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from:- D.G. BORAD

-: Shreenathji Engineering Zone :

DJPatel

CHAPTER 1

Probability

Chapter Outline

1.1 INTRODUCTION

The concept of probability originated from the analysis of the games of chance. Even today, a large number of problems exist which are based on the games of chance, such as tossing of a coin, throwing of dice, and playing of cards. The utility of probability in business and economics is most emphatically revealed in the field of predictions for the future. Probability is a concept which measures the degree of uncertainty and that of certainty as a corollary.

The word *probability* or 'chance' is used commonly in day-to-day life. Daily, we come across the sentences like, 'it may rain today', 'India may win the forthcoming cricket match against Sri Lanka', 'the chances of making profits by investing in shares of Company A are very bright, etc. Each of the above sentences involves an element of uncertainty. A numerical measure of uncertainty is provided by a very important branch of mathematics called *theory of probability*. Before we study the probability theory in detail, it is appropriate to explain certain terms which are essential for the study of the theory of probability.

1.2 SOME IMPORTANT TERMS AND CONCEPTS

1. Random Experiment If an experiment is conducted, any number of times, under identical conditions, there is a set of all possible outcomes associated with it.

If the outcome is not unique but may be any one of the possible outcomes, the experiment is called a random experiment, e.g., tossing a coin, throwing a dice.

2. Outcome The result of a random experiment is called an outcome. For example, consider the following:

(a) Suppose a random experiment is 'a coin is tossed'. This experiment gives two possible outcomes—head or tail.

(b) Suppose a random experiment is 'a dice is thrown'. This experiment gives six possible outcomes—1, 2, 3, 4, 5 or 6—on the uppermost face of a dice.

3. Trial and Event Any particular performance of a random experiment is called a trial and outcome. A combination of outcomes is called an event. For example, consider the following:

(a) Tossing of a coin is a trial, and getting a head or tail is an event.

(b) Throwing of a dice is a trial and getting 1 or 2 or 3 or 4 or 5 or 6 is an event.

4. Exhaustive Event The total number of possible outcomes of a random experiment is called an exhaustive event. For example, consider the following:

(a) In tossing of a coin, there are two exhaustive events, viz., head and tail.

(b) In throwing of a dice, there are six exhaustive events, getting 1 or 2 or 3 or 4 or 5 or 6.

5. Mutually Exclusive Events Events are said to be mutually exclusive if the occurrence of one of them precludes the occurrence of all others in the same trial, i.e., they cannot occur simultaneously. For example, consider the following:

(a) In tossing a coin, the events head or tail are mutually exclusive since both head and tail cannot occur at the same time.

(b) In throwing a dice, all the six events, i.e., getting 1 or 2 or 3 or 4 or 5 or 6 are mutually exclusive events.

6. Equally Likely Events The outcomes of a random experiment are said to be equally likely if the occurrence of none of them is expected in preference to others. For example, consider the following:

(a) In tossing a coin, head or tail are equally likely events.

(b) In throwing a dice, all the six faces are equally likely events.

7. Independent Events Events are said to be independent if the occurrence of an event does not have any effect on the occurrence of other events. For example, consider the following:

(a) In tossing a coin, the event of getting a head in the first toss is independent of getting a head in the second, third, and subsequent tosses.

(b) In throwing a dice, the result of the first throw does not affect the result of the second throw.

8. Favourable Events The favourable events in a random experiment are the number of outcomes which entail the occurrence of the event. For example, consider the following:

In throwing of two dice, the favourable events of getting the sum 5 is (1, 4), (4, 1), (2, 3), (3, 2), i.e., 4.

1.3 DEFINITIONS OF PROBABILITY

1.3.1 Classical Definition of Probability

Let n be the number of equally likely, mutually exclusive, and exhaustive outcomes of a random experiment. Let m be number of the outcomes which are favourable to the occurrence of an event A. The probability of event A occurring, denoted by $P(A)$, is given by

$$P(A) = \frac{\text{Number of outcomes favourable to } A}{\text{Number of exhaustive outcomes}} = \frac{m}{n}$$

1.3.2 Empirical or Statistical Definition of Probability

If an experiment is repeated a large number of times under identical conditions, the limiting value of the ratio of the number of times the event A occurs to the total number of trials of the experiment as the number of trials increase indefinitely is called the probability of occurrence of the event A.

Let $P(A)$ be the probability of occurrence of the event A. Let m be the number of times in which an event A occurs in a series of n trials.

$$P(A) = \lim_{n \to \infty} \frac{m}{n}, \text{ provided the limit is finite and unique.}$$

1.3.3 Axiomatic Definition of Probability

Before discussing the axiomatic definition of probability, it is necessary to explain certain concepts that are necessary to its understanding.

1. Sample Space A set of all possible outcomes of a random experiment is called a sample space. Each element of the set is called a *sample point* or a *simple event* or an *elementary event*.

The sample space of a random experiment is denoted by S. For example, consider the following:

(a) In a random experiment of tossing of a coin, the sample space consists of two elementary events.

$$S = \{H, T\}$$

(b) In a random experiment of throwing of a dice, the sample space consists of six elementary events.

$$S = \{1, 2, 3, 4, 5, 6\}$$

The elements of S can either be single elements or ordered pairs. If two coins are tossed, each element of the sample space consists of the following ordered pairs:

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

2. Event Any subset of a sample space is called an event. In the experiment of throwing of a dice, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Let A be the event that an odd number appears on the dice. Then $A = \{1, 3, 5\}$ is a subset of S. Similarly, let B be the event of getting a number greater than 3. Then $B = \{4, 5, 6\}$ is another subset of S.

Definition of Probability Let S be a sample space of an experiment and A be any event of this sample space. The probability $P(A)$ of the event A is defined as the real-value set function which associates a real value corresponding to a subset A of the sample space S. The probability $P(A)$ satisfies the following three axioms.

Axiom I: $P(A) \geq 0$, i.e., the probability of an event is a nonnegative number.

Axiom II: $P(S) = 1$, i.e., the probability of an event that is certain to occur must be equal to unity.

Axiom III: If $A_1, A_2, ..., A_n$ are finite mutually exclusive events then

$$P(A_1 \cup A_2 \cup ... \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

$$= \sum_{i=1}^{n} P(A_i)$$

i.e., the probability of a union of mutually exclusive events is the sum of probabilities of the events themselves.

Example 1

What is the probability that a leap year selected at random will have 53 Sundays?

Solution

A leap year has 366 days, i.e., 52 weeks and 2 days. These 2 days can occur in the following possible ways:

(i) Monday and Tuesday
(ii) Tuesday and Wednesday
(iii) Wednesday and Thursday
(iv) Thursday and Friday
(v) Friday and Saturday
(vi) Saturday and Sunday
(vii) Sunday and Monday

Number of exhaustive cases $n = 7$

Number of favourable cases $m = 2$

Let A be the event of getting 53 Sundays in a leap year.

$$P(A) = \frac{m}{n} = \frac{2}{7}$$

Example 2

Three unbiased coins are tossed. Find the probability of getting (i) exactly two heads, (ii) at least one tail, (iii) at most two heads, (iv) a head on the second coin, and (v) exactly two heads in succession.

Solution

When three coins are tossed, the sample space S is given by

$$S = \{HHH, HTH, THH, HHT, TTT, THT, TTH, HTT\}$$

$$n(s) = 8$$

(i) Let A be the event of getting exactly two heads.

$$A = \{HTH, THH, HHT\}$$

$$n(A) = 3$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$$

(ii) Let B be the event of getting at least one tail.

$$B = \{HTH, THH, HHT, TTT, THT, TTH, HTT\}$$

$$n(B) = 7$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{7}{8}$$

(iii) Let C be the event of getting at most two heads.

$$C = \{HTH, THH, HHT, TTT, THT, TTH, HTT\}$$

$$n(C) = 7$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{7}{8}$$

(iv) Let D be the event of getting a head on the second coin.

$$D = \{HHH, THH, HHT, THT\}$$

$$n(D) = 4$$

$$P(D) = \frac{n(D)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

(v) Let E be the event of getting two heads in succession.

$$E = \{HHH, THH, HHT\}$$

$$n(E) = 3$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$$

Example 3

A fair dice is thrown. Find the probability of getting (i) an even number, (ii) a perfect square, and (iii) an integer greater than or equal to 3.

Solution

When a dice is thrown, the sample space S is given by

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$n(S) = 6$$

(i) Let A be the event of getting an even number.

$$A = \{2, 4, 6\}$$

$$n(A) = 3$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

(ii) Let B be the event of getting a perfect square.

$$B = \{1, 4\}$$

$$n(B) = 2$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

(iii) Let C be the event of getting an integer greater than or equal to 3.

$$C = \{3, 4, 5, 6\}$$

$$n(C) = 4$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{4}{6} = \frac{2}{3}$$

Example 4

A card is drawn from a well-shuffled pack of 52 cards. Find the probability of (i) getting a king card, (ii) getting a face card, (iii) getting a red card, (iv) getting a card between 2 and 7, both inclusive, and (v) getting a card between 2 and 8, both exclusive.

Solution

Total number of cards = 52

One card out of 52 cards can be drawn in ways.

$$n(S) = {}^{52}C_1 = 52$$

(i) Let A be the event of getting a king card. There are 4 king cards and one of them can be drawn in ${}^{4}C_1$ ways.

$$n(A) = {}^{4}C_1 = 4$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

(ii) Let B be the event of getting a face card. There are 12 face cards and one of them can be drawn in ${}^{12}C_1$ ways.

$$n(B) = {}^{12}C_1 = 12$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{12}{52} = \frac{3}{13}$$

(iii) Let C be the event of getting a red card. There are 26 red cards and one of them can be drawn in ${}^{26}C_1$ ways.

$$n(C) = {}^{26}C_1 = 26$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{26}{52} = \frac{1}{2}$$

(iv) Let D be the event of getting a card between 2 and 7, both inclusive. There are 6 such cards in each suit giving a total of $6 \times 4 = 24$ cards. One of them can be drawn in ${}^{24}C_1$ ways.

$$n(D) = {}^{24}C_1 = 24$$

$$P(D) = \frac{n(D)}{n(S)} = \frac{24}{52} = \frac{6}{13}$$

(v) Let E be the event of getting a card between 2 and 8, both exclusive. There are 5 such cards in each suit giving a total of $5 \times 4 = 20$ cards. One of them can be drawn in ${}^{20}C_1$ ways.

$$n(E) = {}^{20}C_1 = 20$$

$$= \frac{n(E)}{n(S)} = \frac{20}{52} = \frac{5}{13}$$

Example 5

A bag contains 2 black, 3 red, and 5 blue balls. Three balls are drawn at random. Find the probability that the three balls drawn (i) are blue (ii) consist of 2 blue and 1 red ball, and (iii) consist of exactly one black ball.

