PARTIAL DERIVATIVE

→ The derivative of a function w.r.t. more than one independent variables.

→ $\lim_{x \to a} f(x) = l$

for $\varepsilon > 0$ $\exists$ $\delta > 0$ such that

$|x-a| < \delta \Rightarrow |f(x) - l| < \varepsilon$

→ $\lim_{(x,y) \to (a,b)} f(x,y) = l$

for $\varepsilon > 0$ $\exists$ $\delta > 0$ $\ni$ $|(x,y)-(a,b)| < \delta \Rightarrow |f(x,y) - l| < \varepsilon$

Ex: Find the limit of the function.

1) $\lim_{(x,y) \to (3,0)} \frac{3x+y^2}{x+y^3}$

$$\frac{3(3)+(0)^2}{3+(0)^3} = \frac{9}{3} = 3$$

2) $\lim_{(x,y) \to (\frac{\pi}{2},0)} \frac{\cos y + 1}{y - \sin x}$

$$= \frac{\cos(0)+1}{0 - \sin\frac{\pi}{2}} = \frac{1+1}{0-1}$$

$$= -2$$

* Path method.

Path 1 : $x \to 0$ then $y \to 0$

Path 2 : $y \to 0$ then $x \to 0$

Path 3 : $y = mx$ and $x \to 0$

Path 4 : $y = mx^2$ and $x \to 0$

- Limit of the ft^n exists Iff all cases (Paths) have the same limit. Otherwise limit does not exist.

Ex: 1) $\lim_{(x,y)\to(0,0)} \frac{2x^2y^2}{x^4+y^4}$

Path method

Path 1 : $x \to 0$ then $y \to 0$

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

$$\lim_{y\to 0}\left[\lim_{x\to 0} \frac{2x^2y^2}{x^4+y^4}\right]$$

$$\lim_{y\to 0}(0) = 0$$

Path 2: $y \to 0$ then $x \to 0$

$$\lim_{x \to 0} \left[\lim_{y \to 0} \frac{2x^2y^2}{x^4+y^4} \right] = 0$$

Path 3: $y = mx$, $x \to 0$

$$\lim_{x \to 0} \frac{2x^2(mx)^2}{x^4+(mx)^4}$$

$$\lim_{x \to 0} \frac{2m^2x^4}{x^4(1+m^4)}$$

$$\lim_{x \to 0} \frac{2m^2}{1+m^4}$$

Value of is dependend on 'm'

$\therefore$ limit of $f(x,y)$ doesn't exist.

Ex:2 Evaluate or check the limit.

$$\lim_{(x,y) \to (0,0)} \frac{x^2y}{x^4+y^2}$$

Path 1: $x \to 0$ then $y \to 0$

$$\lim_{(x,y) \to (0,0)} f(x,y)$$

$$\lim_{y \to 0} \left[\lim_{x \to 0} \frac{x^2y}{x^4+y^2} \right]$$

$$\lim_{y \to 0} (0) = 0$$

Path 2: $y \to 0$ then $x \to 0$

$$\lim_{x \to 0}\left[\lim_{y \to 0} \frac{x^2 y}{y^2+x^4}\right]$$

$= 0$

Path 3: $y = mx$, $x \to 0$

$$\lim_{x \to 0} \frac{x^2 y}{y^2+x^4}$$

$$\lim_{x \to 0} \frac{x^2(mx)}{(mx)^2+x^4}$$

$$\lim_{x \to 0} \frac{xm}{m^2+x^2} = \underline{\underline{0}}$$

<u>Ex:3</u> $\lim_{(x,y) \to (0,0)} \frac{2x^2 y}{x^4+y^2}$

Path 1 : $x \to 0$ then $y \to 0$

$= 0$

Path 2: $y \to 0$ then $x \to 0$

$= 0$

Path 3 : $y = mx$, $x \to 0$

$$\lim_{x \to 0} \frac{2x^2(mx)}{x^4+(mx)^2}$$

$$\lim_{x \to 0} \frac{2m x^3}{x^2(x^2+m^2)} = \underline{\underline{0}}$$

Path 4: $y = mx^2,\ x \to 0$

$$\lim_{x \to 0} \frac{2x^2(mx^2)}{x^4 + (mx^2)^2}$$

$$\lim_{x \to 0} \frac{2mx^4}{x^4(1+m^2)}$$

$$= \frac{2m}{1+m^2}$$

Ex:4 find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$ using definition

by def^n.

for any $\varepsilon > 0\ \exists\ \delta > 0$ such that

$|(x,y) - (0,0)| < \delta \Rightarrow \left| \frac{4xy^2}{x^2+y^2} - 0 \right| < \varepsilon$

