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Tutorial - 2

Q1) Use the method of a known solution to find another solution y_2 and general solution y of following equations from the given given solution y_1.

1) $(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$; $y_1 = x$.

$\Rightarrow$ Given,

$(x-1)y'' - xy' + y = 0$, $y_1 = x$

By Method of reduction of order :-

$y'' - \frac{x}{x-1}y'' + \frac{y}{x-1} = 0$.

$\therefore y'' + \left(\frac{-x}{x-1}\right)y'' + \left(\frac{1}{x-1}\right)y = 0$

Here, general eq. $\rightarrow$ $\boxed{y'' + Py' + Q}$ $P = \frac{-x}{x-1}$, $Q = \frac{1}{x-1}$

Now, $y_1 = x$

Other solⁿ, $y_2 = uy_1$

where, $u = \int \frac{1}{y_1^2} e^{-\int P dx} dx$

$\therefore u = \int \frac{1}{x^2} e^{-\int \frac{-x}{x-1} dx} dx$

$= \int \frac{1}{x^2} e^{\int \frac{x}{x-1} dx} dx$

$= \int \frac{1}{x^2} e^{\int \left(\frac{x-1+1}{x-1}\right) dx} dx$

$= \int \frac{1}{x^2} e^{\int \left[\frac{x-1}{x-1} + \frac{1}{x-1}\right] dx} dx$

$$= \int \frac{1}{x^2} e^{\int [1+\frac{1}{x-1}] dx} dx$$

$$= \int \frac{1}{x^2} e^{x \cdot \log(x-1)} dx$$

$$= \int \frac{1}{x^2} e^{x} \cdot e^{\log(x-1)} dx$$

$$= \int \frac{1}{x^2} e^{x} \cdot (x-1) dx$$

$$= \int e^{x} \left(\frac{x-1}{x^2}\right) dx$$

{Now, $\int e^{x} [f(x) + f'(x)] dx = e^{x} f(x) + c$}

$$= \int e^{x} \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$$

$$= e^{x} \cdot \frac{1}{x} + c$$

$$\therefore u = \frac{e^{x}}{x} + c$$

Now, $y_2 = u y_1 = \frac{e^{x}}{x} \cdot x = e^{x}$

$\therefore y_2 = e^{x}$.

& General soln :- $y = C_1 y_1 + C_2 y_2$

$\therefore y = C_1 x + C_2 e^{x}$

$\Rightarrow$ $\boxed{y_2 = e^{x}}$

&

$\boxed{y = C_1 x + C_2 e^{x}}$

2) y

$\Rightarrow$

Q2)

1)

$\Rightarrow$

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2) $y'' - y = 0$; $y_1 = e^x$

$\Rightarrow$ Given,

$y'' - 0y' - y = 0$, $y_1 = e^x$

By, Method of reduction of order :-

$y'' - 0y' - y = 0$

Here, comparing with general eq. $\rightarrow$

$P = 0$, $Q = -1$

Now, $y_1 = e^x$

$\therefore$ other solⁿ, $y_2 = uy_1$

where, $u = \int \frac{1}{y_1^2} e^{-\int P dx} dx$

$= \int \frac{1}{(e^x)^2} e^{-\int 0 dx} dx$

$= \int \frac{1}{e^{2x}} \cdot 1 \cdot dx$

$= \int e^{-2x} dx$

$\therefore u = \frac{e^{-2x}}{-2} + c$

Now, $y_2 = uy_1 = \frac{e^{-2x}}{-2} \times e^x = e^{-2x+x} = e^{-x}$

eliminate

& General solⁿ :- $y = C_1 y_1 + C_2 y_2$

$\therefore y = C_1 e^x + C_2 e^{-x}$

$\Rightarrow$ $\boxed{y_2 = e^{-x}}$

&

$\boxed{y = C_1 e^x + C_2 e^{-x}}$.

Q2) Solve the following differential equations.

I) $9y'' - 30y' + 25y = 0$.

