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Ch -

Differential Equation.

Differential Equation.

1st Order | High Order

Order and Degree.

$$\left(\frac{d^n y}{dx^n}\right)^m = f(x.y)$$

$$\boxed{\begin{aligned} n &= \text{Order} \\ m &= \text{degree} \end{aligned}}$$

Ex $\left(\frac{d^2y}{dx^2}\right)^3 + y = 0$

Order = 2
Degree = 3.

Eg:- $\left(\frac{dy}{dx}\right)^3 + \left(\frac{d^2y}{dx^2}\right) + \frac{dy}{dx} + y = 0.$

Order = 2
Degree = 1

$\left(\frac{dy}{dx}\right)^{1/3} = y.$

$\left(\frac{dy}{dx}\right) = y^3$

Order = 1
Degree = 1

Ex:- $\sqrt{\frac{dy}{dx} + y} = \sin x$

$\frac{dy}{dx} + y = (\sin x)^2$

$\boxed{\begin{array}{l} \text{Order} = 1 \\ \text{degree} = 1 \end{array}}$

<u>First</u>	<u>Higher</u>
$\frac{d^1y}{dx^1} + y = 0$	$\frac{d^2y}{dx^2} + y = 0$
$y' + y = e^x$	$y''' + y = e^x$
$\frac{d^1x}{dt^1} + x = e^t$	$\frac{d^4x}{dt^4} + x = e^t$
$x\,dy + y\,d^1x = 0$	$x\,d^2y + y\,d^2x = 0$

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Exact Equation :

$$\boxed{M\,dx + N\,dy = 0}$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Equation is exact

Soln :- $\int M\,dx + \int N\,dy = c$

(terms free from x)

O.TU:
Q.1 ~~...~~ Solve $ye^x\,dx + (2y + e^x)\,dx = 0$

Soln:- Equation is in form of

$$\boxed{M\,dx + N\,dy = 0}$$

$M = ye^x$ | $N = 2y + e^x$

$\frac{\partial M}{\partial y} = e^x$ | $\frac{\partial N}{\partial x} = e^x$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Equation is exact.

Soln $\int M\,dx + \int N\,dy$

(terms free from x)

$$y\int e^x dx + 2\int y\,dy = C.$$

$$y e^x + 2(y^2/2) = c$$

$$\therefore \boxed{y e^x + y^2 = c}$$

Q. Solve $[e^x(x+1) - e^y]dx - x e^y dy = 0$

Soln: $e^x(x+1) - e^y\, dx + (-x e^y)\, dy = 0$

Equation is in form of.

$$\boxed{M dx + N dy = 0}$$

$M = e^x(x+1) - e^y$	$N = -x e^y$
$\frac{\partial M}{\partial y} = -e^y$	$\frac{\partial N}{\partial x} = -e^y$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Equation is exact

Soln $\int M dx + \int N dy = c$ (terms free from x)

$$\int [\underset{u}{(x+1)e^x} - \underset{v}{e^y}]\, dx + 0 = c$$

$$\frac{d}{dx}\sinh x = \sinh x$$

$$\frac{d}{dx}\cosh x = \cosh x.$$

$$[(x+1)e^x - 1e^x] - xe^y = c.$$

$$e^x(x+1-1) - xe^y = c.$$

$$\boxed{xe^x - xe^y = c}$$

solve $\cosh y \cos x \, dx = -\sinh y \sin x \, dy.$

$$\cosh y \cos x \, dx + \sinh y \sin x \, dy = 0.$$

Equation is in form of

$$\boxed{M\,dx + N\,dy = 0}$$

$M = \cosh y \cos x$ | $N = \sinh y \sin x$

$\frac{dM}{dy} = \sinh y \cos x$ | $\frac{\partial N}{dy} = \sinh y \cos x.$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}}$$

Equation is exact.

soln $\int M\,dx + \int N\,dy = c.$ (terms free from x)

$$\cosh y \int \cos x \, dx + 0 = c.$$

$$\boxed{\cosh y \sin x = c}$$

Q. Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$

Soln:

$$\underbrace{(y\cos x + \sin y + y)}_{M}\,dx + \underbrace{(\sin x + x\cos y + x)}_{N}\,dy = 0$$

$$\boxed{M\,dx + N\,dy = 0}$$

$M = y\cos x + \sin y + y.$ | $N = \sin x + x\cos y + x$

$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$ | $\frac{\partial N}{\partial y} = \cos x + \cos y + 1$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Equation is exact.

