PAGE NO..
DATE / /

Unit 2 : Infinite Sequence and Series.

Does the sequence whose n^{th} term is $a_n = \left(\frac{n+1}{n-1}\right)^n$ converge ?

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

$$\lim_{n\to\infty} \left\{\frac{n+1}{n-1}\right\}^n = \lim_{n\to\infty} \left\{\frac{(1+\frac{1}{n})^n}{(1-\frac{1}{n})^n}\right\}$$

$$= \frac{e}{e^{-1}}$$

$$= e^2 \quad \text{(finite value)}$$

Convergent Sequence.

Prove that the sequence $a_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$

$n \in \mathbb{N}$ is monotonic increasing and bounded.

$$a_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$$

$$a_{n+1} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{(n+1)!}$$

$$a_{n+1} - a_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{(n+1)!} - \left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}\right]$$

$$= \frac{1}{(n+1)!} > 0$$

Hence, $\{a_n\}$ is monotonic increasing.

Consider : $k! = 1 \times 2 \times 3 \times \ldots \times k$

$\geq 1 \times 2 \times 2 \times 2 \ldots \times 2$

$= 2^{k-1}$

$k! \geq 2^{k-1} \Rightarrow \frac{1}{k!} \leq \frac{1}{2^{k-1}}$ for all $k = 1, 2, \ldots, n$

$$a_n = \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$$

$$\leq 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n-1}}$$

$$= \frac{1\left[1 + \left(\frac{1}{2}\right)^n\right]}{1 - \frac{1}{2}}$$

~~= 2~~ ($\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n-1}}\right.$ is in geometric progression.

So, $S_n = \frac{a(1-r^n)}{1-r}$, a is first term, r is ratio)

$$= 2\left(1 - \frac{1}{2^n}\right)$$

$$= 2 - \frac{1}{2^{n-1}}$$

$$< 2$$

Thus $\{a_n\}$ is bounded.

$\therefore \{a_n\}$ is convergent.

Scanned by CamScanner

PAGE NO.:
DATE: /

4) Determine whether following series ~~convergent~~ converges or diverge. Find sum of series

$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \ldots$

$$\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} (-1)^n (-1)$$

$$U_n = \frac{(-1)(-1)^n}{2^{n-1}}$$

By root test

$$(U_n)^{1/n} = \frac{(-1)^{1/n}(-1)}{2^{\frac{n-1}{n}}}$$

$$= \frac{(-1)^{1/n}(-1)}{2^{1-1/n}}$$

$$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \frac{(-1)^{1/n}(-1)}{2^{1-1/n}}$$

$$= \frac{(-1)}{2}$$

$$= -1/2 < 1 \Rightarrow |-\tfrac{1}{2}| = \tfrac{1}{2} < 1$$

$\therefore \sum U_n$ is convergent.

$a = 1, \quad r = -1/2$

$$S_n = \frac{a}{1-r} = \frac{1}{1+1/2}$$

$$\boxed{S_n = \frac{2}{3}}$$

PAGE NO..
DATE / /

5) Test the Convergence of $\sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{1}{5^n}\right)$

$$\sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{1}{5^n}\right)=\sum_{n=1}^{\infty}\frac{1}{2^n}+\sum_{n=1}^{\infty}\frac{1}{5^n}$$

$\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)$ & $\sum_{n=1}^{\infty}\frac{1}{5^n}$ are geometric series with $r=\frac{1}{2}$ & $\frac{1}{5}$ respectively, which lies in the interval $(-1,1)$.

Thus, Both the series are cgt & their sums are respectively

$$\frac{a}{1-r}=\frac{1/2}{1-\frac{1}{2}}=1.$$

$$\frac{a}{1-r}=\frac{1/5}{1-\frac{1}{5}}=\frac{1}{4}$$

(In both the case $a=1$).

$$\sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{1}{5^n}\right)=1+\frac{1}{4}=\frac{5}{4}.$$

6) Test Convergence of $\sum_{n=1}^{\infty}\frac{4^n+1}{5^n}$.

$$\sum U_n=\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n+\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n$$

$$=\sum V_n+\sum W_n$$

$$=\left[\frac{4}{5}+\left(\frac{4}{5}\right)^2+\dots\right]+\left[\frac{1}{5}+\left(\frac{1}{5}\right)^2+\dots\right]$$

$a_1=4/5$. $a_2=1/5$,

$r_1=4/5$. $r_2=1/5$

$|r_1| = \frac{4}{5} < 1$ $\qquad$ $|r_2| = \frac{1}{5} < 1$

$\sum V_n$ is cgt. $\qquad$ $\sum W_n$ is cgt.

$\therefore \sum U_n$ is cgt.

$$S_n = \frac{a_1}{1-r_1} + \frac{a_2}{1-r_2}$$

$$= \frac{4/5}{1-4/5} + \frac{1/5}{1-1/5}$$

$$= \frac{4/\cancel{5}}{1/\cancel{5}} + \frac{1/\cancel{5}}{4/\cancel{5}}$$

$$= 4 + 1/4$$

$$= \frac{17}{4}$$

7) Test the Series (By Integral Test).