$\Rightarrow$ Given, $9y'' - 30y' + 25y = 0$ --- ①

Let, $D = \frac{d}{dx}$

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Eq. ① becomes :-

$$(9D^2 - 30D + 25)\,y = 0.$$

The auxiliary equation is :-

$$9m^2 - 30m + 25 = 0.$$

$$9m^2 - 15m - 15m + 25 = 0$$

$$\therefore\ 3m(3m-5) - 5(3m-5) = 0$$

$$\therefore\ (3m-5)(3m-5) = 0.$$

$$\therefore\ m = \frac{5}{3}, \frac{5}{3} \quad \text{(real \& equal)}$$

Hence, general solⁿ :-

$$y = C.F. = y_c = (c_1 + c_2 x)e^{mx}$$

$$\therefore\ y = (c_1 + c_2 x)e^{5/3 x}$$

$$\Rightarrow \boxed{y = [c_1 + c_2 x]\,e^{\frac{5x}{3}}}.$$

2) $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 20y = 0.$

$\Rightarrow$ Given, $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 20y = 0$ --- ①

Let, $D = \frac{d}{dx}$

$\therefore$ Eq. ① becomes :-

$$(D^2 - 9D + 20)\,y = 0$$

The auxiliary equation is :-

$$m^2 - 9m + 20 = 0.$$

$$\therefore\ m^2 - 5m - 4m + 20 = 0$$

$$\therefore\ m(m-5) - 4(m-5) = 0$$

$$\therefore\ (m-5)(m-4) = 0$$

$$\therefore\ m = 4, 5 \quad \text{(real \& distinct)}$$

$$[m_1 = 4,\ m_2 = 5].$$

Hence, general sol^n :-

$$y = C.F. = y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$\therefore y = C_1 e^{4x} + C_2 e^{5x}$$

$$\Rightarrow \boxed{y = C_1 e^{4x} + C_2 e^{5x}}$$

equal).

3) $(D^2 - 6D + 25) y = 0$

$\Rightarrow$ Given, $(D^2 - 6D + 25) y = 0$

$\therefore$ The Auxiliary equation is :-

$$m^2 - 6m + 25 = 0$$

$$\therefore \cancel{m^2 - m - 5m + 25 = 0}$$

$$\therefore \cancel{m(m)}$$

$$\therefore m = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm \sqrt{-64}}{2}$$

$$= \frac{6 \pm \sqrt{-1} \cdot \sqrt{64}}{2} = \frac{6 \pm 8i}{2}$$

$$\therefore m = 3 \pm 4i \quad (\alpha \pm i\beta)$$

i.e. $m = 3 + 4i, \; 3 - 4i$ (complex)

Hence, general sol^n :-

$$y = CF = y_c = e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x]$$

where, $\alpha = 3$, $\beta = 4$

$$\therefore y = e^{3x} [C_1 \cos 4x + C_2 \sin 4x].$$

$$\Rightarrow \boxed{y = e^{3x} [C_1 \cos 4x + C_2 \sin 4x]}$$

4) $4y'' + 4y' + 10y = 0$

$\Rightarrow$ Given, $4y'' + 4y' + 10y = 0$ --- ①

Let, $D = \frac{d}{dx}$

$\therefore$ Eq ① becomes :-

$$(4D^2 + 4D + 10) y = 0$$

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∴ The Auxiliary equation is :-

$$4m^2 + 4m + 10 = 0$$

$$\therefore 2m^2 + 2m + 5 = 0$$

$$\therefore m = \frac{-2 \pm \sqrt{4 - 4(2)(5)}}{2(2)}$$

$$= \frac{-2 \pm \sqrt{4-40}}{4}$$

$$= \frac{-2 \pm \sqrt{-36}}{4} \quad = \frac{-2 \pm \sqrt{-1}\cdot\sqrt{36}}{4}$$

$$= \frac{-2 \pm 6i}{4}$$

$$\therefore m = \frac{-1 \pm 3i}{2} \qquad (\alpha \pm \beta i)$$

i.e. $m = \frac{-1+3i}{2}, \frac{-1-3i}{2}$ (complex)

Hence, general solⁿ :-

$$y = CF = y_c = e^{\alpha x}[C_1 \cos\beta x + C_2 \sin\beta x]$$

where, $\alpha = \frac{-1}{2}$, $\beta = \frac{3}{2}$

$$\therefore y = e^{\frac{-1}{2}x}\left[C_1 \cos\frac{3x}{2} + C_2 \sin\frac{3x}{2}\right]$$

$$\Rightarrow \boxed{y = e^{-\frac{x}{2}}\left[C_1 \cos\frac{3x}{2} + C_2 \sin\frac{3x}{2}\right]}$$

Q3) Solve the following differential equations.

1) $\frac{d^2y}{dx^2} - 4y = (1+e^x)^2 + 3$.

$\Rightarrow$ Given, $\frac{d^2y}{dx^2} - 4y = (1+e^x)^2 + 3$ --①

Let $D = \frac{d}{dx}$

For CF $\Rightarrow$ $\frac{d^2y}{dx^2} - 4y = 0$

i.e. $(D^2-4)y = 0$

$\therefore$ The Auxiliary eq is,

$m^2 - 4 = 0$

$\therefore m^2 = 4$

$\therefore m = \pm 2$ (real & different).

