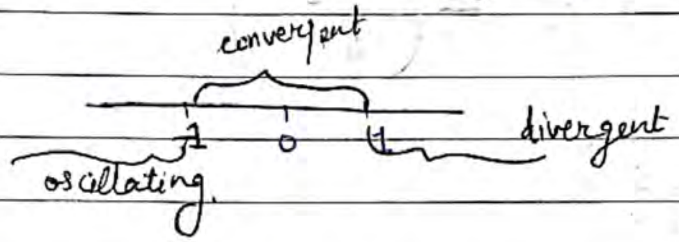
* Geometric Series:-

The series of the form $a + ar + ar^2 + ar^3 + \dots + ar^n + \dots$ is called geometric series where r is the ratio.

1) Series is (i) Σa_n is convergent if $|r| < 1$ & its sum is $\frac{a}{1-r}$

(ii) Σa_n is divergent if $r \geq 1$

(iii) Is oscillating if $r \leq -1$



Q) Test the convergence of $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$

Sol^n $a = 1$, $r = \frac{1}{2}$

$$= \sum_{n=1}^{\infty} ar^{n-1}$$

$$= 1 \sum_{n=1} \left(\frac{1}{2}\right)^{n-1}$$

$|r| = \frac{1}{2}$ $\Rightarrow r = 0.5 < 1$

$$\text{sum} = \frac{a}{1-r}$$

$|r| < 1$ $\therefore$ Series is cvgt.

$$\text{sum} = \frac{1}{1-r} = \frac{1}{1-\frac{1}{2}} = \underline{\underline{2}}$$

Q) Test the convergence of $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^{2n}}$

Sol^n $\sum_{n=0}^{\infty} \left(\frac{3^2}{2^2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{9}{4}\right)^n$

$$= 1 + \frac{9}{4} + \left(\frac{9}{4}\right)^2 + \left(\frac{9}{4}\right)^3 + \dots$$

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$a = 1$, $r = 9/4$

$r > 1$ $\therefore$ series is divergent.

Q) Test the convergence of $3 - 9 + 27 - 81 + \dots$

Sol^n $3 + 3(-3) + 3(-3)^2 + 3(-3)^3 + \dots$

$a = 3$, $r = -3$. oscillating.

$\therefore r < -1$ $\therefore$ Series is oscillating. infly.

Q) Test the convergence of $\sum_{n=0}^{\infty} \left(\frac{3^n - 5^n}{4^n}\right)$

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n - \sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n$$

$a = 1$, $r = \frac{3}{4}$ $\qquad$ $a = 1$, $r = \frac{5}{4}$

$|r| < 1$ $\qquad$ $r > 1$

Convergent series. $\qquad$ Divergent series.

$\therefore \frac{1}{1-r}$

$= \frac{1}{1 - \frac{3}{4}} = \frac{4}{1} = 4 + \infty$

$\therefore$ divergent series.

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imp topics for exam - Comparision, ratio, Root, Power series

Page No.
Date

* n^{th} term test:-

If summation (Σa_n) is divergent if $\lim_{n\to\infty} a_n \neq 0$

If Σa_n converges then $\lim_{n\to\infty} a_n = 0$ but the converse is not true.

i.e. $\lim_{n\to\infty} a_n = 0 \not\Rightarrow \Sigma a_n$ is convergent.

if Σa_n cvt $\Rightarrow \lim_{n\to\infty} a_n = 0$.

Ex: 1) $\sum_{n=1}^{\infty} n^2$

Sol: $a_n = n^2$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty \neq 0$$

$\therefore \Sigma n^2$ is divergent.

19/9/19 Thursday

2) $\sum_{n=1}^{\infty} \frac{-n}{3n+6}$

$$a_n = \frac{-n}{3n+6}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-n}{3n+6}$$

$$= \lim_{n\to\infty} \frac{-\cancel{n}}{\cancel{n}\left(3+\frac{6}{n}\right)} = \frac{-1}{\left(3+\frac{6}{n}\right)}$$

$$\therefore \lim_{n\to\infty} \frac{-1}{\left(3+\frac{6}{n}\right)} = \frac{-1}{3+0} = -\frac{1}{3} \neq 0$$

$\therefore$ Series is divergent.

