Key ♡

For each of exercises 1–10, evaluate the infinite geometric series or show that it diverges.

1. $\sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2}\right)}_{r}^{n} = \frac{\text{first term}}{1-r} = \frac{(1/2)^0}{1-1/2} = \frac{1}{1/2} = \boxed{2}$

2. $\sum_{n=1}^{\infty} \underbrace{\left(\frac{2}{3}\right)}_{r}^{n} = \frac{(2/3)^1}{1-2/3} = \frac{2/3}{1/3} = \boxed{2}$

3. $\sum_{n=0}^{\infty} \frac{3}{5^n} = 3\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n = 3\left(\frac{1}{1-1/5}\right) = \frac{3}{4/5} = \boxed{\frac{15}{4}}$

4. $\sum_{n=0}^{\infty} \frac{5^n}{e^{n+1}} = \sum_{n=0}^{\infty}\frac{5^n}{e^n \cdot e} = \frac{1}{e}\sum_{n=0}^{\infty}\left(\frac{5}{e}\right)^n$

$r = \frac{5}{e} > 1 \Rightarrow \boxed{\text{sum diverges}}$

5. $3 - \frac{3}{5} + \frac{3}{25} - \frac{3}{125} + \cdots$

$= 3\left(1 - \frac{1}{5} + \frac{1}{25} - \frac{1}{125} + \cdots\right) = 3\sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^n = \frac{3}{1+1/5} = \boxed{\frac{5}{2}}$

6. $2\pi + 2\pi^2 + 2\pi^3 + 2\pi^4 + \cdots$

$= 2\pi\left(1 + \pi + \pi^2 + \pi^3 + \cdots\right) = 2\pi\sum_{n=0}^{\infty}\underbrace{\pi}_{r}^{n}$

$r > 1 \Rightarrow \boxed{\text{sum diverges}}$

7. $\sum_{n=1}^{\infty} \frac{e^{n+3}}{\pi^n} = \sum_{n=1}^{\infty}\frac{e^n \cdot e^3}{\pi^n} =$

$e^3\sum_{n=1}^{\infty}\left(\frac{e}{\pi}\right)^n = e^3\left(\frac{e/\pi}{1-\frac{e}{\pi}}\right) = \boxed{e^4\left(\frac{1}{\pi - e}\right)}$

8. $\sum_{n=0}^{\infty} \frac{2^{3n-1}}{7^n} = \sum_{n=0}^{\infty}\frac{2^{3n}\cdot 2^{-1}}{7^n} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{8^n}{7^n} = \frac{1}{2}\sum_{n=0}^{\infty}\underbrace{\left(\frac{8}{7}\right)}_{r}^{n}$

$r > 1 \Rightarrow \boxed{\text{sum diverges}}$

9. $\sum_{n=0}^{\infty} \frac{5^{2n}}{e^{n+1}} = \frac{1}{e}\sum_{n=0}^{\infty}\frac{25^n}{e^n}$ $\quad r = \frac{25}{e} > 1$

$\Rightarrow$ sum diverges

10. $\frac{2}{3} - \frac{2e}{9} + \frac{2e^2}{27} - \frac{2e^3}{81} + \cdots$

$= \frac{2}{3}\left(1 - \frac{e}{3} + \frac{e^2}{9} - \frac{e^3}{27} + \cdots\right) = \frac{2}{3}\sum_{n=0}^{\infty}\left(-\frac{e}{3}\right)^n = \frac{2/3}{1+e/3} = \boxed{\frac{2}{3+e}}$

For each of exercises 11–16, determine the values k for which the geometric series converges.

$|r| < 1$

11. $\sum_{n=1}^{\infty} \left(\frac{2k}{3}\right)^n$ $\quad -1 < \frac{2k}{3} < 1$

$\boxed{-\frac{3}{2} < k < \frac{3}{2}}$

12. $\sum_{n=0}^{\infty} \frac{k^n}{3^{n+2}} = \frac{1}{3^2}\sum_{n=0}^{\infty}\left(\frac{k}{3}\right)^n$

$-1 < \frac{k}{3} < 1 \Rightarrow \boxed{-3 < k < 3}$

13. $\sum_{n=1}^{\infty} \left(k^2 - 6k + 10\right)^n$

$k^2 - 6k + 10 < 1$

$k^2 - 6k + 9 < 0$

$(k-3)^2 < 0$

$|k-3| < 0$

no solutions

$\boxed{\{\ \}}$

14. $\sum_{n=0}^{\infty} \left(\frac{2}{k^2 - 8k + 16}\right)^n$ $\quad \frac{2}{k^2-8k+16} < 1 \Rightarrow \frac{2}{(k-4)^2} < 1$

