Poisson Distribution:

It is a discrete prob. distribution, which was discovered by Simon-Denis Poisson (1781-1840) and published in the year 1838.

∴ Poisson distribution is a limiting case of binomial distri. under the following condns.

1) n, the no. of trials, increases indefinitely, i.e. $n \to \infty$ (n is very large)
2) p, const. prob. of success in each trial, decreases indefinitely, i.e. $p \to 0$. (p is very small)
3) $np = \mu$, which is the expected no. of successes, remains const.

Defn.: If X is a discrete r.v that assumes only non-negative values such that its prob. mass fun is given by

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0,1,2,\dots ; \ \lambda > 0$$
$$= 0, \quad \text{otherwise}$$

then X is said to follow Poisson distribution, with the parameter λ.

Note: (i) $\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^{x}}{x!} = e^{-\lambda}e^{\lambda} = 1$

(ii) Poisson distribution occurs when there are events which do not occur as outcomes of a definite no. of trials of an experiment but which occur at random points of time and space, wherein our interest lies only in the no. of occurrences of the event, not in its non occurrences.

∴ Poisson Distribution as a limiting case of Binomial Distribution

Binomial distri. is

$$P(X=x) = {}^nC_x\, p^x\, q^{n-x}, \quad x = 0, 1, \ldots, n$$

Take $np = \lambda$ is finite

$$\therefore\ p = \frac{\lambda}{n}$$

$$\&\ q = 1-p = 1-\frac{\lambda}{n}$$

Take the limit as $n \to \infty$. Then the limiting case of Binomial distri. is

$$P(X=x) = \lim_{n\to\infty} \binom{n}{x} p^x q^{n-x}$$

$$= \lim_{n\to\infty} {}^nC_x \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n!}{(n-x)!\,x!} \frac{\lambda^x}{n^x} \frac{(1-\lambda/n)^n}{(1-\lambda/n)^x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n!}{(n-x)!\,n^x} \frac{(1-\lambda/n)^n}{(1-\lambda/n)^x}$$

$$= \frac{\lambda^x}{x!} \frac{\left[\lim_{n\to\infty} \frac{n!}{(n-x)!\,n^x}\right]\left[\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right]}{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^x} \quad \text{--- (1)}$$

Here $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^x = 1$ --- (2)

Also $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$ $\left(\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right)$ --- (3)

Moreover, $\lim_{n\to\infty} \frac{n!}{(n-x)!\,n^x}$

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-(x-1))(n-x)!}{(n-x)!\,n^x}$$

Page No.:

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-(x-1))}{n^x}$$

$$= \lim_{n\to\infty} \frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{(n-(x-1))}{n}$$

$$= \lim_{n\to\infty} 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)$$

$$= 1\cdot 1\cdot 1 \cdots \quad x \text{ times}$$

$$= 1^x = 1 \qquad \text{--- (4)}$$

$\therefore$ By result 1,2,3 & 4 we have

$$P(x) = \frac{\lambda^x}{x!}\,\frac{1\cdot e^{-\lambda}}{1}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \qquad x = 0,1,2,\cdots$$

$\therefore$ <u>Mean</u> :

$$\text{Mean} = \mu = E(x)$$

$$= \sum_{x\in R(x)} x\,p(x)$$

$$= \sum_{x=0}^{\infty} x\cdot\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{x\,e^{-\lambda}\,\lambda^{x-1}\cdot\lambda}{x(x-1)!}$$

$$= e^{-\lambda}\cdot\lambda \sum_{x=1}^{\infty} \frac{x\,\lambda^{x-1}}{x(x-1)!}$$

$$= e^{-\lambda}\cdot\lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda}\cdot\lambda\cdot e^{\lambda}$$

$$\therefore \boxed{E(x) = \lambda}$$

∴ **Variance:** $Var(x) = E(x^2) - (E(x))^2$

$$E(x^2) = \sum_{x \in R_x} x^2 P(x)$$

$$= \sum_{x=0}^{\infty} [x(x-1) + x] P(x)$$

$$= \sum_{x=0}^{\infty} x(x-1)P(x) + \sum x P(x)$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!} + E(x).$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-2)!} + E(x)$$

$$= e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2} \cdot \lambda^2}{(x-2)!} + E(x)$$

$$= e^{-\lambda} \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + E(x)$$

$$= e^{-\lambda} \lambda^2 e^{\lambda} + E(x)$$

∴ $E(x^2) = \lambda^2 + \lambda$

∴ $Var(x) = \lambda^2 + \lambda - (\lambda)^2$

$= \lambda$

∴ $S.D = \sqrt{Var(x)}$

$= \sqrt{\lambda}$

Page No.:

