21/2/24

ESE 5030

HW3

① Given that the age distribution is uniform between $a = 42$ and $b = 78$, the pdf is,

$$f(x) = \frac{1}{78-42} = \frac{1}{36}$$

$$P(50 \leq X \leq 70) = \int_{50}^{70} \frac{1}{36} \cdot dx$$

$$= \frac{20}{36}$$

Since the age of each customer is independent of the rest and the probability for each customer to be within a specific age range is the same, the fact that we want the 7^{th} customer will be the

random customer.

same as any random

$\therefore$ p (7^{th} customer in the age range)

$= P(50 \le X \le 70) = \frac{20}{36} = \frac{5}{9}$

$= 0.556$

ⓑ First, we compute the probability of two customers out of the first six customers.

$n = 6$; $k = 2$; $p = 0.556$ (50–70 age range) probability

$P(X=k) = {}^{n}C_{k}\, p^{k} (1-p)^{n-k}$

$= {}^{6}C_{2}\, (0.556)^{2} (1-0.556)^{4}$

$= \frac{6!}{4! \times 2!} \times 0.3091 \times 0.0388$

$= \frac{\cancel{6}^{3} \times 5}{\cancel{2}} \times 0.0120$

$= 0.1802$

... the fact that

We should also include the fa... the 7th customer is also within the age range.

$P($ 7th customer being the 3rd in the age range$)$

$= 0.556 \times 0.1802$

$= 0.1001$

(c) We use the same binomial formula as before.

$n = 7;\ k = 3;\ p = 0.556$

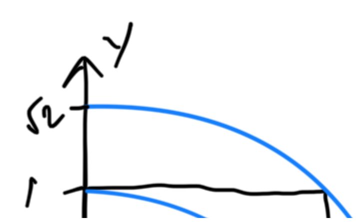
$P(X=3) = {}^{7}C_{3}\,(0.556)^{3}\,(1-0.556)^{4}$

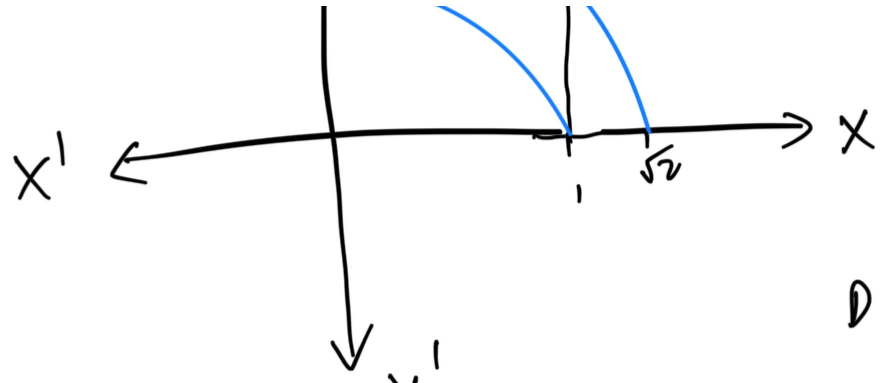
$= \dfrac{7!}{4! \times 3!} \times 0.1719 \times 0.04$

$= \dfrac{7 \times \cancel{6} \times 5}{\cancel{6}} \times 0.006876$

$= 0.24066$

(2)





$$D = \sqrt{x^2+y^2}$$

ⓐ $0 \le L_1 \le 1$: $P(D \le L_1) = \dfrac{\pi L_1^2/4}{1\times 1}$

$$= \frac{\pi L_1^2}{4}$$

We can compute using random values of L_1 from 0 to 1 through simulation.

