INTRODUCTION TO COMPUTER GRAPHICS

Vector, Matrix & Transformations

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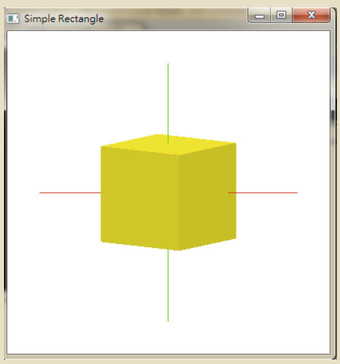
Goal

- Translation
- Rotation
- Scaling



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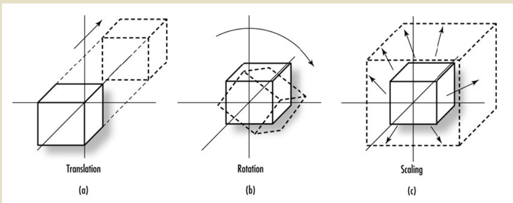
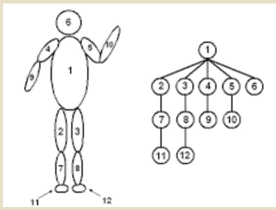
Transformations

- Why use transformations?
  - Create object in convenient coordinates
  - Reuse basic shape multiple times
  - Hierarchical modeling
  - Virtual cameras





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Translation

$$T(t_x, t_y, t_z)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

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Rotation

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The call glRotate(Θ, 1, 0, 0) generates R_x as follows:

$$R_x(\theta)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

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Scaling

$$S(s_x, s_y, s_z)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

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LINEAR ALGEBRA BASIC

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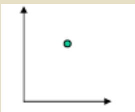
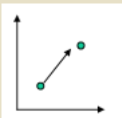
Point v.s Vector

- A 3D point p = [x y z]
  - Represents a location with respect to some coordinate system
- A 3D vector v = [x y z]
  - Represents a displacement from a position
    - Magnitude
    - Direction
      - X-axis(1,0,0)、Y-axis(0,1,0)、Z-axis(0,0,1)



Point



vector

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Unit Vector

- The Euclidean distance of u from the origin is:
  - $||u|| = sqrt(x^2+y^2+z^2)$
  - denoted by $||u||$
- if $||u|| = 1$, then u is a unit vector,
- Normalization:

$$v = \frac{u}{\| u \|}$$ ----- unit vector

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Vector Spaces

- Consists of a set of elements, called vectors
- Two operations are defined on them
  - Addition
  - Multiplication

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Vector Addition

Given V = [X Y Z] and W = [A B C]

- V+W = [X+A Y+B Z+C]

- Properties of Vector addition:
  - 交換律 Commutative law: V+W=W+V
  - 結合律 Associative law : (U+V)+W = U+(V+W)
  - Additive Identity: V+0 = V
  - Additive Inverse: V+W = 0, W=-V

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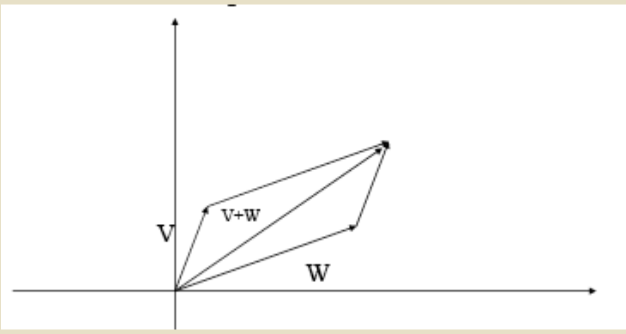
Vector Addition

- Parallelogram Rule (平行四邊形)
  - To visualize what a vector addition is doing, here is a 2D example:



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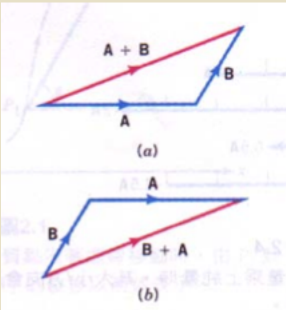
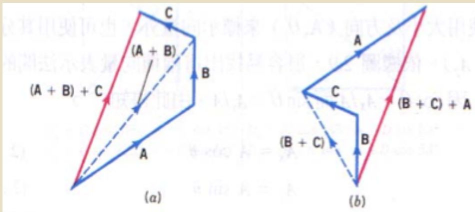
Vector Addition

Commutative law: A+B=B+A



Associative law : (A+B)+C = A+(B+C)



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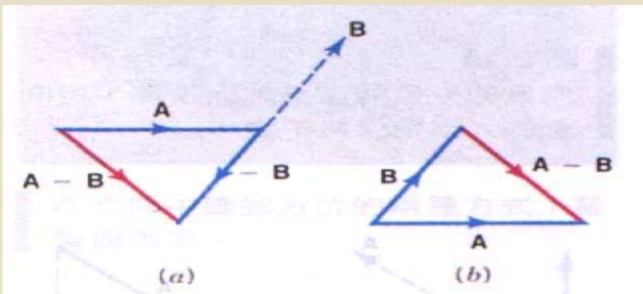
Vector Addition

- Vector Subtraction is a special case (向量的減法可視為加法的特例)
  - A-B=A+(-B)



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Vector Multiplication

Given V = [X Y Z] and a Scalar(純量) s and t

- sV = [sX sY sZ]

- Properties of Vector multiplication:
  - 結合律 Associative: (st)V = s(tV)
  - 純量分配律 Scalar Distribution: (s+t)V = sV+tV
  - 向量分配律 Vector Distribution: s(V+W) = sV+sW
  - Multiplicative Identity: 1V = V

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Dot Product & Distances

Given u = $[x_1\ y_1\ z_1]$ and v = $[x_2\ y_2\ z_2]$

- 內積: $v \bullet u = x_1x_2 + y_1y_2 + z_1z_2$

- The Euclidean distance of u from the origin is:
  - sqrt($x_1^2+y_1^2+z_1^2$)
  - denoted by ||u||
  - ||u|| = sqrt(u•u)
- The Euclidean distance between u and v is:
  - sqrt($(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2$))
  - denoted by || u-v ||

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Dot Product

Properties:

- Given a vector u, v, w and scalar s
  - The result of a dot product is a SCALAR value
  - Commutative: v•w = w•v
  - Non-degenerate: v•v=0 , only when v=0
  - Bilinear: v • (u + sw) = v•u + s(v•w)

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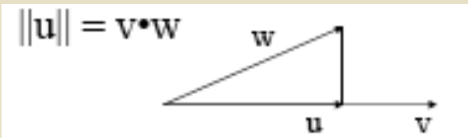
Angles and Projection

Alternative view of the dot product:

- v•w=||v|| ||w|| cos(θ)
  - where θ is the angle between v and w

- If we perpendicularly project w onto v
  - If v is a unit vector (||v|| = 1)
  - Then the projected vector u : ||u|| = v•w



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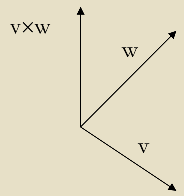
Cross Product

- The cross product of v and w: v x w
  - is a VECTOR, perpendicular to the plane defined by v and w
  - ||v x w||=||v|| ||w|| sinθ
    - θ is the angle between v and w
  - vxw=-(wxv)



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Uses of the Determinant?

