Laplace Transform

time

Let $f(t)$ be a function defined for all $t \geq 0$.

then the Laplace transform of $f(t)$ is ~~de~~ denoted by

$$\mathcal{L}\{f(t)\} = \bar{f}(s) = F(s) = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

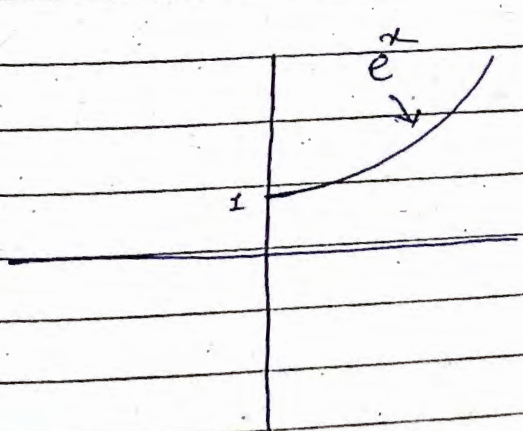
$t \geq 0$

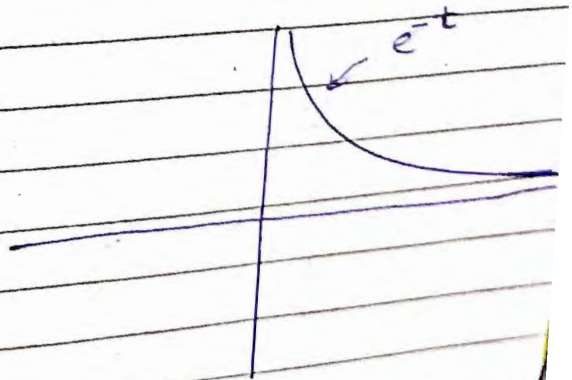
$f(t)$ is a function of time domain.

(For convert the time domain into frequency domain.)

s

frequency Space.





Use → To solve differential eqⁿ.

→ To convert time domain into frequency domain & vice-versa.

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1) $f(t) = k$ $k = \text{const.}$

$$\mathcal{L}\{f(t)\} = \overline{f(s)} = F(s)$$

$$= \int_0^\infty e^{-st} f(t)\, dt.$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} k\, dt$$

$$= \left[\frac{e^{-st}}{-s} k\right]_0^\infty$$

$$= \frac{k}{-s}\left[e^{-st}\right]_0^\infty$$

$$= \frac{k}{-s}\left[0 - 1\right]$$

$$\mathcal{L}\{f(t)\} = \frac{k}{s}$$

$$\mathcal{L}\{k\} = k/s$$

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2) $f(t) = \cos at$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} \cos at \, dt$$

$$\cancel{= \frac{1}{s}\left[e^{-st} \sin \right.}$$

$$= \left[\frac{e^{-st}}{(-s)^2 + a^2} \left(-s(\cos at) + a(\sin at) \right) \right]_0^\infty$$

$$= \left[0 - \frac{1}{(s^2+a^2)}(-s) \right]$$

$$= \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}.$$

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	L. T.
1) $f(t) = k$	$\mathcal{L}\{k\} = k/s$
2) $f(t) = \{e^{kt}\}$	$\mathcal{L}\{e^{kt}\} = \dfrac{1}{s-k}$
3) $f(t) = \cos at$	$\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$
4) $f(t) = \sin at$	$\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$
5) $f(t) = \sinh at$	$\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2-a^2}$
6) $f(t) = \cosh at$	$\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}$
7) $f(t) = t^n$	$\dfrac{\Gamma(n+1)}{s^{n+1}}$; n = non integer $\dfrac{n!}{s^{n+1}}$; n = integer

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3) $f(t) = \sin at$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} \sin at\, dt$$

$$= \left[\frac{e^{-st}}{(-s)^2 + t^2} \left[-s(\sin at) - a(\cos at) \right] \right]_0^\infty$$

$$= \left[0 - \frac{1}{s^2 + a^2} (0 - a) \right]$$

$$\mathcal{L}\{f(t)\} = \frac{+a}{s^2 + a^2}$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$$

~~4) $f(t) = t^n$~~

~~$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$~~

~~$$= \int_0^\infty e^{-st} t^n\, dt$$~~

~~$$= t^n \frac{e^{-st}}{-s} -$$~~

4) $f(t) = e^{kt}$

$$\mathcal{L}\{f(t)\} = \overline{f(s)} = F(s)$$

$$= \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot e^{kt}\,dt$$

$$= \int_0^\infty e^{-t(s-k)}\,dt$$

$$= \left[\frac{-e^{-t(s-k)}}{s-k}\right]_0^\infty$$

$$= \frac{1}{s-k}\left[-e^{-t(s-k)}\right]_0^\infty$$

$$= \frac{-1}{s-k}\left[0 - 1\right]$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s-k}$$

$$\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}$$

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5) $f(t) = \sinh at$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} \cdot \sinh at\, dt$$

$$= \int_0^\infty e^{-st} \cdot \left(\frac{e^{at} - e^{-at}}{2}\right) dt \qquad \left(\because \sinh x = \frac{e^{x} - e^{-x}}{2}\right)$$

$$= \frac{1}{2}\int_0^\infty e^{-st} (e^{at} - e^{-at})\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t(s-a)} - e^{-t(s+a)}\, dt$$

$$= \frac{1}{2}\left\{\left[\frac{-e^{-t(s-a)}}{s-a}\right]_0^\infty - \left[\frac{-e^{-t(s+a)}}{s+a}\right]_0^\infty\right\}$$

$$= \frac{1}{2}\left[0 + \frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{s} + a - \cancel{s} + a}{s^2 - a^2}\right]$$

$$= \frac{a}{s^2 - a^2}$$

$$\mathcal{L}\{\sinh at\} = \frac{a}{s^2 - a^2}$$

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6) $f(t) = \cosh at$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} \cosh at \, dt$$

$$= \int_0^\infty e^{-st} \left(\frac{e^{at} + e^{-at}}{2} \right) dt$$

$$= \frac{1}{2} \int_0^\infty e^{-st} (e^{at} + e^{-at})\, dt$$

$$= \frac{1}{2} \int_0^\infty e^{-t(s-a)}\, dt + \frac{1}{2} \int_0^\infty e^{-t(s+a)}\, dt$$

$$= \frac{1}{2} \left[\frac{-e^{-t(s-a)}}{s-a} \right]_0^\infty + \frac{1}{2} \left[\frac{-e^{-t(s+a)}}{s+a} \right]_0^\infty$$

$$= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[\frac{s + \cancel{a} + s - \cancel{a}}{s^2 - a^2} \right]$$

$$= \frac{s}{s^2 - a^2}$$

$$\mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}$$

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7) $f(t) = t^n$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} t^n \, dt.$$

let $st = x \Rightarrow dt = \frac{1}{s} dx.$

$t = 0 \Rightarrow x = 0, \quad t = \infty \Rightarrow x = \infty$

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \cdot \frac{1}{s} dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{n+1-1} dx$$

Using $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

$$\mathcal{L}\{t^n\} = \frac{\Gamma n+1}{s^{n+1}}$$

Now, if n is a positive integer, then $\Gamma n+1 = n!$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

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1.

