AMATH 584 Homework #1 - Vinsensius

Number 1.1

1.1a

We start with

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

For the first operation, double the first column, we use matrix

$$R1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

So the resultant matrix is

$$Re1 = B * R1 \tag{3}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

For the second operation, half row 3, so we use matrix

$$R2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

So, the resultant matrix is

$$Re1 = Re1 * R2 \tag{6}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

For the third operation, we will use matrix

$$L1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

So, the resultant matrix is

$$Re1 = L1 * Re1 \tag{9}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

For the fourth operation, we will use matrix

$$R4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

Then,the resultant matrix is

$$Re1 = Re1 * R4 \tag{12}$$

$$= \begin{bmatrix} 2 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

For the fifth operation, we will use matrix,

$$L2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{14}$$

Then, the resultant matrix is

$$Re1 = L2 * Re1 \tag{15}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 0 & \frac{1}{2} & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & \frac{1}{2} & 2 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \tag{16}$$

For the sixth operation, we will use matrix

$$R5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

Then the resultant matrix is

$$Re1 = Re1 * R5 \tag{18}$$

$$= \begin{bmatrix} 0 & -1 & \frac{1}{2} & 2 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 & 0 \end{bmatrix} \tag{19}$$

For the last operation, we can use matrix,

$$R6 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{20}$$

The final resultant matrix is

$$Re1 = Re1 * R6 \tag{21}$$

$$= \begin{bmatrix} 0 & -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ -1 & 0 & 0 \end{bmatrix} \tag{22}$$

Thus, we can rewrite everything into a product of 8 matrices

$$Re1 = L2 * L1 * B * R1 * R2 * R4 * R5 * R6 \tag{23}$$

$$= \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ -1 & 0 & 0 \end{bmatrix} \tag{24}$$

1.1b

We can define $A = L2 * L1$, and $C = R1 * R2 * R4 * R5 * R6$. If we do ABC, we should the get the same answer

$$Re1 = ABC \tag{25}$$

$$= \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ -1 & 0 & 0 \end{bmatrix} \tag{26}$$

The calculation is proven in matlab as shown by the code below.

```
B = eye(4);
R1 = [2,0,0,0;0,1,0,0;0,0,1,0;0,0,0,1];
R2 = [1,0,0,0;0,1,0,0;0,0,1/2,0;0,0,0,1];
%R3 = [1,0,1/4,0;0,1,0,0;0,0,1,0;0,0,0,1];
R4 = [0,0,0,1;0,1,0,0;0,0,1,0;1,0,0,0];
L1 = [1,0,1,0;0,1,0,0;0,0,1,0;0,0,0,1];
L2 = [1,-1,0,0;0,1,0,0;0,-1,1,0;0,-1,0,1];
R5 = [1,0,0,0;0,1,0,0;0,0,1,1;0,0,0,0];
R6 = [0,0,0;1,0,0;0,1,0;0,0,1];
A = L2*L1;
C = R1*R2*R4*R5*R6;
Resultant_mat_11a =  L2*L1*B*R1*R2*R4*R5*R6
Resultant_mat_11b = A*B*C
```

```
Resultant_mat_11a =

   -1.0000    0.5000    0.5000
    1.0000         0         0
   -1.0000    0.5000    0.5000
   -1.0000         0         0


Resultant_mat_11b =

   -1.0000    0.5000    0.5000
    1.0000         0         0
   -1.0000    0.5000    0.5000
   -1.0000         0         0
```

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Number 2.1

We will use induction that A is diagonal if A is unitary and triangular. A is unitary if $A^*A = I$. We will assume A to be upper triangular for the proof.

We will start by looking at small case. For example, in a 2-by-2 matrix.

$$A^*A = I \tag{27}$$

$$\begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ 0 & \bar{a}_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{28}$$

When we solve the equation, we found that $|a_{11}| = 1 = |a_{22}|$ and $|a_{12}| = 0$. This shows that A is indeed diagonal for small case.

We now extend the assumption to (n-1)-by-(n-1) matrix that it is a diagonal matrix.

Now we are looking at n-by-n matrix.

$$A^*A = I \tag{29}$$

$$\begin{bmatrix} \bar{a}_{11} & 0 & \cdots & 0 \\ \bar{a}_{12} & \bar{a}_{22} & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ \bar{a}_{1,(n-1)} & \cdots & \bar{a}_{(n-1),(n-1)} & 0 \\ \bar{a}_{1,n} & \cdots & \bar{a}_{(n-1),(n)} & \bar{a}_{n,n} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2,n} \\ \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & a_{(n-1),(n-1)} & a_{(n-1),n} \\ 0 & \cdots & 0 & a_{n,n} \end{bmatrix} = I \tag{30}$$

When we apply the matrix multiplication to get the first column of the identity matrix. We get the condition of the first row of A,$[a_{11}, a_{12}, \cdots, a_{1n}]$ to become $|a_{11}| = 1$ while $|a_{1j}| = 0$ for $1 < j \leq n$. That will simplify the first column of A^* so we just need to find the inner matrix condition which is (n-1)-by-(n-1) matrix multiplication. However, we have assume that the (n-1)-by-(n-1) matrix is diagonal. Thus, by induction A is diagonal if A is triangular and unitary.

