Higher Order Differential Equation

High Order Diff Eq

Homogeneous	Non - Homogeneous
$\frac{d^2y}{dx^2} + y = 0$	$\frac{d^2y}{dx^2} + y = e^x$
$\frac{d^3y}{dx^3} + y = 0$	$\frac{d^3y}{dx^3} + y = \sin x$

* Homogeneous Equation is

$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

$\frac{d}{dx} = D$ ↓ Differentiation operator

$D^2y + Dy + y = 0$

$(D^2 + D + 1)y = 0$

Auxilliary equation is $D^2 + D + 1 = 0$.

$\frac{d^4y}{dx^4} + 4y = 0$

$\frac{d}{dx} = D.$

$D^4y + 4y = 0$

$(D^4 + 4)y = 0$

A.E. is $D^4 + 4 = 0$

$\frac{d^3y}{dx^3} + 3\frac{dy}{dx} + y = 0$

$\frac{d}{dx} = D.$

$(D^3 + 3D + 1)y = 0$

A.E. is

$D^3 + 3D + 1 = 0$

Case I :- If roots of A. E. are real & distinct.

$D = m_1, m_2, m_3, m_4$

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x}$

$(D-1)(D-5)(D+7)y = 0.$

$D = 1, 5, -7$

$y = c_1 e^{1x} + c_2 e^{5x} + c_3 e^{-7x}$

$(D+1/2)(D-3/2) y = 0.$

$D = \frac{-1}{2}, \frac{3}{2}$

$\boxed{y = c_1 e^{-1/2 x} + c_2 e^{3/2 x}}$

Eg :- $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

$\frac{d}{dx} = D.$

$(D^2 - 5D + 6) y = 0$

AE is

$(D^2 - 5D + 6)$

$(D-2)(D-3) = 0$

$\boxed{D = 2, 3}$

$y = c_1 e^{2x} + c_2 e^{3x}$

Case II :- If roots of A. E. are real but same.

$D = m, m, m, m.$

$y = (c_1 + c_2 x^1 + c_3 x^2 + c_4 x^3) e^{mx}$

$(D-1)^7 y = 0.$

$D = 1, 1, 1, 1, 1, 1, 1$

$y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + c_6 x^5 + c_7 x^6). e^{1x}$

$\frac{d^2y}{dx^2} - 6\frac{dy}{dx}$

$\frac{d}{dx} = D.$

$(D^2 - 6D + 9) y = 0$

A.E. is

$(D-3)^2 = 0$

$\boxed{D = 3, 3}$

$y = (c_1 + c_2 x) e^{3x}$

Hint: $L.S. = \frac{(M.S)^2}{4 \times F.S.}$

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* $\left(D - \frac{1}{2}\right)^3 y = 0.$

$D = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

$y = (C_1 + C_2 x + C_3 x^2) e^{1/2 x}$

* $D^2 \cdot (D-1)^3 y = 0$

$D^2 = 0$ | $(D-1)^3 = 0$

$\boxed{D = 0, 0}$ | $\boxed{D = 1, 1, 1}$

$y = (C_1 + C_2 x) e^{0x} + (C_3 + C_4 x + C_5 x^2) e^{x}$

Case III: If roots of A.E. are complex conjugate.

Ex. $(D^2 + D + 1) y = 0.$

A.E. is

$D^2 + D + 1 = 0.$

$D^2 + D = -1$

$D^2 + D + \frac{1}{4} = -1 + \frac{1}{4}$

$\left(D + \frac{1}{2}\right)^2 = \frac{-3}{4}$

$-\frac{3}{4} i^2$

$D + \frac{1}{2} = \pm \frac{\sqrt{3}}{2} i$

$\boxed{D = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i}$

$L.S. = \frac{(M.S)^2}{4 \times D^2}$

$= \frac{D^2}{4D^2}$

$= \frac{1}{4}$

$(D^2 + 4D + 5) y = 0$

A.E. is

$D^2 + 4D + 5 = 0.$

$D^2 + 4D = -5$

$(D^2 + 4D + 4) = -5 + 4$

$(D + 2)^2 = -1$

$= i^2$

$D + 2 = \pm i$

$\boxed{D = -2 \pm 1i}$

$y = e^{-2x}[C_1 \cos x + C_2 \sin x]$

* D

Case IV

Ans

* $D^2+4=0$.

$D^2=-4$

$=4i^2$

$\boxed{D=0\pm 2i}$

$y=c_1\cos 2x+c_2\sin 2x$

$D=\alpha\pm i\beta$

$y=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$

$$\boxed{\begin{aligned}&\text{if } \alpha=0\\ &D=\pm i\beta\\ &y=[c_1\cos\beta x+c_2\sin\beta x]\end{aligned}}$$

Case IV If roots of A. E. are pair of complex:-

Ans-

$$\boxed{\begin{aligned}D&=\alpha\pm i\beta,\ \alpha\pm i\beta\\ y&=e^{\alpha x}[(c_1+c_2x+c_3x^2)\cos\beta x+(c_4+c_5x+c_6x^2)\sin\beta x]\end{aligned}}$$

Eg:- $D=-2\pm i,\ -2\pm i,\ -2\pm i$.

