Application of Partial Derivatives:

* Maxima and Minima : (For two variable)

(i) Local Maximum : Let $z = f(x,y)$ be a continuous function of two independent variables x and y. then,
$f(x,y)$ is said to have a local Maximum at a pt. (a,b), if for all points close to (a,b)

$$f(a+h, b+k) \leq f(a,b)$$

where, h and k are any constant.

(ii) Local Minimum :

$f(x,y)$ is said to have a local Mini. at a pt. (a,b) if,

$$f(a+h, b+k) \geq f(a,b)$$

(iii) Absolute Maximum : The absolute maximum and minimum values of $f(x,y)$ indicates the overall largest and smallest values of f in the D.

(iv) Stationary Points : The points (x,y) of D. for which $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ are called stationary pts. of $f(x,y)$.

(v) Critical Points : The points (x,y) of D for which either one or both $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ does not exist is called critical points.

(vi) Saddle Point : It is a stationary point where the function is neither maximum nor minimum.

⇒ Thm : (The Second Derivative Test for Local Extreme Values):
Suppose that $f(x,y)$ and its first and second-order partial derivatives are continuous through a disk centered at (a,b) such that

$$f_x(a,b) = f_y(a,b) = 0, \text{ then}$$

(i) f has local Maxi at (a,b) if

$$rt - s^2 > 0 \quad \text{and} \quad r < 0$$

(ii) f has local Min. at (a,b) if

$$rt - s^2 > 0 \quad \text{and} \quad r > 0$$

(iii) F has saddle point at (a,b) if
$$rt - s^2 < 0$$

(iv) The test is inconclusive at (a,b) if $rt - s^2 = 0$

Where,
$$r = \frac{\partial^2 f}{\partial x^2}, \quad s = \frac{\partial^2 f}{\partial x \partial y}, \quad t = \frac{\partial^2 f}{\partial y^2}$$

EX-1 Find Maxima and Minima of
$f(x, y) = x^3 + y^3 - 3x - 12y + 20$

Soln

To find Stationary Points.

We have,

$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 - 3 = 0$

$\Rightarrow (x^2 - 1) = 0$

$\Rightarrow x = \pm 1, -1$

&

$\frac{\partial f}{\partial y} = 0 \Rightarrow 3y^2 - 12 = 0$

$\Rightarrow y^2 - 4 = 0$

$\Rightarrow y = 2, -2$

$\therefore$ The Stationary Points are,

$(1, 2), (1, -2), (-1, 2), (-1, -2)$

Now, $r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$

$= \frac{\partial}{\partial x}(3x^2 - 3) = 6x$

$\therefore r = 6x$

$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$= \frac{\partial}{\partial x}(3y^2 - 12)$

$s = 0$

& $t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$

$= \frac{\partial}{\partial y}(3y^2 - 12)$

$t = 6y$

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$\therefore \quad rt - s^2 = (6x)(6y) - (0)$

$$rt - s^2 = 36xy$$

Now,

Points	$rt - s^2$ $(36xy)$	r $6x$	Conclusion
$(1, 2)$	$72 > 0$	$6 > 0$	local min.
$(1, -2)$	$-72 < 0$	—	Saddle pt.
$(-1, 2)$	$-72 < 0$	—	Saddle pt
$(-1, -2)$	$72 > 0$	$-6 < 0$	local Max.

$\therefore$ f has local Min at pt $(1, 2)$ and has local Max. at $(-1, -2)$

$$\therefore \quad f_{min} = f(1, 2) = 2$$

$$f_{max} = f(-1, -2) = 38$$

Ex-2 Find the extreme values of
$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.

Solⁿ: To find stationary points.

