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CHAPTER 6

Applied Statistics: Test of Hypothesis

Chapter Outline

6.1 INTRODUCTION

The main purpose behind the sampling theory is the study of the Tests of Hypothesis or Tests of significance. In many situations, assumptions are made about the population parameters involved in order to arrive at decisions related to population on the basis of sample information. Such an assumption is called statistical hypothesis which may or may not be true. The procedure which enables us to decide on the basis of sample results whether a hypothesis is true or not, is called test of hypothesis or test of significance.

6.2 TERMS RELATED TO TESTS OF HYPOTHESIS

(1) *Parameters*: The statistical constants of population such as mean (μ), standard deviation (σ), correlation coefficient (ρ), population proportion (P) etc. are called the parameters. Greek letters are used to denote the population parameters.

(2) *Statistic*: The statistical constants for the sample drawn from the given population such as mean ($\bar{x}$), standard deviation (s), correlation coefficient (r), sample proportion (p) etc., are called the statistic. Roman letters are used to denote the sample statistic.

(3) *Sampling Distribution*: Consider all possible samples of size 'n' which can be drawn from a population of size 'N'. These samples will give different values of a statistic. The means of the samples will not be identical. If these different means are arranged according to their frequencies, the frequency distribution formed is called sampling distribution of mean. Similarly, the sampling distribution of other statistics can be defined.

(4) *Standard Error*: The standard deviation of the sampling distribution of a statistic is known as its standard error SE. Standard error plays a very important role in the large sample theory and forms the basis of the testing of hypothesis.

(5) *Null Hypothesis*: Null hypothesis is the hypothesis which is tested for possible rejection under the assumption that it is true. It is denoted by H_0. It asserts that there is no significant difference between the statistic and the population parameter and whatever observed difference exists, is merely due to the fluctuations in sampling from the same population.

(6) *Alternative Hypothesis*: Any hypothesis which is complementary to the null hypothesis is called an alternative hypothesis. It is denoted by H_1. It is set in such a way that the rejection of null hypothesis implies the acceptance of alternative hypothesis. For example, if the null hypothesis is that the average height of the students of a college is 166 cm. i.e., $\mu_0 = 166$ cm, say then the null hypothesis is

$$H_0 : \mu = 166 (= \mu_0)$$

and the alternative hypothesis could be

(i) $H_1 : \mu \neq \mu_0$ (i.e., $\mu > \mu_0$ or $\mu < \mu_0$)

(ii) $H_1 : \mu > \mu_0$

(iii) $H_1 : \mu < \mu_0$

Thus, there can be more than one alternative hypothesis.

(7) *Test Statistic*: After setting up the null hypothesis and alternative hypothesis, test statistic is calculated. The test statistic is a statistic based on appropriate

probability distribution. It is used to test whether the null hypothesis should be accepted or rejected. Different probability distribution values are used in appropriate cases while testing the null hypothesis.
For Z-distribution under normal curve for large samples ($n > 30$), the Z-statistic is defined by

$$Z = \frac{t - E(t)}{SE(t)}$$

(8) *Errors in Hypothesis Testing*: The main objective in sampling theory is to draw valid inferences about the population parameters on the basis of the sample results. There is every chance that a decision regarding a null hypothesis may be correct or may not be correct. There are two types of errors.

(i) *Type I error*: It is the error of rejecting the null hypothesis H_0, when it is true. It occurs when a null hypothesis is true, but the difference of means is significant and the hypothesis is rejected. If the probability of making a type I error is denoted by α, the level of significance, then the probability of making a correct decision is $(1 - \alpha)$.

(ii) *Type II error*: It is the error of accepting the null hypothesis H_0, when it is false. It occurs when a null hypothesis is false, but the difference of means is insignificant and the hypothesis is accepted. The probability of making a type II error is denoted by β.

(9) *Level of Significance*: The level of significance is the maximum probability of making a type I error and is denoted by α, i.e., P(Rejecting H_0 when H_0 is true) $= \alpha$. The commonly used level of significance in practice are 5% (0.05) and 1% (0.01). For 5% level of significance ($\alpha = 0.05$), the probability of making type I error is 0.05 or 5% i.e., P(Rejecting H_0 when H_0 is true) = 0.05. This means that there is a probability of making 5 out of 100 type I error. Similarly, 1% level of significance ($\alpha = 0.01$) means that there is a probability of making 1 error out of 100. If no level of significance is given, α is taken as 0.05.

(10) *Critical Region*: The critical region or rejection region is the region of the standard normal curve corresponding to a predetermined level of significance α. The region under the normal curve which is not covered by the rejection region is known as acceptance region. Thus, the statistic which leads to rejection of null hypothesis H_0 gives rejection region or critical region. The value of the test statistic calculated to test the null hypothesis H_0 is known as critical value. Thus, the critical value separates the rejection region from the acceptance region.

(11) *Two Tailed Test and One Tailed Test*: When the test of hypothesis is made on the basis of rejection region represented by both the sides of the standard normal curve, it is called a two tailed test. A test of statistical hypothesis, where the alternative hypothesis H_1 is two sided or two tailed such as:

Null Hypothesis $H_0 : \mu = \mu_0$

Alternative Hypothesis $H_1 : \mu \neq \mu_0$ ($\mu > \mu_0$ and $\mu < \mu_0$), is called two tailed test or two sided test.

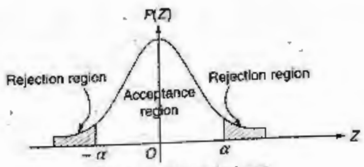

Fig. 6.1 Two tailed test

A test of statistical hypothesis, where the alternative hypothesis is one sided is called one tailed test or one sided test. There are two types of one tailed tests.

(i) *Right Tailed Test:* In the right tailed test, the rejection region or critical region lies entirely on the right tail of the normal curve (Fig. 6.2).

(ii) *Left Tailed Test:* In the left tailed test, the rejection region or critical region lies entirely on the left tail of the normal curve.

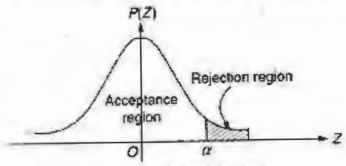

Fig. 6.2 Right tailed test

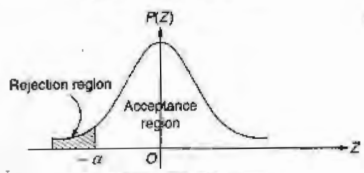

Fig. 6.3 Left tailed test

For example, in a test for testing the mean (μ) of the population
Null Hypothesis $H_0 : \mu = \mu_0$

Alternative Hypothesis $H_1 : \mu > \mu_0$ (Right tailed)

$\mu < \mu_0$ (Left tailed)

A two tailed test is applied in such cases when the difference between the sample mean and population mean is tending to reject the null hypothesis H_0, the difference may be positive or negative.

A one tailed test is applied in such cases when the population mean is at least as large as some specified value of the mean (right tailed test) or at least as small as some specified value of the mean (left tailed test).

Critical value (Z_α)	*Level of significance α*		
	1%	5%	10%
Two tailed test	$\lvert Z_\alpha \rvert = 2.58$	$\lvert Z_\alpha \rvert = 1.96$	$\lvert Z_\alpha \rvert = 1.645$
Right tailed test	$Z_\alpha = 2.33$	$Z_\alpha = 1.645$	$Z_\alpha = 1.28$
Left tailed test	$Z_\alpha = -2.33$	$Z_\alpha = -1.645$	$Z_\alpha = -1.28$

(12) *Confidence Limits:* The limits within which a hypothesis should lie with specified probability are called confidence limits or fiducial limits. Generally, the confidence limits are set up with 5% or 1% level of significance. If the sample value lies between the confidence limits, the hypothesis is accepted, if it does not, then the hypothesis is rejected at the specified level of significance. Suppose that the sampling distribution of a statistic S is normal with mean μ and standard deviation σ. The sample statistic S can be expected to lie in the interval ($\mu - 1.96\sigma$, $\mu + 1.96\sigma$) for 95% times (Fig. 6.29). Because of this, ($S - 1.96\sigma$, $S + 1.96\sigma$) is called the 95% confidence interval for estimation of μ. The ends of this interval, i.e., $S \pm 1.96\sigma$ are called 95% confidence limits for S. Similarly, $S \pm 2.58\sigma$ are 99% confidence limits. The numbers 1.96, 2.58 etc. are called confidence coefficients.

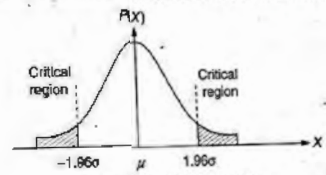

Fig. 6.4 Confidence Limits

6.3 PROCEDURE FOR TESTING OF HYPOTHESIS

The various steps in testing of a statistical hypothesis are as follows:

(i) *Null Hypothesis:* Set up the Null Hypothesis H_0

(ii) *Alternative Hypothesis:* Set up the Alternative Hypothesis H_1. This will decide the use of single-tailed (right or left) or Two tailed test.

(iii) *Level of Significance:* Select the appropriate level of significance (α) depending on the reliability of the estimates and permissible risk. If no level of significance is given, α is selected as 0.05.

(iv) *Test Statistic:* Calculate the test statistic

$$Z = \frac{t - E(t)}{SE(t)} \text{ under } H_0$$

(v) *Critical Value:* Find the significant value (tabulated value) Z_α of Z at the given level of significance α.

(vi) *Decision*: Compare the calculated value of Z with the tabulated value Z_α. If $|Z| < Z_\alpha$ i.e., if the calculated value of Z is less than tabulated value Z_α at the level of significance α, the null hypothesis is accepted. If $|Z| > Z_\alpha$ i.e., if the calculated value of Z is more than tabulated value Z_α at the level of significance α, the null hypothesis is rejected.

6.4 TEST OF SIGNIFICANCE FOR LARGE SAMPLES

If a sample consists of more than 30 items, i.e., $n > 30$, it is considered as large sample. The following assumptions are applied for significance tests of large samples:

(i) The random sampling distribution of statistic has the properties of the normal curve.

(ii) Values (i.e., statistic) given by the samples are sufficiently close to the population values (i.e., parameters) and can be used in its place for calculating the standard error (*SE*) of the estimate.

For example, if SD of the population is not known, SE can be calculated by SD of the sample.

Suppose the hypothesis to be tested is that the probability of success in such trail is p. Assuming it to be true, the mean μ and the standard deviation σ of the sampling distribution of the number of successes are np and $\sqrt{npq}$ respectively as the sampling distribution of number of successes follows a binomial probability distribution.

If x is the observed number of successes in the sample and Z is the standard normal variate then

$$Z = \frac{x-\mu}{\sigma}$$

The tests of significance are as follows:

(i) If $|Z| < 1.96$, the difference between the observed and expected number of successes is not significant.

(ii) If $|Z| > 1.96$, the difference is significant at 5% level of significance.

(iii) If $|Z| > 2.58$, the difference is significant at 1% level of significance.

Example 1

A coin was tossed 960 times and returned heads 183 times. Test the hypothesis that the coin is unbiased. Use a 0.05 level of significance.

Solution

$$n = 960$$

$$p = \text{probability of getting head} = \frac{1}{2}$$

$$q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\mu = np = 960\left(\frac{1}{2}\right) = 480$$

$$\sigma = \sqrt{npq} = \sqrt{960 \times \frac{1}{2} \times \frac{1}{2}} = 15.49$$

$$x = \text{number of sucesses} = 183$$

(i) Null Hypothesis H_0: The coin is unbiased.

(ii) Alternative Hypothesis H_1: The coin is biased.

(iii) Level of significance: $\alpha = 0.05$

(iv) Test statistic: $Z = \dfrac{x-\mu}{\sigma} = \dfrac{183-480}{15.49} = -19.17$

$|Z| = 19.17$

(v) Critical value: $|Z_{0.05}| = 1.96$

(vi) Decision: Since $|Z| > |Z_{0.05}|$, the null hypothesis is rejected at 5%-level of significance, i.e., the coin is biased.

Example 2

A dice is tossed 960 times and it falls with 5 upwards 184 times. Is the dice unbiased at a level of significance of 0.01?

Solution

$$n = 960$$

$$p = \text{Probability of throwing 5 with one die} = \frac{1}{6}$$

$$q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

$$\mu = np = 960\left(\frac{1}{6}\right) = 160$$

$$\sigma = \sqrt{npq} = \sqrt{960 \times \frac{1}{6} \times \frac{5}{6}} = 11.55$$

$$x = \text{number of successes} = 184$$

(i) Null Hypothesis H_0: The dice is unbiased.

(ii) Alternative Hypothesis H_1: The dice is biased.

(iii) Level of significance: $\alpha = 0.01$

(iv) Test statistic: $Z = \dfrac{x-\mu}{\mu} = \dfrac{184-160}{11.55} = 2.08$

$|Z| = 2.08$

(v) Critical value: $|Z_{0.01}| = 2.58$

(vi) Decision: Since $|Z| < |Z_{0.01}|$, the null hypothesis is accepted at 1% level of significance, i.e., the dice is unbiased.

6.5 TEST OF SIGNIFICANCE FOR SINGLE PROPORTION – LARGE SAMPLES

Let p be the sample proportion in a large random sample of size n drawn from a population having proportion P. Also, the population proportion P has a specified value P_0.

Working Rule

(i) Null Hypothesis $H_0: P = P_0$, i.e., the population proportion P has a specified value P_0.

(ii) Alternative Hypothesis $H_1: P \neq P_0$ (i.e., $P > P_0$ or $P < P_0$)
or $H_1: P > P_0$
or $H_1: P < P_0$

(iii) Level of significance: Select the level of significance α

(iv) Test statistic: $Z = \dfrac{p-P}{\sqrt{\dfrac{PQ}{n}}}$, where $Q = 1 - P$

(v) Critical Value: Find the critical value (tabulated value) Z_α of Z at the given level of significance.

(vi) Decision: If $|Z| < Z_\alpha$ at the level of significance α, the null hypothesis is accepted. If $|Z| > Z_\alpha$ at the level of significance α, the null hypothesis is rejected.

Note

1. Null Hypothesis H_0 is rejected when $|Z| > 3$ without mentioning any level of significance.
2. Confidence limits:

   (i) 95% confidence limits $= p \pm 1.96\sqrt{\dfrac{PQ}{n}}$

   (ii) 99% confidence limits $= p \pm 2.58\sqrt{\dfrac{PQ}{n}}$

If the population proportions P and Q are not known, p and q are used in equations.

Example 1

A manufacturer claimed that atleast 95% of the equipment which he supplied to a factory conformed to specification. An examination of a sample of 200 pieces of equipment revealed that 18 were faulty. Test his claim at 5% level of significance.

Solution

$n = 200$

Number of pieces conforming to specification $= 200 - 18 = 182$

p = Sample proportion of pieces conforming to specification $= \dfrac{182}{200} = 0.91$

P = Population proportion of pieces conforming to specification $= 0.95$

$Q = 1 - P = 1 - 0.95 = 0.05$

(i) *Null Hypothesis* $H_0: P = 0.95$ i.e., the proportion of pieces conforming to proportion is 95%.

(ii) *Alternating Hypothesis* $H_1: P < 0.95$ (Left tailed test)

(iii) *Level of significance*: $\alpha = 0.05$

(iv) Test statistic: $Z = \dfrac{p-P}{\sqrt{\dfrac{PQ}{n}}} = \dfrac{0.91-0.95}{\sqrt{\dfrac{(0.95)(0.05)}{200}}} = -2.59$

$|Z| = 2.59$

(v) Critical value: $|Z_{0.05}| = 1.645$

(vi) Decision: Since $|Z| > |Z_{0.05}|$, the null hypothesis is rejected at 5% level of significance, i.e., the manufacturer's claim is rejected.

Example 2

In a hospital 480 female and 520 male babies were born in a week. Do these figures confirm the hypothesis that males and females were born in equal numbers?

Solution

n = Total number of births $= 480 + 520 = 1000$

p = Sample proportion of females born $= \dfrac{480}{1000} = 0.48$

P = Population proportion of females born $= 0.5$

$Q = 1 - P = 1 - 0.5 = 0.5$

(i) Null Hypothesis $H_0: P = 0.5$ i.e., the males and females were born in equal numbers.

(ii) Alternative Hypothesis $H_1: P \neq 0.5$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $Z = \dfrac{p-P}{\sqrt{\dfrac{PQ}{n}}} = \dfrac{0.48-0.5}{\sqrt{\dfrac{(0.5)(0.5)}{1000}}} = -1.265$

$|Z| = 1.265$

(v) Critical value: $|Z_{0.05}| = 1.96$

(vi) Decision: Since $|Z| < |Z_{0.05}|$, the null hypothesis is accepted at 5% level of significance, i.e., males and females were born in equal proportions.

Example 3

In a study designed to investigate whether certain detonators used with explosives in a coal mining meet the requirement that at least 90% will ignite the explosive when charged. It is found that 174 of 200 detonators function properly. Test the null hypothesis $P = 0.9$ against the alternative hypothesis $P < 0.9$ at the 0.05 level of significance.

Solution

$$n = 2000$$

p = Sample proportion of detonators functioning properly $= \frac{174}{200} = 0.87$

P = Population proportion of detonators functioning properly = 0.9

$$Q = 1 - P = 1 - 0.9 = 0.1$$

(i) Null Hypothesis H_0: $P = 0.9$

(ii) Alternative Hypothesis H_1: $P < 0.9$ (Left tailed test)

(iii) Level of significance: $\alpha = 0.05$

(iv) Test statistic: $Z = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{0.87-0.9}{\sqrt{\frac{(0.9)(0.1)}{200}}} = -1.41$

$$|Z| = 1.41$$

(v) Critical value: $|Z_{0.05}| = 1.645$

(vi) Decision: Since $|Z| < |Z_{0.05}|$, the null hypothesis is accepted at 5% level of significance.

Example 4

A salesman in a departmental store claims that at most 60 percent of the shoppers entering the store leave without making a purchase. A random sample of 50 shoppers showed that 35 of them left without making a purchase. Are these sample results consistent with the claim of the salesman? Use a level of significance of 0.05.