$\sqrt{x^2+y^2} < \delta$

$$\left| \frac{4xy^2}{x^2+y^2} \right| = \frac{4|x|y^2}{x^2+y^2} < \varepsilon \quad \text{—— (1)}$$

$$0 < \sqrt{x^2+y^2} < \delta$$

$$y^2 \le x^2 + y^2$$

$$\frac{y^2}{x^2+y^2} \le 1$$

P.T.O.

$$\frac{4|x|y^2}{x^2+y^2} \leq 4|x|$$

$$= 4\sqrt{x^2}$$

$$\leq 4\sqrt{x^2+y^2} \quad (\because \sqrt{x^2} \leq \sqrt{x^2+y^2})$$

$$\frac{4|x|y^2}{x^2+y^2} < 4\delta \text{ —— } (2)$$

from (1) & (2)

$$\varepsilon = 4\delta \Rightarrow \delta = \varepsilon/4$$

$$\therefore \exists\, \delta = \varepsilon/4 > 0 \ni \lim_{(x,y)\to(0,0)} \frac{4x y^2}{x^2+y^2} = 0$$

* For continuity:

1) $f(a,b)$ exist.

2) $\lim_{(x,y)\to(a,b)} f(x,y)$ exist.

3) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

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* Partial Derivatives.

(i) First ordered P.D. of two variables.

Let $f(x,y)$ be the ft^n of two variables x and y then the partial derivative of first order is defined by

$$\frac{\partial f}{\partial x} = f_x$$

$$\frac{\partial f}{\partial y} = f_y$$

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(ii) Second ordered partial derivative of two variables.
The partial derivative of $f(x,y)$ of second order is defined by:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Ex: $f(x,y) = x^2y + 2xy + y^2$

$$f_x = \frac{\partial f}{\partial x} = 2xy + 2y + 0$$

$$f_y = \frac{\partial f}{\partial y} = x^2 + 2x + 2y$$

$$f_{xx} = \frac{\partial^2 f}{\partial x \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= 2y$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= 2$$

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$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= 2x + 2$$

$$f_{yy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= 2x + 2$$

Ex:2 If $u = x^2 y + y^2 z + z^2 x$

Find the value of $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

$$\frac{\partial u}{\partial x} = 2xy + 0 + z^2$$

$$\frac{\partial u}{\partial y} = x^2 + 2yz + 0$$

$$\frac{\partial u}{\partial z} = 0 + y^2 + 2zx$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 2xy + z^2 + x^2 + 2zy + y^2 + 2zx$$

$$= 2(xy + yz + zx) + (x^2 + y^2 + z^2)$$

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Ex:3 If $u = x^3y + e^{xy^2}$ then prove that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[x^3 + e^{xy^2}(2xy)\right]$$

$$= 3x^2 + e^{xy^2}(2y) + (2xy)(e^{xy^2}) \cdot (y^2)$$

$$= 3x^2 + 2ye^{xy^2} + 2xy^3 \cdot e^{xy^2} \text{ --- } (1)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(3x^2y + e^{xy^2} \cdot (y^2)\right)$$

$$= 3x^2 + e^{xy^2} \cdot 2y + y^2(e^{xy^2}) \cdot (2xy)$$

$$= 3x^2 + 2ye^{xy^2} + 2xy^3e^{xy^2} \text{ --- } (2)$$

from (1) & (2)

LHS = RHS / Thus proved.

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Q. If $u = \tan(y+ax) + (y-ax)^{3/2}$

Show that
$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial y^2}$$

→ $$\frac{\partial u}{\partial x} = \sec^2(y+ax)\cdot(0+a) + \frac{3}{2}(y-ax)^{1/2}\cdot(0-a)$$

$$\frac{\partial^2 u}{\partial x^2} = 2\sec(y+ax)\cdot\sec(y+ax)\cdot\tan(y+ax)\cdot a\cdot a$$
$$+\frac{3}{2}\left[\frac{1}{2}(y-ax)^{-1/2}\cdot(0-a)(-a)\right]$$

$$= a^2\left[2\sec(y+ax)\cdot\sec(y+ax)\cdot\tan(y+ax)\right] + \frac{3}{2}\left[\frac{a^2}{2}(y-ax)^{-1/2}\right] \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial y} = \sec^2(y+ax) + \frac{3}{2}(y-ax)^{1/2}$$

$$\frac{\partial^2 u}{\partial y^2} = 2\sec(y+ax)\cdot\sec(y+ax)\cdot\tan(y+ax) + \frac{3}{2}\left[\frac{1}{2}(y-ax)^{-1/2}\right]$$

$$\therefore a^2\cdot\frac{\partial^2 u}{\partial y^2} = 2a^2\sec(y+ax)\cdot\sec(y+ax)\cdot\tan(y+ax)$$
$$+\frac{3}{2}\left[\frac{a^2}{2}(y-ax)^{-1/2}\right] \quad \text{—— (2)}$$