If $L_1 = 0.5$, $P(D \le L_1) = \dfrac{\pi \times (0.5)^2}{4}$

$$= \frac{\pi \times 1}{4\times 4} = \frac{\pi}{16}$$

If $L_1 = 0$, $P(D \le L_1) = 0$

If $L_1 = 1$, $P(D \le L_1) = \dfrac{\pi \times 1^2}{4} = \dfrac{\pi}{4}$

ⓑ $D = \sqrt{x^2+y^2}$, $x, y \sim [0,1)$

$\begin{cases} 0, & x \le -a \end{cases}$

(3) $$f(x) = \begin{cases} b(x+a)/a, & -a \le x \le 0 \\ be^{-\lambda x}, & 0 \le x \end{cases}$$

$$\int_{-\infty}^{-a} f(x) \cdot dx + \int_{-a}^{0} f(x) \cdot dx + \int_{0}^{\infty} f(x) \cdot dx = 1$$

$$\Rightarrow \quad 0 + \int_{-a}^{0} b(x+a)/a \cdot dx + \int_{0}^{\infty} be^{-\lambda x} \cdot dx = 1$$

$$\Rightarrow \quad \frac{b}{a} \int_{-a}^{0} (x+a) \cdot dx + b \int_{0}^{\infty} e^{-\lambda x} \cdot dx = 1$$

$$\Rightarrow \quad \frac{b}{a} \left. \frac{x^2}{2} + ax \right]_{-a}^{0} + b \left. \frac{e^{-\lambda x}}{-\lambda} \right]_{0}^{\infty} = 1$$

$$\Rightarrow \quad \frac{b}{a} \left((0+0) - \left(\frac{a^2}{2} - a^2 \right) \right) + b \left(0 - \frac{1}{-\lambda} \right) = 1$$

$$\Rightarrow \quad \frac{b}{a} \left(0 + \frac{a^2}{2} \right) + \left(b \times \frac{1}{\lambda} \right) = 1$$

$$\Rightarrow \quad \frac{ab}{2} + \frac{b}{\lambda} = 1$$

$$\Rightarrow \frac{b}{\lambda} = 1 - \frac{ab}{2}$$

$$\Rightarrow 1/\lambda = \frac{1}{b}\left(1 - \frac{ab}{2}\right)$$

(b) Let $F(x)$ be the cdf of X.

$$F(x) = \begin{cases} 0, & x \leq -a \\ 0 + \int_{-a}^{x} b(x+a)/a \cdot dx, & -a \leq x \leq 0 \\ 0 + \int_{-a}^{0} b(x+a)/a \cdot dx + \int_{0}^{x} be^{-\lambda x}, & 0 \leq x \end{cases}$$

Simplify

(4) $p_1 = 1/4$; $p_2 = 1/2$; $p_3 = 3/4$

The outcomes where a game ends and a player loses are,

HHT : $(1/4) \times (1/2) \times (1/4) = 1/32$

$\qquad = 3/32$

$1/4 \times 1/2 \times 3/4$

HTH : $1/4 \times 1/2 \times 3/4 = 9/32$

THH : $3/4 \times 1/2 \times 3/4 = 9/32$

TTH : $3/4 \times 1/2 \times 3/4 = 3/32$

THT : $3/4 \times 1/2 \times 1/4 = 1/32$

HTT : $1/4 \times 1/2 \times 1/4 = 1/32$

$$\frac{26}{32} = \frac{13}{16} = 0.8125$$

The probability that the game will end $= 0.8125$

The above probability is that the game will end in the first play, i.e., $x = 1$.

The average number of plays would be

$$\frac{1}{0.8125} = 1.23 \text{ games.}$$

5:

⑤ $Exp(\lambda) \Rightarrow f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$$Ax^2 + Bx + C = 0$$

$B^2 - 4AC \geq 0 \Rightarrow$ roots are real.

$$A, B, C = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

If $x < 0$, $B^2 - 4AC = 0$

If $x > 0$, $(\lambda e^{-\lambda x})^2 - 4(\lambda e^{-\lambda x})^2$

$= \lambda^2 \left((e^{-\lambda x})^2 - 4 (e^{-\lambda x})^2 \right) > 0$, since $\lambda > 0$.

