Unit 2 : Infinite Sequence and Series

1) Prove that the sequence is $a_n = \frac{n}{n^2+1}$ a decreasing and bound below. Is it convergent?

$$U_n = \frac{n}{n^2+1}$$

$$U_{n+1} = \frac{n+1}{n^2+2n+2}$$

$$U_{n+1} - U_n = \frac{n+1}{n^2+2n+2} - \frac{n}{n^2+1}$$

$$= \frac{(n+1)(n^2+1) - n(n^2+2n+2)}{(n^2+2n+2)(n^2+1)}$$

$$= \frac{\cancel{n^3} + n + n^2 + 1 - \cancel{n^3} - 2n^2 - 2n}{(n^2+2n+2)(n^2+)}$$

$$= \frac{-n^2 - n + 1}{(n+1)(n^2+2n+2)} < 0$$

$U_{n+1} < U_n$ $\qquad$ $U_n = \frac{n}{n^2+1} > 0$ $\quad \forall n \in N$

decreasing sequence.

$$\underset{\text{below}}{m} < a_n < \underset{\text{above}}{M}$$

$U_n > 0$

bounded below.

It is convergent.

2) P.T. $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\dots$ Converges and find its ~~sum~~ sum.

$$a = 1$$
$$r = 2/3$$
$$|r| = \frac{2}{3} < 1$$

$\therefore$ Convergent

$$S_n = \frac{a}{1-r}$$
$$= \frac{1}{1-2/3}$$
$$S_n = 3.$$

3) (i) Test the series (by zero Test)

$$\sum_{n=1}^{\infty} \frac{n}{e^{-n}}$$

$$U_n = \frac{n}{e^{-n}}$$

$$U_n = ne^{n}$$

$$\lim_{n\to\infty} ne^{n} \to \infty.$$

$\therefore$ The given series is divergent.

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(ii) Test the series (by Integral Test)

$$\sum_{n=1}^{\infty} \frac{2\tan^{-1} n}{1+n^2}$$

$$f(x) = \frac{2\tan^{-1} n}{1+n^2}$$

This function is decreasing for $x \geq 1$.

$$f(1) = \frac{2\tan^{-1} 1}{1+(1)^2} = 0.78$$

$$f(2) = \frac{2\tan^{-1} 2}{1+(2)^2} = 0.44$$

$$f(3) = \frac{2\tan^{-1} 3}{1+(3)^2} = 0.25$$

$\vdots \quad \vdots \quad \vdots$

$$\cancel{\int_1^{\infty} \frac{2\tan^{-1} n}{1+n^2} = \lim_{b \to n} \int_1^{b} \frac{2\tan^{-1} n}{1+n^2} \, dx}$$

$$\int_1^{\infty} f(x) = \lim_{b \to \infty} \int_1^{b} f(x) = \lim_{b \to \infty} \int_1^{b} \frac{2\tan^{-1} x}{1+x^2}$$

$$= \lim_{b \to \infty} \left[\frac{\cancel{2}(\tan^{-1} x)^2}{\cancel{2}} \right]_1^b$$

$$\cancel{= (\tan^{-1} b)^2 - \tan^{-1}}$$

$$= \lim_{b \to \infty} \left[(\tan^{-1} b)^2 - \tan^{-1} (1)^2 \right]$$

$$= \left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2$$

$$= \frac{\pi}{4} - \frac{\pi}{16}$$

$$= \frac{3\pi^2}{16} \quad \text{is finite value.}$$

$\therefore \Sigma U_n$ is convergent.

