Binary Relations Homework. Dmitry Semenov, M3100, ISU 409537

1. (a) $M_1 = \mathbb{R}$
   $xR_1y \ \leftrightarrow |x-y| \leq 1$

   (b) $M_2 = P(\{a,b,c\})$ = $\{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$
   $R_2 = " \subseteq "$

   (c) $M_3 = \{a,b,c,d\}$

   $$||R_3|| = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

   (d) $M_4 = \{"rock", "scissors", "paper"\}$
   $R_4 = \{\langle x,y \rangle | x \ beats \ y\}$

	R1	R2	R3	R4
Reflexive	$true$	$true$	$false : (x) = (a)$	$false : (x) = ("rock")$
Irreflexive	$false : (x) = (1)$	$false : (x) = (\{a\})$	$true$	$true$
Coreflexive	$false : (x,y) = (1,2)$	$false : (x,y) = (\{a\}, \{a,b\})$	$false : (x,y) = (a,b)$	$false : (x,y) = ("paper", "rock")$
Symmetric	$true$	$false : (x,y) = (\{a\}, \{a,b\})$	$false : (x,y) = (a,d)$	$false : (x,y) = ("paper", "rock")$
Antisymmetric	$false : (x,y) = (1,2)$	$true$	$true$	$true$
Asymmetric	$false : (x,y) = (1,2)$	$false : (x,y) = (\{a\}, \{a\})$	$true$	$true$
Transitive	$false : (x,y,z) = (0,1,2)$	$true$	$false : (x,y,z) = (c,a,d)$	$false : (x,y,z) = ("rock", "scissors", "paper")$
Antitransitive	$false : (x,y,z) = (0,0.5,1)$	$false : (x,y,z) = (\{\varnothing\}, \{a\}, \{a,b\})$	$false : (x,y,z) = (a,b,d)$	$true$
Semiconnex	$false : (x,y) = (0,2)$	$false : (x,y) = (\{a\}, \{c\})$	$true$	$true$
Connex	$false : (x,y) = (0,2)$	$false : (x,y) = (\{a\}, \{c\})$	$false : (x,y) = (a,b)$	$false : (x,y) = ("rock", "rock")$
Left Euclidean	$false : (x,y,z) = (1,0,2)$	$false : (\{a,b\}, \{a\}, \{b\})$	$false : (x,y,z) = (a,b,c)$	$false : (x,y,z) = ("scissors", "rock", "rock")$
Right Euclidean	$false : (x,y,z) = (1,0,2)$	$false : (x,y,z) = (\{a\}, \{a,b\}, \{a,c\})$	$false : (x,y,z) = (c,b,a)$	$false : (x,y,z) = ("rock", "scissors", "scissors")$

$*true - valid \ for \ all \ x,y,z \ values \ from \ M$

2. $R \subseteq M^2 \ and \ S \subseteq M^2$

(a) If R and S are reflexive, then $R \cap S$ is so.

Let $a \in M.$

$(a, a) \in R \wedge (a, a) \in S \Rightarrow$
$(a, a) \in R \cap S \Rightarrow R \cap S \; is \; reflexive$

(b) If R and S are symmetric, then $R \cap S$ is so.

Let $(a, b) \in R \cap S.$

$(a, b) \in R \cap S \Rightarrow$
$(a, b) \in R \wedge (a, b) \in S \Rightarrow$
$(b, a) \in R \wedge (b, a) \in S \Rightarrow$
$(b, a) \in R \cap S \Rightarrow R \cap S \; is \; symmetric$

(c) If R and S are transitive, then $R \cap S$ is so.

Let $(a, b) \in R \cap S \wedge (b, c) \in R \cap S.$

$(a, b) \in R \cap S \wedge (b, c) \in R \cap S \Rightarrow$
$(a, b), (b, c) \in R \wedge (a, b), (b, c) \in S \Rightarrow$
$(a, c) \in R \wedge (a, c) \in S \Rightarrow$
$(a, c) \in R \cap S \Rightarrow R \cap S \; is \; transitive$

(d) If R and S are reflexive, then $R \cup S$ is so.

Let $a \in M.$

$(a, a) \in R \vee (a, a) \in S \Rightarrow$
$(a, a) \in R \cup S \Rightarrow R \cup S \; is \; reflexive$

(e) If R and S are symmetric, then $R \cup S$ is so.