$\int M\,dx + \int N\,dy.$
(terms free from x)

$$\int (y\cos x + \sin y + y)\,dx + 0 = c$$

$$\boxed{y\sin x + x\sin y + xy = c}$$

$\frac{dx}{}$ $\underbrace{\qquad}_{x \text{ form}}$

Ex $\frac{dy}{dx} + (e^{x}) y = \sin x.$

Integrating factor $= e^{\int P dx}$

Solⁿ: $(I.F.) y = \int (I.F.) Q \, dx + c.$

Form-2 $\frac{dx}{dy} + Px = Q.$ $\underbrace{\qquad}_{y \text{ form}}$

$I.F. = e^{\int P dy}$

solⁿ:- $(I.F.) x = \int (I.F.) Q \, dy + c.$

Q.1. Solve $\frac{dy}{dx} + (\tan x) y = \sin 2x$

Given equation is linear
where $P = \tan x$, $Q = \sin 2x$

$$I.F. = e^{\int P dx}$$
$$= e^{\int \tan x \, dx}$$
$$= e^{\log \sec x}$$

$$I.F. = \sec x$$

Soln:-

$$(I.F.)\, y = \int (I.F.)\, Q\, dx + c$$

$$y \sec x = \int \sec x\, (2\sin x \cos x)\, dx + c$$

$$= 2\int \sin x\, dx + c$$

$$\boxed{y \sec x = -2\cos x + c}$$

Q. Solve $\dfrac{dy}{dx} + \left(\dfrac{4x}{x^2+1}\right) y = \dfrac{1}{(x^2+1)^3}$

Given equation is linear

where $P = \dfrac{4x}{x^2+1}$ & $Q = \dfrac{1}{(x^2+1)^3}$

$$I.F. = e^{\int P\, dx}$$

$$= e^{\int \frac{4x}{x^2+1} dx}$$

$$= e^{2\int \frac{2x}{x^2+1} dx}$$

$$= e^{2\log(x^2+1)}$$

$$= e^{\log(x^2+1)^2}$$

$$= (x^2+1)^2$$

Q. 3

Soln

Soln

$(I.F.)\, y = \int (I.F.)\, Q\, dx + c$

$y(x^2+1)^2 = \int (x^2+1)^2 \frac{1}{(x^2+1)^3}\, dx + c$

$= \int \frac{1}{x^2+1}\, dx + c$

$y\,(x^2+1)^2 = \tan^{-1} x + c$

AKTU

Q.3. Solve $\frac{dy}{dx} + (\sin x)\, y = e^{\cos x}$

Soln. Given equation is linear.
where $P = \sin x$, $Q = e^{\cos x}$

$I.F. = e^{\int P dx}$

$= e^{\int \sin x\, dx}$

$= e^{-\cos x}$

Soln.

$(I.F.)\, y = \int (I.F.)\, Q\, dx + c$

$= \int e^{-\cos x} e^{\cos x}\, dx + c$

$= \int dx + c$

$\boxed{y\, e^{-\cos x} = x + c}$

Q Solve $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$

Ans. Divide by $(x+1)$

$$\frac{dy}{dx} + \left(\frac{-1}{x+1}\right)y = e^{3x}(x+1)$$

where $P = \frac{-1}{x+1}$ $\quad Q = e^{3x}(x+1)$

$$I.F. = e^{-\int \frac{1}{x+1}dx}$$
$$= e^{-\log(x+1)}$$
$$= e^{\log(x+1)^{-1}}$$
$$= (x+1)^{-1} = \left(\frac{1}{x+1}\right)$$

$$\boxed{I.F. = \frac{1}{x+1}}$$

Soln :-

$$(I.F.)\,y = \int (I.F.)\,Q\,dx + C$$
$$= \int \left[\frac{1}{(x+1)}\,e^{3x}(x+1)\right]dx + C$$
$$= \int \frac{e^{3x}(x+1)}{(x+1)}\,dx + c.$$

