}\left[\frac{1}{\alpha}\sin \alpha t\right]$$

$$\boxed{f(t) = \frac{\sin \alpha t}{t}}$$ //

Q. $f(s) = \ln\left(1+\frac{\omega^2}{s^2}\right)$

$$= \ln\left(\frac{s^2+\omega^2}{s^2}\right)$$

$$= \ln(s^2+\omega^2) - \log s^2$$

$$f(s) = \ln(s^2+\omega^2) - \alpha \log s.$$

$$f'(s) = \frac{d}{ds} f(s) = \frac{d}{ds}\left[\ln s^2+\omega^2 - \alpha \log s\right]$$

$$= \frac{1}{s^2+\omega^2}\alpha s - \frac{\alpha}{s}$$

$$f'(s) = \frac{\alpha s}{s^2+\omega^2} - \frac{\alpha}{s}$$

$$\cancel{= \frac{\alpha s(s) - \alpha(s^2+\omega^2)}{s(s^2+\omega^2)}}$$

$$\mathcal{L}^{-1} f(s) = \frac{-1}{t}\mathcal{L}^{-1} f'(s)$$

$$= \frac{-1}{t}\left[\mathcal{L}^{-1}\frac{\alpha s}{s^2+\omega^2} - \mathcal{L}^{-1}\frac{\alpha}{s}\right]$$

$$= \frac{-1}{t}\left[-\alpha \mathcal{L}^{-1}\frac{s}{s^2+\omega^2} - \alpha \mathcal{L}^{-1}\frac{1}{s}\right]$$

$$= \frac{-1}{t}$$

Convolution method :-

It is used when two functions are are in multiplication & inverse

eg :- $\left(\mathcal{L}^{-1}\frac{1}{s+1} \cdot \frac{1}{s-2}\right)$

Theorem :- If $\mathcal{L}^{-1} f(s) = f(t)$ &

$\mathcal{L}^{-1} g(s) = g(t)$ then

$$\boxed{\mathcal{L}^{-1} f(s)\, g(s) = \int_0^t f(u)\, g(t-u)\, du}$$

Find out inverse laplace of $\frac{1}{(s+1)(s+2)}$ using convolution.

$$\mathcal{L}^{-1}\frac{1}{(s+1)(s+2)} = \mathcal{L}^{-1}\underset{f(s)}{\frac{1}{(s+1)}} \cdot \underset{(g(s))}{\frac{1}{(s+2)}}$$

$$\mathcal{L}^{-1} f(s) = f(t) = e^{-t}$$
$$\mathcal{L}^{-1} g(s) = g(t) = e^{-2t}$$

$$\mathcal{L}^{-1} \frac{1}{s+1} \frac{1}{s+2} = \int_0^t e^{-u} e^{-2(t-u)} \, du$$

$$= \int_0^t e^{-u} e^{-2t} e^{2u} \, du$$

$$= e^{-2t} \int_0^t e^{u} \, du$$

$$= e^{-2t} \left[e^{u} \right]_0^t$$

$$\boxed{f(t) = e^{-2t} \left[e^{t} - 1 \right]}$$

Q. $f(s) = \dfrac{1}{(s+a)^2}$

$$\mathcal{L}^{-1} \frac{1}{(s+a)(s+a)} = \mathcal{L}^{-1} \underbrace{\frac{1}{(s+a)}}_{f(s)} + \mathcal{L}^{-1} \underbrace{\frac{1}{(s+a)}}_{g(s)}$$

$$\mathcal{L}^{-1} f(s) = f(t) = e^{-at}$$
$$\mathcal{L}^{-1} g(s) = g(t) = e^{-at}$$

$$\mathcal{L}^{-1} \frac{1}{(s+a)} \cdot \frac{1}{(s+a)} = \int_0^t e^{-au} \, e^{-a(t-u)} \, du$$

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$$= \int_0^t e^{-au} e^{-at} e^{au}\, du$$

$$= e^{-at} \int_0^t du$$

$$= e^{-at} \left[u \right]_0^t$$

$$\boxed{f(t) = t e^{-at}}$$

Q. $\dfrac{1}{(s^2+1)(s+1)}$.

$$\mathcal{L}^{-1} \frac{1}{(s^2+1)(s+1)} = \mathcal{L}^{-1} \frac{1}{(s^2+1)} \ \& \ \frac{1}{(s+1)}$$

$$\mathcal{L}^{-1} f(s) = f(t) = \sin t .$$

$$\mathcal{L}^{-1} g(s) = g(t) = e^{-t}$$

$$\mathcal{L}^{-1} \frac{1}{(s^2+1)(s+1)} = \int_0^t \sin u \cdot e^{-(t-u)}\, du$$

$$= e^{-t} \int_0^t e^{u} \sin u \, du.$$

$$= e^{-t} \left[\frac{e^{u}}{1+1} (\sin u - \cos u) \right]_0^t$$

$$= \frac{e^{-t}}{2} \left[e^{t} (\sin t - \cos t) + 1 \right].$$

Imp

* Application :-

→ Solution of Differential Equation by Laplace :-

$$L\frac{d^n y}{dt^n} = L y^n(t) = s^n\bar{y} - s^{n-1}y(0) - s^{n-2}y'(0) - s^{n-3}y''(0) - s^{n-4}y'''(0)$$

$$L\frac{d^4 y}{dt^4} = s^4\bar{y} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''$$