Solution

Total number of balls = 10

3 balls out of 10 balls can be drawn in ${}^{10}C_3$ ways.

$$n(S) = {}^{10}C_3 = 120$$

(i) Let A be the event that the three balls drawn are blue. 3 blue balls out of 5 blue balls can be drawn in ${}^5C_3$ ways.

$$n(A) = {}^5C_3 = 10$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{10}{120} = \frac{1}{12}$$

(ii) Let B be the event that the three balls drawn consist of 2 blue and 1 red ball. 2 blue balls out of 5 blue balls can be drawn in ${}^5C_2$ ways. 1 red ball out of 3 red balls can be drawn in ${}^3C_1$ ways.

$$n(B) = {}^5C_2 \times {}^3C_1 = 30$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{30}{120} = \frac{1}{4}$$

(iii) Let C be the event that three balls drawn consist of exactly one black ball, i.e., remaining two balls can be drawn from 3 red and 5 blue balls. One black ball can be drawn from 2 black balls in ${}^2C_1$ ways and the remaining 2 balls can be drawn from 8 balls in ${}^8C_2$ ways.

$$n(C) = {}^2C_1 \times {}^8C_2 = 56$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{56}{120} = \frac{7}{15}$$

Example 6

A class consists of 6 girls and 10 boys. If a committee of three is chosen at random from the class, find the probability that (i) three boys are selected, and (ii) exactly two girls are selected.

Solution

Total number of students = 16

A committee of 3 students from 16 students can be selected in ${}^{16}C_3$ ways.

$$n(S) = {}^{16}C_3 = 560$$

(i) Let A be the event that 3 boys are selected.

$$n(A) = {}^{10}C_3 = 120$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{120}{560} = \frac{3}{14}$$

(ii) Let B be the event that exactly 2 girls are selected. 2 girls from 6 girls can be selected in ${}^6C_2$ ways and one boy from 10 boys can be selected in ${}^{10}C_1$ ways.

$$n(B) = {}^6C_2 \times {}^{10}C_1 = 150$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{150}{560} = \frac{15}{16}$$

Example 7

From a collection of 10 bulbs, of which 4 are defective, 3 bulbs are selected at random and fitted into lamps. Find the probability that (i) all three bulbs glow, and (ii) the room is lit.

Solution

Total number of bulbs = 10

3 bulbs can be selected from 10 bulbs in ${}^{10}C_3$ ways.

$$n(S) = {}^{10}C_3 = 120$$

(i) Let A be event that all three bulbs glow. This event will occur when 3 bulbs are selected from 6 nondefective bulbs in ${}^6C_3$ ways.

$$n(A) = {}^6C_3 = 20$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{20}{120} = \frac{1}{6}$$

(ii) Let B be the event that the room is lit. Let $\bar{B}$ be the event that the room is dark. The event $\bar{B}$ will occur when 3 bulbs are selected from 4 defective bulbs in ${}^4C_3$ ways.

$$n(\bar{B}) = {}^4C_3 = 4$$

$$P(\bar{B}) = \frac{n(\bar{B})}{n(S)} = \frac{4}{120} = \frac{1}{30}$$

$$\therefore \quad P(B) = 1 - P(\bar{B}) = 1 - \frac{1}{30} = \frac{29}{30}$$

Example 8

There are 20 tickets numbered 1, 2, ..., 20. One ticket is drawn at random. Find the probability that the ticket bears a number which is (i) even, (ii) a perfect square, and (iii) multiple of 3.

Solution

There are 20 tickets numbered from 1 to 20.

$n(S) = 20$

(i) Let A be the event that a ticket bears a number which is even.

$A = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$

$n(A) = 10$

$$P(A) = \frac{n(A)}{n(S)} = \frac{10}{20} = \frac{1}{2}$$

(ii) Let B the event that a ticket bears a number which is a perfect square.

$B = \{1, 4, 9, 16\}$

$n(B) = 4$

$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{20} = \frac{1}{5}$$

(iii) Let C be the event that a ticket bears a number which is a multiple of 3.

$C = \{3, 6, 9, 12, 15, 18\}$

$n(C) = 6$

$$P(C) = \frac{n(C)}{n(S)} = \frac{6}{20} = \frac{3}{10}$$

Example 9

Four letters of the word 'THURSDAY' are arranged in all possible ways. Find the probability that the word formed is 'HURT'.

Solution

Total number of letters in the word 'THURSDAY' = 8

Four letters from 8 letters can be arranged in ${}^8P_4$ ways.

$n(S) = {}^8P_4 = 1680$

Let A be the event that the word formed is 'HURT'. The word 'HURT' can be formed in one way only.

$n(A) = 1$

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{1680}$$

Example 10

A bag contains 5 red, 4 blue, and m green balls. If the probability of getting two green balls when two balls are selected at random is $\frac{1}{7}$, find m.

Solution

Total number of balls = $5 + 4 + m = 9 + m$

2 balls out of $9 + m$ balls can be drawn in ${}^{9+m}C_2$ ways.

$n(S) = {}^{9+m}C_2$

Let A be the event that both the balls drawn are green.

2 green balls out of m green balls can be drawn in ${}^mC_2$ ways.

$n(A) = {}^mC_2$

$$P(A) = \frac{n(A)}{n(S)} = \frac{{}^mC_2}{{}^{9+m}C_2}$$

But $P(A) = \frac{1}{7}$

$$\frac{{}^mC_2}{{}^{9+m}C_2} = \frac{1}{7}$$

$$\frac{m(m-1)}{(m+9)(m+8)} = \frac{1}{7}$$

$$(m+9)(m+8) = 7m(m-1)$$

$$m^2 + 17m + 72 = 7m^2 - 7m$$

$$6m^2 - 24m - 72 = 0$$

$$3m^2 - 12m - 36 = 0$$

$$3m^2 - 18m + 6m - 36 = 0$$

$$3m(m-6) + 6(m-6) = 0$$

$$(3m+6)(m-6) = 0$$

$$3m + 6 = 0 \quad \text{or} \quad m - 6 = 0$$

$$m = -2 \quad \text{or} \quad m = 6$$

But $m \neq -2$

$\therefore$ $m = 6$

EXERCISE 1.1

1. A card is drawn at random from a pack of 52 cards. Find the probability that the card drawn is (i) an ace card, and (ii) a club card.

$\left[\text{Ans.: (i) } \frac{1}{13} \text{ (ii) } \frac{1}{4}\right]$

2. An unbiased coin is tossed twice. Find the probability of (i) exactly one head, (ii) at most one head, (iii) at least one head, and (iv) same face on both the coins.

$$\left[\text{Ans.: (i) } \frac{1}{2} \text{ (ii) } \frac{3}{4} \text{ (iii) } \frac{3}{4} \text{ (iv) } \frac{1}{2}\right]$$

3. A fair dice is thrown thrice. Find the probability that the sum of the numbers obtained is 10.

$$\left[\text{Ans.: } \frac{1}{8}\right]$$

4. A ball is drawn at random from a box containing 12 red, 18 white, 19 blue, and 15 orange balls. Find the probability that (i) it is red or blue, and (ii) it is white, blue, or orange.

$$\left[\text{Ans.: (i) } \frac{2}{5} \text{ (ii) } \frac{43}{55}\right]$$

5. Eight boys and three girls are to sit in a row for a photograph. Find the probability that no two girls are together.

$$\left[\text{Ans.: } \frac{28}{55}\right]$$

6. If four persons are chosen from a group of 3 men, 2 women, and 4 children, find the probability that exactly two of them will be children.

$$\left[\text{Ans.: } \frac{10}{21}\right]$$

7. A box contains 2 white, 3 red, and 5 black balls. Three balls are drawn at random. What is the probability that they will be of different colours?

$$\left[\text{Ans.: } \frac{1}{4}\right]$$

8. Two cards are drawn from a well-shuffled pack of 52 cards. Find the probability of getting (i) 2 king cards, (ii) 1 king card and 1 queen card, and (iii) 1 king card and 1 spade card.

$$\left[\text{Ans.: (i) } \frac{1}{221} \text{ (ii) } \frac{8}{663} \text{ (iii) } \frac{1}{26}\right]$$

9. A four-digit number is to be formed using the digits 0, 1, 2, 3, 4, 5. All the digits are to be different. Find the probability that the digit formed is (i) odd, (ii) greater than 4000, (iii) greater than 3400, and (iv) a multiple of 5.

$$\left[\text{Ans.: (i) } \frac{12}{25} \text{ (ii) } \frac{2}{5} \text{ (iii) } \frac{12}{25} \text{ (iv) } \frac{9}{25}\right]$$

10. 3 books of physics, 4 books of chemistry, and 5 books of mathematics are arranged in a shelf. Find the probability that (i) no physics books are together, (ii) chemistry books are always together, and (iii) books of the same subjects are together.

$$\left[\text{Ans.: (i) } \frac{6}{11} \text{ (ii) } \frac{1}{55} \text{ (iii) } \frac{1}{4620}\right]$$

11. 8 boys and 2 girls are to be seated at random in a row for a photograph. Find the probability that (i) the girls sit together, and (ii) the girls occupy 3^{rd} and 7^{th} seats.

$$\left[\text{Ans.: (i) } \frac{1}{5} \text{ (ii) } \frac{1}{45}\right]$$

12. A committee of 4 is to be formed from 15 boys and 3 girls. Find the probability that the committee contains (i) 2 boys and 2 girls, (ii) exactly one girl, (iii) one particular girl, and (iv) two particular girls.

$$\left[\text{Ans.: (i) } \frac{7}{68} \text{ (ii) } \frac{91}{204} \text{ (iii) } \frac{2}{9} \text{ (iv) } \frac{2}{51}\right]$$

13. If the letters of the word REGULATIONS are arranged at random, what is the probability that there will be exactly four letters between *R* and *E*?

$$\left[\text{Ans.: } \frac{6}{55}\right]$$

14. Find the probability that there will be 5 Sundays in the month of October.

$$\left[\text{Ans.: } \frac{3}{7}\right]$$

1.4 THEOREMS ON PROBABILITY

Theorem 1 The probability of an impossible event is zero, i.e., $P(\phi) = 0$, where ϕ is a null set.

Proof An event which has no sample points is called an impossible event and is denoted by ϕ.