Now as LHS. = RHS. $\therefore$ Thus proved

Ex 5. Prove that $u(x,y) = 2e^x \sin y$ is a solution of laplace eq^n.
i.e. $u_{xx} + u_{yy} = 0$

$$u_{xx} = \frac{\partial^2 u}{\partial x \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(2e^x \sin y\right)$$

$$= 2e^x \sin y \quad \text{— (1)}$$

$$u_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(2e^x \cdot (\cos y)\right)$$

$$= 2e^x \cdot (-\sin y) \quad \text{— (2)}$$

from (1) & (2)

$$u_{xx} + u_{yy} = 2e^x \sin y + (-\sin y \cdot 2e^x)$$

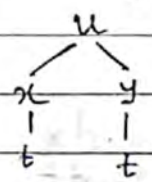
$$= 0$$ | Thus proved.

16/8/19 Friday

Chain Rule

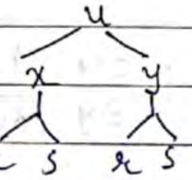
$u = f(x, y)$
$x = x(t), \quad y = y(t)$

Type 1:



$$\Rightarrow \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

Type 2:



$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial u}{\partial x} \cdot \frac{\partial y}{\partial s}$$

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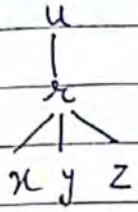
Type 3:

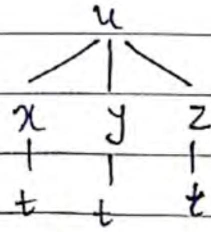


$$\frac{\partial u}{\partial x} = \frac{du}{dr} \cdot \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{du}{dr} \cdot \frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{du}{dr} \cdot \frac{\partial r}{\partial z}$$

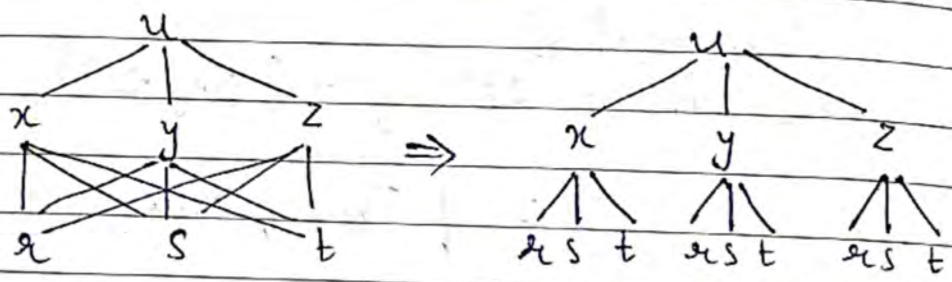
Type 4:



$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

$$\frac{d\cancel{x}}{dt} = /\partial u$$

Type 5:



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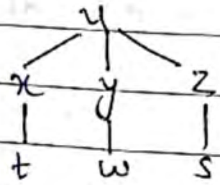

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt}$$

$$\frac{du}{dw} = \frac{\partial u}{\partial y} \cdot \frac{dy}{dw}$$

$$\frac{du}{ds} = \frac{\partial u}{\partial z} \cdot \frac{dz}{ds}$$

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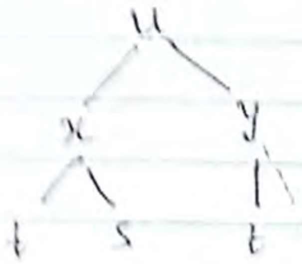
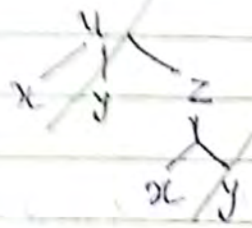
Ex: 

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \quad \frac{\partial y}{\partial t}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \boxed{\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}}$$

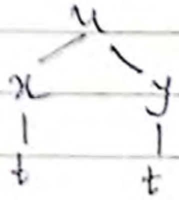
→ 0
has no value
can be considered or not

* Implicite fn

Ex: 

Examples:

1) If $u = \sin^{-1}(x-y)$
$x = 3t, \quad y = 4t^3$, ~~show~~ find ~~that~~ $\frac{du}{dt} = \cancel{\frac{3}{\sqrt{1-t^2}}}$