* Integral test:

→ Let $a_n = f(n)$, $f(x)$
$f(1) + f(2) + \dots + f(n) + \dots$ be a non-negative series and f be a continuous decreasing f^{tn} on the interval $[1, \infty)$ then the series $\sum_{n=1}^{\infty} f(n)$ converges or diverges according as the integral $\int_1^{\infty} f(x)\,dx$ converges or diverges.

1.Q: Test the convergence of $\sum_{n=1}^{\infty} n e^{-n^2}$

Sol^n $a_n = n e^{-n^2}$

$\therefore f(n) = n e^{-n^2}$

$$\Rightarrow \int_1^{\infty} f(x)\,dx = \int_1^{\infty} x e^{-x^2}\,dx$$

$x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{1}{2}dt$

$$\therefore \int_1^{\infty} e^{-t} \frac{dt}{2}$$

$x = 1 \Rightarrow t = 1$

$x = \infty \Rightarrow t = \infty$

$$= \frac{1}{2}\left[\frac{e^{-t}}{-1}\right]_1^{\infty}$$

$$= -\frac{1}{2}\left[e^{-\infty} - e^{-1}\right]$$

$$= -\frac{1}{2}\left[0 - e^{-1}\right] \Rightarrow \frac{1}{2e}$$

$\Rightarrow$ finite

$\therefore \int_1^{\infty} f(x)\,dx$ is convergent.

$\therefore \sum_{n=1}^{\infty} n e^{-n^2}$ is convergent.

Page No.
Date

2) $\sum_{n=1}^{\infty} \frac{1}{n(1+\log^2 n)}$

$a_n = \frac{1}{n(1+\log^2 n)}$

$\therefore f(n) = \frac{1}{n(1+\log^2 n)}$

$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x(1+\log^2 x)}\,dx.$

$\log x = t$

$\Rightarrow \frac{1}{x}\,dx = dt$

$\therefore \int_0^{\infty} \frac{1}{1+t^2}\,dt.$

$x = 1 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$= [\tan^{-1} t]_0^{\infty}$

$\Rightarrow \tan^{-1}\infty - \tan^{-1} 0$

$= \frac{\pi}{2} - 0 \qquad = \frac{\pi}{2}$

3) $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$

$a_n = \frac{\tan^{-1} n}{1+n^2}$

$f(n) = \frac{\tan^{-1} n}{1+n^2}$

$\tan^{-1} x = t.$

$\frac{1}{1+x^2}\,dx = dt.$

$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{\tan^{-1} x}{1+x^2}\,dx$

$x = 1 \Rightarrow t = \frac{\pi}{4}$

$x = \infty \Rightarrow \Rightarrow \frac{\pi}{2}$

$= \int_{\pi/4}^{\pi/2} t\,dt.$

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→ e, log, trigonometric - Integral test.
→ Polynomial form - Comparision test.

Page No.
Date

$$\frac{1}{2}\left[\frac{t^2}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2-\left(\frac{\pi}{4}\right)^2\right]$$

$$=\left[\frac{\pi^2}{4}-\frac{\pi^2}{16}\right]\frac{1}{2}$$

$$=\frac{\pi^2}{2\times 4}\left[\frac{1}{1}-\frac{1}{4}\right]$$

$$=\frac{\pi^2}{2\times 4}\left[\frac{4-1}{4}\right] = \frac{3\pi^2}{16\times 2} = \frac{3\pi^2}{32}$$

* Comparision Test:

for sin, cos, tan

Let Σa_n be a positive term series $\exists\ \Sigma b_n$ such that $\lim_{n\to\infty} \frac{a_n}{b_n} = l$ (finite, non-zero)

then Σa_n & Σb_n both are convergent or divergent.

(i) If Σb_n is divergent $\Rightarrow \Sigma a_n$ is divergent.

(ii) If Σb_n is convergent $\Rightarrow \Sigma a_n$ is convergent.

Note: (Take common & cancel out)

* P-series test:

The series of the form $\Sigma \frac{1}{n^p}$ is called p-series

(i) If $p>1$, $\Sigma \frac{1}{n^p}$ is convergent.

(ii) If $p\le 1$, $\Sigma \frac{1}{n^p}$ is divergent.