4) Test the series. (by Comparision Test)

(i) $$\sum_{n=1}^{\infty} \frac{5n^2 - 3n}{n^2(n-2)(n^2+5)}$$

$$U_n = \frac{5n^2 - 3n}{n^2(n-2)(n^2+5)}$$

$$\cancel{U_n = \frac{n^2(5 - 3/n)}{n^2\left(\frac{1}{n^2} + 1\right)}}$$

$$U_n = \frac{\cancel{n^2}(5 - 3/n)}{\cancel{n^2} \cdot n\left(1 - 2/n\right) \cdot n^2\left(1 + \frac{5}{n^2}\right)}$$

$$U_n = \frac{5 - 3/n}{n^3\left(1 - 2/n\right)\left(1 + \frac{5}{n^2}\right)}$$

$$V_n = \frac{1}{n^3}$$

$$\frac{U_n}{V_n} = \frac{5 - 3/n}{\left(1 - 2/n\right)\left(1 + \frac{5}{n^2}\right)}$$

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$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{5 - 3/n}{\left(1-\frac{2}{n}\right)\left(1+\frac{5}{n^2}\right)}$$

$= 5$ (finite & non zero)

$\Sigma V_n = \Sigma \frac{1}{n^3}$

$P = 3 > 1$

$\therefore \Sigma V_n$ is convergent.

$\therefore \Sigma U_n$ is also convergent. (By Comparision Test)

(ii) $\frac{1\cdot 2}{3^2\cdot 4^2} + \frac{3\cdot 4}{5^2\cdot 6^2} + \frac{5\cdot 6}{7^2\cdot 8^2} + \dots$

$$U_n = \sum_{n=1}^{\infty} \frac{(2n-1)(2n)}{(2n+1)^2(2n+2)^2}$$

$$U_n = \frac{\cancel{n^2}\left(2-\frac{1}{n}\right)(2)}{\cancel{n^2}\left(2+\frac{1}{n}\right)^2 \cdot n^2\left(2+\frac{2}{n}\right)^2}$$

$$= \frac{\left(2-\frac{1}{n}\right)(2)}{n^2\left(2+\frac{1}{n}\right)^2\left(2+\frac{2}{n}\right)^2}$$

let $\Sigma V_n = \Sigma \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{2\left(2-\frac{1}{n}\right)}{\left(2+\frac{1}{n}\right)^2\left(2+\frac{2}{n}\right)^2}$$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{4}{4\cdot 4} = \frac{1}{4} \quad \text{(finite \& non zero)}$$

Now, $\Sigma V_n = \Sigma \frac{1}{n^2}$

$p = 2 > 1$

$\therefore$ ΣV_n is convergent.

$\therefore$ ΣU_n is also convergent.
(By Comparision Test)

5) Test the series. (By ratio Test)

(i) $$\sum_{n=1}^{\infty} \frac{n\, 2^n (n+1)!}{3!\, 3^n}$$

$$U_n = \frac{n\, 2^n (n+1)!}{3!\, 3^n}$$

$$U_{n+1} = \frac{(n+1)\, 2^{(n+1)} (n+2)!}{3!\, 3^{(n+1)}}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{n\, 2^n (n+1)!}{3!\, 3^n} \cdot \frac{3!\, 3^{n+1}}{(n+1) 2^{n+1} (n+2)!}$$

$$= \lim_{n\to\infty} \frac{n \cdot \cancel{2^n} \cdot \cancel{(n+1)!}}{\cancel{3!} \cdot \cancel{3^n}} \cdot \frac{3! \cdot \cancel{3^n} \cdot \cancel{3}}{(n+1)\, \cancel{2^n} \cdot 2 \cdot (n+2) \cancel{(n+1)!}}$$

$$= \lim_{n\to\infty} \frac{3! \cdot n}{\cancel{2}(n+1)(n+2)}$$

$$= \lim_{n\to\infty} \frac{3\, n}{(n+1)(n+2)}$$

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$$= \lim_{n\to\infty} \frac{3\cancel{x}}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$= 0 < 1$$

$\therefore$ ΣU_n is convergent.

