Mémoire M2 Mathématiques Fondamentales

Domino Problem on Groups

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Abstract

We study the decidability of the Domino Problem on groups. We begin by studying the four main strategies of the proof of the undecidability of the domino problem on the infinite grid and examine which of their aspects are generalizable. We then proceed to study the problem on groups with the use of novel concepts in the area such as the concept of graph of groups, one-relator groups and the dual tiling problem. We manage to establish the undecidability of new classes of groups, such as the non-free Artin groups and non-infnite cyclic Generalized Baumslag-Solitar groups. In addition we explore the dual problem of determining which groups can be tiled by a given tileset.

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1 Introduction

The Domino Problem was first introduced by Hao Wang [Wan62] as a model to study the $\forall\exists\forall$ fraction of first order logic. The problem consists on determining if the plane is tileable by a finite set of tiles subject to rules on the placement of adjacent tiles. He conjectured that any set of tiles that successfully tiles the plane, also admits an alternative tilling that is periodic, that is, one that repeats itself in a given direction. This in turn proves the decidability of the domino problem. Nevertheless, in 1966 one of Wang's students, Robert Berger, showed the existence of a set of 104 tiles that tile the plane aperiodically, proving the undecidability of the domino problem [Ber66]. This particular construction was improved upon by Robinson who in addition lowered the number of tiles required [Rob71].
The study of the domino problem is nowadays made in the setting of symbolic dynamics. This was originally introduced by Hadamard [Had99] as a tool to study dynamics through discretization, and expanded upon in the now classic article [MH38] by Morse and Hedlund where they studied subshifts as objects themselves. For a comprehensive study of the classical setting, that is subshifts on $\mathbb{Z}$, we refer the reader to [LM21]. A vast field of study was subsequently opened when the context of symbolic dynamics was expanded to finitely generated groups, including a generalized version of the original domino problem. Each finitely generated group has its own version where one asks if its Cayley graph is tileable by a given tileset.

It has been conjectured that virtually free groups are exactly those that have decidable domino problem, and there are reasons to believe it is true [BS13]. So far, the conjecture has been proven for a number of classes of groups, such as,

- Baumslag-Solitar groups [AK13],
- Polycyclic groups [Jea15a],
- Surface groups [ABM19],
- Direct products of two infinite groups [Jea15b],

as well as many inheritance properties relating the problem to subgroups, quotients, quasi-isometries among others. A survey of these results and further discussions can be found in [ABJ18].

A very peculiar fact is that all proof methods for groups other that $\mathbb{Z}^2$ are mostly directly taken from or inspired by the proofs on that case. At the moment of writing, there are four main strategies to prove the undecidability of the original problem. There are the style of the original proof by Berger and Robinson involving substitutions and embedded computation, the strategy of Aanderaa and Lewis who introduced the notion of distance shift for the proof, the strategy of Kari that uses the encoding via Sturmian sequence of orbits and finally the strategy by Durand, Romashchenko and Shen that involves the concept of simulation and fixed point theorems. Both the first and third strategies have been succesful in proving some classes of groups have undecidable domino problem (most notably the case of Kari's strategy for Baumslag-Solitar and surface groups).

The aim of this thesis is to study the state of the art about the Domino Problem on groups. In particular, we wish to look at the second and fourth of the previously mentioned strategies, and see which group theoretical or structural aspects of $\mathbb{Z}^2$ allow the proofs to work. Also, we would like to incorporate new tools and perspectives to this almost four decade old problem, such as the concept of graph of groups, one-relator groups and the dual tiling problem.
We begin by introducing the concepts in computability theory, geometric group theory and symbolic dynamics that will be necessary for the development of the work. Next, we talk about

the original domino problem and explore the four proof strategies mentioned above. Then, we explore the notion of subshifts of finite type on a more general class of objects including locally finite infinite graphs and finitely generated groups. This section also includes a presentation of all the available tools or inheritance results up to this date, including a new one that uses wreath products. Afterwards, we present two possible generalizations of the ideas used by Aanderaa and Lewis, one which is a brute force generalization and the other involves semi-direct products. Next, we take a brief look at the proof of the undecidability of the problem on Baumslag-Solitar groups, showcasing an example of the application of one of the methods to a different setting. The last three sections involve new lines of inquiry, the study of two-generated one-relator groups, the use of graph of groups for the problem and the study of the dual problem. In each of these sections we obtain new results, new classes of groups where the domino conjecture holds and new perspectives. The report is capped off by a series of questions that arose during the development of the thesis and a conclusion of the work done.

2 Concepts

For the development and exposition of the work, we must introduce three main concepts. The first one is the concept of computability and reductions which will allow us to find a precise notion of what it means for a decision problem to be computable. Next we introduce the notion of group presentations which is the basis of geometric group theory. Finally, we introduce subshifts, which are the main protagonists of the field of study. This objects were originally defined in dynamical terms, but they really became of great interest when their combinatorial aspects where developed.

2.1 Computability

Turing machines where first introduced in 1936 by Alan Turing as a model to capture the way human beings make computations. In fact it is widely believed that the concept he introduced exactly captures the notion of an object being computable. This hypothesis is known as the Church-Turing thesis.

Church-Turing Thesis: Functions that are computable in the intuitive real-world sense, are exactly those that are computable by a Turing machine.

Definition 2.1. A *Turing machine* is a tuple $M = (\Sigma, \Gamma, Q, \delta, q_0, q_a, q_r)$, where

- Q is a finite set of states,
- Γ is the finite tape alphabet,
- $\Sigma \subseteq \Gamma$ is the finite input alphabet,
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{-1, 0, 1\}$ is the transition function,
- $q_0, q_a, q_r \in Q$ are the initial, accepting and rejecting state respectively, with $q_a \neq q_r$.

Definition 2.2. A language L is said to be:

- *Decidable* if there is a Turing machine M such that if $w \in L$, M accepts on w and if $w \notin L$, M rejects on w.
- *Recursively enumerable* if there is a Turing machine M such that M accepts on w if and only if $w \in L$. Otherwise it rejects or loops indefinitely.

Let us introduce the classical example of an undecidable problem. Let HALT be the decision problem that receives a Turing machine M and a word w, and accepts if and only if M reaches a final state on w (accepts or rejects).

In 1936, Turing showed there does not exist a Turing machine that solves the halting problem. A problem such that there exists no Turing machine that decides it is said to be *undecidable*.

Theorem 2.3 (Turing [Tur36])**.** HALT *is undecidable*

A function $f : D \subseteq \{0,1\}^* \to \{0,1\}^*$ is said to be *computable* if there is a Turing machine M that on $w \in D$ accepts with $f(w)$ left on its tape at the end.

Definition 2.4. Let L and L' be two languages. We say L is *many-one reducible* to L', denoted $L \leq_m L'$, is there exists a computable function f such that $x \in L$ if and only if $f(x) \in L'$ for all x. When $L \leq_m L'$ and $L' \leq_m L$ we use the notation $L \equiv_m L'$ and say the too languages are *many-one equivalent*.

Reductions are the backbone of the proofs of undecidability. From the definition we have that $L \leq_m L'$ and L is undecidable, then L' is undecidable.

2.2 Group presentations

Let G be a group and $u, v \in G^*$ two words. $u =_G v$ denotes equality after the group operation of G has been applied on each pair of contiguous symbols for each side.

Definition 2.5. Let G be a group. We say (S, R) is a *presentation* of G if G is isomorphic to $\langle S|R\rangle$, where

$$\langle S|R\rangle = F_S/N_R,$$

F_S is the free group over S, and N_R is the normal (or conjugate) closure of R given by $N_R = \langle\{grg^{-1} \mid g \in F_s, r \in R\}\rangle$.

Definition 2.6. We say a group G is:

- *Recursively presented* if there exists a presentation $\langle S|R\rangle$ such that S is recursive and R is recursively enumerable,
- *Finitely presented* if there exists a presentation $\langle S|R\rangle$ such that both S and R are finite.

In fact, it is possible to characterize recursively presented groups as finitely generated subgroups of finitely presented groups

Theorem 2.7 (Higman's embedding [Hig61])**.** *A finitely generated group G embeds into a finitely presented group if and only if G is recursively presented.*

Definition 2.8. The *word problem* of a group G with respect to a set of generators S is the language

$$\mathrm{WP}(G, S) := \{u \in S^* \mid u =_G 1_G\}.$$

The next Proposition allows us to simple talk about the word problem of a group $\mathrm{WP}(G)$ without referencing the generating set.

Proposition 2.9. *Let S_1 and S_2 be two finite sets of generators for G. Then, we have the equivalence* $\mathrm{WP}(G, S_1) \equiv_m \mathrm{WP}(G, S_2)$.

Remark 2.1. For a given presentation $G = \langle S|R\rangle$ we have that $N_R = \mathrm{WP}(G, S)$.

2.3 Subshifts

For the purposes of this work we define a graph Γ as a tuple (V_Γ, E_Γ), where V_Γ is a set of vertices and $E_\Gamma \subseteq V_\Gamma^2$ is the set of edges, such that the graph is locally finite, i.e., the neighborhood of any vertex $v \in V_\Gamma$ defined by

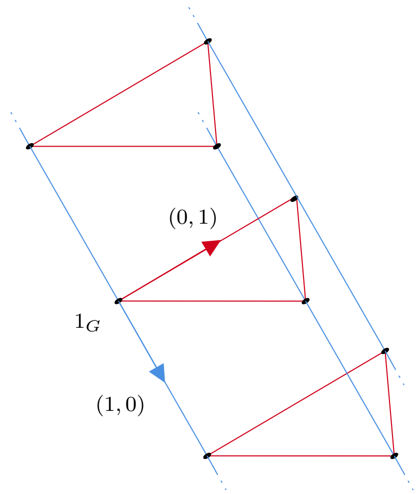
$$N(v) := \{u \in V_\Gamma \mid (v, u) \in E_\Gamma \lor (u, v) \in E_\Gamma\},$$

is uniformly bounded. We also associate the graph with two functions $\mathfrak{i}, \mathfrak{t} : E_\Gamma \to V_\Gamma$ that give the initial and final vertex of en edge, respectively.

We can additionally define a edge-labelling of Γ through a function $L : E_\Gamma \to B$, where B is a finite set

Definition 2.10 (Cayley Graph)**.** Let G be a group and $S \subseteq G$ a finite subset. The *Cayley graph* of G with respect to S is the edge-labeled graph $\Gamma(G, S)$, whose vertex set is G and its edge set is given by all edges of the form (g, gs) where $g \in G$ and $s \in S$. The graph is labeled by $L : \Gamma(G, S) \to S$ by $L(g, gs) = s$.

An example of the Cayley graph of the group $\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ with respect to the generating set $\{(1, 0), (0, 1)\}$ is the following.



To introduce the notion of a subshift, we must first present a notion of patterns of labels in a graph.

Definition 2.11. Let Γ be a graph labeled by L and let S, T be two finite subsets of V_Γ. A mapping $\phi : S \to T$ is a *label preserving isomorphism* if ϕ is a bijection and it satisfies the two following conditions:

- (Morphism) $\forall u, v \in S$, $(u, v) \in E_\Gamma$ if and only if $(\phi(u), \phi(v)) \in E_\Gamma$,
- (Label preservation) $\forall u, v \in S$, $L(u, v) = L(\phi(u), \phi(v))$.

Let A be a finite set and Γ a graph. The full-shift on Γ with respect to A is the set, denoted by A^Γ, of mappings $V_\Gamma \to A$. Each mapping is called a *configuration*. If we endow the full-shift with the product of the discrete topology, it is a compact metrizable space.

Let $P \subseteq V_\Gamma$ be a finite set of connected vertices. A *pattern* with support P is a map $p : P \to A$, and we denote $\operatorname{supp}(p) = P$. We say a pattern p appears in a configuration $x \in A^\Gamma$ if there exists a finite set of vertices $P' \subseteq V_\Gamma$ and a label preserving isomorphism $\phi : P \to P'$ such that $p_u = x_{\phi(u)}$ for all $u \in P$.

Definition 2.12. Given a set of patterns F, a *subshift* $X_F \subseteq A^\Gamma$ is the set of configurations that avoid the set of forbidden patterns F. A subshift is said to be of finite type (SFT) if the set of forbidden patterns F is finite.

In the case where $\Gamma = \Gamma(G, S)$ is the Cayley graph of a finitely generated group $G = \langle S \rangle$, we have a dynamical structure that complements the combinatorial definition of subshifts we just gave. We endow A^G with the left group action $\sigma : G \curvearrowright A^G$ given by $\sigma^g(x)_h = x_{g^{-1}h}$. The product topology in this case has the cylinders,

$$[a]_g := \{x \in A^G \mid x_g = 1\},$$

as a sub-basis. We can give an alternative definition of a subshift through the topology.

Lemma 2.13. *$X \subseteq A^G$ is a subshift if and only if it is a closed σ-invariant subset.*

Given that we have a dynamical structure, we can give a notion of dynamical equivalence between two subshifts.

Definition 2.14. Given two subshifts $X \subseteq A^G$ and $Y \subseteq B^G$, we say that a continuous map $\pi : X \to Y$ is a *shift-morphism* if it conmutes with the action by the group, that is,

$$\pi \circ \sigma_X^g = \sigma_Y^g \circ \pi, \ \forall g \in G.$$

We say the shift-morphism is a *factor* if π is surjective (equivalently Y is a factor of X), and that it is a conjugacy if it is bijective (equiv. X is conjugated to Y).

Once again their is a now classical result that links both the dynamical and combinatorial structure of these objects.

Theorem 2.15 (Curtis-Hedlund-Lyndon)**.** *A map $\varphi : X \to Y$ is a shift morphism if and only if there exists a finite set $S \subseteq G$ and a map $\Phi : A^S \to A$ such that*

$$\varphi(x)_g = \Phi(\sigma^{g^{-1}}(x)|_S).$$

This theorem was originally proved in [Hed69] for the case where $G = \mathbb{Z}$, and a proof of the general case can be found in [CC10].

Definition 2.16. We say a subshift $X \subseteq A^G$ is *sofic* if it is a factor of an SFT.