I. L. T. (Inverse Laplace transform)

1) $L^{-1}\left\{\frac{k}{s}\right\} = k$

2) $L^{-1}\left\{\frac{1}{s-k}\right\} = e^{kt}$

3) $L^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{\sin kt}{k}$

4) $L^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$

5) $L^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{\sinh kt}{k}$

6) $L^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh kt$

7) $L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

$L^{-1}\left\{\frac{\Gamma(n+1)}{s^{n+1}}\right\} = t^n$

Eg. $L^{-1}\left\{\frac{\Gamma(5/2)}{s^{5/2}}\right\} = t^{3/2}$

$L^{-1}\left\{\frac{7!}{s^8}\right\} = t^7$

1) Find Laplace transform of $(2t-1)^2$

$$L\{(2t-1)^2\} = L(4t^2-4t+1)$$

$$= 4L\{t^2\} - 4L\{t\} + L\{1\}$$

$$= 4\left(\frac{2!}{s^3}\right) - 4\left(\frac{1}{s^2}\right) + \frac{1}{s}$$

$$= \frac{8}{s^3} - \frac{4}{s^2} + \frac{1}{s}$$

$$= \frac{8-4s+s^2}{s^3}$$

2) Find $L\{(t^3+e^{-3t}+t^{3/2})\}$

$$L\{t^3+e^{-3t}+t^{3/2})$$

$$= L\{t^3\} + L\{e^{-3t}\} + L\{t^{3/2}\}$$

$$= \frac{4!}{s^4} + \frac{1}{s+3} + \frac{\Gamma 5/2}{s^{5/2}}$$

$$= \frac{4!}{s^4} + \frac{1}{s+3} + \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{s^{5/2}}$$

$$= \frac{4!}{s^4} + \frac{1}{s+3} + \frac{3\sqrt{\pi}}{4\cdot s^{5/2}}$$

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3) Find $\mathcal{L}(\cos^2 at)$

$$\mathcal{L}(\cos^2 at) = \mathcal{L}\left\{\frac{\cos 2t + 1}{2}\right\}$$

$$= \frac{1}{2}\left[\mathcal{L}\{\cos 2at\} + \mathcal{L}\{1\}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+4a^2} + \frac{1}{s}\right]$$

4) Find $\mathcal{L}\left(\frac{e^{at} - e^{bt}}{2}\right)$.

$$\mathcal{L}\left(\frac{e^{at} - e^{bt}}{2}\right) = \frac{1}{2}\left[\mathcal{L}(e^{at}) - \mathcal{L}(e^{bt})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s-b}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{s} - b - \cancel{s} + a}{(s-a)(s-b)}\right]$$

$$= \frac{a-b}{2(s-a)(s-b)}$$

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5) Find $\mathcal{L}(3t^{-1/2} + e^{4t})$

$$\mathcal{L}(3t^{-1/2} + e^{4t}) = 3\mathcal{L}(t^{-1/2}) + \mathcal{L}(e^{4t})$$

$$= 3\left[\frac{\Gamma\frac{1}{2}}{s^{1/2}}\right] + \frac{1}{s-4}$$

$$= \frac{3\sqrt{\pi}}{s^{1/2}} + \frac{1}{s-4}$$

(6) $$= 3\sqrt{\frac{\pi}{s}} + \frac{1}{s-4}$$

* 4) $$\mathcal{L}\{3^t\} = \mathcal{L}\{e^{\ln 3^t}\}$$

$$= \mathcal{L}\{e^{t \ln 3}\}$$

$$= \frac{1}{s - \ln 3} \; ; \; s > \ln 3.$$

~~5) Find $\mathcal{L}^{-1}($~~

$\mathcal{L}$

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★ Find inverse Laplace transform.

1) $$\mathcal{L}^{-1}\left\{\frac{3s+2}{s^2-9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3s}{s^2-9}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s^2-9}\right\}$$

$$= 3\cosh 3t + \frac{2}{3}\sinh 3t$$

$$= 3\cosh 3t + \frac{2}{3}\sinh 3t$$

2) $$\mathcal{L}^{-1}\left\{\frac{6s}{s^2-16}\right\}$$

$$= 6\,\mathcal{L}^{-1}\left\{\frac{s}{s^2-16}\right\}$$

$$= 6\cosh 4t$$

If $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\{f(bt)\} = \frac{1}{b} F\left(\frac{s}{b}\right)$$

[Change of scale]

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$\mathcal{L}\{f(bt)\} = \int_0^\infty e^{-st} f(bt)\, dt$$

let $bt = y$

$dt = \frac{dy}{b}$

$$= \frac{1}{b} \int_0^\infty e^{-st} \cdot f(y)\, dy$$

$$= \frac{1}{b} F\left(\frac{s}{b}\right)$$

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L 18.

* First shifting theorm
If $\mathcal{L}\{f(t)\} = F(s)$ then,

$$\mathcal{L}\{e^{at} \cdot f(t)\} = F(s-a)$$

* 1^{st} Shifting of L.T.

1) $\mathcal{L}\{e^{at} \cdot k\} = \dfrac{k}{s-a}$

2) $\mathcal{L}\{e^{at} \cdot e^{kt}\} = \dfrac{1}{(s-a)-k}$

3) $\mathcal{L}\{\sin kt \cdot e^{at}\} = \dfrac{k}{(s-a)^2+k^2}$

4) $\mathcal{L}\{e^{at} \cdot \cos kt\} = \dfrac{s-a}{(s-a)^2+k^2}$

5) $\mathcal{L}\{e^{at} \cdot \sinh kt\} = \dfrac{k}{(s-a)^2-k^2}$

6) $\mathcal{L}\{e^{at} \cdot \cosh kt\} = \dfrac{s-a}{(s-a)^2-k^2}$

7) $\mathcal{L}\{t^n \cdot e^{at}\} = \dfrac{\Gamma(n+1)}{(s-a)^{n+1}}$ ($n \neq$ integer)

$= \dfrac{n!}{(s-a)^{n+1}}$ ($n =$ integer)

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* 1^{st} shifting of I.L.T.

