ASTR3602: Cosmology Midterm Project

Q4: The Future Universe

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Abstract

In this problem we examine the expansion of the universe under the Benchmark model and how various cosmological constants evolve over long time spans from the present day. Additionally, we explore how a future observer would discover the properties of the universe, as well as how the universe would look to them.

Contents

1 Section 1

In the Benchmark Model of the universe, we expect that the universe will continue to expand at an accelerating rate, as covered in lecture and in the textbook.

The scale factor can be calculated using Equation 5.81:

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2} \tag{1}$$

Which can be solved for the time using the relation $H = \frac{da/dt}{a}$:

$$\frac{\frac{(da/dt)^2}{a^2}}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2} \tag{2}$$

$$\frac{(da/dt)^2}{a^2} = H_0^2 \left(\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2} \right) \tag{3}$$

$$\left(\frac{da}{dt}\right)^2 = a^2 H_0^2 \left(\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2} \right) \tag{4}$$

$$da = (dt) a H_0 \left(\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2} \right)^{\frac{1}{2}} \tag{5}$$

$$\frac{da}{a(\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2})^{\frac{1}{2}}} = (dt) H_0 \tag{6}$$

$$\int_{a_0}^{a} \frac{da}{a(\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2})^{\frac{1}{2}}} = t H_0 \tag{7}$$

Where the limits of integration are $a_0 = 1$, the scale factor at the current date, and a, the scale factor at a future time. Reversing this integration shows the time evolution of the scale factor with time, depicted in figure 1 (the code for which is added in a separate submission).

