SUPPLEMENTARY MATERIAL FOR LOGIC

THE BASICS

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CONTENTS

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FURTHER EXERCISES

1.1 CHAPTER 1: CONSEQUENCES

1. As noted in the chapter 'if and only if' (which is often abbreviated as 'iff') expresses two conditionals: 'A iff B' expresses both of the following conditionals.

   - If A, then B.
   - If B, then A.

   For our purposes, a biconditional 'A iff B' is true so long as A and B are either both true or both false (and such biconditionals are false otherwise). With this in mind, consider the necessary consequence relation. Is the following argument valid (where, here, validity is necessary consequence)? If it is valid – if its conclusion is a necessary consequence of the premises – explain why it is valid. If not, explain why not.

   - Max is happy if and only if Agnes is sleeping.
   - Agnes is sleeping.
   - Therefore, Max is happy.

   What about the following argument?

   - Max is happy if and only if Agnes is sleeping.

- Agnes is not sleeping.
- Therefore, Max is not happy.

2. Consider the 'necessary consequence' relation, which takes cases to be possibilities. Assume, as is reasonable (!), that our actual world is possible – that is, that whatever is true (actually true) is possibly true. Question: on this account of logical consequence, are there any sound arguments that have false conclusions? If so, why? If not, why not?
3. What are counterexamples, and how do they help us to determine whether an argument is valid?
4. Consider the general recipe for logical consequence. If B is a logical consequence of $A_1 \ldots A_n$, is it possible for B to be false? If so, why? If not, why not?
5. Consider the 'necessary consequence' relation, which takes cases to be possibilities. Provide an example of an argument that is valid if 'possible' is taken to mean *physically possible*, but is *invalid* if 'possible' is taken to mean *imaginable*.
6. Using the 'necessary consequence' account of validity, provide a valid argument for each of the following conclusions:

   (a) Either Ray is a raven, or Ray is not a raven.
   (b) The cat is on the mat.
   (c) Kathy is tall, and Trevor is short.
   (d) Some goldfish are orange.

7. Can you make a valid argument invalid by adding more premises? Can you make an invalid argument valid by adding more premises? If so, provide an example. If not, why not?
8. Using the 'necessary consequence' account of validity, provide a conclusion for each of the following arguments that makes it valid.

   (a) i. If it is raining, then the pavement is wet.
       ii. If the pavement is wet, then Sarah slips and falls.
       iii. It is raining.
       iv. Therefore, . . .

(b) i. Either Spain is in Europe or Egypt is not in Africa.
    ii. Egypt is in Africa.
    iii. Therefore, . . .
(c) i. Grass is green, and snow is white.
    ii. Therefore, . . .
(d) i. All tigers are predators.
    ii. All predators are carnivores.
    iii. Therefore, . . .

9. According to the 'necessary consequence' account of validity, which of the following arguments are valid? Which are sound? Justify your answers.
   (a) i. Either Spain is in Europe or Egypt is not in Africa.
       ii. Spain is not in Europe.
       iii. Therefore, Egypt is not in Africa.
   (b) i. All cats are mammals.
       ii. No spiders are mammals.
       iii. Therefore, no spiders are cats.
   (c) i. All physics majors are STEM majors.
       ii. No English majors are physics majors.
       iii. Therefore, no English majors are STEM majors.

1.2 CHAPTER 2: MODELS, MODELED AND MODELING

1. Modeling is, as we've said, incredibly useful. But like most methods of studying the world, modeling comes with limits. Come up with a list of examples of ways we can be led astray when using modeling. It's useful to do this by a two step procedure:
   (a) Pick a particular situation in which one thing, x, serves as a model of another thing, y.
   (b) Think, in this situation, about things one might *falsely* attribute to y on the basis of comparison with x.
2. (This continues the previous exercise.) Given that models can, at times, lead us astray, what does this suggest about ways we might want to think about conclusions we draw on the basis

of models? What does this say, in particular, about the case at hand, where we are using logical theories to model fragments of the natural language consequence relation? (It is worthwhile to revisit this question from time to time as you work through the remainder of the book.)

1.3 CHAPTER 3: LANGUAGE, FORM, AND LOGICAL THEORIES

1. Relying on the informal idea of 'possible circumstance' for our 'cases', and using the 'truth condition' in §3.5 for conjunction, say whether the following argument form is valid: $A \wedge B \therefore B$. Justify your answer by invoking the general definition of 'validity' (or logical consequence) and the given truth condition.
2. Consider the argument form $\neg A \vee B, A \therefore B$. Taking 'cases' to be 'possible circumstances', and using the truth conditions that you provided for disjunction and negation, is the given form valid? Justify your answer. (Your answer may turn, in part, on your philosophy of 'possible circumstances'!)
3. Let us say that a *sentence* is *logically true* if and only if there is no case in which it is not true. Using the truth conditions that you gave for disjunction and negation, say whether the disjunction of 'Agnes is running' and 'Agnes is not running' is logically true. Justify your answer. (Also, what is the logical form of the given sentence? Is it true that, given your truth conditions, *every* sentence of that form is logically true?)
4. Consider the following argument:

   (a) Max is a bachelor.
   (b) Therefore, Max is unmarried.

   Neither sentence has any of our given connectives, and so both sentences are atomic, at least according to our definitions above. As such, atomics have no significant logical form. Instead, following the policy according to which distinct sentences are represented by distinct letters, we would represent the argument form thus: $A \therefore B$. Is this argument form *valid*? If so, why? If not, why not? If there's not enough information to tell, what is the missing information? What premise might be added to make the argument valid?

5. Consider, again, the argument above from 'Max is a bachelor' to 'Max is unmarried'. Is the conclusion a *necessary consequence* of the premise? If so, what, if anything, does this suggest about the role of 'logical form' in the 'necessary consequence' account of validity given in Chapter 1?
6. In your own words, say what it is to give *truth-in-a-case conditions* (or, for our purposes, truth conditions) for sentences. Why, if at all, is this activity – that is, giving so-called truth conditions – essential to an account of *logical consequence* as we've defined it (in Chapter 1)?
7. What is the relationship between logical form and the *validity* and *soundness* of arguments?
8. For each of the following English sentences, state whether it is atomic or molecular, and provide its logical form.

   (a) Joe is a purple cow.
   (b) Joe is purple and Joe is a cow.
   (c) Joe is purple or Joe is a cow.
   (d) Joe is not a cow.
   (e) Joe is not a purple cow and Joe is a cow.
   (f) Joe is not a cow or Joe is purple and a cow.

9. Using the truth conditions you supplied for disjunction and negation and the condition in §3.5 for conjunction, determine which of the following argument forms are valid. Justify your answers. For each invalid argument form, are there premises that could be added to make it a valid form?

   (a) $A \vee B, B \therefore \neg A$
   (b) $A \vee B \therefore C$
   (c) $A \therefore A \wedge (B \vee A)$
   (d) $\neg(A \wedge B) \therefore \neg A \vee B$
   (e) $A \vee (A \wedge B) \therefore A$
   (f) $\neg A \vee \neg B, B \therefore \neg A$

(g) $A \vee (B \vee C), C \therefore \neg A$
(h) $A \vee (B \vee C), \neg C \therefore A \vee B$

10. Suppose there is a unary connective '%' with the following truth-in-a-case conditions: $\%A$ is *true in a possible circumstance* c if and only if either A is true-in-c or A is not true-in-c. Question: Is $A \therefore \%B$ a valid argument form?
11. Suppose there is a binary connective '$\veebar$' with the following truth-in-a-case conditions: $A \veebar B$ is true-in-a-possible-circumstance-c if and only if only A is true-in-c or only B is true-in-c. Is the argument form $A \veebar B, \neg B \therefore \neg A$ valid? What about the argument form $A \veebar B, B \therefore \neg A$? What fragment or word of English might $\veebar$ successfully model?
12. Provide what you think are reasonable truth conditions for the following binary connectives:

    (a) 'But', 'although', 'despite the fact that' (e.g., 'A but B').
    (b) 'Unless' (e.g., 'A unless B').
    (c) 'If ..., then ...' (e.g., 'If A, then B').

1.4 CHAPTER 4: SET-THEORETIC TOOLS

1. Consider the relation of *loves*, which holds between objects x and y if and only if x loves y. Is this relation a function? Justify your answer.
2. Since functions are relations, and all relations have a domain and range, it follows that functions have a domain and range. We say that the *domain* of a function f is the set of f's arguments (or 'inputs'), and the *range* of f is the set of f's values (or 'outputs'). Let the domain of g be $\{1,2,3\}$, where g is defined as follows.

$$g(x) = x + 22$$

   What is the range of g?
3. Let $\mathcal{X} = \{1, 2\}$ and $\mathcal{Y} = \{\text{Max, Agnes}\}$. Specify *all* (non-empty) functions whose domain is $\mathcal{X}$ and range is $\mathcal{Y}$.
4. Specify all (non-empty) subsets of $\{1,2,3\}$.
5. Show why each of the following are true for any sets $\mathcal{X}$ and $\mathcal{Y}$.

(a) If $\mathcal{X} \neq \mathcal{Y}$, then $\mathcal{X} \cap \mathcal{Y} \subset \mathcal{X} \cup \mathcal{Y}$.
(b) If $\mathcal{X} \subset \mathcal{Y}$, then $\mathcal{X} \cup \mathcal{Y} = \mathcal{Y}$.
(c) If $\mathcal{X} \subset \mathcal{Y}$, then $\mathcal{X} \cap \mathcal{Y} \subset \mathcal{Y}$.

Answer to part b We have to show that *if* $\mathcal{X} \subset \mathcal{Y}$, *then* $\mathcal{X} \cup \mathcal{Y} = \mathcal{Y}$. We show this (viz., the given conditional) by so-called conditional proof: we assume that the antecedent is true (viz., that $\mathcal{X} \subset \mathcal{Y}$), and then show – via valid steps (!) – that the consequent is true (usually, we do this simply by invoking definitions involved). So, suppose that $\mathcal{X} \subset \mathcal{Y}$, in which case, by definition of *proper subset* (see Def. 11), it follows that anything in $\mathcal{X}$ is in $\mathcal{Y}$, and that $\mathcal{Y}$ contains something that $\mathcal{X}$ doesn't contain. Now, we need to show the consequent of our target conditional: viz., that $\mathcal{X} \cup \mathcal{Y} = \mathcal{Y}$. This is an identity claim: it claims that the two given sets are identical. How do we show that they're identical? Well, we have to invoke the definition of identity for sets, which tells us that, in this case, $\mathcal{X} \cup \mathcal{Y} = \mathcal{Y}$ iff both $\mathcal{X} \cup \mathcal{Y}$ and $\mathcal{Y}$ contain exactly the same things. In other words, we show that $\mathcal{X} \cup \mathcal{Y} = \mathcal{Y}$ by showing that something (no matter what it is) is in $\mathcal{X} \cup \mathcal{Y}$ if and only if it's in $\mathcal{Y}$. So, in effect, we have to show that two different conditionals are true to show that the two sets are identical:

b.1 If something (no matter what it is) is in $\mathcal{X} \cup \mathcal{Y}$, it is in $\mathcal{Y}$.
b.2 If something (no matter what it is) is in $\mathcal{Y}$, it is in $\mathcal{X} \cup \mathcal{Y}$.

And here, we can just do so-called conditional proofs again for each of (b.1) and (b.2): we assume the given antecedents and show, via valid steps (usually just appealing to the definitions), that the given consequents follow. So, for (b.1), we assume that something – call it (no matter what it is) 'z' – is in $\mathcal{X} \cup \mathcal{Y}$. What we have to show is that z is in $\mathcal{Y}$. Well, by assumption, we have that $z \in \mathcal{X} \cup \mathcal{Y}$, in which case, *by definition of union* (see Def. 8), if $z \in \mathcal{X} \cup \mathcal{Y}$ then *either* $z \in \mathcal{X}$ *or* $z \in \mathcal{Y}$. In the latter case, we have what we want (viz., that $z \in \mathcal{Y}$). What about the former case in which $z \in \mathcal{X}$? Do we also get that $z \in \mathcal{Y}$? Yes: we get this from our initial supposition that $\mathcal{X} \subset \mathcal{Y}$, which assures that anything in $\mathcal{X}$ is in $\mathcal{Y}$. What this tells us is that, either way, if something z (no matter what z may be) is in $\mathcal{X} \cup \mathcal{Y}$, then it's also in $\mathcal{Y}$ (provided that, as we've assumed from the start, $\mathcal{X} \subset \mathcal{Y}$). And this is what we wanted to show for (b.1).

With respect to (b.2), we assume that something z (no matter what z is) is in $\mathcal{Y}$. We need to show that $z \in \mathcal{X} \cup \mathcal{Y}$. But this follows immediately from the definition of union (see Def. 8).[1] According to the definition, something is in $\mathcal{X} \cup \mathcal{Y}$ if and only if it's either in $\mathcal{X}$ or in $\mathcal{Y}$. Hence, given that (by supposition) $z \in \mathcal{Y}$, we have it that $z \in \mathcal{X} \cup \mathcal{Y}$.

Taking stock of Answer 9.b. What we've proved, in showing (b.1) and (b.2), is that, under our assumption that $\mathcal{X} \subset \mathcal{Y}$, something (no matter what it is) is in $\mathcal{X} \cup \mathcal{Y}$ iff it's in $\mathcal{Y}$. By definition of identity for sets (see Def. 7), this tells us that, under our assumption that $\mathcal{X} \subset \mathcal{Y}$, the sets $\mathcal{X} \cup \mathcal{Y}$ and $\mathcal{Y}$ are identical. And this is what (b) required us to show.

6. Let f be some function with $dom(f) = \mathcal{X}$ (i.e., the domain of f is $\mathcal{X}$), for some arbitrary (non-empty) set $\mathcal{X}$. We say that our function f is a function *from* $\mathcal{X}$ *into* $\mathcal{Y}$ if $ran(f) \subseteq \mathcal{Y}$. Given this terminology, specify *all* (non-empty) functions from $\{A, B\}$ *into* $\{1, 2, 3\}$, where A and B are distinct sentences. (Note that any such function must map *every* element of the domain to something in $\{1, 2, 3\}$.)
7. Let $\mathcal{X}$ be an arbitrary set and f an arbitrary function. We say that f is an *operator on* $\mathcal{X}$ if and only if the $dom(f) = \mathcal{X}$ and $ran(f) \subseteq \mathcal{X}$. Consider the following operator on $\{1, 0\}$.

$$g(x) = 1 - x$$

Now, imagine a function v that assigns either 1 or 0 to each atomic sentence of our language, so that, for any atomic sentence A of our language, we have it that $v(A) = 1$ or $v(A) = 0$. Answer the following questions.

(a) Suppose that $v(A) = 1$. What is $g(v(A))$?
(b) Suppose that $v(A) = 0$. What is $g(v(A))$?

(c) If $v(A) = 1$, what is $g(g(v(A)))$?
(d) Is it true that $g(g(x)) = 1$ just when $x = 1$?

8. Let f be some function with $dom(f) = \mathcal{X}$, for some arbitrary, non-empty set $\mathcal{X}$. We say that f is a function *from* $\mathcal{X}$ *onto* a set $\mathcal{Y}$ if $ran(f) = \mathcal{Y}$. Given this terminology, specify all functions from $\{1, 2, 3\}$ *onto* $\{A, B\}$, where A and B are distinct sentences.
9. For any two sets $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{X} = \mathcal{Y}$ if and only if $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{Y} \subseteq \mathcal{X}$. Show why this is true by showing why each of the following are true for any sets $\mathcal{X}$ and $\mathcal{Y}$.

   (a) If $\mathcal{X} = \mathcal{Y}$, then $\mathcal{X} \subseteq \mathcal{Y}$.
   (b) If $\mathcal{X} = \mathcal{Y}$, then $\mathcal{Y} \subseteq \mathcal{X}$.
   (c) If $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{Y} \subseteq \mathcal{X}$, then $\mathcal{X} = \mathcal{Y}$.

10. What are the subsets of the empty set $\emptyset$? What about the set $\{\emptyset\}$? What about $\{\emptyset, \{\emptyset\}\}$?
11. Bertrand Russell showed that (at least in classical set theory) not every predicate can serve as an entry condition in a definition by abstraction. To see this, show why there is no set successfully picked out by the definition by abstraction '$\{x : x$ is not a member of $x\}$'. (You might show this by first assuming that there is such a set and then showing that this assumption has consequences that we know to be false.)
12. Is the Cartesian Product operation a function? Why or why not?
13. Let $dom(f) = \{1, 2, 3, 4\}$, and let $f(x) = x$ for all $x \in dom(f)$. Show that f is an equivalence relation.
14. Suppose $\mathcal{X} \subseteq \mathcal{Y}$. What does this tell us, if anything, about the set $\mathcal{X} \cap \mathcal{Y}$? What about $\mathcal{X} \cup \mathcal{Y}$? Justify your answers.
15. Suppose $\mathcal{X} = \mathcal{Y}$. Which of the following must be true? Which must be false? Justify your answers. If there is not enough information to tell, explain why.

   (a) $\mathcal{X} \subset \mathcal{Y}$ or $\mathcal{Y} \subset \mathcal{X}$
   (b) $\mathcal{X} \cup \mathcal{Y} = \mathcal{X} \cap \mathcal{Y}$
   (c) $\mathcal{X} \subseteq \mathcal{Y}$
   (d) $(\mathcal{X} \cap \mathcal{Y}) \subset \mathcal{Y}$

16. Give *two* definitions by abstraction for each of the following sets.

    (a) $\{4,5,6\}$
    (b) $\{2\}$
    (c) $\{1,3,5,7,9\}$
    (d) $\emptyset$
    (e) $\{\emptyset\}$

17. Which of the following are true, if any? Why?

    (a) $\emptyset \subset \emptyset$
    (b) $\emptyset \in \emptyset$
    (c) $\emptyset \in \{2,4,6\}$
    (d) $\emptyset \in \{\emptyset\}$

18. Suppose $a = b$ and $c = d$. Which of the following, if any, are true for any sets $\mathcal{X}$ and $\mathcal{Y}$? Why?

    (a) If $a \in \mathcal{X}$, then $b \in \mathcal{X}$.
    (b) If $a \in \mathcal{X}$ and $d \in \mathcal{X}$, then $\{a,b,c\} \subseteq \mathcal{X}$.
    (c) If $a \in \mathcal{X}$ and $c \in \mathcal{X}$, then $\{b,c,d\} \subset \mathcal{X}$.
    (d) If $a \in \mathcal{X}$ but $d \notin \mathcal{X}$, then $b \neq c$.
    (e) If $a \neq d$, then at least one of b and c is in $\mathcal{X}$.
    (f) If $a \neq d$, then at least one of b and c is *not* in $\mathcal{X}$.
    (g) If $a \in \mathcal{X}$ and $d \in \mathcal{Y}$, then $\mathcal{X} \neq \mathcal{Y}$.
    (h) If $\mathcal{X} \neq \mathcal{Y}$ and $a \in \mathcal{X}$ and $d \in \mathcal{Y}$, then $b \neq c$.
    (i) If $\mathcal{X} = \mathcal{Y}$ and $a \in \mathcal{X}$, then $b \in \mathcal{Y}$.

19. Given what you've learned about the properties of binary relations, which of the following, if any, are true of the *membership* relation? Why?

    (a) It is reflexive.
    (b) It is symmetric.
    (c) It is transitive.
    (d) It is an equivalence relation.

20. Given what you've learned about the properties of binary relations, which of the following, if any, are true of the *proper subset* relation designated by '$\subset$'? Why?

(a) It is reflexive.
(b) It is symmetric.
(c) It is transitive.
(d) It is an equivalence relation.

21. Given what you've learned about the properties of binary relations, which of the following, if any, are true of the *subset* relation designated by '$\subseteq$'? Why?

    (a) It is reflexive.
    (b) It is symmetric.
    (c) It is transitive.
    (d) It is an equivalence relation.

22. Let f and g be any functions such that $ran(f) = dom(g)$. Show why the function h such that $h(x) = g(f(x))$ for all $x \in dom(f)$ is also a function. What is its range?
23. Show why the following is true for any sets $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$: If $\mathcal{X} \cap \mathcal{Z} = \mathcal{Y} \cap \mathcal{Z}$ and $\mathcal{X} \cup \mathcal{Z} = \mathcal{Y} \cup \mathcal{Z}$, then $\mathcal{X} = \mathcal{Y}$.
24. Represent each of the following unary functions with a table, given that the domain of each is $\{0, 1, 2, 3\}$. What is each function's range?

    (a) $f(x) = x^2$
    (b) $g(x) = |x - 3|$
    (c) $h(x) = f(g(x))$

25. Represent each of the following binary functions with a table, given that the domain of each is $\{0, 1, 2, 3\} \times \{0, 1, 2, 3\}$. What is each function's range?

    (a) $f(x, y) = x^2 + y$
    (b) $g(x, y) = y - |x - 3|$
    (c) $h(x, y) = x \cdot y$

1.5 CHAPTER 5: BASIC CLASSICAL SYNTAX AND SEMANTICS

1. Can you think of a way of defining $\vee$ in terms of $\neg$ and $\wedge$? (Hint: see whether you can come up with a sentence that uses

only $\neg$ and $\wedge$ but has exactly the same 'truth table' as $\vee$.) If so, you've shown that, strictly speaking, we can reduce our number of basic connectives to just $\neg$ and $\wedge$ (and treat $\vee$, like the others, as defined).

2. Related to the previous question, can you think of a way of defining $\wedge$ in terms of $\vee$ and $\neg$?
3. Suppose we have a binary connective '$\veebar$' with the following truth conditions:

$$v \models_1 T \veebar U \text{ iff either } v \models_1 T \text{ and } v \models_0 U \text{ or } v \models_0 T \text{ and } v \models_1 U$$

$$v \models_0 T \veebar U \text{ iff either } v \models_1 T \text{ and } v \models_1 U \text{ or } v \models_0 T \text{ and } v \models_0 U.$$

Do these look like the truth conditions of any connective in English? How might we define '$\veebar$' in terms of '$\wedge$', '$\vee$' and '$\neg$'?

4. Suppose we have a binary connective '$\uparrow$' with the following truth conditions:

$$v \models_1 T \uparrow U \text{ iff } v \models_0 T \text{ or } v \models_0 U \text{ (or both)}$$

$$v \models_0 T \uparrow U \text{ iff } v \models_1 T \text{ and } v \models_1 U.$$

(This connective is called the Sheffer Stroke.) How might we define '$\uparrow$' in terms of '$\wedge$', '$\vee$' and '$\neg$'? How might we define '$\vee$' and '$\neg$' in terms of '$\uparrow$'?

5. Suppose that for arbitrary sentences $A_1, A_2, \ldots, A_n$ and B we have $A_1, A_2, \ldots, A_n \vdash B$. Could there be a sentence S such that $A_1, A_2, \ldots, A_n, S \nvdash B$? Why or why not?
6. Show that '$(A \to B) \vee (B \to A)$', where '$\to$' is defined as in §5.6, is a logical truth of basic classical logic. In this respect, does '$\to$' behave like the connective 'if... then...' of English? In what ways might '$\to$' behave similarly to the English conditional?
7. Which of the following are sentences in the basic classical language?

   (a) $p \vee q \vee r$
   (b) $(p \wedge q) \to q$
   (c) $(\neg((p \wedge q) \vee r))$

(d) $\vee(p\neg q)$
(e) $p_{439} \vee (q_{27} \wedge q_{13})$
(f) $(p \vee r) \leftrightarrow (r \vee \neg\neg\neg(q \wedge r))$
(g) $p \wedge (p \vee \neg p)$

8. Correct the following to make them sentences of the basic classical language.

   (a) $p \wedge q$
   (b) $p \vee q \wedge r$
   (c) $\neg(\neg(r_{67}))$
   (d) $(\neg(q \vee r \vee r)$
   (e) $p \vee ((r \wedge \neg(r)))$

9. Let v be a basic classical case such that $v(p) = v(q) = 1$ and $v(r) = 0$. Which of the following sentences are true-in-v?

   (a) $p \wedge q$
   (b) $r \vee \neg r$
   (c) r
   (d) $p \rightarrow r$

10. Are the following argument forms classically valid? Why or why not?

    (a) $A \rightarrow B, B \therefore A$
    (b) $A, B \vee (B \vee \neg B) \therefore A \wedge B$
    (c) $\neg\neg\neg A \therefore \neg A$
    (d) $A, B \vee (B \wedge \neg B) \therefore A \wedge B$
    (e) $A \rightarrow (B \wedge \neg B) \therefore \neg A$
    (f) $A \rightarrow B, B \rightarrow C, \neg B \therefore \neg(C \wedge A)$
    (g) $A \vee B, B \therefore \neg A$

11. Our classical language is entirely precise. Can you think of examples of where natural language is not precise in this way?
12. Suppose that, instead of functions, we model our classical cases as *sets of sentences*. A *case*, on this approach, is a set $\mathcal{X}$ of sentences. In turn, we say that *truth in a case* is just membership – i.e., being an element – in such a set. Your task is two-fold:

(a) What constraints do we impose on the given cases for them to be *classical* – i.e., 'complete' and 'consistent'?
(b) What are the truth conditions for conjunctions, disjunctions, and negations on this approach?

1.6 CHAPTER 6: BASIC CLASSICAL TABLEAUX

1. Using tableaux, show that each of the following argument forms is *invalid*. Then use an open complete branch to construct a counterexample.

   (a) $A \to B, \neg A \therefore \neg B$
   (b) $A \vee \neg A, \neg B \vee \neg A \therefore A \vee \neg B$
   (c) $A \to B, \neg C \to \neg A, \neg C \therefore B$
   (d) $A \vee (B \wedge C) \therefore A \to (\neg B \vee \neg C)$
   (e) $A \to (\neg B \vee \neg C) \therefore A \vee (B \wedge C)$
   (f) $A \to \neg B, \neg B \vee \neg C, C \therefore \neg A$
   (g) $(A \wedge C) \vee (B \wedge D), \neg D \to \neg(A \vee C) \therefore (B \wedge D) \to \neg(A \wedge C)$
   (h) $(A \vee B) \vee C, \neg A \therefore B$

2. Using tableaux, determine whether each of the following arguments is valid. If it is invalid, use an open complete branch to construct a counterexample.

   (a) $A \therefore B \vee C$
   (b) $A \vee \neg A \therefore A$
   (c) $A, \neg A \therefore B$
   (d) $A \to B, B \to A \therefore B$
   (e) $A \to B, B \to C, A \therefore C$
   (f) $(A \wedge B) \to C \therefore A \to C$
   (g) $A \to B, C \to B \therefore A \to C$
   (h) $A \to B \therefore (A \wedge B) \leftrightarrow A$

3. What is the difference between nodes $\langle A \quad \ominus \rangle$ and $\langle \neg A \quad \oplus \rangle$? Is there a related difference when we talk about truth-in-a-case?
4. Explain in your own words why it is usually better to apply rules that don't branch before applying rules that do.
5. Explain in your own words the relationship between branches of tableaux and cases. What do we show when we produce

a tableau that closes? What do we show when we produce a tableau that doesn't close?

6. Suppose we want to show that an argument from premises $A_1, A_2, \ldots, A_n$ to conclusion B is *invalid*. Why would it be a *bad* strategy to begin a tableau

$$\boxed{\begin{array}{c} \langle A_1 \quad \oplus\rangle \\ \langle A_2 \quad \oplus\rangle \\ \ldots \\ \langle A_n \quad \oplus\rangle \\ \langle B \quad \oplus\rangle \end{array}}$$

and then attempt to show that this tableau closes?

7. Justify the claim that, for any argument with finitely many premises, the procedure to extend the tableau for this argument will end at some point.
8. Justify each tableau rule in Figure 6.3 by appealing to the truth conditions of the connectives given in Chapter 5.

1.7 CHAPTER 7: BASIC CLASSICAL TRANSLATIONS

1. For each of the following natural language arguments, (i) identify the atomic subsentences that occur within it, and assign an atomic sentence of Basic Classical Theory to each; (ii) provide the standard translation of the argument into the language of Basic Classical Theory; and (iii) construct a tableau to determine whether it is valid. Do any of these arguments contain information inside their atomic sentences not captured by the translations? If so, how does that affect their validity. If this makes an argument invalid, what premises might be added to make this argument valid?

   (a) i. Aurore Dupin befriended many artists.
      ii. George Sand wrote many fine novels.
      iii. George Sand is Aurore Dupin.
      iv. Therefore, George Sand wrote many fine novels and befriended many artists.

(b) i. If anyone is napping, then Max is napping.
  ii. Agnes is napping.
  iii. Therefore, Max is napping.

(c) i. Either Max and Agnes are jumping, or neither Max nor Carol is clapping.
  ii. Carol is not clapping.
  iii. Therefore, Max is jumping.

2. For each of the following English sentences, provide a translation into Basic Classical Logic. Be sure to include a translation key. Then use a tableau to determine whether each sentence is logically true, logically false, or contingent.

   (a) Either Jill is not running or Jack is not running.
   (b) Neither Jill nor Jack is running.
   (c) If John is twelve, then he is neither a child nor not a child.
   (d) If John is neither a child nor not a child, then Patty is playing croquet.
   (e) If Abby is a pirate, then she is a sailor, but not an officer.
   (f) If Agnes is sleeping but Ant is awake, then Ant is baking cookies.
   (g) Chip is neither tall nor not tall.
   (h) Agnes tripped over the cat after coming inside.
   (i) Either it is the case that, if Max is playing tennis, then Agnes is eating lunch, or it is the case that, if Agnes is eating lunch, then Max is playing tennis.
   (j) If Sam is playing, then neither Max nor Agnes is playing.
   (k) Neither Max nor Agnes is drinking a soda or eating lunch.

3. Using the key from exercise 1 (p. 46), 'translate' the following sentences from the language of Basic Classical Theory into English.

   (a) $p_1 \rightarrow \neg p_2$
   (b) $(p_1 \wedge q_1) \rightarrow \neg(p_2 \wedge q_2)$
   (c) $\neg(p_2 \vee q_2) \rightarrow (p_1 \vee q_1)$
   (d) $(q_2 \rightarrow p_1) \wedge (q_1 \rightarrow p_2)$
   (e) $\neg(p_1 \vee q_1) \rightarrow (p_2 \vee (q_2 \wedge p_2))$
   (f) $\neg\neg(p_1 \vee q_1) \wedge (\neg p_2 \wedge \neg q_2)$

1.8 CHAPTER 8: ATOMIC INNARDS: UNARY

1. Construct a case in which $Fa \wedge \neg Fb$ is true.[2]

   Sample Answer A case in which $Fa \wedge \neg Fb$ is true as follows. Let $c = \langle D, \delta \rangle$, where $D = \{1,2\}$ and $\delta(a) = 1$, $\delta(b) = 2$, and $F^+ = \{1\}$ and $F^- = \{2\}$. (NB: there are many other cases in which $Fa \wedge \neg Fb$ is true.)

2. Construct a case in which $Fa \rightarrow \neg Fb$ is false.
3. Construct a case in which $(Fa \vee \neg Fa) \wedge Fb$ is false.
4. Consider a case $c = \langle D, \delta \rangle$ where $D = \{1,2\}$, $\delta(a) = 1$, $\delta(b) = 2$, $R^+ = \{1\}$, and $F^+ = D$. Given the requirements for classical cases, what do R^- and F^- have to be? Which of the following are true? Why or why not?

   (a) $c \models_1 Rb$
   (b) $c \models_0 Ra$
   (c) $c \models_1 Fa \rightarrow Ra$
   (d) $c \models_1 Fa \rightarrow Ra$
   (e) $c \models_0 Ra \vee Rb$
   (f) $c \models_0 Ra \leftrightarrow Rb$
   (g) $c \models_1 (Ra \leftrightarrow Rb) \rightarrow Ra$
   (h) $c \models_0 \neg Fb \vee Ra$

   Selected Answers

   $R^- = \{2\}$ and $F^- = \emptyset$.

   (e) $c \models_0 Ra \vee Rb$ iff $c \models_0 Ra$ and $c \models_0 Rb$ iff $\delta(a) \in R^-$ and $\delta(b) \in R^-$. But $R^- = \{2\}$, so $\delta(a) \notin R^-$, so $c \models_0 Ra \vee Rb$ is false.

   (g) $c \models_1 (Ra \leftrightarrow Rb) \rightarrow Ra$ iff either $c \models_0 Ra \leftrightarrow Rb$ or $c \models_1 Ra$. $c \models_1 Ra$ iff $\delta(a) \in R^+$. Since $\delta(a) = 1$ and $R^+ = \{1\}$, $\delta(a) \in R^+$, so $c \models_1 Ra$, and thus $c \models_1 (Ra \leftrightarrow Rb) \rightarrow Ra$.

5. For each of the following *sets* of sentences, construct a case in which every sentence in the set is true. If there is no such case, explain why.

   (a) $\{Fa \vee Fb, \neg Fa\}$
   (b) $\{\neg Fb, Fa \rightarrow Ga, \neg Fb \rightarrow \neg Ga\}$

(c) $\{Ra, \neg(Rb \vee Ra)\}$
(d) $\{Fa \vee Fb, Fa \rightarrow (Fb \wedge Fc), \neg Fc\}$
(e) $\{(Fa \vee Fb) \wedge Fc, Fc \rightarrow \neg Fa\}$

6. For each of the following invalid arguments, construct a case that serves as a counterexample.

(a) $Fa \rightarrow Fb, \neg Fa \therefore \neg Fb$
(b) $Fa \vee Fb, Fb \therefore \neg Fa$
(c) $Ra \rightarrow Rb, \neg Rb \rightarrow Ra, \neg Ra \therefore Ra$
(d) $Fa \vee Ga, Fa \rightarrow Gb, Ga \rightarrow Fb \therefore Fb \wedge Gb$
(e) $\neg(Fa \wedge Fb), Fc \rightarrow (Fa \vee Fb) \therefore \neg Fc$

1.9 CHAPTER 9: EVERYTHING AND SOMETHING

1. Which of the following are true? Why?

(a) $\forall x \neg Px \dashv\vdash \neg \exists x Px$
(b) $\exists x(Px \vee Qx) \dashv\vdash \exists x Px \vee \exists x Qx$
(c) $\forall x Px \vdash \exists x Px$

2. How might we define the universal quantifier in terms of the existential quantifier and the basic sentential connectives? How might we define the existential quantifier in terms of the universal quantifier and the basic sentential connectives? Use the truth conditions of the quantifiers and sentential connectives to justify your answers.
3. Consider a case $c = \langle D, \delta \rangle$ where $D = \{1, 2, 4\}$, $\delta(a) = 1$, $\delta(b) = 2$, $\delta(c) = 4$, $F^+ = \emptyset$, $F^- = D$, $G^+ = D$, and $G^- = \emptyset$. Which of the following is true? Why?