Page No.
Date / /

1) $\sum_{n=1}^{\infty} \frac{2n^2+3n}{5+n^5}$

Sol^n $a_n = \frac{2n^2+5n}{5+n^2}$, $b_n = \frac{1}{n^3}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+3n}{5+n^5} \times \frac{n^3}{1}$

$= \lim_{n\to\infty} \frac{\cancel{n^2}(2+3/n)}{\cancel{n^5}(5/n^5+1)} \times \cancel{n^3}$

$= 2$ $\therefore \Sigma a_n$ & Σb_n are cvgt or dvgt together

$\Sigma b_n = \Sigma \frac{1}{n^3}$

$\therefore p=3$ $\Rightarrow$ By p-series test.
Σb_n is cvgt
$\Rightarrow \Sigma a_n$ is also cvgt.

2) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)} + \dots$

Sol^n $\frac{1}{n(n+1)} = a_n$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n(n+1)} \times \frac{1}{b_n}$ $\therefore b_n = \frac{1}{n^2}$

$\therefore \lim_{n\to\infty} \frac{1}{\cancel{n^2}(1+\frac{1}{n})} \times \cancel{n^2}$

$\therefore = 1$

$\Sigma b_n = \Sigma \frac{1}{n^2}$ $\therefore p=2 \Rightarrow$ By p-series test
Σb_n is convergent.
$\Rightarrow \Sigma a_n$ is also convgt.

3) $\sum_{n=1}^{\infty} \frac{5n^3-3n}{n^2(n-2)(n^2+5)}$

Solⁿ $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{5n^3-3n}{n^2(n-2)(n^2+5)} \times \frac{1}{b_n}$

$= \lim_{n\to\infty} \frac{n^3(5-3/n^2)}{n^2 \times n(1-\frac{2}{n}) \times n^2(1+\frac{5}{n^2})} \times \frac{1}{b_n}$

$= \lim_{n\to\infty} \frac{n^3(5-3/n^2)}{n^5(1-\frac{2}{n})(1+\frac{5}{n^2})} \times n^2$

$\therefore b_n = \frac{1}{n^2}$

$= \frac{5}{1} = 5.$

$\therefore \sum b_n = \sum \frac{1}{n^2}$

$\therefore p = 2.$

By p-series test

$\sum b_n$ is conv.

$\Rightarrow \sum a_n$ is also convg.

4) Test convergence of $\sum \sin\left(\frac{1}{n}\right)$

Solⁿ $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right)$

$\therefore \frac{a_n}{b_n} = \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) \times \frac{1}{b_n}$ $\therefore b_n = \frac{1}{n}$

$\therefore \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \underline{\underline{1}}.$

P.T. $\therefore$ series is divergent.

for if a_n is factorial - ratio test.

Monday

Page No.
Date 23 / 9 / 19

imp (Sum)

D'Alembert's Ratio Test.

Let summation of a_n be a +ve term series, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = l \quad \text{(infinite)}$$

(i) If $0 \le l < 1$, Σa_n is cvgt.

(ii) If $l > 1$, Σa_n is dvgt.

(iii) If $l = 1$, test fails.

Note:

Q) Test the convergence of $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$

Sol^n $a_n = \frac{n!}{(2n+1)!}$ $\qquad a_{n+1} = \frac{(n+1)!}{(2n+3)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \Rightarrow \lim_{n\to\infty} \frac{(n+1)!}{(2n+3)!} \times \frac{(2n+1)!}{(n!)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{(n+1)\cancel{n!}}{(2n+3)(2n+2)\cancel{(2n+1)!}} \times \frac{\cancel{(2n+1)!}}{\cancel{n!}}$$

$$\Rightarrow \lim_{n\to\infty} \cancel{(n+1)} \frac{\cancel{n}(1+1/n)}{n^2(2+3/n)(2+2/n)}$$

$$= \frac{1}{\infty} = 0.$$

$\therefore l = 0$

$\therefore \Sigma a_n$ is cvgt.

formula: $\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right)$

Page No.
Date

Q) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Sol^n $a_n = \frac{n!}{n^n}$, $a_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$= \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!}$

$= \lim_{n\to\infty} \frac{\cancel{(n+1)}\cancel{n!}}{(n+1)^n \cdot \cancel{(n+1)}} \times \frac{n^n}{\cancel{n!}}$

$= \lim_{n\to\infty} \frac{\cancel{n^n} \cdot 1}{\cancel{n^n}\left(1+1/n\right)^n} \times \cancel{n^n}$

$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$

$= \frac{1}{e}$ $\left(\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right)$

< 1

$\therefore \sum a_n$ is cvgt.

