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TUTORIAL 6.

Q1. b) $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dy\, dx.$

Solⁿ.

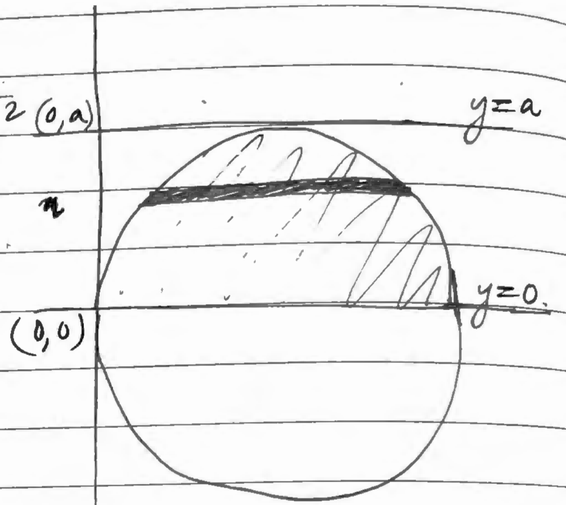
INITIAL LIMITS

$y = 0$ to $y = a$

$x = a - \sqrt{a^2-y^2}$ to $x = a + \sqrt{a^2-y^2}$

$(x-a)^2 = a^2 - y^2$

$(x-a)^2 + y^2 = a^2$ CIRCLE.

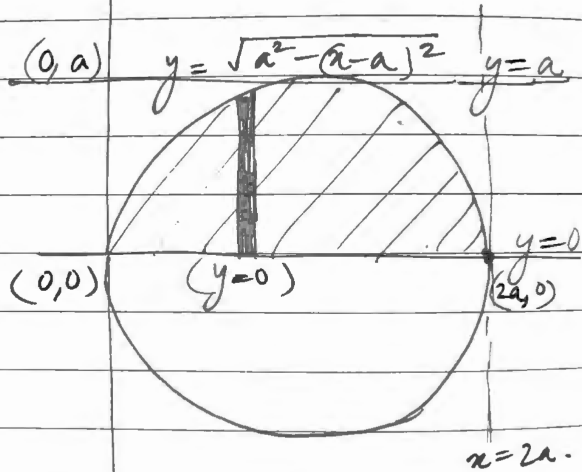


By change of order of intⁿ

$(x-a)^2 = a^2 - y^2$

$y^2 = a^2 - (x-a)^2$

$y = \sqrt{a^2-(x-a)^2}$



FINAL LIMITS

$y = 0$ to $\sqrt{a^2-(x-a)^2}$

$x = 0$ to $x = 2a$

Hence, the integral is

$$\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dy\, dx =$$

$$\Rightarrow \int_0^{2a} \int_0^{\sqrt{a^2-(x-a)^2}} dy\, dx.$$

$$\Rightarrow \int_0^{2a} \left[y \right]_0^{\sqrt{a^2-(x-a)^2}} dx$$

$$\Rightarrow \int_0^{2a} \sqrt{(a^2 - (x-a)^2)}\, dx.$$

$$\Rightarrow \left[\left(\frac{x-a}{2}\right) \sqrt{a^2-(x-a)^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x-a}{a}\right) \right]_0^{2a}$$

continued.

Q1 (b).

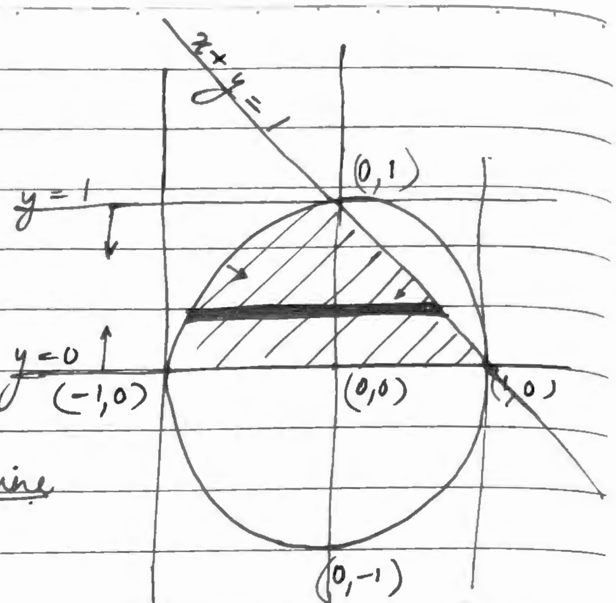
$$\Rightarrow \cancelto{0}{\left(\frac{2a-a}{2}\right)\sqrt{a^2-a^2}} + \frac{a^2}{2}\left(\sin^{-1}\left(\frac{a}{a}\right)\right)$$

$$- \left(\cancelto{0}{-\frac{a}{2}\left(\sqrt{a^2-a^2}\right)} + \frac{a^2}{2}\sin^{-1}\left(-\frac{a}{a}\right)\right)$$

$$\Rightarrow \frac{a^2}{2}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)$$

$$\Rightarrow \frac{\pi a^2}{2}$$

Q1. (c) $\int_0^1 \int_{-\sqrt{1-y^2}}^{1-y} y \, dx \, dy.$



Sol^n:

INITIAL LIMITS | $x = -\sqrt{1-y^2}$ to $x = 1-y$
$y = 0$ to $y = 1$

$x^2 = 1-y^2$
$\underline{x^2+y^2=1}$ $\underline{\text{circle}}$.
$\hookrightarrow y = \sqrt{1-x^2}$

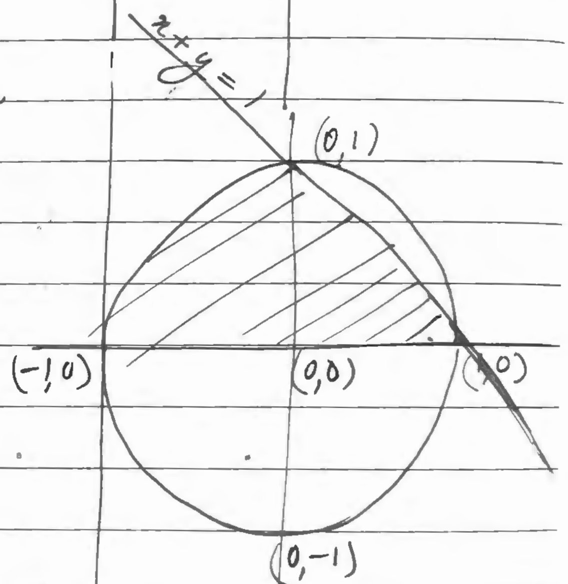
$x = 1-y$
$\underline{x+y=1}$ $\underline{\text{line}}$
$\hookrightarrow y = 1-x$

By change of order of int^n

FINAL LIMITS | $x = 1,$ to $x = 1$
$y = 0$ to $y = \sqrt{1-x^2}$ & $y = 1-x$

$\hookrightarrow$ for $x = -1$ to $x = 0$.
$y = 0$ to $y = \sqrt{1-x^2}$

And for $x = 0$ to $x = 1$
$y = 0$ to $y = 1-x$



Hence, the integral is

$$\int_0^1 \int_{-\sqrt{1-y^2}}^{1-y} y \, dx \, dy. = \int_{-1}^0 \int_0^{\sqrt{1-x^2}} y \, dy \, dx + \int_0^1 \int_0^{1-x} y \, dy \, dx$$

$$\Rightarrow \int_{-1}^0 \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx + \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$\Rightarrow \int_{-1}^0 \frac{1-x^2}{2} \, dx + \int_0^1 \frac{(1-x)^2}{2} \, dx$$

$$\Rightarrow \left[\frac{x}{2} - \frac{x^3}{6}\right]_{-1}^0 + \left[\frac{x}{2} + \frac{x^3}{6} - \frac{x^2}{2}\right]_0^1$$

$$\Rightarrow -\left(\frac{-1}{2} - \left(\frac{-1}{6}\right)\right) + \frac{1}{2} + \frac{1}{6} - \frac{1}{2}$$

