QUOTIENT GRAPHS

1. Introduction

The automorphism group of a graph induces a partition of the vertex set into orbits of its action on the graph. Intuitively, we think of the vertices in one orbit as being "automorphically equivalent". It is natural, therefore, to define the *quotient graph* with respect to the automorphism and study its relation to the original graph. Formally, if Γ is a graph with automorphism group G, and $\diamond$ is the partition of the vertex set $V(G)$ into orbits under the action of G, then the quotient graph Γ/G has Π as its vertex set, and any two distinct vertices X and Y of Γ/G are adjacent whenever there are vertices x and y of Γ such that $x \in X$ and $y \in Y$.

Definition 1.1. If Γ is a graph with automorphism group $\operatorname{Aut}\Gamma = G$, the quotient graph of G is the graph $\Gamma' = \Gamma/G$ having vertex set

$$V(\Gamma') = \{\, X \mid X = \operatorname{Orb}_G(x), \exists x \in V(\Gamma) \,\}$$

and edge set

$$E(\Gamma') = \{\, (X, Y) \mid X = \operatorname{Orb}_G(x), Y = \operatorname{Orb}_G(y), X \neq Y, \exists x, y \in V(G) \,\}.$$

Lemma 1.2. *If X and Y are two orbits of the automorphism group of a graph Γ, and a vertex $x \in X$ is adjacent to a vertex $y \in Y$, then every vertex x' of X is adjacent to some vertex of Y.*

Proof. Let $x' \in X$. Since $X = \operatorname{Orb}(x)$ is the orbit of x under the automorphism group of Γ, there is some automorphism ϕ of Γ such that $\phi(x) = x'$. Then, $\phi(y) = y'$, $\exists y' \in Y = \operatorname{Orb}(y)$. Since $x \sim y$, it follows that $x' \sim y'$, as required. $\square$

Theorem 1.3. *Every cycle in the quotient graph Γ/G corresponds to a cycle in the graph Γ.*