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CHAPTER 2

Random Variables

Chapter Outline

2.1 INTRODUCTION

The outcomes of random experiments are, in general, abstract quantities or, in other words, most of the time they are not in any numerical form. However, the outcomes of a random experiment can be expressed in quantitative terms, in particular, by means of real numbers. Hence, a function can be defined that takes a definite real value corresponding to each outcome of an experiment. This gives a rationale for the concept of random variables about which probability statements can be made. In probability and statistics, a probability distribution assigns a probability to each measurable subset of the possible outcomes of a random experiment. Important and commonly encountered probability distributions include binomial distribution, Poisson distribution, and normal distribution.

2.2 RANDOM VARIABLES

A random variable X is a real-valued function of the elements of the sample space of a random experiment. In other words, a variable which takes the real values, depending on the outcome of a random experiment is called a *random variable*, e.g.,

(i) When a fair coin is tossed, $S = \{H, T\}$. If X is the random variable denoting the number of heads,
$X(H) = 1$ and $X(T) = 0$
Hence, the random variable X can take values 0 and 1.

(ii) When two fair coins are tossed, $S = \{HH, HT, TH, TT\}$. If X is the random variable denoting the number of heads,
$X(HH) = 2, X(HT) = 1, X(TH) = 1, X(TT) = 0.$
Hence, the random variable X can take values 0, 1, and 2.

(iii) When a fair die is tossed, $S = \{1, 2, 3, 4, 5, 6\}$.
If X is the random variable denoting the square of the number obtained,
$X(1) = 1, X(2) = 4, X(3) = 9, X(4) = 16, X(5) = 25, X(6) = 36$
Hence, the random variable X can take values 1, 4, 9, 16, 25, and 36.

Types of Random Variables

There are two types of random variables:
(i) Discrete random variables
(ii) Continuous random variables

Discrete Random Variables A random variable X is said to be discrete if it takes either finite or countably infinite values. Thus, a discrete random variable takes only isolated values, e.g.,

(i) Number of children in a family
(ii) Number of cars sold by different companies in a year
(iii) Number of days of rainfall in a city
(iv) Number of stars in the sky
(v) Profit made by an investor in a day

Continuous Random Variables A random variable X is said to be continuous if it takes any values in a given interval. Thus, a continuous random variable takes uncountably infinite values, e.g.,

(i) Height of a person in cm
(ii) Weight of a bag in kg
(iii) Temperature of a city in degree Celsius
(iv) Life of an electric bulb in hours
(v) Volume of a gas in cc.

Example 1

Identify the random variables as either discrete or continuous in each of the following cases:

(i) A page in a book can have at most 300 words
X = Number of misprints on a page
(ii) Number of students present in a class of 50 students
(iii) A player goes to the gymnasium regularly
X = Reduction in his weight in a month
(iv) Number of attempts required by a candidate to clear the IAS examination
(v) Height of a skyscraper

Solution

(i) X = Number of misprints on a page
The page may have no misprint or 1 misprint or 2 misprint ... or 300 misprints. Thus, X takes values 0, 1, 2, ..., 300. Hence, X is a discrete random variable.
(ii) Let X be the random variable denoting the number of students present in a class. X takes values 0, 1, 2, ..., 50. Hence, X is a discrete random variable.
(iii) Reduction in weight cannot take isolated values 0, 1, 2, etc., but it takes any continuous value.
Hence, X is a continuous random variable.
(iv) Let X be a random variable denoting the number of attempts required by a candidate. Thus, X takes values 1, 2, 3, ... Hence, X is a discrete random variable.
(v) Since height can have any fractional value, it is a continuous random variable.

2.3 PROBABILITY MASS FUNCTION

Probability distribution of a random variable is the set of its possible values together with their respective probabilities. Let X be a discrete random variable which takes the values $x_1, x_2, \dots x_n$. The probability of each possible outcome x_i is $p_i = p(x_i) = P(X = x_i)$ for $i = 1, 2, \dots, n$. The number $p(x_i)$, $i = 1, 2, \dots$ must satisfy the following conditions:

(i) $p(x_i) \geq 0$ for all values of i

(ii) $\sum_{i=1}^{\infty} p(x_i) = 1$

The function $p(x_i)$ is called the probability function or probability mass function of the random variable X. The set of pairs $\{x, p(x_i)\}$, $i = 1, 2, \dots, n$ is called the probability distribution of the random variable which can be displayed in the form of a table as shown below:

$X = x_i$	x_1	x_2	x_3	$\dots x_i$	$\dots x_n$
$p(x_i) = P(X = x_i)$	$p(x_1)$	$p(x_2)$	$p(x_3)$	$\dots p(x_i)$	$\dots p(x_n)$

2.4 DISCRETE DISTRIBUTION FUNCTION

Let X be a discrete random variable which takes the values $x_1, x_2, \dots$ such that $x_1 < x_2 < \dots$ with probabilities $p(x_1), p(x_2) \dots$ such that $p(x_i) \geq 0$ for all values of i and $\sum_{i=1}^{x} p(x_i) = 1$.

The distribution function $F(x)$ of the discrete random variable X is defined by

$$F(x) = P(X \leq x) = \sum_{i=1}^{x} p(x_i)$$

where x is any integer. The function $F(x)$ is also called the cumulative distribution function. The set of pairs $\{x_i, F(x)\}$, $i = 1, 2, \dots$ is called the cumulative probability distribution.

X	x_1	x_2	$\dots$
$F(x)$	$p(x_1)$	$p(x_1) + p(x_2)$	$\dots$

Example 1

A fair die is tossed once. If the random variable is getting an even number, find the probability distribution of X.

Solution

When a fair die is tossed,

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let X be the random variable of getting an even number. Hence, X can take the values 0 and 1.

$$P(X = 0) = P(1, 3, 5) = \frac{3}{6} = \frac{1}{2}$$

$$P(X = 1) = P(2, 4, 6) = \frac{3}{6} = \frac{1}{2}$$

Hence, the probability distribution of X is

$X = x$	0	1
$P(X = 1)$	$\frac{1}{2}$	$\frac{1}{2}$

Also, $\sum P(X = x) = \frac{1}{2} + \frac{1}{2} = 1$

Example 2

Find the probability distribution of the number of heads when three coins are tossed.

Solution

When three coins are tossed,

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Let X be the random variable of getting heads in tossing of three coins. Hence X can take the values 0, 1, 2, 3.

$$P(X = 0) = P(\text{no head}) = P(TTT) = \frac{1}{8}$$

$$P(X = 1) = P(\text{one head}) = P(HTT, THT, TTH) = \frac{3}{8}$$

$$P(X = 2) = P(\text{two heads}) = P(HHT, THH, HTH) = \frac{3}{8}$$

$$P(X = 3) = P(\text{three heads}) = P(HHH) = \frac{1}{8}$$

Hence, the probability distribution of X is

$X = x$	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Also, $\sum P(X = x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$

Example 3

State with reasons whether the following represent the probability mass function of a random variable:

(i)

$X=x$	0	1	2	3
$P(X=x)$	0.4	0.3	0.2	0.1

(ii)

$X=x$	0	1	2	3
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{4}$

(iii)

$X=x$	0	1	2	3
$P(X=x)$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$

Solution

(i) Here, $0 \le P(X=x) \le 1$ is satisfied for all values of X.

$$\sum P(X=x) = P(X=0)+P(X=1)+P(X=2)+P(X=3)$$
$$=0.4+0.3+0.2+0.1$$
$$=1$$

Since $\sum P(X=x)=1$, it represents probability mass function.

(ii) Here, $0 \le P(X=x) \le 1$ is satisfied for all values of X.

$$\sum P(X=x) = P(X=0)+P(X=1)+P(X=2)+P(X=3)$$
$$=\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+\frac{1}{4}$$
$$=\frac{5}{4}>1$$

Since $\sum(P(X=x)>1$, it does not represent a probability mass function.

(iii) Here, $0 \le P(X=x) \le 1$ is not satisfied for all the values of X as $P(X=0)=-\frac{1}{2}$.

Hence, $P(X=x)$ does not represent a probability mass function.

Example 4

Verify whether the following functions can be regarded as the probability mass function for the given values of X:

(i) $P(X=x)=\frac{1}{5}$ *for* $x=0,1,2,3,4$

$=0$ *for otherwise*

(ii) $P(X=x)=\frac{x-2}{5}$ *for* $x=1,2,3,4,5$

$=0$ *for otherwise*

(iii) $P(X=x)=\frac{x^2}{30}$ *for* $x=0,1,2,3,4$

$=0$ *for otherwise*

Solution

(i) $P(X=0)=P(X=1)=P(X=2)=P(X=3)=P(X=4)=\frac{1}{5}$

$P(X=x)\ge 0$ for all values of x

$$\sum P(X=x)=P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)$$
$$=\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$$
$$=1$$

Hence, $P(X=x)$ is a probability mass function.

(ii) $P(X=1)=\frac{1-2}{5}=-\frac{1}{5}<0$

Hence, $P(X=x)$ is not a probability mass function.

(iii) $P(X=0)=0$

$P(X=1)=\frac{1}{30}$

$P(X=2)=\frac{4}{30}$

$P(X=3)=\frac{9}{30}$

$P(X=4)=\frac{16}{30}$

$P(X=x)\ge 0$ for all values of x

$$\sum P(X=x) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)$$
$$= 0+\frac{1}{30}+\frac{4}{30}+\frac{9}{30}+\frac{16}{30}$$
$$= 1$$

Hence, $P(X=x)$ is a probability mass function.

Example 5

A random variable X has the probability mass function given by

X	1	2	3	4
$P(X=x)$	0.1	0.2	0.5	0.2

Find (i) $P(2 \le x < 4)$, (ii) $P(X > 2)$, (iii) P(X is odd), and (iv) P(X is even).

Solution

(i) $P(2 \le X < 4) = P(X=2)+P(X=3)$
$= 0.2+0.5$
$= 0.7$

(ii) $P(X>2) = P(X=3)+P(X=4)$
$= 0.5+0.2$
$= 0.7$

(iii) $P(X \text{ is odd}) = P(X=1)+P(X=3)$
$= 0.1+0.5$
$= 0.6$

(iv) $P(X \text{ is even}) = P(X=2)+P(X=4)$
$= 0.2+0.2$
$= 0.4$

Example 6

If the random variable X takes the value 1, 2, 3, and 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$. Find the probability distribution.

Solution

Let $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = k$

$$P(X=1) = \frac{k}{2}$$
$$P(X=2) = \frac{k}{3}$$

$$P(X=3) = k$$
$$P(X=4) = \frac{k}{5}$$

Since $\sum(P(X=x) = 1$,

$$\frac{k}{2}+\frac{k}{3}+k+\frac{k}{5} = 1$$
$$k = \frac{30}{61}$$

Hence, the probability distribution is

X	1	2	3	4
$P(X=x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

Example 7

A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6	7
$P(X=x)$	a	$4a$	$3a$	$7a$	$8a$	$10a$	$6a$	$9a$

(i) Find the value of a.
(ii) Find $P(X < 3)$.
(iii) Find the smallest value of m for which $P(X \le m) \ge 0.6$.

Solution

(i) Since $P(X=x)$ is a probability distribution function,

$$\sum(P(X=x) = 1$$
$$P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6) + P(X=7) = 1$$
$$a+4a+3a+7a+8a+10a+6a+9a = 1$$
$$a = \frac{1}{48}$$

(ii) $P(X<3) = P(X=0)+P(X=1)+P(X=2)$
$= a+4a+3a$
$= 8a$
$= 8\left(\frac{1}{48}\right)$
$= \frac{1}{6}$

(iii) $P(X \le 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= a + 4a + 3a + 7a + 8a$

$= 23a$

$= 23\left(\frac{1}{48}\right)$

$= 0.575$

$P(X \le 5) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$= a + 4a + 3a + 7a + 8a + 10a$

$= 33a$

$= 33\left(\frac{1}{48}\right)$

$= 0.69$

Hence, the smallest value of m for which $P(X \le m) \ge 0.6$ is 5.

Example 8

The probability mass function of a random variable X is zero except at the points X = 0, 1, 2. At these points, it has the values $P(X = 0) = 3c^3$, $P(X = 1) = 4c - 10c^2$, $P(X = 2) = 5c - 1$. Find (i) c, (ii) $P(X < 1)$, (iii) $P(1 < X \le 2)$, and (iv) $P(0 < X \le 2)$.

Solution

(i) Since $P(X = x)$ is a probability mass function,

$$\sum P(X = x) = 1$$

$$P(X=0) + P(X=1) + P(X=2) = 1$$

$$3c^3 + 4c - 10c^2 + 5c - 1 = 1$$

$$3c^3 - 10c^2 + 9c - 2 = 0$$

$$(3c-1)(c-2)(c-1) = 0$$

$$c = \frac{1}{3},\ 2,\ 1$$

But $c < 1$, otherwise given probabilities will be greater than one or less than zero.

$$\therefore \quad c = \frac{1}{3}$$

Hence, the probability distribution is

X	0	1	2
$P(X=x)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{3}$

(ii) $P(X < 1) = P(X = 0) = \frac{1}{9}$

(iii) $P(1 < X \le 2) = P(X = 2) = \frac{2}{3}$

(iv) $P(0 < X \le 2) = P(X = 1) + P(X = 2)$

$= \frac{2}{9} + \frac{2}{3}$

$= \frac{8}{9}$

Example 9

From a lot of 10 items containing 3 defectives, a sample of 4 items is drawn at random. Let the random variable X denote the number of defective items in the sample. Find the probability distribution of X.

Solution

The random variable X can take the value 0, 1, 2, or 3.

Total number of items = 10

Number of good items = 7

Number of defective items = 3

$$P(X=0) = P(\text{no defective}) = \frac{^7C_4}{^{10}C_4} = \frac{1}{6}$$

$$P(X=1) = P(\text{one defective and three good items}) = \frac{^3C_1\ ^7C_3}{^{10}C_4} = \frac{1}{2}$$

$$P(X=2) = P(\text{two defectives and two good items}) = \frac{^3C_2\ ^7C_2}{^{10}C_4} = \frac{3}{10}$$

$$P(X=3) = P(\text{three defectives and one good item}) = \frac{^3C_3\ ^7C_1}{^{10}C_4} = \frac{1}{30}$$

Hence, the probability distribution of the random variable is

X	0	1	2	3
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

Example 10

Construct the distribution function of the discrete random variable X whose probability distribution is as given below:

X	1	2	3	4	5	6	7
$P(X=x)$	0.1	0.15	0.25	0.2	0.15	0.1	0.05

Solution

Distribution function of X

X	$P(X=x)$	$F(x)$
1	0.1	0.1
2	0.15	0.25
3	0.25	0.5
4	0.2	0.7
5	0.15	0.85
6	0.1	0.95
7	0.05	1

Example 11

A random variable X has the probability function given below:

X	0	1	2
$P(X=x)$	k	$2k$	$3k$

Find (i) k, (ii) $P(X<2)$, $P(X\le 2)$, $P(0<X<2)$, and (iii) the distribution function.

