QUOTIENT GRAPHS

1. Introduction

The automorphism group of a graph induces a partition of the vertex set into orbits of its action on the graph. Intuitively, we think of the vertices in one orbit as being "automorphically equivalent". It is natural, therefore, to define the *quotient graph* with respect to the automorphism and study its relation to the original graph. Formally, if Γ is a graph with automorphism group G, and Π is the partition of the vertex set $V(G)$ into orbits under the action of G, then the quotient graph Γ/G has Π as its vertex set, and any two distinct vertices X and Y of Γ/G are adjacent whenever there are vertices x and y of Γ such that $x \in X$ and $y \in Y$.

Definition 1.1. If Γ is a graph with automorphism group $\operatorname{Aut}\Gamma = G$, the quotient graph of G is the graph $\Gamma' = \Gamma/G$ having vertex set

$$V(\Gamma') = \{\, X \mid X = \operatorname{Orb}_G(x), \exists x \in V(\Gamma) \,\}$$

and edge set

$$E(\Gamma') = \{\, (X, Y) \mid X = \operatorname{Orb}_G(x), Y = \operatorname{Orb}_G(y), X \neq Y, \exists x, y \in V(G) \,\}.$$

Lemma 1.2. *If X and Y are two orbits of the automorphism group of a graph Γ, and a vertex $x \in X$ is adjacent to a vertex $y \in Y$, then every vertex of X is adjacent to some vertex of Y.*

Proof. Let $x' \in X$. Since $X = \operatorname{Orb}(x)$ is the orbit of x under the automorphism group of Γ, there is some automorphism φ of Γ such that $\varphi(x) = x'$. Then, $\varphi(y) = y'$, $\exists y' \in Y = \operatorname{Orb}(y)$. Since $x \sim y$, it follows that $x' \sim y'$, as required. $\square$

Lemma 1.3. *If $X_1, X_2, \ldots, X_k$ is path in the quotient graph $\Gamma' = \Gamma/G$ of a graph G, then there is a path $x_1, x_2, \ldots, x_k$ in Γ with $x_i \in X_i$, $i = 1, 2, \ldots, k$.*

Proof. Since $X_1 \sim_{\Gamma'} X_2$, there exist vertices $x_1 \in X_1$ and $x_2 \in X_2$ such that $x_1 \sim_\Gamma x_2$. Similarly, there exist vertices $x_2' \in X_2$ and $x_3' \in X_3$ such that $x_2' \sim_\Gamma x_3'$. Then by Lemma 1.2 $x_2 \in X_2$ is adjacent to some vertex $x_3 \in X_3$. Thus, we have a sequence x_1, x_2, x_3 with $x_i \in X_i$, $i = 1, 2, 3$. Proceeding similarly, we obtain a sequence of vertices $x_1, x_2, \ldots, x_k$ with $x_i \in X_i$, $i = 1, 2, \ldots, k$. Observe that since each X_i in the Γ'-path is distinct from every X_j, $j \neq i$, each x_i in the sequence must be distinct from each x_j, $j \neq i$, which makes the sequence a path of the required form in Γ. $\square$

Theorem 1.4. *If $X_1, X_2, \ldots, X_k$ is a cycle in the quotient graph $\Gamma' = \Gamma/G$ of a graph G, then the vertices $X_1 \cup \cdots \cup X_k$ of Γ contain an induced cycle.*

Proof. Let $X_1, X_2, \ldots, X_k$ be a cycle of length k in the quotient graph Γ'. Then by Lemma 1.3, there is a path $x_1, x_2, \ldots, x_k$ in Γ, with $x_i \in X_i$, $i = 1, 2, \ldots, k$. Now, since $X_k \sim_{\Gamma'} X_1$, Lemma 1.2 implies that $x_k \sim_{\Gamma} y_1$, for some vertex $y_1 \in X_1$. If $y_1 = x_1$, we obtain a cycle as required. If not, we proceed as before to find a path $y_1, y_2, \ldots, y_k$ with $y_i \in X_i$, $i = 1, 2, \ldots, k$. As there are finitely many vertices, this procedure terminates at a repeated vertex, and we obtain a cycle of the required form in Γ. $\square$

Theorem 1.5. *The quotient graph of a tree is a tree.*

Proof. The quotient graph of a connected graph is clearly connected. By Theorem 1.4, every cycle in a quotient graph corresponds to some cycle in the original graph. Therefore, the quotient graph of an acyclic graph is also acyclic. The result follows. $\square$