So, we can see that it ultimately depends on "x" alone and not on the value of λ if $\lambda > 0$.

getting heads

⑥ ⓐ p_A – Probability of A getting
p_B – Probability of B getting heads
q_A – Probability of A getting tails.
q_B – Probability of B getting tails.

$$P(A \text{ wins}) = p_A + (q_A \times q_B)\,p_A + (q_A \times q_B)^2 p_A + \cdots$$

The above is an infinite geometric series with

$a = p_A$ & $r = q_A \times q_B$.

$$\text{Sum} = \frac{a}{1-r} = \frac{p_A}{1-(q_A \times q_B)}$$

$$P(B \text{ wins}) = q_A p_B + (q_A q_B q_A)\,p_B + (q_A q_B q_A q_B q_A)\,p_B + \cdots$$
$$= q_A p_B + q_A p_B (q_A q_B) + q_A p_B (q_A q_B)^2 + \cdots$$

$$S = \frac{p_B q_A}{1-(q_A \times q_B)}$$

so $P(A \text{ wins}) = P(B \text{ wins})$ for the

If

game to be fair,

$$\frac{p_A}{1 - q_A q_B} = \frac{p_B q_A}{1 - q_A q_B}$$

$$\Rightarrow p_A = p_B (1 - p_A)$$

$$\Rightarrow p_B = \frac{p_A}{1 - p_A}$$

(b) Average numbers of rolls for the game to end is computed based on when we get the first head.

Average numbers of tosses for A to win $= \frac{1}{p_A}$

Average numbers of tosses for B to win $= \frac{1}{q_A p_B}$

$q_A p_B$ $\quad$ $p_A + p_B - p_A p_B$

$$\frac{p_A + q_A p_B}{1 - q_A q_B} = \frac{p_A + p_B - p_A p_B}{1 - q_A q_B}$$

$$= \frac{1 - q_A + 1 - q_B - (1-q_A)(1-q_B)}{1 - q_A q_B}$$

$$= \frac{1 - q_A + 1 - q_B - (1 - q_B - q_A + q_A q_B)}{1 - q_A q_B}$$

$$= \frac{1 - q_A + 1 - q_B - 1 + q_A + q_B - q_A q_B}{1 - q_A q_B}$$

$$= \frac{1 - q_A q_B}{1 - q_A q_B} = 1$$

The above steps shows that the sum of $P(A \text{ wins}) + P(B \text{ wins}) = 1$

So, average number of tosses for the game to end, i.e., either A or B wins

$$= \frac{1}{p_A + p_B}$$

The above formula checks for the fact that both A and B should have a chance at getting heads and winning the game. We see how many attempts it takes for the probability of A and B to reach 1.

The above formula also considers the tosses of A and B as one attempt. If we want the total number of individual attempts rather than individual rounds, then we just multiply it by two. So, the answer would then become,