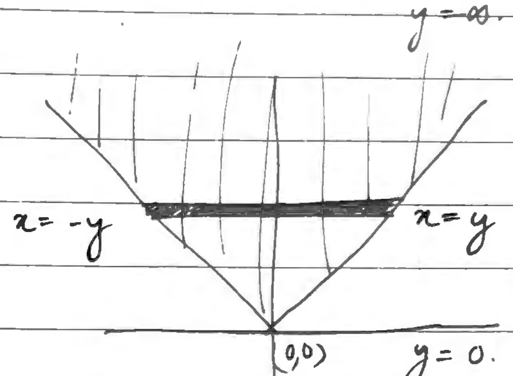
$$\Rightarrow \frac{1}{2} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{6}}$$

$$\Rightarrow 1/2.$$

Q1. (d) $\int_0^{\infty}\int_{-y}^{y} (y^2 - x^2)\, e^{-y}\, dy\, dx$

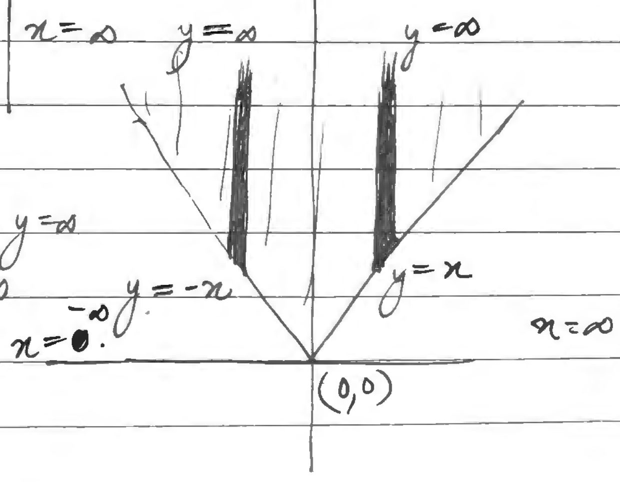
Sol^n:

INITIAL LIMITS | $x = -y$ to $x = y$; $y = 0$ to $y = \infty$



By change of order of int^n:

FINAL LIMITS | $y = -x$ to $y = \infty$ & $y = x$ to ∞; $x = 0$ to $x = \infty$



$\Rightarrow x = -\infty$ to $x = 0$, $y = -x$ to $y = \infty$

$\Rightarrow x = 0$ to $x = \infty$, $y = x$ to $y = \infty$

Hence, the integral is,

$$\int_0^{\infty}\int_{-y}^{y} (y^2 - x^2)\, e^{-y}\, dy\, dx = \int_{-\infty}^{0}\int_{-x}^{\infty} (y^2 - x^2)\, e^{-y}\, dy\, dx + \int_0^{\infty}\int_{x}^{\infty} (y^2 - x^2)\, e^{-y}\, dy\, dx$$

$$= \int_{-\infty}^{0}\int_{-x}^{\infty} \left(y^2 e^{-y} - x^2 e^{-y}\right) dy\, dx + \int_0^{\infty}\int_{x}^{\infty} \left(y^2 e^{-y} - x^2 e^{-y}\right) dy\, dx$$

$$\Rightarrow \int_{-\infty}^{0} \left(\left[-y^2 e^{-y} \right]_{-x}^{\infty} - \int_{-x}^{\infty} 2y\left(-e^{-y}\right) dy - \int_{-x}^{\infty} x^2 e^{-y}\, dy \right) dx$$

$$+ \int_{0}^{\infty} \left(\left[-y^2 e^{-y} \right]_{x}^{\infty} - \int_{x}^{\infty} 2y\left(-e^{-y}\right) dy - \int_{x}^{\infty} x^2 e^{-y}\, dy \right) dx$$

$$\Rightarrow \int_{-\infty}^{0} \left(\left[-y^2 e^{-y} \right]_{-x}^{\infty} - \left[2y e^{-y} \right]_{-x}^{\infty} + \left[-2 e^{-y} \right]_{-x}^{\infty} + \left[x^2 e^{-y} \right]_{-x}^{\infty} \right) dx$$

$$+ \int_{0}^{\infty} \left(\left[-y^2 e^{-y} \right]_{x}^{\infty} - \left[2y e^{-y} \right]_{x}^{\infty} + \left[-2 e^{-y} \right]_{x}^{\infty} + \left[x^2 e^{-y} \right]_{x}^{\infty} \right) dx$$

$$\Rightarrow \int_{-\infty}^{0} (0) - \left(\cancel{-x^2 e^{x}} + 2x e^{x} - 2e^{x} + \cancel{x^2 e^{x}}\right) dx \; +$$

$$\int_{0}^{\infty} (0) - \left(\cancel{-x^2 e^{-x}} - 2x e^{-x} - 2e^{-x} + \cancel{x^2 e^{-x}}\right) dx$$

$$\Rightarrow \int_{-\infty}^{0} \left(-2xe^{x} + 2e^{x}\right) dx \; + \; \int_{0}^{\infty} \left(2xe^{-x} + 2e^{-x}\right) dx$$

$$\Rightarrow \left\{ \left[-2xe^{x}\right]_{-\infty}^{0} + \int_{-\infty}^{0} 2e^{x}\,dx + \int_{-\infty}^{0} 2e^{x}\,dx \right\} + \left\{ \left[-2xe^{-x}\right]_{0}^{\infty} - \int_{0}^{\infty} -2e^{-x}\,dx + \int_{0}^{\infty} 2e^{-x}\,dx \right\}$$

$$\Rightarrow \left[-2xe^{x}\right]_{-\infty}^{0} + 4\left[e^{x}\right]_{-\infty}^{0} \; + \; \left[-2xe^{-x}\right]_{0}^{\infty} - \left[4e^{-x}\right]_{0}^{\infty}$$

$$\Rightarrow 4 - (-4)$$

$$\Rightarrow 8$$

Q2. Evaluate $\iint (y-x)\,dx\,dy$ over the region E in X-Y plane bounded by straight line

$y = x+3, \quad y = x+1$

$3y + x = 5, \quad 3y + x = 7$

Solⁿ. Let $u = y - x$

$v = 3y + x$

$\therefore\ y = x+3 \Rightarrow y - x = 3 \Rightarrow u = 3$

$y = x+1 \Rightarrow y - x = 1 \Rightarrow u = 1$

$3y + x = 5 \Rightarrow v = 5$

$3y + x = 7 \Rightarrow v = 7$

Hence, Limits of as

$u = 1$ to $u = 3$

$v = 5$ to $v = 7$

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$|J| = \begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} = |-3-1| = 4$$

Hence, the new integral will be.

$$\iint_E (y-x)\,dx\,dy = \int_5^7 \int_1^3 u(4)\,du\,dv$$

$$\Rightarrow 4 \int_5^7 \left[\frac{u^2}{2}\right]_1^3 dv$$

$$\Rightarrow 4\left(\frac{9}{2} - \frac{1}{2}\right) \int_5^7 dv$$

$$\Rightarrow 4.4\ [v]_5^7$$

$$\Rightarrow 4.4\ (2)$$

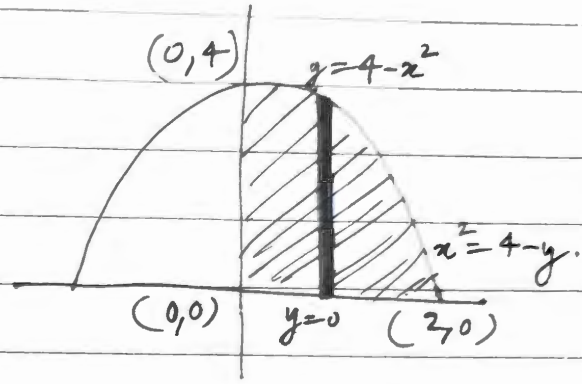
$$\Rightarrow 8 \times 4 = 32$$

Q1.(a) $\int_0^2 \int_0^{4-x^2} \left(\frac{xe^{2y}}{4-y}\right) dy\, dx$

Sol^n.. Initial limits :

$y = 0$ to $y = 4 - x^2$

$x = 0$ to $x = 2$.