$\boxed{y=e^{-2x}[(c_1+c_2x)\cos x+(c_3+c_4x)\sin x]}$

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* Non-Homogeneous Equation :-

Only For mid.

e^{ax}

$\sin x / \cos x$

x^m

$$y = y_c + y_p$$

$$= (C.F.) + (P.I.)$$

$$= \left(\text{Complimentary function}\right) + \left(\text{Particular integral}\right)$$

- Short cut methods :-

Case-I Particular Integral.

$$\frac{e^{ax}}{f(D)}$$

* $(D^2 + 2D + 3)y = e^{2x}$

$$y_p = \frac{e^{2x}}{D^2 + 2D + 3}$$

$$= \frac{e^{2x}}{4+4+3} = \frac{e^{2x}}{11}$$

* $(D^3 + 1)y = e^{-2x}$

$$y_p = \frac{e^{-2x}}{D^3 + 1}$$

$D = -2$

$$= \frac{e^{-2x}}{-8+1} = \frac{e^{-2x}}{7}$$

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Note :- Case failure

$$\frac{e^{ax}}{(D-a)^r} = \frac{x^r}{r!}e^{ax}$$

* $(D-2)^3 y = e^{2x}$

$$y_p = \frac{e^{2x}}{(D-2)^3}$$

$$= \frac{x^3}{3!} e^{2x}$$

* $(D+3)^2 y = e^{-3x}$

$$y_p = \frac{e^{-3x}}{(D+3)^2} = \frac{x^2}{2!} e^{-3x}$$

* $(D^2 - 5D + 6) y = e^{2x}$

$$y_p = \frac{e^{2x}}{D^2 - 5D + 6}$$

$$= \frac{e^{2x}}{(D-3)(D-2)}$$

$$= -\left[\frac{x^1}{1!} e^{2x}\right]$$

$$= -x e^{2x}.$$

GTU Q. Solve $y'' - 3y' + 2y = e^x$.

$(D^2 - 3D + 2)y = e^x$

A.E. is

$D^2 - 3D + 2 = 0$

$(D-1)(D-2) = 0$

$D = 1, 2$.

$$\boxed{y_c = c_1 e^x + c_2 e^{2x}}$$

$$y_p = \frac{e^x}{D^2 - 3D + 2}$$

$$= \frac{e^{x}}{(D-2)(D-1)}$$

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Q. Solve $(D^3+8)y = \cosh 2x$

$$= \left(\frac{e^{2x}+e^{-2x}}{2}\right)$$

soln: The A.E. is

$$D^3+8=0$$
$$D^3=-8$$
$$D=-2,-2,-2$$ X

A.E. is

$$D^3+8=0$$
$$D^3+(2)^3=0$$

$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$

$$(D+2)(D^2-2D+4)=0$$

$(D+2)=0$

$D=-2$

$D^2-2D+4=0$

$D^2-2D=-4$

$D^2-2D+1=-4+1$

$(D-1)^2=-3$

$=3i^2$

$(D-1)=\pm\sqrt{3}i$

$D=-2,\ 1\pm\sqrt{3}i$

$$y_c = c_1e^{-2x}+e^{x}\left[c_2\cos\sqrt{3}x+c_3\sin\sqrt{3}x\right]$$

$$y_p = \frac{1}{2}\left[\frac{e^{2x}+e^{-2x}}{D^3+8}\right]$$

$$= \frac{1}{2}\left[\frac{e^{2x}}{D^3+8}+\frac{e^{-2x}}{D^3+8}\right]$$

$y = y_c + y_p$

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Q. Solve $y''' - 3y'' + 3y' - y = 4e^t$

~~$(D^3 - 3D^2 + 3D - 1)y = 4e^t$~~

Soln: ~~The A.E. is~~

Let $\frac{d}{dt} = D$

$(D^3 - 3D^2 + 3D - 1)y = 4e^t$

A.E. is

$D^3 - 3D^2 + 3D - 1 = 0$

$D^3 - D^2 - 2D^2 + 2D + D - 1 = 0$

$D^2(D-1) - 2D(D-1) + 1(D-1) = 0$

$(D-1)(D^2 - 2D + 1) = 0$

$(D-1)(D-1)^2 = 0$

$(D-1)^3 = 0$

Note :- $D^3 - 6D^2 + 11D - 6 = 0$. (For finding roots)
$D^3 - D^2 - 5D^2 + 5D + 6D - 6 = 0$.
$D^2(D-1) - 5D(D-1) + 6(D-1) = 0$.
$(D-1)(D^2 - 5D + 6) = 0$.
$(D-1)(D-2)(D-3)$

$D = 1, 1, 1$.