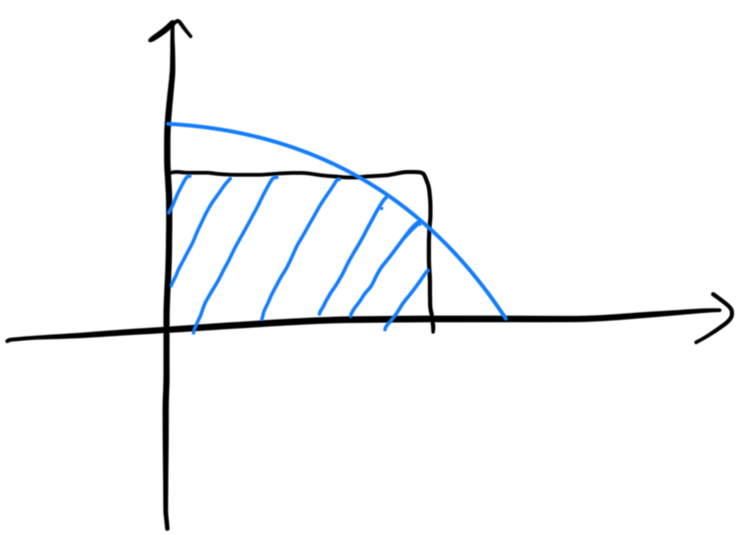
$$= \left(1 \right)$$

$$2 \times \left(\frac{\quad}{p_A + p_B} \right)$$

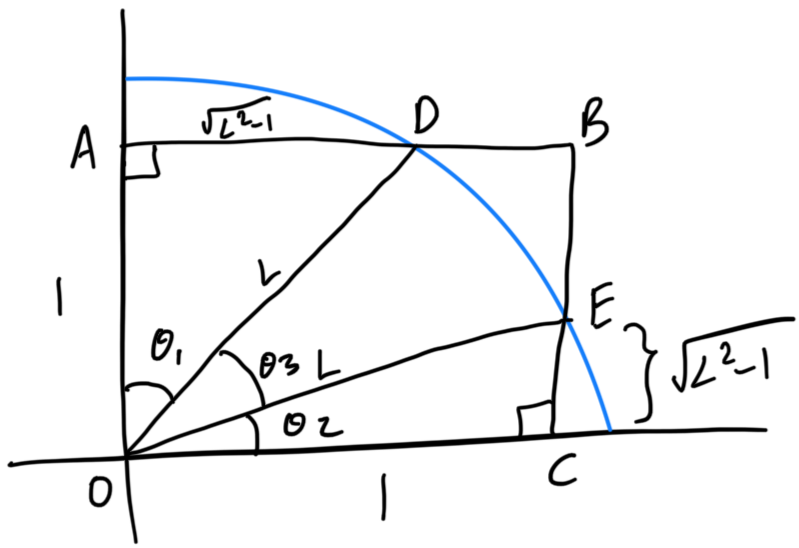
Continuation / Completion of
different answers

② I showed the formula to compute the probability when "L" is from 0 to 1.

Now, I will try to derive a formula for $1 \leq L \leq \sqrt{2}$.



The denominator is 1 because the area of the unit square is 1. We just have to compute the area of the blue shaded region.



$$Ar(\triangle OAD) = \frac{1}{2} \times \sqrt{L^2-1} \times 1$$

$$Ar(\triangle OCE) = \frac{1}{2} \times 1 \times \sqrt{L^2-1}$$

$$\cos\theta_1 = \frac{1}{L} \quad ; \quad \cos\theta_2 = \frac{1}{L}$$

$$\Rightarrow \theta_1 = \theta_2$$

$$\theta_1 + \theta_2 + \theta_3 = 90^\circ \Rightarrow 2\theta_1 + \theta_3 = 90^\circ$$

If $L=\sqrt{2}$, $\sin\theta_1 = \frac{1}{\sqrt{2}} \Rightarrow \theta_1 = \theta_2 = 45$

$$\therefore \theta_3 = 90^\circ - 45^\circ = 45^\circ$$

$$\text{Area of sector ODE} = \frac{45}{360} \times \pi L^2 \quad (L=\sqrt{2})$$

$$= \frac{1}{8} \times \pi \times 2 = \frac{\pi}{4}$$

If $L=1$, $\cos\theta_1 = 1 \Rightarrow \theta_1 = \theta_2 = 0^\circ$ and we get the same answer as shown before.

We use $\cos^{-1}(\ )$ formula to obtain the angle and calculate the area.

In the case of $L=\sqrt{2}$,

$$A = \frac{1}{2}\sqrt{L^2-1} + \frac{1}{2}\sqrt{L^2-1} + \frac{\pi}{4}$$

$$= \sqrt{2-1} + \frac{\pi}{4} \quad (\because L=\sqrt{2})$$

$\frac{\pi}{4} + 1$

$$= \frac{\pi}{4} \pm 1$$

$$p = \frac{\pi/4 \pm 1}{1} = \frac{\pi/4 + 1}{1} \quad (\because p \nless 0)$$

$\because$ p can't be more than 1, we assume the final answer to be 1. We don't

③ ⓐ $$\frac{1}{\lambda} = \frac{1}{b}\left(1 - \frac{ab}{2}\right)$$

$$\Rightarrow \lambda = \frac{2b}{2 - ab}$$

$ab<2$

Since $ab < 2$, λ is always $+$ve, $\because b > 0$.