Solution

$$n = 50$$

p = Sample proportion of shoppers not making a purchase $= \frac{35}{50} = 0.7$

P = Population proportion of shoppers not making a purchase = 0.6

$$Q = 1 - P = 1 - 0.6 = 0.4$$

(i) Null Hypothesis H_0: $P = 0.6$, i.e., the proportion of shoppers not making a purchase is 60%.

(ii) Alternative Hypothesis H_1: $P > 0.6$ (Right tailed test)

(iii) Level of significance: $\alpha = 0.05$

(iv) Test statistic: $Z = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{0.7-0.6}{\sqrt{\frac{(0.6)(0.4)}{50}}} = 1.443$

$$|Z| = 1.443$$

(v) Critical value: $Z_{0.05} = 1.645$

(vi) Decision: Since $|Z| < Z_{0.05}$, the null hypothesis is accepted, i.e., the sample results are consistent with claim of the salesman.

Example 5

The fatality rate of typhoid patients is believed to be 17.26%. In a certain year 640 patients suffering from typhoid were treated in a metropolitan hospital and only 63 patients died. Can you consider the hospital efficient at 1% level of significance?

Solution

$$n = 640$$

p = Sample proportion of typhoid patients died $= \frac{63}{640} = 0.0984$

P = Population proportion of typhoid patients died = 0.1726

$$Q = 1 - P = 1 - 0.1726 = 0.8274$$

(i) Null Hypothesis H_0: $P = 0.1726$, i.e., the hospital is efficient.

(ii) Alternative Hypothesis H_1: $P < 0.1726$ (Left tailed test)

(iii) Level of significance: $\alpha = 0.01$

(iv) Test statistic: $Z = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{0.0984-0.1726}{\sqrt{\frac{(0.1726)(0.8274)}{640}}} = -4.97$

$$|Z| = 4.97$$

(v) Critical value: $|Z_{0.01}| = 2.33$

(vi) Decision: Since $|Z| > |Z_{0.01}|$, the null hypothesis is rejected at 1% level of significance, i.e., the hospital is efficient.

Example 6

In a big city, 325 men out of 600 were found to be smokers. Does this information support the conclusion that the majority of men in this city are smokers?

Solution

$$n = 600$$

$$p = \text{Sample proportion of smokers in city} = \frac{325}{600} = 0.542$$

$$P = \text{Population proportion of smokers in city} = 0.5$$

$$Q = 1 - P = 1 - 0.5 = 0.5$$

(i) Null Hypothesis H_0: $P = 0.5$, i.e., the proportion of smokers in the city is 50%.

(ii) Alternative Hypothesis H_1: $P > 0.5$ (Right tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $Z = \dfrac{p-P}{\sqrt{\dfrac{PQ}{n}}} = \dfrac{0.542-0.5}{\sqrt{\dfrac{(0.5)(0.5)}{600}}} = 2.06$

$$|Z| = 2.06$$

(v) Critical value: $Z_{0.05} = 1.645$

(vi) Decision: Since $|Z| > Z_{0.05}$, the null hypothesis is rejected at 5% level of significance, i.e., proportion of smokers in city is more than 50% and majority of men in the city are smokers.

Example 7

In a random sample of 160 worker exposed to a certain amount of radiation, 24 experienced some ill effects. Construct a 95% confidence interval for the corresponding true percentoge.

Solution

$$n = 160$$

$$p = \text{Sample proportion of workers exposed to radiation} = \frac{24}{160} = 0.15$$

$$q = 1 - p = 1 - 0.15 = 0.85$$

Confidence interval at 95% level of significance is

$$\left(p - 1.96\frac{\sqrt{pq}}{n}, p + 1.96\sqrt{\frac{pq}{n}}\right)$$

i.e., $$\left(0.15 - 1.96\sqrt{\frac{(0.15)(0.85)}{160}}, 0.15 + 1.96\sqrt{\frac{(0.15)(0.85)}{160}}\right)$$

i.e., $$(0.0947, 0.2053)$$

6.6 TEST OF SIGNIFICANCE FOR DIFFERENCE OF PROPORTIONS – LARGE SAMPLES

Let p_1 and p_2 be the sample proportions in two large samples of sizes n_1 and n_2 drawn from two populations having proportions P_1 and P_2.

Working Rule

(i) Null Hypothesis H_0: $P_1 = P_2$, i.e., there is no significant difference in two population proportions P_1 and P_2.

(ii) Alternative Hypothesis H_1: $P_1 \neq P_2$
or H_1: $P_1 > P_2$
or H_1: $P_1 < P_2$

(iii) Level of significance: Select level of significance α

(iv) Test statistic: There are two cases:

(a) When the population proportions P_1 and P_2 are known

$$Z = \frac{P_1 - P_2}{\sqrt{\dfrac{P_1Q_1}{n_1} + \dfrac{P_2Q_2}{n_2}}}$$

(b) When the population proportions P_1 and P_2 are not known but sample proportions p_1 and p_2 are known

There are two methods to estimate P_1 and P_2.

Method of Substitution: In this method, sample proportions p_1 and p_2 are substituted for P_1 and P_2.

$$Z = \frac{p_1 - p_2}{\sqrt{\dfrac{p_1q_1}{n_1} + \dfrac{p_2q_2}{n_2}}}$$

Method of pooling: In this method, the estimated value of two population proportions is obtained by pooling the two sample proportions p_1 and p_2 into a single proportion p.

$$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$

$$Z = \frac{p_1 - p_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

(v) Critical value: Find the critical value (tabulated value) of Z at given level of significance.

(vi) Decision: If $|Z| < Z_\alpha$ at the level of significance, the null hypothesis is accepted. If $|Z| > Z_\alpha$ at the level of significance, the null hypothesis is rejected.

Note

1. Null Hypothesis H_0 is rejected when $|Z| > 3$ without mentioning any level of significance.
2. Confidence limits:

   (i) 95% confidence limits = $(p_1 - p_2) \pm 1.96\sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}}$

   (ii) 99% confidence limits = $(p_1 - p_2) \pm 2.58\sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}}$

If the population proportions P_1 and P_2 are not known, p_1, p_2, q_1 and q_2 are used in equations.

Example 1

Random samples of 400 men and 600 women were asked whether they would like to have a flyover near their residence 200 men and 325 women were in favour of the proposal. Test the hypothesis that proportions of men and women in favour of the proposal are same at 5% level of significance.

Solution

$$n_1 = 400,\ n_2 = 600$$

$$p_1 = \text{Proportion of men} = \frac{200}{400} = 0.5$$

$$p_2 = \text{Proportion of women} = \frac{325}{600} = 0.541$$

$$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{(400)(0.5) + (600)(0.541)}{400 + 600} = 0.525$$

$$q = 1 - p = 1 - 0.525 = 0.475$$

(i) Null Hypothesis H_0: $P_1 = P_2$, i.e., there is no significant difference in proportion of men and women in favour of the proposal.

(ii) Alternative Hypothesis is H_1: $P_1 \neq P_2$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$

(iv) Test statistic: $Z = \frac{p_1 - p_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{0.5 - 0.541}{\sqrt{(0.525)(0.475)\left(\frac{1}{400} + \frac{1}{600}\right)}} = -1.28$

$$|Z| = 1.28$$

(v) Critical value: $|Z_{0.05}| = 1.96$

(vi) Decision: Since $|Z| < |Z_{0.05}|$, the null hypothesis is accepted at 5% level of significance, i.e., there is no significant difference of opinion between men and women in favour of the proposal.

Example 2

In a city A, 20% of a random sample of 900 school boys has a certain slight physical defect. In another city B, 18.5% of a random sample of 1600 school boys has the same defect. Is the difference between the proportions significant at 0.05 level of significance?

Solution

$$n_1 = 900,\ n_2 = 1600$$

p_1 = Proportion of school boys in city $A = 0.2$

p_2 = Proportion of school boys in city $B = 0.185$

$$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{(900)(0.2) + (1600)(0.185)}{900 + 1600} = 0.1904$$

$$q = 1 - p = 1 - 0.1904 = 0.8096$$

(i) Null Hypothesis H_0: $P_1 = P_2$, i.e., there is no significant difference in proportion of two city school boys.

(ii) Alternative Hypothesis H_1: $P_1 \neq P_2$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$

(iv) Test statistic: $Z = \frac{p_1 - p_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{0.2 - 0.185}{\sqrt{(0.1904)(0.8096)\left(\frac{1}{900} + \frac{1}{1600}\right)}} = 0.916$

$$|Z| = 0.916$$

(v) Critical value: $|Z_{0.05}| = 1.96$

(vi) Decision: Since $|Z| < |Z_{0.05}|$, the null hypothesis is accepted at 5% level of significance, i.e., there is no significant difference between the proportions of two city school boys.

Example 3

Before an increase in excise duty on tea, 800 people out of a sample of 1000 were consumers of tea. After an increase in excise duty, 800 people were consumers of tea in a sample of 1200 persons. Find whether there is significant decrease in the consumption of tea after the increase in duty.

Solution

$$n_1 = 1000,\ n_2 = 1200$$

p_1 = Proportion of consumers of tea before increase in excise duty $= \frac{800}{1000} = 0.8$

p_2 = Proportion of consumers of tea after increase in excise duty $= \frac{800}{1200} = 0.67$

$$p = \frac{n_1p_1 + n_2p_2}{n_1 + n_2} = \frac{(1000)(0.8) + (1200)(0.67)}{1000 + 1200} = 0.73$$

$$q = 1 - p = 1 - 0.73 = 0.27$$

(i) Null Hypothesis H_0: $P_1 = P_2$, i.e., there is no significant decrease in the consumption of tea after the increase in duty.

(ii) Alternative Hypothesis H_1: $P_1 > P_2$ (Right tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $$Z = \frac{p_1 - p_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{0.8 - 0.67}{\sqrt{(0.73)(0.27)\left(\frac{1}{1000} + \frac{1}{1200}\right)}} = 6.84$$

$$|Z| = 6.84$$

(v) Critical value: $Z_{0.05} = 1.645$

(vi) Decision: Since $|Z| > Z_{0.05}$, the null hypothesis is rejected at 5% level of significance, i.e., there is significant decrease in the consumption of tea after the increase in duty.

Example 4

15.5% of a random sample of 1600 undergraduates smokers, whereas 20% of a random sample of 900 postgraduates were smokers in a state.

Can we conclude that less number of undergraduates are smokers than the postgraduates?

Solution

$$n_1 = 1600,\ n_2 = 900$$

p_1 = Proportion of undergraduate smokers = 0.155

p_2 = Proportion of postgraduate smokers = 0.2

$$p = \frac{n_1p_1 + n_2p_2}{n_1 + n_2} = \frac{(1600)(0.155) + (900)(0.2)}{1600 + 900} = 0.1712$$

$$q = 1 - p = 1 - 0.1712 = 0.8288$$

(i) Null Hypothesis H_0: $P_1 = P_2$, i.e., there is no significant difference in proportion of undergraduate and postgraduate smokers.

(ii) Alternative Hypothesis H_1: $P_1 < P_2$ (Left tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $$Z = \frac{p_1 - p_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{0.155 - 0.2}{\sqrt{(0.1712)(0.8288)\left(\frac{1}{1600} + \frac{1}{900}\right)}} = -2.87$$

$$|Z| = 2.87$$

(v) Critical value: $|Z_{0.05}| = 1.645$

(vi) Decision: Since $|Z| > |Z_{0.05}|$, the null hypothesis is rejected at 5% level of significance, i.e., less number of undergraduates smokers than the postgraduates.

Example 5

A machine produced 20 defective articles in a batch of 400. After overhauling it produced 10 defective articles in a batch of 300. Has the machine improved?

Solution

$$n_1 = 400,\ n_2 = 300$$

p_1 = Proportion of defective articles before overhauling $= \frac{20}{400} = 0.05$

p_2 = Proportion of defective articles after overhauling $= \frac{10}{300} = 0.033$

$$p = \frac{n_1p_1 + n_2p_2}{n_1 + n_2} = \frac{(400)(0.05) + (300)(0.033)}{400 + 300} = 0.043$$

$$q = 1 - p = 1 - 0.043 = 0.957$$

(i) Null Hypothesis H_0: $P_1 = P_2$, i.e., the proportions of defective articles before and after overhauling are equal.

(ii) Alternative Hypothesis H_1: $P_1 > P_2$ (Right tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $Z = \dfrac{p_1 - p_2}{\sqrt{pq\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)}} = \dfrac{0.05 - 0.033}{\sqrt{(0.043)(0.957)\left(\dfrac{1}{400}+\dfrac{1}{300}\right)}} = 1.097$

$|Z| = 1.097$

(v) Critical value: $Z_{0.05} = 1.645$

(vi) Decision: Since $|Z| < Z_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., proportion of defective articles before and after are equal and machine has not improved.

Example 6

In two large populations, there are 30% and 25% fair haired people respectively. Is this difference likely to be hidden in samples of 1200 and 900 respectively from the two populations?

Solution

$n_1 = 1200, n_2 = 900$

P_1 = Proportion of faired people in the first population = 0.3

$Q_1 = 1 - P_1 = 1 - 0.3 = 0.7$

P_2 = Proportion of faired people in the second population = 0.25

$Q_2 = 1 - P_2 = 1 - 0.25 = 0.75$

(i) Null Hypothesis H_0: $P_1 = P_2$, i.e., the difference in population proportions is likely to be hidden in sampling.

(ii) Alternative Hypothesis H_1: $P_1 \neq P_2$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $Z = \dfrac{P_1 - P_2}{\sqrt{\dfrac{P_1Q_1}{n_1}+\dfrac{P_2Q_2}{n_2}}} = \dfrac{0.3 - 0.25}{\sqrt{\dfrac{(0.3)(0.7)}{1200}+\dfrac{(0.25)(0.75)}{900}}} = 2.56$

$|Z| = 2.56$

(v) Critical value: $|Z_{0.05}| = 1.96$

(vi) Decision: Since $|Z| > |Z_{0.05}|$, the null hypothesis is rejected at 5% level of significance, i.e., the difference in population proportions is not likely to be hidden in sampling.

Example 7

A random sample of 300 shoppers at a supermarket includes 204 who regularly uses cents off coupons. In another sample of 500 shoppers at a supermarket includes 75 who regularly uses cents off coupons. Obtain 95% confidence limits for the difference in the population proportions.

Solution

$n_1 = 300,\ n_2 = 500$

p_1 = Proportion of shoppers who uses cents of coupons in the first sample

$= \dfrac{204}{300} = 0.68$

$q_1 = 1 - p_1 = 1 - 0.68 = 0.32$

p_2 = Proportion of shoppers who uses cents of coupons in the second sample

$= \dfrac{75}{500} = 0.15$

$q_2 = 1 - p_2 = 1 - 0.15 = 0.85$

$$SE = \sqrt{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}} = \sqrt{\frac{(0.68)(0.32)}{300}+\frac{(0.15)(0.85)}{500}} = 0.031$$

95% confidence limits for the difference in population proportion is

$$(p_1 - p_2) - 1.96\sqrt{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}},\ (p_1 - p_2) + 1.96\sqrt{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}$$

i.e., $(0.68 - 0.15) - 1.96\,(0.031),\ (0.68 - 0.15) + 1.96\,(0.031)$

i.e., $(0.469, 0.591)$

EXERCISE 6.1

1. A manufacturer claims at least 95% of the items he produces are failure free. Examinations of a random sample of 600 items showed 39 to be defective. Test the claim at a significance level of 0.05.
   [Ans.: Claim is rejected]

2. In a sample of 400 parts manufactured by a factory, the number of defective parts was found to be 30. The company, however, claim that only 5% of their product is defective. Is the claim tenable?
   [Ans.: Claim is rejected]

3. A sample of 600 persons selected at random from a large city shows that the percentage of male in the sample is 53%. It is believed that male to the total population ratio in the city is $\frac{1}{2}$. Test whether this

belief is confirmed by the observation.

[Ans.: Belief is confirmed by the observation]

4. In a sample of 1000 people in Karnataka, 540 are rice eaters and the rest are wheat eaters. Can we assume that both rice and wheat are equally popular in this state at 1% level of significance?

[Ans.: Both rice and wheat are equally popular in state]

5. In a big city 325 men out of 600 men were found to be smokers. Does this information support the conclusion that the majority of men in this city are smokers?

[Ans.: Majority of men in the city are smokers]

6. A dice was thrown 400 times and 'six' resulted 80 times. Do the data justify the hypothesis of an unbiased dice.

[Ans.: The dice is unbiased]

7. In a random sample of 125 cold drinkers, 68 said they prefer 'Thumsup' to Pepsi'. Test the null hypothesis $P = 0.5$ against the alternative hypothesis $P > 0.5$.

[Ans.: Null hypothesis is accepted]

8. A social worker believes that fewer than 25% of the couples in a certain area have ever used any form of birth control. A random sample of 120 couples was contacted. Twenty of them said they have used. Test the belief of the social worker at 0.05 level.

[Ans.: Belief of the social worker is true]

9. 20 people were attacked by a disease and only 18 survived. Will you reject the hypothesis that the survival rate is attacked by this disease is 85% in favour of the hypothesis that is more at 5% level?

[Ans.: The hypothesis is accepted]

10. A manufacturer of electronic equipment subjects samples of two completing brands of transistors to an accelerated performance test. If 45 of 180 transistors of the first kind and 34 of 120 transistors of second kind fail the test, what can be conclude at the level of significance $\alpha = 0.05$ about the difference between the corresponding sample proportion?

[Ans.: The difference between the proportions is not significant]

11. On the basis of their total scores, 200 candidates of a civil service examination are divided into two groups, the upper 30% and the remaining 70%. Consider the first question of the examination. Among the first group, 40 had the correct answer, whereas among the second group, 80 had the correct answer. On the basis of these results, can one conclude that the first question is not good at discriminating ability of the type being examined here?

[Ans.: The first question is good enough at discriminating ability of the type being examined]

12. A company wanted to introduce a new plan of work and a survey was conducted for this purpose. Out of sample of 500 workers in one group, 62% favoured the new plan and another group of sample of 400 workers, 41% were against the new plan. Is there any significant difference between the two groups in their attitude towards the new plan at 5% level of significance?

