AMATH 515 Homework #4 - Vinsensius
Due: Wednesday, March 15th, by 11:59 pm

Number 1

Prove the following identity for $\alpha \in \mathbb{R}$:

$$\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2.$$

We will start from the LHS of the identity

$$\begin{aligned}
LHS &= \|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 \\
&= \alpha^2\|x\|^2 + 2\alpha(1-\alpha)\langle x, y\rangle + (1-\alpha)^2\|y\|^2 + \alpha(1-\alpha)(\|x\|^2 - 2\langle x, y\rangle + \|y\|^2) \\
&= \alpha^2\|x\|^2 + (1-\alpha)^2\|y\|^2\alpha(1-\alpha)\|x\|^2 + \alpha(1-\alpha)\|y\|^2 \\
&= \alpha\|x\|^2 + (1-\alpha)\|y\|^2 = RHS
\end{aligned}$$

Number 2

An operator T is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all (x, y). For any such nonexpansive operator T, define

$$T_\lambda = (1 - \lambda)I + \lambda T.$$

1. Show that T_λ and T have the same fixed points.
2. Use problem 1 to show

$$\|T_\lambda z - \overline{z}\|^2 \leq \|z - \overline{z}\|^2 - \lambda(1 - \lambda)\|z - Tz\|^2.$$

   where $\overline{z}$ is any fixed point of T, i.e. $T\overline{z} = \overline{z}$.

(a) Let $\overline{x}$ be fixed point of T such that $T\overline{x} = \overline{x}$.

We can prove that $\overline{x}$ is also fixed point of T_λ, i.e. $T_\lambda\overline{x} = \overline{x}$.

$$\begin{aligned}
T_\lambda\overline{x} &= (1 - \lambda)I\overline{x} + \lambda T\overline{x} \\
&= (1 - \lambda)\overline{x} + \lambda\overline{x} \quad \text{(Def. of f.p. of T)} \\
&= \overline{x}
\end{aligned}$$

(b) We start from the LHS of the inequality

$$\begin{aligned}
LHS &= \|T_\lambda z - \overline{z}\|^2 \\
&= \|(1 - \lambda)z + \lambda Tz - T_\lambda\overline{z}\|^2 \quad \text{(From (a) and Def. of f.p.)} \\
&= \|(1 - \lambda)z + \lambda Tz - (1 - \lambda)\overline{z} - \lambda T\overline{z}\|^2 \\
&= \|\lambda(Tz - T\overline{z}) + (1 - \lambda)(z - \overline{z})\|^2 \quad \text{(T is not necessarily linear operator)}
\end{aligned}$$

From problem 1, we can see that $\alpha = \lambda$, $x = Tz - T\overline{z}$, and $y = z - \overline{z}$, so the equation becomes

$$\begin{aligned}
LHS = \|\lambda(Tz - T\overline{z}) + (1 - \lambda)(z - \overline{z})\|^2 =&\lambda\|Tz - T\overline{z}\|^2 + (1 - \lambda)\|z - \overline{z}\|^2 \\
&- \lambda(1 - \lambda)\|Tz - T\overline{z} - (z - \overline{z})\|^2 \\
\leq&\lambda\|z - \overline{z}\|^2 + (1 - \lambda)\|z \\
&- \overline{z}\|^2 - \lambda(1 - \lambda)\|Tz - z\|^2 = RHS \\
&\text{(Def. of nonexpansive of T and f.p.)}
\end{aligned}$$

Number 3

An operator T is *firmly nonexpansive* when it satisfies

$$\|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 \leq \|x-y\|^2.$$

1. Show T is firmly nonexpansive if and only if

$$\langle x-y, Tx-Ty\rangle \geq \|Tx-Ty\|^2.$$

2. Show T is firmly nonexpansive if and only if

$$\langle Tx-Ty, (I-T)x-(I-T)y\rangle \geq 0.$$

3. Suppose that $S = 2T - I$. Let

$$\mu = \|Tx-Ty\|^2 + \|(I-T)x-(I-T)y\|^2 - \|x-y\|^2$$

and let

$$\nu = \|Sx-Sy\|^2 - \|x-y\|^2.$$

Show that $2\mu = \nu$ (you may find it helpful to use problem (1)). Conclude that T is firmly nonexpansive exactly when S is nonexpansive.

