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First Order Differential Equation.

* $$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 = 1$$

This eq^n is 2nd order diff. eq^n.
Degree of this eq^n is 1.

Degree is the ~~power~~ order of highest power in the eq^n.

*

* Variable Seprable:

(i) $\frac{dy}{dx} = F(x, y)$

(ii) Seprate the variable x and y

(iii) Take the integration both the sides.

(iv) General Solution.

Eg. 1) $$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}.$$

$$\frac{dy}{dx} = (e^x + x^2)e^{-y}.$$

$$e^y \frac{dy}{dx} = e^x + x^2$$

$$\int e^y \frac{dy}{dx} = \int e^x + x^2$$

$$\therefore\ e^y = e^x + \frac{x^3}{3} + c$$

2) $2xy\,dx + x^2\,dy = 0$

$$\therefore\ 2xy\,dx = -x^2\,dy$$

$$\therefore\ \frac{dy}{dx} = \int \frac{2}{x}\,dx = \int \frac{-1}{y}\,dy$$

$$2\ln x = -\ln y + C$$

$$2\ln x + \ln y = C$$

3) $xy\dfrac{dy}{dx} = 1 + x + y + xy$

$\cancel{\dfrac{dy}{dx}(xy)(1-}$ $\quad xy\dfrac{dy}{dx} - xy = 1 + x + y$

$$xy\left(\frac{dy}{dx} - 1\right) = 1 + x + y$$

$$\frac{dy}{dx} - 1 = \frac{1}{xy} + \frac{1}{y} + \frac{1}{x}$$

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3) $$xy\frac{dy}{dx} = 1 + x + y + xy$$

$$xy\frac{dy}{dx} = (1+x) + y(1+x)$$

$$xy\frac{dy}{dx} = (y+1)(1+x)$$

$$\frac{y}{y+1}\,dy = \frac{1+x}{x}\,dx$$

$$\int \frac{y+1-1}{y+1}\,dy = \int\left(\frac{1}{x}+1\right)dx$$

$$\therefore \int 1\,dy - \int \frac{1}{1+y}\,dy = \int \frac{1}{x}\,dx + \int 1\,dx$$

$$y - \ln(1+y) = \ln x + x + c$$

4) $$3e^{x}\tan y\,dx + (1+e^{x})\sec^2 y\,dy = 0$$

$$\frac{3e^{x}}{1+e^{x}}\,dx = \cancel{dy}\ \frac{-\sec^2 y\,dy}{\tan y}$$

$$\int \frac{3e^{x}}{1+e^{x}}\,dx = -\int \frac{\sec^2 y}{\tan y}\,dy$$

$$3\ln(1+e^{x}) = -\ln(\tan y) + \ln c$$

$$\ln\left[(1+e^{x})^3\cdot\tan y\right] = \ln c$$

$$(1+e^{x})^3 \cdot \tan y = c$$

Exact differential equation.

A diff. eq^n $M dx + N dy = 0$ is said to be exact if it satisfy the necessary & sufficient condition

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

(i) Test the condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(ii) $\displaystyle\int_{y\ const} M\,dx + \int \left(\begin{matrix}\text{Terms of } N \\ \text{free from } x\end{matrix}\right) dy = c$

↳ General Solution.

1) $(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$

$$M = x^4 - 2xy^2 + y^4$$
$$N = -2x^2y + 4xy^3 - \sin y$$

$$\frac{\partial M}{\partial y} = \cancel{4x^3 - 4xy + 4y^3}$$
$$= \cancel{4y^3 - 4xy}$$
$$\cancel{\frac{\partial N}{\partial x} = 4xy - 4y^3}$$

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$$\frac{\partial M}{\partial y} = -4xy + 4y^3$$

$$\frac{\partial N}{\partial x} = -4xy + 4y^3$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \longrightarrow \text{Exact.}$$

General Solution is

$$\int M\,dx + \int \begin{pmatrix}\text{Terms of } N \\ \text{free from } x\end{pmatrix} dy = c$$

$$\int x^4 - 2xy^2 + y^4\,dx + \int -\sin y\,dy = c$$

$$\frac{x^5}{5} - \frac{2x^2y^2}{2} + y^4x + \cos y = c$$

$$\boxed{\frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c}$$

2) $y e^{x} dx + (2y + e^{x}) dy = 0$

$M = y e^{x}$

$N = 2y + e^{x}$

$\frac{\partial M}{\partial y} = e^{x}$

$\frac{\partial N}{\partial x} = e^{x}$

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \rightarrow$ Exact

$\int M\, dx + \int \begin{matrix}\text{Terms of } N \\ \text{free from } x\end{matrix}\, dy = C$

$\int y e^{x}\, dx + \int 2y\, dy = C$

$\boxed{y e^{x} + y^{2} = c}$

3) $\cancel{\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0}$

$\cancel{\frac{dx}{(\sin x + x\cos y + x)}} + \cancel{\frac{dy}{y\cos}}$

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3) $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

$(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0$

$M = y\cos x + \sin y + y$

$N = \sin x + x\cos y + x$

$\dfrac{\partial M}{\partial y} = \cos x + \cos y + 1$

$\dfrac{\partial N}{\partial x} = \cos x + \cos y + 1$

$\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}} \rightarrow$ Exact

$\int M\,dx + \int \left(\text{Terms of } N \text{ Free from } x\right) dy = c$

$\int (y\cos x + \sin y + y)\,dx + \int 0\,dy = c$

$y\sin x + \sin y\, x + yx + 0 = c$

$\cancel{y(\sin x + x) + x(\sin y + 1) = 0}$

$\cancel{y(\sin x + 1) + x(\sin y + y) = 0}$

$y\sin x + x\sin y + yx = c$

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* Integrating Factor (I.F.):

→ Sometimes diff. eqⁿ is not exact which can be converted to exact diff. eqⁿ by multiplying some suitable factor.

→ Therefore such a factor is called integrating factor (I.F.)

* rules for finding Integrating factors.

⇒ Case 1: If $Mx + Ny \neq 0$ and the eqⁿ is homogeneous then $\frac{1}{Mx+Ny}$ is an integrating factor.

1) Solve $(xy - 2y^2)dx - (x^2 - 3xy)dy = 0$ —— (1)

$\because M\,dx + N\,dy = 0$

$\therefore M = xy - 2y^2$

$N = -x^2 + 3xy$

$$\frac{\partial M}{\partial y} = x - 4y$$

$$\frac{\partial N}{\partial x} = -2x + 3y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{Non exact}$$

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$$Mx + Ny = \cancel{x^2 y} - 2xy^2 - \cancel{x^2 y} + 3xy^2$$
$$= xy^2 \neq 0$$

$Mx + Ny \neq 0$ & eq^n is homogeneous.

$$\frac{1}{Mx+Ny} = \frac{1}{xy^2} \text{ is I.F.}$$

Now, multiplying I.F. with eq^n 1

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

The general solution is

$$\int_{y\ const.} M\,dx + \int \begin{pmatrix}\text{Terms of } N\\ \text{free from } x\end{pmatrix} dy = c$$

$$\int_{y\ const.} \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

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⇒ Case 2 :
If $M x - N y \neq 0$ & the eq^n can be written in the form

$$y f_1(x, y) dx + x f_2(xy) dy = 0$$ then

$$\frac{1}{Mx - Ny}$$ is an I.F.

