5.6 GAMMA DISTRIBUTION

A continuous random variable X is said to follow exponential distribution if its probability function is given by

$$f(x) = \frac{\lambda^r}{\Gamma r} x^{r-1} e^{-\lambda x}, \quad x > 0$$

$$= 0 \quad , \quad x \leq 0$$

Calculate the probability that the engineer will repair a fault ... mean tim

6.5 The Gamma Distribution

Suppose that a system consisting of one original and $(\alpha - 1)$ spare components such that in the case of failure of original component, one of the $(\alpha - 1)$ spare components can be used. The process will continue till we use last component. When last component fails, then the whole system fails. Let $X_1, X_2, \ldots, X_\alpha$ be the lifetimes of the α components. Let each of the random variables $X_1, X_2, \ldots, X_\alpha$ have the same exponential distribution with parameter λ, and also are probabilistically independent. Then the lifetime (time until failure) of the entire system is given by

$$T = \sum_{i=1}^{\alpha} X_i,$$

and which has ***gamma distribution*** with p.d.f.

$$f(t) = \frac{\lambda^{\alpha} t^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)} \quad ; t \geq 0$$

$$= 0 \quad ; \text{otherwise} \qquad ...(6.20)$$

for $\lambda, \alpha > 0$. The parameter α is called the ***shape parameter***, and the parameter λ is called the ***rate parameter*** (as in the case of exponential distribution).

Thus, the sum of α independent exponential random variables has a gamma

5.6.1 Constants of the Gamma Distribution

1. Mean of the Gamma Distribution

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x$$

$$= \int_0^{\infty} x \frac{\lambda^r}{\lceil r} x^{r-1} e^{-\lambda x} \mathrm{d}x$$

$$= \frac{\lambda^r}{\lceil r} \int_0^{\infty} e^{-\lambda x} x^r \mathrm{d}x$$

$$= \frac{\lambda^r}{\lceil r} \frac{\lceil r+1}{\lambda^{r+1}} \qquad \left[\because \int_0^{\infty} e^{-kx} x^{n-1} \mathrm{d}x = \frac{\lceil n}{k^n}\right]$$

$$= \frac{\lambda^r r \lceil r}{\lceil r \cdot \lambda^{r+1}}$$

$$= \frac{r}{\lambda}$$

2. Variance of the Gamma Distribution

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 \qquad \text{...(5.8)}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\mathrm{d}x$$

$$= \int_0^{\infty} x^2 \frac{\lambda^r}{\lceil r} x^{r-1} e^{-\lambda x} \mathrm{d}x$$

$$= \frac{\lambda^r}{\lceil r} \int_0^{\infty} e^{-\lambda x} x^{r+1} \mathrm{d}x$$

$$= \frac{\lambda^r}{\lceil r} \frac{\lceil r+2}{\lambda^{r+2}} \qquad \left[\because \int_0^{\infty} e^{-kx} x^{n-1} \mathrm{d}x = \frac{\lceil n}{k^n}\right]$$

$$= \frac{(r+1) r \lceil r}{\lceil r \cdot \lambda^2}$$

$$= \frac{r^2 + r}{\lambda^2}$$

Substituting in Eq. (5.8),

$$\mathrm{Var}(X) = \frac{r^2 + r}{\lambda^2} - \frac{r^2}{\lambda^2}$$

$$= \frac{r}{\lambda^2}$$

Example 1

Given a Gamma random variable X with r = 3 and λ = 2. Compute E(X), Var(X) and P(X ≤ 1.5 years).

Solution

$$\lambda = 2, \quad r = 3$$

$$f(x) = \frac{\lambda^r}{\Gamma r} x^{r-1} e^{-\lambda x}, \quad x > 0$$

(a) $E(X) = \frac{r}{\lambda} = \frac{3}{2} = 1.5$ years

(b) $\text{Var}(X) = \frac{r}{\lambda^2} = \frac{3}{(2)^2} = 0.75$

(c) $P(X \leq 1.5 \text{ years}) = \int_0^{1.5} f(x)\,dx$

$$= \int_0^{1.5} \frac{2^3}{\Gamma 3} x^2 e^{-2x} dx$$

$$= 4 \left| x^2 \left(\frac{e^{-2x}}{-2} \right) - 2x \left(\frac{e^{-2x}}{4} \right) + 2 \left(\frac{e^{-2x}}{-8} \right) \right|_0^{1.5}$$

$$= 4 \left[(1.5)^2 \left(\frac{e^{-3}}{-2} \right) - 2(1.5) \left(\frac{e^{-3}}{4} \right) + 2 \left(\frac{e^{-3}}{-8} \right) + \frac{1}{4} \right]$$

$$= 0.5768$$

Example 2

The daily consumption of milk in a city, in excess 20000 litres, is approximately distributed as a Gamma variate with parameters $\lambda = \frac{1}{10000}$

and r = 2. The city has a daily stock of 30000 litres. What is the probability that the stock is insufficient on a particular day?

