(6) $$f(x) = \frac{(\alpha+1)(\alpha+\beta+1)}{\beta} x^{\alpha}(1-x^{\beta})$$

$$g(y) = 1, \quad 0 \le y < 1$$

$$\frac{f(x)}{g(x)} = \frac{(\alpha+1)(\alpha+\beta+1)}{\beta} \times x^{\alpha}(1-x^{\beta})$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{(\alpha+1)(\alpha+\beta+1)}{\beta}\left[x^{\alpha}(-\beta x^{\beta-1}) + (1-x^{\beta})\alpha x^{\alpha-1}\right]$$

$$= \frac{(\alpha+1)(\alpha+\beta+1)}{\beta}\left[-\beta x^{\alpha+\beta-1} + \alpha x^{\alpha-1} - \alpha x^{\alpha+\beta-1}\right]$$

$$= \frac{(\alpha+1)(\alpha+\beta+1)}{\beta}\left[-x^{\alpha+\beta-1}(\alpha+\beta) + \alpha x^{\alpha-1}\right] = 0$$

Solve for x, plug it back into $\frac{f(x)}{g(x)}$ to obtain c.

Now, generate two random numbers – R_1 & $R_2 \sim U[0,1]$.

Set $Y = R_1$.

If $R_2 \leq \dfrac{f(x)}{c\,g(y)}$, then set $X = Y$ as our sample.

Else, generate another set of random numbers and repeat.

Since the question only asks for an algorithm, I did not fully solve for c in terms of α and β.

Expected number of trials to get an acceptance is c.

So, the number of trials (expected) to get 1000 samples is $1000c$.

(b) $\alpha = 3$, $\beta = 1$,

$$\frac{(\alpha+1)(\alpha+\beta+1)}{\beta}\left[-x^{\alpha+\beta-1}(\alpha+\beta)+\alpha x^{\alpha-1}\right]=0$$

$$\Rightarrow \quad 4\times 5\left[-x^{3}(4)+3x^{2}\right]=0$$

$$\Rightarrow \quad 20\left(3x^{2}-4x^{3}\right)=0$$

$$\Rightarrow \quad x^{2}(3-4x)=0$$

$$\Rightarrow \quad x=0,\ 3/4$$

$$f(3/4)=4\times 5\times\left(\frac{3}{4}\right)^{3}\left(\frac{1}{4}\right)^{1}$$

$$=\frac{20\times 27}{16\times 16}$$

$$=\frac{540}{256}$$

$$c=2.1093$$

$$E(x)=\int_{-\infty}^{\infty}x\cdot f(x)\cdot dx$$

$$=\int_{0}^{1}x\cdot 20\,x^{3}(1-x^{1})\,dx$$

$$\int^{1} 20\,x^{4}(1-x)\cdot dx$$

$$= \int_0 20x \; ($$

$$= 0.6667$$

$$E(X^2) = \int_0^1 20x^5(1-x) \cdot dx = 0.47619$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= 0.47619 - 0.44448$$

$$= 0.0317$$

The simulated mean and variance are close to this value at 0.6703 and 0.0304 respectively.

The simulated and theoretical standard deviations are 0.17629 and 0.17804 respectively.