University of Cagliari

MSc programs in Computer Engineering, Cybersecurity and Artificial Intelligence, and in Electronic Engineering

Artificial Intelligence

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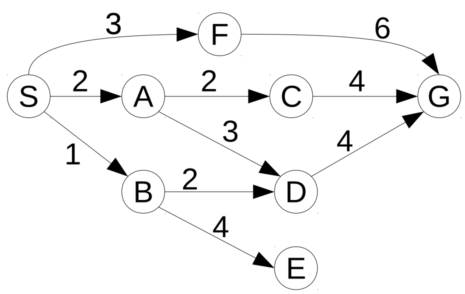
Exercises on search algorithms

1. The graph in the figure below shows the state space of a hypothetical search problem. States are denoted by letters, and the cost of each action is indicated on the corresponding edge. Note that actions are not reversible, since the graph is *oriented*. The table next to the state space shows the value of some admissible heuristic function, considering `G` as the goal state (it is easy to verify that such an heuristic never overestimates the true, minimum path cost from any given state to the goal state `G`).

   state space

   heuristic function (goal state: G)

S	A	B	C	D	E	F	G
6	4	5	2	2	8	4	0

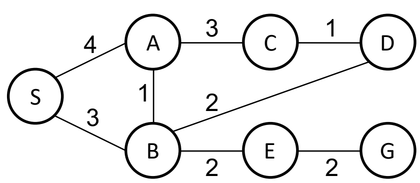
   Considering `S` as the initial state, solve the above search problem using:

   - breadth-first search
   - depth-first search
   - A* search with the heuristic above

   When drawing the search tree you should *clearly* indicate: the order of expansion of each node (e.g., by numbering the expanded nodes according to the order of their expansion); the action corresponding to each edge of the tree; the state, the path cost and the value of the heuristic of each node. In the case of depth-first search you should also indicate which nodes of the search tree are *simultaneously* stored in computer memory (in the case of a computer implementation) during the search process.

2. Consider the state space of a graph search problem shown below, where the number on each edge denotes the cost of the corresponding action (note that actions are reversible), `S` and `G` denote respectively the initial and the goal state, and the table reports the values of an admissible heuristic function.

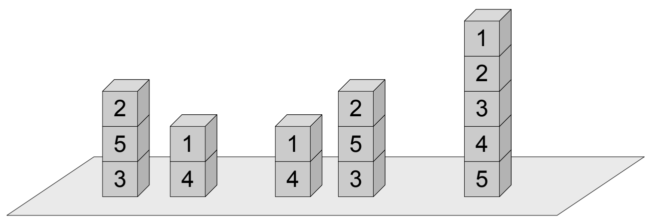
   Show how the depth-first search and A* strategies solve this problem, avoiding repeated states in the same path; in case of tie, the choice of the next node to expand has to be made according to *alphabetical ordering*. For the sake of clarity, draw *separately* the search tree obtained after each node expansion.

S: 5	A: 3	C: 1
D: 2	B: 2	E: 1

3. The *block's world* is a well-known toy domain used in AI for planning problems related to robots. It consists of a set of blocks of various size, shape, and color, possibly identified by a letter or by a number, and placed on a surface (e.g., on a table); the blocks can be stacked into one or more piles, possibly with some constraints (depending, e.g., on their size); the goal is to form a target set of piles starting from a given initial configuration, by moving one block at a time, either to the table or atop another pile of blocks, and only if it is not under another block.

   Consider a simple version of this problem, in which there are five blocks with identical shape and size, numbered from 1 to 5. The figure below shows two possible configurations of such blocks; note that the relative position of the piles does not matter, thus the configuration on the left is equivalent to the one in the middle. Every block can be either on the table (there are no constraints on the number of piles) or atop any another block. The goal is to stack the blocks in a single pile, in the order shown in the rightmost configuration in the figure, starting from any given set of piles, by moving the smallest possible number of blocks. Only blocks at the top of the current piles can be moved, and can be placed only on the table or atop another pile.