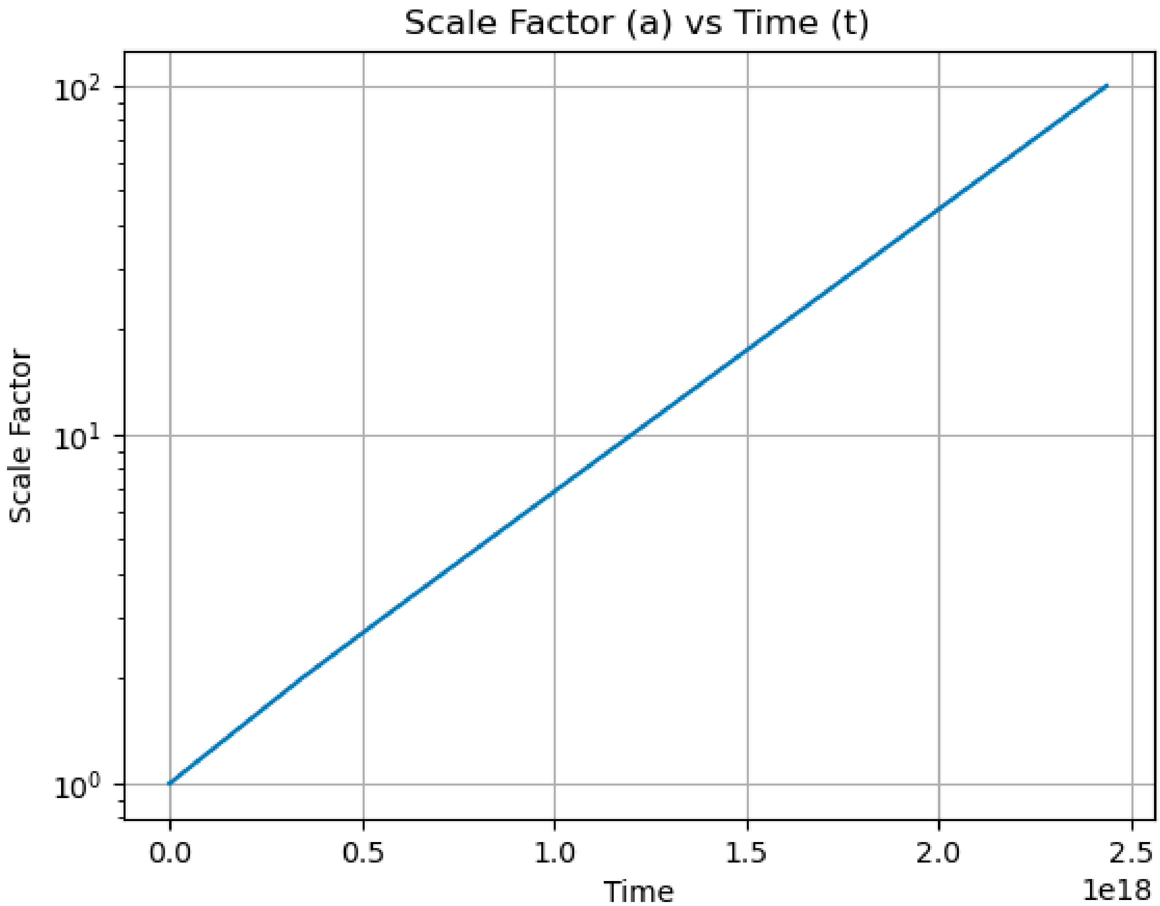

Figure 1: Scale Factor versus Time (s)

This graph shows how the scale factor linearly increases over large time scales on a logarithmic time scale. On a non-logarithmic time scale, the scale factor increases exponentially. Given that the Benchmark Model describes a flat universe, we can use the approximation provided in equation 8 [1]:

$$a(t) = \left[\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}} sinh^2 \left(\frac{3}{2} \sqrt{\Omega_{\Lambda,0}} H_0 t \right) \right]^{\frac{1}{3}} \tag{8}$$

Graphing this equation against time:

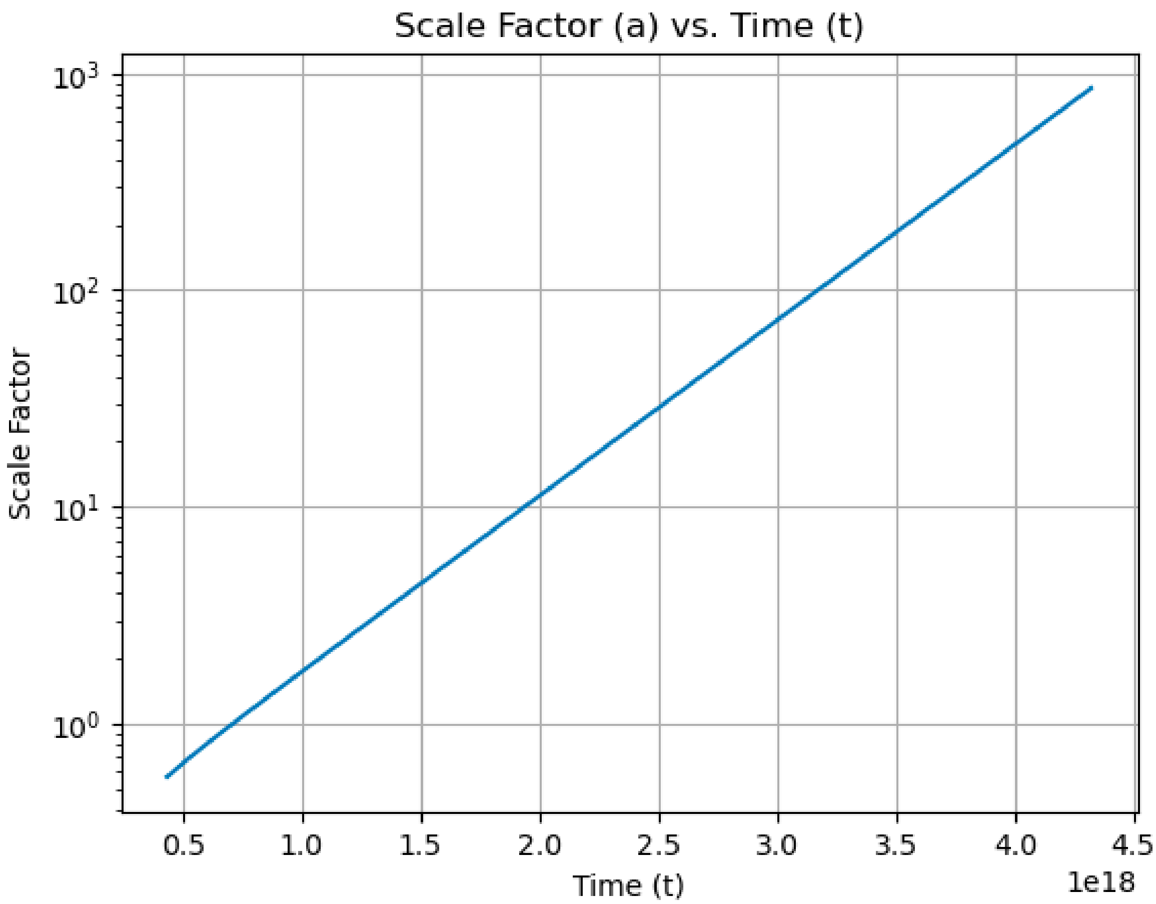

Figure 2: Approximation of Scale Factor versus Time (s)