For a sample space S of an experiment,

$$S \cup \phi = S$$

Taking probability of both the sides,

$$P(S \cup \phi) = P(S)$$

Since S and ϕ are mutually exclusive events,

$$P(S) + P(\phi) = P(S) \qquad \text{[Using Axiom III]}$$

$$\therefore \quad P(\phi) = 0$$

Theorem 2 The probability of the complementary event $\bar{A}$ of A is

$$P(\bar{A}) = 1 - P(A)$$

Proof Let A be an event in the sample space S.

$$A \cup \bar{A} = S$$

$$P(A \cup \bar{A}) = P(S)$$

Since A and $\bar{A}$ are mutually exclusive events,

$$P(A) + P(\bar{A}) = P(S)$$

$$P(A) + P(\bar{A}) = 1 \qquad [\because \ P(S) = 1]$$

$$\therefore \quad P(\bar{A}) = 1 - P(A)$$

Note Since A and $\bar{A}$ are mutually exclusive events,

$A \cup \bar{A} = S$ and $A \cap \bar{A} = \phi$

Corollary Probability of an event is always less than or equal to one, i.e., $P(A) \le 1$

Proof $P(A) = 1 - P(\bar{A})$

$$P(A) \le 1 \qquad [\because \ P(\bar{A}) \ge 0 \text{ by Axiom I}]$$

De Morgan's Laws Since an event is a subset of a sample space, De Morgan's laws are applicable to events.

$$P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B})$$

$$P(\overline{A \cap B}) = P(\bar{A} \cup \bar{B})$$

Theorem 3 For any two events A and B in a sample space S,

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

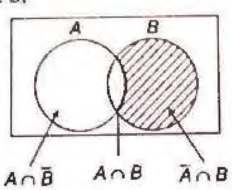

Fig. 1.1

Proof From the Venn diagram (Fig. 1.1),

$$B = (A \cap B) \cup (\bar{A} \cap B)$$

$$P(B) = P\left[(A \cap B) \cup (\bar{A} \cap B)\right]$$

Since $(A \cap B)$ and $(\bar{A} \cap B)$ are mutually exclusive events,

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

Similarly, it can be shown that

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

Theorem 4 Additive Law of Probability (Addition Theorem)

The probability that at least one of the events A and B will occur is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof From the Venn diagram (Fig. 1.1),

$$A \cup B = A \cup (\bar{A} \cap B)$$

$$P(A \cup B) = P\left[A \cup (\bar{A} \cap B)\right]$$

Since A and $(\bar{A} \cap B)$ are mutually exclusive events,

$$P(A \cup B) = P(A) + P(\bar{A} \cap B) \qquad \text{[Using Axiom III]}$$

$$= P(A) + P(B) - P(A \cap B) \qquad \text{[Using Theorem 3]}$$

Remarks

1. If A and B are mutually exclusive events, i.e., $A \cap B = \phi$ then $P(A \cap B) = 0$ according to Theorem 1.
   Hence, $P(A \cup B) = P(A) + P(B)$
2. The event $A \cup B$ (i.e., A or B) denotes the occurrence of either A or B or both. Alternately, it implies the occurrence of at least one of the two events.
   $A \cup B = A + B$
3. The event $A \cap B$ (i.e., A and B) is a compound or joint event that denotes the simultaneous occurrence of the two events.
   $A \cap B = AB$

Corollary 1 From the Venn diagram (Fig. 1.1),

$$P(A \cup B) = 1 - P(\bar{A} \cap \bar{B})$$

where $P(\bar{A} \cap \bar{B})$ is the probability that none of the events A and B occur simultaneously.

Corollary 2 $P(\text{Exactly one of } A \text{ and } B \text{ occurs}) = P\left[(A \cap \bar{B}) \cup (\bar{A} \cap B)\right]$

$$= P(A \cap \bar{B}) + P(\bar{A} \cap B) \qquad \left[\because \ (A \cap \bar{B}) \cap (\bar{A} \cap B) = \phi\right]$$

$$= P(A) - P(A \cap B) + P(B) - P(A \cap B) \qquad \text{[Using Theorem 3]}$$

$$= P(A) + P(B) - 2P(A \cap B)$$

$$= P(A \cup B) - P(A \cap B) \qquad \text{[Using Theorem 4]}$$

$= P$ (at least one of the two events occur)

$- P$ (the two events occur simultaneously)

Corollary 3 The addition theorem can be applied for more than two events. If A, B, and C are three events of a sample space S then the probability of occurrence of at least one of them is given by

$$\begin{aligned}P(A\cup B\cup C) &= P[A\cup(B\cup C)]\\ &= P(A)+P(B\cup C)-P[A\cap(B\cup C)]\\ &= P(A)+P(B\cup C)-P[(A\cap B)\cup(A\cap C)]\\ &= P(A)+P(B)+P(C)-P(B\cap C)-P(A\cap B)-P(A\cap C)+P(A\cap B\cap C)\end{aligned}$$

[Applying Theorem 4 on second and third term]

Alternately, the probability of occurrence of at least one of the three events can also be written as

$$P(A\cup B\cup C)=1-P(\bar{A}\cap\bar{B}\cap\bar{C})$$

If A, B, and C are mutually exclusive events,

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)$$

Corollary 4 The probability of occurrence of at least two of the three events is given by

$$\begin{aligned}P[(A\cap B)\cup(B\cap C)\cup(A\cap C)] &= P(A\cap B)+P(B\cap C)+P(A\cap C)-3P(A\cap B\cap C)\\ &\quad +P(A\cap B\cap C) \qquad \text{[Using Corollary 3]}\\ &= P(A\cap B)+P(B\cap C)+P(A\cap C)-2P(A\cap B\cap C)\end{aligned}$$

Corollary 5 The probability of occurrence of exactly two of the three events is given by

$$\begin{aligned}&P[(A\cap B\cap\bar{C})\cup(A\cap\bar{B}\cap C)\cup(\bar{A}\cap B\cap C)]\\ &= P[(A\cap B)\cup(B\cap C)\cup(A\cap C)]-P(A\cap B\cap C) \qquad \text{[Using Corollary 2]}\\ &= P(A\cap B)+P(B\cap C)+P(A\cap C)-3P(A\cap B\cap C) \qquad \text{[Using Corollary 4]}\end{aligned}$$

Corollary 6 The probability of occurrence of exactly one of the three events is given by

$$P[(A\cap\bar{B}\cap\bar{C})\cup(\bar{A}\cap B\cap\bar{C})\cup(\bar{A}\cap\bar{B}\cap C)]$$

$= P$(at least one of the three event occur) $- P$ (at least two of the three events occur)

$$= P(A)+P(B)+P(C)-2P(A\cap B)-2P(B\cap C)-2P(A\cap C)+3P(A\cap B\cap C)$$

Example 1

A card is drawn from a well-shuffled pack of cards. What is the probability that it is either a spade or an ace?

Solution

Let A and B be the events of getting a spade and an ace card respectively.

$$P(A)=\frac{^{13}C_1}{^{52}C_1}=\frac{13}{52}$$

$$P(B)=\frac{^{4}C_1}{^{52}C_1}=\frac{4}{52}$$

$$P(A\cap B)=\frac{^{1}C_1}{^{52}C_1}=\frac{1}{52}$$

Probability of getting either a spade or an ace card

$$\begin{aligned}P(A\cup B) &= P(A)+P(B)-P(A\cap B)\\ &= \frac{13}{52}+\frac{4}{52}-\frac{1}{52}\\ &= \frac{4}{13}\end{aligned}$$

Example 2

Two cards are drawn from a pack of cards. Find the probability that they will be both red or both pictures.

Solution

Let A and B be the events that both cards drawn are red and pictures respectively.

$$P(A)=\frac{^{26}C_2}{^{52}C_2}=\frac{325}{1326}$$

$$P(B)=\frac{^{12}C_2}{^{52}C_2}=\frac{66}{1326}$$

$$P(A\cap B)=\frac{^{6}C_2}{^{52}C_2}=\frac{15}{1326}$$

Probability that both cards drawn are red or pictures

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$= \frac{325}{1326} + \frac{66}{1326} - \frac{15}{1326}$$
$$= \frac{188}{663}$$

Example 3

The probability that a contractor will get a plumbing contract is $\frac{2}{3}$ and the probability that he will not get an electric contract is $\frac{5}{9}$. If the probability of getting any one contract is $\frac{4}{5}$, what is the probability that he will get both the contracts?

Solution

Let A and B be the events that the contractor will get plumbing and electric contracts respectively.

$$P(A) = \frac{2}{3},\ P(\bar{B}) = \frac{5}{9},\ P(A \cup B) = \frac{4}{5}$$
$$P(B) = 1 - P(\bar{B}) = 1 - \frac{5}{9} = \frac{4}{9}$$

Probability that the contractor will get any one contract

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Probability that the contractor will get both the contracts

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$
$$= \frac{2}{3} + \frac{4}{9} - \frac{4}{5}$$
$$= \frac{14}{45}$$

Example 4

A person applies for a job in two firms A and B, the probability of his being selected in the firm A is 0.7 and being rejected in the firm B is 0.5. The probability of at least one of the applications being rejected is 0.6. What is the probability that he will be selected in one of the two firms?

Solution

Let A and B be the events that the person is selected in firms A and B respectively.

$$P(A) = 0.7,\quad P(\bar{B}) = 0.5,\quad P(\bar{A} \cup \bar{B}) = 0.6$$
$$P(\bar{A}) = 1 - P(A) = 1 - 0.7 = 0.3$$
$$P(B) = 1 - P(\bar{B}) = 1 - 0.5 = 0.5$$
$$P(\bar{A} \cup \bar{B}) = P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B}) \quad \ldots(1)$$

Probability that the person will be selected in one of the two firms

$$P(A \cup B) = 1 - P(\bar{A} \cap \bar{B})$$
$$= 1 - \left[P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cup \bar{B})\right] \quad \text{[Using Eq. (1)]}$$
$$= 1 - (0.3 + 0.5 - 0.6)$$
$$= 0.8$$

Example 5

In a group of 1000 persons, there are 650 who can speak Hindi, 400 can speak English, and 150 can speak both Hindi and English. If a person is selected at random, what is the probability that he speaks (i) Hindi only, (ii) English only, (iii) only of the two languages, and (iv) at least one of the two languages?

Solution

Let A and B be the events that a person selected at random speaks Hindi and English respectively.

$$P(A) = \frac{650}{1000},\quad P(B) = \frac{400}{1000},\ P(A \cap B) = \frac{150}{1000}$$

(i) Probability that a person selected at random speaks Hindi only

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$
$$= \frac{650}{1000} - \frac{150}{1000}$$
$$= \frac{1}{2}$$

(ii) Probability that a person selected at random speaks English only

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$
$$= \frac{400}{1000} - \frac{150}{1000}$$
$$= \frac{1}{4}$$

(iii) Probability that a person selected at random speaks only one of the languages.

$$P\left[(A\cap\bar{B})\cup(\bar{A}\cap B)\right]=P(A)+P(B)-2P(A\cap B)$$

$$=\frac{650}{1000}+\frac{400}{1000}-2\left(\frac{150}{1000}\right)$$

$$=\frac{3}{4}$$

(iv) Probability that a person selected at random speaks at least one of the two languages

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$=\frac{650}{1000}+\frac{400}{1000}-\frac{150}{1000}$$

$$=\frac{9}{10}$$

Example 6

A box contains 4 white, 6 red, 5 black balls, and 5 balls of other colours. Two balls are drawn from the box at random. Find the probability that (i) both are white or both are red, and (ii) both are red or both are black.

Solution

Let A, B, and C be the events of drawing white, red and black balls from the box respectively.

$$P(A)=\frac{^{4}C_2}{^{20}C_2}=\frac{3}{95}$$

$$P(B)=\frac{^{6}C_2}{^{20}C_2}=\frac{3}{38}$$

$$P(C)=\frac{^{5}C_2}{^{20}C_2}=\frac{1}{19}$$

(i) Probability that the both balls are white or both are red

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$=\frac{3}{95}+\frac{3}{38}-0$$

$$=\frac{21}{190}$$

(ii) Probability that both balls are red or both are black

$$P(B\cup C)=P(B)+P(C)-P(B\cap C)$$

$$=\frac{3}{38}+\frac{1}{19}-0$$

$$=\frac{5}{38}$$

Example 7

Three students A, B, C are in a running race. A and B have the same probability of winning and each is twice as likely to win as C. Find the probability that B or C wins.