(a) $c \models_1 \exists x Fx$
(b) $c \models_0 \exists x Fx$
(c) $c \models_1 \forall x \neg Fx$
(d) $c \models_0 \forall x \neg Fx$
(e) $c \models_1 \forall x(Fx \rightarrow Gx)$
(f) $c \models_0 \forall x(Fx \rightarrow Gx)$
(g) $c \models_1 \forall x(Fx \rightarrow \neg Gx)$
(h) $c \models_0 \forall x(Fx \rightarrow \neg Gx)$
(i) $c \models_1 \forall x \neg Fx \rightarrow \neg \forall x Fx$

(j) $c \models_1 \exists x Gx \rightarrow \neg\exists y Fy$
(k) $c \models_0 \exists x Gx \rightarrow \neg\exists y Fy$
(l) $c \models_0 \exists x(Fx \wedge Gx)$

4. What is, in your own words, the difference between a free variable and a bound variable?
5. What is, in your own words, the difference between a (well-formed) formula and a full sentence?

1.10 CHAPTER 10: FIRST-ORDER LANGUAGE WITH ANY-ARITY INNARDS

1. For each of the following invalid arguments, construct a case that serves as a counterexample.

   (a) $Fa \rightarrow Fb, \neg Fa \therefore \neg Fb$
   (b) $Fa \vee Fb, Fb \therefore \neg Fa$
   (c) $Rab \rightarrow Rba, \neg Rba \rightarrow Raa, \neg Raa \therefore Rab$
   (d) $Fa \vee Ga, Fa \rightarrow Gb, Ga \rightarrow Fb \therefore Fb \wedge Gb$
   (e) $\neg(Fa \wedge Fb), Fc \rightarrow (Fa \vee Fb) \therefore \neg Fc$

2. Which of the following are (well-formed) formulas of first-order logic? Which are sentences?

   (a) Fxy
   (b) $Rxy \vee Ga \wedge Rba$
   (c) $(Rab \wedge (Gx \wedge \forall x Hx))$
   (d) $\forall x Fa$
   (e) $\forall x \exists y(Rxy \rightarrow Sxy)$
   (f) $\forall y(Rxy \leftrightarrow Gy)$
   (g) $Fx \vee \forall x Fx$
   (h) $Fa \wedge \exists x(Gbc \wedge Fx)$
   (i) $\exists x Fxx \vee \neg Fxx$
   (j) $\exists x(Fxx \vee \neg Fxx)$

3. Consider the sentence ‘Someone loves everyone’. Interpret the binary predicate L as the English predicate ‘loves’. Using the language of first-order logic, show two ways in which this sentence might be interpreted. Justify your response by appealing to the truth conditions of the quantifiers.

4. Is there any way to accurately represent the ambiguity in 'everyone loves someone' in the *monadic* first-order language? How do you represent the ambiguity – the two different readings of logical form – in the general (polyadic) first-order langauge?
5. Which of the following are true? Why?

   (a) $\exists x \exists y Pxy \dashv\vdash \exists x \exists y Pxy$
   (b) $\exists y \forall x Pxy \dashv\vdash \forall x \exists y Pxy$
   (c) $\forall x \forall y Pxy \dashv\vdash \forall y \forall x Pxy$

1.11 CHAPTER 11: IDENTITY

1. For each of the following sets of sentences, construct a case in which every sentence in the set is true.

   (a) $\{Fa \wedge Fb, a \neq b\}$
   (b) $\{Fa \vee Fb, b = c, \neg Fc\}$
   (c) $\{\neg Fa \rightarrow \neg Fb, a = c, \neg Fc\}$
   (d) $\{Fa, \neg(\neg Fb \vee \neg Fc), a \neq c\}$

2. Consider a case $c = \langle D, \delta \rangle$ where $D = \{1,2\}$, $\delta(a) = \delta(b) = 1$, $\delta(c) = 2$, $F^+ = \{1\}$, and $F^- = \{2\}$.

   (a) What is $\mathcal{E}^+_=$?
   (b) What is $\mathcal{E}^-_=$?
   (c) Is it true that $c \models_1 a = b$? Why?
   (d) Is it true that $c \models_0 a = c$? Why?

3. **Difficult.** Prove that for every sentence S (of arbitrary complexity – not just atomics) and every case c, exactly one of $c \models_1 S$ and $c \models_0 S$ holds. That is, prove that the semantic theory given for sentential logic with innards and identity is a complete and consistent semantic theory.

1.12 CHAPTER 12: TABLEAUX FOR FIRST-ORDER LOGIC WITH IDENTITY

1. Prove, using tableau, that the following pairs of sentences are equivalent:

(a) $\forall x(Pa \rightarrow Qx)$ and $Pa \rightarrow \forall xQx$.
(b) $\exists x(Pa \rightarrow Qx)$ and $Pa \rightarrow \exists xQx$.
(c) $\forall x(Px \rightarrow Qa)$ and $\exists xPx \rightarrow Qa$.
(d) $\exists x(Px \rightarrow Qa)$ and $\forall xPx \rightarrow Qa$.

2. Can you summarize, in a sentence, the result of the previous problem?

1.13 CHAPTER 13: FIRST-ORDER TRANSLATIONS

1. Take a paragraph from a recent news article. Build a translation key and translate the paragraph into our first-order language using your translation key.
2. You may have discovered in the course of the previous problem that our first-order language seems to lack analogs to certain pieces of the vocabulary in the paragraph you chose to translate. Choosing one of these, discuss how you might extend the vocabulary of our first-order language to rectify this.

1.14 CHAPTER 14: ALTERNATIVE LOGICAL THEORIES

1. Given the truth conditions for molecular sentences given on page 182, prove the following:

   - That the consistency constraint on atomics (see page 181) yields exactly one of the following for any paracomplete-but-consistent case c and any (not necessarily atomic) sentence S.

     – $c \models_1 S$ and $c \not\models_1 \neg S$
     – $c \not\models_1 S$ and $c \models_1 \neg S$
     – $c \not\models_1 S$ and $c \not\models_1 \neg S$

   - That the completeness constraint on atomics (see page 182) yields exactly one of the following for any paraconsistent-but-complete case c and any (not necessarily atomic) sentence S.

     – $c \models_1 S$ and $c \not\models_1 \neg S$

  - $c \not\models_1 S$ and $c \models_1 \neg S$
  - $c \models_1 S$ and $c \models_1 \neg S$

- That the general constraint on atomics (see page 182) yields exactly one of the following for any paracomplete-and-paraconsistent case c and any (not necessarily atomic) sentence S.

  - $c \models_1 S$ and $c \not\models_1 \neg S$
  - $c \not\models_1 S$ and $c \models_1 \neg S$
  - $c \not\models_1 S$ and $c \not\models_1 \neg S$
  - $c \models_1 S$ and $c \models_1 \neg S$

2. A notable feature of both of our paracomplete theories is that they admit no logical truths! Recall that, in general, A is a logical truth iff A is true-in-*all cases*. On the paracomplete theory, our cases are either classical *or* incomplete but consistent. As above, the paracomplete theory has a broader range of cases than the classical theory. The result of such broadening is that there are more 'potential counterexamples'.

   Of course, given the philosophical motivation of the paracomplete theory – for example, *unsettledness* or *gaps* – one would expect that Excluded Middle would fail, and it does. To see this, recall that Excluded Middle fails if there's a case in which $A \vee \neg A$ is not true. On the paracomplete theory, there is certainly such a case. After all, just consider a case c that is incomplete with respect to A, that is, a case c such that $c \not\models_1 A$ and $c \not\models_0 A$. By the given truth conditions for negation (see §14.5.2), we have it that if $c \not\models_0 A$ then $c \not\models_1 \neg A$. Hence, in our given case c, we have it that $c \not\models_1 A$ and $c \not\models_1 \neg A$. But, then, by the truth condition for disjunction, we have it that $c \not\models_1 A \vee \neg A$ (since neither disjunct is true-in-c). Hence, there is a paracomplete case in which $A \vee \neg A$ is not true. Hence, Excluded Middle is not a logically true (sentence) form, according to our paracomplete theory.

   That there are *no* logical truths, on the paracomplete theory, is slightly more difficult to see. The basic idea, however, is that there is a paracomplete case in which every *atomic* A is neither true nor false.

Prove, using the truth and falsity conditions for molecular sentences, that such a case will be one in which *no* sentence – atomic or molecular – is true or false. Conclude that there are no K3 or FDE-logical truths.

3. We pointed out above that K3-consequence and LP-consequence are quite different. Despite this, they do agree on some things. Find an example of an argument meeting all three of the following conditions:

   - It is classically valid.
   - It is not K3-valid.
   - It is not LP-valid.

   Prove the argument you've provided meets all three conditions. This will involve three parts as well:

   - You must show no classical case is a counterexample to your argument.
   - You must sketch a K3-case that is a counterexample to your argument.
   - You must sketch a LP-case that is a counterexample to your argument.

4. Using what you learned from previous exercises determine whether the following claim is true or false:

   - If $A \therefore B$ is valid according to both K3 and LP then it is valid according to FDE too.

5. Consider again the example of the vague concept 'child'. We observed that there is a period of time during which we might think that 'child' is neither true of a given person, nor false of them, which motivates paracomplete logics like K3. Might there also be vague boundaries between the period during which it is neither true nor false that someone is a child and the period during which it is clear that it is true that that person is a child? If so, what does this show us, if anything, about paracomplete logics like K3 as models of vague language?

1.15 CHAPTER 15: NONCLASSICAL SENTENTIAL LOGICS

1. Suppose that we define a different, broader sort of 'contingency' thus:

   - A sentence A is *broadly contingent* iff it is true-in-*some case* and *not* true-in-*some case* (i.e., *untrue*-in-some case).

   Which, if any, of the displayed sentences (6a)–(6j), from exercise 3, are broadly contingent? Also, in K3, can a sentence be broadly contingent without being contingent? How about in FDE or LP?
2. *Weak Kleene*. An alternative paracomplete theory, one that is less classical than K3, is so-called Weak Kleene (WK). A WK-case is, like a K3-case, a function from the set of atomic sentences At to the set $\{0, \mathsf{n}, 1\}$.

   For weak Kleene, however, the truth conditions for the connectives are quite a bit different. In particular, if v is a case, we have that:

   - Atomic: if S is atomic, then
     - $v \models_1 S$ iff $v(S) = 1$;
     - $v \models_0 S$ iff $v(S) = 0$.
   - Conjunction: if $S = T \wedge U$, then
     - $v \models_1 S$ iff $v \models_1 T$ and $v \models_1 U$;
     - $v \models_0 S$ iff ($v \models_0 T$ and $v \models_0 U$) or ($v \models_0 T$ and $v \models_1 U$) or ($v \models_1 T$ and $v \models_0 U$).
   - Disjunction: if $S = T \vee U$, then
     - $v \models_1 S$ iff ($v \models_1 T$ and $v \models_1 U$) or ($v \models_1 T$ and $v \models_0 U$) or ($v \models_0 T$ and $v \models_1 U$);
     - $v \models_0 S$ iff $v \models_0 T$ and $v \models_0 U$.
   - Negation: if $S = \neg T$, then
     - $v \models_1 S$ iff $v \models_0 T$;
     - $v \models_0 S$ iff $v \models_1 T$.

The difference, you will notice, involves sentences that contain atomics that are given the semantic value of n. For any sentence S, if c is any case that assigns a semantic value of n to one of the atomic subsentences of S, then $c \not\models_1 S$ and $c \not\models_0 S$. You should check that this is *not* the case for K3.

The question: what, if any, of the argument forms in §5.7 are valid on the WK logical theory? (Consequence, for the WK theory, is defined as usual).

3. On page 64 we proved that the semantics we gave for the basic classical theory was complete and consistent. Modify this proof to show that:
   (a) The semantic theory given for K3 is consistent.
   (b) The semantic theory given for LP is complete.
4. Explain in your own words the roles of designated and antidesignated semantic values. Why do we need these concepts to give a proper formal semantics for our three non-classical logics.
5. Using the truth conditions for the basic sentential connectives and the definition of truth-in-a-case in §14.2, construct truth tables for the basic sentential connectives '$\neg$', '$\wedge$', and '$\vee$' and for the defined connectives '$\rightarrow$' and '$\leftrightarrow$' for K3, LP, and FDE.
6. Explain in your own words the way our understanding of counterexamples has changed in order to accommodate a definition of logical consequence for each of our three non-classical logics.
7. Consider the following definitions of 'logically non-false' and 'logically untrue'. A sentence A is logically non-false if and only if there is no case c such that $c \models_0 A$. A sentence A is logically untrue if and only if there is no case c such that $c \models_1 A$. Is any sentence logically non-false or logically untrue, according to K3?
8. Are there LP-valid arguments that are not K3-valid? Are there K3-valid arguments that are not LP-valid? If so, give an example. If not, show why there are no such arguments.
9. Determine whether the following argument forms are valid according to basic classical logic, K3, LP, and FDE.

(a) $A \to B, \neg B \vee C, \neg C \therefore \neg A$
(b) $\neg A \wedge A, \neg B \to A \therefore \neg B$
(c) $((A \wedge B) \wedge C) \therefore (C \wedge (B \wedge A))$
(d) $A \to (B \to C), A \wedge B \therefore C$
(e) $A \vee (B \wedge C), \neg C \therefore A$

10. Determine whether the following sentences are logically true, logically false, or contingent according to basic classical logic, K3, LP, and FDE.

    (a) $((A \vee \neg A) \wedge B) \to ((B \wedge A) \vee (B \wedge \neg A))$
    (b) $((A \wedge B) \wedge C) \to (C \wedge (B \wedge A))$
    (c) $A \leftrightarrow (A \wedge A)$
    (d) $(A \vee (B \wedge C)) \to (\neg C \to A)$
    (e) $(A \wedge \neg A) \to B$

11. Which of the following claims are true, and which are false? Explain your answers.

    (a) $A \wedge \neg A \vdash_{K3} B$
    (b) $A \wedge \neg A \vdash_{FDE} B \vee \neg B$
    (c) $A \wedge \neg A \vdash_{LP} B \vee \neg B$

12. Construct an argument (other than the ones we've considered so far) that is classically valid and K3-valid, but not LP-valid.
13. Construct an argument (other than the ones we've considered so far) that is classically valid and LP-valid, but not K3-valid.

1.16 CHAPTER 16: NONCLASSICAL FIRST-ORDER THEORIES

1. Which of the following best expresses that *nothing is horrible*? Justify your answer by appealing to the truth and falsity conditions of the quantifiers.

    - $\neg \forall x Hx$
    - $\neg \exists x Hx$

2. In which of our three logical theories do the following two-way-validity claims hold? Justify your answer.

(a) $\forall xFx \dashv\vdash \neg\exists x\neg Fx$
(b) $\exists xFx \dashv\vdash \neg\forall x\neg Fx$

3. Show whether the following arguments are valid in LP.

   (a) $\forall x(Hx \rightarrow Gx), Hb \therefore Gb.$
   (b) $\forall xHx, \forall x(Hx \rightarrow Gx) \therefore \forall xHx.$
   (c) $Gb \therefore \exists xHx \vee \neg\exists xHx.$

4. Show whether the following arguments are valid in K3.

   (a) $\forall x(Hx \rightarrow Gx), Hb \therefore Gb.$
   (b) $\forall xHx, \forall x(Hx \rightarrow Gx) \therefore \forall xHx.$
   (c) $Gb \therefore \exists xHx \vee \neg\exists xHx.$

5. Show whether the following arguments are valid in FDE.

   (a) $\forall x(Hx \rightarrow Gx), Hb \therefore Gb.$
   (b) $\forall xHx, \forall x(Hx \rightarrow Gx) \therefore \forall xHx.$
   (c) $Gb \therefore \exists xHx \vee \neg\exists xHx.$

1.17 CHAPTER 17: NONCLASSICAL TABLEAUX

Most of the exercises from Chapter 15 can be done using tableaux. For each of these exercises, first determine whether it can be done using tableaux. If it can, say how and do so. If it cannot, explain why.

1.18 CHAPTER 18: NONCLASSICAL TRANSLATIONS

Are there additional ways you might make a theory non-classical? Would any of these resolve issues of translation that you noted in the course of doing the exercises in the text?

1.19 CHAPTER 19: SPEAKING FREELY

1. We have said (in this and previous chapters) that existence claims like *b exists* have the form $\exists x(x = b)$. You might be wondering about a different approach: treating 'exists' as a more standard, quantifier-free predicate. How might this go? Is the

predicate to be treated as a logical expression? If so, what are the constraints on its extension and antiextension? If the predicate is non-logical (i.e., its extension and antiextension get no special constraints aside from those imposed on all predicates by the kind of cases involved), how do existence claims like *b exists* relate to existential claims like $\exists x(x = b)$? What, in general, is the logic of your proposed existence predicate? (This question is left wide open as an opportunity for you to construct your own alternative logical theory of existence.)

1.20 CHAPTER 20: POSSIBILITIES

Unless otherwise stated, the mFDE truth conditions (see §20.4.2) are assumed in the following exercises.

1. Is $\Diamond(a = a)$ logically true?
2. Explain in which theories (if any) the following results hold.

   (a) $\Box A \wedge \Box\neg A \vdash B$
   (b) $\vdash \Box A \vee \neg\Box A$
   (c) $\vdash \Box(A \vee \neg A)$
   (d) $\Box A, \Diamond\neg A \vdash B$
   (e) $\vdash \forall x \Diamond \exists y(y = x)$

3. What other connectives might be treated along 'worlds' lines? What if, instead of thinking of the elements in $\mathcal{W}$ as *worlds*, we think of $\mathcal{W}$ as containing *points in time*. Now consider the connectives *it is always true that...* and *it is sometimes true that....* If we treat these connectives along our box and, respectively, diamond lines, what sort of 'temporal logic' (i.e., logic of such temporal connectives) do we get? Related question: what sort of 'ordering' on your points in $\mathcal{W}$ do you need to give in order to add plausible *it will be true that...* and *it was true that...* connectives into the picture? (e.g., do your points of time have to be ordered in the way that, for example, the natural numbers are ordered?)
4. You examined a number of unary connectives that release but fail to capture, and some that capture but fail to release. What connective have you examined that both captures and releases

with respect to all consequence relations canvassed so far? (Hint: it's logical!)

1.21 CHAPTER 21: FREE AND MODAL TABLEAUX

1. (Continuing the previous problem.) The modal systems in this chapter all incorporate 'freed up' clauses for the quantifiers. Consider modal systems exactly like these, except with 'unfree' quantifiers. That is, suppose that each universe $\mathcal{U}$ comes with a single domain D shared by every world of $\mathcal{U}$, and suppose we define truth-in-a-case for the quantifiers just as in Chapter 10. For each of the formulas in problem 3 in the text, determine which of the non-free versions of our modal systems count it as a logical truth (if any).
2. (Continuing the previous problem.) For each of the formulas considered in problem 3 in the text, is it a logical truth of any of the *free* modal systems considered in this chapter? What do you think this tells us about the ability of free (as opposed to unfree) modal logics to model the part of our language that is concerned with possibility and necessity? Justify your answer.
3. We thought about treating moral laws and temporal claims along the same lines as (logical or metaphysical) possibility. Are there other forms of possibility that it would be helpful to analyze (perhaps related to feasibility, action, or beliefs)? How would we change our treatment of worlds in doing so?
4. What other natural language expressions can we think of that have the properties of 'Capture' and 'Release'?
5. Which of the following are true? Justify your answer.

   (a) $\Diamond A \dashv\vdash \Diamond\Diamond A$
   (b) $\Box A \dashv\vdash \Box\Box A$
   (c) $\Box\Diamond A \dashv\vdash \Diamond A$
   (d) $\Diamond\Box A \dashv\vdash \Box A$

6. What do the results of the previous problem tell us about sentences that begin with a string of two or more modal operators?

7. Which of the following argument forms are mFDE-valid? If an argument form is mFDE-invalid, also determine whether it is mCL-valid.

   (a) $\neg\Diamond A \therefore \neg\Box A$
   (b) $\neg\Box A \therefore \Diamond\neg A$
   (c) $\Diamond\neg A \therefore \neg\Box A$
   (d) $\Box\neg A \therefore \neg\Diamond A$
   (e) $\neg\Diamond A \therefore \Box\neg A$
   (f) $\Box A \therefore \Diamond A$

8. We can get by without supplying truth conditions for the sentential connectives '$\rightarrow$' and '$\leftrightarrow$', as we've seen that they can be defined in terms of connectives that we already have. Could we similarly get by without supplying truth conditions for one of the modal operators '$\Box$' and '$\Diamond$', instead defining one in terms of the other (and the other logical expressions we currently have)? Why or why not?
9. Produce an argument that is mK3-valid, but not mLP-valid.
10. Produce an argument that is mLP-valid, but not mK3-valid.
11. Produce a formula that is a logical truth of mCL, but not a logical truth of mLP.

1.22 CHAPTER 22: GLIMPSING DIFFERENT LOGICAL ROADS

1. How does the 'meaningless' approach (see §22.4) compare with Weak Kleene (see Chapter 15 exercises)?
2. Invoking the definitions of 'contingent' and 'broadly contingent' from Chapter 5 exercises, give what you think are the right truth- and falsity-in-a-case conditions for a *contingency* operator in the otherwise mFDE setting. (In other words: add new unary connectives to serve as an *it is contingent that. . .* and *it is broadly contingent that. . .* operators in the otherwise mFDE setting. What is the logic of your connective(s) like? Explore!
3. How might you add a *necessarily consistent* connective to the mFDE? What should the truth conditions for *it is necessarily consistent. . .* be in a broad mFDE setting? What about defining

our target 'necessarily consistent' operator in mFDE thus: just let $\mathbb{C}A$ abbreviate $\Box\neg(A \wedge \neg A)$. What, then, is the logic of $\mathbb{C}$ in mFDE? Are there mFDE cases in which $\mathbb{C}A$ is a *glut* – and, so, true but itself 'inconsistent' (since also false)? Is this a problem for a consistency operator?

4. What other phenomena might motivate different logics? Think and explore!

NOTES

1. Well, we're assuming that so-called Addition is valid, that is, that a disjunction is implied by each of its disjuncts on their own. Some logical theories question this (see, e.g., Chapter 15 in which one such theory is briefly waved at); however, we'll assume it in our reasoning throughout the book.
2. To construct a case, you have to specify the domain, the denotations of the various names, and the extensions and antiextensions of given predicates.

2

SOLUTIONS TO SELECTED EXERCISES

2.1 CHAPTER 1: CONSEQUENCES

1. What is an argument?

   Answer The solution comes from Section 1.3, where the term is defined:

   > Arguments, for our purposes, comprise premises and a conclusion. The latter item is the thesis in question; the former purports to 'support' the conclusion.

2. What is a valid argument?

   Answer The solution comes from Section 1.3, where the term is defined:

   > An argument is said to be valid if its conclusion is a logical consequence of its premises.

3. What is a sound argument?

   Answer The solution is found in Definition 3 of Section 1.3: A sound argument is valid and all its premises are true.

4. What is the general 'recipe' for defining logical consequence (or validity)? What are the two key ingredients that one must specify in defining a consequence relation?

   Answer The solution is found in Definition 1 and the discussion following it in Section 1.2:

   > B is a *logical consequence* of $A_1, \ldots, A_n$ if and only if there is no case in which $A_1, \ldots, A_n$ are all true but B is not true.
   >
   > This 'definition' is really just a recipe. In order to get a proper definition, one needs to specify two key ingredients:
   >
   > – what 'cases' are;
   > – what it is to be *true in a case*.
   >
   > Once these ingredients are specified, one gets an account of logical consequence.

5. Using the 'necessary consequence' account of validity, specify which of the following arguments are valid or invalid. Justify your answer.

   Selected Answers

   Argument 1.

   (i) If Agnes arrived at work on time, then her car worked properly.
   (ii) If Agnes's car worked properly, then the car's ignition was not broken.
   (iii) The car's ignition was not broken.
   (iv) Therefore, Agnes arrived at work on time.

   This argument is not valid. To see this, consider the possible circumstance in which Agnes oversleeps her alarm and ends up rushing to work late in her perfectly-functioning car. Then since her car functioned, it's true that 'if Agnes arrived at work on time, then her car worked properly'. It's also true that if her car worked properly (as it did), then in particular her ignition was not broken. And since in this possible circumstance the car is perfectly functioning, the ignition was not broken. So in this case, (i), (ii), and (iii) are all true. But (iv) is not.

Argument 3.

(i) If Max wins the lottery, then Max will be a millionaire.
(ii) Max will not win the lottery.
(iii) Therefore, Max will not be a millionaire.

This argument is not valid. To see this, consider the possible circumstance in which Max never wins the lottery, but is instead very frugal and invests wisely, saving enough to eventually become a millionaire. In this circumstance it is true that if Max wins the lottery, then Max will be a millionaire. It is also true that Max will not win the lottery. But it is not true that Max will not be a millionaire. Thus, in this case, (i) and (ii) are true but (iii) is not.

2.2 CHAPTER 2: MODELS, MODELED AND MODELING

1. Given the definition of 'model' we've adopted, consider the following list of situations and say, for each of them, (a) what the target system is, (b) what object is being proposed as a *model of* the target system, (c), what the modeling hypotheses are, (d) what an appropriate margin for error is within which the model is in fact a model of the thing it is a model of, and (e) what predictions the model might generate.

   Selected Answers

   - According to the latest weather models, there is a 60 percent chance of rain this weekend.

     (a) The target system is the actual weather.
     (b) The latest weather models (which, presumably, are a family of algorithms being run on a computer somewhere, or perhaps a family of mathematical equations being solved by such algorithms) are being proposed as a model of the actual weather.

(c) The modeling hypothesis is presumably something along the following lines: the portion of the weather models in which rain occurs, given the data, is similar to the actual probability that it rains today.
(d) Answers may vary, but here is a reasonable one: if the margin for error is greater than 10%, this prediction is of little value. The reason is this: suppose for simplicity that the margin for error is 11%. Then, there could be anywhere from a 49% to a 71% chance of rain occurring today. But then the prediction cannot help me decide whether to bring an umbrella: if there is a 49% chance of rain, it's more likely that it *doesn't* rain than that it does; while if there is a 71% chance of rain, it is much more likely that it *does* rain than that it doesn't.
(e) The model might predict various other weather phenomena as well.

- Last weekend I put the finishing touches on the model airplane I was building with my son.

  (a) The target system is the actual airplane.
  (b) The model airplane is being proposed as a model of the actual airplane.
  (c) The modeling hypothesis is that the model airplane is (roughly) *geometrically* similar to the actual airplane: it is similar in shape.
  (d) If it is a *scale* model, then the precision may be very high. That is, it may be that, e.g., within the tolerances of our measuring equipment, the model airplane is a precise 1/50th scale model of the actual airplane.
  (e) The model will predict how the actual airplane will look. It could be used to test mechanical features of the actual airplane as well, for example wind resistance.

2. Why is modeling useful? Why might it be useful to model natural languages?

   Answer In section 2.2 we pointed out that models are useful for scientists 'when they are studying highly complex systems that are too difficult to deal with *in the raw*. Models, being objects the scientists can control, are easier to deal with, understand, and manipulate'.

But it's not only *scientists* who find themselves dealing with highly complex systems. And modeling is useful in *any* such situation for the same reasons as above: models are easier to control, deal with, understand, and manipulate. In particular, since natural language is a highly complex system, when it comes to studying it, models are a very useful tool.

3. Suppose x is similar to y. Why is this not enough to make x a model of y?

   Answer The answer was given in section 2.1: if all it takes for x to be a model of y is that x is similar to y in some way, then everything would be a model of everything else, because any two things are similar in some way.

4. In Chapter 1, we saw a 'basic recipe' for logical consequence. Similarly, in this chapter we saw that there is a recipe for scientific modeling as well. Explain in your own words the components of this recipe.

   Answer Modeling seems to require four components: a target system, an object that models the target system, a modeling hypothesis, and a margin of error.

5. Why might two models x and y both count as good models for a phenomenon z even if, in some cases, x and y make different predictions about z? Can you think of an example of a phenomenon that is usefully modeled in more than one way?

   Answer One way this might happen is if x and y model different *aspects* of z. One example of a phenomenon that is usefully modeled in more than one way is logical consequence! We will work out the details of this modeling in the course of the book.

2.3 CHAPTER 3: LANGUAGE, FORM, AND LOGICAL THEORIES

1. What is a sentential connective? What is a unary connective? What is a binary connective? (What is the degree or arity of a sentential connective?)

   Answer These questions are all answered in the following passage from §3.3:

   > The distinction between atomics and molecular sentences, at least in logic, turns on the idea of *logical connectives*, which are a species of so-called *sentential connectives*. Sentential connectives take sentences and make new (bigger) sentences. Sentential connectives have a 'degree' or 'arity', which marks the number of sentences a given connective requires in order to make a new sentence. For example, '. . . and . . . ' is binary; it takes two (not necessarily distinct) sentences to make a new sentence, while 'it is false that...' is unary, and so takes one sentence to make a new sentence, and so on.

2. In §3.5 we gave a natural truth condition for conjunction. Give what you'd take to be a natural 'truth condition' (strictly, *truth-in-a-case* condition) for disjunction. Do the same for negation.

   Answer Here are natural truth conditions (i.e., more accurately, truth-in-a-case conditions) for disjunction and negation.

   - A disjunction $A \vee B$ is true-in-a-possible-circumstance-c if and only if A is true-in-c or B is true-in-c (or both).
   - A negation $\neg A$ is true-in-a-possible-circumstance-c if and only if A is false-in-c.

3. Consider the argument that takes (6) and the negation of (3) as its premises and (4) as its conclusion. Using the symbolism introduced above, give its argument form. Taking 'cases' to be 'possible circumstances', and using the truth conditions that you provided for disjunction and negation (and, if need be, the condition in §3.5 for conjunction), is the given form valid? Justify your answer.

Answer The form of this argument is $A \vee B, \neg A \therefore B$. This argument is valid. To see this, consider a possible circumstance c in which $A \vee B$ and $\neg A$ are true. Then, since $A \vee B$ is true-in-c, either A is true-in-c or B is true-in-c. But $\neg A$ is true-in-c, so A is false-in-c. Thus, A is not true-in-c. So B must be true-in-c.

2.4 CHAPTER 4: SET-THEORETIC TOOLS

1. Write out $\mathcal{Y} \times \mathcal{Z}$ and $\mathcal{Z} \times \mathcal{Y}$, where $\mathcal{Y} = \{1, 2\}$ and $\mathcal{Z} = \{a, b, c\}$. Are $\mathcal{Y} \times \mathcal{Z}$ and $\mathcal{Z} \times \mathcal{Y}$ the same set? Justify your answer.

   Answer

   - $\mathcal{Y} \times \mathcal{Z} = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 1, c\rangle, \langle 2, a\rangle, \langle 2, b\rangle, \langle 2, c\rangle\}$
   - $\mathcal{Z} \times \mathcal{Y} = \{\langle a, 1\rangle, \langle b, 1\rangle, \langle c, 1\rangle, \langle a, 2\rangle, \langle b, 2\rangle, \langle c, 2\rangle\}$

   These are not the same set. This is because, e.g., $\langle 1, a\rangle \in \mathcal{Y} \times \mathcal{Z}$, but $\langle 1, a\rangle \notin \mathcal{Z} \times \mathcal{Y}$.

2. Using definition by abstraction, give brace-notation names (i.e., names formed using '{' and '}' as per the chapter) for each of the following sets.

   (a) The set of all even numbers.

   Answer $\{x : x \text{ is an even number}\}$

   (b) The set of all felines.

   Answer $\{x : x \text{ is a feline}\}$

   (c) The set of all tulips.

   Answer $\{x : x \text{ is a tulip}\}$

   (d) The set of all possible worlds.

   Answer $\{x : x \text{ is a possible world}\}$

   (e) The set of all people who love cats.

   Answer $\{x : x \text{ is a person and } x \text{ loves cats}\}$

3. Assume that a, b, c, and d are distinct (i.e., non-identical) things. Which of the following relations are functions? (Also, if you

weren't given that the various things are distinct, could you tell whether any of the following are functions? If so, why? If not, why not?)

(a) $\{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle\}$

Answer This is a function.

(b) $\{\langle a, d\rangle, \langle b, d\rangle, \langle c, d\rangle, \langle d, d\rangle\}$

Answer This is a function.

(c) $\{\langle a, b\rangle, \langle a, c\rangle, \langle b, d\rangle, \langle d, d\rangle\}$

Answer This is not a function. Notice that a is paired both with b and c.

(d) $\{\langle b, a\rangle, \langle c, d\rangle, \langle a, a\rangle, \langle b, d\rangle\}$

Answer This is not a function. Notice that b is paired both with a and d

(e) $\{\langle d, d\rangle, \langle d, b\rangle, \langle b, d\rangle, \langle a, d\rangle\}$

Answer This is not a function. Notice that d is paired both with d and b.

For the other part of the question, here is part of the answer: even with out knowing that a, b, c, and d are distinct, we can still tell that (a) is a function because any two of a, b, c, and d that *are* distinct will still get paired with exactly one coordinate.

4. Consider the relation of *biological motherhood*, which holds between objects x and y if and only if y is the biological mother of x. Is this relation a function? Justify your answer.

   Answer Yes, this relation is a function. Anything that has a biological mother has exactly one biological mother.

2.5 CHAPTER 5: BASIC CLASSICAL SYNTAX AND SEMANTICS

1. Show that, on the classical theory, $A \therefore \neg\neg A$ is valid.

Answer Recall that the truth conditions for negation say that $v \vDash_1 \neg T$ iff $v \vDash_0 T$ and $v \vDash_0 \neg T$ iff $v \vDash_1 T$. Now let v be a case. Suppose $v \vDash_1 A$. Then, by the truth conditions we just mentioned, $v \vDash_0 \neg A$, so $v \vDash_1 \neg\neg A$. Thus, any case that makes A true will make $\neg\neg A$ true as well, so the argument $A \therefore \neg\neg A$ is valid.

2. Show that, according to the classical theory, $A, B \therefore A \wedge B$ is valid.

   Answer Recall that the truth conditions for conjunction say that $v \vDash_1 A \wedge B$ iff $v \vDash_1 A$ and $v \vDash_1 B$. Now let v be a case. Suppose that $v \vDash_1 A$ and $v \vDash_1 B$. Then, by the just-mentioned truth conditions, $v \vDash_1 A \wedge B$. Thus, any case that satisfies both A and B will satisfy $A \wedge B$ as well, so the argument $A, B \therefore A \wedge B$ is valid.

3. In addition to our definition of *logical truth* (true-in-*every case*), let us define *contingent* and *logically false* as follows.

   - Sentence A is *logically false* iff it is false-in-*every case*.
   - Sentence A is *contingent* iff it is true-in-*some case*, and false-in-*some case*.

   For each of the following sentences, say whether, according to the classical theory, it is logically true, logically false, or contingent.

   (a) $p \rightarrow p$
   (b) $p \rightarrow \neg p$
   (c) $p \wedge \neg p$
   (d) $q \vee p$
   (e) $q \wedge (p \vee q)$
   (f) $q \vee (p \wedge q)$
   (g) $q \leftrightarrow \neg p$
   (h) $(p \wedge (p \rightarrow q)) \rightarrow q$

   Answer to (b) The sentence $p \rightarrow \neg p$ is contingent: it is true-in-*some case* and false-in-*some case*. Proof: p is atomic, and so there are cases in which p is true, and also cases in which p is false. Let v be any case in which p is true, that is, $v \vDash_1 p$.