Solution

Let Y be the random variable which denotes the daily consumption of milk (in litres) in a city. The random variable $X = Y - 20000$ has a gamma distribution.

$$\lambda = \frac{1}{10000}, r = 2$$

$$f(x) = \frac{\lambda^r}{\Gamma r} x^{r-1} e^{-\lambda x}, \quad x > 0$$

$$= \frac{\left(\frac{1}{10000}\right)^2}{\Gamma 2} x^{2-1} e^{-\frac{1}{10000}x}$$

$$= \frac{x e^{-\frac{1}{1000}x}}{(10000)^2}$$

Probability that the stock is insufficient on a particular day

$$P(Y > 30000) = P(X > 10000)$$

$$= \int_{10000}^{\infty} f(x)\,\mathrm{dx}$$

$$= \int_{1000}^{\infty} \frac{x e^{-\frac{1}{1000}x}}{(10000)^2}\,\mathrm{dx}$$

$$= \frac{1}{10^8} \int_{10^4}^{\infty} x e^{-10^{-4}x}\,\mathrm{dx}$$

$$= \frac{1}{10^8} \left| \frac{x \cdot e^{-10^{-4}x}}{-10^{-4}} - \frac{1 \cdot e^{-10^{-4}x}}{(-10^{-4})^2} \right|_{10^4}^{\infty}$$

$$= \frac{1}{10^8} \left(\frac{e^{-1}}{10^{-8}} + \frac{e^{-1}}{10^{-8}} \right)$$

$$= e^{-1} + e^{-1}$$

$$= 2e^{-1}$$

$$= 0.7358$$

Example 3

In a certain city, the daily consumption of electric power in millions of kilowatt hours can be treated as a random variable having gamma distribution with parameters $\lambda = \frac{1}{2}$ and $r = 3$. If the power plant of this city has a daily capacity of 12 millions kilowatt-hours, what is the probability that this power supply will be inadequate on any given day.

Solution

Let X be a random variable which denotes the daily consumption of electric power in millions killowatt-hours.

$$f(x) = \frac{\lambda^r}{\overline{|r}} x^{r-1} e^{-\lambda x}, \quad x > 0$$

$$= \frac{\left(\frac{1}{2}\right)^3}{\overline{|3}} x^2 e^{-\frac{1}{2}x}$$

$$P(\text{power supply is inadequate}) = P(X > 12)$$

$$= \int_{12}^{\infty} f(x)\,dx$$

$$= \int_{12}^{\infty} \frac{1}{\overline{|3}} \frac{1}{2^3} x^2 e^{-\frac{1}{2}x}\,dx$$

$$= \frac{1}{16}\left| x^2\left(\frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}}\right) - 2x\left(\frac{e^{-\frac{1}{2}x}}{\frac{1}{4}}\right) + 2\left(\frac{e^{-\frac{1}{2}x}}{-\frac{1}{8}}\right)\right|_{12}^{\infty}$$

$$= \frac{1}{16} e^{-6}(288 + 96 + 16)$$

$$= 25e^{-6}$$

$$= 0.062$$

Example 4

f a company employs n sales persons, its gross sales in thousands f rupees may be regarded as a random variable having a gamma stribution with $\lambda = \frac{1}{2}$ and $r = 80\sqrt{n}$. If the sales cost is ₹8000 per *salesperson, how many salespersons should the company employ to maximise the expected profit?*

Solution

Let X be the random variable which denotes the gross sales in rupees by n salespersons.

$$\lambda = \frac{1}{2}, \quad r = 80000\sqrt{n}$$

$$f(x) = \frac{\lambda^r}{\Gamma r} x^{r-1} e^{-\lambda x}, \quad x > 0$$

$$E(X) = \frac{r}{\lambda} = \frac{80000\sqrt{n}}{\frac{1}{2}} = 160000\sqrt{n}$$

