Exponential Distribution:

why do we have to Invent Expo. distri. ?
To predict the amount of waiting time until next event (i.e success, failure, arrival etc)

for eg

The amount of time until the customer finishes browsing and actually purchases something (success)

The amount of time until the hardware fails (failure)

The amount of time you need to wait until the bus arrives (arrival)

why is $\lambda e^{-\lambda t}$ the pdf of the time until the next ~~st~~ event happens ?

Note: If the no. of events per unit time follows a poisson distribution then the amount of time betⁿ events follows the exponential distribution

Exponential Distribution:

A conti. r.v. X is said to have an exponential distribution with parameter λ (rate parameter) if X has the PDF:

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

we can write $X \sim Exp(\lambda)$

Mean and Variance:

$$E(x^n) = \int_0^\infty \lambda e^{-\lambda x} \cdot x^n \, dx$$

$$= \lambda \int_0^\infty x^n \cdot e^{-\lambda x} \, dx$$

$$= \lambda \int_0^\infty x^{(n+1)-1} e^{-\lambda x} \, dx$$

By defn of gamma fn: $\Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx$

$$\frac{\Gamma n}{\lambda^n} = \int_0^\infty e^{-\lambda x} x^{n-1} \, dx$$

$$= \lambda \cdot \frac{\Gamma n+1}{\lambda^{n+1}}$$

$$= \frac{n!}{\lambda^n} \quad \left(\because \Gamma n+1 = n\Gamma n = n! \right)$$

$$\therefore E(x) = \frac{1!}{\lambda^1} = \frac{1}{\lambda} \qquad E(x^2) = \frac{2!}{\lambda^2}$$

$$Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Moment generating fun:

$$M_X(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx$$

$$= \int_{0}^{\infty} e^{tx} \cdot \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_{0}^{\infty} e^{-x(\lambda - t)}\, dx = \lambda \left[\frac{e^{-x(\lambda-t)}}{-(\lambda - t)} \right]_0^{\infty}$$

$$= \lambda \left[\frac{1}{\lambda - t} \right]$$

Examples :

1 The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{3}$. what is the prob. that the repair time exceeds 3 hours ?

:→ X — Time required to repair a machine

$$\lambda = \frac{1}{3}$$

$$f(x) = \begin{cases} \frac{1}{3} e^{-\frac{x}{3}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P(x > 3) = \int_{x=3}^{\infty} f(x)\, dx = \int_{3}^{\infty} \frac{1}{3} e^{-x/3}\, dx$$

$$= \left[-e^{-x/3} \right]_3^{\infty}$$

$$= e^{-1}$$

$$= \frac{1}{e}$$

$$= 0.3679$$

Ex: 2 The life length (in months) x of an electric component follows an exponential distribution with $\lambda = \frac{1}{2}$. What is the prob. that the component survives at least 10 months given that already it had survived for more than 9 months?

$\Rightarrow$

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{2} e^{-\frac{x}{2}} \quad \text{for } x > 0$$

$$\therefore P(x \geq 10 \mid x \geq 9) = P(x \geq 1)$$

$$= \int_1^\infty f(x)\,dx$$

Memoryless prop:

$$P(x > s+t \mid x > s) = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t}$$

$$= P(x > t)$$

$s+t = 10$
$s = 9$
$\therefore t = 1$

$\therefore$ we have

$$P(x \geq 10 \mid x \geq 9) = \int_1^\infty \frac{1}{2} e^{-\frac{x}{2}}\,dx$$

$$= e^{-\frac{1}{2}} = 0.6065$$

Ex: The length of time a person speaks over phone it follows exponential distribution with mean 6. What is the prob. that the person will talk for
(i) more than 8 min?
(ii) betn 4 & 8 min

$\Rightarrow$

$$f(x) = \frac{1}{6} e^{-\frac{x}{6}} \; ; \; x > 0$$

So, the cumulative distri. fun of X is

$$F(x) = P(X \leq x)$$

$$= 1 - e^{-x/6} \quad \text{for } x > 0$$

(i) The prob. that the person will talk for more than 8 min. is

$$P(X>8) = 1-P(X\le 8) = e^{-8/6} = e^{-4/3} = 0.2636$$

(ii) The prob. that the person will talk betⁿ 4 & 8 min. is

$$P(4<X<8) = F(8)-F(4) = e^{-4/6} - e^{-8/6} = 0.2498$$

Ex: If the no. of kilometers that a car can run before its battery wears out is exponentially distributed with an avg. value of 10000 km and if the owner desires to take a 5000 km trip, what is the prob. that he will be able to complete his trip without having to replace the car battery. Assume that the car has been used for sometime.

$\rightarrow$ X - no. of kilometers that a car run before its battery wears out.

$$\text{Mean} = \frac{1}{\lambda} = 10000 \Rightarrow \lambda = \frac{1}{10000}$$

$$\therefore f(x) = \frac{1}{10000} e^{-\frac{x}{10000}}, \quad x \ge 0$$

$$P(X>5000) = \int_{5000}^{\infty} f(x)\,dx$$

$$= \frac{1}{10000}\int_{5000}^{\infty} e^{-\frac{x}{10000}}\,dx$$

$$= e^{-0.5}$$

$$= 0.6065$$

the following.