2 Section 2

To simplify calculations for the Hubble parameter and other values, I'll use the approximation instead, as the approximation very closely matches (if not perfectly matches) the numerical integration of time with the Friedmann equation in the Benchmark model, as shown in Figure 3.

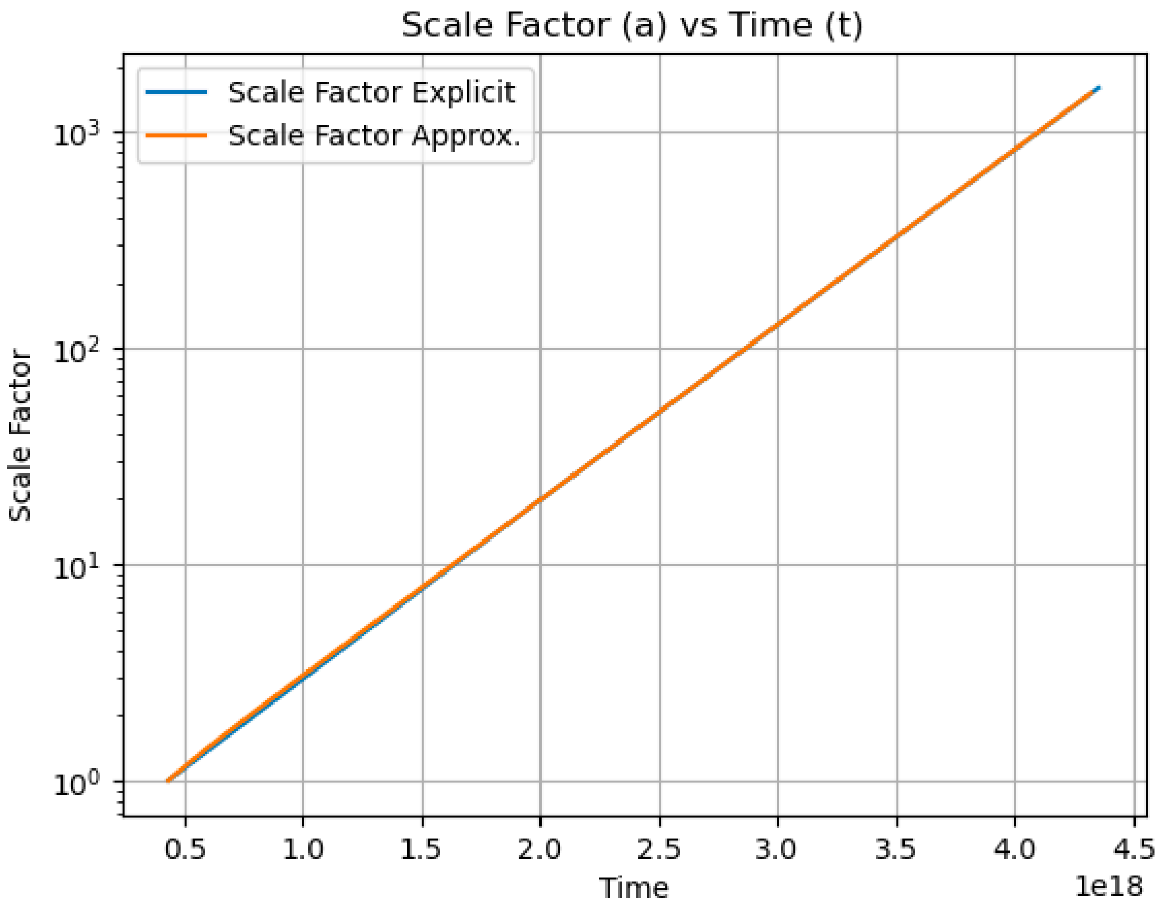

Figure 3: Scale Factor versus Time (s), Approximation and Explicit Integration Comparison