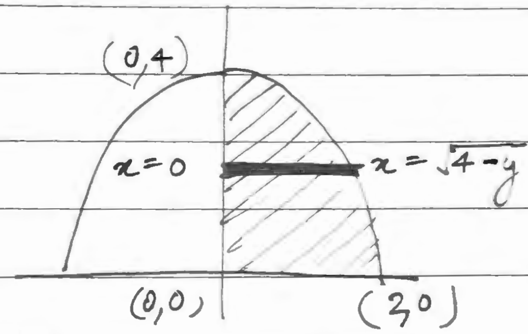


By chng of order of int^n.

Final limits :

$x = 0$ to $x = \sqrt{4-y}$

$y = 0$ to $y = 4$



Hence, the new integral is

$$\int_0^2 \int_0^{4-x^2} \left(\frac{xe^{2y}}{4-y}\right) dy\, dx = \int_0^4 \int_0^{\sqrt{4-y}} \left(\frac{xe^{2y}}{4-y}\right) dx\, dy.$$

$$\Rightarrow \int_0^4 \left[\frac{x^2}{2}\right]_0^{\sqrt{4-y}} \frac{e^{2y}}{4-y}\, dy$$

$$\Rightarrow \frac{1}{2}\int_0^4 \cancel{(4-y)}\, \frac{e^{2y}}{\cancel{(4-y)}}\, dy$$

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$\Rightarrow \frac{1}{2}\left[\frac{e^{2y}}{2}\right]_0^4$

$\Rightarrow \frac{1}{4}\left(e^{8}-1\right)$

$\Rightarrow \frac{e^{8}-1}{4}$

Q3. Under the transformation, $u = 3x+2y$, $v = x+4y$ evaluate the integral $\iint_R 3x^2 + 14xy + 8y^2 \, dx\, dy$ for region R in quadrant I bounded by lines $y = -\frac{3}{2}x+1$, $y = -\frac{3}{2}x+3$, $y = -\frac{1}{4}x$, $y = -\frac{1}{4}x+1$

Solⁿ: $u = 3x+2y$, $v = x+4y$

Limits:

$\Rightarrow y = \left(-\frac{3}{2}\right)x+1 \Rightarrow 3x+2y = 1 \Rightarrow u = 1$

$\Rightarrow y = \left(-\frac{3}{2}\right)x+3 \Rightarrow 3x+2y = 3 \Rightarrow u = 3$

$\Rightarrow y = -\frac{1}{4}x \Rightarrow x+4y = 0 \Rightarrow v = 0$

$\Rightarrow y = -\frac{1}{4}x+1 \Rightarrow x+4y = 1 \Rightarrow v = 1$

$$J^* = \frac{1}{J} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}$$

$J^* = 10 \Rightarrow dx\,dy = 10\, du\, dv.$

Hence, the integral becomes.

$$\iint_R 3x^2 + 14xy + 8y^2 \, dx\, dy = \iint_R (3x+2y)(x+4y)\, dx\, dy$$

$$= \int_0^1\int_1^3 uv(10)\, du\, dv$$

$$= 10\int_0^1 \left[\frac{u^2}{2}\right]_1^3 v\, dv$$

$$= 10 \int_0^1 \left(\frac{9}{2} - \frac{1}{2}\right) v \, dv$$

$$= 40 \left[v^2/2 \right]_0^1$$

$$= \underline{\underline{20}}$$

Q4. By changing into polar coordinates evaluate following

a) $$\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} (x^2 + y^2) \, dy \, dx$$

Sol^n. $x = r\cos\theta$, $y = r\sin\theta$ ($\because$ Polar coordinates)

$J = r$

$dy\,dx = r\,dr\,d\theta$

$y^2 = 2ax - x^2$

$x^2 + y^2 - 2ax = 0$

CIRCLE $(x-a)^2 + y^2 = a^2$

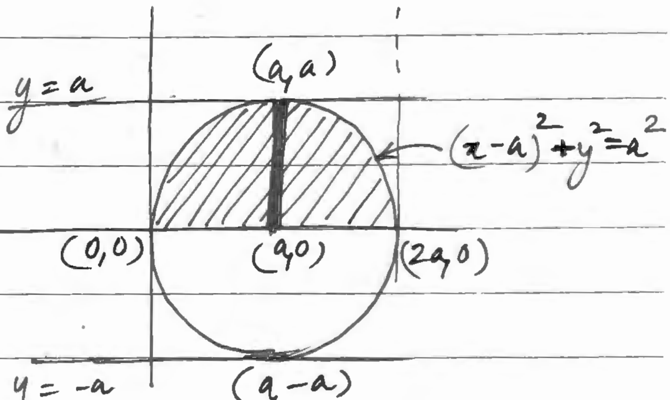
$\cancel{\Rightarrow (a\cos\theta - a)^2 + a^2\sin^2\theta = a^2}$

$\cancel{\Rightarrow a^2(1) + a^2 - 2a^2\cos\theta = a^2}$

$\cancel{\Rightarrow a^2(1 - \cos\theta) = 0}$

$\cancel{\Rightarrow \cos\theta = 1}$

$\cancel{\Rightarrow \theta = 0}$



$\cancel{a\sin\theta = 0}$ $\cancel{(\because y = 0)}$

$\cancel{\Rightarrow \sin\theta = 0}$

$\cancel{\Rightarrow \theta = 0.}$

$\cancel{a\cos\theta = 0}$ $\cancel{(\because x = 0)}$

$\cancel{\Rightarrow \theta = \pi/2}$

$\cancel{a\cos\theta = 2a}$ X

$\cancel{\theta = \cos^{-1}(2)}$

Limits: $\theta = 0$ to $\pi/2$

($\because$ Quad. I)

$r = 0$ to $r = 2a\cos\theta$ [by putting $x = r\cos\theta$ in $x^2 + y^2 - 2ax = 0$. $r = 2a\cos\theta$]

Hence, the new integral becomes,

$$\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} (x^2 + y^2) \, dy \, dx = \int_0^{\pi/2} \int_0^{2a\cos\theta} r^2 \, r \, dr \, d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$\Rightarrow \frac{1}{\cancel{4}}\,\cancel{16}^{\,4} a^4 \int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$\Rightarrow 4a^4 \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta.$$

$$\Rightarrow 4a^4 \int_0^{\pi/2} \left(\frac{1+\cos^2 2\theta + 2\cos 2\theta}{4}\right) d\theta$$

$$\Rightarrow 4a^4 \int_0^{\pi/2} \left(\frac{1}{4} + \frac{2\cos 2\theta}{4} + \frac{\cos^2 2\theta}{4}\right) d\theta$$

$$\Rightarrow 4a^4 \left\{ \left[\frac{\theta}{4}\right]_0^{\frac{\pi}{2}} + \left[\frac{\sin 2\theta}{4}\right]_0^{\pi/2} + \int_0^{\pi/2} \left(\frac{1+\cos 4\theta}{8}\right) d\theta \right\}$$

$$\Rightarrow 4a^4 \left(\frac{\pi}{8} + \cancel{(\ )} + \left[\frac{\theta}{8}\right]_0^{\pi/2} + \cancel{\left[\frac{\sin 4\theta}{32}\right]_0^{\pi/2}} \rightarrow 0. \right)$$