By the classical treatment of negation, $v \models_0 \neg p$. By definition, $p \to \neg p$ is equivalent to $\neg p \lor \neg p$. By the truth conditions for disjunction, $\neg p \lor \neg p$ is true iff one of its disjuncts is true; but $\neg p$ is the only disjunct, and it is not true-in-the-given-case, since $v \models_0 \neg p$. So, v is a case in which $p \to \neg p$ is not true. On the other hand, consider any case v' in which p is false, that is, $v'(p) = 0$. By the truth conditions for negation, $v' \models_1 \neg p$, in which case, by the truth conditions for disjunction, $v' \models_1 \neg p \lor \neg p$, and hence $v' \models_1 p \to \neg p$. So, v' is a case in which $p \to \neg p$ is true. Hence, there are cases in which $p \to \neg p$ is true and cases in which it is false.

4. Prove the validity of each form in Figure 5.1.

   Answer for LEM. To see that LEM is a valid form (i.e., that all of its instances are logically true sentences), we need to show that there's no case in which $A \lor \neg A$ is false (for any sentence A). We do this by Reductio. Suppose, for reductio, that there's some case v such that $v \models_0 A \lor \neg A$ (for some sentence A). The truth conditions for disjunction tell us that $v \models_1 A \lor \neg A$ if and only if $v \models_1 A$ or $v \models_1 \neg A$. Since, by supposition, $v \not\models_1 A \lor \neg A$, we have it that $v \not\models_1 A$ and $v \not\models_1 \neg A$. But since classical cases are complete, we conclude that $v \models_0 A$ and $v \models_0 \neg A$. But this is impossible, since, by truth conditions for negation, $v \models_1 \neg A$ iff $v \models_0 A$. So, we conclude that our initial supposition – namely, that there's some case v in which $A \lor \neg A$ (for some A) is false – is itself untrue. Hence, we conclude that there cannot be a (classical) case in which $A \lor \neg A$ (for some A) is false, which is to say that LEM is valid.

   Answer for Simplification. To see that $A \land B$ implies A in the classical theory, we can use Reductio.[3] Suppose, for reductio, that there's a counterexample to $A \land B \therefore A$, that there's some (classical) case v such that $v \models_1 A \land B$ but $v \models_0 A$. The truth conditions for conjunction tell us that $v \models_1 A \land B$ iff $v \models_1 A$ and $v \models_1 B$. But, then, we have it that $v \models_1 A$, since (by supposition) we have it that $v \models_1 A \land B$. But, by supposition, we also have it that $v \models_0 A$. This is impossible, since classical cases are *consistent*. Hence, we reject our initial assumption that there's a counterexample to Simplification, and conclude that there's no counterexample – and, hence, that the given form is valid.

5. Prove that, where $\vdash$ is our basic classical consequence relation, each of the following are correct (i.e., that the given argument forms are valid in the basic classical theory).

   (a) $A \to B, B \to C \vdash A \to C$.
   (b) $(A \vee B) \wedge C, A \to \neg C \vdash B$.
   (c) $(A \vee B) \wedge C \dashv\vdash (A \wedge C) \vee (B \wedge C)$.
   (d) $(A \wedge B) \vee C \dashv\vdash (A \vee C) \wedge (B \vee C)$.
   (e) $A \to B, \neg A \to B \vdash B$.

   Answer to (e) Let v be a case, and suppose

   (a) $v \models A \to B$; and
   (b) $v \models \neg A \to B$.

   Recall that (i) means that

   (iii) $v \models_0 A$ or $v \models_1 B$ (or both),

   and that (ii) means that

   (iv) $v \models_0 \neg A$ or $v \models_1 B$ (or both).

   Suppose $v \models_0 A$. Then, by the truth conditions for negation, $v \models_1 \neg A$. But then by consistency, $v \not\models_0 \neg A$. But by (iv), we must have that either $v \models_0 \neg A$ or $v \models_1 B$ (or both). Since $v \not\models_0$ $\neg A$, then, we must have that $v \models_1 B$.

   On the other hand, if $v \not\models_0 A$, then by (iii), we must have that $v \models_1 B$.

   Finally, note that for any case v, either $v \models_0 A$ or $v \not\models_0 A$. And we've now seen that, either way, if $v \models_1 A \to B$ and $v \models_1 \neg A \to B$, then $v \models_1 B$. Thus $A \to B, \neg A \to B \vdash B$ is correct.

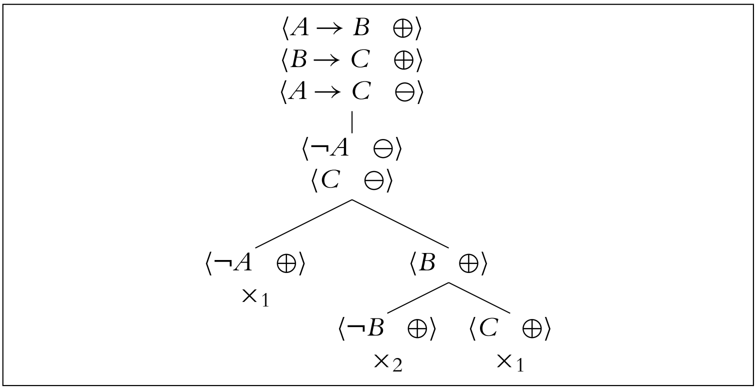
2.6 CHAPTER 6: BASIC CLASSICAL TABLEAUX

1. Using tableaux and letting $\vdash$ be our basic classical consequence relation, prove that each of the following are correct.

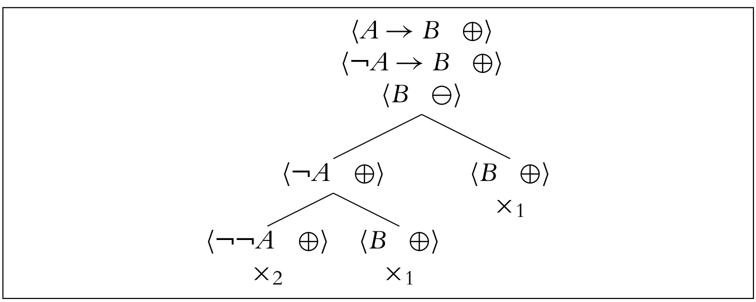
(a) $A \to B, B \to C \vdash A \to C$.
(b) $(A \vee B) \wedge C, A \to \neg C \vdash B$.
(c) $(A \vee B) \wedge C \dashv\vdash (A \wedge C) \vee (B \wedge C)$.
(d) $(A \wedge B) \vee C \dashv\vdash (A \vee C) \wedge (B \vee C)$.
(e) $A \to B, \neg A \to B \vdash B$.

Selected Answers

(a)



(e)



2. We already introduced the notions of *logically true*, *logically false* and *contingent*, which were understood semantically. These can be reintroduced in terms of tableaux as follows:

   - Sentence A is *logically true* iff a tableau that begins with $\langle A \quad \ominus \rangle$ closes.
   - Sentence A is *logically false* iff a tableau that begins with $\langle A \quad \oplus \rangle$ closes.
   - Sentence A is *contingent* iff neither a tableau that begins with $\langle A \quad \ominus \rangle$ nor a tableau that begins with $\langle A \quad \oplus \rangle$ closes.

   For each of the following sentences say whether, according to the classical theory, it is logically true, logically false, or contingent.

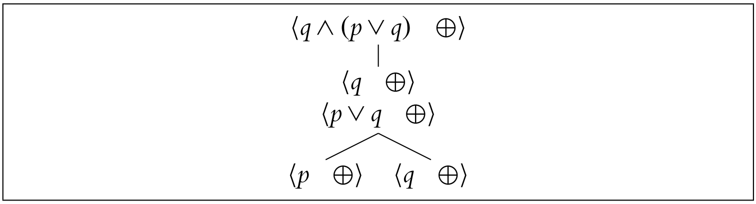
   (a) $p \to p$
   (b) $p \to \neg p$
   (c) $p \wedge \neg p$
   (d) $q \vee p$
   (e) $q \wedge (p \vee q)$
   (f) $q \vee (p \wedge q)$
   (g) $q \leftrightarrow \neg p$
   (h) $(p \wedge (p \to q)) \to q$

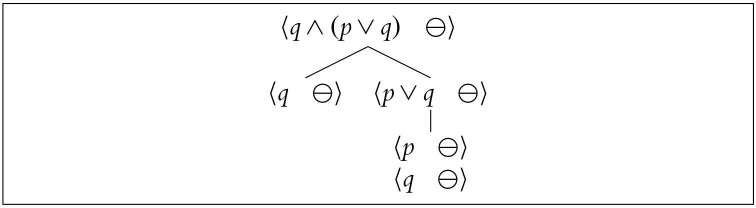
   Selected Answers

   (a) This sentence is logically true, as the following tableau demonstrates:

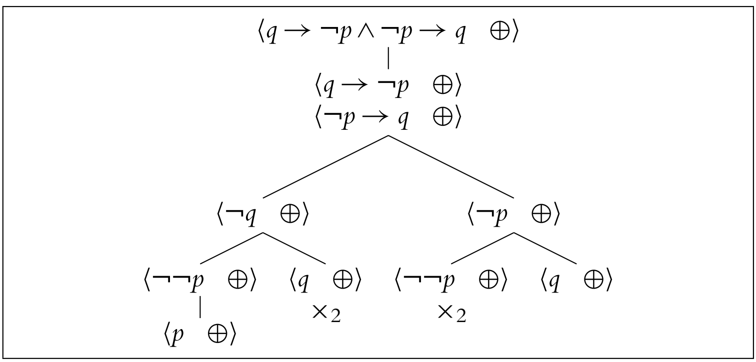
$$
\begin{array}{c}
\langle p \to p \quad \ominus \rangle \\
| \\
\langle \neg p \quad \ominus \rangle \\
\langle p \quad \ominus \rangle \\
\times_3
\end{array}
$$

   (e) This sentence is logically indeterminate, as the following tableaux demonstrate:



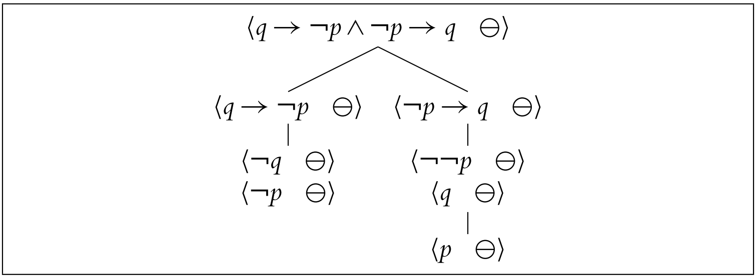


(g) This sentence is logically indeterminate, as the following tableaux demonstrates:



3. For each of the forms in Figure 5.1, give a proof by tableau that it's valid.

   Answer Left to the reader.



2.7 CHAPTER 7: BASIC CLASSICAL TRANSLATIONS

1. Translate the English atomics as follows.

p_1	Max is sleeping.
q_1	Agnes is sleeping.
p_2	Max is getting into trouble.
q_2	Agnes is getting into trouble.

Give the standard translations of the following English sentences into the language of basic classical theory.

(a) Either Agnes is sleeping or not.
(b) Agnes is sleeping if Max is sleeping.
(c) If Max is sleeping then Agnes is sleeping.
(d) Max is sleeping only if Agnes is sleeping.
(e) Neither Max nor Agnes is sleeping.
(f) Max is getting into trouble only if either Agnes is sleeping or Max is not sleeping (or both).

Selected Answers

(b) $p_1 \rightarrow q_1$
(e) $\neg(p_1 \vee q_1)$ *or* $\neg p_1 \wedge \neg q_1$
(f) $p_2 \rightarrow (q_1 \vee \neg p_1)$

2. For each of the following natural language arguments, (i) identify the atomic subsentences that occur within it and

assign an atomic sentence of basic classical theory to each; (ii) provide the standard translation of the argument into the language of basic classical theory; and (iii) construct a tableau to determine whether it is valid. Do any of these arguments contain information inside their atomic sentences not captured by the translations? If so, how does that affect their validity? If this makes an argument invalid, what premises might be added to make this argument valid?

(a) i. If both Max and Agnes are playing tennis, then neither Max nor Agnes is at the store.
    ii. Max is playing tennis.
    iii. Agnes is playing tennis.
    iv. Therefore, neither Max nor Agnes is at the store.

(b) i. All mice are rodents.
    ii. All rodents are mammals.
    iii. Therefore, all mice are mammals.

(c) i. Arthur is King only if he pulls the sword out of the stone.
    ii. Either Arthur pulls the sword out of the stone, or Sir Gawain does.
    iii. Sir Gawain does not pull the sword out of the stone.
    iv. Therefore, Arthur is King.

(d) i. Either Max has his keys and his car is not towed, or he has locked his keys in his car.
    ii. Max locked his keys in his car.
    iii. Therefore, Max does not have his keys, and his car is towed.

(e) i. Both Max and Agnes will visit America this winter, but only one of them will go to Fargo.
    ii. Paul will visit America only if Agnes goes to Fargo.
    iii. Lola will visit America if and only if Paul will visit America too.
    iv. So, if Max goes to Fargo, neither Lola nor Paul will visit America.

Selected Answers

(d)

p_1	Max has his keys.
p_2	Max's car is towed.
p_3	Max locked his keys in his car.

With this key, the argument becomes

$$(p_1 \wedge \neg p_2) \vee p_3, p_2 \vdash \neg p_1 \wedge p_2$$

There is one crucial piece of missing information here: presumably if Max's keys are locked in his car, then Max does not have his keys. This is not reflected in any of the assumptions, but is required in order for the argument to be valid. In terms of the key given, this missing information is $p_3 \rightarrow \neg p_1$. You should check with a tableau that the argument

$$(p_1 \wedge \neg p_2) \vee p_3, p_2, p_3 \rightarrow \neg p_1 \vdash \neg p_1 \wedge p_2$$

is valid.

2.8 CHAPTER 8: ATOMIC INNARDS: UNARY

1. Consider a case $c = \langle D, \delta \rangle$ where $D = \{1, 2, 3\}$, and $\delta(a) = 1$, $\delta(b) = 2$, and $\delta(d) = 3$, and $F^+ = \{1, 2\}$ and $F^- = \{3\}$. For each of the following, say whether it is true or false. If true, say why. If false, say why.

   (a) $c \models_1 Fa$
   (b) $c \models_0 Fa$
   (c) $c \models_1 \neg Fa$
   (d) $c \models_1 Fb \vee Fd$
   (e) $c \models_1 Fb \wedge Fd$
   (f) $c \models_1 \neg(Fb \vee Fd)$
   (g) $c \models_1 Fd \rightarrow Fb$

   Selected Answers

(c) $c \models_1 \neg Fa$ iff $c \models_0 Fa$ iff $\delta(a) \in F^-$. Since our given case c is such that $\delta(a) \notin F^-$ (since $\delta(a)$ is 1, which is not in the antiextension of F in our given case), we conclude that $c \not\models_1 \neg Fa$.

(f) $c \models_1 \neg(Fb \vee Fd)$ iff $c \models_0 Fb \vee Fd$ iff $c \models_0 Fb$ and $c \models_0 Fd$ iff $\delta(b) \in F^-$ and $\delta(d) \in F^-$. But $\delta(b) \notin F^-$, so it is false that $c \models_1 (Fb \vee Fd)$.

2. Consider a (classical) case $c = \langle D, \delta \rangle$ where $F^- = D$. Is there any way to specify D and δ such that $c \models_1 Fa$?

   Answer No. By the definition of a case, D must be non-empty, and $\delta(a) \in D = F^-$. Thus $c \models_0 Fa$, so by consistency, $c \not\models_1 Fa$.

3. For each of the following sentences, construct a case in which it is *true*.

   (a) $\neg Fa \wedge (Fa \vee Fb)$
   (b) $Fa \rightarrow \neg Ga$
   (c) $\neg(Ra \rightarrow Fc)$
   (d) $Fa \wedge \neg(Fb \wedge Fc)$
   (e) $(Ra \wedge Rb) \rightarrow \neg Ra$
   (f) $\neg Fc \wedge (Fc \vee Fa)$
   (g) $(Ra \rightarrow Rb) \rightarrow Ra$
   (h) $(Fa \vee Gb) \wedge (\neg Gb \rightarrow \neg Fa)$
   (i) $(Fc \rightarrow Ra) \wedge (Ra \rightarrow Fb)$
   (j) $(Fa \vee (Fb \vee Fc)) \rightarrow Rc$

   Selected Answers (Many answers are possible to each question; these are just examples.)

   (c) $v \models_1 \neg(Ra \rightarrow Fc)$ iff $v \models_0 Ra \rightarrow Fc$ iff $v \models_1 Ra$ and $v \models_0 Fc$.
   One way to do this is to let $v = \langle D, \delta \rangle$ with $D = \{1, 2\}$, $\delta(a) = 1$, $\delta(c) = (2)$. Also let $R^+ = \{1\}$, $R^- = \{2\}$, $F^+ = \{1\}$ and $F^- = \{2\}$. Then $v \models_1 \neg(Ra \rightarrow Fc)$.

   (i) $v \models_1 (Fc \rightarrow Ra) \wedge (Ra \rightarrow Fb)$ iff $v \models_1 Fc \rightarrow Ra$ and $v \models_1 Ra \rightarrow Fb$. Also $v \models_1 Fc \rightarrow Ra$ iff $v \models_0 Fc$ or $v \models_1 Ra$, and $v \models_1 Ra \rightarrow Fb$ iff $v \models_0 Ra$ or $v \models_1 Fb$.
   One way to do this is to let $v = \langle D, \delta \rangle$ with $D = \{1\}$, $\delta(a) = \delta(b) = \delta(c) = 1$, and to let $R^+ = F^+ = \emptyset$ and $R^- = F^- = \{1\}$. Then since $\delta(c) \in F^-$, $v \models_0 Fc$, so $v \models_1 Fc \rightarrow Ra$.

And since $\delta(a) \in R^-$, $v \models_0 Ra$, so $v \models_1 Ra \rightarrow Fb$. Thus $v \models_1 (Fc \rightarrow Ra) \wedge (Ra \rightarrow Fb)$.

4. For each of the following sentences, construct a case in which it is *false*.

   (a) $Ra \vee Rb$
   (b) $Fa \rightarrow (Fa \rightarrow Fb)$
   (c) $(Ra \rightarrow Rb) \rightarrow Ra$
   (d) $Ra \vee (Fa \vee Fb)$
   (e) $(Fa \rightarrow Ga) \wedge (Ga \rightarrow Fa)$

   Selected Answer (Again, many answers are possible, this is just one example.)

   (b) $v \models_0 Fa \rightarrow (Fa \rightarrow Fb)$ iff $v \models_1 Fa$ and $v \models_0 Fa \rightarrow Fb$. And $v \models_0 Fa \rightarrow Fb$ iff $v \models_1 Fa$ and $v \models_0 Fb$.

   Here's one way to do this: let $v = \langle D, \delta \rangle$ with $D = \{1, 2\}$, $\delta(a) = 1$, $\delta(b) = 2$, and let $F^+ = \{1\}$ and $F^- = \{2\}$.

2.9 CHAPTER 9: EVERYTHING AND SOMETHING

1. Consider a case $c = \langle D, \delta \rangle$ where $D = \{1, 2, 3\}$ and $\delta(a) = 1$, $\delta(b) = 2$, and $\delta(d) = 3$, and $F^+ = \{2, 3\}$ and $F^- = \{1\}$.

   For each of the following, say whether it is true or false. If true, say why. If false, say why.

   (a) $c \models_1 \forall x Fx$
   (b) $c \models_0 \forall x Fx$
   (c) $c \models_1 \exists x Fx$
   (d) $c \models_0 \exists x Fx$

   Answer to (c) $c \models_1 \exists x Fx$ iff $c \models_1 F(\alpha/x)$ for some name α. And, $c \models_1 F(\alpha/x)$ iff $\delta(\alpha) \in F^+$. Notice that $\delta(b) = 2 \in F^+$, so $c \models_1 F(b)$, hence $c \models_1 \exists x Fx$.

2. Which of the following best expresses that *nothing is horrible*, where H represents 'is horrible'? Justify your answer by appealing to the truth and falsity conditions of the quantifiers.

   - $\neg \forall x Hx$
   - $\neg \exists x Hx$

Answer The first of these says that it is not the case that everything is horrible. The second says it is not the case that there is something that is horrible. Of the two options, we expect that the second does the better job of expressing 'nothing is horrible'.

Now we turn to making sure this is correct by appealing to the truth and falsity conditions for these sentences. $\neg\forall xHx$ is true just if $\forall xHx$ is false. But the truth conditions for universal quantifiers say that $\forall xHx$ is false just when $H(\alpha/x)$ is false for some name α. Thus, the first sentence is true just when there is something, α, such that it is false that α is horrible. In other words, when there is something that *isn't* horrible.

On the other hand, $\neg\exists xHx$ is true just if $\exists xHx$ is false. But the truth conditions for existential quantifiers say that $\exists xHx$ is false just when $H(\alpha/x)$ is false for all names α. Thus, the second sentence is true just when, for any thing α, it is false that α is horrible. In other words, when there is nothing that *is* horrible.

3. **Difficult** Prove that for every sentence S (of arbitrary complexity – not just atomics) and every case c, exactly one of $c \models_1 S$ and $c \models_0 S$ holds. That is, prove that the semantic theory given for sentential logic with innards and identity is a complete and consistent semantic theory.

Answer The idea is to use a proof by *induction*. For more on this proof technique, see Chapter 3 of this supplement.

In any event, it is clear that the condition that for every case c, exactly one of $c \models_1 S$ and $c \models_0 S$ holds for *atomic* sentences S. Now let $k \geq 1$ be an integer and suppose that whenever S is a sentence that has k or fewer logical operators and c is a case, exactly one of $c \models_1$ and $c \models_0 S$ holds.

Given this supposition, let T be a sentence with $k+1$ logical operators in it. Then T is either a negation, a disjunction, a conjunction, a universally quantified sentence or an existentially quantified sentence. The first three options are roughly the same; consider for example if $T = A \wedge B$ is a conjunction. Then since T has $k+1$ logical operators, A has at most k logical operators and B has at most k logical operators. Let c be a model. $c \models_1 T$ iff $c \models_1 A$ and $c \models_1 B$, and $c \models_0 T$ iff $c \models_0 A$ and $c \models_0 B$. Since exactly one of $c \models_1 A$ and $c \models_0 A$ happens (by our assumption) and exactly one of $c \models_1 B$ and $c \models_0 B$ happens, it is clear that exactly one of $c \models_1 T$ and $c \models_0 T$ happens.

The universally and existentially quantified options are similar as well. We will examine the case where T is a universal sentence, so $T = \forall x Ax$. $c \models_1 \forall x Ax$ iff for all names α, $c \models_1 A(\alpha/x)$. And $c \models_0 \forall x Ax$ iff for some name α, $c \models_0 A(\alpha/x)$. But for each α, the sentence $A(\alpha/x)$ has k or fewer logical operators. So for each of these sentences, exactly one of $c \models_1 A(\alpha/x)$ and $c \models_0 A(\alpha/x)$ happens. It follows from this that exactly one of $c \models_1 \forall x Ax$ and $c \models_0 \forall x Ax$ happens.

4. **Free Variables**: precisely define the notion of a *bound* variable (i.e., a variable which is *not free*) by finishing the so-called recursive definition below.

   Let u and v be variables. We define what it is for v to be free in a formula A by the following clauses: v *occurs free (or is free or unbound) in A* iff

   (a) A is an atomic formula and v occurs in A; or
   (b) A is a negation $\neg B$ and v occurs free in B; or
   (c) A is a nullation $\dagger B$ and v occurs free in B; or
   (d) A is a conjunction $B \wedge C$ and v occurs free in either B or C; or
   (e) A is a disjunction $B \vee C$ and v occurs free in either B or C; or
   (f) A is an existential $\exists \mathsf{u} B$ and v is not u and v occurs free in B; or
   (g) A is a universal... and ...

   Answer Follow the pattern given in the existential case.

5. Which of the following formulas are well-formed formulas of (monadic) first-order logic? Which among the well-formed formulas are *sentences*?

   (a) Fxy
   (b) $Rx \vee Ga \wedge Rb$
   (c) $(Rb \wedge (Gx \wedge \forall x Hx))$

(d) $\forall x Fa$
(e) $\forall x \exists y(Ry \to Sx)$
(f) $\forall y(Rx \leftrightarrow Gy)$
(g) $Fx \vee \forall x Fx$
(h) $Fa \wedge \exists x(Gb \wedge Fx)$
(i) $\exists x Fx \vee \neg Fx$
(j) $\exists x(Fx \vee \neg Fx)$

Selected Answers

(b) This is not a well-formed formula, and hence not a sentence. Both $(Rx \vee Ga) \wedge Rb$ and $Rx \vee (Ga \wedge Rb)$ are well-formed, in contrast.
(c) This is a well-formed formula, but it is not a sentence, because the first occurrence of x is a free occurrence.

2.10 CHAPTER 10: FIRST-ORDER LANGUAGE WITH ANY-ARITY INNARDS

1. Consider a (classical) case $c = \langle D, \delta \rangle$ where $F^- = D \times D$. Is there any way to specify D and δ such that $c \models_1 Fab$?

   Answer No. The reasoning is essentially the same as that in problem 2 of Chapter 8.

2. Consider a case $c = \langle D, \delta \rangle$ where $D = \{1, 2\}$, $\delta(a) = 1$, $\delta(b) = 2$, $R^+ = \{\langle 1, 1 \rangle, \langle 2, 2 \rangle\}$, and $F^+ = D$. Given the requirements for classical cases, what do R^- and F^- have to be? Which of the following are true? Why or why not?

   (a) $c \models_1 Rbb$
   (b) $c \models_0 Raa$
   (c) $c \models_1 Fa \to Rab$
   (d) $c \models_1 Fa \to Raa$
   (e) $c \models_0 Rab \vee Rba$
   (f) $c \models_0 Rab \leftrightarrow Rba$
   (g) $c \models_1 (Rab \leftrightarrow Rba) \to Raa$
   (h) $c \models_0 \neg Fb \vee Raa$

Selected Answers

(d) $c \models_1 Fa \rightarrow Raa$ iff $c \models_0 Fa$ or $c \models_1 Raa$. $c \models_1 Raa$ iff $\langle \delta(a), \delta(a) \rangle \in R^+$. Since $\delta(a) = 1$ and $\langle 1, 1 \rangle \in R^+$, $c \models_1 Raa$, so $c \models_1 Fa \rightarrow Raa$.

(f) $c \models_0 Rab \leftrightarrow Rba$ iff $c \models_1 Rab$ and $c \models_0 Rba$, or $c \models_0 Rab$ and $c \models_1 Rab$. $c \models_1 Rab$ iff $\langle \delta(a), \delta(b) \rangle \in R^+$. $c \models_1 Rba$ iff $\langle \delta(b), \delta(a) \rangle \in R^+$. Since $R^+ = \{\langle 1, 1 \rangle, \langle 2, 2 \rangle\}$, $\delta(a) = 1$ and $\delta(b) = 2$, neither $c \models_1 Rab$ nor $c \models_1 Rba$ is true. Thus, $c \models_0 Rab \leftrightarrow Rba$ if false.

3. For each of the following sentences, construct a case in which it is *true*.

(a) $\neg Fa \wedge (Fa \vee Fb)$
(b) $Fa \rightarrow \neg Ga$
(c) $\neg(Rab \rightarrow Fc)$
(d) $Fa \wedge \neg(Fb \wedge Fc)$
(e) $(Raa \wedge Rbb) \rightarrow \neg Rab$
(f) $\neg Fc \wedge (Fc \vee Fa)$
(g) $(Rab \rightarrow Rbc) \rightarrow Rac$
(h) $(Fa \vee Gb) \wedge (\neg Gb \rightarrow \neg Fa)$
(i) $(Fc \rightarrow Rab) \wedge (Rac \rightarrow Fb)$
(j) $(Fa \vee (Fb \vee Fc)) \rightarrow Rcba$

Selected Answers

(e) $v \models_1 (Raa \wedge Rbb) \rightarrow \neg Rab$ iff $v \models_0 Raa \wedge Rbb$ or $v \models_1 \neg Rab$. The second option takes less work, so let's do that one.

$v \models_1 \neg Rab$ iff $v \models_0 Rab$ iff $\langle \delta(a), \delta(b) \rangle \in R^+$. Thus, the following will work as a case in which $(Raa \wedge Rbb) \rightarrow \neg Rab$ if true: Let $v = \langle D, \delta \rangle$ with $D = \{1\}$, $\delta(a) = \delta(b) = 1$, $R^+ = \{\langle 1, 1 \rangle\}$, $R^- = \emptyset$.

(g) $v \models_1 (Rab \rightarrow Rbc) \rightarrow Rac$ iff $v \models_0 Rab \rightarrow Rbc$ or $v \models_1 Rac$. Again, the second option is easier.

$v \models_1 Rac$ iff $\langle \delta(a), \delta(c) \rangle \in R^+$. Thus, the following will work as a case in which $(Rab \rightarrow Rbc) \rightarrow Rac$ if true: Let

$v = \langle D, \delta \rangle$ with $D = \{1\}$, $\delta(a) = \delta(b) = \delta(c) = 1$, $R^+ = \{\langle 1,1 \rangle\}$, $R^- = \emptyset$.

4. For each of the following sentences, construct a case in which it is *false*.

   (a) $Raa \vee Rbb$
   (b) $Fa \rightarrow (Fa \rightarrow Fb)$
   (c) $(Rab \rightarrow Rbc) \rightarrow Rac$
   (d) $Rab \vee (Fa \vee Fb)$
   (e) $(Fa \rightarrow Ga) \wedge (Ga \rightarrow Fa)$

   Answer to (d) $v \models_0 Rab \vee (Fa \vee Fb)$ iff $v \models_0 Rab$ and $v \models_0 (Fa \vee Fb)$. $v \models_0 Fa \vee Fb$ iff $v \models_0 Fa$ and $v \models_0 Fb$.

   Thus, a model in which $Rab \vee (Fa \vee Fb)$ is false is the following: let $v = \langle D, \delta \rangle$ with $D = \{1,2\}$, $\delta(a) = 1$, $\delta(b) = 2$, $R^+ = \{\langle 1,1 \rangle, \langle 2,2 \rangle\}$, $R^- = \{\langle 1,2 \rangle, \langle 2,1 \rangle\}$, $F^+ = \emptyset$, $F^- = \{1,2\}$.

5. For each of the following *sets* of sentences, construct a case in which every sentence in the set is true.

   (a) $\{Fa \vee Fb, \neg Fa\}$
   (b) $\{\neg Fb, Fa \rightarrow Ga, \neg Fb \rightarrow \neg Ga\}$
   (c) $\{Raa, \neg(Rba \vee Rab)\}$
   (d) $\{Fa \vee Fb, Fa \rightarrow (Fb \wedge Fc), \neg Fc\}$
   (e) $\{(Fa \vee Fb) \wedge Fc, Fc \rightarrow \neg Fa\}$

   Answer to (d) $v \models_1 Fa \vee Fb$ iff $v \models_1 Fa$ or $v \models_1 Fb$. $v \models_1 Fa \rightarrow (Fb \vee Fc)$ iff $v \models_0 Fa$ or $v \models_1 Fb \vee Fc$. Finally, $v \models_1 \neg Fc$ iff $v \models_0 Fc$.

   So, here's one option: Let $v = \langle D, \delta \rangle$ with $D = \{1,2,3\}$, $\delta(a) = 1$, $\delta(b) = 2$, and $\delta(c) = 3$. Let $F^+ = \{2\}$ and $F^- = \{1,3\}$.

   Then $v \models_1 Fa \vee Fb$ since $\delta(b) = 2 \in F^+$, from which it follows that $v \models_1 Fb$. And $v \models_1 Fa \rightarrow (Fb \vee Fc)$ since $\delta(a) = 1 \in F^-$, from which it follows that $v \models_0 Fa$. And finally, $v \models_1 \neg Fc$ since $\delta(c) = 3 \in F^-$, from which it follows that $v \models_0 Fc$.

2.11 CHAPTER 11: IDENTITY

1. What, in your own words, is the difference between a logical and non-logical expression? Why is the distinction important

for specifying a logical theory versus a theory of non-logical phenomena (a theory that assumes logic at its foundation but goes beyond logical vocabulary)?

Answer Answers here will vary, of course. But it will be very important, regardless of one's answers, to take account of the discussion at the beginning of the chapter, on pp. 133-135, and also the discussion at the start of section 11.3.

2. Discuss the following argument: not everything is identical to itself. After all, I weighed less than 10 pounds when I was born, and I weigh much more than that today. If I were identical to myself, then I'd both weigh less than 10 pounds and more than 10 pounds, but this is impossible.

   Answer Answers will again vary to some extent. Readers should know that in their answers to this question they are liable to tread in deep philosophical waters.

3. Prove that every sentence of the form '$a = b$' is contingent in the theory examined in this chapter. (Recall that a sentence is contingent when it is true in some cases and not true in others).

   Answer This can be proved straightforwardly: Let $v = \langle D, \delta \rangle$ be a case with $D = \{1, 2\}$ and $\delta(a) = 1$, $\delta(b) = 2$. Then $v \models_0 a = b$.

   Now let $v' = \langle D', \delta' \rangle$ be a case with $D' = \{1\}$ and $\delta'(a) = 1$, $\delta'(b) = 1$. Then $v \models_1 a = b$.

   So $a = b$ is true in some cases and false in others, and is hence contingent.

4. Prove that every sentence of the form '$a = a$' is a logical truth in the theory examined in this chapter. (Recall that a sentence is a logical truth when it is true in all cases).

   Answer Left to the reader.

   ⋆ *Special note*: because we're not treating identity as a logical expression it might be prudent to adjust the definition of *logical truth* to something that requires truth in all cases *and* its truth in all cases is in virtue of and only of logical vocabulary. Such an approach is indeed prudent, and in the end the right way to go given the distinction between logical consequence and extra-logical, theoretical consequence relations. Still, for purposes of convenience and simplicity we shall leave the definition of 'logical truth' as it is – trusting that the reader sees the distinctions involved in this special note. *End special note.*

5. Consider the properties of binary relations in §4.3.1. Is identity reflexive? Symmetric? Transitive? An equivalence relation?

   Answer Identity is an equivalence relation. It is, in some sense, the *primordial example* of an equivalence relation.

6. Suppose that objects a and b have exactly the same properties. Does it follow from this that $a = b$? Suppose that $a = b$. Does it follow from this that a and b have exactly the same properties?

   Answer There's room for a little bit of debate on the first answer, but probably the most likely answers are yes and yes. For the first question, suppose that a and b have exactly the same properties. Observe that one property of a is that it is equal to a. Since a and b share all their properties, it must then be the case that b also has this property; that is, it must be the case that b is equal to a! For the other question, notice that if $a = b$, then a and b are literally the same object. So anything we say of a we must also be able to say of b. So they have the same properties.

7. Suppose we introduce another special non-logical predicate I_3, where for every case $c = \langle D, \delta \rangle$

   $$\mathcal{E}^{+}_{I_3} = \{\langle a, a, a \rangle : a \in D\}$$

   and

   $$\mathcal{E}^{-}_{I_3} = \{\langle a, b, c \rangle : a, b, c \in D \text{ and } a \neq b \text{ or } b \neq c\}$$

   How might we define I_3 in terms of the binary identity predicate and the basic sentential connectives? What constraints would we place on the extension of a quaternary identity predicate I_4? How might we define I_4 in terms of the binary identity predicate and the basic sentential connectives?

