AMATH 584 Homework #3 - Vinsensius

Number 6.1

To prove $(I - 2P)$ is unitary we can calculate the inner product of the projector with itself.

$$(I - 2P)^*(I - 2P) = I - 2P - 2P - 4P^2 = I - 2P - 2P - 4P = I \tag{1}$$

The geometric interpretation is that (I-2P) is a reflection action across the axis that is perpendicular to the axis of projection.

Number 6.2

From the definition of action of E on x, we can see that E is

$$E = \frac{1}{2}(I + F)$$

Where

$$F = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & & & 1 & 0 \\ \vdots & & \dots & & \vdots \\ 0 & 1 & & & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Thus, E can be written as

$$E = \frac{1}{2} \begin{bmatrix} 1 & 0 & \dots & 0 & 1 \\ 0 & \ddots & & 1 & 0 \\ \vdots & & 2 & & \vdots \\ 0 & 1 & & \ddots & 0 \\ 1 & 0 & \dots & 0 & 1 \end{bmatrix}$$

We need to check if $E = E^2$ to see if E is a projector.

$$E^2 = \frac{1}{4} \begin{bmatrix} 2 & 0 & \dots & 0 & 2 \\ 0 & \ddots & & 2 & 0 \\ \vdots & & 4 & & \vdots \\ 0 & 2 & & \ddots & 0 \\ 2 & 0 & \dots & 0 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & \dots & 0 & 1 \\ 0 & \ddots & & 1 & 0 \\ \vdots & & 2 & & \vdots \\ 0 & 1 & & \ddots & 0 \\ 1 & 0 & \dots & 0 & 1 \end{bmatrix} = E \tag{2}$$

So, E is a projector. Furthermore, We can see that $E = E^*$ just by examining the matrix. So, E is an orthogonal projector.

Number 6.4

6.4a

$$P = A(A^*A)^{-1}A^* \tag{3}$$

$$= \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 1 \\ 0.5 & 0 & 0.5 \end{bmatrix} \tag{4}$$

The image of P on vector $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^*$ is

$$P \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \tag{5}$$

6.4b

$$P = B(B^*B)^{-1}B^* \tag{6}$$

$$= \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{bmatrix} \tag{7}$$

The image of P on vector $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^*$ is

$$P \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \tag{8}$$

Number 7.1

7.1a

To calculate q_1

$$q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{9}$$

For q_2

$$\tilde{q_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^* \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{10}$$

$$q_2 = \frac{\tilde{q_2}}{||\tilde{q_2}||} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{11}$$

So,

$$\hat{Q} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \tag{12}$$

To find $\hat{R}$, we can rewrite A in terms of columns of $\hat{Q}$

$$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} q_1\sqrt{2} & q_2 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} \tag{13}$$

We can see that $\hat{R}$ is

$$\hat{R} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} \tag{14}$$

Thus, the reduced QR of A is

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{15}$$

To calculate the full QR for A, we need a third independent column vector and then we can create q_3 from that vector. Let the vector be

$$x_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

We will now create q_3

$$\tilde{q_3} = x_3 - (x_3 * q_2)q_2 - (x_3 * q_1)q_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{16}$$

$$q_3 = \frac{\tilde{q_3}}{||\tilde{q_3}||} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{17}$$

So, the full QR decomposition of A is

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{18}$$

7.1b

To calculate q_1

$$q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{19}$$

For q_2

$$\tilde{q_2} = b_2 - (b_2 * q_1)q_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \tag{20}$$

$$q_2 = \frac{\tilde{q_2}}{||\tilde{q_2}||} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \tag{21}$$

So,

$$\hat{Q} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \tag{22}$$

To find $\hat{R}$, we can rewrite B in terms of columns of $\hat{Q}$

$$B = \begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} q_1\sqrt{2} & q_2\sqrt{3} + q_1\sqrt{2} \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix} \tag{23}$$

We can see that $\hat{R}$ is

$$\hat{R} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix} \tag{24}$$

Thus, the reduced QR of B is

$$B = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{25}$$

To calculate the full QR for B, we need a third independent column vector and then we can create q_3 from that vector. Let the vector be

$$x_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

We will now create q_3

$$\tilde{q_3} = x_3 - (x_3 * q_2)q_2 - (x_3 * q_1)q_1 = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \tag{26}$$

$$q_3 = \frac{\tilde{q_3}}{||\tilde{q_3}||} = \sqrt{\frac{3}{2}} \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \tag{27}$$

So, the full QR decomposition of B is

$$B = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{28}$$

Number 8.1

We will examine the number of flops from the inner loop. There are two variable calculations within the inner loop. The calculation to get r_{ij} involves (m-1) addition and m multiplication. The calculation to get v_j involves m subtraction and m multiplication. So the total operation in the inner loop per iteration is (4m-1).

For the outer loop, there are two variable calculations. The calculation to get the norm of v_i involves m multiplication and (m-1) addition. The calculation to get q_i involves m division. Thus, the total flop for outer loop per iteration is (3m-1).