$\therefore$ C.F. $= y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$\therefore y_c = C_1 e^{2x} + C_2 e^{-2x}$

For PI $\Rightarrow$

We can write eq. ① as :-

$$\frac{d^2y}{dx^2} - 4y = 1 + e^{2x} + 2e^{x} + 3$$

i.e. $$\frac{d^2y}{dx^2} - 4y = e^{2x} + 2e^{x} + 4.$$

i.e. $$\frac{d^2y}{dx^2} - 4y = e^{2x} + 2e^{x} + 4e^{0x}$$

$$\therefore PI = y_p = \left(\frac{1}{D^2-4}\right)\left[e^{2x} + 2e^{x} + 4e^{0x}\right]$$

$$= \left[\frac{1}{D^2-4}\right]_{D\to 2} e^{2x} + \left[\frac{1}{D^2-4}\right]_{D\to 1} 2e^{x} + \left[\frac{1}{D^2-4}\right]_{D\to 0} 4e^{0x}$$

$$= x \cdot \frac{1}{(2D)} e^{2x} + \frac{1}{(1-4)} 2e^{x} + \frac{1}{(0-4)} 4.e^{0x}$$

{- case of failure }

$$= x \cdot \left[\frac{1}{2D}\right]_{D\to 2} e^{2x} + \frac{2e^{x}}{(-3)} + \frac{4e^{0x}}{(-4)}$$

$$= x \cdot \frac{1}{2(2)} e^{2x} - \frac{2}{3} e^{x} - 1e^{0x}$$

$$\therefore y_p = \frac{1}{4} xe^{2x} - \frac{2}{3} e^{x} - 1$$

Hence, the general sol^n is :-

$$y = CF + PI = y_c + y_p$$

$$\therefore y = C_1 e^{2x} + C_2 e^{-2x} + \frac{1}{4} x e^{2x} - \frac{2}{3} e^{x} - 1$$

$$\Rightarrow \boxed{y = C_1 e^{2x} + C_2 e^{-2x} + \frac{1}{4} x e^{2x} - \frac{2}{3} e^{x} - 1}$$

2) $y'' + 4y' + 5y = -2\cosh x$

$\Rightarrow$ Given, $y'' + 4y' + 5y = -2\cosh x$

Let $D = \frac{d}{dx}$

For C.F. $\Rightarrow$ $y'' + 4y' + 5y = 0$

i.e. $(D^2 + 4D + 5)y = 0$

$\therefore$ The Auxiliary eq. is :-

$$m^2 + 4m + 5 = 0$$

$$\therefore m = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{-1}\sqrt{4}}{2}$$

$$\therefore m = \frac{-4 \pm 2i}{2}$$

$$\therefore m = -2 \pm i \quad \text{(complex \& different)}$$

$\therefore$ C.F. = $y_c = e^{\alpha x}[C_1 \cos\beta x + C_2 \sin\beta x]$

Here, $\alpha = -2$, $\beta = 1$.

$$\therefore \underline{y_c = e^{-2x}[C_1 \cos x + C_2 \sin x]}$$

For P.I. $\Rightarrow$

$$y'' + 4y' + 5y = -2\cosh x$$

$$(D^2 + 4D + 5)y = -2\cosh x$$

$$\therefore y_p = \frac{1}{(D^2+4D+5)}(-2\cosh x)$$

$$= -2\left(\frac{1}{D^2+4D+5}\right)\left(\frac{e^{x} + e^{-x}}{2}\right)$$

$$= -\underset{D \to 1}{\frac{1}{(D^2+4D+5)}} e^{x} - \underset{D \to -1}{\frac{1}{(D^2+4D+5)}} e^{-x}$$

$$= -\frac{1}{(1+4+5)}e^{x} - \frac{1}{(1-4+5)}e^{-x}$$

$$\therefore y_p = -\frac{1}{10}e^{x} - \frac{1}{2}e^{-x}$$

Hence, the general solⁿ is :-

$$y = CF + P.I. = y_c + y_p$$

$$\therefore y = e^{-2x}(c_1\cos x + c_2\sin x) - \frac{1}{10}e^{x} - \frac{1}{2}e^{-x}$$

$$\Rightarrow \boxed{y = e^{-2x}(c_1\cos x + c_2\sin x) - \frac{1}{10}e^{x} - \frac{1}{2}e^{-x}}$$

3) $(D^2+D)y = x^4 + 2x + 1$.

$\Rightarrow$ Given, $(D^2+D)y = x^4 + 2x + 1$.