2.3.1 The many faces of an SFT

Definition 2.17. Let G be a group and S be a set of generators for G. A subshift $X \subseteq A^G$ is said to be a *nearest neighbor subshift* with respect to S if there exists a set F of forbidden patterns such that $X = X_F$ and every pattern $p \in F$ has support $\text{supp}(p) = \{1_G, s\}$ for some $s \in S$.

Proposition 2.18. *Every SFT is conjugated to a nearest neighbor subshift.*

Another way of interpreting this result is that any SFT can be represented by a set of Wang tiles.

There have been other attemps to generalize the concept of an SFT from the starting point of groups. In [BS20], Bartholdi and Salo introduce an alternative way to understand SFTs from the use of G-sets and labeled graphs.

Definition 2.19. Let Γ be a graph. A *directed* Hom-*shift* (DHS) with carrier Γ is the space $\text{Hom}(\Gamma, \Lambda)$ for some finite graph Λ. We say the DHS is weakly resolving if for a given label in Λ there is at most one edge with that label between two vertices.

Now, consider a group $G = \langle S \rangle$ and a set X on which we have an action of G on the right. We define the *Schreier graph* of X, denoted by $\mathcal{X}$, the graph with vertices X and edges $X \times S$ where $\mathfrak{i}(x, s) = x$ and $\mathfrak{t}(x, s) = xs$.

Definition 2.20. An SFT Ω over X consists of an alphabet A, an integer $n \geq 1$ and a subset $\Pi \subset A^{S^{\leq n}}$ of allowed patterns. It is defined as

$$\Omega = \{\alpha \in A^X \mid \forall x \in X, \ \exists P_x \in \Pi : \ P_x(w) = \alpha(xw), \forall w \in S^{\leq n}\}.$$

Bartholdi and Salo show that this notion of SFT is in fact the same as a weakly resolving DHS on the Schreier graph.

Proposition 2.21. *Let G be a monoid acting on a set X. Then, SFTs on X are equivalent to weakly resolving DHSs with carrier $\mathcal{X}$.*

3 The domino problem

Let us take a close look at the original problem as introduced by Wang.

Definition 3.1. Let C be a finite set of colors. A *Wang tile* is an 4-tuple $t = (t_N, t_E, t_S, t_W) \in C^4$. A set $\tau \subseteq C^4$ of Wang tiles is refered to as a tileset.

We say a configuration $x : \mathbb{Z}^2 \to \tau$ is a *valid tiling* of the plane if $\forall i, j \in \mathbb{Z}$, $x(i,j)_N = x(i, j+1)_S$ and $x(i,j)_E = x(i+1, j)_W$.

Definition 3.2. The *domino problem* is the decision problem that takes as input a finite tileset τ and outputs YES if and only if there exists a valid tiling of the plane by τ.

As stated in the introduction, it was Berger who first showed the undecidability of the domino problem. The proof was later improved upon by Robinson.

Theorem 3.3 ([Ber66])**.** *The Domino Problem is undecidable.*

Definition 3.4. The *origin constrained domino problem* (OCDP) is the decision problem that takes as input a finite tileset τ and a tile $t \in \tau$ and outputs YES if and only if there exists a valid tiling of the plane by τ with t at the origin.

Theorem 3.5 (Kahr, Moore and Wang [KMW62], Büchi [Büc61])**.** *The origin constrained domino problem is undecidable.*

Many of the proofs of the (unconstrained) domino problem rely on somehow reproducing the proof of this theorem.

3.1 The 4 proofs of undecidability

Since the first proof of undecidability by Berger, there have been multiple proof methods developed for the result, which utilize vastly different tools. We will give a brief overview of the methods employed on each variant. An in depth presentation of these proofs can be found in [JV20].

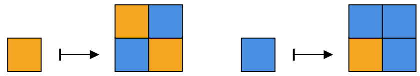
3.1.1 Aperiodic substitutive tiling

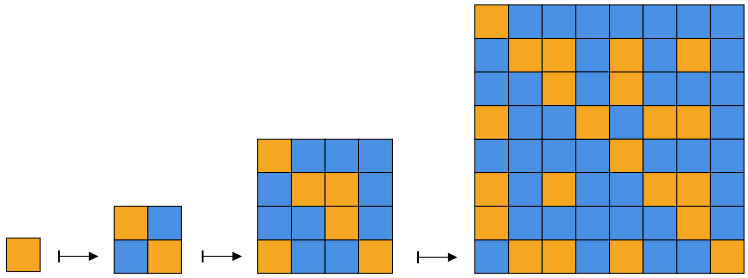
As previously mentioned, the first proof of the undecidability of the domino problem was done by Berger [Ber66] and later simplified by Robinson [Rob71]. As it stands now, the proof consists in defining a substitution that forces all configurations of the subshift it defines to have a hierarchy of arbitrarily large squares.

Definition 3.6. A *square substitution* in $\mathbb{Z}^2$ is a map $\mu : A \to A^{n \times n}$ for some $n > 0$ and a finite alphabet A. The substitution shift defined by μ is defined as,

$$X_\mu := \{x \in A^{\mathbb{Z}^2} \mid \text{ for all pattern } p \sqsubseteq x,\ \exists a \in A, n \in \mathbb{N} \text{ s.t. } p \sqsubseteq \mu^n(a)\}.$$

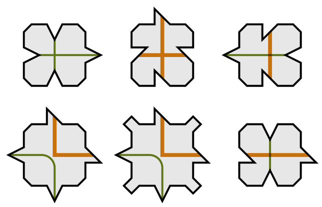
An example of a square substitution is the following,



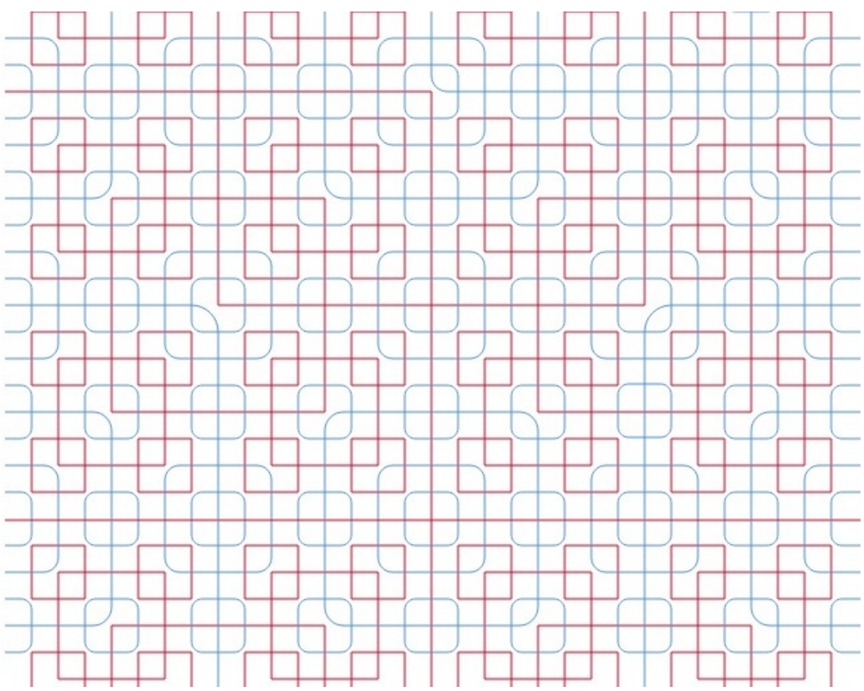


along with an example of repeatedly applying the substitution to the orange letter of the alphabet.

The tilling introduced by Robinson is comprised of the following tiles,



in addition to their rotated versions. It is straight forward how this tiles form a substitution. As mentioned at the beginning these create a hierarchy of ever increasing nested squares.



The proof is done by reducing the halting problem to the domino problem to this substitutive system, through the coding of more and more steps of computation inside bigger and bigger square in the hierarchy. Nevertheless we can identify that the three key concepts at play are those of substitutions, aperiodicity and the ability to make computations through local constrained. All of these aspect will be better explored later on. In addition, due to the fact this is the first proof of undecidability, there are multiple detailed articles explaining its work such as [JV20; Rob71; Gäh13].

3.1.2 Fixed-point simulation

This method was introduced by Durand, Romashchenko and Shen, who besides showing the undecidability of the domino problem, also exhibited a strongly aperiodic tileset [DRS12]. It is important to note that this method also relies on the notion of substitutions. This is due to the fact that the notion of simulation implicitly assumes we have a notion of substituions.

Let us introduce what it means for one tileset to simulate another tileset.

Definition 3.7. Let σ, τ be two tilesets. We sat that σ *simulates* τ *with a zoom factor* N is there exists a one-to-one function $\phi : \tau \to \sigma^{N\times N}$ that satisfies the following conditions,

- For any finite pattern P, $\phi(P)$ is a valid tiling for σ precisely when P is a valid tiling for τ.
- For all tiling of the plane x by σ, there exists y such that $\phi(y) = x$ up to shift. More precisely, there exists a unique pair $(i_0, j_0) \in \{0, ..., N-1\}^2$ and a unique y such that $\phi(y)_{i,j} = x_{i+i_0, j+j_0}$ for all $i, j \in \mathbb{Z}$.

The result that implies the undecidability of the domino problem is the existence of a self-simulating tileset.

Theorem 3.8. *There exists a tileset that simulates itself.*

This proof relies on the use of the following Theorem due to Kleene.

Theorem 3.9 (Kleene's fixed point)**.** *If* $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *is a computable function, then there exists a computable function* $g : \mathbb{N} \to \mathbb{N}$ *such that* $f(g, n) = g(n)$*.*

To apply this result, one must first represent a set of Wang tiles as a computable function. Every tile is a tuple of 4 colors $t = (N, E, S, W)$, so we can design a computable function g that accepts when t represents a tile that belongs to the tileset, and rejects otherwise.

It is also essential to the proof that computations can be done in $\mathbb{Z}^2$, that is, one can explicitly write the computation of a Turing machine in a two dimensional grid. An explicit tileset for such an operation can be observed in the proof of the undecidability of the origin constrained domino problem.

3.1.3 Balanced representation of orbits

This proof strategy comes from Kari who used the mortality problem of Turing machines and representations of numbers through balanced representation of real numbers to create a tileset that proves the undecidability of the domino problem.

Definition 3.10. The *mortality problem of Turing Machines* is the decision problem that takes a deterministic Turing Machines with a halting state and outputs YES if and only if there exists a non-halting configuration.

Theorem 3.11 (Hooper [Hoo66])**.** *The mortality problem of Turing Machines is undecidable.*

The proof by Hooper works even for machines with an alphabet of two letters.

Now, let us have a system of affine maps of the plane $f_1, ..., f_n$ associated with disjoint unit squares with integer corners $U_1, ..., U_n$. We define a partial function $f : \mathbb{R}^2 \to \mathbb{R}^2$ with domain $U = \bigcup_{i=1}^{n} U_i$ by $f(x) = f_i(x)$ when $x \in U_i$. A point $x \in \mathbb{R}^n$ is said to be immortal if $\forall n \in \mathbb{N}$: $f^n(x) \in U$.

Definition 3.12. The *mortality problem of piece-wise affine maps* is the decision problem that takes a system of rational affine transformations of the plane $f_1, ..., f_n$ associated with unit squares $U_1, ..., U_n$ with integer corners and outputs YES if and only if the system has an immortal point.

Theorem 3.13 ([Kar07]). *The mortality problem of piece-wise affine maps is undecidable.*

This is done by encoding Turing machine configurations as unique real numbers through base M expansion for a carefully chosen $M > 0$.

With this at hand, the proof for the domino problem works by coding each successive iteration of the function f, that is $f^n(x)$ for some $x \in U$ through a balanced representation. This creates a tileset such that it tiles the plane if and only if there is an immortal point for f.The fact that the tileset is finite is a combination of the facts that the domains U_i are bounded of integer corners and that each f_i is rational.

A much more detailed explanation of this proof is presented in Section 6 where we see how this method is applied to the class of Baumslag-Solitar groups.

3.1.4 Distance shift

This proof variation was first introduced by Aanderaa and Lewis [AL74; Lew79]. For this we begin by introducing the concept of the distance shift.

Definition 3.14. Let C be a finite set of colors and A a finite alphabet. A *decorated Wang tile* t over C is a tuple (t', a) where t' is a Wang tile over C and $a \in A$ is called the decoration. For $t = (t', a)$ we denote $\pi(t) = a$.

Let τ be a decorated tileset. The sofic shift defined by τ, denoted X_τ, is the set of all bi-infinite words $\pi(x)$, where x is a tiling of $\mathbb{Z}$ by τ.

Definition 3.15. Given a sofic shift $X \subseteq (A \times A)^{\mathbb{Z}}$, the *distance shift* X^Δ corresponding to X is the set of all pairs $(x, y) \in A^{\mathbb{Z}} \times A^{\mathbb{Z}}$ such that $\forall i \in \mathbb{Z}$, $(x, \sigma^i(y)) \in X$.

In other words,

$$(x, y) \in X^\Delta \iff \forall n, m \in \mathbb{Z} : (\sigma^n(x), \sigma^m(y)) \in X.$$

Definition 3.16. Let Dist($\mathbb{Z}$) be the decision problem that takes a sofic shift $X \subseteq (A \times A)^{\mathbb{Z}}$, where A is a finite alphabet, and outputs YES if the distance shift X^Δ not empty.

Let's make a brief outline of the steps followed by the proof.

1. A many-one reduction is made from Dist($\mathbb{Z}$) to the domino problem on $\mathbb{Z}^2$.
2. A many-one reduction is made from the origin constrained domino problem on $\mathbb{Z}^2$ to Dist($\mathbb{Z}$) via the use of almost 1-1 extensions of p-adic integers.
3. The two reductions imply that the origin constrained domino problem is many-one equivalent to the domino problem. We conclude by Theorem 3.5.

The distance problem is reduced to the Domino problem via the following construction.