[Ans.: There is no significant difference between the two groups in their attitude towards the new plan]

13. In a random sample of 1000 persons from town A, 400 are found to be consumers of wheat. In a sample of 800 from town B, 400 are found to be consumers of wheat. Do these data reveal a significant difference between town A and town B, so far as the proportion of wheat consumers is concerned?

[Ans.: There is significant difference between town A and town B as the proportion of wheat consumers is concerned]

14. 100 articles from a factory are examined and 10 are found to be defective. Out of 500 similar articles from a second factory 15 are found to be defective. Test the significance between the difference of two proportions at 5% level.

[Ans.: There is a significant difference between the two proportions]

6.7 TEST OF SIGNIFICANCE FOR SINGLE MEAN – LARGE SAMPLES

Let a random sample size n ($n > 30$) has the sample mean $\bar{x}$ and population has the mean μ. Also, the population mean μ has a specified value μ_0.

Working Rule

(i) Null Hypothesis $H_0 : \mu = \mu_0$, i.e., the population mean μ has a specified value μ_0.

(ii) Alternative Hypothesis $H_1 : \mu \neq \mu_0$.

(iii) Level of significance: Select the level of significance α.

(iv) Test statistic: There are two cases for calculating a test statistic Z.

(a) When the standard deviation σ of population is known

$$Z = \frac{\bar{x} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)}$$

(b) When the standard deviation σ of population is not known

$$Z = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n}}\right)}$$

where s is the sample SD.

(v) Critical value: Find the critical value (tabulated value) Z_α of Z at the given level of significance α.

(vi) Decision: If $|Z| < Z_\alpha$ at the level of significance α, the null hypothesis is accepted. If $|Z| > Z_\alpha$ at the level of significance α, the null hypothesis is rejected.

Note

1. Null Hypothesis H_0 is rejected when $|Z| > 3$ without mentioning any level of significance.
2. Confidence limits:

   (i) 95% confidence limits = $\bar{x} \pm 1.96\left(\frac{\sigma}{\sqrt{n}}\right)$

   (ii) 99% confidence limits = $\bar{x} \pm 2.58\left(\frac{\sigma}{\sqrt{n}}\right)$

If standard deviation σ of population is not known, s is used in equations.

Example 1

A random sample of 100 Indians has an average life span of 71.8 years with standard deviation of 8.9 years. Can it be concluded that the average life span of an Indian is 70 years?

Solution

$$n = 100, \bar{x} = 71.8 \text{ years}, \mu = 70 \text{ years}, s = 8.9 \text{ years}$$

(i) Null Hypothesis H_0: $\mu = 70$ years i.e., the average life span of an Indian is 70 years.

(ii) Alternative Hypothesis H_1: $\mu \neq 70$ years (Two tailed test)

(iii) Level of Significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $Z = \dfrac{\bar{x}-\mu}{\left(\frac{s}{\sqrt{n}}\right)} = \dfrac{71.8-70}{\left(\frac{8.9}{\sqrt{100}}\right)} = 2.02$

$$|Z| = 2.02$$

(v) Critical value: $|Z_{0.05}| = 1.96$

(vi) Decision: Since $|Z| > |Z_{0.05}|$, the null hypothesis is rejected at 5% level of significance, i.e., the average life span of an Indian is not 70 years.

Example 2

A random sample of 50 items gives the mean 6.2 and variance 10.24. Can it be regarded as drawn from a normal population with mean 5.4 at 5% level of significance?

Solution

$$n = 50, \quad \bar{x} = 6.2, \quad \mu = 5.4, \quad s = \sqrt{10.24}$$

(i) Null Hypothesis H_0: $\mu = 5.4$, i.e., the sample is drawn from a normal population with mean 5.4.

(ii) Alternative Hypothesis H_1: $\mu \neq 5.4$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$

(iv) Test statistic: $Z = \dfrac{\bar{x}-\mu}{\left(\frac{s}{\sqrt{n}}\right)} = \dfrac{6.2-5.4}{\left(\frac{\sqrt{10.24}}{\sqrt{50}}\right)} = 1.77$

$$|Z| = 1.77$$

(v) Critical value: $|Z_{0.05}| = 1.96$

(vi) Decision: Since $|Z| < |Z_{0.05}|$, the null hypothesis is accepted at 5% level of significance i.e., the sample is drawn from a normal population with mean 5.4.

Example 3

A random sample of 400 members is found to have a mean of 4.45 cm. Can it be reasonably regarded as a sample from a large population whose mean is 5 cm and variance is 4 cm?

Solution

$$n = 400, \quad \bar{x} = 4.45\text{cm}, \quad \mu = 5\text{cm} \quad \sigma = \sqrt{4} = 2\text{ cm}$$

(i) Null Hypothesis H_0: $\mu = 5$ cm, i.e., the sample is drawn from a large population with mean 5 cm.

(ii) Alternative Hypothesis H_1: $\mu \neq 5$ cm (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $Z = \dfrac{\bar{x}-\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \dfrac{4.45-5}{\left(\frac{2}{\sqrt{400}}\right)} = 5.55$

$$|Z| = 5.55$$

(v) Critical value: $|Z_{0.05}| = 1.96$

(vi) Decision: Since $|Z| > |Z_{0.05}|$, the null hypothesis is rejected at 5% level of significance, i.e., the sample is not drawn from the large population with mean 5 cm.

Example 4

A sample of 900 members has a mean of 3.4 cm and SD 2.61 cm. Is the sample from a large population of mean 3.25 cm and SD 2.61 cm? If the population is normal and its mean is unknown, find the 95% fiducial limits of its true mean.

Solution

$$n = 900,\ \bar{x} = 3.4\text{ cm},\ s = 2.61\text{ cm},\ \mu = 3.25\text{ cm},\ \sigma = 2.61\text{ cm}$$

(i) Null Hypothesis H_0: $\mu = 3.25$ cm, i.e., the sample has been drawn from the population with mean $\mu = 3.25$ cm and SD = 2.61 cm.

(ii) Alternative Hypothesis H_1: $\mu \neq 3.25$ cm (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$

(iv) Test statistic: $Z = \dfrac{\bar{x}-\mu}{\left(\dfrac{\sigma}{\sqrt{n}}\right)} = \dfrac{3.4-3.25}{\left(\dfrac{2.61}{\sqrt{900}}\right)} = 1.72$

$$|Z| = 1.72$$

(v) Critical value: $|Z_{0.05}| = 1.96$

(vi) Decision: Since $|Z| < |Z_{0.05}|$, the null hypothesis is accepted at 5% level of significance i.e., the sample has been drawn from the population with mean $\mu = 3.25$ cm.

95% fiducial limits:

$$\bar{x} \pm 1.96\left(\frac{\sigma}{\sqrt{n}}\right) = 3.4 \pm 1.96\left(\frac{2.61}{\sqrt{900}}\right) = 3.4 \pm 0.1705,$$

i.e., 3.5705 and 3.2295

Example 5

A type company claims that the lives of tyres have mean 42000 km with s.d. of 4000 km. A change in the production process is believed to result in better product. A test sample of 81 new tyres has a mean life of 42500 km. Test at 5% level of significance that the new product is significantly better than the old one.

Solution

$$n = 81,\quad \bar{x} = 42500\text{ km},\quad \mu = 42000\text{ km},\quad \sigma = 4000\text{ km}$$

(i) Null Hypothesis H_0: $\mu = 42000$ km, i.e., the new product is not significantly better than the old one.

(ii) Alternative Hypothesis H_1: $\mu > 42000$ km (Right tailed test)

(iii) Level of significance: $\alpha = 0.05$

(iv) Test statistic: $Z = \dfrac{\bar{x}-\mu}{\left(\dfrac{\sigma}{\sqrt{n}}\right)} = \dfrac{42500-42000}{\left(\dfrac{4000}{\sqrt{81}}\right)} = 1.125$

$$|Z| = 1.125$$

(v) Critical value: $Z_{0.05} = 1.645$

(vi) Decision: Since $|Z| < Z_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., the new product is not significantly better than the old one.

Example 6

The mean breaking strength of cables supplied by a manufacturer is 1800 with standard deviation 100. By a new technique in the manufacturing process it is claimed that the breaking strength of the cable has increased. In order to test the claim a sample of 50 cables is tested. It is found that the mean breaking strength is 1850. Can we support the claim at 1% level of significance?

Solution

$$n = 50,\quad \bar{x} = 1850,\quad \mu = 1800,\quad \sigma = 100$$

(i) Null Hypothesis H_0: $\mu = 1800$, i.e., the mean breaking strength of cables supplied by manufacturer is 1800.

(ii) Alternative Hypothesis H_1: $\mu > 1800$ (Right tailed test)

(iii) Level of significance: $\alpha = 0.01$

(iv) Test statistic: $Z = \dfrac{\bar{x}-\mu}{\left(\dfrac{\sigma}{\sqrt{n}}\right)} = \dfrac{1850-1800}{\left(\dfrac{100}{\sqrt{50}}\right)} = 3.54$

$$|Z| = 3.54$$

(v) Critical value: $Z_{0.01} = 2.33$

(vi) Decision: Since $|Z| > Z_{0.01}$, the null hypothesis is rejected at 1% level of significance, i.e., the mean breaking strength of cables supplied is more than 1800.

Example 7

An ambulance service claims that it takes on the average 10 minutes to reach its destination in emergency calls. A sample of 36 calls has a *mean of 11 minutes and the variance of 16 minutes. Test the claim at 0.05 level of significance.*

Solution

$$n = 36, \quad \bar{x} = 11 \text{ minutes}, \quad \mu = 10 \text{ minutes}, \quad s = \sqrt{16} = 4 \text{ minutes}$$

(i) Null Hypothesis H_0: $\mu = 10$ minutes, i.e., ambulance service takes 10 minutes to reach the destination.

(ii) Alternative Hypothesis H_1: $\mu > 10$ minutes (Right tailed test)

(iii) Level of significance: $\alpha = 0.05$

(iv) Test statistic: $Z = \dfrac{\bar{x}-\mu}{\left(\dfrac{s}{\sqrt{n}}\right)} = \dfrac{11-10}{\left(\dfrac{4}{\sqrt{36}}\right)} = 1.5$

$$|Z| = 1.5$$

(v) Critical value: $Z_{0.05} = 1.645$

(vi) Decision: Since $|Z| < Z_{0.05}$, the null hypothesis is accepted at 5% level of confidence, i.e., the ambulance service takes on the average 10 minutes to reach its destination.

6.8 TEST OF SIGNIFICANCE FOR DIFFERENCE OF MEANS — LARGE SAMPLES

Let $\bar{x}_1$ and $\bar{x}_2$ be the sample means of two independent large random samples with sizes n_1 and n_2 ($n_1 > 30$, $n_2 > 30$) drawn from two populations with means μ_1 and μ_2 and standard deviations σ_1 and σ_2.

Working Rule

(i) Null Hypothesis H_0: $\mu_1 = \mu_2$, i.e., the two samples have been drawn from two different populations having the same means and equal standard deviations.

(ii) Alternative Hypothesis H_1: $\mu_1 \neq \mu_2$ (two tailed test)
or H_1: $\mu_1 < \mu_2$ (one tailed test)
or H_1: $\mu_1 > \mu_2$ (one tailed test)

(iii) Level of significance: Select the level of significance α.

(iv) Test statistic: There are two cases for calculating test statistic.

(a) When the population standard deviations σ_1 and σ_2 are known

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$$

(b) When the population standard deviations σ_1 and σ_2 are not known

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$$

where s_1 and s_2 are sample standard deviations.

(v) Critical Value: Find the critical value (tabulated value) Z_α of Z at the given level of significance.

(vi) Decision: If $|Z| < Z_\alpha$ at the level of significance α, the null hypothesis is accepted. If $|Z| > Z_\alpha$ at the level of significance α, the null hypothesis is rejected.

Note:

1. Null Hypothesis H_0 is rejected when $|Z| > 3$ without mentioning any level of significance.
2. Confidence limits:

(i) 95% confidence limits $= (\bar{x}_1 - \bar{x}_2) \pm 1.96\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}$

(ii) 99% confidence limits $= (\bar{x}_1 - \bar{x}_2) \pm 2.58\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}$

If population standard deviation σ_1 and σ_2 are not known, s_1 and s_2 are used in equations.

Example 1

Test the significance of the difference between the means of two normal population with the same standard deviation from the following data:

	Size	Mean	SD
Sample I	100	64	6
Sample II	200	67	8

Solution

$$n_1 = 100, \quad n_2 = 200, \quad \bar{x}_1 = 64, \quad \bar{x}_2 = 67 \quad s_1 = 6, \quad s_2 = 8$$

(i) Null Hypothesis $H_0 : \mu_1 = \mu_2$ i.e., there is no significant difference between two means.

(ii) Alternative Hypothesis H_1: $\mu_1 \neq \mu_2$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $Z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{64-67}{\sqrt{\dfrac{(6)^2}{200} + \dfrac{(8)^2}{100}}} = -3.31$

$|Z| = 3.31$

(v) Critical value: $|Z_{0.05}| = 1.96$

(vi) Decision: Since $|Z| > |Z_{0.05}|$, the null hypothesis is rejected at 5% level of significance, i.e., the samples do not support the hypothesis that the two population have the same mean although they may have the same standard deviation.

Example 2

The means of simple samples of sizes 1000 and 2000 are 67.5 and 68 cm respectively. Can the samples be regarded as drawn from the same population of S.D. 2.5 cm.

Solution

$n_1 = 1000,\ n_2 = 2000,\ \bar{x}_1 = 67.5$ cm, $\bar{x}_2 = 68$ cm, $\sigma = 2.5$ cm

(i) Null Hypothesis H_0: $\mu_1 = \mu_2$ i.e., the samples have been drawn from the same population of S.D. 2.5 cm

(ii) Alternative Hypothesis H_1 : $\mu_1 \neq \mu_2$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $Z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{\sigma^2}{n_1} + \dfrac{\sigma^2}{n_2}}} = \dfrac{67.5 - 68}{\sqrt{\dfrac{(2.5)^2}{1000} + \dfrac{(2.5)^2}{2000}}} = -5.16$

$|Z| = 5.16$

(v) Critical value: $|Z_{0.05}| = 1.96$

(iv) Decision: Since $|Z| > |Z_{0.05}|$, the null hypothesis is rejected at 5% level of significance, i.e., the samples cannot be regarded as drawn from the same population of SD 2.5 cm.

Example 3

The mean life of a sample of 10 electric bulbs was found to be 1456 hours with SD of 423 hours. A second sample of 17 bulbs chosen from a different batch showed a mean life of 1280 with SD of 398 hours. Is there a significant difference between the means of two batches?

Solution

$n_1 = 10,\ n_2 = 17,\ \bar{x}_1 = 1456$ hours, $\bar{x}_2 = 1280$ hours, $s_1 = 423$ hours, $s_2 = 398$ hours

(i) Null Hypothesis H_0: $\mu_1 = \mu_2$, i.e., there is no significant difference between the means of two batches.

(ii) Alternative Hypothesis H_1: $\mu_1 \neq \mu_2$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $Z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{1456 - 1280}{\sqrt{\dfrac{(423)^2}{10} + \dfrac{(398)^2}{17}}} = 1.07$

$|Z| = 1.07$

(v) Critical value: $|Z_{0.05}| = 1.96$

(vi) Decision: Since $Z < |Z_{0.05}|$, the null hypothesis is accepted at 5% level of significance, i.e., there is no significant difference between the means of two batches.

Example 4

The average of marks scored by 32 boys is 72 with standard deviation 8 while that of 36 girls is 70 with standard deviation 6. Test at 1% level of significance whether the boys perform better than the girls.

Solution

$n_1 = 32,\ n_2 = 36,\ \bar{x}_1 = 72,\ \bar{x}_2 = 70,\ s_1 = 8,\ s_2 = 6$

(i) Null Hypothesis H_0: $\mu_1 = \mu_2$, i.e., there is no significant difference between the performance of boys and girls.

(ii) Alternative Hypothesis H_1: $\mu_1 > \mu_2$ (Right tailed test)

(iii) Level of significance: $\alpha = 0.01$

(iv) Test statistic: $Z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{72 - 70}{\sqrt{\dfrac{(8)^2}{32} + \dfrac{(6)^2}{36}}} = 1.1547$

$|Z| = 1.1547$

(v) Critical value: $Z_{0.01} = 2.33$

(vi) Decision: Since $|Z| < Z_{0.01}$, the null hypothesis is accepted at 1% level of significance, i.e., the boys do not perform better than the girls.

Example 5

A simple sample of heights of 6400 English men has a mean of 170 cm and a s.d. of 6.4 cm, while a simple sample of heights of 1600 Americans has a mean of 172 cm and a s.d. of 63 cm. Do the data indicate that American are, on the average, taller than the English men?

Solution

$n_1 = 1600,\ n_2 = 6400,\ \bar{x}_1 = 172$ cm, $\bar{x}_2 = 170$ cm, $s_1 = 6.3$ cm, $s_2 = 6.4$ cm

(i) Null Hypothesis H_0: $\mu_1 = \mu_2$, i.e., there is no significant difference in heights of Americans and English men.

(ii) Alternative Hypothesis H_1: $\mu_1 > \mu_2$ (Right tailed test)

(iii) Level of significance: $\alpha = 0.01$ (assumption)

(iv) Test statistic: $Z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{172 - 170}{\sqrt{\dfrac{(6.3)^2}{1600} + \dfrac{(6.4)^2}{6400}}} = 11.32$

$|Z| = 11.32$

(v) Critical value: $Z_{0.01} = 2.33$

(vi) Decision: Since $|Z| > Z_{0.01}$, the null hypothesis is rejected at 1% level of significance, i.e., Americans are, on the average, taller than English men.

Example 6

In a certain factory there are two different processes of manufacturing the same item. The average weight in a sample of 250 items produced from one process is found to be 120 gm with a s.d. of 12 gm; the corresponding figures in a sample of 400 items from the other process are 124 gm and 14 gm. Is this difference between the two sample means significant?