For CF $\Rightarrow$ $(D^2+D)y = 0$

$\therefore$ The Auxiliary eq. is :-

$$m^2 + m = 0$$

$$m(m+1) = 0$$

$$\therefore m = 0, -1. \quad \text{(real \& different)}$$

$$\therefore CF = y_c = c_1e^{m_1x} + c_2e^{m_2x}$$

$$\therefore y_c = c_1e^{0x} + c_2e^{-x}$$

$$\therefore y_c = c_1 + c_2e^{-x}$$

For P.I. $\Rightarrow$ $(D^2+D)y = (x^4+2x+1)$

$$\therefore y_p = \frac{1}{(D^2+D)}(x^4+2x+1).$$

$$= \frac{1}{D(1+D)}(x^4+2x+1).$$

$$= \frac{1}{D}(1 - D + D^2 - D^3 + D^4 - D^5 + \cancel{D^6})(x^4+2x+1)$$

$$\left\{\because \frac{1}{1+\phi(x)} = 1 - \phi(x) + [\phi(x)]^2 - [\phi(x)]^3 + \dots\right\}$$

$$= \left(\cancel{\frac{1}{D} - 1 + D - D^2 + D^3 - \dots}\right)(x^4+2x+1)$$

$$= \frac{1}{D}\left(1 - D + D^2 - D^3 + D^4 - D^5 + D^6 - D^7 + \cdots\right)(x^4 + 2x + 1)$$

$$= \left(\frac{1}{D} - 1 + D - D^2 + D^3 - D^4 + D^5 - D^6 + \cdots\right)(x^4 + 2x + 1)$$

$$= \frac{1}{D}(x^4+2x+1) - (x^4+2x+1) + D(x^4+2x+1)$$

$$- D^2(x^4+2x+1) + D^3(x^4+2x+1) - D^4(x^4+2x+1)$$

$$+ D^5(x^4+2x+1) - D^6(x^4+2x+1) + \cdots$$

$$= \int (x^4+2x+1)\,dx - (x^4+2x+1) + (4x^3+2)$$

$$- (12x^2) + (24x) - (24) + 0 - 0 + \cdots$$

$$= \left[\frac{x^5}{5} + \frac{2x^2}{2} + x\right] - x^4 - 2x - 1 + 4x^3 + 2$$

$$- 12x^2 + 24x - 24$$

$$= \frac{x^5}{5} - x^4 + 4x^3 + x^2 - 12x^2 - 2x + 24x + x + 2 - 24$$

$$\therefore y_p = \frac{x^5}{5} - x^4 + 4x^3 - 11x^2 + 23x - 23$$

Hence, general sol^n is:-

$$y = C.F. + P.I. = y_c + y_p$$

$$\therefore y = C_1 + C_2 e^{-x} + \frac{x^5}{5} - x^4 + 4x^3 - 11x^2 + 23x - 23$$

$$\Rightarrow \boxed{y = C_1 + C_2 e^{-x} + \frac{x^5}{5} - x^4 + 4x^3 - 11x^2 + 23x - 23}$$

4) $y'' - 2y' + y = x^2 e^{3x}$

$\Rightarrow$ Given, $y'' - 2y' + y = x^2 e^{3x}$

Let, $D = \frac{d}{dx}$

For C.F $\Rightarrow$ $y'' - 2y' + y = 0$

i.e. $(D^2 - 2D + 1) y = 0$

$\therefore$ The Auxiliary eq. is :-

$$m^2 - 2m + 1 = 0$$
$$m^2 - m - m + 1 = 0$$
$$m(m-1) - 1(m-1) = 0$$
$$(m-1)(m-1) = 0$$
$$\therefore m = 1, 1 \quad (\text{ real \& equal})$$

$\therefore$ C.F. $= y_c = (c_1 + c_2 x) e^{mx}$

$\therefore \underline{y_c = (c_1 + c_2 x) e^{x}}$

For P.I. $\Rightarrow$ $y'' - 2y' + y = x^2 e^{3x}$

i.e. $(D^2 - 2D + 1) y = x^2 e^{3x}$

$$\therefore y_p = \frac{1}{(D^2 - 2D + 1)} x^2 e^{3x}$$

$D \to D + 3$

$$= e^{3x} \cdot \frac{1}{[(D+3)^2 - 2(D+3) + 1]} x^2$$

$$= e^{3x} \cdot \frac{1}{[D^2 + 9 + 6D - 2D - 6 + 1]} x^2$$

$$\cancel{= e^{3x} \frac{1}{[D^2 + 4D + 4]} x^2}$$

$$= e^{3x} \frac{1}{4\left[1 + \left(\frac{D^2 + 4D}{4}\right)\right]} x^2$$

$$= \frac{e^{3x}}{4} \left[1 - \left(\frac{D^2 + 4D}{4}\right) + \left(\frac{D^2 + 4D}{4}\right)^2 - \ldots\right] x^2$$