Let $(a, b) \in R \cup S.$

$(a, b) \in R \cup S \Rightarrow$
$(a, b) \in R \vee (a, b) \in S \Rightarrow$
$(b, a) \in R \vee (b, a) \in S \Rightarrow$
$(b, a) \in R \cup S \Rightarrow R \cup S \; is \; symmetric$

(f) If R and S are transitive, then $R \cup S$ is so.

Counter-example:

Let $(e, a) \in R \wedge (a, b) \in R \wedge (e, b) \in R$

$(b, c) \in S \wedge (c, d) \in S \wedge (b, d) \in S$

So, $R \cup S$ will contain $(a, b), (b, c)$, but not (a, c).

And hense, $R \cup C \; is \; not \; transitive.$

3. An equinumerosity relation $\sim$ over sets is defined as follows: $A \sim B \leftrightarrow |A| = |B|$.

(a) Show that $\sim$ is an equivalence relation over finite sets.

- $(A, A) \in R \, \forall \, (A \in M) \leftrightarrow |A| = |A|$ - $reflexive$
- $(\forall A, B \in M : (A, B) \in R \leftrightarrow (B, A) \in R) \leftrightarrow ((|A| = |B|) \leftrightarrow (|B| = |A|))$ - $symmetric$
- $(\forall A, B, C \in M : (A, B) \in R \wedge (B, C) \in R \rightarrow (A, C) \in R) \leftrightarrow (|A| = |B| \wedge |B| = |C| \rightarrow |A| = |C|)$ - $transitive$

(b) Show that $\sim$ is an equivalence relation over infinite sets.

For infinite sets, $|A| = |B|$ means there is a bijection between A and B.

Thus, in order to prove the equivalence of the equinumerosity relation, it is necessary to specify a bijection between sets for each of the properties described above.

- The identical mapping $id : A \rightarrow A$ is bijection
- If $f : A \rightarrow B$ is a bijection, then the inverse mapping $f^{-1} : B \rightarrow A$ is a bijection.
- If $f : A \rightarrow B$ and $g : B \rightarrow C$ are bijections, then their composition is $f \bullet g : A \rightarrow C$ is bijection.

(c) Find the quotient set of $\mathrm{P}(\{a,b,c,d\})$ by $\sim$.

$Let\ W = \mathrm{P}(a,b,c,d) =$
$\{\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\}$

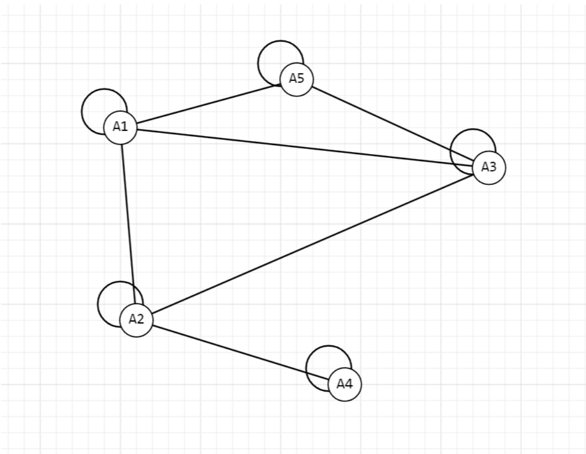
- $[\varnothing]_R = \varnothing$
- $[\{a\}]_R = [\{b\}]_R = [\{c\}]_R = [\{d\}]_R = \{\{a\}, \{b\}, \{c\}, \{d\}\}$
- $[\{a,b\}]_R = [\{a,c\}]_R = [\{a,d\}]_R = [\{b,c\}]_R = [\{b,d\}]_R = [\{c,d\}] =$
  $\{\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}\}$
- $[\{a,b,c\}]_R = [\{a,b,d\}]_R = [\{a,c,d\}]_R = [\{b,c,d\}]_R = \{\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$
- $[\{a,b,c,d\}]_R = \{\{a,b,c,d\}\}$

$W/_R = [W]_R = \{[x]_R : x \in W\} = \{\{a\}, \{a,b\}, \{a,b,c\}, \{a,b,c,d\}\}$

4. Let R_θ be a relation of θ-similarity (clearly, $\theta \in [0;1] \subseteq R$) of finite non-empty sets defined as follows: a set A is said to be θ-similar to B iff the Jaccard index $Jac(A,B)$ = $\frac{|A \cap B|}{|A \cup B|}$ for these sets is at least θ, i.e. $\langle A, B\rangle \in R_\theta \leftrightarrow$ $Jac(A,B) \geq \theta$.