(a) $P(-4 < X < 20)$ (b) $P(|X-8| \geq 6)$

6.4 The Exponential Distribution

Suppose a random variable follow a Poisson distribution as follows.

→ ***For example,***

(1) The number of telephone calls that arrive each day over a period of a year and note that the arrivals follow a poisson distribution with an ***average*** of 4 per day.

(2) The number of hits to your website and note that hits follow a Poisson distribution at a ***rate*** of 5 per day.

(3) The number of customers arriving at a service point and note that arrivals follow a Poisson distribution with an ***average*** of 4 per day.

In the above examples, if we consider T as the time between the events, then we have following situations.

In the first example, T indicates waiting time between calls.

In the second example, T indicates time between hits.

In the third example, T indicates time between customers.

Here, T be the time between the events happening is a ***random variable*** which follow an ***exponential distribution***. Thus, exponential distribution is typically used to model time intervals between random events. In other words, exponential distribution describes waiting time between Poisson occurrence.

It should be noted here that the number of events is a discrete variable, whereas the time between events is a continuous variable.

The exponential distribution is having the probability density function (p.d.f.)

$$\begin{aligned} f(t) &= \lambda e^{-\lambda t} \quad ; t \geq 0 \\ &= 0 \quad ; \text{otherwise} \end{aligned} \qquad \text{...(6.14)}$$

for $\lambda > 0$.

Sometimes the p.d.f. of exponential distribution can also be specified as

$$\begin{aligned} f(t) &= \frac{1}{\lambda} e^{-\frac{t}{\lambda}} \quad ; t \geq 0, \lambda > 0 \\ &= 0 \quad ; \text{otherwise.} \end{aligned} \qquad \text{...(6.15)}$$

In (6.14) λ is known as the ***rate parameter***, whereas in (6.15) λ is known as the ***mean parameter***.

Note 1 When times between random events follow the exponential distribution with rate λ, then the total number of events in a time period of length t follows the Poisson distribution with parameter λt.

Note 2 The exponential distribution is a ***memoryless*** continuous distribution.

It is clear from the definition (6.14) that

(1) $f(t) \geq 0$ for all $t \geq 0$.

(2)

$$\int_{-\infty}^{\infty} f(t)\,dt = \lambda \int_{0}^{\infty} e^{-\lambda t}\,dt$$

$$= \lambda \left(\frac{e^{-\lambda t}}{-\lambda} \right)_0^{\infty}$$

$$= \left(-e^{-\lambda t} \right)_0^{\infty}$$

$$= 1.$$

Thus, $f(t)$ satisfies both the requirements for a function to be a probability density function.

➢ **Distribution Function** In this case the distribution function $F(t)$ is defined as

$$F(t) = P(T \leq t) = \int_0^t f(t)\,dt \qquad \text{(Using (5.19))}$$

$$= \int_0^t \lambda e^{-\lambda t}\,dt \qquad \text{(Using (6.14))}$$

$$= \lambda \left(\frac{e^{-\lambda t}}{-\lambda} \right)_0^{t}$$

$$= -\left(e^{-\lambda t} \right)_0^{t}$$

$$= 1 - e^{-\lambda t}.$$

➢ **Mean and Variance** For any $r \geq 0$,

$$E(T^r) = \int_0^{\infty} t^r f(t)\,dt \qquad \text{(Using (5.21))}$$

ote The exponential distribution occurs most often in applications of Reliability eory and Queuing theory.

xample 6.14 (Lifetime of a Battery)

he lifetime T of an alkaline battery is exponentially distributed with $\lambda = 0.05$ per our.

a) What are the mean and standard deviation of the batterys lifetime ?

b) What are the probabilities for the battery to last between 10 and 15 hours and to ast more than 20 hours ?

Solution

(*a*) As both the mean and standard deviation of the exponential distribution are equal to $1/\lambda$. Therefore,

$$\text{Mean } \mu = \text{S.D. } \sigma = \frac{1}{\lambda} = \frac{1}{0.05} = 20 \text{ hours.}$$

Answer (a)

(*b*)

$$P(10 < T < 15) = \int_{10}^{15} \lambda e^{-\lambda t}\, dt \qquad \text{(Using (6.14)}$$

$$= \int_{10}^{15} 0.05 e^{-0.05t}\, dt$$

$$= 0.05 \left(\frac{e^{-0.05t}}{-0.05} \right)_{10}^{15}$$

$$= -\left[e^{-0.05(15)} - e^{-0.05(10)} \right]$$

$$= 0.1341.$$

$$P(T > 20) = \int_{20}^{\infty} \lambda e^{-\lambda t}\, dt \qquad \text{(Using (6.14))}$$

$$= \int_{20}^{\infty} 0.05 e^{-0.05t}\, dt$$

$$= 0.05\left(\frac{e^{-0.05t}}{-0.05}\right)_{20}^{\infty}$$

$$= -[0 - e^{-0.05(20)}]$$

$$= e^{-0.05(20)}$$

$$= 0.3679.$$

Example 6.15

Accidents occur with a Poisson distribution at an average of 2 per week; that is, $\lambda = 2$.