1) $(x^2y^2 + 2) y \, dx + (2 - 2x^2y^2) x \, dy = 0$ — (1)

~~M_x~~

$M = (x^2y^2 + 2) y = x^2y^3 + 2y$

$N = (2 - 2x^2y^2) x = 2x - 2x^3y^2$

$$\frac{\partial M}{\partial y} = 3x^2y^2 + 2$$

$$\frac{\partial N}{\partial x} = 2 - 6x^2y^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{Non exact.}$$

~~Now multiplying eq^n (1) with~~.

$$Mx - Ny = x^3y^3 + \cancel{2xy} - \cancel{2xy} + 2x^3y^3$$
$$= 3x^3y^3 \neq 0$$

$Mx - Ny \neq 0$ & eq^n can be

Now, multiplying I.F with eq^n ②

$$\left(\frac{x^2y^3}{3x^3y^3} + \frac{2y}{3x^3y^3}\right)dx + \left(\frac{2x}{3x^3y^3} - \frac{2x^3y^2}{3x^3y^3}\right)dy = 0$$

$$\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \left(\frac{2}{3x^2y^3} - \frac{2}{3y}\right)dy = 0$$

The general solution is

$$\int M\,dx + \int \left(\begin{array}{c}\text{Term of } N \\ \text{free from } x\end{array}\right) dy = c$$

$$\int\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \int \frac{-2}{3y}\,dy = c$$

$$\frac{1}{3}\log x + \frac{\cancel{2}}{3y^2}\cdot\left(\frac{-1}{\cancel{2}x^2}\right) + -\frac{2}{3}\log y = c$$

$$\frac{\log x}{3} - \frac{1}{3x^2y^2} - \frac{2}{3}\log y = c$$

Linear Diff. eq^n

The diff. eq^n of the form ~~$\frac{dy}{dx} + P(x)$~~

$$\frac{dy}{dx} + P(x)y = Q(x)$$ is called a first order diff. eq^n.

Working rule:

(i) Write the eq^n in the form of ~~$\frac{dy}{dx}$~~

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{or} \quad \frac{dx}{dy} + P(y)x = Q(y)$$

(ii) Find the integrating factor

$$I.F. = e^{\int P(x)dx} \qquad I.F. = e^{\int P(y)dy}$$

(iii) The general solution is given by

$$y(I.F) = \int Q(x)(I.F.)\,dx + C$$

$$x(I.F.) = \int Q(y)(I.F.)\,dy + C$$

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1) $\frac{dy}{dx} + 2xy = e^{-x^2}$

$P(x) = 2x$
$Q(x) = e^{-x^2}$

$I.F. = e^{\int P(x)\,dx}$

$= e^{\int 2x\,dx}$

$= e^{x^2}$

general solution is

$y(I.F.) = \int Q(x)(I.F.)\,dx + c$

$y(e^{x^2}) = \int e^{-x^2} \cdot e^{x^2}\,dx + c$

$y e^{x^2} = x + c$

2) $\frac{dy}{dx} + y\sin x = e^{\cos x}$

$P(x) = \sin x$
$Q(x) = e^{\cos x}$

$I.F. = e^{\int P(x)\,dx}$

$= e^{\int \sin x\,dx}$

$= e^{-\cos x\,dx}$

$y(I.F.) = \int Q(x)(I.F.)\,dx + C$

$y e^{-\cos x} = x + c$

$y = (x + c)e^{\cos x}$

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3) $(x^2+1)\frac{dy}{dx} + 2xy = 4x^2$

$$\frac{dy}{dx} + \frac{2x}{(x^2+1)}y = \frac{4x^2}{x^2+1}$$

$$P(x) = \frac{2x}{x^2+1}$$

$$Q(x) = \frac{4x^2}{x^2+1}$$

$$I.F. = e^{\int P(x)\,dx}$$

$$= e^{\int \frac{2x}{x^2+1}dx}$$

$$I.F. = e^{\log(x^2+1)} = x^2+1$$

$$y(I.F.) = \int Q(x)(I.F.)\,dx + C$$

$$y(x^2+1) = \int \frac{4x^2}{x^2+1}\cdot\frac{(x^2+1)}{1} + C$$

$$y(x^2+1) = \int 4x^2 + C$$

$$y(x^2+1) = \frac{4}{3}x^3 + C$$

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4) $x \log x \dfrac{dy}{dx} + y = 2\log x$

$$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x}$$

$$P(x) = \frac{1}{x\log x}$$

$$Q(x) = \frac{2}{x}$$

$$I.F. = e^{\int P(x)\,dx}$$

$$= e^{\int \frac{1}{x\log x}}\,dx$$

$$= e^{\log(\log x)}\,dx$$

$$I.F = \log x$$

$$y(I.F.) = \int Q(x)(I.F.)\,dx + c$$

$$y\log x = \int \frac{2}{x}\log x\,dx + c$$

$$y\log x = (\log x)^2 + c$$

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5) $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1)$$

$$P(x) = \frac{-1}{x+1}$$

$$Q(x) = e^{3x}(x+1)$$

$$\text{I.F.} = e^{\int P(x)\,dx}$$

$$= e^{\int \frac{-1}{x+1}\,dx}$$

$$= e^{-\log(x+1)}$$

$$= \frac{1}{(x+1)}$$

$$y(\text{I.F.}) = \int Q(x)(\text{I.F.})\,dx + c$$

$$\frac{y}{x+1} = \int \frac{e^{3x}(x+1)}{x+1}\,dx + c$$

$$\frac{y}{x+1} = \frac{e^{3x}}{3} + c$$

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* Bernouli Differential Equation:

A differential equation of the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$$

where $P(x)$ and $Q(x)$ are the function of x or constant.

Type-1 Divide by y^n.

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x)$$

Suppose $y^{1-n} = v$

Differentiate with respect to x

$$(1-n)\,y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\cancel{\frac{1-n}{1-n}}\;y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\,\frac{dv}{dx}$$

$$\boxed{\frac{1}{1-n}\,\frac{dv}{dx} + P(x)\,v = }$$

Linear equation form.

$$I.F. = e^{\int P(x)}$$

Solⁿ eqⁿ

$$v.\,(I.F.) = \int Q.(I.F.)\,dx$$

Now, Solⁿ ~~of~~ is given by,

$$y^{1-n}\,(I.F.) = \int Q\,(I.F.)\,dx$$

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1) $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

Divide by y^2

$$y^{-2}\frac{dy}{dx} + \frac{1}{yx} = \frac{1}{x^2}$$

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = \frac{1}{x^2}$$

Now, let $y^{-1} = v$

Diff^n w.r.t. x

$$-y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore y^{-2}\frac{dy}{dx} = \frac{-dv}{dx}$$

$$\boxed{-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x^2}} \rightarrow \frac{dv}{dx} - \frac{v}{x} = -\frac{1}{x^2}$$

$$\therefore P(x) = \frac{1}{x} \qquad Q(x) = -\frac{1}{x^2}$$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$$

Sol^n eq^n

$$v(\text{I.F.}) = \int Q(\text{I.F.})\,dx$$

$$\frac{v}{x} = \int \frac{-1}{x^3}\,dx$$

$$\frac{v}{x} = \frac{1}{2x^2} + c$$

$$\boxed{v = \frac{1}{2x} + cx}$$

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$$\boxed{\frac{1}{y} = \frac{1}{2x} + cx}$$

Type-2 $\quad \dfrac{dy}{dx} + P(x)y = Q(x)f(y)$

2) $\dfrac{dy}{dx} + \dfrac{1}{x} = \dfrac{e^y}{x^2}$

divide by $e^y = f(y)$

$\therefore e^{-y}\dfrac{dy}{dx} + \dfrac{1}{x}e^{-y} = \dfrac{1}{x^2}$

Suppose, $e^{-y} = u$

$$-e^{-y}\frac{dy}{dx} = \frac{du}{dx}$$

$$e^{-y}\frac{dy}{dx} = -\frac{du}{dx}$$

$$\therefore -\frac{du}{dx} + \frac{u}{x} = \frac{1}{x^2}$$

$$\frac{du}{dx} - \frac{u}{x} = \frac{-1}{x^2}$$

$P(x) = -\dfrac{1}{x} \qquad Q(x) = -\dfrac{1}{x^2}$

$$\text{I.F.} = e^{\int \frac{-1}{x}dx} = e^{\log x^{-1}} = \frac{1}{x}$$

$$u \cdot \frac{1}{x} = \int \frac{-1}{x^2}\cdot\frac{1}{x}\,dx$$

$$\frac{u}{x} = \int \frac{-1}{x^3}\,dx$$

$$\therefore \frac{y}{x} = \frac{1}{2x^2} + c$$

$$\therefore y = \frac{1}{2x} + cx$$

$$e^{-y} = \frac{1}{2x} + cx$$

3) $$\frac{dy}{dx} = e^{x-y}(e^x - e^y)$$

$$\frac{dy}{dx} = \frac{e^x}{e^y}(e^x - e^y)$$

$$\frac{dy}{dx} = \frac{e^{2x}}{e^y} - e^x$$

$$\frac{dy}{dx} + e^x = \frac{e^{2x}}{e^y}$$

divide by e^{-y}.

