Homework 1

Deadline: 14:20 on October 24, 2025

Exercise 1.

a. (5 pts) Please order the following functions asymptotically:

$$\frac{n}{\log^3 n},\ 64^{\sqrt{n}},\ \log^{10} n,\ \log(n!),\ n\sqrt{n},\ \log(3^n),\ n\log^2 n, 2^n$$

b. (5 pts for each) Please prove or disprove the following statements:

1. $n^{\frac{1}{2}} = O\left(n^{\frac{1}{3}}\right)$
2. $3^n = \Omega\left(27^{\sqrt{n}}\right)$

Exercise 2. Please analyze the time complexity of the following codes

a. (5 pts)

```
for ( int i = 1; i ≤ n; i = i + 1 ) {
        for ( int j = 1; j ≤ √i; j = j + 1) {
                ;
        }
}
```

b. (5 pts)

```
for ( int i = n; i ≥ 1; i = i − 1) {
        int j = i;
        while ( j ≥ 2 ) {
                j = √j;
        }
}
```

Exercise 3. Please analyze the following recursive functions asymptotically:

a. (5 pts)

$$T(n) := \begin{cases} T\left(\frac{n}{6}\right) + T\left(\frac{n}{4}\right) + \frac{n}{2} & n > 1 \\ 1 & n = 1 \end{cases}$$

b. (5 pts)

$$T(n) := \begin{cases} 2 \cdot T(n/2) + n\log n & n > 1 \\ 1 & n = 1 \end{cases}$$

c. (5 pts)

$$T(n) := \begin{cases} 4 \cdot T\left(n^{\frac{1}{4}}\right) + \log_2 n & n > 4 \\ 2 & n \leq 4 \end{cases}$$

Exercise 4. (15 pts) Please analyze the expected number of comparisons involving the smallest element during the execution of *QuickSort* on n distinct numbers.
Hint: Your analysis may use a recursive function or use random variables.

Exercise 5. (10 pts) Consider a set S of n integers in the range $[0, n^{\log_2 \log_2 n} - 1]$. Please describe how to sort S efficiently and analyze the time complexity. Note that the time complexity of your method has to be $o(n \log n)$.

Exercise 6. (10 pts) You have learned the *QuickSort* algorithm and the analysis of its expected time complexity using random variables. Please analyze the expected time complexity of *QuickSelect* using random variables.
Hint: When analyzing the probability of comparing i and j, you have to consider the relative position of k with respect to i and j, where k is the target, e.g., $i < k < j$, $i < j < k$ or $k < i < j$. Please consider under what situation i and j will be compared.

Exercise 7. (15 pts) Consider an n-character sequence X and an m-character sequence Y. Please develop an algorithm to compute the *length* of a *shortest common supersequence* between X and Y. For example, if $X = \langle A, T, C, G, T\rangle$ and $Y = \langle T, G, A, C\rangle$, a shortest common supersequence between X and Y is $\langle A, T, C, G, A, C, T\rangle$, and its length is 7.
Note: You are not allowed to apply the longest common subsequence algorithm. An $\omega(n \cdot m)$-time algorithm will not receive full points.

Exercise 8. (10 pts) Consider an n-character sequence X and an m-character sequence Y. There are three kinds of operations:

- Insert: insert any character at any position.
- Delete: delete a character.
- Replace: replace a character with another character.

The cost of an Insert or a Delete is two, and the cost of a Replace is three. Please develop an algorithm to compute the minimum cost of converting X into Y using these three kinds of operations.
Note: An $\omega(n \cdot m)$-time algorithm will not receive full points.

Recommended Exercises:

Chapter 3 : P 3-2, P 3-3(a), P 3-4.

Chapter 4 : E 4.3-1, E 4.4-1, E 4.4-4., E 4.5-1, P 4-4.

Chapter 6 : E 6.1-8, E 6.3-4, E 6.3-4, P 6-1.

Chapter 7 : E 7.2-5, E 7.3-2, E 7.4-4, P 7-4.

Chapter 8 : E 8.2-6, E 8.3-5, E 8.4-2, P 8-2

Chapter 9 : E 9.1-2, E 9.2-3, E 9.3-3, E 9.3-6, P 9-1.