Solution

$n_1 = 250,\ n_2 = 400,\ \bar{x}_1 = 120$ gm, $\bar{x}_2 = 124$ gm, $s_1 = 12$ gm, $s_2 = 14$ gm

(i) Null Hypothesis H_0: $\mu_1 = \mu_2$, i.e., there is no significant difference between the two sample means.

(ii) Alternative Hypothesis H_1: $\mu_1 \neq \mu_2$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $Z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{120 - 124}{\sqrt{\dfrac{(12)^2}{250} + \dfrac{(14)^2}{400}}} = -3.87$

$|Z| = 3.87$

(v) Critical value: $|Z_{0.05}| = 1.96$

(vi) Decision: Since $|Z| > |Z_{0.05}|$, the null hypothesis is rejected at 5% level of significance, i.e., there is significant difference between two sample means.

Example 7

The mean height of 50 male students who participate in sports is 68.2 inches with a s.d. of 2.5 inches. The mean height of 50 male students who have not participated in sport is 67.2 inches with a s.d. of 2.8 inches. Test the hypothesis that the height of students who have participated in sports is more than the students who have not participated in sports.

Solution

$n_1 = 50,\ n_2 = 50,\ \bar{x}_1 = 68.2$ inch, $\bar{x}_2 = 67.2$ inch, $s_1 = 2.5$ inch, $s_2 = 2.8$ inch

(i) Null Hypothesis H_0: $\mu_1 = \mu_2$, i.e., there is no significant difference in heights of students who have participated in sports or not.

(ii) Alternative Hypothesis H_1: $\mu_1 > \mu_2$ (Right tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $Z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{68.2 - 67.2}{\sqrt{\dfrac{(2.5)^2}{50} + \dfrac{(2.8)^2}{50}}} = 1.88$

$|Z| = 1.88$

(v) Critical value: $Z_{0.05} = 1.645$

(vi) Decision: Since $|Z| > Z_{0.05}$, the null hypothesis is rejected at 5% level of significance, i.e., the height of students who have participated in sports is more than the students who have not participated in sports.

6.9 TEST OF SIGNIFICANCE FOR DIFFERENCE OF STANDARD DEVIATIONS – LARGE SAMPLES

Let s_1 and s_2 be the standard deviations of two independent large random samples with sizes n_1 and n_2 ($n_1 > 30, n_2 > 30$) drawn from two populations with standard deviations σ_1 and σ_2.

Working Rule

(i) Null Hypothesis H_0: $\sigma_1 = \sigma_2$, i.e., the two samples have been drawn from two different populations having same standard deviations.

(ii) Alternative Hypothesis H_1: $\sigma_1 \neq \sigma_2$ (Two tailed test)

or H_1: $\sigma_1 < \sigma_2$ (One tailed test)

or H_1: $\sigma_1 > \sigma_2$ (One tailed test)

(iii) Level of significance: Select the level of significance α.

(iv) Test statistic: There are two cases for calculating test statistic.

(a) When the population standard deviations σ_1 and σ_2 are known

$$Z = \frac{s_1 - s_2}{\sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}}$$

(a) When the population standard deviations σ_1 and σ_2 are not known

$$Z = \frac{s_1 - s_2}{\sqrt{\frac{s_1^2}{2n_1} + \frac{s_2^2}{2n_2}}}$$

where s_1 and s_2 are sample standard deviation.

(v) Critical value: Find the critical vale (tabulated value) Z_α of Z at the given level of significance.

(vi) Decision: If $|Z| < Z_\alpha$ at the level of significance α, the null hypothesis is accepted. If $|Z| > Z_\alpha$ at the level of significance α, the null hypothesis is rejected.

Example 1

The SD of a random sample of 1000 is found to be 2.6 and the SD of another random sample of 500 is 2.7. Assuming the samples to be independent, find whether the two samples could have come from populations with the same SD.

Solution

$$n_1 = 1000,\ n_2 = 500,\ s_1 = 2.6,\ s_2 = 2.7$$

(i) Null Hypothesis H_0: $\sigma_1 = \sigma_2$, i.e., there is no significant difference between two standard deviations.

(ii) Alternative Hypothesis H_1: $\sigma_1 \neq \sigma_2$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $$Z = \frac{s_1 - s_2}{\sqrt{\frac{s_1^2}{2n_1} + \frac{s_2^2}{2n_2}}} = \frac{2.6 - 2.7}{\sqrt{\frac{(2.6)^2}{2(1000)} + \frac{(2.7)^2}{2(500)}}} = -0.97$$

$$|Z| = 0.97$$

(v) Critical value: $|Z_{0.05}| = 1.96$

(vi) Decision: Since $|Z| < |Z_{0.05}|$, the null hypothesis H_0 is accepted at 5% level of significance, i.e., there is no significance difference between two standard deviations and the two samples could have come from populations with the same SD.

Example 2

Random samples drawn from two countries gave the following data relating to the heights of adult males:

	Country A	Country B
Standard deviation (in inches)	2.58	2.50
Number in samples	1000	1200

Is the difference between the standard deviations significant?

Solution

$$n_1 = 1000,\ n_2 = 1200,\ s_1 = 2.58 \text{ inch},\ s_2 = 2.50 \text{ inch}$$

(i) Null Hypothesis H_0: $\sigma_1 = \sigma_2$, i.e., there is no significant difference between two standard deviations.

(ii) Alternative Hypothesis H_1: $\sigma_1 \neq \sigma_2$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $$Z = \frac{s_1 - s_2}{\sqrt{\frac{s_1^2}{2n_1} + \frac{s_2^2}{2n_2}}} = \frac{2.58 - 2.50}{\sqrt{\frac{(2.58)^2}{2(1000)} + \frac{(2.50)^2}{2(1200)}}} = 0.077$$

$$|Z| = 0.077$$

(v) Critical value: $|Z_{0.05}| = 1.96$

(vi) Decision: Since $|Z| < |Z_{0.05}|$, the null hypothesis is accepted at 5% level of significance, i.e., there is no significance difference between the standard deviations.

Example 3

Examine whether the two samples for which the data are given in the following table could have been drawn from populations with the same SD.

	Size	SD
Sample I	100	5
Sample II	200	7

Solution

$$n_1 = 100,\ n_2 = 200,\ s_1 = 5,\ s_2 = 7$$

(i) Null Hypothesis H_0: $\sigma_1 = \sigma_2$, i.e., the two samples could have been drawn from populations with the same SD.

(ii) Alternative Hypothesis H_1: $\sigma_1 \neq \sigma_2$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $$Z = \frac{s_1 - s_2}{\sqrt{\frac{s_1^2}{2n_1} + \frac{s_2^2}{2n_2}}} = \frac{5-7}{\sqrt{\frac{(5)^2}{2(100)} + \frac{(7)^2}{2(200)}}} = -4.02$$

$$|Z| = 4.02$$

(v) Critical value: $|Z_{0.05}| = 1.96$

(vi) Decision: Since $|Z| > |Z_{0.05}|$, the null hypothesis is rejected at 5% level of significance, i.e., the two samples could not have been drawn from populations with the same SD.

EXERCISE 6.2

1. A random sample of 100 students gave a mean weight of 58 kg with a SD of 4 kg. Test the hypothesis that the mean weight in the population is 60 kg.
[Ans.: The mean weight in the population is not 60 kg]

2. A sample of 400 items is taken from a normal population whose mean is 4 and whose variance is also 4. If the sample mean is 4.45, can the sample be regarded as truly random sample?
[Ans.: Sample cannot be regarded as truly random sample]

3. The mean IQ of a sample of 1600 children was 99. Is it likely that this was a random sample from a population with mean IQ 100 and SD 15?
[Ans.: Sample was not drawn from a population with mean 100 and SD 15]

4. In a random sample of 60 workers, the average time taken by them to get to work is 33.8 minutes with a standard deviation of 6.1 minutes. Can we reject the null hypothesis $\mu = 32.6$ minutes in favour of alternative hypothesis $\mu > 32.6$ at $\alpha = 0.025$ level of significance
[Ans.: The null hypothesis is accepted]

5. It is claimed that a random sample of 49 types has a mean life of 15200 km. This sample was drawn from a population whose mean is 15150 km and a standard deviation of 1200 km. Test the significance at 0.05 level.
[Ans.: The null hypothesis is accepted]

6. An ambulance service claims that it takes on the average less than 10 minutes to reach its destination in emergency calls. A sample of 36 calls has a mean of 11 minutes and the variance of 16 minutes. Test the claim at 0.05 level of significance.
[Ans.: The null hypothesis is accepted]

7. Samples of students were drawn from two universities and from their weights in kilograms, the mean and standard deviations are calculated. Make a large sample test to test the significance of the difference between the means.

	Mean	SD	Size of the sample
University A	55	10	400
University B	57	15	100

[Ans.: There is no significant difference between the means]

8. A researchers wants to know the intelligence of students in a school. He selected two groups of students. In the first group, there are 150 students having mean IQ of 75 with a SD of 15. In the second group there are 250 students having mean IQ of 70 with SD of 20. Test the significance that the groups have come from same population.
[Ans.: The groups have not come from same population]

9. Random samples drawn from two places gave the following data relating to the heights of children:

	Mean height in cm	SD in cm	No. of children in sample
Place A	68.50	2.5	1200
Place B	68.58	3.0	1500

Test at 5% level of significance that the mean height is the same for children at two places.
[Ans.: The mean height is same for children at two places]

10. The mean life of a sample of 10 electric bulbs was found to be 1456 hours with SD of 423 hours. A second sample of 17 bulbs chosen from a different batch showed a mean life of 1280 hours with SD of 398 hours. Is there a significant difference between the means of two batches?
[Ans.: There is no difference between the mean life of two batches]

11. The SD of a random sample of 900 members is 4.6 and that of another independent sample of 1600 members is 4.8. Examine if the two samples could have been drawn from a population with SD 4?
[Ans.: Two samples could have been drawn from a population with SD 4]

12. The variability of two sets of plots is as given below:

	Set of 40 plots	Set of 60 Plots
SD per plot	34 kg	28 kg

Examine whether the difference in the variability in yields is significant.

[Ans.: The difference in the variability in yields is significant]

6.10 SMALL SAMPLE TESTS

If the samples are large ($n > 30$) then the sampling distribution of a statistic is normal. But if the samples are small ($n \le 30$) then the above result does not hold good. For estimation of the parameter as well as for testing a hypothesis, following distributions are used:

(i) Student's t-distribution
(ii) Snedecer's F-distribution
(iii) Chi-square (χ^2) distribution

6.11 STUDENT'S t-DISTRIBUTION

The theory of small or exact sample was developed by Irish statistician William S. Gosset who used to write under pen-name of student. The quantity t is defined as

$$t = \frac{\text{Difference of population parameter and the corresponding statistic}}{\text{Standard error of statistic}}$$

with $(n-1)$ degrees of freedom if the sample size is n.

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n ($n \le 30$) drawn from a normal population with mean μ and SD σ. The student's t statistic is defined by

$$t = \frac{\bar{x}-\mu}{\left(\frac{\mu}{\sqrt{n}}\right)} \quad \text{or} \quad t = \frac{\bar{x}-\mu}{\left(\frac{s}{\sqrt{n-1}}\right)}$$

where $\bar{x}$ is sample mean and $s = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}$ is an unbiased estimate of σ^2. The test statistic $t = \frac{\bar{x}-\mu}{\left(\frac{s}{\sqrt{n-1}}\right)}$ is a random variable having t-distribution with $v = n-1$ degrees of freedom and with probability density function $f(t) = c\left(1+\frac{t^2}{v}\right)^{\frac{-(v+1)}{2}}$, where $v = n-1$ and c is a constant required to make the area under the curve unity, i.e., $\int_{-\infty}^{\infty} f(t)dt = 1$.

The t-distribution is used when (i) the sample size is less than or equal to 30, and (ii) population standard deviation is not known.

6.11.1 Assumptions for t-test

(i) Samples are drawn from normal population and are random.
(ii) The population standard deviation may not be known.
(iii) For testing the equality of two population mean, the population variances are regarded as equal.
(iv) In case of two samples, some adjustments in degrees of freedom for t are made.

6.11.2 Properties of t-distribution

(i) The t-distribution is asymptotic to the x-axis, i.e., it extends to infinity on either side.
(ii) The t-distribution is symmetrical about the mean.
(iii) The shape of the curve varies with the degrees of freedom.
(iv) The larger the number of degrees of freedom, the more closely t-distribution resembles standard normal distribution.
(v) Sampling distribution of t does not depend on population parameter but it depends only on degree of freedom v, i.e., on the sample size.

6.11.3 Applications of t-distribution

The t-distribution has following applications in testing of hypotheses for small samples:

(i) To test the significance of the sample mean, when the population variance σ is not known
(ii) To test the significance of the mean of the sample i.e., to test if the sample mean differs significantly from the population mean
(iii) To test the significance of the difference between two sample means, the population variances being equal and unknown
(iv) To test the significance of an observed sample correlation coefficient

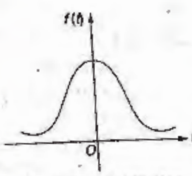

Fig. 6.5 t-distribution curve

6.12 t-TEST: TEST OF SIGNIFICANCE FOR SINGLE MEAN

If $x_1, x_2, \ldots, x_n$ is a random sample of size n ($n \le 30$) drawn from a normal population with mean μ and SD σ and if the sample mean $\bar{x}$ differs significantly from the population mean μ then the student's t statistic is given by

$$t = \frac{\bar{x}-\mu}{\left(\frac{s}{\sqrt{n-1}}\right)}, \quad \text{where } s = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}} \quad \text{with } \nu = n-1$$

Note: Confidence Limit

(i) 95% confidence limits $= \bar{x} \pm t_{0.05}\left(\frac{s}{\sqrt{n-1}}\right)$

where $t_{0.05}$ is the 5% critical value of t for $\nu = n-1$ degree of freedom for a Two tailed test.

(ii) 99% confidence limits $= \bar{x} \pm t_{0.01}\left(\frac{s}{\sqrt{n-1}}\right)$

where $t_{0.01}$ is the 1% critical value of t for $\nu = n-1$ degree of freedom for a Two tailed test.

Example 1

A machinist is making engine parts with axle diameter of 0.7 cm. A random sample of 10 parts shows a mean diameter of 0.742 cm with a standard deviation of 0.04 cm. Compute the statistic you would use to test whether work is meeting the specification at 0.05 level of significance.

Solution

$$n = 10, \quad \bar{x} = 0.742 \text{ cm}, \quad s = 0.04 \text{ cm} \quad \mu = 0.7 \text{ cm}$$

(i) Null Hypothesis H_0: $\mu = 0.7$ cm, i.e., the product is meeting the specification.

(ii) Alternative Hypothesis H_1: $\mu \neq 0.7$ cm (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$

(iv) Test statistic: $t = \frac{\bar{x}-\mu}{\left(\frac{s}{\sqrt{n-1}}\right)} = \frac{0.742-0.7}{\left(\frac{0.04}{\sqrt{10-1}}\right)} = 3.15$

$$|t| = 3.15$$

(v) Critical value: $\nu = n-1 = 10-1 = 9$

$$t_{0.05}(\nu = 9) = 2.262$$

(vi) Decision: Since $|t| > t_{0.05}$, the null hypothesis is rejected at 5% level of significance i.e., the product is not meeting the specification.

Example 2

Ten objects are chosen at random from a large population and their weights are found to be in grams: 63, 63, 64, 65, 66, 69, 69, 70, 70, 71. Discuss the suggestion that the mean weight is 65 g.

Solution

$$n = 10, \mu = 65 \text{ g}$$

$$\left.\begin{array}{l}\bar{x} = 67 \text{ g} \\ s = 2.966 \text{ g}\end{array}\right\} \text{From calculator}$$

(i) Null Hypothesis H_0: $\mu = 65$ g, i.e., there is no significant difference in the mean weight of sample and population.

(ii) Alternate Hypothesis H_1: $\mu \neq 65$ g (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $t = \frac{\bar{x}-\mu}{\left(\frac{s}{\sqrt{n-1}}\right)} = \frac{67-65}{\left(\frac{2.966}{\sqrt{10-1}}\right)} = 2.023$

$$|t| = 2.023$$

(v) Critical value: $\nu = n-1 = 10-1 = 9$

$$t_{0.05}(\nu = 9) = 2.262$$

(vi) Decision: Since $|t| < t_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., the mean weight is 65 g.

Example 3

The mean lifetime of a sample of 25 bulbs is found as 1550 hours with a SD of 120 hours. The company manufacturing the bulbs claims that the average life of their bulbs is 1600 hours. Is the claim acceptance at 5% level of significance?

Solution

$$n = 25, \quad \bar{x} = 1550 \text{ hours}, \quad s = 120 \text{ hours}, \quad \mu = 1600 \text{ hours}$$

(i) Null Hypothesis H_0: $\mu = 1600$ hours, i.e., the average life of bulbs is 1600 hours.

(ii) Alternative Hypothesis H_1: $\mu < 1600$ hours (One tailed test)

(iii) Level of significance: $\alpha = 0.05$

(v) Critical value: $v = n - 1 = 10 - 1 = 9$

$$t_{0.05}\,(v = 9) = 2.262$$

(vi) Decision: Since $|t| < t_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., population has mean IQ of 100.

$$95\%\text{ confidence limits} = \bar{x} \pm t_{0.05}\left(\frac{s}{\sqrt{n-1}}\right)$$

$$= 97.2 \pm 2.262\left(\frac{13.54}{\sqrt{10-1}}\right)$$

$$= 97.2 \pm 10.21$$

$$= 87 \text{ and } 107.41$$

Example 7

The heights of 10 males of a given locality are found to be 175, 168, 155, 170, 152, 170, 175, 160, 160 and 165 cm. Based on this sample, find the 95% confidence limits for the heights of males in that locality.