(ii) $\frac{1}{1\cdot2\cdot3} + \frac{x}{4\cdot5\cdot6} + \frac{x^2}{7\cdot8\cdot9} + \cdots$

$$U_n = \frac{(x)^{n-1}}{(3n-2)(3n-1)(3n)}$$

$$U_{n+1} = \frac{x^n}{(3n+1)(3n+2)(3n+3)}$$

$$\frac{U_n}{U_{n+1}} = \frac{(3n+1)(3n+2)(3n+3)}{x(3n-2)(3n-1)(3n)} \quad \text{---} (1)$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{x^{n-1}}{(3n-2)(3n-1)(3n)} \cdot \frac{(3n+1)(3n+2)(3n+3)}{x^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{x^n}\,(3n+1)(3n+2)(3n+3)}{x(3n-2)(3n-1)(3n)\cdot\cancel{x^n}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^3}\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{3}{n}\right)}{x\cdot\cancel{n^3}\left(3-\frac{2}{n}\right)\left(3-\frac{1}{n}\right)(3)}$$

$$= \frac{1}{x}$$

By ratio test,

$\frac{1}{x} > 1 \Rightarrow x < 1 \Rightarrow$ Convergent.

$\frac{1}{x} < 1 \Rightarrow x > 1 \Rightarrow$ Divergent.

$\frac{1}{x} = 1 \Rightarrow x = 1 \Rightarrow$ test fails.

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(Rabbe's Test)

Now, Put $x = 1$ in eq^n ①

$$\frac{U_n}{U_{n+1}} = \frac{(3n+1)(3n+2)(3n+3)}{(3n-2)(3n-1)(3n)}$$

$$\frac{U_n}{U_{n+1}} - 1 = \frac{(3n+1)(3n+2)(3n+3) - (3n-2)(3n-1)(3n)}{(3n-2)(3n-1)(3n)}$$

$$n\left(\frac{U_n}{U_{n+1}} - 1\right) = \cancel{n}\left[\frac{(3n+1)(3n+2)(3n+3) - (3n-2)(3n-1)(3n)}{(3n-2)(3n-1)(3\cancel{n})}\right]$$

$$= \left[\frac{(9n^2+9n+2)(3n+3) - (9n^2-9n+2)(3n)}{(3n-2)(3n-1)(3)}\right]$$

$$= \frac{\left[27n^3 + 27n^2 + 6n + 27n^2 + 27n + 6 - (27n^3 - 27n^2 + 6n)\right]}{(3n-2)(3n-1)(3)}$$

$$= \left[\frac{27n^3 + 54n^2 + 6n + 27n + 6 - 27n^3 + 27n^2 - 6n}{(3n-2)(3n-1)(3)}\right]$$

$$= \frac{81n^2 + 27n + 6}{(3n-2)(3n-1)(3)}$$

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{n^2\left(81 + \frac{27}{n} + \frac{6}{n^2}\right)}{n^2\left(3 - \frac{2}{n}\right)\left(3 - \frac{1}{n}\right)(3)}$$

$$= \frac{81}{27}$$

$$= 3 > 1$$

$\therefore \Sigma U_n$ is convergent.

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6) Test the Series (By root test)

(i) $\sum_{n=1}^{\infty} \frac{n}{e^{-n}}$

$$U_n = \frac{n}{e^{-n}}$$

$$(U_n)^{1/n} = \left(\frac{n}{e^{-n}}\right)^{1/n}$$

$$= \frac{n^{1/n}}{e^{-1}}$$

$$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \frac{n^{1/n}}{e^{-1}}$$

$$= e \lim_{n\to\infty} n^{1/n}$$

$$= e \cdot 1$$

$$= e.$$

$e > 1$

$\therefore$ By root test,

The given series is divergent.

(ii) $\sum_{n=1}^{\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

$$U_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$$

$$(U_n)^{1/n} = \left[\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}\right]^{1/n}$$

$$(U_n)^{1/n} = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{1/2}}$$

$$= \left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}}$$

$$\lim_{n\to\infty}(U_n)^{1/n} = \lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}}$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$= \frac{1}{e} < 1 \quad \left(\because \lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}} = e\right)$$

$\therefore$ Given Series is convergent. (By root test)

7) Test the series (By Rabbe's Test).