$$\Rightarrow 4a^4 \left(\frac{\pi}{8} + \frac{\pi}{16}\right)$$

$$\Rightarrow \frac{3\pi a^4}{4}$$

(b) $\int_0^1 \int_0^x \sqrt{x^2+y^2} \, dy \, dx$

Sol^n. ~~Initial ti~~

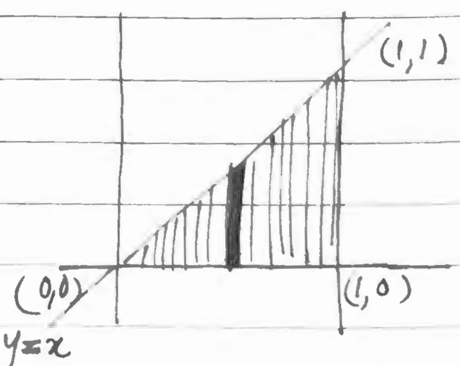
$x = r\cos\theta$, $y = r\sin\theta$

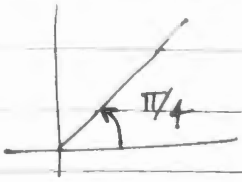
$J = r$

$dx\,dy = r\,dr\,d\theta$

Limits: $\theta = 0$ to $\pi/4$

$r = 0$ to $\sec\theta$





$\frac{r\sin\theta}{r\cos\theta}$ $\left(\theta = \frac{\pi}{4}\right)$

Hence, the new integral is

$$\int_0^1 \int_0^x \sqrt{x^2+y^2}\, dy\, dx = \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} r \; r\, dr\, d\theta.$$

$x = 1 \Rightarrow r\cos\theta = 1$

$r = \sec\theta$

$$\Rightarrow \int_0^{\pi/4} \left[\frac{r^3}{3}\right]_0^{\sec\theta} d\theta$$

$$\Rightarrow \frac{1}{3}\int_0^{\pi/4} \sec^3\theta \, d\theta$$

$$\left[\because \int \sec^3\theta = \frac{1}{2}\left(\ln(\sec\theta+\tan\theta) + \sec\theta\tan\theta\right)\right]$$

$$\Rightarrow \frac{1}{3}\int_0^{\pi/4} \sec\theta\,(1+\tan^2\theta)\, d\theta$$

$$\cancel{\Rightarrow \frac{1}{3}\left(\left[\log(\sec\theta+\tan\theta)\right]_0^{\pi/4} + \int_0^{\pi/4}\tan^2\theta\sec\theta\, d\theta\right)}$$

$$= \int \sec^3\theta$$

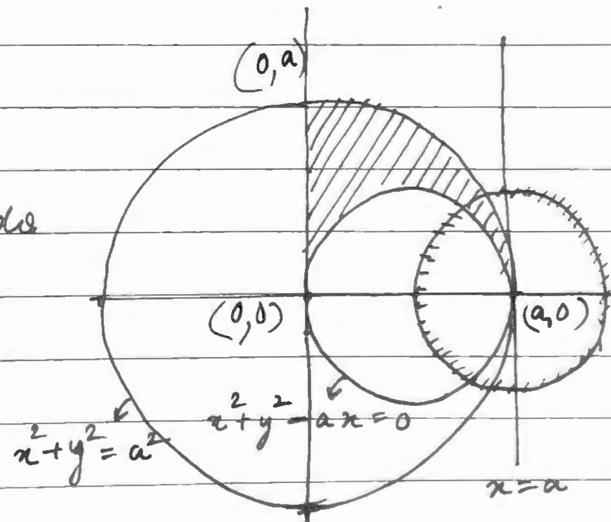
$$\Downarrow \cancel{\left[\int(\sec^3\theta - \sec\theta)\,d\theta + \ln|\sec\theta|\right]}$$

$$\Rightarrow \frac{1}{3}\left(\frac{\log(1+\sqrt{2})}{2} + \left[\frac{\sec\theta\tan\theta}{2}\right]_0^{\pi/4}\right)$$

$$\Rightarrow \frac{1}{3}\left(\frac{\log(1+\sqrt{2})}{2} + \frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow \frac{1}{6}\left(\sqrt{2} + \log(1+\sqrt{2})\right)$$

(c) $$\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dy\,dx}{\sqrt{a^2-x^2-y^2}}$$



Sol^n. $x = r\cos\theta$, $y = r\sin\theta$

$J = r$, $dx\,dy = r\,dr\,d\theta$

Limits:

$\theta = 0$ to $\pi/2$

~~$r = 0$ to a~~

$r = a$ to $a\cos\theta$

$y^2 = -x^2 + a^2 \Rightarrow r^2\sin^2\theta + r^2\cos^2\theta = a^2$

$\Rightarrow r = a$

$x^2+y^2-ax=0 \Rightarrow r^2 - ar\cos\theta = 0$

$\Rightarrow r = a\cos\theta$

Hence, the new integral is

$$\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dy\,dx}{\sqrt{a^2-x^2-y^2}}$$

$$= \int_0^{\pi/2}\int_a^{a\cos\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$$

$a^2 - r^2 = t \Rightarrow -2r\,dr = dt$ $\quad -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\frac{1}{2}(2)\sqrt{t} = -\sqrt{t}$

$$= -\int_0^{\pi/2} \left[\sqrt{a^2-r^2}\right]_a^{a\cos\theta} d\theta$$

$$= -\int_0^{\frac{\pi}{2}} \left(\sqrt{a^2-a^2\cos^2\theta} - 0.\right) d\theta$$

$$= -\int_0^{\frac{\pi}{2}} a\sin\theta \, d\theta$$

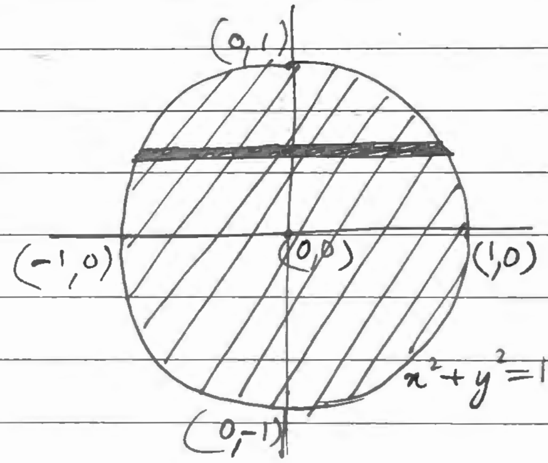
$$= -a\cdot\left[-\cos\theta\right]_0^{\pi/2}$$

$$= a(0-1)$$

$$= -a$$

Q5. Evaluate (i) $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2+y^2)\, dy\, dx$

Sol^n: $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2+y^2)\, dx\, dy$



$\cancel{\Rightarrow \int \left[\frac{x^3}{3} + xy^2\right]}$

Using Polar Coordinates: $x = r\cos\theta$, $y = r\sin\theta$

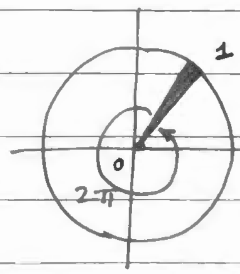
$J = r$, $dx\,dy = r\,dr\,d\theta$

$x^2+y^2=1 \Rightarrow r^2 = 1$

$\Rightarrow r = 1$

Limits: $\theta = 0$ to 2π

$r = 0$ to $r = 1$



$$\Rightarrow \int_0^{2\pi}\int_0^1 r^2\, r\, dr\, d\theta$$

$$\Rightarrow \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^1 d\theta$$

$$\Rightarrow \frac{1}{4}\left[\theta\right]_0^{2\pi}$$

$$\Rightarrow \frac{\pi}{2}$$

(ii) $\int_0^2 \int_0^x y \, dy \, dx$

Sol^n $\Rightarrow \int_0^2 \left[\frac{y^2}{2}\right]_0^x dx$

$\Rightarrow \int_0^2 \left[\frac{x^2}{2}\right] dx$

$\Rightarrow \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2$

$\Rightarrow \frac{1}{6}(8)$

$\Rightarrow \frac{4}{3}$ //.