1) $$\mathcal{L}^{-1}\left\{\frac{k}{s-a}\right\} = e^{at}\cdot k$$

2) $$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)-k}\right\} = e^{at}\cdot e^{kt}$$

3) $$\mathcal{L}^{-1}\left\{\frac{k}{(s-a)^2+k^2}\right\} = \sin kt\cdot e^{at}$$

4) $$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+k^2}\right\} = \cos kt\cdot e^{at}$$

5) $$\mathcal{L}^{-1}\left\{\frac{k}{(s-a)^2-k^2}\right\} = \sinh kt\cdot e^{at}$$

6) $$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2-k^2}\right\} = \cosh kt\cdot e^{at}$$

7) $$\mathcal{L}^{-1}\left\{\frac{\Gamma(n+1)}{(s-a)^{n+1}}\right\} = t^n\cdot e^{at}$$

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\} = t^n\cdot e^{at}.$$

18

* Find inverse laplace transfer by Partial fraction.

1) $L^{-1}\left\{\dfrac{6+s}{s^2+6s+13}\right\}$

$= L^{-1}\left\{\dfrac{6+s}{s^2+6s+9+4}\right\}$

$= L^{-1}\left\{\dfrac{3+(s+3)}{(s+3)^2+(2)^2}\right\}$

$= L^{-1}\left\{\dfrac{3}{(s+3)^2+(2)^2}+\dfrac{s+3}{(s+3)^2+(2)^2}\right\}$

$= L^{-1}\left\{\dfrac{3}{(s+3)^2+(2)^2}\right\}+L^{-1}\left\{\dfrac{s+3}{(s+3)^2+(2)^2}\right\}$

$= \dfrac{3}{2}\sin 2t \cdot e^{-3t} + \cos 2t \cdot e^{-3t}$

$= e^{-3t}\left[\dfrac{3}{2}\sin 2t + \cos 2t\right]$

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$$\frac{3s^2+2}{(s+1)(s+2)(s+3)}$$

$$= \cancel{\frac{3s+0}{(s+1)}} +$$

$$\frac{3s^2+2}{(s+1)(s+2)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{(s+3)}$$

$$3s^2+2 = A(s^2+5s+6) + B(s^2+4s+3) + C(s^2+3s+$$

$$3s^2+0s+2 = s^2(A+B+C) + s(5A+4B+3C) + (6A+3B+2C)$$

$\therefore A+B+C = 3$ ——— (i)

$\therefore 5A+4B+3C = 0$ ——— (ii)

$\therefore 6A+3B+2C = 2$ ——— (iii)

$4B+3C = 5A$

$\therefore \cancel{4B+3C - 5B - 5C = -15}$

$\therefore \cancel{9B+8C = -15}$ $\quad$ $\cancel{-B-2C = -15}$

$\cancel{4B+3C = -5A}$ $\quad$ $\therefore \cancel{B}+2C = 15$

$\cancel{4B+3C =}$

$\cancel{4A+3B+2C = -3}$

$\cancel{4A}$ $\quad$ $6(ii) - 5(iii)$

$0 = 30A + 24B + 18C$

$10 = 30A + 15B + 10C$

$-10 = 9B + 8C$ ——— (x)

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$6(i) - (iii)$

$$6A + 6B + 6C = 18$$
$$\underline{\cancel{6A} + 3B + 2C = 2}$$
$$3B + 4C = 16. \quad \text{— (Y)}$$

$3(Y) - (X)$

$$9B + 12C = 48$$
$$\underline{9B + 8C = -10}$$
$$4C = 58$$

$$C = \frac{58}{4} = \frac{29}{2}$$

$$\boxed{C = \frac{29}{2}}$$

$$3B + 4\left(\frac{29}{2}\right) = 16$$

$$3B + 58 = 16$$

$$\therefore 3B = 16 - 58$$
$$= -42$$

$$\boxed{B = -14}$$

$$5A + 4(-14) + 3\left(\frac{29}{2}\right) = 0$$

$$\therefore -5A = -56 + \frac{87}{2}$$

$$\therefore -10A = -112 + 87$$

$$\therefore -10A = -25$$

$$\boxed{A = 5/2}$$

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$$\frac{3s^2+2}{(s+1)(s+2)(s+3)} = \frac{5/2}{s+1} + \frac{-14}{s+2} + \frac{29/2}{s+3}$$

$$L^{-1}\left\{\frac{3s^2+2}{(s+1)(s+2)(s+3)}\right\}$$

$$= \frac{5}{2} L^{-1}\left\{\frac{1}{s+1}\right\} - 14 L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{29}{2} L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{5}{2} e^{-t} - 14 e^{-2t} + \frac{29}{2} e^{-3t}$$

3) $\dfrac{2s+3}{(s+2)(s+1)^2}$

$$\frac{2s+3}{(s+2)(s+1)^2} = \frac{A}{(s+2)} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2}$$

$$2s+3 = A(s+1)^2 + B(s+2)(s+1) + C(s+2)$$

$\cancel{2s+3} = \cancel{A(s^2}$ Let $s=-2$

$$2(-2)+3 = A(-1)^2$$

$$\boxed{-1 = A}$$

$$2s+3 = (-s^2 - 2s - 1) + B(s^2+3s+2) + C(s+2)$$

$$2s+3 = s^2(B-1) + s(-2+3B+C) + 1(2B+2C-1)$$

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$$B - 1 = 0$$

$$\therefore \boxed{B = 1}$$

$$2B + 2C - 1 = 3$$

$$\therefore 1 + 2C = 3$$

$$\therefore \boxed{C = 1}$$

$$\mathcal{L}^{-1}\left\{\frac{2s+3}{(s+2)(s+1)^2}\right\} = \cancel{\mathcal{L}^{-1}\left\{\frac{-1}{(s+2}\right.}$$

$$= (-1)\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + 1\,\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + 1\,\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$$= (-1)e^{-2t} + e^{-t} + t \cdot e^{-t}$$

$$\left(\because \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)^2 = t \cdot e^{-t}\right)$$

$$= e^{-t}\left(t + 1 - e^{-t}\right)$$

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Find the laplace of $\sin\sqrt{t}$.