Solution

Let A, B, and C be the events that students A, B, and C win the race respectively.

$$P(A)=P(B)=2P(C)$$

$$P(A)+P(B)+P(C)=1$$

$$2P(C)+2P(C)+P(C)=1$$

$$P(C)=\frac{1}{5}$$

$$\therefore\quad P(A)=\frac{2}{5}\text{ and }P(B)=\frac{2}{5}$$

Probability that student B or C wins

$$P(B\cup C)=P(B)+P(C)-P(B\cap C)$$

$$=\frac{2}{5}+\frac{1}{5}-0$$

$$=\frac{3}{5}$$

Example 8

A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.

Solution

Let A, B and C be the events that the card drawn is a king, a heart and a red card respectively.

$$P(A)=\frac{^{4}C_1}{^{52}C_1}=\frac{4}{52}$$

$$P(B)=\frac{^{13}C_1}{^{52}C_1}=\frac{13}{52}$$

$$P(C)=\frac{^{26}C_1}{^{52}C_1}=\frac{26}{52}$$

$$P(A\cap B)=\frac{^{1}C_1}{^{52}C_1}=\frac{1}{52}$$

$$P(B\cap C)=\frac{^{13}C_1}{^{52}C_1}=\frac{13}{52}$$

$$P(A\cap C)=\frac{^{2}C_1}{^{52}C_1}=\frac{2}{52}$$

$$P(A\cap B\cap C)=\frac{^{1}C_1}{^{52}C_1}=\frac{1}{52}$$

Probability that the card drawn is a king or a heart or a red card.

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)$$

$$=\frac{4}{52}+\frac{13}{52}+\frac{26}{52}-\frac{1}{52}-\frac{13}{52}-\frac{2}{52}+\frac{1}{52}$$

$$=\frac{7}{3}$$

Example 9

From a city, 3 newspapers A, B, C are being published. A is read by 20%, B is read by 16%, C is read by 14%, both A and B are read by 8%, both A and C are read by 5%, both B and C are read by 4% and all three A, B, C are read by 2%. What is the probability that a randomly chosen person (i) reads at least one of these newspapers, and (ii) reads one of these newspapers?

Solution

Let A, B, and C be the events that the person reads newspapers A, B, and C respectively.

$P(A)=0.2,\quad P(B)=0.16\quad P(C)=0.14$

$P(A\cap B)=0.08,\quad P(A\cap B)=0.05,\quad P(B\cap C)=0.04$

$P(A\cap B\cap C)=0.02$

(i) Probability that the person reads at least one of these newspapers

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$$

$$=0.2+0.16+0.14-0.08-0.05-0.04+0.02$$

$$=0.35$$

(ii) Probability that the person reads none of these newspapers

$$P(\bar{A}\cap\bar{B}\cap\bar{C})=1-P(A\cup B\cup C)$$

$$=1-0.35$$

$$=0.65$$

Alternatively, the problem can be solved by a Venn diagram (Fig. 1.2).

(i) $P(\text{the person reads at least one paper})=1-\frac{65}{100}=0.35$

(ii) $P(\text{the person reads none of these papers})=0.65$

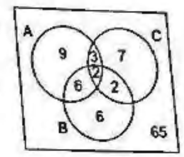

Fig. 1.2

EXERCISE 1.2

1. The probability that a student passes a Physics test is $\frac{2}{3}$ and the probability that he passes both Physics and English tests is $\frac{14}{45}$. The probability that he passes at least one test is $\frac{4}{5}$. What is the probability that the student passes the English test?

   $\left[\text{Ans.: } \frac{4}{9}\right]$

2. What is the probability of drawing a black card or a king from a well-shuffled pack of playing cards?

   $\left[\text{Ans.: } \frac{7}{13}\right]$

3. A pair of unbiased dice is thrown. Find the probability that (i) the sum of spots is either 5 or 10, and (ii) either there is a doublet or a sum less than 6.

   $\left[\text{Ans.: (i) } \frac{7}{36}\ \text{(ii) } \frac{7}{18}\right]$

4. From a pack of well-shuffled cards, a card is drawn at random. What is the probability that the card drawn is a diamond card or a king card?

$\left[\text{Ans.: } \frac{4}{13}\right]$

5. A bag contains 6 red, 5 blue, 3 white, and 4 black balls. A ball is drawn at random. Find the probability that the ball is (i) red or black, and (ii) neither red or black.

$\left[\text{Ans.: (i) } \frac{5}{9} \text{ (ii) } \frac{4}{9}\right]$

6. There are 100 lottery tickets, numbered from 1 to 100. One of them is drawn at random. What is the probability that the number on it is a multiple of 5 or 7?

$\left[\text{Ans.: } \frac{8}{25}\right]$

7. From a group of 6 boys and 4 girls, a committee of 3 is to be formed. Find the probability that the committee will include (i) all three boys or all three girls, (ii) at most two girls, and (iii) at least one girl.

$\left[\text{Ans.: (i) } \frac{1}{5} \text{ (ii) } \frac{29}{30} \text{ (iii) } \frac{5}{6}\right]$

8. From a pack of 52 cards, three cards are drawn at random. Find the probability that (i) all three will be aces or all three kings, (ii) all three are pictures or all three are aces, (iii) none is a picture, (iv) at least one is a picture, (v) none is a spade, (vi) at most two are spades, and (vii) at least one is a spade.

$$\left[\begin{array}{l}\text{Ans.: (i) } \frac{2}{5225} \text{ (ii) } \frac{56}{5225} \text{ (iii) } \frac{38}{85} \text{ (iv) } \frac{47}{85} \\ \text{(v) } \frac{703}{1700} \text{ (vi) } \frac{839}{850} \text{ (vii) } \frac{997}{1700}\end{array}\right]$$

9. From a set of 16 cards numbered 1 to 16, one card is drawn at random. Find the probability that (i) the number obtained is divisible by 3 or 7, and (ii) not divisible by 3 and 7.

$\left[\text{Ans.: (i) } \frac{7}{16} \text{ (ii) } \frac{9}{16}\right]$

10. There are 12 bulbs in a basket of which 4 are working. A person tries to fit them in 3 sockets choosing 3 of the bulbs at random. What is the probability that there will be (i) some light, and (ii) no light in the room?

$\left[\text{Ans.: (i) } \frac{41}{55} \text{ (ii) } \frac{14}{55}\right]$

1.5 CONDITIONAL PROBABILITY

For any two events A and B in a sample space S, the probability of their simultaneous occurrence, i.e., both the events occurrings simultaneously is given by

$$P(A \cap B) = P(A)\, P(B/A)$$

or

$$P(A \cap B) = P(B)\, P(A/B)$$

where $P(B/A)$ is the conditional probability of B given that A has already occurred. $P(A/B)$ is the conditional probability of A given that B has already occurred.

1.6 MULTIPLICATIVE THEOREM FOR INDEPENDENT EVENTS

If A and B are two independent events, the probability of their simultaneous occurrence is given by

$$P(A \cap B) = P(A)\, P(B)$$

$$P(A \cap B) = P(B)\, P(A/B) \quad \text{...(1.1)}$$

Proof $A = (A \cap B) \cup (A \cap \bar{B})$

Since $(A \cap B)$ and $(A \cap \bar{B})$ are mutually exclusive events,

$$P(A) = P(A \cap B) + P(A \cap \bar{B}) \quad \text{[Using Axiom III]}$$

$$= P(B)\, P(A/B) + P(\bar{B})\, P(A/\bar{B})$$

If A and B are independent events, the proportion of A's in B is equal to proportion of A's in $\bar{B}$, i.e., $P(A/B) = P(A/\bar{B})$.

$$P(A) = P(A/B)\left[P(B) + P(\bar{B})\right]$$

$$= P(A/B)$$

Substituting in Eq. (1.1),

$\therefore\ P(A \cap B) = P(A)\, P(B)$

Remark The additive law is used to find the probability of A or B, i.e., $P(A \cup B)$. The multiplicative law is used to find the probability of A and B, i.e., $P(A \cap B)$.

Corollary 1 If A, B and C are three events then

$$P(A \cap B \cap C) = P(A)\,P(B/A)\,P[C/(A \cap B)]$$

If A, B and C are independent events,

$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$$

Corollary 2 If A and B are independent events then A and $\overline{B}$, $\overline{A}$ and B, $\overline{A}$ and $\overline{B}$ are also independent.

Corollary 3 The probability of occurrence of at least one of the events A, B, C is given by

$$P(A \cup B \cup C) = 1 - P(\overline{A} \cap \overline{B} \cap \overline{C})$$

If A, B, and C are independent events, their complements will also be independent.

$$P(A \cup B \cup C) = 1 - P(\overline{A})\,P(\overline{B})\,P(\overline{C})$$

Pairwise Independence and Mutual Independence The events A, B and C are mutually independent if the following conditions are satisfied simultaneously:

$$P(A \cap B) = P(A)\,P(B)$$

$$P(B \cap C) = P(B)\,P(C)$$

$$P(A \cap C) = P(A)\,P(C)$$

and $\quad P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$

If the last condition is not satisfied, the events are said to be pairwise independent. Hence, mutually independent events are always pairwise independent but not vices versa.

Example 1

If A and B are two events such that $P(A) = \frac{2}{3}$, $P(\overline{A} \cap B) = \frac{1}{6}$ and $P(A \cap B) = \frac{1}{3}$, find $P(B)$, $P(A \cup B)$, $P(A/B)$, $P(B/A)$, $P(\overline{A} \cup B)$ and $P(\overline{B})$. Also, examine whether the events A and B are (i) equally likely, (ii) exhaustive, (iii) mutually exclusive, and (iv) independent.

Solution

$$P(B) = P(\overline{A} \cap B) + P(A \cap B) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{2}{3} + \frac{1}{2} - \frac{1}{3} = \frac{5}{6}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{1}{2}\right)} = \frac{2}{3}$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)} = \frac{1}{2}$$

$$P(\overline{A} \cup B) = P(\overline{A}) + P(B) - P(\overline{A} \cap B) = \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{2}{3}$$

$$P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B) = 1 - \frac{5}{6} = \frac{1}{6}$$

$$P(\overline{B}) = 1 - P(B) = 1 - \frac{1}{2} = \frac{1}{2}$$

(i) Since $P(A) \neq P(B)$, A and B are not equally like events.

(ii) Since $P(A \cup B) \neq 1$, A and B are not exhaustive events.

(iii) Since $P(A \cap B) \neq 0$, A and B are not mutually exclusive events.
(iv) Since $P(A \cap B) = P(A)\,P(B)$, A and B are independent events.

Example 2

If A and B are two events such that $P(A) = 0.3$, $P(B) = 0.4$, $P(A \cap B) = 0.2$, find (i) $P(A \cup B)$, (ii) $P(\bar{A}/B)$, and (iii) $P(A/\bar{B})$.

Solution

(i) $$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.4 - 0.2 = 0.5$$

(ii) $$P(\bar{A}/B) = \frac{P(\bar{A} \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)} = \frac{0.4 - 0.2}{0.4} = 0.5$$

(iii) $$P(A/\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} = \frac{P(A) - P(A \cap B)}{1 - P(B)} = \frac{0.3 - 0.2}{1 - 0.4} = \frac{1}{6}$$

Example 3

If A and B are two events with $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$, $P(A \cap B) = \frac{1}{12}$. Find (i) $P(A/B)$, (ii) $P(B/A)$, (iii) $P(B/\bar{A})$, and (iv) $P(A \cap \bar{B})$.