If y denotes the total expected profit of the company,

$$y = \text{total expected sales} - \text{total sales cost}$$
$$= 160000\sqrt{n} - 8000n$$

$$\frac{dy}{dn} = \frac{80000}{\sqrt{n}} - 8000$$

For maximum profits,

$$\frac{dy}{dn} = 0$$

$$\frac{80000}{\sqrt{n}} - 8000 = 0$$

$$\frac{80000}{\sqrt{n}} = 8000$$

$$\sqrt{n} = 10$$

$$n = 100$$

$$\frac{d^2y}{dn^2} = -\frac{40000}{n^{\frac{3}{2}}}$$

When $n = 100$, $\frac{d^2y}{dn^2} = -40 < 0$

$\therefore$ y is maximum when $n = 100$.

Hence, the company should employ 100 salespersons to maximise the expected profit.

Example 6.19

The daily consumption of milk in a city, in excess of 20000 *litres*, is approximately distributed as a gamma variate with parameters $\alpha = 2$ and $\lambda = 1/10000$. The city has a daily stock of 30000 *litres*. What is the probability that the stock is insufficient on a particular day ?

Solution

If X denotes the daily comsumption, in excess of 20000 *litres*, then p.d.f. of X is

$$f(x) = \frac{\left(\frac{1}{10000}\right)^2 x^{2-1} e^{-\frac{1}{10000}x}}{\Gamma(2)} \qquad \text{(Using (6.20))}$$

$$= \frac{x e^{-\frac{1}{10000}x}}{(10000)^2 \Gamma(2)}.$$

The stock of 30000 *litres* will be insufficient on a particular day, if the excess consumption is more than 10000 *litres*.

Therefore,

$$P(X > 10000) = \int_{10000}^{\infty} \frac{xe^{-\frac{1}{10000}x}}{(10000)^2\Gamma(2)} dx. \quad \ldots(i)$$

Let

$$\frac{x}{10000} = t \Rightarrow \frac{dx}{10000} = dt.$$

Therefore, (i) becomes

$$P(X > 10000) = \int_{1}^{\infty} te^{-t} dt \qquad (\text{Since } \Gamma(2) = 1!)$$

$$= t\left(\frac{e^{-t}}{-1}\right) - 1\left(\frac{e^{-t}}{1}\right)\Bigg|_{1}^{\infty}$$

(By Leibniz rule)

$$= 0 - \left(-\frac{1}{e} - \frac{1}{e}\right)$$

$$= \frac{2}{e}$$

$$\approx 0.736.$$

Answer

Example 6.20

The daily consumption of electric power (in millions if kW hours) in a certain city is a random variable X having p.d.f.

$$f(x) = \frac{1}{9}xe^{-\frac{x}{3}} \ ; \ x > 0$$
$$= 0 \qquad ; \ x \leq 0.$$

Find the probability that the power supply is inadequate on any given day if the daily capacity of the power plant is 12 million kW hours.

Solution

The given p.d.f. is for gamma distribution for $\alpha = 2$ and $\lambda = 1/3$.
Now, the power supply is inadequate when $X > 12$. Therefore,

$$P(X>12)=\int_{12}^{\infty}\frac{1}{9}xe^{-\frac{x}{3}}dx. \qquad \text{(Using (6.20)) ...(i)}$$

Let

$$\frac{x}{3}=t \Rightarrow \frac{dx}{3}=dt.$$

Therefore, (i) becomes

$$P(X>12)=\int_{4}^{\infty}\frac{t}{3}e^{-t}\cdot 3dt$$

$$=\int_{4}^{\infty}te^{-t}dt$$

$$=t\left(\frac{e^{-t}}{-1}\right)-1\left(\frac{e^{-t}}{1}\right)\Bigg|_4^{\infty}$$

(Using Leibniz rule)

$$=0-(-4e^{-4}-e^{-4})$$

$$=\frac{4}{e^4}+\frac{1}{e^4}$$

$$=\frac{5}{e^4}$$

$$\approx 0.09158.$$

Answer

Exercises 6.4

01. The survival time in weeks of an animal when subjected to certain exposure of gamma radiation has a gamma distribution with $\alpha = 5$ and $\lambda = 1/10$.
(*a*) What is the mean survival time of a randomly selected animal of the type used in the experiment ?
(*b*) What is the probability that an animal survives more than 30 weeks ?
02. Suppose that the reaction time X has a standard gamma distribution with $\alpha = 2$. Find (*a*) $P(3 \le X \le 5)$ (*b*) $P(X > 4)$