$y_c = (c_1 + c_2 t + c_3 t^2) e^t$

$y_p = \frac{4e^t}{D^3 - 3D^2 + 3D - 1}$

$= \frac{4e^t}{(D-1)^3}$

$= 4\left[\frac{t^3}{3!} e^t\right]$

$$\boxed{y_p = \frac{2}{3} t^3 e^t}$$

$$\boxed{y = y_c + y_p}$$

Q. Solve $(D^4 - 16)y = e^{2x}$.
soln:-

AE is
$(D^4 - 16) = 0$
$(D^2 - 4)(D^2 + 4) = 0$.

$D^2 - 4 = 0$

$D^2 = 4$

$D = \pm 2$

$D^2 + 4 = 0$

$D^2 = -4$

$D = 0 \pm 2i$

$$y_c = (c_1 e^{-2x} + c_2 e^{2x}) + (c_3 \cos 2x + c_4 \sin 2x)$$

$$y_p = \frac{e^{2x}}{D^4 - 16}$$

$$= \frac{e^{2x}}{(D^2-4)(D^2+4)}$$

$$= \frac{e^{2x}}{(D-2)(D+2)(D^2+4)}$$

$$= \frac{1}{4(8)} \left(\frac{e^{2x}}{D-2} \right)$$

$$= \frac{1}{32} \left[\frac{x}{1!} e^{2x} \right]$$

$$\boxed{y_p = \frac{x e^{2x}}{32}}$$

Case-2 $\dfrac{\sin(ax+b) / \cos(ax+b)}{f(D^2)}$.

* $(D^2+20)y = \sin 3x$

$$y_p = \frac{\sin 3x}{D^2+20}.$$

$$\boxed{D^2 = -9.}$$

Note:- After square of 3 ~~keep~~ put minus in its square

$$= \frac{\sin 3x}{-9+20} = \frac{\sin 3x}{11}$$

* $(D^2+15)y = \sin 2x$

$$y_p = \frac{\sin 2x}{D^2+15}$$

$$D^2 = -4.$$

$$= \frac{\sin 2x}{-4+15}$$

$$= \frac{\sin 2x}{11}.$$

$$= \frac{\sin(ax+b) / \cos(ax+b)}{f(-a)^2}$$

$$(D+1)\,y = \sin 2x$$

$$y_p = \frac{\sin 2x}{(D+1)}$$

$$= \frac{\sin 2x\,(D-1)}{(D+1)(D-1)}$$

$$= \frac{2\cos 2x - \sin 2x}{(D^2-1)}$$

$$D^2 = -4$$

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Q. Solve $y'' - 8y' + 9y = 40 \sin 5x$.

Soln:

$$\frac{d}{dx} = D$$

$$(D^2 - 8D + 9)y = 40 \sin 5x.$$

The A.E. is

$$(D^2 - 8D + 9) = 0.$$

$$D^2 - 8D + 16 = -9 + 16.$$

$$(D-4)^2 = 7$$

$$D - 4 = \pm\sqrt{7}$$

$$D = 4 \pm \sqrt{7}$$

$$D = 4 - \sqrt{7},\ 4 + \sqrt{7}.$$

$$y_c = c_1 e^{(4-\sqrt{7})x} + c_2 e^{(4+\sqrt{7})x}$$

$$y_p = \frac{40 \sin 5x}{(D^2 - 8D + 9)}$$

$$D^2 = -25$$

$$= \frac{40 \sin 5x}{-25 - 8D + 9}$$

$$= \frac{40 \sin 5x}{-16 - 8D}$$

$$= \frac{40 \sin 5x}{-8(D+2)}$$

$$= 5 \frac{(\sin 5x)}{(D+2)} \cdot \frac{(D-2)}{(D-2)}$$

$$= -5 \frac{\cos 5x - 2\sin 5x}{D^2-4}$$

$$D^2 = -25$$

$$= -5 \left[\frac{5\cos 5x - 2\sin 5x}{-25-4}\right]$$

$$= \frac{5}{29}\left[5\cos 5x \pm 2\sin 5x\right]$$

$$\boxed{y = y_c + y_p}$$

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Q. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \cos 2x \sin x$

soln:-

$$\frac{d}{dx} = D$$

$$(D^2 - 6D + 9)y = 0$$

$$(D^2 - 6D + 9)y = \frac{1}{2}[\sin 3x - \sin x]$$

$$\boxed{\begin{array}{l} D = 3, 3 \\ y_c = (C_1 + C_2 x)e^{3x} \end{array}}$$

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$$y_p = \frac{1}{2}\left[\frac{\sin 3x - \sin x}{D^2 - 6D + 9}\right]$$

