For each of exercises 1–12, evaluate the integral or show that it diverges.

1. $\displaystyle\int_2^\infty \frac{1}{x+9}\,\mathrm{d}x$ $= \lim_{k\to\infty}\int_2^k \frac{1}{x+9}\,dx$

$= \ln(x+9)\Big]_2^\infty$ $\boxed{\text{diverges}}$!

$\lim_{k\to\infty} \ln(k+9) - \ln(11) \longrightarrow \infty.$

2. $\displaystyle\int_1^\infty \frac{3}{2\sqrt{x}}\,\mathrm{d}x$

$\frac{3}{2}\int_1^\infty x^{-\frac12}dx = \frac{3}{2}\lim_{k\to\infty}\int_1^k x^{-\frac12}dx = 3\lim_{k\to\infty}\sqrt{x}\Big]_1^k$

$= 3\lim_{k\to\infty}\sqrt{k} - 3 \longrightarrow \infty$ ∴ $\boxed{\text{diverges}}$

3. $\displaystyle\int_3^\infty \frac{1}{x\sqrt[4]{x}}\,\mathrm{d}x$

↓ $x^{5/4}$

$\lim_{k\to\infty}\int_3^k x^{-5/4}dx = -4\lim_{k\to\infty} x^{-\frac14}\Big]_3^k = \cancel{-4\lim_{k\to\infty}\frac{1}{\sqrt[4]{k}}}^{\,0} + \frac{4}{\sqrt[4]{3}} = \boxed{\frac{4}{\sqrt[4]{3}}}$

4. $\displaystyle\int_2^5 \frac{1}{x-4}\,\mathrm{d}x$ VA @ $x=4$

$= \underbrace{\lim_{k\to4^-}\int_2^k \frac{1}{x-4}dx} + \lim_{k\to4^+}\int_k^5 \frac{1}{x-4}dx$

$\lim_{k\to4^-} \ln|x-4|\Big]_2^4 = \lim_{k\to4^-} \ln(k-4) - \ln 2$ does not exist ∴ $\boxed{\text{integral diverges}}$

5. $\displaystyle\int_{-\infty}^{\pi/2} \cos(x)\,\mathrm{d}x$

$= \lim_{k\to-\infty}\int_k^{\pi/2}\cos x\,dx = \sin\frac{\pi}{2} - \underbrace{\lim_{k\to-\infty}\sin k}_{\text{does not exist}}$ ∴ $\boxed{\text{integral diverges}}$

6. $\displaystyle\int_0^\infty \frac{1}{(x+3)^2}\,\mathrm{d}x$

$= \lim_{k\to\infty}\int_0^k \frac{1}{(x+3)^2}dx$

$= \lim_{k\to\infty}\frac{-1}{x+3}\Big]_0^k$

$= \cancel{\lim_{k\to\infty}\left(\frac{-1}{k+3}\right)}^{\,0} - \left(-\frac{1}{3}\right) = \boxed{\frac{1}{3}}$

7. $\displaystyle\int_{-2}^4 \frac{1}{(x-1)^4}\,\mathrm{d}x$ VA @ $x=1$

$= \underbrace{\lim_{k\to1^-}\int_{-2}^k \frac{1}{(x-1)^4}dx} + \lim_{k\to1^+}\int_k^4 \frac{1}{(x-1)^4}dx$

$\lim_{k\to1^-}\frac{-1}{(x-1)^3}\Big]_{-2}^k = \underbrace{\lim_{k\to1^-}\frac{-1}{(k-1)^3}}_{\text{does not exist}} + \frac{1}{(-3)^3}$ ∴ $\boxed{\text{integral diverges}}$

8. $\displaystyle\int_0^{\pi/2} \sec(\theta)\tan(\theta)\,\mathrm{d}\theta$ VA @ $\theta = \pi/2$