(a) Draw the graph of a relation $R_\theta \subseteq \{A_i\}^2$, where θ = 0.25, $A_1 = \{1,2,5,6\}, A_2 = \{2,3,4,5,7,9\}, A_3 =$ $\{1,4,5,6\}, A_4 = \{3,7,9\}, A_5 = \{1,5,6,8,9\}$.

- $Jac(A_1, A_2) = \frac{|\{2,5\}|}{|\{1,2,3,4,5,6,7,9\}|} = 0.25 \Rightarrow \langle A_1, A_2\rangle \in R_\theta$
- $Jac(A_1, A_3) = \frac{|\{1,5,6\}|}{|\{1,2,4,5,6\}|} = 0.6 \Rightarrow \langle A_1, A_3\rangle \in R_\theta$
- $Jac(A_1, A_4) = \frac{|\{\}|}{|\{1,2,3,5,6,7,9\}|} = 0 \Rightarrow \langle A_1, A_4\rangle \notin R_\theta$
- $Jac(A_1, A_5) = \frac{|\{1,5,6\}|}{|\{1,2,5,6,8,9\}|} = 0.5 \Rightarrow \langle A_1, A_5\rangle \in R_\theta$
- $Jac(A_2, A_3) = \frac{|\{4,5\}|}{|\{1,2,3,4,5,6,7,9\}|} = 0.25 \Rightarrow \langle A_2, A_3\rangle \in R_\theta$
- $Jac(A_2, A_4) = \frac{|\{3,7,9\}|}{|\{2,3,4,5,7,9\}|} = 0.5 \Rightarrow \langle A_2, A_4\rangle \in R_\theta$
- $Jac(A_2, A_5) = \frac{|\{5,9\}|}{|\{1,2,3,4,5,6,7,8,9\}|} = 0.(2) \Rightarrow \langle A_2, A_5\rangle \notin R_\theta$
- $Jac(A_3, A_4) = \frac{|\{\}|}{|\{1,3,4,5,6,7,9\}|} = 0 \Rightarrow \langle A_3, A_4\rangle \notin R_\theta$
- $Jac(A_3, A_5) = \frac{|\{1,5,6\}|}{|\{1,4,5,6,8,9\}|} = 0.5 \Rightarrow \langle A_3, A_5\rangle \in R_\theta$
- $Jac(A_4, A_5) = \frac{|\{9\}|}{|\{1,3,5,6,7,8,9\}|} = 0.14 \Rightarrow \langle A_1, A_4\rangle \notin R_\theta$
- $Jac(A_i, A_i) = \frac{|A_i|}{|A_i|} = 1 \Rightarrow \langle A_i, A_i\rangle \in R_\theta\ \forall\ i \in [1;5], i \in \mathbb{N}$



(b) Determine whether θ-similarity is a tolerance relation.

- $Jac(A, A) = \frac{|A \cap A|}{|A \cup A|} = \frac{|A|}{|A|} = 1 \Leftrightarrow \forall \theta \in [0; 1]\ Jac(A, A) \geq \theta \Leftrightarrow reflexive$
- $Let\ (A, B) \in R_\theta \Rightarrow Jac(A, B) \geq \theta \Rightarrow \frac{|A \cap B|}{|A \cup B|} \geq \theta \Rightarrow \frac{|B \cap A|}{|B \cup A|} \geq \theta \Rightarrow Jac(B, A) \geq \theta \Rightarrow (B, A) \in R_\theta \Rightarrow symmetric$

$\Rightarrow tolerance\ relation$

(c) Determine whether θ-similarity is an equivalence relation.

- Based on the previous paragraph, we can conclude about reflexive and symmetric properties
- Counterexample to the transitivity property: $A = \{1\}, B = \{1, 2\}, C = \{2\}, \theta = 0.239$. $Jac(A, B) = 0.5 \Rightarrow (A, B) \in R_\theta$. $Jac(B, C) = 0.5 \Rightarrow (B, C) \in R_\theta$. $Jac(A, C) = 0 \Rightarrow (A, B) \notin R_\theta \Rightarrow not\ transitive$
- It is worth noting that $\theta-$similarity can be transitive when $\theta \in \{0\} \cup (0.5; 1]$.