   Answer Left to the reader.

8. Show that for any case c we have that $\mathcal{E}^{+}_{=} \cup \mathcal{E}^{-}_{=} = D \times D$.

   Answer Left to the reader.

9. Show that $a = b, b = c \vdash a = c$.

   Answer Suppose $v \models_1 a = b$ and $v \models_1 b = c$. Then $\langle \delta(a), \delta(b) \rangle \in \mathcal{E}^+_=$ and $\langle \delta(b), \delta(c) \rangle \in \mathcal{E}^+_=$. But then $\delta(a) = \delta(b) = \delta(c)$. So $\delta(a) = \delta(c)$. But this just means that $\langle \delta(a), \delta(c) \rangle \in \mathcal{E}^+_=$, so that $v \models_1 a = c$. So $a = b, b = c \vdash a = c$.

10. For each of the following invalid arguments, construct a case that serves as a counterexample.

    (a) $\neg a = b, Fa \therefore \neg Fb$
    (b) $Fa \rightarrow Gb, b = c, \neg Fa \therefore \neg Gc$
    (c) $Fa, Fb \therefore a = b$
    (d) $a = b \therefore Rab \vee Rba$

    Answer to (c) Let $v = \langle D, \delta \rangle$ with $D = \{1, 2\}$, $\delta(a) = 1$, $\delta(b) = 2$, $F^+ = \{1, 2\}$ and $F^- = \emptyset$. By definition $\mathcal{E}^+_= = \{\langle 1, 1 \rangle, \langle 2, 2 \rangle\}$ and $\mathcal{E}^-_= = \{\langle 1, 2 \rangle, \langle 2, 1 \rangle\}$.

    Since $\delta(a) = 1 \in F^+$, $v \models_1 Fa$. Since $\delta(b) = 2 \in F^+$, $v \models_1 Fb$. But since $\langle \delta(a), \delta(b) \rangle = \langle 1, 2 \rangle \in \mathcal{E}^-_=$, $v \models_0 a = b$. Thus the argument $Fa, Fb \therefore a = b$ is invalid.

2.12 CHAPTER 12: TABLEAUX FOR FIRST-ORDER LOGIC WITH IDENTITY

1. Several times in the chapter we left checking that a given case was actually a counterexample as an exercise. Do these checks now.

   Answer Left to the reader.

2. For each of the following valid forms, give a proof by tableaux that it's valid.

   - $\exists x \exists y Qxy \vdash \exists y \exists x Qxy$
   - $\exists x \forall y Qxy \vdash \forall y \exists x Qxy$

Answer to Part 1

$$
\begin{array}{c}
\langle \exists x \exists y Qxy \quad \oplus \rangle \\
\langle \exists y \exists x Qxy \quad \ominus \rangle \\
| \\
\langle \exists y Qay \quad \oplus \rangle \\
| \\
\langle Qab \quad \oplus \rangle \\
| \\
\langle \exists x Qxb \quad \ominus \rangle \\
| \\
\langle Qab \quad \ominus \rangle \\
\times_1
\end{array}
$$

3. Justify each of the tableau rules in Figure 12.2 by appealing to the truth conditions for the identity predicate and the quantifiers (and negation).

Answer for $\exists$-$\oplus$

The $\exists$-$\oplus$ rule says that if, on some line of a tableau, we find a node of the form $\langle \exists x A(x) \quad \oplus \rangle$, then we may extend the tableau with a node of the form $\langle A(a) \quad \oplus \rangle$, where a is new to the branch.

Recall the truth condition for existential sentences says that $v \models_1 \exists x A(x)$ iff for some name α, $v \models_1 A(\alpha/x)$. Finally, recall that we can view producing a tableau as attempting to build a case that meets the conditions given at the very top of the tableau.

Very roughly, then, we can see that the truth condition warrants the tableau rule by observing that all the truth condition requires is that for *some* name α, $v \models_1 A(\alpha)$. But since we aren't told anything further about whatever α names, we can't add $A(\beta)$ to our tableau for any name β about which we know anything. Thus, we use a *fresh* name α, guaranteeing that all we are adding to our tableau is what is warranted by the truth condition – that for *some* name α, $v \models_1 A(\alpha)$.

4. For each of the following arguments, use a tableau to determine whether it is valid. If it is invalid, use an open branch to construct a counterexample.

   (a) $\forall y \exists x Rxy \therefore \exists x \forall y Rxy$
   (b) $Fa, Fb, a \neq b \therefore \exists x(Fx \wedge x \neq a)$
   (c) $\exists x(Fx \wedge x = b) \therefore Fb$
   (d) $\exists x(Fx \wedge x \neq b) \therefore \neg Fb$
   (e) $\forall x(Fx \rightarrow x \neq b) \therefore \neg Fb$

   Answer to (b)

$$
\begin{array}{c}
\langle Fa \quad \oplus\rangle \\
\langle Fb \quad \oplus\rangle \\
\langle \neg a = b \quad \oplus\rangle \\
\langle \exists x(Fx \wedge x \neq a) \quad \ominus\rangle \\
| \\
\langle Fb \wedge \neg b = a \quad \ominus\rangle \\
\begin{array}{cc}
\langle Fb \quad \ominus\rangle & \langle \neg b = a \quad \ominus\rangle \\
\times_1 & | \\
 & \langle \neg b = b \quad \ominus\rangle \\
 & | \\
 & \langle \neg a = b \quad \ominus\rangle \\
 & \times_1
\end{array}
\end{array}
$$

Since all branches close, this argument is valid.

Note: the last node on the right occurs as an application of the $=$-$\ominus$-rule to the previous node, which itself occurs as an application of the $=$-I-$\ominus$-rule. These two steps are necessary, because none of our closure conditions apply to the pair of nodes $\langle \neg a = b \quad \oplus\rangle$ and $\langle \neg b = a \quad \ominus\rangle$!

Answer to (d) This tableau does not close, so the argument is invalid. We construct a counterexample from the (only) open branch as follows.

$$
\begin{array}{c}
\langle \exists x(Fx \wedge \neg x = b) \quad \oplus \rangle \\
\langle \neg Fb \quad \ominus \rangle \\
| \\
\langle Fa \wedge \neg a = b \quad \oplus \rangle \\
| \\
\langle Fa \quad \oplus \rangle \\
| \\
\langle \neg a = b \quad \oplus \rangle
\end{array}
$$

First: two names occur in our branch: a and b. From the $\langle \neg a = b \quad \oplus \rangle$-node, we see that a and b must denote different elements of the domain. So we will choose a case $v = \langle D, \delta \rangle$, with $D = \{1,2\}$, $\delta(a) = 1$, and $\delta(b) = 2$. Next, from the $\langle \neg Fb \quad \ominus \rangle$-node, we see that $\delta(b) \notin F^-$. So $\delta(b) = 2 \in F^+$. Finally, from the $\langle Fa \quad \oplus \rangle$-node, we see that $\delta(a) = 1 \in Fb$ as well. Thus $F^+ = \{1,2\}$ and $F^- = \emptyset$. The reader should check that this case does in fact counterexample the given argument.

5. Use tableaux to determine whether the following sentences are logically true, logically false, or contingent.

   (a) $\exists x(x = b)$
   (b) $\exists x(x \neq x)$
   (c) $\forall x(x = x)$
   (d) $\forall x \exists y(x = y)$

   Answer to (b) First check whether it is possible for $\exists x(x \neq x)$ to be true:

$$
\begin{array}{c}
\langle \exists x(\neg x = x) \quad \oplus \rangle \\
| \\
\langle \neg a = a \quad \oplus \rangle \\
| \\
\langle a = a \quad \oplus \rangle \\
\times_2
\end{array}
$$

Since the tableau closes, it is not possible for $\exists x(\neg x = x)$ to be true. Thus, this sentence is neither a logical truth nor a contingent sentence, so must be a logical falsehood.

2.13 CHAPTER 13: FIRST-ORDER TRANSLATIONS

1. For each of the following arguments, provide a translation of it into the language of the first-order logic. (Be sure to provide a translation key.) Then construct a tableau to determine whether it is valid or invalid. If it is invalid, use an open branch to construct a case that serves as a counterexample.

   (a) i. All mice are rodents.
       ii. All rodents are mammals.
       iii. Therefore, all mice are mammals.

   (b) i. Every dog is sleeping.
       ii. Rufus is playing.
       iii. Therefore, Rufus is not a dog.

   (c) i. Sherlock Holmes is a fictional character.
       ii. Therefore, Sherlock Holmes does not exist.

   (d) i. Aurore Dupin befriended many artists.
       ii. George Sand wrote many fine novels.
       iii. George Sand is Aurore Dupin.
       iv. Therefore, George Sand wrote many fine novels and befriended many artists.

   (e) i. Everyone confides in someone.
       ii. Therefore, someone is confided in by everyone.

   (f) i. There is some person that everyone confides in.
       ii. Therefore, everyone confides in someone.

   (g) i. If anyone is napping, then Max is napping.
       ii. Agnes is napping.
       iii. Therefore, Max is napping.

   (h) i. Barcan is a logician and Marcus is a person.
       ii. Barcan is identical to Marcus.
       iii. Therefore, some logicians are people.

   (i) i. All dogs are happy.
       ii. If Buster is a dog, then Buster is jumping.
       iii. Buster is jumping.

iv. Therefore, Buster is happy.

Selected Answers

(c) We will use this translation key:

Fx	x is a fictional character.
s	Sherlock Holmes.

Doing so, the argument becomes

$$Fs \therefore \neg\exists x(x = s)$$

Here is a tableau showing the argument to be invalid:

$$\begin{array}{c} \langle Fs \quad \oplus\rangle \\ \langle \neg\exists x(x = s) \quad \ominus\rangle \\ | \\ \langle \forall x \neg x = s \quad \ominus\rangle \\ | \\ \langle \neg a = s \quad \ominus\rangle \\ | \\ \langle a = s \quad \oplus\rangle \\ | \\ \langle Fa \quad \oplus\rangle \end{array}$$

We can construct the following counterexample from this tableau: $D = \{1\}$, $\delta(s) = \delta(a) = 1$, $F^+ = \{1\}$, $F^- = \emptyset$. The reader who is troubled by what this argument's being invalid should think hard about how to correct what seems to be wrong with this model, then have a look at Chapter 19.

(h) We will use this translation key:

Lx	x is a logician.
Px	x is a person.
b	Barcan
m	Marcus

Doing so, the argument becomes

$$Lb \wedge Pm, b = m \therefore \exists x(Lx \wedge Px)$$

Here is a tableau showing the argument to be valid:

$$\begin{array}{c}
\langle Lb \wedge Pm \quad \oplus\rangle \\
\langle b = m \quad \oplus\rangle \\
\langle \exists x(Lx \wedge Px) \quad \ominus\rangle \\
| \\
\langle Lm \wedge Pm \quad \oplus\rangle \\
| \\
\langle Lm \wedge Pm \quad \ominus\rangle \\
\times_1
\end{array}$$

Notes: The $\langle Lm \wedge Pm \quad \oplus\rangle$-node follows from the $\langle Lb \wedge Pm \quad \oplus\rangle$-node and the $\langle b = m \quad \oplus\rangle$-node by the $=$-$\oplus$-rule. On the other hand, the $\langle Lm \wedge Pm \quad \ominus\rangle$-node follows from the $\langle \exists x(Lx \wedge Px) \quad \ominus\rangle$-node by the $\exists$-$\ominus$-rule.

2. For each of the following sentences, provide its translation into the language of first-order logic.

   (a) If there is some cat that loves swimming, then all fish hate swimming.
   (b) At least one dog is a firefighter.
   (c) There is exactly one person who loves all runners.
   (d) Everyone who likes someone likes themselves.
   (e) If some dog is sleeping, then at least two cats are playing.
   (f) If every dog is barking, then there's no sleeping dog.
   (g) If there is one hamster who ate turkey, it's Ernie.
   (h) Every person is either a sibling or an only child.
   (i) Whoever is the leader of the United States can never be the leader of Russia.
   (j) Some cats don't like their owners, and some owners don't like their cats.
   (k) Everyone loves someone who doesn't love them back.
   (l) If anyone is a good musician, Jane is.
   (m) Agnes is taller than Ant, but Ant is the tallest of all.

Selected Answers

(c) We will use this translation key:

Lxy	x loves y.
Px	x is a person.
Rx	x is a runner.

The sentence can be paraphrased as 'there is a person, x, who loves all of those people who are runners, and it turns out that every person who loves all those people who are runners is identical to x'. Paraphrasing it like this *in writing* and *before* turning to translating it makes our life much easier. That said, here is the translation:

$$\exists x\Big(Px \wedge \forall y\big(Ry \rightarrow Lxy\big) \wedge \forall z\big([Pz \wedge \forall y(Ry \rightarrow Lyz)] \rightarrow z = x\big)\Big)$$

(h) We will use this translation key:

Sxy	x is y's sibling.
Px	x is a person.

The sentence can be paraphased 'for every person, x, either there is some person y such that x is y's sibling or there is no person y such that x is y's sibling.' Here is a translation:

$$\forall x(Px \rightarrow (\exists y(Py \wedge Sxy) \vee \neg\exists y(Py \wedge Sxy)))$$

Notice that this is a logical truth! (How could you tell this quickly, without using tableau?)

2.14 CHAPTER 14: ALTERNATIVE LOGICAL THEORIES

1. What linguistic phenomena besides vagueness might motivate a paracomplete logic? Are there some sentences in other parts of our language that we might want to allow to be neither true nor false?

Sample Answer: Other than vagueness, another phenomenon that might motivate paracompleteness would be the presence of non-referring terms. For example, the sentence 'The present king of France is bald' may strike you as the type of sentence that should be neither true nor false, on the grounds that there is no present king of France (as of the writing of this book!) Further parts of our language might be worth considering include, e.g., moral discourse ('What he did was wrong.') or preference statements ('Vanilla is better than chocolate.'). This is a fascinating topic worth exploring!

2. What linguistic phenomena besides the Liar paradox might motivate a paraconsistent logic? Are there some sentences in other parts of our language that we might want to allow to be both true and false? In what parts of our language might we want to deny that explosion ($A, \neg A \therefore B$), which is LP-invalid, is a valid argument form?

   Answer Left to the reader, though see the previous answer for some useful pointers.

3. Which fragments of language seem to be best modeled, not just by a paracomplete logic like K3, not just by a paraconsistent logic like LP, but by a logic that is both paracomplete and paraconsistent, like FDE?

   Answer Again, left to the reader, though see the previous answer for some useful pointers.

4. What, in your own words, is the difference between a paraconsistent and a paracomplete logic? What do such logics have in common?

   Answer In a paraconsistent logic, some sentences are both true and false. In a paracomplete logic, some sentences are neither true nor false. What such logics have in common is their *rejection* of the idea that every sentence has *exactly one* truth value.

5. Provide an example of an argument that is K3-valid, but not FDE-valid. Prove that the argument you've provided meets these two conditions.

   Answer Almost any of the so-called *inference rules* will do; e.g., modus ponens:

$$A \rightarrow B, A \therefore B$$

This is K3-valid: if $A \rightarrow B$ is true, then either A is false or B is true. In K3, if A is true, then A is not false. Thus, if $A \rightarrow B$ is true and A is true, then B must also be true.

On the other hand, it is *not* FDE-valid: it is still the case that if $A \rightarrow B$ is true, then either A is false or B is true. But in FDE, one way for A to be false is for A to be both true and false. Thus, if A is both true and false, but B is simply false, then $A \rightarrow B$ is true because A is false (while also being true), and A is true because A is true (while also being false). But B is false, so the argument $A \rightarrow B, A \therefore B$ is not FDE-valid.

6. Provide an example of an argument that is LP-valid, but not FDE-valid. Prove that the argument you've provided meets these two conditions.

   Answer Recall that LP has logical truths. But since FDE is strictly weaker than K3, and K3 has no logical truths, FDE has no logical truths either. Thus, one possible example of such an argument is the following:

$$\therefore ((A \rightarrow B) \wedge A) \rightarrow B$$

   We leave it to the reader to verify that this *is* LP-valid, but *is not* FDE-valid. (Hint: for the second part, consider what happens with both A and B are neither true nor false.) An extremely valuable exercise is to pause at this point and compare this solution to the solution given in the previous problem.

7. Why, in your own words, are all classical cases also K3, LP, and FDE cases?

   Answer Left to the reader, though see the next problem for some pointers.

8. Why aren't all K3 cases LP cases (and vice versa)?

   Answer Recall that LP cases are *complete*. Thus, nothing that *isn't* complete counts as an LP-case. But some K3-cases are not complete. So not every K3-case counts as an LP-case.

   Similarly, recall that K3 cases are *consistent*. Thus, nothing that *isn't* consistent counts as a K3-case. But some LP-cases are not consistent. So not every LP-case counts as a K3-case.

9. Prove, for any argument α, that if α is FDE-valid, then it is classically valid, K3-valid, and LP-valid.

   Answer Recall that every classical case is an FDE-case, that every LP-case is an FDE-case, and that every K3-case is an FDE-case. So suppose α is FDE-valid. Then, every FDE-case in which all the premises of α are true is an FDE-case in which α's conclusion is true.

   Let c be a classical case. Then c is, we've said, also an *FDE*-case. So if all of α's premises are true-in-c, then α's conclusion is true-in-c. So for any classical case at all, if all of α's premises are true in that case, then α's conclusion is true in that case. But this just means α is classically valid.

   Changing a few words here and there the same argument shows that α is K3-valid and LP-valid.

10. Can an argument be both K3-valid and LP-valid, but FDE-invalid? Explain.

    Answer Yes! Here's the basic idea: take your favorite classical logical truth, e.g., $A \vee \neg A$. We know that this is also an LP-logical truth because, as we said above, every classical logical truth is an LP-logical truth. But we also know it *isn't* a K3-logical truth.

    Now grab your favorite *inference rule*, say modus ponens: $B \to C, B \vdash C$. We know that this is K3-valid, but not LP-valid.

    Now, the idea is to build one argument from these two pieces, one which is K3-valid for 'K3-reasons' and LP-valid for 'LP-reasons'. Then, since neither K3- nor LP-reasons are available in FDE, the combined argument should not be FDE-valid.

    Explicitly, here's such an argument:

$$B \to C, B \therefore C \vee (A \vee \neg A)$$

    This argument is LP-valid because $A \vee \neg A$ is an LP-logical truth. So there's no LP-case in which $A \vee \neg A$ isn't true, and hence no LP-case in which $C \vee (A \vee \neg A)$ isn't true (by the truth conditions for the disjunction), and hence no LP-case in which $B \to C, B$ is true and $C \vee (A \vee \neg A)$ isn't true.

On the other hand, the argument is K3-valid because $B \rightarrow C, B \therefore C$ is K3-valid. So there's no K3-case in which $B \rightarrow C, B$ is true and C is not true, so no case in which $B \rightarrow C, B$ is true and $C \vee (A \vee \neg A)$ is not true, again by the truth conditions for the disjunction.

But the argument is *not* FDE-valid. To see this, what we do is 'squish together' a K3-counterexample to $A \vee \neg A$ (any case in which A is neither true nor false) and a LP-counterexample to $B \rightarrow C, B \therefore C$ (any case in which B is both true and false and C is not true). The result is a case in which A is neither true nor false, B is both true and false, and C is not true. In such a case, $B \rightarrow C$ is true because B is false (while also being true), B is true (while also being false), but $C \vee (A \vee \neg A)$ is not true because neither C nor $A \vee \neg A$ is.

2.15 CHAPTER 15: NONCLASSICAL SENTENTIAL LOGICS

1. We noted that Excluded Middle is not a logically true (sentence) form in K3. Question: is there any K3-case in which $A \vee \neg A$ is *false*? How about a FDE-case? If so, give such a case. If not, say why not.

   Answers There are no K3-cases in which $A \vee \neg A$ is false. We leave showing this is true to the reader. There *are* FDE-cases in which $A \vee \neg A$ is false. We also leave the production of such a case to the reader, though with a hint: remember that you are trying to build a case in which $A \vee \neg A$ is *false*, not a case in which $A \vee \neg A$ is *not true*.

2. Are there any cases in which $A \wedge \neg A$ is true according to LP? How about according to FDE? If so, give an example. If not, say why not.

   Answer We leave this to the reader, although we note that, suitably modified, the hint given in the previous problem will be useful here as well.

3. Are there any cases in which $\neg(A \wedge \neg A)$ is not true according to LP? How about according to FDE? If so, give an example. If not, say why not.

   Answer Again, we leave this to the reader, although we again note that, suitably modified, the hint given in problem 1 will be useful here as well.

4. From the list of notable forms provided in Figure 5.1 on page 68, determine which are K3-valid, which are LP-valid, and which are FDE-valid. For each form that is *in*valid on a given theory, provide a counterexample.

   Partial Answer Here is a list of which forms are valid in each theory. The remainder of the problem is left to the reader.

K3	LP	FDE
Modus Ponens	Excluded Middle	Addition
Modus Tollens	Non-contradiction (!)	Adjunction
Disjunctive Syllogism	Contraposition	Simplification
Contraposition	Addition	De Morgan (1 & 2)
Explosion	Adjunction	Double Negation
Addition	Simplification	
Adjunction	De Morgan (1 & 2)	
Simplification	Double Negation	
De Morgan (1 & 2)		
Double Negation		

5. For each of the following argument forms, determine in which of the four logical theories (CL, K3, LP, and FDE) they are valid.

   (a) $A \rightarrow B, \neg A \rightarrow B \therefore B$
   (b) $(A \vee B) \wedge C, A \rightarrow \neg C \therefore B$
   (c) $A \rightarrow B, B \rightarrow C \therefore A \rightarrow C$

   Answer to (b)

   - The argument is classically valid: $A \rightarrow \neg C$ is true iff A is false or $\neg C$ is true. But $\neg C$ is true iff C is false. So $A \rightarrow \neg C$ is true iff A is false or C is false.

On the other hand, $(A \vee B) \wedge C$ is true iff $A \vee B$ is true and C is true. In a classical case, C cannot be both true and false. So if both $(A \vee B) \wedge C$ and $A \rightarrow \neg C$ are true, then A must be false. But $A \vee B$ must also be true, and $A \vee B$ is true iff A is true or B is true. Since A is false (and thus, since we are dealing with the classical theory, not true), B is true. So the argument is valid.

- The argument is K3-valid: modifying a few words, the argument just given establishes this.
- The argument is *not* LP-valid: let A, B, and C be atomic and consider any case v in which $v(A) = \mathsf{b}$, $v(B) = \mathsf{f}$ and $v(c) = \mathsf{t}$. This case serves to counterexample the given argument (we leave it to the reader to check this).
- Since every LP-case is an FDE-case, the above case also serves as an FDE-counterexample to the given argument. Thus, it is not FDE-valid either.

6. Recall our definitions of *logically true, logically false*, and *contingent*, where A is any sentence.

- A is *logically true* iff it is true-in-*every case*.
- A is *logically false* iff it is false-in-*every case*.
- A is *contingent* iff it is true-in-*some case*, and false-in-*some case*.

Give a (paracomplete) counterexample to each of the following sentences (i.e., a paracomplete case in which the sentence is *not true*). In addition, specify which, if any, of the following sentences are K3-logically false, and which are K3-contingent.

(a) $p \rightarrow p$
(b) $p \rightarrow \neg p$
(c) $p \wedge \neg p$
(d) $q \vee p$
(e) $q \wedge (p \vee q)$
(f) $q \vee (p \wedge q)$
(g) $q \leftrightarrow \neg p$
(h) $(p \wedge (p \rightarrow q)) \rightarrow q$
(i) $p \vee \neg p$
(j) $\neg(p \wedge \neg p)$

Answer None of these are K3-logical truths, and none are K3-logical falsehoods. The same case will demonstrate this every time: the case v in which, for all atoms α, $v(\alpha) = \mathsf{n}$.

However, despite this, not all of them are K3-logical contingencies, either! For example: (a) cannot be false, so despite being neither logically true nor logically false, it is also not contingent! We leave to the reader the determination of which of the remaining sentences are actually contingent.

2.16 CHAPTER 16: NONCLASSICAL FIRST-ORDER THEORIES

1. Consider a case $c = \langle D, \delta \rangle$ where $D = \{1, 2, 3\}$ and $\delta(a) = 1$, $\delta(b) = 2$, and $\delta(d) = 3$, and $F^+ = \{2,3\}$ and $F^- = \{1\}$. Additionally, where R is a binary predicate, let $R^+ = \{\langle 1,2\rangle, \langle 1,1\rangle\}$ and $R^- = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle\}$.

   (a) In which theories can we find such a case? That is, can the given case be an LP-case? A K3-case? A FDE-case?

   (b) For each of the following, say whether it is true or false. If true, say why. If false, say why.

   i. $c \models_1 \forall x Fx$
   ii. $c \models_0 \forall x Fx$
   iii. $c \models_1 \exists x Fx$
   iv. $c \models_0 \exists x Fx$
   v. $c \models_1 \forall x Rxb$
   vi. $c \models_0 \forall x Rxb$
   vii. $c \models_1 \exists x Rax$
   viii. $c \models_0 \exists x Rax$
   ix. $c \models_1 \forall x(Rab \to Fx)$
   x. $c \models_1 \exists x \forall y Rxy$
   xi. $c \models_1 \neg\exists x \forall y Rxy$

   Selected Answers

   (a) Since $R^+ \cap R^- \neq \emptyset$, this is not a classical case nor is it a K3-case. On the other hand, since, e.g., $\langle 2, 2\rangle$ is not in $R^+ \cup R^-$, this is not an LP-case. So it's an FDE-case.

(b)

iii. Recall that $c \models_1 \exists x Fx$ iff for some name α, $c \models_1 F\alpha$ iff for some name α, $\delta(\alpha) \in F^+$. Thus, since $\delta(b) = 2 \in F^+$, $c \models_1 \exists x Fx$. So iii is true.

viii. Recall that $c \models_0 \exists x Rax$ iff $c \models_0 Ra\alpha$ for all names α iff $\langle \delta(a), \delta(\alpha) \rangle \in R^-$ for all names α. The only three names we have on hand are a, b, and d. Observe that $\langle \delta(a), \delta(a) \rangle = \langle 1,1 \rangle \in R^-$, $\langle \delta(a), \delta(b) \rangle = \langle 1,2 \rangle \in R^-$ and $\langle \delta(a), \delta(d) \rangle = \langle 1,3 \rangle \in R^-$. Thus $\langle \delta(a), \delta(\alpha) \rangle \in R^-$ for all names α, so $c \models_0 \exists x Rax$. So viii is true.

2. Construct a case in one of the logical theories investigated here in which $\forall x Gx \vee \forall x \neg Gx$ is not true. Can you construct such a case for any of the other logical theories? For each that you can, do so. For those that you can't, say why.

   Answer Left to the reader, though see the solution to problem 3 for some general guidance.

3. Construct a case in one of the logical theories investigated here in which $\forall x(Gx \rightarrow Hx)$ is true but $\forall x(Gx \wedge Hx)$ is not true. Can you construct such a case for any of the other logical theories? For each that you can, do so. For those that you can't, say why.

   Answer Let $v = \langle D, \delta \rangle$ with $D = \{1,2\}$, $\delta(a) = 1$, $\delta(b) = 2$, and $G^+ = H^+ = \{1\}$, $G^- = H^- = \{2\}$. Observe that this is a classical case. Also observe that $v \models_1 \forall x(Gx \rightarrow Hx)$ and $v \models_0 \forall x(Gx \wedge Hx)$. Since v is a classical case, it is also a K3-case, an LP-case, and an FDE-case, answering all of the questions asked.

4. The argument from §9.1 about Agnes and cats, from (3) and (4) to (5), has the form $\forall x(Cx \rightarrow Sx), Ca \therefore Sa$. In which of our three logical theories is this argument valid? (Give a counterexample for any theory in which the argument is invalid.)

   Answer This is valid in K3, but not in LP or FDE. I will give an LP-counterexample, which is, ipso facto, an FDE-counterexample.

Let $v = \langle D, \delta \rangle$ with $D = \{1\}$, $\delta(a) = 1$, and $C^{+} = C^{-} = S^{-} = \{1\}$ and $S^{+} = \emptyset$. The reader should check, first that this is an LP-case and, second, that it counterexamples the given argument.

2.17 CHAPTER 17: NONCLASSICAL TABLEAUX

Here is an example of how to use a single tableau to determine in which theories a given form is valid. I will examine modus tollens

$$
\begin{array}{c}
\langle A \to B \quad \oplus \rangle \\
\langle \neg B \quad \oplus \rangle \\
\langle \neg A \quad \ominus \rangle \\
\begin{array}{cc}
\langle \neg A \quad \oplus \rangle & \langle B \quad \oplus \rangle \\
\times_1 & \times_2
\end{array}
\end{array}
$$

Since this tableau requires both closure rule 1 and closure rule 2 in order to close, modus tollens is valid only in theories where both of these rules are available, so only in classical logic and in K3.

2.18 CHAPTER 18: NONCLASSICAL TRANSLATIONS

The exercise for this chapter is left to the reader, though it would be wise to revisit the exercises in Chapter 7 for guidance.

2.19 CHAPTER 19: SPEAKING FREELY

1. Is the following argument valid in any of our 'freed up' theories? Explain your answer.

$$\forall x Fx \therefore Pb \to Fb$$

(Hint: don't forget about cases where $\delta(b) \notin E$!)

Answer It is not! Let $v = \langle D, E, \delta \rangle$ with $D = \{1, 2\}$, $E = \{1\}$, $\delta(a) = 1$, $\delta(b) = 2$, $F^{+} = \{1\}$, $F^{-} = \{2\}$, $P^{+} = \{1, 2\}$, and $P^{-} = \emptyset$. The reader can check that this is a free classical counterexample to the given argument. Thus it is a free K3-, a free LP- and a free FDE-counterexample as well.

2. Specify a FDE∗-case in which $Fb \wedge Ga$, $\neg\exists xFx$, and $\neg\exists xGx$ are all true.

   Answer Left to the reader; see the solution to the previous problem for inspiration.

3. Specify for each of the following arguments which freed up theory/ies it is valid in and justify your answer with an argument or counterexample (not just a tableau).

   (a) $\forall xFx \therefore Fa$
   (b) $Fb \wedge Gb \therefore \exists x(Fx \wedge Gx)$
   (c) $\neg\exists xFx \therefore \neg Fb$
   (d) $\neg Fa \therefore \neg\forall xFx$

   Answer a. The argument from $\forall xFx$ to Fa is not valid (according to the current freed up theory). To see this, let c be any of our (current, freed-up) cases in which $\delta(a) \notin E$ (i.e., in which the denotation of name a is not among the objects that, according to c, exist), and let $\delta(a) \notin F^+$ (i.e., the denotation of a is not in the set of objects that, according to c, have the property F), and let everything else in D be in the extension of F (i.e., be in F^+). This case is a counterexample to the given argument.

   Problems 4-6 are much more philosophical. Rather than providing 'solutions' to these problems that will prevent the reader from creatively coming up with their own we encourage instead that (a) a genuine attempt is made to come up with a unique and interesting solution to these difficulties and, after that, that (b) the relevant literature is consulted to determine whether the answer has merit. For a guide to the relevant literature, see the 'further reading' section of the text for this chapter.

2.20 CHAPTER 20: POSSIBILITIES

1. Which of the following arguments are valid in mFDE How about in mCL? Justify your answer (with a proof or counterexample).

(a) $\Box(Fb \wedge Fa) \therefore \Box Fb \wedge \Box Fa$
(b) $\Box Fb \therefore \Diamond Fb$
(c) $\Diamond Fb \therefore \Box Fb$
(d) $\Box(a = a) \therefore \Diamond \exists x(x = a)$
(e) $\Box \neg \exists x(x = a) \therefore \neg(a = a)$
(f) $\Box \Diamond Fa \therefore \Diamond \Box Fa$
(g) $\neg \Diamond \exists x(x = a) \therefore \neg \Box(a = a)$

Answer to (c) This is invalid. To see this, consider the case $c = [F, w]$, with $F = \langle \mathcal{U}, D, E, \delta \rangle$, and $\mathcal{U} = \{w, v\}$, $D = \{1\}$, $E_w = E_v = \{1\}$, $\delta(b) = 1$, $F_w^+ = \{1\}$, $F_w^- = \emptyset$, $F^+v = \emptyset$, $F_v^- = \{1\}$.

Recall that $[F, w] \models_1 \Diamond Fb$ iff for some $w' \in \mathcal{U}$, $[F, w'] \models_1 Fb$, and that $[F, w'] \models_1 Fb$ iff $\delta(b) \in F_{w'}^+$. But $\delta(b) = 1 \in F_w^+$, so there is such a world. Thus, $c \models_1 \Diamond Fb$.

On the other hand recall that $[F, w] \models_1 \Box Fb$ iff for every $w' \in \mathcal{U}$, $[F, w'] \models_1 Fb$, and that $[F, w'] \models_1 Fb$ iff $\delta(b) \in F_{w'}^+$. But $\delta(b) = 1 \notin F_v^+$, so $c \models_0 \Box Fb$.

2. Are there cases in which $a = a$ is not true? If so, provide one. If not, show as much.

   Answer Left to the reader, though see the solution to the next problem for some insight.

3. Are there cases in which $\Box(a = a)$ is not true? If so, provide one. If not, show as much.

   Answer $[F, w] \not\models_1 \Box(a = a)$ iff for some $w' \in \mathcal{U}$, $[F, w'] \not\models_1 a = a$ iff for some $w' \in \mathcal{U}$, $\langle \delta(a), \delta(a) \rangle \notin \mathcal{E}_=^+$. But by the truth conditions for the identity predicate, this is impossible. So there are no cases in which $\Box(a = a)$ is not true.

2.21 CHAPTER 21: FREE AND MODAL TABLEAUX

1. For each of the following sentences of CL*, use tableaux to either show the given sentence is valid or provide a case in which it is false.

   (a) $\forall x Px \to Pt$
   (b) $(\forall x Px \wedge \exists x(x = t)) \to Pt$
   (c) $Pt \to \exists x Px$

(d) $(Pt \wedge \exists x(x = t)) \rightarrow \exists x Px$

Answer to (c)

$$
\begin{array}{c}
\langle Pt \rightarrow \exists x Px \quad \ominus \rangle \\
| \\
\langle \neg Pt \quad \ominus \rangle \\
\langle \exists x Px \quad \ominus \rangle \\
\swarrow \qquad \searrow \\
\langle \mathbb{E}t \quad \ominus \rangle \qquad \langle Pt \quad \ominus \rangle \\
\qquad\qquad\qquad \times_3
\end{array}
$$

Since the tableau does not close we can construct from the open branch the following case v in which the given sentence is not true: $v := \langle D, E, \delta \rangle$ with $D = \{1\}$, $E = \emptyset$, $\delta(t) = 1$, $P^+ = \{1\}$, and $P^- = \emptyset$. We leave it to the reader to check that $Pt \rightarrow \exists x Px$ is not true in this case.