$$\boxed{\frac{y}{x+1} = \frac{e^{3x}}{3} + c}$$

Solve $x\frac{dy}{dx} + (x+1)y = x^{100}$

Divide by x

$$\frac{dy}{dx} + \frac{x+1}{x}y = x^{99}$$

where $P = \frac{x+1}{x}$ $Q = x^{99}$

$$I.F. = e^{\int P dx}$$

$$= e^{\int (1+\frac{1}{x})dx}$$

$$= e^{x+\log x}$$

$$= e^{x} \cdot e^{\log x}$$

$$= xe^{x}$$

$$(I.F.)y = \int (I.F.) Q\, dx + c$$

$$y(xe^{x}) = \int (xe^{x}) x^{99} dx + c$$

$$= \int e^{x} x^{100} dx + c$$

$$= e^{x}[x^{100} - 100x^{99} + 100 \times 99 x^{98} - 100 \times 99 \times 98 \times x^{15} + \dots + 100!]$$

$\int e^x f(x) + \ldots \, dx$

$= e^x f(x)$

Q. Solve $x\dfrac{dy}{dx} - xy = \left(\dfrac{1}{x} + \dfrac{2}{x^2}\right)$

Ans. Divide by x

$\dfrac{dy}{dx} - y = \left(\dfrac{1}{x^2} + \dfrac{2}{x^3}\right)$

where $P = -1 \quad Q = \left(\dfrac{1}{x^2} + \dfrac{2}{x^3}\right)$

$I.F. = e^{\int p\,dx}$

$= e^{\int -1\,dx}$

$\boxed{= e^{-x}}$

Soln

$(I.F.)\, y = \int (I.F.)\, Q\, dx + C$

$= \int e^{-x}\left(\dfrac{1}{x^2} + \dfrac{2}{x^3}\right) dx + C$

Let $-x = t$

$x = -t$

$dx = -dt$

$= -\int e^t \left[\dfrac{1}{(-t)^2} + \dfrac{2}{(-t)^3}\right] dt + c$

$= -\int e^t \left[\dfrac{1}{t^2} + \left(\dfrac{-2}{t^3}\right)\right] dt + c$

$\boxed{= -e^t \dfrac{1}{t^2} + c}$

First Order with higher degrees :-

solvable for P

$$\left(\frac{dy}{dx}\right)^2$$

Solve $x^2\left(\frac{dy}{dx}\right)^2 - 4y^2 = 0$.

:- Let $\frac{dy}{dx} = P$

$$P^2 x^2 - 4y^2 = 0$$

$$P^2 x^2 = 4y^2$$

$$P^2 = \frac{4y^2}{x^2}$$

$$\boxed{P = \pm \frac{2y}{x}}$$

Let $P = \frac{2y}{x}$

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$\frac{dy}{y} = 2\int\frac{dx}{x}$$

$$\log y = 2\log x + \log c$$

$$\log y = \log(x^2 \cdot c)$$

$$\boxed{\therefore\ y = x^2 c}$$

Let $p = \frac{-2y}{x}$

$$\frac{dy}{dx} = \frac{-2y}{x}$$

$$\frac{dy}{y} = -2\int\frac{dx}{x}$$

$$\log y = -2\log x + \log c$$

$$\log y = \log(x^{-2} \cdot c)$$

$$\boxed{y = \frac{c}{x^2}}$$

Solution is

$$(y - x^2c)\left(y - \frac{c}{x^2}\right) = 0.$$

Q. Solve $x^2\left(\frac{dy}{dx}\right)^2 + xy\left(\frac{dy}{dx}\right) - 6y^2 = 0.$

Soln:- Let $\frac{dy}{dx} = p$

$p^2x^2 + pxy - 6y^2 = 0$

$p^2x^2 + 3pxy - 2pxy - 6y^2 = 0.$

$px(px + 3y) - 2y(px + 3y) = 0.$

$(px + 3y)\cdot(px - 2y) = 0.$

$px + 3y = 0$

$px = -3y$

$p = -\frac{3y}{x}$

$px - 2y = 0$

$px = 2y.$

$p = \frac{2y}{x}$

$\frac{dy}{dx} = \frac{-3y}{x}$

$\frac{dy}{y} = -\frac{3dx}{x}$

$\boxed{y = cx^{-3}}$

$\frac{dy}{dx} = \frac{2y}{x}$

$\frac{dy}{y} = \frac{2dx}{x}$

$\boxed{y = cx^2}$

Solution

$\boxed{(y - cx^{-3})(y - cx^2) = 0}$

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Q. Solve $y^2 + xyp - x^2p^2 = 0.$

