Small Categories

VM

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1 | Defining Small Categories

Example 1.1. The category Mat_R has the set of natural numbers $\mathbb{N}$ as its set of objects and the set of all matrices with entries from the ring R as its set of arrows. For $m, n \in \mathbb{N}$, the arrows from n to m are all the $m \times n$ R-matrices, and composition of arrows is matrix multiplication. Given $A \colon n \to m$ and $B \colon m \to p$, the product BA is defined and is a $p \times n$ matrix — i.e., BA is an arrow from n to p. The identity arrow of n is the identity matrix I_n of order n.

2 | Functors

Example 2.1. Let $r \in \mathbb{N}$. In the category Mat_R given in Example 1.1, we can define functors $r \otimes -: \mathrm{Mat}_R \to \mathrm{Mat}_R$ and $- \otimes r: \mathrm{Mat}_R \to \mathrm{Mat}_R$ as follows. For any object n of Mat, $(r \otimes -)n = rn$. For any arrow $A: n \to m$, $(r \otimes -)A = I_r \otimes A$, the Kronecker product of the identity matrix I_r and A. The other functor $- \otimes r$ is defined similarly.

The Kronecker product of matrices satisfies the property $(A \otimes B)(C \otimes D) = AC \otimes BD$ (provided A and C, and similarly B and D are compatible for multiplication). This implies that the functors preserve composition of morphisms. And since $I_r \otimes I_n = I_n \otimes I_r = I_{rn}$, they preserve identity arrows as well.

3 | Adjunctions

Example 3.1. Consider the functor $r \otimes -\colon \mathrm{Mat}_R \to \mathrm{Mat}_R$ defined in Example 2.1. Let $\mathcal{L} = r \otimes -$ and let $\mathcal{R}$ be a right adjoint of $\mathcal{L}$ (if one exists). Then $\mathrm{Hom}(\mathcal{L}(m), n) \cong \mathrm{Hom}(m, \mathcal{R}(n))$ for all $m, n \in \mathbb{N}$. But $L(m) = rm$, so $\mathrm{Hom}(\mathcal{L}(m), n)$ is the set of all $n \times rm$ matrices with entries from R. If R is finite (e.g., $R = \mathbb{Z}_2$), then $|\mathrm{Hom}(rm, n)| = |R|^{rmn}$. This suggests that $\mathrm{Hom}(m, \mathcal{R}(n)) = \mathrm{Hom}(m, rn)$, so that $\mathcal{R}(n) = rn$. This further suggests that $R = r \otimes -$ or $R = - \otimes r$.

Let $R = r \otimes -$. Given an arrow $A\colon \mathcal{L}(m) \to n$, that is, given an $n \times rm$ matrix A, we must define an arrow $A^*\colon m \to \mathcal{R}(n)$, that is, an $rn \times m$ matrix A^*, so that $(\cdot)^*$ is a bijection from $\mathrm{Hom}(\mathcal{L}(m), n)$ to $\mathrm{Hom}(m, \mathcal{R}(n))$. Thus, given an $rn \times m$ matrix B, there must be an $n \times rm$ matrix B^*, such that $(A^*)^* = A$ and $(B^*)^* = B$ for all $A \in \mathrm{Hom}(\mathcal{L}(m), n)$ and $B \in \mathrm{Hom}(m, \mathcal{R}(n))$.

Comparing the orders of A and A^* indicates that the operation we need is some sort of matrix transposition — a "partial" one. Consider a partition of the $n \times rm$ matrix A.

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{32} & \cdots & A_{nm} \end{bmatrix}_{n \times rm}$$

where each A_{ij} is a $1 \times r$ (row) vector. Then we can obtain a matrix A^* of the required order $rn \times m$ by transposing each such block A_{ij} of A.

$$A^* = \begin{bmatrix} A_{11}^T & A_{12}^T & \cdots & A_{1m}^T \\ A_{21}^T & A_{22}^T & \cdots & A_{2m}^T \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}^T & A_{32}^T & \cdots & A_{nm}^T \end{bmatrix}_{rn \times m}$$

Obviously then, given a matrix B of order $rn \times m$, we can partition it similarly into n rows and m columns of $r \times 1$ column vectors, and then transpose each of these vectors to obtain an $n \times rm$ matrix B^*. Both of these operations are well defined, and clearly, $A^{**} = A$ and $B^{**} = B$.

But in order for this bijection to be the natural one between the hom-sets, we must also verify that the following two equations hold for all $\mathcal{L}(m) \xrightarrow{A} n \xrightarrow{C} p$ and $q \xrightarrow{D} m \xrightarrow{B} \mathcal{R}(n)$.

$$\left(\mathcal{L}(m) \xrightarrow{A} n \xrightarrow{C} p \right)^* = m \xrightarrow{A^*} \mathcal{R}(n) \xrightarrow{\mathcal{R}(C)} \mathcal{R}(p)$$

$$\left(q \xrightarrow{D} m \xrightarrow{B} \mathcal{R}(n) \right)^* = \mathcal{L}(q) \xrightarrow{\mathcal{L}(D)} \mathcal{L}(m) \xrightarrow{B^*} n$$