$$e^y \frac{dy}{dx} + e^{x+y} = e^{2x}$$

Let $e^y = v$

$$e^y \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} + ve^x = e^{2x}$$

$P = e^x$ $\quad Q(x) = e^{2x}$

$$I.F. = e^{\int e^x dx} = e^{e^x}$$

Solⁿ is,

$$v(I.F) = \int Q(I.F) d$$

$$y \cdot e^{e^x} = \int e^{2x} \cdot e^{e^x} dx$$

$$y \cdot e^{e^x} = \int e^{2x+e^x} dx$$

$$y \cdot e^{e^x} = \int e^x (e^{e^x}) e^x \, dx$$

Suppose $e^x = t$

$e^x dx = dt$

$$y \cdot e^{e^x} = \int t e^t \cdot dt + c$$

$$= [te^t - e^t] + c$$

$$y e^{e^x} = e^x \cdot e^{e^x} - e^{e^x} + c$$

$$y = e^x - 1 + c e^{-e^x}$$

$$e^y = (e^x - 1) + c e^{-e^x}$$

4) $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

divide by $\cos y$

$$\frac{\tan y}{\cos y} \frac{dy}{dx} + \frac{\tan x}{\cos y} = \cos^2 x$$

$$\frac{\tan y}{\cos y} \frac{dy}{dx} + \tan x \sec y = \cos^2 x$$

Let $\sec y = u$

$$\sec y \tan y \frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{\tan y}{\cos y} \frac{dy}{dx} = \frac{du}{dx}$$

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$$\frac{dy}{dx} + \tan x \, y = \cos^2 x$$

$$I.F. = e^{\int \tan x \, dx}$$

$$= e^{\log|\sec x|}$$

$$= \sec x$$

$$y \cdot \sec x = \int \cos^2 x \sec x \, dx$$

$$y \sec x = \int \cos x \, dx$$

$$y \sec x = \sin x + C$$

$$\sec y \sec x = \sin x + c$$

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I.F.

Case III :

★ ~~Case III Type 3~~

Let $M(x,y)\,dx + N(x,y)\,dy = 0$ is L

if $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ then

$$I.F. = e^{\int f(x)\,dx}$$

~~$I.F. = e^{\int f(y)\,dy}$~~

~~Type 4~~ ★ Case IV : Let $M(x,y)\,dx + N(x,y)\,dy = 0$ is L

if $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y)$ then

$$I.F. = e^{\int f(y)\,dy}$$

General Solution

1) Solve $(2x^3 + 4y)dx - x\,dy = 0$ ——— (1)

$M\,dx + N\,dy = 0$

$M = 2x^3 + 4y$

$N = -x$

$$\frac{\partial M}{\partial y} = 4$$

$$\frac{\partial N}{\partial x} = -1$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}} \Rightarrow \text{Non exact diff eq}^n.$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-1}{x}\left[4-(-1)\right]$$

$$= \frac{-5}{x}$$

$$= f(x)$$

$$I.F. = e^{\int f(x)\,dx}$$

$$= e^{\int -5/x\,dx}$$

$$= e^{-5 \ln x}$$

$$= e^{\ln x^{-5}}$$

$$I.F. = x^{-5}$$

From eqⁿ (1)

$$\frac{1}{x^5}\left[\left(\frac{2x^3+4y}{1}\right)dx - x\,dy\right] = 0$$

$$\left(\frac{2}{x^2}+\frac{4y}{x^5}\right)dx - \frac{1}{x^4}dy = 0 \quad \text{——— (2)}$$

Comparing eqⁿ (2) with $M dx + N dy = 0$

$$M = \frac{2}{x^2}+\frac{4y}{x^5} \qquad N = -\frac{1}{x^4}$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{4}{x^5}} \qquad \boxed{\frac{\partial N}{\partial x} = \frac{1}{x^5}}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{Exact}$$

I.F. $= \frac{1}{x^5}$

$$\int_{y\ \text{const}} M\,dx + \int \begin{matrix}\text{Terms of } N\\ \text{free from } x\end{matrix} = c$$

$$\int\left(\frac{2}{x^2}+\frac{4y}{x^5}\right)dx - \int 0\,dy = c$$

$$-\frac{2}{x} + \frac{4y}{x^4(-4)} = c$$

$$\therefore \frac{-2}{x} - \frac{y}{x^4} = c$$

$$\therefore \frac{2}{x} + \frac{y}{x^4} = c$$

2) solve $2(y^3-4)dx + 5xy^2 dy = 0$

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$M = 2(y^3-4)$ (By case III)

$N = 5xy^2$

$$\frac{\partial M}{\partial y} = 6y^2$$

$$\frac{\partial N}{\partial x} = 5y^2$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}} \longrightarrow$$ ~~Exact~~ Non exact diff. eqⁿ

$$f(x) = \frac{1}{5xy^2}\left(6y^2 - 5y^2\right)$$

$$f(x) = \frac{1}{5x}$$

$$I.F. = e^{\int f(x)\,dx}$$

$$= e^{\int \frac{1}{5x}\,dx}$$

$$= e^{\frac{1}{5}\ln x\, dx}$$

$$I.F. = x^{\frac{1}{5}}$$

$$I.F.\left[M\,dx + N\,dy\right] = 0$$

$$x^{\frac{1}{5}}\left[(2y^3-8)\,dx + 5xy^2\,dy\right] = 0$$

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$$\left(2y^3x^{1/5} - 8x^{1/5}\right)dx + \left(5y^2x^{1/5}\right)dy = 0$$

$$\int\left(2y^3x^{+1/5} - 8x^{+1/5}\right)dx + 0 = C$$

$$\therefore \quad (2y^3-8)\int x^{+1/5}\,dx = C$$

$$\therefore \quad (2y^3-8)\left[\frac{x^{+1/5+1}}{+1/5+1}\right] = C$$

$$(2y^3-8)\left[\frac{x^{6/5}}{6/5}\right] = C$$

$$\boxed{\frac{5(2y^3-8)}{6}\,x^{6/5} = C} \quad \text{or} \quad \boxed{(2y^3-8)x^{6/5} = C'}$$

Solve $2(y^3-4)\,dx + 5xy^2\,dy = 0$

(By case IV)

$M = 2(y^3-4)$

$N = 5xy^2$

$\dfrac{\partial M}{\partial y} = 6y^2$

$\dfrac{\partial N}{\partial x} = 5y^2$

⇒ Solve $2(y^3 - 4)\,dx + 5xy^2\,dy = 0$

(By case IV)

$M = 2y^3 - 8$

$N = 5xy^2$

$\dfrac{\partial M}{\partial y} = 6y^2$

$\dfrac{\partial N}{\partial x} = 5y^2$

$\boxed{\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}}$ → Non exact diff. eqⁿ

$f(y) = \dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$

$= \dfrac{1}{(2y^3-8)}\left(5y^2 - 6y^2\right)$

$= \dfrac{-y^2}{2y^3-8}$ $\quad - \int \dfrac{\quad}{2y^3-8}$

$I.F = e$

$(2y^3-8)^{-1/6}$

$= (2)^{-1/6}(y^3-4)^{-1/6}$

$=$

$$I.F. = e^{\int f(y)\,dy}$$

$$= e^{\int \frac{-y^2}{2y^3-8}\,dy}$$

$$= e^{\frac{+1}{6}\int \frac{-6y^2}{2y^3-8}\,dy}$$

$$= e^{\ln(2y^3-8)^{-1/6}}$$

$$= (2y^3-8)^{-1/6}$$

$$= (2)^{-1/6}(y^3-4)^{-1/6}$$

$$(2y^3-8)^{-1/6}\left[M\,dx + N\,dy\right] = 0$$

$$(2y^3-8)^{-1/6}\left[(2y^3-8)\,dx + 5xy^2\,dy\right] = 0$$

$$(2y^3-8)^{-\frac{1}{6}+1}\,dx + (2y^3-8)^{-1/6}\,5xy^2\,dy = 0$$

$$(2y^3-8)^{5/6}\,dx + 5(2y^3-8)^{-1/6}\,xy^2\,dy = 0$$

$$M = (2y^3-8)^{5/6}\qquad N = 5(2y^3-8)^{-1/6}\,xy^2$$

$$\frac{\partial M}{\partial y} = \frac{5}{\cancel{6}}(2y^3-8)^{-\frac{1}{6}}(\cancel{6}y^2)\qquad \frac{\partial N}{\partial x} = 5(2y^3-8)^{-1/6}\,y^2$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The general Solution is