(*a*) Calculate the probability of more than 3 accidents in any one week.

(*b*) What is the probability that at least two weeks will elapse between accidents?

Solution

(*a*)

$$P(X > 3) = 1 - P(X \leq 3)$$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)]$$

$$= 1 - \left[\frac{e^{-2}(2)^0}{0!} + \frac{e^{-2}(2)^1}{1!} + \frac{e^{-2}(2)^2}{2!} + \frac{e^{-2}(2)^3}{3!}\right]$$

(Using (6.5))

$$= 1 - \left(e^{-2} + 2e^{-2} + 2e^{-2} + \frac{4}{3}e^{-2}\right)$$

$$= 1 - \left(1 + 2 + 2 + \frac{4}{3}\right)e^{-2}$$

$$= 1 - \frac{19}{3}e^{-2}$$

$$\approx 0.14288.$$

***Answer** (a)*

(*b*)

$$P(X > 2) = \int_{2}^{\infty} 2e^{-2t}\,dt$$

(Using (6.14))

$$= 2\left(\frac{e^{-2t}}{-2}\right)_2^{\infty}$$

$$= -[0 - e^{-2(2)}]$$

$$= e^{-4}$$

$$\approx 0.01832.$$

Answer (b)

Example 6.16

The time between breakdowns of a particular machine follows an exponential distribution with a mean of 17 days. Calculate the probability that a machine breaks down in a 15 day period.

Solution

The p.d.f. $f(t)$ is given by

$$f(t) = \frac{1}{17} e^{-\frac{t}{17}}; t \geq 0.$$

The required probability is given by

$$P(0 \leq T \leq 15) = \int_0^{15} f(t)dt$$

$$= \int_0^{15} \frac{1}{17} e^{-\frac{t}{17}} dt$$

$$= \frac{1}{17}\left[\frac{e^{-\frac{t}{17}}}{\left(-\frac{1}{17}\right)}\right]_0^{15}$$

$$= -e^{-\frac{t}{17}}\Big|_0^{15}$$

$$= -e^{-\frac{15}{17}} + 1$$

$$= 0.5862.$$

Thus, there is a 58.62% chance that the machine will breakdown in a 15 day period.

Answer

Example 6.17

A system contains a certain type of component whose lifetime T is exponentially distributed with mean of 5 years. If 8 such components are installed in different systems, what is the probability that at least 3 are still working at the end of 7 years ?

Solution

The p.d.f. $f(t)$ is given by

$$f(t)=\frac{1}{5}e^{-\frac{t}{5}};\, t\geq 0.$$

Therefore,

$$P(T>7)=\frac{1}{5}\int_7^{\infty} e^{-\frac{t}{t}}dt$$

$$=\frac{1}{5}\left[\frac{e^{-t/5}}{(-1/5)}\right]_7^{\infty}$$

$$=-e^{-\frac{t}{5}}\Big|_7^{\infty}$$

$$=e^{-\frac{7}{5}}$$

$$=0.1827.$$

Let n be the number of components out of 8 working after 7 years of instalment, then

$$P(n\geq 3)=\sum_{n=3}^{8}\, {}^{8}C_n(0.1827)^n(1-0.1827)^{8-n}$$

(Using (6.2))

$$=\sum_{n=3}^{8}\, {}^{8}C_n(0.1827)^n(0.8173)^{8-n}$$

$$=1-\sum_{n=0}^{2} {}^{8}C_n(0.1827)^n(0.8173)^{8-n}$$

$$=1-[\ {}^{8}C_0(0.1873)^8+{}^{8}C_1(0.1827)(0.8173)^7$$

$$+{}^{8}C_2(0.1827)^2(0.8173)^6\Big]$$

$$=1-(0.1991+0.3560+0.2786)$$

$$=0.1663.$$

Answer

Example 6.18

The arrival rate of cars at a gas station is $\lambda = 40$ customers per hour.

(*a*) What is the probability of having no arrivals in a 5 minute interval ?

(*b*) What are the mean and variance of the number n of arrivals in 5 minutes ?

(*c*) What is the probability for having 3 arrivals in a 5 minute interval ?

Solution

(*a*)

$$P\left(T>\frac{5}{60}\right)=\int_{5/60}^{\infty} 40e^{-40t}\,dt \qquad \text{(Using (6.14))}$$

$$=40\left(\frac{e^{-40t}}{-40}\right)_{5/60}^{\infty}$$

$$=-\left(e^{-40t}\right)_{5/60}^{\infty}$$

$$=e^{-40\left(\frac{5}{60}\right)}$$

$$\approx 0.03567.$$

Answer (a)

(*b*) Here, the variable n has a Poisson distribution with parameter

$$\mu=\lambda t=40\left(\frac{5}{60}\right)=3.333. \qquad \text{(Using Note 1)}$$

Hence,

$$\text{Mean } E(n)=3.333 \text{ and Variance } V(n)=3.333.$$

Answer (b)

since mean and variance of Poisson distribution are same.