$$\cancel{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\log}} \qquad \left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot 4}{3\cdot 6}\right)^2 + \left(\frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}\right)^2 + \dots$$

$$U_n = \frac{[1\cdot 4\cdot 7\cdots(3n-2)]^2}{[3\cdot 6\cdot 9\cdots(3n)]^2}$$

$$U_{n+1} = \frac{[1\cdot 4\cdot 7\cdots(3n-2)(3n+1)]^2}{[3\cdot 6\cdot 9\cdots(3n)(3n+3)]^2}$$

$$\frac{U_n}{U_{n+1}} = \frac{(3n+3)^2}{(3n+1)^2}$$

$$\cancel{n\left(\frac{U_n}{U_{n+1}}-1\right)} \quad n\left(\frac{U_n}{U_{n+1}}-1\right) = n\left[\frac{(3n+3)^2}{(3n+1)^2}-1\right]$$

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$$n\left(\frac{U_n}{U_{n+1}}-1\right) = n\left[\frac{(3n+3)^2-(3n+1)^2}{(3n+1)^2}\right]$$

$$= n\left[\frac{(3n+3+3n+1)(\cancel{3n}+3-\cancel{3n}-1)}{9n^2+6n+1}\right]$$

$$= n\left[\frac{(6n+4)(2)}{9n^2+6n+1}\right]$$

$$= n\left[\frac{12n+4}{9n^2+6n+1}\right]$$

$$= \frac{\cancel{n^2}\left[12+\frac{4}{n}\right]}{\cancel{n^2}\left[9+\frac{6}{n}+\frac{1}{n^2}\right]}$$

$$\lim_{n\to\infty} n\left[\frac{U_n}{U_{n+1}}-1\right] = \lim_{n\to\infty}\left[\frac{12+4/n}{9+6/n+1/n^2}\right]$$

$$= \frac{12}{9}$$

$$= \frac{4}{3} > 1$$

$\therefore \Sigma U_n$ is convergent. (By Rabbe's Test)

(ii) $\left(\frac{1}{2}\right)^3+\left(\frac{1\cdot 3}{2\cdot 4}\right)^3+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3+\cdots$

$$U_n = \frac{[1\cdot 3\cdot 5\cdots(2n-1)]^3}{[2\cdot 4\cdot 6\cdots(2n)]^3}$$

$$U_{n+1} = \frac{[1\cdot 3\cdot 5\cdots(2n-1)(2n+1)]^3}{[2\cdot 4\cdot 6\cdots(2n)(2n+2)]^3}$$

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$$\frac{U_n}{U_{n+1}} = \frac{(2n+2)^3}{(2n+1)^3}$$

$$n\left(\frac{U_n}{U_{n+1}} - 1\right) = n\left[\frac{(2n+2)^3}{(2n+1)^3} - 1\right]$$

$$= n\left[\frac{(2n+2)^3 - (2n+1)^3}{(2n+1)^3}\right]$$

$$= n\left[\frac{(\cancel{2n}+2-\cancel{2n}-1)(4n^2+8n+4+4n^2+4n+2n+2+4n^2+4n+1)}{8n^3+1+3(2n)(2n+1)}\right]$$

$$\left(\because\ a^3-b^3 = (a-b)^3(a^2+ab+b^2)\right.$$
$$\left.(a+b)^3 = a^3+b^3+3ab(a+b)\right)$$

$$= \frac{n\left[12n^2+18n+7\right]}{\left[8n^3+1+12n^2+6n\right]}$$

$$= \frac{\cancel{n^3}\left[12+\frac{18}{n}+\frac{7}{n^2}\right]}{\cancel{n^3}\left[8+\frac{1}{n^3}+\frac{12}{n}+\frac{6}{n^2}\right]}$$

$$\lim_{n\to\infty}\left[n\left(\frac{U_n}{U_{n+1}}-1\right)\right] = \lim_{n\to\infty}\frac{\left(12+\frac{18}{n}+\frac{7}{n^2}\right)}{\left(8+\frac{1}{n^3}+\frac{12}{n}+\frac{6}{n^2}\right)}$$

$$= \frac{12}{8}$$

$$= \frac{3}{2} > 1$$

$\therefore \Sigma U_n$ is convergent. (By Rabbe's Test)