$$\int M\,dx + \int \left(\begin{array}{l}\text{Terms of } N \\ \text{free from } x\end{array}\right) dy = c$$

$$\int (2y^3 - 8)^{5/6}\, dx + 0 = c$$

$$(2y^3 - 8)^{5/6} \int dx = c$$

$$\boxed{(2y^3 - 8)^{5/6} \cdot x = c}$$

or

$$\boxed{(2y^3 - 8)\, x^{6/5} = c'}$$

3) Solve $\left[\sin(x+y)+\cos(x+y)\right]dx+\cos(x+y)dy=0$

$M=\sin(x+y)+\cos(x+y)$

$N=\cos(x+y)$

$\frac{\partial M}{\partial y}=\cos(x+y)-\sin(x+y)$

$\frac{\partial N}{\partial x}=-\sin(x+y)$

$f(x)=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$

$=\frac{1}{\cancel{\cos(x+y)}}\left[\cancel{\cos(x+y)}\right]$

$f(x)=1$

$I.F.=e^{\int f(x)dx}$

$I.F.=e^{x}$

$e^{x}\left[\sin(x+y)+\cos(x+y)\right]dx+e^{x}\cos(x+y)dx=0$

$M=e^{x}\left[\sin(x+y)+\cos(x+y)\right]$

$\frac{\partial M}{\partial y}=e^{x}\left[\cos(x+y)-\sin(x+y)\right]$

$N=e^{x}\cos(x+y)$

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$$\frac{\partial N}{\partial x} = e^x[-\sin(x+y)] + \cos(x+y)\,e^x$$

$$= e^x\left[\cos(x+y) - \sin(x+y)\right]$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int e^x[\sin(x+y) + \cos(x+y)]\,dx = c$$

$$\cancel{\int e^x \sin(x+y)\,dx + \int e^x \cos(x+y)\,dx = c}$$

$$\cancel{\int e^x} [\quad \int e^x[F(x) + F'(x)]\,dx = e^x F(x)$$

$$\boxed{e^x \sin(x+y) = c}$$

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Case IV :

★ Find the general solution of diff. eqⁿ.

1) $x^2y\,dx - (x^3 + y^3)\,dy = 0$

$M = x^2y$

$N = -x^3 - y^3$

$\frac{\partial M}{\partial y} = x^2$

$\frac{\partial N}{\partial x} = -3x^2$

$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}} \rightarrow$ Non exact diff. eqⁿ.

$f(y) = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$= \frac{1}{x^2y}\left(-3x^2 - x^2\right)$

$= \frac{-4}{y}$

$I.F = e^{\int f(y)\,dy}$

$= e^{\int \frac{-4}{y}\,dy}$

$= e^{\ln y^{-4}}$

$= e$

$$I.F. = \frac{1}{y^4}$$

$$I.F. (M dx + N dy) = 0$$

$$\frac{x^2}{y^3} dx - \left(\frac{x^3}{y^4} + \frac{1}{y}\right) dy = 0$$

$$\frac{\partial M}{\partial y} = \cancel{x^2\left(\frac{y^{-3+1}}{-3+1}\right)} \quad x^2\left(-3y^{-4}\right)$$

$$= \frac{-3x^2}{y^4}$$

$$\frac{\partial N}{\partial x} = \frac{-1}{y^4}(3x^2)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int \frac{x^2}{y^3} dx - \int \frac{1}{y} dy = c$$

$$\frac{1}{y^3}\left(\frac{x^3}{3}\right) - \ln y = c$$

$$\boxed{\frac{x^3}{3y^3} - \ln y = C}$$

★ Case V :

Let $M(x,y)\,dx + N(x,y)\,dy = 0$ is non exact diff. eq^n which can be expressed as.

$$x^a y^b (My\,dx + Nx\,dy) + x^c y^d (Py\,dx + Q$$

where, M, N, P, Q, a, b, c, d are constants & $M, N, P, Q \neq 0$ then I.F. $= x^h y^k$

where h, k are obtaining by solving equations $\dfrac{a+h+1}{M} = \dfrac{b+k+1}{N}$ and

$$\frac{c+h+1}{P} = \frac{d+k+1}{Q}$$

1) Obtain the solution of

$$(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$$

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★ Solution of Non-Linear differential Equation

Case I :

Equation Solvable for p

Consider the eq^n of 1st order with higher degree

$$a_n\left(\frac{dy}{dx}\right)^n + a_{n-1}\left(\frac{\partial y}{\partial x}\right)^{n-1} + \dots + a_1\left(\frac{\partial y}{\partial x}\right) + a_0 = 0$$

Let $\frac{dy}{dx} = p$

$a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0 = 0$;

which is polynomial in terms of p.

Which can be produced as multiplecation of facters as

$$(p-\phi_1)(p-\phi_2)(p-\phi_3)\dots(p-\phi_n) = 0$$

The Solution required only one arbitary constant and since c_i's are arbitary constants.

choose $c_1 = c_2 = c_3 = \dots = c_n$

Thus, the General Solution can be written as

$$F_1(x,y,c_1)\,F_2(x,y,c_2)\dots F(x,y,c_n) = 0$$

1) Solve $\left(\frac{dy}{dx}\right)^2 - (x^2+x)\frac{dy}{dx} + x^3 = 0$ —— (1)

let $\frac{dy}{dx} = p$

$p^2 - (x^2+x)p + x^3 = 0$

$\cancel{p^2 - \frac{2(x^2-x)p}{2} + \frac{(x^2-x)}{2}}$

$p^2 - px^2 - px + x^3 = 0$

$\cancel{p(p-x^2)}$ $\quad p(p-x^2) - x(p-x^2) = 0$

$(p-x)(p-x^2) = 0$ —— (2)

Now,

$p - x = 0$

$p = x$

$\frac{dy}{dx} = x$

$dy = x\,dx$

$y = \frac{x^2}{2} + c_1 \Rightarrow y - \frac{x^2}{2} - c_1 = 0$ —— (I)

$p - x^2 = 0$

$p = x^2$

$\frac{dy}{dx} = x^2$

$y = \frac{x^3}{3} + c_2$

$y - x^3/3 - c_2 = 0$ ——

From (I), (II) & (2)

The required general solution is

$$\left(y - \frac{x^2}{2} - c_1\right)\left(y - \frac{x^3}{3} - c_2\right) = 0$$

2) ~~Solve~~ $(x+2y)\left(\frac{dy}{dx}\right)^3 + (x+3y)\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} = 0$ —— (1)

~~Let $\frac{dy}{dx}$~~ = given eq^n is first order third degree linear eq^n which is solvable for p.

$$(x+2y)p^3 + (x+3y)p^2 + yp = 0$$

$$xp^3 + 2yp^3 + xp^2 + 3yp^2 + yp = 0$$

$$p(xp^2 + 2yp^2 + xp + 3yp + y) = 0$$

$$p[(p+1)(xp + 2yp + y\cancel{p})] = 0$$

$$p(p+1)[p(x+2y) + y] = 0 \quad \text{—— (2)}$$

$\therefore p = 0$

$\frac{dy}{dx} = 0$

$y = c_1$ —— (I)

$p + 1 = 0$

$\frac{dy}{dx} = -1$

$y = -x + c_2$

$y + x - c_2 = 0$ —— (II)

$\frac{dy}{dx}(x+2y) = -y$

$-\frac{(x+2y)}{y}dy = dx$

$-x\ln y - 2y = x + c_3$

$x\ln y + 2y + x = -c_3$

$x(\ln y + 1) + 2y + c_3 = 0$ —— (III)

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~~From (I), (II), (III) & 2~~

$$\cancel{(y-c_1)(y+x-c_2)\left[x(\ln y+1)+2y+c_3\right]=0}$$

$$(x+2y)\frac{dy}{dx}+y=0$$

$$x\frac{dy}{dx}+2y\frac{dy}{dx}+y=0$$

$$x\,dy+2y\,dy+y\,dx=0\,dx$$

$$xy+y^2+xy=c_3$$

$$2xy+y^2-c_3=0 \quad \text{——— (III)}$$

From (I), (II), (III) & 2

$$(y-c_1)(y+x-c_2)(2xy+y^2-c_3)=0$$

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★ Equation Solvable for $\frac{x}{y}$.