Solution

(i) $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3}$$

(ii) $$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{12}}{\frac{1}{3}} = \frac{1}{4}$$

(iii) $$P(B/\bar{A}) = \frac{P(B \cap \bar{A})}{P(\bar{A})} = \frac{P(B) - P(B \cap A)}{1 - P(A)} = \frac{\frac{1}{4} - \frac{1}{12}}{1 - \frac{1}{3}} = \frac{1}{4}$$

(iv) $$P(A \cap \bar{B}) = P(A) - P(A \cap B) = \frac{1}{3} - \frac{1}{12} = \frac{1}{4}$$

Example 4

Find the probability of drawing a queen and a king from a pack of cards in two consecutive draws, the cards drawn not being replaced.

Solution

Let A be the event that the card drawn is a queen.

$$P(A) = \frac{^4C_1}{^{52}C_1} = \frac{4}{52} = \frac{1}{13}$$

Let B be the event that the cards drawn are a king in the second draw given that the first card drawn is a queen.

$$P(B/A) = \frac{^4C_1}{^{51}C_1} = \frac{4}{51}$$

Probability that the cards drawn are a queen and a king

$$P(A \cap B) = P(A)\,P(B/A) = \frac{4}{52} \times \frac{4}{51} = \frac{4}{663}$$

Example 5

A bag contains 3 red and 4 white balls. Two draws are made without replacement. What is the probability that both the balls are red?

Solution

Let A be the event that the ball drawn is red in the first draw.

$$P(A)=\frac{3}{7}$$

Let B be the event that the ball drawn is red in the second draw given that the first ball drawn is red.

$$P(B/A)=\frac{2}{6}$$

Probability that both the balls are red

$$P(A\cap B)=P(A)\,P(B/A)$$
$$=\frac{3}{7}\times\frac{2}{6}$$
$$=\frac{1}{7}$$

Example 6

A bag contains 8 red and 5 white balls. Two successive draws of 3 balls each are made such that (i) the balls are replaced before the second trial, and (ii) the balls are not replaced before the second trial. Find the probability that the first draw will give 3 white and the second, 3 red balls.

Solution

Let A be the event that all 3 balls obtained at the first draw are white, and B be the event that all the 3 balls obtained at the second draw are red.

(i) When balls are replaced before the second trial,

$$P(A)=\frac{{}^{5}C_3}{{}^{13}C_3}=\frac{5}{143}$$

$$P(B)=\frac{{}^{8}C_3}{{}^{13}C_3}=\frac{28}{143}$$

Probability that the first draw will give 3 white and the second, 3 red balls

$$P(A\cap B)=P(A)\,P(B)$$
$$=\frac{5}{143}\times\frac{28}{143}$$
$$=\frac{140}{20449}$$

(ii) When the balls are not replaced before the second trial

$$P(B/A)=\frac{{}^{8}C_3}{{}^{10}C_3}=\frac{7}{15}$$

Probability that the first draw will give 3 white and the second, 3 red balls

$$P(A\cap B)=P(A)\,P(B/A)$$
$$=\frac{5}{143}\times\frac{7}{15}$$
$$=\frac{7}{429}$$

Example 7

From a bag containing 4 white and 6 black balls, two balls are drawn at random. If the balls are drawn one after the other without replacements, find the probability that the first ball is white and the second ball is black.

Solution

Let A be the event that the first ball drawn is white and B be the event that the second ball drawn is black given that the first ball drawn is white.

$$P(A)=\frac{4}{10}$$

$$P(B/A)=\frac{6}{9}$$

Probability that the first ball is white and the second ball is black.

$$P(A\cap B)=P(A)\,P(B/A)$$
$$=\frac{4}{10}\times\frac{6}{9}$$
$$=\frac{4}{15}$$

Example 8

Data on readership of a certain magazine show that the proportion of male readers under 35 is 0.40 and that over 35 is 0.20. If the proportion of readers under 35 is 0.70, find the probability of subscribers that are females over 35 years. Also, calculate the probability that a randomly selected male subscriber is under 35 years of age.

Solution

Let A be the event that the reader of the magazine is a male. Let B be the event that reader of the magazine is over 35 years of age.

$$P(A \cap \overline{B}) = 0.40, \qquad P(A \cap B) = 0.20, \qquad P(\overline{B}) = 0.7$$

$$\begin{aligned} P(B) &= 1 - P(\overline{B}) \\ &= 1 - 0.7 \\ &= 0.3 \end{aligned}$$

(i) Probability of subscribers that are females over 35 years

$$\begin{aligned} P(\overline{A} \cap B) &= P(B) - P(A \cap B) \\ &= 0.3 - 0.2 \\ &= 0.1 \end{aligned}$$

(ii) Probability that a randomly selected male subscriber is under 35 years of age

$$\begin{aligned} P(\overline{B}/A) &= \frac{P(A \cap \overline{B})}{P(A)} \\ &= \frac{P(A \cap \overline{B})}{P(A \cap B) + P(A \cap \overline{B})} \\ &= \frac{0.4}{0.2 + 0.4} \\ &= \frac{0.4}{0.6} \\ &= \frac{2}{3} \end{aligned}$$

Example 9

From a city population, the probability of selecting (a) a male or a smoker is $\frac{7}{10}$, (b) a male smoker is $\frac{2}{5}$, and (c) a male, if a smoker is

already selected, is $\frac{2}{3}$. Find the probability of selecting (i) a nonsmoker, (ii) a male, and (iii) a smoker, if a male is first selected.

Solution

Let A be the event that a male is selected. Let B be the event that a smoker is selected.

$$P(A \cup B) = \frac{7}{10}, \quad P(A \cap B) = \frac{2}{5}, \quad P(A/B) = \frac{2}{3}$$

(i) Probability of selecting a nonsmoker

$$\begin{aligned} P(\overline{B}) &= 1 - P(B) \\ &= 1 - \frac{P(A \cap B)}{P(A/B)} \\ &= 1 - \frac{\left(\frac{2}{5}\right)}{\left(\frac{2}{3}\right)} \\ &= \frac{2}{5} \end{aligned}$$

(ii)

$$\begin{aligned} P(B) &= 1 - P(\overline{B}) \\ &= 1 - \frac{2}{5} \\ &= \frac{3}{5} \end{aligned}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad \text{...(1)}$$

Probability of selecting a male

$$\begin{aligned} P(A) &= P(A \cup B) + P(A \cap B) - P(B) \qquad \text{[Using Eq. (1)]} \\ &= \frac{7}{10} + \frac{2}{5} - \frac{3}{5} \\ &= \frac{1}{2} \end{aligned}$$

(iii) Probability of selecting a smoker if a male is first selected

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$=\frac{\left(\frac{2}{5}\right)}{\left(\frac{1}{2}\right)}$$

$$=\frac{4}{5}$$

Example 10

Sixty per cent of the employees of the XYZ corporation are college graduates. Of these, ten percent are in sales. Of the employee who did not graduate from college, eighty percent are in sales. What is the probability that

(i) an employee selected at random is in sales?

(ii) an employee selected at random is neither in sales nor a college graduate?

Solution

Let A be the event that an employee is a college graduate. Let B be the event that an employee is in sales.

$$P(A)=0.6,\ P(B/A)=0.10,\ P(B/\bar{A})=0.8$$

$$P(\bar{A})=1-P(A)=1-0.60=0.40$$

(i) Probability that an employee is in sales

$$\begin{aligned}P(B)&=P(A\cap B)+P(\bar{A}\cap B)\\&=P(A)\,P(B/A)+P(\bar{A})\,P(B/\bar{A})\\&=(0.6\times0.1)+(0.40\times0.80)\\&=0.38\end{aligned}$$

(ii) Probability that an employee is neither in sales nor a college graduate

$$\begin{aligned}P(\bar{A}\cap\bar{B})&=1-P(A\cup B)\\&=1-[P(A)+P(B)-P(A\cap B)]\\&=1-[P(A)+P(B)-P(A)\,P(B/A)]\\&=1-[0.60+0.38-(0.60\times0.10)]\\&=0.08\end{aligned}$$

Example 11

If A and B are two events such that $P(A)=\frac{3}{8}$, $P(B)=\frac{5}{8}$ and $P(A\cup B)=\frac{3}{4}$, find P(A/B) and P(B/A). Show whether A and B are independent.

Solution

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$\frac{3}{4}=\frac{3}{8}+\frac{5}{8}-P(A\cap B)$$

$$P(A\cap B)=\frac{1}{4}$$

$$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{\left(\frac{1}{4}\right)}{\left(\frac{5}{8}\right)}=\frac{2}{5}$$

$$P(B/A)=\frac{P(A\cap B)}{P(A)}=\frac{\left(\frac{1}{4}\right)}{\left(\frac{3}{8}\right)}=\frac{2}{3}$$

$$P(A)\,P(B)=\frac{3}{8}\times\frac{5}{8}=\frac{15}{64}$$

$$P(A\cap B)\neq P(A)\,P(B)$$

Hence, the events A and B are not independent.

Example 12

The probability that a student A solves a mathematics problem is $\frac{2}{5}$ and the probability that a student B solves it is $\frac{2}{3}$. What is the probability *that (i) the problem is not solved, (ii) the problem is solved, and (iii) both A and B, working independently of each other, solve the problem?*

Solution

Let A and B be events that students A and B solve the problem respectively.

$$P(A)=\frac{2}{5},\ P(B)=\frac{2}{3}$$

Events A and B are independent.

Probability that the student A does not solve the problem

$$P(\bar{A})=1-P(A)=1-\frac{2}{5}=\frac{3}{5}$$

Probability that the student B does not solve the problem

$$P(\bar{B})=1-P(B)=1-\frac{2}{3}=\frac{1}{3}$$

(i) Probability that the problem is not solved

$$P(\bar{A}\cap\bar{B})=P(\bar{A})\,P(\bar{B})=\frac{3}{5}\times\frac{1}{3}=\frac{1}{5}$$

(ii) Probability that the problem is solved

$$P(A\cup B)=1-P(\bar{A}\cap\bar{B})=1-\frac{1}{5}=\frac{4}{5}$$

(iii) Probability that both A and B solve the problem

$$P(A\cap B)=P(A)\,P(B)=\frac{2}{5}\times\frac{2}{3}=\frac{4}{15}$$

Example 13

The probability that the machine A will perform a usual function in 5 years' time is $\frac{1}{4}$, while the probability that the machine B will perform the function in 5 years' time is $\frac{1}{3}$. Find the probability that both machines will perform the usual function.

Solution

Let A and B be the events that machines A and B will perform the usual function respectively.

$$P(A)=\frac{1}{4}$$

$$P(B)=\frac{1}{3}$$

Events A and B are independent.

Probability that both machines will perform the usual function

$$P(A\cap B)=P(A)\,P(B)=\frac{1}{4}\times\frac{1}{3}=\frac{1}{12}$$

Example 14

A person A is known to hit a target in 3 out of 4 shots, whereas another person B is known to hit the same target in 2 out of 3 shots. Find the probability of the target being hit at all when they both try.

[Summer 2015]

Solution

Let A and B be the events that the persons A and B hit the target respectively.

$$P(A)=\frac{3}{4}$$

$$P(B)=\frac{2}{3}$$

Events A and B are independent.