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 + 3y^2 - 6x = 0 \quad \text{—(i)}$$

& $$\frac{\partial f}{\partial y} = 0 \Rightarrow 6xy - 6y = 0 \quad \text{—(ii)}$$

from (ii) we get

$$6y(x-1) = 0 \Rightarrow y = 0 \ \& \ x = 1$$

∴ from eqⁿ (i), if $y = 0$ we get.

$$3x^2 - 6x = 0$$
$$\cancel{3}x(x-2) = 0 \Rightarrow x = 0, x = 2$$

∴ $(0,0)$ & $(2,0)$

also from eqⁿ (i), if $x = 1$ we get

$$3 + 3y^2 - 6 = 0 \Rightarrow 3y^2 - 3 = 0$$
$$\Rightarrow y = 1, -1$$

∴ pts. are $(1,1)$, $(1,-1)$

∴ All stationary pts are

$(0,0)$, $(2,0)$, $(1,1)$ $(1,-1)$

Now

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2 + 3y^2 - 6x\right)$$

$$= 6x - 6$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6y$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6x - 6$$

$$rt - s^2 = (6x-6)(6x-6) - (6y)^2$$

$$= 36(x-1)^2 - 36y^2$$

Points	$rt - s^2$	r	Conclusion
$(1,-1)$	$-36<0$	$-$	Saddle pt
$(2,0)$	$36>0$	$6>0$	local min
$(1,1)$	$-36<0$	$-$	Saddle pt
$(0,0)$	$36>0$	$-6<0$	local max

$\therefore$ f has local min at $(2,0)$ and local max. at $(0,0)$

$\therefore f_{max} = f(0,0) = 4$

$f_{min} = f(2,0) = 0$

Ex-3 Find the Stationary points of the function $f(x,y) = x^2y - xy^2 + 4xy - 4x^2 - 4y^2$ Also, discuss about their nature.

Ex-4 Find the Points of maxima and minima of $f(x,y) = x + y + \frac{1}{x} + \frac{1}{y}$. Also find the maximum and minimum values of the function.

Ex:1 Find the maximum values of $x^2y^3z^4$, given that $2x+3y+4z=a$ using Lagrange's method.

Solⁿ: consider the Lagrange's function.

$$F(x,y,z) = f(x,y,z) + \lambda(g(x,y,z))$$

$$= (x^2y^3z^4) + \lambda(2x+3y+4z-a)$$

To find stationary pts. we get

$dF = 0$

$$\Rightarrow \frac{\partial f}{\partial x} = 0 \Rightarrow 2xy^3z^4 + 2\lambda = 0 \quad \text{—(i)}$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 3x^2y^2z^4 + 3\lambda = 0 \quad \text{—(ii)}$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 4x^2y^3z^3 + 4\lambda = 0 \quad \text{—(iii)}$$

~~Multiply (i) by~~

from (i) $\lambda = -\frac{2xy^3z^4}{2}$

from (ii) $\lambda = -\frac{3x^2y^2z^4}{3}$

from (iii) $\lambda = -\frac{4x^2y^3z^3}{4}$

$\therefore +xy^3\cancel{z^4} = +x^2y^2\cancel{z^4}$ | $-x^2y^2z^4 = -x^2y^3z^3$

$\Rightarrow y = x$ | $z = y$

$\therefore x = y = z$

sub. value in $g(x,y,z)$ a

$2x + 3y + 4z = a$

$2y + 3y + 4y = a$

$\Rightarrow 9y = a$

$\Rightarrow \boxed{y = a/9}$

$\Rightarrow \boxed{x = a/9}$ & $\boxed{z = a/9}$

Thus, Maximum value of $f(x,y,z)$ is $f_{max} = f(a/9, a/9, a/9)$

$= (a/9)^2 (a/9)^3 (a/9)^4$

$= (a/9)^9$

Ex-2 Find the Minimum and Maximum distances from origin to the curve $3x^2 + 4xy + 6y^2 = 140$

Sol^n: Distance from origin to (x, y) pt. is

$$f = d^2 = (x-0)^2 + (y-0)^2.$$

$$\Rightarrow f = d^2 = x^2 + y^2 \quad \text{and}$$

$$g = 3x^2 + 4xy + 6y^2 - 140$$

using Lagrange's multiplier method

$$F = f + \lambda g$$

$$= (x^2 + y^2) + \lambda(3x^2 + 4xy + 6y^2 - 140)$$

Now $dF = 0$

$$\Rightarrow \frac{\partial F}{\partial x} = 0 \Rightarrow 2x + \lambda(6x + 4y) = 0$$

$$\Rightarrow \lambda = \frac{-2x}{6x + 4y} \quad \text{--- (i)}$$

$$\& \quad \frac{\partial F}{\partial y} = 0 \Rightarrow 2y + \lambda(4x + 12y) = 0$$