Solution:

(i) Since $P(X=x)$ is a probability mass function,

$$\sum P(X=x)=1$$
$$k+2k+3k=1$$
$$6k=1$$
$$k=\frac{1}{6}$$

Hence, the probability distribution is

X	0	1	2
$P(X=x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

(ii) $P(X<2)=P(X=0)+P(X=1)=\frac{1}{6}+\frac{2}{6}=\frac{1}{2}$

$$P(X\le 2)=P(X=0)+P(X=1)+P(X=2)=\frac{1}{6}+\frac{2}{6}+\frac{3}{6}=1$$

$$P(0<X<2)=P(X=1)=\frac{1}{3}$$

(iii) Distribution function

X	$P(X=x)$	$F(x)$
0	$\frac{1}{6}$	$\frac{1}{6}$
1	$\frac{2}{6}$	$\frac{1}{2}$
2	$\frac{3}{6}$	1

Example 12

A random variable X takes the values −3, −2, −1, 0, 1, 2, 3, such that $P(X=0)=P(X>0)=P(X<0)$, $P(X=-3)=P(X=-2)=P(X=-1)=P(X=1)=P(X=2)=P(X=3)$. Obtain the probability distribution and the distribution function of X.

Solution

Let $P(X=0)=P(X>0)=P(X<0)=k_1$

Since $\sum P(X=x)=1$

$$k_1+k_1+k_1=1$$
$$\therefore \quad k_1=\frac{1}{3}$$
$$P(X=0)=P(X>0)=P(X<0)=\frac{1}{3}$$

Let $P(X=1)=P(X=2)=P(X=3)=k_2$

$$P(X>0)=P(X=1)+P(X=2)+P(X=3)$$
$$\frac{1}{3}=k_2+k_2+k_2$$
$$\therefore \quad k_2=\frac{1}{9}$$
$$P(X=1)=P(X=2)=P(X=3)=\frac{1}{9}$$

Similarly, $P(X=-3)=P(X=-2)=P(X=-1)=\frac{1}{9}$

Probability distribution and distribution function

X	$P(X=x)$	$F(x)$
-3	$\frac{1}{9}$	$\frac{1}{9}$
-2	$\frac{1}{9}$	$\frac{2}{9}$
-1	$\frac{1}{9}$	$\frac{3}{9}$
0	$\frac{1}{3}$	$\frac{6}{9}$
1	$\frac{1}{9}$	$\frac{7}{9}$
2	$\frac{1}{9}$	$\frac{8}{9}$
3	$\frac{1}{9}$	1

Example 13

A discrete random variable X has the following distribution function:

$$F(x)=\begin{cases}0 & x<1\\ \frac{1}{3} & 1\le x<4\\ \frac{1}{2} & 4\le x<6\\ \frac{5}{6} & 6\le x<10\\ 1 & x\ge 10\end{cases}$$

Find (i) $P(2<X\le 6)$, (ii) $P(X=5)$, (iii) $P(X=4)$, (iv) $P(X\le 6)$, and (v) $P(X=6)$.

Solution

(i) $P(2<X\le 6)=F(6)-F(2)=\frac{5}{6}-\frac{1}{3}=\frac{3}{6}=\frac{1}{2}$

(ii) $P(X=5)=P(X\le 5)-P(X<5)=F(5)-P(X<5)=\frac{1}{2}-\frac{1}{2}=0$

(iii) $P(X=4)=P(X\le 4)-P(X<4)=F(4)-P(X<4)=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$

(iv) $P(X\le 6)=F(6)=\frac{5}{6}$

(v) $P(X=6)=P(X\le 6)-P(X<6)=F(6)-P(X<6)=\frac{5}{6}-\frac{1}{2}=\frac{1}{3}$

EXERCISE 2.1

1. Verify whether the following functions can be considered as probability mass functions:

   (i) $P(X=x)=\frac{x^2+1}{18}, x=0,1,2,3$ [Ans.: Yes]

   (ii) $P(X=x)=\frac{x^2-2}{8}, x=1,2,3$ [Ans.: No]

   (iii) $P(X=x)=\frac{2x+1}{18}, x=0,1,2,3$ [Ans.: No]

2. The probability mass function of a random variable X is

X	0	1	2	3	4	5	6
$P(X=x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find $P(X<4)$ and $P(3<X\le 6)$.

$\left[\text{Ans.: } \frac{16}{49}, \frac{33}{49}\right]$

3. A random variable X has the following probability distribution:

X	1	2	3	4	5	6	7
$P(X=x)$	k	$2k$	$3k$	k^2	k^2+k	$2k^2$	$4k^2$

Find (i) k, (ii) $P(X<5)$, (iii) $P(X>5)$, and (iv) $P(0\le X\le 5)$

$\left[\text{Ans.: } \frac{1}{8} \text{ (ii) } \frac{49}{64} \text{ (iii) } \frac{3}{32} \text{ (iv) } \frac{29}{32}\right]$

4. A discrete random variable X has the following probability distribution:

X	-2	-1	0	1	2	3
$P(X = x)$	0.1	k	0.2	$2k$	0.3	$3k$

Find (i) k, (ii) $P(X \geq 2)$, and (iii) $P(-2 < X < 2)$.

$$\left[\text{Ans.: } \frac{1}{15} \text{ (ii) } \frac{1}{2} \text{ (iii) } \frac{2}{5}\right]$$

5. Given the following probability function of a discrete random variable X:

X	0	1	2	3	4	5	6	7
$P(X = x)$	0	c	$2c$	$2c$	$3c$	c^2	$2c^2$	$7c^2 + c$

Find (i) c, (ii) $P(X \geq 6)$, (iii) $P(X < 6)$, and (iv) Find k if $P(X \leq k) > \frac{1}{2}$, where k is a positive integer.

[**Ans.:** (i) 0.1 (ii) 0.19 (iii) 0.81 (iv) 4]

6. A random variable X assumes four values with probabilities $\frac{1+3x}{4}, \frac{1-x}{4}, \frac{1+2x}{4}$ and $\frac{1-4x}{4}$. For what value of x do these values represent the probability distribution of X?

$$\left[\text{Ans.: } -\frac{1}{3} \leq X \leq \frac{1}{4}\right]$$

7. Let X denote the number of heads in a single toss of 4 fair coins. Determine (i) $P(X < 2)$, and (ii) $P(1 < X \leq 3)$.

$$\left[\text{Ans.: (i) } \frac{5}{16} \text{ (ii) } \frac{5}{8}\right]$$

8. If 3 cars are selected from a lot of 6 cars containing 2 defective cars, find the probability distribution of the number of defective cars.

Ans.:

X	0	1	2
$P(X = x)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{2}{5}$

9. Five defective bolts are accidentally mixed with 20 good ones. Find the probability distribution of the number of defective bolts, if four bolts are drawn at random from this lot.

Ans.:

X	0	1	2	3	4
$P(X = x)$	$\frac{969}{2530}$	$\frac{1140}{2530}$	$\frac{380}{2530}$	$\frac{40}{2530}$	$\frac{1}{2530}$

10. Two dice are rolled at once. Find the probability distribution of the sum of the numbers on them.

Ans.:

X	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

11. A random variable X takes three values 0, 1, and 2 with probabilities $\frac{1}{3}, \frac{1}{6}$, and $\frac{1}{2}$ respectively. Obtain the distribution function of X.

$$\left[\text{Ans.: } F(0) = \frac{1}{3}, F(1) = \frac{1}{2}, F(2) = 1\right]$$

12. A random variable X has the following probability function:

X	0	1	2	3	4
$P(X = x)$	k	$3k$	$5k$	$7k$	$9k$

Find (i) the value of k, (ii) $P(X < 3)$, $P(X \geq 3)$, $P(0 < X < 4)$, and (iii) distribution function of X.

$$\left[\begin{array}{l}\text{Ans.: (i) } \frac{1}{25}, \text{ (ii) } \frac{9}{25}, \frac{16}{25}, \frac{3}{5} \\ \text{(iii) } F(0) = \frac{1}{25}, F(1) = \frac{4}{25}, F(2) = \frac{9}{25}, F(3) = \frac{16}{25}, F(4) = 1\end{array}\right]$$

13. A random variable X has the probability function

X	-2	-1	0	1	2	3
$P(X = x)$	0.1	k	0.2	$2k$	0.3	k

Find (i) k, (ii) $P(X \leq 1)$, (iii) $P(-2 < X < 1)$, and (iv) obtain the distribution function of X.

[**Ans.:** (i) 0.1 (ii) 0.6 (iii) 0.3]

14. The following is the distribution function $F(x)$ of a discrete random variable X:

X	-3	-2	-1	0	1	2	3
$P(X = x)$	0.08	0.2	0.4	0.65	0.8	0.9	1

Find (i) the probability distribution of X, (ii) $P(-2 \le X \le 1)$, and (iii) $P(X \ge 1)$.

Ans.: (i)

X	-3	-2	-1	0	1	2	3
$P(X = x)$	0.08	0.12	0.2	0.25	0.15	0.1	0.1

(ii) 0.72 (ii) 0.35

2.5 PROBABILITY DENSITY FUNCTION

Let X be a continuous random variable such that the probability of the variable X falling in the small interval $x-\frac{1}{2}\mathrm{d}x$ to $x+\frac{1}{2}\mathrm{d}x$ is $f(x)\,\mathrm{d}x$, i.e.,

$$P\left(x-\frac{1}{2}\mathrm{d}x \le X \le x+\frac{1}{2}\mathrm{d}x\right)=f(x)\mathrm{d}x$$

The function $f(x)$ is called the probability density function of the random variable X and the continuous curve $y = f(x)$ is called the probability curve.

Properties of Probability Density Function

(i) $f(x) \ge 0, \quad -\infty < x < \infty$

(ii) $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$

(iii) $P(a < x < b) = \int_a^b f(x)\,\mathrm{d}x$

2.6 CONTINUOUS DISTRIBUTION FUNCTION

If X is a continuous random variable having the probability density function $f(x)$ then the function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\mathrm{d}x, \quad -\infty < x < \infty$$

is called the distribution function or cumulative distribution function of the random variable X.

Properties of Cumulative Distribution Function

(i) $F(-\infty) = 0$

(ii) $F(\infty) = 1$

(iii) $0 \le F(x) \le 1, \quad -\infty < x < \infty$

(iv) $P(a < X < b) = F(b) - F(a)$

(v) $F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x), \quad f(x) \ge 0$

Example 1

Show that the function $f(x)$ defined by

$$f(x) = \frac{1}{7} \quad 1 < x < 8$$
$$= 0 \quad \textit{otherwise}$$

is a probability density function for a random variable. Hence, find $P(3 < X < 10)$.

Solution

$f(x) \ge 0 \quad$ in $\quad 1 < x < 8$

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \int_{-\infty}^{1} f(x)\mathrm{d}x + \int_{1}^{8} f(x)\mathrm{d}x + \int_{8}^{\infty} f(x)\mathrm{d}x$$
$$= 0 + \int_1^8 \frac{1}{7}\,\mathrm{d}x + 0$$
$$= \frac{1}{7}|x|_1^8$$
$$= \frac{1}{7}(8-1)$$
$$= 1$$

Hence, $f(x)$ is a probability density function.

$$P(3 < X < 10) = \int_3^{10} f(x)\mathrm{d}x$$
$$= \int_3^8 f(x)\mathrm{d}x + \int_8^{10} f(x)\mathrm{d}x$$
$$= \int_3^8 \frac{1}{7}\mathrm{d}x + 0$$
$$= \frac{1}{7}|x|_3^8$$
$$= \frac{1}{7}(8-3)$$
$$= \frac{5}{7}$$

Example 2

Is the function $f(x)$ defined by

$$f(x) = e^{-x} \quad x \ge 0$$
$$= 0 \quad x < 0$$

is a probability density function. If so, find the probability that the variate having this density falls in the interval $(1, 2)$.

Solution

$f(x) \ge 0$ in $(0, \infty)$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$
$$= 0 + \int_{0}^{\infty} e^{-x}\,dx$$
$$= \left|-e^{-x}\right|_0^{\infty}$$
$$= -e^{-\infty} + 1$$
$$= 1$$

Hence, $f(x)$ is a probability density function.

$$P(1 \le X \le 2) = \int_{1}^{2} f(x)\,dx$$
$$= \int_{1}^{2} e^{-x}\,dx$$
$$= \left|-e^{-x}\right|_1^{2}$$
$$= -e^{-2} + e^{-1}$$
$$= 0.233$$

Example 3

If a random variable has the probability density function $f(x)$ as

$$f(x) = 2e^{-2x} \quad x > 0$$
$$= 0 \quad x \le 0$$

Find the probabilities that it will take on a value (i) between 1 and 3, and (ii) greater than 0.5.

Solution

(i) Probability that the variable will take a value between 1 and 3

$$P(1 < X < 3) = \int_{1}^{3} f(x)\,dx$$
$$= \int_{1}^{3} 2e^{-2x}\,dx$$
$$= 2\left|\frac{e^{-2x}}{-2}\right|_1^{3}$$
$$= -(e^{-6} - e^{-2})$$
$$= e^{-2} - e^{-6}$$

(ii) Probability that the variable will take a value greater than 0.5

$$P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,dx$$
$$= \int_{0.5}^{\infty} 2e^{-2x}\,dx$$
$$= 2\left|\frac{e^{-2x}}{-2}\right|_{0.5}^{\infty}$$
$$= -(e^{-\infty} - e^{-1})$$
$$= e^{-1}$$

Example 4

Find the constant k such that the function

$$f(x) = kx^2 \quad 0 < x < 3$$
$$= 0 \quad \textit{otherwise}$$

is a probability density function and compute (i) $P(1 < x < 2)$, (ii) $P(X < 2)$, and (iii) $P(X \ge 2)$.

Solution

Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$0+\int_0^3 kx^2\,dx+0=1$$

$$k\left|\frac{x^3}{3}\right|_0^3=1$$

$$\frac{k}{3}(27-0)=1$$

$$9k=1$$

$$k=\frac{1}{9}$$

Hence, $f(x)=\frac{1}{9}x^2 \quad 0<x<3$

$= 0 \quad$ otherwise

(i) $P(1<X<2)=\int_1^2 f(x)dx$

$$=\int_1^2 \frac{1}{9}x^2\,dx$$

$$=\frac{1}{9}\left|\frac{x^3}{3}\right|_1^2$$

$$=\frac{1}{27}(8-1)$$

$$=\frac{7}{27}$$

(ii) $P(X<2)=\int_{-\infty}^2 f(x)dx$

$$=\int_{-\infty}^0 f(x)dx+\int_0^2 f(x)dx$$

$$=0+\int_0^2 \frac{1}{9}x^2\,dx$$

$$=\frac{1}{9}\int_0^2 x^2\,dx$$

$$=\frac{1}{9}\left|\frac{x^3}{3}\right|_0^2$$

$$=\frac{1}{27}(8-0)$$

$$=\frac{8}{27}$$

(iii) $P(X\geq 2)=1-P(X<2)$

$$=1-\frac{8}{27}$$

$$=\frac{19}{27}$$

Example 5

If the probability density function of a random variable is given by

$f(x)=k(1-x^2) \quad 0<x<1$

$=0 \quad$ *otherwise*

Find the value of k and the probabilities that a random variable having this probability density will take on a value (i) between 0.1 and 0.2, and (ii) greater than 0.5.