$$\left\{ \because \frac{1}{1 + \phi(x)} = 1 - \phi(x) + [\phi(x)]^2 - [\phi(x)]^3 + \ldots \right\}$$

$$= \frac{e^{3x}}{4}\left[x^2 - \left(\frac{D^2+4D}{4}\right)x^2 + \left(\frac{D^4+16D^2+8D^3}{16}\right)x^2\right]$$

$$= \frac{e^{3x}}{4}\left[x^2 - \frac{1}{4}D^2x^2 - \frac{4}{4}Dx^2 + \frac{1}{16}D^4x^2 + \frac{16}{16}D^2x^2 + \frac{8}{16}D^3x^2\right]$$

$$= \frac{e^{3x}}{4}\left[x^2 - \frac{\cancel{2}^{1}}{\cancel{4}2} - 2x + 0 + 2 + 0\right]$$

$$= \frac{e^{3x}}{4}\left[x^2 - 2x + \frac{3}{2}\right]$$

$$\therefore y_p = \frac{e^{3x}}{4}(x^2-2x+3/2)$$

Hence, general solⁿ :-

$$y = C.F. + P.I. = y_c + y_p$$

$$\therefore y = (c_1+c_2x)e^x + \frac{e^{3x}}{4}(x^2-2x+3/2)$$

$$\Rightarrow \boxed{y = (c_1+c_2x)e^x + \frac{e^{3x}}{4}\left(x^2-2x+\frac{3}{2}\right)}$$

5) $\frac{d^2y}{dx^2} + 16y = x\sin 2x$.

$\Rightarrow$ Given, $\frac{d^2y}{dx^2} + 16y = x\sin 2x$

Let, $D = \frac{d}{dx}$

For C.F. $\Rightarrow$ $\frac{d^2y}{dx^2} + 16y = 0$

ie. $[D^2+16]y = 0$.

$\therefore$ The Auxiliary eq. is :-

$$m^2+16=0$$

$$m^2 = -16$$

$\therefore m = \pm 4i$ (complex & different).

$\therefore C.F. = y_c = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$.

Here, $\alpha = 0$, $\beta = 4$.

$\therefore y_c = e^{0x}[c_1 \cos 4x + c_2 \sin 4x]$.

$\therefore \underline{y_c = c_1 \cos 4x + c_2 \sin 4x}$.

For P.I. $\Rightarrow$ $\dfrac{d^2y}{dx^2} + 16y = x \sin 2x$

ie. $[D^2+16]y = x \sin 2x$

$$\therefore y_p = x \cdot \frac{1}{f(D)} \sin 2x - \frac{f'(D)}{\{f(D)\}^2} \sin 2x.$$

$$= x \cdot \frac{1}{(D^2+16)}\sin 2x - \frac{2D}{(D^2+16)^2}\sin 2x.$$
$$D^2 \to -4 \qquad\qquad D^2 \to -4$$

$$= x \cdot \frac{1}{(-4+16)}\sin 2x - \frac{2D}{(-4+16)^2}\sin 2x.$$

$$= x \cdot \frac{1}{12}\sin 2x - \frac{2}{(12)^2} D \sin 2x$$

$$= \frac{x \sin 2x}{12} - \frac{2}{144}\frac{d}{dx}(\sin 2x)\,dx$$

$$= \frac{x \sin 2x}{12} - \frac{1}{72} \cdot 2 \cos 2x$$

$$\underline{y_p = \frac{x \sin 2x}{12} - \frac{\cos 2x}{36}}.$$

Hence, general sol^n :-

$y = C.F. + P.I. = y_c + y_p$

$$\therefore y = c_1 \cos 4x + c_2 \sin 4x + \frac{x \sin 2x}{12} - \frac{\cos 2x}{36}$$

$$\Rightarrow \boxed{y = c_1 \cos 4x + c_2 \sin 4x + \frac{x \sin 2x}{12} - \frac{\cos 2x}{36}}.$$

6) $(D^2 - 4D + 4)y = e^{2x} + \cos 2x + x^3$.

$\Rightarrow$ Given, $(D^2 - 4D + 4)y = e^{2x} + \cos 2x + x^3$.

For C.F. $\Rightarrow$ $(D^2 - 4D + 4)y = 0$.

$\therefore$ The Auxiliary eq. is :-

$m^2 - 4m + 4 = 0$.