$$\mathcal{L}\{\sin\sqrt{t}\} = \mathcal{L}\{\sin t^{1/2}\}$$

$$f(t) = \sin\sqrt{t}.$$

$$\mathcal{L}\{\sin\sqrt{t}\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} \sin\sqrt{t}\, dt.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin\sqrt{t} = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots$$

$$\mathcal{L}\{\sin\sqrt{t}\} = \frac{\Gamma\frac{3}{2}}{s^{3/2}} - \frac{\Gamma\frac{5}{2}}{3!\, s^{5/2}} + \frac{\Gamma\frac{7}{2}}{5!\, s^{7/2}} - \frac{\Gamma\frac{9}{2}}{7!\, s^{9/2}} \cdots$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{3!} \cdot \frac{\left(\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\right)}{s^{5/2}} + \frac{1}{5!} \cdot \frac{\left(\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\right)}{s^{7/2}} + \cdots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} \left[1 - \left(\frac{1}{4s}\right) + \left(\frac{1}{4s}\right)^2 - \cdots \right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} \left[e^{-\frac{1}{4s}} \right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\, e^{-\frac{1}{4s}}$$

$$\mathcal{L}\left\{\sinh\tfrac{t}{2}\sin\tfrac{\sqrt{3}}{2}t\right\} = ?$$

$$\sinh\frac{t}{2} = \frac{e^{t/2} - e^{-t/2}}{2}$$

$$\sinh\frac{t}{2}\cdot\sin\frac{\sqrt{3}}{2}t$$

$$= \left(\frac{e^{t/2} - e^{-t/2}}{2}\right)\sin\frac{\sqrt{3}}{2}t$$

$$= \frac{e^{t/2}}{2}\sin\frac{\sqrt{3}}{2}t - \frac{e^{-t/2}}{2}\sin\frac{\sqrt{3}}{2}t$$

$$\mathcal{L}\left\{\frac{e^{t/2}}{2}\sin\frac{\sqrt{3}}{2}t - \frac{e^{-t/2}}{2}\sin\frac{\sqrt{3}}{2}t\right\}$$

$$= \frac{1}{2}\mathcal{L}\left\{\frac{e^{t/2}}{2}\sin\frac{\sqrt{3}}{2}t\right\} - \mathcal{L}\left\{\frac{e^{-t/2}}{2}\sin\frac{\sqrt{3}}{2}t\right\}$$

$$= \frac{1}{2}\left[\frac{\sqrt{3}/2}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{\sqrt{3}/2}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$= \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$= \frac{\sqrt{3}}{4}\left[\frac{\left[\left(s+\frac{1}{2}\right)^2+\frac{3}{4}\right] - \left[\left(s-\frac{1}{2}\right)^2+\frac{3}{4}\right]}{\left(\left(s-\frac{1}{2}\right)^2+\frac{3}{4}\right)\left(\left(s+\frac{1}{2}\right)^2+\frac{3}{4}\right)}\right]$$

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$$= \frac{\sqrt{3}}{4}\left[\frac{\cancel{s^2} + s + \cancel{\tfrac{1}{4}} - \cancel{s^2} + s - \cancel{\tfrac{1}{4}}}{\left(s^2 - s + \tfrac{1}{4} + \tfrac{3}{4}\right)\left(s^2 + s + \tfrac{1}{4} + \tfrac{3}{4}\right)}\right]$$

$$\cancel{= \frac{\sqrt{3}}{2}\left[\frac{2s}{s^4 + s^3}\right]}$$

$$= \frac{\sqrt{3}}{2}\left[\frac{s}{(s^2 - s + 1)(s^2 + s + 1)}\right]$$

$$= \frac{\sqrt{3}}{2}\left[\frac{s}{s^4 + \cancel{s^3} + \cancel{s^2} - \cancel{s^3} - \cancel{s^2} - \cancel{s} + s^2 + \cancel{s} + 1}\right]$$

$$= \frac{\sqrt{3}\, s}{2(s^4 + s^2 + 1)}$$

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* Laplace transform of

$$\mathcal{L}\{t\,f(t)\} = -\frac{d}{ds}F(s)$$

*1) Find $\mathcal{L}\{t \sin \omega t\}$

$$= -\frac{d}{ds}\frac{\omega}{s^2+\omega^2}$$

$$= \frac{-\cancel{\omega}}{\cancel{\omega}}\tan^{-1}\left(\frac{s}{\omega}\right)$$

$$= -\tan^{-1}\left(\frac{s}{\omega}\right)$$

$$= -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$= -\frac{(s^2+\omega^2)(0)-\omega(2s)}{(s^2+\omega^2)^2}$$

$\frac{d\omega}{ds} = 0$ ($\because \omega$ = const)

$$= \frac{2\omega s}{(s^2+\omega^2)^2}$$

* 2) $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$

1) $\mathcal{L}\{t^2 \sin \pi t\} = \frac{d^2}{ds^2}\left(\frac{\pi}{s^2+\pi^2}\right)$

$$= \frac{d}{ds}\left[\frac{(s^2+\pi^2)(0)-(\pi)(2s)}{(s^2+\pi^2)^2}\right]$$

$$= \frac{d}{ds}\left(\frac{-2s\pi}{(s^2+\pi^2)^2}\right)$$

$$= \frac{(s^2+\pi^2)^2(-2\pi) - (-2s\pi)\left(2(s^2+\pi^2)(2s)\right)}{(s^2+\pi^2)^4}$$

$$= \frac{-2\pi}{(s^2+\pi^2)^4}\left[(s^2+\pi^2)^2 - 4s^2(s^2+\pi^2)\right.$$

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$$= \frac{-2\pi(s^2+\pi^2)}{(s^2+\pi^2)^4} \cdot \left[(s^2+\pi^2) - 4s^2\right]$$

$$= \frac{-2\pi}{(s^2+\pi^2)^3}\left(-3s^2+\pi^2\right)$$

$$= \frac{2\pi(\pi^2-3s^2)}{(s^2+\pi^2)^3}$$

2) $\mathcal{L}\left\{t\, e^{4t} \cos 2t\right\} = ?$

$$\mathcal{L}\left\{t\, e^{4t}\cos 2t\right\} = -\frac{d}{ds}\left(\frac{(s-4)}{(s-4)^2+(2)^2}\right)$$

$$= -\frac{d}{ds}\left(\frac{s-4}{(s-4)^2+4}\right)$$

$$= -\frac{((s-4)^2+4)(1) - (s-4)(2(s-4))}{((s-4)^2+4)^2}$$

$$= -\frac{(s-4)^2+4-2(s-4)^2}{[(s-4)^2+4]^2}$$

$$= -\left[\frac{4-(s-4)^2}{[(s-4)^2+4]^2}\right]$$

$$= \frac{(s-4)^2-4}{[(s-4)^2+4]^2} = \frac{s^2-8s+12}{[(s-4)^2+4]^2}$$

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3) $\mathcal{L}\{t^2 \cdot e^{4t}\}$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s-4}\right)$$

$$= \frac{d}{ds} \; \frac{-1}{(s-4)^2}$$

$$= \frac{d}{ds} \cdot \frac{2}{(s-4)^3}$$

~~4) $\mathcal{L}$~~

★ If $\mathcal{L}\{f(t)\} = F(s)$ then,

$$\cancel{\mathcal{L}\left\{\int_0^t f(t)\, dt\right\} = \int_0^\infty e^{-su} \int_0^t f(u)\, du\, dt}$$

$$\mathcal{L}\left\{\int_0^t f(u)\, du\right\} = \frac{F(s)}{s}$$

$$\mathcal{L}\left\{ n \text{ times} \ldots \int_0^t \int_0^t \int_0^t f(u)\, du\, du \ldots n \text{ times} \right\} = \frac{F(s)}{s^n}$$