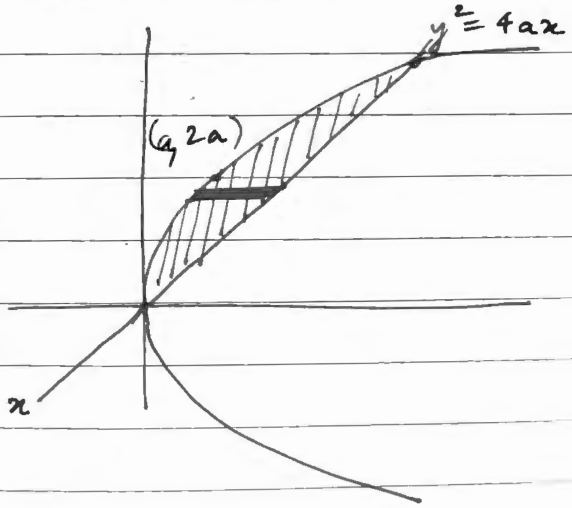
(iii) $\int_0^{4a} \int_{y^2/4a}^{y} \left(\frac{x^2-y^2}{x^2+y^2}\right) dx\, dy$

Sol^n: ~~$\int \frac{x^2-y^2}{x^2+y^2} dx$~~ ~~$t = (x^2+y^2) \Rightarrow dt = 2x\, dx$~~

~~$\Rightarrow \frac{x^2+y^2}{x^2+y^2}$~~

$x = r\cos\theta$

$y = r\sin\theta$



$y^2 = 4ax$

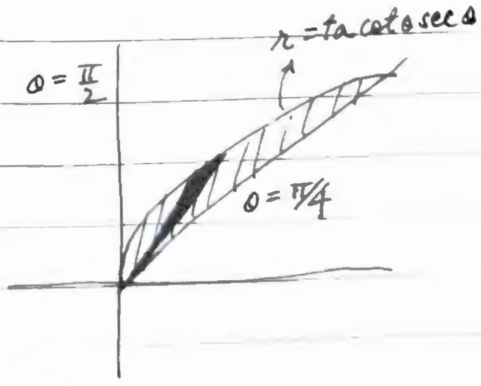
$r^{\cancel{2}}\sin^2\theta = 4a\cancel{r}\cos\theta$

$r = 4a\cot\theta\,\mathrm{cosec}\,\theta$

Limits: $r = 0$ to $4a\cot\theta\sec\theta$

$\theta = \pi/4$ to $\theta = \pi/2$

Hence, the new integral is



$\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2} dx\, dy =$

$\int_{\pi/4}^{\pi/2} \int_0^{4a\cot\theta\,\mathrm{cosec}\,\theta} \frac{\cancel{r^2}(\cos^2\theta - \sin^2\theta)}{\cancel{r^2}(1)} r\, dr\, d\theta$

$(\because J = r)$

$\Rightarrow \int_{\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\, d\theta \left[\frac{r^2}{2}\right]_{}^{4a\cot\theta\,\mathrm{cosec}\,\theta}$

$$\Rightarrow \int_{\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\, d\theta \cdot \frac{\cancel{16}^{8}}{\cancel{2}} a^2 \cot^2\theta \,\mathrm{cosec}^2\theta$$

$$\Rightarrow 8a^2 \int_{\pi/4}^{\pi/2} (1 - 2\sin^2\theta)(\cot^2\theta\,\mathrm{cosec}^2\theta)\, d\theta$$

$$\Rightarrow 8a^2 \int_{\pi/4}^{\pi/2} \left[(\cot^2\theta\,\mathrm{cosec}^2\theta) - 2\cot^2\theta\right] d\theta$$

$$\Rightarrow 8a^2 \int_{\pi/4}^{\pi/2} \left[\cancel{-}(\underbrace{\cot^2\theta}_{u})(\underbrace{-\mathrm{cosec}^2\theta}_{v}) - 2(\mathrm{cosec}^2\theta - 1)\right] d\theta$$

$$\Rightarrow \cancel{8a^2 \left[\frac{-\cot^3\theta}{} - \int \cot\theta \right.}$$

$$\Rightarrow t = \cot\theta, \quad dt = -\mathrm{cosec}^2\theta\, d\theta$$

$$\Rightarrow 8a^2 \left[-\cancel{}\frac{\cot^3\theta}{3}\right]_{\pi/4}^{\pi/2} + \left[2\cot\theta + 2\theta\right]_{\pi/4}^{\pi/2}$$

$$\Rightarrow 8a^2\left[\left(-0 - \left(-\frac{1}{3}\right)\right) + 0 - 2 + 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right]$$

$$\Rightarrow 8a^2\left(\frac{1}{3} - 2 + \cancel{2}\,\pi/2\right)$$

$$\Rightarrow 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right)$$

Q6. Evaluate $\iint_D e^{-x^2-y^2}\, dA$, where D is region bounded by semi-circle $x = \sqrt{4-y^2}$ & y-axis

Solⁿ. Let, $x = r\cos\theta$

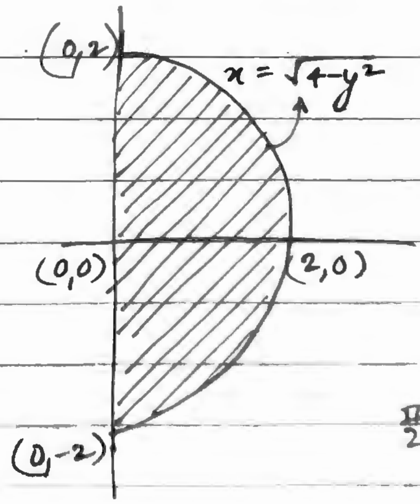
$y = r\sin\theta$

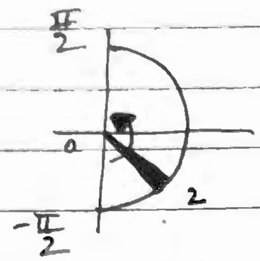
$J = r$

$dA = r\, dr\, d\theta$

Limits: $\theta = -\frac{\pi}{2}$ to $\frac{\pi}{2}$

$r = 0$ to $r = 2$





Hence, the new integral is $\iint_D e^{-x^2-y^2}\, dA =$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 e^{-(r^2)}\, r\, dr\, d\theta$$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} -\frac{1}{2}\left[e^{-r^2}\right]_0^2 d\theta$$

$$\Rightarrow \frac{-1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^{-4} - 1)\, d\theta$$

$$\Rightarrow -\frac{1}{2}\left(\frac{1}{e^4} - 1\right) [\theta]_{-\pi/2}^{\pi/2}$$

$$\Rightarrow -\frac{\pi}{2}\left(\frac{1}{e^4} - 1\right)$$

$$\Rightarrow \frac{\pi}{2}\left(1 - \frac{1}{e^4}\right) \text{ square units.}$$

Q7. Evaluate (i) $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz\, dy\, dx$