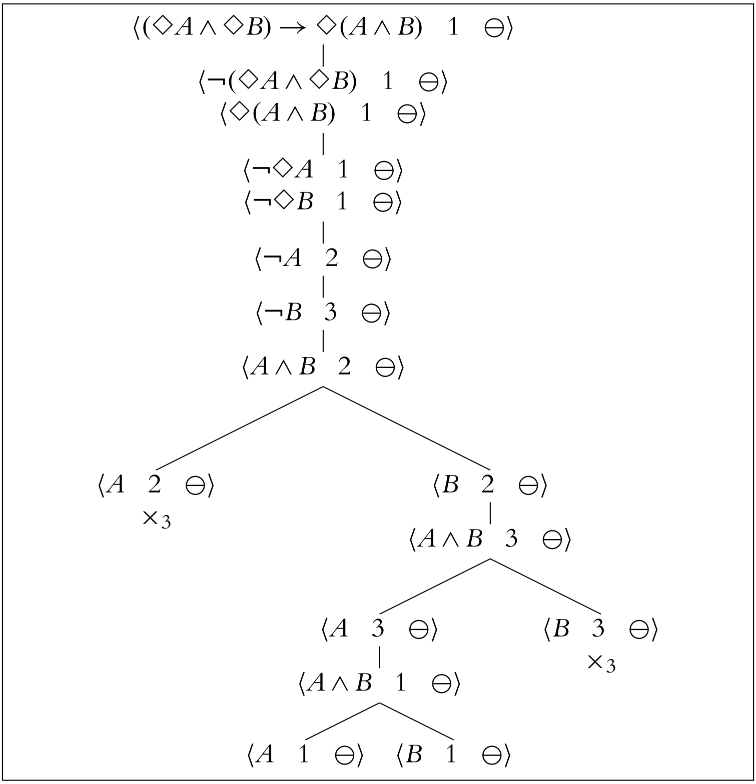
2. For each of the following sentences of mCL, use tableaux to either show the given sentence is valid or provide a case in which it is false.

(a) $\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$
(b) $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$
(c) $\Box(A \vee B) \rightarrow (\Box A \vee \Box B)$
(d) $(\Box A \vee \Box B) \rightarrow \Box(A \vee B)$
(e) $\Diamond(A \wedge B) \rightarrow (\Diamond A \wedge \Diamond B)$
(f) $(\Diamond A \wedge \Diamond B) \rightarrow \Diamond(A \wedge B)$
(g) $\Diamond(A \vee B) \rightarrow (\Diamond A \vee \Diamond B)$
(h) $(\Diamond A \vee \Diamond B) \rightarrow \Diamond(A \vee B)$
(i) $\Box(A \rightarrow A)$
(j) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
(k) $\Box(A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$
(l) $\Box A \rightarrow \Diamond A$
(m) $\Box A \rightarrow A$
(n) $A \rightarrow \Diamond A$

(o) $A \rightarrow \Box\Diamond A$
(p) $\Diamond\Box A \rightarrow A$
(q) $\Box A \rightarrow \Box\Box A$
(r) $\Diamond\Diamond A \rightarrow \Diamond A$
(s) $\Diamond A \rightarrow \Box\Diamond A$

Selected Answers

(f)



The tableau does not close, so the sentence is not a logical truth. From the open branch, we construct a case c in which it is false as follows. First, let A be the sentence Ga and B be the sentence Gb. We specify the case as follows

$c = \langle F, 1\rangle$, where $F = \langle U, D, E, \delta\rangle$, and $U = \{1,2,3\}$, $D = E_1 = E_2 = E_3 = \{1,2\}$, $\delta(a) = 1$, $\delta(b) = 2$, and G has the following (world-relative) extensions and anti-extensions:

$G_1^+ = \emptyset$
$G_1^- = \{1,2\}$
$G_2^+ = \{1\}$
$G_2^- = \{2\}$
$G_3^+ = \{2\}$
$G_3^- = \{1\}$

We leave it to the reader to check that this is in fact a case in which the given sentence is false.

(h)

$$
\begin{array}{c}
\langle ((\Diamond A \vee \Diamond B) \rightarrow \Diamond (A \vee B) \quad 1 \quad \ominus\rangle \\
| \\
\langle \neg(\Diamond A \vee \Diamond B) \quad 1 \quad \ominus\rangle \\
\langle \Diamond (A \vee B) \quad 1 \quad \ominus\rangle \\
| \\
\langle A \vee B \quad 1 \quad \ominus\rangle \\
| \\
\langle A \quad 1 \quad \ominus\rangle \\
\langle B \quad 1 \quad \ominus\rangle \\
\begin{array}{cc}
\langle \neg\Diamond A \quad 1 \quad \ominus\rangle & \langle \neg\Diamond B \quad 1 \quad \ominus\rangle \\
| & | \\
\langle \neg A \quad 2 \quad \ominus\rangle & \langle \neg B \quad 2 \quad \ominus\rangle \\
| & | \\
\langle A \vee B \quad 2 \quad \ominus\rangle & \langle A \vee B \quad 2 \quad \ominus\rangle \\
| & | \\
\langle A \quad 2 \quad \ominus\rangle & \langle B \quad 2 \quad \ominus\rangle \\
\times_3 & \times_3
\end{array}
\end{array}
$$

Since the tableau closes, any sentence of this form is a mCL-logical truth.

3. Provide a translation of each of the following formulas into English, interpreting '$\Box$' as expressing logical necessity and '$\Diamond$' as expressing logical possibility. Which seem to be plausible candidates for logical truths? Which do not? Why?

   (a) $\forall x \Box Fx \to \Box \forall x Fx$
   (b) $\Diamond \exists x Fx \to \exists x \Diamond Fx$
   (c) $\Box \forall x Fx \to \forall x \Box Fx$
   (d) $\exists x \Diamond Fx \to \Diamond \exists x Fx$

   Answer to (b) Let's translate 'Fx' as 'x is a firefighter.' Then, this sentence says 'if it's logically possible that something is a firefighter, then something exists that is (logically) possibly a firefighter'.

   This doesn't *intuitively* strike me as a plausible candidate for a logical truth. Why not? Because it seems to me there could be a (logically possible) world in which there is something that is a firefighter without there being anything *that exists in this world* that is a firefighter in some (logically possible) world. Perhaps for all the things that exist in this world, it is metaphysically impossible for them to be firefighters, yet in some other logically possible world, there is some thing – something that doesn't exist in this world – that not only is not so restricted, but in fact *is*, in that world, a firefighter.

2.22 CHAPTER 22: GLIMPSING DIFFERENT LOGICAL ROADS

In line with the spirit of the chapter – exploring *different logical roads* – *we leave the solution of these problems to the reader and ask that the answers provided be as creative as possible!*

1. Does the actuality operator Capture? Does it Release? Prove that your answers are correct.
2. Do we have that $\Box A \vdash \nabla A$ in the mFDE theory (expanded with ∇ as above)? If so, prove it. If not, give a counterexample.
3. Using the revised truth conditions (see page 270), give a countermodel to η-Explosion: viz., $A, \eta A \therefore B$.

4. With respect to the 'meaningless' approach to disjunction (see §22.4), provide a case in which A is true but $A \vee B$ not true (for some A and B).
5. Fill out the 'meaningless' approach (see §22.4) by adding predicates, quantifiers, and a necessity operator. (NB: there may be more than one way of doing this that is consistent with the basic 'meaningless' idea.)

NOTE

1. NB: we certainly do not need to use Reductio, since the answer falls directly out of the truth conditions for conjunction. We do so, however, to give a useful example of Reductio reasoning.

3

SOUNDNESS AND COMPLETENESS PROOFS

In this chapter we prove, for each tableau system considered in the text, two important results that jointly entail these tableau systems are *correct* – in a technical sense to be made clearer below – with respect to the semantic theories also given in the text. The chapter begins with a discussion of the *method* of proof we will use and an examination of some generalities about the two results. The proofs that follow are loosely based on the proofs found in Graham Priest's *Introduction to Non-Classical Logic*.

3.1 INDUCTION

The proofs in this appendix are by and large what are known as *inductive proofs*. The idea of proof by induction is straightforward: suppose we can divide what we need to prove into steps, and we can take the first step. Suppose also we can prove that for any step we take, we can take the next step, too. Then since we can take the first step, we can take the second step. And since we can take the second step, we can take the third step. And since we can take the third step...

Mathematical induction gives a powerful way to learn things about the formal languages we've examined in *Logic: The Basics* because of the way those languages are constructed. In particular, a very common way to prove things about a formal language (a method you will see repeatedly in the course of this appendix) is to show that something is true of all atomic sentences in the language, and that if it is true of sentences with n or fewer connectives (or connectives and quantifiers or connectives and quantifiers and modalities, etc.) then it is true of sentences with $n+1$ connectives. Using these two facts, we then can conclude whatever it is we're concluding is true for every sentence in whatever language we're discussing

More formally, here is the statement of the *Principle of (strong) Mathematical Induction* that we will rely on in this appendix:

Principle of Mathematical Induction Suppose $P_0, P_1, P_2, \ldots$ is an infinite list of sentences. In order to prove the truth of every member of this list, it suffices to do the following:

- Prove that P_0 is true, and
- Prove that for any n, *if* P_k were true for all $k \leq n$, *then* P_{n+1} would also be true.

An example of the method will help make it more concrete. Suppose we want to prove the following:

Example Theorem Sentences in our propositional languages have the same number of left and right parentheses.

Proof We can think of this as an infinite list of propositions we want to prove the truth of:

P_0: Sentences in our propositional languages with exactly 0 connectives (that is, any atomic sentence) have the same number of left and right parentheses.

P_1: Sentences in our propositional languages with exactly 1 connective have the same number of left and right parentheses.

P_2: Sentences in our propositional languages with exactly 2 connectives have the same number of left and right parentheses.

$\vdots$

P_n: Sentences in our propositional languages with exactly n connectives have the same number of left and right parentheses.

$\vdots$

Rather than setting out to prove every member of the list one-by-one (which would take us (literally!) forever), we use mathematical induction. To do this, we first notice that P_0 is true: atomic sentences contain the same number (zero, as it turns out) of left and right parentheses. Now we want to prove that for any n, *if* P_k were true for all $k \leq n$, *then* P_{n+1} would also be true.

To that end, suppose P_k is true for all k not greater than n; that is, suppose that for each $k \leq n$ any sentence with exactly k connectives has the same number of left and right parentheses. Now suppose someone hands us a sentence ψ with $n+1$ connectives. There are three ways ψ might be constructed:

(i) ψ could be the negation of some sentence, so that $\psi = \neg\phi$ for some sentence ϕ.
(ii) ψ could be the conjunction of two sentences, so that $\psi = (A \wedge B)$ for some sentences A and B.
(iii) ψ could be the disjunction of two sentences, so that $\psi = (A \vee B)$ for some sentences A and B.

Notice in the first case that ϕ has n connectives. So since we are assuming P_n is true, ϕ has the same number of left and right parentheses. So in the first case, ψ will also have the same number of left and right parentheses. In the second and third cases, each of A and B has n or fewer connectives. So each of them has the same number of left and right parentheses. But then in each case, the number of left parentheses in ψ is the number of left parentheses in A plus the number of left parentheses in B plus one, which is the same thing as the number of right parentheses in A plus the number of right parentheses in B plus one. So ψ has the same number of left and right parentheses.

By mathematical induction, this completes the proof. Voila! $\square$

It's important to understand this proof technique before continuing on, so if you are uncomfortable with it, it's worth your while to skip at this point to the end of the appendix where the exercises are located to try your hand at induction a few times. That said, if you're perfectly happy with mathematical induction, we now turn to explaining the two senses in which a tableau system might be *correct*: soundness and completeness.

3.2 GENERAL DISCUSSION

In the text we claimed we could use tableaux to check for validity. But how can we be certain that the various tableau systems we gave *correctly* capture their corresponding notions of validity? To demonstrate this, we must first distinguish the following two relations:

- The relation that holds between a set of sentences X and an individual sentence ϕ exactly when the tableau that represents the argument $X \therefore \phi$ in langauge $\mathcal{L}$ closes; and
- The relation that holds between a set of sentences X and an individual sentence ϕ exactly when every $\mathcal{L}$-case that satisfies X also satisfies ϕ.

We write '$X \stackrel{\mathcal{L}}{\Rightarrow} \phi$' when the first relation holds and use (as we used in the text) '$X \models^{\mathcal{L}} \phi$' when the second relation holds.

Definition *A tableau system is sound when for all sets of sentences X and sentences ϕ, if $X \stackrel{\mathcal{L}}{\Rightarrow} \phi$ then $X \models^{\mathcal{L}} \phi$.*

Definition *A tableau system is complete when for all sets of sentences X and sentences ϕ, if $X \models^{\mathcal{L}} \phi$ then $X \stackrel{\mathcal{L}}{\Rightarrow} \phi$.*

Sound systems don't prove too much – if $X \not\models^{\mathcal{L}} \phi$, then $X \stackrel{\mathcal{L}}{\not\Rightarrow} \phi$. Complete systems don't prove too little – it never happens that $X \models^{\mathcal{L}} \phi$ and $X \stackrel{\mathcal{L}}{\not\Rightarrow} \phi$. A system that is sound and complete proves exactly the right amount – it proves all and only those arguments that are valid in the given semantic theory. In the remainder we will show that for each language $\mathcal{L}$ considered in the text, the corresponding tableau system is sound and complete with respect to the semantics for $\mathcal{L}$.

3.3 PROPOSITIONAL CASE

Since this is our first soundness proof, there will be quite a bit of discussion in between results to let you know what's happening and why we're doing it. In the later soundness proofs, there will be less of this discussion, as it will be taken to be understood. Also, many of the inductive proofs you see involve checking each of the different ways a sentence can be constructed. It is natural to say that we are checking each of the possible *cases*. This, unfortunately, clashes with the way we used the word 'case' in the rest of the book. To prevent confusion, then, we use here 'model' where we used 'case' in the rest of the text. Thus, for example, rather than saying '$X \overset{\mathcal{L}}{\Rightarrow} \phi$ exactly when every $\mathcal{L}$-case that satisfies X also satisfies ϕ', we will say '$X \overset{\mathcal{L}}{\Rightarrow} \phi$ exactly when every $\mathcal{L}$-*model* that satisfies X also satisfies ϕ'. This rather minor difference will allow us to sidestep a great many possible confusions.

For simplicity, it is useful to think of all our propositional models as functions from the set of atomics At to the set $\{1, 0, \mathsf{b}, \mathsf{n}\}$; that is, as FDE-models. From this point of view, a CL-model is a propositional model whose range does not contain b or n, a K3-model is a propositional model whose range does not contain b, and an LP-model is a propositional model whose range does not contain n.

3.3.1 SOUNDNESS

We will prove in this section essentially the following: if some model of language $\mathcal{L}$ is described by a branch on a tableau we are working on, then when we extend that branch, there will still be some model of language $\mathcal{L}$ that is described by at least one of the resulting branches. Then, we will use this result to demonstrate the soundness of our propositional tableau systems. First, a definition:

Definition *Given a tableau (from any of our systems) T and a branch b of T, we say a model v is faithful to b when for every sentence S, if a node of the form $\langle S \quad \oplus \rangle$ is on b, then $v \vDash_1 S$, and if a node of the* *form* $\langle S \quad \ominus \rangle$ *is on* b*, then* $v \not\models_1 S$*. Let* $f(b)$ *be the set of models that are faithful to* b*. (We leave it to context to determine which type of model is being considered.)*

A model is *faithful* to a branch when, roughly, the branch describes the model accurately. Given this and what we said above, it is no surprise that the next result we prove is the following:

Lemma 1. (Faithfulness) *If* T *is a propositional tableau,* b *is a branch in* T *and* v *is a propositional model that is faithful to* b*, then if we extend* b *using one of our tableau rules,* v *will be faithful to at least one of the resulting branches.*

The proof proceeds by examining each rule individually, taking the positive rules first, then the negative rules. Not all the cases will be taken up here – only a few examples are examined and the remaining rules are left as exercises.

Proof Suppose v is a propositional model that is faithful to the branch b.

POSITIVE RULES:
Suppose b contains a node of the form $\langle A \wedge B \quad \oplus \rangle$. Recall the $\wedge$-$\oplus$-rule:

$$\begin{array}{c} \langle A \wedge B \quad \oplus \rangle \\ | \\ \langle A \quad \oplus \rangle \\ \langle B \quad \oplus \rangle \end{array}$$

By the truth conditions for conjunction, $v \models_1 A \wedge B$ iff $v \models_1 A$ and $v \models_1 B$. Since v is faithful to b, $v \models_1 A \wedge B$. Thus, $v \models_1 A$ and $v \models_1 B$. So if we extend b using the $\wedge$-$\oplus$-rule, v will be faithful to the resulting branch.

Suppose b contains a node of the form $\langle A \vee B \quad \oplus \rangle$. Recall the $\vee$-$\oplus$-rule:

$$\begin{array}{c} \langle A \vee B \quad \oplus \rangle \\ \diagup \quad \diagdown \\ \langle A \quad \oplus \rangle \quad \langle B \quad \oplus \rangle \end{array}$$

By the truth conditions for disjunction, $v \vDash_1 A \vee B$ iff $v \vDash_1 A$ or $v \vDash_1 B$. Since v is faithful to b, $v \vDash_1 A \vee B$. Thus, either $v \vDash_1 A$ or $v \vDash_1 B$. In the first case v is faithful to the left-hand branch, in the second case v is faithful to the right-hand branch. So if we extend b using the $\vee$-$\oplus$-rule, v will be faithful to at least one of the resulting branches.

Suppose b contains a node of the form $\langle \neg(S \to T) \quad \oplus \rangle$. Recall the $\neg$-$\to$-$\oplus$-rule:

$$\begin{array}{c} \langle \neg(A \to B) \quad \oplus \rangle \\ | \\ \langle A \quad \oplus \rangle \\ \langle \neg B \quad \oplus \rangle \end{array}$$

By the truth conditions for negation, $v \vDash_1 \neg(S \to T)$ iff $v \vDash_0 S \to B$. Then, by the truth conditions for implication, $v \vDash_0 S \to B$ iff $v \vDash_0 \neg A$ and $v \vDash_0 B$. Finally, by the truth conditions for negation again, $v \vDash_0 \neg A$ iff $v \vDash_1 A$ and $v \vDash_0 B$ iff $v \vDash_1 \neg B$. Since v is faithful to b, $v \vDash_1 \neg(S \to T)$. Thus, $v \vDash_1 A$ and $v \vDash_1 \neg B$. So if we extend b using the $\neg$-$\vee$-$\oplus$-rule, v will be faithful to the resulting branch.

NEGATIVE RULES:
Suppose b contains a node of the form $\langle A \to B \quad \ominus \rangle$. Recall the $\to$-$\ominus$-rule:

$$\begin{array}{c} \langle A \to B \quad \ominus \rangle \\ | \\ \langle \neg A \quad \ominus \rangle \\ \langle B \quad \ominus \rangle \end{array}$$

By the truth conditions for $\to$, $v \vDash_1 A \to B$ iff $v \vDash_1 \neg A$ or $v \vDash_1 B$. Since v is faithful to b, $v \nvDash_1 A \to B$. So $v \nvDash_1 \neg A$ and $v \nvDash_1 B$. Thus if we extend b using the $\rightarrow$-$\ominus$-rule, v will be faithful the resulting branch.

Suppose b contains a node of the form $\langle \neg(A \vee B) \quad \ominus \rangle$. Recall the $\neg$-$\vee$-$\ominus$-rule:

$$\begin{array}{c}
\langle \neg(A \vee B) \quad \ominus \rangle \\
\diagup \qquad \diagdown \\
\langle \neg A \quad \ominus \rangle \qquad \langle \neg B \quad \ominus \rangle
\end{array}$$

By the truth conditions for negation, $v \vDash_1 \neg(A \vee B)$ iff $v \vDash_0 A \vee B$. Then, by the truth conditions for disjunction, $v \vDash_0 A \vee B$ iff $v \vDash_0 A$ and $v \vDash_0 B$. Finally, by the truth conditions for negation again, $v \vDash_0 A$ iff $v \vDash_1 \neg A$ and $v \vDash_0 B$ iff $v \vDash_1 \neg B$. Since v is faithful to b, $v \nvDash_1 \neg(A \vee B)$. So either $v \nvDash_1 \neg A$ or $v \nvDash_1 \neg B$. In the first case v is faithful to the left-hand branch, in the second case v is faithful to the right-hand branch. Thus if we extend b using the $\neg$-$\vee$-$\ominus$-rule, v will be faithful to one of the resulting branches.

Suppose b contains a node of the form $\langle \neg\neg A \quad \ominus \rangle$. Recall the $\neg\neg$-$\ominus$-rule:

$$\begin{array}{c}
\langle \neg\neg A \quad \ominus \rangle \\
| \\
\langle A \quad \ominus \rangle
\end{array}$$

By the truth conditions for negation, $v \vDash_1 \neg\neg A$ iff $v \vDash_0 \neg A$ iff $v \vDash_1 A$. Since v is faithful to b, $v \nvDash_1 \neg\neg A$. So $v \nvDash_1 A$. Thus if we extend b using the $\neg\neg$-$\ominus$-rule, v will be faithful to the resulting branch.

For the remaining rules, the result can be shown by similar means. These parts of the demonstration are left as exercises. We conclude by induction that the result follows for every sentence in the language. □

With Lemma 1 in hand, we turn to considering what happens when branches close. First, recall our closure conditions and which theories we use them in.

Closure Rules

- $\times_1$: The branch contains $\langle A \quad \oplus\rangle$ and $\langle A \quad \ominus\rangle$ for some sentence A.
- $\times_2$: The branch contains $\langle A \quad \oplus\rangle$ and $\langle \neg A \quad \oplus\rangle$ for some sentence A.
- $\times_3$: The branch contains $\langle A \quad \ominus\rangle$ and $\langle \neg A \quad \ominus\rangle$ for some sentence A.

Theories and Their Closure Rules

Logical Theory	Branches close with
CL	$\times_1$, $\times_2$, or $\times_3$,
K3	$\times_1$ or $\times_2$
LP	$\times_1$ or $\times_3$
FDE	$\times_1$

That these closure conditions are the correct conditions and that they are paired with the correct theories can be seen from the following lemmas.

Lemma 2. (Soundness – Closure) *If b is a branch that closes with the $\times_1$-rule, then no* FDE*-model (and hence, no* LP*-model or* K3*-model or* CL*-model) v is faithful to b.*

Proof Since b closes with the $\times_1$-rule, b contains nodes of the form $\langle A \quad \oplus\rangle$ and $\langle A \quad \ominus\rangle$ for some sentence A. If v were faithful to b, then this would mean that $v \vDash_1 A$ and $v \nvDash_1 A$, which is a contradiction. Thus v must not be faithful to b. □

Lemma 3. (Soundness – Closure) *If b is a branch that closes with the $\times_2$-rule, then no* K3*-model (and hence no* CL*-model) v is faithful to b.*

Proof Since b closes with the $\times_2$-rule, b contains nodes of the form $\langle A \quad \oplus\rangle$ and $\langle \neg A \quad \oplus\rangle$ for some sentence A. If v were a K3–model faithful to b, then this would mean that $v \vDash_1 A$ and $v \vDash_1 \neg A$.

But we proved in Exercise 3. of Chapter 15 that K3 models are consistent, so this is a contradiction. Thus v must not be faithful to b. □

Lemma 4. (Soundness – Closure *If b is a branch that closes with the $\times_3$-rule, then no* LP*-model (and hence no* CL*-model) v is faithful to b.*

Proof Essentially the same as Lemma 3; left as an exercise. □

We are now in position to prove the soundness of our tableau systems:

Theorem 1. (Soundness of Propositional FDE Tableau System)
If $X \overset{\mathsf{FDE}}{\Longrightarrow} \phi$, then $X \vDash^{\mathsf{FDE}} \phi$.

Proof Suppose $X \nvDash^{\mathsf{FDE}} \phi$. We will prove that $X \overset{\mathsf{FDE}}{\not\Longrightarrow} \phi$.

Since $X \nvDash^{\mathsf{FDE}} \phi$, there is an FDE–model v such that v satisfies X but $v \nVdash_1 \phi$. Such a model v is faithful to the initial tableau T for $X \therefore \phi$. Then by Lemma 1, no matter how we extend T, v will remain faithful to some branch b of T. Suppose now that $X \overset{\mathsf{FDE}}{\Longrightarrow} \phi$. Given this, each branch of T must end with $\times_1$. Thus, b ends with $\times_1$. But this contradicts Lemma 2, so we must have that $X \overset{\mathsf{FDE}}{\not\Longrightarrow} \phi$. □

Theorem 2. (Soundness of Propositional K3 Tableau System)
If $X \overset{\mathsf{K3}}{\Longrightarrow} \phi$, then $X \vDash^{\mathsf{K3}} \phi$.

Proof Suppose $X \nvDash^{\mathsf{K3}} \phi$. We will prove that $X \overset{\mathsf{K3}}{\not\Longrightarrow} \phi$.

Since $X \nvDash^{\mathsf{K3}} \phi$, there is a K3–model v such that v satisfies X but $v \nVdash_1 \phi$. Such a model v is faithful to the initial tableau T for $X \therefore \phi$. Then by Lemma 1, no matter how we extend T, v will remain faithful to some branch b of T. Suppose now that $X \overset{\mathsf{K3}}{\Longrightarrow} \phi$. Given this, each branch of T must end with either $\times_1$ or $\times_2$. So b ends with either $\times_1$ or $\times_2$. But since every K3 model is an FDE–model, Lemma 2 shows that b must not end with $\times_1$. Thus, b ends with $\times_2$. But this contradicts Lemma 3, so we must have that $X \overset{\mathsf{K3}}{\not\Longrightarrow} \phi$. □

Theorem 3. (Soundness of the Propositional LP Tableau System)
If $X \overset{\mathsf{LP}}{\Longrightarrow} \phi$, *then* $X \models^{\mathsf{LP}} \phi$.

Proof Essentially the same as for Theorem 2; left as an exercise. □

Theorem 4. (Soundness of the Propositional CL Tableau System)
If $X \overset{\mathsf{CL}}{\Longrightarrow} \phi$, *then* $X \models^{\mathsf{CL}} \phi$.

Proof Only slightly different from Theorem 2; left as an exercise. □

3.3.2 COMPLETENESS

Next we prove completeness for our propositional tableau systems. First, two definitions:

Definition *If T is a tableau and b is a branch on T, then b is called saturated when it satisfies the following conditions:*

- *If $\langle \phi \quad \circ \rangle$ (where '$\circ$' is either '$\oplus$' or '$\ominus$') is on b and*

$$\begin{array}{c} \langle \phi \quad \circ \rangle \\ | \\ \langle A_1 \quad \circ \rangle \\ \langle A_2 \quad \circ \rangle \\ \vdots \\ \langle A_n \quad \circ \rangle \end{array}$$

Is an inference licensed by some rule in our tableau system, then each of $\langle A_1 \quad \circ \rangle, \langle A_2 \quad \circ \rangle, \ldots,$ and $\langle A_n \quad \circ \rangle$ is on b.

- *If $\langle \phi \quad \circ \rangle$ (where '$\circ$' is either '$\oplus$' or '$\ominus$') is on b and*

$$\begin{array}{ccc} & \langle \phi \quad \circ \rangle & \\ \swarrow & & \searrow \\ \langle A_1 \quad \circ \rangle & & \langle B_1 \quad \circ \rangle \\ \langle A_2 \quad \circ \rangle & & \langle B_2 \quad \circ \rangle \\ \vdots & & \vdots \\ \langle A_n \quad \circ \rangle & & \langle B_n \quad \circ \rangle \end{array}$$

is a step licensed by some rule in our tableau system, then either each of $\langle A_1 \quad \circ\rangle$, $\langle A_2 \quad \circ\rangle, \ldots,$ *and* $\langle A_n \quad \circ\rangle$ *is on b or each of* $\langle B_1 \quad \circ\rangle$, $\langle B_2 \quad \circ\rangle, \ldots,$ *and* $\langle B_n \quad \circ\rangle$ *is on b .*

In other words, b is called saturated when every rule that can be applied to a node on b has been applied.[2]

Definition *If T is a propositional tableau and b is a branch of T then the cannonical relation induced by b,* i_b*, is the relation between atomic sentences and semantic values defined as follows:*

- $\langle \phi, 1\rangle \in i_b$ *iff* $\langle \phi \quad \oplus\rangle$ *but not* $\langle \neg\phi \quad \oplus\rangle$ *is on b.*
- $\langle \phi, \mathsf{b}\rangle \in i_b$ *iff* $\langle \phi \quad \oplus\rangle$ *and* $\langle \neg\phi \quad \oplus\rangle$ *occur on b.*
- $\langle \phi, \mathsf{n}\rangle \in i_b$ *iff* $\langle \phi \quad \ominus\rangle$ *and* $\langle \neg\phi \quad \ominus\rangle$ *is on b.*
- $\langle \phi, 0\rangle \in i_b$ *iff none of the above conditions hold.*

The relation i_b may seem an odd thing to study. The next lemma explains why it matters to us:

Lemma 5. (Induced Models) *For saturated branches b,* i_b *is a function (and hence an* FDE*-model) iff b does not close with* $\times_1$.

Proof We first establish a mini-lemma:

Mini-Lemma 1. (Pairing) *If b contains nodes of the form* $\langle A \quad \ominus\rangle$ *and* $\langle A \quad \oplus\rangle$ *for some sentence A then for some atomic* ϕ*, b either contains nodes of the form* $\langle \phi \quad \oplus\rangle$ *and* $\langle \phi \quad \ominus\rangle$ *or of the form* $\langle \neg\phi \quad \oplus\rangle$ *and* $\langle \neg\phi \quad \ominus\rangle$.

Proof The proof will be by induction. First, notice the result is trivial if b contains nodes of the form $\langle \phi \quad \oplus\rangle$ and $\langle \phi \quad \ominus\rangle$ or of the form $\langle \neg\phi \quad \oplus\rangle$ and $\langle \neg\phi \quad \ominus\rangle$ for some atomic sentence ϕ. Now suppose the following:

- Whenever b has nodes of the form $\langle A \quad \ominus\rangle$ and $\langle A \quad \oplus\rangle$ and A has n or fewer connectives, b also contains nodes either of the form $\langle \phi \quad \oplus\rangle$ and $\langle \phi \quad \ominus\rangle$ or of the form $\langle \neg\phi \quad \oplus\rangle$ and $\langle \neg\phi \quad \ominus\rangle$ for some atomic sentence ϕ; and
- Whenever b contains nodes of the form $\langle \neg A \quad \ominus\rangle$ and $\langle \neg A \quad \oplus\rangle$ and A has n or fewer connectives, b also contains nodes either

of the form $\langle \phi \quad \oplus \rangle$ and $\langle \phi \quad \ominus \rangle$ or of the form $\langle \neg\phi \quad \oplus \rangle$ and $\langle \neg\phi \quad \ominus \rangle$ for some atomic sentence ϕ.

Let ψ have $n+1$ connectives, and suppose b contains nodes of the form $\langle \psi \quad \ominus \rangle$ and $\langle \psi \quad \oplus \rangle$. There are seven different ways ψ could be formed; we examine three illustrative examples here:

1. Let ψ be a conjunction, so that $\psi = S \wedge T$ for some S and T. Since b is saturated, the $\wedge$-$\oplus$-rule guarantees that $\langle S \quad \oplus \rangle$ and $\langle T \quad \oplus \rangle$ occur on b. Similarly, since b is saturated, the $\wedge$-$\ominus$-rule guarantees that either $\langle S \quad \ominus \rangle$ or $\langle T \quad \ominus \rangle$ is on b. Thus, either b contains nodes of the form $\langle S \quad \oplus \rangle$ and $\langle S \quad \ominus \rangle$ or of the form $\langle T \quad \oplus \rangle$ and $\langle T \quad \ominus \rangle$. But since $\psi = S \wedge T$ has $n+1$ connectives, S and T each have n or fewer connectives, so in either case we can conclude that for some atomic sentence ϕ, b contains nodes either of the form $\langle \phi \quad \oplus \rangle$ and $\langle \phi \quad \ominus \rangle$ or of the form $\langle \neg\phi \quad \oplus \rangle$ and $\langle \neg\phi \quad \ominus \rangle$.
2. Let ψ be a disjunction, so that $\psi = S \vee T$ for some S and T. Since b is saturated, the $\vee$-$\oplus$-rule guarantees that either $\langle S \quad \oplus \rangle$ or $\langle T \quad \oplus \rangle$ is on b. Similarly, since b is saturated, the $\vee$-$\ominus$-rule guarantees that $\langle S \quad \ominus \rangle$ and $\langle T \quad \ominus \rangle$ occur on b. Thus, either b contains nodes of the form $\langle S \quad \oplus \rangle$ and $\langle S \quad \ominus \rangle$ or of the form $\langle T \quad \oplus \rangle$ and $\langle T \quad \ominus \rangle$. But since $\psi = S \vee T$ has $n+1$ connectives, S and T each have n or fewer connectives, so in either case we can conclude that for some atomic sentence ϕ, b contains nodes either of the form $\langle \phi \quad \oplus \rangle$ and $\langle \phi \quad \ominus \rangle$ or of the form $\langle \neg\phi \quad \oplus \rangle$ and $\langle \neg\phi \quad \ominus \rangle$.
3. Let ψ be a negated implication, so that $\psi = \neg(S \rightarrow T)$ for some S and T. Since b is saturated, the $\neg$-$\rightarrow$-$\oplus$-rule guarantees that $\langle S \quad \oplus \rangle$ and $\langle \neg T \quad \oplus \rangle$ occur on b. Similarly, since b is saturated, the $\neg$-$\rightarrow$-$\ominus$-rule guarantees that either $\langle S \quad \ominus \rangle$ or $\langle \neg T \quad \ominus \rangle$ is on b. Thus, either b contains nodes of the form $\langle S \quad \oplus \rangle$ and $\langle \neg S \quad \ominus \rangle$ or of the form $\langle \neg T \quad \oplus \rangle$ and $\langle \neg T \quad \ominus \rangle$. But since $\psi = \neg(S \rightarrow T)$ has $n+1$ connectives, $\neg S$ and $\neg T$ each have n or fewer connectives, so in either case we can conclude that for some atomic sentence ϕ, b contains nodes either of the form $\langle \phi \quad \oplus \rangle$ and $\langle \phi \quad \ominus \rangle$ or of the form $\langle \neg\phi \quad \oplus \rangle$ and $\langle \neg\phi \quad \ominus \rangle$.

The remaining four cases follow similarly and are left as exercises. By induction, the mini-lemma follows. $\square$

Now we return to proving the lemma. Note that $\langle \phi \quad \oplus \rangle$ either does or isn't on b and, if it does, then $\langle \neg\phi \quad \oplus \rangle$ also either does or isn't on b. So if $\langle \phi \quad \oplus \rangle$ is on b, then exactly one of $\langle \phi, 1 \rangle$ and $\langle \phi, \mathsf{b} \rangle$ is an element of i_b. On the other hand if $\langle \phi \quad \ominus \rangle$ is on b, a similar argument shows that exactly one of $\langle \phi, 0 \rangle$ and $\langle \phi, \mathsf{n} \rangle$ is an element of i_b.