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s^n}\right\} = n \text{ times} \int_0^t f(u)\, du \ldots$$

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1) Obtain $\mathcal{L}\left\{\int_0^t (t^4 + \sin 3t)\, dt\right\}$

$$\mathcal{L}\left\{\int_0^t t^4\, dt\right\} + \mathcal{L}\left\{\int_0^t \sin 3t\, dt\right\}$$

$$= \frac{1}{s}\left[\frac{4!}{s^5}\right] + \frac{1}{s}\left[\frac{3}{s^2+9}\right]$$

$$= \frac{1}{s}\left[\frac{4!}{s^5} + \frac{3}{s^2+9}\right]$$

2) Find $\mathcal{L}\left\{\int_0^t e^u (u + \sin u)\, du\right\}$

$$= \mathcal{L}\left\{\int_0^t e^t\, t\, dt\right\} + \mathcal{L}\left\{\int_0^t e^t \sin t\, dt\right\}$$

$$= \frac{1}{s}\left[\frac{1}{(s-1)^2}\right] + \frac{1}{s}\left[\frac{1}{(s-1)^2 + (1)^2}\right]$$

$$= \frac{1}{s}\left[\frac{1}{(s-1)^2} + \frac{1}{(s-1)^2 + 1}\right]$$

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* Applications of laplace transform.

$$\mathcal{L}\left\{\frac{d^3y}{dt^3}\right\} = s^3\mathcal{L}\{y(t)\} - s^2y(0) - s^1y'(0) - s^0y''(0)$$

$$\mathcal{L}\left\{\frac{d^ny}{dt^n}\right\} = s^n\mathcal{L}\{y(t)\} - s^{n-1}y(0) - \dots - s^0y^{(n-1)}$$

* Solve the initial value problem (IVP) by using Laplace transform.

1) $y'' + 4y = 0$

$y(0) = 1 \quad , \quad y'(0) = 6.$

~~$y'' + 4y = 0$~~

We know that

$$\mathcal{L}\left\{\frac{d^ny}{dt^n}\right\} = \left\{s^n\mathcal{L}\{y(t)\} - s^{n-1}y(0) \dots\right\}$$

$$\cancel{\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\}} =$$

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + 4\mathcal{L}\{y\} = 0$$

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$$\left[s^2\mathcal{L}\{y(t)\} - s\,y(0) - y'(0)\right] + 4\left[\mathcal{L}\{y(t)\}\right] = 0$$

$$\therefore s^2\mathcal{L}\{y(t)\} - s(1) - (6) + 4\mathcal{L}\{y(t)\} = 0$$

$$\therefore \mathcal{L}\{y(t)\}(s^2+4) - s - 6 = 0$$

$$\therefore y(t) = \mathcal{L}^{-1}\left\{\frac{s+6}{s^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+(2)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{s^2+(2)^2}\right\}$$

$$\boxed{y(t) = \cos 2t + 3\sin 2t}$$

2) $y'' + 6y = 1$; $y(0) = 2$, $y'(0) = 0$

$$\mathcal{L}\{y'' + 6y\} = \mathcal{L}\{1\}$$

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + 6\mathcal{L}\{y\} = \frac{1}{s}$$

$$\left[s^2\mathcal{L}\{y(t)\} - s\,y(0) - y'(0)\right] + 6\left[\mathcal{L}\{y(t)\}\right] = \frac{1}{s}$$

$$\mathcal{L}\{y(t)\}(s^2+6) = \frac{1}{s} + 2s$$

$$\mathcal{L}\{y(t)\} = \frac{2s^2+1}{s(s^2+6)}$$

$$\mathcal{L}\{y(t)\} = \frac{2s}{s^2+6} + \frac{1}{s(s^2+6)}$$

$$\boxed{\begin{aligned} \mathcal{L}\{f(t)\} &= F(s) \\ \frac{F(s)}{s} &= \int_0^t \mathcal{L}^{-1}\{f(t)\}\,dt \end{aligned}}$$

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+6}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+6)}\right\}$$

$$= 2\cos\sqrt{6}t + \int_0^t \frac{\sin\sqrt{6}t}{\sqrt{6}}\,dt$$

$$= 2\cos\sqrt{6}t + \frac{1}{6}\left[-\cos\sqrt{6}t\right]_0^t$$

$$= 2\cos\sqrt{6}t - \frac{1}{6}\left[\cos\sqrt{6}t - 1\right]$$

$$= \cos\sqrt{6}t\left[2 - \frac{1}{6}\right] + \frac{1}{6}$$

$$= \cos\sqrt{6}t\left[\frac{11}{6}\right] + \frac{1}{6}$$

$$= \frac{11}{6}\cos\sqrt{6}t + \frac{1}{6}$$

$$= \frac{1}{6}\left[11\cos\sqrt{6}t + 1\right]$$

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3) $\ddot{y} + 9y = 6$; $t \geq 0$,

$y(0) = 2$, $\frac{dy}{dt} = 0$, $t=0$

Taking laplace both ~~the~~ sides.

$L\{y'' + 9y\} = L\{6\}$

$L\left\{\frac{d^2y}{dt^2}\right\} + 9L\{y(t)\} = \frac{6}{s}$

$[s^2 L\{y(t)\} - s\,y(0) - y'(0)] + 9L\{y(t)\} = \frac{6}{s}$

$L\{y(t)\}(s^2+9) = 2s + \frac{6}{s}$

$L\{y(t)\} = \frac{2s^2+6}{s(s^2+9)}$

$y(t) = L^{-1}\left\{\frac{2s}{s^2+9}\right\} + 6L^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$

$= 2\cos 3t + \frac{6}{3}\int_0^t \sin 3t \, dt$

$= 2\cos 3t + \frac{2}{3}\left[-\cos 3t\right]_0^t$

$= 2\cos 3t - \frac{2}{3}\cos 3t + \frac{2}{3}$

$=$ ②

$= \cos 3t\left[\frac{4}{3}\right] + \frac{2}{3}$

$= \frac{4}{3}\cos 3t + \frac{2}{3}$

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4) $y'' + y' = e^{2t}$ $\quad y(0) = y'(0) = 0$

Taking laplace both sides.