$\Rightarrow not\ always\ equivalence\ relation$

5. The characteristic function f_S of a set S is defined as follows:

$$f_S(x) = \begin{cases} 1 \ \ if\ x \in S \\ 0 \ \ if\ x \notin S \end{cases}$$

Let A and B be finite sets. Show that for all $x \in \mathbb{U}$:

(a) $f_{\overline{A}}(x) = 1 - f_A(x)$

- Let $x \in A \Rightarrow x \notin \overline{A}$. $f_{\overline{A}}(x) = 0 = (1 - f_A(x) = 1 - 1 = 0)$
- Let $x \notin A \Rightarrow x \in \overline{A}$. $f_{\overline{A}}(x) = 1 = (1 - f_A(x) = 1 - 0 = 1)$

(b) $f_{A \cap B}(x) = f_A(x) * f_B(x)$

- Let $x \in A \wedge x \in B \Rightarrow x \in (A \cap B). f_{A \cap B} = 1 = (f_A(x) * f_B(x) = 1 * 1 = 1)$
- Let $x \in A \wedge x \notin B \Rightarrow x \notin (A \cap B). f_{A \cap B} = 0 = (f_A(x) * f_B(x) = 1 * 0 = 0)$
- Let $x \notin A \wedge x \notin B \Rightarrow x \notin (A \cap B). f_{A \cap B} = 0 = (f_A(x) * f_B(x) = 0 * 0 = 0)$
- Let $x \notin A \wedge x \in B \Rightarrow x \notin (A \cap B). f_{A \cap B} = 0 = (f_A(x) * f_B(x) = 0 * 1 = 0)$

(c) $f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x) * f_B(x)$

- Let $x \in A \wedge x \in B \Rightarrow x \in (A \cup B). f_{A \cup B} = 1 = (f_A(x) + f_B(x) - f_A(x) * f_B(x) = 1 + 1 - 1 = 1)$
- Let $x \in A \wedge x \notin B \Rightarrow x \in (A \cup B). f_{A \cup B} = 1 = (f_A(x) + f_B(x) - f_A(x) * f_B(x) = 1 + 0 - 0 = 1)$
- Let $x \notin A \wedge x \in B \Rightarrow x \in (A \cup B). f_{A \cup B} = 1 = (f_A(x) + f_B(x) - f_A(x) * f_B(x) = 0 + 1 - 0 = 1)$
- Let $x \notin A \wedge x \notin B \Rightarrow x \notin (A \cup B). f_{A \cup B} = 0 = (f_A(x) + f_B(x) - f_A(x) * f_B(x) = 0 + 0 - 0 = 0)$

(d) $f_{A \oplus B}(x) = f_A(x) + f_B(x) - 2 * f_A(x) * f_B(x)$

- Let $x \in A \wedge x \in B \Rightarrow x \notin (A \oplus B). f_{A \oplus B} = 0 = (f_A(x) + f_B(x) - 2 * f_A(x) * f_B(x) = 1 + 1 - 2 = 0)$
- Let $x \in A \wedge x \notin B \Rightarrow x \in (A \oplus B). f_{A \oplus B} = 1 = (f_A(x) + f_B(x) - 2 * f_A(x) * f_B(x) = 1 + 0 - 0 = 1)$
- Let $x \notin A \wedge x \in B \Rightarrow x \in (A \oplus B). f_{A \oplus B} = 1 = (f_A(x) + f_B(x) - 2 * f_A(x) * f_B(x) = 0 + 1 - 0 = 1)$
- Let $x \notin A \wedge x \notin B \Rightarrow x \notin (A \oplus B). f_{A \oplus B} = 0 = (f_A(x) + f_B(x) - 2 * f_A(x) * f_B(x) = 0 + 0 - 0 = 0)$

6. Find the error in the “proof” of the following “theorem”.

“Theorem”: Let R be a relation on a set A that is symmetric and transitive. Then R is reflexive.
“Proof ”: Let $a \in A$. Take an element $b \in A$ such that $\langle a, b \rangle \in R$. Because R is symmetric, we also have $\langle b, a \rangle \in R$. Now using the transitive property, we can conclude that $\langle a, a \rangle \in R$ because $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$.