Solution

$$n = 10$$

$$\left.\begin{array}{l}\bar{x} = 165\\ s = 7.6\end{array}\right\}\text{ From calculator}$$

$$v = n - 1 = 10 - 1 = 9$$

From t-table

$$t_{0.05}(v = 9) = 2.262 \text{ (Two tailed test)}$$

The 95% confidence limits for μ are

$$\left[\bar{x} - t_{0.05}\left(\frac{s}{\sqrt{n-1}}\right),\ \bar{x} + t_{0.05}\left(\frac{s}{\sqrt{n-1}}\right)\right]$$

i.e.,
$$\left[165 - \frac{2.262(7.6)}{\sqrt{10-1}},\ 165 + \frac{2.262(7.6)}{\sqrt{10-1}}\right]$$

i.e., [159.27, 170.73]

i.e., the heights of males in the locality are likely to be in limits 159.27 cm and 170.73 cm.

6.13 *t*-TEST: TEST OF SIGNIFICANCE FOR DIFFERENCE OF MEANS

Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be two independent samples of sizes n_1 and n_2 ($n_1 \le 30$, $n_2 \le 30$) with means $\bar{x}$ and $\bar{y}$ and standard deviations s_1 and s_2 from a

normal population with means μ_1 and μ_2 and same standard deviations. The student's t statistic is given by

$$t = \frac{\bar{x} - \bar{y}}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \quad \text{with } v = n_1 + n_2 - 2$$

where
$$\bar{x} = \frac{\Sigma x}{n_1}$$

$$\bar{y} = \frac{\Sigma y}{n_2}$$

and
$$s = \sqrt{\frac{\Sigma(x-\bar{x})^2 + \Sigma(y-\bar{y})^2}{n_1 + n_2 - 2}}$$

In terms of standard deviations s_1 and s_2,

$$t = \frac{\bar{x} - \bar{y}}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

where
$$s = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}}$$

and
$$s_1 = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n_1}}$$

$$s_2 = \sqrt{\frac{\Sigma(y-\bar{y})^2}{n_2}}$$

Note

1. If $n_1 = n_2 = n$ and the samples are independent, i.e., the observations in the two samples are not all related then test statistic is given by

$$t = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{s_1^2 + s_2^2}{n-1}}} \quad \text{with } v = 2n - 2$$

2. If $n_1 = n_2 = n$ and if the pairs of values of x and y are associated or correlated in some way (or not independent), the above formula for testing of hypothesis cannot be used.

   Let $d_i = x_i - y_i$ denote the difference (with proper sign) in the values of x and y for the ith pair ($i = 1, 2, \ldots, n$).

   The test statistics is given by

$$t = \frac{\bar{d}}{\left(\frac{s}{\sqrt{n-1}}\right)} \quad \text{with } v = n - 1$$

where $\bar{d}$ and s denote the mean and standard deviation of the difference d_i respectively, i.e.,

$$\bar{d} = \frac{\sum d_i}{n}$$

$$s = \sqrt{\frac{\sum (d_i - \bar{d})^2}{n}} = \frac{\sum d_i^2}{n} - \left(\frac{\sum d_i}{n}\right)^2$$

(3) Confidence Limits

(i) 95% confidence limits $= (\bar{x} - \bar{y}) \pm t_{0.05}\left(\frac{1}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}\right)$

where $t_{0.05}$ is the 5% critical value of t for $v = n_1 + n_2 - 2$ degree of freedom for a Two tailed test.

(ii) 99% confidence limits $= (\bar{x} - \bar{y}) \pm t_{0.01}\left(\frac{1}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}\right)$

where $t_{0.01}$ is the 1% critical value of t for $v = n_1 + n_2 - 2$ degree of freedom for a Two tailed test.

Example 1

The means of two random samples of size 9 and 7 are 196.42 and 198.82 respectively. The sum of squares of the deviation from the mean are 26.94 and 18.73 respectively. Can the sample be considered to have been drawn from the same population?

Solution

$$n_1 = 9,\ n_2 = 7,\ \bar{x} = 196.42,\ \bar{y} = 198.82$$

$$\sum (x - \bar{x})^2 = 26.94,\ \sum (y - \bar{y})^2 = 18.73$$

$$s = \sqrt{\frac{\sum (x - \bar{x})^2 + \sum (y - \bar{y})^2}{n_1 + n_2 - 2}} = \sqrt{\frac{26.94 + 18.73}{9 + 7 - 2}} = 1.806$$

(i) Null Hypothesis $H_0: \mu_1 = \mu_2$, i.e., the samples are drawn from the same population.

(ii) Alternative Hypothesis $H_1: \mu_1 \neq \mu_2$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $t = \frac{\bar{x} - \bar{y}}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{196.42 - 198.82}{1.8061\sqrt{\frac{1}{9} + \frac{1}{7}}} = -2.6368$

$$|t| = 2.6368$$

(v) Critical value: $v = n_1 + n_2 - 2 = 9 + 7 - 2 = 14$

$$t_{0.05}(v = 14) = 2.145$$

(vi) Decision: Since $|t| > t_{0.05}$, the null hypothesis is rejected at 5% level of significance, i.e., the samples are not drawn from the same population.

Example 2

Samples of two types of electric bulbs were tested for length of life and the following data were obtained.

	Size	Mean	SD
Sample 1	8	1234 hr	36 hr
Sample 2	7	1036 hr	40 hr

Is the difference in the means sufficient to warrant that type 1 bulbs are superior to type 2 bulbs?

Solution

$$n_1 = 8,\ n_2 = 7,\ \bar{x}_1 = 1234\text{ hr},\ \bar{x}_2 = 1036\text{ hr}$$

$$s_1 = 36\text{ hr},\ s_2 = 40\text{ hr}$$

$$s = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{(8)(36)^2 + (7)(40)^2}{8 + 7 - 2}} = 40.73\text{ hr}$$

(i) Null Hypothesis $H_0: \mu_1 = \mu_2$, i.e., the type 1 bulbs are not superior to type 2 bulbs.

(ii) Alternative Hypothesis $H_1: \mu_1 > \mu_2$ (One tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $t = \frac{\bar{x}_1 - \bar{x}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{1234 - 1036}{40.73\sqrt{\frac{1}{8} + \frac{1}{7}}} = 9.39$

$$|t| = 9.39$$

(v) Critical value: $v = n_1 + n_2 - 2 = 8 + 7 - 2 = 13$

$$t_{0.05}(v = 13) = 1.771$$

(vi) Decision: Since $|t| > t_{0.05}$, the null hypothesis is rejected at 5% level of significance, i.e., the type 1 bulbs are superior to type 2 bulbs.

Example 3

The mean height and SD height of 8 randomly chosen soldiers are 166.9 cm and 8.29 cm respectively. The corresponding values of 6 randomly chosen sailors are 170.3 cm and 8.50 cm respectively. Based on this data, can we conclude that soldiers are, in general, shorter than sailors?

Solution

$$n_1 = 8,\ n_2 = 6,\ \bar{x}_1 = 166.9 \text{ cm},\ \bar{x}_2 = 170.3 \text{ cm}$$
$$s_1 = 8.29 \text{ cm},\ s_2 = 8.50 \text{ cm}$$
$$s = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{(8)(8.29)^2 + (6)(8.50)^2}{8+6-2}} = 9.05 \text{ cm}$$

(i) Null Hypothesis H_0: $\mu_1 = \mu_2$, i.e., there is no significant difference between the heights of soldiers and sailors.

(ii) Alternative Hypothesis H_1: $\mu_1 < \mu_2$ (One tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $t = \dfrac{\bar{x}_1 - \bar{x}_2}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \dfrac{166.9 - 170.3}{9.05\sqrt{\dfrac{1}{8} + \dfrac{1}{6}}} = -0.696$

$|t| = 0.696$

(v) Critical value: $v = n_1 + n_2 - 2 = 8 + 6 - 2 = 12$

$t_{0.05}\,(v = 12) = 1.782$

(vi) Decision: Since $|t| < t_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., there is no significant difference between the heights of soldiers and sailors and we cannot conclude that sailors are, in general, shorter than sailors.

Example 4

Two types of batteries are tested for their length of life and the following data are obtained:

	No. of Samples	Mean life in hours	Variance
Type A	9	600	121
Type B	8	640	144

Is there a significant difference in the two means? Find 95% confidence limits for the difference in means.

Solution

$$n_1 = 9,\ n_2 = 8,\ \bar{x}_1 = 600 \text{ hours},\ \bar{x}_2 = 640 \text{ hours}$$
$$s_1 = \sqrt{121} = 11 \text{ hours},\ s_2 = \sqrt{144} = 12 \text{ hours}$$
$$s = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{(9)(121) + (8)(144)}{9+8-2}} = 12.22 \text{ hours}$$

(i) Null Hypothesis H_0: $\mu_1 = \mu_2$, i.e., there is no significant difference in two means.

(ii) Alternative Hypothesis H_1: $\mu_1 \neq \mu_2$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $t = \dfrac{\bar{x}_1 - \bar{x}_2}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \dfrac{600 - 640}{12.22\sqrt{\dfrac{1}{9} + \dfrac{1}{8}}} = -6.74$

$|t| = 6.74$

(v) Critical value: $v = n_1 + n_2 - 2 = 9 + 8 - 2 = 15$

$t_{0.05}\,(v = 15) = 2.132$

(vi) Decision: Since $|t| > t_{0.05}$, the null hypothesis is rejected at 5% level of confidence, i.e., there is significant difference in the two means.

$$\begin{aligned}
\text{95\% confidence limits for } (\mu_1 - \mu_2) &= (\bar{x}_1 - \bar{x}_2) \pm t_{0.05}\left(s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right) \\
&= (600 - 640) \pm 2.132\left(12.22\sqrt{\frac{1}{9} + \frac{1}{8}}\right) \\
&= -40 \pm 12.66 \\
&= -27.34 \text{ and } -52.66
\end{aligned}$$

Example 5

A group of 5 patients treated with medicine A weigh 42, 39, 48, 60 and 41 kg. Second group of 7 patients from the same hospital treated with *medicine B weigh 38, 42, 56, 64, 68, 69 and 62 kg. Do you agree with the claim that medicine B increases the weight significantly?*

Solution

$$\left.\begin{aligned} n_1 &= 5, n_2 = 7 \\ \bar{x} &= 46 \text{ kg} \\ \bar{y} &= 57 \text{ kg} \\ s_1 &= 7.62 \text{ kg} \\ s_2 &= 11.5 \text{ kg} \end{aligned}\right\} \text{From calculator}$$

$$s = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{(5)(7.62)^2 + (7)(11.5)^2}{5+7-2}} = 11.03 \text{ kg}$$

(i) Null Hypothesis $H_0 : \mu_1 = \mu_2$, i.e., there is no significant difference between the medicines A and B as regards their effect on the increase in weight.

(ii) Alternative Hypothesis $H_1 : \mu_1 > \mu_2$ (One tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic $t = \dfrac{\bar{x} - \bar{y}}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \dfrac{46 - 57}{11.03\sqrt{\dfrac{1}{5} + \dfrac{1}{7}}} = -1.7$

$|t| = 1.7$

(v) Critical value: $v = n_1 + n_2 - 2 = 5 + 7 - 2 = 10$

$t_{0.05}\,(v = 10) = 1.812.$

(vi) Decision: Since $|t| < t_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., the medicines A and B do not differ significantly as regards their effect on increase in weight.

Example 6

The following data represent the biological values of protein from cow's milk and buffalo's milk at a certain level:

Cow's milk	1.82	2.02	1.88	1.61	1.81	1.54
Buffalo's milk	2.00	1.83	1.86	2.03	2.19	1.88

Examine if the average values of protein in the two samples significantly differ.

Solution

Here, $n_1 = n_2 = 6$ and two samples are independent.

$$\left.\begin{aligned} n &= 6 \\ \bar{x}_1 &= 1.78 \\ \bar{x}_2 &= 1.965 \\ s_1 &= 0.16 \\ s_2 &= 0.124 \end{aligned}\right\} \text{From calculator}$$

(i) Null Hypothesis H_0: $\mu_1 = \mu_2$, i.e., there is no significant difference in average values of proteins in two milk samples.

(ii) Alternative Hypothesis H_1: $\mu_1 \neq \mu_2$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $t = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2 + s_2^2}{n-1}}} = \dfrac{1.78 - 1.965}{\sqrt{\dfrac{(0.16)^2 + (0.124)^2}{6-1}}} = -2.043$

$|t| = 2.043$

(v) Critical value: $v = 2n - 2 = 2(6) - 2 = 10$

$t_{0.05}\,(v = 10) = 2.228$

(vi) Decision: Since $|t| < t_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., there is no significant difference in average values of proteins in two milk samples.

Example 7

A certain injection administered to 12-patients resulted in the following changes of blood pressure:

5, 2, 8, −1, 3, 0, 6, −2, 1, 5, 0, 4

Can it be concluded that the injection will be in general accompanied by an increase in blood pressure?

Solution

Here, 'the changes' $d = x - y$ in blood pressure are given, i.e., x is the final blood pressure after administering the injection and y is the initial blood pressure. It is required to test whether the mean blood pressure has increased, i.e., μ_1 is greater than μ_2.

$$n = 12, \quad \sum d_i = 31, \quad \sum d_i^2 = 185$$

$$\bar{d} = \frac{\sum d_i}{n} = \frac{31}{12} = 2.58$$

$$s = \sqrt{\frac{\sum d_i^2}{n} - \left(\frac{\sum d_i}{n}\right)^2} = \sqrt{\frac{185}{12} - \left(\frac{31}{12}\right)^2} = 2.96$$

(i) Null Hypothesis H_0: $\mu_1 = \mu_2$, i.e., mean blood pressure has not increased.

(ii) Alternative Hypothesis H_1: $\mu_1 > \mu_2$ (One tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $t = \dfrac{\bar{d}}{\left(\dfrac{s}{\sqrt{n-1}}\right)} = \dfrac{2.58}{\left(\dfrac{2.96}{\sqrt{12-1}}\right)} = 2.89$

$$|t| = 2.89$$

(v) Critical value: $v = n - 1 = 12 - 1 = 11$

$$t_{0.05}\,(v = 11) = 1.796$$

(vi) Decision: Since $|t| > t_{0.05}$, the null hypothesis is rejected, i.e., injection will in general accompanied by an increase in blood pressure.

Example 8

Scores obtained in a shooting competition by 10 soldiers before and after intensive training are given below:

Score before training	67	24	57	55	63	54	56	68	33	43
Score after training	70	38	58	58	56	67	68	75	42	38

Test whether the intensive training is useful at 0.05 level of significance.

Solution

Since both the scores belongs to same set of soldiers, scores can be regarded as correlated and no longer independent. Paired t-test is applied to check hypothesis.

$$n_1 = n_2 = n = 10$$

Calculation of paired-t

x	67	24	57	55	63	54	56	68	33	43
y	70	38	58	58	56	67	68	75	42	38
$d = x - y$	−3	−14	−1	−3	7	−13	−12	−7	−9	5
d^2	9	196	1	9	49	169	144	49	81	25

$$\sum d_i = -50, \quad \sum d_i^2 = 732$$

$$\bar{d} = \frac{\sum d_i}{n} = \frac{-50}{10} = -5$$

$$s = \sqrt{\frac{\sum d_i^2}{n} - \left(\frac{\sum d_i}{n}\right)^2} = \sqrt{\frac{732}{10} - \left(\frac{-50}{10}\right)^2} = 6.94$$

(i) Null Hypothesis H_0: $\mu_1 = \mu_2$, i.e., there is no significant effect of intensive training.

(ii) Alternative Hypothesis H_1: $\mu_1 < \mu_2$ (One tailed test)

(iii) Level of significance: $\alpha = 0.05$

(iv) Test statistic: $t = \dfrac{\bar{d}}{\left(\dfrac{s}{\sqrt{n-1}}\right)} = \dfrac{-5}{\left(\dfrac{6.94}{\sqrt{10-1}}\right)} = -2.16$

$$|t| = 2.16$$

(v) Critical value: $v = n - 1 = 10 - 1 = 9$

$$t_{0.05}\,(v = 9) = 1.96$$

(vi) Decision: Since $|t| > t_{0.05}$, the null hypothesis is rejected at 5% level of significance, i.e., intensive training is useful.

6.14 t-TEST: TEST OF SIGNIFICANCE FOR CORRELATION COEFFICIENTS

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be n pairs of observations of a random sample from a bivariate normal population and let r be the observed correlation coefficient in the sample. It is required to test if this sample correlation coefficient is significant of any correlation in the population, i.e., whether the value of the population correlation coefficient ρ is zero and the observed value of r has arisen due to fluctuation of sampling. The student's t statistic is given by

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \text{ with } v = n - 2$$

Example 1

A random sample of 18 pairs of observations from a bivariate normal population gives a correlation coefficient of 0.3. Is it likely that variables are uncorrelated in the population?

Solution

$$n = 18, \quad r = 0.3$$

(i) Null Hypothesis H_0: $\rho = 0$, i.e., the variables are uncorrelated.

(ii) Alternative Hypothesis H_1: $\rho \neq 0$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $t = \dfrac{r\sqrt{n-2}}{\sqrt{1-r^2}} = \dfrac{0.3\sqrt{18-2}}{\sqrt{1-(0.3)^2}} = 1.26$

$$|t| = 1.26$$

(v) Critical value: $v = n - 2 = 18 - 2 = 16$

$t_{0.05}(v = 16) = 2.12$

(vi) Decision: Since $|r| < t_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., the variables are uncorrelated in the population.

Example 2

A random sample of 10 nations gives a correlation coefficient of 0.5 between literacy rate and political stability. Is the relationship significant?

Solution

$n = 10, \quad r = 0.5$

(i) Null Hypothesis $H_0 : \rho = 0$, i.e., there is no relationship between literacy rate and political stability.

(ii) Alternative Hypothesis $H_1 : \rho \neq 0$ (Two tailed test)

(iii) Level of significance $\alpha = 0.5$ (assumption)

(iv) Test statistic: $t = \dfrac{r\sqrt{n-2}}{\sqrt{1-r^2}} = \dfrac{0.5\sqrt{10-2}}{\sqrt{1-(0.5)^2}} = 1.63$

$|t| = 1.63$

(v) Critical value: $v = n - 2 = 10 - 2 = 8$

$t_{0.05}(v = 8) = 2.306$

(vi) Decision: Since $|t| < t_{0.05}$, the null hypothesis is accepted at 5% level of significance i.e., there is no relationship between literacy rate and political stability.