$\mathcal{L}\{y'' + y'\} = \mathcal{L}\{e^{2t}\}$

$\left[s^2 \mathcal{L}\{y(t)\} - s\,y(0) - y'(0)\right] + \left[s\,\mathcal{L}\{y(t)\} - y(0)\right]$

$= \frac{1}{s-2}$

$\therefore \mathcal{L}\{y(t)\}(s^2 + s) = \frac{1}{s-2}$

$\therefore \mathcal{L}\{y(t)\} = \frac{1}{(s-2)(s^2+s)}$

$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s^2+s)}\right\}$

$\frac{1}{(s-2)(s^2+s)} = \cancel{\frac{1}{s(s+1)(s-2)}}\ \frac{A}{(s-2)} + \frac{Bs+C}{(s^2+s)}$

$1 = A(s^2 + s) + (Bs + c)(s-2)$

$= A(s^2 + s) + Bs^2 - 2Bs + Cs - 2C$

$1 = s^2(A+B) + s(A - 2B + c) - 2C$

$\therefore A + B = 0$

$A - 2B + C = 0$

$-2C = 1$

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$$\therefore \quad C = -\frac{1}{2}$$

$$A - 2B + C = 0$$

$$\therefore \quad A - 2B = \frac{1}{2}$$

$$2A + 2B = 0$$

$$3A = \frac{1}{2}$$

$$\therefore \quad A = \frac{1}{6}$$

$$B = -\frac{1}{6}$$

$$\therefore \quad \frac{1}{(s-2)(s^2+s)} = \frac{1}{6(s-2)} + \frac{-\frac{1}{6}s - \frac{1}{2}}{(s^2+s)}$$

$$= \frac{1}{6(s-2)} - \frac{1}{6}\frac{s}{s^2+s} - \frac{1}{2(s^2+s)}$$

$$= \frac{1}{6(s-2)} - \frac{1}{6}\left(\frac{1}{s+1}\right) - \frac{1}{2s(s+1)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{6(s-2)} - \frac{1}{6}\left(\frac{1}{s+1}\right) - \frac{1}{2s(s+1)}\right\}$$

$$= \frac{1}{6}e^{2t} - \frac{e^{-t}}{6} - \frac{1}{2}\int_0^t e^{-t}\,dt$$

$$= \frac{e^{2t}}{6} - \frac{1}{6e^t} - \frac{1}{2}\left[-e^{-t}\right]_0^t$$

$$= \frac{e^{2t}}{6} - \frac{1}{6e^t} - \frac{1}{2}\left[-e^{-t} + e^0\right]$$

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$$y(t) = \frac{e^{2t}}{6} - \frac{1}{6e^t} + \frac{1}{2e^t} - \frac{1}{2}$$

$$= \frac{e^{2t}}{6} - \frac{\cancel{2}}{\cancel{6}e^t} - \frac{1}{2}$$

$$y(t) = \frac{1}{2}\left[\frac{e^{2t}}{3} - \frac{2}{3e^t} - 1\right]$$

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$$\mathcal{L}^{-1}\left\{\frac{s}{(s-1)(s^2+2s+2)}\right\} = ?$$

$$\frac{s}{(s-1)(s^2+2s+2)} = \frac{A}{(s-1)} + \frac{Bs+C}{s^2+2s+2}$$

$$s = A(s^2+2s+2) + (Bs+C)(s-1)$$

$$s = As^2 + 2As + 2A + Bs^2 - Bs + Cs - C$$

$$s = s^2(A+B) + s(2A-B+C) + (2A-C)$$

$$A + B = 0$$
$$2A - C = 0$$
$$2A - B + C = 1.$$

$$2A + 2B = 0$$
$$2A - C = 0$$
$$2B = -C$$

$$2A - B - 2B = 1$$
$$\therefore 2A - 3B = 1.$$
$$2A + 2B = 0$$
$$-5B = 1$$

$$B = -\frac{1}{5}$$

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$$A = \frac{1}{5}$$

$$C = \frac{2}{5}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-1)(s^2+2s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{5(s-1)} + \frac{-\frac{1}{5}(s)+\frac{2}{5}}{s^2+2s+2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{5(s-1)}\right\} + \mathcal{L}^{-1}\left\{\frac{-s}{5((s+1)^2+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{5((s+1)^2+1)}\right\}$$

$$= \frac{1}{5}e^{t} - \frac{1}{5}\,\cancel{\cosh}\left\{\frac{(s+1)-3}{(s+1)^2+1}\right\}$$

$$= \frac{e^{t}}{5} - \frac{1}{5}\,\overset{e^{-t}\cos t}{\cancel{\cosh(t+1)te}} + \frac{3}{5}e^{-t}\sin t.$$

★ ~~Solve~~. $\mathcal{L}^{-1}\left\{\frac{s+3}{(s^2+1)(s^2+9)}\right\} = ?$

$$\frac{s+3}{(s^2+1)(s^2+9)} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+9)}$$

$$s+3 = (As+B)(s^2+9) + (Cs+D)(s^2+1)$$

$$= As^3 + 9As + Bs^2 + 9B + Cs^3 + Ds^2 + Cs + D$$

$$s+3 = s^3(A+C) + s^2(B+D) + s(9A+C) + (9B+D)$$

$A + C = 0$ $\qquad$ $9A + C = 1$

$B + D = 0$ $\qquad$ $9B + D = 3$

$$A + C = 0$$
$$9A + C = 1$$
$$\overline{-8A = -1}$$
$$A = \frac{1}{8}$$
$$C = -\frac{1}{8}$$

$$B + D = 0$$
$$9B + D = 3$$
$$\overline{-8B = -3}$$
$$B = \frac{3}{8}$$
$$D = -\frac{3}{8}$$

$$\mathcal{L}^{-1}\left\{\frac{s+3}{(s^2+1)(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{8}s + \frac{3}{8}}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-\frac{1}{8}s - \frac{3}{8}}{s^2+9}\right\}$$

$$= \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{s+3}{s^2+1}\right\} + \frac{-1}{8}\mathcal{L}^{-1}\left\{\frac{+s+3}{s^2+9}\right\}$$

$$= \frac{1}{8}\left[\cos t + 3\sin t\right] - \frac{1}{8}\left[\cos 3t + 3\sin 3t\right]$$

$$= \frac{1}{8}\left[\cos t - \cos 3t\right] + \frac{3}{8}\left[\sin t - \sin 3t\right]$$

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* Convolution Theorm

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$$f^*g = \int_0^t f(u)\, g(t-u)\, du$$

