Simulation 8

(a) $X_1, Y_1, X_2, Y_2 \sim N(\mu, \sigma^2)$

$$X \sim N(\mu, \sigma^2) \Rightarrow f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Then, $X_2 - X_1$ and $Y_2 - Y_1$ are also normally distributed albeit with different means and variances.

$$E[X_2 - X_1] = E[X_2] - E[X_1] = \mu - \mu = 0$$

$$Var(X_2 - X_1) = Var(X_2) + Var(X_1) = \sigma^2 + \sigma^2$$
$$= 2\sigma^2$$

So, $X_2 - X_1 \sim N(0, 2\sigma^2)$ and $Y_2 - Y_1 \sim N(0, 2\sigma^2)$

$$D = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}$$

Let $Z = (X_2 - X_1)^2 + (Y_2 - Y_1)^2$

$X_2 - X_1$ and $Y_2 - Y_1$ are both normal variables

Since $X_2 - X_1$ and $Y_2 - Y_1$

with mean $= 0$ and variance $= 2\sigma^2$, $\dfrac{X_2 - X_1}{\sqrt{2\sigma^2}}$ and $\dfrac{Y_2 - Y_1}{\sqrt{2\sigma^2}}$

would become standard normal variables.

All this from online sources confirmed that the sum of the squares of two standard normal variables follows a chi-squared distribution with two degrees of freedom.

Then, it seems $\sqrt{Z}$ would follow a rayleigh distribution with parameter $\sigma' = \sqrt{2\sigma^2}$
$= \sigma\sqrt{2}$

The cdf is given by,

$$F_D(d) = 1 - e^{-\frac{d^2}{2(\sigma\sqrt{2})^2}}$$

$$= 1 - e^{-\frac{d^2}{4\sigma^2}}$$

$$\Rightarrow P(D \leq R_1 + R_2) = 1 - e^{-\frac{(R_1+R_2)^2}{4\sigma^2}}$$

$$\therefore\ P(D > R_1+R_2) = 1 - P(D \le R_1+R_2)$$
$$= 1 - \left(1 - e^{-\frac{(R_1+R_2)^2}{4\sigma^2}}\right)$$
$$= e^{-\frac{(R_1+R_2)^2}{4\sigma^2}} = e^{-\rho^2/2}$$

I implemented the simulation for various μ, σ, R_1 and R_2 and obtained the following results,

$\mu = 4;\ \sigma = 9,\ R_1 = 3,\ R_2 = 4\ (D = 7)$: $P(R_1+R_2 > D) = 0.865$

$\mu = 2,\ \sigma = 4,\ R_1 = 5,\ R_2 = 10\ (D = 15)$: $P(R_1+R_2 > D) = 0.026$

$\mu = 5,\ \sigma = 4,\ R_1 = 2,\ R_2 = 6\ (D = 8)$: $P(R_1+R_2 > D) = 0.357$

The simulation goes above and below the theoretical result obtained and shows that they are correct.