Probability that the person A will not hit the target $=P(\bar{A})=1-P(A)=1-\frac{3}{4}=\frac{1}{4}$

Probability that the person B will not hit the target $= P(\bar{B}) = 1 - P(B) = 1 - \frac{2}{3} = \frac{1}{3}$

Probability that the target is not hit at all

$$P(\bar{A} \cap \bar{B}) = P(\bar{A})\, P(\bar{B})$$
$$= \frac{1}{4} \times \frac{1}{3}$$
$$= \frac{1}{12}$$

Probability that the target is hit at all when they both try

$$P(A \cup B) = 1 - P(\bar{A} \cap \bar{B})$$
$$= 1 - \frac{1}{12}$$
$$= \frac{11}{12}$$

Aliter

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= P(A) + P(B) - P(A)\, P(B) \quad [\because A \text{ and } B \text{ independent}]$$
$$= \frac{3}{4} + \frac{2}{3} - \frac{3}{4} \times \frac{2}{3}$$
$$= \frac{11}{12}$$

Example 15

The odds against A speaking the truth are 4 : 6 while the odds in favour of B speaking the truth are 7 : 3. What is the probability that A and B contradict each other in stating the same fact?

Solution

Let A and B be events that A and B speak the truth respectively.

$$P(A) = \frac{6}{10}$$
$$P(B) = \frac{7}{10}$$

Events A and B are independent.

Probability that A speaks a lie $= P(\bar{A}) = 1 - P(A) = 1 - \frac{6}{10} = \frac{4}{10}$

Probability that B speaks a lie $= P(\bar{B}) = 1 - P(B) = 1 - \frac{7}{10} = \frac{3}{10}$

Probability that A and B contradict each other

$$P\left[(A \cap \bar{B}) \cup (\bar{A} \cap B)\right] = P(A \cap \bar{B}) + P(\bar{A} \cap B) \quad \left[\because (A \cap \bar{B}) \text{ and } (\bar{A} \cap B) \text{ are mutually exclusive events}\right]$$
$$= P(A)\, P(\bar{B}) + P(\bar{A})\, P(B)$$
$$= \frac{6}{10} \times \frac{3}{10} + \frac{4}{10} \times \frac{7}{10}$$
$$= \frac{23}{50}$$

Example 16

An urn contains 10 red, 5 white and 5 blue balls. Two balls are drawn at random. Find the probability that they are not of the same colour.

Solution

Let A, B, and C be the events that two balls drawn at random be of the same colour, i.e., red, white, and blue respectively.

$$P(A) = \frac{{}^{10}C_2}{{}^{20}C_2} = \frac{9}{38}$$
$$P(B) = \frac{{}^{5}C_2}{{}^{20}C_2} = \frac{1}{19}$$
$$P(C) = \frac{{}^{5}C_2}{{}^{20}C_2} = \frac{1}{19}$$

Events A, B, and C are independent.

Probability that both balls drawn are of same colour

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$= \frac{9}{38} + \frac{1}{19} + \frac{1}{19}$$
$$= \frac{13}{38}$$

Probability that both balls drawn are not of the same colour

$$P(\bar{A} \cap \bar{B} \cap \bar{C}) = 1 - P(A \cup B \cup C)$$
$$= 1 - \frac{13}{38}$$
$$= \frac{25}{38}$$

Example 17

A problem in statistics is given to three students A, B and C, whose chances of solving it independently are $\frac{1}{2}, \frac{1}{3},$ and $\frac{1}{4}$ respectively. Find the probability that

(i) the problem is solved

(ii) at least two of them are able to solve the problem

(iii) exactly two of them are able to solve the problem

(iv) exactly one of them is able to solve the problem

Solution

Let A, B, and C be the events that students A, B, and C solve the problem respectively.

$$P(A)=\frac{1}{2},\ P(B)=\frac{1}{3},\ P(C)=\frac{1}{4}$$

Events A, B, and C are independent.

(i) Probability that the problem is solved or at least one of them is able to solve the problem is same.

$$\begin{aligned}P(A\cup B\cup C) &= P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)\\ &\quad +P(A\cap B\cap C)\\ &= P(A)+P(B)+P(C)-P(A)\,P(B)-P(A)\,P(C)-P(B)\,P(C)\\ &\quad +P(A)\,P(B)\,P(C)\\ &= \frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\left(\frac{1}{2}\times\frac{1}{3}\right)-\left(\frac{1}{2}\times\frac{1}{4}\right)-\left(\frac{1}{3}\times\frac{1}{4}\right)+\left(\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\right)\\ &= \frac{3}{4}\end{aligned}$$

(ii) Probability that at least two of them are able to solve the problem

$$\begin{aligned}P[(A\cap B)\cup(B\cap C)\cup(A\cap C)] &= P(A\cap B)+P(B\cap C)+P(A\cap C)-2P(A\cap B\cap C)\\ &= P(A)\,P(B)+P(B)\,P(C)+P(A)\,P(C)\\ &\quad -2P(A)\,P(B)\,P(C)\\ &= \left(\frac{1}{2}\times\frac{1}{3}\right)+\left(\frac{1}{3}\times\frac{1}{4}\right)+\left(\frac{1}{2}\times\frac{1}{4}\right)-2\left(\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\right)\\ &= \frac{7}{24}\end{aligned}$$

(iii) Probability that exactly two of them are able to solve the problem

$$\begin{aligned}&P[(A\cap B\cap \bar{C})\cup(A\cap \bar{B}\cap C\cup(\bar{A}\cap B\cap C]\\ &= P(A\cap B)+P(B\cap C)+P(A\cap C)-3P(A\cap B\cap C)\\ &= P(A)\,P(B)+P(B)\,P(C)+P(A)\,P(C)-3P(A)\,P(B)\,P(C)\\ &= \left(\frac{1}{2}\times\frac{1}{3}\right)+\left(\frac{1}{3}\times\frac{1}{4}\right)+\left(\frac{1}{2}\times\frac{1}{4}\right)-3\left(\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\right)\\ &= \frac{1}{4}\end{aligned}$$

(iv) Probability that exactly one of them is able to solve the problem

$$\begin{aligned}&P[A\cap \bar{B}\cap \bar{C})\cup(\bar{A}\cap B\cap \bar{C})\cup(\bar{A}\cap \bar{B}\cap C)]\\ &= P(A)+P(B)+P(C)-2P(A\cap B)-2P(B\cap C)-2P(A\cap C)+3P(A\cap B\cap C)\\ &= \frac{1}{2}+\frac{1}{3}+\frac{1}{4}-2\left(\frac{1}{2}\times\frac{1}{3}\right)-2\left(\frac{1}{3}\times\frac{1}{4}\right)-2\left(\frac{1}{2}\times\frac{1}{4}\right)+3\left(\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\right)\\ &= \frac{11}{24}\end{aligned}$$

Example 18

A husband and wife appeared in an interview for two vacancies in an office. The probability of the husband's selection is $\frac{1}{7}$ and that of the wife's selection is $\frac{1}{5}$. Find the probability that (i) both of them are selected, (ii) only one of them is selected, (iii) none of them is selected, and (iv) at least one of them is selected.

Solution

Let A and B be the events that the husband and wife are selected respectively.

$$P(A)=\frac{1}{7},\quad P(B)=\frac{1}{5}$$

Events A and B are independent.

(i) Probability that both of them are selected

$$\begin{aligned}P(A\cap B) &= P(A)\,P(B)\\ &= \frac{1}{7}\times\frac{1}{5}\\ &= \frac{1}{35}\end{aligned}$$

(ii) Probability that at least one of them is selected

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{1}{7} + \frac{1}{5} - \frac{1}{35}$$
$$= \frac{11}{35}$$

(iii) Probability that none of them is selected

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$$
$$= 1 - \frac{11}{35}$$
$$= \frac{24}{35}$$

(iv) Probability that only one of them is selected

$$P\left[A \cap \bar{B}) \cup (\bar{A} \cap B)\right] = P(A \cup B) - P(A \cap B)$$
$$= \frac{11}{35} - \frac{1}{35}$$
$$= \frac{10}{35}$$
$$= \frac{2}{7}$$

Example 19

There are two bags. The first contains 2 red and 1 white ball, whereas the second bag has only 1 red and 2 white balls. One ball is taken out at random from the first bag and put in the second. Then a ball is chosen at random from the second bag. What is the probability that this last ball is red?

Solution

There are two mutually exclusive cases.

Case I: A red ball is transferred from the first bag to the second bag and a red ball is drawn from it.

Case II: A white ball is transferred from the first bag to the second bag and then a red ball is drawn from it.

Let A be the event of transferring a red ball from the first bag, and B be the event of transferring a white ball from the first bag.

$$P(A) = \frac{2}{3}$$

$$P(B) = \frac{1}{3}$$

Let E be the event of drawing a red ball from the second bag.

$$P(E/A) = \frac{2}{4}$$
$$P(E/B) = \frac{1}{4}$$
$$P(\text{Case I}) = P(A \cap E)$$
$$= P(A)\,P(E/A)$$
$$= \frac{2}{3} \times \frac{2}{4}$$
$$= \frac{1}{3}$$
$$P(\text{Case II}) = P(B \cap E)$$
$$= P(B)\,P(E/B)$$
$$= \frac{1}{3} \times \frac{1}{4}$$
$$= \frac{1}{12}$$
$$P\left[(A \cap E) \cup (B \cap E)\right] = P(A \cap E) + P(B \cap E)$$
$$= \frac{1}{3} + \frac{1}{12}$$
$$= \frac{5}{12}$$

Example 20

An urn contains four tickets marked with numbers 112, 121, 211, and 222, and one ticket is drawn. Let A_i ($i = 1, 2, 3$) be the event that the i^{th} digit of the ticket drawn is 1. Show that the events A_1, A_2, A_3 are pairwise independent but not mutually independent.

Solution

$$A_1 = \{112, 121\},\ A_2 = \{112, 211\},\ A_3 = \{121, 211\}$$
$$A_1 \cap A_2 = \{112\},\ A_1 \cap A_3 = \{121\},\ A_2 \cap A_3 = \{211\}$$
$$P(A_1) = \frac{2}{4} = \frac{1}{2} = P(A_2) = P(A_3)$$
$$P(A_1 \cap A_2) = \frac{1}{4} = P(A_1 \cap A_3) = P(A_2 \cap A_3)$$

$$P(A_1 \cap A_2) = P(A_1)\, P(A_2) = \frac{1}{4}$$

$$P(A_2 \cap A_3) = P(A_2)\, P(A_3) = \frac{1}{4}$$

$$P(A_1 \cap A_3) = P(A_1)\, P(A_3) = \frac{1}{4}$$

Hence, events A_1, A_2, and A_3 are pairwise independent.

$$P(A_1 \cap A_2 \cap A_3) = P(\phi) = 0$$

$$P(A_1 \cap A_2 \cap A_3) \neq P(A_1)\, P(A_2)\, P(A_3)$$

Hence, events A_1, A_2, and A_3 are not mutually independent.

EXERCISE 1.3

1. Find the probability of drawing 2 red balls in succession from a bag containing 4 red and 5 black balls when the ball that is drawn first is (i) not replaced, and (ii) replaced.

$\left[\text{Ans.: (i) } \frac{1}{6} \text{ (ii) } \frac{16}{81}\right]$

2. Two aeroplanes bomb a target in succession. The probability of each correctly scoring a hit is 0.3 and 0.2 respectively. The second will bomb only if the first misses the target. Find the probability that (i) the target is hit, and (ii) both fail to score hits.