Recurrence formula:

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$\therefore P(x+1) = \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}$$

$$= \frac{e^{-\lambda}\lambda\lambda^{x}}{(x+1)x!} = \frac{\lambda}{x+1} \cdot \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$\therefore \boxed{P(x+1) = \frac{\lambda}{x+1} P(x)}$$

∴ Moment generating funⁿ of Poisson Distribution:

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad x = 0,1,2,\ldots$$

The moment generating funⁿ is given by

$$M_X(t) = E(e^{tX})$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^{t}\lambda)^{x}}{x!}$$

$$= e^{-\lambda} e^{\lambda e^{t}} = e^{\lambda(e^{t}-1)}$$

$$\boxed{M_X(t) = e^{\lambda(e^{t}-1)}}$$

$$\text{Mean} = E(X) = \mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(e^{-\lambda} e^{\lambda e^{t}}\right)\right]_{t=0}$$

$$= \left[e^{-\lambda} e^{\lambda e^{t}} \lambda e^{t}\right]_{t=0} = e^{-\lambda} e^{\lambda} \lambda = \lambda$$

Page No.:

$$\mu_2' = E(x^2) = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0}$$

$$= \left[e^{-\lambda} e^{\lambda e^t} (\lambda e^t)^2 + e^{-\lambda} e^{\lambda e^t} \lambda e^t\right]_{t=0}$$

$$= e^{-\lambda} e^{\lambda} \lambda^2 + e^{-\lambda} e^{\lambda} \lambda = \lambda^2 + \lambda$$

$$Var = E(x^2) - (E(x))^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

∴ Recurrence formula for central moment of poisson distri.

$$E(x) = \lambda$$

∴ we have $\mu_r = E(x-\lambda)^r$

$$= \sum_{x=0}^{\infty} (x-\lambda)^r \frac{\lambda^x e^{-\lambda}}{x!} \quad \text{--- (1)}$$

Diffing (1) w.r.t λ

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} \frac{1}{x!}\left[x\lambda^{x-1} e^{-\lambda} (x-\lambda)^r + \lambda^x(-e^{-\lambda})(x-\lambda)^r - \lambda^x e^{-\lambda} r (x-\lambda)^{r-1}\right]$$

$$= \sum_{x=0}^{\infty} \frac{1}{x!}\left[\lambda^{x-1} e^{-\lambda} (x-\lambda)^r (x-\lambda) - \lambda^x e^{-\lambda} r (x-\lambda)^{r-1}\right]$$

$$= \sum_{x=0}^{\infty} \frac{1}{x!} \lambda^{x-1} e^{-\lambda} (x-\lambda)^r (x-\lambda) - \sum_{x=0}^{\infty} \frac{1}{x!} \lambda^x e^{-\lambda} r (x-\lambda)^{r-1}$$

$$\therefore \lambda \frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} (x-\lambda)^{r+1} - \lambda r \sum_{x=0}^{\infty} \frac{1}{x!} \lambda^x e^{-\lambda} (x-\lambda)^{r-1}$$

$$\Rightarrow \lambda \frac{d\mu_r}{d\lambda} = \mu_{r+1} - \lambda r \mu_{r-1}$$

Hence $\mu_{r+1} = \lambda r \mu_{r-1} + \lambda \frac{d\mu_r}{d\lambda}$

$$\mu_{r+1} = \lambda\left(r\mu_{r-1} + \frac{d\mu_r}{d\lambda}\right)$$

Central moments μ_2, μ_3 & μ_4.