Example 3

Find the least value of r in samples of 18 pairs of observations from a bivariate normal population, which is significant at 5% level.

Solution

The value of r for $n = 18$ will be significant at 5% level if

$$\left|\frac{r\sqrt{n-2}}{\sqrt{1-r^2}}\right| \geq t_{0.05}(v = 16)$$

$$\left|\frac{r\sqrt{n-2}}{\sqrt{1-r^2}}\right| \geq 2.12$$

Squaring both the sides and putting $n = 18$,

$$\frac{r^2(18-2)}{1-r^2} \geq 4.5$$

$$16r^2 \geq 4.5 - 4.5r^2$$

$$20.5r^2 \geq 4.5$$

$$r^2 \geq 0.22$$

$$|r| \geq 0.47$$

Hence, the least value of r is 0.47 (numerically).

EXERCISE 6.3

1. A sample of 26 bulbs gives a mean life of 990 hours with a SD of 20 hours. The manufacturer claims that the mean life of bulbs is 1000 hours. Is the sample not up to standard?

   [Ans.: The sample is not up to the standard]

2. The average breaking strength of the steel rods is specified to be 18.5 thousand pounds. To test this, sample of 14 rods were tested. The mean and SD obtained were 17.85 and 1.955 respectively. Is the result of experiment significant?

   [Ans.: The result of experiment is not significant]

3. A random sample of six steel beams has a mean compressive strength of 58392 psi (pounds per square inch) with a SD of 648 psi. Use this information and level of significance $\alpha = 0.05$ to test whether the true average compressive strength of the steel from which this sample came is 58000 psi. Assume normality.

   [Ans.: The average compressive strength of the steel beam is not equal to 58000 psi]

4. A sample of 155 members has a mean of 67 and SD of 52. Is this sample has been taken from a large population of mean 70?

   [Ans.: The sample has not been taken from the given population]

5. The heights of 10 males of a given locality are found to be 70, 67, 62, 68, 61, 68, 70, 64, 64, 66 inches. Is it reasonable to believe that the average height is greater than 64 inches? Test at 5% significance level assuming that for 9 degrees of freedom $t = 1.833$ at $\alpha = 0.05$.

   [Ans.: The average height is greater than 64 inches]

6. A random sample from a company's very extensive files shows that the orders for a certain kind of machinery were filled respectively in 10, 12, 19, 14, 15, 18, 11 and 13 days. Use the level of significance $\alpha = 0.01$ to test the claim that on the average such orders are filled in 10.5 days.

Choose the alternative hypothesis so that rejection of null hypothesis $\mu = 10.5$ days implies that it takes longer than indicated.

[Ans.: The orders on average are filled in more than 10.5 days]

7. Producer of gutkha claims that the nicotine content in his gutkha on the average is 1.83 mg. Can this claim be accepted if a random sample of 8 gutkha of this type have the nicotine contents of 2, 1.7, 2.1, 1.9, 2.2, 2.1, 2, 1.6 mg? Use a 0.05 level of significance.

[Ans.: The null hypothesis is accepted]

8. Two horses *A* and *B* were tested according to the time (in seconds) to run a particular track with the following results:

Horse A	28	30	32	33	33	29	34
Horse B	29	30	30	24	27	29	

Test whether the two horses have the same running capacity.

[Ans.: The two horses do not have the same running capacity]

9. To examine the hypothesis that the husbands are more intelligent than the wives, an investigator took a sample of 10 couples and administered them a test which measures the IQ. The results are as follows:

Husbands	117	105	97	105	123	109	86	78	103	107
Wives	106	98	87	104	116	95	90	69	108	85

Test the hypothesis with a reasonable test at the level of significance of 0.05.

[Ans.: These is no significant difference in IQs]

10. Two independent samples of 8 and 7 items respectively had the following values:

Sample I	11	11	13	11	15	9	12	14
Sample II	9	11	10	13	9	8	10	-

Is the difference between the means of samples significant?

[Ans.: The difference between the mean of samples is not significant]

11. Random samples of specimens of coal from two mines A and B are drawn and their heat-producing capacity (in millions of calories/ton) were measured yielding the following results:

Mine A	8350	8070	8340	8130	8260	-
Mine B	7900	8140	7920	7840	7890	7950

Is there significant difference between the means of these two samples at 0.01 level of significance?

[Ans.: There is significant difference between the means of two samples]

12. A random sample of 27 pairs of observations from a bivariate normal population gives a correlation coefficient of 0.42. Is it likely that the variables are uncorrelated in the population?

[Ans.: correlated]

13. Find the least value of r in a sample of 27 pairs from a bivariate normal population which is significant at 5% level.

[Ans.: $|r| = 0.487$]

6.15 SNEDECOR'S *F*-TEST FOR RATIO OF VARIANCES

Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be the values of two independent random samples of sizes n_1 and n_2 ($n_1 \le 30$, $n_2 \le 30$) with means $\bar{x}$ and $\bar{y}$ drawn from the normal population with mean μ and standard deviation σ. The test statistic of Snedecor's *F*-test in terms of unbiased estimates of standard deviations S_1 and S_2 of population is given by

$$F = \frac{S_1^2}{S_2^2}, \quad \text{where } S_1^2 > S_2^2$$

and

$$S_1^2 = \frac{\sum (x-\bar{x})^2}{n_1 - 1}$$

$$S_2^2 = \frac{\sum (y-\bar{y})^2}{n_2 - 1}$$

with numerator degree of freedom $v_1 = n - 1$ and denominator degree of freedom $v_2 = n_2 - 1$.

If s_1 and s_2 are standard deviations of samples then

$$s_1^2 = \frac{\sum (x-\bar{x})^2}{n_1}$$

$$s_2^2 = \frac{\sum (y-\bar{y})^2}{n_2}$$

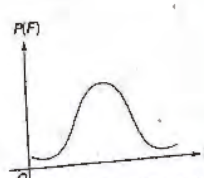

Fig. 6.6 F-distribution curve

$$\therefore \quad \sum (x-\bar{x})^2 = n_1 s_1^2$$

$$\sum (y-\bar{y})^2 = n_2 s_2^2$$

Substituting in S_1^2 and S_2^2,

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1}$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1}$$

The Snedecor's F-distribution is defined by

$$P(F) = cF^{\left(\frac{\nu_1-2}{2}\right)}\left(1+\frac{\nu_1}{\nu_2}F\right)^{-\left(\frac{\nu_1+\nu_2}{2}\right)}$$

where the constant c depends on ν_1 and ν_2. It is so chosen that the area under the curve is unity.

6.15.1 Properties of F-distribution

(i) F-distribution curve lies entirely in the first quadrant and is unimodal.
(ii) F-distribution is independent of the population variance σ^2 and depends on ν_1 and ν_2 only.
(iii) The mode of F-distribution is less than unity.
(iv) $F_{1-\alpha}(\nu_1, \nu_2) = \dfrac{1}{F_\alpha(\nu_2, \nu_1)}$
where $F_\alpha(\nu_2, \nu_1)$ is the value of F with ν_2 and ν_1 degrees of freedom such that the area under the F-distribution curve right of F_α is α.
(v) F-test is one tailed test (right tailed test).

6.15.2 Test of Significance for Ratio of Variances

Significant test is performed by means of Snedecer's F-table which provides 5% and 1% of points of significance for F. 5% points of F means that the area under the F-curve, to the right of the ordinate at a value of F, is 0.05. Further, F-table gives only single tail test. F-distribution is very useful for testing the equality of population means by comparing sample variances.

Working Rule

(i) Null Hypothesis $H_0: \sigma_1^2 = \sigma_2^2$
(ii) Alternative Hypothesis $H_1: \sigma_1^2 > \sigma_2^2$
(iii) Level of significance: Select the level of significance
(iv) Test statistic: $F = \dfrac{S_1^2}{S_2^2}$ where $S_1^2 > S_2^2$
(v) Critical value: Find the critical value (tabulated value) F_α at the given level of significance at degree of freedoms,

$$\nu_1 = n_1 - 1$$
$$\nu_2 = n_2 - 1$$

(vi) Decision: If $F < F_\alpha$ at the level of significance α, the null hypothesis is accepted. If $F > F_\alpha$ at the level of significance α, the null hypothesis is rejected.

Example 1

In two independent samples of sizes 8 and 10, the sum of squares of deviations of the sample values from the respective means were 84.4 and 102.6. Test whether the difference of variances of the population is significant or not. Use a 0.05 level of significance.

Solution

$$n_1 = 8, \quad n_2 = 10$$
$$\Sigma(x-\bar{x})^2 = 84.4, \quad \Sigma(y-\bar{y})^2 = 102.6,$$
$$S_1^2 = \frac{\Sigma(x-\bar{x})^2}{n_1-1} = \frac{84.4}{8-1} = 12.057$$
$$S_2^2 = \frac{\Sigma(y-\bar{y})^2}{n_2-1} = \frac{102.6}{10-1} = 11.4$$

(i) Null Hypothesis $H_0: \sigma_1^2 = \sigma_2^2$, i.e., the variances of two populations are equal.
(ii) Alternative Hypothesis $H_1: \sigma_1^2 > \sigma_2^2$
(iii) Level of significance: $\alpha = 0.05$
(iv) Test statistic: $F = \dfrac{S_1^2}{S_2^2} = \dfrac{12.057}{11.4} = 1.057$
(v) Critical value: $\nu_1 = n_1 - 1 = 8 - 1 = 7$

$$\nu_2 = n_2 - 1 = 10 - 1 = 9$$
$$F_{0.05}(\nu_1 = 7, \nu_2 = 9) = 3.29$$

(vi) Decision: Since $F < F_{0.05}$, the null hypothesis is accepted at 0.05 level of significance, i.e., there is no significant difference in variances of the population.

Example 2

The standard deviations calculated from two random samples of sizes 9 and 13 are 2.1 and 1.8 respectively. Can the samples be regarded as drawn from normal populations with the same SD?

Solution

$$n_1 = 9, \; n_2 = 13, \; s_1 = 2.1, \; s_2 = 1.8$$
$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{9(2.1)^2}{9-1} = 4.96$$
$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{13(1.8)^2}{13-1} = 3.51$$

(i) Null Hypothesis H_0: $\sigma_1^2 = \sigma_2^2$, i.e., variances of two populations are equal.

(ii) Alternative Hypothesis H_1: $\sigma_1^2 > \sigma_2^2$

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $F = \dfrac{S_1^2}{S_2^2} = \dfrac{4.96}{3.51} = 1.41$

(v) Critical value: $v_1 = n_1 - 1 = 9 - 1 = 8$

$$v_2 = n_2 - 1 = 13 - 1 = 12$$
$$F_{0.05}(v_1 = 8,\ v_2 = 12) = 2.85$$

(vi) Decision: Since $F < F_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., the samples can be regarded as drawn from normal population with same SD.

Example 3

Two random samples are drawn from two populations and the following results were obtained:

Sample I	16	17	18	19	20	21	22	24	26	27		
Sample II	19	22	25	25	26	28	29	30	31	32	35	36

Find the variances of the two samples and test whether the two populations have the same variances.

Solution

$$n_1 = 10, \quad n_2 = 12$$

$$\left.\begin{aligned} \bar{x}_1 &= 21 \\ \bar{x}_2 &= 28 \\ s_1 &= 3.55 \\ s_2 &= 4.98 \end{aligned}\right\} \text{From calculator}$$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{10(3.55)^2}{10-1} = 14$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{12(4.98)^2}{12-1} = 27.05$$

(i) Null Hypothesis H_0: $\sigma_1^2 = \sigma_2^2$, i.e., two populations have the same variances.

(ii) Alternative Hypothesis H_1: $\sigma_1^2 > \sigma_2^2$

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: Since $S_2^2 > S_1^2$,

$$F = \frac{S_2^2}{S_1^2} = \frac{27.05}{14} = 1.93$$

(v) Critical value: $v_1 = n_1 - 1 = 10 - 1 = 9$

$$v_2 = n_2 - 1 = 12 - 1 = 11$$
$$F_{0.05}(v_2 = 11,\ v_1 = 9) = 3.10$$

(vi) Decision: Since $F < F_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., two populations have the same variances.

Example 4

In a test given to two groups of students drawn from two normal populations, the marks obtained were as follows:

Group I	18	20	36	50	49	36	34	49	41
Group II	29	28	26	35	30	44	46		

Examine at 5% level, whether the two populations have the same variances.

Solution

$$n_A = 9$$
$$n_B = 7$$

$$\left.\begin{aligned} \bar{x} &= 37 \\ \bar{y} &= 34 \\ s_1 &= 11.225 \\ s_2 &= 7.426 \end{aligned}\right\} \text{From calculator}$$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{9(11.2225)^2}{9-1} = 141.75$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{7(7.426)^2}{7-1} = 64.3$$

(i) Null Hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$, i.e., the two populations have same variances.

(ii) Alternative Hypothesis $H_1 : \sigma_1^2 \neq \sigma_2^2$

(iii) Level of significance: $\alpha = 0.05$

(iv) Test statistic: $F=\dfrac{S_1^2}{S_2^2}=\dfrac{141.75}{64.33}=2.203$

(v) Critical value: $\nu_1=n_1-1=9-1=8$

$\nu_2=n_2-1=7-1=6$

$F_{0.05}(\nu_1=8, \nu_2=6)=4.15$

(vi) Decision: Since $F<F_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., the two populations have the same variances.

Example 5

A group of 10 rats fed on diet A and another group of 8 rats fed on diet B recorded following increase in weight:

Diet A	5	6	8	1	12	4	3	9	6	10	gm
Diet B	2	3	6	8	1	10	2	8			gm

Find, if the variances are significantly different?

Solution

$n_1=10, \quad n_2=8$

$\left.\begin{aligned} s_1&=3.2 \\ s_2&=3.23 \end{aligned}\right\}$ From calculator

$$S_1^2=\frac{n_1s_1^2}{n_1-1}=\frac{10(3.2)^2}{10-1}=11.38$$

$$S_2^2=\frac{n_2s_2^2}{n_2-1}=\frac{8(3.23)^2}{8-1}=11.92$$

(i) Null Hypothesis H_0: $\sigma_1^2=\sigma_2^2$, i.e., the there is no significant difference in variances.

(ii) Alternative Hypothesis H_1: $\sigma_1^2>\sigma_2^2$

(iii) Level of significance: $\alpha=0.05$ (assumption)

(iv) Test statistic: Since $S_2^2>S_1^2$,

$$F=\frac{S_2^2}{S_1^2}=\frac{11.92}{11.33}=1.05$$

(v) Critical value: $\nu_1=n_1-1=10-1=9$

$\nu_2=n_2-1=8-1=7$

$F_{0.05}(\nu_2=7, \nu_1=9)=3.29$

(vi) Decision: Since $F<F_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., the two variances are not significantly different.

Example 6

Two random samples gave the following data:

	Size	*Mean*	*Variance*
Sample I	8	9.6	1.2
Sample II	11	16.5	2.5

Can we conclude that the two samples have been drawn from the same normal population?

Solution

A normal distribution has two parameters, mean μ and variance σ^2. To conclude that the two samples have been drawn from the same normal population, we have to test for

(i) Equality of two means $H_0\ (\mu_1=\mu_2)$ by t-test

(ii) Equality of two variances $H_0\ (\sigma_1^2=\sigma_2^2)$ by F-test.

F-test:

$n_1=8, \quad n_2=11, \quad \bar{x}_1=9.6, \quad \bar{x}_2=16.5, \quad s_1^2=1.2 \quad s_2^2=2.5$

$$S_1^2=\frac{n_1s_1^2}{n_1-1}=\frac{8(1.2)}{8-1}=1.37$$

$$S_2^2=\frac{n_2s_2^2}{n_2-1}=\frac{11(2.5)}{11-1}=2.75$$

(i) Null Hypothesis H_0: $\sigma_1^2=\sigma_2^2$, i.e., variances of two populations are equal.

(ii) Alternative Hypothesis H_1: $\sigma_1^2>\sigma_2^2$

(iii) Level of significance: $\alpha=0.05$ (assumption)

(iv) Test statistic: Since $S_2^2>S_1^2$,

$$F=\frac{S_2^2}{S_1^2}=\frac{2.75}{1.37}=2.007$$

(v) Critical value: $\nu_1=n_1-1=8-1=7$

$\nu_2=n_2-1=11-1=10$

$F_{0.05}(\nu_2=10, \nu_1=7)=3.64$

(vi) Decision: Since $F<F_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., two populations have the same variances.

t-test:

$$s = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{8(1.2) + 11(2.5)}{8 + 11 - 2}} = 1.48$$

(i) Null Hypothesis H_0: $\mu_1 = \mu_2$, i.e., means of two populations are equal.

(ii) Alternative Hypothesis H_1: $\mu_1 \neq \mu_2$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $t = \dfrac{\bar{x}_1 - \bar{x}_2}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \dfrac{9.6 - 16.5}{1.48\sqrt{\dfrac{1}{8} + \dfrac{1}{11}}} = -10.03$

$|t| = 10.03$

(v) Critical value: $v_1 = n_1 + n_2 - 2 = 8 + 11 - 2 = 17$

$t_{0.05}(v = 17) = 2.11$

(vi) Decision: Since $|t| > t_{0.05}$, the null hypothesis is rejected at 5% level of significance, i.e., two populations have not same means.

Hence, the two samples could not have been drawn from the same normal population.

Example 7

Two nicotine contents in two random samples of tobacco are given below:

Sample I	21	24	25	26	27	
Sample II	22	27	28	30	31	36

Can we say that two samples came from the same population?