(c)

$$P(N=3)=\frac{e^{-3.333}(3.333)^3}{3!} \quad \text{(Using (6.5))}$$

$$\approx 0.2202.$$

***Answer* (c)**

Exercises 6.3

01. If jobs arrive every 15 seconds on average; that is, $\lambda=4$ per minute. What is the probability of waiting less than or equal to 30 seconds ?
02. If on the average three trucks arrive per hour to be unloaded at a warehouse, using exponential distribution find the probabilities that the time between the arrival of successive trucks will be
    (*a*) less than 5 minutes, (*b*) at least 45 minutes.
03. The length of time for one person to be served at a cafeteria is a random variable T having an exponential distribution with a mean of 4 minutes. Find the probability that a person is served in less than 3 minutes on at least 4 of the next 6 days.
04. The mean time taken by an engineer to repair an electrical fault in a system is 2.7 hours. Calculate the probability that the engineer will repair a fault in less than the mean time.

5.5 EXPONENTIAL DISTRIBUTION

A continuous random variable X is said to follow exponential distribution if its probability function is given by

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$
$$= 0 \quad , \quad x \leq 0$$

where $\lambda > 0$ is called the rate of the distribution.

5.5.1 Memoryless Property of the Exponential Distribution

The exponential distribution has the memoryless (forgetfulness) property. This property indicates that the distribution is independent of its part, that means future happening of an event has no relation to whether or not this even has happened in the past. This property is as follows:

If X is exponentially distributed, and s, t are two positive real numbers then

$$P[(X > s+t)/(X > s)] = P(X > t)$$

Proof: $P[(X > s+t)/(X > s)] = \dfrac{P\left[(X > s+t) \cap (X > s)\right]}{P(X > s)}$

[using conditional probability]

$$= \frac{P(X > s+t)}{P(X > s)}$$

$$= \frac{\int_{s+t}^{\infty} \lambda e^{-\lambda x} dx}{\int_{s}^{\infty} \lambda e^{-\lambda x} dx}$$

$$= \frac{\lambda \left| \frac{e^{-\lambda x}}{-\lambda} \right|_{s+t}^{\infty}}{\lambda \left| \frac{e^{-\lambda x}}{-\lambda} \right|_{s}^{\infty}}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t} \qquad \text{...(5.5)}$$

$$P(X > t) = \int_t^{\infty} \lambda e^{-\lambda x} \mathrm{d}x$$

$$= \lambda \left| \frac{e^{-\lambda x}}{-\lambda} \right|_t^{\infty}$$

$$= e^{-\lambda t} \qquad \text{...(5.6)}$$

From Eq. (5.5) and Eq. (5.6),

$$P[(X > s+t)/(X > s)] = P(X > t), \qquad \text{for } s, t > 0$$

5.5.2 Constants of the Exponential Distribution

1. Mean of the Exponential Distribution

$$E(X) = \int_{-\infty}^{\infty} x\, f(x) \mathrm{d}x$$

$$= \int_0^{\infty} x\, \lambda e^{-\lambda x} \mathrm{d}x$$

$$= \lambda \left| x \cdot \frac{e^{-\lambda x}}{-\lambda} - 1 \cdot \frac{e^{-\lambda x}}{\lambda^2} \right|_0^{\infty}$$

$$= \lambda \cdot \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda}$$

2. Variance of the Exponential Distribution

$$\text{Var}(X) = E(X^2) - [E(X)]^2 \qquad \text{...(5.7)}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \mathrm{d}x$$

$$= \int_0^{\infty} x^2 \lambda e^{-\lambda x} \mathrm{d}x$$

$$= \lambda \left| x^2 \frac{e^{-\lambda x}}{-\lambda} - 2x \frac{e^{-\lambda x}}{\lambda^2} + 2 \frac{e^{-\lambda x}}{-\lambda^3} \right|_0^\infty$$

$$= \lambda \left(\frac{2}{\lambda^3} \right)$$

$$= \frac{2}{\lambda^2}$$

Substituting in Eq (5.7),

$$\text{Var}(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \qquad \left[\because \mu = \frac{1}{\lambda} \right]$$

3. Standard Deviation of the Exponential Distribution

$$\text{SD} = \sqrt{\text{Var}(X)} = \sqrt{\frac{1}{\lambda^2}} = \frac{1}{\lambda}$$

4. Mode of the Exponential Distribution

Mode is the value of x for which $f(x)$ is maximum.

$$\begin{aligned} f(x) &= \lambda e^{-\lambda x}, \quad x > 0 \\ &= 0 \quad\quad\;\;, \quad x \le 0 \end{aligned}$$

$f(x)$ will be maximum when $e^{-\lambda x}$ is maximum.

