10.1 — Overview of Infinite Sequences and Series

In this section, we will introduce infinite series and sequences, and we will define convergence and divergence.

A sequence is a list of numbers, while a series is a sum of numbers. An example of a sequence is

$$\{a_1, a_2, a_3, a_4, \dots a_k\},$$

which we can model as $\{a_n\}_1^k$ to represent a sequence a_n starting at $n = 1$ and ending at $n = k$. In calculus, we will focus on infinite sequences—sequences that never terminate. A series, however, is the sum of a sequence; i.e., consider

$$a_1 + a_2 + a_3 + a_4 + \cdots a_k,$$

which can be written using summation notation as $\sum_{n=1}^{k} a_n$. It is important to note that we may pick our starting index to be any integer greater than or equal to 0 (sequences and series do not have to just start at $n = 1$.)

Example 1

Find the general term a_n that models the sequence $\{2, 5, 8, 11, \dots\}$.

For this problem, we notice that the first term is 2, the second term is 5, and the third term is 8. We notice that each incremental term increases by 3, so, starting at $n = 1$, the general form is given by $a_n = \ 2 + 3(n-1)$. Alternatively, if we chose to start at $n = 0$, then $a_n = \ 2 + 3n$.

Let us now define the concept of convergence. If the terms of a sequence $\{a_n\}$ approach a certain finite value L; i.e., if $\lim_{n\to\infty} a_n = L$, where L is finite, then $\{a_n\}$ converges to L. However, if the limit does not exist, then $\{a_n\}$

is divergent (it does not converge to a finite value.)

For example, the sequence in Example 1, $a_n = 2 + 3(n-1)$, diverges because $\lim\limits_{n\to\infty} a_n$ approaches infinity. This result is intuitive because we can see that, as we write out more terms, the terms will continuously increase, approaching infinity. Consider, in contrast, a sequence such as $a_n = \dfrac{1}{n}$. The terms of this sequence (starting at $n = 1$), are given by $\left\{1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots\right\}$. It is clear that the terms in this sequence approach 0, a finite value. Thus, this sequence converges to 0.

CONVERGENCE OF A SEQUENCE $\{a_n\}$

I. $\{a_n\}$ converges to L if $\lim\limits_{n\to\infty} a_n = L$, where L is a finite value.

II. $\{a_n\}$ diverges if $\lim\limits_{n\to\infty} a_n$ does not exist.

Consider Fig. 10-1.1, in which a convergent sequence $\{a_n\}$ is shown. We see that the sequence converges to a value L, as the terms of $\{a_n\}$ tend to the value L. We can verify that $\lim\limits_{n\to\infty} a_n = L$.

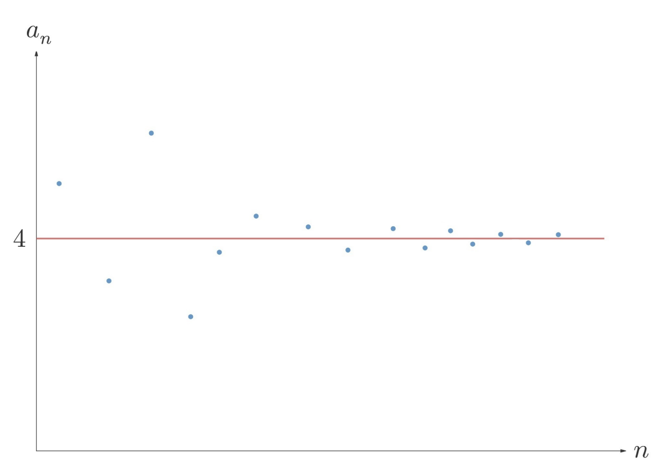

Fig. 10-1.1.

Now consider Fig. 10-1.2, in which a divergent sequence $\{a_n\}$ is shown. We see that the terms of $\{a_n\}$ will grow to infinity and is therefore divergent. We may verify this result because $\lim_{n\to\infty} a_n = \infty$ (does not exist).

Fig. 10-1.2.

The last scenario of divergence is shown in Fig. 10-1.3, in which a sequence $\{a_n\}$ is shown. We see that the terms of $\{a_n\}$ oscillate between the values $(L-k)$ and $(L+k)$. Specifically, $\lim_{n\to\infty} a_n$ does not exist, so the sequence diverges. These oscillating sequences are typically in the form $a_n = (-1)^n b_n$, where b_n is non-oscillating.

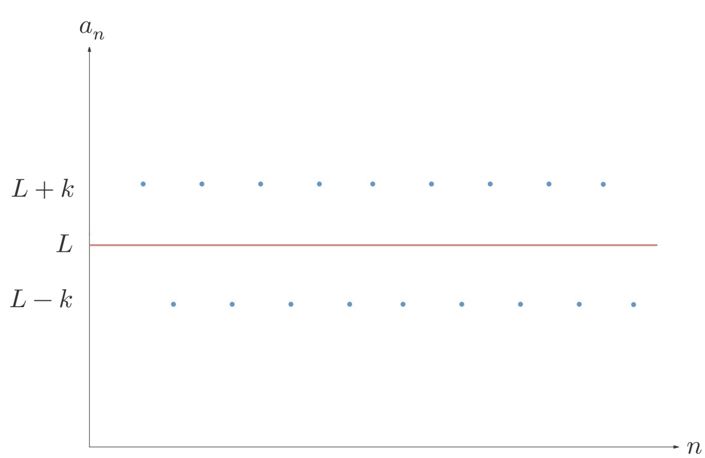

Fig. 10-1.3.