Thus the Hubble parameter is obtained through the relation used in Equation 2:

$$H(t) = \frac{\dot{a}}{a} \tag{9}$$

where

$$a = \left[\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}} sinh^2\left(\frac{3}{2}\sqrt{\Omega_{\Lambda,0}}H_0 t\right)\right]^{\frac{1}{3}} \tag{10}$$

and

$$\dot{a} = \left[\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}} \right)^{\frac{1}{3}} \left(\frac{3}{2} \sqrt{\Omega_{\Lambda,0}} H_0 \right) sinh \left(\frac{3}{2} \sqrt{\Omega_{\Lambda,0}} H_0 t \right)^{-\frac{1}{3}} cosh \left(\frac{3}{2} \sqrt{\Omega_{\Lambda,0}} H_0 t \right) \right] \tag{11}$$

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The Hubble parameter is graphed in Figure 4. The Hubble parameter will decrease until approximately 32 Gyr, at which point it plateaus to around $1.88 * 10^{-18} s^{-1}$.

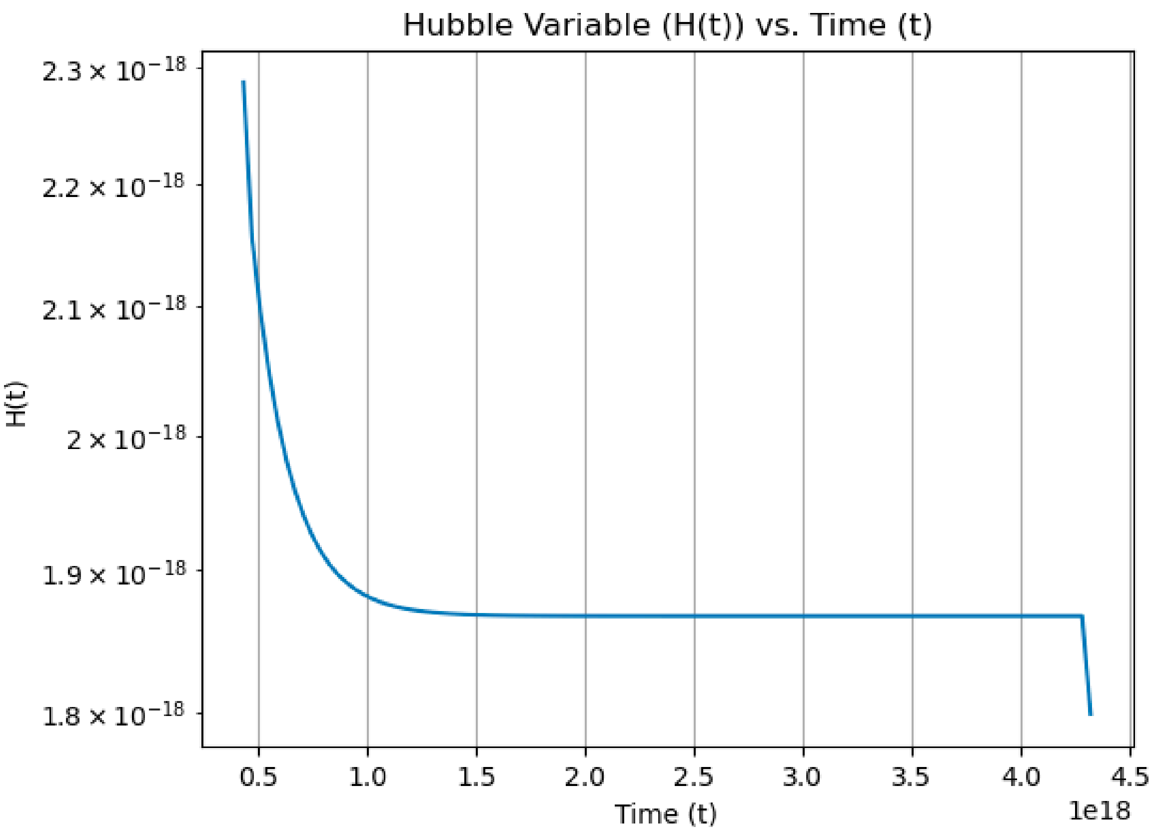

Figure 4: Hubble Parameter versus Time (s)