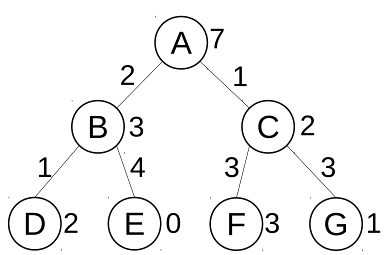


   (a) Formulate the above version of the block's world as a search problem, by *precisely* defining the state space, the initial state, the goal test, the set of possible actions and the path cost.

   (b) Assume that the initial configuration is the one shown on the left in the figure above, and consider a heuristic function defined as the number of blocks that are not in the highest pile (or in one of the highest piles). Under this setting, show how the A* search strategy *expands* the first four nodes of the search tree, avoiding repeated states. As in the previous exercise, when drawing the search tree you should *clearly* indicate: the order of expansion of each node; the action corresponding to each edge of the tree; the state, the path cost and the value of the heuristic of each node.

   (c) Define another possible admissible heuristic, and prove its admissibility.

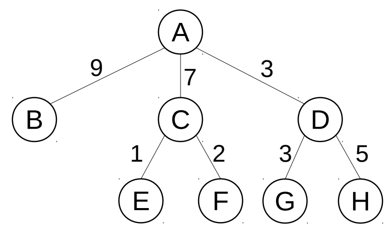
4. Assume that the search tree below has been obtained for some search problem after the first three expansion steps by the A* strategy. Numbers on the edges denote the cost of the corresponding actions, letters inside each node denote the corresponding state, and numbers next to each node denote the value of the heuristic function (which is assumed to be admissibile) for the same state. Which node will A* select for expansion in the next step?



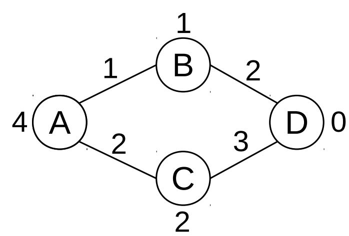
5. Assume that the search tree below has been obtained for some search problem after the first three expansion steps by some *unknown* uninformed strategy. The meaning of letters and numbers is the same as in the previous exercise. Determine whether the underlying search strategy can be *any* of the following:

   (a) depth-first search

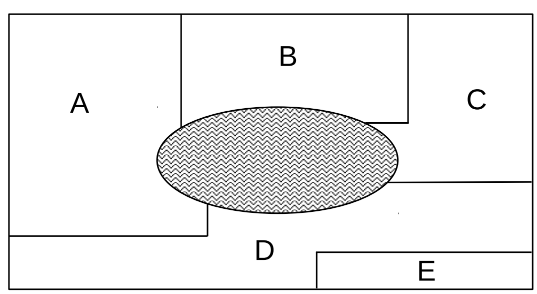
   (b) uniform cost search



6. Consider the *state space* of a given search problem in the figure below, where `D` is the goal state, numbers on each edge denote the cost of the corresponding action, and numbers next to each state denote the corresponding value of some heuristic function. Is the considered heuristic function *admissible*?



7. Consider the map colouring problem below, involving five regions, denoted by `A`, ..., `E`, around a lake, and assume that four colours are available: yellow, green, orange and cyan. Assume that regions `A` and `B` have already been assigned the colours yellow and green, respectively. Which region would the depth-first backtracking search algorithm select, and which colour would it assign to such a region, in the next step, taking into account the *minimum remining values* (MRV), *degree* and *least constraining value* (LCV) heuristics?