Solution

F-test:

$n_1 = 5,\ n_2 = 6$

$$\left.\begin{aligned} \bar{x}_1 &= 24.6 \\ \bar{x}_2 &= 29 \\ s_1 &= 2.06 \\ s_2 &= 4.24 \end{aligned}\right\} \text{From calculator}$$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{5(2.06)^2}{5 - 1} = 5.30$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{6(4.24)^2}{6 - 1} = 21.57$$

(i) Null Hypothesis H_0: $\sigma_1^2 = \sigma_2^2$, i.e., variances of two populations are equal.

(ii) Alternative Hypothesis H_1: $\sigma_1^2 > \sigma_2^2$

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: Since $S_2^2 > S_1^2$,

$$F = \frac{S_2^2}{S_1^2} = \frac{21.57}{5.30} = 4.07$$

(v) Critical value: $v_1 = n_1 - 1 = 5 - 1 = 4$

$v_2 = n_2 - 1 = 6 - 1 = 5$

$F_{0.05}(v_2 = 5, v_1 = 4) = 6.26$

(vi) Decision: Since $F < F_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., the two populations have the same variances.

t-test:

$$s = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{5(2.06)^2 + 6(4.24)^2}{5 + 6 - 2}} = 14.34$$

(i) Null Hypothesis H_0: $\mu_1 = \mu_2$, i.e., means of two populations are equal.

(ii) Alternative Hypothesis H_1: $\mu_1 \neq \mu_2$ (Two tailed test)

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: $t = \dfrac{\bar{x}_1 - \bar{x}_2}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \dfrac{24.6 - 29}{14.34\sqrt{\dfrac{1}{5} + \dfrac{1}{6}}} = -0.51$

$|t| = 0.51$

(v) Critical value: $v = n_1 + n_2 - 2 = 5 + 6 - 2 = 9$

$t_{0.05}(v = 9) = 2.262$

(vi) Decision: Since $|t| < t_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., two populations have same means.

Hence, two samples came from the same population.

EXERCISE 6.4

1. If two independent samples of sizes $n_1 = 13$ and $n_2 = 7$ are taken from a normal population. What is the probability that the variance of the first sample will be at least four times as large as that of the second sample? **[Ans.: 0.05]**

2. The standard deviations calculated from two random samples of size 9 and 13 are 2 and 1.9 respectively. Can the samples be regarded as drawn from the normal populations with the same standard deviation?
[Ans.: The samples can be regarded as drawn from the normal populations with the same standard deviation]

3. Two samples are drawn from two normal populations. From the following data test whether the two samples have the same variance at 5% level?

Sample I	60	65	71	74	76	82	85	87		
Sample II	61	66	67	85	78	63	85	86	88	91

[Ans.: Two samples have the same variances]

4. The time taken by workers in performing a job by method I and method II is given below.

Method I	20	16	26	27	22		
Method II	27	33	42	35	32	34	38

Do the data show that the variances of time distribution in a population from which these samples are drawn do not differ significantly?
[Ans.: The variances of time distribution in a population from which the samples are drawn do not differ significantly]

5. Following results were obtained from two samples, each drawn from two different population A and B:

Population	A	B
Sample	I	II
Sample size	25	17
Sample SD	3	2

Test the hypothesis that the variance of brand A is more than that of B.
[Ans.: Variance of brand A is not more than the variance of brand B]

6. In a laboratory experiment two samples gave the following results:

Sample	Size	Sample mean	Sum of squares of deviation from the mean
1	10	15	90
2	12	14	108

Test the equality of sample variances at 5% level of significance.
[Ans.: The two population have the same variances]

6.16 CHI-SQUARE (χ^2) TEST

The chi-square (χ^2) test is a useful measure of comparing experimentally obtained results with those expected theoretically and based on hypothesis. It is used as a test statistic in testing a hypothesis that provides a set of theoretical frequencies with which observed frequencies are compared. The magnitude of discrepancy between observed and theoretical frequencies is given by the quantity χ^2 (pronounced as chi-square). If $\chi^2 = 0$, the observed and expected frequencies completely coincide. As the value of χ^2 increases, the discrepancy between the observed and theoretical frequency decreases.

If $f_{o_1}, f_{o_2}, \ldots, f_{o_n}$ be a set of observed frequencies and $f_{e_1}, f_{e_2}, \ldots, f_{e_n}$ be the corresponding set of expected (or theoretical) frequencies then χ^2 is defined by

$$\chi^2 = \frac{(f_{o_1} - f_{e_1})^2}{f_{e_1}} + \frac{(f_{o_2} - f_{e_2})^2}{f_{e_2}} + \cdots + \frac{(f_{o_n} - f_{e_n})^2}{f_{e_n}} = \sum \frac{(f_o - f_e)^2}{f_e}$$

with $n - 1$ degrees of freedom.

Note

If the data is given in a series of n numbers then degrees of freedom $v = n - 1$
In case of binomial distribution, $v = n - 1$
In case of Poisson distribution, $v = n - 2$
In case of normal distribution, $v = n - 3$

6.16.1 Chi-Square Distribution

If $x_1, x_2, \ldots, x_n$ are n independent normal variates with mean zero and standard deviation unity then $x_1^2 + x_2^2 + \ldots + x_n^2$ is a random variate having χ^2 distribution with probability density function given by

$$P(\chi^2) = y_0 (\chi^2)^{\frac{v-1}{2}} e^{-\frac{\chi^2}{2}}$$

where v = degrees of freedom = $n - 1$ and y_0 = constant depending on the degrees of freedom

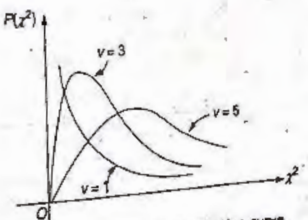

Fig. 6.7 Chi-square distribution curve

6.16.2 Properties of χ^2-Distribution

(i) Chi-Square test is always positively skewed.
(ii) The mean of chi-square distribution is the number of degrees of freedom.
(iii) The standard deviation of chi-square distribution = $\sqrt{2v}$.
(iv) Chi-square values increases with the increase in degrees of freedom.
(v) The value of χ^2 lies between zero and infinity.
(vi) For different values of degrees of freedom, the shape of the curve will be different.

6.17 CHI-SQUARE TEST: GOODNESS OF FIT

The values of χ^2 is used to test whether the deviations of the observed frequencies from the expected frequencies are significant or not. It is also used to fit a set of observations to a given distribution. Hence, chi-square test provides a test of goodness of fit and may be used to examine the validity of some hypothesis about an observed frequency distribution.

Test of Significance

Let $f_{o_1}, f_{o_2}, \ldots, f_{o_n}$ be a set of observed frequencies and $f_{e_1}, f_{e_2}, \ldots, f_{e_n}$ be the corresponding set of expected or theoretical frequencies. The χ^2 statistic is given by

$$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e}$$

Working Rule

(i) Set up a null hypothesis.
(ii) Set up an alternative hypothesis.
(iii) Set a level of significance α.
(iv) Calculate χ^2.
(v) Find the degree of freedom and find the corresponding value of χ^2 at given level of significance α.
(vi) If the calculated value of χ^2 is less than tabulated value of χ^2 at the level of significance α, the null hypothesis is accepted. If calculated value of χ^2 is more than tabulated value of χ^2 at the level of significance α, the null hypothesis is rejected.

Example 1

A dice was thrown 132 times and the following frequencies were observed:

No. obtained	1	2	3	4	5	6	Total
Frequency	15	20	25	15	29	28	132

Test the hypothesis that the dice is unbiased.

Solution

$n = 6$

(i) Null Hypothesis H_0: The dice is unbiased.
(ii) Alternative Hypothesis H_1: The dice is biased.
(iii) Level of significance: $\alpha = 0.05$ (assumption)
(iv) Test statistic:

Expected frequency of each number $f_e = \frac{132}{6} = 22$

No. obtained	Observed frequency, f_o	Expected frequency, f_e	$\frac{(f_o - f_e)^2}{f_e}$
1	15	22	2.23
2	20	22	0.18
3	25	22	0.41
4	15	22	2.23
5	29	22	2.23
6	28	22	1.64
			$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e} = 8.92$

(v) Critical value: $v = n - 1 = 6 - 1 = 5$

$\chi^2_{0.05}(v = 5) = 11.07$

(vi) Decision: Since $\chi^2 < \chi^2_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., the dice is unbiased.

Example 2

The number of car accidents in a metropolitan city was found to be 20, 17, 12, 6, 7, 15, 8, 5, 16 and 14 per month respectively. Use χ^2 test to check whether these frequencies are in agreement with the belief that the occurrence of accidents was the same during 10 months period. Test at 5% level of significance.

Solution

$n = 10$

(i) Null Hypothesis H_0: Occurrence of accident was same during 10 months period.

(ii) Alternative Hypothesis H_1: Occurrence of accidents was not same during 10 months period.

(iii) Level of significance: $\alpha = 0.05$

(iv) Test statistic: If occurrence of accidents is same, the expected frequency of accidents per month

$$f_e = \frac{20+17+12+6+7+15+8+5+16+14}{10} = 12$$

Observed frequency, f_o	Expected frequency, f_e	$\frac{(f_o - f_e)^2}{f_e}$
20	12	5.33
17	12	2.08
12	12	0
6	12	3
7	12	2.08
15	12	0.75
8	12	1.33
5	12	4.08
16	12	1.33
14	12	0.33
		$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e} = 20.31$

(v) Critical value: $\nu = n - 1 = 10 - 1 = 9$

$$\chi^2_{0.05} \ (\nu = 9) = 16.92$$

(vi) Decision: Since $\chi^2 > \chi^2_{0.05}$, the null hypothesis is rejected at 5% level of significance, i.e., occurrence of accidents was not same during 10 months period.

Example 3

200 digits were chosen at random from a set of tables, The frequency of the digits are shown below:

Digits	0	1	2	3	4	5	6	7	8	9
Frequency	18	19	23	21	16	25	22	20	21	15

Use the χ^2-test to access the correctness of the hypothesis that the digits were distributed in equal number in the tables from which these were chosen.

Solution

$$n = 10$$

(i) Null Hypothesis H_0: The digits were distributed in equal number in the tables.

(ii) Alternative Hypothesis H_1: The digits were not distributed in equal number in the tables.

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: Expected frequency of each digit $f_e = \frac{200}{10} = 20$

Observed frequency, f_o	Expected frequency, f_e	$\frac{(f_o - f_e)^2}{f_e}$
18	20	0.2
19	20	0.05
23	20	0.45
21	20	0.05
16	20	0.8
25	20	1.25
22	20	0.2
20	20	0
21	20	0.05
15	20	1.25
		$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e} = 4.3$

(v) Critical value: $\nu = n - 1 = 10 - 1 = 9$

$$\chi^2_{0.05} \ (\nu = 9) = 16.92$$

(vi) Decision: Since $\chi^2 < \chi^2_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., the digits were distributed in equal number in the table.

Example 4

Theory predicts that the proportion of beans in the four groups A, B, C, D should be 9 : 3 : 3: 1. In an experiment among 1600 beans, the numbers in the four groups were 882, 313, 287 and 118. Does the experimental results support the theory?

Solution

$$n = 4$$

(i) Null Hypothesis H_0: The proportion of the beans in the four groups A, B, C, D is $9:3:3:1$.

(ii) Alternative Hypothesis H_1: The proportion of the beans in the four groups A, B, C, D is not $9:3:3:1$.

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic:

Group	Observed Frequency, f_o	Expected frequency, f_e	$\frac{(f_o-f_e)^2}{f_e}$
A	882	$\frac{9}{16}\times 1600 = 900$	0.36
B	313	$\frac{3}{16}\times 1600 = 300$	0.56
C	287	$\frac{3}{16}\times 1600 = 300$	0.56
D	118	$\frac{1}{16}\times 1600 = 100$	3.24
			$\chi^2 = \sum\frac{(f_o-f_e)^2}{f_e} = 4.72$

(v) Critical value: $\nu = n - 1 = 4 - 1 = 3$

$\chi^2_{0.05}\ (\nu = 3) = 7.81$

(vi) Decision: Since $\chi^2 < \chi^2_{0.05}$ the null hypothesis is accepted at 5% level of significance, i.e., experimental results support the theory and the proportion of the beans is $9:3:3:1$.

Example 5

The following mistakes per page were observed in a book:

No. of mistakes per page	0	1	2	3	4
No. of pages	211	90	19	5	0

Fit a Poisson distribution and test the goodness of fit.

Solution

(i) Null Hypothesis H_0: The mistakes follow Poisson distribution and Poisson distribution can be fitted to the data.

(ii) Alternative Hypothesis H_1: The mistakes do not follow Poisson distribution.

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic: The expected frequencies by Poisson distribution are given by

$$\text{Expected frequency } f_e = Np = N\left(\frac{e^{-\lambda}\lambda^x}{x!}\right),\ x = 0,1,2,3,4$$

$$\lambda = \frac{\Sigma fx}{N} = \frac{211(0)+90(1)+19(2)+5(3)+0(4)}{211+90+19+5+0} = 0.44$$

$$f_e = Np = 325\left(\frac{e^{-0.44}0.44^x}{x!}\right),\quad x = 0,1,2,3,4$$

Expected or Theoretical frequency

x	0	1	2	3	4
f_e	209.31	92.1	20.26	2.97	0.33

When expected frequencies are less than 10, classes are grouped together.

No. of mistakes	Observed frequency f_o	Expected frequency f_e	$f_o - f_e$	$\frac{(f_o-f_e)^2}{f_e}$
0	211	209.31	1.69	0.014
1	90	92.10	−2.1	0.048
2	19	20.26		
3	5	2.97	0.44	0.008
4	0	0.33		
				$\chi^2 = \sum\frac{(f_o-f_e)^2}{f_e} = 0.07$

(v) Critical value: The number of degrees of freedom is 1 for each class. There are 5 classes originally. Hence, the degrees of freedom originally is 5. Since the classes are reduced by 2, the degrees of freedom is reduced by 2. Further, while calculating the parameter λ, two sums Σfx and Σf are used. Hence, the degrees of freedom is again reduced by 2.

Hence, the number of degrees of freedom $\gamma = 5 - (2+2) = 1$

$\chi^2_{0.05} = 3.84$

(vi) Decision: Since $\chi^2 < \chi^2_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., the mistakes follow Poisson's distribution.

Example 6

A set of five similar coins is tossed 320 times and result is obtained as follows:

No. of heads	0	1	2	3	4	5
Frequency	6	27	72	112	71	32

Test the hypothesis that the data follow a binomial distribution.

Solution

(i) Null Hypothesis H_0: The data follow a binomial distribution.
(ii) Alternative Hypothesis H_1: The data do not follow binomial distribution.
(iii) Level of significance: $\alpha = 0.05$
(iv) Test statistic: Probability of getting a head $p = \frac{1}{2}$

Probability of getting a tail $q = \frac{1}{2}$

By binomial distribution,

$$p(x) = {}^nC_x p^x q^{n-x} = {}^5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}, \quad x = 0,1,2,3,4,5$$

$$N = 320$$

$$\text{Expected frequency } f_e = Np(x) = 320\left[{}^5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}\right], x = 0,1,2,3,4,5$$

Expected or theoretical frequency

x	0	1	2	3	4	5
f_e	10	50	100	100	50	10

No. of heads	Observed frequency f_o	Expected frequency f_e	$f_o - f_e$	$\frac{(f_o - f_e)^2}{f_e}$
0	6	10	–4	1.6
1	27	50	–23	10.58
2	72	100	–28	7.84
3	112	100	12	1.44
4	71	50	21	8.82
5	32	10	22	48.4
				$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e} = 78.68$

(v) Critical value: $v = n - 1 = 6 - 1 = 5$

$$\chi^2_{0.05} = 11.07$$

(vi) Decision: Since $\chi^2 > \chi^2_{0.05}$ at 5% level of significance, the null hypothesis is rejected, i.e., the data do not follow the binomial distribution.

Example 7

Fit the equation of the best fitting normal curve to the following data:

x	135	145	155	165	175	185	195	205	Total
f	2	14	22	25	19	13	3	2	100

Compare the theoretical and observed frequencies. Using χ^2 test find goodness of fit. Given that $\mu = 165.6$ and $\sigma = 15.02$.

Solution

$$\mu = 165.6, \sigma = 15.02, N = \Sigma f = 100$$

The data is first converted into class intervals with inclusive series

Class interval	Lower class X	$Z = \frac{X-\mu}{\sigma}$	Area from 0 to Z	Area in class interval	Expected frequencies
130–140	130	–2.37	0.4911	0.0357	3.57 ≈ 4
140–150	140	–1.70	0.4554	0.1046	10.46 ≈ 11
150–160	150	–1.04	0.3508	0.2065	20.65 ≈ 21
160–170	160	–0.37	0.1443	0.2584	25.84 ≈ 26
170–180	170	0.29	0.1141	0.2174	21.74 ≈ 21
180–190	180	0.96	0.3315	0.1159	11.59 ≈ 12
190–200	190	1.62	0.4474	0.0416	4.16 ≈ 4
200–210	200	2.29	0.4890	0.0095	0.95 ≈ 1
210–220	210	2.96	0.4985		

Calculation of χ^2

When expected frequencies are less than 10, classes are grouped together.

x	Observed frequency f_o	Expected frequency f_e	$f_o - f_e$	$\frac{(f_o - f_e)^2}{f_e}$
135	2	4	1	0.067
145	14	11		
155	22	21	1	0.048
165	25	26	–1	0.038
175	19	21	–2	0.19
185	13	12	1	0.0588
195	3	4		
205	2	1		
				$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e} = 0.4018$

Critical value: There are 5 frequencies. While calculating mean and standard deviation, three sums Σf, Σfx, and Σfx^2 are used. Hence, the number of degrees of freedom $v = 5 - 3 = 2$

$\chi^2_{0.05} = 5.99$

Since $\chi^2 < \chi^2_{0.05}$ at 5% level of significance, the fit is good and the distribution is nearly normal.