Let X be a sofic shift over the alphabet $A \times A$. We define the two-dimensional shift X_2 as

$$X_2 = \{z \in A^{\mathbb{Z}^2} \mid z_{ij} = (x_i, y_{j+i}),\ x, y \in A^{\mathbb{Z}},\ (z_{ij})_{i\in\mathbb{Z}} \in X\}.$$

Proposition 3.17. *X_2 is sofic and each row of X_2 is composed of (x, y) such that $\forall j\ (x, \sigma^j(y)) \in X$. Furthermore, there exists and algorithm that can obtain a decorated set of Wang tiles for X_2 from a set of decorated tiles for X.*

Proposition 3.18. *X_2 is empty if and only if X^Δ is empty.*

For the second step, we have

Theorem 3.19. *There is no algorithm that decides, given a sofic shift X, whether X^Δ is empty.*

The proof of Theorem 3.19 relies on the use of Toeplitz subshifts, that is almost 1-1 extensions of odometers. This is because, in an heuristic sense, the difference between two elements of such a shift is contained in the shift.

Let $(\mathbb{Z}_p, +1)$ be the odometer composed of the set of p-adic integers $\mathbb{Z}_p$, for $p \in \mathbb{Z}$, and the addition by 1 with carry. We define the function $a_p : \mathbb{Z}_p \to \{1, ..., p-1\}$ by

$$a_p(m) = \min_{i\geq 0}\{m_i : m_i \neq 0\}.$$

Let S_p be the subshift generated by the sequence u where $u_i = a_p(i)$. It is possible to show that S_p is composed exactly of sequences of the form $x_i = a_p(m+i)$ for some $m \in \mathbb{Z}_p$.

The map defined by

$$\begin{array}{rccc} f : & S_p & \to & \mathbb{Z}_p \\ & x & \mapsto & f(x) \end{array}$$

satisfies $f(\sigma(x)) = f(x) + 1$.

Proposition 3.20. *There exists a sofic subshift X such that the distance shift X^Δ is exactly $S_p \times S_p$. In particular the distance shift has no periodic point and therefore gives rise to an aperiodic 2-dimensional SFT.*

The proof of the Theorem is concluded by showing that an instance of the origin constrained domino problem can be many-one maped to the particular kind of sofic shift mentioned in the previous proposition.

4 Generalizing the setting of the domino problem

Let us generalize the setting on which we can define the domino problem.

Definition 4.1. Let Γ be a graph. The *domino problem for* Γ is the decision problem that given a finite set of patterns F decides if $X_F \neq \varnothing$. We denote $\mathrm{DP}(\Gamma)$ the set of finite sets of patterns where this is the case.

With this framework Aubrun, Barbieri and Moutot showed that the domino problem is undecidable on any orbit graph of a non-deterministic substitution with an expanding eigenvalue [ABM19].

In the case where Γ is the Cayley graph of a finitely generated group, $\Gamma = \Gamma(G, S)$, we write $\mathrm{DP}(G, S)$. In fact, because of the equivalence of SFTs and nearest neighbor subshifts in this particular case, we can define $\mathrm{DP}(G, S)$ on the set of nearest neighbor forbidden patterns. The following result allows us to speak of *the* domino problem of a finitely generated group G, $\mathrm{DP}(G)$.

Proposition 4.2. *Let S and S' be two finite sets of generators of a group G. Then* $\mathrm{DP}(G,S)$ *is many-one equivalent to* $\mathrm{DP}(G,S')$.

A variant of the domino problem that is of interest is the origin constrained domino problem, as we saw for the case of $\mathbb{Z}^2$. This variant is also generalized:

Definition 4.3. Let Γ be a graph. The *origin constrained domino problem for* Γ is the decision problem that given a finite set of patterns F and a letter $a \in A$ decides if there exists a configuration $x \in X_F$ and a vertex $v \in V_\Gamma$ such that $x_v = a$. We denote $\mathrm{OCDP}(\Gamma)$ the set of finite sets of patterns and letters where this is the case.

4.1 The group toolbox

The following set of results allow us to relate the decidability of the problem to group theoretical properties. These results are all presented and proven in [ABJ18].

The first result links the decidability of the word problem to the domino problem. This automatically tells us that to find a group with decidable domino problem, we must look among those with decidable word problem.

Theorem 4.4. *For any group G,* $\mathrm{WP}(G) \leq_m \overline{\mathrm{DP}(G)}$. *In particular, undecidable word problem implies undecidable domino problem.*

The results we will make the most use of in the next sections are the following, which related the problem to the subgroups and quotients.

Proposition 4.5. *For every finitely generated subgroup $H \leq G$, we have* $\mathrm{DP}(H) \leq_m \mathrm{DP}(G)$.

As a consequence of this proposition, any group containing $\mathbb{Z}^2$ as a subgroup has undecidable domino problem.

Proposition 4.6. *For every finitely generated normal subgroup $N \trianglelefteq G$, we have that* $\mathrm{DP}(G/N)$ *many-one reduces to* $\mathrm{DP}(G)$.

Proposition 4.7. *Let $H \leq G$ be a subgroup such that $[G:H] < +\infty$. Then,* $\mathrm{DP}(H) \equiv_m \mathrm{DP}(G)$.

Theorem 4.8 ([Jea15b])**.** *Let G_1 and G_2 be two infinite, finitely generated groups. Then, $G_1 \times G_2$ has an undecidable domino problem.*

4.1.1 Geometry and actions

The large scale structure of groups can be formally studied through the concept of lipschitz maps and quasi-isometries.

Definition 4.9. Let G and H be two finitely generated groups, with S a finite set generators for G.

- A map $f : G \to H$ is said to be *Lipschitz* if there exists a finite set $T \subseteq H$ such that $\forall g \in G$ and $s \in S$: $f(g)^{-1}f(gs) \in T$.
- A map $f : G \to G$ is said to be *at a bounded distance from the identity* if there exists a finite set $F \subseteq G$ such that for all $g \in G$, $g^{-1}f(g) \in F$.

Definition 4.10. We say two groups G and H are *quasi-isometric*, denoted as $G \sim_{QI} H$, if there exists two Lipschitz maps $f : G \to H$ and $g : H \to G$ such that $f \circ g$ and $g \circ f$ are at a bounded distance from the identity. In this case we say g is a quasi-inverse of f.

Theorem 4.11 ([Coh17])**.** *Let G and H be two finitely presented groups. Suppose $G \sim_{QI} H$, then G has undecidable domino problem if and only if H has undecidable domino problem.*

A similar result can be obtained for a similar type of action, originally introduce by Whyte [Why99] and Seward [Sew14] independently.

Definition 4.12. Let G be a group such that it acts on the right on another group H. We say the action is *free* if $h.g = h$ for some $h \in H$, then $g = 1_G$. We say the action is *transition-like* if it is free and for every $g \in G$, the map $h \mapsto h.g$ is at a bounded distance from the identity.

Theorem 4.13 ([Jea15b])**.** *Let G be a finitely presented group with undecidable domino problem and G a finitely generated group. If G acts translation-like on H, then H has undecidable domino problem.*

Finally, there is a generalization of Jeandel's result in the context of graphs. Heuristically the notion of a group automata-containing another group means that the edges on the first are given by graph-walking finite state automata that simulate the edges of the second group.

Theorem 4.14 ([BS20])**.** *Let G, H be two finitely generated groups. If G is automata-contained in H, then* $\mathrm{DP}(G) \leq_m \mathrm{DP}(H)$.

4.1.2 Groups with decidable domino problem

Proposition 4.15. *Let F be a free group of finite rank. Then,* $\mathrm{DP}(F)$ *is decidable.*

Theorem 4.16. *Every virtually free group has a decidable word problem.*

This theorem was also directly proven in [MS85] and [KL05] by the following reasoning:

- The domino problem can be expressed in Monadic Second Order Logic,
- A group is virtually free if and only if its Cayley graph has finite tree-width,
- Graphs with finite tree-width are exactly the ones with decidable MSO logic.

This makes a compelling case for the fact that these are the only groups with decidable domino problem.

Conjecture 4.17 (Ballier, Stein [BS13])**.** A finitely generated group G has a decidable domino problem if and only if G is virtually free.

The conjecture is motivated by the fact that if a group is not virtually free, its Cayley graph has unbounded tree-width which in turn implies that it has arbitrarily large grids as minors (Halin's Theorem). This suggests that one could embed computation on these grids.

4.1.3 Wreath products

Using the previous tools we can state a result for certain cases of wreath products through the use of the Krasner–Kaloujnine universal embedding theorem.

Definition 4.18. Let G and H be two groups. The *wreath product $G \wr H$* is defined as the semidirect product

$$G \wr H := \left(\bigoplus_{h \in H} G \right) \rtimes H,$$

where $k \in H$ acts on $(g_h)_{h \in H} \in \bigoplus_{h \in H} G$ by $k(g_h) = (g_{k^{-1}h})_{h \in H}$.

The standard example of a wreath product is the Lamplighter group,

$$L = \langle a, b \mid (a^n b^{-n})^2, \ \forall n \in \mathbb{Z} \rangle = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}.$$

The next theorem allows us to see a group embedded on the wreath product of one of its normal subgroups with its quotient.

Theorem 4.19 (Krasner–Kaloujnine universal embedding [KK51])**.** *Let G be a group and let $N \trianglelefteq G$ be a normal subgroup. Then, there is a monomorphism $\theta : G \to N \wr G/N$ such that N maps surjectively onto* $\mathrm{Im}(\theta) \cap N^{G/N}$.

As a consequence of this Theorem and the fact that the wreath product of two finitely generated group is finitely generated, we can state the following result.

Proposition 4.20. *Let G be a group and $N \trianglelefteq G$ a normal subgroup. Then, we have that* $\mathrm{DP}(G) \leq_m \mathrm{DP}(N \wr G/N)$.

Nevertheless, this result does not allow us to establish the decidability of the domino problem on the lamplighter group because in this case $G = \mathbb{Z} \ltimes \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{DP}(G)$ is decidable by virtue of being virtually free. However Bartholdi and Salo have shown in [BS20] that the origin constrained domino problem on the group is undecidable. The unconstrained version remains an open problem.

5 Possible generalizations of the Distance Shift method

To use this proof's techniques on other groups, we must generalize all 3 steps mentioned in the previous section.

5.1 Direct generalization

Step 1 is possible to generalize directly, as we show in what follows.

Definition 5.1. Let $X \subseteq (A \times A)^G$ be a sofic shift. The *distance shift* X^Δ corresponding to X is the set of pairs $(x, y) \in A^G \times A^G$ such that $\forall g \in G$, $(x, \sigma^g(y)) \in X$. In other words,

$$(x, y) \in X^\Delta \iff \forall g, h \in G : (\sigma^g(x), \sigma^h(y)) \in X.$$

Disclaimer: there is a detail concerning the fact that the group action gives the inverse of the shift in the case of $G = \mathbb{Z}$.

Definition 5.2. For a finitely generated group G, we define the decision problem $\mathrm{Dist}(G)$ as the set of all sofic subshifts X such that $X^\Delta \neq \varnothing$.

Definition 5.3. Let G be a finitely generated group, S a finite set of generators, C a finite set of colors and A a finite alphabet.

- A *Wang tile* for (G, S) is an element $t \in C^{S \cup S^{-1}}$.
- Let $\tau \subseteq C^{S \cup S^{-1}}$ be a tileset. We say a configuration $x : G \to \tau$ is a *valid tiling* of G if $\forall g \in G$ we have
$$x(g)_s = x(gs)_{s^{-1}}.$$
- A *decorated Wang tile* t is a tuple (t', a) where t' is a Wang tile and $a \in A$ is the decoration. We denote $\pi(t) = a$.
- Let τ be a decorated tileset. The sofic shift defined by τ, denoted $X_\tau \subseteq A^G$, is the set of all configurations $\pi(x)$, where x is a tiling of G by τ.

The following Proposition is a result of a direct generalization of the proof for $\mathbb{Z}$. Nevertheless, the result is not of much use because Theorem 4.8 already establishes the undecidability of the domino problem for the product of two infinite finitely generated groups.

Proposition 5.4. $\mathrm{Dist}(G) \leq_m \mathrm{DP}(G \times G)$.

Proof. Let $X \subseteq (A \times A)^G$ be a sofic subshift. As in the case of $G = \mathbb{Z}$ we define a new subshift X_2 by all the configurations $z \in (A \times A)^{G \times G}$ such that

$$z_{(g,h)} = (x_g, y_{h^{-1}g}),\ x, y \in A^G,$$

and for every $h \in G$, $(z_{(g,h)})_{g \in G}$ in a configuration in X. X_2 is then composed of pairs of configurations (x, y) such that for any $g \in G$, $(x, \sigma^g(y)) \in X$. From the fact that X is sofic, we deduce that X_2 must also be.

Now, suppose X is defined from a set of decorated Wang tiles τ, that is, $X = X_\tau$. There is a straight forward algorithmic procedure to generate a decorated tileset for X_2.

Let $\star$ be a fixed color. Take $t = (t', a) \in \tau$ and define $p = (p', b)$ a decorated Wang tile for $G \times G$ over $A \times A$ by setting $b = a$ and $\forall s \in S$: $p'_{(s^{\pm 1}, 0)} = t'_{s^{\pm 1}}$ and $p'_{(0, s^{\pm 1})} = \star$. The tileset created by this procedure, Π, satisfies $X_2 = X_\Pi$.

Finally, by definition of X_2 we have that is is empty if and only if X^Δ is empty. □

Let us move on to Step 2 and examine what are the critical steps to achieve the result on $\mathbb{Z}$, on how these could be generalized.

As previously mentioned, the Toeplitz subshifts S_p are used on this step because, excluding some details, the difference of two configurations of S_p are still in the shift. A first attempt at generalizing this aspect could be the use of G-odometers and G-Toeplitz arrays introduced by Cortez and Petite [CP08]. The difficulty with this approach lies within the following.

Let $x, y \in A^{\mathbb{Z}}$. We define the the difference of x and y by the configuration,

$$\mathrm{diff}(x, y)_n = \begin{cases} x_n & \text{if } x_n = y_n \\ \_ & \text{otherwise} \end{cases}.$$

The proof relies on being able to eliminate the symbol $\_$ from $\mathrm{diff}(x, y)$ and still obtain an element of the subshift. How to perform this operation is not evident in the case of a group.

In case we are able to recreate the proof, the conclusion obtained would be the many-one equivalence $\mathrm{OCDP}(G \times G) \equiv_m \mathrm{DP}(G \times G)$.