3 Section 3

The energy density of the universe at 50 Gyr is given by the equation:

$$\frac{H(t)^2}{H_0^2} = \frac{\epsilon(t)}{\epsilon_{c,0}} + \frac{1-\Omega_0}{a(t)^2} \tag{12}$$

Now we need to solve this equation for the time dependent critical energy density term:

$$\frac{H(t)^2}{H_0^2} = \frac{\epsilon(t)}{\epsilon_{c,0}} + \frac{1-\Omega_0}{a(t)^2} \tag{13}$$

$$\frac{\epsilon(t)}{\epsilon_{c,0}} = \frac{H(t)^2}{H_0^2} - \frac{1-\Omega_0}{a(t)^2} \tag{14}$$

$$\epsilon(t) = \epsilon_{c,0}\left(\frac{H(t)^2}{H_0^2} - \frac{1-\Omega_0}{a(t)^2}\right) \tag{15}$$

At 50 Gyr, the critical energy density and Hubble constant need to be reevaluated for that time. The Hubble constant can be calculated with equations 9, 10, and 11, which was performed in Python and found to be $1.06 * 10^{-18} s^{-1}$. The critical energy density is given by the following equation:

$$\epsilon_{c,0} = \frac{3c^2 H_0^2}{8\pi G} \tag{16}$$

Where G, the gravitational constant, is $6.67 * 10^{-11} \frac{N*m^2}{kg^2}$, and we use the H_0 for 50 Gyr found above. Solving for $\epsilon_{c,0}$ in the above equation we get $1.81 * 10^{-10}$. Plugging this into the equation for $\epsilon(t)$:

$$\epsilon(t) = 1.81 * 10^{-10}\left(\frac{(1.06 * 10^{-18} s^{-1})^2}{(1.06 * 10^{-18} s^{-1})^2} - \frac{1-0.3-0.0-0.7}{9^2}\right) \tag{17}$$

The two terms in parentheses resolve to 1 so the energy density is equivalent to the critical energy density, which is confirmed in the textbook by Equation 4.29 for a flat universe as described by the Benchmark Model. Thus the energy density at 50 Gyr is $1.06 * 10^{-18} s^{-1}$.

The relationship between the CMB temperature and time is given by the following equation [2]:

$$T(t) = \frac{T_0}{a(t)} \tag{18}$$

Here a(t) is calculated using equation 8. The temperature is graphed in figure 5, showing an inverse relationship between time and temperature. The CMB temperature at 50 Gyr is calculated to be approximately 0.303 K at 50 Gyr, using the approximation for the scale factor.

Under the assumption the universe is homogeneous and isotropic, the CMB would remain visible at 50 Gyr as it decreases in proportion to the scale factor, and thus will be evenly redshifted, as shown in Figure 5. Thus the same techniques used for observing the CMB would be valid in the far future, although increasingly more sensitive equipment would be needed to confirm its measurement (up until a certain point, as far enough into the future the CMB would be redshifted far past any sensitivity threshold).

4 Section 4

In the future, the distance of standard candles could be solve using the Taylor approximation for luminosity distance (since as seen in problem set 2, this very closely matched the integration form of luminosity distance even for large redshifts) shown in the following equation:

$$d_L = z\frac{c}{H_0}\left(1 + z\frac{1-q_0}{2}\right) \tag{19}$$

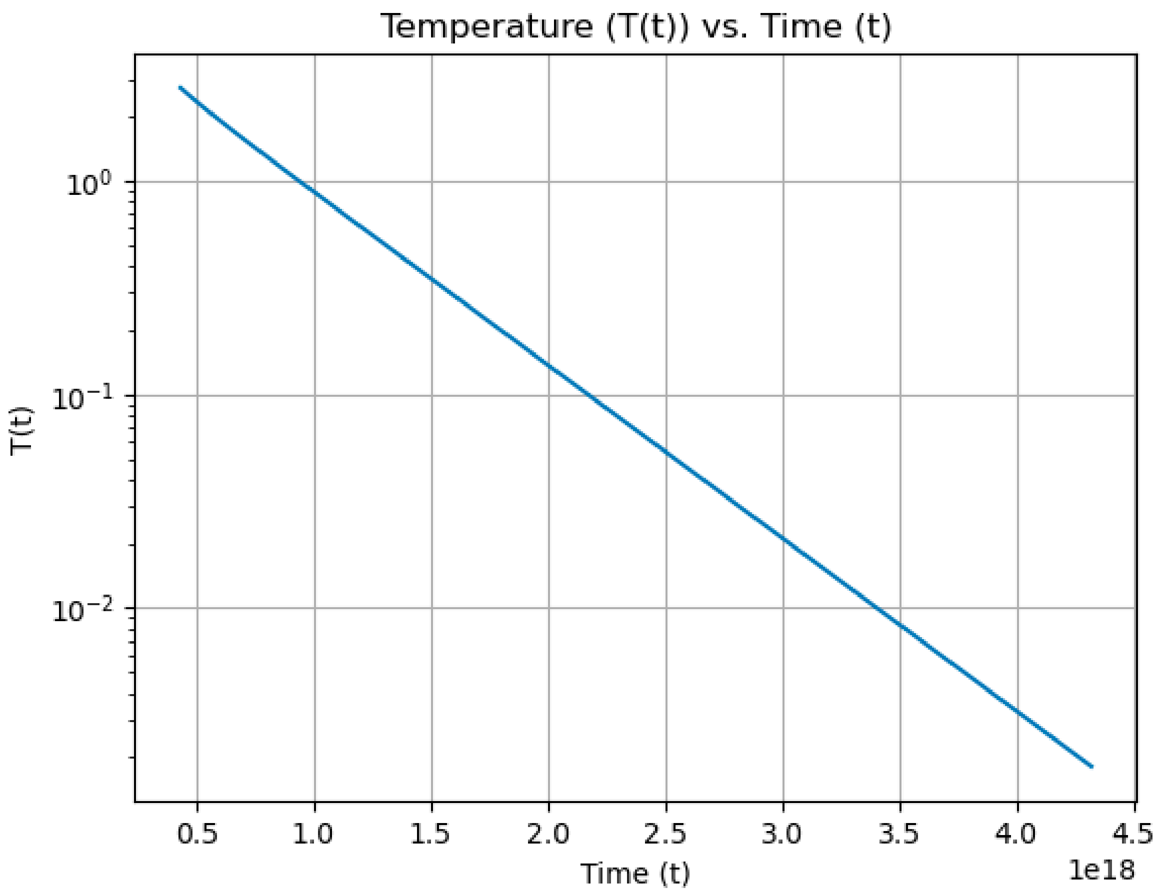

Figure 5: CMB Temperature versus Time

The luminosity distance is graphed for the Hubble constant H_0 values at the times 50 Gyr, 70 Gyr, and 90 Gyr. These values are graphed in Figure 6, which show that for future times, the luminosity distance for objects as a given redshift z will be greater than at previous times, indicating an expansion of the distance between objects and an observer.

As the universe expands under the Benchmark Model, the redshift for objects in the universe increases. At significantly high redshifts, the motion of the object due to the Hubble-flow dominates as opposed to its velocity. Thus in a universe far in the future, where the Hubble-flow might dominate the movement of objects, recession velocity becomes a less