$= \lim_{k\to\frac{\pi}{2}^-}\int_0^k \frac{\sin\theta}{\cos^2\theta}d\theta = \lim_{k\to\frac{\pi}{2}}\sec\theta\Big]_0^k = \underbrace{\lim_{k\to\frac{\pi}{2}}\sec k}_{\text{does not exist}} - 1$ ∴ $\boxed{\text{integral diverges}}$

9. $\displaystyle\int_{\pi/3}^\infty 2(\sin(\theta)+2)\cos(\theta)\,\mathrm{d}\theta$

let $u = \sin\theta + 2 \implies du = \cos\theta\,d\theta$

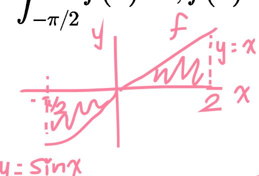
$\lim_{k\to\infty}(\sin\theta+2)^2\Big]_{\pi/3}^k = \underbrace{\lim_{k\to\infty}(\sin k+2)^2}_{\text{does not exist}} - \left(\frac{\sqrt{3}}{2}+2\right)^2$ ∴ $\boxed{\text{integral diverges}}$

10. $\displaystyle\int_4^\infty \frac{3}{x\ln(x)}\,\mathrm{d}x$

let $u = \ln x \implies du = \frac{1}{x}dx$

$\int_{x=4}^{x=\infty}\frac{3}{u}du = \ln(\ln x)\Big]_4^\infty = \underbrace{\lim_{k\to\infty}\ln(\ln k)}_{\infty} - \ln(\ln 4)$ ∴ $\boxed{\text{integral diverges}}$

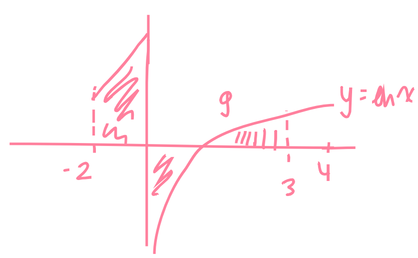
11. $\displaystyle\int_{-\pi/2}^2 f(x)\,\mathrm{d}x,\ f(x) = \begin{cases}\sin(x) & \text{for } x \le 0\\ x & \text{for } x > 0\end{cases}$



$\int_{-\pi/2}^0 \sin x\,dx + \int_0^2 x\,dx$

$(0-(-1)) + (2-0) = \boxed{3}$

12. $\displaystyle\int_{-2}^3 g(x)\,\mathrm{d}x,\ g(x) = \begin{cases}x+5 & \text{for } -2 \le x \le 0\\ \ln(x) & \text{for } 0 < x \le 4\end{cases}$



$\lim_{k\to0} k\ln k =$ l'hopital's rule

$\underbrace{\lim_{k\to0}\frac{\ln k}{1/k}}_{\infty/\infty} = \lim_{k\to0}\frac{1/k}{-1/k^2} = \lim_{k\to0} k = 0$

$\underbrace{\int_{-2}^0 (x+5)\,dx}_{8} + \int_0^3 \ln x\,dx = 8 + \lim_{k\to0^+}\Big[x\ln x - x\Big]_k^3 =$

$8 + (3\ln 3 - 1) - \cancel{\lim_{k\to0^+} k(\ln k - 1)}^{\,0}$

$\boxed{5 + 3\ln 3}$

For exercises 13–21, determine whether the following integrals are proper or improper. If an integral is improper, classify the type (i.e., Type I or Type II).