Solution

Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)dx=1$$

$$\int_{-\infty}^{0} f(x)dx+\int_0^1 f(x)dx+\int_1^{\infty} f(x)dx=1$$

$$0+\int_0^1 k(1-x^2)dx+0=1$$

$$k\left|x-\frac{x^3}{3}\right|_0^1=1$$

$$k\left(1-\frac{1}{3}\right)=1$$

$$k=\frac{3}{2}$$

Hence, $f(x)=\frac{3}{2}(1-x^2) \quad 0<x<1$

$=0 \quad$ otherwise

(i) Probability that the variable will take on a value between 0.1 and 0.2

$$P(0.1<X<0.2)=\int_{0.1}^{0.2} f(x)dx$$

$$=\int_{0.1}^{0.2}\frac{3}{2}(1-x^2)dx$$

$$= \frac{3}{2}\left| x - \frac{x^3}{3} \right|_{0.1}^{0.2}$$

$$= \frac{3}{2}\left[\left(0.2 - \frac{0.008}{3}\right) - \left(0.1 - \frac{0.001}{3}\right)\right]$$

$$= 0.1465$$

(ii) Probability that the variable will take on a value greater than 0.5 .

$$P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,dx$$

$$= \int_{0.5}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= \int_{0.5}^{1} \frac{3}{2}(1 - x^2)\,dx + 0$$

$$= \frac{3}{2}\left| x - \frac{x^3}{3} \right|_{0.5}^{1}$$

$$= \frac{3}{2}\left[\left(1 - \frac{1}{3}\right) - \left(0.5 - \frac{0.125}{3}\right)\right]$$

$$= 0.3125$$

Example 6

If X is a continuous random variable with pdf

$$f(x) = x^2 \quad 0 \le x \le 1$$
$$= 0 \quad \textit{otherwise}$$

If $P(a \le X \le 1) = \dfrac{19}{81}$, *find the value of a.*

Solution

$$P(a \le X \le 1) = \frac{19}{81}$$

$$\int_a^1 f(x)\,dx = \frac{19}{81}$$

$$\int_a^1 x^2\,dx = \frac{19}{81}$$

$$\left| \frac{x^3}{3} \right|_a^1 = \frac{19}{81}$$

$$\frac{1}{3}(1 - a) = \frac{19}{81}$$

$$1 - a = \frac{19}{27}$$

$$a = \frac{46}{27}$$

Example 7

Let X be a continuous random variable with pdf

$$f(x) = kx\,(1 - x),\ 0 \le x \le 1$$

Find k and determine a number b such that $P(X \le b) = P(X \ge b)$.

Solution

Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x) = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 1$$

$$0 + \int_0^1 kx(1 - x)\,dx + 0 = 1$$

$$k\int_0^1 (x - x^2)\,dx = 1$$

$$k\left| \frac{x^2}{2} - \frac{x^3}{3} \right|_0^1 = 1$$

$$k\left[\left(\frac{1}{2} - \frac{1}{3}\right) - (0 - 0)\right] = 1$$

$$k\left(\frac{1}{6}\right) = 1$$

$$k = 6$$

Hence, $f(x) = 6(x - x^2) \quad 0 \le x \le 1$

Since total probability is 1 and $P(X \le b) = P(X \ge b)$,

$$P(X \le b) = \frac{1}{2}$$

$$\int_0^b f(x)\,dx = \frac{1}{2}$$

$$6\int_0^b (x - x^2)\,dx = \frac{1}{2}$$

$$6\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^b = \frac{1}{2}$$

$$\frac{b^2}{2} - \frac{b^3}{3} = \frac{1}{12}$$

$$6b^2 - 4b^3 = 1$$

$$4b^3 - 6b^2 + 1 = 0$$

$$(2b-1)(2b^2 - 2b - 1) = 0$$

$$b = \frac{1}{2} \text{ or } b = \frac{1 \pm \sqrt{3}}{2}$$

b lies in (0, 1).

$\therefore \quad b = \frac{1}{2}$

Example 8

The length of time (in minutes) that a certain lady speaks on the telephone is found to be a random phenomenon, with a probability function specified by the function

$$f(x) = A e^{-\frac{x}{5}} \quad x \ge 0$$
$$= 0 \quad \textit{otherwise}$$

(i) Find the value of A that makes $f(x)$ a probability density function.
(ii) What is the probability that the number of minutes that she will take over the phone is more than 10 minutes?

Solution

(i) For $f(x)$ to be a probability density function,

$$\int_{-\infty}^{\infty} f(x)x = 1$$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = 1$$

$$0 + \int_0^{\infty} A e^{-\frac{x}{5}} dx = 1$$

$$A\left|\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right|_0^{\infty} = 1$$

$$-5A(e^{-\infty} - e^{-0}) = 1$$

$$-5A(0-1) = 1$$

$$5A = 1$$

$$A = \frac{1}{5}$$

Hence, $f(x) = \frac{1}{5}e^{-\frac{x}{5}} \quad x \ge 0.$

$= 0 \quad$ otherwise

(ii) $P(X > 10) = \int_{10}^{\infty} f(x)dx$

$$= \int_{10}^{\infty} \frac{1}{5} e^{-\frac{x}{5}} dx$$

$$= \frac{1}{5}\left|\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right|_{10}^{\infty}$$

$$= -(e^{-\infty} - e^{-2})$$

$$= -(0 - e^{-2})$$

$$= e^{\frac{1}{2}}$$

Example 9

A continuous random variable X has a pdf $f(x)^2 = 3x^2, 0 \le x \le 1$. Find a and b such that

(i) $P(X \le a) = P(X > a)$ and

(ii) $P(X > b) = 0.05$

Solution

Since total probability is 1 and $P(X \le a) = P(X > a)$,

$$P(X \le a) = \frac{1}{2}$$

$$\int_0^a f(x)\,dx = \frac{1}{2}$$

$$\int_0^a 3x^2 dx = \frac{1}{2}$$

$$3\left|\frac{x^3}{3}\right|_0^a = \frac{1}{2}$$

$$a^3 = \frac{1}{2}$$

$$a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

$$P(X > b) = 0.05$$

$$\int_b^1 f(x)\,dx = 0.05$$

$$\int_b^1 3x^2\,dx = 0.05$$

$$3\left|\frac{x^3}{3}\right|_b^1 = 0.05$$

$$1 - b^3 = 0.05$$

$$b^3 = \frac{19}{20}$$

$$b = \left(\frac{19}{20}\right)^{\frac{1}{3}}$$

Example 10

Let the continuous random variable X have the probability density function

$$f(x) = \frac{2}{x^3} \qquad 1 < x < \infty$$
$$= 0 \qquad \textit{otherwise}$$

Find F(x).

Solution

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$
$$= \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$
$$= 0 + \int_1^x \frac{2}{x^3}\,dx$$
$$= 2\left|\frac{x^{-2}}{-2}\right|_1^x$$
$$= -\left|\frac{1}{x^2}\right|_1^x$$
$$= -\left(\frac{1}{x^2} - 1\right)$$
$$= 1 - \frac{1}{x^2}$$

Hence, $F(x) = 1 - \frac{1}{x^2} \qquad 1 < x < \infty$

$= 0 \qquad$ otherwise

Example 11

Verify that the function F(x) is a distribution function.

$$F(x) = 0 \qquad x < 0$$
$$= 1 - e^{-\frac{x}{4}} \qquad x \ge 0$$

Also, find the probabilities $P(X \le 4), P(X \ge 8), P(4 \le X \le 8)$.

Solution

For the function $F(x)$,

(i) $F(-\infty) = 0$

(ii) $F(\infty) = 1 - e^{-\infty} = 1 - 0 = 1$

(iii) $0 \le F(x) \le 1 \qquad -\infty < x < \infty$

If $f(x)$ is the corresponding probability density function,

$$f(x) = F'(x) = 0 \qquad x < 0$$
$$= \frac{1}{4}e^{-\frac{x}{4}} \qquad x \ge 0$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$
$$= 0 + \int_{0}^{\infty} \frac{1}{4} e^{-\frac{x}{4}}\,dx$$
$$= \frac{1}{4}\left|\frac{e^{-\frac{x}{4}}}{-\frac{1}{4}}\right|_0^{\infty}$$
$$= -\left|e^{-\frac{x}{4}}\right|_0^{\infty}$$
$$= -(0-1)$$
$$= 1$$

Hence, $F(x)$ is a distribution function.

$$P(X \le 4) = F(4)$$
$$= 1 - e^{-1}$$
$$= 1 - \frac{1}{e}$$
$$= \frac{e-1}{e}$$

$$P(X \ge 8) = 1 - P(X \le 8)$$
$$= 1 - F(8)$$
$$= 1 - (1 - e^{-2})$$
$$= e^{-2}$$
$$= \frac{1}{e^2}$$

$$P(4 \le X \le 8) = F(8) - F(4)$$
$$= (1 - e^{-2}) - (1 - e^{-1})$$
$$= e^{-1} - e^{-2}$$
$$= \frac{1}{e} - \frac{1}{e^2}$$
$$= \frac{e-1}{e^2}$$

Example 12

The troubleshooting capacity of an IC chip in a circuit is a random variable X whose distribution function is given by

$$F(x) = 0 \qquad x \le 3$$
$$= 1 - \frac{9}{x^2} \qquad x > 3$$

where x denotes the number of years. Find the probability that the IC chip will work properly (i) less than 8 years, (i) beyond 8 years, (iii) between 5 to 7 years, and (iv) anywhere from 2 to 5 years.

Solution

(i) $P(X \le 8) = F(8)$
$$= 1 - \frac{9}{8^2}$$
$$= 0.8594$$

(ii) $P(X > 8) = 1 - P(X \le 8)$
$$= 1 - F(8)$$
$$= 1 - 0.8594$$
$$= 0.1406$$

(iii) $P(5 \le X \le 7) = F(7) - F(5)$
$$= \left(1 - \frac{9}{7^2}\right) - \left(1 - \frac{9}{5^2}\right)$$
$$= 0.1763$$

(iv) $P(2 \le X \le 5) = F(5) - F(2)$
$$= \left(1 - \frac{9}{5^2}\right) - 0$$
$$= 0.64$$

Example 13

The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} ax & 0 \le x \le 1 \\ a & 1 \le x \le 2 \\ 3a - ax & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find the value of a, and (ii) find the cdf of X.

Solution

(i) Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx = 1$$

$$0 + \int_{0}^{1} ax\,dx + \int_{1}^{2} a\,dx + \int_{2}^{3} (3a - ax)\,dx = 1$$

$$a\left|\frac{x^2}{2}\right|_0^1 + a\left|x\right|_1^2 + \left|3ax - \frac{ax^2}{2}\right|_2^3 = 1$$

$$a\left(\frac{1}{2} - 0\right) + a(2-1) + \left[\left(9a - \frac{9a}{2}\right) - (6a - 2a)\right] = 1$$

$$\frac{1}{2}a + a + \frac{9a}{2} - 4a = 1$$

$$2a = 1$$

$$a = \frac{1}{2}$$

(ii) $F(x) = \int_{-\infty}^{x} f(x)\,dx$

For $0 \le x \le 1$,

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} ax\,dx$$

$$= a\left|\frac{x^2}{2}\right|_0^x$$

$$= \frac{ax^2}{2}$$

For $1 \le x \le 2$,

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} ax\,dx + \int_{1}^{x} a\,dx$$

$$= a\left|\frac{x^2}{2}\right|_0^1 + a\left|x\right|_1^x$$

$$= a\left(\frac{1}{2} - 0\right) + a(x-1)$$

$$= \frac{a}{2} + ax - a$$

$$= ax - \frac{a}{2}$$

For $2 \le x \le 3$,

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} ax\,dx + \int_{1}^{2} a\,dx + \int_{2}^{x} (3a - ax)\,dx$$

$$= a\left|\frac{x^2}{2}\right|_0^1 + a\left|x\right|_1^2 + \left|3ax - \frac{ax^2}{2}\right|_2^x$$

$$= a\left(\frac{1}{2} - 0\right) + a(2-1) + \left[\left(3ax - \frac{ax^2}{2}\right) - (6a - 2a)\right]$$

$$= \frac{a}{2} + a + 3ax - \frac{ax^2}{2} - 4a$$

$$= 3ax - \frac{ax^2}{2} - \frac{5a}{2}$$

Hence,
$$F(x) = \begin{cases} \dfrac{ax^2}{2} & 0 \le x \le 1 \\ ax - \dfrac{a}{2} & 1 \le x \le 2 \\ 3ax - \dfrac{ax^2}{2} - \dfrac{5a}{2} & 2 \le x \le 3 \end{cases}$$

Example 14

The pdf of a continuous random variable X is

$$f(x) = \frac{1}{2}e^{-|x|}$$

Find cdf F(x).

Solution

$$f(x)=\frac{1}{2}e^{x} \qquad -\infty<x<0$$
$$=\frac{1}{2}e^{-x} \qquad 0<x<\infty$$

$$F(x)=\int_{-\infty}^{x} f(x)\,dx$$

For $x\le 0$,

$$F(x)=\int_{-\infty}^{x}\frac{1}{2}e^{x}\,dx$$
$$=\frac{1}{2}\left|e^{x}\right|_{-\infty}^{x}$$
$$=\frac{1}{2}(e^{x}-e^{-\infty})$$
$$=\frac{1}{2}e^{x}$$

For $x>0$,

$$F(x)=\int_{-\infty}^{0} f(x)\,dx+\int_{0}^{x} f(x)\,dx$$
$$=\int_{-\infty}^{0}\frac{1}{2}e^{x}\,dx+\int_{0}^{x}\frac{1}{2}e^{-x}\,dx$$
$$=\frac{1}{2}\left|e^{x}\right|_{-\infty}^{0}+\frac{1}{2}\left|-e^{-x}\right|_{0}^{x}$$
$$=\frac{1}{2}(1-e^{-\infty})+\frac{1}{2}(-e^{-x}+e^{0})$$
$$=\frac{1}{2}-\frac{1}{2}e^{-x}+\frac{1}{2}$$
$$=1-\frac{1}{2}e^{-x}$$

Hence, $F(x)=\frac{1}{2}e^{x} \qquad x\le 0$

$$=1-\frac{1}{2}e^{-x} \qquad x>0$$

Example 15

Find the value of k and the distribution function F(x) given the probability density function of a random variable X as

$$f(x)=\frac{k}{x^2+1} \qquad -\infty<x<\infty$$

Solution

Since $f(x)$ is the probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx=1$$
$$\int_{-\infty}^{\infty}\frac{k}{x^2+1}\,dx=1$$
$$k\int_{-\infty}^{\infty}\frac{1}{x^2+1}\,dx=1$$
$$k\left|\tan^{-1}x\right|_{-\infty}^{\infty}=1$$
$$k\left[\tan^{-1}\infty-\tan^{-1}(-\infty)\right]=1$$
$$k\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]=1$$
$$k\pi=1$$
$$k=\frac{1}{\pi}$$

Hence, $f(x)=\frac{1}{\pi}\frac{1}{x^2+1} \qquad -\infty<x<\infty$

$$F(x)=\int_{-\infty}^{x} f(x)\,dx$$
$$=\frac{1}{\pi}\int_{-\infty}^{x}\frac{1}{x^2+1}\,dx$$
$$=\frac{1}{\pi}\left|\tan^{-1}x\right|_{-\infty}^{x}$$
$$=\frac{1}{\pi}\left[\tan^{-1}x-\tan^{-1}(-\infty)\right]$$
$$=\frac{1}{\pi}\left(\tan^{-1}x+\frac{\pi}{2}\right)$$

Example 16

Find the constant k such that

$$f(x) = kx^2 \quad 0 < x < 3$$
$$= 0 \quad \text{otherwise}$$

is a probability function. Also, find the distribution function $F(x)$ and $P(1 < X \le 2)$.

