QUOTIENT GRAPHS

1. Introduction

The automorphism group of a graph induces a partition of the vertex set into orbits of its action on the graph. Intuitively, we think of the vertices in one orbit as being "automorphically equivalent". It is natural, therefore, to define the *quotient graph* with respect to the automorphism and study its relation to the original graph. Formally, if Γ is a graph with automorphism group G, and $\diamond$ is the partition of the vertex set $V(G)$ into orbits under the action of G, then the quotient graph Γ/G has Π as its vertex set, and any two distinct vertices X and Y of Γ/G are adjacent whenever there are vertices x and y of Γ such that $x \in X$ and $y \in Y$.

Definition 1.1. If Γ is a graph with automorphism group $\operatorname{Aut}\Gamma = G$, the quotient graph of G is the graph $\Gamma' = \Gamma/G$ having vertex set

$$V(\Gamma') = \{\, X \mid X = \operatorname{Orb}_G(x), \exists x \in V(\Gamma) \,\}$$

and edge set

$$E(\Gamma') = \{\, (X, Y) \mid X = \operatorname{Orb}_G(x), Y = \operatorname{Orb}_G(y), X \neq Y, \exists x, y \in V(G) \,\}.$$

Lemma 1.2. *If X and Y are two orbits of the automorphism group of a graph Γ, and a vertex $x \in X$ is adjacent to a vertex $y \in Y$, then every vertex x' of X is adjacent to some vertex of Y.*

Proof. Let $x' \in X$. Since $X = \operatorname{Orb}(x)$ is the orbit of x under the automorphism group of Γ, there is some automorphism φ of Γ such that $\varphi(x) = x'$. Then, $\varphi(y) = y'$, $\exists y' \in Y = \operatorname{Orb}(y)$. Since $x \sim y$, it follows that $x' \sim y'$, as required. $\square$

Theorem 1.3. *Every cycle in the quotient graph $\Gamma' = \Gamma/G$ of a graph G corresponds to a cycle in the graph Γ.*

Proof. Let $X_1, X_2, \ldots, X_k$ be a cycle of length k in the quotient graph Γ'. Then, $X_1 \sim_{\Gamma'} X_2$ implies that there exist vertices $x_1 \in X_1$ and $x_2 \in X_2$ of Γ, with $x_1 \sim_\Gamma x_2$. Again, $X_2 \sim_{\Gamma'} X_3$ implies that $x_2' \sim_\Gamma x_3'$, for some vertices $x_2' \in X_2$ and $x_3' \in X_3$. But then by Lemma 1.2, x_2 itself is adjacent to some vertex $x_3 \in X_3$, in Γ. Similarly, we may find a sequences of vertices $x_4, x_5, \ldots, x_k$, such that $x_1, x_2, \ldots, x_k$ is a path in Γ. Now, since $X_k \sim_{\Gamma'} X_1$, we must have $x_k \sim_\Gamma y_1$, for some vertex $y_1 \in X_1$. If $y_1 = x_1$, we obtain a cycle as required. If not, we proceed as before to find a vertex $y_2 \in X_2$, with $y_1 \sim_\Gamma y_2$, and so on, thus obtain a longer path in Γ consisting of vertices of the orbits $X_1, \ldots, X_k$. As there are finitely many vertices, this procedure terminates at a repeated vertex, and we obtain a cycle in Γ. $\square$