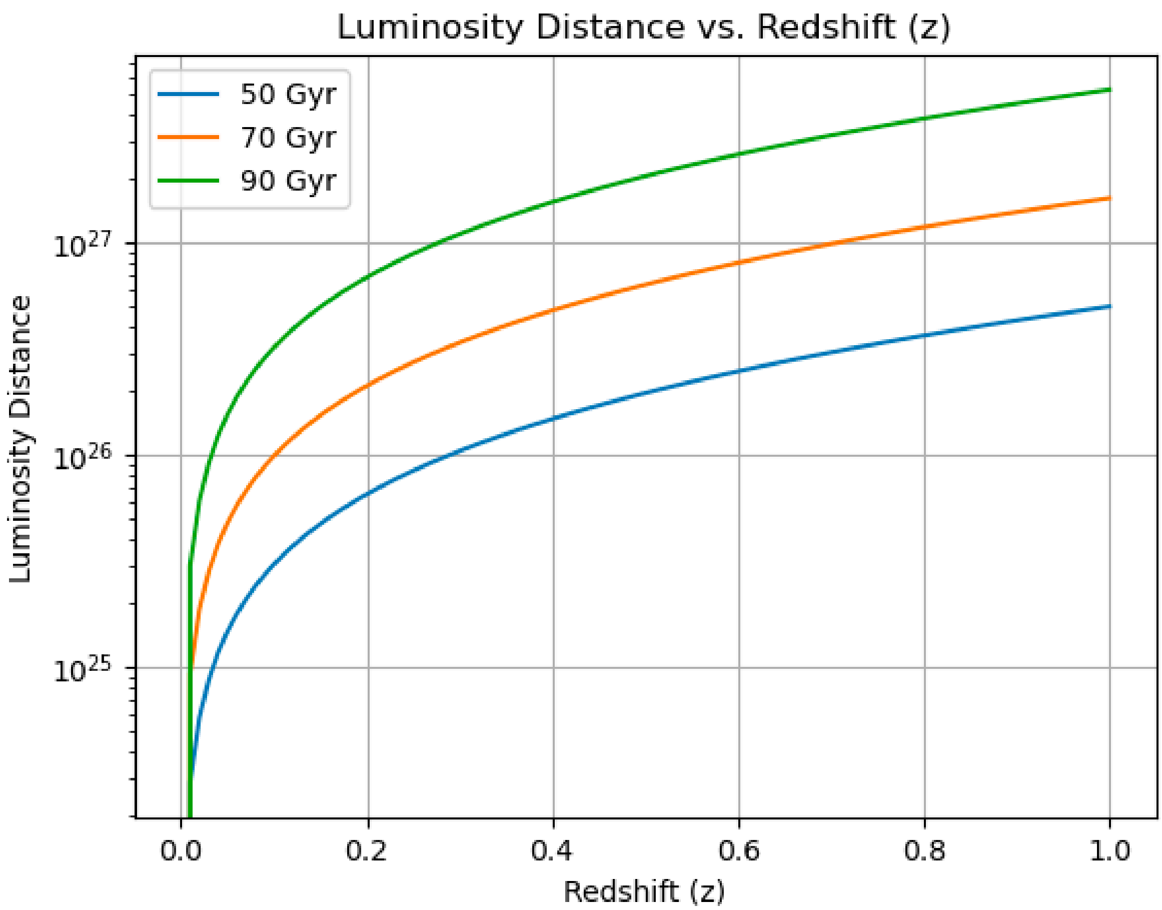

Figure 6: Luminosity Distance versus Redshift

and less useful parameter. Co-moving distance is still possible to calculate, with increasingly less accuracy, although for far greater lengths of time eventually all objects will become invisible in this universe except for those outside of the local cluster and thus no meaningful measurements of standard candles can be performed, nor could future civilizations make meaningful calculations about the kind of universe they live in.

Galaxies outside of the local cluster of an observer would experience greater redshift over time. This can be seen with Figure 6, as luminosity distance is proportional to the redshift, and luminosity distance increases with time. Within a local cluster objects will appear blue shifted as they are gravitationally bound to each other, and will move closer together over time.

For future civilizations, it becomes gradually harder to determine what universe they live in as objects become more and more redshifted. Far into the future, as the universe continues to expand, the horizon distance (Equation 5.115) - the furthest distance which an object can be observed - will also change. The horizon distance is given by:

$$d_{hor}(t_0) = \frac{3.20c}{H_0} \tag{20}$$

Thus equation describes how the horizon distance is inversely proportional to H_0, which decreases over time. The horizon distance is thus increasing over time, meaning that for a given distance, one would observe later galaxies pass that distance at a smaller velocity than earlier galaxies. Galaxies outside of the Hubble sphere described by the horizon distance are receding from the observer faster than the speed of light. Thus is would be more difficult for future civilizations to determine what universe they live in, as there would be less galaxies to observe and those which are observed would have different velocities than what was expected.

References

[1] benrg (https://physics.stackexchange.com/users/56188/benrg). *What is the equation for the scale factor of the universe, a(t), for the best fit of data to the ΛCDM Model of Cosmology?* Physics Stack Exchange. URL:https://physics.stackexchange.com/q/263725 (version: 2016-06-20). eprint: `https://physics.stackexchange.com/q/263725`. URL: `https://physics.stackexchange.com/q/263725`.

[2] Pulsar (https://physics.stackexchange.com/users/24142/pulsar). *CMBR temperature over time?* Physics Stack Exchange. URL:https://physics.stackexchange.com/q/76768 (ver-

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