Suppose eq^n

$$(p-\phi_1)(p-\phi_2)\cdots(p-\phi_n)=0 \text{ can be written}$$

as $\frac{x}{y} = g(\frac{y}{x}, p)$

Differentiate the eq^n w.r.t. $\frac{y}{x}$,

$$\frac{dx/y}{dy/x} = h\left(\frac{y}{x}, p, \frac{dp}{dy/x}\right) \Rightarrow \frac{1}{p} = h\left(\frac{x}{y}, p, \frac{dp}{dy/x}\right)$$

After simplifing, eq^n turns into

$$f\left(\frac{x}{y}, p, \frac{dp}{dy/x}\right) = 0 ;$$

which is the 1st order, 1st degree diff. eq^n with dependent variable ~~variable~~ p and independent variable $\frac{y}{x}$.

Suppose it's solution is $G(\frac{y}{x}, p, c) = 0$
which is 1st ~~degree~~ order 1st degree diff. eq^n with dependent variable $\frac{x}{y}$ and independent variable $\frac{x}{y}$.

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1) Find general Solution of $xe^{-y} = p^2$ ——— (2)

$$xe^{-y} = p^2$$

$$x = \frac{p^2}{e^{-y}}$$

$$x = p^2 e^{y} \quad \text{———} \quad (*)$$

derivation w.r.t. y.

$$\frac{dx}{dy} = e^{y}\left(2p\frac{dp}{dy}\right) + p^2 e^{y}.$$

$$\frac{1}{p} = e^{y}\left(2p\frac{dp}{dy} + p^2\right)$$

$$1 = e^{y}p^2\left(2\frac{dp}{dy} + p\right)$$

$$e^{y}p^2\left(2\frac{dp}{dy} + p\right) - 1 = 0$$

$$e^{y}p^3 + 2e^{y}p^2\frac{dp}{dy} - 1 = 0 \quad \text{———} \quad (2)$$

Assuming $p^3 = v$

$$3p^2\frac{dp}{dy} = \frac{dv}{dy}$$

$$e^{y}\cdot v + \frac{2}{3}e^{y}\frac{dv}{dy} - 1 = 0$$

1) Find general Solution of $x e^{-y} - p^2 = 0$ ——— (3)

$$x e^{-y} = p^2$$

$$x = \frac{p^2}{e^{-y}}$$

$$x = p^2 e^{y} \quad \text{———} \quad (*)$$

derivation w.r.t. y

$$\frac{dx}{dy} = e^{y}\left(2p\frac{dp}{dy}\right) + p^2 e^{y}$$

$$\frac{1}{p} = e^{y}\left(2p\frac{dp}{dy} + p^2\right)$$

$$1 = e^{y} p^2\left(2\frac{dp}{dy} + p\right)$$

$$e^{y} p^2\left(2\frac{dp}{dy} + p\right) - 1 = 0$$

$$e^{y} p^3 + 2e^{y} p^2 \frac{dp}{dy} - 1 = 0 \quad \text{———} \quad (2)$$

Assuming $p^3 = v$

$$3p^2 \frac{dp}{dy} = \frac{dv}{dy}$$

$$e^{y} \cdot v + \frac{2}{3} e^{y} \frac{dv}{dy} - 1 = 0$$

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$$\therefore \frac{2}{3} e^{y} \frac{dx}{dy} + e^{y} x - 1 = 0$$

$$\frac{dx}{dy} + \frac{3}{2} x - \frac{3}{2} e^{-y} = 0$$

$$\boxed{\frac{dy}{dx} + py = Q}$$

$$\frac{dx}{dy} + \frac{3}{2} x = \frac{3}{2} e^{-y}$$

which is linear diff. eqⁿ

General Solution :

$$(I.F.)\, x = \int (I.F.)\, Q \, dy + c$$

$$I.F. = e^{\int P\, dy}$$

$$= e^{\int \frac{3}{2} dy}$$

$$I.F. = e^{\frac{3}{2} y}$$

$$G.S. \rightarrow \left(e^{\frac{3}{2} y}\right) x = \int e^{\frac{3}{2} y} \cdot \frac{3}{2} e^{-y} \, dy + c$$

$$e^{\frac{3}{2} y} \cdot x = \frac{3}{2} \int e^{y\left(\frac{3}{2} - 1\right)} dy + c$$

$$e^{\frac{3}{2} y} \cdot x = \frac{3}{2} \frac{e^{y\left(\frac{3}{2} - 1\right)}}{\left(\frac{3}{2} - 1\right)} + c$$

$$\cancel{e^{\frac{3}{2}y}\cdot y} = \frac{3}{2}\cancel{e^{\frac{3}{2}y}\cdot e^{-y}}$$

$$e^{\frac{3}{2}y}\, y = \frac{3}{2}\,\frac{e^{\frac{1}{2}y}}{\frac{1}{2}} + c$$

$$e^{\frac{3}{2}y}\, y = 3e^{\frac{1}{2}y} + c$$

$$e^{\frac{3}{2}y}\cdot p^3 = 3e^{\frac{1}{2}y} + c$$

$$p^3 = \frac{3e^{\frac{1}{2}y}}{e^{\frac{3}{2}y}} + ce^{-\frac{3}{2}y}.$$

$\frac{1}{2} - \frac{3}{2} \quad -\frac{2}{2}$

$$p^2 = \left(3e^{-y} + ce^{-\frac{2}{3}y}\right)^{2/3}$$

From (*)

$$\cancel{\frac{x}{e^y} = \left(3e^{-y} + ce^{-\frac{2}{3}y}\right)^{2/3}}$$

$$\frac{x}{e^y} = \left(3e^{-y} + ce^{-\frac{2}{3}y}\right)^{2/3}$$ is required general solution of given eq^n.

$f(x, y, c)$

2) Find the singular Solution & general Solution of

$$y + px = x^4 p^2$$

$$y = x^4 p^2 - px$$

derivation w.r.t. x.

$$\frac{dy}{dx} = 4x^3 p^2 + 2p\frac{dp}{dx}(x^4) - \left(p + x\frac{dp}{dx}\right)$$

$$P = 4x^3 p^2 + 2p\frac{dp}{dx} x^4 - p - x\frac{dp}{dx}$$

$$2P = 4x^3 p^2 + \frac{dp}{dx}(2px^4 - x)$$

$$\cancel{0} = 4x^3 p^2 + (2px^4 - x)\frac{dp}{dx} - 2P = 0$$

* Higher Order linear differential equation (Homogeneous System)

General form:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 y + a_0 = R_x \text{ (non homogeneous)}$$

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 y + a_0 = 0 \text{ (homogeneous)}$$

Solution method for homogeneous diff. eq^n.

tep: 1) $\frac{d}{dx} = D$ ⟶ Operator form.

2) Replace D by m → Auxillary Equation.

3) Make the factors & get values of m

4) Use the chart. ⟶ *

* General Solution of higher order diff. eq^n

$= y_c = $ C.F. (Complimentary factor)

General solution of non-homogeneous diff. eq^n

$= y_c + y_p$

y_p = Particular Integral

~~Values~~ of
For Complementary Solution
values of m's are

Real

($\neq$) Non-Repeated
(Let $m_1, m_2, m_3, \ldots, m_n$ are different roots)

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots$$

(=) Repeated
(Let $m_1 = m_2 = \ldots = m_n = m$ are n equal roots)

$$y_c = e^{mx}(c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \ldots)$$

Complex Conjugate

($\neq$) Non-Repeated
(Let $m_1 = \alpha_1 \pm i\beta_1$
$m_2 = \alpha_2 \pm i\beta_2$
$m_3 = \alpha_3 \pm i\beta_3$
$\vdots$
are different roots)

$$y_c = e^{\alpha_1 x}(c_1 \cos\beta_1 x + c_2 \sin\beta_1 x) + e^{\alpha_2 x}(c_3 \cos\beta_2 x + c_4 \sin\beta_2 x) + e^{\alpha_3 x}(c_5 \cos\beta_3 x + c_6 \sin\beta_3 x) + \ldots$$