If b closes with $\times_1$, then by the mini-lemma, there is an atomic ϕ so that $\langle \phi \quad \oplus \rangle$ and $\langle \phi \quad \ominus \rangle$ is on b. Thus, i_b contains one of $\langle \phi, 1 \rangle$ and $\langle \phi, \mathsf{b} \rangle$ and one of $\langle \phi, 0 \rangle$ and $\langle \phi, \mathsf{n} \rangle$, so fails to be a function.

If b does not close with $\times_1$, then for no sentence ϕ and thus, in particular, no atomic sentence ϕ do $\langle \phi \quad \oplus \rangle$ and $\langle \phi \quad \ominus \rangle$ both occur on b. So for each atomic ϕ, exactly one of $\langle \phi, 1 \rangle$, $\langle \phi, \mathsf{b} \rangle$, $\langle \phi, 0 \rangle$ and $\langle \phi, \mathsf{n} \rangle$ is in i_b. Thus i_b is a function. $\square$

Despite the fact that i_b is not generally a function, it is useful to think of it as being given by the following definition:

$$i_b(\phi) = \begin{cases} 1 & \text{if } \langle \phi \quad \oplus \rangle \text{ but not } \langle \neg\phi \quad \oplus \rangle \text{ occurs on } b \\ \mathsf{b} & \text{if } \langle \phi \quad \oplus \rangle \text{ and } \langle \neg\phi \quad \oplus \rangle \text{ occur on } b \\ \mathsf{n} & \text{if } \langle \phi \quad \ominus \rangle \text{ and } \langle \neg\phi \quad \ominus \rangle \text{ occur on } b \\ 0 & \text{otherwise} \end{cases}$$

Lemma 6. (Completeness) *If b is a saturated open branch on a tableau T then i_b is faithful to b.*

Proof We prove this by induction.

First, notice that if ϕ is an atomic sentence or the negation of an atomic sentence, then by the definition of i_b, if $\langle \phi \quad \oplus \rangle$ is on b, then $i_b \vDash_1 \phi$ and if $\langle \phi \quad \ominus \rangle$ is on b, then $i_b \nvDash_1 \phi$.

Now suppose the following all hold:

- if ϕ has n or fewer connectives and $\langle \phi \quad \oplus \rangle$ is on a saturated branch b, then $i_b \vDash_1 \phi$;
- if ϕ has n or fewer connectives and $\langle \phi \quad \ominus \rangle$ is on a saturated branch b, then $i_b \nvDash_1 \phi$;

- if ϕ has n or fewer connectives and $\langle\neg\phi \quad \oplus\rangle$ is on a saturated branch b, then $i_b \vDash_1 \neg\phi$; and
- if ϕ has n or fewer connectives and $\langle\neg\phi \quad \ominus\rangle$ is on a saturated branch b, then $i_b \nvDash_1 \neg\phi$.

Suppose now that ψ has $n+1$ connectives and occurs in some node on b. There are 14 total cases to consider. Here are six illustrative examples:

POSITIVE CASES:
Suppose $\langle\psi \quad \oplus\rangle$ is on b.

1. Let ψ be an implication, so that $\psi = S \to T$ for some S and T. Then, since b is saturated, the $\to$-$\oplus$-rule guarantees that either $\langle\neg S \quad \oplus\rangle$ or $\langle T \quad \oplus\rangle$ is on b. But since $\psi = S \to T$ has $n+1$ connectives, S and T each have n or fewer connectives. Thus either $i_b \vDash_1 \neg S$ or $i_b \vDash_1 T$. In either case, truth conditions for implication give $i_b \vDash_1 S \to T$.
2. Let ψ be a negated disjunction, so that $\psi = \neg(S \vee T)$ for some S and T. Then, since b is saturated, $\langle\neg S \quad \oplus\rangle$ and $\langle\neg T \quad \oplus\rangle$ occur on b. But since $\psi = \neg(S \vee T)$ has $n+1$ connectives, $\neg S$ and $\neg T$ each have n or fewer connectives. Thus $i_b \vDash_1 \neg S$ and $i_b \vDash_1 \neg T$. But then truth conditions for negation and disjunction give that $i_b \vDash_1 \neg(S \vee T)$.
3. Let ψ be a double negation, so that $\psi = \neg\neg S$ for some S. Then, since b is saturated, the $\neg\neg$-$\oplus$-rule guarantees that $\langle S \quad \oplus\rangle$ is on b. But S has n or fewer connectives, so $i_b \vDash_1 S$. But then truth conditions for negation give that $i_b \vDash_1 \neg\neg S$.

NEGATIVE CASES:
Suppose $\langle\psi \quad \ominus\rangle$ is on b. We examine each way ψ could be constructed:

1. Let ψ be a conjunction, so that $\psi = S \wedge T$ for some S and T. Then since b is saturated, the $\wedge$-$\ominus$-rule guarantees that either $\langle S \quad \ominus\rangle$ or $\langle T \quad \ominus\rangle$ is on b. But since $\psi = S \wedge T$ has $n+1$ connectives, S and T each have n or fewer connectives. Thus either $i_b \nvDash_1 S$ or $i_b \nvDash_1 T$. In either case, truth conditions for conjunction give $i_b \nvDash_1 S \wedge T$.

2. Let ψ be a negated conjunction, so that $\psi = \neg(S \wedge T)$ for some S and T. Then, since b is saturated, the $\neg$-$\wedge$-$\ominus$-rule guarantees that $\langle \neg S \quad \ominus \rangle$ and $\langle \neg T \quad \ominus \rangle$ occur on b. But since $\psi = \neg(S \wedge T)$ has $n+1$ connectives, $\neg S$ and $\neg T$ each have n or fewer connectives. Thus $i_b \not\models_1 \neg S$ and $v \not\models_1 \neg T$. But then truth conditions for negation and conjunction give that $i_b \not\models_1 \neg(S \wedge T)$.
3. Let ψ be a negated implication, so that $\psi = \neg(S \rightarrow T)$ for some S and T. Then, since b is saturated, the $\neg$-$\rightarrow$-$\ominus$-rule guarantees that either $\langle S \quad \ominus \rangle$ or $\langle \neg T \quad \ominus \rangle$ is on b. But since $\psi = \neg(S \rightarrow T)$ has $n+1$ connectives, S and T each have n or fewer connectives. Thus either $i_b \not\models_1 S$ or $i_b \not\models_1 \neg T$. In either case, truth conditions for negation and implication give that $i_b \not\models_1 \neg(S \rightarrow T)$.

The remaining cases are left as exercises. By mathematical induction we conclude that if b is a saturated open branch on a tableau T i_b is faithful to b. □

Lemma 7. (Completeness – Closure) *For saturated branches b, if b does not close with $\times_1$ or $\times_2$, then i_b is a* K3*-model.*

Proof First we need a result analogous to Mini-Lemma 1:

Mini-Lemma 2. (Pairing) *If b contains nodes of the form $\langle A \quad \oplus \rangle$ and $\langle \neg A \quad \oplus \rangle$ for some sentence A then for some atomic ϕ, b contains nodes of the form $\langle \phi \quad \oplus \rangle$ and $\langle \neg \phi \quad \oplus \rangle$.*

Proof The proof is essentially the same as the proof of Mini-Lemma 1, so left as an exercise. □

From here the result follows easily: Since b does not close with $\times_1$, i_b is a model. Since b does not close with $\times_2$, b does not contain any nodes of the form $\langle \phi \quad \oplus \rangle$ and $\langle \neg \phi \quad \oplus \rangle$. So, in particular, b contains no such nodes with ϕ atomic. But then the range of i_b does not include b, so i_b is a K3-model. □

Lemma 8. (Completeness – Closure) *For saturated branches b, if b does not close with $\times_1$ or $\times_3$, then i_b is a* LP*-model.*

Proof This proof is essentially the same as the proof for Lemma 7, so is left as an exercise. □

Lemma 9. (Completeness – Closure) *For saturated branches b, if b does not close with $\times_1$, $\times_2$, or $\times_3$, then i_b is a* CL*-model.*

Proof Since i_b does not close with $\times_1$ or $\times_2$, i_b is a K3-model. Since i_b does not close with $\times_1$ or $\times_3$, i_b is a LP-model. So by definition, i_b is a CL-model. □

From here, completeness is trivial:

Theorem 5. (Completeness of Propositional FDE-Tableau System) *If $X \vDash^{\mathsf{FDE}} \phi$ then $X \stackrel{\mathsf{FDE}}{\Longrightarrow} \phi$.*

Proof Suppose $X \stackrel{\mathsf{FDE}}{\not\Longrightarrow} \phi$. Then if T is a saturated tableau for the argument $X \therefore \phi$, there is a branch b that does not close with $\times_1$. Thus, i_b is a model and by Lemma 6, it must be faithful to b. Thus, in particular i_b must satisfy X but not satisfy ϕ. So $X \nvDash^{\mathsf{FDE}} \phi$. □

Theorem 6. (Completeness of Propositional K3-Tableau System) *If $X \vDash^{\mathsf{K3}} \phi$ then $X \stackrel{\mathsf{K3}}{\Longrightarrow} \phi$.*

Proof Essentially the same as for Theorem 5, left as an exercise. □

Theorem 7. (Completeness of Propositional LP-Tableau System) *If $X \vDash^{\mathsf{LP}} \phi$ then $X \stackrel{\mathsf{LP}}{\Longrightarrow} \phi$.*

Proof Essentially the same as for Theorem 5, left as an exercise. □

Theorem 8. (Completeness of Propositional CL-Tableau System) *If $X \vDash^{\mathsf{CL}} \phi$ then $X \stackrel{\mathsf{CL}}{\Longrightarrow} \phi$.*

Proof Essentially the same as for Theorem 5, left as an exercise. □

3.4 FIRST-ORDER LOGIC WITHOUT IDENTITY

Recall that a first-order FDE-model is a pair $\langle D, \delta \rangle$ with D a set and δ a function such that:

- δ assigns to each name α an element $\delta(\alpha)$ of D;
- δ assigns to each n-ary predicate Π a pair $\langle \delta^+(\Pi), \delta^-(\Pi) \rangle$ with $\delta^+(\Pi) \subseteq D^n$ and $\delta^-(\Pi) \subseteq D^n$;[2]
- For each $d \in D$ there is some name α such that $\delta(\alpha) = d$.

Given this, we can characterize the K3-, LP-, and CL-models as follows:

- A first-order K3-model is a first-order FDE-model such that for every n-ary predicate Π, $\delta^+(\Pi) \cap \delta^-(\Pi) = \emptyset$;
- A first-order LP-model is a first-order FDE-model such that for every n-ary predicate Π, $\delta^+(\Pi) \cup \delta^-(\Pi) = D^n$; and
- A first-order CL-model is a first-order FDE-model that is both a first-order K3-model and a first-order LP-model.

3.4.1 SOUNDNESS

In general pattern, the proofs of soundness and completeness for our first-order theories are essentially the same as our proofs of soundness and completeness for our propositional theories, with the exception of our needing a few extra lemmas here and there. The following two lemmas are examples of this:

Lemma 10. (Denotation) *Let $A(x_1, \ldots, x_n)$ be an open formula with n free variables. Let $v_1 = \langle D, \delta_1 \rangle$ and $v_2 = \langle D, \delta_2 \rangle$ be models and $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be lists of names. Suppose that*

(a) For every predicate Π occurring in A, $\delta_1^+(\Pi) = \delta_2^+(\Pi)$ and $\delta_1^-(\Pi) = \delta_2^-(\Pi)$;
(b) For every name a occurring in A, $\delta_1(a) = \delta_2(a)$; and
(c) $\delta_1(\alpha_1) = \delta_2(\beta_1), \delta_1(\alpha_2) = \delta_2(\beta_2), \ldots, \delta_1(\alpha_n) = \delta_2(\beta_n)$.

Then $v_1 \vDash_1 A(\alpha_1/x_1, \ldots, \alpha_n/x_n)$ iff $v_2 \vDash_1 A(\beta_1/x_1, \ldots, \beta_n/x_n)$.

Proof As usual, we proceed by induction. If A is an atomic formula or the negation of an atomic formula, then the result follows for any value of n by our assumptions and the definition of the relation $\vDash_1$. Now suppose that (brace yourself; very long inductive hypothesis coming) for every value of n, if

- $A(x_1, \dots, x_n)$ is an open formula with n free variables and k or fewer connectives and quantifiers, and
- $v_1 = \langle D, \delta_1 \rangle$ and $v_2 = \langle D, \delta_2 \rangle$ are models and
- $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ are lists of names such that

  (a) For every predicate Π occurring in A, $\delta_1^+(\Pi) = \delta_2^+(\Pi)$ and $\delta_1^-(\Pi) = \delta_2^-(\Pi)$;
  (b) For every name a occurring in A, $\delta_1(a) = \delta_2(a)$; and
  (c) $\delta_1(\alpha_1) = \delta_2(\beta_1), \delta_1(\alpha_2) = \delta_2(\beta_2), \dots, \delta_1(\alpha_n) = \delta_2(\beta_n)$;

Then $v_1 \vDash_1 A(\alpha_1/x_1, \dots, \alpha_n/x_n)$ iff $v_2 \vDash_1 A(\beta_1/x_1, \dots, \beta_n/x_n)$.

Let A have $k+1$ connectives and quantifiers. The cases involving the connectives are straightforward and left as exercises. Here we examine the case where A is a universally quantified formula. The case in which A is existentially quantified is only slightly different, and also left as an exercise.

Let $A(x_1, \dots, x_n) = \forall y B(y, x_1, \dots, x_n)$. Truth conditions for the univeral quantifier give that $v_1 \vDash_1 A(\alpha_1/x_1, \dots, \alpha_n/x_n)$ iff for every name γ, $v_1 \vDash_1 B(\gamma/y, \alpha_1/x_1, \dots, \alpha_n/x_n)$. Notice that since A has $k+1$ connectives and quantifiers, B has k connectives and quantifiers. Also, since every object in D has a name in every model, there is some name ν_γ such that $\delta_2(\nu_\gamma) = \delta_1(\gamma)$. But then notice that the formula B, models v_1 and v_2, and lists of names $\gamma, \alpha_1, \dots, \alpha_n$ and $\nu_\gamma, \beta_1, \dots, \beta_n$ satisfy the inductive hypothesis. So, by induction, $v_1 \vDash_1 B(\gamma/y, \alpha_1/x_1, \dots, \alpha_n/x_n)$ iff $v_2 \vDash_1 B(y/\nu_\gamma, \beta_1/x_1, \dots, \beta_n/x_n)$. But γ was an arbitrary name, so this will hold for any name γ, thus if $v_1 \vDash_1 A(\alpha_1/x_1, \dots, \alpha_n/x_n)$, then $v_2 \vDash_1 A(\beta_1/x_1, \dots, \beta_n/x_n)$. Essentially the same argument gives if $v_2 \vDash_1 A(\beta_1/x_1, \dots, \beta_n/x_n)$, then $v_1 \vDash_1 A(\alpha_1/x_1, \dots, \alpha_n/x_n)$, establishing the required biconditional. □

Lemma 11. (Locality) *Let A be a sentence and suppose $v_1 = \langle D, \delta_1 \rangle$ and $v_2 = \langle D, \delta_2 \rangle$ are models that satisfy the following conditions:*

(a) For every predicate Π *occurring in* A*,* $\delta_1^+(\Pi) = \delta_2^+(\Pi)$ *and* $\delta_1^-(\Pi) = \delta_2^-(\Pi)$.
(b) For every name α *occurring in* A*,* $\delta_1(\alpha) = \delta_2(\alpha)$.

Then $v_1 \models_1 A$ *iff* $v_2 \models_1 A$.

Proof We proceed by induction. Suppose v_1 and v_2 satisfy (a) and (b). If A is atomic, then for some number n, predicate Π, and names $\alpha_1, \ldots, \alpha_n$, $A = \Pi\alpha_1 \ldots \alpha_n$. But since v_1 and v_2 assign the same extension and anti-extension to Π and give the same denotation to each of the α_i, $v_1 \models A$ iff $v_2 \models_1 A$. If A is the negation of an atomic, essentially the same argument gives the result.

Now suppose that for all models v_1 and v_2 satisfying (a) and (b) and all sentences ϕ with n or fewer connectives and quantifiers in total, $v_1 \models_1 \phi$ iff $v_2 \models_1 \phi$ and $v_1 \models_1 \neg\phi$ iff $v_2 \models_1 \neg\phi$. Let ψ be a sentence with $n+1$ connectives and quantifiers in total. The cases involving the connectives are straightforward and left as exercises. Here we examine the cases where A is a quantified formula.

Let $\psi = \forall x\phi(x)$, for some open sentence ϕ. Then $v_1 \models_1 \psi$ iff for all names α, $v_1 \models_1 \phi(\alpha/x)$. Choose an arbitrary name α, and let $\delta_1(\alpha) = d \in D$. Since every object has a name in every model, there is a name β such that $\delta_2(\beta) = d$. Then the models v_1 and v_2, the formula $\phi(x)$, and the names α and β together satisfy the hypotheses of the denotation lemma, so $v_1 \models_1 \phi(\alpha/x)$ iff $v_2 \models_1 \phi(\beta/x)$. Since α was arbitrary, this holds for every name, so $v_1 \models_1 \psi$ iff $v_2 \models_1 \psi$.

Let $\psi = \exists x\phi(x)$, for some open sentence ϕ. Then $v_1 \models_1 \psi$ iff for some name α, $v_1 \models_1 \phi(\alpha/x)$. Choose such an α, and let $\delta_1(\alpha) = d \in D$. Since every object has a name in every model, there is a name β such that $\delta_2(\beta) = d$. Then the models v_1 and v_2, the formula $\phi(x)$, and the names α and β together satisfy the hypotheses of the denotation lemma, so $v_1 \models_1 \phi(\alpha/x)$ iff $v_2 \models_1 \phi(\beta/x)$. Thus, $v_1 \models_1 \psi$ iff $v_2 \models_1 \psi$. $\square$

From here we return to the path paved by our proof of completeness for the propositional tableau systems.

Lemma 12. (Faithfulness) *If T is a predicate tableau, b is a branch in T, and $f(b)$ is non-empty, then when we extend b using one of our tableau rules, there will be a $v \in f(b)$ that is faithful to at least one of the resulting branches.*

Here is a more simple statement of the lemma: if there is at least one model that is faithful to b, then when we extend the tableau by one of the rules, there is at least one model that is faithful to at least one of the resulting branches.

Proof Suppose $v = \langle D, \delta \rangle$ is a model that is faithful to the branch b. We examine each rule of our tableau system individually. For the propositional rules, the arguments are as in the propositional case. Thus, we need only examine the new rules. The arguments are all roughly similar; here I will show half of them, leaving the other half as exercises.

POSITIVE RULES:

Suppose b contains a node of the form $\langle \forall x A(x) \quad \oplus \rangle$. Recall the $\forall$-$\oplus$-rule:

$$\begin{array}{c} \langle \forall x A(x) \quad \oplus \rangle \\ | \\ \langle A(t) \quad \oplus \rangle \end{array}$$

By the truth conditions for the universal quantifier, $v \models_1 \forall x A(x)$ iff $v \models_1 A(\alpha/x)$ for every name α. Since v is faithful to b, $v \models_1 \forall x A(x)$. So for every name α, $v \models_1 A(\alpha/x)$. So, no matter what name t we use when we extend b using the $\forall$-$\oplus$-rule, v itself will be faithful to the resulting branch.

Suppose b contains a node of the form $\langle \neg\forall x A(x) \quad \oplus \rangle$. Recall the $\neg$-$\forall$-$\oplus$-rule:

$$\begin{array}{c} \langle \neg\forall x A(x) \quad \oplus \rangle \\ | \\ \langle \exists x \neg A(x) \quad \oplus \rangle \end{array}$$

By the truth conditions for negation, $v \vDash_1 \neg\forall x A(x)$ iff $v \vDash_0 \forall x A(x)$. By the truth conditions for the universal quantifier, $v \vDash_0 \forall x A(x)$ iff for some name α, $v \vDash_0 A(\alpha/x)$. By the truth conditions for negation again, for some name α, $v \vDash_0 A(\alpha/x)$ iff for some name α, $v \vDash_1 \neg A(\alpha/x)$. Finally, by the truth conditions for the existential quantifier, $v \vDash_1 \neg A(\alpha/x)$ for some name α iff $v \vDash_1 \exists x \neg A(x)$. Since v is faithful to b, $v \vDash \neg\forall x A(x)$. Thus $v \vDash \exists x \neg A(x)$. So when we extend b using the $\neg$-$\forall$-$\oplus$-rule, v itself will be faithful to the resulting branch.

NEGATIVE RULES:

Suppose b contains a node of the form $\langle \forall x A(x) \quad \ominus \rangle$. Recall the $\forall$-$\ominus$-rule:

$$\begin{array}{c} \langle \forall x A(x) \quad \ominus \rangle \\ | \\ \langle A(a) \quad \ominus \rangle \end{array}$$

By the truth conditions for the universal quantifier, $v \vDash_1 \forall x A(x)$ iff $v \vDash_1 A(\alpha/x)$ for every name α. Since v is faithful to b, $v \nvDash_1 \forall x A(x)$. So for some name α, $v \nvDash_1 A(\alpha/x)$. Let a be a name that has not occurred on b. Let $v' = \langle D, \delta' \rangle$ be a model that agrees with v on every name and predicate occurring on b, and for which $\delta'(a) = \delta(\alpha)$. By locality, v' is faithful to b, and by construction, $v \nvDash_1 A(a)$. Thus, $v' \in f(b)$, and when we extend b using the $\forall$-$\ominus$-rule, v' is faithful to the resulting branch.

Suppose b contains a node of the form $\langle \neg\forall x A(x) \quad \ominus \rangle$. Recall the $\neg$-$\forall$-$\ominus$-rule:

$$\begin{array}{c} \langle \neg\forall x A(x) \quad \ominus \rangle \\ | \\ \langle \exists x \neg A(x) \quad \ominus \rangle \end{array}$$

By the truth conditions for negation, $v \vDash_1 \neg\forall x A(x)$ iff $v \vDash_0 \forall x A(x)$. By the truth conditions for the universal quantifier, $v \vDash_0 \forall x A(x)$

iff for some name α, $v \vDash_0 A(\alpha/x)$. By the truth conditions for negation again, for some name α, $v \vDash_0 A(\alpha/x)$ iff for some name α, $v \vDash_1 \neg A(\alpha/x)$. Finally, by the truth conditions for the existential quantifier, $v \vDash_1 \neg A(\alpha/x)$ for some name α iff $v \vDash_1 \exists x \neg A(x)$. Since v is faithful to b, $v \nvDash \neg\forall x A(x)$. Thus $v \nvDash \exists x \neg A(x)$. So when we extend b using the $\neg$-$\forall$-$\ominus$-rule, v itself will be faithful to the resulting branch. □

Lemma 13. (Soundness – Closure) *If b is a branch that closes with the $\times_1$-rule, then no model v is faithful to b.*

Proof Since b closes with the $\times_1$ rule, b contains nodes of the form $\langle A \quad \oplus\rangle$ and $\langle A \quad \ominus\rangle$ for some sentence A. If v were faithful to b, then this would mean that $v \vDash_1 A$ and $v \nvDash_1 A$, which is a contradiction. Thus v must not be faithful to b. □

Lemma 14. (Soundness – Closure) *If b is a branch that closes with the $\times_2$-rule, then no* K3*-model (and hence no* CL*-model) v is faithful to b.*

Proof Since b closes with the $\times_2$ rule, b contains nodes of the form $\langle A \quad \oplus\rangle$ and $\langle \neg A \quad \oplus\rangle$ for some sentence A. If v were a K3–model faithful to b, then this would mean that $v \vDash_1 A$ and $v \vDash_1 \neg A$. But in exercise 4 of Chapter 1.15 of this supplementary material we proved that K3 models are consistent, so this is a contradiction. Thus v must not be faithful to b. □

Lemma 15. (Soundness – Closure) *If b is a branch that closes with the $\times_3$-rule, then no* LP*-model (and hence no* CL*-model) v is faithful to b.*

Proof Essentially the same as Lemma 14; left as an exercise. □

We are now in position to prove the soundness of our tableau systems:

Theorem 9 (Soundness: Predicate FDE Tableau Sans Identity)**.**
If $X \overset{\text{FDE}}{\Longrightarrow} \phi$, then $X \vDash^{\text{FDE}} \phi$.

Proof Suppose $X \nvDash^{\text{FDE}} \phi$. We will prove that $X \overset{\text{FDE}}{\not\Longrightarrow} \phi$.

Since $X \not\models^{\mathsf{FDE}} \phi$, there is an FDE-model v such that v satisfies X but $v \not\models_1 \phi$. Such a model v is faithful to the initial tableau T for $X \therefore \phi$. Then by Lemma 1, no matter how we extend T, v will remain faithful to some branch b of T. Suppose now that $X \xRightarrow{\mathsf{FDE}} \phi$. Given this, each branch of T must end with $\times_1$. Thus, b ends with $\times_1$. But this contradicts Lemma 13, so we must have that $X \overset{\mathsf{FDE}}{\not\Longrightarrow} \phi$. □

Theorem 10. (Soundness: Predicate K3 Tableau Sans Identity)
If $X \xRightarrow{\mathsf{K3}} \phi$, then $X \models^{\mathsf{K3}} \phi$.

Proof Suppose $X \not\models^{\mathsf{K3}} \phi$. We will prove that $X \overset{\mathsf{K3}}{\not\Longrightarrow} \phi$.

Since $X \not\models^{\mathsf{K3}} \phi$, there is a K3-model v such that v satisfies X but $v \not\models_1 \phi$. Such a model v is faithful to the initial tableau T for $X \therefore \phi$. Then by Lemma 1, no matter how we extend T, v will remain faithful to some branch b of T. Suppose now that $X \xRightarrow{\mathsf{K3}} \phi$. Given this, each branch of T must end with either $\times_1$ or $\times_2$. So b ends with either $\times_1$ or $\times_2$. But by Lemma 13, b must not end with $\times_1$, and by Lemma 14, b must not end with $\times_2$. So we must have that $X \overset{\mathsf{K3}}{\not\Longrightarrow} \phi$. □

Theorem 11. (Soundness: Predicate LP Tableau Sans Identity)
If $X \xRightarrow{\mathsf{LP}} \phi$, then $X \models^{\mathsf{LP}} \phi$.

Proof Essentially the same as for Theorem 10; left as an exercise. □

Theorem 12. (Soundness: Predicate CL Tableau Sans Identity)
If $X \xRightarrow{\mathsf{CL}} \phi$, then $X \models^{\mathsf{CL}} \phi$.

Proof Only slightly different from Theorem 10; left as an exercise. □

3.4.2 COMPLETENESS

Next we prove completeness. First, two definitions:

Definition *If T is a first-order tableau, b is a saturated branch on T, and a is the first name that is on b then the canonical positive model i_b^+ induced by b is the pair $\langle D, \delta \rangle$ such that* *(i)* *D is the set of all names that occur on b, and δ is defined by*

$$\delta(\alpha) = \begin{cases} \alpha & \text{if } \alpha \text{ occurs on } b \\ a & \text{otherwise} \end{cases}$$

(ii) $\delta^+(\Pi) = \{\langle \alpha_1, \ldots, \alpha_n \rangle : \langle \Pi\alpha_1 \ldots \alpha_n \quad \oplus \rangle$ *occurs on* $b\}$
(iii) $\delta^-(\Pi) = \{\langle \alpha_1, \ldots, \alpha_n \rangle : \langle \neg\Pi\alpha_1 \ldots \alpha_n \quad \oplus \rangle$ *occurs on* $b\}$

Definition *If T is a first-order tableau, b is a saturated branch on T, and a is the first name that is on b then the canonical negative model i_b^- induced by b is the pair $\langle D, \delta \rangle$ such that*

(i) *D is the set of all names that occur on b, and δ is defined by*

$$\delta(\alpha) = \begin{cases} \alpha & \text{if } \alpha \text{ occurs on } b \\ a & \text{otherwise} \end{cases}$$

(ii) $\delta^+(\Pi) = \{\langle \alpha_1, \ldots, \alpha_n \rangle : \langle \Pi\alpha_1 \ldots \alpha_n \quad \ominus \rangle$ *does not occur on* $b\}$
(iii) $\delta^-(\Pi) = \{\langle \alpha_1, \ldots, \alpha_n \rangle : \langle \neg\Pi\alpha_1 \ldots \alpha_n \quad \ominus \rangle$ *does not occur on* $b\}$

It is clear both i_b^+ and i_b^- are models. Also note that in i_b^+ and i_b^-, every name α occurring on b is itself an element of the domain. In fact, in a model induced by b (whether positive or negative), every name on b is a name for itself. It is important to notice that this makes induced models rather unusual among the models.

We now turn to establishing the important properties of models induced by b.

Lemma 16. (Completeness) *For saturated branches b, i_b^+ and i_b^- are faithful to b iff b does not close with $\times_1$.*

We prove the case for i_b^+. The case involving i_b^- is similar and left as an exercise.

Proof Just as before, we first establish a mini-lemma:

Mini-Lemma 3. (Pairing) *If b contains nodes of the form $\langle A \quad \ominus \rangle$ and $\langle A \quad \oplus \rangle$ for some sentence A then for some n-ary predicate Π, b either contains nodes of the form $\langle \Pi\alpha_1, \ldots, \alpha_n \quad \oplus \rangle$ and $\langle \Pi\alpha_1, \ldots, \alpha_n \quad \ominus \rangle$ or $\langle \neg\Pi\alpha_1, \ldots, \alpha_n \quad \oplus \rangle$ and $\langle \neg\Pi\alpha_1, \ldots, \alpha_n \quad \ominus \rangle$.*

Proof First, notice the result is trivial if b contains nodes of the form $\langle \Pi\alpha_1,\ldots,\alpha_n \quad \oplus\rangle$ and $\langle \Pi\alpha_1,\ldots,\alpha_n \quad \ominus\rangle$ or of the form $\langle \neg\Pi\alpha_1,\ldots,\alpha_n \quad \oplus\rangle$ and $\langle \neg\Pi\alpha_1,\ldots,\alpha_n \quad \ominus\rangle$. Now suppose the following:

- If b contains a pair of nodes of the form $\langle A \quad \ominus\rangle$, $\langle A \quad \oplus\rangle$ and A has n or fewer connectives, then b contains nodes either of the form $\langle \Pi\alpha_1,\ldots,\alpha_n \quad \oplus\rangle$, $\langle \Pi\alpha_1,\ldots,\alpha_n \quad \ominus\rangle$ or $\langle \neg\Pi\alpha_1,\ldots,\alpha_n \quad \oplus\rangle$, $\langle \neg\Pi\alpha_1,\ldots,\alpha_n \quad \ominus\rangle$; and
- If b contains nodes of the form $\langle \neg A \quad \ominus\rangle$, $\langle \neg A \quad \oplus\rangle$ and A has n or fewer connectives, then b contains nodes either of the form $\langle \Pi\alpha_1,\ldots,\alpha_n \quad \oplus\rangle$, $\langle \Pi\alpha_1,\ldots,\alpha_n \quad \ominus\rangle$ or $\langle \neg\Pi\alpha_1,\ldots,\alpha_n \quad \oplus\rangle$, $\langle \neg\Pi\alpha_1,\ldots,\alpha_n \quad \ominus\rangle$.

Let ψ have $n+1$ connectives and quantifiers, and suppose b contains nodes of the form $\langle \psi \quad \ominus\rangle$ and $\langle \psi \quad \oplus\rangle$. We examine each way ψ could be formed. For the propositional cases, the arguments are exactly as in Section 3, so we need only examine the quantificational cases. Of these, the cases involving negated quantifiers follow easily from the cases for non-negated quantifiers, so we examine only the latter.

1. Let ψ be a universal sentence, so that $\psi = \forall x A(x)$ for some open formula $A(x)$. Since b is saturated, the $\forall$-$\ominus$-rule guarantees for some name a, $\langle A(a) \quad \ominus\rangle$ is on b. But then by the $\forall$-$\oplus$-rule, $\langle A(a) \quad \oplus\rangle$ also is on b. But each of these is a sentence with n or fewer connectives and quantifiers, so by the inductive hypothesis, for some n-ary predicate Π, b either contains nodes of the form $\langle \Pi\alpha_1,\ldots,\alpha_n \quad \oplus\rangle$ and $\langle \Pi\alpha_1,\ldots,\alpha_n \quad \ominus\rangle$ or of the form $\langle \neg\Pi\alpha_1,\ldots,\alpha_n \quad \oplus\rangle$ and $\langle \neg\Pi\alpha_1,\ldots,\alpha_n \quad \ominus\rangle$.
2. Let ψ be a universal sentence, so that $\psi = \exists x A(x)$ for some open formula $A(x)$. Since b is saturated, the $\exists$-$\oplus$-rule guarantees for some name a, $\langle A(a) \quad \oplus\rangle$ is on b. But then by the $\exists$-$\ominus$-rule, $\langle A(a) \quad \ominus\rangle$ also is on b. But each of these is a sentence with n or fewer connectives and quantifiers, so by the inductive hypothesis, for some n-ary predicate Π, b either contains nodes of the form $\langle \Pi\alpha_1,\ldots,\alpha_n \quad \oplus\rangle$ and $\langle \Pi\alpha_1,\ldots,\alpha_n \quad \ominus\rangle$ or of the form $\langle \neg\Pi\alpha_1,\ldots,\alpha_n \quad \oplus\rangle$ and $\langle \neg\Pi\alpha_1,\ldots,\alpha_n \quad \ominus\rangle$. $\square$

Now suppose b closes with $\times_1$. Then there must be nodes of the form $\langle A \quad \oplus\rangle$ and $\langle A \quad \ominus\rangle$ on b for some sentence A. It follows from Mini-Lemma 3 that for some n-ary predicate Π, b either contains nodes of the form $\langle \Pi\alpha_1, \ldots, \alpha_n \quad \oplus\rangle$ and $\langle \Pi\alpha_1, \ldots, \alpha_n \quad \ominus\rangle$ or of the form $\langle \neg\Pi\alpha_1, \ldots, \alpha_n \quad \oplus\rangle$ and $\langle \neg\Pi\alpha_1, \ldots, \alpha_n \quad \ominus\rangle$.

In the first case, since b contains $\langle \Pi\alpha_1 \ldots \alpha_n \quad \oplus\rangle$, the n-tuple $\langle \alpha_1, \ldots, \alpha_n\rangle$ is in $\delta^+(\Pi)$. So $i_b^+ \vDash_1 \Pi\alpha_1 \ldots \alpha_n$. But since b contains a node of the form$\langle \Pi\alpha_1, \ldots, \alpha_n \quad \ominus\rangle$, i_b^+ is not faithful to b. A similar argument establishes the result in the second case.

On the other hand, suppose b does not close with $\times_1$. If $\langle \Pi\alpha_1 \ldots \alpha_n \quad \oplus\rangle$ is on b, then from condition (ii) in the definition of i_b^+, we see that $i_b^+ \vDash_1 \Pi\alpha_1 \ldots \alpha_n$. Similarly, if b contains the node $\langle \neg\Pi\alpha_1 \ldots \alpha_n \quad \oplus\rangle$, then from condition (iii) in the definition of i_b^+, we see that $i_b^+ \vDash_1 \neg\Pi\alpha_1 \ldots \alpha_n$.