13. $\displaystyle\int_1^3 \frac{2}{x-3}\,\mathrm{d}x$

VA @ $x = 3$

improper — type II

14. $\displaystyle\int_0^e \ln(x+1)\,\mathrm{d}x$

$\ln(x+1)$ defined & continuous for all $x > -1$

proper

15. $\displaystyle\int_{-3}^{-2} \frac{1}{x^3+1}\,\mathrm{d}x$

VA at $x = -1$

improper — type II

16. $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$

infinite bounds

improper — type I

17. $\displaystyle\int_2^4 \frac{3}{\underbrace{x+1}}\,\mathrm{d}x$

cont. for all $x \neq -1$

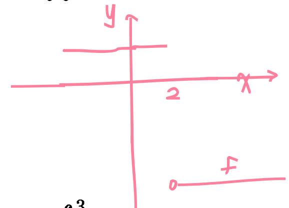
proper

18. $\displaystyle\int_0^{1/2} \frac{-2}{\underbrace{x^9-1}}\,\mathrm{d}x$

cont. for all $x \neq 1$

proper

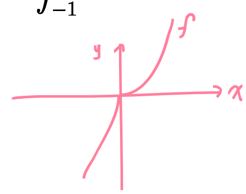
19. $\displaystyle\int_0^8 f(x)\,\mathrm{d}x, f(x) = \begin{cases} 3 & \text{for } 0 \leq x \leq 2 \\ -7 & \text{for } x > 2 \end{cases}$



discontinuity at $x = 2$;

improper — type II

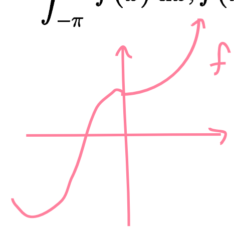
20. $\displaystyle\int_{-1}^3 f(x)\,\mathrm{d}x, f(x) = \begin{cases} 2x & \text{for } x \leq 0 \\ x^2 & \text{for } x > 0 \end{cases}$



f cont. at $x = 0$;

proper

21. $\displaystyle\int_{-\pi}^{\infty} f(x)\,\mathrm{d}x, f(x) = \begin{cases} \cos(x)+1 & \text{for } x \leq 0 \\ x^2+1 & \text{for } x > 0 \end{cases}$



improper — type I

22. Find all the values of n for which $\displaystyle\int_2^{\infty} \frac{3}{x^{1-3n}}\,\mathrm{d}x$ converges.

p-integral: $\displaystyle\int_a^{\infty} \frac{1}{x^p}\,dx$ $\quad = 3\displaystyle\int_2^{\infty} \frac{1}{x^{1-3n}}\,dx$ $\quad \underbrace{1-3n}_{p} > 1$

if $p \leq 1$ then diverges; if $p > 1$ then converges

$1-3n > 1$

$3n < 0 \Rightarrow \boxed{n < 0}$

23. Find all the values of n for which $\displaystyle\int_1^{\infty} \frac{1}{x^{5n-1}}\,\mathrm{d}x$ converges.

$5n-1 > 1$

$\boxed{n > 2/5}$

24. Find all the values of n for which $\int_1^{\infty} \frac{1}{x^{n^2-4n+4}}\,dx$ diverges.

$n^2-4n+4<1$ $\quad$ $1<n<3$ $\quad$ OR $\quad$ ~~$n<1$ and $n>3$~~

$n^2-4n+3<0$ $\quad$ let $n=0$: $0^2-4(0)+4<1$

$(n-3)(n-1)<0$ $\quad$ X not true

25. It is known that $0<f(x)\le \frac{1}{x^{4/3}}$, where f is continuous for all x. Show that $\int_1^{\infty} f(x)\,dx$ converges and is no greater than 3.

$0<\underbrace{\int_1^{\infty} f(x)dx}_{\text{converges too}} < \underbrace{\int_1^{\infty}\frac{1}{x^{4/3}}dx}_{\text{Converges }(p>1)}$

$\int_1^{\infty} x^{-4/3}dx = \lim_{k\to\infty}\int_1^k x^{-4/3}dx = -3\lim_{k\to\infty} x^{-1/3}\Big]_1^k$

$= -3\left[\cancel{\lim_{k\to\infty}\frac{1}{\sqrt[3]{k}}}^{\,0} - \frac{1}{\sqrt[3]{1}}\right]$