Solution

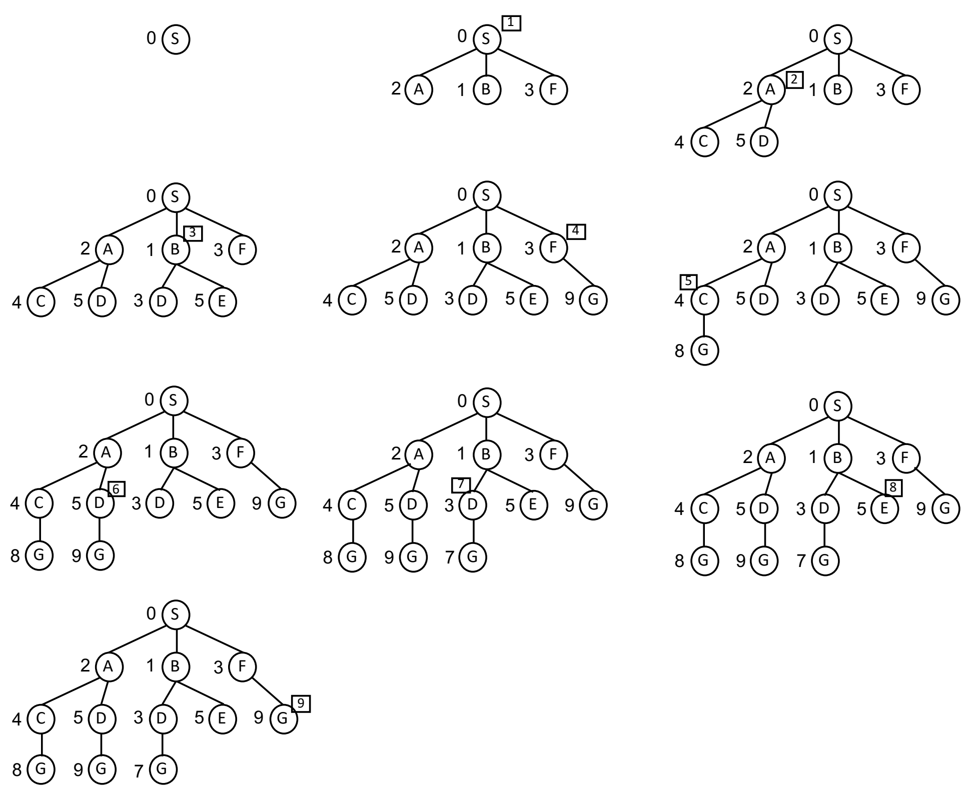
Exercise 1

The search trees built by the three strategies are shown below. The number of the left of each node denotes its path cost for breadth- and depth-first search (note however that the path cost is not used by these strategies to choose the next leaf node to expand), and the sum of its path cost and heuristic for A*-search. The order of expansion is denoted by the small numbers inside squares. For the sake of clarity, the search tree obtained after each expansion is shown.

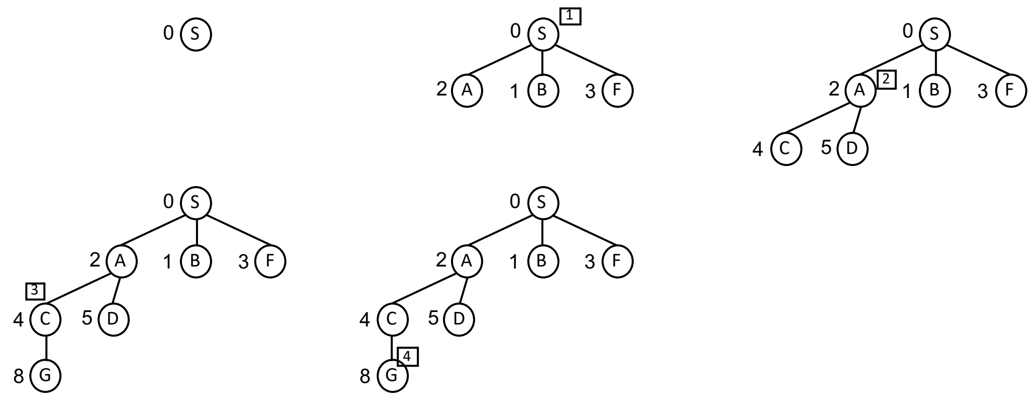
Remember that when several leaf nodes can be chosen for expansion (i.e., when there is more than one leaf node at the lowest depth for breadth-first search, or at the highest depth for depth-first search, or with the lowest value of the evaluation function for A*-search – the sum of the path cost and the heuristic function) a random choice should be made among them; in this exercise we assume that the choice is made by considering the alphabetical ordering among the corresponding states (e.g., if the choice is among nodes `A`, `B` and `F`, node `A` is chosen). This happens at step $\boxed{9}$ in the search tree below.