Solution

Since $f(x)$ is probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{0}^{3} kx^2\,dx + 0 = 1$$

$$k\left|\frac{x^3}{3}\right|_0^3 = 1$$

$$k(9 - 0) = 1$$

$$k = \frac{1}{9}$$

Hence, $f(x) = \frac{1}{9}x^2 \quad 0 < x < 3$

$= 0 \quad$ otherwise

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_0^x \frac{1}{9}x^2\,dx$$

$$= \frac{1}{9}\left|\frac{x^3}{3}\right|_0^x$$

$$= \frac{1}{27}x^3$$

Hence, $F(x) = \frac{1}{27}x^3 \quad 0 < x < 3$

$= 0 \quad$ otherwise

$$P(1 < x \le 2) = \int_{1}^{2} f(x)\,dx$$

$$= \int_{1}^{2} \frac{1}{9}x^2\,dx$$

$$= \frac{1}{9}\left|\frac{x^3}{3}\right|_1^2$$

$$= \frac{1}{27}(8 - 1)$$

$$= \frac{7}{27}$$

EXERCISE 2.2

1. Verify whether the following functions are probability density functions:

   (i) $f(x) = k\,e^{-kx} \qquad x \ge 0, k > 0$

   (ii) $f(x) = \frac{1}{2}e^{-|x|} \qquad -\infty < x < \infty$

   (iii) $f(x) = \frac{2}{9}x\left(2 - \frac{x}{2}\right) \qquad 0 \le x \le 3$

   [Ans.: (i) Yes (ii) Yes (iii) Yes]

2. Find the value of k if the following are probability density functions:

   (i) $f(x) = k(1 + x) \qquad 2 \le x \le 5$

   (ii) $f(x) = k(x - x^2) \qquad 0 \le x \le 1$

   (iii) $f(x) = kx\,e^{-4x^2} \qquad 0 \le x \le \infty$

   (iv) $f(x) = kx\,e^{-\frac{x^2}{4}} \qquad 0 \le x \le \infty$

   $\left[\text{Ans.: (i) } \frac{2}{27} \text{ (ii) } 6 \text{ (iii) } 8 \text{ (iv) } \frac{1}{2}\right]$

3. A function is defined as

$$f(x)=\begin{cases}0 & x<2\\ \dfrac{2x+3}{18} & 2\le x\le 4\\ 0 & x>4\end{cases}$$

Show that $f(x)$ is a probability density function and find $P(2<X<3)$.

$\left[\text{Ans.: }\frac{4}{9}\right]$

4. Let X be a continuous random variable with probability distribution

$$f(x)=\begin{cases}\dfrac{x}{6}+k & 0\le x\le 3\\ 0 & \text{otherwise}\end{cases}$$

Find k, and $P(1\le X\le 2)$.

$\left[\text{Ans.: }1, \frac{1}{3}\right]$

5. Find the value of k such that $f(x)$ is a probability density function. Find also, $P(X\le 1.5)$.

$$f(x)=\begin{cases}kx & 0\le x\le 1\\ k & 1\le x\le 2\\ k(3-x) & 2\le x\le 3\end{cases}$$

$\left[\text{Ans.: }\frac{1}{2}, \frac{1}{2}\right]$

6. If X is a continuous random variable whose probability density function is given by

$f(x)=k(4x-2x^2)$ $\quad 0<x<2$

$=0$ $\quad$ otherwise

Find (i) the value of k, and (ii) $P(X>1)$.

$\left[\text{Ans.: (i) }\frac{3}{8}\text{ (ii) }\frac{1}{2}\right]$

7. If a random variable has the probability density function

$f(x)=k(x^2-1)$ $\quad -1\le x\le 3$

$=0$ $\quad$ otherwise

Find (i) the value of k, and (ii) $P\left(\frac{1}{2}\le X\le\frac{5}{2}\right)$.

$\left[\text{Ans.: (i) }\frac{3}{28}\text{ (ii) }\frac{19}{56}\right]$

8. The probability density function is

$f(x)=k(3x^2-1)$ $\quad -1\le x\le 2$

$=0$ $\quad$ otherwise

Find (i) the value of k, and (ii) $P(-1\le X\le 0)$.

$\left[\text{Ans.: (i) }\frac{1}{6}\text{ (ii) }0\right]$

9. Is the function defined by

$f(x)=0$ $\quad x<2$

$=\frac{1}{18}(2x+3)$ $\quad 2\le x\le 4$

$=0$ $\quad x>4$

a probability density function? Find the probability that a variate having $f(x)$ as density function will fall in the interval $2\le X\le 3$.

$\left[\text{Ans.: Yes, }\frac{4}{9}\right]$

10. A random variable X gives measurements x between 0 and 1 with a probability function

$f(x)=12x^3-21x^2+10x$ $\quad 0\le x\le 1$

$=0$ $\quad$ otherwise

(i) Find $P\left(X\le\frac{1}{2}\right)$ and $P\left(X>\frac{1}{2}\right)$.

(ii) Find a number k such that $P(X\le k)=\frac{1}{2}$.

$\left[\text{Ans.: (i) }\frac{7}{16}\text{ (ii) }0.452\right]$

11. The distribution function of a random variable X is given by

$$F(x)=\begin{cases}1-e^{-x^2} & x>0\\ 0 & \text{otherwise}\end{cases}$$

Find the probability density function.

$\left[\text{Ans.: }f(x)=2xe^{-x^2}\ \ x>0;\ =0\ \ \text{otherwise}\right]$

12. The cdf of a continuous random variable X is given by

$$F(x)=\begin{cases}0 & x<0\\ x^2 & 0\le x\le 1\\ 1 & x>1\end{cases}$$

Find the pdf and $P\left(\frac{1}{2}\le X\le\frac{4}{5}\right)$.

[Ans.: 0.195]

13. Find the distribution function corresponding to the following probability density functions:

(i) $f(x)=\begin{cases}\frac{1}{2}x^2e^{-x} & 0\le x<\infty\\ 0 & \text{otherwise}\end{cases}$

(ii) $f(x)=x \quad 0\le x\le 1$
$\quad =2-x \quad 1\le x\le 2$
$\quad =0 \quad \text{otherwise}$

(iii) $f(x)=\lambda(x-1)^4 \quad 1\le x\le 3, \lambda>0$
$\quad =0 \quad \text{otherwise}$

Ans.: (i) $F(x)=\begin{cases}1-e^{-x}\left(1+x+\frac{x^2}{2}\right) & x\ge 0\\ 0 & \text{otherwise}\end{cases}$

(ii) $F(x)=\begin{cases}0 & x<0\\ \frac{x^2}{2} & 0\le x\le 1\\ 2x-0.5x^2-1 & 1\le x\le 2\\ 1 & x>2\end{cases}$

(iii) $\lambda=\frac{5}{32}$, $F(x)=\begin{cases}0 & x\le 1\\ \frac{5}{32}(x-1)^4 & 1\le x\le 3\\ 1 & x\ge 3\end{cases}$

14. A continuous random variable X has the following probability density function

$$f(x)=\frac{a}{x^5} \quad 2\le x\le 10$$

Determine the constant a, distribution function of X, and find the probability of the event $4\le X\le 7$.

$\left[\text{Ans.: }\frac{2500}{39}, F(x)=\frac{625}{39}\left(\frac{1}{16}-\frac{1}{x^4}\right), 0.056\right]$

2.7 TWO-DIMENSIONAL DISCRETE RANDOM VARIABLES

In one-dimensional random variable, the outcome of any experiment had only one characteristic. In many situations, the outcome of a random experiment depends on two or more characteristics e.g., both voltage and current are measured in certain experiment.

Let X and Y be two random variables defined on the same sample space S, then the function (X, Y) that assigns a point in R^2 is called a two-dimensional random variable.

A two-dimensional random variable is said to be discrete if it takes at most a countable number of points in R^2. When (X, Y) is a two-dimensional discrete random variable, the possible values of (X, Y) may be represented as (x_i, y_j), $i = 1, 2, \ldots, m, \ldots$; $j = 1, 2, \ldots, n, \ldots$

2.7.1 Joint Probability Mass Function

If (X, Y) is a two-dimensional discrete random variable, then the joint discrete function of X, Y, also called the joint probability mass function of X, Y, denoted by p_{XY} is defined by

$$p_{XY}(x_i, y_j)=P(X=x_i, Y=y_j) \text{ for a value of } (x_i, y_j) \text{ of } (X, Y)$$

and $\quad p_{XY}(x_i, y_j)=0, \quad \text{otherwise}$

Following conditions should be satisfied for a function to be a probability mass function:

(i) $p_{XY}(x_i, y_j)\ge 0$, for all i and j

(ii) $\sum_{j=1}^{n}\sum_{i=1}^{m}p_{XY}(x_i, y_j)=1$

2.7.2 Cumulative Distribution Function

If (X, Y) is a two-dimensional discrete random variable, then $F_{XY}(x, y) = P(X\le x, Y\le y)$ is called the cumulative distribution function (cdf) of (X, Y) and is defined by

$$F_{XY}(x,y)=\sum_{j=1}^{n}\sum_{i=1}^{m}p_{XY}(x_i, y_j)=\Sigma\Sigma p_{ij}$$

Properties of cdf

(i) $F(-\infty, y) = 0 = F(x, \infty)$ and $F(\infty, \infty) = 1$

(ii) $P(a < X < b, Y \le y) = F(b, y) - F(a, y)$

(iii) $P(X \le x, c < Y < d) = F(x, d) - F(x, c)$

(iv) $P(a < X < b, c < Y < d) = F(b, d) - F(a, d) - F(b, c) + F(a, c)$

(v) For a discrete random variable, $F_{XY}(x,y)$ will have step discontinuities. Derivatives at such discontinuities are not defined. At points of continuity,

$$\frac{\partial^2 F}{\partial x\, \partial y} = f(x, y)$$

2.7.3 Marginal Probability function

Let (X, Y) be a two-dimensional discrete random variable which takes up countable number of values (x_i, y_j). Then the probability distribution of X is given by

$$\begin{aligned} p_X(x_i) &= P(X = x_i) \\ &= P(X = x_i, Y = y_1) + P(X = x_i, Y = y_2) + \dots + P(X = x_i, Y = y_m) \\ &= p_{i1} + p_{i2} + \cdots + p_{ij} + \cdots + p_{im} \\ &= \sum_{j=1}^{m} p_{ij} \\ &= \sum_{j=1}^{m} p(x_i, y_j) \\ &= p_{i*} \end{aligned}$$

and is known as marginal probability mass function or discrete marginal density function of X.

Similarly,

$$p_Y(y_j) = P(Y = y_j) = \sum_{i=1}^{n} p_{ij} = \sum_{i=1}^{n} p(x_i, y_j) = p_{*j}$$

is the marginal probability mass function of Y.

2.7.4 Conditional Probability Function

Let (X, Y) be a two-dimensional discrete random variable. Then the conditional discrete density function or conditional probability mass function of X, given $Y = y$, denoted by $p_{X/Y}(x/y)$ is defined as

$$p_{X/Y} = P(X = x / Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}, \text{ provided } P(Y = y) \neq 0.$$

The conditional probability mass function of Y, given $X = x$, denoted by $p_{Y/X}(y/x)$ is defined as:

$$p_{Y/X} = P(Y = y / X = x) = \frac{P(X = x, Y = y)}{P(X = x)}, \text{ provided } P(X = x) \neq 0.$$

A necessary and sufficient condition for the discrete random variables X and Y to be independent is

$$P(X = x_i, Y = y_j) = P(X = x_i)\, P(Y = y_j) \text{ for all values } (x_i, y_j) \text{ of } (X, Y)$$

Example 1

From the following table for bivariate distribution of (X, Y), find (i) $P(X \le 1)$ (ii) $P(Y \le 3)$ (iii) $P(X \le 1, Y \le 3)$, (iv) $P(X \le 1/Y \le 3)$ (v) $P(Y \le 3/X \le 1)$ (vi) $P(X + Y \le 4)$

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Solution

Marginal distributions

X \ Y	1	2	3	4	5	6	$p_X(x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$
$p_Y(y)$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	$\Sigma p(x) = 1$ $\Sigma p(y) = 1$

(i) $P(X \le 1) = P(X=0) + P(X=1)$

$$= \frac{8}{32} + \frac{10}{16}$$

$$= \frac{7}{8}$$

(ii) $P(Y \le 3) = P(Y=1) + P(Y=2) + P(Y=3)$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64}$$

$$= \frac{23}{64}$$

(iii) $P(X \le 1, Y \le 3) = P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3) + P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3)$

$$= 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$$

$$= \frac{9}{32}$$

(iv) $P(X \le 1 / Y \le 3) = \dfrac{P(X \le 1, Y \le 3)}{P(Y \le 3)}$

$$= \frac{\frac{9}{32}}{\frac{23}{64}}$$

$$= \frac{18}{23}$$

(v) $P(Y \le 3 / X \le 1) = \dfrac{P(X \le 1, Y \le 3)}{P(X \le 1)}$

$$= \frac{\frac{9}{32}}{\frac{7}{8}}$$

$$= \frac{9}{28}$$

(vi) $P(X + Y \le 4) = P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3) + P(X=0, Y=4) + P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3) + P(X=2, Y=1) + P(X=2, Y=2)$

$$= 0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{32} + \frac{1}{32}$$

$$= \frac{13}{32}$$

Example 2

For the following joint distribution of X and Y, find the marginal distributions:

Y \ X	0	1	2
0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$
1	$\frac{3}{14}$	$\frac{3}{14}$	0
2	$\frac{1}{28}$	0	0

Solution

Marginal distributions

Y \ X	0	1	2	$p_Y(y)$
0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{6}{14}$
2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
$p_X(x)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$	$\Sigma p(x) = 1$ $\Sigma p(y) = 1$

Marginal distributions of X

$$P(X=0) = P(X=0, Y=0) + P(X=0, Y=1) + P(X=0, Y=2)$$

$$= \frac{3}{28} + \frac{3}{14} + \frac{1}{28}$$

$$= \frac{10}{28}$$

$$P(X=1)=P(X=1,Y=0)+P(X=1,Y=1)+P(X=1,Y=2)$$
$$=\frac{9}{28}+\frac{3}{14}+0$$
$$=\frac{15}{28}$$

$$P(X=2)=P(X=2,Y=0)+P(X=2,Y=1)+P(X=2,Y=2)$$
$$=\frac{3}{28}+0+0$$
$$=\frac{3}{28}$$

Marginal distributions of Y

$$P(Y=0)=P(X=0,Y=0)+P(X=1,Y=0)+P(X=2,Y=0)$$
$$=\frac{3}{28}+\frac{9}{28}+\frac{3}{28}$$
$$=\frac{15}{28}$$

$$P(Y=1)=P(X=0,Y=1)+P(X=1,Y=1)+P(X=2,Y=1)$$
$$=\frac{3}{14}+\frac{3}{14}+0$$
$$=\frac{6}{14}$$

$$P(Y=2)=P(X=0,Y=2)+P(X=1,Y=2)+P(X=2,Y=2)$$
$$=\frac{1}{28}+0+0$$
$$=\frac{1}{28}$$

Example 3

The joint distribution of X and Y is given by

$$f(x,y)=\frac{x+y}{21},\quad x=1,2,3;\quad y=1,2$$

Find the marginal distributions.