$=\boxed{3}$

26. It is known that $0<\frac{1}{x^{5/9}}\le g(x)$, where g is continuous for all x. Show that $\int_1^{\infty} g(x)\,dx$ is divergent. $\quad 0<\int_1^{\infty} f(x)dx<3$

$0<\underbrace{\int_1^{\infty}\frac{1}{x^{5/9}}dx}_{\text{diverges }(p<1)} < \underbrace{\int_1^{\infty} g(x)dx}_{\text{diverges too}}$ ("larger than infinity")

27. For a continuous function $g(x)$, it is known that $g(0)=0$ and $g(4)=12$. Evaluate $\int_0^4 \frac{g'(x)}{g(x)}\,dx$ or show that it diverges.

$\int_0^4 \frac{g'(x)dx}{g(x)} = \int_0^{12}\frac{1}{u}du$ $\quad$ let $u=g(x)$ $\quad$ $u_{\text{lower}}=g(0)=0$

$du=g'(x)dx$ $\quad$ $u_{\text{upper}}=g(4)=12$

$=\lim_{k\to 0^+}\int_k^{12}\frac{1}{u}du = \underbrace{\lim_{k\to 0^+}\ln k}_{-\infty} - \ln 12$ $\quad$ $\boxed{\therefore \int_0^4 \frac{g'(x)}{g(x)}dx \text{ diverges}}$

28. It is known that $g(x)\le q(x)$, where g and q are continuous for all x. It is known that $\int_2^{\infty} q(x)\,dx$ is finite. Explain why $\int_2^{\infty} g(x)\,dx$ could be divergent.

Convergent

$\int_2^{\infty} g(x)dx \le \underbrace{\int_2^{\infty} q(x)dx}_{\text{Convergent}}$

this could diverge to $-\infty$.

29. Indicate whether the following statement is true: If $f(x)$ is continuous, positive, and decreasing for all $x\ge 1$, and $\lim_{x\to\infty} f(x)=0$, then $\int_1^{\infty} f(x)$ converges.

False — Consider $\underbrace{f(x)=\frac{1}{x}}$ — $\int_1^{\infty}\frac{1}{x}dx$ diverges

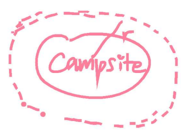
Cont., >0, and decreasing for all $x\ge 1$, and $\lim_{x\to\infty}\frac{1}{x}=0$

30. The rate at which people like an Instagram post is given by $i(t) = 200e^{-t/10}$, where t is time, measured in hours. What is the maximum number of likes that the Instagram post will receive?

$$\int_0^\infty i(t)\,dt = \int_0^\infty 200e^{-t/10}\,dt = 200\lim_{k\to\infty}\int_0^k e^{-t/10}\,dt =$$

$$-2000\lim_{k\to\infty}\left[e^{-t/10}\right]_0^k = -2000\left[\cancel{\lim_{k\to\infty}\frac{1}{e^{k/10}}}^{\,0} - \frac{1}{1}\right] = \boxed{2000}$$

31. The density of trees in a forest, T, in terms of the distance, r, surrounding a campsite is given by $T(r) = \dfrac{290}{r^3 + 8r^2 + 16r}$, where r is measured in meters and $T(r)$ is measured in trees per square meter. The number of trees within r meters of the campsite is given by $2\pi\displaystyle\int_0^r uT(u)\,du$. Explain why the total number of trees in the forest is less than 500.



$$2\pi\int_0^\infty rT(r)\,dr = 2\pi\int_0^\infty \frac{290}{r^2+8r+16}\,dr$$

$$2\pi\lim_{k\to\infty}\int_0^k \frac{290}{(r+4)^2}\,dr = 580\pi\lim_{k\to\infty}\left(\frac{-1}{r+4}\right)\Big]_0^k = 580\pi\left(\cancel{\lim_{k\to\infty}\frac{-1}{k+4}}^{\,0}\right) - \left(-\frac{1}{4}\right) = \frac{580\pi}{4} = 145\pi < 500$$