5.2 Expanding symmetries

Let us have two finitely generated groups G and H such that H acts on G by bijective functions. Let $\varphi : H \to S_G$ be the action and $X \subseteq (A \times A)^G$ a sofic subshift. For a configuration $x \in A^G$ we use the notation $\varphi_h(x)_g := x_{\varphi_h(g)}$. We define a new subshift X_2 by all the configurations $z \in (A \times A)^{G \times H}$ such that

$$z_{(g,h)} = (x_g, \varphi_h(y)_g),\ x, y \in A^G,$$

and for every $h \in H$, $(z_{(g,h)})_{g \in G}$ is a configuration in X. With this in hand we can say that X_2 is composed by pairs (x, y) of configurations such that for any $h \in H$, $(x, \varphi_h(y)) \in X$.

In addition, we must also introduce a generalized version of the distance shift, which will test the robustness of the configurations under a greater number of symmetries.

Definition 5.5. Let $X \subseteq (A \times A)^G$ be a shift and let $F \leq S_G$ be a subgroup of the group of bijective functions over G. The *symmetric distance shift with respect to F*, $X^{\Delta(F)}$ corresponding to X is the set of pairs $(x, y) \in A^G \times A^G$ such that $\forall \phi \in F$, $(x, \phi(y)) \in X$. In other words,

$$(x, y) \in X^{\Delta(F)} \iff \forall \phi, \psi \in F : (\phi(x), \psi(y)) \in X.$$

We define $\mathrm{Dist}_F(G)$ to be the problem of determining, given a sofic shift $X \subseteq (A \times A)^G$, whether $X^{\Delta(F)}$ is empty.

The original distance shift is recovered when we take the subset

$$\Sigma = \{\phi_g \in S_G \mid \phi_g(h) = g^{-1}h,\ \forall h \in G\}.$$

Remark 5.1. Notice that if we have two subgroups $F_1, F_2 \leq S_G$ such that $F_1 \subseteq F_2$, then $X^{\Delta(F_2)} \subseteq X^{\Delta(F_1)}$.

Proposition 5.6. *Let G and H finitely generated groups such that H acts on G by bijections. Then, we have that* $\mathrm{Dist}_{\varphi(H)}(G) \leq_m \mathrm{DP}(G \times H)$.

Proof. Let us have $G = \langle S \rangle$ and $H = \langle S' \rangle$. Suppose X is defined from a set of decorated Wang tiles τ, that is, $X = X_\tau$. There is a straight forward algorithmic procedure to generate a decorated tileset for X_2.

Let $\star$ be a fixed color. Take $t = (t', a) \in \tau$ and define $p = (p', b)$ a decorated Wang tile for $G \times H$ over $A \times A$ by setting $b = a$ and $\forall s \in S$: $p'_{(s^{\pm 1}, 0)} = t'_{s^{\pm 1}}$ and $\forall s \in S'$: $p'_{(0, s^{\pm 1})} = \star$. The tileset created by this procedure, Π, satisfies $X_\Pi = X_2$, because for every $h \in H$, $(z_{(g,h)})_{g \in G}$ is a configuration in X.

By our definition of X_2, it is straightforward that if and only if $X^{\Delta(\varphi(H))}$ is empty. □

Corollary 5.7. *For G and H two finitely generated groups,* $\mathrm{Dist}_{\phi(H)}(G) \leq_m \mathrm{DP}(G \ltimes_\phi H)$.

Proof. The proof is direct from the fact that a semidirect product comes from an action by automorphisms. $\phi : H \to \mathrm{Aut}(G)$. □

6 Comments on the Fixed-Point method

As hinted at before, if we want to generalize the proof strategy by Durand, Romashchenko and Shen to different groups, we would at least need two main ingredients:

1. Being able to connect the border of a connected support to its interior. This suggests, for example, that the proof might work on one-ended groups. These are groups such that removing any finite connected finite subgraph does not disconnect the Cayley graph of the group.

2. The ability to write computations through local constraints. In particular, we would need for the group to have decidable word problem in order to understand the structure of the ball of a given radius.

If the reader is aware of the literature surrounding the existence of strongly aperiodic SFTs, these conditions might seem familiar. Managing to successfully generalize the proof strategy could also mean that we manage to create strongly aperiodic SFT on groups. This is in accordance with important results in the area of aperiodicity.

Theorem 6.1 ([Coh17])**.** *Groups with two or more ends cannot have strongly aperiodic SFTs*

Theorem 6.2 ([Jea15a])**.** *If G has a strongly aperiodic SFT, then it has decidable word problem.*

In particular, this suggest that using this method to prove the undecidability of the domino problem may only work on groups that admit strongly aperiodic SFTs.

7 Baumslag-Solitar groups

Baumslag-Solitar groups were introduced as we know them now in [BS62], to provide examples of non-Hopfian groups, although cases of them were defined some years prior by Higman in [Hig51]. They are defined by the presentation:

$$\mathrm{BS}(m,n) = \langle a, b \mid a^m b = b a^n \rangle.$$

The first things to note are that $\mathrm{BS}(1,1) = \mathbb{Z}^2$ and $\mathrm{BS}(m,n) \simeq \mathrm{BS}(-m,-n)$. The main result of this section is the following Theorem due to Aubrun and Kari.

Theorem 7.1 ([AK13], [AK21])**.** *$\mathrm{DP}(\mathrm{BS}(m,n))$ is undecidable for all n, m.*

To prove this result, we begin by defining a projection from the group to the Euclidean plane $\Phi_{m,n} : \mathrm{BS}(m,n) \to \mathbb{R}^2$.

Note that any element in the group can be represented by a word over the alphabet $A = \{a, b, a^{-1}, b^{-1}\}$. This representation is not unique. For $w \in A^*$ and $x \in A$ we represent the number of occurrences of x in w by $|w|_x$. The contribution of x to w is defined as $\|w\|_x = |w|_x - |w|_{x^{-1}}$.

Definition 7.2. The *height coordinate* $\beta : A^* \to \mathbb{R}$ is defined as $\beta(w) = \|w\|_{b^{-1}}$. We also define the *displacement coordinate* $\alpha : A^* \to \mathbb{R}$ by induction over the length of words through the properties:

- $\alpha(\epsilon) = 0$,
- $\alpha(w.b) = \alpha(w.b^{-1}) = \alpha(w)$,
- $\alpha(w.a) = \alpha(w) + \left(\frac{m}{n}\right)^{-\beta(w)}$,
- $\alpha(w.a^{-1}) = \alpha(w) - \left(\frac{m}{n}\right)^{-\beta(w)}$.

The following lemma can be easily proven by induction.

Lemma 7.3. *$\forall u, v \in A^*$:*

$$\alpha(u.v) = \alpha(u) + \left(\frac{m}{n}\right)^{-\beta(u)} \alpha(v).$$

With this function at hand, we can proceed to define the projection. Let $g \in \mathrm{BS}(m,n)$ be an element represented by the word $w \in A^*$. Its projection is defined as,

$$\Phi_{m,n}(g) = (\alpha(w), \beta(w)).$$

Proposition 7.4. *$\Phi_{m,n}$ is well defined on $\mathrm{BS}(m,n)$.*

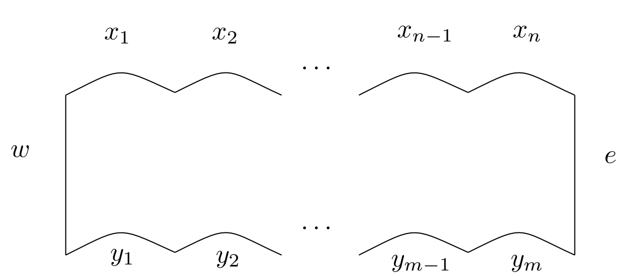
We will use tiles defined the following sequence of edges of the Cayley graph:

$$1 \to a \to a^2 \to \ \ldots\ \to a^m \to a^m b = b a^n$$

and

$$b \to ba \to ba^2 \to \ \ldots\ \to ba^{n-1} \to ba^n = a^m b.$$

We say one of this tiles computes an affine function $f : \mathbb{R}^2 \to \mathbb{R}^2$ if the colors satisfy the expression:



$$\frac{1}{n} f\left(\sum_{i=1}^{n} x_i\right) + w = \frac{1}{m}\sum_{i=1}^{n} y_i + e.$$

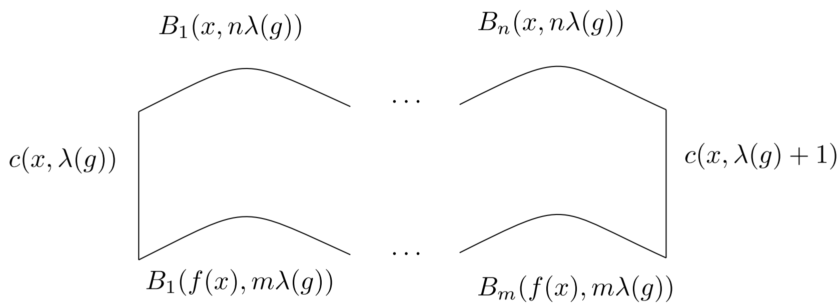
We define the translated balanced representation of $x \in \mathbb{R}^2$ as the bi-infinite sequence:

$$B_k(x, z) = \lfloor (z + k)\, x \rfloor - \lfloor (z + (k-1))\, x \rfloor .$$

To make use of the balanced representation, we define another function $\lambda : \mathrm{BS}(m, n) \to \mathbb{R}^2$ by

$$\lambda(g) = \frac{1}{m}\left(\frac{m}{n}\right)^{-\beta(g)} \alpha(g).$$

Let $f(x) = Mx + b$ be an affine rational map over $\mathbb{R}^2$. We define the tiles corresponding to f by



where,

$$c(x, z) = \frac{1}{n} f(\lfloor nzx \rfloor) - \frac{1}{m}(\lfloor mzf(x) \rfloor) + \left\lfloor z - \frac{1}{2} \right\rfloor b.$$

One can show that these tiles are consistent: $B_k(x, \lambda(g)) = B_{k-1}(x, \lambda(g.a))$.

Using the fact that $x - 1 \le \lfloor x \rfloor \le x$, we can show that for all $k \in \mathbb{Z}$ and $g \in \mathrm{BS}(m, n)$,

$$x - 1 \le B_k(x, z) \le x + 1.$$

Because x is taken in the domain of f, which is bounded, the values that $B_k(x, z)$ can take are finite independent of z. In addition, due to the same inequality we arrive at,

$$\frac{2 - 3n}{2n} b - \frac{1}{n} M \cdot 1 \le c(x, \lambda(g)) \le \frac{2 - n}{2n} b + \frac{1}{m}.$$

Because b and M are rational, we can take q to be the least common multiple between m and the denominators of the coordinates of both $\frac{1}{2n} b$ and $\frac{1}{n} M \cdot 1$, with which we have $p_1, p_2 \in \mathbb{Z}^2$ such that

$$\frac{p_1}{q} \le c(x, \lambda(g)) \le \frac{p_2}{q},$$

for every $g \in \mathrm{BS}(m, n)$. Finally, because of the numbers that appear in the expression of $c(x, \lambda(g))$, we can write it as p/q for $p \in \mathbb{Z}^2$. This means that $p_1 \le p \le p_2$ and therefore there are only a finite number of possibilities for the values of $c(x, \lambda(g))$. The case of $c(x, \lambda(g) + 1)$ is analogous.

The proof is concluded by showing that an immortal point exists for a system of piece-wise affine maps if and only if there is a tiling of the Cayley graph by the defined tileset.

8 Domino Problem for one-relator groups

8.1 Preliminaries

It is natural to try and extend the Baumslag-Solitar case to the more general class of one-relator groups. As the name suggests, these are finitely generated groups with only one relation $G_w = \langle S \mid w \rangle$, where $w \in F_S$.

This class of groups contain many interesting groups such as the surface groups for both oriented and non-oriented manifolds or special cases of Artin groups. Furthermore, by virtue of Prop 4.5 the study of one-relator groups allows us to we have the following proof structure:

Lemma 8.1. *Let $\Sigma(w)$ be the set of letters appearing in the word w. Let us have a finitely presented group $G = \langle S \mid w_1, ..., w_n \rangle$, such that $\Sigma(w_1) \neq \Sigma(w_j)$ for all $j \neq 1$. Then, there is a many-one reduction* $\mathrm{DP}(\langle \Sigma(w_1) \mid w_1 \rangle) \leq_m \mathrm{DP}(G)$.

It is in fact this proof structure that will allow us to prove that all non-free Artin groups have undecidable domino problem in Section 8.5. Another example is the case of the Heisenberg group, defined by the presentation

$$\mathcal{H} = \langle x, y, z \mid [x, z], [y, z], [x, y]z^{-1} \rangle,$$

which has an undecidable domino problem because it contains the one-relator subgroup $\mathbb{Z}^2 = \langle x, z \mid [x, z] \rangle$. Furthermore, Higman's group, defined by the presentation,

$$\langle a, b, c, d \mid a^{-1}bab^{-2}, b^{-1}cbc^{-2}, c^{-1}dcd^{-2}, d^{-1}ada^{-2} \rangle,$$

has undecidable domino problem becuase the one-relator group $\langle a, b \mid a^{-1}bab^{-2} \rangle \simeq \mathrm{BS}(1, 2)$ has.

Here we will focus on the case of 2-generated 1-relator groups.

8.1.1 Magnus' Freiheitsatz

Statements about properties of one-relator groups can be proved by induction on the length of the relator through a technique introduced by Magnus in [Mag30]. This is done in the proof of the following theorem:

Theorem 8.2 (Freiheitsatz)**.** *Let $G = \langle S \mid w \rangle$ be a one-relator group and let $T \subseteq S$ be such that w cannot be written as a word in T. Then, T is a basis for a free subgroup of G.*

In the proof, he shows that any one-relator group can be expressed as the HNN-extension of another one-relator group. The same techniques can be applied to prove the following theorems.

