Key ♡

For exercises 1–20, evaluate the following integrals.

1. $\int \sin(x)\cos^2(x)\,dx$

let $u = \cos x$, $du = -\sin x\,dx$

$= -\int u^2\,du$

$= -\frac{u^3}{3} + C = \boxed{-\frac{\cos^3 x}{3} + C}$

2. $\int \sin^3(x)\cos^4(x)\,dx$

$= \int \sin x \sin^2 x \cos^4 x\,dx$

$= \int \sin x (1-\cos^2 x)\cos^4 x\,dx$ let $u = \cos x$, $du = -\sin x\,dx$

$= -\int (1-u^2)u^4\,du = \frac{u^7}{7} - \frac{u^5}{5} + C = \boxed{\frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C}$

3. $\int \sin^2(x)\,dx$

$= \int \frac{1}{2}(1-\cos 2x)\,dx$

$= \boxed{\frac{1}{2}x - \frac{1}{4}\sin 2x + C}$

4. $\int \cos^3(x)\sin^4(x)\,dx$

$= \int \cos x \cos^2 x \sin^4 x\,dx$ let $u = \sin x$, $du = \cos x\,dx$

$= \int \cos x (1-\sin^2 x)\sin^4 x\,dx$

$= \int (1-u^2)u^4\,du = \frac{u^5}{5} - \frac{u^7}{7} + C$

$= \boxed{\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C}$

5. $\int \sec^2(x)\tan(x)\,dx$

$= \int \sec x \sec x \tan x\,dx$ let $u = \sec x$, $du = \sec x \tan x\,dx$

$= \int u\,du$

$= \frac{u^2}{2} + C = \boxed{\frac{1}{2}\sec^2 x + C}$

6. $\int \sec^3(x)\tan^3(x)\,dx$

$= \int \sec x \sec^2 x \tan x \tan^2 x\,dx$

$= \int \sec x \tan x \sec^2 x(\sec^2 x - 1)\,dx$ let $u = \sec x$, $du = \sec x \tan x\,dx$

$= \int u^2(u^2-1)\,du = \frac{u^5}{5} - \frac{u^3}{3} + C$

$= \boxed{\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C}$

7. $\int \sin^5(x)\sec^6(x)\,dx$

$= \int \tan^5 x \sec x\,dx = \int \sec x \tan x \tan^4 x\,dx$

$= \int \sec x \tan x (\sec^2 x - 1)^2\,dx$ let $u = \sec x$, $du = \sec x \tan x\,dx$

$= \int (u^2-1)^2\,du = \int u^4 - 2u^2 + 1\,du = \frac{u^5}{5} - \frac{2u^3}{3} + u + C$

8. $\int \csc^4(x)\cot(x)\,dx$

$= \int \csc^3 x \csc x \cot x\,dx$ let $u = \csc x$, $du = -\csc x \tan x\,dx$

$= -\int u^3\,du$

$= -\frac{u^4}{4} + C = \boxed{\frac{-1}{4}\csc^4 x + C}$

$= \frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C$

9. $\int \csc^8(x)\cot^2(x)\,dx$

$= \int \csc^2 x \csc^6 x \cot^2 x\,dx$

$= \int \csc^2 x (1+\cot^2 x)^3 \cot^2 x\,dx$ let $u = \cot x$, $du = -\csc^2 x\,dx$

$= -\int (1+u^2)^3\,du = -\int u^8 + 3u^6 + 3u^4 + u^2\,du$

$= \boxed{-\frac{1}{9}\cot^9 x - \frac{3}{7}\cot^7 x - \frac{3}{5}\cot^5 x - \frac{1}{3}\cot^3 x + C}$

10. $\int \cot^3(x)\csc^4(x)\,dx$

$= \int \cot x \cot^2 x \csc x \csc^3 x\,dx$

$= \int \csc x \cot x \cot^2 x \csc^3 x\,dx$ let $u = \csc x$, $du = -\csc x \cot x\,dx$

$= \int \csc x \cot x (\csc^2 x - 1)\csc^3 x\,dx = -\int (u^2-1)u^3\,du =$

$\boxed{\frac{1}{4}\csc^4 x - \frac{1}{6}\csc^6 x + C}$

11. $\int \sec^5(x)\,dx$

$\int fg'\,dx = fg - \int f'g\,dx$

$= \int \sec^3 x \sec^2 x\,dx$ let $f = \sec^3 x$ and $g' = \sec^2 x$

$= \sec^3 x \tan x - \int 3\sec^3 x \tan^2 x\,dx$ then $f' = 3\sec^3 x \tan x\,dx$ and $g = \tan x$