The above proof is incorrect for the following reason.
The case that does not take into account the proof is an empty relation, $\forall x, y \in A : (x, y) \notin R$.

Then the relation R is symmetric, since the left part of the implication is always false: $\forall x, y \in A : (xRy) \to (yRx)$.
Similarly, the relation R is transitive: $\forall x, y, z \in A : (xRy) \wedge (yRz) \to (xRz)$.

At the same time, the relation R is not reflexive, since $\forall x \in A : \neg(xRx)$

7. Give an example of a relation R on the set $\{a, b, c\}$ such that the symmetric closure of the reflexive closure of the transitive closure of R is not transitive.

$Let\ R = \{\langle a, b\rangle, \langle a, c\rangle\}$

$R^t = \{\langle a, b\rangle, \langle a, c\rangle\}$

$(R^t)^r = \{\langle a, b\rangle, \langle a, c\rangle, \langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle\}$

$((R^t)^r)^s) = \{\langle a, b\rangle, \langle a, c\rangle, \langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle b, a\rangle, \langle c, a\rangle\}$

$((R^t)^r)^s)$ is not transitive. Counterexample to the transitivity property: $\langle b, a\rangle \in ((R^t)^r)^s \wedge \langle a, c\rangle \in ((R^t)^r)^s$, but $\langle b, c\rangle \notin ((R^t)^r)^s$.

8. Prove or disprove the following statements about the functions f and g:

(a) If f and g are injections, then $g \circ f$ is also an injection.

- Let $f : A \to B,\ g : B \to C,\ g \circ f = g(f(x)) : A \to C$
- $\forall a, b : f(a) = f(b) \Leftrightarrow a = b$
- $\forall a, b : g(a) = g(b) \Leftrightarrow a = b$
- $\forall a, b : g(f(a)) = g(f(b)) \Rightarrow f(a) = f(b) \Rightarrow a = b \Rightarrow g \circ f\ is\ injection$

(b) If f and g are surjections, then $g \circ f$ is also a surjection.

- Let $f : A \to B,\ g : B \to C,\ g \circ f = g(f(x)) : A \to C$
- $\forall c \in C\ \exists\ b \in B : g(b) = c$
- $\forall b \in B\ \exists\ a \in A : f(a) = b$
- $\forall c \in C\ \forall\ b \in B\ \exists\ a \in A : g(f(a)) = g(b) = c \Rightarrow g \circ f\ is\ surjection$

(c) If f and $f \circ g$ are injections, then g is also an injection.

- Let $g : A \to B,\ f : B \to C,\ f \circ g = f(g(x)) : A \to C$
- $\forall a, b : f(a) = f(b) \Leftrightarrow a = b$
- $\forall a, b : f(g(a)) = f(g(b)) \Leftrightarrow g(a) = g(b)$
- g must be injective, otherwise, it would contradict the fact that $g \circ f$ is injective.

(d) If f and $f \circ g$ are surjections, then g is also a surjection

- Let $g : A \to B,\ f : B \to C,\ f \circ g = f(g(x)) : A \to C$
- $\forall c \in C\ \exists\ b \in B : f(b) = c$
- A counterexample about the surjectivity of g: $A = \{1, 2\}, B = \{1, 2\}, C = \{0\}$.

$g(1) = g(2) = f(1) = f(2) = 0$. Since, f and $f \circ g$ are surjections, but g is not.

9. Let $H = \{1, 2, 4, 5, 10, 12, 20\}$. Consider a divisibility relation $R \subseteq H^2$ defined as follows: $xRy \leftrightarrow y \vdots x$.

(a) Sort R (as a set of pairs) lexicographically.

$R = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 4\rangle, \langle 1, 5\rangle, \langle 1, 10\rangle, \langle 1, 12\rangle, \langle 1, 20\rangle, \langle 2, 2\rangle, \langle 2, 4\rangle, \langle 2, 10\rangle, \langle 2, 12\rangle,$
$\langle 2, 20\rangle, \langle 4, 4\rangle, \langle 4, 12\rangle, \langle 4, 20\rangle, \langle 5, 5\rangle, \langle 5, 10\rangle, \langle 5, 20\rangle, \langle 10, 10\rangle, \langle 10, 20\rangle, \langle 12, 12\rangle, \langle 20, 20\rangle\}$

(b) Show that R is a partial order.