(i) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$.

$$f(x) = \frac{1}{x \log x}$$

$$\lim_{b\to\infty} \int_2^b f(x) = \lim_{b\to\infty} \int_2^b \frac{1}{x \log x} dx$$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u} du \qquad (\log x = u)$$

PAGE NO..
DATE / /

$$= \lim_{b\to\infty} [\ln y]_{\ln 2}^{\ln b}$$

$$= \lim_{b\to\infty} [\ln(\ln b) - \ln(\ln 2)]$$

$$= \infty$$

$\therefore$ Given Series is divergent.

(ii) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \log n \sqrt{(\log n)^2 - 1}}$

$$f(x) = \frac{1}{x \log x \sqrt{(\log x)^2 - 1}}$$

is continuous & positive.

It is decreasing for $x \geq 3$ as shown below.

$$n = 3 \Rightarrow f(3) = \frac{1}{3(\log 3)\sqrt{(\log 3)^2 - 1}} \simeq -0.6820$$

$n = 4$

$$f(4) = \frac{1}{4(\log 4)\sqrt{(\log 4)^2 - 1}} \simeq -0.7877$$

$\vdots \qquad \vdots \qquad \vdots$

$$\lim_{b\to\infty} \int_3^b f(x)\,dx = \lim_{b\to\infty} \int_3^b \frac{1}{x(\log x)\sqrt{(\log x)^2 - 1}}\,dx$$

$$= \lim_{b\to\infty} \int_{\ln 3}^{\ln b} \frac{1}{y\sqrt{y^2 - 1}}\,dy \qquad (\because y = \ln x)$$

$$= \lim_{b\to\infty} (\sec^{-1} y)_{\ln 3}^{\ln b}$$

$$= \lim_{b\to\infty} \left[\sec^{-1}(\log b) - \sec^{-1}(\log 3)\right]$$

$$= \frac{\pi}{2} - \sec^{-1}(\log 3)$$

(finite value)

$\therefore$ Given Series is convergent.

8) Test the Series (By Comparisicon Test).

(i) $$\sum_{n=1}^{\infty} \frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}}$$

$$U_n = \frac{n^{2/3}\left(2-\frac{1}{n^2}\right)^{1/3}}{n^{3/4}\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{1/4}}$$

let $$\sum V_n = \frac{n^{2/3}}{n^{3/4}}$$

$$= n^{2/3 - 3/4}$$

$$= \frac{1}{n^{1/12}}$$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\left(2-\frac{1}{n^2}\right)^{1/3}}{\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{1/4}}$$

$$= \frac{(2)^{1/3}}{(3)^{1/4}}$$ (finite & non zero)

Now $$\sum V_n = \sum \frac{1}{n^{1/12}}$$

$$p = \frac{1}{12} < 1$$

$\therefore \sum V_n$ is divergent

$\therefore \sum U_n$ is alsee dgt. (By Comparision Test)

(ii) $\sum_{n=1}^{\infty} \frac{1}{1+2^2+3^2+4^2+\dots}$

$$U_n = \sum_{n=1}^{\infty} \frac{1}{1^2+2^2+3^2+\dots+n^2}$$

$$= \frac{1}{\frac{n(n+1)(2n+1)}{6}}$$

$$= \frac{6}{n^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

let $\sum V_n = \sum \frac{1}{n^3}$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{6}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

$$= \frac{6}{2}$$

$= 3$ (finite and non zero)

here $\sum V_n = \sum \frac{1}{n^3}$

$$p = 3 > 1$$

$\sum V_n$ is cgt.

$\sum U_n$ is also convergent.

(By Comparision Test)

$\cancel{\text{(iii)} \sum_{n=1}^{\infty} \sqrt[3]{n^3+1} - n}$

$=$

(iii) $\sum_{n=1}^{\infty} \sqrt[3]{n^3+1} - n$

$$U_n = \sqrt[3]{n^3+1} - n$$

$$= (n^3+1)^{\frac{1}{3}} - n$$

$$= n\left[\left(1+\frac{1}{n^3}\right)^{\frac{1}{3}}\right] - n$$

$\cancel{\therefore \sum V_n = \sum n}$

$$= n\left[1 + \frac{1}{3}\cdot\frac{1}{n^3} + \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\frac{1}{2!}\,\frac{1}{n^6} + \dots\right] - n$$

$$= \left[\cancel{n} + \frac{1}{3n^2} - \frac{1}{9n^5} + \dots\right] - \cancel{n}$$

$$\cancel{\frac{U_n}{V_n} = \frac{n\left[\left(1+\frac{1}{n^3}\right)^{\frac{1}{3}} - 1\right]}{n}}$$

$$\cancel{= \left(1+\frac{1}{n^3}\right)^{1/3} - 1}$$

$$\cancel{\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty}\left[1+\frac{1}{n^3}\right]^{\frac{1}{3}} - 1}$$

$$\cancel{= 0}$$

$$U_n = \frac{1}{3n^2} - \frac{1}{9n^5} + \dots$$

$$U_n = \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9n^3} + \dots\right]$$

$$\sum V_n = \sum \frac{1}{n^2} \implies p = 2 > 1$$

$\therefore \sum V_n$ is convergent.