$\therefore m^2 - 2m - 2m + 4 = 0$

$\therefore m(m-2) - 2(m-2) = 0$

$\therefore (m-2)(m-2) = 0$

$\therefore m = 2, 2$ (real & equal).

$\therefore C.F = y_c = (c_1 + c_2 x)e^{mx}$

$\therefore \underline{y_c = (c_1 + c_2 x) e^{2x}}$.

For P.I. $\Rightarrow$ $(D^2 - 4D + 4)y = e^{2x} + \cos 2x + x^3$.

$\therefore y_p = \dfrac{1}{(D^2 - 4D + 4)} (e^{2x} + \cos 2x + x^3)$

$= \dfrac{1}{(D^2 - 4D + 4)} e^{2x} + \dfrac{1}{(D^2 - 4D + 4)} \cos 2x + \dfrac{1}{(D^2 - 4D + 4)} x^3$

$D \to 2$ $\quad$ $D^2 \to -4$

(Case of failure)

$= x \cdot \dfrac{1}{(2D - 4)} e^{2x} + \dfrac{1}{(-4 - 4D + 4)} \cos 2x + \underbrace{\dfrac{1}{4\left[1 + \left(\frac{D^2 - 4D}{4}\right)\right]} x^3}$

$D \to 2$

(Case of failure)

$= x \cdot x \cdot \dfrac{1}{2} e^{2x} + \dfrac{1}{-4D} \cos 2x + Y$

$= \dfrac{x^2}{2} e^{2x} - \dfrac{1}{4} \int \cos 2x \, dx + Y$

$= \dfrac{x^2}{2} e^{2x} - \dfrac{1}{4} \cdot \dfrac{\sin 2x}{2} + Y$

$= \dfrac{x^2}{2} e^{2x} - \dfrac{1}{8} \sin 2x + Y$ $\quad$ --(1)

Now, $y = \frac{1}{4\left[1+\left(\frac{D^2-4D}{4}\right)\right]} x^3$

$$= \frac{1}{4}\left[1-\left(\frac{D^2-4D}{4}\right)+\left(\frac{D^2-4D}{4}\right)^2-\left(\frac{D^2-4D}{4}\right)^3+\cdots\right]x^3$$

$\left\{\because \frac{1}{1+\phi(x)} = 1-\phi(x)+[\phi(x)]^2-[\phi(x)]^3+\cdots\right\}$

$$= \frac{1}{4}\left[1-\frac{D^2}{4}+D+\frac{(D^4+16D^2-8D^3)}{16}-\frac{(D^6-64D^3-12D^5+48D^4)}{64}\right]x^3$$

$\left\{\because (a-b)^3 = a^3-3a^2b+3ab^2-b^3\right\}$

$$= \frac{1}{4}\left[x^3-\frac{1}{4}D^2x^3+Dx^3+\frac{1}{16}D^4x^3+\frac{16}{16}D^2x^3-\frac{8}{16}D^3x^3 - \frac{(D^6x^3-64D^3x^3-12D^5x^3+48D^4x^3)}{64}\right]$$

$$= \frac{1}{4}\left[x^3-\frac{1}{4}(6x)+3x^2+0+6x-\frac{8}{16}(6) - \frac{[0-64(6)-0+0]}{64}\right]$$

$$= \frac{1}{4}\left[x^3-\frac{3}{2}x+3x^2+6x-3+6\right]$$

$$\therefore y = \frac{1}{4}\left[x^3+3x^2+\frac{9}{2}x+3\right]$$

$\therefore$ From ① :-

$$\therefore y_p = \frac{x^2}{2}e^{2x}-\frac{1}{8}\sin 2x+\frac{1}{4}\left(x^3+3x^2+\frac{9}{2}x+3\right)$$

Hence, general sol^n :-

$$y = C.F + P.I. = y_c + y_p$$

$$\therefore y = (C_1+C_2x)e^{2x}+\frac{x^2}{2}e^{2x}-\frac{1}{8}\sin 2x+\frac{1}{4}\left(x^3+3x^2+\frac{9}{2}x+3\right)$$

$$\Rightarrow \boxed{y = (c_1+c_2x)e^{2x} + \frac{x^2}{2}e^{2x} - \frac{1}{8}\sin 2x + \frac{1}{4}\left(x^3+3x^2+\frac{9}{2}x+3\right).}$$