Thus, summing the total operation over loop, we get the total operation is

$$\begin{aligned}\sum_{i=1}^{n}(3m-1) + \sum_{i=1}^{n}\sum_{j=i+1}^{n}(4m-1) &= n(3m-1) + (4m-1)\sum_{i=1}^{n} i \qquad (29)\\ &= n(3m-1) + (4m-1)\frac{n(n+1)}{2} \qquad (30)\end{aligned}$$

Number 11.3 (skip b and c)

Normal	QR	Backslash	SVD
9.99999999890956e-01	1.00000000099661e+00	1.00000000099661e+00	1.00000000099661e+00
-5.01789344641309e-07	-4.22743235664312e-07	-4.22743295861238e-07	-4.22743239310182e-07
-7.99997193687411e+00	-7.99998123567914e+00	-7.99998123567742e+00	-7.99998123567947e+00
-5.25906565965866e-04	-3.18763315735749e-04	-3.18763333746966e-04	-3.18763307380820e-04
1.06714520076276e+01	1.06694307964672e+01	1.06694307965635e+01	1.06694307963850e+01
-2.45841124422283e-02	-1.38202903078750e-02	-1.38202906047977e-02	-1.38202898770879e-02
-5.61259231454890e+00	-5.64707562132252e+00	-5.64707562077651e+00	-5.64707562267255e+00
-1.44734116074438e-01	-7.53160345859420e-02	-7.53160351574602e-02	-7.53160319249800e-02
1.78208809297431e+00	1.69360697489656e+00	1.69360697515560e+00	1.69360697156720e+00
-6.32859834274373e-02	6.03210077138836e-03	6.03210084144789e-03	6.03210333889887e-03
-3.43783281833645e-01	-3.74241700253419e-01	-3.74241700377927e-01	-3.74241701366866e-01
8.22944295621049e-02	8.80405755143123e-02	8.80405755522258e-02	8.80405757221269e-02

The QR, backslash and SVD methods are the same up to about the seventh or eighth digit before they start to differ. However, normal method, there are some values which are different at the first digit compared to the other three methods. So, it is likely that the normal method has some instabilities with the matrix A that we forms.

Least Square Problem

a

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} r_1 + c \\ r_2 + c \\ r_3 + c \\ r_4 + c \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 6 \\ 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 6 \\ 3 \\ 7 \end{bmatrix}$$

$$c \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 6 \\ 3 \\ 7 \end{bmatrix}$$

So, adding constants to the r_i terms does not change the original least squares problem.

b

The reason that the solution to the least squares problem of the 6 equations solve exactly fit the last equation because adding the last equation will make the solution of the problem unique. If there are only 5 equations the matrix of coefficient will not have full rank and hence the result will not be unique.

c

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 6 \\ 3 \\ 7 \\ 20 \end{bmatrix}$$

The result is

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 5.25 \\ 4.625 \\ 9.125 \\ 1 \end{bmatrix}$$

So, the team order from first rank to last is T3, T1,T2 and T4.

```
format long e
% 6.4
A = [1,0;0,1;1,0];
ans_6a_1 = A*(A'*A)^(-1)*A';
x = [1,2,3]';
ans_6a_2 = ans_6a_1*x;
B = [1,2;0,1;1,0];
ans_6b_1 = B*(B'*B)^(-1)*B';
ans_6b_2 = ans_6b_1*x;

%%%%% 11.3
t = linspace(0,1,50);
Atil = fliplr(vander(t));
A = Atil(:,1:12);
b = cos(4*t');

% Part a - least square via normal eq
R = chol(A'*A); % cholesky factorization
w_a = R'\(A'*b); % solve lower triangular sys R*w = A*b
x_a = R\w_a; % solve upper triangular sys Rx = w
% part d - least square via qr
[q,r] = qr(A,'econ');
x_d = r\(q'*b);
% part e - least square via \
x_e = A\b;
% part f - least squart via svd
[u,sig,v] = qr(A,'econ');
w_f = sig\(u'*b); % solve diagonal system sw= u*b
x_f = v*w_f;

%%%% Ranking question
A_rank_6 = [1,-1,0,0;-1,0,1,0;1,0,0,-1;0,0,1,-1;0,1,0,-1;1,1,1,1];
b_rank_6 = [4,9,6,3,7,20]';
x_rank_6 = A_rank_6\b_rank_6;

A_rank_5 = [1,-1,0,0;-1,0,1,0;1,0,0,-1;0,0,1,-1;0,1,0,-1];
b_rank_5 = [4,9,6,3,7]';
x_rank_5 = A_rank_5\b_rank_5;

A_rank_61 = [1,-1,0,0;-1,0,1,0;1,0,0,-1;0,0,1,-1;0,1,0,-1;1,1,1,1];
b_rank_61 = [4,9,6,3,7,30]';
x_rank_61 = A_rank_61\b_rank_61;
```

```
Warning: Rank deficient, rank = 3, tol =  1.922963e-15.
```

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