④ ⓐ $P(Y \leq a) = F(a)$

Since "X" is a random variable, when we replace "a" with "X", we need to account for the probability of "X" occuring.

"X" has a range space "R_X" and pmf $p(x)$.
To include this in the cdf of Y, we will have to account for all the samples - "x" in R_X and also its probability - $p(x)$.

$$P(Y \leq x) = F(x)\, p(x)$$

$$P(Y \leq X) = \sum_{x \in R_X} F(x)\, p(x)$$

A more mathematical way to show the above would be,

$$P(Y \le X) = \sum_{x \in R_X} P(Y \le x \cap X = x)$$

$$= \sum_{x \in R_X} P(Y \le x) \times P(X = x)$$

$$= \sum_{x \in R_X} F(x) \times p(x)$$

(b) $T \to$ Lifetime of bulb ; $f(t) = \begin{cases} \lambda e^{-\lambda x} & , x \ge 0 \\ 0 & , x < 0 \end{cases}$

$$\mu = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{\mu} = \frac{1}{1000}$$

$X \to$ Checking time; $p(x) = p(1-p)^{x-1}$, $x \to$ number when first success occurs

$$F(t) = 1 - e^{-\lambda t} = 1 - e^{-t/\mu}$$

$$p = \frac{1}{\mu_2} = \frac{1}{1500}$$

From our previous result,

$$P(Y \le X) = \sum_{x \in R_X} F(x) p(x)$$

we show,

$$P(T \leq X) = \sum_{x=1}^{\infty} F(x)\, p(x)$$

$$= \sum_{x=1}^{\infty} \left(1 - e^{-x/1000}\right)\left(1 - \frac{1}{1500}\right)^{x-1} \times \frac{1}{1500}$$

$$= \sum_{x=1}^{\infty} \left(1 - e^{-x/1000}\right)\left(\frac{1499}{1500}\right)^{x-1} \times \frac{1}{1500}$$

$$= \frac{1}{1500}\left(\sum_{x=1}^{\infty} \left(1499/1500\right)^{x-1} - \sum_{x=1}^{\infty} e^{-x/1000}\left(1499/1500\right)^{x-1}\right)$$

$$= \frac{1}{1500} \times \frac{1500}{1499}\left(\sum_{x=1}^{\infty} \left(1499/1500\right)^{x} - \sum_{x=1}^{\infty} e^{-x/1000}\left(\frac{1499}{1500}\right)^{x}\right)$$

$$= \frac{1}{1499}\left(\frac{1499/1500}{\frac{1500-1499}{1500}} - \sum_{x=1}^{\infty} e^{-x/1000}\left(\frac{1499}{1500}\right)^{x}\right)$$

$$= \frac{1}{1499}\left(1499 - \sum_{x=1}^{\infty} e^{-x/1000}\left(\frac{1499}{1500}\right)^{x}\right)$$

$$= 1 - \frac{1}{1499}\sum_{x=1}^{\infty} e^{-x/1000}\left(\frac{1499}{1500}\right)^{x}$$

$$= 1 - \frac{599}{1499} = 1 - 0.4$$

$$= 0.6$$

(c) It matches with the excel simulation of 0.58