Theorem 8.3 (Magnus, [Mag32])**.** *The word problem is decidable in all one-relator groups.*

Theorem 8.4 (Karras, Solitar, Magnus, [KMS60])**.** *For some $k \geq 1$, let $G = \langle S \mid w^k \rangle$ be a one-relator group such that $r \in F_S$ is not a proper power. Then $\bar{r} \in G$, where $r =_G \bar{r}$ has order k and every torsion element of G is conjugate to $\bar{r}$.*

We note the following result which is a consequence of Newman's Spelling Theorem, which is proved by the same scheme.

Theorem 8.5 ([New68])**.** *All one-relator groups with torsion are hyperbolic.*

8.2 Presentation Symmetries

There are a number of operations one can apply on the unique relation that can leave the group unchanged. The first of these is the fact that one can choose a cyclically reduced word.

Lemma 8.6. *Let $\langle S|R\rangle$ be a group presentation. For all $\phi \in \mathrm{Inner}(F_S)$, that is, there exists $g \in F_S$ such that $\phi(h) = ghg^{-1}$, we have that $\langle S|R\rangle = \langle S|\phi(R)\rangle$.*

In particular, for a word $w = w_1 w_2 \ldots w_n$, applying a conjugation by w_n corresponds to cyclically permuting the letters in the relation.

More symmetries can be identified from the fact that the generators are "mute":
Let us define an substitution $\tau_a : F_S \to F_S$, where $a \in S$ as

$$\tau_a : \begin{cases} a \mapsto a^{-1} \\ s \mapsto s, \ \forall s \neq a \end{cases}$$

In addition, let $\mu : F_S \to F_S$ defined by $\mu(w) = w^{-1}$. Also, for a permutation $\sigma \in S_n$, let $P_\sigma : F_S \to F_S$ be such that $P_\sigma(s_i) = s_{\sigma(i)}$. Applying these functions does not change the group.

Lemma 8.7. *Let $\varphi \in \langle \{\tau_a\}_{a\in S} \cup \{\mu\} \cup \{P_\sigma\}_{\sigma \in S_n}\rangle$. Then, $G_w \simeq G_{\varphi(w)}$.*

There are further isomorphisms we can find between the groups defined by different words that do not depend on the words explicitly. We do this through Tietze transformations of the presentation of the groups.

Lemma 8.8. *Let $w \in S^*$ be cyclically reduced and $\tau_{a,b}$ the substitution defined by*

$$\tau_{a,b} : \begin{cases} a \mapsto ab^{-1} \\ s \mapsto s, \ \forall s \neq a \end{cases},$$

where $a, b \in S$ are distinct generators. Then $G_w \simeq G_{\tau_{a,b}(w)}$.

Although this applies for any finite set of generators, the proof is done for the case $|S| = 2$.

Proof. Let us decompose our word in the following way,

$$w = a^{n_1}(ab)^{m_1}b^{k_1} \circ \ \ldots \ \circ a^{n_N}(ab)^{m_N}b^{k_N},$$

where $n_i, m_i, k_i \in \mathbb{Z}$. By applying $\tau_{a,b}$ we arrive at,

$$\tau_{a,b}(w) = (ab^{-1})^{n_1}a^{m_1}b^{k_1} \circ \ \ldots \ \circ (ab^{-1})^{n_N}a^{m_N}b^{k_N}$$

We prove the isomorphism through Tietze transformations:

$$\begin{aligned} \langle a, b \mid a^{n_1}(ab)^{m_1}b^{k_1} \circ \ \ldots \ \circ a^{n_N}(ab)^{m_N}b^{k_N}\rangle &\simeq \langle a, b, c \mid c = ab, \ a^{n_1}(ab)^{m_1}b^{k_1} \circ \ \ldots \ \circ a^{n_N}(ab)^{m_N}b^{k_N}\rangle, \\ &\simeq \langle a, b, c \mid c = ab, \ (cb^{-1})^{n_1}c^{m_1}b^{k_1} \circ \ \ldots \ \circ (cb^{-1})^{n_N}c^{m_N}b^{k_N}\rangle, \\ &\simeq \langle b, c \mid (cb^{-1})^{n_1}c^{m_1}b^{k_1} \circ \ \ldots \ \circ (cb^{-1})^{n_N}c^{m_N}b^{k_N}\rangle, \\ &\simeq \langle a, b \mid (ab^{-1})^{n_1}a^{m_1}b^{k_1} \circ \ \ldots \ \circ (ab^{-1})^{n_N}a^{m_N}b^{k_N}\rangle. \end{aligned}$$

□

8.2.1 Closed HNN-extensions

Let denote $\mathcal{G}$ the set of all 2-generated 1-relator groups, and define two operations $H : \mathcal{G} \to \mathcal{G}$ and $h : F_2 \to F_2$ in the following way.

For $G = \langle a, b \mid w \rangle \in \mathcal{G}$, we define the isomorphism $\phi : \langle a \rangle \to \langle b \rangle$ by $\phi(a) = b$, and $H(G) = G*_\phi$. The image of w by h will be the relator of the image by H:

$$H(G) = \langle a, b \mid h(w) \rangle.$$

Note that h can be defined as the substitution:

$$h : \begin{matrix} a & \mapsto & a \\ b & \mapsto & bab^{-1} \end{matrix}.$$

Lemma 8.9. *H is well defined and* $\mathrm{DP}(G) \leq_m \mathrm{DP}(H(G))$*.*

Proof. If we take $G = \langle a, b \mid w \rangle \in \mathcal{G}$,

$$\begin{aligned} H(G) = G*_\phi &= \langle a, b, t \mid w, b = tat^{-1} \rangle = \langle a, b, t \mid \hat{h}(w), b = tat^{-1} \rangle \\ &\simeq \langle a, t \mid h(w) \rangle \in \mathcal{G}, \end{aligned}$$

where $\hat{h}(w)$ is the word obtained by replacing every instance of b in w by tat^{-1}. Because G embeds into its HNN-extension $H(G)$, we have that $\mathrm{DP}(G) \leq_m \mathrm{DP}(H(G))$. $\square$

It is also straightforward to see that, $h(w^k) = h(w)^k$ for all $k \in \mathbb{Z}$ and $H(F_2) = F_2$.

Example 8.1. $H(\mathbb{Z}^2) = \langle a, b \mid a^2(ab)^{-1}b^2(ba)^{-1} \rangle$ has undecidable word problem.

Question 1. Does there exist $N > 0$ such that for almost all $G_w \in \mathcal{G}$ with $|w| > N$ there is a $G' \in \mathcal{G}$ such that $H(G') = G_w$?

We can make a necessary condition inspired by Magnus' Freiheitsatz. Let us define the homomorphism $\sigma_a : F_2 \to \mathbb{Z}$ by

$$\sigma_a(s) = \begin{cases} 1 & \text{if } s = a \\ 0 & \text{if } s = b \end{cases}$$

We analogously define σ_b.

Lemma 8.10. *Let $G, G' \in \mathcal{G}$ be two groups. If $G = H(G') = \langle a, b \mid w \rangle$, then $\sigma_b(w) = 0$, up to exchanging a and b in the presentation.*

Proof. Let w' be the word such that $w = h(w')$. Then, every occurence of b in w is followed by ab^{-1}. Therefore, $\sigma_b(w) = 0$. $\square$

Note that this particular HNN-extension can be applied to n-generated groups:

If we denote the set of all n-generated 1-relator groups by $\mathcal{G}_n$, we define $H_{a,b}$ and $h_{a,b}$ analogously. Let $G = \langle S \mid w \rangle \in \mathcal{G}_n$ and $a, b \in S$ distinct. Define $\phi : \langle a \rangle \to \langle b \rangle$ by $\phi(a) = b$, and $H_{a,b}(G) = G*_\phi \in \mathcal{G}_n$. In this case, $h_{a,b}$ is the substitution,

$$h_{a,b} : \begin{cases} b \ \mapsto \ bab^{-1} \\ s \ \mapsto \ s, \ \forall s \neq b \end{cases}.$$

Then for all distinct $a, b \in S$, $\mathrm{DP}(G) \leq_m \mathrm{DP}(H_{a,b}(G))$.

8.3 Results

The main result is a classification of the domino problem on groups defined by words with a particular structure.

Proposition 8.11. *Let $n, m, k \in \mathbb{Z}$ with $n \neq 0$. Let $w = x^n y^m x^k$ be a word where $x, y \in \{a, b, a^{-1}, b^{-1}\}$. Then* $\mathrm{DP}(G_w)$ *is decidable if and only if* $(m \in \{0, 1\} \ \wedge \ n + k = \pm 1)$.

This Proposition allows us to find an alternative proof of the undecidability of the domino problem for Baumslag-Solitar groups.

Corollary 8.12. $\mathrm{BS}(m, n)$ *has undecidable domino problem.*

Proof. By using the previous proposition we have that $\langle a, b \mid a^m b^{-n} \rangle$ has undecidable domino problem. By using Lemma 8.9 along with the fact that

$$H(\langle a, b \mid a^m b^{-n} \rangle) = \mathrm{BS}(m, n),$$

we conclude. □

It also allows us to see where the undecidability appears in terms of the length of the defining word.

Corollary 8.13. *Let $w \in \{a, b, a^{-1}, b^{-1}\}^*$ be cyclically reduced. If $|w| \leq 3$, then* $\mathrm{DP}(G_w)$ *is decidable.*

Proof. We do this by exhausting all possible cases with the additional aid of Lemmas 8.6 and 8.7.

- For $n = 1$ we just have $w = a$, and therefore $G_w \simeq \mathbb{Z}$.
- For $n = 2$ we have the two possibilities $w_1 = ab$ and $w_2 = a^2$. By Lemma 8.14 we know $G_{w_1} \simeq \mathbb{Z}$ has decidable domino problem and by Prop. 8.11, $G_{w_2} \simeq \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ also does.
- For $n = 3$ our possibilities are $w_1 = ab^2$, $w_2 = a^3$, exactly as above, with $G_{w_2} \simeq \mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. All other possibilities are exhausted by the before mentioned lemmas.

□

When $|w| = 4$ we get the first appearance of groups with undecidable domino problem problem. These words are (modulo variants by Lemmas 8.6 and 8.7):

- $w = aba^{-1}b^{-1}$: In this case $G_w = \mathbb{Z}^2$, which is the original Domino Problem.
- $w = a^2b^2$ and $w = abab^{-1}$: Both these words define the same group, namely the fundamental group of the Klein Bottle, $\pi_1(K)$. Which by Prop 8.11 also has undecidable domino problem.

The other possibilities for words of length four are: a^4 defining $\mathbb{Z} * \mathbb{Z}/4\mathbb{Z}$, $(ab)^2$ defining $\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and ab^3 defining $\mathbb{Z}$.

8.3.1 Proof of Prop 8.11

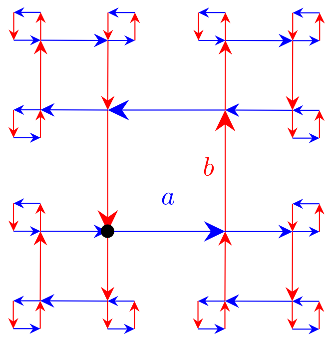
Let us begin by looking at the simplest case.

Lemma 8.14. *For every $n \in \mathbb{Z}$, the groups defined by the word ab^n has a decidable word problem.*

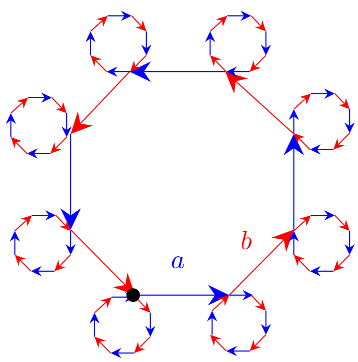
Proof. For $w = ab^n$, we have that $G_w \simeq \mathbb{Z} = \langle 1, -n \rangle$. □

If we increase the complexity of the word, we begin having more elaborate Cayley graphs, as can be seen in the following examples.

- $w = (ab)^2$



- $w = (ab)^3$



Remark 8.1. As we can see, the Cayley graph of the group defined by the word $(ab)^n$ is composed of $2n$-gons with a $2n$-gon on each vertex.

To show that the groups defined by this family of words have decidable domino problem, we will look at a more general class whose Cayley graphs can be seen as sticking two alternating geometric figures.

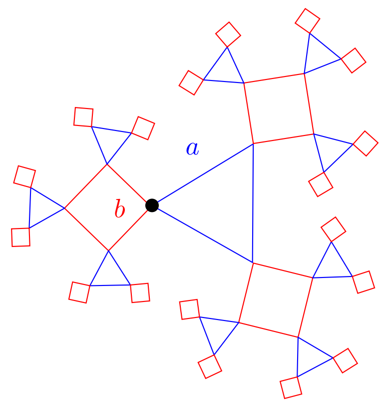
Definition 8.15. Let $m, n \in \mathbb{N} \setminus \{0\}$. We define the group,

$$G_{n,m} = \langle a, b \mid a^n, b^m \rangle = \mathbb{Z}/n\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}.$$

Some of these groups are well know. They are a particular case of triangle groups defined by the tuple (n, m, ∞). When $n = 2$, the groups are called Hecke groups. One notable example is the Modular group $\mathrm{PSL}(2, \mathbb{Z}) = G_{2,3}$.

Remark 8.2. The Cayley graph of the group $G_{n,m}$ can be generated by iterating the following process: take an n-gon, and adding an m-gon to each vertex. We take 2-gons to be a wedge with two edges.

For example, here is a section of the Cayley graph for $G_{3,4}$:



We use the following well known result:

Proposition 8.16. *If A and B are finite groups, then $A * B$ contains a normal subgroup of index $|A| \cdot |B|$ that is free of rank $(|A|-1)(|B|-1)$.*

Using this Proposition in conjunction with Theorem 4.16, we obtain the following corollary.

Corollary 8.17. $\mathrm{DP}(G_{n,m})$ *is decidable for all $n, m \geq 1$.*

We can therefore determine the decidability of the domino problem for this particular case.

Lemma 8.18. *Let $n \geq 2$. Then the groups defined by the words $(ab)^n$ and a^n have decidable word problem.*

Proof. We begin by noting that the group defined by a^n and $(ab)^n$ are in fact isomorphic. This can be seen through Lemma 8.8 because $\tau_{a,b}((ab)^n) = a^n$.