Maximum value of $e^{-\lambda x} = 1$, which is at $x = 0$.

Hence, $x = 0$ is the mode of the exponential distribution

5. Median of the Exponential Distribution

If M is the median of the exponential distribution,

$$\int_{-\infty}^{M} f(x)\, dx = \frac{1}{2}$$

$$\int_{0}^{M} \lambda e^{-\lambda x}\, dx = \frac{1}{2}$$

$$\lambda \left| \frac{e^{-\lambda x}}{-\lambda} \right|_0^M = \frac{1}{2}$$

$$-(e^{-\lambda M} - 1) = \frac{1}{2}$$

$$-e^{-\lambda M}=\frac{1}{2}-1=-\frac{1}{2}$$

$$e^{-\lambda M}=\frac{1}{2}$$

$$-\lambda M\log e=\log\frac{1}{2}=-\log 2$$

$$\lambda M=\log 2$$

$$M=\frac{1}{\lambda}\log 2$$

Example 1

Let X be a random variable with pdf

$$f(x)=\begin{cases}\frac{1}{5}e^{-\frac{x}{5}} & x>0\\ 0 & \text{otherwise}\end{cases}$$

Find (i) $P(X>5)$ (ii) $P(3\le X\le 6)$ (iii) mean (iv) variance.

Solution

$$\lambda=\frac{1}{5}$$

(i) $P(X>5)=\int_5^\infty f(x)\,dx$

$$=\int_5^\infty \frac{1}{5}e^{-\frac{x}{5}}\,dx$$

$$=\frac{1}{5}\left|\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right|_5^\infty$$

$$=-\left|e^{-\frac{x}{5}}\right|_5^\infty$$

$$=-(e^{-\infty}-e^{-1})$$

$$=e^{-1}$$

(ii) $P(3 \le X \le 6) = \int_3^6 f(x)\,dx$

$$= \int_3^6 \frac{1}{5} e^{-\frac{x}{5}}\,dx$$

$$= \frac{1}{5}\left|\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right|_3^6$$

$$= -\left|e^{-\frac{x}{5}}\right|_3^6$$

$$= -\left(e^{-\frac{6}{5}} - e^{-\frac{3}{5}}\right)$$

$$= e^{-\frac{3}{5}} - e^{-\frac{6}{5}}$$

$$= 0.2476$$

(iii) Mean $\mu = \frac{1}{\lambda} = \frac{1}{\left(\frac{1}{5}\right)} = 5$

(iv) Variance = Var$(X) = \frac{1}{\lambda^2} = \frac{1}{\left(\frac{1}{5}\right)^2} = 25$

Example 2

A random variable has pdf $f(x) = ce^{-2x}$ for $x > 0$. Find (i) $P(X > 2)$ (ii) $P\left(X < \frac{1}{c}\right)$.

Solution

Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^{\infty} ce^{-2x}\,dx = 1$$

$$\left|\frac{ce^{-2x}}{-2}\right|_0^{\infty} = 1$$

$$-\frac{c}{2}\left|e^{-2x}\right|_0^\infty = 1$$

$$-\frac{c}{2}(e^{-\infty} - e^0) = 1$$

$$\frac{c}{2} = 1$$

$$c = 2$$

$$\therefore\ f(x) = 2e^{-2x}, \qquad x > 0$$

(i) $P(X > 2) = \int_2^\infty f(x)\mathrm{d}x$

$$= \int_2^\infty 2e^{-2x}\mathrm{d}x$$

$$= 2\left|\frac{e^{-2x}}{-2}\right|_2^\infty$$

$$= -\left|e^{-2x}\right|_2^\infty$$

$$= -(e^{-\infty} - e^{-4})$$

$$= e^{-4}$$

$$= 0.0183$$

(ii) $P\left(X < \frac{1}{c}\right) = P\left(X < \frac{1}{2}\right)$

$$= \int_0^{\frac{1}{2}} f(x)\mathrm{d}x$$

$$= \int_0^{\frac{1}{2}} 2e^{-2x}\mathrm{d}x$$

$$= 2\left|\frac{e^{-2x}}{-2}\right|_0^{\frac{1}{2}}$$

$$= -\left|e^{-2x}\right|_0^{\frac{1}{2}}$$

$$= -(e^{-1} - e^0)$$

$$= -e^{-1} + 1$$

$$= 0.6321$$

Example 3

If X is random variable which follows an exponential distribution with parameter λ with $P(X \le 1) = P(X > 1)$, find $\text{Var}(X)$.