Finally, remember that the goal test is applied when a node is *selected for expansion*: therefore a solution is found not when a node containing a goal state is *generated* (by expanding it parent node), but when it is selected for expansion.

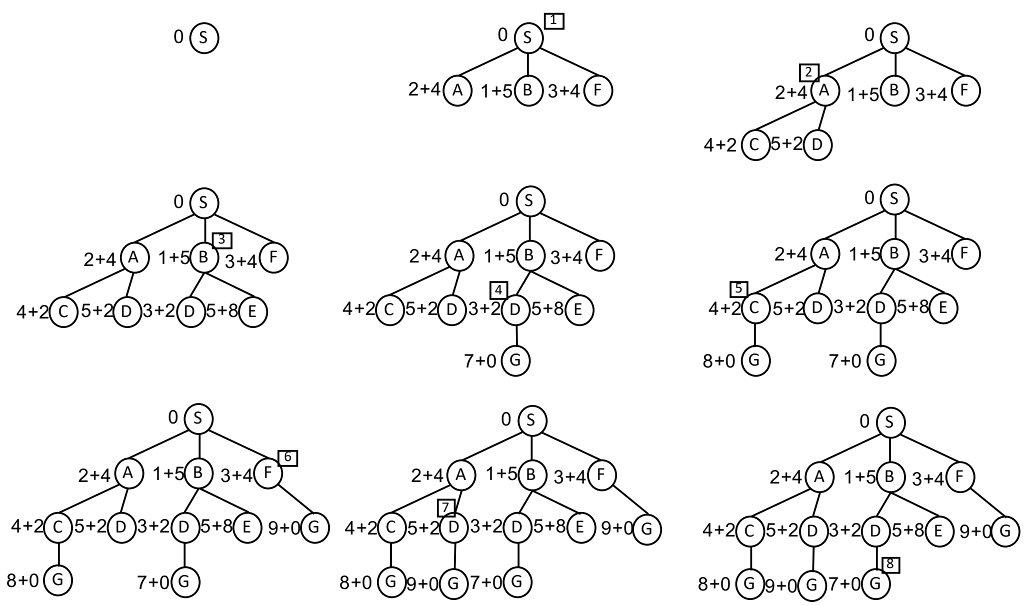
The search tree built by **breadth-first search**, step by step, is shown in the figure below (from top to bottom and from left to right). Note that different nodes can correspond to the same state (this is the case of states `D` and `G`): this means that such a state can be reached from the initial state through different paths in the state space (i.e., different sequences of actions), possibly with different path costs: therefore such nodes should be kept distinct in the search tree. Note also that node `E` has no successors (no other state can be reached from state `E`, according to the state space): therefore when it is selected for expansion (in step $\boxed{8}$) no children nodes are added to the tree. Finally, note that the solution found by breadth-first search (`S`→`F`→`G`) is not the one with minimum path cost, as it can happen due to the fact that this search strategy is not optimal.



The search tree built by **depth-first search** is shown below. In this case a solution is found along the path which is explored first (when the node corresponding to state G is selected for expansion at step $\boxed{4}$). Therefore no backtracking step is carried out and no already expanded node along "dead ends" is deleted. Note that also in this case the solution found is not the minimum-cost one (depth-first search is not optimal).