Sol^{n}. $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz\, dy\, dx$

$$\Rightarrow \int_0^1 \int_0^{2-x} [z]_0^{2-x-y}\, dy\, dx$$

$$\Rightarrow \int_0^1 \int_0^{2-x} (2-x-y)\, dy\, dx$$

$$\Rightarrow \int_0^1 \left[2y - xy - \frac{y^2}{2}\right]_0^{2-x} dx$$

$$\Rightarrow \int_0^1 \left(2(2-x) - x(2-x) - \frac{(2-x)^2}{2}\right) dx$$

$$\Rightarrow \int_0^1 \left[4 - 2x - 2x + x^2 - \left(\frac{x^2 - 4x + 4}{2}\right)\right] dx$$

$$\Rightarrow \int_0^1 \left[(x-2)^2 - \frac{(x-2)^2}{2}\right] dx$$

$$\Rightarrow \frac{1}{2}\int_0^1 (x-2)^2\, dx$$

$$\Rightarrow \frac{1}{2}\int_0^1 (x^2 - 4x + 4)\, dx$$

$$\Rightarrow \frac{1}{2}\left[\frac{x^3}{3} - \frac{4x^2}{2} + 4x\right]_0^1$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{3} - 2 + 4\right)$$

$$\Rightarrow 7/6.$$

Q7. (ii) $\int_0^1\int_0^{\sqrt{2}}\int_0^{2\pi} (r^2\cos^2\theta + z^2)\, r\, d\theta\, dr\, dz$

Sol^n. $\int_0^1\int_0^{\sqrt{2}}\int_0^{2\pi} (r^2\cos^2\theta + z^2)\, r\, d\theta\, dr\, dz$

$$\Rightarrow \int_0^1\int_0^{\sqrt{2}}\int_0^{2\pi}\left(rz^2 + r^2\left(\frac{1+\cos 2\theta}{2}\right)r\right) d\theta\, dr\, dz$$

$$\Rightarrow \int_0^1\int_0^{\sqrt{2}}\left[\theta rz^2 + \frac{\theta r^3}{2} + \frac{1}{2}r^3\frac{\sin 2\theta}{2}\right]_0^{2\pi} dr\, dz$$

$$\Rightarrow \int_0^1\int_0^{\sqrt{2}}\left(2\pi\left(rz^2 + r^3/2\right) + 0\right) dr\, dz$$

$$\Rightarrow 2\pi\int_0^1\left[\frac{r^2}{2}z^2 + \frac{r^4}{8}\right]_0^{\sqrt{2}} dz$$

$$\rightarrow 2\pi\int_0^1\left(\frac{z^3}{2} + \frac{z^2}{8}\right) dz$$

$$\rightarrow 2\pi\left[\frac{z^4}{8} + \frac{z^3}{24}\right]_0^1$$

$$\rightarrow 2\pi\left(\frac{1}{8} + \frac{1}{24}\right)$$

$$\rightarrow \frac{\pi}{3}$$

Q7(iii) $\displaystyle\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx$

Sol^n: $\displaystyle\int_0^{\log 2}\int_0^{x}\left[e^{x+y+z}\right]_0^{x+\log y} dy\,dx$

$$\Rightarrow \int_0^{\log 2}\int_0^{x}\left(e^{2x+y+\log y} - e^{x+y}\right) dy\,dx$$

$$\Rightarrow \int_0^{\log 2}\int_0^{x} e^{x}.e^{y}\left(e^{x}e^{\log y} - 1\right) dy\,dx$$

$$\Rightarrow \int_0^{\log 2}\int_0^{x} e^{x}.e^{y}\left(e^{x}y - 1\right) dy\,dx$$

$$\Rightarrow \int_0^{\log 2}\int_0^{x}\left(e^{2x}ye^{y} - e^{x}e^{y}\right) dy\,dx$$

$$\Rightarrow \int_0^{\log 2}\left[e^{2x}\left(ye^{y} - e^{y}\right) - e^{x}e^{y}\right]_0^{x} dx$$

$$\Rightarrow \int_0^{\log 2}\left[\left(e^{2x}\left(xe^{x} - e^{x}\right) - e^{2x}\right) - \left(e^{2x}(-1) - e^{x}\right)\right] dx$$

$$\Rightarrow \int_0^{\log 2}\left(xe^{3x} - e^{3x} - \cancel{e^{2x}} + \cancel{e^{2x}} + e^{x}\right) dx$$

$$\Rightarrow \left[\frac{xe^{3x}}{3} - \frac{e^{3x}}{3} + e^{x} - \frac{e^{3x}}{9}\right]_0^{\log 2}$$

$$\Rightarrow \cancel{\frac{\log 2\, e^{3\log 2}}{3} - 0 - \left(\frac{2e^{3\log 2}}{3} - \frac{2}{3}\right) + \left(e^{\log 2} - 1\right)}$$

$$\Rightarrow \cancel{\frac{8\log 2}{3} - \frac{16}{3} + \frac{2}{3} - 1 + 2}$$

$$\Rightarrow \cancel{8\log 2}$$

$$\Rightarrow \left[\frac{xe^{3x}}{3} - e^{3x}\left(\frac{1}{3} + \frac{1}{9}\right) + e^{x}\right]_0^{\log 2}$$

$$\Rightarrow \frac{\log 2\, e^{3\log 2}}{3} - e^{3\log 2}\left(\frac{4}{9}\right) + e^{\log 2} -$$

$$\left(0 - 1\left(\frac{1}{3} + \frac{1}{9}\right) + 1\right)$$

$$\Rightarrow \frac{8\log 2}{3} - 8\left(\frac{4}{9}\right) + 2 - 1 + \frac{4}{9}$$

$$\Rightarrow \frac{24\log 2}{9} - \frac{19}{9}$$

$$\Rightarrow \frac{1}{3}\left(8\log 2 - \frac{19}{3}\right)$$

Q7. (iv) $\int_{1}^{3}\int_{1/x}^{1}\int_{0}^{\sqrt{xy}} xyz \, dz \, dy \, dx$

Sol^n: $\int_{1}^{3}\int_{1/x}^{1} xy\left[\frac{z^2}{2}\right]_{0}^{\sqrt{xy}} dy \, dx$

$$\Rightarrow \int_{1}^{3}\int_{1/x}^{1} xy \, \frac{xy}{2} \, dy \, dx$$

$$\Rightarrow \int_{1}^{3}\int_{1/x}^{1} \frac{x^2}{2} y^2 \, dy \, dx$$

$$\Rightarrow \int_{1}^{3} \frac{x^2}{2}\left[\frac{y^3}{3}\right]_{1/x}^{1} dx$$

$$\Rightarrow \int_{1}^{3} \frac{x^2}{2}\left(\frac{1}{3} - \frac{1}{3x^3}\right) dx$$

$$\Rightarrow \frac{1}{6}\int_{1}^{3}\left(x^2 - \frac{1}{x}\right) dx$$

$$\Rightarrow \frac{1}{6}\left[\frac{x^3}{3} - \log x\right]_{1}^{3}$$

$$\Rightarrow \frac{1}{6}\left(9 - \log 3 - \frac{1}{3} + 0\right)$$

$$\Rightarrow \frac{1}{6}\left(\frac{26}{3} - \frac{3\log 3}{3}\right)$$

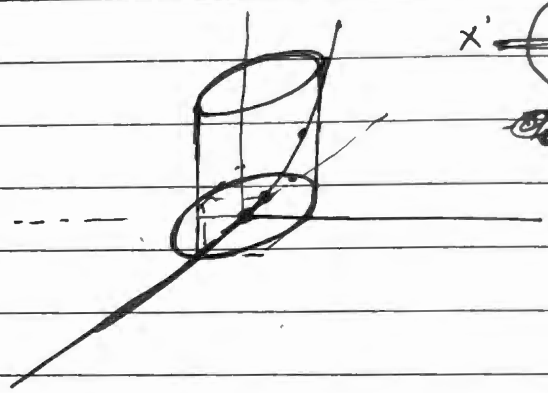
$$\Rightarrow \frac{1}{18}\left(26 - 3\log 3\right)$$

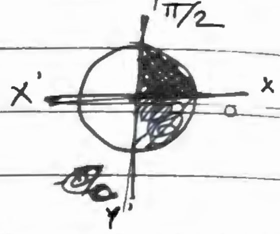
Q 8. Evaluate using cylindrical coordinates:

i) Find the volume bounded by the X-Y plane, the paraboloid $2z = x^2+y^2$ & the cylinder $x^2+y^2=4$

Sol^n: $x = r\cos\theta,\ y = r\sin\theta,\ z = z,\ J = r$

- $x^2+y^2=2z \Rightarrow r^2 = 2z \Rightarrow z = \frac{r^2}{2}$
- $x^2+y^2 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$
- X-Y plane $\Rightarrow z = 0$.