$$= \frac{1}{2}\left[\frac{\sin 3x}{D^2 - 6D + 9} - \frac{\sin x}{D^2 - 6D + 9}\right]$$

$$= \frac{1}{2}\left[\frac{\sin 3x}{D^2 - 6D + 9} - \frac{\sin x}{D^2 - 6D + 9}\right]$$

$$= \frac{1}{2}\left[\frac{\sin 3x}{-9 - 6D + 9} - \frac{\sin x}{-1 - 6D + 9}\right]$$

$$= \frac{1}{2}\left[\frac{-1}{6}\,\frac{\sin 3x}{D} - \frac{\sin x}{8 - 6D}\right]$$

$$= \frac{1}{2}\left[\frac{-1}{6}\int \sin 3x \, dx - \frac{1}{2}\left(\frac{\sin x}{(4 - 3D)}\,\frac{(4 + 3D)}{(4 + 3D)}\right)\right]$$

$$= \frac{1}{2}\left[\frac{-1}{6}\left(\frac{-\cos 3x}{3}\right) - \frac{1}{2}\left(\frac{4\sin x + 3\cos x}{16 - 9D^2}\right)\right]$$

$$D^2 = -1$$

$$= \frac{1}{2}\left[\frac{1}{18}\cos 3x - \frac{1}{50}(4\sin x + 3\cos x)\right]$$

$$\boxed{y = y_c + y_p}$$

$$y = (c_1 + c_2 x)e^{3x} + \left[\frac{1}{18}\cos 3x - \frac{1}{50}(4\sin x + 3\cos x)\right]$$

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Case:3 $\frac{x^m}{f(D)} =$

$1+D^{-1} = 1-D+D^2-D^3 \ldots$

$1-D^{-1} = 1+D+D^2+D^3 \ldots$

Q. $(D+5)y = x^2$

$y_p = \frac{x^2}{D+5}$

$= \frac{x^2}{5(1+D/5)}$

$= \frac{x^2}{5}\left[1-\left(\frac{D}{5}\right)+\left(\frac{D}{5}\right)^2+ \ldots\right]$

$= \frac{x^2}{5}\left[1-\frac{D}{5}+\frac{D^2}{25}+\ldots\right]$

$= \frac{1}{5}\left[x^2-\frac{2x}{5}+\frac{2}{25}\right]$

Q. $(D^4-16)y = x^4$

$y_p = \frac{x^4}{D^4-16}$

$= \frac{x^4}{-16(1-D^4/16)}$

$= -\frac{1}{16}x^4\left(1-\frac{D^4}{16}\right)^{-1}$

$= -\frac{1}{16}x^4\left[1+\left(\frac{D^4}{16}\right)+\ldots\right]$

$= -\frac{1}{16}\left[x^4+\frac{24}{16}\right]$

$= -\frac{1}{16}\left[x^4+\frac{3}{2}\right]$

$(D^4-16)y = e^{2x}+x^4$

$= \frac{e^{2x}+x^4}{D^4-16}$

$= \left(\frac{e^{2x}}{D^4-16}\right)+\left(\frac{x^4}{D^4 \ldots}\right.$

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Q. Solve

$(D^3 - D^2 - 6D)\, y = x^2 + 1.$

Soln:-

$$y_p = \frac{x^2+1}{D^3 - D^2 - 6D}$$

$$= \frac{x^2+1}{-6D\left[1 + \left(\frac{D^3 - D^2}{-6D}\right)\right]}$$

$$= \frac{-1}{6D} \frac{(x^2+1)}{\left[1 + \left(\frac{D}{6} - \frac{D^2}{6}\right)\right]}$$

$$= \frac{-1}{6D} (x^2+1) \left[1 - \left(\frac{D}{6} - \frac{D}{6}\right)^2\right]^{-1}$$

$$= \frac{-1}{6D} (x^2+1) \left[1 + \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2\right]$$

$$= \frac{-1}{6D} (x^2+1) \left[1 - \frac{D}{6} + \frac{D^2}{6} + \frac{D^2}{36} + \ldots\right]$$

$$= \frac{-1}{6D} (x^2+1) \left[1 - \frac{D}{6} + \frac{7}{36} D^2 + \ldots\right]$$

$$= \frac{-1}{6D} \left[(x^2+1) - \frac{1}{6}(2x) + \frac{7}{36}(2)\right]$$

$$= \frac{-1}{60}\left[x^2 - \frac{x}{3} + \left(1 + \frac{7}{18}\right)\right]$$

$$= -\frac{1}{6}\int\left(x^2 - \frac{x}{3} + \frac{25}{18}\right)dx.$$

$$= \frac{1}{6}\left[\frac{x^3}{3} - \frac{1}{3}\cdot\left(\frac{x^2}{2}\right) + \frac{25}{18}x\right]$$