For $r=1$ $\quad \mu_2 = \lambda\left(\mu_0 + \frac{d\mu_1}{d\lambda}\right) = \lambda \quad (\mu_0 = 1, \mu_1 = 0)$

$r=2 \quad \mu_3 = \lambda\left(2\mu_1 + \frac{d\mu_2}{d\lambda}\right) = \lambda$

$r=3 \quad \mu_4 = \lambda\left(3\mu_2 + \frac{d\mu_3}{d\lambda}\right) = \lambda(3\lambda+1) = 3\lambda^2 + \lambda$

Examples:

Ex:1 The MGF of a r.v X is given by $M_X(t) = e^{3(e^t-1)}$. Find $P(X=1)$.

$\therefore \rightarrow$ We know that MGF for poisson distri is

$$M_Y(t) = e^{\mu(e^t-1)}$$

Given $\quad M_X(t) = e^{3(e^t-1)}$

Hence by uniqueness thm. of MGF r.v X has a Poisson distri with $\mu = 3$

Thus, $\quad P(X=x) = \frac{e^{-3}\, 3^x}{x!}, \quad x = 0, 1, 2, \ldots$

& so

$$P(X=1) = \frac{e^{-3} \cdot 3^1}{1!}$$

$$= 3e^{-3}$$

$$= 0.1494$$

Page No.:

Ex: 2 Suppose the no. of accidents occuring weekly on a particular stretch of a highway follow a poisson distribution with mean 3. Calculate the prob. that there is at least one accident this week.

:→ Given: $\lambda = 3$

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0,1,2,\ldots$$

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - P(X=0)$$

$$= 1 - \frac{e^{-3}3^{0}}{0!} = 1 - e^{-3} = 0.9502.$$

Ex: 3 - If X and Y are Independent poisson variables such that $P(X=1) = P(X=2)$ and $P(Y=2) = P(Y=3)$. Find the Variance of $(X-2Y)$.

:→

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0,1,2,\ldots$$

$$P(Y=y) = \frac{e^{-\lambda}\lambda^{y}}{y!}, \quad y = 0,1,2,\ldots$$

Given: $P(X=1) = P(X=2)$

ie $\frac{e^{-\lambda}\lambda}{1!} = \frac{e^{-\lambda}\lambda^{2}}{2!} \Rightarrow \frac{\lambda}{1} = \frac{\lambda^{2}}{2} \Rightarrow \lambda = 2$

$\therefore \mathrm{Var}(X) = 2$

Again $P(Y=2) = P(Y=3)$

$$\frac{e^{-\lambda}\lambda^{2}}{2!} = \frac{e^{-\lambda}\lambda^{3}}{3!} \Rightarrow \frac{\lambda^{2}}{2} = \frac{\lambda^{3}}{6} \Rightarrow \lambda = 3$$

$\therefore \mathrm{Var}(Y) = 3$

$$\mathrm{var}(X-2Y) = \mathrm{var}(X) + 4\,\mathrm{var}(Y)$$

$$= 2 + 4(3)$$

$$= 14$$

Page No.:

Ex:4 If the MGF of the r.v is $e^{4(e^t-1)}$, Find $P(X=\mu+\sigma)$

$\Rightarrow$ $M_x(t) = e^{\lambda(e^t-1)}$

$\therefore$ Mean $= \lambda = 4 =$ vari

$S.d = \sqrt{var} = 2$

$P(X=x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$

$\therefore P(X=\mu+\sigma) = P(X=6) = \dfrac{e^{-4}4^6}{6!} = 0.1042$

Ex:5 The atoms of a radioactive elt. of are randomly disintegrating. If every gram of this elt., on avg emits 3.9 alpha particles per second, what is the prob. that during the next second the no. of alpha particles emitted from 1g is (1) at most 6 2) at least 2 and 3) at least 3 and at most 6

$\Rightarrow$ Given: mean $= \lambda = 3.9$

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-3.9}(3.9)^x}{x!}, \quad x=0,1,2,\ldots$$

(1) P(at most 6)

$P(X\le 6) = P(X=0)+P(X=1)+\cdots+P(X=6)$

$$= e^{-3.9}\left[\frac{(3.9)^0}{0!}+\frac{(3.9)^1}{1!}+\frac{(3.9)^2}{2!}+\cdots+\frac{(3.9)^6}{6!}\right]$$

$= 0.899$

(2) P(at least 2)

$P(X\ge 2) = 1-[P(x=0)+P(x=1)]$

$$= 1-e^{-3.9}\left[\frac{(3.9)^0}{0!}+\frac{(3.9)}{1!}\right]$$

$= 0.901$

3) P(at least 3 and at most 6) $= P(3\le x\le 6)$

$= 0.646$

Page No.:

Ex: At a busy traffic junction the prob. of an individual having an accident is $p = 0.0001$. However, during a certain part of the day 1000 cars pass through the junction. What is the prob. that two or more accidents occur during that period? ($e^{-0.1} = 0.9048$)

:→

$p = 0.0001$

$n = 1000$

Mean $= \lambda = n \times p = 0.1$

X = no. of accidents during a certain part of the day.