Using the divisibility properties of real numbers:

- $\forall x \in H : (x, x) \in R \Leftrightarrow (x \vdots x \Leftrightarrow \frac{x}{x} = 1) \Rightarrow reflexive$
- $\forall x, y \in H : (x, y) \in R \land (y, x) \in R \Rightarrow (x \vdots y \land y \vdots x) \Rightarrow (x = y) \Rightarrow antisymmetric$
- $\forall x, y, z \in H : (x, y) \in R \land (y, z) \in R \Rightarrow (y \vdots x \land z \vdots y)$

$y \vdots x \Rightarrow$ all exponents appearing in x prime factorization contains in y prime factorization

$z \vdots y \Rightarrow$ all exponents appearing in y prime factorization contains in z prime factorization

$\Rightarrow$ all exponents appearing in x prime factorization contains in z prime factorization $\Rightarrow z \vdots x \Rightarrow (x, z) \in R \Rightarrow transitive$
$\Rightarrow R\, is\, a\, partial\, order$

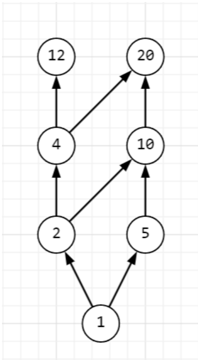
(c) Determine whether R is a linear (total) order.

- Counterexample to the connex property: $x = 10,\ y = 12 : (x, y) \notin R \land (y, x) \notin R$

$\Rightarrow R\, is\, not\, a\, total\, order$

(d) Draw the Hasse diagram for a graded poset $\langle H, R, \rho \rangle$, where $\rho : H \to \mathbb{N}_0$ is a grading function which maps a number $n \in H$ to the sum of all exponents appearing in its prime factorization, e.g., ρ (20) = $\rho(2^2 * 5^1) = 2 + 1 = 3$, so the number 20 would have the 3rd rank (bottom-up).

- $\rho(1) = \rho(1^0) = 0$
- $\rho(2) = \rho(2^1) = 1$
- $\rho(5) = \rho(5^1) = 1$
- $\rho(4) = \rho(2^2) = 2$
- $\rho(10) = \rho(2^1 * 5^1) = 2$
- $\rho(12) = \rho(2^2 * 3^1) = 3$
- $\rho(20) = \rho(2^2 * 5^1) = 3$



Hasse diagram for $\langle H, R, \rho \rangle$

(e) Find the minimal, minimum (least), maximal and maximum (greatest) elements in the poset $\langle H, R \rangle$. If there are multiple or none, explain why.

- $1 \in H\, is\, minimal \leftrightarrow \forall b \neq 1 : (b, 1) \notin R$
- $12 \in H\, is\, maximal \leftrightarrow \forall b \neq 12 : (12, b) \notin R$, at the same time, $20 \in H\, is\, maximal \leftrightarrow \forall b \neq 20 : (20, b) \notin R$
- $greatest$ element doesn't exist, because $(12, 20) \notin R$
- $1 \in H\, is\, least \leftrightarrow \forall b : (1, b) \in R$

(f) Perform a topological sort of the poset $\langle H, R \rangle$

$H_{topological\_sorted} = \{1, 5, 2, 10, 4, 20, 12\}$

10. Prove that the transitive closure R^+ is in fact transitive.

Definition. $R^+ = \cup_{n \in \mathbb{N}^+} R^n$ is a transitive closure of $R \subseteq M^2$, where

- $R^{k+1} = R^k \circ R$ is a compositional (functional) power,

- $R^1 = R$,
- $S \circ R = \{\langle x, y\rangle | \exists z : (xRz) \wedge (zSy)\}$ is a composition (relative product) of relations R and S.