Solution

Since X is random variable which follows an exponential distribution,

$$f(x) = \lambda e^{-\lambda x}, \quad x \ge 0$$

$$P(X \le 1) = P(X > 1)$$

$$1 - P(X > 1) = P(X > 1)$$

$$2P(X > 1) = 1$$

$$P(X > 1) = \frac{1}{2}$$

$$\int_1^\infty f(x)\,dx = \frac{1}{2}$$

$$\int_1^\infty \lambda e^{-\lambda x}\,dx = \frac{1}{2}$$

$$\lambda \left| \frac{e^{-\lambda x}}{-\lambda} \right|_1^\infty = \frac{1}{2}$$

$$-\left| e^{-\lambda x} \right|_1^\infty = \frac{1}{2}$$

$$-(e^{-\infty} - e^{-\lambda}) = \frac{1}{2}$$

$$e^{-\lambda} = \frac{1}{2}$$

$$\frac{1}{e^{\lambda}} = \frac{1}{2}$$

$$e^{\lambda} = 2$$

$$\lambda = \log_e 2$$

$$\text{Var}(X) = \frac{1}{\lambda^2} = \frac{1}{(\log_e 2)^2}$$

Example 4

If X is a exponentially distributed random variable with parameter λ, find the value of k such that $\dfrac{P(X > k)}{P(X \le k)} = a.$

Solution

$$\frac{P(X>k)}{P(X\le k)}=a$$

$$\frac{P(X>k)}{1-P(X>k)}=a$$

$$P(X>k)=a[1-P(X>k)]$$

$$P(X>k)(1+a)=a$$

$$P(X>k)=\frac{a}{1+a}$$

$$\int_k^{\infty} f(x)\,dx=\frac{a}{1+a}$$

$$\int_k^{\infty} \lambda e^{-\lambda x}\,dx=\frac{a}{1+a}$$

$$\lambda\left|\frac{e^{-\lambda x}}{-\lambda}\right|_k^{\infty}=\frac{a}{1+a}$$

$$-\left|e^{-\lambda x}\right|_k^{\infty}=\frac{a}{1+a}$$

$$-(e^{-\infty}-e^{-\lambda k})=\frac{a}{1+a}$$

$$e^{-\lambda k}=\frac{a}{1+a}$$

$$\frac{1}{e^{\lambda k}}=\frac{a}{1+a}$$

$$e^{\lambda k}=\frac{1+a}{a}$$

$$\lambda k=\log\left(\frac{1+a}{a}\right)$$

$$k=\frac{1}{\lambda}\log\left(\frac{1+a}{a}\right)$$

Example 5

If the density function of a continuous random variable X is $f(x) = ce^{-b(x-a)}$, $a \le x$ where a, b, c are constants. Show that $b = c = \frac{1}{\sigma}$ and $a = \mu - \sigma$, where $\mu = E(X)$ and $\sigma^2 = \mathrm{Var}(X)$.

Solution

Since $f(x)$ is a density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{a}^{\infty} ce^{-b(x-a)}\,dx = 1$$

$$c\left|\frac{e^{-b(x-a)}}{-b}\right|_a^{\infty} = 1$$

$$-\frac{c}{b}\left|e^{-b(x-a)}\right|_a^{\infty} = 1$$

$$-\frac{c}{b}(e^{-\infty} - e^{0}) = 1$$

$$\frac{c}{b} = 1$$

$$b = c \qquad \text{...(1)}$$

$$\mu = E(X) = \int_a^{\infty} bxe^{-b(x-a)}\,dx$$

$$= be^{ab}\left|x\left(\frac{e^{-bx}}{-b}\right) - \frac{e^{-bx}}{b^2}\right|_a^{\infty}$$

$$= be^{ab}\left(\frac{a}{b}e^{-ab} + \frac{1}{b^2}e^{-ab}\right)$$

$$= a + \frac{1}{b} \qquad \text{...(2)}$$

$$E(X^2) = \int_a^{\infty} bx^2 e^{-b(x-a)}\,dx$$

$$= be^{ab}\left|x^2\left(\frac{e^{-bx}}{-b}\right) - 2x\left(\frac{e^{-bx}}{-b^2}\right) + 2\left(\frac{e^{-bx}}{-b^3}\right)\right|_a^{\infty}$$

$$= b\left(\frac{a^2}{b} + \frac{2a}{b^2} + \frac{2}{b^3}\right)$$

$$= \frac{1}{b^2}(a^2b^2 + 2ab + 2)$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$\sigma^2 = \frac{1}{b^2}(a^2b^2 + 2ab + 2) - \left(a^2 + \frac{2a}{b} + \frac{1}{b^2}\right)$$

$$= \frac{1}{b^2}$$

$$\sigma = \frac{1}{b} \qquad \text{...(3)}$$

From Eq. (1) and (3),

$$b = c = \frac{1}{\sigma}$$

Subtracting Eq. (3) from Eq. (2),

$$\mu - \sigma = a$$

$$\therefore \quad a = \mu - \sigma$$

Example 6

The mileage which car owners get with a certain kind of radial tire is a random variable having an exponential distribution with mean 4000 km. Find the probabilities that one of these tires will last (i) at least 2000 km (ii) at most 3000 km.