Also, since b does not close with $\times_1$, if $\langle \Pi\alpha_1 \ldots \alpha_n \quad \ominus\rangle$ is on b, then also$\langle \Pi\alpha_1 \ldots \alpha_n \quad \oplus\rangle$ isn't on b, so $i_b^+ \nvDash_1 \Pi\alpha_1 \ldots \alpha_n$. Similarly, if $\langle \neg\Pi\alpha_1 \ldots \alpha_n \quad \ominus\rangle$ is on b, then $\langle \neg\Pi\alpha_1 \ldots \alpha_n \quad \oplus\rangle$ isn't on b, so $i_b^+ \nvDash_1 \neg\Pi\alpha_1 \ldots \alpha_n$. Thus, if ϕ is and atomic sentence or the negation of an atomic sentence, then if $\langle \phi \quad \oplus\rangle$ is on b, $i_b^+ \vDash_1 \phi$ and if $\langle \phi \quad \ominus\rangle$ is on b, $i_b^+ \nvDash_1 \phi$.

Now suppose that

- if ϕ has n or fewer connectives and quantifiers and $\langle \phi \quad \oplus\rangle$ is on a saturated branch b, then $i_b^+ \vDash_1 \phi$,
- if ϕ has n or fewer connectives and quantifiers and $\langle \phi \quad \ominus\rangle$ is on a saturated branch b, then $i_b^+ \nvDash_1 \phi$.
- if ϕ has n or fewer connectives and quantifiers and $\langle \neg\phi \quad \oplus\rangle$ is on a saturated branch b, then $i_b^+ \vDash_1 \neg\phi$,
- if ϕ has n or fewer connectives and quantifiers and $\langle \neg\phi \quad \ominus\rangle$ is on a saturated branch b, then $i_b^+ \nvDash_1 \neg\phi$.

Next suppose ψ has $n+1$ connectives and quantifiers in total and occurs in some node on b. If the main connective in ψ is a propositional connective, we can argue in exactly the same way as in the propositional case. We thus only consider the cases that are new to the first-order setting. This leaves us with eight total cases to consider. We present the following four; the remaining cases being similar and left as exercises:

POSITIVE CASES:

Suppose $\langle \psi \quad \oplus \rangle$ is on b. We examine each way ψ could be constructed.

1. Let ψ be an existential sentence, so that $\psi = \exists x A(x)$ for some open formula $A(x)$. Since b is saturated, the $\exists$-$\oplus$-rule tells us that for some name a, $\langle A(a/x) \quad \oplus \rangle$ is on b. But this is a closed sentence with n or fewer quantifiers and connectives, so $i_b^+ \vDash_1 A(a/x)$. Thus by the truth conditions for existential sentences, $i_b^+ \vDash_1 \exists x A(x)$.
2. Let ψ be a negated existential sentence, so that $\psi = \neg\exists x A(x)$ for some open formula $A(x)$. Then since b is saturated, the $\neg$-$\exists$-$\oplus$ rule ensures that $\langle \forall x \neg A(x) \quad \oplus \rangle$ is on b. Then, again by saturation, the $\forall$-$\oplus$-rule ensures that for every name a that has occurred anywhere on b, $\langle \neg A(a/x) \quad \oplus \rangle$ is on b. But each of these is a closed sentence with n or fewer quantifiers and connectives, so $i_b^+ \vDash_1 \neg A(a/x)$. Thus, by the truth conditions for negations, $i_b^+ \vDash_0 A(a/x)$. So by the truth conditions for existential sentences, $i_b^+ \vDash_0 \exists x A(x)$. By a final appeal to the truth conditions for negations, we then see $i_b^+ \vDash_1 \neg\exists x A(x)$.

NEGATIVE CASES:

Suppose $\langle \psi \quad \ominus \rangle$ is on b. We examine each way ψ could be constructed.

1. Let ψ be an existential sentence, so that $\psi = \exists x A(x)$ for some open formula $A(x)$. Let α be a name. Since v is a model, $\delta(\alpha) \in D$; say $\delta(\alpha) = d$. But D consists of all the names that have occurred in b, so d must have occurred on b at some point. Then, since b is saturated, the $\exists$-$\ominus$-rule tells us that $\langle A(d/x) \quad \ominus \rangle$ occurs somewhere on b. But this is a closed sentence with n or fewer quantifiers and connectives, so by assumption this gives that $i_b^+ \nvDash_1 A(d/x)$. Thus by the truth conditions for existential sentences, $i_b^+ \nvDash_1 \exists x A(x)$.
2. Let ψ be a negated existential sentence, so that $\psi = \neg\exists x A(x)$ for some open formula $A(x)$. Then since b is saturated, the

$\neg$-$\exists$-$\ominus$ rule ensures that $\langle \forall x \neg A(x) \quad \ominus \rangle$ is on b. Then, again by saturation, the $\forall$-$\ominus$-rule ensures that for some name a, $\langle \neg A(a/x) \quad \ominus \rangle$ is on b. But this is a closed sentence with n or fewer quantifiers and connectives, so $i_b^+ \not\models_1 \neg A(a/x)$. Thus, by the truth conditions for negations, $i_b^+ \not\models_0 A(a/x)$, so for some name α, $i_b^+ \not\models_0 A(\alpha/x)$, thus by the truth conditions for existential sentences, $i_b^+ \not\models_0 \exists x A(x)$. So by a final appeal to the truth conditions for negations, $i_b^+ \not\models_1 \neg \exists x A(x)$. □

Next we establish some features of canonical models.

Lemma 17. (Completeness – Closure) *For saturated branches b, if b does not close with $\times_1$ or $\times_2$, then i_b^+ is a* K3*-model.*

Proof First, we need a mini-lemma analogous to Mini-Lemma 3:

Mini-Lemma 4. (Pairing) *If b contains nodes of the form $\langle A \quad \oplus \rangle$ and $\langle \neg A \quad \oplus \rangle$ for some sentence A then for some atomic ϕ, b contains nodes of the form $\langle \phi \quad \oplus \rangle$ and $\langle \neg \phi \quad \oplus \rangle$.*

Proof The proof is essentially the same as the proof of Mini-Lemma 3, so left as an exercise. □

Now suppose b does not close with $\times_1$ or $\times_2$. Then since b doesn't close with $\times_1$, i_b^+ is a model. Also, since b does not close with $\times_2$, by Mini-Lemma 4 for no atomic ϕ, does b contain nodes of the form $\langle \phi \quad \oplus \rangle$ and $\langle \neg \phi \quad \oplus \rangle$. Thus, for every predicate Π, $\delta^+(\Pi) \cap \delta^-(\Pi) = \emptyset$, so i_b^+ is a K3-model. □

Lemma 18. (Completeness – Closure) *For saturated branches b, if b does not close with $\times_1$ or $\times_3$, then i_b^- is an* LP*-model.*

Proof Analogous to the proof for Lemma 18; left as an exercise. □

Lemma 19. (Completeness – Closure) *Suppose b is saturated. Define i_b to be the union of i_b^+ and i_b^- – that is, let i_b be the pair $\langle D, \delta \rangle$ such that*

(i) *D is the set of all names that occur on b, and δ is defined by*

$$\delta(\alpha) = \begin{cases} \alpha & \text{if } \alpha \text{ occurs on } b \\ a & \text{otherwise} \end{cases}$$

(ii) $\delta^+(\Pi) = \{\langle \alpha_1, \ldots, \alpha_n \rangle : \langle \Pi\alpha_1 \ldots \alpha_n \quad \oplus \rangle$ *occurs on* $b\} \cup$
$\{\langle \alpha_1, \ldots, \alpha_n \rangle : \langle \neg\Pi\alpha_1 \ldots \alpha_n \quad \ominus \rangle$ *does not occur on* $b\}$, *and*

(iii) $\delta^-(\Pi) = \{\langle \alpha_1, \ldots, \alpha_n \rangle : \langle \neg\Pi\alpha_1 \ldots \alpha_n \quad \oplus \rangle$ *occurs on* $b\} \cup$
$\{\langle \alpha_1, \ldots, \alpha_n \rangle : \langle \Pi\alpha_1 \ldots \alpha_n \quad \ominus \rangle$ *does not occur on* $b\}$.

Then i_b is faithful to b iff b does not close with $\times_1$, $\times_2$ or $\times_2$.

Proof Analogous to the proof for Lemma 18; left as an exercise. □

At long last we are in position to prove completeness for our first-order theories without identity:

Theorem 13. (Completeness: Predicate FDE Tableau Sans Identity) *If* $X \models^{\mathsf{FDE}} \phi$ *then* $X \overset{\mathsf{FDE}}{\Longrightarrow} \phi$.

Proof Suppose $X \overset{\mathsf{FDE}}{\not\Longrightarrow} \phi$. Then if T is a saturated tableau for the argument $X \therefore \phi$, there is a branch b that does not close with $\times_1$. So by Lemma 16, both i_b^+ and i_b^- are faithful to b. Thus, they each must satisfy X but not satisfy ϕ. So $X \not\models^{\mathsf{FDE}} \phi$. □

Theorem 14. (Completeness: Predicate K3 Tableau Sans Identity) *If* $X \models^{\mathsf{K3}} \phi$ *then* $X \overset{\mathsf{K3}}{\Longrightarrow} \phi$.

Proof Suppose $X \overset{\mathsf{K3}}{\not\Longrightarrow} \phi$. Then if T is a saturated tableau for the argument $X \therefore \phi$, there is a branch b that does not close with $\times_1$ or $\times_2$. By Lemma 17, then, i_b^+ is a K3-model that is faithful to b. Thus it must satisfy X but not satisfy ϕ. So $X \not\models^{\mathsf{K3}} \phi$. □

Theorem 15. (Completeness: Predicate LP Tableau Sans Identity) *If* $X \models^{\mathsf{LP}} \phi$ *then* $X \overset{\mathsf{LP}}{\Longrightarrow} \phi$.

Proof Essentially the same as for Theorem 14, left as an exercise. □

Theorem 16. (Completeness: Predicate CL Tableau Sans Identity) *If* $X \models^{\mathsf{CL}} \phi$ *then* $X \overset{\mathsf{CL}}{\Longrightarrow} \phi$.

Proof Essentially the same as for Theorem 14, left as an exercise. □

3.5 CLASSICAL LOGIC WITH IDENTITY

Though identity is classified in this text as a nonlogical piece of the vocabulary, we nonetheless should ensure that the rules we've added to our tableau system to handle identity claims behave as they ought. In this chapter we prove that they do. Soundness of the tableau system with identity is essentially the same as soundness of the tableau system without identity. To prove completeness, however, a somewhat substantial shift is required. In particular, while in the previous section we were able to construct a model from an open branch on a tableau simply by using the set of names occurring on the branch as the domain of the model, the addition of equality requires something somewhat more subtle: rather than using the names themselves as the elements of our domain, we use *equivalence classes* of names.

3.5.1 EQUIVALENCE CLASSES

Recall from Chapter 4 that we can identify a binary relation on a set D with a set R of ordered pairs of elements of D. Recall also that an equivalence relation is a binary relation that satisfies the following three conditions:

- For all $x \in D$, $\langle x, x\rangle \in R$
- If $\langle x, y\rangle \in R$ then $\langle y, x\rangle \in R$.
- If $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then $\langle x, z\rangle \in R$.

Given an equivalence relation R on D and an element d of D, we define the equivalence class of d – which we write $[d]$ – to be the following set

$$[d] := \{x \in D | \langle x, d\rangle \in R\}$$

In other words, if we say x and y are *equivalent* when $\langle x, y\rangle \in R$, then the equivalence class of d is the set containing all and only those elements of D that are equivalent to d.

An important thing to note about equivalence classes is that they come with many different names. Indeed, if $e \in [d]$, then in fact $[e] = [d]$. We leave it as an exercise to prove this.

For the purposes of the soundness proof for classical logic with identity, you can forget about equivalence classes. When it comes to the completeness proof, however, you will have to think carefully about how they work.

3.5.2 SOUNDNESS

Note first that Denotation and Locality still hold. For convenience, we restate these important lemmas here, modifying their language slightly to reflect the new setting. The reader should check that the proofs of these lemmas given in Section 4 still hold.

Lemma 20. (Denotation) *Let $A(x_1, \ldots, x_n)$ be an open formula with n free variables. Let $v_1 = \langle D, \delta_1\rangle$ and $v_2 = \langle D, \delta_2\rangle$ be models and $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be lists of names such that*

(a) For every predicate (including identity) Π occurring in A, $\delta_1^+(\Pi) = \delta_2^+(\Pi)$ and $\delta_1^-(\Pi) = \delta_2^-(\Pi)$.
(b) For every name a occurring in A, $\delta_1(a) = \delta_2(a)$.
(c) $\delta_1(\alpha_1) = \delta_2(\beta_1), \delta_1(\alpha_2) = \delta_2(\beta_2), \ldots, \delta_1(\alpha_n) = \delta_2(\beta_n)$.

Then $v_1 \vDash_1 A(\alpha_1/x_1, \ldots, \alpha_n/x_n)$ iff $v_2 \vDash_1 A(\beta_1/x_1, \ldots, \beta_n/x_n)$.

Lemma 21. (Locality) *Let A be a sentence and suppose $v_1 = \langle D, \delta_1\rangle$ and $v_2 = \langle D, \delta_2\rangle$ are models that satisfy the following conditions:*

(a) For every predicate Π occurring in A, $\delta_1^+(\Pi) = \delta_2^+(\Pi)$ and $\delta_1^-(\Pi) = \delta_2^-(\Pi)$.
(b) For every name α occurring in A, $\delta_1(\alpha) = \delta_2(\alpha)$.

Then $v_1 \vDash_1 A$ iff $v_2 \vDash_1 A$.

Next we prove a lemma analogous to Lemmas 1 and 12:

Lemma 22. (Faithfulness) *If T is a tableau for classical first-order logic with identity, b is a branch in T and $f(b)$ is non-empty, then when we extend b using one of our tableau rules, there will be a $v \in f(b)$ that is faithful to at least one of the resulting branches.*

Proof As usual, we prove this by examining each rule in the tableau system one by one. The cases involving the propositional and quantificational rules are exactly as in Sections 3 and 4 (and thus require denotation and locality, which we could otherwise omit), so are omitted. We turn to the only new rules: those for equality.

Suppose v is a model for first-order classical logic with identity that is faithful to the branch b.

1. Suppose b contains a node in which the name t occurs. Recall the =-I-⊕-rule:

$$\begin{array}{c} \cdot \\ | \\ \langle t = t \quad \oplus \rangle \end{array}$$

   For any model v, and any name t, $\langle \delta(t), \delta(t) \rangle$ is in the extension of equality. Thus, if v was faithful to b, then when we extend b using the =-I-⊕-rule, v will be faithful to the resulting branch.
2. Suppose b contains the nodes $\langle t = u \quad \oplus \rangle$ and $\langle A(t) \quad \oplus \rangle$. Recall the =-⊕-rule:

$$\begin{array}{c} \langle t = u \quad \oplus \rangle \\ \langle A(t) \quad \oplus \rangle \\ | \\ \langle A(u) \quad \oplus \rangle \end{array}$$

   By the truth conditions for atomic sentences, $v \vDash_1 A(t)$ iff $\delta(t)$ is in the extension of A and $v \vDash_1 t = u$ iff $\langle \delta(t), \delta(u) \rangle$ is in the extension of equality. But the extension of equality consists of only the identity pairs, so $\langle \delta(t), \delta(u) \rangle$ is in the extension of equality iff $\delta(t)$ and $\delta(u)$ are the very same object. So $\delta(t)$ is in the extension of A iff $\delta(u)$ is in the extension of A, and thus (again by the truth conditions for atomic sentences) iff $v \vDash_1 A(u)$. Since v is faithful to b, $v \vDash_1 A(t)$ and $v \vDash_1 t = u$. Thus, $v \vDash_1 A(u)$. So when we extend b using the $=$-$\oplus$-rule, v will be faithful to the resulting branch.

3. Suppose b contains a node of the form $\langle \neg t = u \quad \ominus \rangle$ Recall the $=$-$\ominus/\oplus$-rule:

$$\begin{array}{c} \langle \neg t = u \quad \ominus \rangle \\ | \\ \langle t = u \quad \oplus \rangle \end{array}$$

By the truth conditions for negation, $v \vDash_1 \neg t = u$ iff $v \vDash_0 t = u$. Since v is faithful to b, $v \nvDash_1 \neg t = u$. So $v \nvDash_0 t = u$. Thus, $\langle \delta(t), \delta(u) \rangle \notin \delta^-(=)$. But since v is complete, this means $\langle \delta(t), \delta(u) \rangle \in \delta^+(=)$, so $v \vDash_1 t = u$. So when we extend the branch using the $=$-$\oplus/\ominus$-rule, v will remain faithful to the resulting branch.

The remaining rules admit proofs quite similar to these, as we leave it to the reader to check. □

Theorem 17. (Soundness: Predicate CL Tableau with Identity) *If* $X \overset{\mathsf{CL}=}{\Longrightarrow} \phi$, *then* $X \vDash^{\mathsf{CL}=} \phi$.

Proof *Essentially the same as the proofs of the other soundness theorems; left as an exercise.* □

3.5.3 COMPLETENESS

As usual, we begin our completeness proof by defining the induced models. However, before doing this, we must note the following:

Lemma 23. (Equivalence) *Let T be a tableau for first-order classical logic with identity, b be a saturated branch on T, and N be the set of* *names occurring (anywhere) on T. Define the relation R on N by letting $\langle s,t\rangle \in R$ iff $\langle s=t \quad \oplus\rangle$ is on b. Then R is an equivalence relation.*

Proof Since b is saturated, for every name t that is on b, the =-I-$\oplus$-rule guarantees that there is a node of the form $\langle t=t \quad \oplus\rangle$ on b. So $\langle t,t\rangle \in R$. Since N consists exactly of the names that occur on b, this shows that R is reflexive.

Suppose $\langle s,t\rangle \in R$. Then the node $\langle s=t \quad \oplus\rangle$ is on b. So, since s is on b, there is a node of the form $\langle s=s \quad \oplus\rangle$ on b. But then since b is saturated, the =-$\oplus$-rule guarantees that there is a node of the form $\langle t=s \quad \oplus\rangle$ on b. This, in turn gives that $\langle t,s\rangle \in R$, so that R is symmetric.

Suppose $\langle s,t\rangle \in R$ and $\langle t,u\rangle \in R$. Then, by the definition of R, there are nodes of the form $\langle s=t \quad \oplus\rangle$ and $\langle t=u \quad \oplus\rangle$ on R. But then the =-$\oplus$-rule guarantees that there is a node of the form $\langle s=u \quad \oplus\rangle$ on b. So $\langle s,u\rangle \in R$. Thus R is transitive. □

Definition *Let T be a tableau for first-order classical logic with identity, b be a saturated branch on T, and a be the first name occurring on b. Then the canonical model i_b induced by b is the model $\langle D,\delta\rangle$ such that*

- *D is the set of equivalence classes of names that occur on b under the equivalence relation defined above, and δ is defined by*

$$\delta(\alpha) = \begin{cases} [\alpha] & \textit{if } \alpha \textit{ occurs on } b \\ [a] & \textit{otherwise} \end{cases}$$

- *For non-identity predicates, $\langle[\alpha_1],\ldots,[\alpha_n]\rangle \in \delta^+(\Pi)$ if and only if $\langle \Pi\alpha_1\ldots\alpha_n \quad \oplus\rangle$ or $\langle \neg\Pi\alpha_1\ldots\alpha_n \quad \ominus\rangle$ is on b, and $\langle[\alpha_1],\ldots,[\alpha_n]\rangle \in \delta^-(\Pi)$ otherwise.*
- *$\delta^+(=)$ is the set of identity pairs of elements of D, $\delta^-(=)$ is the set of non-identity pairs of elements of D.*

There is one thing to worry about with this definition. Notice we said $\langle[\alpha_1],\ldots,[\alpha_n]\rangle \in \delta^+(\Pi)$ if and only if $\langle \Pi\alpha_1\ldots\alpha_n \quad \oplus\rangle$ or $\langle \neg\Pi\alpha_1\ldots\alpha_n \quad \ominus\rangle$ is on b. The problem is that we need to be sure that this never gets us in trouble; that is, taking the unary case for simplicity, that it nevery happens that for some names α and β, then $[\alpha]=[\beta]$, but $\langle \Pi\alpha \quad \oplus\rangle$ or $\langle \neg\Pi\alpha \quad \ominus\rangle$ is on b while neither

$\langle \Pi\beta \quad \oplus\rangle$ nor $\langle \neg\Pi\beta \quad \ominus\rangle$ do. This would be bad – in such a case, the definition would demand both that $[\alpha] \in \Pi^+$ and that $[\beta] \notin \Pi^+$, which can't happen since $[\alpha] = [\beta]$.

So, suppose $\langle [\alpha_1], \ldots, [\alpha_n]\rangle = \langle [\beta_1], [\beta_2], \ldots, [\beta_n]\rangle$. Then each of $\langle \alpha_1 = \beta_1 \quad \oplus\rangle, \ldots, \langle \alpha_n = \beta_n \quad \oplus\rangle$ must occur on b. Thus, $\langle \Pi\alpha_1 \ldots \alpha_n \quad \oplus\rangle$ is on b iff (by n applications of the $=$-$\oplus$-rule) $\langle \Pi\beta_1\beta_2 \ldots \beta_n \quad \oplus\rangle$ is on b. On the other hand, since each of $\langle \alpha_1 = \beta_1 \quad \oplus\rangle, \ldots, \langle \alpha_n = \beta_n \quad \oplus\rangle$ must be on b, $=$-$\oplus/\ominus$ guarantees that each of $\langle \neg\alpha_1 = \beta_1 \quad \ominus\rangle, \ldots, \langle \neg\alpha_n = \beta_n \quad \ominus\rangle$ must be on b. Thus, $\langle \neg\Pi\alpha_1 \ldots \alpha_n \quad \ominus\rangle$ is on b iff (by n applications of the $=$-$\ominus$-rule) $\langle \neg\Pi\beta_1\beta_2 \ldots \beta_n \quad \ominus\rangle$ is on b. Thus the choice of particular name for $[a]$ does not matter in our statement of this definition.

Lemma 24. (Faithfulness) *If b is a saturated open branch on a tableau for first-order classical logic with identity T, then i_b is faithful to b.*

Proof We'll first show the lemma is true if ϕ is an atomic sentence or the negation of an atomic sentence. This splits into the following cases:

NON-IDENTITY CASES: Suppose ϕ is of the form $\Pi\alpha_1 \ldots \alpha_n$ for some n-ary non-identity predicate Π and n names α_i.

- Suppose $\langle \phi \quad \oplus\rangle$ is on b. Then since $\langle \phi \quad \oplus\rangle$ occurs on b, $\langle [\alpha_1], \ldots, [\alpha_n]\rangle = \langle \delta(\alpha_1), \ldots, \delta(\alpha_n)\rangle \in \delta^+(\Pi)$, so $i_b \vDash_1 \phi$.
- Suppose $\langle \phi \quad \ominus\rangle$ is on b. Then since $\langle \phi \quad \ominus\rangle$ is on b, and b is does not close with $\times_1$, $\langle \phi \quad \oplus\rangle$ isn't on b, and since b does not close with $\times_3$, $\langle \neg\phi \quad \ominus\rangle$ isn't on b. Thus, the n-tuple $\langle [\alpha_1], \ldots, [\alpha_n]\rangle = \langle \delta(\alpha_1), \ldots, \delta(\alpha_n)\rangle \notin \delta^+(\Pi)$, so $i_b \nvDash_1 \phi$.
- Suppose $\langle \neg\phi \quad \oplus\rangle$ is on b. Then since b does not close with $\times_1$, $\langle \neg\phi \quad \ominus\rangle$ isn't on b, and since b does not close with $\times_2$, $\langle \phi \quad \oplus\rangle$ isn't on b. Thus,

  $$\langle [\alpha_1], \ldots, [\alpha_n]\rangle = \langle \delta(\alpha_1), \ldots, \delta(\alpha_n)\rangle \in \delta^-(\Pi)$$

  and so $i_b \vDash_0 \phi$. By the truth conditions for negations, it follows that $i_b \vDash_1 \neg\phi$.
- Suppose $\langle \neg\phi \quad \ominus\rangle$ is on b. Then since $\langle \neg\phi \quad \ominus\rangle$ is on b, the n-tuple $\langle [\alpha_1], \ldots, [\alpha_n]\rangle = \langle \delta(\alpha_1), \ldots, \delta(\alpha_n)\rangle \in \delta^+(\Pi)$, so $\nu \vDash_1 \phi$. By the truth conditions for negations, it follows that $i_b \nvDash_1 \neg\phi$.

IDENTITY CASES

- Suppose $\langle s = t \quad \oplus \rangle$ is on b. Then $[s] = [t]$, so the pair $\langle [s], [t] \rangle = \langle \delta(s), \delta(t) \rangle \in \delta^+(=)$. Thus $i_b \models_1 s = t$.
- Suppose $\langle s = t \quad \ominus \rangle$ is on b. Then since b does not close with $\times_1$, $\langle s = t \quad \oplus \rangle$ isn't on b. So $[s] \neq [t]$, and thus the pair $\langle [s], [t] \rangle = \langle \delta(s), \delta(t) \rangle \notin \delta^+(=)$. Thus $i_b \not\models_1 s = t$.
- Suppose $\langle \neg s = t \quad \oplus \rangle$ is on b. Then since b does not close with $\times_2$, $\langle s = t \quad \oplus \rangle$ isn't on b. So $[s] \neq [t]$, and thus the pair $\langle [s], [t] \rangle = \langle \delta(s), \delta(t) \rangle \in \delta^-(=)$. Thus $i_b \models_0 s = t$, so by the truth conditions for negation, $i_b \models_1 \neg s = t$.
- Suppose $\langle \neg s = t \quad \ominus \rangle$ is on b. Then by the $=$-$\ominus/\oplus$-rule, $\langle s = t \quad \oplus \rangle$ is on b. So the pair $\langle [s], [t] \rangle = \langle \delta(s), \delta(t) \rangle \in \delta^+(=)$. Thus $i_b \models_1 s = t$. So by truth conditions for negation, $i_b \not\models_1 \neg s = t$.

For non-atomic sentences, the proof proceeds exactly as in the first-order case without identity, so is left as an exercise. □

From here Completeness follows easily:

Theorem 18. (Soundness: Predicate CL Tableau with Identity) *If* $X \overset{\mathsf{CL}=}{\models} \phi$ *then* $X \overset{\mathsf{CL}=}{\Longrightarrow} \phi$.

Proof Essentially the same as for, e.g., Theorem 14, left as an exercise. □

3.6 FREE LOGICS WITHOUT IDENTITY

Recall that an FDE*-model is a triple $\langle D, E, \delta \rangle$ with D a set, $E \subseteq D$, and δ a function such that

- δ assigns to each name α an element $\delta(\alpha)$ of D,
- δ assigns to each n-ary predicate Π a pair $\langle \delta^+(\Pi), \delta^-(\Pi) \rangle$ with $\delta^+(\Pi) \subseteq D^n$ and $\delta^-(\Pi) \subseteq D^n$.
- For each $d \in D$ there is some name α such that $\delta(\alpha) = d$.

Given this, we can characterize the K3$*$-, LP$*$-, and CL$*$-models as follows:

- A K3$*$-model is an FDE$*$-model such that for every n-ary predicate Π, $\delta^+(\Pi) \cap \delta^-(\Pi) = \emptyset$,

- An LP$*$-model is an FDE$*$-model such that for every n-ary predicate Π, $\delta^+(\Pi) \cup \delta^-(\Pi) = D^n$, and
- A CL$*$-model is an FDE$*$-model that is both a K3$*$-model and an LP$*$-model.

3.6.1 SOUNDNESS

We begin with the by-now-familiar denotation and locality lemmas.

Lemma 25. (Denotation) *Let $A(x_1, \ldots, x_n)$ be an open formula with n free variables. Let $v_1 = \langle D, E, \delta_1 \rangle$ and $v_2 = \langle D, E, \delta_2 \rangle$ be* FDE$*$*-models and $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be lists of names such that*

(a) For every predicate Π occurring in A, $\delta_1^+(\Pi) = \delta_2^+(\Pi)$ and $\delta_1^-(\Pi) = \delta_2^-(\Pi)$.
(b) For every name a occurring in A, $\delta_1(a) = \delta_2(a)$.
(c) $\delta_1(\alpha_1) = \delta_2(\beta_1), \delta_1(\alpha_2) = \delta_2(\beta_2), \ldots, \delta_1(\alpha_n) = \delta_2(\beta_n)$.

Then $v_1 \vDash_1 A(\alpha_1/x_1, \ldots, \alpha_n/x_n)$ iff $v_2 \vDash_1 A(\beta_1/x_1, \ldots, \beta_n/x_n)$.

Proof As usual, we proceed by induction. The base case, inductive hypothesis, and propositional cases are as in the previous iterations of the lemma. The only interesting difference is in the quantified formulas. We show here the case for the existential quantifier; the case for the universal quantifier is left as an exercise.

Let $A(x_1, \ldots, x_n) = \exists y B(y, x_1, \ldots, x_n)$. Then by truth conditions for the existential quantifier, $v_1 \vDash_1 A(\alpha_1/x_1, \ldots, \alpha_n/x_n)$ iff for some name γ, $\delta_1(\gamma) \in E$ and $v_1 \vDash_1 B(\gamma/y, \alpha_1/x_1, \ldots, \alpha_n/x_n)$. Notice that since A has $k+1$ connectives and quantifiers, B has k connectives and quantifiers. Also, since every object in D has a name in every model, there is some name ν_γ such that $\delta_2(\nu_\gamma) = \delta_1(\gamma)$. But then notice that for v_1 and v_2 the lists of names $\gamma, \alpha_1, \alpha_2, \ldots, \alpha_n$ and $\nu_\gamma, \beta_1, \ldots, \beta_n$ satisfy the hypotheses of the lemma. So, by induction, $v_1 \vDash_1 B(\gamma/y, \alpha_1/x_1, \ldots, \alpha_n/x_n)$ iff $v_2 \vDash_1 B(y/\nu_\gamma, \beta_1/x_1, \ldots, \beta_n/x_n)$. Also, since $\delta_2(\nu_\gamma)$ and $\delta_1(\gamma)$ are the very same object, $\delta_2(\nu_\gamma) \in E$ iff $\delta_1(\gamma) \in E$. It follows from these observations and the truth conditions for existentially quantified sentences, then, that $v_1 \vDash_1 A(\alpha_1/x_1, \ldots, \alpha_n/x_n)$ iff $v_2 \vDash_1 A(\beta_1/x_1, \ldots, \beta_n/x_n)$. $\square$

Lemma 26. (Locality) *Let A be a sentence and suppose $v_1 = \langle D, E, \delta_1 \rangle$ and $v_2 = \langle D, E, \delta_2 \rangle$ are* FDE∗*-models that satisfy the following conditions:*

(a) For every predicate Π occurring in A, $\delta_1^+(\Pi) = \delta_2^+(\Pi)$ and $\delta_1^-(\Pi) = \delta_2^-(\Pi)$.
(b) For every name α occurring in A, $\delta_1(\alpha) = \delta_2(\alpha)$.

Then $v_1 \vDash_1 A$ iff $v_2 \vDash_1 A$.

Proof As usual, we proceed by induction. The base case, inductive hypothesis, and propositional cases are as in the previous iterations of the lemma. The only interesting difference is in the quantified formulas. Here we will show the case involving the universal quantifier; the case involving the existential is left as an exercise.

Let $\psi = \forall x \phi(x)$, for some open sentence ϕ. Then $v_1 \vDash_1 \psi$ iff for all names α, if $\delta_1(\alpha) \in E$, then $v_1 \vDash_1 \phi(\alpha/x)$. Choose an arbitrary name α, and let $\delta_1(\alpha) = d \in D$. Since every object has a name in every model, there is a name β such that $\delta_2(\beta) = d$. Since $\delta_1(\alpha)$ and $\delta_2(\beta)$ are the very same object, $\delta_1(\alpha) \in E$ iff $\delta_2(\beta) \in E$. Also, the models v_1 and v_2, the formula $\phi(x)$, and the names α and β together satisfy the hypotheses of the denotation lemma, so $v_1 \vDash_1 \phi(\alpha/x)$ iff $v_2 \vDash_1 \phi(\beta/x)$. Since α was arbitrary, these two facts are true for every name, so $v_1 \vDash_1 \psi$ iff $v_2 \vDash_1 \psi$. □

Finally, we show that the free logic tableau rules behave well with the respect to models for free logic. Of course, '$\mathbb{E}$' is not an official part of the vocabulary of any of our languages. So in order to dercide when a model is fauthful to a branch containing nodes of the form $\langle \mathbb{E}a \quad \oplus \rangle$ or $\langle \mathbb{E}a \quad \ominus \rangle$, we must interpret these expressions appropriately. It will suffice to do this in a very natural way: translate $\mathbb{E}t$ as $\exists x(x = t)$.

Lemma 27. (Faithfulness) *If T is a tableau for free first-order logic without identity, b is a branch in T, and $f(b)$ is non-empty, then when we extend b using one of our tableau rules, there will be a $v \in f(b)$ that is faithful to at least one of the resulting branches.*

Proof Suppose $\nu = \langle D, E, \delta \rangle$ is a model that is faithful to the branch b. We examine a pair of illustrative examples, leaving the examination of the remaining rules as an exercise.

Suppose b contains a node of the form $\langle \forall x A(x) \quad \oplus \rangle$. Recall the $\forall$-$\oplus$-rule:

$$\begin{array}{c} \langle \forall x A(x) \quad \oplus \rangle \\ \diagup \qquad \diagdown \\ \langle \mathbb{E}t \quad \ominus \rangle \qquad \langle A(t) \quad \oplus \rangle \end{array}$$

By the truth conditions for the universal quantifier, $\nu \vDash_1 \forall x A(x)$ iff $\nu \vDash_1 A(\alpha/x)$ for every name α such that $\delta(\alpha) \in E$. Let t be a name on b, and consider $\delta(t)$. Either $\delta(t) \in E$ or $\delta(t) \notin E$. In the first case, since $\nu \vDash_1 A(\alpha/x)$ for every name α such that $\delta(\alpha) \in E$, $\nu \vDash_1 A(t/x)$. Thus, ν is faithful to the right-hand branch. In the second case, recall that $\nu \vDash_1 \exists x(x = t)$ iff for some name α, $\delta(\alpha) \in E$ and $\langle \delta(\alpha), \delta(t) \rangle \in \delta^+(=)$. Since $\delta^+(=)$ consists of all and only the identity pairs, $\langle \delta(\alpha), \delta(t) \rangle \in \delta^+(=)$ iff $\delta(\alpha)$ and $\delta(t)$ are exactly the same object. But if $\delta(\alpha)$ and $\delta(t)$ are the very same object, then $\delta(\alpha) \notin E$, iff $\delta(t) \notin E$. Altogether, then, if $\delta(t) \notin E$, then $\nu \nvDash_1 \exists x(x = t)$; that is, $\nu \nvDash_1 \mathbb{E}t$. So ν is faithful to the left-hand branch. Thus, in either case, when we extend b, ν will be faithful to one of the resulting branches.