P(2 or more accident occur)

$P(X \geq 2) = 1 - P(X < 2)$

$= 1 - [P(X=0) + P(X=1)]$

$= 1 - \left[\frac{e^{-0.1}(0.1)^0}{0!} + \frac{e^{-0.1}(0.1)^1}{1!}\right]$

$= 0.0952$

Ex: Wireless sets are manufactured with 25 soldered joints each on the avg. 1 defective joint in 500. How many sets can be expected to be free from defective joints in a consignment of 10,000 sets.

:→

$p = \frac{1}{500}$

$n = 25$

$\lambda = np = 0.05$

X = no. of defective joints in a set

P(no joint is defective) $= P(X=0)$

$= \frac{e^{-(0.05)}(0.05)^0}{0!} = 0.95122$

∴ The expected no. of sets free from defects in 10000 sets is $10000 \times 0.95122 = 9512$

Ex: Find the prob. that at most 5 defective fuses will be found in a box of 200 fuses if experience shows that 2% of such fuses are defective.

Sol→ $n = 200$

$p = 2\% = 0.02$

Mean $\lambda = np = 4$

X – detective fuse

$P(\text{at most } 5) = P(X \leq 5)$

$= P(X=0) + \cdots + P(X=5)$

$= e^{-4}(42.866) = 0.785$

Ex: Fit a poisson distribution to the following data and calculate the theoretical frequencies.

Deaths:	0	1	2	3	4
Frequencies:	122	60	15	2	1

Sol→

$N = \sum f = 122+60+15+2+1 = 200$

$\sum fx = 0\times122 + 1\times60 + 2\times15 + 3\times2 + 4\times1 = 100$

Mean $= \frac{\sum fx}{N} = \frac{100}{200} = 0.5 = \lambda$

$\therefore P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.5}(0.5)^x}{x!}, \quad x = 0,1,2,\ldots$

Hence the theoretical frequencies are given by

$f(x) = N.P(X=x)$

$= 200 \times \frac{e^{-0.5}(0.5)^x}{x!}$

Deaths	0	1	2	3	4
Observed freq	122	60	15	2	1
Expected freq.	121	61	15	3	0

Page No.:

Ex: Fit a poisson distribution for the following distribution

x	0	1	2	3	4	5
f	142	156	69	27	5	1

→ Mean $\mu = \frac{\Sigma fx}{\Sigma f} = \frac{400}{400} = 1$

$N \times f(0) = N e^{-\lambda} = 147.1518$

$f(x+1) = \frac{\lambda}{x+1} f(x)$

$N f(x+1) = [N f(x)] \cdot \frac{\lambda}{x+1}$

∴ $N f(1) = 147.1518 \times \lambda = 147.1518$

$N f(2) = 147.1518 \times \frac{\lambda}{2} = 73.5759$ & so on

No. of accidents	0	1	2	3	4	5
observed	142	156	69	27	5	1
Expected	147.1	147.15	73.57	24.52	6.13	1.22

Ex: The no. of planes landing at an airport in a 30 minutes interval obeys the Poisson law with mean 25. Use Chebyshev's inequality to find the least chance that the no. of planes landing within the 30 minutes interval will be between 15 and 35.

→ X – no. of planes landing at an airport in a 30 min interval

$X \sim P(25)$ ∴ Mean $\lambda = 25$ Var $= 25 = \sigma^2$

By chebyshev's Inequality

$P(|X-\mu| \le c) \ge \frac{\sigma^2}{c^2}$

ie $P(15 \le X \le 35) = P(|X-\mu| \le 10) \ge \frac{25}{10^2} = \frac{1}{4}$