32. A particle continuously moves along the x-axis at a velocity given by $v(t) = \dfrac{44}{t^{2/3}}$ for $t > 1$, where $v(t)$ is measured in meters per second and t is measured in seconds. Show that the particle will travel an infinite distance.

distance traveled: $\displaystyle\int_1^t |v(u)|\,du = \int_1^t \frac{44}{u^{2/3}}\,du$

Consider $$\int_1^\infty \frac{44}{t^{2/3}}\,dt = 44\lim_{k\to\infty}\int_1^k t^{-2/3}\,dt = 132\lim_{k\to\infty}\left(t^{1/3}\right)\Big]_1^k =$$

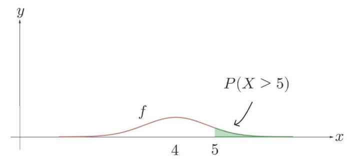
$$\underbrace{132\lim_{k\to\infty}\sqrt[3]{k}}_{\infty} - 132(1)$$

because $\displaystyle\int_1^\infty v(t)\,dt$ diverges, the particle will travel an infinite distance.

A graphing calculator is required for exercise 33.

33. In statistics, a Normal distribution has many applications—one of which is probability modeling. The integral of a probability density function (PDF) yields the probability of a random variable X falling between the bounds; that is, $\int_a^b f(x)\,\mathrm{d}x = P(a \le X \le b)$, where f is a PDF. Suppose that waiting times at the grocery store checkout lane are Normally distributed, with a mean of 4 minutes and a standard deviation of 0.8 minute. The PDF that models this scenario is given by $f(x) = \frac{1}{0.8\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-4}{0.8}\right)^2}$. What is the probability that a customer will wait in line for more than 5 minutes? (Hint: Large numbers, such as 9999, suffice for approximations of infinity.)

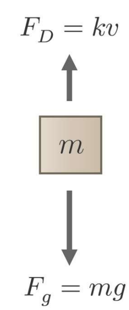
$P(X>5) = \int_5^{\infty} f(x)dx \approx \int_5^{9999} f(x)dx$



Using calculator: $\int_5^{9999} \frac{1}{0.8\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-4}{0.8}\right)^2} dx \approx 0.1056$

the customer will have approximately %10.56 percent chance of waiting longer than 5 minutes.

34. A freefalling object will accelerate until it reaches a terminal velocity—the maximum speed at which the object can fall. A drag force, which can be modeled by $F_D = kv$, where k is the object's drag coefficient and v is the object's velocity, acts on the object as it falls. The force due to gravity is given by $F_g = mg$, where m is the object's mass and g is the constant of gravitational acceleration, $9.8\mathrm{m/s^2}$ on Earth. Newton's Second Law gives the equation $mg - kv = m\frac{\mathrm{d}v}{\mathrm{d}t}$. Performing separation of variables, an equation that can be used to solve for v_T, the terminal velocity of the object, is $\int_0^{v_T} \frac{1}{g - \frac{k}{m}v}\,\mathrm{d}v = \int_0^{\infty} \mathrm{d}t$. Show that $v_T = \frac{mg}{k}$. (Hint: m, k, and g are constants.)



$\int_0^{V_T} \frac{1}{g-\frac{k}{m}V} = \int_0^{\infty} dt$

$-\frac{m}{k}\left(\ln\left|g-\frac{kV_T}{m}\right| - \ln g\right) = t\Big]_0^{\infty}$

$\ln\left|1-\frac{kV_T}{mg}\right| = \frac{-k}{m} t\Big]_0^{\infty} \Longrightarrow 1-\frac{kV_T}{mg} = e^{-\frac{kt}{m}}\Big]_0^{\infty}$

$V_T = \frac{mg}{k}\left(1-e^{-kt/m}\right)\Big]_0^{\infty} = \frac{mg}{k} - \cancel{\lim_{t\to\infty} e^{-kt/m}}^{0} = \frac{mg}{k}.$