Third report
Financial Mathematics for data science
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The aim of this report is to investigate the convergence of the Cox-Ross-Rubinstein and the Leisen-Reimer discrete model to the Black and Scholes model but first a quick introduction to those models.

The Black Scholes model is probably the most popular option pricing framework due to its easily implementation.
The formula presents a theoretical estimate of the price of European-style options independently of the risk of the underlying security while future payoffs from options can be discounted using the risk-neutral rate.

$$
\begin{aligned}
c &= S_0 e^{-qT} N(d_1) - X e^{-rT} N(d_2) \\
p &= X e^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1) \\
\text{where} \quad d_1 &= \frac{\ln(S_0 / X) + (r - q + \sigma^2 / 2)T}{\sigma \sqrt{T}} \\
d_2 &= \frac{\ln(S_0 / X) + (r - q - \sigma^2 / 2)T}{\sigma \sqrt{T}} \\
d_2 &= d_1 - \sigma \sqrt{T}
\end{aligned}
$$

Where c = Call option value, p = Put option value, S=Current stock (or other underlying) price, K *or* X=Strike price, r=Risk-free interest rate, q = dividend yield, T=Time to maturity and N denotes taking the normal cumulative probability, the cost of carry b=r-q has been replaced with r.

The **Cox Ross and Rubinstein** model for American/European Options.
The binomial options pricing model furnishes a numerical method for the valuation of options. The model implements a "discrete-time" (lattice based) method to approximate closed-form Black–Scholes formula.
It can be tuned to estimate the American analogue the key difference between the two main styles of options relates to exercise behaviour.
When the CRR estimation has been set to European there should be evidence of CRR converging to Black Scholes as we increase the step size, it is trivial to catch that convergence is oscillatory and the subsequent rate of convergence tends to rapidly slow down with the number of steps.

$$u = e^{\sigma \sqrt{\Delta t}} \quad \text{and} \quad d = e^{-\sigma \sqrt{\Delta t}}$$

Fundamental equations to construct the binomial tree following the CRR method.
The rate of convergence was later studied by Leisen (1998) who showed that the CRR tree converges with order 1.

The **Leisen Reimer** model is known for its rapid converge, it is also possible to deduce from the plots the higher quality of the convergence itself that, instead of its CRR counterpart, is not oscillatory.

$$u = e^{(r-q)\Delta t} \cdot \frac{p'}{p}$$

$$d = e^{(r-q)\Delta t} \cdot \frac{1-p'}{1-p}$$

The above formulas work for the Leisen-Reimer method, the exponent term $e^{(r-q)\Delta t}$ has already been studied, with the cost of carry b=r-q. It can be interpreted as net cost of holding the underlying security over one step, as Δt is the duration of one step in years, calculated as *t/n*.
In each formula this term is multiplied by a ratio of two probabilities.
We can calculate p' using the Peizer-Pratt formula as *p*, using the Black-Scholes d_1 (instead of d_2) as argument.

$$p' = \mathrm{h}^{-1}(d_1)$$

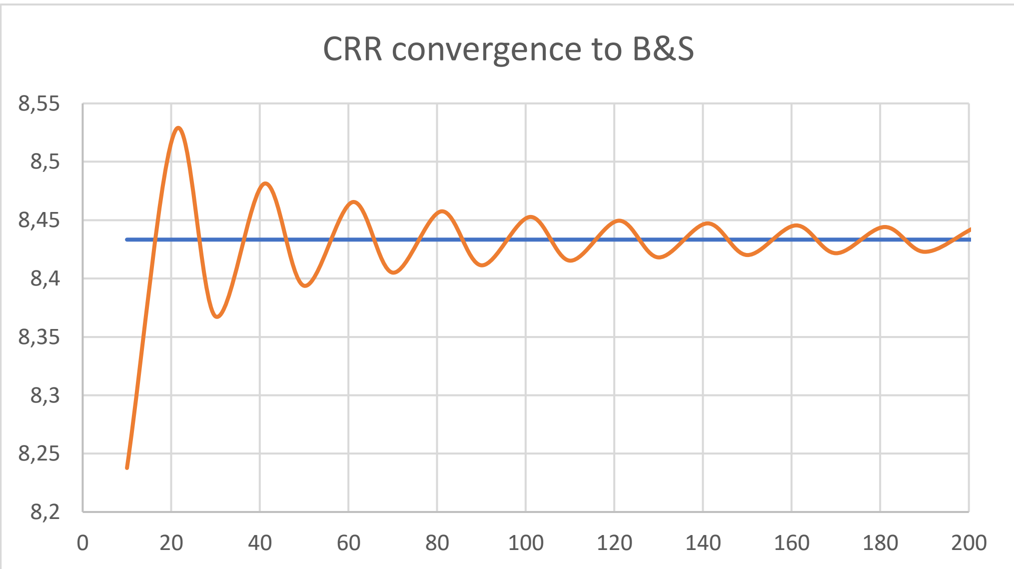
The Leisen-Reimer macro got a major optimization thanks to the truncation of the zero region of the tree that becomes a substantial compontents of the binomial lattice as the step number increases.