Now, let $w = (ab)^n$. The result follows from Theorem 4.11, the previous Corollary and the fact that $G_w \sim_{QI} G_{2n,2n}$ (in fact, they are isometric). □

Lemma 8.19. *Let $n, m \geq 1$. Then the group defined by the word $w = (ab^n)^m$ has a decidable domino problem.*

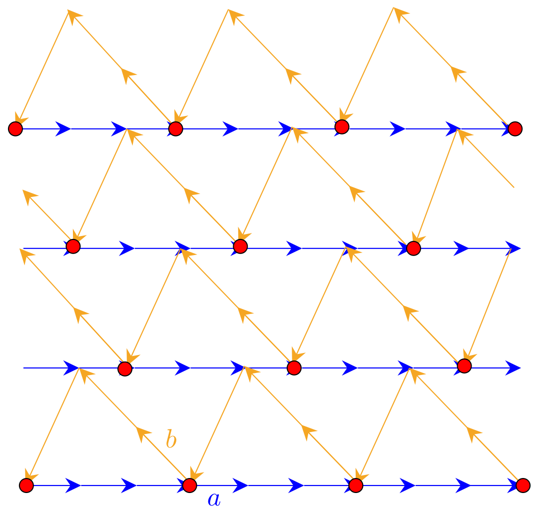
Proof. From Lemma 8.8, we have that $\tau^n_{a,b}\left((ab^n)^m\right) = a^m$. We conclude by Lemma 8.18. □

The final piece of the puzzle are words of the form $a^n b^m$.

Lemma 8.20. *Let $n, k \geq 1$ and $m \geq 2$. Then, the group defined by the word $a^n b^m a^k$ has undecidable world problem.*

Proof. We begin by noticing that due to Lemma 8.6, the group defined by the word $a^n b^m a^k$ is the same as the one defined $a^{n+k} b^m$. Therefore, we can consider words of the form $a^n b^m$ for $n, m \geq 2$.

Let us begin by looking at the case $a^3 b^3$. Take a look at a slice of its Cayley graph:



In this slice we can see that the vertices in red define a lattice. Formally, the subgroup generated by a^3 and $b^{-1}a$ is isomorphic to $\mathbb{Z}^2$.

For the general case, let $H = \langle a^n, b^{-1}a\rangle$. We have that $a^n \notin \langle b^{-1}a\rangle$ and

$$\begin{aligned}[a^n, b^{-1}a] &= a^n b^{-1} a a^{-n} a^{-1} b = a^n b^{-1} a^{-n} b \\ &= a^n b^{-1} b^m b = a^n b^m = 1_G.\end{aligned}$$

Thus, $H \simeq \mathbb{Z}^2$. By Prop 4.5, we conclude. □

Remark 8.3. The group defined by $a^n b^m$ can be seen as the amalgamated free product $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$ with the injections $1 \mapsto n$ and $1 \mapsto -m$.

Proof of Prop 8.11. Due to Lemmas 8.6 and 8.7, we can take $x = a$ and $y = b$. If $m = 0$, we have $w = a^{n+k}$, which by Lemma 8.18 means that $\mathrm{DP}(G_w)$ is decidable. If $m = 1$ or $n+k = \pm 1$, we have the word in Lemma 8.14. All other cases are undecidable due to Lemma 8.20. □

8.3.2 Further classification

For words with four "terms" we have a partial classification.

Lemma 8.21. *Let $n, m, k, p \in \mathbb{Z}\backslash\{0\}$. Let $w = x^n y^m x^k y^p$ be a word where $x, y \in \{a, b, a^{-1}, b^{-1}\}$. Then, we have the following:*

- *If $(m = -p)$, $(n = -k)$, $(n = k \;\wedge\; |m - p| > 1)$ or $(m = p \;\wedge\; |n - k| > 1)$, then $\mathrm{DP}(G_w)$ is undecidable.*
- *If $(n = k = 1 \wedge m = p)$ or $(m = p = 1 \wedge n = k)$, then $\mathrm{DP}(G_w)$ is decidable.*

Proof. Let us begin by $(m = -p)$ and $(n = -k)$. These are equivalent through $P_{(1\ 2)}$, as defined in Lemma 8.7, so we only have to consider the first case. In this case, $w = a^n b^m a^k b^{-m}$. It is easy to see that the subgroup generated by $\langle a, b^m\rangle$ is isomorphic to $\mathrm{BS}(n, -k)$. We conclude by Prop 4.5 and Theorem 7.1.

Once again $(n = k \;\wedge\; |m - p| > 1)$ and $(m = p \;\wedge\; |n - k| > 1)$ equivalent through $P_{(1\ 2)}$. We have $w = a^n b^m a^n b^p$. Let $H = \langle a^n, b\rangle$. By the defining relation we have that

$$H \simeq \langle a, b \mid ab^m ab^p\rangle.$$

By applying Lemma 8.8 this group can also be presented as $\langle a, b \mid a^2 b^{p-m} \rangle$, which as we saw on Lemma 8.20 has an undecidable word problem. We conclude by Prop 4.5.

Finally, for $(n = k = 1 \wedge m = p)$ and $(m = p = 1 \wedge n = k)$ we have equivalence through S. Then, the group defined by $(ab^m)^2$ has decidable problem by Lemma 8.19. $\square$

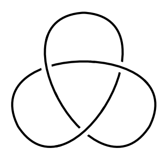
The remaining cases for a complete classification in this context are $(n = k \wedge m = p)$, that is, $w = (a^n b^m)^2$ and $(n \neq \pm k \wedge m \neq \pm p)$.

8.4 Knot Groups

Definition 8.22. A *knot* K is an embedding of the 1-sphere $\mathbb{S}^1$ in the 3-dimensional Euclidean space $\mathbb{R}^3$. A *knot group* is the fundamental group given by the (open) complement of the knot, $\pi_1(\mathbb{R}^3 \setminus K)$.

Knot groups are of interest because they are knot invariants, that is, two equivalent knots have isomorphic knot groups. As we will later see, a particular class of knot groups can be seen as part of a larger class that generalizes Baumslag-Solitar groups.

An example of a knot, the trefoil knot, is the following:



which has the knot group given by $\langle a, b \mid a^2 b^3 \rangle$.

Proposition 8.23. *The knot groups for the (p, q)-torus knots have undecidable domino problem. In particular, the knot group of the trefoil knot has undecidable domino problem.*

The presentation of the knot group of the (p, q)-torus knot is $\langle a, b \mid a^p b^{-q} \rangle$. We can see that the trefoil know is the particular case when $p = 2$ and $q = -3$. This groups have undecidable domino problem by virtue of Proposition 8.11.

It remains to see if the knot group of the figure eight knot has undecidable domino problem. Its presentations is:

$$\langle a, b \mid bab^{-1}ab = aba^{-1}ba \rangle.$$

Although this group at a first glance looks like a closed HNN-extension, by virtue of Lemma 8.10 we can see that it is not. Specifically, we have

$$\sigma_b(bab^{-1}aba^{-1}b^{-1}ab^{-1}a^{-1}) = -1,$$

and

$$\sigma_a(bab^{-1}aba^{-1}b^{-1}ab^{-1}a^{-1}) = 1.$$

8.5 Artin Groups

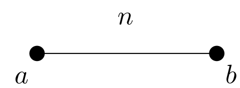
Let us introduce a class of groups called Artin groups. These groups where first introduced by Emil Artin and Jacques Tits as a natural generalization of braid groups [Art47; Tit66].

Let $\Gamma = (V, E, \lambda)$ be a labeled graph with labels $\lambda : E \to \{2, 3, ...\}$. We define the *Artin group* of Γ through the presentation:

$$A(\Gamma) := \langle V \mid \underbrace{abab...}_{\lambda(e)} = \underbrace{baba...}_{\lambda(e)},\ \forall e = (a,b) \in E \rangle.$$

Let us call Γ_n be the graph of 2 vertices a and b and the edge connecting them labeled by n.

Γ_n :



Notice that $A(\Gamma_2) \simeq \mathbb{Z}^2$.

Proposition 8.24. *All non-free Artin groups have undecidable domino problem.*

Proof. Let $A(\Gamma)$ be an Artin group defined from $\Gamma = (V, E, \lambda)$ and $e = (a, b) \in E$. Notice that $A(\Gamma_n) \simeq \langle a, b \rangle \leq A(\Gamma)$. This means that by Prop. 4.5, it suffices to show the undecidability for $A(\Gamma_n)$ for every $n \in \mathbb{N}$. We have two cases:

- Case 1: $n = 2k$, $k \geq 1$.

  We have that $A(\Gamma_{2k})$ is the 2-generated 1-relator group:

  $$A(\Gamma_{2k}) = \langle a, b \mid (ab)^k = (ba)^k \rangle = \langle a, b \mid (ab)^k = b(ab)^{k-1}a \rangle.$$

  By applying Lemma 8.8, this group is also presented by

  $$A(\Gamma_{2k}) \simeq \langle a, b \mid a^k = ba^k b^{-1} \rangle = \langle a, b \mid a^k ba^{-k} b^{-1} \rangle.$$

  Because $|k - 1 + 1 + k| = 2k > 1$, we conclude by Lemma 8.21 that the domino problem for $A(\Gamma_{2k})$ is undecidable.

- Case 2: $n = 2k + 1$, $k \geq 1$.

  Once again, $A(\Gamma_{2k+1})$ is the 2-generated 1-relator group:

  $$A(\Gamma_{2k+1}) = \langle a, b \mid (ab)^k a = (ba)^k b \rangle = \langle a, b \mid (ab)^k a = b(ab)^k \rangle.$$

  By applying Lemma 8.8 we arrive at

  $$A(\Gamma_{2k+1}) \simeq \langle a, b \mid a^{k+1} b^{-1} = ba^k \rangle = \langle a, b \mid a^k ba^{-(1+k)} b \rangle.$$

  Due to the fact that $|k - (-1 - k)| = 2k + 1 > 1$, by Lemma 8.21, $A(\Gamma_{2k+1})$ has undecidable domino problem.

This concludes our proof.

$\square$

Corollary 8.25. *All Right-Angled Artin groups and all Braid groups have undecidable domino problem.*

9 Graph of Groups

Definition 9.1. A *graph of groups* $(\Gamma, \mathcal{G})$ is a connected graph Γ, along with a collection of groups and monomorphisms $\mathcal{G}$ that includes:

- a vertex group G_v for each $v \in V_\Gamma$,
- an edge group G_e for each $e \in E_\Gamma$, where $G_e = G_{\bar{e}}$,
- a set of injections $\{\alpha_e : G_e \to G_{\mathfrak{t}(e)} \mid e \in E_\Gamma\}$, where $\mathfrak{t}(e)$ is the terminal vertex of e.

The main interest of theses objects is their fundamental group. This group is an extension of the definition of the fundamental group for graphs. Luckily there is a result that gives us an explicit expression for the fundamental group, which allows us to skip the formal definition. A complete treatment of the concept can be found in [Lym20].

Theorem 9.2. *Let $T_0 \subseteq \Gamma$ be a spanning tree. The group $\pi_1(\Gamma, \mathcal{G}, T_0)$ is isomorphic to a quotient of the free product of the vertex groups, with the free group on the set E_Γ of oriented edges. That is,*

$$\mathop{\Large *}_{v \in V_\Gamma} G_v * F(E_\Gamma)/R,$$

where R is the normal closure of the subgroup generated by the following relations

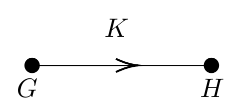
- $\alpha_{\bar{e}}(h)e = e\alpha_e(h)$*, where e is an oriented edge of Γ, $h \in G_e$,*
- $\bar{e} = e^{-1}$*, where e is an oriented edge of E_Γ,*
- $e = 1$ *if e is an oriented edge of T_0.*

We will omit both $\mathcal{G}$ and T_0 if the context allows it. As mentioned before, a proof of this Theorem can be found in [Lym20]. Even better, the fundamental group does not depend on the spanning tree.

Proposition 9.3 ([Ser80])**.** *The fundamental group of a graph of groups does not depend on the spanning tree.*

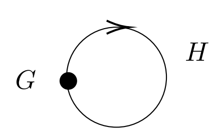
Let us look at some examples of fundamental groups of graph of groups. First, we have traditional operations of geometric group theory viewed in this light. The amalgamated free product $G *_K H$ is viewed as the fundamental group $\pi_1(\Gamma_1)$

Γ_1



Similarly, an HNN extension $G*_\phi$ can be seen as the fundamental group $\pi_1(\Gamma_2)$:

Γ_2



$\alpha_e = \mathrm{id}$ and $\alpha_{\bar{e}} = \phi$. In this sense, the concept of graph of groups can be seen as the natural generalization of these concepts.

The following result characterizes virtually free groups through their graph of groups. It is possible that this opens an alternative way to approach the Domino Conjecture.

Theorem 9.4 ([Bas93]). *A group G is virtually free if and only if G is the fundamental group of a finite graph of groups with finite vertex groups of bounded order.*

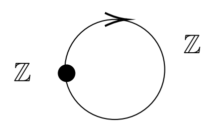
For example we can see that the group $\mathrm{SL}(2,\mathbb{Z}) = \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$ is virtually free as we see from graph Γ_1.

With this formalism we can prove that a class of groups that contain the Baumslag-Solitar groups, have all undecidable domino problem.

Definition 9.5. A group G is said to be a *Generalized Baumslag-Solitar group* (GBS) if it is the fundamental group of a finite graph of groups where all the vertex and edge groups are $\mathbb{Z}$.

These groups where introduced as a natural extension of the class of Baumslag-Solitar groups and Torus knot groups. Usual Baumslag-Solitar groups have the following graph of groups:

Γ_{mn}

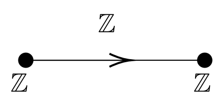


where $\alpha_e(1) = m$ and $\alpha_{\bar{e}}(1) = n$. This gives,

$$\mathrm{BS}(m,n) = \pi_1(\Gamma_{mn}).$$

The other basic example is given by the knot groups introduced in the previous section,

Λ_{mn}



where $\alpha_e(1) = m$ and $\alpha_{\bar{e}}(1) = -n$. This time we have,

$$\pi_1(\Lambda_{mn}) = \langle a, b \mid a^m b^n \rangle.$$

From the previous section we know that both these types of groups have undecidable domino problem. Because both these graphs are the building blocks for the GBS's graph of groups (in conjunction with Prop 4.5), we arrive at the result.