Example 2

Determine which of the following sequences converge. If a sequence converges, state the value to which it converges.

I. $a_n = \dfrac{2n+3}{n^2+8}$

II. $\{a_n\} = \{-1, 1, -1, 1, -1, 1 \ldots\}$

III. $a_n = \dfrac{6n^3}{n+3}$

IV. $a_n = \dfrac{3n^2-4}{7n^2+1}$

I. We notice that $\lim_{n \to \infty} \dfrac{2n+3}{n^2+8} = 0$, so $\{a_n\}$ converges to 0.

II. This sequence oscillates between -1 and 1, and it will never converge to a final value. Therefore, $\{a_n\}$ diverges.

III. $\lim_{n \to \infty} = \dfrac{6n^3}{n+3}$ does not exist (it approaches infinity.) Therefore, the sequence diverges.

IV. $\lim_{n \to \infty} \dfrac{3n^2-4}{7n^2+1} = \dfrac{3}{7}$, so $\{a_n\}$ converges to $\dfrac{3}{7}$.

Now let us discuss infinite series. Infinite series are in the form

$$\sum_{n=k}^{\infty} a_n = a_k + a_{k+1} + a_{k+2} + a_{k+3} \cdots,$$

and, in calculus, we analyze these series to determine whether they converge or diverge. At first, this idea seems paradoxical—it would be logical to assume that the sum would approach infinity as more terms are added.

However, if each term decreases (i.e., $a_{n+1} < a_n$), then there is a possibility of the sum converging. Consider, for example, a sum such as

$$1 + 2 + 3 + 4 + \cdots.$$

It is clear that, as more terms are added, the sum will approach infinity. Conversely, consider a sum such as

$$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots,$$

whose terms are decreasing. This sum happens to be convergent, and its exact value is $\frac{4}{3}$! We will analyze these forms of series in section 10.2—infinite geometric series. From these points, we introduce the *n*th term test (also known as the divergence test), which permits us to determine immediately whether a sum diverges.

CONVERGENCE OF A SERIES $\sum a_n$

$$\text{If } \lim_{n\to\infty} a_n \neq 0, \text{ then } \sum a_n \text{ diverges.}$$

This test states that if the terms of $\{a_n\}$ are not shrinking in absolute value to 0, then $\sum a_n$ diverges without question. However, it is important to know that this test CANNOT establish convergence; i.e., if $\lim_{n\to\infty} a_n = 0$, then the test is inconclusive—we must therefore use other methods to determine whether $\sum a_n$ converges or diverges.

Consider, as an example, the series $\sum_{n=1}^{\infty} \frac{1}{n}$, which is called the harmonic series. Writing some of its terms, we get

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots,$$

and this series happens to diverge! This result is non-intuitive, and it demonstrates a case in which $\lim_{n\to\infty} a_n = 0$ but $\sum a_n$ diverges.

Example 3

Using the divergence test, determine which of the following series diverge.

I. $\sum_{n=0}^{\infty} \sin(n)$

II. $\sum_{n=4}^{\infty} \frac{1}{n-3}$

III. $\sum_{n=1}^{\infty} e^{n+2}$

IV. $\sum_{n=3}^{\infty} \frac{2n^2+10}{3n^2-2}$

V. $\sum_{n=3}^{\infty} \frac{n!}{n^{999}}$

I. Because $\lim_{n\to\infty} \sin(n)$ does not exist (sine continuously oscillates and will never approach a final value), the divergence test states that $\sum_{n=0}^{\infty} \sin(n)$ diverges.

II. $\lim_{n\to\infty} \frac{1}{n-3} = 0$, so the divergence test is inconclusive.

III. $\lim_{n\to\infty} e^{n+2}$ does not exist (it approaches infinity), so, by the divergence test, $\sum_{n=1}^{\infty} e^{n+2}$ diverges.

IV. $\lim_{n\to\infty} \frac{2n^2+10}{3n^2-2} = \frac{2}{3} \neq 0$, so $\sum_{n=3}^{\infty} \frac{2n^2+10}{3n^2-2}$ diverges.

V. $\lim_{n\to\infty} \frac{n!}{n^{999}}$ approaches infinity (factorials always grow faster than exponential functions), so $\sum_{n=3}^{\infty} \frac{n!}{n^{999}}$ diverges.

We now have an idea of what we will work with in this chapter. The first half of this chapter will focus on methods for determining whether a series converges or diverges. In the second half, we will apply those methods to power series—infinite polynomials that can represent various functions such as $\sin(x)$, $\cos(x)$, and e^x.

SECTION SUMMARY

- A sequence is a list of numbers, an example of which is $\{a_n\} = \{a_1, a_2, a_3, a_4, \ldots\}$.
- A series is a list of numbers, an example of which is $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$.
- A sequence $\{a_n\}$ converges to the value L if $\lim_{n\to\infty} a_n = L$.
- A sequence $\{a_n\}$ diverges if $\lim_{n\to\infty} a_n$ does not exist.
- The divergence test: A series $\sum a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$. If $\lim_{n\to\infty} a_n = 0$, then the test is indeterminate (may or may not converge).