$$f(x) = \begin{cases} 0, & x \le -a \\ b(x+a)/a, & -a \le x \le 0 \\ be^{-\frac{2bx}{2-ab}}, & 0 \le x \end{cases}$$

$$F(x) = \begin{cases} 0, & x \le -a \\ \int_{-a}^{x} \frac{b(x+a)}{a} \cdot dx, & -a \le x \le 0 \\ \int_{-a}^{0} \frac{b(x+a)}{a} \cdot dx + \int_{0}^{x} be^{-\frac{2bx}{2-ab}} \cdot dx, & 0 \le x \end{cases}$$

$$\int_{-a}^{0} \frac{b(x+a)}{a} \cdot dx = \frac{b}{a} \int_{-a}^{0} (x+a) \cdot dx$$

$$= \frac{b}{a} \left. \frac{x^2}{2} + ax \right]_{-a}^{0}$$

$$= \frac{b}{a}\left(0 - \left(\frac{a^2}{2} - a^2\right)\right)$$

$$= \frac{b}{\cancel{a}} \times \frac{a^{\cancel{2}}}{2} = \frac{ab}{2}$$

$0, \quad x \le -a$

$$\therefore \quad F(x) = \begin{cases} 0, & x \leq \\ \displaystyle\int_{-a}^{x} \frac{b(x+a)}{a} \cdot dx, & -a \leq x \leq 0 \\ \displaystyle\frac{ab}{2} + \int_{0}^{x} be^{-2bx/2-ab} \cdot dx, & 0 \leq x \end{cases}$$

$$\int_{-a}^{x} \frac{b(x+a)}{a} \cdot dx = \frac{b}{a} \int_{-a}^{x} x+a \cdot dx$$

$$= \frac{b}{a} \left. \frac{x^2}{2} + ax \right]_{-a}^{x}$$

$$= \frac{b}{a} \left(\frac{x^2}{2} + ax - \left(\frac{a^2}{2} - a^2 \right) \right)$$

$$= \frac{b}{a} \left(\frac{x^2}{2} + ax + \frac{a^2}{2} \right)$$

$$= \frac{b}{2a} \left(x^2 + 2ax + a^2 \right)$$

$$= \frac{b\,(x+a)^2}{2a}$$

$$\int^{x} be^{-2bx/2-ab} \cdot dx = b \int_{0}^{x} e^{-(2b/2-ab) \cdot x} \cdot dx$$

$$0$$

$$= b \left. \frac{e^{-(2b/2-ab)x}}{-2b/2-ab} \right]_0^x$$

$$= \frac{b(2-ab)}{-2b} \left[e^{-(2b/2-ab)\cdot x} - e^0 \right]$$

$$= \frac{b(2-ab)}{-2b} \times \left(e^{-(2b/2-ab)x} - 1 \right)$$

$$= \frac{ab-2}{2} \left(e^{-(2b/2-ab)x} - 1 \right)$$

$$F(x) = \begin{cases} 0, & x \leq -a \\ \dfrac{b(x+a)^2}{2a}, & -a \leq x \leq 0 \\ \dfrac{ab}{2} + \dfrac{(ab-2)}{2}\left(e^{-(2b/2-ab)x} - 1 \right), & 0 \leq x \end{cases}$$

To generate a random sample from $R \sim [0,1)$,

(1) $x < -a$, $F(x) = 0$, So no random

variable because the pdf and cdf is zero.

② $-a \le x \le 0, \quad F(x) = \dfrac{b(x+a)^2}{2a}$

$F(x) = R \Rightarrow x = F^{-1}(R)$

$$\frac{b(x+a)^2}{2a} = R$$

$$(x+a)^2 = \frac{2aR}{b}$$

$$x = \sqrt{\frac{2aR}{b}} - a$$

③ $0 \le x, \quad F(x) = \dfrac{ab}{2} + \left(\dfrac{ab-2}{2}\right)\left(e^{-\left(\frac{2b}{2-ab}\right)x} - 1\right)$

$F(x) = R$

$$\Rightarrow \frac{ab}{2} + \left(\frac{ab-2}{2}\right)\left(e^{-(2b/2-ab)x} - 1\right) = R$$

$$\Rightarrow \left(e^{-(2b/2-ab)x} - 1\right) = \frac{(R - ab/2) \times 2}{ab-2}$$

$(2b/2-ab)\, x$

$$\Rightarrow e^{-(\cdots)} = \frac{2R-ab}{ab-2}+1$$

$$= \frac{2R-ab+ab-2}{ab-2}$$

$$\Rightarrow e^{\left(-2b/2-ab\right)x} = \frac{2R-2}{ab-2}$$

$$\Rightarrow \left(\frac{-2b}{2-ab}\right)x = \ln\left(\frac{2R-2}{ab-2}\right)$$

$$\Rightarrow x = \frac{2-ab}{2b} \times \ln\left(\frac{ab-2}{2(R-1)}\right)$$

Simulation 1

Simulation 1 :-

I create a user defined function because I want to experiment with various values of γ.