The search tree built by **A*-search** is shown below. Next to each node the corresponding path cost g and heuristic h are shown as $g + h$. Remember that at each step the leaf node with the minimum value of $g + h$ is chosen for expansion (ties are broken randomly, as in all search strategies). The solution found at step $\boxed{8}$ is guaranteed to be the minimum-cost one, according to the fact that A* is optimal when an admissible heuristic is used.



Exercise 2

The solution is left as an exercise.

Exercise 3

(a) The state space is made up of the set of distinct arrangements of the five blocks into one or more (up to five) piles. A state can be represented by a *set*[1] of piles, and each pile can be represented as a sequence of block numbers, where the numbers from left to right correspond, e.g., to block from the top to the bottom of the pile. For instance, the two equivalent configurations of piles (states) shown on the left and in the middle of the above figure are represented by the set $\{(2,5,3),(1,4)\}$. Accordingly, the goal state is represented by $(1,2,3,4,5)$.

The actions can be formally described as follows: given a state $\{(b_{1,1},\ldots,b_{1,n_1}),\ldots,(b_{p,1},\ldots,b_{p,n_p})\}$, where p denotes the number of piles ($1 \leq p \leq 5$) and n_k the number of blocks in the k-th pile ($n_k \geq 1$ for each k, and $\sum_{k=1}^{p} n_k = 5$), the actions consist of moving one of the p blocks $b_{k,1}$ (on the top of one of the piles) either to the table, thus generating a new pile (only if the original pile contains more then one block, i.e., $n_k > 1$), or atop one of the other $p-1$ piles (if any).

Actions can be implemented as a *successor function SF*: it receives as an argument the description s of a state, and returns all the pairs (s', a) where s' is one of the states obtained as described above, and a the description of the corresponding action. Each action can be described by indicating the number of the block that has been moved, $b_{k,1}$, and its new position, i.e., the number of the block atop of which it has been placed, or the table (which can be denoted by the number 0). For instance, from state $\{(2,5,3),(1,4)\}$ the action $(2,0)$ consists of moving the block 2 to the table, leading to the state $\{(5,3),(2),(1,4)\}$, and $(1,2)$ consists of moving the block 1 atop the block 2, leading to the state $\{(1,2,5,3),(4)\}$.

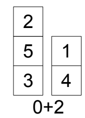
Finally, since the goal is to reach the target configuration by moving the smallest possible number of blocks, it follows that the path cost is given by the number of actions that have been carried out to reach a given state from the initial one, and thus each action has a cost equal to 1.

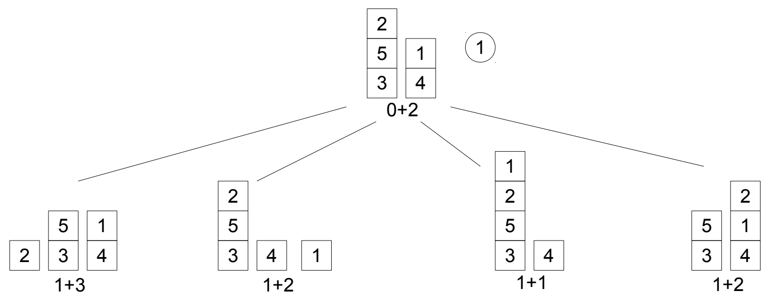
(b) The search tree obtained by the A* search strategy after the expansion of the first four nodes, using the heuristic given in the text and avoiding repeated states, is shown in the figure below. For the sake of clarity, the tree obtained after the expansion of each node is shown.