Repeated
(Let $\alpha_1 \pm i\beta_1 = \alpha_2 \pm i\beta_2 = \ldots = \alpha_n \pm i\beta_n = \alpha \pm i\beta$ are n equal roots)

$$y_c = e^{\alpha x}\left[(c_1 + c_2 x + \ldots)\cos\beta x + (c_{n+1} + c_{n+2} x + \ldots)\sin\beta x\right]$$

1) Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

The differential operator is

$$D^2y + Dy - 6y = 0 \qquad \left(\because \frac{d}{dx} = D\right)$$

$$A.E \Rightarrow (m^2 + m - 6)y = 0$$

$$m^2 + m - 6 = 0$$

$$(m+3)(m-2) = 0$$

$$m = +2, \; m = -3$$

real and non repeated roots.

$$y_c = c_1 e^{2x} + c_2 e^{-3x}$$

2) Solve. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = 0$

The differential operator is

$$D^2y - 2Dy - y = 0$$

$$A.E. \Rightarrow (m^2 - 2m - 1)y = 0$$

$$m^2 - 2m - 1 = 0$$

$$(m^2 - 2m + 1) - 2 = 0$$

$$(m-1)^2 - (\sqrt{2})^2 = 0$$

$$(m - 1 - \sqrt{2})(m - 1 + \sqrt{2}) = 0$$

$$m = \sqrt{2} + 1 \quad \text{or} \quad m = 1 - \sqrt{2}$$

$$m = 1 \pm \sqrt{2}$$

real & non repeated roots

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$$y_c = c_1 e^{1+\sqrt{2}\,x} + c_2 e^{1-\sqrt{2}\,x}$$

3) $3\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 0$

differential operator is

$$3D^3y - 7D^2y + 4Dy = 0.$$

A.E. $\Rightarrow (3m^3 - 7m^2 + 4m)y = 0$

$$m(3m^2 - 7m + 4) = 0$$
$$m(3m(m-1) - 4(m-1)) = 0$$
$$m(3m-4)(m-1) = 0$$
$$m = 0, \quad m = \frac{4}{3}, \quad m = 1.$$

real and non repeated roots

$$y_c = c_1 + c_2 e^{\frac{4}{3}x} + c_3 e^{x}$$

4) $2\frac{dy^3}{dx^3} + 3\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 3y = 0$

The differential eq^n is

$$2D^3y + 3D^2y - 8Dy + 3y = 0$$

~~A.E $\Rightarrow$ ym (2m² + 3m − 8) + 3 = 0~~

A.E. $\Rightarrow (2m^3 + 3m^2 - 8m + 3)y = 0$

$$(m-1)(2m^2 + 5m - 3) = 0$$

$$(m-1)(2m^2 + 6m - m - 3) = 0$$

$$(m-1)(m+3)(2m-1) = 0$$

$$m = 1, \quad m = -3, \quad m = \frac{1}{2}$$

real & non repeated

$$y_c = c_1 e^{x} + c_2 e^{-3x} + c_3 e^{\frac{1}{2}x}.$$

$$\frac{d^4y}{dx^4} - 5\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0$$

differential operator is

$$D^4y - 5D^3y + 5D^2y + 5Dy - 6y = 0$$

A.E. $= (m^4 - 5m^3 + 5m^2 + 5m - 6)y = 0$

$$\cancel{= m^2(m^2-5) - 5m(m^2-1) - 6 = 0}$$

$$\cancel{= (m+1)(m^3 - 6m^2 + m + 4}$$

$$= (m-1)(m^3 - 4m^2 - 9m - 4$$

$= m^4 - m^3 - 4m^3 + 4m^2 +$ [illegible] $m^2 +$ [illegible] $m +$ [illegible]

$= m^3(m-1) - 4m^2(m-1) + m(m-1) + 6$ [illegible]

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A.E $\Rightarrow m^4 - 5m^3 + 5m^2 + 5m - 6 = 0$

$\Rightarrow (m-1)(m^3 - 4m^2 + m + 6) = 0$

~~$\Rightarrow (m-1)(m-1)(m^2 - 3m - 2$~~

$m - 1 = 0$ or $m^3 - 4m^2 + m + 6 = 0$

$m = 1$ or $(m+1)(m^2 - 5m + 6) = 0$

$(m+1)(m^2 - 3m - 2m + 6) = 0$

$(m+1)(m-3)(m-2) = 0$

$\boxed{m = 1, -1, 3, 2}$

real & non repeted

$y_c = c_1 e^{x} + c_2 e^{-x} + c_3 e^{3x} + c_4 e^{2x}$.

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Homogeneous G.S. $\rightarrow y = y_c$

Non Homogeneous

$$a_n \frac{d^n y}{dx^n} + \cdots + a_0 y = R(x)$$

$$a_n D^n y + a_{n-1} D^{n-1} y + \cdots = 0$$

$$f(D) y = R(x)$$

$$y = y_c + y_p$$

y_p → General Method; Special forms; Special Methods

$f(D) y = R(x)$

Undetermined Coefficient.

For y_P = Particular Integral

General $\rightarrow$ $y_P = \frac{1}{f(D)} R(x)$

$(R(x))$ = right side.

$\rightarrow$ P. I. $= y_P = \frac{1}{D-a} R(x) = e^{ax} \int R(x) e^{-ax} dx$

$= \frac{1}{D+a} R(x) = e^{-ax} \int R(x) e^{ax} dx$

$\boxed{\begin{array}{l} D = \text{differential equation} \\ \frac{1}{D} = \text{Integration.} \end{array}}$

1) $(4D^2 - 4D + 1) y = 4$

A.E. $\rightarrow$ $(2m-1)^2 = 0$

$m = \frac{1}{2}, \frac{1}{2}$

$y_c = (c_1 + c_2 x) e^{\frac{1}{2}x}$

Now, $4 = R(x) = 4e^{0x}$

Also $f(D) = (2D-1)^2$

$f(0) = (-1)^2$

$= 1 \neq 0 = f(a)$

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$$\therefore \; y_p = \frac{1}{f(a)} 4e^{0x}$$

$$= \frac{4}{1}$$

$$y_p = 4$$

General Solution

$$y = y_c + y_p$$

$$y = (c_1 + c_2)e^{\frac{1}{2}x} + 4$$

1) $\frac{d^2y}{dx^2} + \frac{3dy}{dx} + y = e^{2x} + e^{-x}$

here homogeneous equation is

$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$

$D^2y + 3Dy + y = 0$

$(D^2 + 3D + 1)y = 0$

A.E. $\rightarrow (m^2 + 3m + 1) = 0$

$m = \frac{-3 \pm \sqrt{9-4}}{2}$

$\boxed{m = \frac{-3 \pm \sqrt{5}}{2}}$

real & non repeated.

$y_c = c_1 e^{\left(\frac{-3+\sqrt{5}}{2}\right)x} + c_2 e^{\left(\frac{-3-\sqrt{5}}{2}\right)x}$

here $R(x) = e^{2x} + e^{-x}$.

$e^{ax+b} \Rightarrow a = 2, b = 0$

$e^{ax+b} \Rightarrow a = -1, b = 0$

for $y_p \rightarrow R(x) = e^{2x} + e^{-x}$

$y_p = \frac{1}{f(D)}(R(x))$

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$$y_p = \frac{1}{D^2+3D+1}(e^{2x}+e^{-x})$$

$$y_p = \frac{e^{2x}}{D^2+3D+1} + \frac{e^{-x}}{D^2+3D+1}$$

$$= \frac{e^{2x}}{(2)^2+6+1} + \frac{e^{-x}}{(-1)^2+3(-1)+1}$$

$$y_p = \frac{e^{2x}}{11} - e^{-x}$$

So, general solution is,

$$y = y_c + y_p$$

$$= c_1 e^{\left(\frac{-3+\sqrt{5}}{2}\right)x} + c_2 e^{\left(\frac{-3-\sqrt{5}}{2}\right)} + \frac{e^{2x}}{11} - e^{-x}$$