Proposition 9.6. *All GBS groups non-isomorphic to $\mathbb{Z}$ have undecidable domino problem.*

Proof. Let G be a GBS with its corresponding graph of groups Γ. Because G is not $\mathbb{Z}$, at least one edge, $e \in E_\Gamma$, satisfies $\alpha_e \not\equiv \pm 1$. If this edge is a loop, from the previous remarks we know that G contains a non-$\mathbb{Z}$ Baumslag-Solitar group and we are done. Similarly, if the edge is in the spanning tree $T \subseteq \Gamma$ such that $G = \pi_1(\Gamma, T)$, then G contains a knot group as mentioned in the previous remarks.

The last case is when all edges in the spanning tree satisfy $\alpha_{e'} \equiv \pm 1$, and there are no loops. Let a and b be the two vertices of e. Because T is spanning, we know that $a, b \in V_T$ and therefore in G, $a = b^{\pm 1}$. Then, the relation given by the edge e is,

$$a^{\alpha_e(1)} e = e b^{\alpha_{\bar{e}}(1)} \iff a^{\alpha_e(1)} e = e a^{\pm \alpha_{\bar{e}}(1)}.$$

This means G contains the non-$\mathbb{Z}$ Baumslag Solitar group $\mathrm{BS}(\alpha_e(1), \pm\alpha_{\bar{e}}(1))$, which by virtue of Theorem 7.1, concludes our proof. $\square$

Remark 9.1. As we see in the last part of the proof, we can in fact contract all edges such that $\alpha_e \equiv \pm 1 \equiv \alpha_{\bar{e}}$ into a single vertex without changing the fundamental group of the graph of groups.

9.1 Adyan-Rabin Theorem

By using the concepts from graph of groups there have been numerous results establishing the decidability of properties of groups.

Definition 9.7. A *group property* $\mathcal{P}$ is a map ϕ from the set of all groups to the set $\{0, 1\}$, such that if $G_1 \simeq G_2$, then $\phi(G_1) = \phi(G_2)$.

A property $\mathcal{M}$ of finitely presented groups is *Markovian* if the following conditions are satisfied:

1. $\phi \neq 0$, i.e. some finitely presented group satisfies $\mathcal{M}$,
2. There exists a finitely presented group that does not embed into any finitely presented group that satisfies $\mathcal{M}$.

Examples of Markov group properties include being abelian, being virtually free, being simple, being torsion free among others.

Proposition 9.8. *The property $\mathcal{M}$ of finitely presented groups such that $\mathcal{M}(G) = 1$ if and only if $\mathrm{DP}(G)$ is decidable, is a Markov group property.*

Proof. It is clear that $\mathcal{M}$ is a group property. Because of Theorem 4.16, we know that any virtually free group satisfies property $\mathcal{M}$, so the first condition is satisfied. Finally, we cannot embed $\mathbb{Z}^2$ into any group with decidable domino problem due to Prop. 4.5. Therefore, $\mathcal{M}$ is a Markov group property. $\square$

Theorem 9.9 (Adyan [Ady55], Rabin [Rab58])**.** *There is no algorithm that decides if a finitely presented group satisfies a given Markov property.*

Corollary 9.10. *There is no algorithm that decides if a finitely presented group has decidable domino problem.*

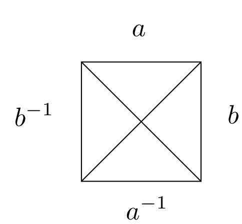
The proof of the Adyan-Rabin Theorem relies heavily on the existence of a finitely presented group with undecidable word problem, as given by the Novikov-Boone Theorem:

Theorem 9.11 (Novikov [Nov58], Boone [Boo59])**.** *There exists some finitely presented group G such that the word problem for G is undecidable.*

This theorem, in turn, is a consequence of Higman's Embedding Theorem and the existence of recursively enumerable sets that are not recursive.

10 The dual problem

Notice that if we take from Definition 5.3 the concept of Wang tiles on groups, locally the tiles for an n-generated group are all the same. For example, for all 2-generated groups they look like this:



Question 2. Is it possible to find a set of Wang tiles τ that tile $\mathbb{Z}^2$ but not F_2, or vice versa?

Question 3. Given a set of Wang tiles for n-generated groups, is there an algorithm to determine what groups can be tiled by the tileset?

We can say some things about this question. To cement the ideas we will make use of the concept of marked groups. Let us fix $n \geq 2$.

Definition 10.1. A *marked group* consists of (G, S) where G is a group with a prescribed family $S = (s_1, ..., s_n)$ of generators. It is important that the family is ordered, and that repetitions can occur.

This notion of marked group has its own notion of isomorphism.

Definition 10.2. We say two marked groups $(G, (s_1, ..., s_n))$ and $(G', (s'_1, ..., s'_n))$ are *isomorphic* if the morphism defined on the generators, $s_i \mapsto s'_i$ extend to a group isomorphism. The set of all n-marked groups $\mathcal{G}_n$ is the set of n-marked groups up to marked group isomorphism.

Remark 10.1. Notice that isomorphic groups can be non-isomorphic when viewed as marked groups. For example, $(\langle a, b \mid a = 1\rangle, (a, b))$ and $(\langle a, b \mid b = 1\rangle, (a, b))$ are non-isomorphic as marked groups.

A marked group (G, S) has a natural labeling of the edges of its Cayley graph given by $\{1, ..., n\}$.

There are two alternative ways to see the set $\mathcal{G}_n$:

- Epimorphisms: Consider the free group F_n marked by the free basis $(s_1, ..., s_n)$. We know that generated sets of cardinality n of a group G are in one-to-one correspondence with an epimorphism $h : F_n \to G$.
- Normal subgroups: $\mathcal{G}_n$ may be seen as the set of normal subgroups of F_n. Each normal subgroups represents the kernel of an epimorphism, namely, the natural epimorphism to the quotient.

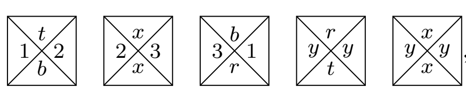
10.1 Tilings

We define a Wang tile for a finitely generated group G in Definition 5.3. Because we are using marked groups, there is no ambiguity in the definition of Wang tiles, they work for any n-marked group. Given a tileset τ as defined, we denote the set of groups tileable by τ as $\mathrm{TP}(\tau) \subseteq \mathcal{G}_n$.

We have an alternative way to see this problem from a graph theory perspective.

Definition 10.3. Let τ be a tileset. We define the *tileset graph* $\Gamma_\tau = (V, E)$ as a labeled graph, labeled by a set of n-generators S where $V = \tau$ and there is an edge labeled by s from t_1 to t_2 if $(t_1)_s = (t_2)_{s^{-1}}$.

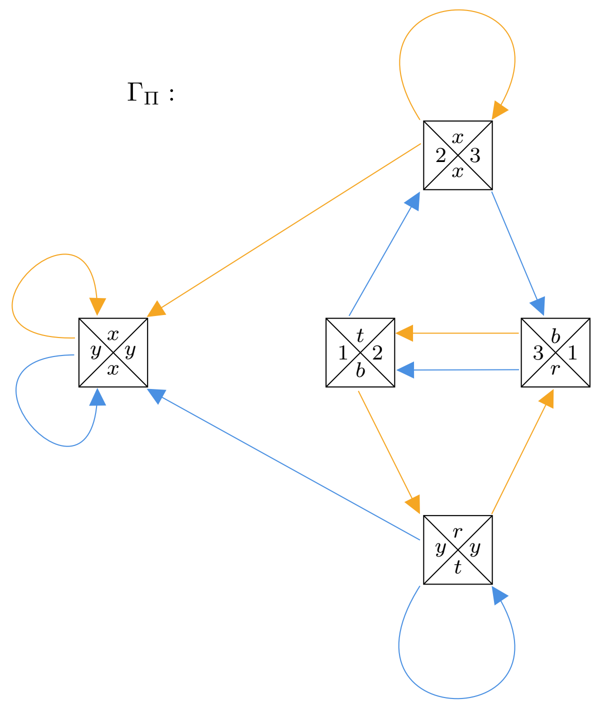
An example of a tileset graph is the following. If we have the 2-tileset Π given by



,

then the corresponding graph is,

Γ_Π :



where the blue edges represent the first generator and the orange ones the second generator. One can check that $\langle a, b \mid a^3b^3\rangle \in \mathrm{TP}(\Pi)$.

With this formalism the set $\mathrm{TP}(\tau)$ can be seen as all groups whose Cayley graph has a graph morphism to Γ_τ. Symbolically,

$$\mathrm{TP}(\tau) = \{G \in \mathcal{G}_n \mid \mathrm{Hom}(\Gamma(G, S), \Gamma_\tau) \neq \varnothing\}.$$

Definition 10.4. We say a tileset τ is *complete* if for every vertex $v \in V_{\Gamma_\tau}$, there are edges $\{e_s\}_{s\in S} \cup \{e_{s^{-1}}\}_{s\in S}$ such that $L(e_{s^{\pm 1}}) = s$ and $\mathfrak{i}(e_s) = v$ and $\mathfrak{t}(e_{s^{-1}}) = v$. Notice that we are allowed to have $e_s = e_{s^{-1}}$. We also say the tileset is connected if Γ_τ is connected.

Completeness is also know as Piantodosi's condition [Pia08] or condition $(\star)$ as presented in [MC19].

Let us see that from a tileset that tiles at least one group, we can always extract a complete sub-tileset from it. This will also allow us to partially answer Question 2.

Lemma 10.5. *Let τ be a tileset. If there exists $G \in \mathcal{G}_n$ such that $G \in \mathrm{TP}(\tau)$, then $F_n \in \mathrm{TP}(\tau)$. In other words, $\mathrm{TP}(\tau) \neq \varnothing$ if and only if $F_n \in \mathrm{TP}(\tau)$.*

Proof. Let us create a tiling x of F_n from τ. Let $y \in \tau^G$ be a tiling of G by τ. For $w \in F_n$ let us take $g \in G$ such that $g =_G w$. We set $x_w := y_g$. Because for every $s \in S \cup S^{-1}$ we have $ws =_G gs$ and no forbidden pattern appears on y. Therefore, $x \in \tau^{F_n}$. □

This can be re-stated in the language of complete tilesets.

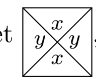
Lemma 10.6. *Let τ be a tileset such that $\mathrm{TP}(\tau) \neq \varnothing$. Then, there exists a complete tileset τ' such that $\tau' \subseteq \tau$.*

Proof. By Lemma 10.5, we know that τ tiles $F_n = \langle s_1, ..., s_n \rangle$. Let τ' the set of tiles that appear on configurations of τ^{F_n}. Let $t \in \tau'$. Then we have $x \in (\tau')^{F_n}$ and $g \in F_n$ such that $x_g = t$. Then, in $\Gamma_{\tau'}$ there is an edge labeled s_i from the tile t to the tile x_{gs_i} and one from the tile $x_{gs^{-1}}$ to t. Therefore τ' is complete and contained in τ. $\square$

Remark 10.2. There is a certain monotonicity to the problem: if $\tau_1 \subseteq \tau_2$, then $\mathrm{TP}(\tau_1) \subseteq \mathrm{TP}(\tau_2)$. It is also straight forward to see that if we have any two tilesets τ_1 and τ_2, then they satisfy $\mathrm{TP}(\tau_1) \cup \mathrm{TP}(\tau_2) \subseteq \mathrm{TP}(\tau_1 \cup \tau_2)$.

Definition 10.7. A tileset τ is said to be *minimal* if it does not properly contain complete tilesets.

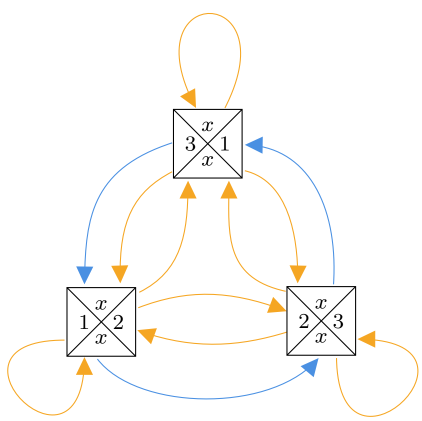
Example 10.1. We call the class of sigleton tilests that are complete, uni-tileset. Each of this tilesets is characterized by n colors $c_1, ..., c_n$ such that the tile satisfies $t_{s_i} = t_{s_i^{-1}} = c_i$. Notice that all uni-tilesets are minimal, and if $\{t\}$ is such a tileset, $\mathrm{TP}(\{t\}) = \mathcal{G}_n$.

We can see that the tileset Π defined above contains the uni-tileset 

, and therefore, by monotonicity $\mathrm{TP}(\Pi) = \mathcal{G}_2$.

From this we can see that a tileset contains a uni-tileset if and only if its graph has an n-rose as an induced subgraph.

Sadly, not all minimal tilesets tile every group. This can be seen with the tileset on 2 generators shown by the graph:



This tileset is minimal, can tile F_2, but cannot tile the group $\mathrm{PSL}(2, \mathbb{Z}) = \langle a, b \mid a^2, b^3 \rangle$. This example suggests that the cycle structure of the graph influences the groups τ can tile.

Luckily we can characterize all tilesets that tile every n-marked group.

Proposition 10.8. *Let τ be a tileset. Then $\mathrm{TP}(\tau) = \mathcal{G}_n$ if and only if there exists $t_0 \in \tau$ such that $\{t_0\}$ is a uni-tileset.*

Proof. We already saw that for tilesets that contain uni-tilesets $\mathrm{TP}(\tau) = \mathcal{G}_n$. For the converse, it suffices to see that the trivial group can be seen as the n-marked group

$$\langle s_1, ..., s_n \mid s_i = 1, \ \forall i \rangle,$$

and therefore it requires a uni-tileset. $\square$

Let us look at conditions on tilesets that arise from trying to tile different classes of groups such as infinite groups, virtually free or amenable.