7) $y'' - 6y' + 13y = 8e^{3x}\sin 4x + x\sin 2x - x + e^x$

$\Rightarrow$ Given, $y'' - 6y' + 13y = 8e^{3x}\sin 4x + x\sin 2x - x + e^x$

Let $D = d/dx$

For C.F $\Rightarrow$ $y'' - 6y' + 13y = 0$

$\therefore (D^2 - 6D + 13)y = 0$

$\therefore$ The Auxiliary eq. is :-

$m^2 - 6m + 13 = 0.$

$\therefore m = \dfrac{6 \pm \sqrt{36-52}}{2} = \dfrac{6 \pm \sqrt{-16}}{2}$

$\therefore m = \dfrac{6 \pm 4i}{2} = 3 \pm 2i$

$\therefore m = 3 \pm 2i$ (complex & different)

$\therefore$ C.F $= y_c = e^{\alpha x}[c_1\cos\beta x + c_2\sin\beta x]$

Here, $\alpha = 3$, $\beta = 2$

$\therefore \underline{y_c = e^{3x}[c_1\cos 2x + c_2\sin 2x]}$.

For P.I $\Rightarrow$ $y'' - 6y' + 13y = 8e^{3x}\sin 4x + x\sin 2x - x + e^x$

i.e. $(D^2 - 6D + 13)y = 8e^{3x}\sin 4x + x\sin 2x - x + e^x$

$\therefore y_p = \dfrac{1}{(D^2-6D+13)}(8e^{3x}\sin 4x + x\sin 2x - x + e^x).$

$\therefore y_p = \dfrac{1}{(D^2-6D+13)}8e^{3x}\sin 4x + \dfrac{1}{(D^2-6D+13)}x\sin 2x - \dfrac{1}{(D^2-6D+13)}x + \dfrac{1}{(D^2-6D+13)}e^x$

$\therefore$ Let $y_p = Y_1 + Y_2 - Y_3 + Y_4$

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$$Y_1 = \frac{1}{(D^2-6D+13)} 8e^{3x}\sin 4x$$

$$= 8 \cdot \frac{1}{(D^2-6D+13)} e^{3x} \sin 4x$$

$$D \to D+3$$

$$= 8e^{3x} \frac{1}{[(D+3)^2-6(D+3)+13]} \sin 4x$$

$$= 8e^{3x} \cdot \frac{1}{[D^2+9+6D-6D-18+13]} \sin 4x$$

$$= 8e^{3x} \cdot \frac{1}{(D^2+4)} \sin 4x.$$

$$D^2 \to -16$$

$$= 8e^{3x} \cdot \frac{1}{(-16+4)} \sin 4x$$

$$= 8e^{3x} \cdot \frac{1}{(-12)} \sin 4x$$

$$\therefore Y_1 = -\frac{2}{3} e^{3x} \cdot \sin 4x \quad \text{-- (1)}$$

$$Y_2 = \frac{1}{(D^2-6D+13)} x \sin 2x.$$

$$\therefore y_2 = x \cdot \frac{1}{f(D)} \sin 2x - \frac{f'(D)}{\{f(D)\}^2} \sin 2x$$

$$= x \cdot \frac{1}{(D^2-6D+13)} \sin 2x - \frac{(2D-6)}{(D^2-6D+13)^2} \sin 2x$$

$$D^2 \to -4 \qquad D^2 \to -4$$

$$= x \cdot \frac{1}{(-4-6D+13)} \sin 2x - \frac{(2D-6)}{(-4-6D+13)^2} \sin 2x$$

$$= x \cdot \frac{1}{(9-6D)} \sin 2x - \frac{(2D-6)}{(9-6D)^2} \sin 2x.$$

$$= x \cdot \frac{1}{(9-6D)} \cdot \frac{(9+6D)}{(9+6D)} \sin 2x - \frac{(2D-6)}{[81+36D^2-108D]} \sin 2x$$

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$$= x \cdot \frac{(9+6D)}{(81-36D^2)} \sin 2x - \frac{(2D-6)}{(36D^2-108D+81)} \sin 2x$$

$D^2 \to -4$ $\qquad$ $D^2 \to -4$

$$= x \cdot \frac{(9+6D)}{[81-36(-4)]} \sin 2x - \frac{(2D-6)}{[36(-4)-108D+81]} \sin 2x$$

$$= x \cdot \frac{(9+6D)}{(81+144)} \sin 2x - \frac{(2D-6)}{[-144-108D+81]} \sin 2x$$