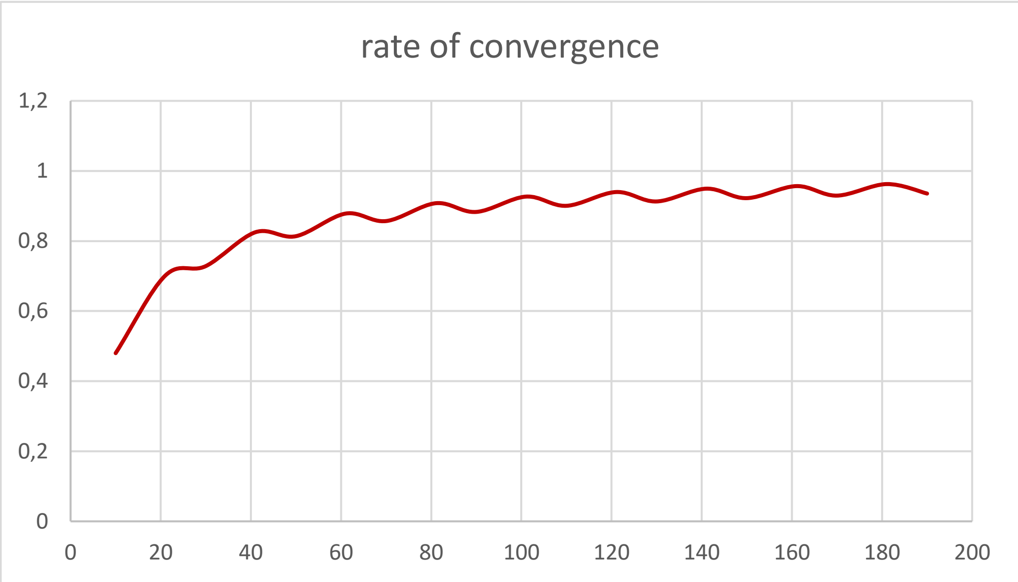
S	r	vol	T	K	n	opt	ex
100	1%	20%	1	100	10	C	E