Definition 10.9. Let $\Gamma = (V, E)$ be a graph. A *cycle* is a path $\gamma = v_0e_0v_1e_1...v_{n-1}e_{n-1}$ such that for all $i \in \{0, ..., n-1\}$ $v_i \in V$, $e_i \in E$, $v_i = \mathfrak{i}(v_i) = \mathfrak{t}(e_{i-1})$ and $\mathfrak{t}(e_{n-1}) = v_0$, where all vertices are distinct. The length of a cycle, denoted $|\gamma|$, is defined as the amount of edges it contains. We also denote by $\gamma_v \in \{0, 1\}$ if vertex v appears on γ.

In the case where $\Gamma = \Gamma_\tau$ for a tileset τ, we denote $\mathcal{C}(s)$ the set of all cycles labeled by s.

Proposition 10.10. *Let τ be a complete tileset such that $\mathrm{TP}(\tau) \neq \varnothing$ and let $\ell(s) = \{|\gamma| \mid \gamma \in \mathcal{C}(s)\}$ be the set of lengths of cycles labeled by s. If there exists a generator $s \in S$ such that: $1 \notin \ell(s)$ and $\{2, 3\} \nsubseteq \ell(s)$, then $\mathrm{TP}(\tau)$ does not include every virtually free group.*

Proof. Suppose we have a generator s as in the hypothesis. We will show the that the group $G = \langle S \mid s^N \rangle$ is not tilable, where $N \in \mathbb{N}$ is such that it is not in the semigroup generated by $\ell(s)$, namely $\langle \ell(s) \rangle_{\mathbb{N}}$.

First let us see that N exists. If $2 \in \ell(s)$ and $3 \notin \ell(s)$, then $N = 3$ cannot be obtained as a positive combination of elements in $\ell(s)$. The same goes for $N = 2$ if $2 \notin \ell(s)$.

Suppose τ tiles G, and take $x \in \tau^G$. Then, in Γ_τ we have an edge, e_i labeled s from $x_{s^{i-1}}$ to x_{s^i}. Because $x_{1_G} = x_{s^N}$, we have the loop of length N, $e_1...e_N$, in Γ_τ labeled s, which is a contradiction because this loop is not a cycle and also cannot be obtained by concatenating cycles because of the definition of N. $\square$

We have additional constraints on tilesets that tile every n-generated infinite group originally introduced in [CGG14], and later expanded upon by Hellouin and Maturana [MC19]. This conditions arises in particular when tiling amenable groups.

Definition 10.11. Let τ be a tileset over the colors C. For every generator $s \in S$ we define a matrix $M^s \in \mathbb{Z}^{\tau \times C}$ by

$$M^s_{t,c} = \begin{cases} 1 \text{ if } t_s = c \neq t_{s^{-1}} \\ -1 \text{ if } t_{s^{-1}} = c \neq t_s \\ 0 \text{ otherwise} \end{cases}.$$

We say τ satisfies condition $(\star\star)$ if there exists a non-trivial $\vec{x} \in \mathbb{R}^\tau_+$ such that for every $s \in S$, $M^s\vec{x} = 0$.

Theorem 10.12 ([MC19])**.** *Let τ be a tileset over the colors C and enumerate the cycles $\mathcal{C}(s_i) = \{\gamma^i_j\}_{j=1}^{|\mathcal{C}(s_i)|}$. Then, τ satisfies condition $(\star\star)$ if and only if there exists a non-trivial positive solution to the equation,*

$$\forall t \in \tau : \sum_{j=1}^{|\mathcal{C}(s_1)|} x_{1j}|\gamma^1_j|_t = \sum_{j=1}^{|\mathcal{C}(s_2)|} x_{2j}|\gamma^2_j|_t = \ ... \ = \sum_{j=1}^{|\mathcal{C}(s_n)|} x_{nj}|\gamma^n_j|_t.$$

Hellouin and Maturana present a necessary condition for tiling amenable groups, which we have modified for the notation in use.

Theorem 10.13 ([MC19])**.** *Let $G \in \mathcal{G}_n$ be an amenable group and τ a tileset. If $G \in \mathrm{TP}(\tau)$, then τ satisfies condition $(\star\star)$.*

Because $\mathbb{Z}^2$ is amenable, this Theorem finishes answering Question 2, due to the existence of complete tilesets that do not satisfy $(\star\star)$.

Question 4. Does a complete tileset τ satisfying $(\star\star)$ and for all $s \in S$, $1 \in \ell(s)$ or $\{2, 3\} \subseteq \ell(s)$, tile every amenable group?

Hellouin and Maturana show that being complete and satisfying ($\star\star$) in not enough to tile a non-free group. Nevertheless, their example does not satisfy our new condition.

Question 5. Does there exist a set of tilesets $(\tau_i)_{i\in I}$ such that $\bigcup_{i\in I} \mathrm{TP}(\tau_i) = \mathcal{G}_n$ and for $i \neq j$, $\mathrm{TP}(\tau_i) \cap \mathrm{TP}(\tau_j) = \{F_n\}$?

10.2 Decidability

In this particular section we will take only finitely presented n-marked groups. Asking decidability questions is only sensible if the input is finite.

Lemma 10.14. *Given a tileset τ and denoting $W\mathcal{G}_n \subseteq \mathcal{G}_n$ the set of groups with decidable word problem in $\mathcal{G}_n$, then $\mathrm{TP}(\tau) \cap W\mathcal{G}_n$ is co-recursively enumerable.*

Proof. Let $G \in W\mathcal{G}_n$. The semi-algorithm takes G and for each $N \in \mathbb{N}$, tries to tile the ball of radius N with respect to the word metric on G, $B(1_G, N)$. We can know explicity the elements in $B(1_G, N)$ because G has decidable word problem. By a standard compactness argument we know that if G is not tileable by τ the exists an N such that $B(1_G, N)$ is not tileable by τ. This means that the algorithm will stop if the given group is not tilable by τ. $\square$

Proposition 10.15. *We have that there exists a tileset τ_0 such that $\mathrm{TP}(\tau_0)$ is undecidable or for every tileset τ, $\mathrm{TP}(\tau)$ is decidable by an algorithm A_τ and the function $f : \tau \mapsto A_\tau$ is not computable.*

Proof. Suppose that for every tileset τ, $\mathrm{TP}(\tau)$ is decidable by an algorithm A_τ but the function $f(\tau) = A_\tau$ is computable. Then given $G \in \mathcal{G}_n$, we create the algorithm B_G that receives as input a tileset τ and computes if $G \in \mathrm{TP}(\tau)$ through A_τ. In other words, $B_G(\tau) = f(\tau)(G)$. Therefore, B_G is an algorithm that decides $\mathrm{DP}(G)$, which can be done for any $G \in \mathcal{G}_n$. We conclude from the fact that $\mathrm{DP}(\mathbb{Z}^n)$ is undecidable for $n \geq 2$. $\square$

Because the domino problem for free groups is decidable, we can use Lemma 10.5 and Prop. 10.8 to state the following.

Proposition 10.16. *For a given tileset τ, there is an algorithm to decide if $\mathrm{TP}(\tau)$ is empty or tiles every group.*

This way, we can see that the difficulty lies on tilesets that are in between, that is, they are non-empty but do not tile every group.

10.3 Topology

Let us take a brief look at the topology of $\mathcal{G}_n$. It will be obtained by looking at $\mathcal{G}_n$ in its normal subgroup representation. Let 2^{F_n} be the set of subsets of F_n, and for any two subsets $A, A' \in 2^{F_n}$ consider the radius of the largest ball on which both subsets coincide:

$$\rho(A, A') = \max\{R \in \mathbb{N} \mid A \cap B_{F_n}(R) = A' \cap B_{F_n}(R)\}.$$

We define the metric $d(A, A') = e^{-\rho(A,A')}$. Endowing 2^{F_n} with this metric makes it totally discontinuous, and by Tychonoff's Theorem, a compact space. By looking at $\mathcal{G}_n$ as the set of normal subgroups of F_n, we have the following.

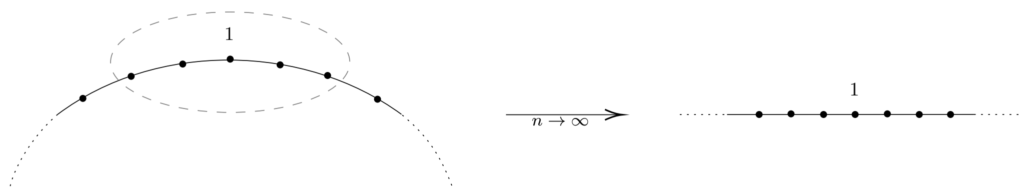
Theorem 10.17 ([Gri84])**.** *$\mathcal{G}_n$ is compact.*

We can further understand this topology in terms of the Cayley graph. Notice that a relation on a marked group (G, S) is a word in S^*. Thus, with the previous topology, two marked groups (G, S) and (G', S') are at a distance e^{-R} if the have the same relations up to length R (and up to renaming of the generating set).

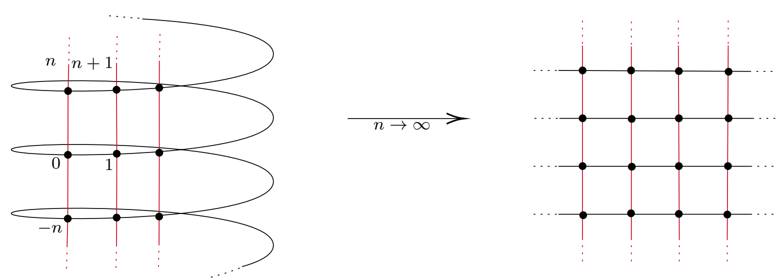
Because the set of relations of (G, S) of length at most $2R+1$ contains the same information as the ball of radius R of its Cayley graph, the metric can be expressed in terms of the Cayley graph: two marked groups (G, S) and (G', S') are at a distance $e^{-(2L+1)}$ is their labeled Cayley graphs have the same ball of radius L. This allows us to state the following result.

Let us look at two classic examples of convergence in this topology.

- $(\mathbb{Z}/n\mathbb{Z}, 1) \to (\mathbb{Z}, 1)$:



- $(\mathbb{Z}, (1, n)) \to (\mathbb{Z}^2, ((1, 0), (0, 1)))$:



Proposition 10.18. $\mathrm{TP}(\tau)$ *is compact in* $\mathcal{G}_n$.

Proof. We will show that $\mathrm{TP}(\tau)$ is closed. Let $\{G_n\}_{n\in\mathbb{N}} \subseteq \mathrm{TP}(\tau)$ be a sequence of groups, such that $G_n \to G$. Due to the topology, this means that for any radius $R > 0$ there exists a sufficiently large $N(R)$ such that $B_{G_n}(R) = B_G(R)$ for all $n \geq N(R)$.

With this we create the following $x \in \tau^G$. We know that for every n there exists $y^n \in \tau^{G_n}$. Now, set

$$x_g = y_g^{N(R)}, \quad \text{for} \quad |g|_S = R.$$

With this, x is a valid tiling of G by τ, and therefore $G \in \mathrm{TP}(\tau)$. $\square$

11 Conclusion

Throughout the text we have explored different aspects and ideas at the forefront of the study of the domino problem on groups. From its roots in logic to its modern setting in symbolic dynamics and group theory the domino problem has been explored from multiple angles and has been a fruitful area of research.

We began this work by looking at the four proofs of the undecidability of the domino problem on $\mathbb{Z}^2$ and their applicability to other groups. Each of these use particular properties of $\mathbb{Z}^2$ to work. Although the study of the methods gave no new classes of groups with undecidable domino problem or new proofs for classes with known results, we did manage to isolate some structural elements that the proofs require.

From our analysis of the first two methods, the aperiodic tiling and the fixed point simulation methods, we notice that for their generalization it would greatly aid to find a way to generalize the notion of substitutions to a more general class of structures. Even though the first proof has been utilized to show the existence of strongly aperiodic SFTs on groups whose Cayley graph contains infinite grids, such as the Heisenberg group [SSU20], there still exist potential to apply the technique to further groups.

Question 6. Is there a way to generalize the notion of substitutions to groups or infinite graphs?

This was done for Baumslag-Solitar groups [Sil20], but it remains to find a definition that works for a wider class of groups.

Even if there is a way to generalize substitutions, generalizing the fixed-point method also requires the ability to transmit the colors from the border of the simulation and a way to make computation through local constraints.

Question 7. On which groups can we apply the proof strategy of Durand, Romashchenko and Shen?

Moving to the next strategy, because Kari's proof scheme has been successfully generalized to Baumslag-Solitar groups we can take aspects of the generalization and see their applicability to new groups.

Question 8. What properties of G allow us to define a projection function $\Phi : G \mapsto \mathbb{R}^2$ as was done for Baumslag-Solitar groups?

Finally, for the proof strategy of Aanderaa and Lewis, although out attempts at generalizing the results allowed us to establish new connections between decision problems, it did prove undecidability for new groups.

Question 9. Given a group G and a subgroup of symmetries $F \leq S_G$, what can we say about the decidability of $\mathrm{Dist}_F(G)$?

We proceeded to study the decidability of the domino problem for two-generated one-relator groups. By looking at what aspects of the word defining the group we managed to characterize the decidability of the domino problem of words of the form $x^n y^m x^k$. As a consequence of this result we managed to prove that non-free Artin groups and non-$\mathbb{Z}$ Generalized Baumslag Solitar groups have undecidable domino problem. Nevertheless, it remains to characterize the decidability of other word structures.

Question 10. Does the knot group of the Figure Eight Knot, given by the presentation $\langle a, b \mid bab^{-1}ab = aba^{-1}ba \rangle$, have undecidable domino problem?

In addition, from the limited classification obtained we see that all groups of the form with torsion have decidable domino problem. Nielsen's result (Theorem 8.5) suggests that this might be the case for all one-relator groups with torsion.

Question 11. Do all one-relator groups with torsion have decidable domino problem?

Finally, we took a look at the problem of determining which groups are tilable by a given tileset. By using the framework of marked groups we were able to establish some necessary conditions for the tilesets to tile different classes of groups. We also studied the decidability of the problem. It remains to see which conditions are satisfied by well known tilesets, such as the Robinson tileset.

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