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{1}{3} \quad (\text{finite \& non zero})$$

$\therefore \sum V_n$ is cgt.

$\therefore \sum U_n$ is dgt.

Scanned by CamScanner

PAGE NO..
DATE / /

(iv) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{4!} + \ldots$$

$$\sum U_n = \sin\left(\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{n^3 . 3!} + \frac{1}{n^5 . 5!} - \frac{1}{n^7 4!} + \ldots$$

$$\sum U = \frac{1}{n}\left[1 - \frac{1}{n^2 3!} + \frac{1}{n^4 5!} - \frac{1}{n^6 7!} + \ldots\right]$$

let $\sum V_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty}\left[1 - \frac{1}{n^2 3!} + \frac{1}{n^4 5!} - \frac{1}{n^6 7!} + \ldots\right]$$

$= 1$ (finite & non zero)

here $\sum V_n = \sum \frac{1}{n}$
which is Harmonic series and it is always divergent.
$\therefore \sum U_n$ is also divergent
(By Comparision Test)

(v) $\sum_{n=1}^{\infty} \frac{n+1}{n^3 - 3n + 2}$.

$$U_n = \frac{n+1}{n^3 - 3n + 2}$$

$$= \frac{n\left(1 + \frac{1}{n}\right)}{n^3\left(1 - \frac{3}{n^2} + \frac{2}{n^3}\right)}$$

$$U_n = \frac{1}{n^2} \frac{\left(1+\frac{1}{n}\right)}{\left(1-\frac{3}{n^2}+\frac{2}{n^3}\right)}$$

$$V_n = \frac{1}{n^2}$$

$$\Sigma V_n = \Sigma \frac{1}{n^2}$$

$$\frac{U_n}{V_n} = \frac{\left(1+\frac{1}{n}\right)}{\left(1-\frac{3}{n^2}+\frac{2}{n^3}\right)}$$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)}{\left(1-\frac{3}{n^2}+\frac{2}{n^2}\right)}$$

$$= 1 \quad \text{(finite \& non zero)}$$

here $\Sigma V_n = \Sigma \frac{1}{n^2}$.

$p = 2 > 1$

ΣV_n is convergent.

$\therefore$ ΣU_n is also convergent.

(By Comparisision Test)

9) Test the Series (By ratio Test).

(i) $\sum_{n=1}^{\infty} \frac{n^3+1}{2^n+2}$

$$U_n = \frac{n^3+1}{2^n+2}$$

$$U_{n+1} = \frac{(n+1)^3+1}{2^{n+1}+2}$$

Page
Date

$$\frac{U_n}{U_{n+1}} = \frac{n^3+1}{2^n+2} \cdot \frac{2^{n+1}+2}{(n+1)^3+1}$$

$$= \frac{(n^3+1)(2^n \cdot 2+2)}{(2^n+2)[(n+1)^3+1]}$$

$$= \frac{\cancel{n^3} \cdot \cancel{2^n}\left[1+\frac{1}{n^3}\right]\left[2+\frac{2}{2^n}\right]}{\cancel{n^3} \cdot \cancel{2^n}\left[1+\frac{2}{2^n}\right]\left[\left(1+\frac{1}{n}\right)^3+\frac{1}{n^3}\right]}$$

$$= 1$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n^3}\right)\left(2+\frac{2}{2^n}\right)}{\left(1+\frac{2}{2^n}\right)\left[\left(1+\frac{1}{n}\right)^3+\frac{1}{n^3}\right]}$$

$$= \frac{(1)(2)}{(1)(1)}$$

$$= 2 > 1$$

$\therefore$ ΣU_n is convergent.

PAGE NO..
DATE .

(ii) $x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots \quad (x > 0)$

$$U_n = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} (-1)^{n-1}$$

$$U_{n+1} = \sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1} (-1)^{n}$$

$$\frac{U_n}{U_{n+1}} = \frac{x^{2n-1}(-1)^{n-1}}{2n-1} \cdot \frac{2n+1}{x^{2n+1}(-1)^n}$$

$$= \frac{\cancel{x^{2n}}\,\cancel{(-1)^n}\,(2n+1)}{x(2n-1)\cdot(-1)\,\cancel{(x)^{2n}}\cdot x\,\cancel{(-1)^n}}$$

$$= \frac{2n+1}{-x^2(2n-1)}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{2n+1}{-x^2(2n-1)}$$

$$\lim_{n\to\infty} \frac{\cancel{n}(2+\frac{1}{n})}{-x^2\cdot\cancel{n}(2-\frac{1}{n})}$$

$$= \frac{\cancel{2}}{-\cancel{2}x^2}$$

$$= \frac{-1}{x^2}$$

$\therefore x$

10) Test the series (By Root test)