$$= x \cdot \frac{(9+6D)}{225} \sin 2x - \frac{(2D-6)}{(-108D-63)} \sin 2x$$

$$= x \cdot \frac{3(3+2D)}{225} \sin 2x - \frac{2(D-3)}{(-3)(36D+21)} \sin 2x.$$

$$= x \cdot \frac{(3+2D)}{75} \sin 2x + \frac{2(D-3)}{9(12D+7)} \sin 2x.$$

$$= \frac{x}{75} \cdot (3 \sin 2x + 2D \sin 2x)$$

$$+ \frac{2}{9} \cdot \frac{(D-3)(12D-7)}{(12D+7)(12D-7)} \sin 2x$$

$$= \frac{x}{75} \left(3 \sin 2x + 2 \frac{d}{dx} \sin 2x \, dx\right)$$

$$+ \frac{2}{9} \frac{(12D^2-7D-36D+21)}{(144D^2-49)} \sin 2x$$

$$= \frac{x}{75} (3 \sin 2x + 2 \cdot 2 \cos 2x)$$

$$+ \frac{2}{9} \frac{(12D^2-43D+21)}{(144D^2-49)} \sin 2x$$

$D^2 \to -4$

$$= \frac{x}{75} (3 \sin 2x + 4 \cos 2x) + \frac{2}{9} \frac{(-48-43D+21)}{(-576-49)} \sin 2x$$

$$= \frac{x}{75}(3\sin 2x + 4\cos 2x) + \frac{2}{9}\frac{(-43D-27)}{(-625)}\sin 2x.$$

$$= \frac{x}{75}(3\sin 2x + 4\cos 2x) + \frac{2}{9(625)}(43D+27)\sin 2x$$

$$= \frac{x}{75}(3\sin 2x + 4\cos 2x) + \frac{2}{5625}(43D\sin 2x + 27\sin 2x).$$

$$= \frac{x}{75}(3\sin 2x + 4\cos 2x) + \frac{2}{5625}\left(43\frac{d}{dx}\sin 2x dx + 27\sin 2x\right)$$

$$\therefore y_2 = \frac{x}{75}(3\sin 2x + 4\cos 2x) + \frac{2}{5625}(86\cos 2x + 27\sin 2x) \quad \text{-- (2)}$$

$$\odot\ y_3 = \frac{1}{(D^2-6D+13)}x.$$

$$= \frac{1}{13\left[1+\left(\frac{D^2-6D}{13}\right)\right]}x.$$

$$= \frac{1}{13}\left[1-\left(\frac{D^2-6D}{13}\right) + \left(\frac{D^2-6D}{13}\right)^2 - \ldots\right]x.$$

$$\left\{\because \frac{1}{1+\phi(x)} = 1-\phi(x) + [\phi(x)]^2 - [\phi(x)]^3 + \ldots\right\}$$

$$= \frac{1}{13}\left[1-\frac{D^2}{13} + \frac{6D}{13} + \frac{(D^4+36D^2-12D^3)}{169} - \ldots\right]x$$

$$= \frac{1}{13}\left[x - \frac{1}{13}D^2x + \frac{6}{13}Dx + \frac{(D^4x+36D^2x-12D^3x)}{169} - \ldots\right]$$

$$= \frac{1}{13}\left[x - 0 + \frac{6}{13} + 0 + 0 \ldots\right]$$

$$\therefore y_3 = \frac{1}{13}\left[x+\frac{6}{13}\right] = \frac{13x+6}{169} \quad \text{-- (3)}$$

$$\odot\ y_4 = \frac{1}{(D^2-6D+13)}e^x$$

$$D \to 1$$

$$= \frac{1}{[(1)^2 - 6 + 13]} e^x$$

$$\therefore y_4 = \frac{1}{8} e^x \quad \text{---} (4)$$

Now, $y_p = y_1 + y_2 + y_3 + y_4$

$\therefore$ From (1), (2), (3) & (4); we get :-

$$\therefore y_p = -\frac{2}{3} e^{3x} \sin 4x + \frac{x}{75} (3 \sin 2x + 4 \cos 2x) + \frac{2}{5625} (27 \sin 2x + 86 \cos 2x) - \frac{(13x+6)}{169} + \frac{e^x}{8}$$

$\therefore$ Hence, general sol^n :-

$$y = C.F. + P.I. = y_c + y_p$$

$$\therefore y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x) - \frac{2}{3} e^{3x} \sin 4x + \frac{x}{75} (3 \sin 2x + 4 \cos 2x) + \frac{2}{5625} (27 \sin 2x + 86 \cos 2x) - \frac{(13x+6)}{169} + \frac{e^x}{8}$$

$$\Rightarrow \boxed{y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x) - \frac{2}{3} e^{3x} \sin 4x + \frac{x}{75} (3 \sin 2x + 4 \cos 2x) + \frac{2}{5625} (27 \sin 2x + 86 \cos 2x) - \frac{(13x+6)}{169} + \frac{e^x}{8}}$$

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