Let $(a, b) \in R^+ \wedge (b, c) \in R^+$. Then exist are $j, k \in [1, n] \in \mathbb{N} \,|(a, b) \in R^j, (b, c) \in R^k$. Hence $(a, c) \in R^j \circ R^k = R^{j+k}$. Since $R^{j+k} \subseteq R^+, (a, c) \in R^+ \Rightarrow R^+ \ is\ transitive.$

11. Prove that a set S is infinite if and only if there is a proper subset $A \subset S$ such that there is a one to-one correspondence (bijection) between A and S.

- Let S in infinite set. Since all infinite sets have counable infinite subset, we can construct countable infinite subset of S. $A = \{a_1, a_2, a_3, \ldots\}$.
- Let's define two partitions of A as follows: $A_1 = \{a_2, a_4, a_6, a_8, \ldots\},\ A_2 = \{a_1, a_3, a_5, a_7, \ldots\}$
- We can establish a one-to-one correspondence between A and A_1 like that : $a_n = a_{2n}$.
- Then we can assert that exist a bijection between $A \cup (S \backslash A) = S$ and $A_1 \cup (S \backslash A) = S \backslash A_2$ (mapping each element in $S \backslash A$ to itself). Since $S \backslash A_2 \subset S$, there is bijection between infinite set and it's proper subset.
- Let $A \subset S$, $f : A \to S$ is bijection. Using theorem about no bijection between finite set and proper subset, S must be infinite.

12. Given a set S and two partitions P_1 and P_2 of S, we define the relation $P_1 \preceq P_2$ as follows: partition P_1 is considered a refinement of the partition P_2 if every set in P_1 is a subset of one of the sets in P_2. Show that the set of all partitions of a set S with the refinement relation $\preceq$ is a lattice.

- To show that the set of all partitions of a set S with the refinement relation $\preceq$ is a lattice, we need to show that it has a least upper bound and a greatest lower bound for every pair of partitions.
- Least Upper Bound: Let P_1 and P_2 are two partitions of S. To find the supremum of P_1 and P_2, we need to find a partition P such that $P_1 \preceq P$ and $P_2 \preceq P$, and for any other partition Q, if $P_1 \preceq Q$ and $P_2 \preceq Q$, then $P \preceq Q$.
  - We can construct P by taking the union of each pair of corresponding sets from P_1 and P_2. For every set A in P_1 and every set B in P_2, we take the union of A and B and add it to P. $\Rightarrow$ every set in P_1 and P_2 is a subset of some set in P.
  - To show that $P_1 \preceq P$ and $P_2 \preceq P$, we can observe that every set in P_1 is a subset of itself and is also a subset of the union of corresponding sets in P_1 and P_2. Every set in P_2 is a subset of itself and is also a subset of the union of corresponding sets in P_1 and P_2.
  - To show that if $P_1 \preceq Q$ and $P_2 \preceq Q$, then $P \preceq Q$ for any other partition Q, we can observe that every set in P_1 and P_2 is a subset of some set in Q. Since P is the union of corresponding sets from P_1 and P_2, every set in P is also a subset of some set in Q. Hence, $P \preceq Q$.
  - $\Rightarrow$ the supremum of P_1 and P_2 exists.
- Greatest Lower Bound: Let P_1 and P_2 are two partitions of S. To find the infimum of P_1 and P_2, we need to find a partition P such that $P \preceq P_1$ and $P \preceq P_2$, and for any other partition Q, if $Q \preceq P_1$ and $Q \preceq P_2$, then $Q \preceq P$.
  - We can construct P by taking the intersection of each pair of corresponding sets from P_1 and P_2. For every set A in P_1 and every set B in P_2, we take the intersection of A and $B$and add it to P. $\Rightarrow$ every set in P is a subset of some set in P_1 and P_2.
  - To show $P \preceq P_1$ and $P \preceq P_2$, we can observe that every set in P is a subset of the corresponding sets in P_1 and P_2, since it is the intersection of those sets.
  - To show that if $Q \preceq P_1$ and $Q \preceq P_2$, then $Q \preceq P$ for any other partition Q, we can observe that since $Q \preceq P_1$ and $Q \preceq P_2$, every set in Q is a subset of some set in both P_1 and P_2. Since P is the intersection of corresponding sets from P_1 and P_2, every set in P is also a subset of some set in Q. Hence, $Q \preceq P$.
  - $\Rightarrow$ the infimum of P_1 and P_2 exists.
- $\Rightarrow$ the set of all partitions of a set S with the refinement relation ≤ is a lattice.