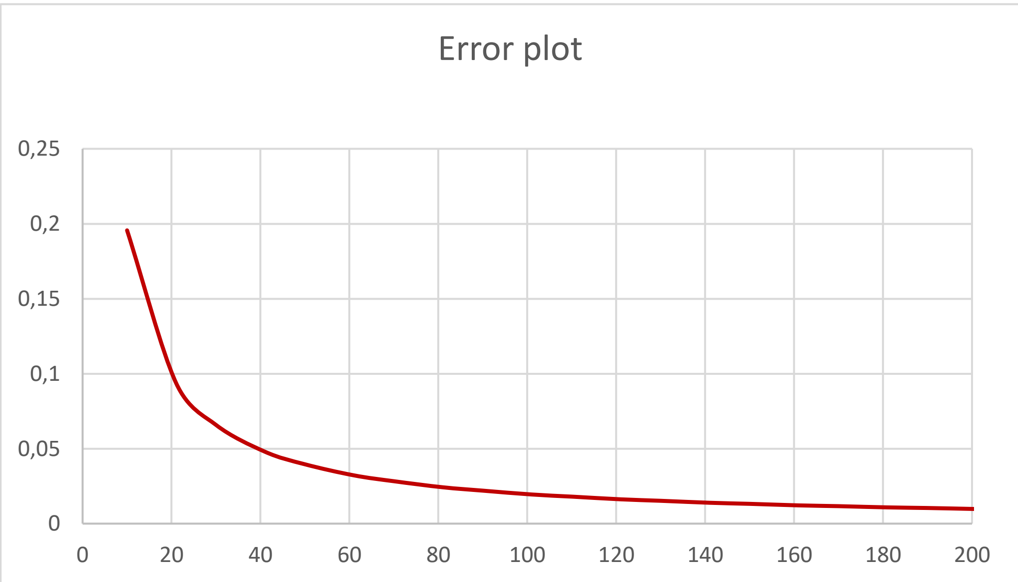
CRRTree(Spot, K, T, rf, vol, n, OpType, ExType)

BSCall(Stock, Exercise, Rate, Sigma, Time)

n	CRRTree	B&S	rate	error
10	8,2377065	8,433319	0,479987	0,195612
21	8,52721	8,433319	0,701271	0,093891
30	8,3674755	8,433319	0,728492	0,065843
41	8,4812849	8,433319	0,825089	0,047966
50	8,3937423	8,433319	0,813856	0,039576
61	8,4655282	8,433319	0,878311	0,032209
70	8,4050287	8,433319	0,857013	0,02829
81	8,4575636	8,433319	0,907917	0,024245
90	8,4113064	8,433319	0,88306	0,022012
101	8,4527569	8,433319	0,92677	0,019438
110	8,4153039	8,433319	0,900489	0,018015
121	8,4495408	8,433319	0,939828	0,016222
130	8,4180727	8,433319	0,912969	0,015246
141	8,4472378	8,433319	0,949408	0,013919
150	8,4201038	8,433319	0,922347	0,013215
161	8,4455074	8,433319	0,956735	0,012189
170	8,4216573	8,433319	0,92965	0,011661
181	8,4441597	8,433319	0,96252	0,010841
190	8,422884	8,433319	0,9355	0,010435
201	8,4430803	8,433319		0,009762



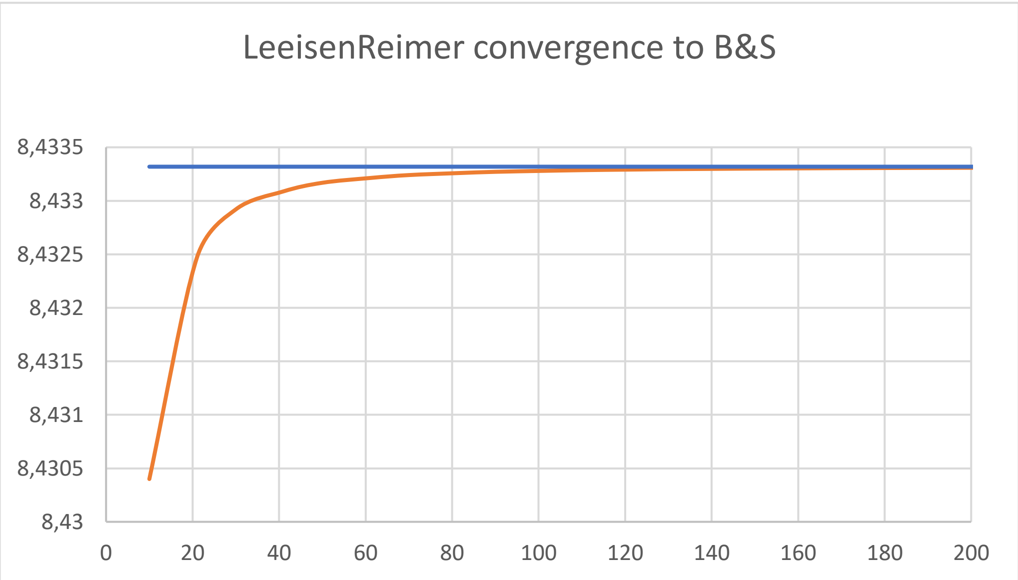

rate of convergence
1,2
1
0,8
0,6
0,4
0,2
0
0
20
40
60
80
100
120
140
160
180
200