$\Rightarrow$ Property : $1^* 1 = t$

1) Find the convolution of t & e^{at}, $a>0$

t & e^{at}

$f(t) = t$

$g(t) = e^{at}$

$g(t-u) = e^{a(t-u)}$

$$f^*g = \int_0^t u\, e^{at-au}\, du$$

$$= e^{at}\int_0^t u\, e^{-au}\, du$$

$f(t) = e^{at}$

$g(t) = t$

$g(t-u) = t-u$

$$f^*g = \int_0^t e^{au}(t-u)\, du$$

$$= t\int_0^t e^{au}\, du - \int_0^t u\, e^{au}\, du$$

$$= e^{at}\left[\frac{u\, e^{-au}}{-a} + \frac{1}{a}\,\frac{e^{-au}}{-a}\right]_0^t$$

$$= e^{at}\left[\left\{-\frac{t\, e^{-at}}{a} - \frac{1}{a^2}e^{-at}\right\} - \left(-\frac{1}{a^2}\right)\right]$$

$$= e^{at}\left[-\frac{e^{-at}}{a}\left[\frac{1}{a} + t\right] + \frac{1}{a^2}\right]$$

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$$= -\frac{1}{a}\left(t+\frac{1}{a}\right) + \frac{e^{at}}{a^2}$$

$$= \frac{1}{a}\left[\frac{e^{at}}{a} - \left(t+\frac{1}{a}\right)\right]$$

2) Using Convolution theorem determine

$$L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$$

Let $F(s) = \frac{1}{s} \Rightarrow f(t) = 1$

$$G(s) = \frac{1}{s^2+4}$$

~~Let~~ $= \frac{1}{2}\cdot\frac{2}{s^2+4} \Rightarrow g(t) = \frac{1}{2}\sin 2t$

By using convolution theorem of L.T.

$$L^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = f * g$$

$$\left(\because L^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \int_0^t L^{-1}F(s)\ldots\right.$$

$$= \int_0^t 1\cdot\frac{1}{2}\sin 2u \, du$$

$$= \frac{1}{2}\left[-\frac{\cos 2u}{2}\right]_0^t$$

$$= -\frac{1}{4}\cos 2t + \frac{1}{4}$$ [illegible]

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$$= \frac{1}{4} - \frac{1}{4}\cos 2t$$

$$= \frac{1}{2}\left[\frac{1-\cos 2t}{2}\right]$$

$$= \frac{1}{2}\sin^2 t$$

3) Using Convolution theorm of L.T

$$L^{-1}\left\{\frac{1}{s^2 - s - 12}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s-4)(s+3)}\right\}$$

$$F(s) = \frac{1}{s-4} \Rightarrow f(t) = e^{4t}$$

$$G(s) = \frac{1}{s+3} \Rightarrow g(t) = e^{-3t} \Rightarrow g(t-u) = e^{-3(t-u)}$$

By using convolution theorm of L.T.

$$L^{-1}\left\{\frac{1}{(s-4)(s+3)}\right\} = f * g$$

$$= \int_0^t e^{4u} \cdot e^{-3(t-u)}\, du$$

$$= \int_0^t e^{4u} e^{-3t} \cdot e^{3u}\, du$$

$$= \int_0^t e^{-3t} e^{7u} du$$

$$= e^{-3t} \int_0^t e^{7u} du$$

$$= e^{-3t} \left[\frac{e^{7u}}{7} \right]_0^t$$

$$= \frac{e^{-3t+7t}}{7} - \frac{e^{-3t}}{7}$$

$$= \frac{1}{7} \left[e^{4t} - e^{-3t} \right]$$

4) Using Convolution theorm determine

$$L^{-1} \left\{ \frac{1}{s^4 + 4s^3 + 4s^2} \right\}$$

$$L^{-1} \left\{ \frac{1}{s^4 + 4s^3 + 4s^2} \right\} = L^{-1} \left\{ \frac{1}{s^2 (s^2 + 4s + 4)} \right\}$$

$$= L^{-1} \left\{ \frac{1}{s^2 (s+2)^2} \right\}$$

$$F(s) = \frac{1}{(s+2)^2} \Rightarrow f(t) = e^{-2t} t^1$$

$$G(s) = \frac{1}{s^2} \rightarrow g(t) = t^1$$

$\therefore f(u) = u e^{-2u}$

$g(t) = t$

$g(t-u) = t-u$

By using convolution theorm of Laplace Transform

$$L^{-1}\left\{\frac{1}{s^4+4s^3+4s^2}\right\} = L^{-1}\left\{\frac{1}{s^2(s+2)^2}\right\} = f^* g$$

$$= \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t u e^{-2u}(t-u)\, du$$

$$= t\int_0^t e^{-2u} u\, du - \int_0^t u^2 e^{-2u}\, du$$

$$= t\left[u\frac{e^{-2u}}{-2} + \frac{e^{-2u}}{4}\right]_0^t - \left[\frac{u^2e^{-2u}}{-2} + \frac{2ue^{-2u}}{4} + \frac{2e^{-2u}}{-8}\right]_0^t$$

$$= t\left[\frac{te^{-2t}}{-2} + \frac{e^{-2t}}{4} - \frac{1}{4}\right] - \left[\frac{t^2e^{-2t}}{2} + \frac{2te^{-2t}}{4} + \frac{2e^{-2t}}{-8} + \frac{1}{4}\right]$$

$$= \left[\frac{t^2e^{-2t}}{-2} + \frac{e^{-2t}t}{4} - \frac{t}{4} - \frac{t^2e^{-2t}}{2} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4} + \frac{1}{4}\right]$$

$$= -t^2e^{-2t} - \frac{e^{-2t}t}{4} - t/4 + 1/4$$

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Q.9. Find I.L.T.

1) $L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = ?$

2) $L^{-1}\left\{\frac{1}{s(s^2-3s+3)}\right\} = ?$

3) $L^{-1}\left\{\frac{1}{(s^2-1)(s^2+1)}\right\} = ?$

4) $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = ?$

5) $L^{-1}\left\{\log\left(1+\frac{\omega^2}{s^2}\right)\right\} = ?$

$$F(s) = \log\left(1+\frac{\omega^2}{s^2}\right) = \log\left(\frac{s^2+\omega^2}{s^2}\right)$$

$$= \log(s^2+\omega^2) - \log s^2$$

$$\frac{d}{ds}[F(s)] = F'(s) = \frac{2s}{s^2+\omega^2} - \frac{2}{s}$$

$$-\frac{d}{ds}[F(s)] = \frac{-2s}{s^2+\omega^2} + \frac{2}{s}$$

$$t\,f(t) = L^{-1}\left(\frac{2}{s} - \frac{2s}{s^2+\omega^2}\right)$$

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$$f(t) = +\frac{1}{t}\left[\mathcal{L}^{-1}\left(\frac{2}{s} - \frac{2s}{s^2+\omega^2}\right)\right]$$