Solution

Marginal distributions

Y \ X	1	2	3	$p_Y(y)$
1	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{9}{21}$
2	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{12}{21}$
$p_X(x)$	$\frac{5}{21}$	$\frac{7}{21}$	$\frac{9}{21}$	$\Sigma p(x)=1$ $\Sigma p(y)=1$

Marginal distributions of X

$$P(X=1)=P(X=1,Y=1)+P(X=1,Y=2)$$
$$=\frac{2}{21}+\frac{3}{21}$$
$$=\frac{5}{21}$$

$$P(X=2)=P(X=2,Y=1)+P(X=2,Y=2)$$
$$=\frac{3}{21}+\frac{4}{21}$$
$$=\frac{7}{21}$$

$$P(X=3)=P(X=3,Y=1)+P(X=3,Y=2)$$
$$=\frac{4}{21}+\frac{5}{21}$$
$$=\frac{9}{21}$$

Marginal distributions of Y

$$P(Y=1)=P(X=1,Y=1)+P(X=2,Y=1)+P(X=3,Y=1)$$
$$=\frac{2}{21}+\frac{3}{21}+\frac{4}{21}$$
$$=\frac{9}{21}$$

$$P(Y=2)=P(X=1,Y=2)+P(X=2,Y=2)+P(X=3,Y=2)$$
$$=\frac{3}{21}+\frac{4}{21}+\frac{5}{21}$$
$$=\frac{12}{21}$$

Example 4

Given is the joint distribution of X and Y

Y \ X	0	1	2
0	0.02	0.08	0.1
1	0.05	0.2	0.25
2	0.03	0.12	0.15

Find (i) marginal distributions (ii) the conditional distributions of X given Y = 0.

Solution

Marginal distributions

Y \ X	0	1	2	$p_Y(y)$
0	0.02	0.08	0.1	0.2
1	0.05	0.2	0.25	0.5
2	0.03	0.12	0.15	0.3
$p_X(x)$	0.1	0.4	0.5	$\Sigma p(x)=1$ $\Sigma p(y)=1$

Marginal distributions of X

$$P(X=0)=P(X=0,Y=0)+P(X=0,Y=1)+P(X=0,Y=2)$$
$$=0.02+0.05+0.03$$
$$=0.1$$

$$P(X=1)=P(X=1,Y=0)+P(X=1,Y=1)+P(X=1,Y=2)$$
$$=0.08+0.2+0.12$$
$$=0.4$$

$$P(X=2)=P(X=2,Y=0)+P(X=2,Y=1)+P(X=2,Y=2)$$
$$=0.1+0.25+0.15$$
$$=0.5$$

Marginal distributions of Y

$$P(Y=0)=P(X=0,Y=0)+P(X=1,Y=0)+P(X=2,Y=0)$$
$$=0.02+0.08+0.1$$
$$=0.2$$

$$P(Y=1)=P(X=0,Y=1)+P(X=1,Y=1)+P(X=2,Y=1)$$
$$=0.05+0.2+0.25$$
$$=0.5$$

$$P(Y=2)=P(X=0,Y=2)+P(X=1,Y=2)+P(X=2,Y=2)$$
$$=0.03+0.12+0.15$$
$$=0.3$$

Conditional distributions of X for Y = 0

$$P(X=0/Y=0)=\frac{P(X=0,Y=0)}{P(Y=0)}=\frac{0.02}{0.2}=0.1$$

$$P(X=1/Y=0)=\frac{P(X=1,Y=0)}{P(Y=0)}=\frac{0.08}{0.2}=0.4$$

$$P(X=2/Y=0)=\frac{P(X=2,Y=0)}{P(Y=0)}=\frac{0.1}{0.2}=0.5$$

$X=x$	0	1	2
$P(X=x/y=0)$	0.1	0.4	0.5

Example 5

The joint probability distribution of two random variables X and Y is given by

$$P(X=0,Y=1)=\frac{1}{3},\ P(X=1,Y=-1)=\frac{1}{3}\text{ and }P(X=1,Y=1)=\frac{1}{3}.$$

Find (i) marginal distributions of X and Y and (ii) the conditional probability distributions of X given Y = 1.

Solution

Marginal distributions

Y \ X	−1	0	1	Marginal Y $P_Y(y)$
−1	0	0	$\frac{1}{3}$	$\frac{1}{3}$
0	0	0	0	0
1	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
Marginal X $P_X(x)$	0	$\frac{1}{3}$	$\frac{2}{3}$	$\Sigma p(y) = 1$ $\Sigma p(x) = 1$

Marginal distributions of X

$$P(X=-1) = P(X=-1, Y=-1) + P(X=-1, Y=0) + P(X=-1, Y=1)$$
$$= 0$$

$$P(X=0) = P(X=0, Y=-1) + P(X=0, Y=0) + P(X=0, Y=1)$$
$$= 0 + 0 + \frac{1}{3}$$
$$= \frac{1}{3}$$

$$P(X=1) = P(X=1, Y=-1) + P(X=1, Y=0) + P(X=1, Y=1)$$
$$= \frac{1}{3} + 0 + \frac{1}{3}$$
$$= \frac{2}{3}$$

Marginal distributions of Y

$$P(Y=-1) = P(X=-1, Y=-1) + P(X=0, Y=-1) + P(X=1, Y=-1)$$
$$= 0 + 0 + \frac{1}{3}$$
$$= \frac{1}{3}$$

$$P(Y=0) = P(X=-1, Y=0) + P(X=0, Y=0) + P(X=1, Y=0)$$
$$= 0$$

$$P(Y=1) = P(X=-1, Y=1) + P(X=0, Y=1) + P(X=1, Y=1)$$
$$= 0 + \frac{1}{3} + \frac{1}{3}$$
$$= \frac{2}{3}$$

Conditional Probability distributions of X given $Y = 1$ is

$$P(X = x / Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

$$P(X = -1 / Y = 1) = \frac{P(X = -1, Y = 1)}{P(Y = 1)} = 0$$

$$P(X = 0 / Y = 1) = \frac{P(X = 0, Y = 1)}{P(Y = 1)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$$P(X = 1 / Y = 1) = \frac{P(X = 1, Y = 1)}{P(Y = 1)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

Example 6

If the joint probability mass function of (X, Y) is given by

$$P(x, y) = k(2x + 3y),\ x = 0, 1, 2;\ y = 1, 2, 3$$

Find all the marginal probability distribution. Also, find the probability distribution of (X + Y).

Solution

$$P(x, y) = k(2x + 3y)$$

Marginal distributions

Y \ X	0	1	2	$p_Y(y)$
1	3 k	5 k	7 k	15 k
2	6 k	8 k	10 k	24 k
3	9 k	11 k	13 k	33 k
$p_X(x)$	18 k	24 k	30 k	72 k

Solution

Marginal distributions

Y \ X	−1	0	1	Marginal Y $P_Y(y)$
−1	0	0	$\frac{1}{3}$	$\frac{1}{3}$
0	0	0	0	0
1	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
Marginal X $p_X(x)$	0	$\frac{1}{3}$	$\frac{2}{3}$	$\Sigma p(y) = 1$ $\Sigma p(x) = 1$

Marginal distributions of X

$$P(X=-1) = P(X=-1, Y=-1) + P(X=-1, Y=0) + P(X=-1, Y=1)$$
$$=0$$

$$P(X=0) = P(X=0, Y=-1) + P(X=0, Y=0) + P(X=0, Y=1)$$
$$=0+0+\frac{1}{3}$$
$$=\frac{1}{3}$$

$$P(X=1) = P(X=1, Y=-1) + P(X=1, Y=0) + P(X=1, Y=1)$$
$$=\frac{1}{3}+0+\frac{1}{3}$$
$$=\frac{2}{3}$$

Marginal distributions of Y

$$P(Y=-1) = P(X=-1, Y=-1) + P(X=0, Y=-1) + P(X=1, Y=-1)$$
$$=0+0+\frac{1}{3}$$
$$=\frac{1}{3}$$

$$P(Y=0) = P(X=-1, Y=0) + P(X=0, Y=0) + P(X=1, Y=0)$$
$$=0$$

$$P(Y=1) = P(X=-1, Y=1) + P(X=0, Y=1) + P(X=1, Y=1)$$
$$=0+\frac{1}{3}+\frac{1}{3}$$
$$=\frac{2}{3}$$

Conditional Probability distributions of X given $Y = 1$ is

$$P(X=x/Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$P(X=-1/Y=1) = \frac{P(X=-1, Y=1)}{P(Y=1)} = 0$$

$$P(X=0/Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$$P(X=1/Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

Example 6

If the joint probability mass function of (X, Y) is given by

$$P(x, y) = k(2x+3y),\ x = 0, 1, 2;\ y = 1, 2, 3$$

Find all the marginal probability distribution. Also, find the probability distribution of $(X + Y)$.

Solution

$$P(x, y) = k(2x+3y)$$

Marginal distributions

Y \ X	0	1	2	$p_Y(y)$
1	3 k	5 k	7 k	15 k
2	6 k	8 k	10 k	24 k
3	9 k	11 k	13 k	33 k
$p_X(x)$	18 k	24 k	30 k	72 k

$$\because \quad \Sigma p(x) = \Sigma p(y) = 1$$

$$72k = 1$$

$$k = \frac{1}{72}$$

Marginal distributions of X and Y

Y \ X	0	1	2	$p_Y(y)$
1	$\frac{3}{72}$	$\frac{5}{72}$	$\frac{7}{72}$	$\frac{15}{72}$
2	$\frac{6}{72}$	$\frac{8}{72}$	$\frac{10}{72}$	$\frac{24}{72}$
3	$\frac{9}{72}$	$\frac{11}{72}$	$\frac{13}{72}$	$\frac{33}{72}$
$p_X(x)$	$\frac{18}{72}$	$\frac{24}{72}$	$\frac{30}{72}$	1

Probability distribution of $(X + Y)$

$X + Y$	P
1	$p_{01} = \frac{3}{72}$
2	$p_{02} + p_{11} = \frac{11}{72}$
3	$p_{03} + p_{12} + p_{21} = \frac{24}{72}$
4	$p_{13} + p_{22} = \frac{21}{72}$
5	$p_{23} = \frac{13}{72}$
	Total = 1

Example 7

Let X and Y have the following marginal probability distributions:

X \ Y	0	1	2	$p_X(x)$
0	0.1	0.04	0.06	0.2
1	0.2	0.08	0.12	0.4
2	0.2	0.08	0.12	0.4
$p_Y(y)$	0.5	0.2	0.3	$\Sigma p(x) = 1$ $\Sigma p(y) = 1$

Solution

X and Y are independent, if $p_{ij} = p_{i*}\, p_{*j}$ for all i and j.

$$p_{0*} = 0.1 + 0.04 + 0.06 = 0.2$$

$$p_{1*} = 0.2 + 0.08 + 0.12 = 0.4$$

$$p_{2*} = 0.2 + 0.08 + 0.12 = 0.4$$

$$p_{*0} = 0.1 + 0.2 + 0.2 = 0.5$$

$$p_{*1} = 0.04 + 0.08 + 0.08 = 0.2$$

$$p_{*2} = 0.06 + 0.12 + 0.12 = 0.3$$

Now,

$$p_{0*}\, p_{*0} = (0.2)(0.5) = 0.1 = p_{00}$$

$$p_{0*}\, p_{*1} = (0.2)(0.2) = 0.04 = p_{01}$$

$$p_{0*}\, p_{*2} = (0.2)(0.3) = 0.06 = p_{02}$$

Similarly, it can be verified that

$$p_{1*}\, p_{*0} = p_{10};\ p_{1*}\, p_{*1} = p_{11};\ p_{1*} p_{*2} = p_{12}$$

$$p_{2*}\, p_{*0} = p_{20};\ p_{2*}\, p_{*1} = p_{21};\ p_{2*} p_{*2} = p_{22}$$

Hence, the random variables X and Y are independent.

EXERCISE 2.3

1. Find the marginal distributions of X and Y from the bivariate distribution of (X, Y) given below:

X \ Y	1	2
1	0.1	0.2
2	0.3	0.4

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Ans.:

$X = x$	1	2
$P(X = x)$	0.3	0.7

$Y = y$	1	2
$P(Y = y)$	0.4	0.6

2. For the joint probability distribution of two random variables X and Y given below:

X \ Y	1	2	3	4	Total
1	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{10}{36}$
2	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{9}{36}$
3	$\frac{5}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{8}{36}$
4	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{9}{36}$
Total	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	1

Find (i) marginal distributions of X and Y.

(ii) Conditional distributions of X given the value of $Y = 1$ and that of Y given the value of $X = 2$.

Ans.: (i)

Value of $X = x$	1	2	3	4
$P(X = x)$	$\frac{10}{36}$	$\frac{9}{36}$	$\frac{8}{36}$	$\frac{9}{36}$

Value of $Y = y$	1	2	3	4
$P(Y = y)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{9}{36}$

(ii)

$Y = y$	1	2	3	4
$P(X = x/Y = 1)$	$\frac{4}{11}$	$\frac{1}{11}$	$\frac{5}{11}$	$\frac{1}{11}$

$Y = y$	1	2	3	4
$P(Y = y/X = 2)$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{9}$

3. A two-dimensional random variable (X, Y) has the joint probability mass function $p(x, y) = \frac{2x + y}{27}$, where x and y can assume only the integer values 0, 1 and 2. Find the conditional distributions of Y for $X = x$.

Y \ X	0	1	2
0	0	$\frac{1}{3}$	$\frac{2}{3}$
1	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{4}{9}$
2	$\frac{4}{15}$	$\frac{5}{15}$	$\frac{6}{15}$

4. Let X and Y have the following joint probability distribution:

Y \ X	0	1
0	0.1	0.15
1	0.2	0.3
2	0.1	0.15

Show that X and Y are independent.

5. The joint probability distribution of X and Y is given below:

Y \ X	1	2	3
2	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$
4	$\frac{1}{4}$	$\frac{1}{4}$	0
6	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$

Find $P(X < 4)$, $P(Y > 1)$, $P(X < 4/Y > 1)$, $P(2 \le X \le 5, Y > 1)$, $P(Y = 3/X = 2)$, $P(X + Y \le 7)$.

$$\left[\textbf{Ans.:} \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{1}{6}, \frac{19}{24}\right)\right]$$

6. For the following joint probability distribution of X and Y, find (i) marginal distributions of X and Y, (ii) conditional distributions of X given $Y = 2$, (iii) Are X and Y independent?