(i) $\sum_{n=1}^{\infty} \dfrac{2^{3n}}{3^{2n}}$

$$U_n = \frac{2^{3n}}{3^{2n}}$$

$$(U_n)^{1/n} = \frac{2^3}{3^2}$$

$$= \frac{8}{9} < 1$$

$\therefore \Sigma U_n$ is convergent. (By root test)

(ii) $\sum_{n=1}^{\infty} \dfrac{[(n+1)x]^n}{(n)^{n+1}}$

$$U_n = \frac{[(n+1)x]^n}{(n)^{n+1}}$$

$$(U_n)^{1/n} = \frac{\{[(n+1)x]^n\}^{1/n}}{\{(n)^{n+1}\}^{1/n}}$$

$$= \frac{(n+1)x}{(n)^{\cancel{n}+1-\cancel{n}}}$$

$$= \frac{\cancel{n}x}{\cancel{n}} + \frac{x}{n}$$

$$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \left(x + \frac{x}{n}\right)$$

$= x$ if $x < 1 \Rightarrow U_n$ is ~~dgt~~ cgt
$x > 1 \Rightarrow U_n$ is dgt

PAGE NO..
DATE . /

11) Test the Series: (By Rabbe's Test)

(i) $\sum_{n=1}^{\infty} \frac{1}{1^2+2^2+3^2+\dots+n^2}$

$$U_n = \frac{1}{1^2+2^2+3^2+\dots+n^2}$$

$$\cancel{U_{n+1} = \frac{1}{1^2+2^2+\dots+n^2+(n+1)^2}}$$

$$U_n = \frac{1}{\frac{n(n+1)(2n+1)}{6}}$$

$$\cancel{\frac{U_n}{U_{n+1}} - 1} =$$

$$U_n = \frac{6}{n(n+1)(2n+1)}$$

$$U_{n+1} = \frac{6}{(n+1)(n+2)(2n+3)}$$

$$\frac{U_n}{U_{n+1}} = \frac{\cancel{6}}{n\cancel{(n+1)}(2n+1)} \cdot \frac{\cancel{(n+1)}(n+2)(2n+3)}{\cancel{6}}$$

$$n\left(\frac{U_n}{U_{n+1}} - 1\right) = n\left(\frac{(n+2)(2n+3)}{n(2n+1)} - 1\right)$$

$$= \cancel{n}\left[\frac{\cancel{2n^2}+4n+3n+6-\cancel{2n^2}-n}{\cancel{n}(2n+1)}\right]$$

$$= \frac{6n+6}{(2n+1)}$$

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{6n+6}{2n+1}$$

PAGE NO.:
DATE / /

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}}-1\right) = \lim_{n\to\infty} \frac{n\left(6+6/n\right)}{n\left(2+1/n\right)}$$

$$= \frac{6}{2}$$

$$= 3 > 1$$

$\therefore \Sigma U_n$ is convergent.

(ii) $\frac{2}{7} + \frac{2\cdot5}{7\cdot10} + \frac{2\cdot5\cdot8}{7\cdot10\cdot13} + \ldots$

$$U_n = \frac{2\cdot5\cdot8\cdots(3n-1)\,\cancel{(3n+2)}}{7\cdot10\cdot13\cdots(3n+4)}$$

$$U_{n+1} = \frac{2\cdot5\cdot8\cdots(3n-1)(3n+2)}{7\cdot10\cdot13\cdots(3n+4)(3n+7)}$$

$$\frac{U_n}{U_{n+1}} = \frac{3n+7}{3n+2}$$

$$n\left(\frac{U_n}{U_{n+1}}-1\right) = n\left(\frac{\cancel{3n}+7-\cancel{3n}-2}{3n+2}\right)$$

$$= n\left(\frac{5}{3n+2}\right)$$

$$n\left(\frac{U_n}{U_{n+1}}-1\right) = \frac{5n}{3n+2} = \frac{5}{3+2/n}$$

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}}-1\right) = \lim_{n\to\infty} \frac{5}{3+2/n}$$

$$= \frac{5}{3} > 1$$

$\therefore \Sigma U_n$ is convergent.

Scanned by CamScanner

PAGE NO..
DATE : /

12) Test the Series (By Leibnitz's Test)

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots$$

$$U_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(5n)}{(3)^{n-1}}$$

$$|U_n| = \frac{5n}{(3)^{n-1}}$$

$$|U_{n+1}| = \frac{5(n+1)}{3^n}$$

$$|U_n| - |U_{n+1}| = \frac{5n}{3^{n-1}} - \frac{5n+5}{3^n}$$

$$= \frac{15n}{3^n} - \frac{5n+5}{3^n}$$

$$= \frac{10n-5}{3^n} > 0$$

$$\therefore |U_n| > |U_{n+1}|$$

$$\lim_{n\to\infty} |U_n| = \lim_{n\to\infty} \frac{5n}{(3)^{n-1}}$$

$$= 0$$

$\therefore$ ΣU_n is convergent.