3) $\sum_{n=1}^{\infty} \frac{n!\,2^n}{n^n}$

Sol^n $a_n = \frac{n!\,2^n}{n^n}$ $\quad a_{n+1} = \frac{(n+1)!\,2^{(n+1)}}{(n+1)^{n+1}}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \Rightarrow \lim_{n\to\infty} \frac{(n+1)!\,2^{(n+1)}}{(n+1)^{n+1}} \times \frac{n^n}{n!\,2^n}$

$\Rightarrow \lim_{n\to\infty} \frac{\cancel{(n+1)}\,\cancel{n!}\,\cancel{2^n}\cdot 2}{(n+1)^n\cdot\cancel{(n+1)}} \times \frac{n^n}{\cancel{n!}\,\cancel{2^n}}$

$\Rightarrow \lim_{n\to\infty} \frac{2\cdot n^n}{(n+1)^n}$

$\Rightarrow \lim_{n\to\infty} \frac{2\,\cancel{n^n}}{\cancel{n^n}\left(1+\frac{1}{n}\right)^n} = \frac{2}{e+0} = \frac{2}{e}$

$\cancel{\Rightarrow} < 1$

$\therefore \Sigma a_n$ is cvgt.

4) Test the cvgs of.

$x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$

$a_n = \frac{x^n}{n}$ $\quad a_{n+1} = \frac{x^{n+1}}{(n+1)}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \Rightarrow \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)} \cdot \frac{n}{x^n}$

$\rightarrow \lim_{n\to\infty} \frac{\cancel{x^n}\cdot x \cdot n}{(n+1)\;\cancel{x^n}}$ $\quad \therefore \frac{\cancel{n}\,x}{\cancel{n}\left(1+\frac{1}{n}\right)} = x$

cont.

(i) If $0 \le x < 1$, Σa_n is cvgt.
(ii) If $x > 1$, Σa_n is dvgt.
(iii) If $x = 1$ Ratio test fails.

$\therefore x = 1$, $\Sigma a_n = \Sigma \frac{1}{n}$

$\therefore$ By p-series test $\therefore$, $p = 1$

$\therefore \Sigma a_n$ is dvgt.

Q) Test the convergence of the series $\frac{1}{1\cdot2\cdot3} + \frac{x}{4\cdot5\cdot6} + \frac{x^2}{7\cdot8\cdot9} + \dots$

Sol^n $a_n = \frac{x^{n-1}}{n(n+1)(n+2)}$ $\quad a_{n+1} = \frac{x^n}{(n+1)(n+2)(n+3)}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \Rightarrow \lim_{n\to\infty} \frac{\cancel{x^n}\, x}{\cancel{(n+1)}\cancel{(n+2)}(n+3)} \cdot \frac{n\cancel{(n+1)}\cancel{(n+2)}}{\cancel{x^n}}$

$\Rightarrow \lim_{n\to\infty} \frac{x \cdot n}{(n+3)}$

$\Rightarrow \lim_{n\to\infty} \frac{x \cdot \cancel{n}}{\cancel{n}(1 + 3/n)} = x$.

Sol^n $a_n = \frac{x^{n-1}}{(3n-2)(3n-1)(3n)}$

$a_{n+1} = \frac{x^n}{(3n+1)(3n+2)(3n+3)}$

$$\lim_{n\to\infty} \frac{\cancel{x^n}\cdot x}{(3n+1)(3n+2)(3n+3)} \cdot \frac{(3n-2)(3n-1)(3n)}{\cancel{x^n}}$$

$$\lim_{n\to\infty} \frac{\cancel{n^3}\left(3-\frac{2}{n}\right)\left(3-\frac{1}{n}\right)3\cdot x}{\cancel{n^3}\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{3}{n}\right)}$$

$$\lim_{n\to\infty} \frac{(3)(3)(3)x}{(3)(3)(3)} = \underline{\underline{x}}$$

(i) If $0 \le x < 1$ Σa_n is cvgt.
(ii) If $x > 1$, Σa_n is dvgt.

(iii) If $x = 1$
Ratio test fail.
$\therefore x = 1$, $\Sigma a_n = \Sigma \frac{1}{n}$

$\therefore$ By p-series test, $p = 1$.

Σa_n is dvgt.