The conditional probability density function of Y, given $X = x$, denoted by $f(y/x)$ is defined as:

$$f(y/x) = \frac{f(x,y)}{f_X(x)}$$

A necessary and sufficient condition for the continuous random variables X and Y to be independent is

$$f(x,y) = f_X(x)\, f_Y(y)$$

Example 1

The joint probability density function of a two dimensional random variable is

$$f(x,y) = \frac{1}{2}xe^{-y}, \quad 0 < x < 2, y > 0$$
$$= 0, \quad \text{otherwise}$$

Find the cumulative distribution function.

Solution

The cumulative distribution function is given by

$$F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy$$
$$= \int_0^y\int_0^x \frac{1}{2}xe^{-y}dx\,dy$$
$$= \frac{1}{2}\int_0^y e^{-y}\left|\frac{x^2}{2}\right|_0^x dy$$
$$= \frac{1}{4}x^2\left|-e^{-y}\right|_0^y$$
$$= \frac{1}{4}x^2(-e^{-y}+e^0)$$
$$= \frac{1}{4}x^2(-e^{-y}+1)$$
$$F(x,y) = \frac{1}{4}x^2(1-e^{-y}), \quad 0 < x < 2, y > 0$$
$$= 0 \quad , \text{otherwise}$$

Example 2

The joint probability density function of a two dimensional random variable (X, Y) is $f(x,y) = xe^{-x(y+1)}, x > 0, y > 0$. *Examine whether the variables X and Y are independent.*

Solution

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$
$$= \int_0^{\infty} xe^{-x(y+1)}dy$$
$$= x\left|\frac{e^{-x(y+1)}}{-x}\right|_0^{\infty}$$
$$= -(e^{-\infty}-e^{-x})$$
$$= e^{-x}, x > 0$$
$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$
$$= \int_0^{\infty} xe^{-x(y+1)}dx$$
$$= \left|x\frac{e^{-x(y+1)}}{-(y+1)} - 1\frac{e^{-x(y+1)}}{(y+1)^2}\right|_0^{\infty}$$
$$= \frac{1}{(y+1)^2}, y > 0$$
$$f_X(x)\cdot f_Y(y) = e^{-x}\frac{1}{(y+1)^2}$$
$$f(x,y) = xe^{-x(y+1)}$$
$$f(x,y) \neq f_X(x)\cdot f_Y(y)$$

Hence, X and Y are not independent.

Example 3

Two random variables X and Y have the joint pdf

$$f(x,y) = Ae^{-(2x+y)}, \quad x, y \geq 0$$
$$= 0 \quad , \text{otherwise}$$

Find (i) A (ii) marginal pdf of X and Y (iii) f(y/x)

Solution

(i) Since $f(x, y)$ is a pdf,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{d}x\,\mathrm{d}y=1$$

$$\int_0^{\infty}\int_0^{\infty} Ae^{-(2x+y)}\mathrm{d}x\,\mathrm{d}y=1$$

$$A\int_0^{\infty}\left[\int_0^{\infty} e^{-2x}\mathrm{d}x\right]e^{-y}\mathrm{d}y=1$$

$$A\int_0^{\infty}\left|\frac{e^{-2x}}{-2}\right|_0^{\infty} e^{-y}\mathrm{d}y=1$$

$$\frac{A}{-2}\int_0^{\infty}(e^{-\infty}-e^{0})e^{-y}\mathrm{d}y=1$$

$$\frac{A}{-2}\int_0^{\infty}(-1)e^{-y}\mathrm{d}y=1$$

$$\frac{A}{2}\left|-e^{-y}\right|_0^{\infty}=1$$

$$\frac{A}{2}(-e^{-\infty}+e^{0})=1$$

$$A=2$$

(ii) $f_X(x)=\int_{-\infty}^{\infty} f(x,y)\mathrm{d}y$

$$=\int_0^{\infty} Ae^{-(2x+y)}\mathrm{d}y$$

$$=2\left|-e^{-(2x+y)}\right|_0^{\infty}$$

$$=2(-e^{-\infty}+e^{-2x})$$

$$=2e^{-2x}, x\geq 0$$

$\therefore$ $f_X(x)=2e^{-2x}, \quad x\geq 0$

$\quad\quad =0 \quad , \quad x<0$

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}x$$

$$=\int_0^{\infty} Ae^{-(2x+y)}\mathrm{d}x$$

$$=2\left|\frac{e^{-(2x+y)}}{-2}\right|_0^{\infty}$$

$$=-\frac{2}{2}(e^{-\infty}-e^{-y})$$

$$=-(0-e^{-y})$$

$$=e^{-y}, y\geq 0$$

$\therefore f_Y(y)=e^{-y}, \ y\geq 0$

$\quad\quad =0 \ , \ y<0$

(iii) $f(y/x)=\dfrac{f(x,y)}{f_X(x)}$

$$=\frac{2e^{-(2x+y)}}{2e^{-2x}}$$

$$=e^{-y}, \ y\geq 0$$

Example 4

The joint probability distribution of X and Y is given by

$$f(x,y)=\frac{6-x-y}{8}, \quad 0<x<2, \ 2<y<4$$

$$=0, \quad \text{otherwise}$$

Find $f(y/x=2)$.

Solution

$$f(y/x)=\frac{f(x,y)}{f_X(x)}$$

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y$$

$$=\int_2^4 \frac{6-x-y}{8}\mathrm{d}y$$

$$=\frac{1}{8}\left|6y-xy-\frac{y^2}{2}\right|_2^4$$

$$= \frac{1}{8}[(24-4x-8)-(12-2x-2)]$$

$$= \frac{1}{8}(6-2x)$$

$$f(y/x) = \frac{6-x-y}{6-2x}$$

Putting $x = 2$,

$$f(y/x=2) = \frac{4-y}{2}$$

Example 5

The joint pdf of a two dimensional variable (X, Y) is given by

$$f(x,y) = kxye^{-(x^2+y^2)},\quad x>0, y>0.$$

Find the value of k and prove that X and Y are independent.

Solution

Since $f(x, y)$ is a pdf,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_0^{\infty}\int_0^{\infty} k\,x\,y\,e^{-(x^2+y^2)}dx\,dy = 1$$

$$k\int_0^{\infty} ye^{-y^2}dy \cdot \int_0^{\infty} xe^{-x^2}dx = 1 \quad \text{...(1)}$$

Putting $x^2 = t$, $x = \sqrt{t}$, $dx = \frac{1}{2\sqrt{t}}dt$

When $x = 0$, $t = 0$

When $x = \infty$, $t = \infty$

$$\int_0^{\infty} xe^{-x^2}dx = \int_0^{\infty}\sqrt{t}e^{-t}\frac{1}{2\sqrt{t}}dt$$

$$= \frac{1}{2}\left|-e^{-t}\right|_0^{\infty}$$

$$= \frac{1}{2}(-e^{-\infty}+e^0)$$

$$= \frac{1}{2}$$

Similarly, $\int_0^{\infty} ye^{-y^2}dy = \frac{1}{2}$

Putting both integral values in Eq. (1),

$$k\cdot\frac{1}{2}\cdot\frac{1}{2} = 1$$

$$k = 4$$

If X and Y are independent,

$$f_X(x)\cdot f_Y(y) = f(x,y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^{\infty} k\,x\,y\,e^{-(x^2+y^2)}dy$$

$$= k\,x\,e^{-x^2}\int_0^{\infty} y\,e^{-y^2}dy$$

$$= 4x\,e^{-x^2}\cdot\frac{1}{2}$$

$$= 2x\,e^{-x^2}, x>0$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^{\infty} k\,x\,y\,e^{-(x^2+y^2)}dx$$

$$= k\,y\,e^{-y^2}\int_0^{\infty} xe^{-x^2}dx$$

$$= 4y\,e^{-y^2}\cdot\frac{1}{2}$$

$$= 2ye^{-y^2}, y>0$$

$$f_X(x)\cdot f_Y(y) = 2x\,e^{-x^2}\cdot 2y\,e^{-y^2}, x,y>0$$

$$= 4xy\,e^{-(x^2+y^2)}$$

$$= f(x,y),\quad x>0, y>0$$

Hence, X and Y are independent.

Example 6

The joint probability density function of a two dimensional random variable (X, Y) is

$$f(x,y) = kx(x-y), \quad 0<x<2, -x<y<x$$
$$= 0 \quad , \quad \text{elsewhere}$$

Find (i) k (ii) $f_X(x)$ (iii) $f_Y(y)$ (iv) $f(y/x)$

Solution

Since $f(x, y)$ is a probability density function,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_0^2\int_{-x}^{x} kx(x-y)\,dy\,dx = 1$$

$$k\int_0^2 x\left|xy-\frac{y^2}{2}\right|_{-x}^{x} dx = 1$$

$$k\int_0^2 x\left[\left(x^2-\frac{x^2}{2}\right)-\left(-x^2-\frac{x^2}{2}\right)\right]dx = 1$$

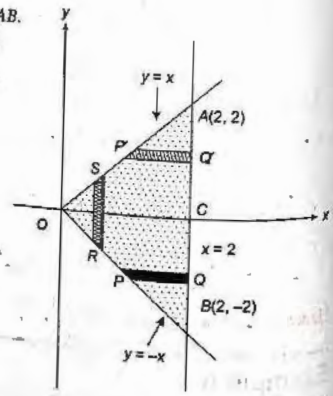
$$k\int_0^2 2x^3 dx = 1$$

$$k\left|\frac{2x^4}{4}\right|_0^2 = 1$$

$$k(8) = 1$$

$$k = \frac{1}{8}$$

(ii) The region of integration is ΔOAB.
In ΔOAB, along vertical strip RS,
Limits of y : $y = -x$ to $y = x$
and x varies from $x = 0$ to $x = 2$.



$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{-x}^{x} kx(x-y)\,dy$$

$$= kx\left|xy-\frac{y^2}{2}\right|_{-x}^{x}$$

$$= kx\left[\left(x^2-\frac{x^2}{2}\right)-\left(-x^2-\frac{x^2}{2}\right)\right]$$

$$= kx\,(2x^2)$$

$$= \frac{1}{8}(2x^3)$$

$$= \frac{x^3}{4}, \quad 0<x<2$$

(iii) For limits of x, ΔOAB is divided into two parts, ΔOBC and ΔOAC.
In ΔOBC, along horizontal strip PQ,
Limits of x: $x = -y$ to $x = 2$ and y varies from $y = -2$ to $y = 0$.
In ΔOAC, along horizontal strip $P'Q'$,
Limits of x: $x = y$ to $x = 2$ and y varies from $y = 0$ to $y = 2$.

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_{-y}^{2} kx(x-y)\,dx, \quad -2\le y\le 0$$

$$= \int_{y}^{2} kx(x-y)\,dx, \quad 0\le y\le 2$$

Now,

$$\int_{-y}^{2} kx(x-y)\,dx = k\left|\frac{x^3}{3}-\frac{x^2y}{2}\right|_{-y}^{2}$$

$$= k\left[\left(\frac{8}{3}-2y\right)-\left(-\frac{y^3}{3}-\frac{y^3}{2}\right)\right]$$

$$= \frac{1}{8}\left(\frac{8}{3}-2y+\frac{5y^3}{6}\right)$$

$$= \frac{1}{3}-\frac{y}{4}+\frac{5}{48}y^3$$

Also,

$$\int_{y}^{2} kx(x-y)\,dx = k\left|\frac{x^3}{3}-\frac{x^2y}{2}\right|_{y}^{2}$$

$$= k\left[\left(\frac{8}{3}-2y\right)-\left(\frac{y^3}{3}-\frac{y^3}{2}\right)\right]$$

$$= \frac{1}{8}\left(\frac{8}{3}-2y+\frac{y^3}{6}\right)$$

$$= \frac{1}{3}-\frac{y}{4}+\frac{y^3}{48}$$

Hence, $f_Y(y) = \frac{1}{3}-\frac{y}{4}+\frac{5}{48}y^3, \quad -2\le y\le 0$

$$= \frac{1}{3}-\frac{y}{4}+\frac{y^3}{48}, \quad 0\le y\le 2$$

(iv) $f(y/x)=\dfrac{f(x,y)}{f_X(x)}$

$$=\frac{\frac{1}{8}x(x-y)}{\frac{x^3}{4}}$$

$$=\frac{x-y}{2x^2},\quad -x<y<x$$

Example 7

The joint pdf of a two-dimensional random variable (X, Y) is given by

$$f_{XY}(x,y)=\frac{8}{9}xy,\qquad 1\le y\le 2,\ \ 1\le x\le y$$
$$=0\ ,\qquad \text{otherwise}$$

Find the marginal density function of X and Y.

Solution

The region of integration is ΔABC.
In ΔABC, along vertical strip PQ, limits of y: $y=x$ to $y=2$ and x varies from $x=1$ to $x=2$.
Marginal density function of X is

$$f_X(x)=\int_{-\infty}^{\infty}f(x,y)\mathrm{d}y$$
$$=\int_x^2\frac{8}{9}xy\ \mathrm{d}y$$
$$=\frac{8}{9}x\left|\frac{y^2}{2}\right|_x^2$$
$$=\frac{4}{9}x(4-x^2),\qquad 1\le x\le 2$$

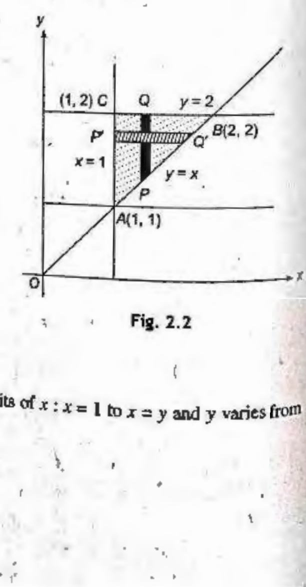


Fig. 2.2

In ΔABC, along horizontal strip $P'Q'$, limits of x : $x=1$ to $x=y$ and y varies from $y=1$ to $y=2$.
Marginal density function of Y is

$$f_Y(y)=\int_{-\infty}^{\infty}f(x,y)\mathrm{d}x$$
$$=\int_1^y\frac{8}{9}xy\ \mathrm{d}x$$

$$=\frac{8}{9}y\left|\frac{x^2}{2}\right|_1^y$$
$$=\frac{4}{9}y(y^2-1),\qquad 1\le y\le 2$$

Example 8

If the joint distribution function of X and Y is given by

$$F(x,y)=(1-e^{-x})(1-e^{-y}),\quad x>0,y>0$$
$$=0\qquad ,\quad \text{otherwise}$$

Find $f_X(x), f_Y(y)$ (ii) *Are X and Y are independent* (iii) *Find* $P(1<X<3,\ 1<Y<2)$.