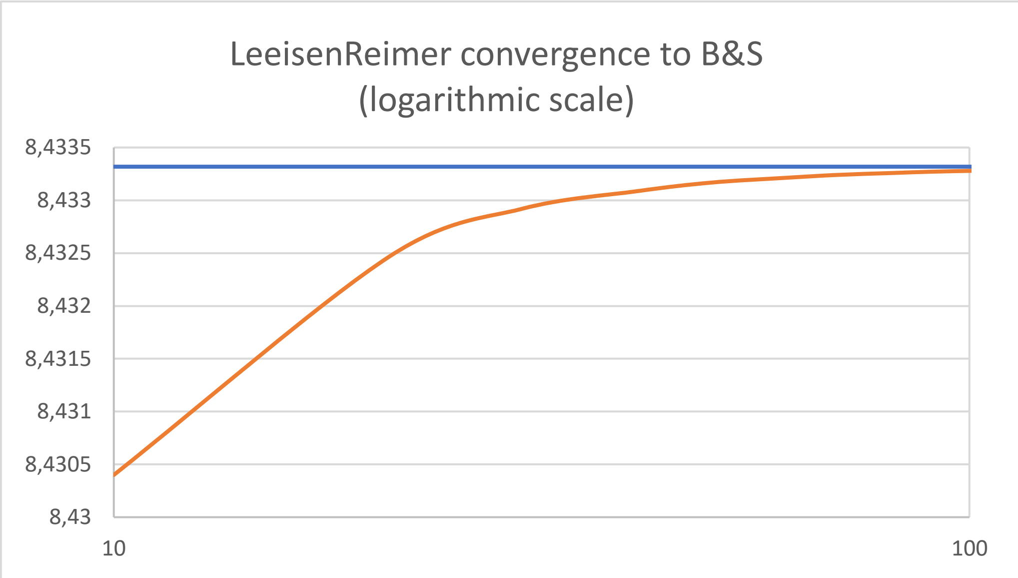

Error plot
0,25
0,2
0,15
0,1
0,05
0
0
20
40
60
80
100
120
140
160
180
200

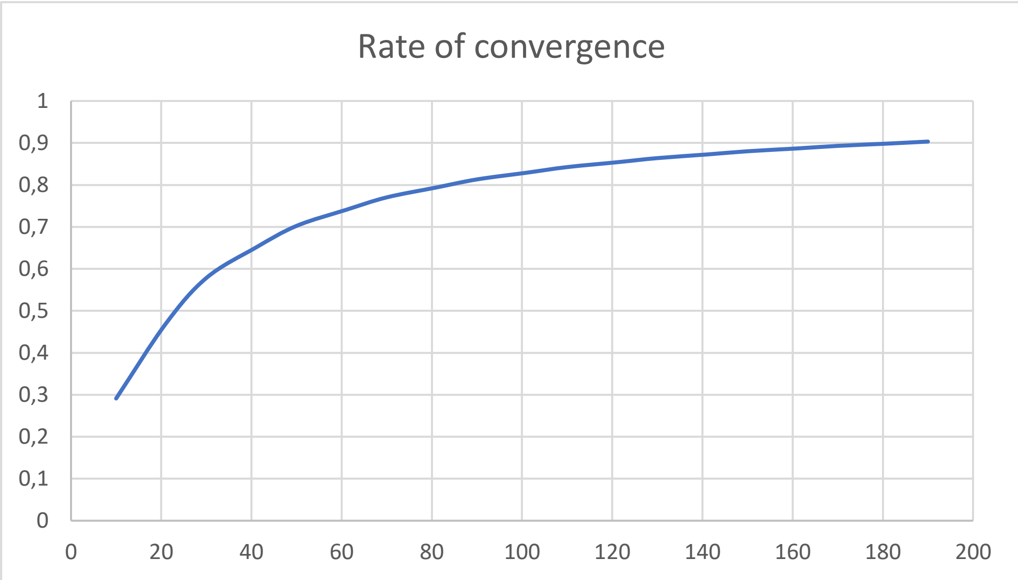
b = cost of carry

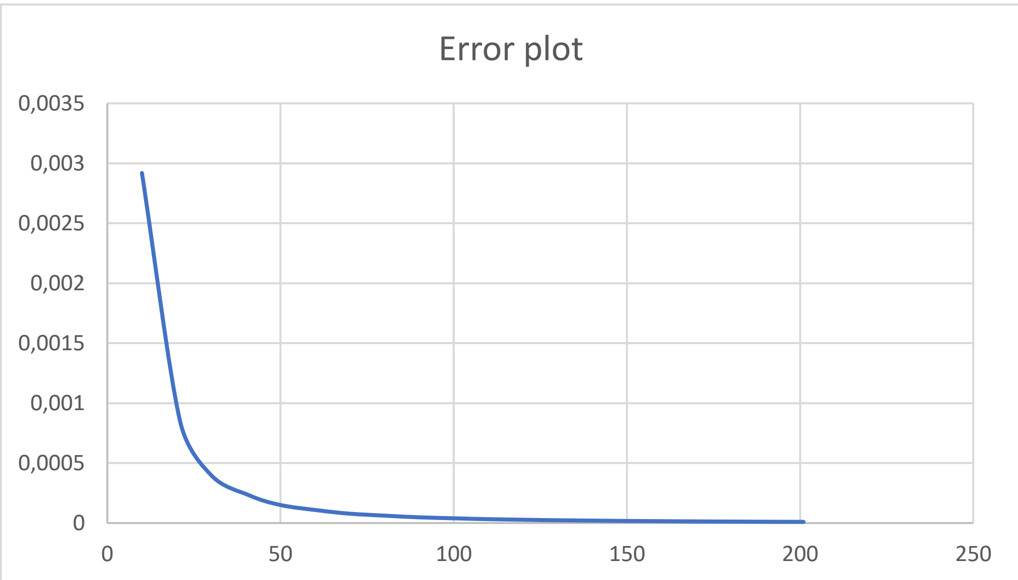
LeisenReimerTrunc(AmeEur, CallPut, S , K , T , r , b, v , n)

n	leis-reim	B&S	rate	error
10	8,4303998	8,433319	0,291226	0,002919
21	8,4324686	8,433319	0,469248	0,00085
30	8,4329198	8,433319	0,578395	0,000399
41	8,433088	8,433319	0,650945	0,000231
50	8,4331685	8,433319	0,702411	0,00015
61	8,4332132	8,433319	0,740741	0,000105
70	8,4332405	8,433319	0,770368	7,81E-05
81	8,4332585	8,433319	0,793941	6,02E-05
90	8,4332709	8,433319	0,813138	4,78E-05
101	8,4332798	8,433319	0,829071	3,89E-05
110	8,4332865	8,433319	0,842505	3,22E-05
121	8,4332915	8,433319	0,853985	2,71E-05
130	8,4332955	8,433319	0,863907	2,32E-05
141	8,4332987	8,433319	0,872568	2E-05
150	8,4333012	8,433319	0,880193	1,75E-05
161	8,4333033	8,433319	0,886959	1,54E-05
170	8,433305	8,433319	0,893001	1,36E-05
181	8,4333065	8,433319	0,898431	1,22E-05
190	8,4333077	8,433319	0,903337	1,09E-05
201	8,4333088	8,433319		9,89E-06




LeeisenReimer convergence to B&S
(logarithmic scale)
8,4335
8,433
8,4325
8,432
8,4315
8,431
8,4305
8,43
10
100


Rate of convergence
1
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
0
20
40
60
80
100
120
140
160
180
200


Error plot
0,0035
0,003
0,0025
0,002
0,0015
0,001
0,0005
0
0
50
100
150
200
250