$$\Rightarrow \lambda = \frac{-2y}{4x + 12y} \quad \text{--- (ii)}$$

$\therefore$ From (i) & (ii)

$$\frac{2x}{6x + 4y} = \frac{2y}{4x + 12y}$$

$$\frac{2x}{6x+4y} = \frac{2y}{4x+12y}$$

$$\Rightarrow 2x(4x+12y) - 2y(6x+4y) = 0$$

$$\Rightarrow 8x^2 + 24xy - 12xy - 8y^2 = 0$$

$$\Rightarrow 8x^2 + 12xy - 8y^2 = 0$$

$$\Rightarrow 2x^2 + 3xy - 2y^2 = 0$$

Ex 3 Find the numbers x, y, z such that $xyz = 8$ and $xy + yz + zx$ is maximum using the Lagrange's method of undetermined multipliers.

Sol^n: Here, $f(x,y,z) = xy + yz + zx$
& $g(x,y,z) = xyz - 8$

$\therefore$ By Lagrange's multiplier

$$F(x,y,z) = f(x,y,z) + \lambda(g(x,y,z))$$
$$= (xy + yz + zx) + \lambda(xyz - 8)$$

$\therefore \frac{\partial F}{\partial x} = 0 \Rightarrow (y+z) + \lambda yz = 0$

$$\Rightarrow \lambda = -\frac{y+z}{yz} \quad \text{——— (i)}$$

$\frac{\partial F}{\partial y} = 0 \Rightarrow (x+z) + \lambda(xz) = 0$

$$\Rightarrow \lambda = -\frac{x+z}{xz} \quad \text{——— (ii)}$$

& $\frac{\partial F}{\partial z} = 0 \Rightarrow (x+y) + \lambda(xy) = 0$

$$\Rightarrow \lambda = -\frac{x+y}{xy} \quad \text{——— (iii)}$$

from (i) & (ii)

$$-\frac{y+z}{y\cancel{z}} = -\frac{x+z}{x\cancel{z}} \Rightarrow x\cancel{y} + xz = \cancel{xy} + z\ldots$$

$$\Rightarrow \boxed{x = y}$$

From (ii) & (iii)

$$\frac{x+z}{\cancel{x}z} = \frac{x+y}{\cancel{x}y}$$

$$\Rightarrow xy + \cancel{yz} = xz + \cancel{yz}$$

$$\Rightarrow \boxed{y = z}$$

Sub. value of x & z in $g(x,y$
we get

$$\cancel{g(x,y,z)} \cdot xyz = 8$$

$$\Rightarrow \cancel{0}' y^3 = 8$$

$$= y = 2$$

$$\Rightarrow x = 2, \quad y = 2 \quad \& \quad z = 2$$

Ex-4 The Pressure P at any point (x, y, z) in Space is $P = 400\, xyz^2$. Find the highest Pressure on the surface of the unit Sphere $x^2 + y^2 + z^2 = 1$.

Soln: Here,

$$f(x, y, z) = 400\, xyz^2$$

$$g(x, y, z) = x^2 + y^2 + z^2 = 1$$

∴ using Lagrange's Multiplier.

$$F(x, y, z) = f(x, y, z) + \lambda\, g(x, y, z)$$
$$= 400\, xyz^2 + \lambda\,(x^2 + y^2 + z^2 - 1)$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 400\, yz^2 + 2x\lambda = 0$$

$$\Rightarrow \lambda = -\frac{400\, yz^2}{2x} \quad \text{---} (i)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 400\, xz^2 + 2y\lambda = 0$$

$$\Rightarrow \lambda = -\frac{400\, xz^2}{2y} \quad \text{---} (ii)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 800\, xyz + 2z\lambda = 0$$

$$\Rightarrow \lambda = -\frac{800\, xyz}{2z} \quad \text{---} (iii)$$

from (i) & (ii) we get

$$-\frac{400\, y\cancel{z^2}}{2x} = -\frac{400\, x\cancel{z^2}}{2y}$$

$$\Rightarrow y^2 = x^2 \Rightarrow x = y$$

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from (ii) & (iii)

$$-\frac{400\cancel{x}z^2}{\cancel{x}y} = -\frac{800\cancel{x}y\cancel{z}}{\cancel{x}\cancel{z}}$$