27/9/19 Friday

* n^{th} Root Test: (Cauchy's Root Test)

→ If Σu_n is a +ve term series and if $\lim_{n\to\infty} (u_n)^{1/n} = l$

then (i) Σu_n is cvgt if $l < 1$.
(ii) Σu_n is dgt if $l > 1$
(iii) Test fails if $l = 1$

Page No.
Date

Q.1) Test convergence of series $\sum \frac{1}{(\log n)^n}$

Sol^n $a_n = \frac{1}{(\log n)^n}$

$\lim_{n\to\infty} (u_n)^{1/n}$

$\lim_{n\to\infty} \left(\frac{1}{(\log n)^n}\right)^{1/n}$

$\lim_{n\to\infty} \frac{1}{\log n}$

$= \frac{1}{\log\infty} = \frac{1}{\infty} = \underline{\underline{0}}$

$\therefore$ By root test $\sum a_n$ is cvgt.

Q.2) Test convergence of $\sum \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$

Sol^n $a_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$

$\lim_{n\to\infty} (u_n)^{1/n}$

$\therefore \lim_{n\to\infty} \left(\frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}\right)^{1/n}$

$\therefore \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$

$\therefore \lim_{n\to\infty} \frac{1}{e} \longrightarrow \frac{1}{e} < 1$

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$\therefore$ By root test Σa_n is cvgt.

Q.3) Test convergence of series $\Sigma \frac{(n+\sqrt{n})^n}{3^n \cdot n^{n+1}}$

Sol^n $a_n = \frac{(n+\sqrt{n})^n}{3^n \cdot n^{n+1}}$

$$\lim_{n\to\infty} (u_n)^{1/n}$$

$$\lim_{n\to\infty} \left(\frac{(n+\sqrt{n})^n}{3^n \cdot n^n \cdot n^1}\right)^{1/n}$$

$$\therefore \lim_{n\to\infty} \left(\frac{n+\sqrt{n}}{3\cdot n^1 \cdot n^{1/n}}\right)^{\cancel{n}/\cancel{n}}$$

$$\therefore \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{\sqrt{n}}{n}\right)}{\cancel{n}\left(3(n^{1/n})\right)}$$

$$\therefore = \frac{1}{3(1)} \qquad \left(\because \lim_{n\to\infty} n^{1/n} = 1\right)$$

$$\therefore \frac{1}{3} < 1$$

$\therefore$ By root test, Σa_n is cvgt.

Q.4) Test convergence of series: $\frac{x}{\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^3}{4\sqrt{3}} + \dots$

Sol^n $u_n = \frac{x^n}{n^{1/n+1}}$

$$\therefore \lim_{n\to\infty} (u_n)^{1/n}$$

Divergent test/ zero test:

$$\lim_{n\to\infty} u_n \neq 0 \Rightarrow \text{dgt}$$
$$u_n = 0 \not\Rightarrow \text{cvgt}.$$

Page No.

Date

$$\lim_{n\to\infty} \left(\frac{x^n}{n^{1/n+1}} \right)^{1/n}$$

$$\cancel{\lim_{n\to\infty} \left(\frac{x \cdot n^{n+1}}{n} \right)^{1/n}}$$

$$\cancel{\therefore \lim_{n\to\infty} \left(\frac{x \cdot n^n \cdot n}{n} \right)^{1/n}}$$

$$\cancel{\therefore \lim_{n\to\infty} x^{1/n} \cdot n}$$

$$\therefore \lim_{n\to\infty} \frac{(x^n)^{1/n}}{\left(n^{\frac{1}{n+1}}\right)^{1/n}}$$

$$\therefore \lim_{n\to\infty} \frac{x}{n^{\frac{1}{n(n+1)}}}$$

$$\therefore x \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n(n+1)}}} = x\left(\frac{1}{1}\right) = x$$

$$\left(\because \lim_{n\to\infty} n^{\frac{1}{n(n+1)}} = 1 \right)$$

By root test,

Σu_n is cgt if $x<1$

Σu_n is dgt if $x>1$

Test fails if $x=1$

For $x=1$

$$u_n = \frac{1}{n^{1/n+1}}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n+1}}} = 1 \neq 0$$

$\therefore$ Series is dgt.

Page No.
Date

* Alternating series: (Lebnitz's test).

