**AMATH 515 Homework #0 - Vinsensius**
**Due: Wednesday, January 11th, by 11 pm**

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# Number 1

**Calculus primer**. For a function $f : \mathbb{R}^n \to \mathbb{R}$, we define the *gradient* to be the vector of partial derivatives:

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

and the *Hessian* to be the matrix of second partial derivatives:

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \dots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

Compute the gradients and hessians of the following functions, with $x \in \mathbb{R}^4$ in all three examples.

1. $f(x) = \sin(x_1 + x_2 + x_3 + x_4)$

2. $f(x) = \|x\|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$

3. $f(x) = \ln(x_1 x_2 x_3 x_4)$.

1.

$$\nabla f(x) = \begin{bmatrix} \cos(x_1 + x_2 + x_3 + x_4) \\ \cos(x_1 + x_2 + x_3 + x_4) \\ \cos(x_1 + x_2 + x_3 + x_4) \\ \cos(x_1 + x_2 + x_3 + x_4) \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} -\sin(x_1+x_2+x_3+x_4) & -\sin(x_1+x_2+x_3+x_4) & -\sin(x_1+x_2+x_3+x_4) & -\sin(x_1+x_2+x_3+x_4) \\ -\sin(x_1+x_2+x_3+x_4) & -\sin(x_1+x_2+x_3+x_4) & -\sin(x_1+x_2+x_3+x_4) & -\sin(x_1+x_2+x_3+x_4) \\ -\sin(x_1+x_2+x_3+x_4) & -\sin(x_1+x_2+x_3+x_4) & -\sin(x_1+x_2+x_3+x_4) & -\sin(x_1+x_2+x_3+x_4) \\ -\sin(x_1+x_2+x_3+x_4) & -\sin(x_1+x_2+x_3+x_4) & -\sin(x_1+x_2+x_3+x_4) & -\sin(x_1+x_2+x_3+x_4) \end{bmatrix}$$

2.

$$\nabla f(x) = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \\ 2x_4 \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

3.

$$\nabla f(x) = \begin{bmatrix} \frac{1}{x_1} \\ \frac{1}{x_2} \\ \frac{1}{x_3} \\ \frac{1}{x_4} \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} -\frac{1}{x_1^2} & 0 & 0 & 0 \\ 0 & -\frac{1}{x_2^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{x_3^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{x_4^2} \end{bmatrix}$$

# Number 2

**Linear algebra primer.**

1. What are the eigenvalues of the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \pi & 2 & 0 & 0 \\ 64 & -15 & 3 & 0 \\ 321 & 0 & 0 & 5 \end{bmatrix}$$

2. Write down bases for the range and nullspace of the following matrix, written as the outer product of two vectors:

$$A = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

3. Let $A$ be a $10 \times 5$ matrix, and $b$ a vector in $\mathbb{R}^{10}$. The notation $A^T$ denotes the *transpose* of $A$, where the columns of $A$ are rows of $A^T$.

   - What is the size of $A^T A$? What is the size of $A^T b$?
   - How many solutions might there be to the system $Ax = b$?
   - How many solutions might there be to the system $A^T Ax = A^T b$?
   - Suppose the columns of $A$ are linearly independent. How many solutions might there be to the system $Ax = b$? To the system $A^T Ax = A^T b$?

1. Since the matrix is lower diagonal, the eigenvalues are 1, 2,3, and 5.
2. After multiplication, A is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Since there is only 1 independent column, the range of A is

$$\text{Range}(A) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

So, the rank of A is 1, we expect the nullity of A to be 2 by rank-nullity theorem. So, we expect the null space of A to have 2 vectors.

To find the null space, we find vector v such that Av = 0.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the equation above, we found that $v_1 = -v_2 - v_3$. So, the null space of A is

$$\text{Null}(A) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

3.
- Size of $A^T A$ is 5x5, while size $A^T b$ is 5x1
- It will depend on rank of A. If A has full rank, the solution will be unique. If rank of A is less than 5, there will be infinite solutions because there will be free variable. If there are some inconsistency in the system of equation, there will be no solution to the problem Ax = b.
- If $A^T A$ is invertible, the solution to the problem will be unique, and there will be no solution.
- If A has linearly independent columns, there will be a unique solution to Ax=b and $A^T A$x = $A^T b$ problems.