PAGE NO..
DATE / /

13) Check for the convergence of the series.

$$\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{7\cdot 8} + \dots$$

$$U_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n)}$$

$$U_n = \frac{(-1)^{n-1}}{(2n-1)(2n)}$$

$$U_{n+1} = \frac{(-1)^n}{(2n+1)(2n+2)}$$

Ratio Test,

$$\frac{U_n}{U_{n+1}} = \frac{(-1)^{n-1}}{(2n-1)(2n)} \cdot \frac{(2n+1)(2n+2)}{(-1)^n}$$

$$= \frac{\cancel{(-1)^n}(2n+1)(2n+2)}{(-1)\cancel{(-1)^n}(2n-1)(2n)}$$

$$= \frac{(2n+1)(2n+2)}{(-1)(2n-1)(2n)}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(-1)(2n-1)(2n)}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}(2+\frac{1}{n})(2+\frac{2}{n})}{\cancel{n^2}(2-\frac{1}{n})(2)(-1)}$$

$$= \frac{(4)}{(-1)(4)}$$

$$= -1 < 0$$

$\therefore \sum U_n$ is convergent.

Check for absolute or conditional convergence

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$$

$$U_n = (-1)^n \cdot \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

$$|U_n| = \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

$$|U_{n+1}| = \frac{1}{\sqrt{n+1}+\sqrt{n+2}}$$

$$|U_n| - |U_{n+1}| = \frac{1}{\sqrt{n}+\sqrt{n+1}} - \frac{1}{\sqrt{n+1}+\sqrt{n+2}}$$

$$= \frac{\cancel{\sqrt{n+1}} + \sqrt{n+2} - \sqrt{n} - \cancel{\sqrt{n+1}}}{(\sqrt{n}+\sqrt{n+1})(\sqrt{n+1}+\sqrt{n+2})}$$

$$= \frac{\sqrt{n+2} - \sqrt{n}}{(\sqrt{n}+\sqrt{n+1})(\sqrt{n+1}+\sqrt{n+2})} > 0$$

$$|U_n| > |U_{n+1}|$$

$$\lim_{n\to\infty} |U_n| = \lim_{n\to\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}\left[1+\left(1+\frac{1}{\sqrt{n}}\right)\right]} = 0$$

By Leibnitz's test ΣU_n is cgt.

Here, $|U_n| = \dfrac{1}{\sqrt{n}+\sqrt{n+1}}$

$$= \frac{1}{\sqrt{n}\left[1+\left(1+\frac{1}{\sqrt{n}}\right)\right]}$$

PAGE NO..
DATE / /

Now, let $\Sigma V_n = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{|U_n|}{V_n} = \lim_{n\to\infty} \frac{1}{\left[1+\left(1+\frac{1}{\sqrt{n}}\right)\right]}$$

$= \frac{1}{2}$ (finite & non zero)

$\Sigma |U_n|$ and ΣV_n both are cgt. or dgt.

$$\Sigma V_n = \frac{1}{n^{1/2}}$$

$$p = \frac{1}{2} < 1$$

$\therefore \Sigma V_n$ is dgt.

$\therefore \Sigma |U_n|$ is also dgt.

ΣU_n is conditionally cgt.

15) For what values of x the series converges absolutely or conditional.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3x-1)^n}{(n)^2}$$

$$U_n = \frac{(-1)^{n-1}(3x-1)^n}{(n)^2}$$

$$|U_n| = \frac{(3x-1)^n}{n^2}$$

$$|U_{n+1}| = \frac{(3x-1)^{n+1}}{(n+1)^2}$$

Scanned by CamScanner

PAGE NO.:
DATE: /

$$\left|\frac{U_n}{U_{n+1}}\right| = \frac{(3x-1)^n}{n^2} \cdot \frac{(n+1)^2}{(3x-1)^{n+1}}$$

$$= \frac{\cancel{(3x-1)^n}}{n^2} \cdot \frac{(n+1)^2}{\cancel{(3x-1)^n}\cdot(3x-1)}$$

$$= \frac{(n+1)^2}{n^2(3x-1)}$$

$$= \frac{\cancel{n^2}(1+\frac{1}{n})^2}{\cancel{n^2}(3x-1)}$$

$$\lim_{n\to\infty}\left|\frac{U_n}{U_{n+1}}\right| = \lim_{n\to\infty}\frac{(1+\frac{1}{n})^2}{(3x-1)}$$

$$= \frac{1}{3x-1}$$

here $\left|\frac{U_n}{U_{n+1}}\right|$ is converges for $\left|\frac{U_n}{U_{n+1}}\right| > 1$

and diverges for $\left|\frac{U_n}{U_{n+1}}\right| < 1$

$$\therefore \left|\frac{1}{3x-1}\right| > 1$$

$$\therefore |3x-1| < 1$$

$$\therefore -1 < 3x-1 < 1$$

$$0 < 3x < 2$$

$$\therefore 0 < x < 2/3$$

PAGE NO..
DATE / /

$\therefore$ Given Series is absolute convergent for $x \in (0, \frac{2}{3})$.