- Linear Independence of columns in a matrix
- Cross Product
  - Given 2 vectors v=[v_1 v_2 v_3], w=[w_1 w_2 w_3], the cross product is defined to be the determinant of

$$\begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$$

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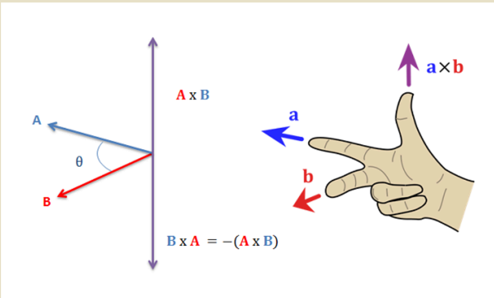
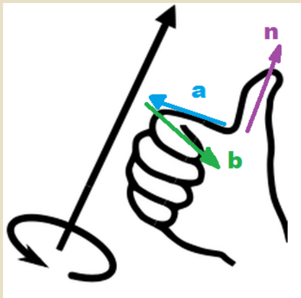
Right hand rule





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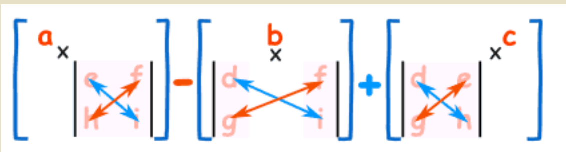
Determinant of a Matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\det A = |A| = \sum_{i=1}^{n} (-1)^{1+i} A_{1i}$$

|A| = a(ei - fh) - b(di - fg) + c(dh - eg)



http://www.mathsisfun.com/algebra/matrix-determinant.html

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$$C = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

$$\begin{aligned} |\mathbf{C}| &= 6\times(-2\times7 - 5\times8) - 1\times(4\times7 - 5\times2) + 1\times(4\times8 - (-2)\times2) \\ &= 6\times(-54) - 1\times(18) + 1\times(36) \\ &= \mathbf{-306} \end{aligned}$$

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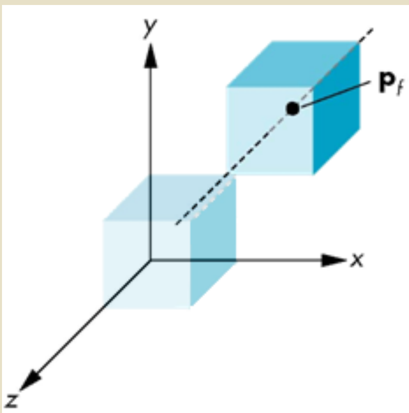
Translation

- **void glTranslatef(tx , ty , tz);**



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Translation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

$$T(t_x,t_y,t_z)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

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Properties of Translation

$$\boxed{T(t_x,t_y,t_z)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}}$$

$$T(0,0,0)\,\mathbf{v} = \mathbf{v}$$

$$T(s_x,s_y,s_z)\,T(t_x,t_y,t_z)\mathbf{v} = T(s_x+t_x,s_y+t_y,s_z+t_z)\,\mathbf{v}$$

$$T(s_x,s_y,s_z)\,T(t_x,t_y,t_z)\mathbf{v} = T(t_x,t_y,t_z)\,T(s_x,s_y,s_z)\mathbf{v}$$

$$T^{-1}(t_x,t_y,t_z)\,\mathbf{v} = T(-t_x,-t_y,-t_z)\,\mathbf{v}$$

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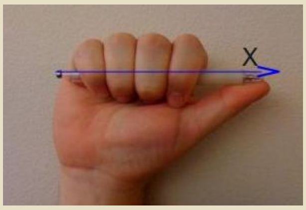
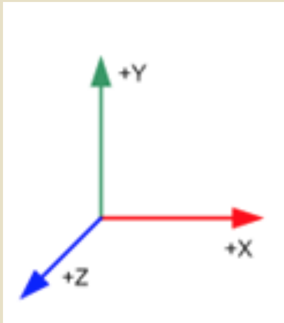
Rotation Matrix Direction

- OpenGL is right hand rule (counter clockwise)





http://www.youtube.com/watch?v=GbhTCg0FCr4

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Rotations (3D)

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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