[Ans.: (i) 0.44 (ii) 0.56]

3. Box A contains 5 red and 3 white marbles and Box B contains 2 red and 6 white marbles. If a marble is drawn from each box, what is the probability that they are both of the same colour?

[Ans.: 0.109]

4. Two marbles are drawn in succession from a box containing 10 red, 30 white, 20 blue, and 15 orange marbles, with replacement being made after each draw. Find the probability that (i) both are white, and (ii) the first is red and the second is white.

$\left[\text{Ans.: (i) } \frac{4}{25} \text{ (ii) } \frac{4}{75}\right]$

5. A, B, C are aiming to shoot a balloon. A will succeed 4 times out of 5 attempts. The chance of B to shoot the balloon is 3 out of 4, and that of C is 2 out of 3. If the three aim the balloon simultaneously, find the probability that at least two of them hit the balloon.

$\left[\text{Ans.: } \frac{5}{6}\right]$

6. There are 12 cards numbered 1 to 12 in a box. If two cards are selected, what is the probability that the sum is odd (i) with replacement, and (ii) without replacement?

$\left[\text{Ans.: (i) } \frac{1}{2} \text{ (ii) } \frac{6}{11}\right]$

7. Two cards are drawn from a well-shuffled pack of 52 cards. Find the probability that they are both aces if the first card is (i) replaced, and (ii) not replaced.

$\left[\text{Ans.: (i) } \frac{1}{169} \text{ (ii) } \frac{1}{221}\right]$

8. A can hit a target 2 times in 5 shots; B, 3 times in 4 shots; and C, 2 times in 3 shots. They fire a volley. What is the probability that at least 2 shots hit the target?

$\left[\text{Ans.: } \frac{2}{3}\right]$

9. There are two bags. The first bag contains 5 red and 7 white balls and the second bag contains 3 red and 12 white balls. One ball is taken out at random from the first bag and is put in the second bag. Now, a ball is drawn from the second bag. What is the probability that this last ball is red?

$\left[\text{Ans.: } \frac{41}{192}\right]$

10. In a shooting competition, the probability of A hitting the target is $\frac{1}{2}$; of B, is $\frac{2}{3}$; and of C, is $\frac{3}{4}$. If all of them fire at the target, find the probability that (i) none of them hits the target, and (ii) at least one of them hits the target.

$\left[\text{Ans.: (i) } \frac{1}{24} \text{ (ii) } \frac{23}{24}\right]$

11. The odds against a student X solving a statistics problem are 12 to 10 and the odds in favour of a student Y solving the problem are 6 to 9.

What is the probability that the problem will be solved when both try independently of each other?

$\left[\text{Ans.: } \frac{37}{55}\right]$

12. A bag contains 6 white and 9 black balls. Four balls are drawn at random twice. Find the probability that the first draw will give 4 white balls and the second draw will give 4 black balls if (i) the balls are replaced, and (ii) the balls are not replaced before the second draw.

$\left[\text{Ans.: (i) } \frac{6}{5915} \text{ (ii) } \frac{3}{715}\right]$

13. An urn contains 10 white and 3 black balls. Another urn contains 3 white and 5 black balls. Two balls are transferred from the first urn to the second urn and then one ball is drawn from the latter. What is the probability that the ball drawn is white?

$\left[\text{Ans.: } \frac{5}{26}\right]$

14. A man wants to marry a girl having the following qualities: fair complexion—the probability of getting such a girl is $\frac{1}{20}$, handsome dowry—the probability is $\frac{1}{50}$, westernized manners and etiquettes—the probability of this is $\frac{1}{100}$. Find the probability of his getting married to such a girl when the possessions of these three attributes are independent.

$\left[\text{Ans.: } \frac{1}{100000}\right]$

15. A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98 and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the fire engine and ambulance will be available.

[Ans.: 0.9016]

16. In a certain community, 36% of the families own a dog and 22% of the families that own a dog also own a cat. In addition, 30% of the families own a cat. What is the probability that (i) a randomly selected family owns both a dog and a cat, and (ii) a randomly selected family owns a dog given that it owns a cat?

[Ans.: (i) 0.0792 (ii) 0.264]

1.7 BAYES' THEOREM

Let $A_1, A_2, \ldots, A_n$ be n mutually exclusive and exhaustive events with $P(A_i) \neq 0$ for $i = 1, 2, \ldots, n$ in a sample space S. Let B be an event that can occur in combination with any one of the events $A_1, A_2, \ldots, A_n$ with $P(B) \neq 0$. The probability of the event A_i when the event B has actually occurred is given by

$$P(A_i/B) = \frac{P(A_i)\, P(B/A_i)}{\sum_{i=1}^{n} P(A_i)\, P(B/A_i)}$$

Proof Since $A_1, A_2, \ldots, A_n$ are n mutually exclusive and exhaustive events of the sample space S,

$$S = A_1 \cup A_2 \cup \ldots \cup A_n$$

Since B is another event that can occur in combination with any of the mutually exclusive and exhaustive events $A_1, A_2, \ldots, A_n$,

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$$

Taking probability of both the sides,

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$$

The events $(A_1 \cap B), (A_2 \cap B)$, etc., are mutually exclusive.

$$P(B) = \sum_{i=1}^{n} P(A_i \cap B) = \sum_{i=1}^{n} P(A_i)\, P(B/A_i)$$

The conditional probability of an event A given that B has already occurred is given by

$$P(A_i/B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)\, P(B/A_i)}{P(B)} = \frac{P(A_i)\, P(B/A_i)}{\sum_{i=1}^{n} P(A_i)\, P(B/A_i)}$$

Example 1

A company has two plants to manufacture hydraulic machines. Plant I manufactures 70% of the hydraulic machines, and Plant II manufactures 30%. At Plant I, 80% of hydraulic machines are rated standard quality; and at Plant II, 90% of hydraulic machines are rated standard quality. A machine is picked up at random and is found to be of standard quality. What is the chance that it has come from Plant I? **[Summer 2015]**

Solution

Let A_1 and A_2 be the events that the hydraulic machines are manufactured in Plant I and Plant II respectively. Let B be the event that the machine picked up is found to be of standard quality.

$$P(A_1) = \frac{70}{100} = 0.7$$

$$P(A_2) = \frac{30}{100} = 0.3$$

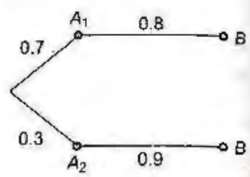

Fig. 1.3

Probability that the machine is of standard quality given that it is manufactured in Plant I

$$P(B/A_1) = \frac{80}{100} = 0.8$$

Probability that the machine is of standard quality given that it is manufactured in Plant II

$$P(B/A_2) = \frac{90}{100} = 0.9$$

Probability that a machine is manufactured in Plant I given that it is of standard quality

$$P(A_1/B) = \frac{P(A_1)\,P(B/A_1)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2)}$$

$$= \frac{0.7 \times 0.8}{0.7 \times 0.8 + 0.3 \times 0.9}$$

$$= 0.6747$$

Example 2

A bag A contains 2 white and 3 red balls, and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that the red ball is drawn from the bag B.

Solution

Let A_1 and A_2 be the events that the ball is drawn from bags A and B respectively. Let B be the event that the ball drawn is red.

$$P(A_1) = \frac{1}{2}$$

$$P(A_2) = \frac{1}{2}$$

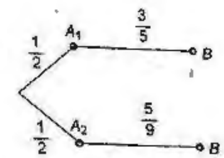

Fig. 1.4

Probability that the ball drawn is red given that it is drawn from the bag A

$$P(B/A_1) = \frac{3}{5}$$

Probability that the ball drawn is red given that it is drawn from the bag B

$$P(B/A_2) = \frac{5}{9}$$

Probability that the ball is drawn from the bag B given that it is red

$$P(A_2/B) = \frac{P(A_2)\,P(B/A_2)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2)}$$

$$= \frac{\frac{1}{2} \times \frac{5}{9}}{\left(\frac{1}{2} \times \frac{3}{5}\right) + \left(\frac{1}{2} \times \frac{5}{9}\right)}$$

$$= \frac{25}{52}$$

Example 3

The chances that Doctor A will diagnose a disease X correctly is 60%. The chances that a patient will die by his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of Doctor A, who had the disease X, died. What is the chance that his disease was diagnosed correctly?

Solution

Let A_1 be the event that the disease X is diagnosed correctly by Doctor A. Let A_2 be the event that the disease X is not diagnosed correctly by Doctor A. Let B be the event that a patient of Doctor A who has the disease X, dies.

$$P(A_1)=\frac{60}{100}=0.6$$
$$P(A_2)=P(\bar{A}_1)=1-P(A_1)=0.4$$

Probability that the patient of Doctor A who has the disease X dies given that the disease X is diagnosed correctly

$$P(B/A_1)=\frac{40}{100}=0.4$$

Probability that the patient of Doctor A who has the disease X dies given that the disease X is not diagnosed correctly

$$P(B/A_2)=\frac{70}{100}=0.7$$

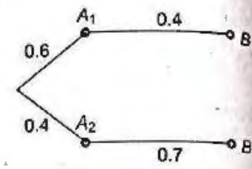

Fig. 1.5

Probability that the disease X is diagnosed correctly given that a patient of Doctor A who has the disease X dies

$$P(A_1/B)=\frac{P(A_1)\,P(B/A_1)}{P(A_1)\,P(B/A_1)+P(A_2)\,P(B/A_2)}$$
$$=\frac{0.6\times 0.4}{(0.6\times 0.4)+(0.4\times 0.7)}$$
$$=\frac{6}{13}$$

Example 4

In a bolt factory, machines A, B, C manufacture 25%, 35%, and 40% of the total output and out of the total manufacturing, 5%, 4%, and 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. Find the probabilities that it is manufactured from (i) Machine A, (ii) Machine B, and (iii) Machine C.

Solution

Let A_1, A_2 and A_3 be the events that bolts are manufactured by machines A, B, and C respectively. Let B be the event that the bolt drawn is defective.

$$P(A_1)=\frac{25}{100}=0.25$$
$$P(A_2)=\frac{35}{100}=0.35$$
$$P(A_3)=\frac{40}{100}=0.4$$

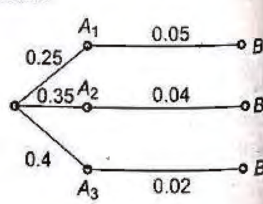

Fig. 1.6

Probability that the bolt drawn is defective given that it is manufactured from Machine A

$$P(B/A_1)=\frac{5}{100}=0.05$$

Probability that the bolt drawn is defective given that it is manufactured from Machine B

$$P(B/A_2)=\frac{4}{100}=0.04$$

Probability that the bolt drawn is defective given that it is manufactured from Machine C

$$P(B/A_3)=\frac{2}{100}=0.02$$

(i) Probability that a bolt is manufactured from Machine A given that it is defective

$$P(A_1/B)=\frac{P(A_1)\,P(B/A_1)}{P(A_1)\,P(B/A_1)+P(A_2)\,P(B/A_2)+P(A_3)\,P(B/A_3)}$$
$$=\frac{0.25\times 0.05}{(0.25\times 0.05)+(0.35\times 0.04)+(0.4\times 0.02)}$$
$$=0.3623$$

(ii) Probability that a bolt is manufactured from Machine B given that it is defective

$$P(A_2/B)=\frac{P(A_2)\,P(B/A_2)}{P(A_1)\,P(B/A_1)+P(A_2)\,P(B/A_2)+P(A_3)\,P(B/A_3)}$$
$$=\frac{0.35\times 0.04}{(0.25\times 0.05)+(0.35\times 0.04)+(0.4\times 0.02)}$$
$$=0.4058$$

(iii) Probability that a bolt is manufactured from Machine C given that it is defective

$$P(A_3/B)=\frac{P(A_3)\,P(B/A_3)}{P(A_1)\,P(B/A_1)+P(A_2)\,P(B/A_2)+P(A_3)\,P(B/A_3)}$$
$$=\frac{0.4\times 0.02}{(0.25\times 0.05)+(0.35\times 0.04)+(0.4\times 0.02)}$$
$$=0.2319$$

Example 5

A businessman goes to hotels X, Y, Z for 20%, 50%, 30% of the time respectively. It is known that 5%, 4%, 8% of the rooms in X, Y, Z hotels have faulty plumbings. What is the probability that the businessman's room having faulty plumbing is assigned to Hotel Z?