Sol. → 

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}.$$

$$\therefore \frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}}\cdot 3 + \frac{1}{\sqrt{1-(x-y)^2}}\cdot(-1)\cdot 12t^2$$

$$= \frac{1}{\sqrt{1-(x-y^2)}}\left(3 - 12t^2\right)$$

$x^2 - 2xy + y^2$
$= (3t)^2 - 2(3t)(4t^3) + (4t^3)^2$
$9t^2 - 24t^4 + 16t^6$
$t^2(9 - 24t^2 + 16t^3)$

2) If $u = x^2 e^y$

$x = \sin t,\ y = t^3$

Find $\frac{du}{dt}$

Sol^n

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

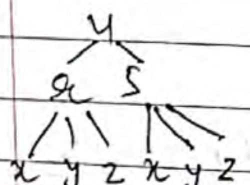
$$= 2xe^y \cdot \cos t + x^2 e^y \cdot 3t^2$$

$$= xe^y(2\cos t + 3xt^2)$$

3) If $u = f(x^2 + 2yz,\ y^2 + 2zx)$

P.T. $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$

Sol^n →



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot(2x) + \frac{\partial u}{\partial s}\cdot(2z)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$= \frac{\partial u}{\partial r}\cdot(2z) + \frac{\partial u}{\partial s}\cdot(2y)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}$$

$$= \frac{\partial u}{\partial r}\cdot(2y) + \frac{\partial u}{\partial s}\cdot(2x)$$

$$(y^2 - zx)\left[\frac{\partial u}{\partial r}\cdot(2x) + \frac{\partial u}{\partial s}(2z)\right] + (x^2 - yz)\left[\frac{\partial u}{\partial r}(2z) + \frac{\partial u}{\partial s}(2y)\right]$$

$$+ (z^2 - xy)\left[\frac{\partial u}{\partial r}\cdot(2y) + \frac{\partial u}{\partial s}\cdot(2x)\right]$$

$$= \frac{\partial u}{\partial r} 2xy^2 + \frac{\partial u}{\partial s} 2zy^2 - \frac{\partial u}{\partial r} 2x^2 z - \frac{\partial u}{\partial s} 2z^2 x + \frac{\partial u}{\partial s}(2zx^2)$$

$$+ \frac{\partial u}{\partial r} 2y(x^2) - \frac{\partial u}{\partial r} 2z^2 y - \frac{\partial u}{\partial s} 2y^2 z + \frac{\partial u}{\partial r} 2yz^2$$

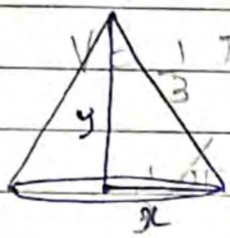
$$+ \frac{\partial u}{\partial s} 2xz^2 - \frac{\partial u}{\partial r} 2y^2 x - \frac{\partial u}{\partial s} 2x^2 y$$

$$= 0$$

4) The altitude of a right circular cone is 15 cm and is increasing at 0.4 cm/s. The radius of the base is 10 cm and is decreasing at 0.6 cm/s. Find rate of change of volume.

$r = 10,\ h = 15$

Sol^n



$$V = \frac{1}{3}\pi r^2 h \qquad \frac{dr}{dt} = (-0.6),\ \frac{dh}{dt} = (0.4)$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\, 2r \cdot \frac{dr}{dt} \cdot h + \frac{1}{3}\pi r^2 \cdot \frac{dh}{dt}$$

$$= \frac{1}{3}\pi\left(2(rh)\cdot\frac{dr}{dt} + r^2 \cdot \frac{dh}{dt}\right)$$

$$= \frac{1}{3}\pi\left(2\times10\times15\times(-0.6) + (10)^2 \cdot 0.4\right)$$

$$= \frac{1}{3}\pi\left(\frac{2\times10\times15\times(-6)}{10} + \frac{100\times4}{10}\right)$$

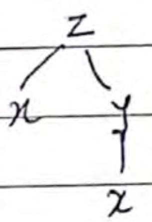
$$= \frac{1}{3}\pi\left(-180 + 40\right)$$

$$= \frac{-140\pi}{3}$$

* Implicit function:

→ Let $f(x,y)=0$ represent the implicit ft^n. suppose $z=f(x,y)$ & $y=g(x)$ then $\frac{dy}{dx}=\frac{-f_x}{f_y}$ gives the differentiation of implicit ft^n.

{ only if $z=0$ in RHS



$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$$

$$0=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{-\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}=\frac{-z_x}{z_y}=\frac{-f_x}{f_y}$$

eg $\frac{dy}{dz}=\frac{-f_z}{f_y}$

1) Find $\frac{dy}{dx}$ if $x^3+y^3=6xy$.

$x^3+y^3-6xy=0$

$$\frac{dy}{dx}=\frac{-f_x}{f_y}$$

$\frac{-3x^2+6y}{3y^2-6x}=0$

$3y^2-6x$