6.18 CHI-SQUARE TEST FOR INDEPENDENCE OF ATTRIBUTES

In statistics, sometimes we have to deal with attributes or qualitative characters, which cannot be measured accurately, although items can be divided into two or more categories w.r.t. the attributes. Let A and B be two attributes of the population. A can be divided into m categories $A_1, A_2 \ldots, A_m$ and B can be divided into n categories $B_1, B_2, \ldots, B_n$. The data can be shown in the form of a two-way table with m rows and n columns, as in a bivariate frequency distribution. This two-way frequency table for attributes is known as $m \times n$ contingency table. The frequency of observations belonging to both the categories A_i and B_j simultaneously is shown in the cell at the i-th row and j-th column and denoted by (A_iB_j). Similarly (A_i) and (B_j) denote the frequency of items belonging to categories A_i and B_j respectively and N, the total frequency.

(3 × 4) contingency table

		Attribute B				Total
		B_1	B_2	B_3	B_4	
Attribute A	A_1	$(A_1 B_1)$	$(A_1 B_2)$	$(A_1 B_3)$	$(A_1 B_4)$	(A_1)
	A_2	$(A_2 B_1)$	$(A_2 B_2)$	$(A_2 B_3)$	$(A_2 B_4)$	(A_2)
	A_3	$(A_3 B_1)$	$(A_3 B_2)$	$(A_3 B_3)$	$(A_3 B_4)$	(A_3)
	Total	(B_1)	(B_2)	(B_3)	(B_4)	N

Independence of Attributes

Two attributes A and B are said to be independent if they are not related to each other. If two attributes A and B are not independent, they are associated on the basis of cell frequencies. It is required to test whether two attributes A and B are associated or not. Under null hypothesis H_0 (attributes are independent), the expected frequency f_e of any cell is given by

$$f_e = \frac{(\text{Row total}) \times (\text{Column total})}{\text{Total frequency}} = \frac{(A_i)(B_j)}{N}$$

Then test statistic is given by

$$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e}$$

with degree of freedom v = (number of row − 1) (number of columns − 1)

If the calculated value of χ^2 is less than tabulated value of χ^2 at the given level of significance α for degree of freedom v, the null hypothesis is accepted and attributes are said to be independent. If calculated value of χ^2 is more than tabulated value of χ^2 at given level of significance α for degree of freedom v, the null hypothesis is rejected.

Yate's Correction

In a 2 × 2 table, there is only one degree of freedom. If any of the expected frequency is less than 10, Yate's correction is applied in chi-square formula.

$$\chi^2 = \sum \left[\frac{\{|f_o - f_e| - 0.5\}^2}{f_e} \right]$$

Example 1

A total of 3759 individual were interviewed in a public opinion survey on a political proposal. Of them 1872 were men and the rest were women. A total of 2257 individuals were in favour of the proposal and 917 were opposed to it. A total of 243 men were undecided and 442 women were opposed to it. Do you justify or contradict the hypothesis that there is no association between sex and attitude at 5% level of significance?

Solution

$$N = 3759$$

Opinion about political proposal

	Favoured	Opposed	Undecided	Total
Men	1154	475	243	1872
Women	1103	442	342	1887
Total	2257	917	585	3759

(i) Null Hypothesis H_0: There is no association between sex and attitude i.e., sex and attitude are independent.

(ii) Alternative Hypothesis H_1: There is association between sex and attitude.

(iii) Level of significance: $\alpha = 0.05$

(iv) Test statistic:

Calculation of χ^2

Observed Frequency f_o	Expected Frequency $f_e = \frac{(A_i)(B_j)}{N}$	$\frac{(f_o - f_e)^2}{f_e}$
1154	$\frac{1872 \times 2257}{3759} \approx 1124$	0.8
475	$\frac{1872 \times 917}{3759} \approx 457$	0.71
243	$\frac{1872 \times 585}{3759} \approx 291$	7.92
1103	$\frac{1887 \times 2257}{3759} \approx 1133$	0.79
442	$\frac{1887 \times 917}{3759} \approx 460$	0.70
342	$\frac{1887 \times 585}{3759} \approx 294$	7.84
		$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e} = 18.76$

(v) Critical value: $v = (r-1)(c-1) = (2-1)(3-1) = 2$

$$\chi^2_{0.05} \ (v = 2) = 5.99$$

(vi) Decision: Since $\chi^2 > \chi^2_{0.05}$, the null hypothesis is rejected at 5% level of significance, i.e., there is association between sex and attitude.

Example 2

A sample of 400 students of undergraduate and 400 students of post-graduate classes was taken to know their opinion about autonomous colleges. 290 of the undergraduate and 310 of the postgraduate students favoured the autonomous status. Present these facts in the form of a table and test at 5% level of significance, that the opinion regarding autonomous status of colleges is independent of the level of classes of students.

Solution

$N = 800$

Opinion about autonomous colleges

	Favoured	Not favoured	Total
Undergraduate	290	110	400
Postgraduate	310	90	400
Total	600	200	800

(i) Null Hypothesis H_0: There is no relation between the classes of students and opinion, i.e., two attributes are independent.

(ii) Alternative Hypothesis H_1: There is relation between the classes of students and opinion.

(iii) Level of significance: $\alpha = 0.05$

(iv) Test statistic:

Observed Frequency f_o	Expected frequency $f_e = \frac{(A_i)(B_j)}{N}$	$\frac{(f_o - f_e)^2}{f_e}$
290	$\frac{400 \times 600}{800} = 300$	0.33
110	$\frac{400 \times 200}{800} = 100$	1.00
310	$\frac{400 \times 600}{800} = 300$	0.33
90	$\frac{400 \times 200}{800} = 100$	1.00
		$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e} = 2.66$

(v) Critical value: $v = (r-1)(c-1) = (2-1)(2-1) = 1$

$$\chi^2_{0.05} \ (v-1) = 3.81$$

(vi) Decision: Since $\chi^2 < \chi^2_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., there is no relation between the classed of students are opinion.

Example 3

In an experiment on immunisation of cattle from tuberculosis the following results were obtained:

	Affected	Not affected	Total
Inoculated	267	27	294
Not inoculated	757	155	912
Total	1024	182	1206

Use χ^2-test to determine the efficiency of vaccine in preventing the tuberculosis.

Solution

$N = 1206$

(i) Null Hypothesis H_0: There is no relation between inoculation and effect on disease, i.e., two attributes are independent.

(ii) Alternative Hypothesis H_1: There is relation between inoculation and effect on disease.

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic:

Observed Frequency f_o	Expected frequency $f_e = \frac{(A_i)(B_j)}{N}$	$\frac{(f_o - f_e)^2}{f_e}$
267	$\frac{294 \times 1024}{1206} \approx 250$	1.156
27	$\frac{294 \times 182}{1206} \approx 44$	6.568
757	$\frac{912 \times 1024}{1206} \approx 774$	0.37
155	$\frac{912 \times 182}{1206} \approx 138$	2.09
		$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e} = 10.19$

(v) Critical value: $v = (r-1)(c-1) = (2-1)(2-1) = 1$

$\chi^2_{0.05} (v = 1) = 3.84$

(vi) Decision: Since $\chi^2 > \chi^2_{0.05}$, the null hypothesis is rejected at 5% level of significance, i.e., vaccine is effective in preventing tuberculosis.

Example 4

Given the following contingency table for hair colour and eye colour. Find the value of χ^2. Is there good association between the two?

Eye colour	Hair colour			Total
	Fair	Brown	Black	
Blue	15	5	20	40
Grey	20	10	20	50
Brown	25	15	20	60
Total	60	30	60	150

Solution

$N = 150$

(i) Null Hypothesis H_0: There is no association between two attributes, hair and eye colours.

(ii) Alternative Hypothesis H_1: There is association between two attributes, hair and eye colours.

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic:

Observed Frequency f_o	Expected frequency $f_e = \frac{(A_i)(B_j)}{N}$	$\frac{(f_o - f_e)^2}{f_e}$
15	$\frac{40 \times 60}{150} = 16$	0.0625
5	$\frac{40 \times 30}{150} = 8$	1.125
20	$\frac{40 \times 60}{150} = 16$	1
20	$\frac{50 \times 60}{150} = 20$	0
10	$\frac{50 \times 30}{150} = 10$	0
20	$\frac{50 \times 60}{150} = 20$	0
25	$\frac{60 \times 60}{150} = 24$	0.042
15	$\frac{60 \times 30}{150} = 12$	0.75
20	$\frac{60 \times 60}{150} = 24$	0.666
		$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e} = 3.6465$

(v) Critical value: $v = (r-1)(c-1) = (3-1)(3-1) = 4$

$\chi^2_{0.05} (v = 4) = 9.49$

(vi) Decision: Since $\chi^2 < \chi^2_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., there is no association between two attributes, hair and eye colours.

Example 5

Two researchers adopted different sampling techniques while investigating some group of students to find the number of students falling into different intelligence level. The results are as follows:

Researchers	*Below average*	*Average*	*Above average*	*Genius*	*Total*
X	86	60	44	10	200
Y	40	33	25	2	100
Total	126	93	69	12	300

Would you say that the sampling techniques adopted by the two researchers are significantly different?

Solution

$N = 300$

(i) Null Hypothesis H_0: There is no significant difference in the sampling techniques adopted by the two researchers.

(ii) Alternative Hypothesis H_1: There is significant difference in the sampling techniques adopted by the two researchers.

(iii) Level of significance: $\alpha = 0.05$ (assumption)

(iv) Test statistic:

Observed frequency f_o	*Expected frequency* $f_e = \frac{(A_i)(B_j)}{N}$	$\frac{(f_o - f_e)^2}{f_e}$
86	$\frac{200\times126}{300} = 84$	0.0476
60	$\frac{200\times93}{300} = 62$	0.0645
44	$\frac{200\times69}{300} = 46$	0.0869
10	$\frac{200\times12}{300} = 8$	0.5
40	$\frac{100\times126}{300} = 42$	0.0952
33	$\frac{100\times93}{300} = 31$	0.129
25	$\frac{100\times69}{300} = 23$	0.1739
2	$\frac{100\times12}{300} = 4$	1
		$\chi^2 = \sum\frac{(f_o - f_e)^2}{f_e} = 2.0971$

(v) Critical value: $\nu = (r-1)(c-1) = (2-1)(4-1) = 3$

$\chi^2_{0.05}(\nu = 3) = 7.81$

(vi) Decision: Since $\chi^2 < \chi^2_{0.05}$, the null hypothesis is accepted at 5% level of significance, i.e., there is no significant difference in the sampling techniques adopted by the two researchers.

Example 6

The following table gives the level of education and the marriage adjustment score for a sample of married women:

Level of education	*Marriage adjustment*				*Total*
	Very low	*Low*	*High*	*Very high*	
College	24	97	62	58	241
High school	22	28	30	41	121
Middle school	32	10	11	20	73
Total	78	135	103	119	435

Can you conclude from the above data the higher the level of education, the greater is the degree of adjustment in marriage?

Solution

$N = 435$

(i) Null Hypothesis H_0: There is no relation between the level of education and adjustment in marriage, i.e., two attributes are independent.

(ii) Alternative Hypothesis H_1: There is relation between level of education and adjustment in marriage.

(iii) Level of significance: $\alpha = 0.05$ (assumption)
(iv) Test statistic:

Observed frequency f_o	Expected frequency $f_e = \frac{(A_i)(B_j)}{N}$	$\frac{(f_o - f_e)^2}{f_e}$
24	$\frac{241\times78}{435} \approx 43$	8.3953
97	$\frac{241\times135}{435} \approx 75$	6.4533
62	$\frac{241\times103}{435} \approx 57$	0.4386
58	$\frac{241\times119}{435} \approx 66$	0.9697
22	$\frac{121\times78}{435} \approx 22$	0
28	$\frac{121\times135}{435} \approx 37$	2.1892
30	$\frac{121\times103}{435} \approx 29$	0.0345
41	$\frac{121\times119}{435} \approx 33$	1.9394
32	$\frac{73\times78}{435} \approx 13$	27.7692
10	$\frac{73\times135}{435} \approx 23$	7.3478
11	$\frac{73\times103}{435} \approx 17$	2.1176
20	$\frac{73\times119}{435} \approx 20$	0
		$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e} = 57.713$

(v) Critical value: $v = (r-1)(c-1) = (3-1)(4-1) = 6$

$\chi^2_{0.05}\ (v = 6) = 12.59$

(vi) Decision: Since $\chi^2 > \chi^2_{0.05}$, the null hypothesis is rejected at 5% level of significance i.e., level of education and adjustment in marriage are related and higher the level of education, the greater is the degree of adjustment in marriage.

Example 7

Two batches each of 12 animals are taken for test of inoculation. One batch was inoculated and the other batch was not inoculated. The number of dead and surviving animals are given in the following table for both the cases. Can the inoculation be regarded as effective against the disease. Make Yate's correction for continuity of χ^2?

	Dead	Survived	Total
Inoculated	2	10	12
Not inoculated	8	4	12
Total	10	14	24

Solution

$N = 24$

(i) Null hypothesis H_0: There is no relation between inoculation and death i.e., inoculation and effect on disease are independent.
(ii) Alternative Hypothesis H_1: There is relation between inoculation and death.
(iii) Level of significance: $\alpha = 0.05$ (assumption)
(iv) Test statistic: Yate's correction is used only when $v = 1$ and when some expected frequencies are small, i.e., less than 10. Here, expected frequencies are less than 10 each.

Observed frequency f_o	Expected frequency $f_e = \frac{(A_i)(B_j)}{N}$	$\frac{\{\lvert f_o - f_e\rvert - 0.5\}^2}{f_e}$
2	$\frac{12\times10}{25} = 5$	1.25
10	$\frac{12\times14}{24} = 7$	0.89
8	$\frac{12\times10}{24} = 5$	1.25
4	$\frac{12\times14}{24} = 7$	0.89
		$\chi^2 = \sum \frac{\{\lvert f_o - f_e\rvert - 0.5\}^2}{f_e} = 4.28$

(v) Critical value: $v = (2-1)(2-1) = 1$

$$\chi^2_{0.05}\ (v=1) = 3.84$$

(vi) Decision: Since $\chi^2 > \chi^2_{0.05}$, the null hypothesis is rejected at 5% level of significance, i.e., there is association between inoculation and death and inoculation is regarded as effective against the disease.

EXERCISE 6.5

1. A dice is thrown 264 times with the following results: Show that the dice is biased [Given $\chi^2_{0.05} = 11.07$ for 5 df]

No. appeared on the dice	1	2	3	4	5	6
Frequency	40	32	28	58	54	52

2. A pair of dice are thrown 360 times and frequency of each sum is given below:

Sum	2	3	4	5	6	7	8	9	10	11	12
Frequency	8	24	35	37	44	65	51	42	26	14	14

would you say that the dice are fair on the basis of the chi-square test at 0.05 level of significance?

[Ans.: The dice are fair]

3. 4 coins are tossed 160 times and the following results were obtained:

No. of heads	0	1	2	3	4
Observed frequencies	17	52	54	31	6

Under the assumption that coins are balanced, find the expected frequencies of 0, 1, 2, 3 or 4 heads, and test the goodness of fit ($\alpha = 0.05$).

[Ans.: Expected frequencies: 10, 40, 60, 40, 10, the data do not follow binomial distribution]

4. Fit a Poisson distribution to the following data and for its goodness of fit at level of significance 0.05:

x	0	1	2	3	4
f	419	352	154	56	19

5. The following table gives the number of accidents in a city during a week. Find whether the accidents are uniformly distributed over a week.

Day	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total
No. of accidents	13	15	9	11	12	10	14	84

[Ans.: The accidents are uniformly distributed over a week]

6. Weights in kilograms of 10 students are given below: 38, 40, 45, 53, 47, 43, 55, 48, 52, 49

Can we say that the variance of the normal distribution from which the above sample is drawn is 20 kg?

[Ans.: The sample is drawn from the normal population with variance 20]

7. Five dice are thrown 192 times and the number of times 4, 5 or 6 are obtained are as follows:

No. of dice showing 4, 5, 6	5	4	3	2	1	0
Frequency	6	46	70	48	20	2

Calculate χ^2.

[Ans.: 16.94]

8. The distribution of defects in printed circuit board is hypothesised to follow Poisson distribution. A random sample of 60 printed boards shows the following data:

No. of defects	0	1	2	3
Observed frequency	32	15	9	4

Does the hypothesis of Poisson distribution appropriate?

[Ans.: The defects follow Poisson distribution]

9. Based on the following data, determine if there is a relation between literacy and smoking.

	Smokers	Non-smokers
Literates	83	57
Illiterates	45	68

[Ans.: $\chi^2 = 9.19$, yes]

10. Table below shows the performances of students in mathematics and physics. Test the hypothesis that the performance in mathematics is independent of performance in physics.

Grades in Physics	Grades in Mathematics		
	High	Medium	Low
High	56	71	12
Medium	47	163	38
Low	14	42	81

[Ans.: $\chi^2 = 132.31$, Hypothesis is rejected]

11. Investigate the association between the darkness of eye colour in father and son from the following data:

Colour of son's eyes	*Colour of father's eyes*		*Total*
	Dark	*Not dark*	
Dark	48	90	138
Not dark	80	782	862
Total	128	872	1000

[Ans.: $\chi^2 = 3.84$, There is association between two attributes]

12. From the following data, find whether there is any significant linking in the habit of taking soft drinks among the categories of employees.

Soft drink	*Employees*		
	Clerks	*Teachers*	*Officers*
Pepsi	10	25	65
Thumsup	15	30	65
Fanta	50	60	30

[Ans.: $\chi^2 = 60.24$, Two attributes are not independent]

13. 1000 students at college level were graded according to their IQ and the economic conditions of their home. Use χ^2-test to find out whether there is any association between condition at home and IQ.

Economic condition	*IQ*		
	High	*Low*	*Total*
Rich	460	140	600
Poor	240	160	400
Total	700	300	1000

[Ans.: $\chi^2 = 31.733$, These is no association between two attributes]

14. A random sample of 500 students were classified according to economic condition of their family and also according to merit as shown below:

Merit	*Economic condition*			*Total*
	Rich	*Middleclass*	*Poor*	
Meritorious	42	137	61	240
Not-meritorious	58	113	89	260
Total	100	250	150	500

Test whether the two attributes merit and economic condition are associated or not.

[Ans.: $\chi^2 = 9.30$, The two attributes are associated]