I declare the variables and gammas from 0,1 with 20 values, i.e., 0, 0.5, 1, 1.5, 2, etc.

I generate a random demand within a range given inside a for loop that runs for 2000 days.

I find the sold items which will be ... demand and inventory because the minimum of

demand can be higher than the inventory.

I decrement the sold items from the inventory.

I calculate the profit based on sold items.
If the demand was more than the sold items, I compute the difference to find the profit that could have been made and decrement it from the daily profit.

I append the daily profit to a list called daily-profit.

Then, I check if the inventory is less

than γM and there are no existing orders pending.

I generate a random lead time within a range.

I append an order to the list of orders in the form of a tuple that contains day + lead_time + 1 and the number of supplies to order.

I check the order list and if the current day $\geq$ the order day, I will increment the inventory with M.

If the inventory stock is greater than M, ... by computing the cost to

I find the loss by computing

the company and decrementing it from the daily profit.

I reinitialize the inventory to the value of "M" and remove the order.

I return the mean of the daily profits over 2000 days.

(5) $A \sim Exp(\lambda) \Rightarrow A = \begin{cases} \lambda e^{-a\lambda}, & a \geq 0 \\ 0, & a < 0 \end{cases}$

If roots are real, $B^2 - 4AC \geq 0$

$$P(B^2 - 4AC \geq 0)$$

$$= P(A \leq B^2/4C)$$

$$= 1 - P\left(A > \frac{B^2}{4C}\right)$$

$$\left(\qquad \qquad 4c \right)$$

$$= \int_0^\infty \int_0^\infty \int_{B^2/4C}^\infty \lambda^3 e^{-\lambda(a+b+c)} \cdot da\, db\, dc$$

$$= \int_0^\infty \int_0^\infty \left[\frac{\lambda^3 \; e^{-\lambda(a+b+c)}}{-\lambda} \right]_{B^2/4C}^\infty \cdot db\, dc$$

$$= \int_0^\infty \int_0^\infty 0 - \frac{\lambda^3 e^{-\lambda\left(B^2/4C + B + C\right)}}{-\lambda} \cdot db\, dc$$

$$= \int_0^\infty \int_0^\infty \lambda^2 e^{-\lambda\left(\frac{B^2+4BC+4C^2}{4C}\right)} \cdot dB\, dC$$

$$= \int_0^\infty \int_0^\infty \lambda^2 e^{-\lambda \frac{(b+2c)^2}{4c}} \cdot db\, dc$$

$$= \lambda^2 \int_0^\infty \int_0^\infty e^{-\left(\frac{b\sqrt{\lambda}}{2} + 2c\sqrt{\lambda}/2\right)^2/c} \cdot db\, dc$$

$$= \lambda^2 \times \frac{1}{3 \times \frac{8 \times \lambda\sqrt{\lambda}}{8} \times \frac{\sqrt{\lambda}}{2}}$$

$$= \lambda^2 \cdot \frac{1}{\quad}$$

$$= \lambda \times \frac{1}{\frac{3 \times \lambda^2}{2}}$$

$$= \frac{2}{3}$$

$$P\left(A \leq B^2/4C\right) = 1 - P\left(A > B^2/4C\right)$$

$$= 1 - \frac{2}{3} = \frac{1}{3}$$

Hence, we proved that the roots being real is independent of λ.

② (b) Average distance from origin $= \int_0^1 \int_0^1 \sqrt{x^2+y^2}\, .dx\,dy$

Sir had used online tools in class, so, using online tools to save time, I found the answer to be 0.7652 which is close to the simulated value of 0.7668.