(a) Start by the firmly nonexpansive definition,

$$\begin{aligned}
\|Tx-Ty\|^2 + & \\
\|(I-T)x-(I-T)y\|^2 \leq & \|x-y\|^2 \\
\|Tx-Ty\|^2 \leq & \|x-y\|^2 - \|(I-T)x-(I-T)y\|^2 \\
\|Tx-Ty\|^2 \leq & \|x-y\|^2 - \|(x-y)-(Tx-Ty)\|^2 \\
\|Tx-Ty\|^2 \leq & \|x-y\|^2 - \\
& (\|x-y\|^2 - 2\langle x-y, Tx-Ty\rangle + \|Tx-Ty\|^2) \\
\|Tx-Ty\|^2 \leq & \langle x-y, Tx-Ty\rangle
\end{aligned}$$

(b) Start by the firmly nonexpansive definition,

$$\begin{aligned}
\|Tx-Ty\|^2 + \|(I-T)x-(I-T)y\|^2 \leq \|x-y\|^2 \\
\|Tx-Ty\|^2 + 2\langle Tx-Ty, (I-T)x-(I-T)y\rangle + \|(I-T)x-(I-T)y\|^2 \\
\leq \|x-y\|^2 + 2\langle Tx-Ty, (I-T)x-(I-T)y\rangle \text{(Completing the square)} \\
\|Tx-Ty+(I-T)x-(I-T)y\| \leq \|x-y\|^2 + 2\langle Tx-Ty, (I-T)x-(I-T)y\rangle
\end{aligned}$$

$$\|x-y\|^2 \leq \|x-y\|^2 + 2\langle Tx-Ty, (I-T)x-(I-T)y\rangle$$
$$0 \leq \langle Tx-Ty, (I-T)x-(I-T)y\rangle$$

(c) (a) Show that $2\mu - \nu$. We will begin from expression of ν

$$\begin{aligned}
\|Sx-Sy\|^2 - \|x-y\|^2 =& \|(2T-I)x-(2T-I)y\|^2 - \|x-y\|^2 \\
=& \|(2Tx-2Ty)\| - (x-y)\|^2 - \|x-y\|^2 \\
=& 4\|Tx-Ty\|^2 - 4\langle Tx-Ty, x-y\rangle + \|x-y\|^2 - \|x-y\|^2 \\
=& 2\|Tx-Ty\|^2 + 2(\|Tx-Ty\|^2 - 2\langle Tx-Ty, x-y\rangle + \|x-y\|^2) \\
& -2\|x-y\|^2 \quad \text{(By completing square)} \\
=& 2\|Tx-Ty\|^2 + 2\| - (Tx-Ty) + (x-y)\|^2 - 2\|x-y\|^2 \\
=& 2\|Tx-Ty\|^2 + 2\|(I-T)x-(I-T)y\|^2 - 2\|x-y\|^2 \\
&= 2\mu
\end{aligned}$$

(b) Based on the definition of nonexpansiveness of S

$$\begin{aligned}
\|Sx-Sy\|^2 \leq& \|x-y\|^2 \\
\|(2T-I)x-(2T-I)y\|^2 \leq& \|x-y\|^2 \\
4\|Tx-Ty\|^2 - 4\langle Tx-Ty, x-y\rangle + \|x-y\|^2 \leq& \|x-y\|^2 \\
\|Tx-Ty\|^2 \leq \langle Tx-Ty, x-y\rangle&
\end{aligned}$$

Thus, T is firmly nonexpansive from part(a).

Number 4

Coding Assignment

Please download `515Hw4_Coding.ipynb` **and** `solvers.py` **to complete problem (4)**

Implement an interior point method to solve the problem

$$\min_x \frac{1}{2}\|Ax - b\|^2 \quad \text{s.t.} \quad Cx \le d.$$

Let the user input A, b, C, and d. Test your algorithm using a box constrained problem (where you can apply the prox-gradient method).

Similar to IPdemo, just follow them, and try to read the notes on the IP.