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8) Test the Series (by Leibnitz's Test)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)}$$

$$U_n = \frac{(-1)^{n+1}}{\log(n+1)}$$

$$|U_n| = \frac{1}{\log(n+1)}$$

$$|U_{n+1}) = \frac{1}{\log(n+2)}$$

$$|U_n| - |U_{n+1}| = \frac{1}{\log(n+1)} - \frac{1}{\log(n+2)}$$

$$= \frac{\log(n+2) - \log(n+1)}{\log(n+1)\log(n+2)}$$

$$> 0$$

$$\lim_{n\to\infty} |U_n| = \lim_{n\to\infty} \frac{(-1)^{n+1}}{\log(n+1)}$$

$$= \frac{1}{\infty}$$

$$= 0 < 1$$

$\therefore \Sigma U_n$ is cgt.

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9) Determine Absolute or Conditional Convergence of $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^2}{n^3+1}$

$$U_n = (-1)^n \cdot \frac{n^2}{n^3+1}$$

$$|U_n| = \frac{n^2}{n^3+1}$$

$$= \frac{n^2}{n^3\left(1+\frac{1}{n^3}\right)}$$

$$= \frac{1}{n\left(1+\frac{1}{n^3}\right)}$$

$$\Sigma V_n = 1/n$$

$$\lim_{n\to\infty} \frac{|U_n|}{V_n} = \lim_{n\to\infty} \frac{1}{1+1/n^3} = 1 \text{ (finite \& non zero)}$$

Here $\Sigma V_n = 1/n$, which is harmonic series.

ΣV_n is divergent.

$\Sigma |U_n|$ is divergent.

Here, ΣU_n is cgt but $\Sigma |U_n|$ is dgt

ΣU_n is conditionally cgt.

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10) Find ROC and Interval of Convergence of the series $\sum_{n=1}^{\infty} \frac{(-3)^n (x)^n}{\sqrt{n+1}}$

$$U_n = \frac{(-1)^n (3)^n (x)^n}{\sqrt{n+1}}$$

$$|U_n| = \frac{3^n \cdot x^n}{\sqrt{n+1}}$$

$$|U_{n+1}| = \frac{3^{n+1}\, x^{n+1}}{\sqrt{n+2}}$$

$$\lim_{n\to\infty} \left| \frac{U_n}{U_{n+1}} \right| = \lim_{n\to\infty} \frac{\cancel{3^n} \cdot \cancel{x^n}}{\sqrt{n+1}} \cdot \frac{\sqrt{n+2}}{\cancel{3^n} \cdot 3 \cdot \cancel{x^n} \cdot x}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+2}}{\sqrt{n+1}} \frac{1}{3x}$$

$$= \frac{1}{|3x|}$$

By Ratio test,

$$\left| \frac{1}{3x} \right| > 1 \Rightarrow |x| < 1/3$$

$$-1/3 < x < 1/3 \quad \text{(convergent)}$$

$$\left| \frac{1}{3x} \right| < 1 \Rightarrow |x| > 1/3 \quad \text{(divergent)}$$

Now, Put $x = 1/3$

$$U_n = \frac{(-1)^n 3^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}}$$

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$$U_n = \frac{(-1)^n}{\sqrt{n+1}}$$

$$|U_n| = \frac{1}{\sqrt{n+1}} \Rightarrow |U_{n+1}| = \frac{1}{\sqrt{n+2}}$$

$$|U_n| - |U_{n+1}| = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}$$

$$= \frac{\sqrt{n+2} - \sqrt{n+1}}{\sqrt{n+1}\sqrt{n+2}} > 0, \quad \forall n \in N$$

$$|U_n| > |U_{n+1}|$$

$$\lim_{n\to\infty} |U_n| = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$$

ΣU_n is convergent.

$x = 1/3$ $\quad \Sigma U_n$ is convergent.

Now, Put $x = -1/3$

$$U_n = \frac{(-1)^n 3^n (-\frac{1}{3})^n}{\sqrt{n+1}}$$

$$= \frac{1}{\sqrt{n+1}}$$

$$U_n = \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{n}\left(1+\frac{1}{\sqrt{n}}\right)}$$

$$V_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{1}{1+1/\sqrt{n}} = 1$$ (finite and non zero)

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ΣV_n and ΣU_n both are convergent or divergent.