2) Solve

$$(D^2-4)y = 1+e^x$$

homogeneous operator form

$$(D^2-4)y = 0$$

A.E. $\rightarrow$ $m = \pm 2$

real, different

$$y_c = c_1 e^{-2x} + c_2 e^{2x}$$

Now, for $y_p \longrightarrow R(x) = 1+e^x$

$$y_p = \frac{1}{f(D)}\left[R(x)\right]$$

$$y_p = \frac{1}{D^2-4}(1+e^x)$$

$$= \frac{1}{D^2-4}(e^{0x}+e^{1x})$$

$$= \frac{1}{D^2-4}e^{0x} + \frac{1}{D^2-4}e^{1x}$$

$$= \frac{-1}{4}e^{0x} - \frac{1}{3}e^{x}$$

$$y_p = \frac{-1}{4} - \frac{1}{3}e^{x}$$

3) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = e^{x+5}$

here homogeneous operator form

$$D^2y - 5Dy + 4y = 0$$

$$(D^2-5D+4)y = 0$$

A.E. $m^2 - 5m + 4 = 0$

$m^2 - 4m - m + 4 = 0$

$(m-1)(m-4) = 0$

$m = 1, 4$

real & distince

$$y_c = c_1e^{x} + c_2e^{4x}$$

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for y_p

$$R(x) = e^{x+5}$$

$$y_p = \frac{1}{f(D)} (R(x))$$

$$= \frac{1}{D^2-5D+4} (e^{x+5})$$

$$= \frac{\cancel{\quad}}{} e^{x+5} \Rightarrow a=1,\ b=5$$

$$(D-a)^r \Rightarrow f^r(a) \Rightarrow y_p = \frac{x^r}{f^r(a)} e^{ax}$$

$$D^2 - 5D + 4 = 0$$

$$2D - 5 = 0$$

here $a = 1. \Rightarrow -3 = f'(a)$

$$y_p = \frac{x^1}{f'(a)} (e^{x+5})$$

$$y_p = \frac{x}{-3} e^{x+5}.$$

* Case (II)

$R(x) = \sin(ax+b)$ or $\cos(ax+b)$

$f(-a^2) \neq 0$

$$y_p = \frac{1}{f(-a^2)} \sin(ax+b)$$

or $$y_p = \frac{1}{f(-a^2)} \cos(ax+b)$$

$f(-a^2) = 0$

↳ (D^2+a^2) is a factor

Let $(D^2+a^2)^r$ is repeated for r times

$R(x) = \sin(ax+b)$

$$y_p = \frac{(-1)^r x^r}{(2a)^r r!} \sin\left[(ax+b) + \frac{r\pi}{2}\right]$$

$R(x) = \cos(ax+b)$

~~$y_p = \cos(ax+$~~

$$y_p = \frac{(-1)^r x^r}{(2a)^r r!} \cos\left[(ax+b) + \frac{r\pi}{2}\right]$$

1) Solve $\dfrac{d^2y}{dx^2} + 6y = \cos^2 x$

here homogeneous operator form

$D^2y + 6y = 0$

$(D^2+6)y = 0$

A.E → $m^2 + 6 = 0$

$m^2 = -6$

$m = \pm\sqrt{6}\, i$

$m = 0 \pm \sqrt{6}\, i$

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$$y_c = e^{0x}\left[c_1 \cos\sqrt{6}\,x + c_2 \sin\sqrt{6}\,x\right]$$

$$R(x) = \cos^2 x$$

$$R(x) = \frac{1+\cos 2x}{2}$$

$$= \frac{1}{2} + \frac{\cos 2x}{2}$$

$$y_p = \frac{1}{f(D)}\left[R(x)\right]$$

$$= \frac{1}{D^2+6}\cos^2 x$$

$$= \frac{1}{D^2+6}\left[\frac{1}{2} + \frac{1}{2}\cos 2x\right]$$

$$= \frac{1}{2}\left(\frac{e^{0x}}{D^2+6}\right) + \frac{1}{2}\left(\frac{\cos 2x}{D^2+6}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{6} + \frac{1}{2}\left[\frac{\cos 2x}{-4+6}\right]$$

$$y_p = \frac{1}{12} + \frac{1}{4}\cos 2x$$

General Solution

$$y = y_c + y_p$$

$$= e\left[c_1 \cos\sqrt{6}x + c_2 \sin\sqrt{6}\,x\right] + \left[\frac{1}{12} + \frac{1}{4}\cos 2x\right]$$

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2) Solve $(D^2+9)y = 2\sin 3x + \cos 3x$

homogeneous operator form.

$$(D^2+9)y = 0$$

A.E. $\Rightarrow m^2 + 9 = 0$

$$m^2 = -9$$

$$m = \pm 3i$$

$$m = 0 \pm 3i$$

$$y_c = e^{0x}\left[c_1 \cos 3x + c_2 \sin 3x\right]$$

for y_p

$$R(x) = 2\sin 3x + \cos 3x$$

$$y_p = \frac{1}{f(D)} R(x)$$

$$= \frac{1}{D^2+9}\left[2\sin 3x + \cos 3x\right]$$

$$= 2\left[\frac{\sin 3x}{D^2+9}\right] + \frac{\cos 3x}{D^2+9} \qquad \left(D^2 + a^2 = 0\right)$$

here $r = 1$.

$$= 2\left[\frac{(-1)x}{6\cdot 1}\sin\left((3x)+\frac{\pi}{2}\right)\right] + \frac{(-1)x}{6\cdot 1}\cos\left(3x+\frac{\pi}{2}\right)$$

$$= 2\left[\frac{-x}{6}\cos 3x\right] + \frac{x}{6}\sin 3x$$

$$y_p = \frac{x}{6}\sin 3x - \frac{x}{3}\cos 3x$$

$$= \frac{x}{6}\left[\sin 3x - 2\cos 3x\right]$$

$$y = y_c + y_p$$

$$= c_1 \cos 3x + c_2 \sin 3x + \frac{x}{6}\sin 3x - \frac{x}{3}\cos 3x$$

$$y = \cos 3x\left[c_1 - \frac{x}{3}\right] + \sin 3x\left[c_2 + \frac{x}{6}\right]$$

3) Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin x \cos 3x.$

homogeneous operator form

$$D^2y + 3Dy + 2y = 0$$

$$(D^2 + 3D + 2)y = 0$$

$$A.E. \Rightarrow m^2 + 3m + 2 = 0$$

$$m^2 + 2m + m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -1, -2$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

for y_p

$$R[x] = \frac{1}{2}\left[2\cos 3x \sin x\right]$$

$$R(x) = \frac{1}{2}\left[\sin(4x) - \sin(2x)\right]$$

$$y_p = \frac{1}{f(D)} R(x)$$

$$= \frac{1}{D^2+3D+2}\left[\frac{\sin 2x}{2} - \frac{\sin x}{2}\right]$$

$$\cancel{= \frac{1}{2}\left[\frac{\sin 2x}{D^2+3D+2}\right] - \frac{1}{2}\left[\frac{\sin x}{D^2+3D+2}\right]}$$

$$= \frac{1}{2}\left[\frac{1}{-4+3D+2}\sin 2x - \frac{1}{-1+3D+2}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{3D-2}\sin 2x - \frac{1}{3D+1}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{(3D+2)}{9D^2-4}\sin 2x - \frac{(3D-1)}{9D^2-1}\sin x\right]$$

$$= \frac{1}{2}\left[(3D+2)\left(\frac{-1}{40}\right)\sin 2x - (3D-1)\left(\frac{-1}{10}\right)\sin x\right]$$

$$= \frac{-1}{40}\sin 2x - \frac{3D}{80}\sin 2x + \frac{1}{20}3D\sin x - \frac{1}{20}\sin x$$

$$= \frac{-1}{40}\sin 2x - \frac{3}{40}\cos 2x + \frac{3}{20}\cos x - \frac{1}{20}\sin x$$

$$\cancel{= \frac{-1}{40}\left[\sin 2x + 3\cos 2x\right] + \frac{1}{2}}$$

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$$y_p = \frac{-1}{40}\left[\sin 2x + 3\cos 2x\right] + \frac{1}{20}\left[3\cos x - \sin x\right]$$

General solution.

$$y = y_c + y_p$$

$$y = \left(c_1 e^{-x} + c_2 e^{-2x}\right) + \left[\frac{-1}{40}(\sin 2x + 3\cos 2x) + \frac{1}{20}(3\cos x - \sin x)\right]$$

4) Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = \cosh x + e^{2x} + \sin 3x$.