Number 2.2

2.2a

By using the definition of inner product and norm we can write $||x_1 + x_2||^2$ as

$$||x_1 + x_2||^2 = (x_1 + x_2) * (x_1 + x_2) \tag{31}$$

$$\tag{32}$$

By distribution of the inner product, we get

$$||x_1 + x_2||^2 = x_1 * x_1 + x_1 * x_2 + x_2 * x_1 + x_2 * x_2 \tag{33}$$

The terms $x_1 * x_2$ and $x_2 * x_1$ will be 0 since x_1 and x_2 are orthogonal. Thus, the equation becomes

$$||x_1 + x_2||^2 = ||x_1||^2 + ||x_2||^2 \tag{34}$$

2.2b

We have shown that the theorem works for small case, i.e. n = 2. We now extend the assumption to (n-1) orthogonal vectors, that the statement

$$\left|\left|\sum_{i=1}^{n-1} x_i\right|\right|^2 = \sum_{i=1}^{n-1} ||x_i||^2 \tag{35}$$

is true. We will now try to prove that the theorem works for n orthogonal vectors.

$$\left|\left|\sum_{i=1}^{n} x_i\right|\right|^2 = (x_1 + \cdots + x_{n-1} + x_n) * (x_1 + \cdots + x_{n-1} + x_n) \tag{36}$$

$$\begin{aligned} &= (x_1 + \cdots + x_{n-1}) * (x_1 + \cdots + x_{n-1}) + x_n * (x_1 + \cdots + x_{n-1}) \\ &\quad +(x_1 + \cdots + x_{n-1}) * x_n + x_n * x_n \end{aligned} \tag{37}$$

Based on the (n-1) case, we can simplify to

$$\left|\left|\sum_{i=1}^{n} x_i\right|\right|^2 = \sum_{i=1}^{n-1} ||x_i||^2 + ||x_n||^2 = \sum_{i=1}^{n} ||x_i||^2 \tag{38}$$

By induction, the theorem works for general case of n orthogonal vectors.

Number 2.3

2.3a

We start with the equation $Ax = \lambda x$.

$$Ax = \lambda x \tag{39}$$

We will left-multiply by x^* to both sides of the equation.

$$x^*(Ax) = x^* \lambda x \tag{40}$$

$$x^*(Ax) = \lambda ||x|| \tag{41}$$

Now we will take the conjugate transpose of both sides

$$(x^*(Ax))^* = (\lambda ||x||)^* \tag{42}$$

$$x^* A^* x = \lambda^* ||x|| \tag{43}$$

Since A is hermitian then $A^* = A$ so the equation 43 becomes

$$x^* A x = \lambda^* ||x|| \tag{44}$$

Equating equation 41 and 44, we get

$$\lambda^* ||x|| = \lambda ||x|| \tag{45}$$

$$\lambda^* = \lambda \tag{46}$$

Therefore, λ is real.

2.3b

Assume that α and β are the eigenvalues of eigenvectors x and y respectively. We will look at the inner product of x and y.

$$x^* y = 0 \tag{47}$$

$$(\alpha x)^* y = 0 \tag{48}$$

We know that $\alpha x = Ax$, thus the equation becomes

$$(Ax)^* y = 0 \tag{49}$$

$$x^*(A^* y) = 0 \tag{50}$$

$$x^* \beta y = 0 \tag{51}$$

$$\beta x^* y = 0 \tag{52}$$

Setting the equations 48 and 52, we get

$$\alpha x^* y = \beta x^* y \tag{53}$$

$$(\alpha - \beta) x^* y = 0 \tag{54}$$

Since the eigenvalues are non-zero, thus the only condition for the equation above to be true is that $x^* y = 0$. Therefore, x and y are orthogonal vectors.

Number 2.6

We start with $A^{-1}A = I$ to get the α

$$A^{-1}A = I \tag{55}$$
$$(I + uv^*)(I + \alpha uv^*) = I \tag{56}$$
$$\alpha uv^* + uv^* + \alpha uv^* uv^* = 0 \tag{57}$$
$$uv^*(1 + \alpha + \alpha v^* u) = 0 \tag{58}$$
$$\alpha = \frac{-1}{1 + v^* u} \tag{59}$$

If A is singular, then A^{-1} is undefined. For that to happen, α needs to be undefined. This means that the requirement for the A to be singular is when $v^*u = -1$ assuming that u, and v are nonzero vectors

We are now looking at the null(A) for which A is singular.

$$\text{null}(A) = \{x|Ax = 0\} \tag{60}$$
$$= \{x|(I + uv^*)x = 0\} \tag{61}$$
$$= \{x|x + uv^*x = 0\} \tag{62}$$
$$= \{x|x = (-v^*x)u\} \tag{63}$$

Since $(-v^*x)$ is a constant so, we can rewrite null(A) as

$$\text{null}(A) = \text{span}(u) \tag{64}$$