A Probabilistic Approach to Goldbach's Conjecture Using Exponential Sums

Abstract

We present a hybrid analytic and probabilistic method to approach the famous Goldbach Conjecture. By using exponential sums over primes, combined with a heuristic probabilistic model for primality via the Prime Number Theorem, we derive an expected count of Goldbach pairs and show it is always positive for even integers greater than two. This offers intuitive evidence for the conjecture's validity.

1. Circle Method for Goldbach Pairs

Let N be an even integer. Define the exponential sum over primes up to N:

$$S(\alpha) = \sum_{p \leq N} e^{2\pi i p \alpha}$$

Then the number of representations of N as a sum of two primes is captured by:

$$G(N) = \int_0^1 S(\alpha)^2 e^{-2\pi i N \alpha} \, d\alpha$$

Expanding the square:

$$G(N) = \int_0^1 \sum_{p_1 \leq N} \sum_{p_2 \leq N} e^{2\pi i (p_1 + p_2 - N)\alpha} \, d\alpha$$

Using orthogonality of exponentials:

$$\int_0^1 e^{2\pi i k \alpha} \, d\alpha = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

So only terms where $p_1 + p_2 = N$ survive:

$$G(N) = \sum_{\substack{p_1 + p_2 = N \\ p_1, p_2 \leq N}} 1$$

This exactly counts the number of prime pairs that sum to N.

2. Probabilistic Relaxation of Primality

To derive an analytic approximation, we now introduce a probabilistic interpretation. Using the Prime Number Theorem, the probability that a given number n is prime is roughly:

$$\mathbb{P}(n \text{ is prime}) \approx \frac{1}{\ln n}$$

Assuming primality is independent for m and $N-m$, the expected number of Goldbach pairs becomes:

$$H_{\text{approx}}(N) = \sum_{m=2}^{N-2} \frac{1}{\ln m} \cdot \frac{1}{\ln(N-m)}$$

This is a weighted sum over all candidate decompositions of N, using prime-likelihoods as weights.

3. Why the Expected Count is Always Positive

Every term in the sum above is strictly positive for $2 \leq m \leq N-2$. Since $\ln x > 0$ for all $x > 1$, we have:

$$H_{\text{approx}}(N) > 0 \quad \forall N \geq 4, \text{ even}$$

Thus, the probabilistic model always predicts at least some expected Goldbach pairs.

4. Asymptotic Estimate

We now approximate the sum using an integral:

$$H_{\text{approx}}(N) \approx \int_{2}^{N-2} \frac{1}{\ln x \cdot \ln(N-x)} \, dx$$

Make the substitution $x = tN$, where $0 < t < 1$, and $dx = N dt$. This gives:

$$H_{\text{approx}}(N) \approx N \int_{\epsilon}^{1-\epsilon} \frac{1}{\ln(tN) \cdot \ln((1-t)N)} \, dt$$

Since $\ln(tN) \sim \ln N$ for most t, this integral behaves like:

$$H_{\text{approx}}(N) \sim \frac{N}{\ln^2 N}$$

This is consistent with classical asymptotics for Goldbach pair counts.

5. Conclusion

By combining the circle method with a probabilistic model based on the Prime Number Theorem, we construct a compelling heuristic for the number of Goldbach pairs. The expected count is always positive and asymptotically behaves like $\frac{N}{\ln^2 N}$, in agreement with well-known estimates. While not a formal proof, this supports the conjecture with intuitive and quantitative backing.