Suppose b contains a node of the form $\langle \exists x A(x) \quad \oplus \rangle$. Recall the $\exists$-$\oplus$-rule:

$$\begin{array}{c} \langle \exists x A(x) \quad \oplus \rangle \\ | \\ \langle \mathbb{E}a \quad \oplus \rangle \\ \langle A(a) \quad \oplus \rangle \end{array}$$

By the truth conditions for the existential quantifier, $\nu \vDash_1 \exists x A(x)$ iff $\nu \vDash_1 A(\alpha/x)$ for some name α such that $\delta(\alpha) \in E$. Since ν is faithful to b, $\nu \vDash_1 \exists x A(x)$. Thus, for some name α, $\nu \vDash_1 A(\alpha/x)$

and $\delta(\alpha) \in E$. Let a be a name that has not occurred on b. Let $v' = \langle D, E, \delta' \rangle$ be a model that agrees with v on every name and predicate occurring on b, and for which $\delta'(a) = \delta(\alpha)$. By locality, v' is faithful to b, and by construction, $v \vDash_1 A(a)$. Also, since $\delta'(a)$ and $\delta(\alpha)$ are the very same object and $\delta(\alpha) \in E$, $\delta'(a) \in E$ as well. Thus, by the truth conditions for existential quantification and equality, $v' \vDash_1 \exists x(x = a)$; that is, $v' \vDash_1 \mathbb{E}a$. Altogether then, $v' \in f(b)$, and when we extend b using the $\exists$-$\oplus$-rule, v' is faithful to the resulting branch. $\square$

The proofs of the following lemmas are exactly the same as the proofs of the corresponding lemmas in Section 4.1, so are omitted.

Lemma 28. (Soundness – Closure) *If b is a branch that closes with the $\times_1$-rule, then no model v is faithful to b.*

Lemma 29. (Soundness – Closure) *If b is a branch that closes with the $\times_2$-rule, then no* K3∗*-model (and hence no* CL∗*-model) v is faithful to b.*

Lemma 30. (Soundness – Closure) *If b is a branch that closes with the $\times_3$-rule, then no* LP∗*-model (and hence no* CL∗*-model) v is faithful to b.*

Soundness follows in the by-now-expected way, where 'FO' abbreviates 'first-order'

Theorem 19. (Soundness: FO FDE∗ Tableau Sans Identity)
If $X \xRightarrow{\mathsf{FDE*}} \phi$, then $X \vDash^{\mathsf{FDE*}} \phi$.

Proof Suppose $X \nvDash^{\mathsf{FDE*}} \phi$. We will prove that $X \not\xRightarrow{\mathsf{FDE*}} \phi$.

Since $X \nvDash^{\mathsf{FDE*}} \phi$, there is an FDE∗-model v such that v satisfies X but $v \nvDash_1 \phi$. Such a model v is faithful to the initial tableau T for $X \therefore \phi$. Then by Lemma 1, no matter how we extend T, v will remain faithful to some branch b of T. Suppose now that $X \xRightarrow{\mathsf{FDE*}} \phi$. Given this, each branch of T must end with $\times_1$. Thus, b ends with $\times_1$. But this contradicts Lemma 28, so we must have that $X \not\xRightarrow{\mathsf{FDE*}} \phi$. $\square$

Theorem 20. (Soundness: FO K3∗ Tableau Sans Identity)
If $X \overset{\mathsf{K3*}}{\Longrightarrow} \phi$, *then* $X \overset{\mathsf{K3*}}{\models} \phi$.

Proof Omitted, left as an exercise. □

Theorem 21. (Soundness: FO LP∗ Tableau Sans Identity)
If $X \overset{\mathsf{LP*}}{\Longrightarrow} \phi$, *then* $X \overset{\mathsf{LP*}}{\models} \phi$.

Proof Essentially the same as for Theorem 20; left as an exercise. □

Theorem 22. (Soundness: FO CL∗ Tableau Sans Identity)
If $X \overset{\mathsf{CL*}}{\Longrightarrow} \phi$, *then* $X \overset{\mathsf{CL*}}{\models} \phi$.

Proof Only slightly different from Theorem 20; left as an exercise □

3.6.2 COMPLETENESS

Next we prove completeness. First, two definitions:

Definition *If T is a free first-order tableau, b is a saturated branch on T, and a is the first name that is on b then the canonical positive model i_b^+ induced by b is the triple $\langle D, E, \delta\rangle$ such that*

(i) *D is the set of all names that occur on b,*
(ii) *E is the subset of D consisting of all names t such that $\langle \mathbb{E}t \quad \ominus\rangle$ isn't on b, and*
(iii) *δ is defined by*

$$\delta(\alpha) = \begin{cases} \alpha & \text{if } \alpha \text{ occurs on } b \\ a & \text{otherwise} \end{cases}$$

(iv) $\delta^+(\Pi) = \{\langle \alpha_1, \ldots, \alpha_n\rangle : \langle \Pi\alpha_1 \ldots \alpha_n \quad \oplus\rangle \text{ occurs on } b\}$
(v) $\delta^-(\Pi) = \{\langle \alpha_1, \ldots, \alpha_n\rangle : \langle \neg\Pi\alpha_1 \ldots \alpha_n \quad \oplus\rangle \text{ occurs on } b\}$

Definition *If T is a free first-order tableau, b is a saturated branch on T, and a is the first name that is on b then the canonical negative model i_b^- induced by b is the triple $\langle D, E, \delta\rangle$ such that*

(i) *D is the set of all names that occur on b,*
(ii) *E is the set of names t such that $\langle \mathbb{E}t \quad \ominus\rangle$ isn't on b, and*
(iii) *δ is defined by*

$$\delta(\alpha) = \begin{cases} \alpha & \text{if } \alpha \text{ occurs on } b \\ a & \text{otherwise} \end{cases}$$

(iv) $\delta^{+}(\Pi) = \{\langle \alpha_1, \ldots, \alpha_n\rangle : \langle \Pi\alpha_1 \ldots \alpha_n \quad \ominus\rangle$ *does not occur on* $b\}$
(v) $\delta^{-}(\Pi) = \{\langle \alpha_1, \ldots, \alpha_n\rangle : \langle \neg\Pi\alpha_1 \ldots \alpha_n \quad \ominus\rangle$ *does not occur on* $b\}$

Lemma 31. (Faithfulness) *For saturated branches b, i_b^{+} and i_b^{-} are faithful to b iff b does not close with $\times_1$.*

We prove the case for i_b^{+}. The case involving i_b^{-} is similar and left as an exercise.

Proof Just as before, we first establish a mini-lemma:

Mini-Lemma 5. (Pairing) *If b contains nodes of the form $\langle A \quad \ominus\rangle$ and $\langle A \quad \oplus\rangle$ for some sentence A then one of the following occurs:*

- *For some n-ary predicate Π, b either contains nodes of the form $\langle \Pi\alpha_1, \ldots, \alpha_n \quad \oplus\rangle$ and $\langle \Pi\alpha_1, \ldots, \alpha_n \quad \ominus\rangle$ or of the form $\langle \neg\Pi\alpha_1, \ldots, \alpha_n \quad \oplus\rangle$ and $\langle \neg\Pi\alpha_1, \ldots, \alpha_n \quad \ominus\rangle$; or*
- *For some name a, b contains nodes of the form $\langle \mathbb{E}a \quad \oplus\rangle$, $\langle \mathbb{E}a \quad \ominus\rangle$.*

Proof The base cases and inductive hypothesis are exactly as before. The inductive step requires examination of each way A could be constructed. Here we examine the case where the sentence is universally quantified, the other cases being similar either to this or to previously-examined cases.

- Suppose $\langle \forall x A(x) \quad \oplus\rangle$ and $\langle \forall x A(x) \quad \ominus\rangle$ occur on b for some open formula $A(x)$. Since b is saturated, the $\forall$-$\ominus$-rule guarantees for some name a, $\langle \mathbb{E}a \quad \oplus\rangle$ and $\langle A(x/a) \quad \ominus\rangle$ occur on b. Also, by the $\forall$-$\oplus$-rule, either $\langle \mathbb{E}a \quad \ominus\rangle$ or $\langle A(x/a) \quad \oplus\rangle$ also is on b. In the first case, b contains the nodes $\langle \mathbb{E}a \quad \oplus\rangle$ and $\langle \mathbb{E}a \quad \ominus\rangle$, so the result holds. In the second case, b contains nodes of the form $\langle A(x/a) \quad \oplus\rangle$ and $\langle A(x/a) \quad \ominus\rangle$. But $A(x/a)$ is a sentence with n or fewer connectives and quantifiers, so by the inductive hypothesis the result also holds. □

Now suppose b closes with $\times_1$. Then there must be nodes of the form $\langle A \quad \oplus\rangle$ and $\langle A \quad \ominus\rangle$ on b for some sentence A. It follows from Mini-Lemma 5 that either for or some n-ary predicate Π, b either contains nodes of the form $\langle \Pi\alpha_1,\ldots,\alpha_n \quad \oplus\rangle$ and $\langle \Pi\alpha_1,\ldots,\alpha_n \quad \ominus\rangle$ or $\langle \neg\Pi\alpha_1,\ldots,\alpha_n \quad \oplus\rangle$ and $\langle \neg\Pi\alpha_1,\ldots,\alpha_n \quad \ominus\rangle$; or for some name a, b contains nodes of the form $\langle \mathbb{E}a \quad \oplus\rangle$, $\langle \mathbb{E}a \quad \ominus\rangle$. In the first case we can argue in exactly the same way as in Section 4.2. In the second case, recall that we translate '$\mathbb{E}a$' as '$\exists x(x = a)$'. It follows that v is faithful to a branch containing both $\langle \mathbb{E}a \quad \oplus\rangle$ and $\langle \mathbb{E}a \quad \ominus\rangle$ iff it is true both that for some name α, $\delta(\alpha) \in E$ and $\langle \delta(\alpha), \delta(a)\rangle \in \delta^+(=)$ and for no name β is $\delta(\beta) \in E$ and $\langle \delta(\beta), \delta(a)\rangle \in \delta^+(=)$, which is clearly a contradiction.

On the other hand, suppose b does not close with $\times_1$. If $\langle \Pi\alpha_1 \ldots \alpha_n \quad \oplus\rangle$ is on b, then from condition (ii) in the definition of i_b^+, $i_b^+ \vDash_1 \Pi\alpha_1 \ldots \alpha_n$. Similarly, if $\langle \neg\Pi\alpha_1 \ldots \alpha_n \quad \oplus\rangle$ is on b, then from condition (iii) in the definition of i_b^+, we see that $i_b^+ \vDash_1 \neg\Pi\alpha_1 \ldots \alpha_n$.

Also, since b does not close with $\times_1$, if $\langle \Pi\alpha_1 \ldots \alpha_n \quad \ominus\rangle$ is on b, then $\langle \Pi\alpha_1 \ldots \alpha_n \quad \oplus\rangle$ isn't on b, so $i_b^+ \nvDash_1 \Pi\alpha_1 \ldots \alpha_n$. Similarly, if $\langle \neg\Pi\alpha_1 \ldots \alpha_n \quad \ominus\rangle$ is on b, then $\langle \neg\Pi\alpha_1 \ldots \alpha_n \quad \oplus\rangle$ isn't on b, so $i_b^+ \nvDash_1 \neg\Pi\alpha_1 \ldots \alpha_n$. Thus, if ϕ is an atomic sentence or the negation of an atomic sentence, then if $\langle \phi \quad \oplus\rangle$ is on b, $i_b^+ \vDash_1 \phi$ and if $\langle \phi \quad \ominus\rangle$ is on b, $i_b^+ \nvDash_1 \phi$.

In addition, since b does not close with $\times_1$, if $\langle \mathbb{E}a \quad \oplus\rangle$ is on b, then $\langle \mathbb{E}a \quad \ominus\rangle$ does not, and vice-versa. This guarantees that v is well-defined and faithful to all nodes of this form.

Now suppose that

- if ϕ has n or fewer connectives and quantifiers and $\langle \phi \quad \oplus\rangle$ is on a saturated branch b, then $i_b^+ \vDash_1 \phi$,
- if ϕ has n or fewer connectives and quantifiers and $\langle \phi \quad \ominus\rangle$ is on a saturated branch b, then $i_b^+ \nvDash_1 \phi$.
- if ϕ has n or fewer connectives and quantifiers and $\langle \neg\phi \quad \oplus\rangle$ is on a saturated branch b, then $i_b^+ \vDash_1 \neg\phi$,
- if ϕ has n or fewer connectives and quantifiers and $\langle \neg\phi \quad \ominus\rangle$ is on a saturated branch b, then $i_b^+ \nvDash_1 \neg\phi$.

Now suppose ψ has $n+1$ connectives and quantifiers in total and occurs in some node on b. We examine only the following case, the remaining cases being similar or essentially the same as previously examined cases and left as exercises:

- Let ψ be a universal sentence, so that $\psi = \forall x A(x)$ for some open formula $A(x)$. Suppose $\langle \psi \quad \oplus \rangle$ is on b. Let α be a name. Since v is a model, $\delta(\alpha) \in D$; say $\delta(\alpha) = d$. But D consists of all the names that have occurred in b, so d must have occurred on b at some point. Then, since b is saturated, the $\forall$-$\oplus$-rule tells us that either $\langle A(d/x) \quad \oplus \rangle$ occurs somewhere on b or $\langle \mathbb{E}d \quad \ominus \rangle$ occurs somewhere on b. In other words, if $\langle \mathbb{E}d \quad \ominus \rangle$ isn't on b, then $\langle A(d/x) \quad \oplus \rangle$ does. But $A(d/x)$ is a closed sentence with n or fewer quantifiers and connectives, so in this case, $i_b^+ \vDash_1 A(d/x)$. Thus, if $\langle \mathbb{E}d \quad \ominus \rangle$ isn't on b, then $i_b^+ \vDash_1 A(d/x)$. But E consists of exactly those names a for which $\langle \mathbb{E}a \quad \ominus \rangle$ isn't on b, so by the truth conditions for universal sentences, $i_b^+ \vDash \forall x A(x)$.

By induction, we conclude that i_b is faithful to b. □

Next we establish some features of canonical models. The proofs of these results are exactly as in the case for non-free first-order logic without identity, so omitted.

Lemma 32. (Completeness – Closure) *For saturated branches b, if b does not close with $\times_1$ or $\times_2$, then i_b^+ is a* K3∗*-model.*

Lemma 33. (Completeness – Closure) *For saturated branches b, if b does not close with $\times_1$ or $\times_3$, then i_b^- is an* LP∗*-model.*

Lemma 34. (Completeness – Closure) *Suppose b is saturated. Define i_b to be the union of i_b^+ and i_b^- – that is, let i_b be the pair $\langle D, \delta \rangle$ such that*

(i) D is the set of all names that occur on b, and δ is defined by

$$\delta(\alpha) = \begin{cases} \alpha & \textit{if } \alpha \textit{ occurs on } b \\ a & \textit{otherwise} \end{cases}$$

(ii) $\delta^+(\Pi) = \{\langle \alpha_1, \ldots, \alpha_n \rangle : \langle \Pi\alpha_1 \ldots \alpha_n \quad \oplus \rangle$ occurs on $b\} \cup$ $\{\langle \alpha_1, \ldots, \alpha_n \rangle : \langle \neg\Pi\alpha_1 \ldots \alpha_n \quad \ominus \rangle$ does not occur on $b\}$, and

(iii) $\delta^-(\Pi) = \{\langle\alpha_1,\ldots,\alpha_n\rangle : \langle\neg\Pi\alpha_1\ldots\alpha_n \quad \oplus\rangle$ *occurs on* $b\} \cup$
$\{\langle\alpha_1,\ldots,\alpha_n\rangle : \langle\Pi\alpha_1\ldots\alpha_n \quad \ominus\rangle$ *does not occur on* $b\}$.

Then i_b *is faithful to* b *iff* b *does not close with* $\times_1$, $\times_2$ *or* $\times_2$.

Completeness follows from these results immediately. Proofs are omitted, as they follow the same pattern as before.

Theorem 23. (Completeness: FO FDE$*$ Tableau sans Identity) *If* $X \models^{\mathsf{FDE}*} \phi$ *then* $X \overset{\mathsf{FDE}*}{\Longrightarrow} \phi$.

Theorem 24. (Completeness: FO FDE$*$ Tableau sans Identity) *If* $X \models^{\mathsf{K3}*} \phi$ *then* $X \overset{\mathsf{K3}*}{\Longrightarrow} \phi$.

Theorem 25. (Completeness: FO FDE$*$ Tableau sans Identity) *If* $X \models^{\mathsf{LP}*} \phi$ *then* $X \overset{\mathsf{LP}*}{\Longrightarrow} \phi$.

Theorem 26. (Completeness: FO FDE$*$ Tableau sans Identity) *If* $X \models^{\mathsf{CL}*} \phi$ *then* $X \overset{\mathsf{CL}*}{\Longrightarrow} \phi$.

3.7 CLASSICAL FREE LOGIC WITH IDENTITY

As the proof of the soundness and completeness for classical free logic with identity is only slightly different from the proof of soundness and completeness for classical non-free logic with identity, in this section we only highlight the small differences between the two proofs, leaving the production of the actual proofs of soundness and completeness to the reader.

3.7.1 SOUNDNESS

Denotation and Locality still hold, with essentially the same proofs.

A Faithfulness Lemma analogous to Lemmas 1, 12, and 22 holds as well:

Lemma 35. (Faithfulness) *If* T *is a tableau for classical free first-order logic with identity,* b *is a branch in* T *and* $f(b)$ *is non-empty, then when we extend* b *using one of our tableau rules, there will be a* $v \in f(b)$ *that is faithful to at least one of the resulting branches.*

From here, soundness follows immediately:

Theorem 27. (Soundness: FO CL$*$ Tableau Plus Identity)
If $X \overset{\mathsf{CL}*=}{\Longrightarrow} \phi$, *then* $X \vDash^{\mathsf{CL}*=} \phi$.

3.7.2 COMPLETENESS

Here we come to the only significant difference between the free and non-free cases:

Definition *Let* T *be a tableau for free first-order classical logic with identity,* b *be a saturated branch on* T*, and* a *be the first name occurring on* b*. Then the canonical model* i_b *induced by* b *is the model* $\langle D, E, \delta \rangle$ *such that*

- D *is the set of equivalence classes of names that occur on* b *under the equivalence relation defined above, and* δ *is defined by*

$$\delta(\alpha) = \begin{cases} [\alpha] & \textit{if } \alpha \textit{ occurs on } b \\ [a] & \textit{otherwise} \end{cases}$$

- E *is the subset of* D *consisting of all those* $[\alpha] \in E$ *such that for no* $\beta \in [\alpha]$ *does* $\langle \mathbb{E}\beta \quad \ominus \rangle$ *occur on* b.
- *For predicates other than identity,* $\langle [\alpha_1], \ldots, [\alpha_n] \rangle \in \delta^+(\Pi)$ *if and only if* $\langle \Pi\alpha_1 \ldots \alpha_n \quad \oplus \rangle$ *or* $\langle \neg\Pi\alpha_1 \ldots \alpha_n \quad \ominus \rangle$ *is on* b*, and* $\langle [\alpha_1], \ldots, [\alpha_n] \rangle \in \delta^-(\Pi)$ *otherwise.*
- $\delta^+(=)$ *is the set of identity pairs of elements of* D*,* $\delta^-(=)$ *is the set of non-identity pairs of elements of* D.

Lemma 36. (Faithfulness) *If* b *is a saturated open branch on a tableau for free first-order classical logic with identity* T*, then* i_b *is faithful to* b.

Theorem 28. (Completeness: FO CL$*$ Tableau Plus Identity)
If $X \vDash^{\mathsf{CL}=*} \phi$ *then* $X \overset{\mathsf{CL}=*}{\Longrightarrow} \phi$.

3.8 MODAL LOGICS WITHOUT IDENTITY

The addition of *worlds* and *universes* makes the complexity of our soundness and completeness proofs increase substantially, but the overall structure of the proofs remains the same. Here, in outline, is the barest sketch of how these results are proved, leaving almost all details to the reader.

3.8.1 SOUNDNESS

We begin, as always, by proving that the tableau rules preserve faithfulness. Most details are omitted, as it is assumed that by this point the reader has gained sufficient familiarity with the process of proving soundness and completeness to fill in the remaining details with ease.

Lemma 37. (Denotation) *Let $A(x_1,\ldots,x_n)$ be an open formula with n free variables. Let $\nu_1 = \langle \mathcal{U}, D, \mathcal{E}, \delta_1\rangle$ and $\nu_2 = \langle \mathcal{U}, D, \mathcal{E}, \delta_2\rangle$ be models and $\alpha_1,\ldots,\alpha_n$ and $\beta_1,\ldots,\beta_n$ be lists of names such that*

(a) For every predicate Π occurring in A, $\delta_1^+(\Pi) = \delta_2^+(\Pi)$ and $\delta_1^-(\Pi) = \delta_2^-(\Pi)$ at each world w,
(b) For every name a occurring in A, $\delta_1(a) = \delta_2(a)$.
(c) $\delta_1(\alpha_1) = \delta_2(\beta_1), \delta_1(\alpha_2) = \delta_2(\beta_2), \ldots, \delta_1(\alpha_n) = \delta_2(\beta_n)$.

Then $\nu_1 \vDash_1 A(\alpha_1/x_1,\ldots,\alpha_n/x_n)$ iff $\nu_2 \vDash_1 A(\beta_1/x_1,\ldots,\beta_n/x_n)$.

Proof As usual, we proceed by induction. The atomic cases and the inductive hypothesis are as in the previous iterations of the lemma. Here we examine the case involving necessity, the other cases being left to the reader.

- Suppose $A(x_1,\ldots,x_n) = \Box B(x_1,\ldots,x_n)$. Then by truth conditions for the necessity operator, $\nu_1 \vDash_1 A(x_1,\ldots,x_n)$ iff for each $w \in \mathcal{U}$, $[\mathcal{U}, w] \vDash_1 B(x_1,\ldots,x_n)$. But B is less complex than A, so the inductive hypothesis applies. Thus $\nu_1 \vDash_1 A(x_1,\ldots,x_n)$ iff $\nu_2 \vDash_1 A(x_1,\ldots,x_n)$. □

Lemma 38. (Locality) *Let A be a sentence and suppose $\nu_1 = \langle \mathcal{U}, D, \mathcal{E}, \delta_1\rangle$ and $\nu_2 = \langle \mathcal{U}, D, \mathcal{E}, \delta_2\rangle$ are models that satisfy the following conditions:*

(a) For every predicate Π occurring in A, $\delta_1^+(\Pi) = \delta_2^+(\Pi)$ and $\delta_1^-(\Pi) = \delta_2^-(\Pi)$ at each world w.
(b) For every name α occurring in A, $\delta_1(\alpha) = \delta_2(\alpha)$.

Then $\nu_1 \vDash_1 A$ iff $\nu_2 \vDash_1 A$.

Proof Omitted, left as an exercise. □

Lemma 39. (Faithfulness) *If T is a tableau for free first-order modal logic without identity, b is a branch in T, and $f(b)$ is non-empty, then when we extend b using one of our tableau rules, there will be a $v \in f(b)$ that is faithful to at least one of the resulting branches.*

Proof Omitted, left as an exercise. □

The proofs of the following lemmas are exactly the same as the proofs of the corresponding lemmas in Section 4.1, so are omitted.

Lemma 40. (Soundness – Closure) *If b is a branch that closes with the $\times_1$-rule, then no model v is faithful to b.*

Lemma 41. (Soundness – Closure) *If b is a branch that closes with the $\times_2$-rule, then no* K3*-model (and hence no* CL*-model) v is faithful to b.*

Lemma 42. (Soundness – Closure) *If b is a branch that closes with the $\times_3$-rule, then no* LP*-model (and hence no* CL*-model) v is faithful to b.*

Soundness follows in the by-now-expected way, where 'FOFM' abbreviates 'first-order free modal':

Theorem 29. (Soundness: FOFM FDE Tableau Sans Identity)
If $X \overset{\mathsf{FDE}}{\Longrightarrow} \phi$, then $X \vDash^{\mathsf{FDE}} \phi$.

Proof Suppose $X \nvDash^{\mathsf{FDE}} \phi$. We will prove that $X \overset{\mathsf{FDE}}{\not\Longrightarrow} \phi$.

Since $X \nvDash^{\mathsf{FDE}} \phi$, there is an FDE-model v such that v satisfies X but $v \nvDash_1 \phi$. Such a model v is faithful to the initial tableau T for $X \therefore \phi$. Then by Lemma 1, no matter how we extend T, v will remain faithful to some branch b of T. Suppose now that $X \overset{\mathsf{FDE}}{\Longrightarrow} \phi$. Given this, each branch of T must end with $\times_1$. Thus, b ends with $\times_1$. But this contradicts Lemma 28, so we must have that $X \overset{\mathsf{FDE}}{\not\Longrightarrow} \phi$. □

Theorem 30. (Soundness: FOFM K3 Tableau Sans Identity)
If $X \overset{\mathsf{K3}}{\Longrightarrow} \phi$, then $X \vDash^{\mathsf{K3}} \phi$.

Proof Omitted, left as an exercise. □

Theorem 31. (Soundness: FOFM LP Tableau Sans Identity)
If $X \overset{\mathsf{LP}}{\Longrightarrow} \phi$, then $X \vDash^{\mathsf{LP}} \phi$.

Proof Omitted, left as an exercise. $\square$

Theorem 32. (Soundness: FOFM CL Tableau Sans Identity)
If $X \overset{\mathsf{CL}}{\Longrightarrow} \phi$*, then* $X \overset{\mathsf{CL}}{\models} \phi$.

Proof Omitted, left as an exercise. $\square$

3.8.2 COMPLETENESS

Next we prove completeness. First, two definitions:

Definition *If T is a free first-order modal tableau, b is a saturated branch on T, and a is the first name that is on b, then the canonical positive model i_b^+ induced by b is the quadruple $\langle \mathcal{U}, D, \mathcal{E}, \delta \rangle$ such that*

- *(i)* *$\mathcal{U}$ is the set of all indices i such that a node of the form $\langle A \quad i \quad \oplus \rangle$ or $\langle A \quad i \quad \ominus \rangle$ is on b for some sentence A.*
- *(ii)* *D is the set of all names that occur on b,*
- *(iii)* *E_w is the subset of D consisting of all names t such that $\langle \mathbb{E}t \quad w \quad \ominus \rangle$ isn't on b, and*
- *(iv)* *δ is defined by*

$$\delta(\alpha) = \begin{cases} \alpha & \text{if } \alpha \text{ occurs on } b \\ a & \text{otherwise} \end{cases}$$

- *(v)* $\delta_w^+(\Pi) = \{\langle \alpha_1, \ldots, \alpha_n \rangle : \langle \Pi\alpha_1 \ldots \alpha_n \quad w \quad \oplus \rangle \text{ occurs on } b\}$
- *(vi)* $\delta_w^-(\Pi) = \{\langle \alpha_1, \ldots, \alpha_n \rangle : \langle \neg\Pi\alpha_1 \ldots \alpha_n \quad w \quad \oplus \rangle \text{ occurs on } b\}$

Definition *If T is a free first-order modal tableau, b is a saturated branch on T, and a is the first name that is on b then the canonical negative model i_b^- induced by b is the quadruple $\langle \mathcal{U}, D, \mathcal{E}, \delta \rangle$ such that*

- *(i)* *$\mathcal{U}$ is the set of all indices i such that a node of the form $\langle A \quad i \quad \oplus \rangle$ or $\langle A \quad i \quad \ominus \rangle$ is on b for some sentence A.*
- *(ii)* *D is the set of all names that occur on b,*
- *(iii)* *E_w is the subset of D consisting of all names t such that $\langle \mathbb{E}t \quad w \quad \ominus \rangle$ isn't on b, and*
- *(iv)* *δ is defined by*

$$\delta(\alpha) = \begin{cases} \alpha & \text{if } \alpha \text{ occurs on } b \\ a & \text{otherwise} \end{cases}$$

(v) $\delta^{+}_{w}(\Pi) = \{\langle\alpha_1,\ldots,\alpha_n\rangle : \langle\neg\Pi\alpha_1\ldots\alpha_n \quad w \quad \ominus\rangle$ *does not occur on* $b\}$

(vi) $\delta^{-}_{w}(\Pi) = \{\langle\alpha_1,\ldots,\alpha_n\rangle : \langle\Pi\alpha_1\ldots\alpha_n \quad w \quad \ominus\rangle$ *does not occur on* $b\}$

Lemma 43. (Faithfulness) *For saturated branches* b*,* i^{+}_{b} *and* i^{-}_{b} *are faithful to* b *iff* b *does not close with* $\times_1$*.*

Proof Omitted, left as an exercise. □

The proofs of the following results are exactly as in the case for non-free first-order logic without identity, so omitted.

Lemma 44. (Completeness – Closure) *For saturated branches* b*, if* b *does not close with* $\times_2$*, then* i^{+}_{b} *is a free, modal* K3*-model.*

Lemma 45. (Completeness – Closure) *For saturated branches* b*, if* b *does not close with* $\times_3$*, then* i^{-}_{b} *is a free, modal* LP*-model.*

Lemma 46. (Completeness – Closure) *Suppose* b *is saturated. Define* i_b *to be the union of* i^{+}_{b} *and* i^{-}_{b} *in the by-now obvious way. Then* i_b *is faithful to* b *iff* b *does not close with* $\times_1$*,* $\times_2$ *or* $\times_2$*.*

Completeness follows from these results immediately.

Theorem 33. (Completeness: FOFM FDE Tableau Sans Identity)
If $X \models^{\mathsf{FDE}} \phi$ *then* $X \overset{\mathsf{FDE}}{\Longrightarrow} \phi$*.*

Theorem 34. (Completeness: FOFM K3 Tableau Sans Identity)
If $X \models^{\mathsf{K3}} \phi$ *then* $X \overset{\mathsf{K3}}{\Longrightarrow} \phi$*.*

Theorem 35. (Completeness: FOFM LP Tableau Sans Identity)
If $X \models^{\mathsf{LP}} \phi$ *then* $X \overset{\mathsf{LP}}{\Longrightarrow} \phi$*.*

Theorem 36. (Completeness: FOFM CL Tableau Sans Identity)
If $X \models^{\mathsf{CL}} \phi$ *then* $X \overset{\mathsf{CL}}{\Longrightarrow} \phi$*.*

3.9 FREE FIRST-ORDER CLASSICAL MODAL LOGIC WITH IDENTITY

By this point, the reader has hopefully caught on to the pattern to be followed. As in the previous iterations, again the only difference to be accounted for here involves (a) keeping track of the identity rules in the soundness proof and (b) using equivalence classes of names in the canonical models. Everything goes exactly as one expects (though a dedicated reader really ought to check this claim!)

3.10 FURTHER READING

One could be forgiving for drawing the following slightly-mistaken conclusion from soundness and completeness results: Given any argument (in one of the languages we've examined in *Logic: The Basics*), one can use a tableau to determine whether or not it is valid.

As it turns out, we've in fact only shown half of this: given any argument, the tableau for that argument will close if, and only if, that argument is valid. However, what we have *not* given is any procedure for determining whether a given tableau will close *other than by simply trying*. But here's the problem: suppose we've been working on a tableau for a few hours, chugging out new nodes and using up old ones as we go. Suppose we have, by this point, produced a couple of hundred lines, dozens of branches, and only a handful of these have closed. We might be able to survey our work and *suspect* that the tableau in fact does not close. But unless *every branch* in the tableau is saturated, we cannot be sure of this. And one can easily come up with arguments that give rise to tableau with infinitely long branches – that is, branches that are neither saturated nor closed after any finite number of steps.

Thus, there is a sense in which tableaux are sometimes more reticent than we might wish. For any argument that *is* valid, a tableau will tell us that it's valid, given enough time (and patience, elbow grease, etc.). On the other hand, it is not *always* the case that a tableau for an *invalid* argument will give rise in a finite number of steps to saturated open branches from which counterexamples can be constructed.

This discussion has been very general. It turns out some (but not most!) of the tableau systems we've given in this book do in fact suffice both to show when arguments are valid and when arguments are not valid (and always in a finite number of steps). But discussing this in any rigorous way would take up more space than we have available. Good starting points for results in this area (results known as *decidability theorems*) include Smullyan (1968), which also presents logic using tableau systems similar to ours but covers a different range of topics. On the other hand, Jeffrey (1967) is a book directly devoted to the *meta*theory of (classical) formal logic, and makes a great place for a beginner to start examining such results.

In addition to decidability, readers may wish to see proofs of soundness and completeness for systems other than those in this book. The near-indespensible Priest (2008) is a good resource to that end. For more advanced topics, see Anderson and Belnap (1975) and Anderson *et al.* (1992).

3.11 EXERCISES

0. Fill in the details left out in the appendix. If your instructor assigns you this problem, be aware that it is likely to take you several days of continuous work to do it, as there is a large amount of information to be supplied. You're probably justified in organizing a small uprising at this point, and demanding that you only be assigned a portion of this work.
1. Prove by induction that any well-formed-formula of any of our modal languages has the same number of left and right parentheses.
2. Prove by induction that if P is a propositional FDE-sentence with no negations and v is a propositional FDE-model such that $v \vDash_0 A$ for every atomic sentence A that occurs in P, then $v \vDash_0 P$.
3. Look back at our proofs of the consistency of K3 and of the soundness of LP. Notice that they are loose inductions. With the tools you now know, do a more rigorous induction in each case.
4. Extend the proofs of soundness and completeness for first-order classical logic with identity to proofs of soundness and completeness for each of the other first-order logics with identity.

5. One might think that identity should be treated differently in the different logics. Explore this. Here are some pointers:

    - One might expect that in K3, identity would be allowed to be incomplete – perhaps the antiextension of identity oughtn't include every non-identity pair. Show that not all the identity rules are *sound* if we allow this.
    - Similarly, one might think the LP-antiextension of identity should be allowed to include some identity pairs. Again, show some of the identity rules are not sound if we allow this.

6. A natural way around these difficulties is to only allow, in each logic, the identity rules that are sound for that logic. Explore this. Here are some pointers:

    - In the previous problem you saw that some of the identity rules are unsound with respect to a logic that allows identity to be an incomplete predicate. Show that the tableau system that results from restricting to only those rules that *are* sound with respect to such a system is not *complete* with respect to that system.
    - Do a similar examination for LP.

7. Think about the previous two problems and write a thoughful essay about what they say about the relation of identity.

NOTES

1. We use 'saturated' rather than, e.g., 'completed' for the following reasons:

    - Graham Priest uses this word in (Priest, 2008), and Priest is a worthy expert to defer to in such matters.
    - 'Completed' has connotations that 'saturated' does not. For example, one may think that a branch is *completed* once it has closed. Whether or not this is the case, it is clear a branch can be closed and yet fail to be *saturated*.
    - At least one of the authors appreciates that 'saturated' has a chemical feel to it. When water is saturated with salt, it

contains as much salt as it can, all broken down into its smallest atomic components. This very closely parallels what happens to the sentences that occur on a saturated branch of a tableau.

2. Previously we wrote the extension and antiextension of predicates using the notation $\mathcal{E}^{+}_{\Pi}$ and $\mathcal{E}^{-}_{\Pi}$. That notation will be inconvenient in this appendix because sometimes we will need to consider two different models at the same time. So, just for the appendix, we adopt this alternative notation for extensions and antiextensions.

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