$(k-4)^2 > 2 \Rightarrow |k-4| > \sqrt{2}$

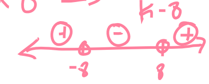
$k - 4 < -\sqrt{2},\ k - 4 > \sqrt{2}$

$\boxed{k < 4 - \sqrt{2},\ k > 4 + \sqrt{2}}$

15. $\sum_{n=0}^{\infty} \left(\frac{2k}{k-8}\right)^n$

$\frac{2k}{k-8} < 1 \Rightarrow \frac{2k - (k-8)}{k-8} < 0 \Rightarrow \frac{k+8}{k-8} < 0$

$\frac{2k}{k-8} < 1$



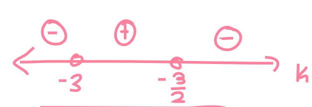
$\boxed{-8 < k < 8}$

16. $\sum_{n=10}^{\infty} \left(\frac{k}{2k+3}\right)^n$

$\frac{k}{2k+3} < 1$

$\frac{k - (2k+3)}{2k+3} < 0$

$\frac{-k-3}{2k+3} < 0$



$\boxed{k < -3,\ k > -\frac{3}{2}}$

For each of exercises 17–22, express the repeating decimal as an infinite geometric series.

17. $0.387\overline{2}$

$0.387 + 0.0002 + 0.00002 + 0.000002 + \cdots$

$= 0.387 + 0.0002(1 + 0.1 + 0.01 + \cdots)$

$= 0.387 + 0.0002\sum_{n=0}^{\infty}(0.1)^n$

18. $0.\overline{4}$

$0.4 + 0.04 + 0.004 + 0.0004 + \cdots$

$= 0.4(1 + 0.1 + 0.01 + 0.001 + \cdots)$

$= 0.4\sum_{n=0}^{\infty}(0.1)^n$

19. $5.3\overline{2}$

$5.3 + 0.02 + 0.002 + 0.0002 + \cdots$

$= 5.3 + 0.02(1 + 0.1 + 0.01 + \cdots)$

$= 5.3 + 0.02\sum_{n=0}^{\infty}(0.1)^n$

20. $6.23\overline{1}$

$6.23 + 0.001 + 0.0001 + 0.00001 + \cdots$

$= 6.23 + 0.001(1 + 0.1 + 0.01 + 0.001 + \cdots)$

$= 6.23 + 0.001\sum_{n=0}^{\infty}(0.1)^n$

21. $2.8\overline{5}$

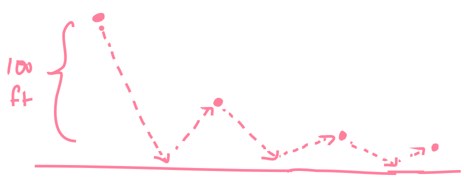
$2.8 + 0.05 + 0.005 + 0.0005 + \cdots$

$= 2.8 + 0.05(1 + 0.1 + 0.01 + 0.001 + \cdots)$

$= 2.8 + 0.05\sum_{n=0}^{\infty}(0.1)^n$

22. $4.6\overline{7}$

23. A ball is dropped and bounces on the floor. Each time the ball hits the floor, it rebounds to a height one-third of that from which it fell. A ball is dropped from a height of 100 feet and follows this model. What is the total vertical distance traveled by the ball?



distance: $100 + \frac{100}{3} + \frac{100}{9} + \frac{100}{27} + \cdots$

$= 100\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$

$= \frac{100}{1-\frac{1}{3}} = \frac{100}{2/3} = \boxed{150 \text{ ft.}}$

24. In economics, individuals' marginal propensity to consume (MPC) is a number, from 0 to 1, that measures the proportion of their income, on average, that they spend. For example, a country with a MPC of 0.7 indicates that the population, on average, spends 70 percent of their income and saves 30 percent. The total economic activity produced by a purchase of 1000 dollars would be represented by the geometric series $1000 + 1000(0.7) + 1000(0.7)^2 + 1000(0.7)^3 + \cdots$, as money will change hands indefinitely under this model. Fiscal policy utilizes this number to close recessionary gaps. Suppose that the United States is in a recessionary gap of 400 billion dollars and its citizens have a MPC of 0.5. How much money must the government spend to create 400 billion dollars of economic activity, thus closing the recessionary gap?

let x = \$ spent by gov.

400 billion $= x + MPC\,x + MPC^2 x + MPC^3 x + \cdots$

$= x\sum_{n=0}^{\infty} MPC^n = \frac{x}{1-MPC} = \frac{x}{0.5}$

$\boxed{\therefore\ x = \$200 \text{ billion}}$