Solution

Let X be the random variable which denotes the mileage obtained with the tire.

$$\text{Mean } \mu = \frac{1}{\lambda} = 4000 \text{ km}$$

$$f(x) = \lambda e^{-\lambda x}, \qquad x > 0$$

$$= \frac{1}{4000} e^{-\frac{1}{4000}x}, \qquad x > 0$$

(i) $$P(X \geq 2000) = \int_{2000}^{\infty} f(x)\,dx$$

$$= \int_{2000}^{\infty} \frac{1}{4000} e^{-\frac{1}{4000}x}\,dx$$

$$= \frac{1}{4000} \left| \frac{e^{-\frac{1}{4000}x}}{-\frac{1}{4000}} \right|_{2000}^{\infty}$$

$$= -\left| e^{-\frac{1}{4000}x} \right|_{2000}^{\infty}$$

$$= -(e^{-\infty} - e^{-0.5})$$

$$= e^{-0.5}$$

$$= 0.6065$$

(ii) $P(X \leq 3000) = \int_0^{3000} f(x)\,dx$

$$= \int_0^{3000} \frac{1}{4000} e^{-\frac{1}{4000}x}\,dx$$

$$= \frac{1}{4000} \left| \frac{e^{-\frac{1}{4000}x}}{-\frac{1}{4000}} \right|_0^{3000}$$

$$= -\left| e^{-\frac{1}{4000}x} \right|_0^{3000}$$

$$= -(e^{-0.75} - e^0)$$

$$= -e^{-0.75} + 1$$

$$= 0.5270$$

Example 7

If the number of kilometers that a car can run before its battery wears out is exponentially distributed with an average value of 10000 km and if the owner desires to take a 5000 km trip, what is the probability that he will be able to complete his trip without having to replace the car battery. Assume that the car has been used for same time.

Solution

Let X be the random variable which denotes the number of kilometers that a car can run before its battery wears out.

$$\text{Mean } \mu = \frac{1}{\lambda} = 10000$$

$$f(x) = \lambda e^{-\lambda x}, \qquad x > 0$$

$$= \frac{1}{1000} e^{-\frac{1}{10000}x}, \ x > 0$$

$$P(X > 5000) = \int_{5000}^{\infty} f(x)\,dx$$

$$= \int_{5000}^{\infty} \frac{1}{10000} e^{-\frac{1}{10000}x}\,dx$$

$$= \frac{1}{10000}\left|\frac{e^{-\frac{1}{10000}x}}{-\frac{1}{10000}}\right|_{5000}^{\infty}$$

$$= -\left|e^{-\frac{1}{10000}x}\right|_{5000}^{\infty}$$

$$= -(e^{-\infty} - e^{-0.5})$$

$$= e^{-0.5}$$

$$= 0.6065$$

Example 8

The average time it takes to serve a customer at a petrol pump is 6 minutes. The service time follows exponential distribution. Calculate the probability that

(i) *A customer will take less than 2 minutes to complete the service.*
(ii) *A customer will take between 4 and 5 minutes to get the service.*
(iii) *A customer will take more than 10 minutes for his service.*

Solution

Let X be the random variable which denotes the service time.

$$\text{Mean } \mu = \frac{1}{\lambda} = 6$$

$$f(x) = \lambda e^{-\lambda x},\ x > 0$$

$$= \frac{1}{6} e^{-\frac{1}{6}x},\ x > 0$$

(i) $P(X < 2) = \int_{0}^{2} f(x)\,dx$

$$= \int_{0}^{2} \frac{1}{6} e^{-\frac{1}{6}x}\,dx$$

$$= \frac{1}{6}\left|\frac{e^{-\frac{1}{6}x}}{-\frac{1}{6}}\right|_0^2$$

$$= -\left|e^{-\frac{1}{6}x}\right|_0^2$$

$$= -(e^{-\frac{1}{3}} - e^0)$$

$$= -e^{-\frac{1}{3}} + 1$$

$$= 0.2835$$

(ii) $P(4 < X < 5) = \int_4^5 f(x)\,dx$

$$= \int_4^5 \frac{1}{6} e^{-\frac{1}{6}x}\,dx$$

$$= \frac{1}{6}\left|\frac{e^{-\frac{1}{6}x}}{-\frac{1}{6}}\right|_4^5$$

$$= -\left|e^{-\frac{1}{6}x}\right|_4^5$$

$$= -\left(e^{-\frac{5}{6}} - e^{-\frac{2}{3}}\right)$$

$$= 0.0788$$

(iii) $P(X > 10) = \int_{10}^{\infty} f(x)\,dx$

$$= \int_{10}^{\infty} \frac{1}{6} e^{-\frac{1}{6}x}\,dx$$

$$= \frac{1}{6}\left|\frac{e^{-\frac{1}{6}x}}{-\frac{1}{6}}\right|_{10}^{\infty}$$

$$= -\left|e^{-\frac{1}{6}x}\right|_{10}^{\infty}$$

$$= -\left(e^{-\infty} - e^{-\frac{10}{6}}\right)$$

$$= e^{-\frac{10}{6}}$$

$$= 0.1889$$

Example 9

The length of time X to complete a job is exponentially distributed with $E(X) = \mu = \frac{1}{\lambda} = 10$ *hours. (i) Compute the probability of job completion between two consecutive jobs exceeding 20 hours. (ii) The cost of job completion is given by $C = 4 + 2X + 2X^2$. Find the expected value of C.*

