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Tutorial

4. $\int_{2}^{\infty} e^{-h^2 x^2} \cdot dx$

Let $x = \sqrt{t}$

$\therefore dx = \frac{1}{2\sqrt{t}} \cdot dt$

$x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$\int_{2}^{\infty} e^{-h^2 t} \cdot \frac{1}{2\sqrt{t}} \cdot dt$

$= \frac{1}{2} \int_{2}^{\infty} e^{-t \cdot h^2} \cdot t^{(1/2 - 1)} \cdot dt$

$= \frac{1}{2} \times \frac{\Gamma 1/2}{h^{2 \times 1/2}}$ $\left(\because \int_{2}^{\infty} e^{-ax} \cdot x^{n-1} = \frac{\Gamma n}{a^n} \right)$

$= \frac{1}{2} \times \frac{\sqrt{\pi}}{h} = \frac{\sqrt{\pi}}{2h}$

1* $\beta(m, n) = \beta(n, m)$

$\int_{0}^{1} x^{m-1} (1-x)^{n-1} \cdot dx$

Let $1 - x = t$

$\therefore x = 1 - t$

$\therefore dx = -dt$

$x = 0 \Rightarrow t = 1$

$x = 1 \Rightarrow t = 0$

$$\int_1^0 (1-t)^{m-1} \cdot t^{n-1} \cdot -dt$$

$$= -\int_0^1 (1-t)^{m-1} \cdot t^{n-1} \cdot -dt$$

$$= \int_0^1 (1-t)^{m-1} \cdot t^{n-1} \cdot dt$$

$$= \beta(n, m)$$

2. $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x \cdot dx$

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \cdot dx$$

Let $x = \sin^2\theta$

$\therefore dx = 2\sin\theta\cos\theta \cdot d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\int_0^{\pi/2} \sin\theta^{2(m-1)} \cdot \cos\theta^{2(n-1)} \cdot 2\sin\theta\cos\theta \cdot d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2+1}\theta \cdot \cos^{2n-2+1}\theta \cdot d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2m-1} \cdot \theta$$

3. Prove $\left(\Gamma\tfrac{1}{2}\right)^2 = \pi$

we have $\beta(m,n) = \dfrac{\Gamma m\, \Gamma n}{\Gamma m+n}$

Let $m = n = \frac{1}{2}$, we get

$$\beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{\Gamma\tfrac{1}{2}\,\Gamma\tfrac{1}{2}}{\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\right)}$$

$$= \frac{\left(\Gamma\tfrac{1}{2}\right)^2}{\Gamma 1}$$

$$= \left(\Gamma\tfrac{1}{2}\right)^2 \quad \text{--- (1)}$$

$$\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta$$

let $m = n = \frac{1}{2}$, we get.

$$\beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2\times\frac{1}{2}-1}\theta \cdot \cos^{2\times\frac{1}{2}-1}\theta \cdot d\theta$$

$$= 2\int_0^{\pi/2} \sin^0\theta \cdot \cos^0\theta \cdot d\theta$$

$$= 2\int_0^{\pi/2} 1 \cdot d\theta$$

$$= 2\left[\theta\right]_0^{\pi/2}$$

$$= 2 \times \frac{\pi}{2} - 0$$

$$= \pi \quad \text{--- (2)}$$

From (1) & (2)

$$\boxed{\left(\Gamma\tfrac{1}{2}\right)^2 = \pi}$$

4. $\int_0^\infty e^{-x^2} \cdot dx = \frac{\sqrt{\pi}}{2}$

we know that, $\Gamma n = 2 \int_0^\infty e^{-x^2} \cdot x^{2n-1} \cdot dx$

let $n = \frac{1}{2}$, we get

$$\Gamma \tfrac{1}{2} = 2 \int_0^\infty e^{-x^2} \cdot x^{2 \times \frac{1}{2} - 1} \cdot dx$$

$$\therefore \sqrt{\pi} = 2 \int_0^\infty e^{-x^2} \cdot x^0 \cdot dx$$

$$\therefore \sqrt{\pi} = 2 \int_0^\infty e^{-x^2} \cdot dx$$

$$\therefore \frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-x^2} \cdot dx$$

$$\therefore \int_0^\infty e^{-x^2} \cdot dx = \frac{\sqrt{\pi}}{2}$$

5. Prove that $\int_0^\infty e^{-x^4} \cdot x^2 \cdot dx = \frac{1}{4} \Gamma \frac{3}{4}$

let $x^4 = t$

$\therefore x^2 = \sqrt{t}$

$\therefore x = \sqrt[4]{t}$

$\therefore dx = \frac{1}{4} \cdot t^{-3/4} \cdot dt$

$$= \int_0^\infty e^{-t} \cdot \sqrt{t} \cdot \frac{1}{4} \cdot t^{-3/4} \cdot dt$$

$$= \frac{1}{4} \int_0^\infty e^{-t} \cdot t^{1/2 - 3/4} \cdot dt$$

$$= \frac{1}{4} \int_0^\infty e^{-t} \, t^{-1/4} \cdot dt$$

$$= \frac{1}{4} \int_0^\infty e^{-t} \cdot t^{\frac{3}{4}-1} \cdot dt$$

$$= \frac{1}{4} \times \Gamma 3/4 \qquad \left(\because \int_0^\infty e^{-x} \cdot x^{n-1} = \Gamma n \right)$$

$$= \frac{1}{4} \Gamma \frac{3}{4}$$