$= \sec^3 x \tan x - \int 3\sec^3 x(\sec^2 x - 1)\,dx$

$= \sec^3 x \tan x - 3\int \sec^5 x\,dx + 3\int \sec^3 x\,dx \Longrightarrow$

$4\int \sec^5 x\,dx = \sec^3 x \tan x + 3\int \sec^3 x\,dx$

$\therefore \int \sec^5 x\,dx = \frac{1}{4}\left[\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x|\right] + C$

12. $\int \sin^4(x)\,dx$

$= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx$

$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx$

$= \frac{1}{4}\int 1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\,dx = \boxed{\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C}$

$\int fg'dx = fg - \int f'g dx$

13. $\int \csc^3(x)\,dx$

let $f = \csc x$ and $g' = \csc^2 x$
then $f' = -\csc x \cot x$ and $g = -\cot x$

$\int \csc x \csc^2 x\,dx$

$= -\csc x \cot x - \int \csc x \cot^2 x\,dx$

$= -\csc x \cot x - \int \csc x(\csc^2 x - 1)\,dx = -\csc x \cot x - \int \csc^3 x\,dx + \int \csc x\,dx$

$\therefore 2\int \csc^3 x = -\csc x \cot x + \int \csc x\,dx \Rightarrow \int \csc^3 x\,dx = \boxed{-\frac{1}{2}\csc x \cot x - \frac{1}{2}\ln|\csc x + \cot x| + C}$

14. $\int \tan^3(x)\,dx$

$= \int \tan x \tan^2 x\,dx = \int \tan x(\sec^2 x - 1)\,dx$

$= \underbrace{\int \tan x \sec^2 x\,dx}_{\int u\,dv} - \underbrace{\int \tan x\,dx}_{\ln|\sec x|}$

let $u = \sec x$
$du = \sec x \tan x\,dx$

$\frac{u^2}{2} = \frac{1}{2}\sec^2 x$

$= \boxed{\frac{1}{2}\sec^2 x - \ln|\sec x| + C}$

15. $\int \sec^4(x)\,dx$

$= \int \sec^2 x(1 + \tan^2 x)\,dx$

let $u = \tan x$
$du = \sec^2 x\,dx$

$= \int (1 + u^2)\,du$

$= u + \frac{u^3}{3} + C$

$= \boxed{\tan x + \frac{1}{3}\tan^3 x + C}$

16. $\int \tan^4(x)\,dx$

$= \int \tan^2 x(\sec^2 x - 1)\,dx$

$= \underbrace{\int \tan^2 x \sec^2 x\,dx}_{\int u^2\,du} - \underbrace{\int \tan^2 x\,dx}_{\int(1 - \sec^2 x)\,dx}$

let $u = \tan x$
$du = \sec^2 x\,dx$

$= \frac{u^3}{3}$

$= \frac{1}{3}\tan^3 x$

$= x - \tan x$

$= \boxed{\frac{1}{3}\tan^3 x - \tan x + x + C}$

17. $\int \cos(11x)\cos(2x)\,dx$

$= \int \frac{1}{2}[\cos(11x + 2x) + \cos(11x - 2x)]\,dx$

$= \frac{1}{2}\int (\cos 13x + \cos 9x)\,dx$

$= \boxed{\frac{1}{26}\sin 13x + \frac{1}{18}\sin 9x + C}$

18. $\int \sin(13x)\sin(9x)\,dx$

$= \int \frac{1}{2}[\cos(13x - 9x) - \cos(13x + 9x)]\,dx$

$= \frac{1}{2}\int (\cos 4x - \cos 22x)\,dx$

$= \boxed{\frac{1}{8}\sin 8x - \frac{1}{44}\sin 22x + C}$

19. $\int \cos(11x)\sin(3x)\,dx$

$= \int \frac{1}{2}[\sin(11x + 3x) - \sin(11x - 3x)]\,dx$

$= \frac{1}{2}\int (\sin 14x - \sin 8x)\,dx$

$= \boxed{\frac{-1}{28}\cos 14x + \frac{1}{16}\cos 8x + C}$

20. $\int (\sin(x)\cos^2(8x) - \sin^2(8x)\sin(x))\,dx$

$= \int \sin x \underbrace{(\cos^2 8x - \sin^2 8x)}_{\cos 16x}\,dx$

$= \int \sin x \cos 16x\,dx$

$= \int \frac{1}{2}[\sin(x + 16x) + \sin(x - 16x)]\,dx$

$= \frac{1}{2}\int (\sin 17x - \sin 15x)\,dx$

$= \boxed{\frac{-1}{34}\cos 17x + \frac{1}{30}\cos 15x + C}$