Solution

$$F(x,y)=(1-e^{-x})(1-e^{-y})$$

The joint pdf is given by

$$f(x,y)=\frac{\partial^2 F}{\partial x\partial y}$$
$$=\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$
$$=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}(1-e^{-x})(1-e^{-y})\right]$$
$$=\frac{\partial}{\partial x}(1-e^{-x})(e^{-y})$$
$$=e^{-x}e^{-y}$$
$$=e^{-(x+y)},x>0,y>0$$

$$\therefore\quad f(x,y)=e^{-(x+y)},\quad x>0,y>0$$
$$=0,\qquad ,\ \text{otherwise}$$

(i) $f_X(x)=\int_{-\infty}^{\infty}f(x,y)\,\mathrm{d}y$

$$=\int_0^{\infty}e^{-(x+y)}\mathrm{d}y$$
$$=\left|-e^{-(x+y)}\right|_0^{\infty}$$
$$=(-e^{-\infty}+e^{-x})$$
$$=e^{-x},x>0$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$
$$= \int_0^{\infty} e^{-(x+y)}dx$$
$$= \left|-e^{-(x+y)}\right|_0^{\infty}$$
$$= (-e^{-\infty} + e^{-y})$$
$$= e^{-y}, x > 0$$

(ii) $f_X(x)\cdot f_Y(y) = e^{-x}\cdot e^{-y}$
$$= e^{-(x+y)}, x>0, y>0$$
$$= f(x,y)$$

Hence, X and Y are independent.

(iii) Since X and Y are independent,

$$P(1<X<3, 1<Y<2) = P(1<X<3)\cdot P(1<Y<2)$$
$$= \int_1^3 f_X(x)\,dx\cdot\int_1^2 f_Y(y)\,dy$$
$$= \int_1^3 e^{-x}\,dx\cdot\int_1^2 e^{-y}\,dy$$
$$= \left|-e^{-x}\right|_1^3\cdot\left|-e^{-y}\right|_1^2$$
$$= (-e^{-3}+e^{-1})\cdot(-e^{-2}+e^{-1})$$
$$= e^{-5} - e^{-4} - e^{-3} - e^{-2}$$

Example 9

The joint probability density of two random variables is given by

$$f(x,y) = 15e^{-3x-5y}, \quad x>0, y>0$$
$$= 0 \quad , \text{ elsewhere}$$

Find (i) $P(1<X<2, 0.2<Y<0.3)$ (ii) $P(X<2, Y>0.2)$ (iii) *marginal probability density functions of X and Y.*

Solution

(i) $P(1<X<2, 0.2<Y<0.3) = \int_{0.2}^{0.3}\int_1^2 f(x,y)\,dx\,dy$

$$= \int_{0.2}^{0.3}\int_1^2 15e^{-3x-5y}\,dx\,dy$$
$$= 15\int_{0.2}^{0.3} e^{-5y}\left[\int_1^2 e^{-3x}dx\right]dy$$

$$= 15\int_{0.2}^{0.3} e^{-5y}\left|\frac{e^{-3x}}{-3}\right|_1^2 dy$$
$$= -5\int_{0.2}^{0.3} e^{-5y}(e^{-6}-e^{-3})dy$$
$$= -5(e^{-6}-e^{-3})\left|\frac{e^{-5y}}{-5}\right|_{0.2}^{0.3}$$
$$= (e^{-6}-e^{-3})(e^{-1.5}-e^{-1.0})$$
$$= 6.84\times10^{-3}$$

(ii) $P(X<2, Y>0.2) = \int_{0.2}^{\infty}\int_0^2 f(x,y)\,dx\,dy$

$$= \int_{0.2}^{\infty}\int_0^2 15e^{-3x-5y}\,dx\,dy$$
$$= 15\int_{0.2}^{\infty}\left[\int_0^2 e^{-3x}dx\right]e^{-5y}dy$$
$$= 15\int_{0.2}^{\infty}\left|\frac{e^{-3x}}{-3}\right|_0^2 e^{-5y}dy$$
$$= -5\int_{0.2}^{\infty}(e^{-6}-1)e^{-5y}dy$$
$$= -5(e^{-6}-1)\left|\frac{e^{-5y}}{-5}\right|_{0.2}^{\infty}$$
$$= (e^{-6}-1)(e^{-\infty}-e^{-1.0})$$
$$= (e^{-6}-1)(-e^{-1.0})$$
$$= 0.367$$

(iii) The region of integration is the first quadrant. Hence, x and y both varies from 0 to ∞.

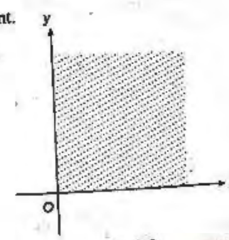


Fig. 2.3

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$
$$= \int_0^{\infty} 15e^{-3x-5y}dy$$
$$= 15e^{-3x}\left|\frac{e^{-5y}}{-5}\right|_0^{\infty}$$

$$= -3e^{-3x}(e^{-\infty} - e^{0})$$
$$= 3e^{-3x}, \ x > 0$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$
$$= \int_{0}^{\infty} 15e^{-3x-5y} dy$$
$$= 15e^{-5y} \left| \frac{e^{-3x}}{-3} \right|_0^{\infty}$$
$$= -5e^{-5y}(e^{-\infty} - e^{0})$$
$$= 5e^{-5y}, \ y > 0$$

Example 10

The joint pdf of (X, Y) is given by

$$f(x, y) = \frac{1}{4} e^{-|x|-|y|}, \ -\infty < x < \infty, -\infty < y < \infty$$

(i) Are X and Y independent?
(ii) Find the probability that $X \le 1$ and $Y < 0$.

Solution

$$|x| = -x, \quad -\infty < x \le 0$$
$$= x, \quad 0 \le x < \infty$$

Similarly,
$$|y| = -y, \quad -\infty < y \le 0$$
$$= y, \quad 0 \le y < \infty$$

$\therefore$
$$f(x, y) = \frac{1}{4} e^{-|x|-|y|}$$
$$= \frac{1}{4} e^{x+y}, -\infty < x \le 0, -\infty < y \le 0$$
$$= \frac{1}{4} e^{-x-y}, 0 \le x < 0, 0 \le y < \infty$$

(i)
$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$
$$= \int_{-\infty}^{\infty} \frac{1}{4} e^{-|x|-|y|} dy$$

$$= \frac{1}{4} e^{-|x|} \int_{-\infty}^{\infty} e^{-|y|} dy$$
$$= \frac{1}{4} e^{-|x|} \left[\int_{-\infty}^{0} e^{y} dy + \int_{0}^{\infty} e^{-y} dy \right]$$
$$= \frac{1}{4} e^{-|x|} \left[\left| e^{y} \right|_{-\infty}^{0} + \left| -e^{-y} \right|_{0}^{\infty} \right]$$
$$= \frac{1}{4} e^{-|x|}(1+1)$$
$$= \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{4} e^{-|x|-|y|} dx$$
$$= \frac{1}{4} e^{-|y|} \int_{-\infty}^{\infty} e^{-|x|} dx$$
$$= \frac{1}{4} e^{-|y|} \left[\int_{-\infty}^{0} e^{x} dx + \int_{0}^{\infty} e^{-x} dx \right]$$
$$= \frac{1}{4} e^{-|y|} \left[\left| e^{x} \right|_{-\infty}^{0} + \left| -e^{-x} \right|_{0}^{\infty} \right]$$
$$= \frac{1}{4} e^{-|y|}(1+1)$$
$$= \frac{1}{2} e^{-|y|}, \quad -\infty < y < \infty$$

$$f_X(x) \cdot f_Y(y) = \frac{1}{2} e^{-|x|} \cdot \frac{1}{2} e^{-|y|}$$
$$= \frac{1}{4} e^{-|x|-|y|}, -\infty < x < \infty, -\infty < y < \infty$$
$$= f(x, y)$$

Hence, X and Y are independent.

(ii)
$$P(X \le 1, Y < 0) = \int_{-\infty}^{0} \int_{-\infty}^{1} f(x, y)\, dx\, dy$$
$$= \int_{-\infty}^{0} \int_{-\infty}^{1} \frac{1}{4} e^{-|x|-|y|} dx\, dy$$

$$=\frac{1}{4}\int_{-\infty}^{0} e^{-|y|}\left[\int_{-\infty}^{0} e^{x}\,dx+\int_{0}^{1} e^{-x}\,dx\right]dy$$

$$=\frac{1}{4}\int_{-\infty}^{0} e^{-|y|}\left[\left|e^{x}\right|_{-\infty}^{0}+\left|-e^{-x}\right|_{0}^{1}\right]dy$$

$$=\frac{1}{4}\int_{-\infty}^{0} e^{-|y|}(1-e^{-1}+1)\,dy$$

$$=\frac{1}{4}(2-e^{-1})\int_{-\infty}^{0} e^{y}\,dy$$

$$=\frac{1}{4}(2-e^{-1})\left|e^{y}\right|_{-\infty}^{0}$$

$$=\frac{1}{4}(2-e^{-1})(1)$$

$$=\frac{1}{4}(2-e^{-1})$$

Example 11

The joint pdf of (X, Y) is given by

$$f(x,y)=ke^{-x}\cos y,\quad 0\le x\le 2,\ 0\le y\le\frac{\pi}{2}$$
$$=0\qquad,\qquad \text{otherwise}$$

Find (i) k (ii) $P\left(X+Y\ge\frac{\pi}{2}\right)$.

Solution

(i) Since $f(x, y)$ is a pdf,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy=1$$

$$\int_{0}^{\frac{\pi}{2}}\int_{0}^{2} k\,e^{-x}\cos y\,dx\,dy=1$$

$$k\int_{0}^{\frac{\pi}{2}}\cos y\left|-e^{-x}\right|_{0}^{2}dy=1$$

$$k\int_{0}^{\frac{\pi}{2}}\cos y\,(-e^{-2}+1)\,dy=1$$

$$k(1-e^{-2})\left|\sin y\right|_{0}^{\frac{\pi}{2}}=1$$

$$k(1-e^{-2})(1)=1$$

$$k=\frac{1}{1-e^{-2}}$$

(ii) $P\left(X+Y\ge\frac{\pi}{2}\right)=1-P\left(X+Y<\frac{\pi}{2}\right)$

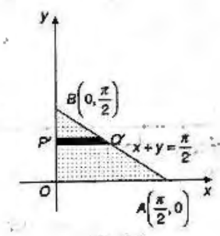


Fig. 2.4

The region of integration $x + y < 1$ is the ΔOAB. In ΔOAB, along horizontal strip $P'Q'$,

Limits of $x: x=0$ to $x=\frac{\pi}{2}-y$

Limits of $y: y=0$ to $y=\frac{\pi}{2}$

$$P(X+Y<1)=\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}-y} ke^{-x}\cos y\,dx\,dy$$

$$=k\int_{0}^{\frac{\pi}{2}}\cos y\left|-e^{-x}\right|_{0}^{\frac{\pi}{2}-y}dy$$

$$=k\int_{0}^{\frac{\pi}{2}}\cos y\left[-e^{-\left(\frac{\pi}{2}-y\right)}+e^{0}\right]dy$$

$$=k\int_{0}^{\frac{\pi}{2}}\cos y\left[-e^{-\frac{\pi}{2}}e^{y}+1\right]dy$$

$$=k\left[-e^{-\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}}e^{y}\cos y\,dy+\int_{0}^{\frac{\pi}{2}}\cos y\,dy\right]$$

$$=k\left[-e^{-\frac{\pi}{2}}\left|\frac{e^{y}}{1+1}(\cos y+\sin y)\right|_{0}^{\frac{\pi}{2}}+\left|\sin y\right|_{0}^{\frac{\pi}{2}}\right]$$

$$=k\left[-e^{-\frac{\pi}{2}}\left\{\frac{e^{\frac{\pi}{2}}}{2}\left(\cos\frac{\pi}{2}+\sin\frac{\pi}{2}\right)-\frac{1}{2}\right\}+\sin\frac{\pi}{2}\right]$$

$$=k\left[-e^{-\frac{\pi}{2}}\left\{\frac{e^{\frac{\pi}{2}}}{2}(1)-\frac{1}{2}\right\}+1\right]$$

$$=k\left(-\frac{1}{2}+\frac{e^{-\frac{\pi}{2}}}{2}+1\right)$$

$$=\frac{k}{2}\left(1+e^{-\frac{\pi}{2}}\right)$$

$$P\left(X+Y\geq\frac{\pi}{2}\right)=1-\frac{k}{2}\left(1+e^{-\frac{\pi}{2}}\right)$$

$$=1-\frac{\left(1+e^{-\frac{\pi}{2}}\right)}{2(1-e^{-2})}$$

Example 12

The joint p.d.f of a two-dimensional random variable (X, Y) is given by

$$f(x,y)=\frac{1}{8}(6-x-y),\quad 0<x<2,\ 2<y<4$$
$$=0\quad,\quad \text{otherwise}$$

Find (i) $P(X<1, Y<3)$ (ii) $P(X<1/Y<3)$.

Solution

(i) $P(X<1, Y<3)=\int_2^3\int_0^1 f(x,y)\,dx\,dy$

$$=\int_2^3\int_0^1\frac{1}{8}(6-x-y)\,dx\,dy$$

$$=\frac{1}{8}\int_2^3\left|6x-\frac{x^2}{2}-xy\right|_0^1 dy$$

$$=\frac{1}{8}\int_2^3\left(6-\frac{1}{2}-y\right)dy$$

$$=\frac{1}{8}\int_2^3\left(\frac{11}{2}-y\right)dy$$

$$=\frac{1}{8}\left|\frac{11}{2}y-\frac{y^2}{2}\right|_2^3$$

$$=\frac{1}{8}\left[\left(\frac{33}{2}-\frac{9}{2}\right)-(11-2)\right]$$

$$=\frac{3}{8}$$

(ii) $P(X<1/Y<3)=\dfrac{P(X<1,Y<3)}{P(Y<3)}$ ...(1)

$$P(Y<3)=\int_2^3\int_0^2 f(x,y)\,dx\,dy$$

$$=\int_2^3\int_0^2\frac{1}{8}(6-x-y)\,dx\,dy$$

$$=\frac{1}{8}\int_2^3\left|6x-\frac{x^2}{2}-xy\right|_0^2 dy$$

$$=\frac{1}{8}\int_2^3(12-2-2y)\,dy$$

$$=\frac{1}{8}\int_2^3(10-2y)\,dy$$

$$=\frac{1}{8}\left|10y-y^2\right|_2^3$$

$$=\frac{1}{8}[(30-9)-(20-4)]$$

$$=\frac{5}{8}$$

Substituting in Eq (1),

$$P(X<1/Y<3)=\frac{\left(\frac{3}{8}\right)}{\left(\frac{5}{8}\right)}=\frac{3}{5}$$

Example 13

The joint pdf of a two-dimensional random variable (X, Y) is given by

$$f_{XY}(x,y)=\begin{cases}xy^2+\dfrac{x^2}{8}, & 0<x<2,\ 0<y<1\\ 0\quad, & \text{Otherwise}\end{cases}$$

Find (i) $P(X>1)$ (ii) $P\left(Y<\frac{1}{2}\right)$ (iii) $P\left(X>1/Y<\frac{1}{2}\right)$ (iv) $P\left(Y<\frac{1}{2}/X>1\right)$ (v) $P(X<Y)$ (vi) $P(X+Y\leq 1)$