$\Rightarrow z^2 = 2y^2$

$\Rightarrow z = \pm\sqrt{2}\,y$

$x = y \,, \quad z = \pm\sqrt{2}\,y$

sub. values in $g(x,y,z)$ we get

$x^2+y^2+z^2 = 1$

$\Rightarrow y^2+y^2+2y^2 = 1$

$\Rightarrow 4y^2 = 1$

$\Rightarrow y^2 = 1/4 \Rightarrow \boxed{y = \pm 1/2}$

$\therefore x = \pm 1/2 \,, \quad z = \pm\sqrt{2}\cdot\frac{1}{2}$

$z = \pm\frac{1}{\sqrt{2}}$

$\therefore$ The max. value of P is

$P = 400\,x y z^2$

$= 400\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)^2$

$\boxed{P = 50}$

* Tangent Planes and Normal lines :

The equation of the tangent plane to the surface $f(x,y,z)=0$ at the point $P(x_0,y_0,z_0)$ is given by.

$$(x-x_0)\frac{\partial f}{\partial x_0}+(y-y_0)\frac{\partial f}{\partial y_0}+(z-z_0)\frac{\partial f}{\partial z_0}=0$$

The equation of the normal line to the surface $f(x,y,z)=0$ at the point P is given by,

$$\frac{x-x_0}{\partial f/\partial x_0}=\frac{y-y_0}{\partial f/\partial y_0}=\frac{z-z_0}{\partial f/\partial z_0}$$

Ex-1 Find the equations of the tangent plane and normal line to the surface $x^2+y^2+z^2=1$ at $(2,2,1)$

Sol^n $\cancel{f(x,y,z) = 1 - x^2 - y^2 - z^2 = 0}$

$$F(x,y,z) = x^2+y^2+z^2-1 = 0$$

$$(x_0, y_0, z_0) = (2,2,1)$$

$\cancel{\frac{\partial f}{\partial x}}$

$$\frac{\partial f}{\partial x} = 2x \,, \quad \left.\frac{\partial f}{\partial x}\right|_{(2,2,1)} = 4$$

$$\frac{\partial f}{\partial y} = 2y \,, \quad \left.\frac{\partial f}{\partial y}\right|_{(2,2,1)} = 4$$

$$\frac{\partial f}{\partial z} = 2z \,, \quad \left.\frac{\partial f}{\partial z}\right|_{(2,2,1)} = 2$$

$\therefore$ The equation of tangent line is

$$(x-x_0)\frac{\partial f}{\partial x} + (y-y_0)\frac{\partial f}{\partial y} + (z-z_0)\frac{\partial f}{\partial z} = 0$$

$$\Rightarrow (x-2)(4) + (y-2)(4) + (z-1)(2) = 0$$

$$\Rightarrow 4x + 4y + 2z - 18 = 0$$

$$\Rightarrow 2x + 2y + z = 9$$

And The equation of normal line is

$$\frac{(x-x_0)}{f_x} = \frac{(y-y_0)}{f_y} = \frac{(z-z_0)}{f_z}$$

$$= \frac{x-2}{4} = \frac{y-2}{4} = \frac{z-1}{2}$$

Ex-2 Find the equation of tangent plane and normal line to the surface $\cos \pi x - x^2 y + e^{xz} + yz = 4$ at the point P (0, 1, 2).

Sol^n: $\therefore \; f(x,y,z) = \cos \pi x - x^2 y + e^{xz} + yz - 4 = 0$

$$\frac{\partial f}{\partial x} = -\pi \sin \pi x - 2xy + e^{xz} \cdot (z)$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,1,2)} = 2$$

$$\frac{\partial f}{\partial y} = -x^2 + z$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,1,2)} = 2$$

$$\frac{\partial f}{\partial z} = x e^{xz} + y \, , \quad \left.\frac{\partial f}{\partial z}\right|_{(0,1,2)} = 1$$

The eq^n of Tangent Plane is.

$$2(x-0) + 2(y-1) + 1(z-2) = 0$$

$$\Rightarrow 2x + 2y + z = 4$$

The eq^n of Normal line is

$$\frac{x}{2} = \frac{y-1}{2} = \frac{z-2}{1}$$