$\therefore$ The limits are:

$\theta = 0$ to $\pi/2$

$r = 0$ to 2

$z = 0$ to $\frac{r^2}{2}$

Hence, the volume of this region is given by

$$\int_0^{\frac{\pi}{2}}\int_0^{2}\int_0^{\frac{r^2}{2}} dz\, dr\, d\theta\ r$$

$$\Rightarrow \int_0^{\frac{\pi}{2}}\int_0^{2} [z]_0^{r^2/2}\, r\, dr\, d\theta$$

$$\Rightarrow \int_0^{\pi/2}\int_0^{2} \frac{r^3}{2}\, dr\, d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{8}\right]_0^2 d\theta$$

$$\Rightarrow 2\,[\theta]_0^{\pi/2}$$

$\rightarrow$ π) cubic units.

Q8 ii) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} x^2 \, dz \, dy \, dx$

~~$x = r\cos\theta, \; y = r\sin\theta, \; z = z$~~

Using spherical coordinates

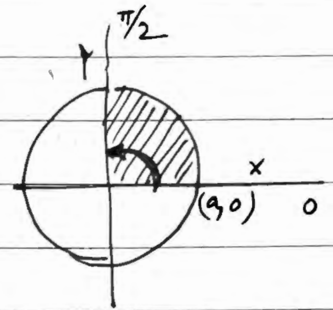
$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$J = r^2\sin\theta$

Limits of $z \Rightarrow 0$ to $\sqrt{a^2-x^2-y^2}$

Limits of $y \Rightarrow 0$ to $\sqrt{a^2-x^2}$

Limits of $x \Rightarrow 0$ to a



(Base Angle) $\phi \Rightarrow 0$ to $\pi/2$

Region : positive octant of sphere $x^2+y^2+z^2=a^2$

(New) Limits : $r = 0$ to $r = a$

$\theta = 0$ to $\theta = \pi/2$

$\phi = 0$ to $\phi = \pi/2$

Hence, the spherical form of the integral is

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a r^2\sin^2\theta\cos^2\phi \; r^2\sin\theta \, dr \, d\theta \, d\phi$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a r^4 \sin^3\theta \cos^2\phi \, dr \, d\theta \, d\phi$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left[\frac{r^5}{5}\right]_0^a \sin^3\theta \cos^2\phi \, d\theta \, d\phi$$

$$\Rightarrow \frac{a^5}{5} \int_0^{\frac{\pi}{2}} \cos^2\phi \int_0^{\frac{\pi}{2}} \sin^3\theta \, d\theta \, d\phi$$

$$\Rightarrow \frac{a^5}{5} \int_0^{\frac{\pi}{2}} \sin\theta(1-\cos^2\theta) \, d\theta \int_0^{\frac{\pi}{2}} \cos^2\phi \, d\phi$$

$$\Rightarrow \frac{a^5}{5} \left\{ \left(\left[-\cos\theta\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (-\sin\theta)\cos^2\theta \, d\theta \right) \int_0^{\frac{\pi}{2}} \cos^2\phi \, d\phi \right\}$$

$$\Rightarrow \frac{a^5}{5}\int_0^{\frac{\pi}{2}}\left(\left[-\cos\theta\right]_0^{\frac{\pi}{2}} + \left[\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}}\right)\cos^2\phi \, d\phi$$

$$\Rightarrow \frac{a^5}{5}\int_0^{\frac{\pi}{2}}\left(-0-(-1)+0-\tfrac{1}{3}\right)\cos^2\phi \, d\phi$$

$$\Rightarrow \frac{2a^5}{15}\int_0^{\pi/2}\cos^2\phi \, d\phi$$

$$\Rightarrow \frac{\cancel{2}a^5}{15}\left(\frac{1}{\cancel{2}}\right)\int_0^{\frac{\pi}{2}}(1+\cos 2\phi)\, d\phi$$

$$\Rightarrow \frac{a^5}{15}\left(\left[\phi\right]_0^{\pi/2} + \left[\frac{\sin 2\phi}{2}\right]_0^{\pi/2}\right)$$

$$\Rightarrow \frac{a^5}{15}\left(\frac{\pi}{2}+0\right)$$

$$\Rightarrow \frac{\pi a^5}{30}$$

Q8. Use triple integral to find volume of solid within cylinder $x^2+y^2=9$ between planes $z=1$, $x+z=1$

Sol^n.

$x+z=1$, $x^2+y^2=9$

$\therefore z=1-x$, $y=\pm\sqrt{9-x^2}$

At $y=0$, $x=\pm 3$

Hence, the limits are:

$z = 1$ to $z = 1 - x$

$y = -\sqrt{9-x^2}$ to $y = \sqrt{9-x^2}$

$x = -3$ to $x = 3$.

Hence, $V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{1}^{1-x} dz\, dy\, dx$

$$V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}} \cancel{(-1 + (1-x))\, dy\, dx} \; [z]_{1}^{1-x} \, dy\, dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \cancel{1} - x - \cancel{1} \; dy\, dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} -x \, dy\, dx$$

$$= \int_{-3}^{3} [-xy]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \, dx$$

$$= \int_{-3}^{3} -x\left(\sqrt{9-x^2} - \left(-\sqrt{9-x^2}\right)\right) dx$$

$$= \int_{-3}^{3} -2x\sqrt{9-x^2} \, dx$$

$I = \int -2x\sqrt{9-x^2}\, dx$, let $t = 9 - x^2 \Rightarrow dt = -2x\,dx$

$\therefore I = \int \sqrt{t}\, dt \Rightarrow I = \frac{2}{3} t^{3/2}$

$$\therefore V = \left[\frac{2}{3} (9-x^2)^{3/2} \right]_{-3}^{3}$$

$$V = 2 \cdot \left[\frac{2}{3} (9-x^2)^{3/2} \right]_{0}^{3}$$

$$V = \left| 2\left(0 - \frac{2}{3} \times 27\right) \right|$$

$V = 2 \times 2 \times 9$

$V = 36$ cubic units.

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Q9. Evaluate $\iiint_B (x^2+y^2+z^2)\,dV$, where B is region bounded by spheres $x^2+y^2+z^2=a^2$ & $x^2+y^2+z^2=b$

$a > b > 0$

Sol^n: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$

$z = r\cos\theta$, $J = r^2\sin\theta$

$x^2+y^2+z^2=a^2 \Rightarrow r^2=a^2 \Rightarrow r=a$

$x^2+y^2+z^2=b^2 \Rightarrow r^2=b^2 \Rightarrow r=b$

For complete sphere

$\theta = 0$ to π , $\phi = 0$ to 2π

Hence, spherical form of given integral is

$$\int_0^{2\pi}\int_0^{\pi}\int_b^{a} r^2 \; r^2\sin\theta \; dr\, d\theta\, d\phi$$

$$\Rightarrow \int_0^{2\pi}\int_0^{\pi}\int_b^{a} r^4\, dr \quad \sin\theta\, d\theta\; d\phi$$

$$\Rightarrow \int_0^{2\pi}\int_0^{\pi}\left[\frac{r^5}{5}\right]_b^a \sin\theta\, d\theta\; d\phi$$

$$\Rightarrow \left(\frac{a^5-b^5}{5}\right)\int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\, d\theta$$

$$\Rightarrow \left(\frac{a^5-b^5}{5}\right)(2\pi)\left[-\cos\theta\right]_0^{\pi}$$