Solution

(i) $$P(X>1)=\int_0^1\int_1^2 f(x,y)\,dx\,dy$$

$$=\int_0^1\int_1^2\left(xy^2+\frac{x^2}{8}\right)dx\,dy$$

$$=\int_0^1\left|\frac{x^2y^2}{2}+\frac{x^3}{24}\right|_1^2 dy$$

$$=\int_0^1\left(2y^2+\frac{1}{3}\right)-\left(\frac{y^2}{2}+\frac{1}{24}\right)dy$$

$$=\int_0^1\left(\frac{3y^2}{2}+\frac{7}{24}\right)dy$$

$$=\left|\frac{y^3}{2}+\frac{7y}{24}\right|_0^1$$

$$=\frac{1}{2}+\frac{7}{24}$$

$$=\frac{19}{24}$$

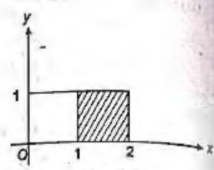


Fig. 2.5

(ii) $$P\left(Y<\frac{1}{2}\right)=\int_0^{\frac{1}{2}}\int_0^2 f(x,y)\,dx\,dy$$

$$=\int_0^{\frac{1}{2}}\int_0^2\left(xy^2+\frac{x^2}{8}\right)dx\,dy$$

$$=\int_0^{\frac{1}{2}}\left|\frac{x^2y^2}{2}+\frac{x^3}{24}\right|_0^2 dy$$

$$=\int_0^{\frac{1}{2}}\left(2y^2+\frac{1}{3}\right)dy$$

$$=\left|\frac{2y^3}{3}+\frac{1}{3}y\right|_0^{\frac{1}{2}}$$

$$=\frac{1}{12}+\frac{1}{6}$$

$$=\frac{1}{4}$$

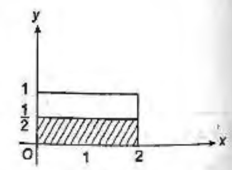


Fig. 2.6

(iii) $$P\left(X>1,Y<\frac{1}{2}\right)=\int_0^{\frac{1}{2}}\int_1^2 f(x,y)\,dx\,dy$$

$$=\int_0^{\frac{1}{2}}\int_1^2\left(xy^2+\frac{x^2}{8}\right)dx\,dy$$

$$=\int_0^{\frac{1}{2}}\left|\frac{x^2y^2}{2}+\frac{x^3}{24}\right|_1^2 dy$$

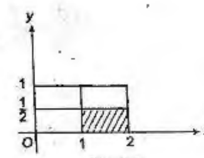


Fig. 2.7

$$=\int_0^{\frac{1}{2}}\left(2y^2+\frac{1}{3}\right)-\left(\frac{y^2}{2}+\frac{1}{24}\right)dy$$

$$=\int_0^{\frac{1}{2}}\left(\frac{3y^2}{2}+\frac{7}{24}\right)dy$$

$$=\left|\frac{y^3}{2}+\frac{7y}{24}\right|_0^{\frac{1}{2}}$$

$$=\frac{1}{16}+\frac{7}{48}$$

$$=\frac{5}{24}$$

$$P\left(X>1/Y<\frac{1}{2}\right)=\frac{P\left(X>1,Y<\frac{1}{2}\right)}{P\left(Y<\frac{1}{2}\right)}=\frac{\frac{5}{24}}{\frac{1}{4}}=\frac{5}{6}$$

(iv) $$P\left(Y<\frac{1}{2}/X>1\right)=\frac{P\left(X>1,Y<\frac{1}{2}\right)}{P(X>1)}=\frac{\frac{5}{24}}{\frac{19}{24}}=\frac{5}{19}$$

(v) $$P(X<Y)=\int_0^1\int_0^y f(x,y)\,dx\,dy$$

$$=\int_0^1\int_0^y\left(xy^2+\frac{x^2}{8}\right)dx\,dy$$

$$=\int_0^1\left|\frac{x^2y^2}{2}+\frac{x^3}{24}\right|_0^y dy$$

$$=\int_0^1\left(\frac{y^4}{2}+\frac{y^3}{24}\right)dy$$

$$=\left|\frac{y^5}{10}+\frac{y^4}{96}\right|_0^1$$

$$=\frac{1}{10}+\frac{1}{96}$$

$$=\frac{53}{480}$$

(vi) $$P(X+Y\le 1)=\int_0^1\int_0^{1-y} f(x,y)\,dx\,dy$$

$$=\int_0^1\int_0^{1-y}\left(xy^2+\frac{x^2}{8}\right)dx\,dy$$

$$=\int_0^1\left|\frac{x^2y^2}{2}+\frac{x^3}{24}\right|_0^{1-y}dy$$

$$=\int_0^1\left\{\frac{(1-y)^2y^2}{2}+\frac{(1-y)^3}{24}\right\}dy$$

$$=\int_0^1\left\{\frac{(1-2y+y^2)y^2}{2}+\frac{(1-y)^3}{24}\right\}dy$$

$$=\int_0^1\left\{\frac{1}{2}(y^2-2y^3+y^4)+\frac{1}{24}(1-y)^3\right\}dy$$

$$=\left|\frac{1}{2}\left(\frac{y^3}{3}-\frac{y^4}{2}+\frac{y^5}{5}\right)+\frac{1}{24}\cdot\frac{(1-y)^4}{(-4)}\right|_0^1$$

$$=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right)+\frac{1}{24}\cdot\frac{1}{4}$$

$$=\frac{13}{480}$$

Example 14

The joint pdf of a two dimensional random variable (X, Y) is given by

$$f(x,y)=\frac{1}{2\pi a^2}e^{-\frac{x^2+y^2}{2a^2}},\ -\infty<x,y<\infty$$

Find $P(X^2+Y^2\le 4)$.

Solution

$$P(X^2+Y^2\le 4)=\iint_{x^2+y^2\le 4} f(x,y)\,dx\,dy$$

$$=\iint_{x^2+y^2\le 4}\frac{1}{2\pi a^2}e^{-\frac{x^2+y^2}{2a^2}}\,dx\,dy$$

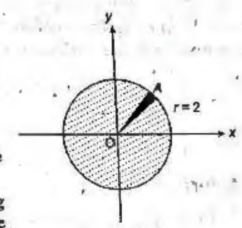


Fig. 2.8

The region of integration is the interior of the circle $x^2+y^2=4$.

Converting to polar coordinates by putting $x=\cos\theta, y=r\sin\theta, dxdy=rdrd\theta$, equation of the circle $x^2+y^2=4$ reduces to $r=2$.

In the region, along elementary radial strip OA,

Limits of r : $r=0$ to $r=2$

and in the region

Limits of θ: $\theta=0$ to $\theta=2\pi$

$$P(X^2+Y^2\le 4)=\int_0^{2\pi}\int_0^2\frac{1}{2\pi a^2}e^{-\frac{r^2}{2a^2}}\cdot r\,dr\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\int_0^2 -e^{-\frac{r^2}{2a^2}}\left(-\frac{r}{a^2}\right)dr\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\left|-e^{-\frac{r^2}{2a^2}}\right|_0^2 d\theta \qquad \left[\because \int_0 e^{f(x)}f'(x)dx=e^{f(x)}\right]$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\left(-e^{-\frac{2}{a^2}}+1\right)d\theta$$

$$=\frac{1}{2\pi}\left(1-e^{-\frac{2}{a^2}}\right)|\theta|_0^{2\pi}$$

$$=\frac{1}{2\pi}\left(1-e^{-\frac{2}{a^2}}\right)(2\pi)$$

$$=1-e^{-\frac{2}{a^2}}$$

Example 15

A gun is aimed at a certain point (origin of the co-ordinate system). Because of the random factors, the actual hit point can be any point (X, Y) in a circle of radius 'a' about the origin. Assume that the joint density of X and Y is constant in this circle and is given by

$$f(x,y)=c, \quad x^2+y^2 \le a^2$$
$$=0, \quad \text{otherwise}$$

Find (i) c (ii) $f_X(x)$.

Solution

(i) Since $f(x, y)$ is a probability density function,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy=1$$

$$\iint_{x^2+y^2\le a^2} c\,dx\,dy=1$$

$$c\iint_{x^2+y^2\le a^2} dx\,dy=1$$

$$c(\text{area of circle } x^2+y^2=a^2)=1$$

$$c(\pi a^2)=1$$

$$c=\frac{1}{\pi a^2}$$

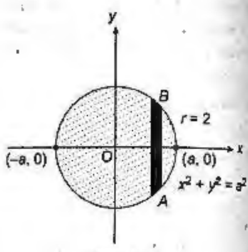


Fig. 2.9

(iii) The region of integration is the interior of the circle $x^2 + y^2 = a^2$.

In the region along the vertical strip AB, Limits of $y=-\sqrt{a^2-x^2}$ to $y=\sqrt{a^2-x^2}$ and x varies from $x=-a$ to $x=a$.

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy$$

$$=\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} c\,dy$$

$$=c\,|y|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}$$

$$=c\left(\sqrt{a^2-x^2}+\sqrt{a^2-x^2}\right)$$

$$=\frac{1}{\pi a^2}\left(2\sqrt{a^2-x^2}\right)$$

$$=\frac{2}{\pi a^2}\sqrt{a^2-x^2}, \qquad -a \le x \le a$$

EXERCISE 2.4

1. The joint pdf of a two dimensional random variable (X, Y) is given by

   $f(x,y)=2,\ 0<x<1,\ 0<y<1$
   $=0,\ \text{otherwise}$

   Find (i) marginal density function (ii) conditional density function.

   [Ans.: (i) $f(x)=2x,\ 0<x<1$; $=0$, otherwise; $f(y)=2(1-y),\ 0<y<1$; $=0$, otherwise (ii) $f(y/x)=\frac{1}{x},\ 0<x<1$; $f(x/y)=\frac{1}{1-y},\ 0<y<1$]

2. The joint pdf of a two dimensional random variable (X, Y) is given by

   $f(x,y)=\frac{1}{ab},\ 0<x<a,\ 0<y<b$
   $=0$, otherwise

   Find the joint probability distribution function and marginal density functions. Are X and Y independent?

   [Ans.: $F(x,y)=\frac{xy}{ab},\ 0<x\le a,\ 0<y\le b$; $=0$, otherwise; $f_X(x)=\frac{1}{a}, f_Y(y)=\frac{1}{b}$, Yes]

3. The joint pdf of (X, Y) is given by $f(x,y)=k, 0\le x<y\le 2$.

   Find (i) k (ii) $f_X(x)$, $f_Y(y)$ (iii) $f(y/x)$, $f(x/y)$

   [Ans.: (i) $\frac{1}{2}$, (ii) $f_X(x)=\frac{1}{2}(2-x), 0\le x\le 2; f_Y(y)=\frac{1}{2}y, 0\le y\le 2$ (iii) $f(y/x)=\frac{1}{2-x}, x<y<2;\ f(x/y)=\frac{1}{y}, 0<x<y$]

4. The joint pdf of (X, Y) is given by

   $f(x,y)=k(x^3y+xy^3), 0\le x\le 2, 0\le y\le 2$

   Find (i) k (ii) $f_X(x)$, $f_Y(y)$ (iii) $f(y/x)$, $f(x/y)$

$$\left[\begin{array}{l}\text{Ans.: (i) } k=\frac{1}{16}, \text{(ii)} f_X(x)=\frac{1}{8}(x^3+2x), 0\le x\le 2; f_Y(y)=\frac{1}{8}(y^3+2y), 0\le y\le 2\\ \text{(iii) } f(y/x)=\frac{y(x^2+y^2)}{2(x^2+2)}, 0\le y\le 2; f(x/y)=\frac{x(x^2+y^2)}{2(y^2+2)}, 0\le x\le 2.\end{array}\right]$$

5. The joint pdf of (X, Y) is given by

$$f(x,y)=\frac{1}{3}(3x^2+xy),\quad 0<x\le 1,\quad 0<y\le 2$$
$$=0, \qquad \text{, otherwise}$$

Find $P(X+Y\ge 1)$.

$\left[\text{Ans.: } \frac{65}{72}\right]$

6. The joint pdf of (X, Y) is given by

$f(x,y)=k(6-x-y),\ 0<x<2,\ 2<y<4$

Find (i) k (ii) $P(X<1, Y<3)$, (iii) $P(X+Y<3)$, (iv) $P(X<1/y<3)$

$\left[\text{Ans.: (i) } \frac{1}{8}, \text{(ii) } \frac{3}{8}, \text{(iii) } \frac{5}{24}, \text{(iv) } \frac{3}{5}\right]$

7. The joint pdf of (X, Y) is given by $f(x,y)=e^{-y},\quad x>0,\quad y>x$
$=0\quad,\quad \text{otherwise}$

Find (i) $P(X>1 / Y<5)$ (ii) marginal distributions of X and Y.

$\left[\text{Ans.: (i) } \frac{e^4-5}{e^5-6}, \text{(ii) } f_X(x)=e^{-x}, x>0; f_Y(y)=ye^{-y}, y>0\right]$

8. The joint pdf of (X, Y) is given by $f(x,y)=\frac{1}{12}e^{-\frac{x}{4}-\frac{y}{3}}, x\ge 0, y\ge 0$
$=0,\qquad \text{otherwise}$

(i) Find conditional density functions of X and Y.

(ii) Are X, Y independent?

$\left[\text{Ans.: (i) } f(y/x)=\frac{1}{3}e^{-\frac{y}{3}}, y\ge 0; f(x/y)=\frac{1}{4}e^{-\frac{x}{4}}, x\ge 0, \text{ (ii) Yes}\right]$

9. The joint pdf of (X, Y) is given by $f(x,y)=\frac{2}{(1+x+y)^3},\ x>0, y>0$
$=0\qquad,\ \text{otherwise}$

Find (i) $F(x, y)$ (ii) $f_X(x)$ (iii) $f(y/x)$.

$$\left[\begin{array}{l}\text{Ans.: (i) } F(x,y)=1-\frac{1}{1+x}+\frac{1}{1+x+y}-\frac{1}{1+y}\\ \text{(ii) } f_X(x)=\frac{1}{(1+x)^2},\ x>0\\ \qquad =0\quad, \text{ otherwise}\\ \text{(iii) } f(y/x)=\frac{2(1+x)^2}{(1+x+y)^3}\end{array}\right]$$

10. The joint pdf of (X, Y) is given by $f(x,y)=x^2+\frac{xy}{3},\quad 0<x<1, 0<y<2$
$=0\qquad,\ \text{otherwise}$

Find (i) $P\left(X>\frac{1}{2}\right)$ (ii) $P(Y<X)$ (iii) $P\left(Y<\frac{1}{2}/X<\frac{1}{2}\right)$

$\left[\text{Ans.: (i) } \frac{5}{6}, \text{(ii) } \frac{7}{24}, \text{(iii) } \frac{5}{32}\right]$

11. The joint pdf of (X, Y) is given by $f(x,y)=\frac{1}{4}(1+xy),\ |x|<1,\ |y|<1$
$=0\qquad,\ \text{otherwise}$

Show that X and Y are independent.

12. The joint pdf of (X, Y) is given by $f(x,y)=Ae^{-|x|-2|y|}$. Show that X and Y are independent.