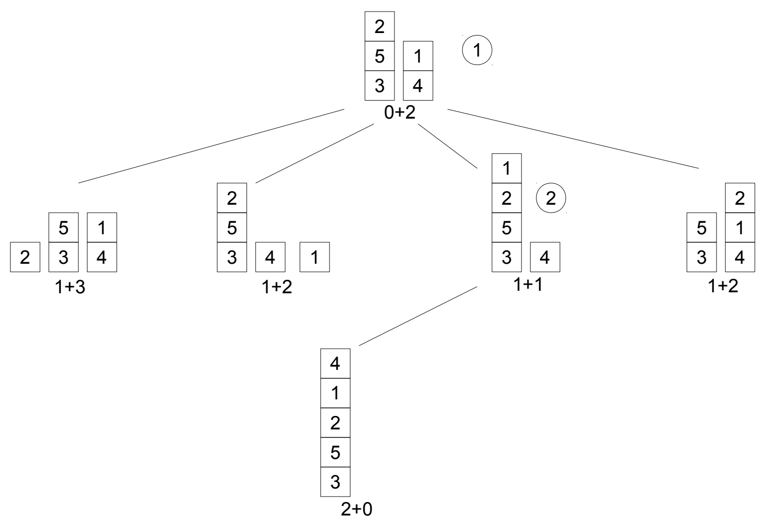
The A* evaluation function (path cost + heuristic) is shown below each node. The numbers inside circles denote the order in which the corresponding node has been expanded. Note that the third node has no successors, since the only action that can be carried out on its state leads to a repeated state (the one of its parent node). Note also that the are two candidate leaf nodes to be expanded at the fourth iteration of A* (the second and fourth node from left in the figure, at depth 1), since they exhibit the same value of the evaluation function, which is lower than the one of the remaining leaf nodes; in this case a random choice has to be made between the candidate nodes: in the figure below the node on the right is chosen for expansion.

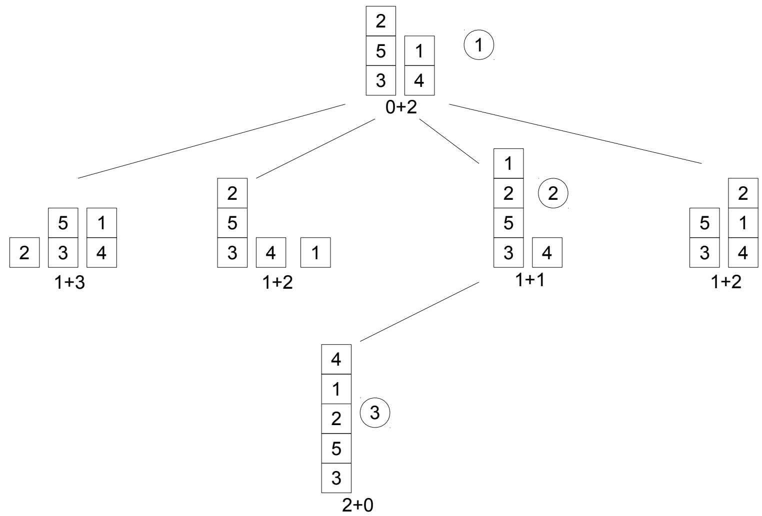
[1] According to the problem definition, the ordering between piles is not relevant.

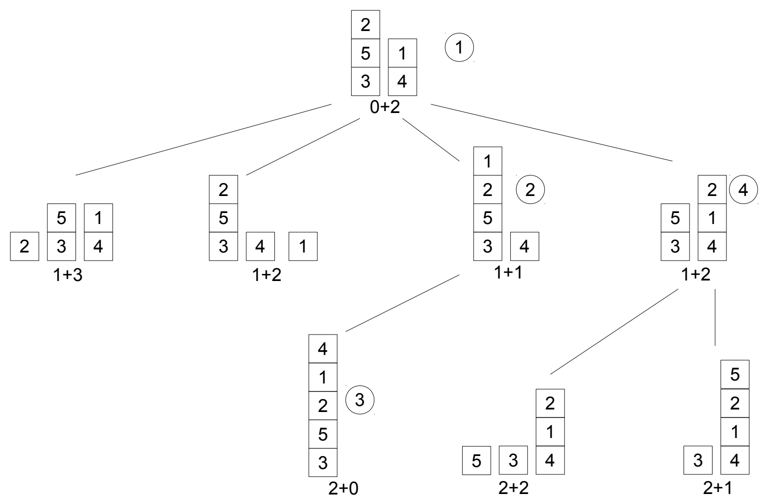
root node:











(c) Another possible admissible heuristic is the following: let n be the number of blocks above which there is a block different from the one in the goal state; then the heuristic is defined as $\lfloor n/2 \rfloor$. For instance, if the goal state is $\{(1, 2, 3, 4, 5)\}$, in the state $\{(2, 5, 3), (1, 4)\}$:

- there is no block atop block 1, which counts as 0;
- there is no block atop block 2, contrary to the goal state: this counts as 1;
- block 5, instead of block 2, is atop block 3: this counts as 1;
- block 1, instead of block 3, is atop block 4: this counts as 1;
- block 2, instead of block 4, is atop block 5: this counts as 1;

the value of the heuristic function for the state $\{(2, 5, 3), (1, 4)\}$ is therefore $\lfloor 4/2 \rfloor = 2$; in other words, at least two blocks have to be moved to reach the goal state.

It is not difficult to see that this heuristic is admissible: by moving one block from any state s', at most two other blocks will have a different block atop in the resulting state s'', and therefore in s'' at most two more blocks than in s' can have the correct block atop as in the goal state.

Exercise 4

A* selects for expansion the leaf node n with the lowest value of the evaluation function $f(n) = g(n)+h(n)$, where g and h denote the path cost of n (i.e., the sum of of the step costs – costs of the actions – in the path from the root node to n) and the value of the heuristic function for the corresponding state, respectively. Ties are broken randomly. It immediately follows that A* will select (randomly) either the left-most or the right-most leaf node, whose evaluation function equals 5.

Exercise 5

The search strategy used to obtain the considered search tree cannot be depth-first: indeed, after the expansion of the root node, either node `C` or node `D` has been expanded in the second step; in the former case, either node `E` or node `F` would have been expanded in the third step, instead of node `D`; analogously, in the latter case either node `G` or node `H` would have been expanded in the third step, instead of node `C`.

The search strategy cannot be uniform-cost, either: indeed, in the second step node `D` (the one with the lowest path cost among `B`, `C` and `D`) would have been expanded, but in the third step uniform-cost would have expanded node `G` instead of `C`, whose path cost (6) is lower than the ones of nodes `B` (9), `C` (7) and `H` (8).

Exercise 6

A heuristic function is admissible if it never overestimates the minimum cost from any state to the goal state. It is easy to see that this condition is *not* fulfilled by the considered heuristic:

- The minimum cost from states `B` and `C` to the goal state `D` is 2 and 3, respectively, which is higher than the corresponding heuristic (respectively, 1 and 2);
- The minimum cost from state `A` to `D` is 3, which corresponds to the path `A` → `B` → `D`: it is however *lower* than the corresponding heuristic (4).

Exercise 7

The considered partial assignment is {`A` = `yellow`, `B` = `green`}. The MRV heuristic is used first, to choose the next variable to assign. Variables `C` and `D` have only three values left, since they neighbour with `B` and `A`, respectively, whereas `E` has all four values left; therefore there is a tie between `C` and `D`.

The degree heuristic is then used to break the tie: it chooses variable `D`, which has two constraints with unassigned variables (`C` and `E`), whereas `C` has a constraint with only one of them (`D`).

The LCV heuristic is finally used to choose a value for `D`. Among the remaining values (`green`, `orange` and `cyan`), it chooses `D` = `green`, which leaves three feasible values for both `C` and `E`, whereas `orange` and `cyan` would leave only two feasible values for both `C` and `E`.