16) Find ROC and IOC of convergence of the series.

(i) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x)^{2n-1}}{2n-1}$

$$U_n = \frac{(-1)^{n-1} (x)^{2n-1}}{2n-1}$$

$$|U_n| = \frac{(x)^{2n-1}}{2n-1}$$

$$|U_{n+1}| = \frac{(x)^{2n+1}}{2n+1}$$

$$\lim_{n \to \infty} \left| \frac{U_n}{U_{n+1}} \right| = \lim_{n \to \infty} \frac{(x)^{2n-1}}{2n-1} \cdot \frac{2n+1}{(x)^{2n+1}}$$

$$= \lim_{n \to \infty} \frac{\cancel{x^{2n}}}{x(2n-1)} \cdot \frac{2n+1}{\cancel{x^{2n}} \cdot x}$$

$$= \lim_{n \to \infty} \frac{2n+1}{x^2(2n-1)}$$

$$= \lim_{n \to \infty} \frac{\cancel{n}(2 + \frac{1}{n})}{x^2 \cdot \cancel{n}(2 - \frac{1}{n})}$$

$$= \frac{1}{x^2}$$

PAGE NO.:
DATE: /

By ratio test

$\left|\frac{1}{x^2}\right| > 1$ $\qquad |x^2| < 1$

$0 < x^2 < 1$

$-1 < x < 1$ cgt.

$\left|\frac{1}{x^2}\right| < 1 \Rightarrow |x^2| > 1$ dgt.

$x > 1.$

$\left|\frac{1}{x^2}\right| = 1 \Rightarrow x^2 = 1 \Leftrightarrow x = 1$ test fails.

For $x = 1.$

$$U_n = \frac{(-1)^{n-1}(1)^{2n-1}}{2n-1}$$

$$= \frac{(-1)^{n-1}}{2n-1}$$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{(-1)^{n-1}}{n\left(2-\frac{1}{n}\right)}$$

$$= \infty.$$

$\therefore U_n$ is dgt.

$\therefore$ IOC : $-1 < x < 1$ or $(-1, 1)$

ROC $= 1.$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+3)^{2n+1}}{n!}$

PAGE NO..
DATE / /

$$\cancel{U_n = \frac{(2x+3)^{2n+1}}{n!}}$$

$$\cancel{U_{n+1} = \frac{(2x+3)^{2n+3}}{(n+1)!}}$$

$$\cancel{\frac{U_n}{U_{n+1}} = \frac{(2x+3)^{2n+1}}{n!} \cdot \frac{(n+1)(n!)}{(2x+3)^{2n+1}(2x+3)^2}}$$

$$\cancel{\frac{U_n}{U_{n+1}} = \frac{n+1}{(2x+3)^2}}$$

$$U_n = \frac{(2x+3)^{2n+1}}{n!}$$

$$(U_n)^{1/n} = \frac{(2x+3)^{2+\frac{1}{n}}}{(n!)^{1/n}}$$

$$= \frac{n}{n} \frac{(2x+3)^{2+\frac{1}{n}}}{(n!)^{1/n}}$$

$$(U_n)^{1/n} = \frac{e\,(2x+3)^{2+1/n}}{n}$$

$$\lim_{n\to\infty} (U_n)^{1/n} = 0 < 1$$

$\therefore \sum_{n=1}^{\infty} U_n$ is cgt.

for all x $\sum_{n=1}^{\infty} U_n$ is cgt.

$R = \infty$

PAGE NO.:
DATE: /

Find ROC and Interval Of Convergence of the series ~~ex~~ $\sum_{n=1}^{\infty}$

$$1 - \frac{1}{2}(x-2) + \frac{1}{2^2}(x-2)^2 + \dots + \left(\frac{-1}{2}\right)^n (x-2)^n + \dots$$

$$U_n = (-1)^{n-1}\frac{(x-2)^{n-1}}{2^{n-1}}$$

$$\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{(x-2)^{n-1}}{2^{n-1}}$$

$$|U_n| = \frac{(x-2)^{n-1}}{2^{n-1}}$$

$$|U_{n+1}| = \frac{(x-2)^n}{2^n}$$

$$\left|\frac{U_n}{U_{n+1}}\right| = \frac{(x-2)^{n-1}}{2^{n-1}} \cdot \frac{2^n}{(x-2)^n}$$

$$= \frac{\cancel{(x-2)^n} \cdot 2}{(x-2)\cancel{2^n}} \cdot \frac{\cancel{2^n}}{\cancel{(x-2)^n}}$$

$$\left|\frac{U_n}{U_{n+1}}\right| = \left|\frac{2}{x-2}\right|$$

$$\lim_{n\to\infty}\left|\frac{U_n}{U_{n+1}}\right| = \left|\frac{2}{x-2}\right|$$

By ratio test,

$$\left|\frac{2}{x-2}\right| > 1 \quad \cancel{\Rightarrow} \quad \cancel{|2| > |x-2|}$$

PAGE NO.
DATE / /

$\frac{2}{\cancel{\quad}} \times |x-2|$

$-2 < x-2 < 2$

$0 < x < 4$

18) Expand log sin x in the power of (x − 2) up to four terms.