$$\Rightarrow 2\pi\,\frac{a^5-b^5}{5}(2)$$

$$\Rightarrow \frac{4\pi}{5}\left(a^5-b^5\right)$$

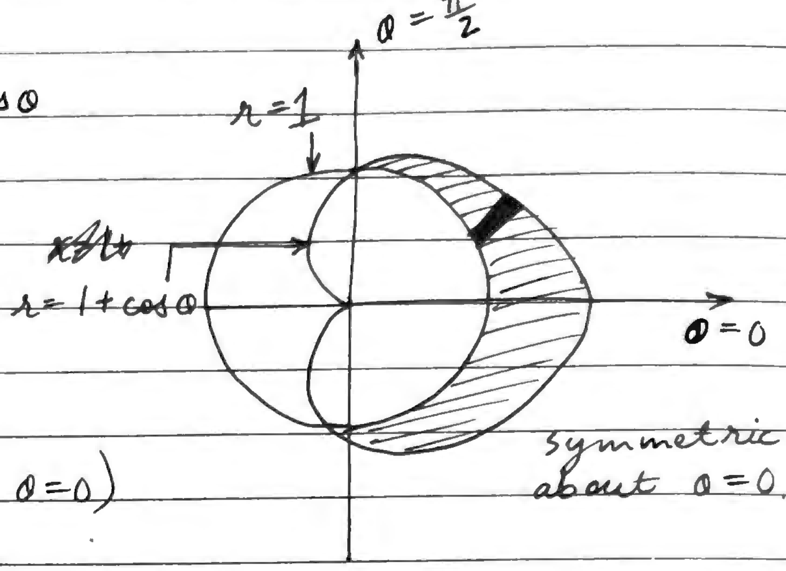
Q 10. i) Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r=1$

Sol^n: $r = 1$, $r = 1 + \cos\theta$

$1 = 1 + \cos\theta$

$\cos\theta = 0$

$\theta = \pm \frac{\pi}{2}$



Total area = 2 (area above $\theta=0$)

Limits: $r = 1$ to $r = 1+\cos\theta$

$\theta = 0$ to $\theta = \pi/2$

$$A = 2\int_0^{\pi/2}\int_1^{1+\cos\theta} r \, dr \, d\theta$$

$$A = 2\int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_1^{1+\cos\theta} d\theta$$

$$A = 2\int_0^{\frac{\pi}{2}} \left(\frac{(1+\cos\theta)^2}{2} - \frac{1}{2}\right) d\theta$$

$$A = 2\int_0^{\frac{\pi}{2}} \cancel{\frac{1}{2}} + \frac{\cos^2\theta}{2} + \frac{\cancel{2}\cos\theta}{\cancel{2}} - \cancel{\frac{1}{2}} \, d\theta$$

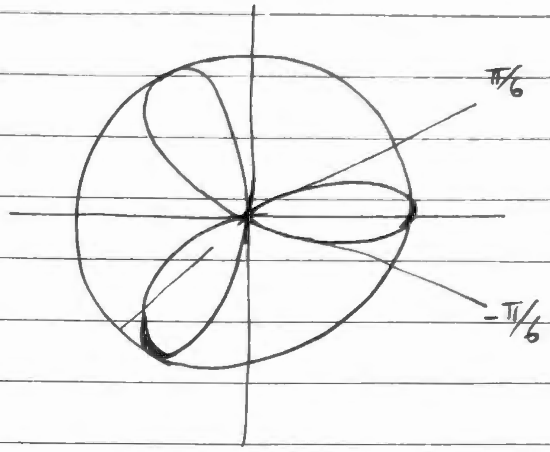
$$A = 2\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{4} + \cos\theta\right) d\theta$$

$$A = 2\left[\frac{\theta}{4} + \frac{\sin 2\theta}{8} + \sin\theta\right]_0^{\pi/2}$$

$$A = 2\left(\frac{\pi}{8} + 0 + 1\right)$$

$$A = \frac{\pi}{4} + 2 \text{ square units}$$

ii) Find the area of the region, one loop of the rose $r = \cos 3\theta$.



Sol^n $\theta = \frac{-\pi}{6}$ to $\frac{\pi}{6}$

$r = 0$ to $\cos 3\theta$

$$A = 3\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_{0}^{\cos 3\theta} r\,dr\,d\theta$$

$$A = 3\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left[\frac{r^2}{2}\right]_0^{\cos 3\theta} d\theta$$

$$A = 3\int_{-\pi/6}^{\pi/6} \frac{\cos^2 3\theta}{2}\, d\theta$$

$$A = 3\int_{-\pi/6}^{\frac{\pi}{6}} \left(\frac{1}{2} + \frac{\cos 6\theta}{2}\right) d\theta$$

$$A = 3\left[\frac{\theta}{2} + \frac{\sin 6\theta}{12}\right]_{-\pi/6}^{\pi/6}$$

$$A = 3\left(\frac{\pi}{12} + 0 - \left(\frac{-\pi}{12}\right) - 0\right)$$

$$A = 3\left(\frac{\pi}{6}\right)$$

$A = \frac{\pi}{2}$ unit square.

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Q. 11. Evaluate $\iiint_V 2x\,dV$, where V is the solid region under the plane $2x + 3y + z = 6$ that lie in the first octant.

Sol^n: $2x + 3y + z = 6 \Rightarrow z = 6 - 2x - 3y$

$z = 0 \rightarrow 2x + 3y + z = 6 \Rightarrow y = \frac{6-2x}{3}$

$z = 0$
$y = 0 \rightarrow 2x + 3y + z = 6 \Rightarrow x = 3.$

Hence, the limits are:

$z = 0$ to $z = 6 - 2x - 3y$

$y = 0$ to $y = \frac{6-2x}{3}$

$x = 0$ to $x = 3.$

Hence, the integral is

$$\int_0^3 \int_0^{\frac{6-2x}{3}} \int_0^{6-2x-3y} 2x \, dz\, dy\, dx$$

$$\Rightarrow \int_0^3 \int_0^{\frac{6-2x}{3}} 2x \left[z\right]_0^{6-2x-3y} dy\, dx$$

$$\Rightarrow \int_0^3 \int_0^{\frac{6-2x}{3}} 2x\left(6 - 2x - 3y\right) dy\, dx$$

$$\Rightarrow \int_0^3 \int_0^{\frac{6-2x}{3}} \left(12x - 4x^2 - 6xy\right) dy\, dx$$

$$\Rightarrow \int_0^3 \left[12xy - 4x^2y - \frac{6xy^2}{2}\right]_0^{\frac{6-2x}{3}} dx$$

$$\Rightarrow \int_0^3 \left(12x\left(\frac{6-2x}{3}\right) - 4x^2\left(\frac{6-2x}{3}\right) - 3x\left(\frac{6-2x}{3}\right)^2\right) dx$$

$$\Rightarrow \int_0^3 \left(24x - 8x^2 - \frac{24x^2}{3} + \frac{8x^3}{3} - \cancel{x} \, x \left(\frac{36 + 4x^2 - 24x}{3}\right)\right) dx$$

$$\Rightarrow \int_0^3 \left(24x - 12x - \cancel{8x^2} + \cancel{8x^2} - \cancel{\frac{24x^2}{3}} \, 8x^2 + \frac{8x^3}{3} - \frac{4x^3}{3}\right) dx$$

$$\Rightarrow \int_0^3 \left(12x - 8x^2 + \frac{4x^3}{3}\right) dx$$

$$\Rightarrow \left[\cancel{12}^{6} \frac{x^2}{\cancel{2}} - \frac{8x^3}{3} + \frac{\cancel{4} x^4}{3 \cdot \cancel{4}}\right]_0^3$$

$$\Rightarrow 6(9) - \frac{8(3)^{\cancel{3}\,2}}{\cancel{3}} + \frac{3^{\cancel{4}\,3}}{\cancel{3}}$$

$$\Rightarrow 54 - 72 + 27$$

$$\Rightarrow 9$$