Now $\Sigma V_n = \Sigma \frac{1}{n^{1/2}}$ where $p = 1/2 < 1$

ΣV_n is divergent.

$\therefore \Sigma U_n$ is also divergent.

$x = -1/3$

$\therefore \Sigma U_n$ is divergent.

ROC $= \frac{1}{3}$, IOC $= \left(-\frac{1}{3}, \frac{1}{3}\right]$

11) Expand log (sec x) in powers of x.

$f(x) = \log(\sec x)$

let $y = \log(\sec x)$

$\frac{dy}{dx} = \frac{1}{\sec x}(\sec x \tan)$

$= \tan x$

$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots$

$\int \frac{dy}{dx} = y = \frac{x^2}{2} + \frac{x^4}{12} + \frac{2}{15}\frac{x^6}{6} + \dots + c$

$y = \log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \dots + c$

Put $x = 0$

$0 = 0 + c$

$\boxed{c = 0}$

$\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \dots$

12) Prove that

$$\tan\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$$

Let $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$\cancel{\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\sqrt{\ }\right.}$$

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1 - \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}}{\frac{2\tan\theta/2}{1+\tan^2\theta/2}}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan^2\theta/2 - 1 + \tan^2\theta/2}{2\tan\theta/2}\right)$$

$$= \tan^{-1}\left(\tan\theta/2\right)$$

$$= \theta/2$$

$$= \frac{1}{2}\tan^{-1}x$$

$$= \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right)$$

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13) Expand $\log x$ in power of $(x-1)$ by Taylor's theorem and hence find the value of $\log_e 1.1$.

$f(x) = \log x$ $\quad (x-1)$ $\quad \log_e(1.1) = ?$

By the taylor's series expansion

$$f(x) = f(a) + \frac{(x-a)f'(a)}{1!} + \frac{(x-a)^2 f''(a)}{2!} + \frac{(x-a)^3 f'''(a)}{3!} + \ldots$$

$$f(x) = f(1) + \frac{(x-1)f'(1)}{1!} + \frac{(x-1)^2 f''(1)}{2!} + \frac{(x-1)^3 f'''(1)}{3!} + \ldots$$

Here, $f(x) = \log x$ $\qquad f(1) = 0$

$f'(x) = \frac{1}{x}$ $\qquad f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$ $\qquad f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $\qquad f'''(1) = 2$

$f^{IV}(x) = -\frac{6}{x^4}$ $\qquad f^{IV}(1) = -6$

⋮ $\qquad$ ⋮

and so on $\qquad$ and so on

$$\log x = 0 + (x-1) + \frac{(x-1)^2(-1)}{2!} + \frac{(x-1)^3(2)}{3!} + \frac{(x-1)^4(-6)}{4!} + \ldots$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

Put $x = 1.1$

$$\log_e(1.1) = (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \ldots$$

$\log_e(1.1) = 0.09532$

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14) Express $(x-1)^4 + 2(x-1)^3 + 5(x-1) + 2$ in ascending power of x.

$f(x-1) = (x-1)^4 + 2(x-1)^3 + 5(x-1) + 2$

$f(x) = x^4 + 2x^3 + 5x + 2$

$h = -1$

$f(x) = x^4 + 2x^3 + 5x + 2$ $\qquad$ $f(-1) = 1 \cancel{-2} - 5 \cancel{+2} = -4$

$f'(x) = 4x^3 + 6x + 5$ $\qquad$ $f'(-1) = -4 + 6 + 5 = 7$.

$f''(x) = 12x^2 + 6$ $\qquad$ $f''(-1) = 12 - 12 = 0$

$f'''(x) = 24x$ $\qquad$ $f'''(-1) = -24 + 12 = -12$

$f^{IV}(x) = 24$ $\qquad$ $f^{IV}(-1) = 24$

$$f(x+h) = f(h) + \frac{x f'(h)}{1!} + \frac{x^2 f''(h)}{2!} + \cdots + \frac{x^n f^n(h)}{n!} + \cdots$$

$$f(x-1) = -4 + 7x - \frac{12x^3}{3!} + \frac{24x^4}{4!}$$

$$= -4 + 7x - 2x^3 + x^4$$

$$= x^4 - 2x^3 + 7x - 4$$

15) Expand $3x^3 + 8x^2 + x - 2$ in power of $x-3$

$a = 3$.