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

here homogeneous operator form,

$$D^2y + 4Dy = 0$$

$$(D^2 + 4D)y = 0$$

A. E. $\Rightarrow (m^2 + 4m) = 0$

$\Rightarrow m(m+4) = 0$

$\Rightarrow m = 0$ & $m = -4$

Real & non repeated

$$y_c = c_1 e^{0x} + c_2 e^{-4x}$$

$$= c_1 + c_2 e^{-4x}.$$

For y_p.

$$R(x) = \cosh x + e^{2x} + \sin 3x$$

$$= \frac{e^x + e^{-x}}{2} + e^{2x} + \sin 3x.$$

$$y_p = \frac{1}{f(D)}\left[R(x)\right]$$

$$y_p = \frac{1}{D^2+4D}\left[\frac{e^x+e^{-x}}{2} + e^{2x} + \sin 3x\right].$$

$$y_p = \frac{1}{2}\left[\frac{e^x}{D^2+4D} + \frac{e^{-x}}{D^2+4D}\right] + \frac{e^{2x}}{D^2+4D} + \frac{\cos 3x}{D^2+4D}.$$

$$= \frac{1}{2}\left[\frac{e^x}{1+4} + \frac{e^{-x}}{1-4}\right] + \frac{e^{2x}}{4+8} + \frac{\cos 3x}{9+4D}.$$

$$= \frac{1}{2}\left[\frac{e^x}{5} + \frac{e^{-x}}{-3}\right] + \frac{e^{2x}}{12} + \frac{1}{4D+9}\cos 3x$$

$$= \frac{e^x}{10} + \frac{e^{-x}}{-6} + \frac{e^{2x}}{12} + \frac{4D-9}{16D^2-81}\cos 3x$$

$$= \frac{e^x}{10} + \frac{e^{-x}}{-6} + \frac{e^{2x}}{12} + \frac{4D\cos 3x}{\cancel{603}-207} + \frac{\cancel{9}1}{\cancel{7}23}\cos 3x$$

$$= \frac{e^x}{10} + \frac{e^{-x}}{-6} + \frac{e^{2x}}{12} + \frac{4(-\sin 3x)}{\cancel{603}-207} + \frac{1}{23}\cos 3x$$

$$= \frac{e^x}{10} + \frac{e^{-x}}{-6} + \frac{e^{2x}}{12} + \frac{4}{207}\sin 3x + \frac{1}{23}\cos 3x$$

$y =$

Case (iii)

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$$R(x) = x^m$$

$$\frac{1}{f(D)} R(x)$$

$$\frac{1}{f(D)} = \frac{1}{1+\phi(D)}$$

$$= 1 - \phi(D) + [\phi(D)]^2 - [\phi(D)]^3 + \cdots$$

$$\frac{1}{f(D)} = \frac{1}{1-\phi(D)}$$

$$= 1 + \phi(D) + [\phi(D)]^2 + [\phi(D)]^3 + \cdots$$

1) $\frac{d^4 y}{dx^4} - 16y = e^{2x} + x^4$

here homogeneous operator form

$$D^4 y - 16y = 0$$
$$(D^4 - 16)y = 0$$

A.E. $\Rightarrow m^4 - 16 = 0$

$\Rightarrow (m^2 - 4)(m^2 + 4) = 0$

$\Rightarrow \boxed{m = \pm 2}$ or $\boxed{m = \pm 2i}$

$$y_c = c_1 e^{2x} + c_2 e^{-2x} + e^{0x}[c_3 \cos 2x + c_4 \sin 2x$$

For y_p:

$$R(x) = e^{2x} + x^4$$

$$y_p = \frac{1}{f(D)}[R(x)]$$

$$y_p = \frac{1}{D^4-16}(e^{2x}+x^4)$$

$$= \frac{1}{D^4-16} e^{2x} + \frac{1}{D^4-16} x^4$$

$$= \frac{x^1}{4(2)^3} e^{2x} + \frac{1}{-16}\left[\frac{1}{1-\left(\frac{D^4}{16}\right)}\right] x^4$$

$$= \frac{xe^{2x}}{32} - \frac{1}{16}\left[1+\left(\frac{D^4}{16}\right)+\dots\right] x^4$$

$$y_p = \frac{xe^{2x}}{32} - \frac{1}{16}\left(x^4 + \frac{24}{16}\right)$$

General Solution

$$y = y_c + y_p$$

$$= c_1 e^{2x} + c_2 e^{-2x} + e^{0x}[c_3 \cos 2x + c_4 \sin 2x]$$

$$+ \frac{xe^{2x}}{32} - \frac{1}{16}\left(x^4 + \frac{24}{16}\right)$$

2) $$\frac{d^3y}{dx^3} + 8y = x^4 + 2x + 1 + e^{5x} + 5\sin(3x+2)$$

here homogeneous operator form

$$D^3y + 8y = 0$$

$$(D^3+8)y = 0$$

$$D^3 + 8 = 0$$

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② A.E. ⇒ $m^3+8=0$

$(m^3)+(2)^3=0$

$(m+2)(m^2-2m+4)=0$

$(m+2)=0$ or

$m^2-2m+4=0$

$\Delta = b^2-4ac$

$= 4-16$

$\Delta = -12$

$m = \frac{-b \pm \sqrt{\Delta}}{2a}$

$m = -2, \frac{+2 \pm \sqrt{-12}}{2}$

$= -2, 1 \pm \sqrt{-3}$

$m = -2, 1+\sqrt{3}i, 1-\sqrt{3}i$

$y_c = c_1 e^{-2x} + e^{x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)$

For y_p,

$R(x) = x^4+2x+1+e^{5x}+\sin(3x+2)$

$y_p = \frac{1}{f(D)}[R(x)]$

$y_p = \frac{1}{D^3+8}\left[x^4+2x+1+e^{5x}+\sin(3x+2)\right]$

$= \frac{1}{D^3+8}x^4 + \frac{2}{D^3+8}x + \frac{1}{D^3+8} + \frac{e^{5x}}{D^3+8} + \frac{1}{D^3+8}\sin(3x+2)$

$\cancel{= \frac{1}{8}\left[\frac{1}{\left(\frac{D^3}{8}\right)+(1)}\right]x^4 + \frac{2}{8}\frac{1}{\left[1+\left(\frac{D^3}{8}\right)\right]}x}$

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$$= \frac{1}{8\left(1+\left(\frac{D}{2}\right)^3\right)}\left(x^4 + \frac{1}{8\left[1+\left(\frac{D}{2}\right)^3\right]} x\right) + \frac{e^{0x}}{D^3+8} + \frac{e^{5x}}{D^3+8}$$

$$+ \frac{1}{D^3+8} \sin(3x+2)$$

$$\cancel{\frac{1}{8\left(1+\frac{D^3}{8}\right)}} = \frac{1}{8}\left[1 - \frac{D^3}{8} + \dots\right] x^4 + \frac{1}{8}\left[1 - \frac{D^3}{8} + \dots\right] x$$

$$+ \cancel{1}$$

$$= \frac{1}{8}\left(1 + \frac{D^3}{8} + \dots\right)\left(x^4 + 2x + 1\right) + \cancel{\frac{1}{8}} \frac{1}{D^3+8} e^{5x} + \frac{1}{D^3+8} \sin(3x+2)$$

$$= \frac{1}{8}\left(x^4 + 3x + 2x + 1\right) + \frac{1}{133} e^{5x} + \frac{D^3-8}{D^6-64} \sin(3x+2)$$

$$(D^2)^3$$
$$-9$$

$$= \frac{1}{8}\left(x^4 + 5x + 1\right) + \frac{1}{133} e^{5x} + \left(D^3 - 8\right) \frac{1}{-793} \sin(3x+2)$$

$$y_p = \frac{1}{8}\left(x^4 + 5x + 1\right) + \frac{e^{5x}}{133} + \frac{\sin(3x+2)(27)}{793} + \frac{8 \sin(3x+2)}{793}$$

$$\cancel{\frac{1}{8}}$$

$$y = y_c + y_p$$

$$= c_1 e^{-2x} + e^{x}\left[c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right] + \frac{1}{8}\left(x^4 + 5x + 1\right)$$

$$\begin{array}{r} 729 \\ +\ 64 \\ \hline 793 \end{array}$$

$$+ \frac{e^{5x}}{133} + \frac{27\sin(3x+2)}{793} + \frac{8\sin(3x+2)}{793}$$