$$L\frac{d^3 y}{dt^3} = s^3\bar{y} - s^2 y(0) - s y'(0) - y''(0)$$

$$L\frac{d^2 y}{dt^2} = s^2\bar{y} - s y(0) - y'(0)$$

$$L\frac{dy}{dt} = s\bar{y} - y(0)$$

$$L y(t) = \bar{y}.$$

IVP = initial value problem

Q. Solve IVP $y'' + 4y = 0$ where $y(0) = 1$ $y(0)$

Soln:-

$$\frac{d^2y}{dt^2} + 4y = 0$$

Step 1:- Laplace transform is

$$\mathcal{L}\frac{d^2y}{dt^2} + 4\mathcal{L}y = 0.$$

Step 2:- $[s^2\bar{y} - s y_{(0)} - y'_{(0)}] + 4\bar{y} = 0.$

$$(s^2+4)\bar{y} - s - (-1) = 0.$$

$$(s^2+4)\bar{y} - s + 1 = 0$$

$$(s^2+4)\bar{y} = s - 1$$

Step 3:- $\bar{y} = \frac{s}{s^2+4} - \frac{1}{s^2+4}.$

Step 4:- Inverse Laplace

$$\mathcal{L}^{-1}\bar{y} = \mathcal{L}^{-1}\frac{s}{s^2+4} - \mathcal{L}^{-1}\frac{1}{s^2+4}$$

$$\boxed{y_{(t)} = \cos 2t - \frac{1}{2}\sin 2t}$$

Q. Solve IVP $y'' + y = t$ where $y_{(0)} = 1$ $y'_{(0)} = 1.$

Soln:-

$$\frac{d^2y}{dt^2} + y = t$$

Step 1:- Laplace transform.

$$\mathcal{L}\frac{d^2y}{dt^2} + \mathcal{L}y = \mathcal{L}t$$

Step 3:- $\frac{s}{s^2+1} + \cancel{\frac{1}{s^2+1}} + \left(\frac{1}{s^2} - \cancel{\frac{1}{s^2+1}}\right)$

Step 4:- Inverse Laplace

$$\bar{y} = L^{-1}\frac{s}{s^2+1} + L^{-1}\frac{1}{s^2}$$

$$\boxed{y(t) = \cos t + t^2}$$

Note: Trick

$$\frac{1}{(s+2)(s+4)} = \frac{1}{2}\left(\frac{1}{s+2} - \frac{1}{s+4}\right)$$

$$\frac{1}{(2s^2+1)(2s^2+10)} = \frac{1}{9}\left(\frac{1}{2s^2+1} - \frac{1}{2s^2+10}\right)$$

$$\frac{1}{(3s^3+1)(3s^3+5)} = \frac{1}{4}\left(\frac{1}{3s^3+1} - \frac{1}{3s^3+5}\right)$$

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Q. $y'' + y = \sin 2t$ where $y(0) = 2$; $y'(0) = 1$.
Solve IVP.

Solnⁿ:-

$$\frac{d^2y}{dt^2} + y = \sin 2t$$

Step 1st: Laplace transform.

$$L\frac{d^2y}{dt^2} + Ly = L\sin 2t$$

$$[s^2\bar{y} - sy(0) - y'(0)] + \bar{y} = \frac{2}{s^2+4}$$

Step 2nd: $(s^2+1)\bar{y} - 2s - 1 = \frac{2}{s^2+4}$

$$(s^2+1)\bar{y} = 2s + 1 + \frac{2}{s^2+4}$$

$$\bar{y} = \frac{2s}{s^2+1} + \frac{1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}$$

Step 3rd: $= 2\frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{2}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$

$$2\frac{s}{s^2+1} + \left(1 + \frac{2}{3}\right)\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4}$$

$$2\frac{s}{s^2+1} + \frac{5}{3}\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4}$$

Step 4: Inverse Laplace.

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{2}{3}\frac{1}{2}\sin 2t$$

$$\boxed{y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t}$$

Q. Solve $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$ where $x(0) = 0$, $x'(0) = 1$.

Soln :-

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0.$$

Step 1: Laplace transform

$$L\frac{d^2x}{dt^2} + 5L\frac{dx}{dt} + 6Lx = 0.$$

$$[s^2\bar{x} - s\,x(0) - x'(0)] + 5[s\bar{x} - x(0)] + 6\bar{x} = 0.$$

Step 2:

$$(s^2 + 5s + 6)\bar{x} - 1 = 0.$$

$$(s^2 + 5s + 6)\bar{x} = 1$$

$$\bar{x} = \frac{1}{(s^2 + 5s + 6)}$$

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3) $$\bar{x} = \frac{1}{(s+2)(s+3)}$$

$$= \left(\frac{1}{s+2} - \frac{1}{s+3} \right)$$

p4: Inverse Laplace.

$$\boxed{x_{(t)} = e^{-2t} - e^{-3t}}$$

Solve $\frac{d^2y}{dt^2} + a^2y = K \sin at$.

Q:

Step 1: Laplace transform

$$L\frac{d^2y}{dt^2} + a^2 L\bar{y} = K\, L \sin at$$

$$[s^2\bar{y} - sy_{(0)} - y'_{(0)}] + a^2\bar{y} = \frac{Ka}{s^2+a^2}$$

Let $y_{(0)} = A$

$y'_{(0)} = B$

2: $$(s^2+a^2)\bar{y} - As - B = Ka \frac{1}{s^2+a^2}$$

$$(s^2+a^2)\bar{y} = As + B + Ka \frac{1}{s^2+a^2}$$

3: $$\bar{y} = A \frac{s}{\quad} + B \frac{1}{s^2+a^2} + Ka \frac{1}{(s^2+a^2)^2}$$

Step 4: Inverse Laplace

$$\boxed{y_{(t)} = A \sin at + B \frac{1}{a} \sin at + \frac{1}{2} a^{s}.}$$