Solution

Let A_1, A_2 and A_3 be the events that the businessman goes to hotels X, Y, Z respectively. Let B be the event that the rooms have faulty plumbings.

$$P(A_1) = \frac{20}{100} = 0.2$$

$$P(A_2) = \frac{50}{100} = 0.5$$

$$P(A_3) = \frac{30}{100} = 0.3$$

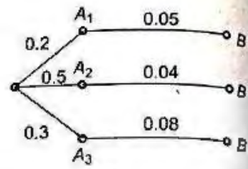

Fig. 1.7

Probability that rooms have faulty plumbings given that rooms belong to Hotel X

$$P(B/A_1) = \frac{5}{100} = 0.05$$

Probability that rooms have faulty plumbing given that rooms belong to Hotel Y

$$P(B/A_2) = \frac{4}{100} = 0.04$$

Probability that rooms have faulty plumbings given that rooms belong to Hotel Z

$$P(B/A_3) = \frac{8}{100} = 0.08$$

Probability that the businessman's room belongs to Hotel Z given that the room has faulty plumbing

$$P(A_3/B) = \frac{P(A_3)\,P(B/A_3)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + P(A_3)\,P(B/A_3)}$$

$$= \frac{0.3 \times 0.08}{(0.2 \times 0.05) + (0.5 \times 0.04) + (0.3 \times 0.08)}$$

$$= \frac{4}{9}$$

Example 6

Of three persons the chances that a politician, a businessman, or an academician would be appointed the Vice Chancellor (VC) of a university are 0.5, 0.3, 0.2 respectively. Probabilities that research is promoted by these persons if they are appointed as VC are 0.3, 0.7, 0.8 respectively.

(i) Determine the probability that research is promoted.

(ii) If research is promoted, what is the probability that the VC is an academician?

Solution

Let A_1, A_2 and A_3 be the events that a politician, a businessman or an academician will be appointed as the VC respectively. Let B be the event that research is promoted by these persons if they are appointed as VC.

$$P(A_1) = 0.5$$

$$P(A_2) = 0.3$$

$$P(A_3) = 0.2$$

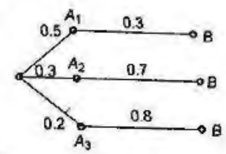

Fig. 1.8

Probability that research is promoted given that a politician is appointed as VC

$$P(B/A_1) = 0.3$$

Probability that research is promoted given that a businessman is promoted as VC

$$P(B/A_2) = 0.7$$

Probability that research is promoted given that an academician is appointed as VC

$$P(B/A_3) = 0.8$$

(i) Probability that research is promoted

$$P(B) = P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + P(A_3)\,P(B/A_3)$$

$$= (0.5 \times 0.3) + (0.3 \times 0.7) + (0.2 \times 0.8)$$

$$= 0.52$$

(ii) Probability that the VC is an academician given that research is promoted by him

$$P(A_3/B) = \frac{P(A_3)\,P(B/A_3)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + P(A_3)\,P(B/A_3)}$$

$$= \frac{0.2 \times 0.8}{0.52}$$

$$= \frac{4}{13}$$

Example 7

The contents of urns I, II, and III are as follows:

1 white, 2 red, and 3 black balls,

2 white, 3 red, and 1 black ball, and

3 white, 1 red, and 2 black balls.

One urn is chosen at random and two balls are drawn. They happen to be white and red. Find the probability that they came from (i) Urn I, (ii) Urn II, and (iii) Urn III.

Solution

Let A_1, A_2, and A_3 be the events that urns I, II and III are chosen respectively. Let B be the event that 2 balls drawn are white and red.

$$P(A_1)=\frac{1}{3}$$

$$P(A_2)=\frac{1}{3}$$

$$P(A_3)=\frac{1}{3}$$

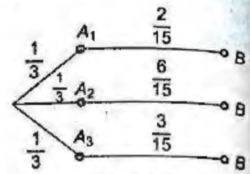

Fig. 1.9

Probability that 2 balls drawn are white and red given that they are chosen from the urn I

$$P(B/A_1)=\frac{{}^1C_1\times{}^2C_1}{{}^6C_2}=\frac{1\times 2}{15}=\frac{2}{15}$$

Probability that 2 balls drawn are white and red given that they are chosen from the urn II

$$P(B/A_2)=\frac{{}^2C_1\times{}^3C_1}{{}^6C_2}=\frac{2\times 3}{15}=\frac{6}{15}$$

Probability that 2 balls drawn are white and red given that they are chosen from the urn III

$$P(B/A_3)=\frac{{}^3C_1\times{}^1C_1}{{}^6C_2}=\frac{3\times 1}{15}=\frac{3}{15}$$

(i) Probability that 2 balls came from the urn I given that they are white and red

$$P(A_1/B)=\frac{P(A_1)\,P(B/A_1)}{P(A_1)\,P(B/A_1)+P(A_2)\,P(B/A_2)+P(A_3)\,P(B/A_3)}$$

$$=\frac{\frac{1}{3}\times\frac{2}{15}}{\left(\frac{1}{3}\times\frac{2}{15}\right)+\left(\frac{1}{3}\times\frac{6}{15}\right)+\left(\frac{1}{3}\times\frac{3}{15}\right)}$$

$$=\frac{2}{11}$$

(ii) Probability that 2 balls came from the urn II given that they are white and red

$$P(A_2/B)=\frac{P(A_2)\,P(B/A_2)}{P(A_1)\,P(B/A_1)+P(A_2)\,P(B/A_2)+P(A_3)\,P(B/A_3)}$$

$$=\frac{\frac{1}{3}\times\frac{6}{15}}{\left(\frac{1}{3}\times\frac{2}{15}\right)+\left(\frac{1}{3}\times\frac{6}{15}\right)+\left(\frac{1}{3}\times\frac{3}{15}\right)}$$

$$=\frac{6}{11}$$

(iii) Probability that 2 balls came from the urn III given that they are white and red

$$P(A_3/B)=\frac{P(A_3)\,P(B/A_3)}{P(A_1)\,P(B/A_1)+P(A_2)\,P(B/A_2)+P(A_3)\,P(B/A_3)}$$

$$=\frac{\frac{1}{3}\times\frac{3}{15}}{\left(\frac{1}{3}\times\frac{2}{15}\right)+\left(\frac{1}{3}\times\frac{6}{15}\right)+\left(\frac{1}{3}\times\frac{3}{15}\right)}$$

$$=\frac{3}{11}$$

EXERCISE 1.4

1. There are 4 boys and 2 girls in Room A and 5 boys and 3 girls in Room B. A girl from one of the two rooms laughed loudly. What is the probability the girl who laughed was from Room B?

   $\left[\text{Ans.: } \frac{9}{17}\right]$

2. The probability of X, Y, and Z becoming managers are $\frac{4}{9}, \frac{2}{9}$, and $\frac{1}{3}$ respectively. The probabilities that the bonus scheme will be introduced if X, Y, and Z become managers are $\frac{3}{10}, \frac{1}{2}$, and $\frac{4}{5}$ respectively. (i) What is the probability that the bonus scheme will be introduced? (ii) If the bonus scheme has been introduced, what is the probability that the manager appointed was X?

   $\left[\text{Ans.: (i) } \frac{23}{45} \text{ (ii) } \frac{6}{23}\right]$

3. A factory has two machines, A and B. Past records show that the machine A produces 30% of the total output and the machine B, the remaining 70%. Machine A produces 5% defective articles and Machine B produces 1% defective items. An item is drawn at random and found to be defective. What is the probability that it was produced (i) by the machine A, and (ii) by the Machine B?

   [Ans.: (i) 0.682 (ii) 0.318]

4. A company has two plants to manufacture scooters. Plant I manufactures 80% of the scooters, and Plant II manufactures 20%. At Plant I, 85 out of 100 scooters are rated standard quality or better. At Plant II, only 65 out of 100 scooters are rated standard quality or better. What is the probability that a scooter selected at random came from (i) Plant I, and (ii) Plant II if it is known that the scooter is of standard quality?

[Ans.: (i) 0.84 (ii) 0.16]

5. A new pregnancy test was given to 100 pregnant women and 100 non-pregnant women. The test indicated pregnancy in 92 of the 100 pregnant women and in 12 of the 100 non-pregnant women. If a randomly selected woman takes this test and the test indicates she is pregnant, what is the probability she was not pregnant?

$\left[\text{Ans.: } \frac{3}{26}\right]$

6. An insurance company insured 2000 scooter drivers, 4000 car drivers, and 6000 truck drivers. The probability of an accident is 0.01, 0.03, and 0.15 in the respective category. One of the insured drivers meets with an accident. What is the probability that he is a scooter driver?

$\left[\text{Ans.: } \frac{1}{52}\right]$

7. Consider a population of consumers consisting of two types. The upper-income class of consumers comprise 35% of the population and each member has a probability of 0.8 of purchasing Brand *A* of a product. Each member of the rest of the population has a probability of 0.3 of purchasing Brand *A* of the product. A consumer, chosen at random, is found to be the buyer of Brand *A*. What is the probability that the buyer belongs to the middle-income and lower-income classes of consumers?

$\left[\text{Ans.: } \frac{39}{95}\right]$

8. There are two boxes of identical appearance, each containing 4 spark plugs. It is known that the box I contains only one defective spark plug, while all the four spark plugs of the box II are non-defective. A spark plug drawn at random from a box, selected at random, is found to be non-defective. What is the probability that it came from the box I?

$\left[\text{Ans.: } \frac{3}{7}\right]$

9. Vijay has 5 one-rupee coins and one of them is known to have two heads. He takes out a coin at random and tosses it 5 times—it always falls head upward. What is the probability that it is a coin with two heads?

$\left[\text{Ans.: } \frac{8}{9}\right]$

10. Stores *A*, *B*, and *C* have 50, 75, and 100 employees and, respectively 50, 60, 70 per cent of these are women. Resignations are equally likely among all employees, regardless of sex. One employee resigns and this is a woman. What is the probability that she works in Store *C*?

[Ans.: 0.5]