$f(x) = \log \sin x \qquad (x-2)$

By taylor's Series Expansion

$$f(x) = f(a) + (x-a)\frac{f'(a)}{1!} + (x-a)^2\frac{f''(a)}{2!} + (x-a)^3\frac{f'''(a)}{3!} + \dots$$

$$f(x) = f(2) + (x-2)\frac{f'(2)}{1!} + (x-2)^2\frac{f''(2)}{2!} + (x-2)^3\frac{f'''(2)}{3!} + \dots$$

$f(x) = \log \sin x \qquad f(2) = \log \sin 2$

$f'(x) = \frac{\cos x}{\sin x} \qquad f'(2) = \cot 2$

$f''(x) = -\operatorname{cosec}^2 x \qquad f''(2) = -\operatorname{cosec}^2 2$

$f'''(x) = -2\operatorname{cosec}^2 x \cot x \qquad f'''(2) = -2\operatorname{cosec}^2 2 \cot 2$

$$\log \sin x = \log \sin 2 + \frac{(x-2)\cot 2}{1!} + (x-2)^2\frac{(-\operatorname{cosec}^2 2)}{2!} + (x-2)^3\frac{(-2\operatorname{cosec}^2 2 \cot 2)}{3!} + \dots$$

19) Using Maclaurin's series, prove that

$$\tanh^{-1} x = x + \frac{x^2}{3} + \frac{x^5}{5} + \dots$$

$$\cancel{f(x) = \tan^{-1} x}$$

$$\cancel{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots}$$

$$\cancel{f(x) = \tan^{-1} x} \qquad \cancel{f(0) = 0}$$

$$\cancel{f'(x) = \frac{1}{1+x^2}} \qquad \cancel{f'(0) = \frac{1}{1}}$$

$$f''(x) = \cancel{\log(1+x^2)} \qquad f''(0) =$$

$$\cancel{= -1(1+x^2)^{-2}(2x)}$$

$$\cancel{= \frac{-2x}{(1+x^2)^2}}$$

$$\cancel{f'''(x) = 4x}$$

$$\tanh^{-1} x = \frac{1}{2} \ln\left[\frac{1+x}{1-x}\right]$$

$$= \frac{1}{2}\left[\ln(1+x) - \ln(1-x)\right]$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\therefore \tanh^{-1} x = \frac{1}{2}\left[\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} + \dots\right)\right]$$

$$= \frac{1}{2}\left[x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \cancel{\frac{x^4}{4}} + \dots + x + \cancel{\frac{x^2}{2}} + \frac{x^3}{3} + \cancel{\frac{x^4}{4}} + \dots\right]$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

Scanned by CamScanner

PAGE NO..
DATE / /

20) Find the Maclaurin's series of $\log(1+x)$ & hence find the series of $\log\left(\frac{1+x}{1-x}\right)$ & obtain value of $\log\left(\frac{11}{9}\right)$

$f(x) = \log(1+x)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{1+x}$ $\qquad f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2}$ $\qquad f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$ $\qquad f'''(0) = 2$

$f^{iv}(x) = \frac{-6}{(1+x)^4}$ $\qquad f^{iv}(0) = -6$

~~and so on~~ $\qquad$ and so on

$$\log(1+x) = x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \dots$$

$$\boxed{\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots}$$

$$\boxed{\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots}$$

$$\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots\right)$$

$$= 2\left(x + \frac{x^3}{3} + \dots\right)$$

PAGE NO..
DATE : /

$$\log\left(\frac{11}{9}\right) = \log\left(\frac{10+1}{10-1}\right)$$

$$= 2\left(10 + \frac{(10)^3}{3} + \cdots\right)$$

$$\log\left(\frac{11}{9}\right) = 0.087150$$

21) Express $(x-1)^4 - 3(x-1)^3 + 4(x-1)^2 + 5$ in ascending power of x.

$$f(x-1) = (x-1)^4 - 3(x-1)^3 + 4(x-1)^2 + 5$$

$$f(x) = x^4 - 3x^3 + 4x^2 + 5$$

$$h = -1$$

$f(x) = x^4 - 3x^3 + 4x^2 + 5$ | $f(-1) = \cancel{1} + 5 \; \cancel{8} +3 + 4 = 13$

$f'(x) = 4x^3 - 9x^2 + 8x$ | $f'(-1) = -9 - 4 - 8 = -21$

$f''(x) = 12x^2 - 18x + 8$ | $f''(-1) = 12 + 18 + 8 = 38$

$f'''(x) = 24x - 18$ | $f'''(-1) = -24 - 18 = -42$

$f^{IV}(x) = 24$ | $f^{IV}(-1) = 24$

$$f(x+h) = f(h) + \frac{x f'(h)}{1!} + \frac{x^2 f''(h)}{2!} + \frac{x^3 f'''(h)}{3!} + \cdots$$

$$f(x) = 13 - 21x + \frac{38x^2}{2!} - \frac{42x^3}{3!} + \frac{24x^4}{4!}$$

$$f(x) = x^4 - 7x^3 + 19x^2 - 21x + 13$$

22) *Expand the polynomial*

$f(x) = x^5 + 2x^4 - x^2 + x + 1$ *in powers of* $x+1$.