Soln:-

$p^2x^2 - xyp - y^2 = 0$

$p^2x^2 (-xy)p + (-y)^2 = 0.$

$\boxed{ap^2 + bp + c = 0}$

$a = x^2 \quad \boxed{b = -xy} \quad \boxed{c = -y^2}$

$\Delta = b^2 - 4ac$

$= x^2y^2 - 4x^2(-y^2)$

$= x^2y^2 + 4x^2y^2$

$= 5x^2y^2$

$\sqrt{\Delta} = \sqrt{5}\, xy$

$p = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

$= \dfrac{xy \pm \sqrt{5}\, xy}{2x^2}$

$= xy \left(\dfrac{1 \pm \sqrt{5}}{2x^2} \right)$

$p = y/x \left(\dfrac{1 \pm \sqrt{5}}{2} \right)$

Now

$$p = \frac{y}{x}\left(\frac{1+\sqrt{5}}{2}\right)$$

$$\frac{dy}{dx} = \frac{y}{x}\left(\frac{1+\sqrt{5}}{2}\right)$$

$$\frac{dy}{y} = \left(\frac{1+\sqrt{5}}{2}\right)\frac{dx}{x}$$

$$y = cx^{\left(\frac{1+\sqrt{5}}{2}\right)}$$

$$p = \frac{y}{x}\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\frac{dy}{dx} = \frac{y}{x}\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\frac{dy}{dx} = \left(\frac{1-\sqrt{5}}{2}\right)\cdot\frac{dx}{x}.$$

$$y = cx^{\left(\frac{1-\sqrt{5}}{2}\right)}$$

solution

$$\boxed{\left(y - cx^{\left(\frac{1+\sqrt{5}}{2}\right)}\right)\left(y - cx^{\left(\frac{1-\sqrt{5}}{2}\right)}\right) = 0}$$

Type 3:

Q. Solve $y + px = p^2x^4$.

Soln:- $y = p^2x^4 - px$

w.r.t x

$$\frac{dy}{dx} = \left(4x^3p^2 + 2px^4\frac{dp}{dx}\right)$$

$$- \left(p + x\frac{dp}{dx}\right)$$

$$p - 4x^3p^2 + 2px^4\frac{dp}{dx} - p - x\frac{dp}{dx}$$

$$2px^4\frac{dp}{dx} - x\frac{dp}{dx} - x\frac{dp}{dx} + 4x^3p^2 - 2p = 0$$

$$x\frac{dp}{dx}(2px^3 - 1) + 2p(2px^3 - 1) = 0$$

$$\therefore (2px^3 - 1)\left(x\frac{dp}{dx} + 2p\right) = 0$$

$$\therefore x\frac{dp}{dx} + 2p = 0.$$

$$\therefore x\frac{dp}{dx} = -2p$$

$$\frac{dp}{p} = \frac{-2dx}{x}$$

$$\downarrow$$

$$\boxed{p = cx^{-2} = \frac{c}{x^2}}$$

Put value of p in ①

$$\boxed{y = x^4\left(\frac{c}{x^2}\right)^2 - x\left(\frac{c}{x^2}\right)}$$

Type-3 Equation solvable ~~too~~ for x.

$$X = p + p^4$$

w.r.t y.

$$\frac{dx}{dy} = \frac{dp}{dy} + 4p^3\frac{dp}{dy}$$

$$\frac{1}{p} = \frac{dp}{dy}\left(1 + 4p^3\right)$$

$$dy = p\,(1 + 4p^3)\,dp$$

$$dy = \int(p + 4p^4)\,dp$$

$$\boxed{y = \frac{p^2}{2} + \frac{4p^5}{5} + c}$$

Where p is any paremeter.