An infinite series in which ~~to~~ terms are alternatively +ve or -ve is called an alternating series.

i.e. $u_1 - u_2 + u_3 - u_4 + \ldots$

(i) it is cvgt if each term is numerically less than its preciding term.

i.e. $|u_n| > |u_{n+1}|$ $\therefore u_{n+1} - u_n < 0$

(ii) ~~ea~~ $\lim_{n\to\infty} |u_n| = 0$

When $\lim_{n\to\infty} |u_n| \neq 0$ then series is oscillatory

30/9/19 Monday.

→ If (i) & (ii) both are satisfied then series u_n is cvgt and if one of them is not satisfied then the series is oscillating series.

Q) Examine the convergence of $\frac{1}{1\cdot2} - \frac{1}{3\cdot4} + \frac{1}{5\cdot6} - \frac{1}{7\cdot8} + \ldots$

Sol^n, $\Sigma a_n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)2n}$.

Here,

$$u_n = \frac{1}{(2n-1)(2n)}$$

(i) $u_{n+1} - u_n < 0$

(ii) $\therefore u_{n+1} - u_n = \frac{1}{(2n+1)(2n+2)} - \frac{1}{(2n-1)(2n)}$

Page No.
Date

$$\frac{(2n-1)(2n)-(2n+1)(2n+2)}{(2n+1)(2n+2)(2n-1)(2n)}$$

$$= \frac{4n^2-2n-4n^2-6n-2}{(2n+1)(2n+2)(2n-1)(2n)}$$

$$= \frac{-8n-2}{(2n+1)(2n+2)(2n-1)(2n)}$$

$\therefore u_{n+1}-u_n<0$

$\Rightarrow u_{n+1}<u_n$

(ii) $\lim_{n\to\infty} u_n$

$= \lim_{n\to\infty} \frac{1}{(2n-1)(2n)}$

$= 0$

$\therefore \Sigma a_n$ is cvgt.

Q 2) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

Soln: $u_n = \frac{1}{n^2}$

(i) $u_{n+1}-u_n = \frac{1}{(n+1)^2} - \frac{1}{n^2}$

$= \frac{n^2-n^2-2n-1}{n^2(n-1)^2}$

<0

$\therefore u_{n+1}<u_n$

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$

$\therefore$ By Lebnitz's test Σa_n is cvgt.

Q.3) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2(n+1)}$

Sol^n Σa_n: $a_n = \frac{1}{n^2(n+1)}$

(i) $a_{n+1} - a_n = \frac{1}{(n+1)^2(n+2)} - \frac{1}{n^2(n+1)}$

$= \frac{n^2(n+1) - (n+1)^2(n+2)}{(n+1)^2(n+2)\, n^2(n+1)}$

$= \frac{n^3+n^2-(n^2+2n+1)(n+2)}{(n+1)^3(n+2)(n^2)}$

$= \frac{\cancel{n^3}+n^2-\cancel{n^3}-2n^2-n-2n^2-4n-2}{(n+1)^3(n+2)(n^2)}$

$= \frac{3n^2-5n-2}{(n+1)^3(n+2)(n^2)}$

< 0

$\therefore a_{n+1} < a_n$

(ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2(n+1)} = 0$

$\therefore$ By Lebnitz's test Σa_n is cvgt.

4) $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\sqrt{n}+\sqrt{n+1}}$

5) $\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{n^2+1}$

6) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(n+1)^n}{(2n)^n}$

* Radius and inverval of convergence

1) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{2n+1}$

→ By ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+2} x^{2n+3}}{2n+3} \times \frac{2n+1}{(-1)^{n+1} x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{x^{2n}} \cdot x^{3}}{\cancel{n}(2+3/n)} \times \frac{\cancel{n}(2+\frac{1}{n})}{\cancel{x^{2n}} \cdot x} \right|$$

$$= |x^2|$$

if $0 \le |x^2| < 1$, Σa_n cvgt.

$\Rightarrow -1 < x < 1$, Σa_n cvgt

if $|x^2| > 1$, Σa_n dvgt.