$f(x) = 3x^3 + 8x^2 + x - 2$ $\qquad$ $f(3) = 81 + 72 + 3 - 2 = 154$

$f'(x) = 9x^2 + 16x + 1$ $\qquad$ $f'(3) = 81 + 48 + 1 = 130$

$f''(x) = 18x + 16$ $\qquad$ $f''(3) = 54 + 16 = 70$

$f'''(x) = 18$ $\qquad$ $f'''(3) = 18$

$f^{IV}(x) = 0$ $\qquad$ $f^{IV}(3) = 0$

Using Taylor's Series

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$$

$$= 154 + (x-3)(130) + \frac{(x-3)^2}{2!}(70) + \frac{(x-3)^3}{3!}(18)$$

$$= 154 + 130(x-3) + 35(x-3)^2 + 3(x-3)^3$$

16) Find the taylor's series expansion of $\tan\left(x + \frac{\pi}{4}\right)$ in power of x, showing at least four nonzero terms. Hence find the value of $\tan 46°$ and $\tan 43°$.

$$f(x+h) = \tan(x + \pi/4)$$

$$\therefore h = \pi/4$$

$f(x) = \tan x$ — $f(\pi/4) = 1$

$f'(x) = \sec^2 x$ — $f'(\pi/4) = 2$

$f''(x) = 2\sec^2 x \tan x$ — $f''(\pi/4) = 4$

$f'''(x) = 2\sec^4 x + 4\tan^2 x \sec^2 x$ — $f'''(\pi/4) = 8 + 8 = 16$

$f^{iv}(x) = 8\sec^4 x \tan x + 8\tan^3 x \sec^2 x + 8\sec^4 x \tan x$ — $f^{iv}(\pi/4) = 32 + 16 + 32 = 80$

⋮

and so on

$$f(x+h) = f(h) + \frac{x f'(h)}{1!} + \frac{x^2 f''(h)}{2!} + \dots + \frac{x^n f^n(h)}{n!} + \dots$$

$$f\left(x+\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) + \frac{x f'\left(\frac{\pi}{4}\right)}{1!} + \frac{x^2 f''\left(\frac{\pi}{4}\right)}{2!} + \dots + \frac{x^n f^n\left(\frac{\pi}{4}\right)}{n!} + \dots$$

$$f\left(x+\frac{\pi}{4}\right) = 1 + 2x + \frac{4x^2}{2!} + \frac{16x^3}{3!} + \frac{80x^4}{4!} + \dots$$

$\rightarrow$ $\tan(43^\circ) = \tan\left(\frac{\pi}{4} - 2^\circ\right)$

$$x = -2^\circ = -\frac{\pi}{90} \Rightarrow -0.0349$$

$$\tan\left(\frac{\pi}{4} - \frac{\pi}{90}\right) = \tan 43^\circ = 1 + 2(-0.0349) + \frac{4(-0.0349)^2}{2} + \frac{16(-0.0349)^3}{3!} + \frac{80(-0.0349)^4}{4!}$$

$$\tan(43^\circ) \cong 0.932515$$

$\rightarrow$ $\tan(46^\circ) = \tan\left(\frac{\pi}{4} + 1^\circ\right)$

$$x = 1^\circ = \frac{\pi}{180} \Rightarrow 0.01745$$

$$\tan\left(\frac{\pi}{4} + \frac{\pi}{180}\right) = \tan(46^\circ) = 1 + 2(0.01745) + \frac{4(0.01745)^2}{2} + \frac{16(0.01745)^3}{3\times 2} + \frac{80}{4!}(0.01745)^4$$

$$\tan(46^\circ) \cong 1.035530$$