$\therefore$

6) $\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

7) $\log\left(\frac{s^2+a^2}{(s+b)^2}\right)$

8) $\log\sqrt{\frac{(s^2-a^2)}{s^2}}$

9) $\log\sqrt{\frac{s-1}{s+2}}$

10) $\log\sqrt{\frac{(s^2+1)}{s(s+1)}}$

11) $\tan^{-1} s \longrightarrow t\,f(t) = -\mathcal{L}^{-1}\left\{\frac{1}{1+s^2}\right\}$

$f(t) = -\frac{1}{t}\sin t$

12) $\tan^{-1}\left(\frac{s+a}{b}\right)$

13) $\frac{s}{(s+1)(s-1)^2}$

14) $\frac{s^3}{s^4-a^4}$

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* State the convolution theo...
$f * g$. ______ (7)

* By using Partial fraction find out inverse laplace transform ______ (5) to (7)

* Solve differential eq^n by laplace transform ______ (3) to (5)

* Find laplace transform by using def^n.

* Find laplace transform of unit step function or derac delta function. ______ (3) to (5)

* unit step function
$u(t)$

Case 1 : L.T. of unit step function.

(1) $u(t) = \begin{cases} 0 & ; t < a \\ 1 & ; t > a \end{cases} \Rightarrow \mathcal{L}\{u(t)\}$

$$= \int_0^\infty e^{-st} u(t)\, dt$$

(2) $\cancel{u(t)}$ $u(t-a) = \begin{cases} 0 & ; t < a \\ 1 & ; t > a. \end{cases} \Rightarrow \mathcal{L}\{u(t-a)\}$

$$= \int_0^\infty e^{-st} u(t-a)\, dt$$

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* State the convolution theorm and find $f*g$. —— (7)

* By using Partial fraction find out inverse laplace transform —— (5) to (7)

* Solve differential eqn by laplace transform —— (3) to (5)

* Find laplace transform by using defn.

* Find laplace transform of unit step function or derac delta function. —— (3) to (5)

* unit step function

$u(t)$

Case 1: L.T. of unit step function.

(1) $u(t) = \begin{cases} 0 & ; t < a \\ 1 & ; t > a \end{cases} \Rightarrow \mathcal{L}\{u(t)\}$

$$= \int_0^\infty e^{-st} u(t)\, dt$$

(2) $u(t-a) = \begin{cases} 0 & ; t < a \\ 1 & ; t > a. \end{cases} \Rightarrow \mathcal{L}\{u(t-a)\}$

$$= \int_0^\infty e^{-st} u(t-a)\, dt$$

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(3) $f(t)\,u(t-a) = \begin{cases} 0 & ; t<a \\ f(t) & ; t>a. \end{cases}$

$$\mathcal{L}\{f(t)u(t-a)\} = \int_0^\infty e^{-st} f(t)\,u(t-a)\,dt$$

$$= e^{-as} F(s+a)$$

(4) $f(t-a)\,u(t-a) = \begin{cases} 0 & ; t<a \\ f(t-a) & ; t>a. \end{cases}$

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_0^\infty e^{-st} f(t-a)\,u(t-a)\,dt$$

$$= e^{-as} F(s)$$

$t - a = v \Rightarrow dt = dv.$

$t = v + a.$

* Find the L.T. of $t^2 \cdot u(t-2)$

$f(t) = t^2.$

$u(t-a) = u(t-2)$

$a = 2$

$$\mathcal{L}\{f(t)u(t-a)\} = \int_0^\infty e^{-st}\, t^2 \cdot u(t-2)\,dt$$

$$= e^{-2s} \cdot \frac{2}{(s+2)^3}$$

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* ~~Find L.T. of~~ $\mathcal{L}\{t^2 \cdot u(t-2)\}$

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as} F(s+a)$$

$$= \cancel{e^{-2s} \frac{2}{(s+2)^3}}$$

$$= e^{-as} \mathcal{L}\{f(t+a)\}$$

$$= e^{-as} \mathcal{L}\{(t+2)^2\}$$

$$= e^{-as} \mathcal{L}\{t^2 + 4t + 4\}$$

$$= e^{-as} \left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]$$

* Find L.T.

$$\mathcal{L}\{e^{-3t} u(t-2)\}$$

$$f(t) = e^{-3t}$$

$$u(t-a) = u(t-2)$$

$$a = 2$$

$$\mathcal{L}\{e^{-3t} u(t-2)\} = \int_0^\infty e^{-st} e^{-3t} u(t-2)\, dt$$

$$\cancel{= e^{-2s} \frac{1}{\ }}$$ 1.

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$$\mathcal{L}\{e^{-3t}u(t-2)\} = e^{-as}\mathcal{L}\{e^{-3(t+a)}\}$$

$$= e^{-2s}\mathcal{L}\{e^{-3t}\cdot e^{-6}\}$$

$$= \frac{e^{-2s}}{e^{6}}\left[\frac{1}{s+3}\right]$$

Q

Laplace transform of periodic function.

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)\,dt$$

Find the L.T. of $f(t) = e^t \quad 0 < t < 2\pi$

$\cancel{\mathcal{L}\{e^t\} = 1}$

$$\boxed{\begin{array}{c}\text{Condition of periodic function} \\ f(t) = f(t+2\pi)\end{array}}$$

$\cancel{\mathcal{L}\{e^t\} = \frac{1}{\quad}}$ $\quad P = \frac{2\pi - 0}{\cancel{2}}$

$\cancel{P = \pi}$

$$\mathcal{L}\{e^t\} = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st}\, e^t\,dt$$

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$$\mathcal{L}\{e^t\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{t(-s+1)}\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{t(-s+1)}}{-s+1} \right]_0^{2\pi}$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{2\pi(-s+1)}}{-s+1} - \frac{1}{-s+1} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{-2\pi s} \cdot e^{2\pi} - 1}{-s+1} \right]$$

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First Order Differential Equation.

* $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 = 1$

This eqⁿ is 2nd order diff. eqⁿ.
Degree of this eqⁿ is 1.

Degree is the ~~power~~ order of highest power in the eqⁿ.

*

* Variable Seprable:

(i) $\frac{dy}{dx} = F(x, y)$

(ii) Seprate the variable x and y

(iii) Take the integration both the sides.

(iv) General Solution.

Eg. 1) $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

$$\frac{dy}{dx} = (e^x + x^2) e^{-y}$$

$$e^y \frac{dy}{dx} = e^x + x^2$$

$$\int e^y \frac{dy}{dx} = \int e^x + x^2$$

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