$f(x) = x^5 + 2x^4 - x^2 + x + 1 \qquad (x+1)$

$\cancel{f(x+1) = (x+1)^5 + 2(x+1)^4 - (x+1)^2 + (x+1) + 1}$

let $a = -1$

$f(x) = x^5 + 2x^4 - x^2 + x + 1 \qquad f(-1) = -1 + 2 - 1 - 1 + 1 = 0$

$f'(x) = 5x^4 + 8x^3 - 2x + 1 \qquad f'(-1) = 5 - 8 + 2 + 1 = 0$

$f''(x) = 20x^3 + 24x^2 - 2 \qquad f''(-1) = -20 + 24 - 2 = 2$

$f'''(x) = 60x^2 + 48x \qquad f'''(-1) = 60 - 48 = 12$

$f^{iv}(x) = 120x + 48 \qquad f^{iv}(-1) = -120 + 48 = -72$

$f^{v}(x) = 120 \qquad f^{v}(-1) = 120$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{iv}(a) + \cdots$$

$$f(x) = 0 + 0 + \frac{(x+1)^2 \cdot 2}{2 \times 1} + \frac{(x+1)^3 \cdot 12}{3 \times 2 \times 1} + \frac{(x+1)^4 \cdot (-72)}{4 \times 3 \times 2} + \frac{(x+1)^5 (120)}{5 \times 4 \times 3 \times 2}$$

$$f(x) = (x+1)^2 + 2(x+1)^3 - 3(x+1)^4 + (x+1)^5.$$

PAGE NO.
DATE. / /

22) State Taylor's series for one variable and find $\sqrt{36.12}$

Let $x = 36$ & $h = 0.12$

$f(x+h) = \sqrt{x+h} \Rightarrow f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-1/2}$

$f''(x) = -\frac{1}{4} x^{-3/2}$

$f'''(x) = \frac{3}{8} x^{-5/2}$

$f(x+h) = f(x) + hf'(x) + \frac{h^2 f''(x)}{2!} + \frac{h^3 f'''(x)}{3!} + \ldots$

$\sqrt{x+h} = x^{1/2} + \frac{h}{2} x^{-1/2} - \frac{h^2}{8} x^{-3/2} + \frac{h^3}{16} x^{-5/2} + \ldots$

Putting
$x = 36$ and $h = 0.12$,

$\sqrt{36.12} = (36)^{1/2} + \frac{0.12}{2}(36)^{-1/2} - \frac{(0.12)^2}{8}(36)^{-3/2}$
$+ \frac{(0.12)^3}{16}(36)^{-5/2} - \ldots$

$\cong 6 + 0.001 - 0.000008 + \ldots$

$\cong 6.009991$

PAGE NO..
DATE. / /

24) Expand $\sin\left(x+\frac{\pi}{4}\right)$ in power of x. Hence find the value of $\sin 46^\circ$.

$$f\left(x+\pi/4\right) = \sin\left(x+\pi/4\right)$$

$$f(x) = \sin x$$

$$h = \pi/4$$

$f(x) = \sin x$	$f(\pi/4) = 1/\sqrt{2}$
$f'(x) = \cos x$	$f'(\pi/4) = 1/\sqrt{2}$
$f''(x) = -\sin x$	$f''(\pi/4) = -1/\sqrt{2}$
$f'''(x) = -\cos x$	$f'''(\pi/4) = -1/\sqrt{2}$
$\vdots$	$\vdots$
and so on	and so on.

$$\sin(x+h) = f(h) + \frac{x f'(h)}{1!} + \frac{x^2 f''(h)}{2!} + \frac{x^3 f'''(h)}{3!} + \ldots$$

$$f\left(x+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + x\left(\frac{1}{\sqrt{2}}\right) - \frac{x^2}{2}\left(\frac{1}{\sqrt{2}}\right) - \frac{x^3}{6}\left(\frac{1}{\sqrt{2}}\right) + \ldots$$

$$= \frac{1}{\sqrt{2}}\left(x - \frac{x^2}{2} - \frac{x^3}{6} + \ldots\right)$$

$$f\left(1+\pi/4\right) = \frac{1}{\sqrt{2}}\left(1 - \frac{1}{2} - \frac{1}{6} + \ldots\right)$$

$$= 0.7193.$$

Navneet 7/12/[illegible]