Solution

Let X be a random variable which denotes the length of time to complete a job.

$$E(X) = \mu = \frac{1}{\lambda} = 10$$

$$f(x) = \lambda e^{-\lambda x}$$

$$= \frac{1}{10} e^{-\frac{1}{10}x}$$

(i) $P(X > 20) = \int_{20}^{\infty} f(x)\,dx$

$$= \int_{20}^{\infty} \frac{1}{10} e^{-\frac{1}{10}x}\,dx$$

$$= \frac{1}{10}\left|\frac{e^{-\frac{1}{10}x}}{-\frac{1}{10}}\right|_{20}^{\infty}$$

$$= -\left|e^{-\frac{1}{10}x}\right|_{20}^{\infty}$$

$$= -(e^{-\infty} - e^{-2})$$

$$= e^{-2}$$

$$= 0.1353$$

(ii) For an exponential random variable,

$$E(X) = \mu = \frac{1}{\lambda} = 10$$

$$\text{Var}(X) = \frac{1}{\lambda^2}$$

$$\text{Var}(X) = E(X^2) - \mu^2$$

$$\begin{aligned} E(X^2) &= \text{Var}(X) + \mu^2 \\ &= \frac{1}{\lambda^2} + \frac{1}{\lambda^2} \\ &= \frac{2}{\lambda^2} \\ &= 200 \end{aligned}$$

$$\begin{aligned} E(C) &= E(4 + 2X + 2X^2) \\ &= E(4) + 2E(X) + 2E(X^2) \\ &= 4 + 2(10) + 2(200) \\ &= 424 \end{aligned}$$

Example 10

The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$.

(i) *What is the probability that the repair time exceeds 2 hours?*

(ii) *What is the conditional probability that a repair takes at least 11 hours given that its direction exceeds 8 hours?*

Solution

Let X be the random variable which denotes the time to repair the machine.

$$\lambda = \frac{1}{2}$$

$$\begin{aligned} f(x) &= \lambda e^{-\lambda x}, & x > 0 \\ &= \frac{1}{2} e^{-\frac{1}{2}x}, & x > 0 \end{aligned}$$

(i) $P(X > 2) = \int_2^\infty f(x)dx$

$$= \int_2^\infty \frac{1}{2} e^{-\frac{1}{2}x} dx$$

$$= \frac{1}{2} \left| \frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}} \right|_2^\infty$$

$$= -\left| e^{-\frac{1}{2}x} \right|_2^\infty$$

$$= -(e^{-\infty} - e^{-1})$$

$$= e^{-1}$$

$$= 0.3679$$

(ii) $P(X \geq 11/X > 9) = P(X > 3)$ (By the memoryless property)

$$= \int_3^\infty f(x)dx$$

$$= \int_3^\infty \frac{1}{2} e^{-\frac{1}{2}x} dx$$

$$= \frac{1}{2} \left| \frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}} \right|_3^\infty$$

$$= -\left| e^{-\frac{1}{2}x} \right|_3^\infty$$

$$= -(e^{-\infty} - e^{-1.5})$$

$$= e^{-1.5}$$

$$= 0.2231$$

Example 11

The daily consumption of milk in excess of 20000 gallons is approximately exponentially distributed with $\lambda = \frac{1}{3000}$. *The city has a daily stock of 35000 gallons. What is the probability that of 2 days selected at random, the stock is insufficient for both the days.*

Solution

Let Y be a random variable which denotes the daily consumption of milk consumed in a day. The random variable $X = Y - 20000$ has an exponential distribution.

$$\lambda = \frac{1}{3000}$$

$$f(x) = \lambda e^{-\lambda x}, \qquad x > 0$$

$$= \frac{1}{3000} e^{-\frac{1}{3000}x}, \qquad x > 0$$

Probability that the stock is insufficient for both days

$$P(Y > 35000) = P(X > 15000)$$

$$= \int_{15000}^{\infty} f(x)\,dx$$

$$= \int_{15000}^{\infty} \frac{1}{3000} e^{-\frac{1}{3000}x}\,dx$$

$$= \frac{1}{3000}\left|\frac{e^{-\frac{1}{3000}x}}{-\frac{1}{3000}}\right|_{15000}^{\infty}$$

$$= -\left|e^{-\frac{1}{3000}x}\right|_{15000}^{\infty}$$

$$= -(e^{-\infty} - e^{-5})$$

$$= e^{-5}$$

$$= 0.0067$$

EXERCISE 5.4

1. If X is exponentially distributed, prove that probability that X exceeds its expected value is less than 0.5.

2. The amount of time that a watch will run without having to be reset is a random variable having an exponential distribution with mean 120 days. Find the probability that such a watch will
   (a) have to be set in less than 24 days.
   (b) not have to be reset in at least 180 days.

   [Ans.: (a) 0.1813, (b) 0.2231]