$|x^2| = 1$, test fails

Page No.
Date

$|x^2| = 1$

$|x|^2 = 1$

$|x| = 1 \Rightarrow x = \pm 1$

for $x = 1$

$$\Sigma a_n = \Sigma \frac{(-1)^{n+1} (1)^{2n+1}}{2n+1} = \Sigma \frac{(-1)^{n+1}}{2n+1}$$

By lebnitz's test.

$$\Sigma a_n = \Sigma (-1)^{n+1} u_n$$

$$u_n = \frac{1}{2n+1}$$

(i) $u_{n+1} - u_n < 0$

$\Rightarrow \frac{1}{2n+3} - \frac{1}{2n+1} < 0$

(ii) $\lim_{n \to \infty} u_n = 0$

$\Rightarrow \lim_{n \to \infty} \frac{1}{2n+1} = 0$

$\therefore \Sigma a_n$ is cvgt for $x=1$

if $x = -1$

$$\Sigma a_n = \Sigma \frac{(-1)^{n+1} (-1)^{2n+1}}{2n+1}$$

$$= \Sigma \frac{(-1)^{3n+2}}{2n+1}$$

By lebnitz test.

$$\Sigma a_n = \Sigma (-1)^{3n+2} u_n$$

$$u_n = \frac{1}{2n+1}$$

By lebnitz test

$$\Sigma a_n = \Sigma (-1)^{3n+2} u_n$$

$$u_n = \frac{1}{2n+1}$$

Σa_n is cvgt for $x = -1$

Page No.
Date

$\sum a_n$ is cvgt for $-1 \le x \le 1$
Interval of conv = $[-1, 1]$

Radius of conv.

$$ROC = \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{1}{2n+3} \cdot 2n+1 \right|$$

$$= \lim_{n \to \infty} \left| \frac{\cancel{n}(2 + 1/n)}{\cancel{n}(2+3/n)} \right|$$

$$= 2/2$$

$R = ROC = 1$

Page No.
Date

* Power series:

→ A series of the form $\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \dots$ is called power series in terms of $(x-a)$

if $a=0$, then the series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \text{ is called}$$

power series in terms of x.

Q) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} x^{n+1}$

Sol^n By ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\left| \frac{(-1)^{n+1} x^{n+2}}{2n+1} \times \frac{2n-1}{(-1)^n x^{n+1}} \right| \right]$$

$$= \lim_{n\to\infty} \left[\left| \frac{\cancel{(-1)^n} \cdot (-1) \cdot \cancel{x^n} \cdot x^2 \cdot (2n-1)}{(2n+1) \cdot \cancel{(-1)^n} \cdot \cancel{x^n} \cdot x} \right| \right]$$

$$= \lim_{n\to\infty} \left[\left| \frac{(-1)(2n-1)x}{(2n+1)} \right| \right]$$

$$= \lim_{n\to\infty} \left| \frac{(-2n+1)x}{2n+1} \right|$$

$$= \lim_{n\to\infty} \frac{n}{n} \left[\left| \frac{(-2+\frac{1}{n})x}{2+\frac{1}{n}} \right| \right]$$

$$= \lim_{n\to\infty} |-1(x)| \qquad = \lim_{n\to\infty} |-1||x|$$

if $0 \le |x| < 1$, Σa_n is cvgt.

if $|x| > 1$, Σa_n is dvgt.

if $|x| = 1$, test fails.

for $|x| = 1$

$\Rightarrow x = \pm 1$

if $x = 1$, $\Sigma a_n = \Sigma \frac{(-1)^n}{2n-1}$

By alternating series,

$u_n = \frac{1}{2n-1}$

$u_{n+1} - u_n = \frac{1}{2n+1} - \frac{1}{2n-1} < 0$

(ii) $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{2n-1}$

$= 0$

$\therefore$ By Leibnitz's test, Σa_n is cvgt for $x = 1$

for $x = -1$

$\Sigma a_n = \frac{(-1)^n \cdot x^{n+1}}{2n-1}$

$= \frac{\Sigma (-1)^{2n+1}}{2n-1}$

$= \frac{(-1)^{2n} \ (-1)}{2n-1}$

$$= \frac{(-1)}{(-1)(1-2n)}$$

$$\Sigma a_n = \frac{1}{1-2n}$$

By comparision test

$$a_n = \frac{1}{1-2n}, \quad b_n = \frac{1}{n}$$

$$\lim_{n \to \infty} \left| \frac{a_n}{b_n} \right| = \frac{1}{\frac{1-2n}{\cancel{1}}} = \frac{1}{\cancel{n}(1/n - 2)} \times \frac{\cancel{n}}{1}$$

$$= -\frac{1}{2}$$

Comparing by p series

$$b_n = \frac{1}{n} = \frac{1}{n^p}$$

$$p = 1.$$

Σa_n is divergent for $x = -1$

Σa_n is ~~divergent~~ convergent for $-1 < x \leq 1$